ADVANCES AND APPLICATIONS OF STOCHASTIC ITÔ-TAYLOR APPROXIMATION AND CHANGE OF TIME METHOD:

IN THE FINANCIAL SECTOR

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ABSTRACT

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IN THE FINANCIAL SECTOR

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In this thesis, we discuss two different approaches for the solution of stochastic differential equations (SDEs): Itô-Taylor method (IT-M) and change of time method (CT-M). First approach is an approximation in space-domain and the second one is a probabilistic transformation in time-domain. Both approaches may be considered to substitute SDEs for more “practical” representations and solutions. IT-M was most studied for one-dimensional SDEs. The main aim of this work is to extend the theory of one-dimensional IT-M to the higher-dimensional SDEs. After covering IT-M for systems of SDEs with uncorrelated Brownian motions, we also consider the systems of SDEs with correlated Brownian motions. Then, discretization schemes are given and prepared to solve the systems of SDEs. As for the second approach, CT-M is discussed briefly. After this, applications of CT-M and IT-M are considered, especially, for most famous models, e.g., Cox-Ingersoll-Ross model and Ornstein-Uhlenbeck model. As an application of IT-M, stochastic control problems are also considered. In order to get an expression for the gradient of sensitivity, Malliavin calculus is used. Throughout the thesis we provide examples from the financial sector. This thesis ends with a conclusion and an outlook to future studies.

Keywords: Change of time method, Itô-Taylor expansions, Systems of SDEs, Correlated Brownian motions, Financial mathematics, Malliavin calculus

ÖZ

STOKASTİK ITÔ-TAYLOR YAKLAŞIMLARININ VE ZAMANI DEĞİŞTİRME YÖNTEMİNİN GELİŞTİRİLMESİ VE FİNANSAL SEKTÖRE UYGULAMALARI

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Bu tezde, stokastik diferansiyel denklemlerin (SDD) çözümleri için iki farklı yaklaşım tartışılmaktadır: Itô-Taylor metodu (IT-M) ve zamanı değiştirme metodu. Birinci metot uzay bölgesinde bir yaklaşım ve ikinci metot ise zaman bölgesinde olasılıksal bir dönüşümdür. Her iki metot da SDD'leri ve onların çözümlerini daha "pratik" gösterimlerle temsil etme imkanı verir. Şimdiye kadar IT-M birboyutlu SDD'ler için yeterince incelenmiştir. Bu çalışmanın asıl amacı bir-boyutlu IT-M'nun teorisini çok-boyutlu SDD'lere genişletmektir. IT-M'nu aralarında korelasyon olmayan Brownian hareketleri ile ifade edilen SDD sistemleri için ele aldıktan sonra, aralarında korelasyon olan Brownian hareketleri ile ifade edilen SDD sistemleri için de ele alınmıştır. Sonra ayrıklaştırma şemaları verilmiştir ve SDD sistemlerini çözmek için kullanılmıştır. İkinci yaklaşım olan zamanı değiştirme metodu kısaca anlatılmıştır. Daha sonra, çok önemli modeller olan Cox-Ingersoll-Ross modeli ve Ornstein-Uhlenbeck modeli için uygulamalar yapılmıştır. IT-M'nın bir uygulaması olarak, stokastik kontrol problemleri de incelenmiştir. Hassasiyet duyarlılığı gradyantı için bir ifade elde edebilmek için Malliavin kalkülüs kullanılmıştır. Tez boyunca finansal sektörden örnekler verilmiştir. Bu çalışma bir değerlendirme ve gelecek çalışmalara bir bakış ile sonuçlandırılmıştır.

Anahtar Kelimeler: Zamanı değiştirme metodu, Itô-Taylor açılımları, SDD sistemleri, Korelasyonlu Brownian Hareketleri, Finansal matematik, Malliavin kalkülüs

To My Family

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CHAPTER 1

INTRODUCTION

During the past few decades, stochastic differential equations (SDEs) have become quite popular models in a variety of areas such as financial mathematics, actuarial sciences, physics, biology, geology, mechanics, astronomy and other fields of science and engineering. For example, in financial mathematics, fluctuating stock prices and option prices can be modeled by SDEs [13], or in physics, they are used to model the effect of thermal noise in electrical circuits and numerous kinds of disturbances in telecommunications systems [25, 34].

These random events began to be modeled with the discovery of Brownian motion by Robert Brown in 1827. He observed the motion of pollen particles in water, noting that the particles moved through the water. But he was not able to determine the mechanism that caused this motion. Thorvald N. Thiele explained Brownian motion in mathematical terms in a paper on the method of least squares published in 1880. Independently of Thorvald N. Thiele, Louis Bachelier, a French mathematician, modeled the stochastic process, Brownian motion, in his PhD thesis *"The Theory of Speculation"* (published in 1900). Since it is historically the first paper to use advanced mathematics in finance, he can be considered as a pioneer in the study of financial mathematics and stochastic processes. Then, Norbert Wiener (1894 – 1964) described Brownian motion as a continuous-time stochastic process, which is the reason why it is also called a Wiener process. However, the history of stochastic differential equations can be considered to have started with the Einstein's famous paper *"On the Motion of Small Particles Suspended in a Stationary Liquid, as Required by the Molecular Kinetic Theory of Heat"* (1905), where he presented a mathematical connection between microscopic random motion of particles and the macroscopic diffusion equation.

In modeling of stochastic integral equations, a problem arises from the integration based on the Brownian motion. As a consequence of irregular paths of Brownian motion, it is nowhere differentiable. This causes that one can not apply directly usual calculus rules and formulas to it. However, Kiyoshi Itô extended these rules and formulas from usual calculus to stochastic processes, which is called *Itô (stochastic) calculus* (1951). He published many papers on stochastic calculus [15–19]. His famous formula, *Itô Lemma*, [19] helps us to solve the SDEs. There is also other kind of stochastic calculus, *Stratonovich (stochastic) calculus*, prepared by Russian physicist R.L. Stratonovich in 1966 [35]. Stratonovich stochastic integral is the most common alternative to Itô stochastic integral. Unlike Itô integral, it leads to the chain rule and further famous formulas of ordinary calculus as special cases. However, in many real-world applications, such as modeling of stock prices under the non-arbitrage principle or pricing of options [38], it is more appropriate to use Itô calculus.

Unfortunately, many stochastic integrals can not be solved explicitly, emerging the need of stochastic numerical integration schemes [26]. So far, different approaches are proposed to solve SDEs numerically. Discretization of both time and space variables was proposed by Kushner (1977). But this was inefficient in higher-dimensional cases. In 1978, Boyce suggested the simulation of general random systems by Monte-Carlo methods. However, since this method does not use the special structure of SDEs, it is less useful. Kushner's Markov chain approach was extended in higher-order approximations in 1992 by Platen.

Taylor-series expansion is one of the way for finding approximate solutions. In literature, it can be seen that various studies have been conducted on the stochastic generalization of the Taylor formula, Itô-Taylor method (IT-M). The first generalization of this extension was firstly presented by Platen and Wagner [32]. Later, Platen and Kloeden derived and investigated stochastic Taylor expansion; for details, cf. [23]. For a further study in the light of the Platen and Kloeden, we may refer to [41].

Moreover, there is also a probabilistic method of transformation to solve SDEs: Change of Time Method (CT-M). History of CT-M can be considered to begin in 1940 by Doeblin although his sealed envelope was opened 60 years after when it was sent. CT-M was also introduced by Bochner (1949, 1955). Dambis and Dubins-Schwarz developed a theory of random time changes for semimartingales in the 1960s [21, 33]. Johnson and Shanno [20] studied pricing of options using time changed *stochastic volatility model* (SVM). German, Carr, Madan and Yor [6] used subordinated process to construct SVM for Lévy process.

In this work, we study two different methods to solve SDEs: IT-M and CT-M. After investigating IT-M for one-dimensional SDEs [23, 41], we extend the theory of IT-M to the systems of SDEs [42], which is our main purpose. In the multi-dimensional case, correlation between Brownian motions is also taken into account. We transform the systems of SDEs with correlated Brownian motions to the uncorrelated case. Then, we apply IT-M. In order to get numerical solutions of a system of SDEs, we focus on discretization schemes. After this, some basic results from CT-M are presented [1,2,36,41]. To price variance and volatility swaps, we apply CT-M to Cox-Ingersoll-Ross (CIR) model. As for IT-M applications, we consider CIR model, Ornstein-Uhlenbeck (OU) model and stochastic control problems. For CIR model and OU model, we obtain discretization schemes. IT-M is also used to approximate state equations of stochastic control problems. Before estimating the so-called sensitivity gradient, we apply Malliavin calculus to get an expression for the gradient of the cost functional.

In the preliminaries, presented in Chapter 2, we give the fundamental definitions and theorems of stochastic calculus. In Chapter 3, we explain how the Itô-Taylor schemes are constructed for one-dimensional SDEs. The Itô-Taylor schemes for the systems of SDEs and discretization schemes for both one-dimensional and higher-dimensional SDEs are covered in Chapter 4. Chapter 5 includes the CT-M for SDEs. In Chapter 6, after giving some applications of random time change to SDEs, we apply IT-M to systems of SDEs and stochastic control problems. We derive the gradient of the cost functional by using sensitivity with the help of Malliavin calculus. In the last chapter, we conclude and give an outlook to future studies.

CHAPTER 2

MATHEMATICAL PRELIMINARIES

Before we start with the main topic of this thesis, we cover some basic definitions and theorems of stochastic calculus [25, 30]. Throughout the thesis, we let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space. Here, Ω denotes a sample space, $\mathcal{F}$ is a σ-algebra, $(\mathcal{F}_t)_{t\geq 0}$ is a filtration and $\mathbb{P}$ is a probability measure [24]. We may also restrict us to smaller time intervals $[0, T]$ for some "maturity time" $T > 0$.

2.1 Brownian Motion

Definition 2.1. A *stochastic process* is a collection of random variables $(X_t : t \in \mathcal{T})$ defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

When the *parameter set* $\mathcal{T}$ is countable, i.e., $\mathcal{T} = \mathbb{N} = \{0, 1, 2, \ldots\}$, we say that the process is a *discrete parameter process.* If $\mathcal{T}$ is not countable, the process is said to have a *continuous parameter process.* Mostly, we take $\mathcal{T}$ as $\mathbb{R}_+ = [0, \infty)$. The index t represents time, and X_t can be considered as the "state" or the "position" of the process at time t.

Definition 2.2. *Brownian motion* is a real-valued stochastic process $(W_t : t \in [0, T])$ for some $T > 0$ with the following properties:

- *Independent increments*: For all $0 \leq t_0 < t_1 < \ldots < t_m \leq T$, the increments
$$W_{t_1} - W_{t_0}, W_{t_2} - W_{t_1}, \ldots, W_{t_m} - W_{t_{m-1}}$$
are independent.
- *Stationary increments*: If $0 \leq s < t \leq T$, the increment $W_t - W_s$ and $W_{t-s} - W_0$ have the same probability law.
- *Continuity of paths*: $\mathbb{P}$ a.s. the map $t \mapsto W_t$ is continuous.

Remark 2.1. A Brownian motion is called *standard Brownian motion* or *Wiener process* if

$$W_0 = 0 \;\; \mathbb{P} \text{ a.s.}, \qquad \mathbb{E}(W_t) = 0, \qquad \mathbb{E}(W_t^2) = t.$$

From now on, we assume that the Brownian motion is standard if nothing else is mentioned.

Definition 2.3. [5, 25] A *standard n-dimensional Brownian motion* $(\mathbb{Z}_t)_{t\geq 0}$ is an $\mathbb{R}^n$-valued stochastic process

$$\mathbb{Z}_t = (Z_t^1, Z_t^2, \ldots, Z_t^n)^T,$$

where Z_t^i and Z_t^j, for all $i \neq j$, are uncorrelated Brownian motions with the following properties of normalization:

$$(dZ_t^i)^2 = dt, \; dZ_t^i dt = 0, \; (dt)^2 = 0.$$

In fact, this implies the following rule for different components:

$$dZ_t^i dZ_t^j = \delta_{ij} dt \;\; (i, j = 1, 2, \ldots, n),$$

where δ_{ij} is the *Kronecker delta*:

$$\delta_{ij} = \begin{cases} 1 \text{ if } i = j, \\ 0 \text{ if } i \neq j. \end{cases}$$

Definition 2.4. [5] A process defined as $\mathbb{W}_t := (W_t^1, W_t^2, \ldots, W_t^n)^T$ is a *correlated Brownian motion* if

$$dW_t^i dW_t^j = \rho_{ij} dt \;\; (i, j = 1, 2, \ldots, n),$$

for a positive semi-definite matrix $\rho = (\rho_{ij})_{1\leq i,j\leq n}$ satisfying the following property:

$$\rho_{ii} = 1, \text{ and } \rho_{ij} = \rho_{ji} \in [-1, 1] \text{ for all } i \neq j.$$

2.2 Construction of the Itô Integral

We assume a simple population growth model as

$$\frac{dN_t}{dt} = a_t N_t, \qquad N_0 = c, \tag{2.1}$$

where N_t is the size of population at time t, a_t is the relative rate of growth at time t, and c is a given constant. In most cases, we completely do not know a_t since it is subject to some random environmental effects. For this reason, we can write a_t as

$$a_t = r_t + noise,$$

where r_t is nonrandom function and the exact behavior of the "noise" term is not known, only its probability distribution is known. So we can rewrite Eqn. (2.1) as

$$\frac{dN_t}{dt} = (r_t + noise) \cdot N_t$$

or, more generally, in equations of the form

$$\frac{dX_t}{dt} = a(t, X_t) + h(t, X_t) \cdot noise, \tag{2.2}$$

where a and h are some given functions. If we denote the *noise* term by a stochastic process, $(\xi_t)_{t\geq 0}$, we get

$$\frac{dX_t}{dt} = a(t, X_t) + h(t, X_t)\xi_t, \tag{2.3}$$

where ξ_t satisfies the following properties:

(i) $t_1 \neq t_2 \Longrightarrow \xi_{t_1}$ and ξ_{t_2} are independent $(t_1, t_2 \geq 0)$,

(ii) $\{\xi_t\}$ is stationary for all $t \geq 0$,

(iii) $E(\xi_t) = 0$ for all $t \geq 0$.

Let $0 \leq t_0 < t_1 < \ldots < t_m \leq T$ for any given $T > 0$, then a discrete version of Eqn. (2.3)

$$X_{\nu+1} - X_\nu = a(t_\nu, X_\nu)\Delta t_\nu + h(t_\nu, X_\nu)\xi_\nu \Delta t_\nu \quad (\nu = 0, 1, \ldots, m-1), \tag{2.4}$$

where

$$\xi_\nu = \xi_{t_\nu}, \qquad \Delta t_\nu = t_{\nu+1} - t_\nu.$$

For simplicity, we consider $\xi_\nu \Delta t_\nu =: \Delta V_\nu = \Delta V_{t_{\nu+1}} - \Delta V_{t_\nu}$, with $V_{t_0} = 0$, where $(V_t)_{t\geq 0}$ is some suitable stochastic process. It is clear that V_t satisfies the assumptions (i)-(iii) and a Brownian motion (Z_t) is the only such process with continuous paths. So, if we replace V_t by Z_t in Eqn. (2.4), we get

$$X_\nu = X_0 + \sum_{j=0}^{\nu-1} a(t_j, X_j)\Delta t_j + \sum_{j=0}^{\nu-1} h(t_j, X_j)\Delta Z_j. \tag{2.5}$$

Since the limit of the right hand side of the Eqn. (2.5) exists as $\Delta t_j \longrightarrow 0$, it can be rewritten as

$$X_t = X_{t_0} + \int_{t_0}^{t} a(s, X_s)ds + \text{“} \int_{t_0}^{t} h(s, X_s)dZ_s\text{”}. \tag{2.6}$$

The problem with this is that a Brownian motion (Z_t) is nowhere differentiable, so that we need to define

$$\int_S^T f(t, \omega)dZ_t(\omega) \quad \text{for all} \ \ 0 \leq S \leq T.$$

Firstly, we give some definitions.

Definition 2.5. Assume $\mathcal{C} = \mathcal{C}(S, T)$ to be the class of functions f with the variable (t, ω),

$$f : [0, \infty) \times \Omega \longrightarrow \mathbb{R},$$

such that

(i) $(t,\omega) \mapsto f(t,\omega)$ is $(\mathcal{B}\otimes\mathcal{F})-$measurable, where $\mathcal{B}$ denotes the Borel σ-algebra on $[0,\infty)$,

(ii) $f(t,\omega)$ is $\mathcal{F}_t-$adapted,

(iii) $E\big(\int_S^T f(t,\omega)^2 dt\big) < \infty$.

Definition 2.6. Assume that $(0 = t_0 < t_1 < \ldots < t_m = T)$ is a partition of the interval $[0,T]$. A function $\phi \in \mathcal{C}$ is called *elementary function* if it is of the form

$$\phi(t,\omega) = \sum_{j=0}^{m-1} e_j(\omega)\chi_{[t_j,t_{j+1})}(t),$$

where the random variables $e_j(\omega)$ $(j = 0,1,\ldots,m-1)$ are $\mathcal{F}_{t_j}$-measurable and $\chi_{[t_j,t_{j+1})(t)}$ is the indicator (characteristic) function.

Now, we can define **Itô integral for elementary function** as follows:

Definition 2.7. (**Itô integral for elementary function**) Let $(S = t_1 < t_2 < \ldots < t_m = T)$ be a partition of the interval $[S,T]$, then:

$$\int_S^T \phi(t,\omega)dZ_t(\omega) := \sum_{j=1}^{m-1} e_j(\omega)(Z_{t_{j+1}}(\omega) - Z_{t_j}(\omega)).$$

Lemma 2.1. (***The Itô isometry for elementary functions***) *If $\phi(t,\omega)$ is bounded and elementary, then it holds:*

$$E\bigg(\Big(\int_S^T \phi(t,\omega)dZ_t(\omega)\Big)^2\bigg) = E\Big(\int_S^T \phi^2(t,\omega)dt\Big).$$

The aim is now to extend the definition of Itô integral from elementary functions to all functions in $\mathcal{C}$. It can be done in several steps:

Step 1. *Let $g \in \mathcal{C}$ be bounded and $g(\cdot,\omega)$ continuous for each ω. Then, there exist elementary functions $(\phi_\upsilon)_{\upsilon\in\mathbb{N}} \in \mathcal{C}^{\mathbb{N}}$ defined as $\phi_\upsilon(t,\omega) = \sum_{j=1}^{m-1} g(t_j,\omega)\chi_{[t_j,t_{j+1})(t)}$ such that*

$$E\Big(\int_S^T (g-\phi_\upsilon)^2 dt\Big) \longrightarrow 0 \quad \textit{as } \upsilon \longrightarrow \infty.$$

Step 2. *Let $h \in \mathcal{C}$ be bounded. Then, there exist bounded functions $(g_\upsilon)_{\upsilon\in\mathbb{N}} \in \mathcal{C}^{\mathbb{N}}$ such that $g_\upsilon(\cdot,\omega)$ is continuous for all $\omega \in \Omega$ and $\upsilon \in \mathbb{N}$, and*

$$E\Big(\int_S^T (h-g_\upsilon)^2 dt\Big) \longrightarrow 0 \quad \textit{as } \upsilon \longrightarrow \infty.$$

Step 3. *Let $f \in \mathcal{C}$. Then there exist a sequence $(h_\upsilon)_{\upsilon\in\mathbb{N}} \in \mathcal{C}^{\mathbb{N}}$ such that h_υ is bounded for all $\upsilon \in \mathbb{N}$ and*

$$E\Big(\int_S^T (f-h_\upsilon)^2 dt\Big) \longrightarrow 0 \quad \textit{as } \upsilon \longrightarrow \infty.$$

Now using these results, it is time to define Itô integral for all functions in $\mathcal{C}$.

Definition 2.8. (**The Itô integral**) Let $f \in \mathcal{C}(S,T)$. Then, the **Itô integral** of f (from S to T) is defined by

$$\int_S^T f(t,\omega)dZ_t(\omega) = \lim_{\upsilon \to \infty} \int_S^T \phi_\upsilon(t,\omega)dZ_t(\omega),$$

where $(\phi_\upsilon)_{\upsilon\in\mathbb{N}}$ is a sequence of elementary functions such that

$$E\left(\int_S^T (f(t,\omega) - \phi_\upsilon(t,\omega))^2 dt\right) \longrightarrow 0 \quad \text{as } \upsilon \longrightarrow \infty.$$

Theorem 2.2. (***The Itô isometry***)

$$E\left(\left(\int_S^T f(t,\omega)dZ_t(\omega)\right)^2\right) = E\left(\int_S^T f^2(t,\omega)dt\right),$$

for each $f \in \mathcal{C}$.

Now, Eqn.(2.6), i.e.,

$$X_t = X_{t_0} + \int_{t_0}^t a(s,X_s)ds + \int_{t_0}^t h(s,X_s)dZ_s, \tag{2.7}$$

is well-defined and called a *one-dimensional stochastic* (*or Itô*) *integral equation.* Here, the coefficient functions $a(t,X_t)$ and $h(t,X_t)$ are called *drift* and *diffusion coefficients*, respectively. These functions need to be sufficiently smooth real-valued functions satisfying a linear growth bound. The first integral is the *Riemann integral* and the second integral is called the *Itô stochastic integral.* The differential form of Eqn. (2.7),

$$dX_t = a(t,X_t)dt + h(t,X_t)dZ_t \quad (t_0 \le t \le T), \tag{2.8}$$

is called a *one-dimensional stochastic* (*or Itô*) *differential equation* (*SDE*).

2.3 Itô Lemma

The Riemann integrals can be evaluated by using Fundamental Theorem of Calculus, Chain rule and Taylor series. For the Itô integral, it is needed to have similar rules and formulas. However, Itô Lemma, also called *chain rule of stochastic calculus*, acts in the capacity of three calculus theorems. It is based on the substitution rule, providing a methodology for the solution of Eqn. (2.8). Itô Lemma also implies that a full linearization is not generally possible if the variable of a smooth function f is a stochastic process, but a particular quadratic term stays as a remainder due to the processes' uncertainity or (co-)variation.

Lemma 2.3. (***The one-dimensional Itô Lemma*** [30]) *Let (X_t) be Itô process satisfying Eqn. (2.8) and $f : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ be a given bounded function in $C^2([0,\infty) \times \mathbb{R})$. Then,*

$$df(t, X_t) = \frac{\partial f}{\partial t}(t, X_t)dt + \frac{\partial f}{\partial x}(t, X_t)dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, X_t)(dX_t)^2, \qquad (2.9)$$

where $(dX_t)^2 = (dX_t) \cdot (dX_t)$ is computed according to the rules

$$dt \cdot dt = dt \cdot dZ_t = dZ_t \cdot dt = 0 \quad \text{and} \quad dZ_t \cdot dZ_t = dt.$$

Before we give the multi-dimensional Itô Lemma, we need to extend the theory of one-dimensional stochastic processes and equations to the higher-dimensional ones. For this reason, we consider the process $(\mathbb{X}_t)_{t\geq 0}$ in $\mathbb{R}^d$. Let $(\mathbb{Z}_t)_{t\geq 0}$ be a standard n-dimensional Brownian motion defined as $\mathbb{Z}_t = (Z_t^1, Z_t^2, \ldots, Z_t^n)^T$. Then, the k^{th} component of the vector-valued SDE is given by

$$dX_t^k = a_k dt + \sum_{j=1}^{n} h_{kj} dZ_t^j \qquad (k = 1, 2, \ldots, d),$$

where $a_k(t, \mathbb{X}_t)$ and $h_{kj}(t, \mathbb{X}_t)$ are *drift* and *diffusion coefficients*, respectively. We shortly put $\mathbb{A} := \mathbb{A}(t, \mathbb{X}_t) = (a_1(t, \mathbb{X}_t), \ldots, a_d(t, \mathbb{X}_t))^T$, $\mathbb{X}_t = (X_t^1, \ldots, X_t^d)^T$ and

$$\mathbb{H} := \mathbb{H}(t, \mathbb{X}_t) = \begin{pmatrix} h_{11}(t, \mathbb{X}_t) & \ldots & h_{1n}(t, \mathbb{X}_t) \\ \vdots & \ddots & \vdots \\ h_{d1}(t, \mathbb{X}_t) & \ldots & h_{dn}(t, \mathbb{X}_t) \end{pmatrix},$$

to get the following compact matrix formulation:

$$d\mathbb{X}_t = \mathbb{A}dt + \mathbb{H}d\mathbb{Z}_t. \qquad (2.10)$$

Lemma 2.4. (***The multi-dimensional Itô Lemma*** [30]) *Let $\mathbb{X}_t = (X_t^1, \ldots, X_t^d)^T$ be a vector-valued Itô process satisfying Eqn. (2.10). Let $g : [0,\infty) \times \mathbb{R}^d \to \mathbb{R}^p$ be a given bounded function in $C^2([0,\infty) \times \mathbb{R}^d)$. Then,*

$$dg(t, \mathbb{X}_t) = \frac{\partial g}{\partial t}(t, \mathbb{X}_t)dt + \sum_{i=1}^{d} \frac{\partial g}{\partial x_i}(t, \mathbb{X}_t)dX_t^i + \frac{1}{2}\sum_{i,j=1}^{d} \frac{\partial^2 g}{\partial x_i x_j}(t, \mathbb{X}_t)dX_t^i dX_t^j, \quad (2.11)$$

where $dZ_t^i dZ_t^j = \delta_{ij} dt$ and $dZ_t^i dt = dt dZ_t^i = 0$.

CHAPTER 3

STOCHASTIC TAYLOR EXPANSIONS

Taylor-series expansion is one of the way for finding approximate solutions. In literature, it can be seen that various studies have been conducted on the stochastic generalization of the Taylor formula. The first generalization of this extension was firstly presented by Platen and Wagner in [32]. Later, Platen and Kloeden derived and investigated stochastic Taylor expansion; in detail, [23]. For a further study in the light of the Platen and Kloeden, we may refer to [41].

In the first section of this chapter, we derive deterministic Taylor expansions in detail to find approximate solutions of deterministic ordinary differential equations (ODEs). Once we understand Taylor series for deterministic case, it is easier to see the stochastic version of the Taylor series which we derive in the second section to understand how stochastic integration methods are designed.

3.1 Taylor Expansions for ODEs

In this section, we review how we can obtain a deterministic Taylor expansion [23]. We consider the solution (X_t) of the following one-dimensional ODE:

$$\frac{dX_t}{dt} = a(t, X_t) \quad (t_0 \leq t \leq T), \tag{3.1}$$

with initial value X_{t_0} for some $t_0 \in [0, T]$, where the function a is sufficiently smooth and has a linear growth bound. Eqn. (3.1) can be written in the equivalent integral equation form as

$$X_t = X_{t_0} + \int_{t_0}^{t} a(s, X_s) ds. \tag{3.2}$$

Lemma 3.1. *Let $\mathcal{L}$ be an operator defined as*

$$\mathcal{L} := \frac{\partial}{\partial t} + a\frac{\partial}{\partial x}. \tag{3.3}$$

Then, Taylor series expansion of Eqn. (3.2) has the following form:

$$\begin{aligned} X_t = X_{t_0} &+ a(t_0, X_{t_0})(t - t_0) + \frac{1}{2!}\mathcal{L}a(t_0, X_{t_0})(t - t_0)^2 \\ &+ \frac{1}{3!}\mathcal{L}^2 a(t_0, X_{t_0})(t - t_0)^3 + \ldots. \end{aligned}$$

Proof. Suppose $f : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function, then the evolution of the function f is governed by

$$\frac{df(t, X_t)}{dt} = \frac{\partial f(t, X_t)}{\partial t} + a(t, X_t)\frac{\partial f(t, X_t)}{\partial x} \tag{3.4}$$

via the chain rule. The integral form of this differential equation is

$$f(t, X_t) = f(t_0, X_{t_0}) + \int_{t_0}^{t} \left(\frac{\partial f(s, X_s)}{\partial s} + a(s, X_s)\frac{\partial f(s, X_s)}{\partial x} \right) ds, \tag{3.5}$$

where $f(t_0, X_{t_0})$ is the given initial condition. By defining a linear operator

$$\mathcal{L} = \frac{\partial}{\partial t} + a(t, X_t)\frac{\partial}{\partial x}, \tag{3.6}$$

we can rewrite Eqn. (3.5) as

$$f(t, X_t) = f(t_0, X_{t_0}) + \int_{t_0}^{t} \mathcal{L}f(s, X_s)ds, \tag{3.7}$$

for $t_0 \le t \le T$. Obviously, for $f(t, X_t) = X_t$ we have $\mathcal{L}f = a, \mathcal{L}^2 f = \mathcal{L}a, \ldots$, and Eqn. (3.7) reduces the original Eqn. (3.2):

$$X_t = X_{t_0} + \int_{t_0}^{t} a(s, X_s)ds. \tag{3.8}$$

If we now apply the relation of Eqn. (3.7) to the function $f = a$, we get

$$a(t, X_t) = a(t_0, X_{t_0}) + \int_{t_0}^{t} \mathcal{L}a(s, X_s)ds, \tag{3.9}$$

and substituting into Eqn. (3.8), we obtain

$$\begin{aligned} X_t &= X_{t_0} + \int_{t_0}^{t} \left(a(t_0, X_{t_0}) + \int_{t_0}^{s} \mathcal{L}a(z, X_z)dz \right) ds \\ &= X_{t_0} + a(t_0, X_{t_0}) \int_{t_0}^{t} ds + \int_{t_0}^{t} \int_{t_0}^{s} \mathcal{L}a(z, X_z)dzds, \end{aligned} \tag{3.10}$$

which is the simplest nontrivial Taylor expansion for X_t. We can also apply Eqn. (3.7) to the function $f = \mathcal{L}a$, herewith getting

$$\mathcal{L}a(t, X_t) = \mathcal{L}a(t_0, X_{t_0}) + \int_{t_0}^{t} \mathcal{L}^2 a(s, X_s)ds, \tag{3.11}$$

and substituting Eqn. (3.11) into Eqn. (3.10) leads to

$$
\begin{aligned}
X_t = X_{t_0} &+ a(t_0, X_{t_0}) \int_{t_0}^{t} ds + \mathcal{L}a(t_0, X_{t_0}) \int_{t_0}^{t} \int_{t_0}^{s} dz ds \\
&+ \int_{t_0}^{t} \int_{t_0}^{s} \int_{t_0}^{z} \mathcal{L}^2 a(u, X_u) du dz ds,
\end{aligned}
$$

for $t_0 \leq t \leq T$. If we continue this procedure to infinitum, we obtain the following Taylor series expansion for the solution of Eqn. (3.2):

$$
\begin{aligned}
X_t = X_{t_0} &+ a(t_0, X_{t_0})(t - t_0) + \frac{1}{2!}\mathcal{L}a(t_0, X_{t_0})(t - t_0)^2 \\
&+ \frac{1}{3!}\mathcal{L}^2 a(t_0, X_{t_0})(t - t_0)^3 + \ldots .
\end{aligned}
$$

□

3.2 Itô-Taylor Expansions

In this section, we refer to the one-dimensional SDE (2.8):

$$dX_t = a(t, X_t)dt + h(t, X_t)dZ_t \quad (t_0 \leq t \leq T), \tag{3.12}$$

with the integral form of Eqn. (3.12):

$$X_t = X_{t_0} + \int_{t_0}^{t} a(s, X_s)ds + \int_{t_0}^{t} h(s, X_s)dZ_s. \tag{3.13}$$

To construct Itô-Taylor Expansions for SDEs, we need to give some notations.

3.2.1 Multiple Stochastic Integrals

Definition 3.1. [23] For $m = 1, 2, 3, \ldots$, $j_i \in \{1, 2, \ldots, m\}$, and $i \in \{1, 2, \ldots, k\}$, a row vector

$$\alpha = (j_1, j_2, \ldots, j_k)$$

is called a *multi-index* of length

$$l := l(\alpha) \in \{1, 2, \ldots\}.$$

For example,

$$l((2, 0, 1)) = 3, \quad l((0, 1, 0, 3, 4)) = 5.$$

We note that the multi-index of length zero is denoted by v.
Let $\mathcal{M}$ denote the set of all multi-indices. For $\alpha \in \mathcal{M}$ with $l(\alpha) \geq 1$, $-\alpha$ and $\alpha-$ can be obtained by deleting first and last component, respectively, of α.
For example,

$$-(1, 3, 4, 0) = (3, 4, 0) \text{ and } (1, 3, 4, 0)- = (1, 3, 4).$$

Definition 3.2. [23] Let $f = (f(t) : t \geq 0)$ denote any adapted right-continuous stochastic process with left-hand limits existing. Then, we define the sets $\mathcal{H}_v$, $\mathcal{H}_{(0)}$ and $\mathcal{H}_{(1)}$ such that

- The set $\mathcal{H}_v$ represent all such processes f with
$$|f(t,\omega)| < \infty.$$
- The set $\mathcal{H}_{(0)}$ contains all those processes f with
$$\int_0^t |f(s,\omega)|ds < \infty.$$
- The set $\mathcal{H}_{(1)}$ contains all those processes f with
$$\int_0^t |f(s,\omega)|^2 ds < \infty.$$

Moreover, we define
$$\mathcal{H}_{(j)} := \mathcal{H}_{(1)}$$
for $j \in \mathbb{N}$ with $j \geq 2$.

Definition 3.3. [23] Let $\mathcal{H}_\alpha$ be a set for multi-indices $\alpha = (j_1, j_2, \ldots, j_k) \in \mathcal{M}$ with length $l(\alpha) > 1$. Assume ρ and τ be two stopping times with $0 \leq \rho(\omega) \leq \tau(\omega) \leq T$. Then, for a multi-index $\alpha \in \mathcal{M}$ and a process $f \in \mathcal{H}_\alpha$, *multiple Itô integral* $I_\alpha[f(\cdot)]_{\rho,\tau}$ is defined recursively by
$$I_\alpha[f(\cdot)]_{\rho,\tau} := \begin{cases} f(\tau), & \text{if } k = 0, \\ \int_\rho^\tau I_{\alpha^-}[f(\cdot)]_{\rho,s} ds, & \text{if } k \geq 1 \text{ and } j_k = 0, \\ \int_\rho^\tau I_{\alpha^-}[f(\cdot)]_{\rho,s} dZ_s^{j_k}, & \text{if } k \geq 1 \text{ and } j_k \geq 1; \end{cases}$$
i.e., integration takes place with respect to ds if $j_k = 0$, or $dZ_s^{j_k}$ if $j_k \neq 0$.

Here, the index i in dZ_s^i represents different Brownian motions. If there is only one Brownian motion, we can only say dZ_s. To understand better this definition, let us look at following examples:
$$\begin{aligned} I_v[f(\cdot)]_{t_0,t} &= f(t), \\ I_{(1)}[f(\cdot)]_{\tau_i,\tau_{i+1}} &= \int_{\tau_i}^{\tau_{i+1}} f(s) dZ_s^1, \\ I_{(0)}[f(\cdot)]_{\rho,\tau} &= \int_\rho^\tau f(s) ds, \\ I_{(0,2)}[f(\cdot)]_{t_0,t} &= \int_{t_0}^t \int_{t_0}^{s_2} f(s_1) ds_1 dZ_{s_2}^2, \\ I_{(1,2,0)}[f(\cdot)]_{t_0,t} &= \int_{t_0}^t \int_{t_0}^{s_3} \int_{t_0}^{s_2} f(s_1) dZ_{s_1}^1 dZ_{s_2}^2 ds_3, \end{aligned}$$
for an appropriate process f. We note that for a simpler notation we use $I_{i_1 i_2 \ldots i_k}$ taking $f(t) \equiv 1$.

3.2.2 Itô-Taylor Expansions for SDEs

Now, we are able to derive the Taylor series based on the solutions of SDEs, which are called *Itô-Taylor Expansions* [23]. It is similar to the Taylor series expansion for ODEs in the previous section. The only difference is the application of Itô Lemma. For this reason, it can be considered as an extension of deterministic Taylor series expansions.

Lemma 3.2. *Let $\mathcal{L}^0$ and $\mathcal{L}^1$ be operators defined as*

$$\mathcal{L}^0 := \frac{\partial}{\partial t} + a\frac{\partial}{\partial x} + \frac{1}{2}h^2\frac{\partial^2}{\partial x^2}$$

and

$$\mathcal{L}^1 := h\frac{\partial}{\partial x}.$$

Then, the Itô-Taylor approximation of Eqn. (3.13) looks as follows:

$$\begin{aligned} X_t = X_{t_0} &+ a(t_0, X_{t_0})I_0 + h(t_0, X_{t_0})I_1 + \mathcal{L}^0 a(t_0, X_{t_0})I_{00} + \mathcal{L}^1 a(t_0, X_{t_0})I_{10} \\ &+ \mathcal{L}^0 h(t_0, X_{t_0})I_{01} + \mathcal{L}^1 h(t_0, X_{t_0})I_{11} + \mathcal{L}^0\mathcal{L}^0 a(t_0, X_{t_0})I_{000} \\ &+ \mathcal{L}^1\mathcal{L}^0 a(t_0, X_{t_0})I_{100} + \mathcal{L}^0\mathcal{L}^1 a(t_0, X_{t_0})I_{010} + \mathcal{L}^1\mathcal{L}^1 a(t_0, X_{t_0})I_{110} \\ &+ \mathcal{L}^0\mathcal{L}^0 h(t_0, X_{t_0})I_{001} + \mathcal{L}^1\mathcal{L}^0 h(t_0, X_{t_0})I_{101} + \mathcal{L}^0\mathcal{L}^1 h(t_0, X_{t_0})I_{011} \\ &+ \mathcal{L}^1\mathcal{L}^1 h(t_0, X_{t_0})I_{111} + R_t, \end{aligned}$$

with remainder

$$\begin{aligned} R_t \;=\; & I_{0000}[\mathcal{L}^0\mathcal{L}^0\mathcal{L}^0 a]_{t_0,t} + I_{1000}[\mathcal{L}^1\mathcal{L}^0\mathcal{L}^0 a]_{t_0,t} + I_{0100}[\mathcal{L}^0\mathcal{L}^1\mathcal{L}^0 a]_{t_0,t} \\ &+ I_{1100}[\mathcal{L}^1\mathcal{L}^1\mathcal{L}^0 a]_{t_0,t} + I_{0010}[\mathcal{L}^0\mathcal{L}^0\mathcal{L}^1 a]_{t_0,t} + I_{1010}[\mathcal{L}^1\mathcal{L}^0\mathcal{L}^1 a]_{t_0,t} \\ &+ I_{0110}[\mathcal{L}^0\mathcal{L}^1\mathcal{L}^1 a]_{t_0,t} + I_{1110}[\mathcal{L}^1\mathcal{L}^1\mathcal{L}^1 a]_{t_0,t} + I_{0001}[\mathcal{L}^0\mathcal{L}^0\mathcal{L}^0 h]_{t_0,t} \\ &+ I_{1001}[\mathcal{L}^1\mathcal{L}^0\mathcal{L}^0 h]_{t_0,t} + I_{0101}[\mathcal{L}^0\mathcal{L}^1\mathcal{L}^0 h]_{t_0,t} + I_{1101}[\mathcal{L}^1\mathcal{L}^1\mathcal{L}^0 h]_{t_0,t} \\ &+ I_{0011}[\mathcal{L}^0\mathcal{L}^0\mathcal{L}^1 h]_{t_0,t} + I_{1011}[\mathcal{L}^1\mathcal{L}^0\mathcal{L}^1 h]_{t_0,t} + I_{0111}[\mathcal{L}^0\mathcal{L}^1\mathcal{L}^1 h]_{t_0,t} \\ &+ I_{1111}[\mathcal{L}^1\mathcal{L}^1\mathcal{L}^1 h]_{t_0,t}. \end{aligned} \tag{3.14}$$

Proof. Now, application of Lemma 2.3 to Eqn. (3.13) gives

$$\begin{aligned} f(t, X_t) \;=\; & f(t_0, X_{t_0}) + \int_{t_0}^{t} \frac{\partial f}{\partial s}(s, X_s)ds + \int_{t_0}^{t} \frac{\partial f}{\partial x}(s, X_s)dX_s \\ &+ \int_{t_0}^{t} \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(s, X_s)(dX_s)^2 \\ =\; & f(t_0, X_{t_0}) \\ &+ \int_{t_0}^{t} \left(\frac{\partial f}{\partial s}(s, X_s) + a(s, X_s)\frac{\partial f}{\partial x}(s, X_s) + \frac{1}{2}h^2(s, X_s)\frac{\partial^2 f}{\partial x^2}(s, X_s) \right) ds \\ &+ \int_{t_0}^{t} h(s, X_s)\frac{\partial f}{\partial x}(s, X_s)dZ_s. \end{aligned} \tag{3.15}$$

If we define the operators

$$\mathcal{L}^0 = \frac{\partial}{\partial t} + a\frac{\partial}{\partial x} + \frac{1}{2}h^2\frac{\partial^2}{\partial x^2}$$

and

$$\mathcal{L}^1 = h\frac{\partial}{\partial x},$$

then, we can rewrite Eqn. (3.15) as

$$f(t, X_t) = f(t_0, X_{t_0}) + \int_{t_0}^{t} \mathcal{L}^0 f(s, X_s)ds + \int_{t_0}^{t} \mathcal{L}^1 f(s, X_s)dZ_s. \tag{3.16}$$

We continue with the usage of Itô Lemma for the terms in the integrals. In Eqn. (3.16), if we choose $f(t, X_t) = X_t$, then Eqn. (3.16) becomes

$$X_t = X_{t_0} + \int_{t_0}^{t} a(s, X_s)ds + \int_{t_0}^{t} h(s, X_s)dZ_s, \tag{3.17}$$

which is the original SDE of Eqn. (3.13). For $f(t, X_t) = a(t, X_t)$, Eqn. (3.16) becomes

$$a(t, X_t) = a(t_0, X_{t_0}) + \int_{t_0}^{t} \mathcal{L}^0 a(s, X_s)ds + \int_{t_0}^{t} \mathcal{L}^1 a(s, X_s)dZ_s. \tag{3.18}$$

Similarly, after choosing $f(t, X_t) = h(t, X_t)$, Eqn. (3.16) follows in the form

$$h(t, X_t) = h(t_0, X_{t_0}) + \int_{t_0}^{t} \mathcal{L}^0 h(s, X_s)ds + \int_{t_0}^{t} \mathcal{L}^1 h(s, X_s)dZ_s. \tag{3.19}$$

Substituting Eqns. (3.18) and (3.19) into Eqn. (3.17) implies that

$$\begin{aligned} X_t = X_0 \quad &+ \int_{t_0}^{t} \Big[a(t_0, X_{t_0}) + \int_{t_0}^{s} \mathcal{L}^0 a(\tau, X_\tau)d\tau + \int_{t_0}^{s} \mathcal{L}^1 a(\tau, X_\tau)dZ_\tau\Big]ds \\ &+ \int_{t_0}^{t} \Big[h(t_0, X_{t_0}) + \int_{t_0}^{s} \mathcal{L}^0 h(\tau, X_\tau)d\tau + \int_{t_0}^{s} \mathcal{L}^1 h(\tau, X_\tau)dZ_\tau\Big]dZ_s. \end{aligned}$$

Similarly, we will apply Itô Lemma for $f(t, X_t) = \mathcal{L}^0 a(t, X_t)$, $f(t, X_t) = \mathcal{L}^1 a(t, X_t)$,

$f(t, X_t) = \mathcal{L}^0 h(t, X_t)$ and $f(t, X_t) = \mathcal{L}^1 h(t, X_t)$ to get

$$\begin{aligned}
X_t = X_{t_0} &+ a(t_0, X_{t_0}) \int_{t_0}^{t} ds + h(t_0, X_{t_0}) \int_{t_0}^{t} dZ_s \\
&+ \mathcal{L}^0 a(t_0, X_{t_0}) \int_{t_0}^{t} \int_{t_0}^{s} d\tau ds + \mathcal{L}^1 a(t_0, X_{t_0}) \int_{t_0}^{t} \int_{t_0}^{s} dZ_\tau ds \\
&+ \mathcal{L}^0 h(t_0, X_{t_0}) \int_{t_0}^{t} \int_{t_0}^{s} d\tau dZ_s + \mathcal{L}^1 h(t_0, X_{t_0}) \int_{t_0}^{t} \int_{t_0}^{s} dZ_\tau dZ_s \\
&+ \int_{t_0}^{t} \int_{t_0}^{s} \Big[\int_{t_0}^{\tau} \mathcal{L}^0 \mathcal{L}^0 a(z, X_z) dz + \int_{t_0}^{\tau} \mathcal{L}^1 \mathcal{L}^0 a(z, X_z) dZ_z \Big] d\tau ds \\
&+ \int_{t_0}^{t} \int_{t_0}^{s} \Big[\int_{t_0}^{\tau} \mathcal{L}^0 \mathcal{L}^1 a(z, X_z) dz + \int_{t_0}^{\tau} \mathcal{L}^1 \mathcal{L}^1 a(z, X_z) dZ_z \Big] dZ_\tau ds \\
&+ \int_{t_0}^{t} \int_{t_0}^{s} \Big[\int_{t_0}^{\tau} \mathcal{L}^0 \mathcal{L}^0 h(z, X_z) dz + \int_{t_0}^{\tau} \mathcal{L}^1 \mathcal{L}^0 h(z, X_z) dZ_z \Big] d\tau dZ_s \\
&+ \int_{t_0}^{t} \int_{t_0}^{s} \Big[\int_{t_0}^{\tau} \mathcal{L}^0 \mathcal{L}^1 h(z, X_z) dz + \int_{t_0}^{\tau} \mathcal{L}^1 \mathcal{L}^1 h(z, X_z) dZ_z \Big] dZ_\tau dZ_s.
\end{aligned}$$

If we continue by iterating with the functions $\mathcal{L}^0\mathcal{L}^0 a(t, X_t), \mathcal{L}^1\mathcal{L}^0 a(t, X_t)$, $\mathcal{L}^0\mathcal{L}^1 a(t, X_t)$, $\mathcal{L}^1\mathcal{L}^1 a(t, X_t)$, $\mathcal{L}^0\mathcal{L}^0 h(t, X_t)$, $\mathcal{L}^1\mathcal{L}^0 h(t, X_t)$, $\mathcal{L}^0\mathcal{L}^1 h(t, X_t)$ and $\mathcal{L}^1\mathcal{L}^1 h(t, X_t)$, we obtain

$$\begin{aligned}
X_t = X_{t_0} &+ a(t_0, X_{t_0}) I_0 + h(t_0, X_{t_0}) I_1 + \mathcal{L}^0 a(t_0, X_{t_0}) I_{00} + \mathcal{L}^1 a(t_0, X_{t_0}) I_{10} \\
&+ \mathcal{L}^0 h(t_0, X_{t_0}) I_{01} + \mathcal{L}^1 h(t_0, X_{t_0}) I_{11} + \mathcal{L}^0\mathcal{L}^0 a(t_0, X_{t_0}) I_{000} \\
&+ \mathcal{L}^1\mathcal{L}^0 a(t_0, X_{t_0}) I_{100} + \mathcal{L}^0\mathcal{L}^1 a(t_0, X_{t_0}) I_{010} + \mathcal{L}^1\mathcal{L}^1 a(t_0, X_{t_0}) I_{110} \\
&+ \mathcal{L}^0\mathcal{L}^0 h(t_0, X_{t_0}) I_{001} + \mathcal{L}^1\mathcal{L}^0 h(t_0, X_{t_0}) I_{101} + \mathcal{L}^0\mathcal{L}^1 h(t_0, X_{t_0}) I_{011} \\
&+ \mathcal{L}^1\mathcal{L}^1 h(t_0, X_{t_0}) I_{111} + R_t,
\end{aligned}$$

where R_t denotes the remainder term which can be expressed as in Eqn. (3.14).

Here, we note that $I_{i_1 i_2 \ldots i_k}$ represents the multiple Itô integrals with constant integrands and the terms $I_{i_1 i_2 \ldots i_{k+1}}[\mathcal{L}^{i_1}\mathcal{L}^{i_2} \ldots \mathcal{L}^{i_k} a]_{t_0,t}$ or $I_{i_1 i_2 \ldots i_{k+1}}[\mathcal{L}^{i_1}\mathcal{L}^{i_2} \ldots \mathcal{L}^{i_k} h]_{t_0,t}$ in R_t stand for the multiple Itô integrals with non-constant integrands.

□

Remark 3.1. Multiple Itô integrals appearing in the remainder terms can be shown to converge in the mean-square limit [23]. Moreover, convergence, how well the approximate solution converges to the true solution, can be found with the help of the moment estimation of the product of multiple Itô integrals.

CHAPTER 4

ITÔ-TAYLOR EXPANSIONS FOR SYSTEMS OF STOCHASTIC DIFFERENTIAL EQUATIONS AND APPROXIMATIONS

4.1 Introduction

In this chapter, Itô-Taylor expansions for the systems of SDEs are given. We consider both the systems of SDEs with standard Brownian Motions and the systems of SDEs having correlated Brownian Motions. We directly apply Itô-Taylor formula to the systems of SDEs with standard Brownian Motions. But the correlated ones are first transformed to the ones having uncorrelated standard Brownian motions, and then Itô Lemma is applied to get the Itô-Taylor expansions [42]. Finally, we give related discretization schemes to get an approximate solution. In order to measure the order of convergence, we use special forms of Itô-Taylor expansions such as Euler scheme and Milstein scheme. Throughout this chapter we refer to Eqn. (2.10), i.e.,

$$d\mathbb{X}_t = \mathbb{A}dt + \mathbb{H}d\mathbb{Z}_t. \tag{4.1}$$

We consider now the k^{th} component of the system of SDEs given by Eqn. (4.1):

$$dX_t^k = a_k(t, \mathbb{X}_t)dt + \sum_{j=1}^{n} h_{kj}(t, \mathbb{X}_t)dZ_t^j \quad (k = 1, 2, \ldots, d),$$

where $a_k(t, \mathbb{X}_t) = a_k(t, X_t^1, \ldots, X_t^d)$ and $h_{kj}(t, \mathbb{X}_t) = h_{kj}(t, X_t^1, \ldots, X_t^d)$.

For applying the Itô Lemma of Eqn. (2.11), we let $\mathbb{Y}_t = (Y_t^1, Y_t^2, \ldots, Y_t^p)^T$ defined by $Y_t = g(t, \mathbb{X}_t)$ and $g(t, \mathbb{X}_t) = (g^1(t, \mathbb{X}_t), \ldots, g^p(t, \mathbb{X}_t))^T$.

Then, the component Y_t^ℓ is given by

$$dY_t^\ell = \frac{\partial g^\ell}{\partial t}dt + \sum_{i=1}^{d} \frac{\partial g^\ell}{\partial x^i}dX_t^i + \frac{1}{2}\sum_{i,j=1}^{d} \frac{\partial^2 g^\ell}{\partial x^i x^j}dX_t^i dX_t^j, \tag{4.2}$$

where

$$dX_t^i dX_t^j = \sum_{i,j=1}^{d} \sum_{p=1}^{n} h_{jp}h_{ip}dt. \tag{4.3}$$

After substituting Eqn. (4.3) into Eqn. (4.2), we get

$$dY_t^\ell = \left(\frac{\partial g^\ell}{\partial t} + \sum_{i=1}^{d} a_i \frac{\partial g^\ell}{\partial x^i} + \frac{1}{2}\sum_{i,j=1}^{d}\sum_{p=1}^{n} h_{jp}h_{ip}\frac{\partial^2 g^\ell}{\partial x^i x^j}\right)dt + \sum_{j=1}^{d}\sum_{p=1}^{n} h_{jp}\frac{\partial g^\ell}{\partial x^j} dZ_t^p, \quad (4.4)$$

where all derivatives of g^ℓ are to be evaluated in $(t, \mathbb{X}_t)$ and the Brownian motions are uncorrelated.

4.2 Itô-Taylor Approximation for Standard Brownian Motions

In this section, we let $\mathbb{Z}_t = (Z_t^1, Z_t^2, \ldots, Z_t^n)^T$ be a standard n-dimensional Brownian motion. Eqn. (4.1) can be written in integral form as

$$\mathbb{X}_t = \mathbb{X}_{t_0} + \int_{t_0}^{t} \mathbb{A}(s, \mathbb{X}_s)ds + \int_{t_0}^{t} \mathbb{H}(s, \mathbb{X}_s)d\mathbb{Z}_s. \quad (4.5)$$

The system of SDEs can be classified with respect to the criterion of (non-)shared states and (un-)correlated (or (un-)standardized) Brownian motions. We give Itô-Taylor approximation for each classification in the following subsections.

4.2.1 Completely Decoupled Systems

We consider the following system

$$dX_t^k = a_k(t, X_t^k)dt + h_{kk}(t, X_t^k)dZ_t^k \quad (k = 1, 2, \ldots, d), \quad (4.6)$$

where all equations have their own Brownian motions and their own states; this also implies $d = n$.

Lemma 4.1. *Let $\mathcal{L}^0$ and $\mathcal{L}^j$ be operators defined as*

$$\mathcal{L}^0 := \frac{\partial}{\partial t} + \sum_{i=1}^{d} a_i \frac{\partial}{\partial x^i} + \frac{1}{2}\sum_{i,j=1}^{d}\sum_{p=1}^{n} h_{jp}h_{ip}\frac{\partial^2}{\partial x^i x^j}$$

and

$$\mathcal{L}^j := \sum_{p=1}^{d} h_{pj}\frac{\partial}{\partial x^p} \quad (j = 1, 2, \ldots, n).$$

Then, the Itô-Taylor approximation of Eqn. (4.6) can be written as:

$$
\begin{aligned}
X_t^k = X_{t_0}^k &+ a_k(t_0, X_{t_0}^k) I_0 + h_{kk}(t_0, X_{t_0}^k) I_k \\
&+ \mathcal{L}^0 a_k(t_0, X_{t_0}^k) I_{00} + \sum_{j=1}^{n} \mathcal{L}^j a_k(t_0, X_{t_0}^k) I_{j0} \\
&+ \mathcal{L}^0 h_{kk}(t_0, X_{t_0}^k) I_{0k} + \sum_{j=1}^{n} \mathcal{L}^j h_{kk}(t_0, X_{t_0}^k) I_{jk} \\
&+ \mathcal{L}^0 \mathcal{L}^0 a_k(t_0, X_{t_0}^k) I_{000} + \sum_{j=1}^{n} \mathcal{L}^j \mathcal{L}^0 a_k(t_0, X_{t_0}^k) I_{j00} \\
&+ \sum_{j=1}^{n} \mathcal{L}^0 \mathcal{L}^j a_k(t_0, X_{t_0}^k) I_{0j0} + \sum_{j,p=1}^{n} \mathcal{L}^j \mathcal{L}^p a_k(t_0, X_{t_0}^k) I_{jp0} \\
&+ \mathcal{L}^0 \mathcal{L}^0 h_{kk}(t_0, X_{t_0}^k) I_{00k} + \sum_{j=1}^{n} \mathcal{L}^j \mathcal{L}^0 h_{kk}(t_0, X_{t_0}^k) I_{j0k} \\
&+ \sum_{j=1}^{n} \mathcal{L}^0 \mathcal{L}^j h_{kk}(t_0, X_{t_0}^k) I_{0jk} + \sum_{j,p=1}^{n} \mathcal{L}^j \mathcal{L}^p h_{kk}(t_0, X_{t_0}^k) I_{jpk} + R_t,
\end{aligned}
$$

with

$$
\begin{aligned}
R_t \quad = \quad & I_{0000}[\mathcal{L}^0 \mathcal{L}^0 \mathcal{L}^0 a_k]_{t_0,t} + \sum_{j=1}^{n} I_{j000}[\mathcal{L}^j \mathcal{L}^0 \mathcal{L}^0 a_k]_{t_0,t} \\
&+ \sum_{j=1}^{n} I_{0j00}[\mathcal{L}^0 \mathcal{L}^j \mathcal{L}^0 a_k]_{t_0,t} + \sum_{j,p=1}^{n} I_{pj00}[\mathcal{L}^p \mathcal{L}^j \mathcal{L}^0 a_k]_{t_0,t} \\
&+ \sum_{j=1}^{n} I_{00j0}[\mathcal{L}^0 \mathcal{L}^0 \mathcal{L}^j a_k]_{t_0,t} + \sum_{j,p=1}^{n} I_{p0j0}[\mathcal{L}^p \mathcal{L}^0 \mathcal{L}^j a_k]_{t_0,t} \\
&+ \sum_{j,p=1}^{n} I_{0jp0}[\mathcal{L}^0 \mathcal{L}^j \mathcal{L}^p a_k]_{t_0,t} + \sum_{j,p,l=1}^{n} I_{ljp0}[\mathcal{L}^l \mathcal{L}^j \mathcal{L}^p a_k]_{t_0,t} \\
&+ I_{000k}[\mathcal{L}^0 \mathcal{L}^0 \mathcal{L}^0 h_{kk}]_{t_0,t} + \sum_{j=1}^{n} I_{j00k}[\mathcal{L}^j \mathcal{L}^0 \mathcal{L}^0 h_{kk}]_{t_0,t} \\
&+ \sum_{j=1}^{n} I_{0j0k}[\mathcal{L}^0 \mathcal{L}^j \mathcal{L}^0 h_{kk}]_{t_0,t} + \sum_{j,p=1}^{n} I_{pj0k}[\mathcal{L}^p \mathcal{L}^j \mathcal{L}^0 h_{kk}]_{t_0,t} \\
&+ \sum_{j=1}^{n} I_{00jk}[\mathcal{L}^0 \mathcal{L}^0 \mathcal{L}^j h_{kk}]_{t_0,t} + \sum_{j,p=1}^{n} I_{p0jk}[\mathcal{L}^p \mathcal{L}^0 \mathcal{L}^j h_{kk}]_{t_0,t} \\
&+ \sum_{j,p=1}^{n} I_{0jpk}[\mathcal{L}^0 \mathcal{L}^j \mathcal{L}^p h_{kk}]_{t_0,t} + \sum_{j,p,l=1}^{n} I_{ljpk}[\mathcal{L}^l \mathcal{L}^j \mathcal{L}^p h_{kk}]_{t_0,t}. \qquad (4.7)
\end{aligned}
$$

Proof. For Eqn. (4.6), Eqn. (4.4) becomes

$$dY_t^\ell = (\frac{\partial g^\ell}{\partial t} + \sum_{i=1}^n a_i \frac{\partial g^\ell}{\partial x^i} + \frac{1}{2}\sum_{i=1}^n h_{ii}h_{ii}\frac{\partial^2 g^\ell}{\partial x^i x^i})dt + \sum_{i=1}^n h_{ii}\frac{\partial g^\ell}{\partial x^i}dZ_t^i,$$

where $g(t, \mathbb{X}_t) = (X_t^1, \ldots, X_t^n)^T$. Herewith,

$$dX_t^k = a_k(t, X_t^k)dt + h_{kk}(t, X_t^k)dZ_t^k \tag{4.8}$$

and, in here, we note that ℓ and k represent the ℓ^{th} component of the function $g(t, \mathbb{X}_t)$ and the k^{th} component of the system of SDEs of Eqn. (4.8), respectively. The integral form of Eqn. (4.8) is

$$X_t^k = X_{t_0}^k + \int_{t_0}^t a_k(s, X_s^k)ds + \int_{t_0}^t h_{kk}(s, X_s^k)dZ_s^k. \tag{4.9}$$

We assume the operators $\mathcal{L}^0$ and $\mathcal{L}^j$ as stated in Lemma 4.1. This allows us to express the multi-dimensional version of the Itô Lemma in a compact way:

$$Y_t^\ell = Y_{t_0}^\ell + \int_{t_0}^t \mathcal{L}^0 g^\ell ds + \sum_{j=1}^n \int_{t_0}^t \mathcal{L}^j g^\ell dZ_s^j. \tag{4.10}$$

Now, we apply Eqn. (4.10) to the terms in Eqn. (4.9). If we choose $g^\ell = a_\ell(t, X_t^\ell)$, then Eqn. (4.10) becomes

$$a_k(t, X_t^k) = a_k(t_0, X_{t_0}^k) + \int_{t_0}^t \mathcal{L}^0 a_k(s, X_s^k)ds + \sum_{j=1}^n \int_{t_0}^t \mathcal{L}^j a_k(s, X_s^k)dZ_s^j. \tag{4.11}$$

Similarly, for $g^\ell = h_{\ell\ell}(t, X_t^\ell)$ Eqn. (4.10) becomes

$$h_{kk}(t, X_t^k) = h_{kk}(t_0, X_{t_0}^k) + \int_{t_0}^t \mathcal{L}^0 h_{kk}(s, X_s^k)ds + \sum_{j=1}^n \int_{t_0}^t \mathcal{L}^j h_{kk}(s, X_s^k)dZ_s^j. \tag{4.12}$$

After substituting Eqns. (4.11) and (4.12) into Eqn. (4.9), we get

$$\begin{aligned}
X_t^k = X_{t_0}^k &+ \int_{t_0}^t \big[a_k(t_0, X_{t_0}^k) + \int_{t_0}^s \mathcal{L}^0 a_k(\tau, X_\tau^k)d\tau \\
&+ \sum_{j=1}^n \int_{t_0}^s \mathcal{L}^j a_k(\tau, X_\tau^k)dZ_\tau^j\big]ds + \int_{t_0}^t \big[h_{kk}(t_0, X_{t_0}^k) \\
&+ \int_{t_0}^s \mathcal{L}^0 h_{kk}(\tau, X_\tau^k)d\tau + \sum_{j=1}^n \int_{t_0}^s \mathcal{L}^j h_{kk}(\tau, X_\tau^k)dZ_\tau^j\big]dZ_s^k, \\
= X_{t_0}^k &+ a_k(t_0, X_{t_0}^k)I_0 + h_{kk}(t_0, X_{t_0}^k)I_k \\
&+ \int_{t_0}^t \big[\int_{t_0}^s \mathcal{L}^0 a_k(\tau, X_\tau^k)d\tau + \sum_{j=1}^n \int_{t_0}^s \mathcal{L}^j a_k(\tau, X_\tau^k)dZ_\tau^j\big]ds \qquad (4.13) \\
&+ \int_{t_0}^t \big[\int_{t_0}^s \mathcal{L}^0 h_{kk}(\tau, X_\tau^k)d\tau + \sum_{j=1}^n \int_{t_0}^s \mathcal{L}^j h_{kk}(\tau, X_\tau^k)dZ_\tau^j\big]dZ_s^k.
\end{aligned}$$

We can continue with an application of Itô Lemma of Eqn. (4.10) to the functions $\mathcal{L}^0 a_k$, $\sum_{j=1}^n \mathcal{L}^j a_k$, $\mathcal{L}^0 h_{kk}$ and $\sum_{j=1}^n \mathcal{L}^j h_{kk}$ in (4.13) and then, to the functions $\mathcal{L}^0\mathcal{L}^0 a_k$, $\sum_{j=1}^n \mathcal{L}^j\mathcal{L}^0 a_k$, $\sum_{j=1}^n \mathcal{L}^0\mathcal{L}^j a_k$, $\sum_{j,p=1}^n \mathcal{L}^j\mathcal{L}^p a_k$, $\mathcal{L}^0\mathcal{L}^0 h_{kk}$, $\sum_{j,p=1}^n \mathcal{L}^j\mathcal{L}^0 h_{kk}$, $\sum_{j=1}^n \mathcal{L}^0\mathcal{L}^j h_{kk}$, $\sum_{j,p=1}^n \mathcal{L}^j\mathcal{L}^p h_{kk}$, to obtain the Itô-Taylor expansion:

$$
\begin{aligned}
X_t^k = X_{t_0}^k &+ a_k(t_0, X_{t_0}^k) I_0 + h_{kk}(t_0, X_{t_0}^k) I_k \\
&+ \mathcal{L}^0 a_k(t_0, X_{t_0}^k) I_{00} + \sum_{j=1}^n \mathcal{L}^j a_k(t_0, X_{t_0}^k) I_{j0} \\
&+ \mathcal{L}^0 h_{kk}(t_0, X_{t_0}^k) I_{0k} + \sum_{j=1}^n \mathcal{L}^j h_{kk}(t_0, X_{t_0}^k) I_{jk} \\
&+ \mathcal{L}^0\mathcal{L}^0 a_k(t_0, X_{t_0}^k) I_{000} + \sum_{j=1}^n \mathcal{L}^j\mathcal{L}^0 a_k(t_0, X_{t_0}^k) I_{j00} \\
&+ \sum_{j=1}^n \mathcal{L}^0\mathcal{L}^j a_k(t_0, X_{t_0}^k) I_{0j0} + \sum_{j,p=1}^n \mathcal{L}^j\mathcal{L}^p a_k(t_0, X_{t_0}^k) I_{jp0} \\
&+ \mathcal{L}^0\mathcal{L}^0 h_{kk}(t_0, X_{t_0}^k) I_{00k} + \sum_{j=1}^n \mathcal{L}^j\mathcal{L}^0 h_{kk}(t_0, X_{t_0}^k) I_{j0k} \\
&+ \sum_{j=1}^n \mathcal{L}^0\mathcal{L}^j h_{kk}(t_0, X_{t_0}^k) I_{0jk} + \sum_{j,p=1}^n \mathcal{L}^j\mathcal{L}^p h_{kk}(t_0, X_{t_0}^k) I_{jpk} + R_t,
\end{aligned}
$$

where R_t stands for the remainder term which can be written as Eqn. (4.7).

Here, we note that $I_{i_1 i_2 \ldots i_k}$ represents the multiple Itô integrals with constant integrands, and the terms $I_{i_1 i_2 \ldots i_{k+1}}[\mathcal{L}^{i_1}\mathcal{L}^{i_2} \ldots \mathcal{L}^{i_k} a_k]_{t_0,t}$ or $I_{i_1 i_2 \ldots i_{k+1}}[\mathcal{L}^{i_1}\mathcal{L}^{i_2} \ldots \mathcal{L}^{i_k} h_{kk}]_{t_0,t}$ in R_t denote the multiple Itô integrals with nonconstant integrands.

□

4.2.2 Systems with Common States

Now, we consider the following system:

$$dX_t^k = a_k(t, \mathbb{X}_t) + h_{kk}(t, \mathbb{X}_t) dZ_t^k \quad (k = 1, 2, \ldots, d), \tag{4.14}$$

where all equations may have any states in common while they have their own Brownian motions. We obtain Itô-Taylor approximation of Eqn. (4.14) in the following Lemma.

Lemma 4.2. *Let $\mathcal{L}^0$ and $\mathcal{L}^j$ be operators defined as in Lemma 4.1. Then, Itô-Taylor approximation of Eqn. (4.14) can be written as:*

$$
\begin{aligned}
X_t^k = X_{t_0}^k \quad &+ \quad a_k(t_0,\mathbb{X}_{t_0})I_0 + h_{kk}(t_0,\mathbb{X}_{t_0})I_k \\
&+ \quad \mathcal{L}^0 a_k(t_0,\mathbb{X}_{t_0})I_{00} + \sum_{j=1}^{n}\mathcal{L}^j a_k(t_0,\mathbb{X}_{t_0})I_{j0} \\
&+ \quad \mathcal{L}^0 h_{kk}(t_0,\mathbb{X}_{t_0})I_{0k} + \sum_{j=1}^{n}\mathcal{L}^j h_{kk}(t_0,\mathbb{X}_{t_0})I_{jk} \\
&+ \quad \mathcal{L}^0\mathcal{L}^0 a_k(t_0,\mathbb{X}_{t_0})I_{000} + \sum_{j=1}^{n}\mathcal{L}^j\mathcal{L}^0 a_k(t_0,\mathbb{X}_{t_0})I_{j00} \\
&+ \quad \sum_{j=1}^{n}\mathcal{L}^0\mathcal{L}^j a_k(t_0,\mathbb{X}_{t_0})I_{0j0} + \sum_{j,p=1}^{n}\mathcal{L}^j\mathcal{L}^p a_k(t_0,\mathbb{X}_{t_0})I_{jp0} \\
&+ \quad \mathcal{L}^0\mathcal{L}^0 h_{kk}(t_0,\mathbb{X}_{t_0})I_{00k} + \sum_{j=1}^{n}\mathcal{L}^j\mathcal{L}^0 h_{kk}(t_0,\mathbb{X}_{t_0})I_{j0k} \\
&+ \quad \sum_{j=1}^{n}\mathcal{L}^0\mathcal{L}^j h_{kk}(t_0,\mathbb{X}_{t_0})I_{0jk} + \sum_{j,p=1}^{n}\mathcal{L}^j\mathcal{L}^p h_{kk}(t_0,\mathbb{X}_{t_0})I_{jpk} + R_t,
\end{aligned}
$$

with

$$
\begin{aligned}
R_t \quad = \quad & I_{0000}[\mathcal{L}^0\mathcal{L}^0\mathcal{L}^0 a_k]_{t_0,t} + \sum_{j=1}^{n} I_{j000}[\mathcal{L}^j\mathcal{L}^0\mathcal{L}^0 a_k]_{t_0,t} \\
+ \quad & \sum_{j=1}^{n} I_{0j00}[\mathcal{L}^0\mathcal{L}^j\mathcal{L}^0 a_k]_{t_0,t} + \sum_{j,p=1}^{n} I_{pj00}[\mathcal{L}^p\mathcal{L}^j\mathcal{L}^0 a_k]_{t_0,t} \\
+ \quad & \sum_{j=1}^{n} I_{00j0}[\mathcal{L}^0\mathcal{L}^0\mathcal{L}^j a_k]_{t_0,t} + \sum_{j,p=1}^{n} I_{p0j0}[\mathcal{L}^p\mathcal{L}^0\mathcal{L}^j a_k]_{t_0,t} \\
+ \quad & \sum_{j,p=1}^{n} I_{0jp0}[\mathcal{L}^0\mathcal{L}^j\mathcal{L}^p a_k]_{t_0,t} + \sum_{j,p,l=1}^{n} I_{ljp0}[\mathcal{L}^l\mathcal{L}^j\mathcal{L}^p a_k]_{t_0,t} \\
+ \quad & I_{000k}[\mathcal{L}^0\mathcal{L}^0\mathcal{L}^0 h_{kk}]_{t_0,t} + \sum_{j=1}^{n} I_{j00k}[\mathcal{L}^j\mathcal{L}^0\mathcal{L}^0 h_{kk}]_{t_0,t} \\
+ \quad & \sum_{j=1}^{n} I_{0j0k}[\mathcal{L}^0\mathcal{L}^j\mathcal{L}^0 h_{kk}]_{t_0,t} + \sum_{j,p=1}^{n} I_{pj0k}[\mathcal{L}^p\mathcal{L}^j\mathcal{L}^0 h_{kk}]_{t_0,t} \\
+ \quad & \sum_{j=1}^{n} I_{00jk}[\mathcal{L}^0\mathcal{L}^0\mathcal{L}^j h_{kk}]_{t_0,t} + \sum_{j,p=1}^{n} I_{p0jk}[\mathcal{L}^p\mathcal{L}^0\mathcal{L}^j h_{kk}]_{t_0,t} \\
+ \quad & \sum_{j,p=1}^{n} I_{0jpk}[\mathcal{L}^0\mathcal{L}^j\mathcal{L}^p h_{kk}]_{t_0,t} + \sum_{j,p,l=1}^{n} I_{ljpk}[\mathcal{L}^l\mathcal{L}^j\mathcal{L}^p h_{kk}]_{t_0,t}.
\end{aligned}
$$

Proof. It follows a similar logic and formulation as for the completely decoupled systems which we studied in Subsection 4.2.1. The only difference is that the drift and diffusion coefficients, a_k and h_{kk}, respectively, are evaluated in $(t, \mathbb{X}_t)$, not in (t, X_t), now.

□

After giving Itô-Taylor approximations for the special cases of systems of SDEs, i.e., completely decoupled systems of SDEs and systems with common states, we also need to consider more general case of systems of SDEs.

4.2.3 General Case

By allowing all equations to have in common both Brownian motions and states, we refer to the following system:

$$dX_t^k = a_k(t, \mathbb{X}_t) + \sum_{j=1}^{n} h_{kj}(t, \mathbb{X}_t) dZ_t^j \qquad (k = 1, 2, \ldots, d), \tag{4.15}$$

for which we obtain the subsequent result with the same logic as for the previous two cases..

Lemma 4.3. *Let $\mathcal{L}^0$ and $\mathcal{L}^j$ be operators defined as in Lemma 4.1. Then, the Itô-Taylor approximation of Eqn. (4.15) can be written as:*

$$\begin{aligned} X_t^k = X_{t_0}^k &+ a_k(t_0, \mathbb{X}_{t_0}) I_0 + \sum_{j=1}^{n} h_{kj}(t_0, \mathbb{X}_{t_0}) I_j \\ &+ \mathcal{L}^0 a_k(t_0, \mathbb{X}_{t_0}) I_{00} + \sum_{j=1}^{n} \mathcal{L}^j a_k(t_0, \mathbb{X}_{t_0}) I_{j0} \\ &+ \sum_{j=1}^{n} \mathcal{L}^0 h_{kj}(t_0, \mathbb{X}_{t_0}) I_{0j} + \sum_{l,j=1}^{n} \mathcal{L}^l h_{kj}(t_0, \mathbb{X}_{t_0}) I_{lj} \\ &+ \mathcal{L}^0 \mathcal{L}^0 a_k(t_0, \mathbb{X}_{t_0}) I_{000} + \sum_{j=1}^{n} \mathcal{L}^j \mathcal{L}^0 a_k(t_0, \mathbb{X}_{t_0}) I_{j00} \\ &+ \sum_{j=1}^{n} \mathcal{L}^0 \mathcal{L}^j a_k(t_0, \mathbb{X}_{t_0}) I_{0j0} + \sum_{j,p=1}^{n} \mathcal{L}^j \mathcal{L}^p a_k(t_0, \mathbb{X}_{t_0}) I_{jp0} \\ &+ \sum_{j=1}^{n} \mathcal{L}^0 \mathcal{L}^0 h_{kj}(t_0, \mathbb{X}_{t_0}) I_{00j} + \sum_{j,l=1}^{n} \mathcal{L}^l \mathcal{L}^0 h_{kj}(t_0, \mathbb{X}_{t_0}) I_{l0j} \\ &+ \sum_{j,l=1}^{n} \mathcal{L}^0 \mathcal{L}^l h_{kj}(t_0, \mathbb{X}_{t_0}) I_{0lj} + \sum_{j,p,l=1}^{n} \mathcal{L}^l \mathcal{L}^p h_{kj}(t_0, \mathbb{X}_{t_0}) I_{lpj} + R_t, \end{aligned} \tag{4.16}$$

with

$$
\begin{aligned}
R_t \;=\;& I_{0000}[\mathcal{L}^0\mathcal{L}^0\mathcal{L}^0 a_k]_{t_0,t} + \sum_{j=1}^{n} I_{j000}[\mathcal{L}^j\mathcal{L}^0\mathcal{L}^0 a_k]_{t_0,t} \\
&+ \sum_{j=1}^{n} I_{0j00}[\mathcal{L}^0\mathcal{L}^j\mathcal{L}^0 a_k]_{t_0,t} + \sum_{j,p=1}^{n} I_{pj00}[\mathcal{L}^p\mathcal{L}^j\mathcal{L}^0 a_k]_{t_0,t} \\
&+ \sum_{j=1}^{n} I_{00j0}[\mathcal{L}^0\mathcal{L}^0\mathcal{L}^j a_k]_{t_0,t} + \sum_{j,p=1}^{n} I_{p0j0}[\mathcal{L}^p\mathcal{L}^0\mathcal{L}^j a_k]_{t_0,t} \\
&+ \sum_{j,p=1}^{n} I_{0jp0}[\mathcal{L}^0\mathcal{L}^j\mathcal{L}^p a_k]_{t_0,t} + \sum_{j,p,l=1}^{n} I_{ljp0}[\mathcal{L}^l\mathcal{L}^j\mathcal{L}^p a_k]_{t_0,t} \\
&+ \sum_{j=1}^{n} I_{000j}[\mathcal{L}^0\mathcal{L}^0\mathcal{L}^0 h_{kj}]_{t_0,t} + \sum_{j,p=1}^{n} I_{p00j}[\mathcal{L}^p\mathcal{L}^0\mathcal{L}^0 h_{kj}]_{t_0,t} \\
&+ \sum_{j,l=1}^{n} I_{0l0j}[\mathcal{L}^0\mathcal{L}^l\mathcal{L}^0 h_{kj}]_{t_0,t} + \sum_{j,p,l=1}^{n} I_{pl0j}[\mathcal{L}^p\mathcal{L}^l\mathcal{L}^0 h_{kj}]_{t_0,t} \\
&+ \sum_{j=1}^{n} I_{00lj}[\mathcal{L}^0\mathcal{L}^0\mathcal{L}^l h_{kj}]_{t_0,t} + \sum_{j,p,l=1}^{n} I_{p0lj}[\mathcal{L}^p\mathcal{L}^0\mathcal{L}^l h_{kj}]_{t_0,t} \\
&+ \sum_{j,p,l=1}^{n} I_{0lpj}[\mathcal{L}^0\mathcal{L}^l\mathcal{L}^p h_{kj}]_{t_0,t} + \sum_{j,p,l,r=1}^{n} I_{rlpj}[\mathcal{L}^r\mathcal{L}^l\mathcal{L}^p h_{kj}]_{t_0,t}. \quad (4.17)
\end{aligned}
$$

Proof. We consider the integral form of Eqn. (4.15):

$$
X_t^k = X_{t_0}^k + \int_{t_0}^t a_k(s,\mathbb{X}_s)ds + \sum_{j=1}^{n}\int_{t_0}^t h_{kj}(s,\mathbb{X}_s)dZ_s^j. \quad (4.18)
$$

Again, we first choose $g^\ell := a_\ell(t,\mathbb{X}_t)$ to apply the Itô Lemma and, then, we let $g^\ell := h_{\ell j}(t,\mathbb{X}_t)$ to get

$$
\begin{aligned}
X_t^k = X_{t_0}^k \;&+\; \int_{t_0}^t \Bigg[a_k(t_0,\mathbb{X}_{t_0}) + \int_{t_0}^s \mathcal{L}^0 a_k(\tau,\mathbb{X}_\tau)d\tau \\
&+ \sum_{l,j=1}^{n}\int_{t_0}^s \mathcal{L}^j a_k(\tau,\mathbb{X}_\tau)dZ_\tau^j\Bigg]ds + \sum_{j=1}^{n}\int_{t_0}^t \Bigg[h_{kj}(t_0,\mathbb{X}_{t_0}) \\
&+ \int_{t_0}^s \mathcal{L}^0 h_{kj}(\tau,\mathbb{X}_\tau)d\tau + \sum_{l=1}^{n}\int_{t_0}^s \mathcal{L}^l h_{kj}(\tau,\mathbb{X}_\tau)dZ_\tau^l\Bigg]dZ_s^j.
\end{aligned}
$$

We follow the same arguments as for the completely decoupled case, with the only difference that a_k and h_{kj} depend on $(t, \mathbb{X}_t)$ now. Then,

$$\begin{aligned} X_t^k = X_{t_0}^k &+ a_k(t_0, \mathbb{X}_{t_0})I_0 + \sum_{j=1}^{n} h_{kj}(t_0, \mathbb{X}_{t_0})I_j \\ &+ \mathcal{L}^0 a_k(t_0, \mathbb{X}_{t_0})I_{00} + \sum_{j=1}^{n} \mathcal{L}^j a_k(t_0, \mathbb{X}_{t_0})I_{j0} \\ &+ \sum_{j=1}^{n} \mathcal{L}^0 h_{kj}(t_0, \mathbb{X}_{t_0})I_{0j} + \sum_{l,j=1}^{n} \mathcal{L}^l h_{kj}(t_0, \mathbb{X}_{t_0})I_{lj} \\ &+ \mathcal{L}^0\mathcal{L}^0 a_k(t_0, \mathbb{X}_{t_0})I_{000} + \sum_{j=1}^{n} \mathcal{L}^j\mathcal{L}^0 a_k(t_0, \mathbb{X}_{t_0})I_{j00} \\ &+ \sum_{j=1}^{n} \mathcal{L}^0\mathcal{L}^j a_k(t_0, \mathbb{X}_{t_0})I_{0j0} + \sum_{j,p=1}^{n} \mathcal{L}^j\mathcal{L}^p a_k(t_0, \mathbb{X}_{t_0})I_{jp0} \\ &+ \sum_{j=1}^{n} \mathcal{L}^0\mathcal{L}^0 h_{kj}(t_0, \mathbb{X}_{t_0})I_{00j} + \sum_{j,l=1}^{n} \mathcal{L}^l\mathcal{L}^0 h_{kj}(t_0, \mathbb{X}_{t_0})I_{l0j} \\ &+ \sum_{j,l=1}^{n} \mathcal{L}^0\mathcal{L}^l h_{kj}(t_0, \mathbb{X}_{t_0})I_{0lj} + \sum_{j,p,l=1}^{n} \mathcal{L}^l\mathcal{L}^p h_{kj}(t_0, \mathbb{X}_{t_0})I_{lpj} + R_t, \end{aligned} \tag{4.19}$$

where R_t represents the remainder term which can be expressed as Eqn. (4.17).

□

4.3 Itô-Taylor Approximation for Correlated Brownian Motions

In the previous section, the systems of stochastic processes was driven by multi-dimensional standard Brownian motion. However, much more realistically, there is often a correlation between the Brownian motions. In that case, a transformation of the given systems of SDEs into an equivalent systems of SDEs driven by standard multi-dimensional Brownian motion is very useful. In this section, we present such a transformation method to be able to apply IT-M on the transformed systems.

We use the symbol $(\mathbb{W}_t)_{t\geq 0}$ instead of $(\mathbb{Z}_t)_{t\geq 0}$ in order to point out the difference from the earlier standard Brownian motions. Now, we consider the system

$$dX_t^k = a_k(t, \mathbb{X}_t) + \sum_{j=1}^{n} h_{kj}(t, \mathbb{X}_t)dW_t^j \quad (k = 1, 2, \ldots, d = n), \tag{4.20}$$

where $dW_t^i dW_t^j = \rho_{ij}dt$.

Then, the correlation matrix ρ can be written as:

$$\rho := \begin{pmatrix} 1 & \rho_{12} & \dots & \rho_{1n} \\ \rho_{21} & 1 & \dots & \rho_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{n1} & \rho_{n2} & \dots & 1 \end{pmatrix}, \quad \rho_{ij} = \rho_{ji} \in [-1, 1].$$

Here, ρ is positive semi-definite matrix which means that $\rho = \rho^T$ and

$$\sum_{i,j=1}^{n} \rho_{ij} x_i x_j \geq 0,$$

for all $x = (x_1, \dots, x_n)^T \in \mathbb{R}^n$.

By using some basic standard Linear Algebra, one can find an $n \times n$ matrix $\mathbb{B} = (b_{ij})_{1 \leq i,j \leq n}$ such that

$$\rho = \mathbb{B}\mathbb{B}^T.$$

Moreover, using Cholesky Decomposition [5], we make $\mathbb{B}$ as an upper (or lower) triangular matrix.

Correlated Brownian motions can be interpreted as a linear combination of uncorrelated ones such that

$$\mathbb{W}_t = \mathbb{B}\mathbb{Z}_t,$$

where $\mathbb{Z}_t = (Z_t^1, \dots, Z_t^n)^T$ and $\mathbb{W}_t = (W_t^1, \dots, W_t^n)^T$ are a standard n-dimensional Brownian motion and a correlated Brownian motion, respectively.
In componentwise notation,

$$W_t^i = \sum_{j=1}^{n} b_{ij} Z_t^j \quad (i = 1, 2, \dots, n). \tag{4.21}$$

Now, after substituting Eqn. (4.21) into Eqn. (4.20), we obtain a system of SDEs as in Eqn. (4.15) so that we can apply a similar procedure of argumentation as in the general case.

4.4 Discretization

Multiple Itô integrals, $I_{i_1 i_2 \dots i_k}$, have to be evaluated and expressed in terms of different random variables to construct numerical schemes. This section covers the discretization schemes for strong Taylor approximations.

4.4.1 Discretization Schemes with Strong Taylor Approximations

In order to judge the quality of a numerical scheme, it is necessary to have some sort of measure of how well the approximate solution converge to the true solution.

For this reason, special forms of Itô-Taylor expansions whose orders are known is used for obtaining the solutions. We consider the strong convergence criteria of measuring convergence.

Definition 4.1. Let $X_\Delta(T)$ be a discrete time approximation of a continuous-time process X and X_T be a true solution at time T. Then, there exists positive constant c, independent of maximum step-size Δ, and numbers $\Delta_0, p > 0$ such that

$$E(|X_T - X_\Delta(T)|) \leq c\Delta^p, \quad \forall \Delta \in (0, \Delta_0).$$

In this case, we say $X_\Delta(T)$ *converge strongly* with order p.

Consider an equispaced discretization $t_0 \leq \tau_0 < \tau_1 < \ldots < \tau_\nu < \ldots < \tau_m = T$ of the time interval $[t_0, T]$. Let $\Delta = T/m$ denote the increments (step-size); then, for all $\nu \in \{0, 1, \ldots, m-1\}$ it holds:

$$\begin{aligned}
I_0 &= \int_{\tau_\nu}^{\tau_{\nu+1}} ds = \Delta = \tau_{\nu+1} - \tau_\nu, \\
I_j &= \int_{\tau_\nu}^{\tau_{\nu+1}} dZ_s^j = \Delta Z^j = Z_{\tau_{\nu+1}}^j - Z_{\tau_\nu}^j, \\
I_{j0} &= \int_{\tau_\nu}^{\tau_{\nu+1}} \int_{\tau_\nu}^{s} dZ_u^j ds = \Delta\tilde{W}, \\
I_{0j} &= \int_{\tau_\nu}^{\tau_{\nu+1}} \int_{\tau_\nu}^{s} du dZ_s^j = (\Delta Z^j)\Delta - \Delta\tilde{W}, \\
I_{jj} &= \frac{1}{2}\big((\Delta Z^j)^2 - \Delta\big), \\
I_{jj0} &= I_{0jj} = I_{j0j} = \frac{1}{6}\Delta\big((\Delta Z^j)^2 - \Delta\big), \\
I_{jjj} &= \frac{1}{6}\Delta\big((\Delta Z^j)^2 - 3\Delta(\Delta Z^j)\big), \\
I_{j00} &= I_{0j0} = I_{00j} = \frac{1}{6}\Delta^2\Delta Z_s^j,
\end{aligned}$$

where $\Delta\tilde{W}$ and ΔZ^j are Gaussian random variables with $\Delta Z^j \sim N(0, \Delta)$, $\Delta\tilde{W} \sim N(0, \frac{1}{3}\Delta^3)$ and $E\big(\Delta Z^j \Delta\tilde{W}\big) = \frac{1}{2}\Delta^2$.

Now, we shall use the above relations to propose some strong approximations.

4.4.1.1 The Euler-Maruyama Scheme

The simplest example of a strong Taylor approximation X of the solution of Eqn. (3.12) and X^k of the solution of Eqn. (4.15) is the *Euler-Maruyama* or *Euler method* attaining the order of strong convergence 0.5. In the one-dimensional case, Eqn. (3.12), *Euler scheme* is of the form

$$X_{\nu+1} = X_\nu + a\Delta + h\Delta Z, \tag{4.22}$$

for $\nu = 0, 1, 2, \ldots, m-1$. In the multi-dimensional case, Eqn. (4.15), the k^{th} component of the *Euler scheme* looks as follows:

$$X_{\nu+1}^k = X_\nu^k + a_k \Delta + \sum_{j=1}^{n} h_{kj} \Delta Z^j, \tag{4.23}$$

for $\nu = 0, 1, 2, \ldots, m-1$.

In the cases where the drift and diffusion coefficients, a_k and h_{kj}, respectively, are nearly constant, this method generally gives us good numerical results. However, whenever the coefficients are nonlinear, the method can provide a poor estimate of the solution. So, higher-order schemes, which we introduce and evaluate in the following subsubsections, should be used to obtain more satisfactory schemes.

4.4.1.2 The Milstein Scheme

Milstein scheme can be obtained by adding second-order terms from the Itô-Taylor expansion to the Euler scheme, which increases the strong convergence order from 0.5 to 1.0. In the one-dimensional case, Eqn. (3.12), *Milstein scheme* has the following look:

$$X_{\nu+1} = X_\nu + a\Delta + h\Delta Z + \mathcal{L}^1 h I_{11}, \tag{4.24}$$

for $\nu = 0, 1, 2, \ldots, m-1$. In the multi-dimensional case, Eqn. (4.15), the k^{th} component of the *Milstein scheme* has the form

$$X_{\nu+1}^k = X_\nu^k + a_k \Delta + \sum_{j=1}^{n} h_{kj} \Delta Z^j + \sum_{j_1, j_2 = 1}^{n} \mathcal{L}^{j_1} h_{kj_2} I_{j_1 j_2}, \tag{4.25}$$

for $\nu = 0, 1, 2, \ldots, m-1$. We note that Milstein scheme is identical to the Euler scheme when the diffusion term does not contain any component of the $\mathbb{X}_t$ variable.

4.4.1.3 The Order 1.5 Strong Taylor Scheme

We can get more accurate strong Taylor schemes by including further multiple stochastic integrals from the stochastic Taylor approximation. In the one-dimensional case, Eqn. (3.12), the *order* 1.5 *strong Taylor scheme* is of the form

$$X_{\nu+1} = X_\nu + a\Delta + h\Delta Z + \mathcal{L}^1 h I_{11} + \mathcal{L}^1 a I_{10} + \mathcal{L}^0 a I_{00} + \mathcal{L}^0 h I_{01} + \mathcal{L}^1 \mathcal{L}^1 h I_{111}, \tag{4.26}$$

for $\nu = 0, 1, 2, \ldots, m-1$. In the multi-dimensional case, Eqn. (4.15), the k^{th} component of the *order* 1.5 *strong Taylor scheme* is given by

$$\begin{aligned} X_{\nu+1}^k = X_\nu^k &+ a_k\Delta + \frac{1}{2}\mathcal{L}^0 a_k \Delta^2 + \sum_{j=1}^{n}(h_{kj}\Delta Z^j + \mathcal{L}^0 h_{kj} I_{0j} + \mathcal{L}^j a_k I_{j0}) \\ &+ \sum_{j_1,j_2=1}^{n} \mathcal{L}^{j_1} h_{kj_2} I_{j_1 j_2} + \sum_{j_1,j_2,j_3=1}^{n} \mathcal{L}^{j_1}\mathcal{L}^{j_2} h_{kj_3} I_{j_1 j_2 j_3}, \end{aligned} \tag{4.27}$$

for $\nu = 0, 1, 2, \ldots, m-1$.

4.4.1.4 The Order 2.0 Strong Taylor Scheme

The numerical results which was given by the *order 2.0 strong Taylor scheme* is better than other three method. In the one-dimensional case, Eqn. (3.12), the *order* 1.5 *strong Taylor scheme* is of the form

$$\begin{aligned} X_{\nu+1} = X_\nu &+ a\Delta + h\Delta Z + \mathcal{L}^1 h I_{11} + \mathcal{L}^1 a I_{10} + \mathcal{L}^0 a I_{00} \\ &+ \mathcal{L}^0 h I_{01} + \mathcal{L}^1\mathcal{L}^1 h I_{111} + \mathcal{L}^1\mathcal{L}^0 a I_{100} + \mathcal{L}^0\mathcal{L}^1 a I_{010} \\ &+ \mathcal{L}^1\mathcal{L}^1 a I_{110} + \mathcal{L}^0\mathcal{L}^0 h I_{001} + \mathcal{L}^1\mathcal{L}^0 h I_{101} + \mathcal{L}^0\mathcal{L}^1 h I_{011} \\ &+ \mathcal{L}^1\mathcal{L}^1 h I_{111} + \mathcal{L}^1\mathcal{L}^1\mathcal{L}^1 h I_{1111}, \end{aligned} \tag{4.28}$$

for $\nu = 0, 1, 2, \ldots, m-1$. In the multi-dimensional case, Eqn. (4.15), the k^{th} component of the *order 2.0 strong Taylor scheme* takes the following form:

$$\begin{aligned} X_{\nu+1}^k = X_\nu^k &+ a_k\Delta + \frac{1}{2}\mathcal{L}^0 a_k \Delta^2 + \sum_{j=1}^{n}(h_{kj}\Delta Z^j + \mathcal{L}^0 h_{kj} I_{0j} + \mathcal{L}^j a_k I_{j0}) \\ &+ \sum_{j_1,j_2=1}^{n} \big(\mathcal{L}^{j_1} h_{kj_2} I_{j_1 j_2} + \mathcal{L}^0\mathcal{L}^{j_1} h_{kj_2} I_{0 j_1 j_2} + \mathcal{L}^{j_1}\mathcal{L}^0 h_{kj_2} I_{j_1 0 j_2} \\ &+ \mathcal{L}^{j_1}\mathcal{L}^{j_2} a_k I_{j_1 j_2 0}\big) + \sum_{j_1,j_2,j_3=1}^{n} \mathcal{L}^{j_1}\mathcal{L}^{j_2} h_{kj_3} I_{j_1 j_2 j_3} \\ &+ \sum_{j_1,j_2,j_3,j_4=1}^{n} \mathcal{L}^{j_1}\mathcal{L}^{j_2}\mathcal{L}^{j_2} h_{kj_4} I_{j_1 j_2 j_3 j_4}, \end{aligned} \tag{4.29}$$

for $\nu = 0, 1, 2, \ldots, m-1$.

CHAPTER 5

CHANGE OF TIME METHOD

With the IT-M, we can approximate all SDEs, whose closed form or analytic solution are not known. However, in some situations, using IT-M is not very useful or impractical, causing that we need another methodology. *Change of Time Method* (*CT-M*) is one of such probabilistic methods to obtain a "simple" representation for a given SDE (maybe with a "complicated" structure), using the idea of changing the scales of time [4]. Scaling of time is sometimes done in a deterministic manner, but it is more often used in a *random* manner. It is central to the work of Doeblin [7]. Dambis and Dubins-Schwartz developed a theory of random time changes for semimartingales in the 1960s [21, 33]. Time change was chosen to be a subordinator by Feller [8]. In fact, in the finance literature, the terms *time change* and *subordinator* are sometimes used synonymously. In order to construct stochastic volatility for Lévy processes, a subordinated process is used [6]. In [1,3,31,37], the class of time changes are formulated. A special feature and an advantage of CT-M consist in the possibility that the time change can be defined by direct reference to *risk*, defined by a quadratic variation. Herewith, CT-M can become a strong tool of risk management.

In this chapter, we give a brief introduction of random time change to solve SDEs. We mostly refer to [1, 37, 41].

5.1 Change of Time for Martingales

Theorem 5.1. (***Dambis, Dubins-Schwartz Theorem*** [7,21,33]) *Let $(M_t)_{t\geq 0}$ be a continuous local martingale on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. Suppose the quadratic variation of M_t exists and denoted by $[M]_t$ such that*

$$\lim_{t\to\infty} [M]_t = \infty \quad a.s..$$

Then, if we define a stopping time $\tau_t := \inf\{u \geq 0 : [M]_u > t\}$ and $\tilde{\mathcal{F}}_t := \mathcal{F}_{\tau_t}$ $(t \geq 0)$, the time change process $Z_t := M_{\tau_t}$ $(t \geq 0)$ is an $(\mathcal{F}_{\tau_t})$-Brownian motion, and

$$M_t = Z_{[M]_t} \quad (t \geq 0).$$

We note that the local martingale $(M_t)_{t\geq 0}$ can be expressed by $(Z_t)_{t\geq 0}$ and an $(\tilde{\mathcal{F}}_t)$-stopping time since $\{[M]_t \leq u\} = \{\tau_u \geq t\}$.

5.2 Change of Time for Itô Integral

Definition 5.1. [14,41] Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space and I be the class of functions

$$\begin{array}{rcl} \psi : [0,\infty) & \longrightarrow & [0,\infty), \\ t & \mapsto & \psi_t, \end{array}$$

which satisfy the following conditions:

- $\psi_0 = 0$.
- ψ is continuous and strictly increasing.
- $\psi_t \to \infty$ as $t \to \infty$.

Obviously, if ψ^{-1} is the inverse function of $\psi \in I$ then $\psi^{-1} \in I$. Each $\psi \in I$ defines a transformation T^ψ of $C := C([0,\infty))$ (the set of continuous functions w defined on $[0,\infty)$ with values in $\mathbb{R}$) into itself by

$$\begin{array}{rcl} T^\psi : C & \longrightarrow & C, \\ w & \mapsto & (T^\psi w), \end{array}$$

where

$$(T^\psi w)(t) := w(\psi_t^{-1}) \quad (t \in [0,\infty)).$$

Here, T^ψ is called the *time change* defined by $\psi \in I$ and $\psi = \psi_t(\omega)$ is called a *process of the time change* for $\omega \in \Omega$. It is clear that $\psi = \psi_t(\omega) \in \Omega$ is an $(\mathcal{F}_t)$-adapted increasing process, so that the inverse function ψ_t^{-1} of ψ_t is an $(\mathcal{F}_t)$-stopping time for each fixed $t \in [0,\infty)$. We note that often in literature, the event or scenario ω is in the role of a continuous function w, indeed.

Let $\tilde{M}_t := \int_0^t h(s)dZ(s)$ $(t \geq 0)$ be a family of Itô integrals with

$$\lim_{t\to\infty} [\tilde{M}]_t = \lim_{t\to\infty} \int_0^t h^2(s)ds = +\infty \quad \text{and} \quad \psi_t := \inf\{u \geq 0 : [\tilde{M}]_u > t\}.$$

Then, $(B_t) = (\tilde{M}_{\psi_t})$ is a Brownian motion. Here, the change of time is

$$\psi_t^{-1} = [\tilde{M}]_t = \int_0^t h^2(s)ds.$$

Thus, an SDE in $\mathbb{R}^1$ of the form

$$X_t = X_0 + \int_0^t h(s, X_s)dZ_s + \int_0^t a(s, X_s)ds$$

can be rephrased in the following way:

$$\begin{aligned} X_t - X_0 &= B_{[\tilde{M}]_t} + \int_0^t a(s, X_s)ds \\ &= B_{\int_0^t h^2(s,X_s)ds} + \int_0^t a(s, X_s)ds, \end{aligned}$$

with a Brownian motion $(B_t)_{t\geq 0}$. Then, one-*dimensional Itô Lemma* takes the form [3, 41]

$$f(t, X_t) - f(0, X_0) = B_{\int_0^t h^2(s,X_s)f'(s,X_s)^2 ds} + \int_0^t \big(\frac{\partial f}{\partial s} + \mathcal{L}f\big)(s, X_s)ds, \tag{5.1}$$

with $\mathcal{L}f := \frac{1}{2}h^2 f'' + af'$. Now, let us verify this result.

If $X_t = X_0 + \int_0^t h(s, X_s)dZ_s + \int_0^t a(s, X_s)ds$, then by applying Itô Lemma we get:

$$\begin{aligned} f(t, X_t) - f(0, X_0) &= \int_0^t \frac{\partial f}{\partial s}ds + \int_0^t \frac{\partial f}{\partial x}\big(a(s, X_s)ds + h(s, X_s)dZ_s\big) \\ &\quad + \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}h^2(s, X_s)ds \\ &= \int_0^t \frac{\partial f}{\partial s}ds + \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}h^2(s, X_s)ds + \int_0^t \frac{\partial f}{\partial x}a(s, X_s)ds \\ &\quad + \int_0^t \frac{\partial f}{\partial x}h(s, X_s)dZ_s \\ &= B_{\int_0^t h^2(s,X_s)f'(s,X_s)^2 ds} + \int_0^t \big(\frac{\partial f}{\partial s} + \mathcal{L}f\big)(s, X_s)ds. \end{aligned}$$

5.3 Change of Time for SDEs

We consider the SDE given in the following form (without drift):

$$dX_t = h(t, X_t)dZ_t, \tag{5.2}$$

where $(Z_t)_{t\geq 0}$ is a Brownian motion, $h(t, X_t)$ is a continuous and measurable function, and $(X_t)_{t\geq 0}$ is a continuous process on $[0, \infty)$. If we can solve Eqn. (5.2), then we can also resolve the equations having drift term $a(t, X_t)dt$ by the method of *transformation of drift* or *Girsanov transformation*. The following theorem provides us to solve Eqn. (5.2).

Theorem 5.2. [31, 37] *Let $\tilde{Z} = (\tilde{Z}_t)_{t\geq 0}$ be a 1-dimensional $(\mathcal{F}_t)_{t\geq 0}$-Brownian motion with $\tilde{Z}_0 = 0$ for a given filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ and let X_0 be an $(\mathcal{F}_0)$-adapted random variable. We define a continuous process $V := (V_t)_{t\geq 0}$ by*

$$V_t := X_0 + \tilde{Z}_t \quad (t \geq 0).$$

Let $(\psi_t)_{t\geq 0}$ be the change of time process such that

$$\psi_t := \int_0^t h^{-2}(\psi_s, X_0 + \tilde{Z}_s)ds \quad (t \geq 0).$$

If $X_t := V_{\psi_t^{-1}} = X_0 + \tilde{Z}_{\psi_t^{-1}}$ and $\tilde{\mathcal{F}}_t := \mathcal{F}_{\psi_t^{-1}}$ $(t \geq 0)$, then there exists an $(\tilde{\mathcal{F}}_t)$-adapted Brownian motion $Z = (Z_t)_{t\geq 0}$ such that (X_t, Z_t) is a solution of Eqn. (5.2) on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Remark 5.1. The converse of this theorem also holds [31].

Proof. By definition of the time change, $M_t := \tilde{Z}_{\psi_t^{-1}}$ is a martingale with quadratic variation $[M]_t = \psi_t^{-1}$, where (suppressing quantifiers for the ease of notation)

$$\begin{aligned}
\psi_t = \int_0^t h^{-2}(\psi_s, V_s)ds \quad &\Rightarrow \quad d\psi_t = h^{-2}(\psi_t, V_t)dt \\
&\Rightarrow \quad dt = h^2(\psi_t, V_t)d\psi_t \\
&\Rightarrow \quad \int_0^t ds = \int_0^t h^2(\psi_s, V_s)d\psi_s \\
&\Rightarrow \quad t = \int_0^t h^2(\psi_s, V_s)d\psi_s \\
&\Rightarrow \quad \psi_s^{-1} = \int_0^{\psi_s^{-1}} h^2(\psi_u, V_u)d\psi_u \\
&\Rightarrow \quad d\psi_s^{-1} = h^2(\psi_{\psi_s^{-1}}, V_{\psi_s^{-1}})\psi'_{\psi_s^{-1}}d\psi_s^{-1} \\
&\Rightarrow \quad d\psi_s^{-1} = h^2(s, V_{\psi_s^{-1}})d\psi_{\psi_s^{-1}} \\
&\Rightarrow \quad \int_0^t d\psi_s^{-1} = \int_0^t h^2(s, V_{\psi_s^{-1}})d\psi_{\psi_s^{-1}} \\
&\Rightarrow \quad \psi_t^{-1} = \int_0^t h^2(s, V_{\psi_s^{-1}})ds.
\end{aligned}$$

Hence, ψ_t^{-1} satisfies the equation

$$\psi_t^{-1} = \int_0^t h^2(s, X_0 + \tilde{Z}_{\psi_s^{-1}})ds \quad (t \geq 0).$$

For any $t \geq 0$, we set $Z_t := \int_0^t h^{-1}(s, X_s)dM_s$. Then,

$$\begin{aligned}
[Z]_t &= \int_0^t h^{-2}(s, X_s)d[M]_s \\
&= \int_0^t h^{-2}(s, X_s)h^2(s, X_s)ds. \\
&= \int_0^t ds = t.
\end{aligned}$$

This result implies that $(Z_t)_{t\geq 0}$ is an $(\mathcal{F}_t)$-Brownian motion. Since $M_t = \tilde{Z}_{\psi_t^{-1}} = X_t - X_0 = \int_0^t h(s, X_s) dZ_s$ $(t \geq 0)$, (X_t, Z_t) is a solution of Eqn. (5.2). $\square$

CHAPTER 6

APPLICATIONS

This chapter covers several applications of both CT-M and IT-M.

6.1 Applications of Change of Time Method

Sometimes, it is more useful to solve SDEs with the change of time method. For example, CT-M can be used for Cox-Ingersoll-Ross (CIR) model when valuing *variance* and *volatility swaps* [1]. We shall present examples in Subsections 6.1.1 and 6.1.2.

6.1.1 Cox-Ingersoll-Ross Model

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space. The CIR model is based on the following stochastic process:

$$d\sigma_t^2 = k(\theta^2 - \sigma_t^2)dt + \gamma\sigma_t dZ_t \quad (t \geq 0), \tag{6.1}$$

where σ_0 and θ are short and long volatility, respectively. Furthermore, k is a reversion speed, $\gamma > 0$ is a volatility parameter, $(Z_t)_{t\geq 0}$ is a standard Brownian motion.

Lemma 6.1. [1] *A solution of the Eqn. (6.1) has the following form:*

$$\sigma_t^2 = e^{-kt}\big(\sigma_0^2 - \theta^2 + \tilde{Z}_{\psi_t^{-1}}\big) + \theta^2,$$

where $\tilde{Z}_{\psi_t^{-1}}$ *is an* $\mathcal{F}_{\psi_t^{-1}}$*-measurable one-dimensional Brownian motion. Here,* ψ_t^{-1} *is the inverse of* ψ_t*, defined as:*

$$\psi_t := \gamma^{-2}\int_0^t \Big(e^{k\psi_s}\big(\sigma_0^2 - \theta^2 + \tilde{Z}_s\big) + \theta^2 e^{2k\psi_s}\Big)^{-1} ds \quad (t \geq 0).$$

Proof. We define the following process:

$$V_t = e^{kt}\big(\sigma_t^2 - \theta^2\big) \quad (t \geq 0).$$

Then, using the Itô product rule (Itô Lemma for the product of Itô processes) [30], we obtain:

$$\begin{aligned} dV_t &= ke^{kt}\big(\sigma_t^2 - \theta^2\big)dt + e^{kt}d\sigma_t^2 \\ &= ke^{kt}\big(\sigma_t^2 - \theta^2\big)dt + e^{kt}\big(k(\theta^2 - \sigma_t^2) + \gamma\sigma_t dZ_t\big) \\ &= \gamma e^{kt}\sigma_t dZ_t^2 \\ &= \gamma e^{kt}\sqrt{e^{-kt}V_t + \theta^2}dZ_t. \end{aligned}$$

Applying CT-M to the general equation, we get

$$dX_t = h(t, X_t)dZ_t,$$

where $h(t, X_t) = e^{kt}\sqrt{e^{-kt}V_t + \theta^2}$. $X_t = V_t$ implies that $X_0 = \sigma_0^2 - \theta^2$ and $V_t = \sigma_0^2 - \theta^2 + \tilde{Z}_{\psi_t^{-1}}$. Then,

$$\begin{aligned} e^{kt}\big(\sigma_t^2 - \theta^2\big) &= \sigma_0^2 - \theta^2 + \tilde{Z}_{\psi_t^{-1}} \\ &\Rightarrow \sigma_t^2 = e^{-kt}\big(\sigma_0^2 - \theta^2 + \tilde{Z}_{\psi_t^{-1}}\big) + \theta^2. \end{aligned}$$

We note that $(\tilde{Z}_{\psi_t^{-1}})$ is an $(\mathcal{F}_{\psi_t^{-1}})$-measurable one-dimensional Brownian motion and ψ_t^{-1} is the inverse of ψ_t:

$$\psi_t = \gamma^{-2}\int_0^t \big(e^{k\psi_s}\big(\sigma_0^2 - \theta^2 + \tilde{Z}_s\big) + \theta^2 e^{2k\psi_s}\big)^{-1}ds.$$

□

6.1.2 Variance and Volatility Swaps

A *variance swap* is a forward contract on annualized variance, the square of realized volatility. Its payoff at expiration is given by

$$N(\sigma_R^2(S) - K_{\text{var}}),$$

where $V = \sigma_R^2(S)$ is the *realized stock variance* over the life of the contract,

$$\sigma_R^2(S) := \frac{1}{T}\int_0^T \sigma_s^2 ds.$$

Here, K_{var} is strike (delivery) price for variance, N is the notional amount of the swap in dollars per annualized volatility point squared. The holder of a variance swap at expiration gains N dollars for every point of $\sigma_R^2(S) - K_{\text{var}}$. Moreover, the price of a forward contract P on the future realized variance is the expected present value of the future payoff in the risk-neutral world:

$$P_{\text{var}} = E(e^{-rT}(\sigma_R^2(S) - K_{\text{var}})),$$

where r is the risk-free discount rate. We remark that

$$\begin{aligned} E(\sigma_t^2) &= E(e^{-kt}\big(\sigma_0^2 - \theta^2 + \tilde{Z}^2(\psi_t^{-1})\big) + \theta^2) \\ &= e^{-kt}\big(\sigma_0^2 - \theta^2\big) + e^{-kt}E(\tilde{Z}^2(\psi_t^{-1})) + \theta^2 \\ &= e^{-kt}\big(\sigma_0^2 - \theta^2\big) + \theta^2. \end{aligned}$$

In fact,

$$\begin{aligned} E(V) &= \frac{1}{T}\int_0^T E(\sigma_s^2)ds \\ &= \frac{1}{T}\int_0^T \big(e^{-ks}\big(\sigma_0^2 - \theta^2\big) + \theta^2\big)ds \\ &= \frac{1}{T}\bigg(\frac{e^{-ks}}{-k}\big(\sigma_0^2 - \theta^2\big) + \theta^2 s\bigg)\bigg|_0^T \\ &= \frac{1}{T}\bigg(\frac{e^{-kT}}{-k}\big(\sigma_0^2 - \theta^2\big) + \theta^2 T + \frac{1}{k}(\sigma_0^2 - \theta^2)\bigg) \\ &= \frac{1 - e^{-kT}}{kT}(\sigma_0^2 - \theta^2) + \theta^2, \end{aligned}$$

so that $P_{\text{var}} = e^{-rT}\bigg(\frac{1-e^{-kT}}{kT}(\sigma_0^2 - \theta^2) + \theta^2 - K_{\text{var}}\bigg)$.

A *volatility swap* is a forward contract on annualized variance. Its payoff at expiration is given by

$$N(\sigma_R(S) - K_{\text{vol}}),$$

where $\sigma_R(S)$ is the *realized stock volatility* over the life of the contract,

$$\sigma_R(S) := \frac{1}{T}\sqrt{\int_0^T \sigma_s^2 ds}.$$

In a similar way, a *volatility swap* can be studied [1]; its value (price) can be represented by

$$\begin{aligned} P_{\text{vol}} &= e^{-rT}\bigg(\Big(\frac{1 - e^{-kT}}{kT}(\sigma_0^2 - \theta^2) + \theta^2\Big)^{1/2} - \frac{\gamma^2 e^{-2kT}}{2k^3T^2}\bigg[(2e^{2kT} \\ &\quad -4e^{kT}kT - 2)(\sigma_0^2 - \theta^2) + (2e^{2kT} - 3e^{2kT} + 4e^{kT} - 1)\theta^2\bigg] \\ &\quad \Big/\bigg[8\bigg(\frac{1 - e^{-kT}}{kT}(\sigma_0^2 - \theta^2) + \theta^2\bigg)^{3/2}\bigg] - K_{\text{vol}}\bigg). \end{aligned}$$

6.2 Applications of Itô-Taylor Expansions

We will consider the stochastic control problems and systems of SDEs in this section. Before addressing the CIR model, we concentrate on stochastic control problems.

6.2.1 Optimal Stochastic Control with Malliavin-Based Approach

In this subsection, state equations will be approximated by IT-M. Furthermore, the expectation of the gradient of the cost functional will be stated by Malliavin calculus.

6.2.1.1 Introduction

Firstly, we recall some basic notations and results of Malliavin calculus. For a detailed explanation, we may refer to [28,29,43]. Suppose $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space.

Definition 6.1. Let $\mathcal{H}$ be a real separable Hilbert space with inner product denoted by $\langle \cdot, \cdot \rangle_{\mathcal{H}}$. For $f := f(t) \in \mathcal{H}$, we let $\|f\|_{\mathcal{H}} := \sqrt{\langle f, f \rangle_{\mathcal{H}}}$. We say that $W = (W(f) : f \in \mathcal{H})$ is an *isonormal Gaussian process* if W is a centered Gaussian (normally distributed) random variable (i.e., $E(W(f)) = 0$ with variance $\|f\|^2_{\mathcal{H}}$) such that $E(W(f)W(h)) = \langle f, h \rangle_{\mathcal{H}}$ $\forall (f, h) \in \mathcal{H}$.

From now on, we let $\mathcal{H} := L^2([0, T], \mathbb{R}^n)$ for some $n \in \mathbb{N}$.

Example 6.1. The *Wiener stochastic integral* $W(f)$ is defined as

$$W(f) := \int_0^T f(s) dW_s \quad \forall f \in \mathcal{H}.$$

is an isonormal Gaussian process.

Definition 6.2. Let $C^\infty(\mathbb{R}^n)$ be the set of all infinitely often continuously differentiable functions $h : \mathbb{R}^n \to \mathbb{R}$ such that h and its partial derivatives have polynomial growth property. For $n \in \mathbb{N}$ and $h \in C^\infty(\mathbb{R}^n)$, we denote $\mathcal{S}$ as the set of all smooth random variables $F : \Omega \to \mathbb{R}$ such that

$$F = h(W(f_1), \dots, W(f_n)),$$

where $f_i \in \mathcal{H}$ for $i = 1, 2, \dots, n$.

Definition 6.3. The *Malliavin derivative* $\mathcal{D}_t F$ of a smooth random variable of $F \in \mathcal{S}$ is a $\mathcal{H}$-valued random variable given by

$$\mathcal{D}_t F = \sum_{i=1}^n \partial_{x_i} h(W(f_1), \dots, W(f_n)) f_i.$$

For example: $\mathcal{D}(W(f)) = f$.
For a given $p \in \mathbb{N}$, the domain of $\mathcal{D}$ in $L^p(\Omega)$ will be denoted by $\mathbb{D}^{1,p}$ with respect to the norm

$$\|F\|_{1,p} := (E(F^p) + E\|\mathcal{D}F\|^p_{\mathcal{H}})^{1/p}.$$

Proposition 6.2. *Assume that $f : \mathbb{R}^k \to \mathbb{R}$ is a continuously differentiable function with bounded partial derivatives. For a random vector $F = (F^1, F^2, \ldots, F^k)^T$ with $F^i \in \mathbb{D}^{1,p}$ $(i = 1, 2, \ldots, k)$, one has $f(F) \in \mathbb{D}^{1,p}$ $(p \in \mathbb{N})$ and*

$$\mathcal{D}_t(f(F)) = \sum_{i=1}^{k} \partial_{x_i} f(F) \mathcal{D}_t F^i.$$

Definition 6.4. The *divergence operator* δ, being the adjoint operator of $\mathcal{D}$, is an unbounded operator on $L^2([0,T] \times \Omega, \mathbb{R}^n)$ with values in $L^2(\Omega)$ such that:

- The domain of δ, denoted by Dom(δ), is the set of $\mathcal{H}$-valued square-integrable random variables $u \in L^2([0,T] \times \Omega, \mathbb{R}^n)$ such that there exists a constant $c(u)$ satisfying:

$$|E\big(\int_0^T \mathcal{D}_t F \cdot u_t dt\big)| \leq c(u) ||F||_{L^2} \quad \forall F \in \mathbb{D}^{1,2}.$$

- If $u \in$Dom(δ), then $\delta(u) \in L^2(\Omega)$ with the integration-by-parts formula

$$E(F\delta(u)) = E\bigg(\int_0^T \mathcal{D}_t F \cdot u_t dt\bigg) \quad \forall F \in \mathbb{D}^{1,2}.$$

Remark 6.1. If u is an adapted process, then *Skorohod integral* and Itô integral coincide:

$$\delta(u) = \int_0^T u_t dW_t \quad \forall u \in L^2([0,T] \times \Omega, \mathbb{R}^n).$$

Proposition 6.3. *Let $F \in \mathbb{D}^{1,2}$ and $u \in$Dom(δ) such that $E(F^2 \int_0^T \|u_t\|^2_{L^2} dt) < \infty$, then the product Fu is Skorohod integrable:*

$$\delta(Fu) = F\delta(u) - \int_0^T \mathcal{D}_t F \cdot u_t dt$$

6.2.1.2 Sensitivity Analysis

We consider the following system of stochastic differential equations in $\mathbb{R}^n$ with an n-dimensional Brownian motion:

$$\mathbb{X}_t = x + \int_0^t \mathbb{A}(s, \mathbb{X}_s, \mu) ds + \int_0^t \mathbb{H}(s, \mathbb{X}_s, \mu) d\mathbb{B}_s, \tag{6.2}$$

where $\mathbb{A}(t, \mathbb{X}_t, \mu) = (a_1(t, \mathbb{X}_t, \mu), \ldots, a_n(t, \mathbb{X}_t, \mu))^T$, $\mathbb{X}_t = (X^1_t, \ldots, X^n_t)^T$, $\mathbb{B}_t = (B^1_t, \ldots, B^n_t)^T$, $\mu = (\mu_1, \ldots, \mu_n)^T$ and

$$\mathbb{H}(t, \mathbb{X}_t, \mu) = \begin{pmatrix} h_{11}(t, \mathbb{X}_t, \mu) & \ldots & h_{1n}(t, \mathbb{X}_t, \mu) \\ \vdots & \ddots & \vdots \\ h_{n1}(t, \mathbb{X}_t, \mu) & \ldots & h_{nn}(t, \mathbb{X}_t, \mu) \end{pmatrix}.$$

For a given instantaneous cost function g and a terminal cost function f, the *sensitivity* with respect to μ of the *expected cost functional* is defined by

$$J(\mu) := E\bigg(\int_0^T g(s, \mathbb{X}_s, \mu)ds + f(\mathbb{X}_T) \bigg).$$

Now, we introduce a stochastic optimal control problem by

$$\underset{\mu}{\text{minimize}}\ J(\mu) \ \text{ subject to } \ \text{Eqn. (6.2)}. \tag{6.3}$$

There are various methods to get the numerical solutions of the problem stated in Eqn. (6.3). A classical approach for an optimal control problem is to derive necessary optimality conditions. Standard optimization algorithms require the gradient computation of the cost functional given. There are some approaches for such computations such as adjoint approach and sensitivities. We will derive the gradient of the cost functional by using the sensitivity with the help of Malliavin calculus.

The well-known *Monte-Carlo method* can be employed to simulate the sensitivity $\nabla_\mu J$ [10]. At this point, we shall use the emerging approach of Malliavin calculus in order to get an expression for $\nabla_\mu J$. This method is based on the integration-by-parts formula. It can be considered as a generalization of the well-known *likelihood ratio method* [10]. We can write Eqn. (6.2) as

$$\mathbb{X}_t = x + \int_0^t \mathbb{A}(s, \mathbb{X}_s, \mu)ds + \sum_{j=1}^n \int_0^t h_j(s, \mathbb{X}_s, \mu)dB_s^j,$$

where h_j is the j^{th} column of $\mathbb{H}$. If we write the Jacobian matrix as $\mathbb{Y}_t := \nabla_x \mathbb{X}_t$ and denote the inverse of the Jacobian matrix by $\mathbb{Z}_t := \mathbb{Y}_t^{-1}$, then [9, 10]:

$$\begin{aligned}
\mathbb{Y}_t &= \mathbb{I} + \int_0^t \mathbb{A}'\mathbb{Y}ds + \sum_{j=1}^n \int_0^t h_j'\mathbb{Y}_s dB_s^j, \\
\mathbb{Z}_t &= \mathbb{I} - \int_0^t \mathbb{Z}_s(\mathbb{A}' - \sum_{j=1}^n (h_j')^2)ds - \sum_{j=1}^n \int_0^t \mathbb{Z}_s h_j' dB_s^j, \\
\dot{\mathbb{X}}_t &= \int_0^t (\dot{\mathbb{A}} + \mathbb{A}'\dot{\mathbb{X}}_s)ds + \sum_{j=1}^n \int_0^t (\dot{h}_j + h_j'\dot{\mathbb{X}}_s)dB_s^j,
\end{aligned}$$

where $\mathbb{I}$ is the $(n \times n)$-identity matrix and $\dot{\mathbb{X}}$ is the derivative of $\mathbb{X}$ with respect μ. Also, $\mathbb{A}'$ and h_j' denote the Jacobian matrices of $\mathbb{A}$ and h_j, respectively.

Proposition 6.4. [10] *We assume the following conditions (i)-(iii):*

(i) The functions $\mathbb{A}$ and $\mathbb{H}$ are continuously differentiable with respect to the variables t, x, μ, and for some $\eta > 0$ and $\mathcal{A} \subset \mathbb{R}^n$, the Hölder continuity condition is satisfied such that

$$\underset{(t,x,\mu,\mu') \in [0,T] \times \mathbb{R}^n \times \mathcal{A} \times \mathcal{A}}{sup} \frac{|g(t,x,\mu) - g(t,x,\mu')|}{|\mu - \mu'|^\eta} < \infty,$$

for both $g = \partial_\mu \mathbb{A}$ and $g = \partial_\mu \mathbb{H}$. Furthermore, for any $\mu \in \mathcal{A}$, the functions $\mathbb{A}(\cdot,\cdot,\mu)$, $\mathbb{H}(\cdot,\cdot,\mu)$, $\partial_\mu\mathbb{A}(\cdot,\cdot,\mu)$ and $\partial_\mu\mathbb{H}(\cdot,\cdot,\mu)$ are continuously differentiable with respect to t and their first-order and second-order derivatives with respect to x both exist and are continuous; the functions $\partial_\mu\mathbb{A}$ and $\partial_\mu\mathbb{H}$ are uniformly bounded in (t,x,μ), and the derivatives of $\mathbb{A}$, $\mathbb{H}$, $\partial_\mu\mathbb{A}$, $\partial_\mu\mathbb{H}$ with respect to (t,x) are uniformly bounded as well.

(ii) The (squared) matrix $\mathbb{H}$ satisfies a uniform ellipticity condition:

$$[\mathbb{H}\mathbb{H}^*](t,x,\mu) \geq c\mathbb{I} \quad \forall (t,x) \in [0,T] \times \mathbb{R}^n,$$

for a real number $c > 0$.

(iii) Be f a bounded measurable function.

Then,

$$\dot{J}(\mu) := \frac{1}{T} f(\mathbb{X}_T)\delta([\mathbb{H}^{-1}\mathbb{Y}\mathbb{Z}_T\dot{\mathbb{X}}_T]^*).$$

6.2.1.3 Approximation Procedure

We apply Itô-Taylor approximation to the quantities $(\mathbb{X}_t)_{t\geq 0}$, $(\mathbb{Y}_t)_{t\geq 0}$ and $(\mathbb{Z}_t)_{t\geq 0}$.

Lemma 6.5. *Let $(\mathbb{X}_t)$ be the stochastic process defined in Eqn. (6.2). Furthermore, $(\mathbb{Y}_t)$ and $(\mathbb{Z}_t)$ denote the Jacobian of $(\mathbb{X}_t)$ and the inverse of the Jacobian, respectively. Then, we get the approximate solutions as:*

$$\begin{aligned}
X_t^k = X_0^k \quad &+ \quad a_k(0,\mathbb{X}_0,\mu_0)I_0 + \sum_{j=1}^n h_{kj}(0,\mathbb{X}_0,\mu)I_j \\
&+ \quad \mathcal{L}^0 a_k(0,\mathbb{X}_0,\mu)I_{00} + \sum_{j=1}^n \mathcal{L}^j a_k(0,\mathbb{X}_0,\mu)I_{j0} \\
&+ \quad \sum_{j=1}^n \mathcal{L}^0 h_{kj}(0,\mathbb{X}_0,\mu)I_{0j} + \sum_{l,j=1}^n \mathcal{L}^l h_{kj}(0,\mathbb{X}_0,\mu)I_{lj} \\
&+ \quad \mathcal{L}^0\mathcal{L}^0 a_k(0,\mathbb{X}_0,\mu)I_{000} + \sum_{j=1}^n \mathcal{L}^j\mathcal{L}^0 a_k(0,\mathbb{X}_0,\mu)I_{j00} \\
&+ \quad \sum_{j=1}^n \mathcal{L}^0\mathcal{L}^j a_k(0,\mathbb{X}_0,\mu)I_{0j0} + \sum_{j,p=1}^n \mathcal{L}^j\mathcal{L}^p a_k(0,\mathbb{X}_0,\mu)I_{jp0} \\
&+ \quad \sum_{j=1}^n \mathcal{L}^0\mathcal{L}^0 h_{kj}(0,\mathbb{X}_0,\mu)I_{00j} + \sum_{j,l=1}^n \mathcal{L}^l\mathcal{L}^0 h_{kj}(0,\mathbb{X}_0,\mu)I_{l0j} \\
&+ \quad \sum_{j,l=1}^n \mathcal{L}^0\mathcal{L}^l h_{kj}(0,\mathbb{X}_0,\mu)I_{0lj} + \sum_{j,p,l=1}^n \mathcal{L}^l\mathcal{L}^p h_{kj}(0,\mathbb{X}_0,\mu)I_{lpj} + R_{1t}
\end{aligned}$$

and

$$
\begin{aligned}
\mathbb{Y}_t^{k\ell} = \mathbb{I}_{k\ell} &+ (\mathbb{A}'\mathbb{Y})_{k\ell}(0,\mathbb{X}_0,\mu)I_0 + \sum_{j=1}^{n}(h_j'\mathbb{Y}_s)_{k\ell}(0,\mathbb{X}_0,\mu)I_j \\
&+ \mathcal{L}^0(\mathbb{A}'\mathbb{Y})_{k\ell}(0,\mathbb{X}_0,\mu)I_{00} + \sum_{j=1}^{n}\mathcal{L}^j(\mathbb{A}'\mathbb{Y})_{k\ell}(0,\mathbb{X}_0,\mu)I_{j0} \\
&+ \sum_{j=1}^{n}\mathcal{L}^0(h_j'\mathbb{Y}_s)_{k\ell}(0,\mathbb{X}_0,\mu)I_{0j} + \sum_{l,j=1}^{n}\mathcal{L}^l(h_j'\mathbb{Y}_s)_{k\ell}(0,\mathbb{X}_0,\mu)I_{lj} \\
&+ \mathcal{L}^0\mathcal{L}^0(\mathbb{A}'\mathbb{Y})_{k\ell}(0,\mathbb{X}_0,\mu)I_{000} + \sum_{j=1}^{n}\mathcal{L}^j\mathcal{L}^0(\mathbb{A}'\mathbb{Y})_{k\ell}(0,\mathbb{X}_0,\mu)I_{j00} \\
&+ \sum_{j=1}^{n}\mathcal{L}^0\mathcal{L}^j(\mathbb{A}'\mathbb{Y})_{k\ell}(0,\mathbb{X}_0,\mu)I_{0j0} + \sum_{j,p=1}^{n}\mathcal{L}^j\mathcal{L}^p(\mathbb{A}'\mathbb{Y})_{k\ell}(0,\mathbb{X}_0,\mu)I_{jp0} \\
&+ \sum_{j=1}^{n}\mathcal{L}^0\mathcal{L}^0(h_j'\mathbb{Y}_s)_{k\ell}(0,\mathbb{X}_0,\mu)I_{00j} + \sum_{j,l=1}^{n}\mathcal{L}^l\mathcal{L}^0(h_j'\mathbb{Y}_s)_{k\ell}(0,\mathbb{X}_0,\mu)I_{l0j} \\
&+ \sum_{j,l=1}^{n}\mathcal{L}^0\mathcal{L}^l(h_j'\mathbb{Y}_s)_{k\ell}(0,\mathbb{X}_0,\mu)I_{0lj} + \sum_{j,p,l=1}^{n}\mathcal{L}^l\mathcal{L}^p(h_j'\mathbb{Y}_s)_{k\ell}(0,\mathbb{X}_0,\mu)I_{lpj} + R_{2t},
\end{aligned}
$$

where $\mathbb{Y}_t^{k\ell}$ stands for the $(k,\ell)^{th}$ component of the matrix $(\mathbb{Y}_t)_{t\geq 0}$ and, R_{1t} and R_{2t} represent the remainder terms.

Remark 6.2. Similar formulations remain valid for both $(\mathbb{Z}_t)_{t\geq 0}$ and $(\dot{\mathbb{X}}_t)_{t\geq 0}$ as for $(\mathbb{Y}_t)_{t\geq 0}$.

Consider an equispaced discretization $0 \leq \tau_0 < \tau_1 < \ldots < \tau_\nu < \ldots < \tau_m = T$ of the time interval $[0,T]$.

Lemma 6.6. [9,10] *Let U_T be the random variable defined as $U_T := \delta([\mathbb{H}^{-1}\mathbb{Y}\mathbb{Z}_T\dot{\mathbb{X}}_T]^*)$ (of Proposition 6.4). Assume that $\begin{pmatrix} \mathbb{X}_t \\ \dot{\mathbb{X}}_t \end{pmatrix}$ is $\mathbb{R}^{2n}$-valued stochastic differential equation with Jacobian $\hat{\mathbb{Y}}_t$ and the inverse of the Jacobian is called $(\hat{\mathbb{Z}}_t)$ $(t \geq 0)$. Then, U_T is approximated by*

$$
\begin{aligned}
U_T &\approx \sum_{i=1}^{n}[\mathbb{Z}_T^m\ \dot{\mathbb{X}}_T^m]_i \int_0^T [\mathbb{H}^{-1}(s,\mathbb{X}_s^m)\mathbb{Y}_s^m]_i^* dB_s \\
&- \sum_{i=1}^{n}\int_0^T \left(\sum_{j=1}^{n} P^m_{\beta(j,i),T}R^m_{\beta(j,i),s}\right)[\mathbb{H}^{-1}(s,\mathbb{X}_s^m)\mathbb{Y}_s^m]_i ds,
\end{aligned}
$$

where $\mathcal{D}_s([\mathbb{Z}_T\dot{\mathbb{X}}_T]_i) = 1_{s\leq T}\sum_{j=1}^n P_{\beta(j,i),T}R_{\beta(j,i),s}$ with $P_{\beta(j,i),T}$ and $R_{\beta(j,i),s}$ are being by some appropriate coordinates of the processes $\hat{\mathbb{Y}}_t$ and $\hat{\mathbb{Z}}_t$, respectively.

For full details of this lemma, we refer to [9, 10].

6.2.2 Cox-Ingersoll-Ross Model

Example 6.2. (CIR model) Consider the system of Eqn. (6.1):

$$d\sigma_t^2 = k(\theta^2 - \sigma_t^2)dt + \gamma\sigma_t dZ_t. \tag{6.4}$$

For Eqn. (6.4), Lemma 3.2 gives us the Itô-Taylor approximation as

$$\begin{aligned}
\sigma_t^2 = \sigma_0^2 &+ k(\theta^2 - \sigma_0^2)I_0 + \gamma\sigma_0 I_1 - k^2(\theta^2 - \sigma_0^2)I_{00} - k\gamma\sigma_0 I_{10} \\
&+ \frac{\gamma}{\sigma_0}\left(\frac{k(\theta^2 - \sigma_0^2)}{2} - \frac{\gamma^2}{8}\right)I_{01} + \frac{\gamma^2}{2}I_{11} + k^3(\theta^2 - \sigma_0^2)I_{000} \\
&+ k^2\gamma\sigma_0 I_{100} - \frac{k\gamma}{\sigma_0}\left(\frac{k(\theta^2 - \sigma_0^2)}{2} - \frac{\gamma^2}{8}\right)I_{010} - \frac{k\gamma^2}{2}I_{110} \\
&+ \left[\frac{k(\theta^2 - \sigma_0^2)}{4\sigma_0}\left(\frac{\gamma^3 - 4k\theta^2}{4\sigma_0^2} - k\right) + \frac{\gamma^2}{16\sigma_0}\left(\frac{3(4k\theta^2 - \gamma^3)}{4\sigma_0^2} + k\right)\right]I_{001} \\
&+ \left[\frac{\gamma}{4\sigma_0^2}\left(\frac{\gamma^3}{4} - k\theta^2\right) - \frac{\gamma k}{4}\right]I_{101} + R_t,
\end{aligned}$$

where R_t denotes the remainder term.

6.2.3 Ornstein-Uhlenbeck Model

Example 6.3. We consider 2-dimensional version of a *weakly-coupled Ornstein-Uhlenbeck (OU) model*:

$$\begin{aligned}
dX_t^1 &= \alpha_1(\theta_1 - X_t^1)dt + \sigma_1 dW_t^1, \\
dX_t^2 &= \alpha_2(\theta_2 - X_t^2)dt + \sigma_2 dW_t^2,
\end{aligned} \tag{6.5}$$

where $dW_t^1 dW_t^2 = \rho dt$, with real coefficients $\alpha_1,\ \alpha_2,\ \sigma_1,\ \sigma_2,\ \theta_1,\ \theta_2 > 0$ and $\rho \in (-1, 1)$.

Then, the correlation matrix ρ can be written as:

$$\rho = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{pmatrix} \begin{pmatrix} 1 & \rho \\ 0 & \sqrt{1-\rho^2} \end{pmatrix},$$

by Cholesky Decomposition.

Thus,

$$\mathbb{W}_t = \mathbb{B}\mathbb{Z}_t = \begin{pmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{pmatrix} \begin{pmatrix} Z_t^1 \\ Z_t^2 \end{pmatrix}.$$

So, Eqn. (6.5) becomes

$$\begin{aligned}
dX_t^1 &= \alpha_1(\theta_1 - X_t^1)dt + \sigma_1 dZ_t^1, \\
dX_t^2 &= \alpha_2(\theta_2 - X_t^2)dt + \sigma_2\rho dZ_t^1 + \sigma_2\sqrt{1-\rho^2}dZ_t^2.
\end{aligned} \tag{6.6}$$

In integral form, we state Eqn. (6.6) as

$$\begin{aligned}
X_t^1 = X_{t_0}^1 &+ \alpha_1\theta_1 \int_{t_0}^t ds - \alpha_1 \int_{t_0}^t X_s^1 ds + \sigma_1 \int_{t_0}^t dZ_s^1, \\
X_t^2 = X_{t_0}^2 &+ \alpha_2\theta_2 \int_{t_0}^t ds - \alpha_2 \int_{t_0}^t X_s^2 ds + \sigma_2\rho \int_{t_0}^t dZ_s^1 + \sigma_2\sqrt{1-\rho^2} \int_{t_0}^t dZ_s^2.
\end{aligned}$$

Applying the procedure of argumentation from the previous chapters, we get

$$\begin{aligned}
X_t^1 = X_{t_0}^1 &+ \alpha_1(\theta_1 - X_{t_0}^1)I_0 + \sigma_1 I_1 - \alpha_1^2(\theta_1 - X_{t_0}^1)I_{00} \\
&- \alpha_1\sigma_1 I_{10} + \alpha_1^3(\theta_1 - X_{t_0}^1)I_{000} + \alpha_1^2\sigma_1 I_{100} + R_{1t}, \\
X_t^2 = X_{t_0}^2 &+ \alpha_2(\theta_2 - X_{t_0}^2)I_0 + \sigma_2\rho I_1 + \sigma_2\sqrt{1-\rho^2} I_2 \\
&- \alpha_2^2(\theta_2 - X_{t_0}^2)I_{00} - \rho\alpha_2\sigma_2 I_{10} - \alpha_2\sigma_2\sqrt{1-\rho^2} I_{20} \\
&+ \alpha_2^3(\theta_2 - X_{t_0}^2)I_{000} + \rho\alpha_2^2\sigma_2 I_{100} + \sqrt{1-\rho^2}\alpha_2^2\sigma_2 I_{200} + R_{2t},
\end{aligned}$$

with remainder terms R_{1t} and R_{2t}.

Example 6.4. We state the 2-dimensional version of *strongly-coupled Ornstein-Uhlenbeck (OU) model*:

$$\begin{aligned}
dX_t^1 &= (-\alpha_{11}X_t^1 - \alpha_{12}X_t^2)dt + \sigma_1 dW_t^1, \\
dX_t^2 &= (-\alpha_{21}X_t^1 - \alpha_{22}X_t^2)dt + \sigma_2 dW_t^2,
\end{aligned} \tag{6.7}$$

where $dW_t^1 dW_t^2 = \rho dt$ and $\theta_1, \theta_2 = 0$.

In terms of standard Brownian motions, we write Eqn. (6.7) as

$$\begin{aligned}
dX_t^1 &= (-\alpha_{11}X_t^1 - \alpha_{12}X_t^2)dt + \sigma_1 dZ_t^1, \\
dX_t^2 &= (-\alpha_{21}X_t^1 - \alpha_{22}X_t^2)dt + \sigma_2\rho dZ_t^1 + \sigma_2\sqrt{1-\rho^2} dZ_t^2.
\end{aligned} \tag{6.8}$$

In a similar way, we obtain:

$$\begin{aligned}
X_t^1 = X_{t_0}^1 &- (\alpha_{11}X_{t_0}^1 + \alpha_{12}X_{t_0}^2)I_0 + \sigma_1 I_1 - (\sigma_1\alpha_{11} + \rho\sigma_2\alpha_{12})I_{10} \\
&+ \big[\alpha_{11}(\alpha_{11}X_{t_0}^1 + \alpha_{12}X_{t_0}^2) + \alpha_{12}(\alpha_{21}X_{t_0}^1 + \alpha_{22}X_{t_0}^2)\big] I_{00} \\
&- \sigma_2\alpha_{12}\sqrt{1-\rho^2} I_{20} - \big[(\alpha_{11}^2 + \alpha_{12}\alpha_{21})(\alpha_{11}X_{t_0}^1 + \alpha_{12}X_{t_0}^2) \\
&+ (\alpha_{11}\alpha_{12} + \alpha_{12}\alpha_{22})(\alpha_{21}X_{t_0}^1 + \alpha_{22}X_{t_0}^2)\big] I_{000} \\
&+ \big[\sigma_1(\alpha_{11}^2 + \alpha_{12}\alpha_{21}) + \sigma_2\rho(\alpha_{11}\alpha_{12} + \alpha_{12}\alpha_{22})\big] I_{100} \\
&+ \sigma_2(\alpha_{11}\alpha_{12} + \alpha_{12}\alpha_{22})\sqrt{1-\rho^2} I_{200} + R_{1t},
\end{aligned}$$

and

$$\begin{aligned}
X_t^2 = X_{t_0}^2 &- (\alpha_{21}X_{t_0}^1 + \alpha_{22}X_{t_0}^2)I_0 + \sigma_2\rho I_1 + \sigma_2\sqrt{1-\rho^2} I_2 \\
&- (\sigma_1\alpha_{21} + \rho\sigma_2\alpha_{22})I_{10} + \big[\alpha_{21}(\alpha_{11}X_{t_0}^1 + \alpha_{12}X_{t_0}^2) \\
&+ \alpha_{22}(\alpha_{21}X_{t_0}^1 + \alpha_{22}X_{t_0}^2)\big] I_{00} - \sigma_2\alpha_{22}\sqrt{1-\rho^2} I_{20} \\
&- \big[(\alpha_{21}\alpha_{11} + \alpha_{22}\alpha_{21})(\alpha_{11}X_{t_0}^1 + \alpha_{12}X_{t_0}^2) + (\alpha_{21}\alpha_{12} \\
&+ \alpha_{22}^2)(\alpha_{21}X_{t_0}^1 + \alpha_{22}X_{t_0}^2)\big] I_{000} + \big[\sigma_1(\alpha_{11}\alpha_{21} + \alpha_{22}\alpha_{21}) \\
&+ \sigma_2\rho(\alpha_{21}\alpha_{12} + \alpha_{22}^2)\big] I_{100} + \sigma_2(\alpha_{12}\alpha_{21} + \alpha_{22}^2)\sqrt{1-\rho^2} I_{200} + R_{2t},
\end{aligned}$$

where R_{1t} and R_{2t} are remainder terms.

Let $\alpha_{12} = \alpha_{21} = \alpha_{22} = 1, \alpha_{11} = 2, \sigma_1 = \sigma_2 = 1$ and $\rho = 0.6$, then Eqn. (6.8) becomes

$$\begin{aligned} dX_t^1 &= (-2X_t^1 - X_t^2)dt + dZ_t^1, \\ dX_t^2 &= (-X_t^1 - X_t^2)dt + 0.6dZ_t^1 + 0.8dZ_t^2, \end{aligned} \tag{6.9}$$

and we get:

$$\begin{aligned} X_t^1 = X_{t_0}^1 &- (2X_{t_0}^1 + X_{t_0}^2)I_0 + I_1 + (5X_{t_0}^1 + 3X_{t_0}^2)I_{00} - 2.6I_{10} \\ &- 0.8I_{20} - (13X_{t_0}^1 + 8X_{t_0}^2)I_{000} + 6.8I_{100} + 2.4I_{200} + R_{1t} \\ X_t^2 = X_{t_0}^2 &- (X_{t_0}^1 + X_{t_0}^2)I_0 + 0.6I_1 + 0.8I_2 - 1.6I_{10} - 0.8I_{20} \\ &+ (3X_{t_0}^1 + 2X_{t_0}^2)I_{00} - (8X_{t_0}^1 + 5X_{t_0}^2)I_{000} + 4.2I_{100} + 1.6I_{200} + R_{2t}. \end{aligned}$$

Then, the Euler scheme reads the system of Eqn. (6.9) as:

$$\begin{aligned} X_{\nu+1}^1 &= X_\nu^1 - (X_\nu^2 + 2X_\nu^1)\Delta + \Delta Z^1, \\ X_{\nu+1}^2 &= X_\nu^2 - (X_\nu^2 + X_\nu^1)\Delta + 0.6\Delta Z^1 + 0.8\Delta Z^2 \quad (\nu \in \mathbb{N}). \end{aligned}$$

Applying the Milstein scheme to the system of Eqn. (6.9), we obtain

$$\begin{aligned} X_{\nu+1}^1 &= X_\nu^1 - (X_\nu^2 + 2X_\nu^1)\Delta + \Delta Z^1, \\ X_{\nu+1}^2 &= X_\nu^2 - (X_\nu^2 + X_\nu^1)\Delta + 0.6\Delta Z^1 + 0.8\Delta Z^2 \quad (\nu \in \mathbb{N}). \end{aligned}$$

Applying the order 1.5 strong Taylor scheme to Eqn. (6.9), we get:

$$\begin{aligned} X_{\nu+1}^1 = X_\nu^1 &- (X_\nu^2 + 2X_\nu^1)\Delta + \frac{1}{2}(3X_\nu^2 + 5X_\nu^1)\Delta^2 + \Delta Z^1 - 2.6I_{10} - 0.8I_{20}, \\ X_{\nu+1}^2 = X_\nu^2 &- (X_\nu^2 + X_\nu^1)\Delta + \frac{1}{2}(2X_\nu^2 + 3X_\nu^1)\Delta^2 + 0.6\Delta Z^1 + 0.8\Delta Z^2 \\ &- 1.6I_{10} - 0.8I_{20} \quad (\nu \in \mathbb{N}). \end{aligned}$$

Finally, for Eqn. (6.9) the order 2.0 strong Taylor scheme can be reduced to:

$$\begin{aligned} X_{\nu+1}^1 = X_\nu^1 &- (X_\nu^2 + 2X_\nu^1)\Delta + \frac{1}{2}(3X_\nu^2 + 5X_\nu^1)\Delta^2 + \Delta Z^1 - 2.6I_{10} - 0.8I_{20}, \\ X_{\nu+1}^2 = X_\nu^2 &- (X_\nu^2 + X_\nu^1)\Delta + \frac{1}{2}(2X_\nu^2 + 3X_\nu^1)\Delta^2 + 0.6\Delta Z^1 + 0.8\Delta Z^2 \\ &- 1.6I_{10} - 0.8I_{20} \quad (\nu \in \mathbb{N}). \end{aligned}$$

6.2.3.1 Numerical Results and Implementation Details

In this part, we consider numerical examples for both the systems with uncorrelated and correlated Brownian motions. In order to implement the discrete scheme, we use MATLAB. There are many documentations that describes the main features of MATLAB commands related to SDE. Some numerical interpretations can be found in the SDEs' MATLAB packages [11]. However, numerical examples implemented in MATLAB are mostly in the one-dimensional case. Some coupled SDEs are considered, but having symmetric coefficients allowing easy computations that arise from multiple Itô integrals. Our first example demonstrates some triple SDEs having uncorrelated Brownian motions.

Example Run 1. The system of SDEs consisting of three equations proposed in Hofmann, Platen and Schweizer [12] is considered as:

$$\begin{cases} dX_t^1 = X_t^1 X_t^2 dZ_t^1, \\ dX_t^2 = -(X_t^2 - X_t^3)dt + 0.3X_t^2 dZ_t^2, \\ dX_t^3 = \frac{1}{\alpha}(X_t^2 - X_t^3)dt, \end{cases}$$

where X_t^1, X_t^2 and X_t^3 represent the asset price, the instantenous volatility, and the averaged volatility, respectively, and, Z_t^1 and Z_t^2 are uncorrelated Brownian motions. As in [11], the *Milstein scheme* is obtained as:

$$\begin{aligned} X_{\nu+1}^1 = X_\nu^1 \quad &+ \quad X_\nu^1 X_\nu^2 \Delta Z^1 + \frac{1}{2} X_\nu^1 (X_\nu^2)^2 \{(\Delta Z^1)^2 - \Delta\} \\ &+ \quad 0.3 X_\nu^1 X_\nu^2 \int_{t_\nu}^{t_{\nu+1}} \int_{t_\nu}^{t} dZ_s^2 dZ_s^1, \\ X_{\nu+1}^2 = X_\nu^2 \quad &- \quad (X_\nu^2 - X_\nu^3)\Delta + 0.3X_\nu^2 \Delta Z^2 + 0.045 X_\nu^2 \{(\Delta Z^1)^2 - \Delta\}, \\ X_{\nu+1}^3 = X_\nu^3 \quad &+ \quad \frac{1}{\alpha}(X_\nu^2 - X_\nu^3)\Delta \quad (\nu \in \mathbb{N}). \end{aligned}$$

We take $\alpha = 1$, $T = 1$, $X_0^1 = 1$, $X_0^2 = 0.1$ and $X_0^3 = 0.1$ as the initiation data; Δ is considered as 2^{-9}.

The scheme has the double integral $\int_{t_\nu}^{t_{\nu+1}} \int_{t_\nu}^{t} dZ_s^2 dZ_s^1$. In [11], this integral is approximated by Euler method. Although it is a bit challenging, we approximate such integrals by using the following representations from [23]:

$$\begin{aligned} &I_0^p = \Delta, \quad I_j^p = \sqrt{\Delta}\xi_j, \quad I_{00}^p = \frac{1}{2}\Delta^2, \\ &I_{j0}^p = \frac{1}{2}\Delta\big(\sqrt{\Delta}\xi_j + a_{j0}\big), \quad I_{0j}^p = \frac{1}{2}\Delta\big(\sqrt{\Delta}\xi_j - a_{j0}\big). \end{aligned}$$

Here,

$$\begin{aligned} a_{j0} &= -\frac{1}{\pi}\sqrt{2\Delta}\sum_{r=1}^{p}\frac{1}{i}\zeta_{ji} - 2\sqrt{\Delta\rho_p}\mu_{jp}, \\ I_{j_1 j_2}^p &= \frac{1}{2}\Delta\xi_{j_1}\xi_{j_2} - \frac{1}{2}\sqrt{\Delta}(a_{j_2 0}\xi_{j_1} - a_{j_1 0}\xi_{j_2}) + \Delta A_{j_1 j_2}^p, \\ A_{j_1 j_2}^p &= \frac{1}{2\pi}\sum_{i=1}^{p}\frac{1}{i}(\zeta_{j_1 i}\eta_{j_2 i} - \zeta_{j_2 i}\eta_{j_1 i}), \end{aligned}$$

with

$$\xi_j = \frac{1}{\sqrt{\Delta}} W^j, \quad \zeta_{ji} = \sqrt{\frac{2}{\Delta}} \pi i a_{ji}, \qquad \eta_{ji} = \sqrt{\frac{2}{\Delta}} \pi i b_{ji},$$
$$\mu_{jp} = \frac{1}{\sqrt{\Delta \rho_p}} \sum_{i=p+1}^{\infty} a_{ji}, \quad \rho_p = \frac{1}{12} - \frac{1}{2\pi^2} \sum_{i=1}^{p} \frac{1}{i^2},$$

where $j = 1, 2, \ldots, m$, and $i = 1, 2, \ldots, p$, for number $p > 0$ with the property

$$p = p(\Delta) \geq \frac{K}{\Delta^2},$$

with an appropriate constant $K > 0$, to ensure the convergence order of the numerical scheme.

We note that ζ_{ji}, η_{ji} and μ_{jp} are uncorrelated Gaussian random variables. Now, we use the *Polar Marsaglia method* to generate pairs of random variables.

The following lines show the implementation of this method in MATLAB:

```
%Polar Marsaglia Method
function     [z1,z2]= Polar
l=0.5;
while l>0
u1 = rand;u2 = rand;
v1 = 2*u1 - 1;v2 = 2*u2 - 1;
V = (v1.*v1)+(v2.*v2);
if (V<=1)&&(V>0)
    break;
end
end
z1 = v1.*sqrt(-2*log(V)./V);
z2 = v2.*sqrt(-2*log(V)./V).
```

We can approximate the iterated integral $\int_{\nu_r}^{t_{\nu+1}} \int_{\nu_r}^{t} dZ_s^2 dZ_s^1$ as:

```
%Approximation of I_ij
function I_ij=ito_ij(p,Delta,G1,G2,mu1_j,mu2_j,ro)
a_ij=0;
for i=1:p
    [zeta1, zeta2]= Polar;[eta1 ,eta2 ]=Polar;
    a_ij=a_ij+(1/i)*(zeta1*(sqrt(2)*G2+eta2)...
    -zeta2*(sqrt(2)*G1+eta1));
end
I_ij=a_ij*Delta/(pi);
I_ij=I_ij+Delta*(G1*G2/2+sqrt(ro)*(mu1_j*G2-mu2_j*G1)).
```

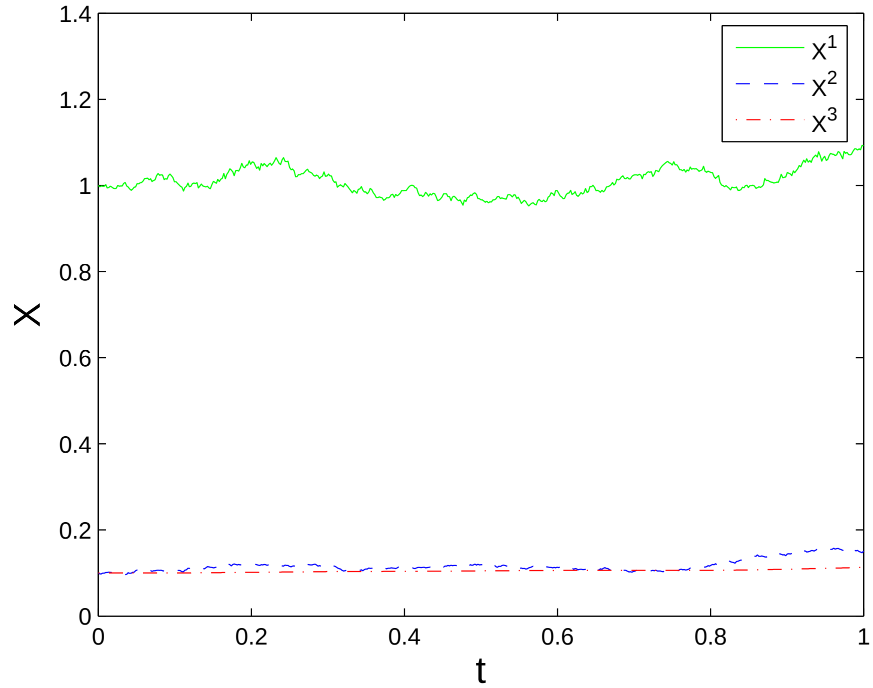

Figure 6.1: Numerical result of Example Run1 with Milstein approximation.

In Figure 6.1, we give the numerical result of Example Run1. We note that we have completely the same results for X_t^1, X_t^2 and X_t^3 as in [11] with the same initial data.

Example Run 2. *(Correlated Brownian motions)* We recall the strongly-coupled OU process of Eqn. (6.7):

$$\begin{aligned} dX_t^1 &= (-\alpha_{11}X_t^1 - \alpha_{12}X_t^2)dt + \sigma_1 dW_t^1, \\ dX_t^2 &= (-\alpha_{21}X_t^1 - \alpha_{22}X_t^2)dt + \sigma_2 dW_t^2, \end{aligned}$$

where $dW_t^1 dW_t^2 = \rho dt$ and the transformed form of Eqn. (6.7) is

$$\begin{aligned} dX_t^1 &= (-2X_t^1 - X_t^2)dt + dZ_t^1, \\ dX_t^2 &= (-X_t^1 - X_t^2)dt + 0.6dZ_t^1 + 0.8dZ_t^2. \end{aligned}$$

Our Taylor scheme with order 1.5 gives

$$\begin{aligned} X_{\nu+1}^1 = X_\nu^1 \quad &+ \quad (-X_\nu^2 - 2X_\nu^1)\Delta + \frac{1}{2}(3X_\nu^2 + 5X_\nu^1)\Delta^2 + \Delta Z^1 - 2.6I_{10} - 0.8I_{20}, \\ X_{\nu+1}^2 = X_\nu^2 \quad &+ \quad (-X_\nu^2 - X_\nu^1)\Delta + \frac{1}{2}(2X_\nu^2 + 3X_\nu^1)\Delta^2 + 0.6\Delta Z^1 \\ &+ \quad 0.8\Delta Z^2 - 1.6I_{10} - 0.8I_{20} \quad (\nu \in \mathbb{N}). \end{aligned}$$

We compute the integrals I_{10} and I_{20} numerically as stated in the following lines:

```
%Approximation of I_j0 and I_0j
function [I_10,I_20]=ito_j0(p,Delta,G1,G2,mu1_j,mu2_j,ro)
a_10=0;a_20=0;
for i=1:p
   [eta1,eta2]=Polar;
    a_10=a_10+(1/i)*eta1;
    a_20=a_20+(1/i)*eta2;
end
I_10=a_10*(1/pi)*sqrt(Delta*2)+2*sqrt(Delta*ro)*mu1_j;
I_10=(1/2)*Delta*I_10+(1/2)*Delta*sqrt(Delta)*G1;
I_20=a_20*(1/pi)*sqrt(Delta*2)+2*sqrt(Delta*ro)*mu2_j;
I_20=(1/2)*Delta*I_20+(1/2)*Delta*sqrt(Delta)*G2.
```

The main file can be run as:

```
clf
randn('state',1)
T = 1; Delta = 2^(-9); delta = Delta^2;
L = T/Delta; K = Delta/delta;
X1 = zeros(1,L+1); X2 = zeros(1,L+1);
X1(1) = 1;X2(1) = 0.1;
p=2;ro=0;
for i=1:p
   ro=ro+1/(i*i);ro=(pi*pi)/6-ro;ro=ro/(2*pi*pi);
end
for j = 1:L
     G1 = randn;   G2 = randn;
Winc2 = sqrt(Delta)*G2;Winc1 = sqrt(Delta)*G1;
 [mu1, mu2 ]=Polar;

[I10,I20]=ito_j0(p,Delta,G1,G2,mu1,mu2,ro);
     X1(j+1) = X1(j) +(-2*X1(j)-X2(j))*Delta +...
     (0.5)*(3*X2(j)+5*X1(j))*Delta^2+Winc1-(2.6)*I10-(0.8)*I20;
    X2(j+1) = X2(j) + (-X2(j)-X1(j))*Delta+(0.5)*(2*X2(j)...
    +3*X1(j))*Delta^2+ (0.6)*Winc1 +(0.8)*Winc2-(1.6)*I10-(0.8)*I20;
end
plot([0:Delta:T],X1,'r-'), hold on
plot([0:Delta:T],X2,'bl--')
xlabel('t','FontSize',16), ylabel('X','FontSize',16)
legend('X^1','X^2').
```

We can also find the exact solution of the system of Eqn. (6.7). We can rewrite Eqn. (6.7) as

$$d\mathbb{X}_t = -\mathbb{A}\mathbb{X}_t dt + \mathbb{B} d\mathbb{Z}_s, \tag{6.10}$$

where $\mathbb{A} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ and $\mathbb{B} = \begin{pmatrix} 1 & 0 \\ 0.6 & 0.8 \end{pmatrix}$.

Multiplying both sides of Eqn. (6.10) by $\exp(t\mathbb{A})$, we get

$$\exp(t\mathbb{A})d\mathbb{X}_t = -\exp(t\mathbb{A})\mathbb{A}\mathbb{X}_t dt + \exp(t\mathbb{A})\mathbb{B}d\mathbb{Z}_t. \tag{6.11}$$

We can arrange Eqn. (6.11) as

$$\exp(t\mathbb{A})d\mathbb{X}_t + \exp(t\mathbb{A})\mathbb{A}\mathbb{X}_t dt = \exp(t\mathbb{A})\mathbb{B}d\mathbb{Z}_t. \tag{6.12}$$

It can be easily seen that the left-hand side of Eqn. (6.12) is equal to the derivative of $\exp(t\mathbb{A})\mathbb{X}_t$ with respect to t. So, we rewrite Eqn. (6.12) in the following way:

$$d(\exp(t\mathbb{A})\mathbb{X}_t) = \exp(t\mathbb{A})\mathbb{B}d\mathbb{Z}_t. \tag{6.13}$$

Integrating Eqn. (6.13) over the interval $[0, t]$, we obtain

$$\exp(t\mathbb{A})\mathbb{X}_t - \mathbb{X}_0 = \int_0^t \exp(s\mathbb{A})\mathbb{B}d\mathbb{Z}_s.$$

The *exact* solution of the system of Eqn. (6.7) is obtained in matrix formulation as:

$$\mathbb{X}_t = \mathbb{X}_0 \exp(-t\mathbb{A}) + \int_0^t \exp((s-t)\mathbb{A})\mathbb{B}d\mathbb{Z}_s. \tag{6.14}$$

We perform the exact numerical simulation for the system of Eqn. (6.7). Firstly, we compute the matrix multiplications in Eqn. (6.14), and then we approximate componentwise.

The Matlab code for the exact numerical solution is:

```
randn('state',1)
%parameters
t_start = 0;          %simulation start time
t_end = 1;          %simuation end time
dt = 2^(-9);            %time step
tau = 1;            %relaxation time
c = 1;                %diffusion constant
x0 = 1;               %initial value for stochastic variable x
mu = 0;               %mean of stochatic process x
y0 = 0.1;               %initial value for integral x
start_dist = 0;    %start of OU pdf
end_dist = 1;       %end of OU pdf
k1=1/(1+sqrt(2));
k2=1/(-1+sqrt(2));
%time
T = t_start:dt:t_end;
%compute x and y
i = 1;
```

```
x(1) = x0;
y(1) = y0;
for t=t_start+dt:dt:t_end
s1=(0.5)*(sqrt((k1*0.5)*(1-(exp(-dt/k1))^2))
+sqrt((k2*0.5)*(1-(exp(-dt/k2))^2)));
s2=sqrt(2)*(0.5)*(-sqrt((k2*0.5)*(1-(exp(-dt/k2))^2))
+sqrt((k1*0.5)*(1-(exp(-dt/k1))^2)));
c1=(0.25)*sqrt(2)*(-sqrt((k2*0.5)*(1-(exp(-dt/k2))^2))
+sqrt((k1*0.5)*(1-(exp(-dt/k1))^2)));
c2=(0.5)*(sqrt((k2*0.5)*(1-(exp(-dt/k2))^2))
+sqrt((k1*0.5)*(1-(exp(-dt/k1))^2)));
i = i + 1;
   r1 = randn;
   r2 = randn;
   x(i) = x(i-1)*(0.5)*(exp(-dt/k1)+exp(-dt/k2))
    +y(i-1)*(0.5)*sqrt(2)*(exp(-dt/k1)-exp(-dt/k2))+...
       (s1+(0.6)*s2)*r1+(0.8)*s2*r2;
   y(i) =  y(i-1)*(0.5)*(exp(-dt/k1)+exp(-dt/k2))
    +x(i-1)*(0.25)*sqrt(2)*(exp(-dt/k1)-exp(-dt/k2))+...
       (c1+(0.6)*c2)*r1+(0.8)*c2*r2;
end.
```

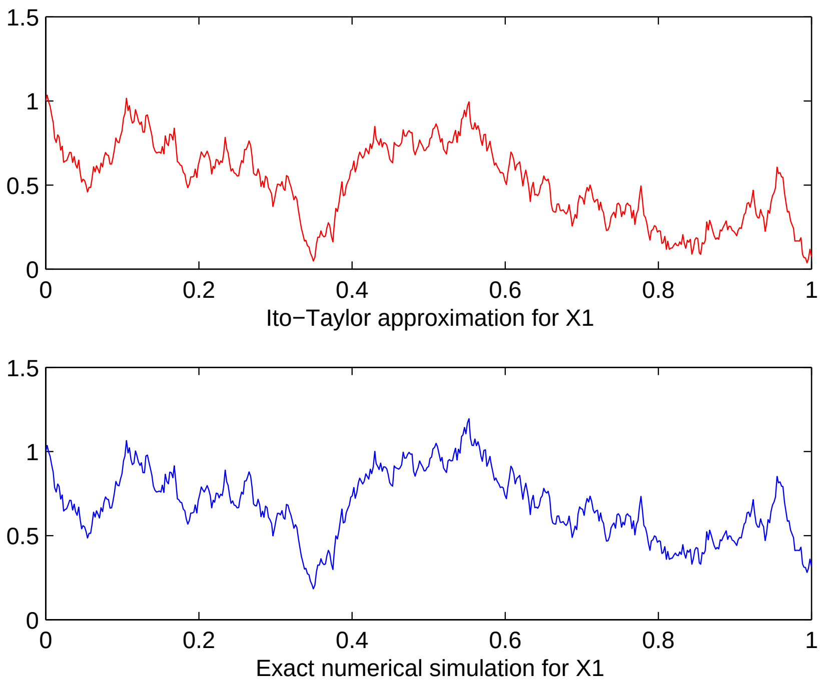

Figure 6.2: Comparison of the exact numerical solution of Run2 with Taylor Scheme of order 1.5 for X_1.

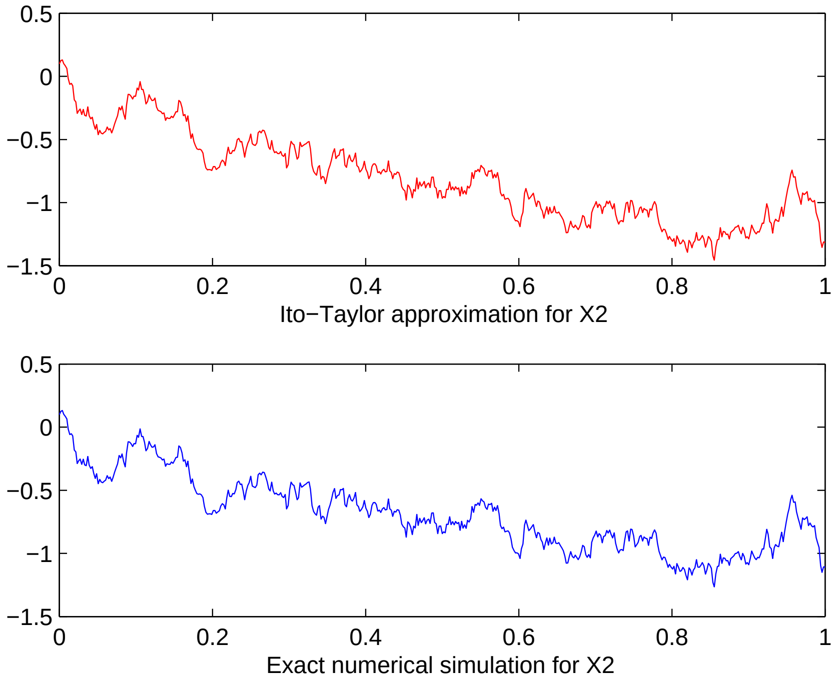

Figure 6.3: Comparison of the exact numerical solution of Run2 with Taylor Scheme of order 1.5 for X_2.

In Figures 6.2-6.3, we compare the obtained results. It can be easily seen that the approximate solution and the exact solution are almost the same.

CHAPTER 7

CONCLUSION AND OUTLOOK

In this study, the solutions of SDEs were studied and discussed by means of two different methods of transformations. The first one, based on the idea of finding an approximate solution, is IT-M. Since the Itô-Taylor expansion is the stochastic version of Taylor series expansion for ODEs, we have firstly got solutions for ODEs by discretization. Then, by using the similar terminology as in the case of ODEs, we obtained the Itô-Taylor approximation for one-dimensional SDEs.

As for the deterministic case, there also exists a multi-variable version of the Itô calculus which means that we can extend the theory of the IT-M from one-dimensional SDEs to systems of SDEs. We classified those systems of SDEs by cases with respect to the criterion of (un-)correlated (or (un-)standardized) Brownian motions. For the systems of SDEs with standard Brownian motion, we considered states and Brownian motions with respect to shared and non-shared cases. After obtaining Itô-Taylor expansions in each cases with standard Brownian motions, we focused on the systems of SDEs with correlated Brownian motions. Since it is not possible to apply directly Itô Lemma to the systems of SDEs with correlated Brownian motions, it is needed to transform them to ones having standard Brownian motion to obtain the Itô-Taylor approximation. Then, in order to get the approximate solutions of SDEs and systems of SDEs, related discretization schemes were given. We considered strong convergence criterion to approximate the true solution.

As a second way of finding solutions of SDEs, we proposed a probabilistic method of representation: CT-M, which can be regarded as a standard tool for building financial models. The idea was to change the scales of time before applying the Itô Lemma to SDEs, so that we could more easily get representations of the solution for SDEs. Additionally, CT-M, by definition, keeps tracks of financial risk.

After that, we gave some applications of both IT-M and CT-M. We obtained the discrete forms of some well-known financial models such as CIR model and OU model. For CIR model, we also used CT-M, which is helpful, e.g., for pricing variance and volatility swaps.

Also, we used the Itô-Taylor theory to get approximate solutions of stochastic control problems. When computing the derivative of the cost functional, we benefited from Malliavin calculus. As for future work, deep relations of Malliavin calculus with control problems could be worked out. Moreover, Itô-Taylor approximation procedure can be considered for nonlinear stochastic control problems [40] and *hybrid systems* [22]. Discrete forms of the state equations may be obtained by methods presented in this thesis. The Itô-Taylor approximation provides the needed linearization and discretization. In fact, we could proceed to use IT-M in the financial context of stochastic control, e.g., in portfolio optimization. Finally, we may use CT-M for those problems, too, and could also permit the existence of *fractional Brownian motions* [27, 39] that we can back to Brownian motions through our transformation approach.

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