# SIMULATING STOCHASTIC DIFFERENTIAL EQUATIONS USING ITÔ-TAYLOR SCHEMES

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# ABSTRACT

SIMULATING STOCHASTIC DIFFERENTIAL EQUATIONS USING ITÔ-TAYLOR SCHEMES

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The aim of this work is to understand the stochastic Taylor schemes and to measure the accuracy of them by comparing their closeness to the exact solutions at the discretization points. Our assumption is that when we use the stochastic Taylor schemes with higher orders, we obtain better approximation processes to exact solutions of the stochastic differential equations. We give the stochastic Taylor schemes with different orders by regarding the convergence criteria for the stochastic differential equations. While Euler-Maruyama and Milstein schemes are derived by using the derivatives of stochastic Taylor expansion, stochastic Runge-Kutta schemes do not need these derivatives in their constructions. Therefore, we have the chance to get higher order stochastic Taylor schemes with less computational difficulties in Runge-Kutta schemes. Moreover, in the application part of the thesis, we observe that the stochastic Runge-Kutta schemes supply the best approximate processes to the exact solutions, for instance, in simulating Orsntein-Uhlenbeck process and in Monte Carlo method for option pricing models.

*Keywords*: Stochastic Taylor Schemes, Euler-Maruyama, Milstein, Runge-Kutta, Stochastic Simulations, Monte Carlo

# ÖZ

STOKASTİK DİFERANSİYEL DENKLEMLERİN ITÔ-TAYLOR METOTLARI KULLANILARAK SİMÜLASYONU

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Bu çalışmadaki asıl amacımız, stokastik Taylor metotlarını anlamak ve bölme noktalarındaki gerçek çözüme olan yakınlıklarını karşılaştırarak doğruluklarını ölçmek. Bizim varsayımımız şudur ki, daha yüksek mertebeli stokastik Taylor metotları kullandığımızda, stokastik diferansiyel denklemlerin gerçek çözümlerine daha iyi yaklaşım gösteren süreçler elde ederiz. Stokastik diferansiyel denklemler için yakınsama kriterlerini göz önünde bulundurarak farklı mertebelerden stokastik Taylor metotlarını verdik. Euler-Maruyama ve Milstein metotları, stokastik Taylor açılımındaki türevler kullanılarak elde edilirken, stokastik Runge-Kutta metodunu oluştururken bu türevlere ihtiyaç duyulmuyor. Bu nedenle, Runge-Kutta metotlarda daha az hesaplama zorluklarıyla daha yüksek mertebeli stokastik Taylor metotları elde edebiliriz. Ayrıca, tezin uygulama bölümünde, stokastik Runge-Kutta metotlarının gerçek çözümlere en iyi yaklaşan süreçleri sağladığını gözlemledik, örnek olarak Orsntein-Uhlenbeck sürecinin simülasyonunda ve opsiyon fiyatlama modellerinin Monte Carlo metotlarında.

*Anahtar Kelimeler*: Stokastik Taylor metotları, Euler-Maruyama, Milstein, Runge-Kutta, Stokastik Simülasyonlar, Monte Carlo

*To My Family*

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| | |
|---|---|
| $\Omega$ | Sample space |
| $\mathcal{F}$ | $\sigma$-algebra |
| $\mathcal{F}_t$ | Filtration |
| $\mathbb{P}$ | Probability measure |
| $W_t$ | Brownian Motion with $t > 0$ |
| $\mathbb{R}$ | Set of real numbers |
| $\mathbb{Z}^+$ | Set of non-negative integer numbers |
| $E(X)$ | Expectation of the process $X$ |
| $Var(X)$ | Variance of the process $X$ |
| $\mathbb{W}_t$ | Multi-dimensional Brownian Motion |
| $\mathrm{Id}_d$ | $d \times d$ identity matrix |
| $X_t^k$ | $k$-th component of process $\mathbb{X}_t$ |
| $\mathcal{M}$ | Set of multi-indices |
| $\alpha$ | Multi-index |
| $l(\alpha)$ | Length of multi-index $\alpha$ |
| $n(\alpha)$ | Number of component of multi-index $\alpha$ |
| $\mathcal{A}$ | Hierarchical set |
| $\mathcal{R}(\mathcal{A})$ | Remainder set |
| $I_\alpha$ | Multiple Itô integral |
| $J_\alpha$ | Multiple Stratonovich integral |
| $\Lambda_p$ | Set of multi-indices used to define strong Taylor schemes of order $p$ |
| $\Gamma_q$ | Set of multi-indices used to define weak Taylor schemes of order $q$ |
| $\mathbb{Q}$ | Risk-neutral probability measure |
| a.s. | almost surely |
| w.p.1 | with probability 1 |

# CHAPTER 1

# INTRODUCTION

The desire of obtaining more realistic models emerges the need of modeling with the stochastic differential equations instead of ordinary differential equations in the fields of financial mathematics, actuarial sciences, physics, biology, chemistry, economics, and so on. Especially, the stochastic differential equations are used for modeling stock and option prices in financial mathematics.

Stochastic differential equations come into the picture when a system decribed by ordinary differential equation is influenced by a random term. So, it makes sense to consider that by adding an extra term to the ordinary differential equations, they can be obtained. This extra term includes *Brownian Motion*, named after Robert Brown in 1827, and brings the *randomness* to the model. The key point of the stochastic differential equation is that we cannot compute the random part by using known ordinary calculus rules, since the Brownian Motion is nowhere differentiable. Therefore, we are interested in the integral form of the stochastic differential equations, and the most known versions of the stochastic calculus are *Itô* and *Stratonovich stochastic calculus*. In that area, Kiyoshi Itô was the first mathematician who defined related rules and formulas (1951) for the stochastic differential and integral equations by generalizing them from the ordinary calculus. His most important contribution in financial mathematics is the so-called *Itô Lemma* which enables us to solve the stochastic integral equations. Alternatively, Ruslen L. Stratonovich introduced the Stratonovich stochastic integral by modifying the Itô stochastic integral (1966). Thus, we can convert one stochastic calculus to the other according to the studied field such as physics or biology [22]. While Itô stochastic calculus plays an important role in modeling the stock and option prices in financial mathematics or determining the growth of a population in biological science, Stratonovich stochastic calculus is commonly used in physical science and engineering to determine the movements of particles, or to make a stability analysis for the stochastic dynamical systems.

Unlike the ordinary differential equations [3], most of the stochastic differential equations don't have an exact solution process; therefore, numerical approximation schemes have been developed by truncating the stochastic Taylor expansion of the process at some point. Peter E. Kloeden and Eckhard Platen widely state these numerical approximation schemes in the book of *Numerical Solution of Stochastic Differential Equations* [11]. Regarding the convergence criteria,

solution processes strongly or weakly converge to the true solution with an order defined in the Taylor expansion of the process. Leonhard Euler and Gisiro Maruyama obtained the basic approximation scheme, called *Euler-Maruyama scheme*, by generalizing the Euler method for the ordinary case to the stochastic differential equations (1955). So, this method is based on the the first differentials of the process. In the following years, Grigori N. Milstein introduced *Milstein scheme* for the numerical approximation schemes by using the second order stochastic Taylor expansion (1974) [19]. By including more terms from the stochastic Taylor expansion, we get higher orders of the approximation schemes such as *order of 3/2 strong Taylor scheme*, *order of 2 weak Taylor scheme* and so on in order to obtain more satisfactory methods. Unfortunately, adding higher order terms of stochastic Taylor expansion to the numerical approximation schemes is getting more complex and causes some computational mistakes; therefore, instead of them, Runge-Kutta schemes, named after two German mathematicians Carl Runge and Martin Kutta, become more preferable to propose stochastic numerical approximations (1901). In these schemes, we do not need the differentiations of the terms available in Taylor expansion and we can get higher order approximation processes by applying the stochastic Runge-Kutta schemes. For detailed researches about the scheme, we may refer to the book of Andreas Rößler entitled *Runge-Kutta Methods for the Numerical Solution of Stochastic Differential Equations* in which he investigated many types of stochastic Runge-Kutta schemes [26].

Our main objective on this thesis is to understand the theoretical as well as the computational aspects of the numerical methods for simulating stochastic differential equations. The stochastic Taylor expansions are introduced, hereby different types of approximation schemes could be defined for the stochastic differential equations [30]. We investigate both strong and weak Taylor approximation schemes such as Euler-Maruyama scheme for the strong order $1/2$ and the weak order 1, Milstein scheme for the strong order 1, weak Taylor scheme of order 2 and strong Taylor scheme of order $3/2$. For higher orders, after we examine the ordinary Runge-Kutta schemes, we move on the stochastic version of Runge-Kutta methods and give the formulations of second, third and fourth stages Runge-Kutta schemes for the stochastic differential equations. At the end, we support the importance and the necessity of these schemes with an error analysis by providing the applications of different stochastic processes, for example, *Geometric Brownian Motion* and *Orsntein-Uhlenbeck process*.

Actually, this may not be enough to understand the accuracy of the approximation methods; therefore, we are also interested in modeling the price of European options. Fisher Black, Myron Scholes and Robert Merton developed a formula to calculate the price of a European type option (1973) [1]. The price is calculated by *Black & Scholes model* while the approximation methods are used to determine the behaviour of a stock price. The weakness of this model is that the volatility is assumed to be constant which does not capture may essential aspects of volatility structure, such as volatility smile. Hence, it is more meaningful to study on the stochastic volatility models in which the volatility is not constant and both the stock price dynamics and the volatility are modeled by the methods

stated above. A finance professor Steven Heston describes the *Heston model* to analyze the bond and currency options (1993) [4]. Instead of calculating the price of the option at only one time, the price can be repeatedly calculated by using *Monte Carlo method*, and the average of them gives more realistic result at the end. Monte Carlo simulation was invented when working on a nuclear weapon project in the late 1940s by Stanislaw Ulam. This simulation is the most powerful way to measure the accuracy of the approximation processes.

In the preliminary research of this work, presented in Chapter 2, we define fundamental definitions of Brownian Motion, Itô and Stratonovich stochastic integrals and then we mention the stochastic Itô-Taylor expansions. In the next chapter of the thesis, we cover classical approximation schemes with different orders according to the convergence criteria of stochastic calculus. In the application part, we compare the methods for various kind of stochastic processes by the error analysis and give the option pricing models in Monte Carlo simulation to demonstrate the success of the approximation methods. At the end of the study, we conclude and give an outlook to future studies.

For the stochastic simulations, we use MATLAB programming [5, 12, 9, 31, 25] and its toolboxes to help financial engineers and practitioners who benefit the most from these numerical solutions. The book written by Ömür Uğur, *An Introduction to Computational Finance*, is very beneficial to understand how to write MATLAB codes or create MATLAB functions [34]. Besides, some of the MATLAB scripts and functions used to implement the methods are given in the appendix at the end of the thesis.

# CHAPTER 2

# PRELIMINARIES

In this chapter, we give some basic definitions and theorems of stochastic calculus [2, 8, 11, 14, 17, 22]. We consider a *filtered probability space* $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ where $\Omega$ is a sample space, $\mathcal{F}$ is a $\sigma$-algebra, $(\mathcal{F}_t)_{t\geq 0}$ is a filtration and $\mathbb{P}$ is a probability measure [17]. The $\sigma$-algebra contains empty set, complements of the sets of the algebra and the union of the sets of the algebra. Here, $(\mathcal{F}_t)_{t\geq 0}$ represents a right-continuous filtration and $\mathcal{F}_0$ contains all $\mathbb{P}$-negligible events in $\mathcal{F}$. The probability measure $\mathbb{P}$ has the properties that $\mathbb{P}(\emptyset) = 0$, $\mathbb{P}(\Omega) = 1$ and $0 \leq \mathbb{P}(A) \leq 1$, for any set $A \in \Omega$. We can define *continuous-time stochastic process* as a set of random variables $X_t$ in the time interval $I = [t_0, T]$ where $t_0 \geq 0$ is the initial time and $0 < T < \infty$ is the maturity time.

**Definition 2.1.** [8, 17] A one-dimensional *Brownian motion* $(W_t)_{t\geq 0}$ is a continuous and $\mathbb{R}$-valued adapted process which is defined on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, with the following properties:

(*i*) $W_0 = 0$ w.p.1.

(*ii*) the increments $W_t - W_s$ are independent of $\mathcal{F}_s$ for each $0 \leq s < t$.

(*iii*) $W_t - W_s$ is normally distributed with zero mean and $t - s$ variance.

We call $W_t$ as a *standard Brownian motion* if the Brownian Motion is normally distributed with the properties that

$$E(W_t) = 0, \quad Var(W_t) = t$$

and $W_0 = 0$ w.p.1. We can also call this process as *Wiener process*, so we denote it by "$W_t$". Wiener process helps us to constitute the random part of a stochastic process. Throughout the thesis, we will consider the standard Brownian motions if nothing else mentioned. A path of the standard Brownian motion is depicted in Figure 2.1; similarly, several paths are sown in Figure 2.2.

We can also define multi-dimensional Brownian motion $\mathbb{W}_t = (W_t^1, W_t^2, \ldots, W_t^d)$ with $d \in \mathbb{Z}^+$.

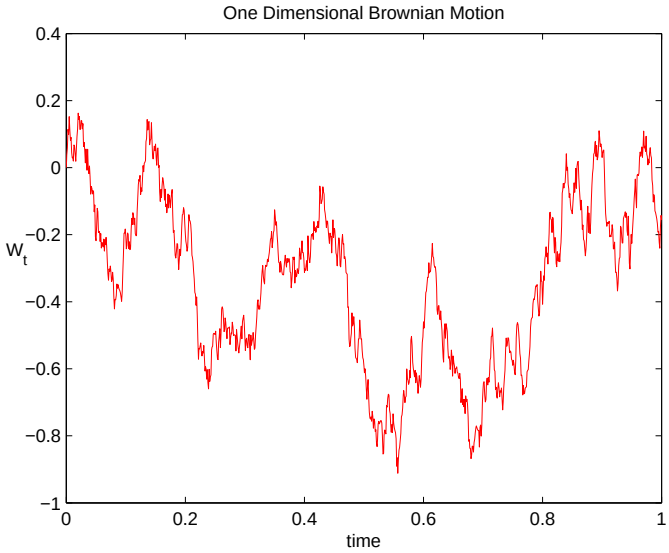


Figure 2.1: Standard Brownian Motion $W_t$ for $t \in [0, 1]$.

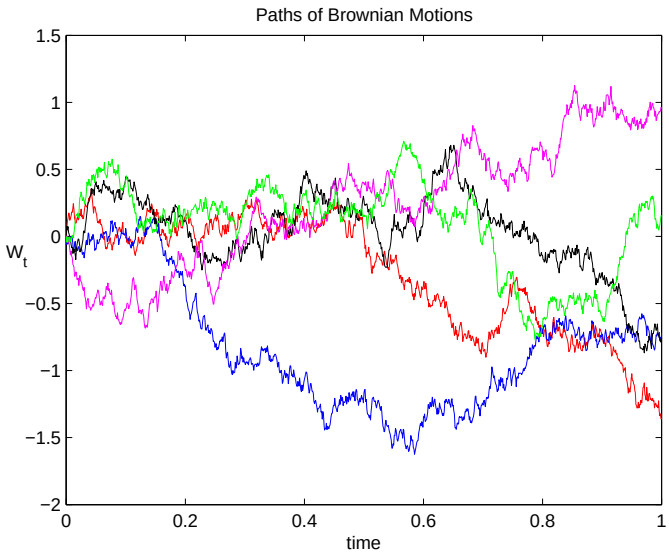


Figure 2.2: Constructing several paths of Brownian motion $\mathbb{W}_t$ for the time interval $[0, 1]$.

**Definition 2.2.** A $d$-dimensional Brownian motion is a continuous and adapted process $\mathbb{W}_t = (W_t^1, W_t^2, \ldots, W_t^d)$ whose values are in $\mathbb{R}^d$, defined on a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, and has the following properties:

$(i)$ $\mathbb{W}_t = 0$ with probability 1,

$(ii)$ the increments $\mathbb{W}_t - \mathbb{W}_s$ are independent of $\mathcal{F}_s$ for each $0 \leq s < t$,

$(iii)$ $\mathbb{W}_t - \mathbb{W}_s$ is normally distributed with 0 mean and $(t-s)\mathrm{Id}_d$ variance.

In finance, we will deal with stochastic differential equations of the form

$$dX_t = a(t, X_t)dt + b(t, X_t)dW_t, \quad X_{t_0} = x_0 \in \mathbb{R} \tag{2.1}$$

in the interval $[t_0, T]$ where the first term is called *drift term* and the second term is called *diffusion term* [29]. However, as the Wiener process is nowhere differentiable, we define the integral form of the equation (2.1) as follows:

$$X_t = X_0 + \int_{t_0}^{t} a(s, X_s)ds + \int_{t_0}^{t} b(s, X_s)dW_s. \tag{2.2}$$

The first integral is a *Riemann integral* which is calculated by the help of ordinary calculus. However, the second integral includes a stochastic term and has unbounded first variation. Therefore, we should define the stochastic integrals such as *Itô* and *Stratonovich stochastic integrals*.

## 2.1 Itô and Stratonovich Integrals

In this part, we will state the most known stochastic integrals, Itô and Stratonovich integrals, with their definitions. We will also mention about their similarities with and the differences between each other [14, 29]. At the end of this section, *Itô* and *Stratonovich* Formulas are given as in the theorems stated in [11, 24].

In order to define stochastic integrals, we consider a partition $0 \leq t_0 < t_1 < \ldots < t_N = T$ of an integration interval $[t_0, T]$ with the property that

$$\max_{0\leq i\leq N-1} (t_{i+1} - t_i) \to 0 \quad \text{as} \quad N \to \infty.$$

Let $\tau_i = \theta t_{i+1} + (1-\theta)t_i$ for a fixed $\theta \in [0, 1]$. Then, the series of random variables

$$\sum_{i=0}^{N-1} X_{\tau_i} \left(W_{t_{i+1}} - W_{t_i}\right) \quad \text{as} \quad N \to \infty \tag{2.3}$$

converge in probability if the process $X_t$ satisfies

$$E\left(\int_{t_0}^{t} X_s^2 ds\right) < \infty, \quad \forall t > 0, \tag{2.4}$$

or converge in the mean-square sense if the process $X_t$ satisfies

$$\mathbb{P}\left(\int_{t_0}^{t} X_s^2 ds < \infty\right) = 1, \quad \forall t > 0. \tag{2.5}$$

For different values of $\theta$, we denote a fixed point of the subinterval $[t_i, t_{i+1}]$ by the corresponding $\tau_i$. For example, for $\theta = 0$, $\tau_i$ describes the left endpoints of intervals $[t_i, t_{i+1}]$ and limit of the series of random variables (2.3) describes the *Itô integral* as

$$\lim_{\Delta t_i \to 0} \sum_{i=0}^{N-1} X_{t_i} \left(W_{t_{i+1}} - W_{t_i}\right) := \int_{t_0}^{t} X_s dW_s.$$

For $\theta = 1/2$, $\tau_i$ represents the midpoints of $[t_i, t_{i+1}]$ and the *Stratonovich integral* by taking the limit of the series of random variables is constructed. We use the notation "$\circ$" to denote the Stratonovich calculus:

$$\lim_{\Delta t_i \to 0} \sum_{i=0}^{N-1} X_{\frac{t_i + t_{i+1}}{2}} \left(W_{t_{i+1}} - W_{t_i}\right) := \int_{t_0}^{t} X_s \circ dW_s.$$

Therefore, $(X_t)_{t \in I}$ is a solution of the Itô stochastic differential equation

$$dX_t = a(t, X_t)dt + b(t, X_t)dW_t, \tag{2.6}$$

where $(W_t)_{t \in I}$ is a Brownian motion, $a(t, X_t)$ and $b(t, X_t)$ are real valued functions.

Moreover, $(X_t)_{t \in I}$ is also a solution of the Stratonovich stochastic differential equation

$$dX_t = \underline{a}(t, X_t)dt + b(t, X_t) \circ dW_t \tag{2.7}$$

with respect to Stratonovich calculus, where $\underline{a}(t, X_t)$ and $b(t, X_t)$ are real valued functions and $\underline{a}(t, X_t)$ is defined as

$$\underline{a}(t, x) := a(t, x) - \frac{1}{2} b(t, x) \frac{\partial b}{\partial x}(t, x). \tag{2.8}$$

We have seen that both Itô and Stratonovich stochastic integrals are used for the same processes, but the only difference between them is that we are looking for the midpoints or the left endpoints of the discretizations for Stratonovich and Itô integrals, respectively [11, 14]. Therefore, there exist some relations and differences between the properties of these two types of integrals because of the transformation of the drift term (2.8).

We can say for the relation of the integrals that we can get the same solution of a stochastic differential equation by using Itô integral as well as Stratonovich integral, so we can always switch one type of the integral to the corresponding integral as seen in the following example [11]: consider a process $X_t$ such as

$$X_t = X_{t_0} \exp\left(2W_t - 2W_{t_0}\right),$$

which is the solution of both Itô stochastic differential equation of the form

$$dX_t = 2X_t dt + 2X_t dW_t,$$

and the Stratonovich stochastic differential equation is

$$dX_t = 2X_t \circ dW_t.$$

The Itô integral has a different property than the Stratonovich integral, namely the *Itô isometry*, which helps us connect *Itô integration* to *Lebesgue integration*.

**Definition 2.3.** Let $f : I \times \Omega \to \mathbb{R}$ be a Borel-measurable function and $X_t : I \times \Omega \to \mathbb{R}$ be an $\mathcal{F}_t$-adapted process with $E\left(\int_{t_0}^{t} X_s^2 ds\right) < \infty$, for all $t \in I$ holds. Then, the following is called *Itô isometry*

$$E\left(\left(\int_{t_0}^{t} f(s, X_s) dW_s\right)^2\right) = E\left(\int_{t_0}^{t} f^2(s, X_s) ds\right). \tag{2.9}$$

When $W_t$ is a Brownian motion with respect to the filtration $(\mathcal{F}_t)_{t\in I}$, the processes $\left(\int_{t_0}^{t} f(s, X_s) dW_s\right)_{t\in I}$ and $W_t$ are martingales with respect to $(\mathcal{F}_t)_{t\in I}$. Therefore, the following property

$$E\left(\int_{t_0}^{t} f(s, X_s) dW_s\right) = 0$$

holds for all $t \in I$.

On the other hand, Stratonovich calculus has another property that is different from Itô calculus. For example, we can easily calculate the following integral with Stratonovich calculus because of the property that we can use similar rules with the ordinary calculus for the Stratonovich case:

$$\int_0^t W_s \circ dW_s = \frac{1}{2} W_t^2.$$

However, we have different result for the same integration by using Itô calculus, which is

$$\int_0^t W_s dW_s = \frac{1}{2} W_t^2 - \frac{1}{2} t.$$

Since Itô and Stratonovich integrals have different rules and structures, we prefer one of them to work with regarding interested studying field. After these properties we have covered, we are ready to give the definition of Itô and Stratonovich processes.

**Definition 2.4.** Let $a(t, X_t)$ and $b(t, X_t)$ be real valued $(\mathcal{F}_t)$-adapted functions, and $b(t, X_t)$ is also Borel-measurable. Then, the stochastic process

$$X_t = X_{t_0} + \int_{t_0}^{t} a(s, X_s) ds + \int_{t_0}^{t} b(s, X_s) dW_s \tag{2.10}$$

is called an *Itô process* if the process satisfies the following two conditions of linear growth and boundedness:

$$\mathbb{P}\left(\int_{t_0}^{t} |a(s, X_s)|ds < \infty \text{ for all } t \geq 0\right) = 1 \tag{2.11}$$

and

$$\mathbb{P}\left(\int_{t_0}^{t} b^2(s, X_s)ds) < \infty \text{ for all } t \geq 0\right) = 1. \tag{2.12}$$

This definition shows that the stochastic integral equation (2.2) is well-defined now.

Furthermore, we can also define the $m$-dimensional stochastic differential equation as follows:

$$d\mathbb{X}_t = \mathbb{A}(t, \mathbb{X}_t)dt + \mathbb{B}(t, \mathbb{X}_t)d\mathbb{W}_t, \quad \mathbb{X}_{t_0} \in \mathbb{R}^m, \tag{2.13}$$

where $\mathbb{W}_t = (W_t^1, W_t^2, \ldots, W_t^d)$ are $d$-dimensional Brownian motion, $\mathbb{A}(t, \mathbb{X}_t) := (a_1(t, \mathbb{X}_t), a_2(t, \mathbb{X}_t), \ldots, a_m(t, \mathbb{X}_t))$ is $\mathbb{R}^m$-valued function and $\mathbb{B}(t, \mathbb{X}_t)$ is $\mathbb{R}^{m\times d}$-valued function which can be written as

$$\mathbb{B}(t, \mathbb{X}_t) = \begin{pmatrix} b_1^1(t, \mathbb{X}_t) & \ldots & b_1^d(t, \mathbb{X}_t) \\ \vdots & \ddots & \vdots \\ b_m^1(t, \mathbb{X}_t) & \ldots & b_m^d(t, \mathbb{X}_t) \end{pmatrix}$$

with $j = 1, 2, \ldots, d$ and $m > 1$.

The $k$th term of a solution $\mathbb{X}_t := (X_t^1, X_t^2, \ldots, X_t^m) \in \mathbb{R}^m$ of a given Itô stochastic differential equation (2.13) has the form

$$X_t^k = X_{t_0}^k + \int_{t_0}^{t} a_k(s, \mathbb{X}_s)ds + \sum_{j=1}^{d} \int_{t_0}^{t} b_k^j(s, \mathbb{X}_s)dW_s^j, \quad X_{t_0}^k = x_0 \in \mathbb{R} \tag{2.14}$$

where $a_k(t, \mathbb{X}_t)$ and $b_k^j(t, \mathbb{X}_t)$ are $\mathbb{R}^m$-valued functions for $k = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, d$.

Similarly, we can define Stratonovich stochastic process in one- and multi-dimensions regarding the transformation of the drift term (2.8).

**Definition 2.5.** Let $\underline{a}(t, X_t)$ and $b(t, X_t)$ be real valued $(\mathcal{F}_t)$-adapted functions, and $b(t, X_t)$ is also Borel-measurable. We assume that $\underline{a}(t, X_t)$ is defined as before in (2.8). Then, the stochastic process

$$X_t = X_{t_0} + \int_{t_0}^{t} \underline{a}(s, X_s)ds + \int_{t_0}^{t} b(s, X_s) \circ dW_s \tag{2.15}$$

is called a *Stratonovich process* if the process satisfies the following two conditions of linear growth and boundedness:

$$\mathbb{P}\left(\int_{t_0}^{t} |\underline{a}(s, X_s)|ds < \infty \text{ for all } t \geq 0\right) = 1 \tag{2.16}$$

and

$$\mathbb{P}\left(\int_{t_0}^{t} b^2(s, X_s)ds) < \infty \text{ for all } t \geq 0\right) = 1. \tag{2.17}$$

In addition, in the multi-dimensional case, the $k$th term of a solution $\mathbb{X}_t := (X_t^1, X_t^2, \ldots, X_t^m) \in \mathbb{R}^m$ of the Stratonovich stochastic integral equation (2.15) takes the following form

$$X_t^k = X_{t_0}^k + \int_{t_0}^{t} \underline{a}_k(s, \mathbb{X}_s)ds + \sum_{j=1}^{d} \int_{t_0}^{t} b_k^j(s, \mathbb{X}_s) \circ dW_s^j, \quad X_{t_0}^k = x_0 \in \mathbb{R} \tag{2.18}$$

with

$$\underline{a}_k = a_k - \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{d} b_i^j \frac{\partial b_k^j}{\partial x_i},$$

where $\underline{a}_k(t, \mathbb{X}_t)$ and $b_k^j(t, \mathbb{X}_t)$ are $\mathbb{R}^m$-valued functions for $k, i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, d$.

In the ordinary calculus, we compute Riemann (or Lebesgue) integrals with the help of Fundamental Theorem of Calculus, Chain Rule and Taylor series. However, in the stochastic calculus, we have diffusion terms with Brownian Motions; therefore, we need some new formulas which are generalized from the ordinary calculus to the stochastic calculus in order to calculate the stochastic integrals. At that point, Itô Formula and Stratonovich Formula, chain rule for random variables, help us to deduce the randomness of continuous-time stochastic processes.

**Theorem 2.1. (*Itô Formula* [11, 22])** *Let $X_t$ be an Itô process defined in (2.10). Suppose that each of the processes $a(t, X_t)$ and $b(t, X_t)$ satisfy the conditions (2.11) and (2.12). Assume that $f(t, x)$ is a twice continuously differentiable function. Then, $Y_t = f(t, X_t)$ is again an Itô process given by*

$$\begin{aligned} dY_t &= \left(\frac{\partial f}{\partial t}(t, X_t) + a(t, X_t)\frac{\partial f}{\partial x}(t, X_t) + \frac{1}{2}b^2(t, X_t)\frac{\partial^2 f}{\partial x^2}(t, X_t)\right) dt \\ &+ b(t, X_t)\frac{\partial f}{\partial x}(t, X_t)dW_t. \end{aligned} \tag{2.19}$$

Similarly, we can give the definition of Stratonovich Formula as follows:

**Theorem 2.2. (*Stratonovich Formula* [11])** *Let $X_t$ be a Stratonovich process defined in (2.15). Suppose that each of the processes $\underline{a}(t, X_t)$ and $b(t, X_t)$ satisfy the conditions (2.16) and (2.17). Let $f(t, x)$ be a twice continuously differentiable function. Then, $Y_t = f(t, X_t)$ is again a Stratonovich process given by*

$$dY_t = \left(\frac{\partial f}{\partial t}(t, X_t) + \underline{a}(t, X_t)\frac{\partial f}{\partial x}(t, X_t)\right) dt + b(t, X_t)\frac{\partial f}{\partial x}(t, X_t) \circ dW_t. \tag{2.20}$$

*Proof.* Proofs of Theorems 2.1 and 2.2 can be found in Kloeden & Platen [11]. □

Up to now, we deal with the stochastic integrals and their properties. At the end of this section, Itô and Stratonovich formulas are given in order to obtain the stochastic Taylor expansions in the next section.

## 2.2 Stochastic Taylor Expansions

In this section, we will define Taylor series expansions and use them to find an approximate solution of a given stochastic differential equation. In fact, stochastic Taylor expansions can be considered as an extension of ordinary Taylor expansions. In this area, general Taylor expansion formulas were firstly introduced by Platen and Wagner [23], then Kloeden and Platen gave the most known versions of the stochastic Taylor expansions in detail in their book [11].

We will evaluate the stochastic Taylor expansions in two subsections. Initially, Itô-Taylor expansions will be discussed, then we will reformulate and generalize the Itô-Taylor expansion to the case of Stratonovich-Taylor.

### 2.2.1 Itô-Taylor Expansion

In this subsection, we try to find the stochastic Taylor expansion of an Itô integral at some point. It can be thought that the process is a generalization of an ordinary Taylor expansion by using Itô formula (2.19).

Firstly, let $X_t$ be the solution of an Itô stochastic differential equation (2.1) which is in the integral form

$$X_t = X_{t_0} + \int_{t_0}^{t} a(s, X_s)ds + \int_{t_0}^{t} b(s, X_s)dW_s, \tag{2.21}$$

where real-valued functions $a(t, X_t)$ and $b(t, X_t)$ are defined as before. Let $Y_t = f(t, X_t)$ where $f : I \times \mathbb{R} \to \mathbb{R}$ is a twice continuously differentiable function as defined in Theorem 2.1. In order to expand Itô-Taylor series recursively, we consider the integral form of the Itô formula (2.19), given by the following,

$$\begin{aligned} Y_t = Y_{t_0} \quad &+ \quad \int_{t_0}^{t} \left\{ \frac{\partial f}{\partial t}(s, X_s) + a(s, X_s)\frac{\partial f}{\partial x}(s, X_s) + \frac{1}{2}b^2(s, X_s)\frac{\partial^2 f}{\partial x^2}(s, X_s) \right\} ds \\ &+ \quad \int_{t_0}^{t} b(s, X_s)\frac{\partial f}{\partial x}(s, X_s)dW_s. \end{aligned} \tag{2.22}$$

We define two operators $L^0$ and $L^1$ to make this equation simpler:

$$L^0 := \frac{\partial}{\partial t} + a\frac{\partial}{\partial x} + \frac{1}{2}b^2\frac{\partial^2}{\partial x^2} \tag{2.23}$$

and

$$L^1 := b\frac{\partial}{\partial x}. \tag{2.24}$$

Now, we can write Itô formula (2.22) in terms of the operator functions as in [11]:

$$Y_t = Y_{t_0} + \int_{t_0}^{t} L^0 f(s, X_s)ds + \int_{t_0}^{t} L^1 f(s, X_s)dW_s.$$

When we apply the Itô formula (2.22) to the function $f = a$ in (2.21), we obtain

$$\begin{aligned} X_t = X_{t_0} &+ \int_{t_0}^{t} \left\{ a(t_0, X_{t_0}) + \int_{t_0}^{s} L^0 a(u, X_u)du + \int_{t_0}^{s} L^1 a(u, X_u)dW_u \right\} ds \\ &+ \int_{t_0}^{t} b(s, X_s)dW_s. \end{aligned} \tag{2.25}$$

If we seperate the extra terms and reorganize the last equation in order to make it similar to (2.21), we have

$$X_t = X_{t_0} + \int_{t_0}^{t} a(t_0, X_{t_0})ds + \int_{t_0}^{t} b(s, X_s)dW_s + R_t$$

where the remainder term $R_t$ is given by

$$R_t = \int_{t_0}^{t}\int_{t_0}^{s} L^0 a(u, X_u)duds + \int_{t_0}^{t}\int_{t_0}^{s} L^1 a(u, X_u)dW_u ds.$$

We can continue applying the Itô formula (2.22) to the function $f = L^0 a$ in (2.25) to get

$$\begin{aligned} X_t &= X_{t_0} + \int_{t_0}^{t} a(t_0, X_{t_0})ds + \int_{t_0}^{t} b(s, X_s)dW_s + \int_{t_0}^{t}\int_{t_0}^{s} L^1 a(u, X_u)dW_u ds \\ &+ \int_{t_0}^{t}\int_{t_0}^{s} \left\{ L^0 a(t_0, X_{t_0}) + \int_{t_0}^{u} L^0 L^0 a(v, X_v)dv + \int_{t_0}^{u} L^1 L^0 a(v, X_v)dW_v \right\} duds. \end{aligned}$$

When we arrange the above equation again, we have

$$X_t = X_{t_0} + \int_{t_0}^{t} a(t_0, X_{t_0})ds + \int_{t_0}^{t}\int_{t_0}^{s} L^0 a(t_0, X_{t_0})duds + \int_{t_0}^{t} b(s, X_s)dW_s + R_t$$

with the remainder

$$\begin{aligned} R_t &= \int_{t_0}^{t}\int_{t_0}^{s}\int_{t_0}^{u} L^0 L^0 a(v, X_v)dvduds + \int_{t_0}^{t}\int_{t_0}^{s}\int_{t_0}^{u} L^1 L^0 a(v, X_v)dW_v duds \\ &+ \int_{t_0}^{t}\int_{t_0}^{s} L^1 a(u, X_u)dW_u ds. \end{aligned}$$

Or, at the first, we can also apply the Itô formula (2.22) to both the functions $f = a$ and $f = b$ in (2.21) at the same time, and we have

$$\begin{aligned} X_t &= X_{t_0} + \int_{t_0}^{t} \left\{ a(t_0, X_{t_0}) + \int_{t_0}^{s} L^0 a(u, X_u)du + \int_{t_0}^{s} L^1 a(u, X_u)dW_u \right\} ds \\ &+ \int_{t_0}^{t} \left\{ b(t_0, X_{t_0}) + \int_{t_0}^{s} L^0 b(u, X_u)du + \int_{t_0}^{s} L^1 b(u, X_u)dW_u \right\} dW_s. \end{aligned} \tag{2.26}$$

We can then rewrite the equation (2.21) and obtain

$$X_t = X_{t_0} + \int_{t_0}^{t} a(t_0, X_{t_0})ds + \int_{t_0}^{t} b(t_0, X_{t_0})dW_s + R_t,$$

where $R_t$ stands for the remainder term again, and which can be expressed as

$$\begin{aligned} R_t &= \int_{t_0}^{t}\int_{t_0}^{s} L^0 a(u, X_u)duds + \int_{t_0}^{t}\int_{t_0}^{s} L^1 a(u, X_u)dW_u ds \\ &\quad + \int_{t_0}^{t}\int_{t_0}^{s} L^0 b(u, X_u)dudW_s + \int_{t_0}^{t}\int_{t_0}^{s} L^1 b(u, X_u)dW_u dW_s. \end{aligned}$$

Recursively, we can keep applying the Itô formula (2.22) to the function $f = L^1 b$ in (2.26),

$$\begin{aligned} X_t = X_{t_0} &+ \int_{t_0}^{t} a(t_0, X_{t_0})ds + \int_{t_0}^{t} b(t_0, X_{t_0})dW_s \\ &+ \int_{t_0}^{t}\int_{t_0}^{s} L^1 b(t_0, X_{t_0})dW_u dW_s + R_t \end{aligned}$$

with a more complex form of the remainder

$$\begin{aligned} R_t &= \int_{t_0}^{t}\int_{t_0}^{s} L^0 a(u, X_u)duds + \int_{t_0}^{t}\int_{t_0}^{s} L^1 a(u, X_u)dW_u ds \\ &\quad + \int_{t_0}^{t}\int_{t_0}^{s} L^0 b(u, X_u)dudW_s + \int_{t_0}^{t}\int_{t_0}^{s}\int_{t_0}^{u} L^0 L^1 b(v, X_v)dvdW_u dW_s \\ &\quad + \int_{t_0}^{t}\int_{t_0}^{s}\int_{t_0}^{u} L^1 L^1 b(v, X_v)dW_v dW_u dW_s. \end{aligned}$$

If we continue in this way by applying the Itô formula (2.22) to the functions $f = L^i a$ and $f = L^i b$ for $i = 0, 1$ in the next steps, we will have the *Itô-Taylor expansion*. To formalize this process, we need to solve and generalize the above equations to the multi-dimensional cese. However, this far is not enough yet to define general stochastic Taylor expansions. Firstly, we need to introduce multi-indices and hierarchical sets [11, 27].

**Definition 2.6.** [26] Let $\mathcal{M}$ be the set of all *multi-indices* defined as

$$\mathcal{M} := \{\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_l) : l \in \mathbb{N}\} \cup \{v\}.$$

where $\alpha_i \in \{0, 1, \ldots, m\}$, $m = 1, 2, \ldots$, $i = 1, 2, \ldots, l$ and $v$ is the multi-index of length 0. We denote the length of a multi-index by $l(\alpha)$ and the number of components of a multi-index which are equal to 0 by $n(\alpha)$.

For example,

$$\begin{aligned} \alpha = (0, 1, 0) \quad &\Rightarrow \quad l(0, 1, 0) = 3, \quad n(0, 1, 0) = 2, \\ \alpha = (1, 2, 0) \quad &\Rightarrow \quad l(1, 2, 0) = 3, \quad n(1, 2, 0) = 1. \end{aligned}$$

We will use the notations $-\alpha = (\alpha_2, \alpha_3, \ldots, \alpha_l)$ and $\alpha- = (\alpha_1, \alpha_2, \ldots, \alpha_{l-1})$ for the indices whose lengths are greater than 1 in order to denote the indexes by deleting the first and the last component of $\alpha$, respectively. For example,

$$\begin{aligned} \alpha = (0,1,0) \quad &\Rightarrow \quad -(0,1,0) = (1,0), \quad (0,1,0)- = (0,1), \\ \alpha = (1,2,0) \quad &\Rightarrow \quad -(1,2,0) = (2,0), \quad (1,2,0)- = (1,2). \end{aligned}$$

**Definition 2.7.** [26] Let $\mathcal{A}$ be a nonempty subset of $\mathcal{M}$. $\mathcal{A}$ is called as a *hierarchical set* if it satisfies

$$\sup_{\alpha \in \mathcal{A}} l(\alpha) < \infty \quad \text{and} \quad -\alpha \in \mathcal{A} \qquad \text{for each} \quad \alpha \in \mathcal{A} \setminus \{v\}$$

and $\mathcal{R}(\mathcal{A})$ is called the *remainder set* if it consists of all next following multi-indices with respect to the given hierarchical set $\mathcal{A}$, which is defined as follows:

$$\mathcal{R}(\mathcal{A}) := \{\alpha \in \mathcal{M} \setminus \mathcal{A} : -\alpha \in \mathcal{A}\}.$$

Moreover, we have to define some classes for adapted right continuous stochastic process $(f_t)_{t \in I}$ with left hand limits [11, 26]. For each $t \geq 0$,

- $f \in \mathcal{H}_v \quad$ if $\quad |f(t, X_t)| < \infty \quad \mathcal{P}$-a.s.,
- $f \in \mathcal{H}_{(0)} \quad$ if $\quad \int_0^t |f(s, X_s)| ds < \infty \quad$ w.p.1,
- $f \in \mathcal{H}_{(1)} \quad$ if $\quad \int_0^t |f(s, X_s)|^2 ds < \infty \quad$ w.p.1.

Thus, we can write $\mathcal{H}_{(i)} = \mathcal{H}_{(1)}$, for all $i = 2, 3, \ldots, l$ where $l \geq 2$ as well as $\mathcal{H}_{(\alpha)}$ for multi-indices $\alpha$ with $l(\alpha) > 1$.

We need multiple Itô integrals to solve these stochastic processes $(f_t)_{t \in I}$ in the integration interval.

**Definition 2.8.** [11, 27] Let $\rho$ and $\tau$ be two stopping times with $t_0 \leq \rho \leq \tau \leq T$. For a multi-index $\alpha \in \mathcal{M}$ and a process $f \in \mathcal{H}_{(\alpha)}$, *multiple Itô integral* $I_\alpha[f(\cdot)]_{\rho,\tau}$ with respect to the $d$-dimensional Brownian motion $W_t$ recursively is defined as follows:

$$I_\alpha[f(\cdot)]_{\rho,\tau} := \begin{cases} f(\tau) & \text{if} \quad l = 0, \\ \int_\rho^\tau I_{\alpha-}[f(\cdot)]_{\rho,s} ds & \text{if} \quad l \geq 1 \text{ and } \alpha_l = 0, \\ \int_\rho^\tau I_{\alpha-}[f(\cdot)]_{\rho,s} dW_s^{\alpha_l} & \text{if} \quad l \geq 1 \text{ and } \alpha_l \geq 1. \end{cases}$$

In order to be more clear about the usage of multiple Itô integrals $I_\alpha[f(\cdot)]$, we give some examples for a constant function $f \equiv 1$ with the condition $0 \leq t_0 \leq t \leq T$

as follows:

$$
\begin{aligned}
I_{(0)}[f(\cdot)]_{t_0,t} &= \int_{t_0}^{t} 1du = t - t_0, && (2.27)\\
I_{(1)}[f(\cdot)]_{t_0,t} &= \int_{t_0}^{t} 1dW_s = W_t - W_{t_0}, && (2.28)\\
I_{(1,0)}[f(\cdot)]_{t_0,t} &= \int_{t_0}^{t} I_{(1)}[f(\cdot)]_{t_0,s}ds = \int_{t_0}^{t} (W_s - W_{t_0})ds, && (2.29)\\
I_{(0,1)}[f(\cdot)]_{t_0,t} &= \int_{t_0}^{t} I_{(0)}[f(\cdot)]_{t_0,s}dW_s = \int_{t_0}^{t} (s - t_0)dW_s, && (2.30)\\
I_{(1,0,2)}[f(\cdot)]_{t_0,t} &= \int_{t_0}^{t} I_{(1,0)}[f(\cdot)]_{t_0,s}dW_s^2 = \int_{t_0}^{t}\int_{t_0}^{s} I_{(1)}[f(\cdot)]_{t_0,u}dudW_s^2\\
&= \int_{t_0}^{t}\int_{t_0}^{s}\int_{t_0}^{u} 1dW_v^1 dudW_s^2\\
&= \int_{t_0}^{t}\left(\int_{t_0}^{s}(W_u^1 - W_{t_0}^1)du\right)dW_s^2. && (2.31)
\end{aligned}
$$

While we have one-dimensional Brownian motion $W_t$ in the equations (2.27)-(2.30), there is 2-dimensional Brownian motion $\mathbb{W}_t = (W_t^1, W_t^2)$ in the last equation (2.31). The upper numbers of $W_t$ represent which Brownian motion is used. We note that if there is only one-dimensional Brownian motion, we use the notation $W_t$ without any sup-script.

We want to define the Itô-Taylor expansion; therefore, we need the Itô coefficient functions as well. Let us generalize the operators which were defined for the one-dimensional case in the equations (2.23) and (2.24) at the beginning of the subsection. For the $k$th component of the generalized operators for $m$-dimensional Itô process with $d$-dimensional Brownian motion, we have the following equations

$$L^0 := \frac{\partial}{\partial t} + \sum_{k=1}^{m} a_k \frac{\partial}{\partial x_k} + \frac{1}{2}\sum_{k,i=1}^{m}\sum_{j=1}^{d} b_k^j b_i^j \frac{\partial^2}{\partial x_k \partial x_i} \tag{2.32}$$

and

$$L^j := \sum_{i=1}^{m} b_i^j \frac{\partial}{\partial x_i} \tag{2.33}$$

for $i, k = 1, 2, \ldots, m$ and $j = 0, 1, \ldots, d$ [11].

For each multi-index $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_l) \in \mathcal{M}$ and a continuously differentiable function $f(t, \mathbb{X}_t)$, the *Itô coefficient function* will be defined in the following look:

$$f_\alpha := \begin{cases} f & \text{if} \quad l = 0,\\ L^{\alpha_1}\cdots L^{\alpha_l} f & \text{if} \quad l \geq 1. \end{cases}$$

Now, we are ready to state the Itô-Taylor expansion theorem:

**Theorem 2.3.** (***Itô-Taylor Expansion***[11, 27]) *Let $\mathcal{A} \subseteq \mathcal{M}$ be a hierarchical set, $\rho$ and $\tau$ be two stopping times with $t_0 \leq \rho \leq \tau \leq T$ and let $f(t, \mathbb{X}_t) : I \times \mathbb{R}^m \to \mathbb{R}$. Then for the solution $(\mathbb{X}_t)_{t \in I}$ of the Itô stochastic differential equation (2.13), the Itô-Taylor expansion*

$$f(\tau, \mathbb{X}_\tau) = \sum_{\alpha \in \mathcal{A}} I_\alpha[f_\alpha(\rho, \mathbb{X}_\rho)]_{\rho,\tau} + \sum_{\alpha \in \mathcal{R}(\mathcal{A})} I_\alpha[f_\alpha(\cdot, \mathbb{X}_\cdot)]_{\rho,\tau} \tag{2.34}$$

*holds, provided that all of the derivatives of $f(t, \mathbb{X}_t)$, $a(t, \mathbb{X}_t)$ and $b(t, \mathbb{X}_t)$ and all of the multiple Itô integrals appearing in (2.34) exist.*

*Proof.* Proof of the theorem can be found in [11]. □

### 2.2.2 Stratonovich-Taylor Expansion

By using a similar way with Itô-Taylor expansion, we can now derive Stratonovich-Taylor expansion for stochastic processes. We always remember that we can turn one of the stochastic calculus into the other. Therefore, some equations which show the relations between Itô and Stratonovich calculus are given in this subsection.

Let us recall the process $X_t$, defined in (2.15), be the solution of Stratonovich stochastic differential equation in the integral form

$$X_t = X_{t_0} + \int_{t_0}^{t} \underline{a}(s, X_s)ds + \int_{t_0}^{t} b(s, X_s) \circ dW_s \tag{2.35}$$

with $\underline{a} = a - (1/2)bb'$, where $b'$ denotes the partial derivative with respect to the second argument, namely, $b'(t, x) = \frac{\partial b}{\partial x}(t, x)$. Let $f : I \times \mathbb{R} \to \mathbb{R}$ be a continuously twice differentiable function, then we get the integral form of *Stratonovich formula* (2.20) in the following equation:

$$\begin{aligned} f(t, X_t) = f(t_0, X_{t_0}) \quad &+ \quad \int_{t_0}^{t} \left\{ \frac{\partial f}{\partial t}(s, X_s) + \underline{a}(s, X_s)\frac{\partial f}{\partial x}(s, X_s) \right\} ds \\ &+ \quad \int_{t_0}^{t} b(s, X_s)\frac{\partial f}{\partial x}(s, X_s) \circ dW_s. \end{aligned} \tag{2.36}$$

It can be seen that the first integral has 2 terms, not 3 terms as in the Itô formula (2.19). This situation leads an opportunity to use the chain rule of ordinary (deterministic) calculus to the Stratonovich stochastic differential equation.

Now, we define the operators of Stratonovich-Taylor expansion as follows:

$$\underline{L}^0 := \frac{\partial}{\partial t} + \underline{a}\frac{\partial}{\partial x} \tag{2.37}$$

and

$$\underline{L}^1 := b\frac{\partial}{\partial x} \tag{2.38}$$

for $\underline{a} = a - (1/2)bb'$. Then, the equation (2.36) can be written by using these operators as follows:

$$f(t, X_t) = f(t_0, X_{t_0}) + \int_{t_0}^{t} \underline{L}^0 f(s, X_s) ds + \int_{t_0}^{t} \underline{L}^1 f(s, X_s) \circ dW_s.$$

When we apply the Stratonovich formula (2.36) to the functions $f = a$ and $f = b$ at the same time, we obtain

$$\begin{aligned} X_t = X_{t_0} &+ \int_{t_0}^{t} \left\{ \underline{a}(t_0, X_{t_0}) + \int_{t_0}^{s} \underline{L}^0 \underline{a}(u, X_u) du + \int_{t_0}^{s} \underline{L}^1 \underline{a}(u, X_u) \circ dW_u \right\} ds \\ &+ \int_{t_0}^{t} \left\{ b(t_0, X_{t_0}) + \int_{t_0}^{s} \underline{L}^0 b(u, X_u) du + \int_{t_0}^{s} \underline{L}^1 b(u, X_u) \circ dW_u \right\} \circ dW_s. \quad (2.39) \end{aligned}$$

We can write the integral form of (2.35) with a remainder term

$$X_t = X_{t_0} + \int_{t_0}^{t} \underline{a}(t_0, X_{t_0}) ds + \int_{t_0}^{t} b(t_0, X_{t_0}) \circ dW_s + R_t,$$

where

$$\begin{aligned} R_t &= \int_{t_0}^{t} \int_{t_0}^{s} \underline{L}^0 \underline{a}(u, X_u) du ds + \int_{t_0}^{t} \int_{t_0}^{s} \underline{L}^1 \underline{a}(u, X_u) \circ dW_u ds \\ &+ \int_{t_0}^{t} \int_{t_0}^{s} \underline{L}^0 b(u, X_u) du \circ dW_s + \int_{t_0}^{t} \int_{t_0}^{s} \underline{L}^1 b(u, X_u) \circ dW_u \circ dW_s. \end{aligned}$$

If we continue in this way and apply the Stratonovich formula (2.36) to a function, for example, $f = \underline{L}^1 b$ in (2.39), we obtain

$$\begin{aligned} X_t = X_{t_0} &+ \int_{t_0}^{t} \underline{a}(t_0, X_{t_0}) ds + \int_{t_0}^{t} b(t_0, X_{t_0}) \circ dW_s \\ &+ \int_{t_0}^{t} \int_{t_0}^{s} \underline{L}^1 b(t_0, X_{t_0}) \circ dW_u \circ dW_s + R_t, \end{aligned}$$

where $R_t$ stands for the new remainder term, expressed as

$$\begin{aligned} R_t &= \int_{t_0}^{t} \int_{t_0}^{s} \underline{L}^0 \underline{a}(u, X_u) du ds + \int_{t_0}^{t} \int_{t_0}^{s} \underline{L}^1 \underline{a}(u, X_u) \circ dW_u ds \\ &+ \int_{t_0}^{t} \int_{t_0}^{s} \underline{L}^0 b(u, X_u) du \circ dW_s + \int_{t_0}^{t} \int_{t_0}^{s} \int_{t_0}^{u} \underline{L}^0 \underline{L}^1 b(v, X_v) dv \circ dW_u \circ dW_s \\ &+ \int_{t_0}^{t} \int_{t_0}^{s} \int_{t_0}^{u} \underline{L}^1 \underline{L}^1 b(v, X_v) \circ dW_v \circ dW_u \circ dW_s. \end{aligned}$$

We can recursively continue with the usage of the Stratonovich formula (2.36) to the functions $f = \underline{L}^i \underline{a}$ and $f = \underline{L}^i b$ for $i = 0, 1$ in the next steps to obtain the

*Stratonovich-Taylor expansion.* We do not need to state similar examples again and again, we can recommend the book [11] for detailed explanations about this process.

We again define some classes for adapted right continuous stochastic process $(\underline{f}_t)_{t\in I}$ with left hand limits according to the Stratonovich calculus. For each $t \geq 0$,

- $\underline{f} \in \underline{\mathcal{H}}_v$ if $|\underline{f}(t, X_t)| < \infty$ $\mathcal{P}$-a.s.,
- $\underline{f} \in \underline{\mathcal{H}}_{(0)}$ if $\int_0^t |\underline{f}(s, X_s)|ds < \infty$ w.p.1,
- $\underline{f} \in \underline{\mathcal{H}}_{(1)}$ if $\int_0^t |\underline{f}(s, X_s)|^2 ds < \infty$ w.p.1.

Thus, we can write $\underline{\mathcal{H}}_{(i)} = \underline{\mathcal{H}}_{(1)}$, for all $i = 2, 3, \ldots, l$ where $l \geq 2$ as well as $\underline{\mathcal{H}}_{(\alpha)}$ for multi-indices $\alpha$ with $l(\alpha) > 1$.

We have multiple Stratonovich integral in order to solve these stochastic processes $(\underline{f}_t)_{t\in I}$ in a time interval.

**Definition 2.9.** [11, 27] Let $\rho$ and $\tau$ be two stopping times with $t_0 \leq \rho \leq \tau \leq T$. For a multi-index $\alpha \in \mathcal{M}$ and a process $\underline{f} \in \underline{\mathcal{H}}_{(\alpha)}$, *multiple Stratonovich integral* $J_\alpha[\underline{f}(\cdot)]_{\rho,\tau}$ with respect to the $d$-dimensional Brownian motion $W_t$ recursively is defined as follows:

$$J_\alpha[\underline{f}(\cdot)]_{\rho,\tau} := \begin{cases} \underline{f}(\tau) & \text{if} \quad l = 0, \\ \int_\rho^\tau J_{\alpha-}[\underline{f}(\cdot)]_{\rho,s} ds & \text{if} \quad l \geq 1 \text{ and } \alpha_l = 0, \\ \int_\rho^\tau J_{\alpha-}[\underline{f}(\cdot)]_{\rho,s} \circ dW_s^{\alpha_l} & \text{if} \quad l \geq 1 \text{ and } \alpha_l \geq 1. \end{cases}$$

For a constant function $\underline{f} \equiv 1$ with the condition $0 \leq t_0 \leq t \leq T$, we demonstrate the usage of multiple Stratonovich integrals in below examples

$$J_{(0)}[\underline{f}(\cdot)]_{t_0,t} = \int_{t_0}^t 1 du = t - t_0, \tag{2.40}$$

$$J_{(1)}[\underline{f}(\cdot)]_{t_0,t} = \int_{t_0}^t 1 \circ dW_s = W_t - W_{t_0}, \tag{2.41}$$

$$\begin{aligned} J_{(1,1)}[\underline{f}(\cdot)]_{t_0,t} &= \int_{t_0}^t J_{(1)}[\underline{f}(\cdot)]_{t_0,s} \circ dW_s \\ &= \int_{t_0}^t (W_s - W_{t_0}) \circ dW_s = \frac{1}{2}(W_t - W_{t_0})^2, \end{aligned} \tag{2.42}$$

$$J_{(0,1)}[\underline{f}(\cdot)]_{t_0,t} = \int_{t_0}^t J_{(0)}[\underline{f}(\cdot)]_{t_0,s} \circ dW_s = \int_{t_0}^t (s - t_0) \circ dW_s, \tag{2.43}$$

$$\begin{aligned} J_{(1,0,2)}[\underline{f}(\cdot)]_{t_0,t} &= \int_{t_0}^t J_{(1,0)}[\underline{f}(\cdot)]_{t_0,s} \circ dW_s^2 = \int_{t_0}^t \int_{t_0}^s J_{(1)}[\underline{f}(\cdot)]_{t_0,u} du \circ dW_s^2 \\ &= \int_{t_0}^t \int_{t_0}^s \int_{t_0}^u 1 \circ dW_v^1 du \circ dW_s^2, \end{aligned} \tag{2.44}$$

Repeatedly, the numbers above of $W_t$'s represent the Brownian motion used and if we do not write any number above, then we understand that we have one-dimensional Brownian motion.

We want to define the Stratonovich-Taylor expansion; therefore, we need to introduce Stratonovich coefficient functions. Let us generalize the operators in the equations (2.37) and (2.38) from one-dimensional case to the multi-dimensional case. The $k$-th component of the generalized operators for $m$-dimensional Stratonovich process with $d$-dimensional Brownian motion:

$$\underline{L}^0 := \frac{\partial}{\partial t} + \sum_{k=1}^{m} \underline{a}_k \frac{\partial}{\partial x_k} \tag{2.45}$$

and

$$\underline{L}^j := \sum_{i=1}^{m} b_i^j \frac{\partial}{\partial x_i} \tag{2.46}$$

where

$$\underline{a}_k := a_k - \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{d} b_i^j \frac{\partial b_k^j}{\partial x_i}$$

for $k, i = 1, 2, \ldots, m$ and $j = 0, 1, \ldots, d$. For each multi-index $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_l) \in \mathcal{M}$ and a continuously differentiable function $\underline{f}(t, \mathbb{X}_t)$, the *Stratonovich coefficient function* is defined to be

$$\underline{f}_{\alpha} := \begin{cases} \underline{f} & \text{if} \quad l = 0, \\ \underline{L}^{j_l} \underline{f}_{-\alpha} & \text{if} \quad l \geq 1. \end{cases}$$

Now, we can define the Stratonovich-Taylor expansion in the following theorem.

**Theorem 2.4.** (***Stratonovich-Taylor Expansion***[11, 27]) *Let $\mathcal{A} \subseteq \mathcal{M}$ be a hierarchical set, $\rho$ and $\tau$ be two stopping times with $t_0 \leq \rho \leq \tau \leq T$ and let $\underline{f}(t, \mathbb{X}_t) : I \times \mathbb{R}^m \to \mathbb{R}$. Then, for $(\mathbb{X}_t)_{t \in I}$, defined in (2.2), the Stratonovich-Taylor expansion is introduced as follows:*

$$f(\tau, \mathbb{X}_\tau) = \sum_{\alpha \in \mathcal{A}} J_\alpha[\underline{f}_\alpha(\rho, \mathbb{X}_\rho)]_{\rho,\tau} + \sum_{\alpha \in \mathcal{R}(\mathcal{A})} J_\alpha[\underline{f}_\alpha(\cdot, \mathbb{X}_\cdot)]_{\rho,\tau} \tag{2.47}$$

*provided that all of the derivatives of $\underline{f}(t, \mathbb{X}_t)$, $\underline{a}(t, \mathbb{X}_t)$ and $b(t, \mathbb{X}_t)$ and all of the multiple Stratonovich integrals appearing in (2.47) exist.*

*Proof.* Proof of the theorem can be found in [11]. □

So far, we have reviewed both Itô-Taylor and Stratonovich-Taylor expansions. Since every Itô integral has an equivalent Stratonovich representation, transformation of coefficient functions to each other is possible, as well. To illustrate, we can analyze in three cases [11]:

- for $l = 0$, we have

$$J_\alpha[f(\cdot, X_\cdot)]_{\rho,\tau} = I_\alpha[f(\cdot, X_\cdot)]_{\rho,\tau} = f(\tau, X_\tau),$$

- for $l = 1$, we have

$$J_{(j_1)}[f(\cdot, X_\cdot)]_{\rho,\tau} = I_{(j_1)}[f(\cdot, X_\cdot)]_{\rho,\tau} + I_{\{j_1 \neq 0\}} I_{(0)} \left[ \frac{1}{2} \underline{L}^{j_1} f(\cdot, X_\cdot) \right]_{\rho,\tau},$$

- for $l \geq 2$, we have

$$\begin{aligned} J_\alpha[f(\cdot, X_\cdot)]_{\rho,\tau} &= I_{(j_1)} \left[ J_{\alpha-}[f(\cdot, X_\cdot)]_{\rho,\cdot} \right]_{\rho,\tau} \\ &+ I_{\{j_l = j_{l-1} \neq 0\}} I_{(0)} \left[ \tfrac{1}{2} J_{(\alpha-)-}[f(\cdot, X_\cdot)]_{\rho,\cdot} \right]_{\rho,\tau}, \end{aligned}$$

where $t_0 \leq \rho \leq \tau \leq T$ are stopping times and $\alpha \in \mathcal{M}$ are multi-indices.

By using the above equations, we might switch one stochastic calculus to the other one and it supplies an opportunity to study on any field with the appropriate stochastic calculus. In finance, we are mostly interested in Itô stochastic differential equations; therefore, from now on, we continue to study on Itô-Taylor approximations instead of analyzing both types of the Taylor approximations. However, if it is needed, we know how to interchange the type of the stochastic calculus from one to another.

# CHAPTER 3

# STOCHASTIC TAYLOR SCHEMES

This chapter aims to analyze strong Taylor approximations and weak Taylor approximations by using Itô-Taylor expansions defined in the previous chapter and in [7, 16, 36, 37, 38]. Before we cover the main topic of the chapter, we give a time discretization on the integration interval. we take into account the convergence criteria which leads us to give the special forms of Itô-Taylor expansions of the stochastic differential equations to measure the order of Itô-Taylor schemes at the discretization points [11]. Firstly, Euler-Maruyama scheme, the simplest form of Itô-Taylor expansions, will be investigated according to both strong and weak convergence criteria. Secondly, adding an additional term from Itô-Taylor expansion to Euler-Maruyama scheme, we get another approximation method, so-called Milstein scheme. Then, we derive higher order Taylor schemes by taking more terms from Itô-Taylor expansion; for example, order 2 weak Taylor scheme and order 3/2 strong Taylor scheme. At the last section of this chapter, we analyze related Runge-Kutta schemes in different stages in deterministic and stochastic sense.

## 3.1 Introduction

In this section, we give some basic definitions about Itô processes and then convergence criteria are stated at the end.

Throughout this chapter, we refer to the Itô process satisfying the stochastic integral equation (2.10)

$$X_t = X_{t_0} + \int_{t_0}^{t} a(s, X_s)ds + \int_{t_0}^{t} b(s, X_s)dW_s \tag{3.1}$$

for $t \in [t_0, T]$. Let us introduce a partition of the time interval $I = [t_0, T]$ such as

$$t_0 \leq t_1 < \cdots < t_n < \cdots < t_N = T. \tag{3.2}$$

The choice of the discretization points, which can be left endpoints or midpoints of the subintervals $[t_n, t_{n+1}]$ with $n = 0, 1, \ldots, N-1$, affects the type of stochastic calculus studied, Itô calculus or Stratonovich calculus, respectively. However,

we will continue to utilize Itô calculus only while defining the approximation methods.

Let $(Y_t)_{t\in I}$ be an approximation process to the exact solution $X_t$ as follows:

$$Y_t = Y_{t_0} + \int_{t_0}^{t} a(s, X_s)ds + \int_{t_0}^{t} b(s, X_s)dW_s, \quad t \in [t_0, T] \tag{3.3}$$

We denote the approximation process by $Y_{t_n} := Y_n$ at the discretization points for $n = 0, 1, \ldots, N$. The process $Y_{t_{n+1}}$ can be written recursively

$$Y_{n+1} = Y_n + \int_{t_n}^{t_{n+1}} a(s, X_s)ds + \int_{t_n}^{t_{n+1}} b(s, X_s)dW_s. \tag{3.4}$$

We can make some simplifications of multiple Itô integrals, defined in equations (2.27)–(2.30) in the previous chapter, so as to use them in the aprroximation schemes in the following sections. Let the notation $\Delta_n$ indicate the increment of subintervals $[t_n, t_{n+1}]$ of the time discretization (3.2) as follows:

$$\Delta_n := \int_{t_n}^{t_{n+1}} dt = t_{n+1} - t_n \tag{3.5}$$

which is also equal to $I_{(0)}[f(\cdot)]_{t_n,t_{n+1}}$ in (2.27).

The increment of the one-dimensional standard Brownian motion $W_t$ on the time interval $[t_n, t_{n+1}]$ is denoted by the following:

$$\Delta W_n := \int_{t_n}^{t_{n+1}} dW_t = W_{t_{n+1}} - W_{t_n}, \tag{3.6}$$

which is equal to $I_{(1)}[f(\cdot)]_{t_n,t_{n+1}}$ in (2.28) as well.

Moreover, in the next sections, the terms $I_{(1,0),\Delta_n}$ and $I_{(0,1),\Delta_n}$ in (2.29) and (2.30), respectively, come into the picture in the stochastic Taylor approximation methods. However, the computations of these terms are expensive. When we realize that these terms are also random variables, we define $\Delta Z_n$ as a new random variable, which is equal to the multiple Itô integral $I_{(1,0),\Delta_n}$, with the following properties:

- the mean is $E(\Delta Z_n) = 0$,
- the variance is $Var(\Delta Z_n) = \frac{1}{3}\Delta_n^3$,
- the correlation with $\Delta W_n$ is $E(\Delta W_n, \Delta Z_n) = \frac{1}{2}\Delta_n^2$.

Then, we can introduce these two multiple Itô integrals in terms of $\Delta Z_n$ in the following

$$I_{(1,0),\Delta_n} = \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} dW_u ds = \int_{t_n}^{t_{n+1}} (dW_s - dW_{t_n})ds = \Delta Z_n \tag{3.7}$$

and

$$I_{(0,1),\Delta_n} = \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} du dW_s = \int_{t_n}^{t_{n+1}} (s - t_n) dW_s = \Delta W_n \Delta_n - \Delta Z_n. \tag{3.8}$$

Also, we will face with the following multiple Itô integrals; therefore, we want to indicate them before the derivations of the stochastic Taylor approximations as well [11]:

$$I_{(1,1),\Delta_n} = \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} dW_u dW_s = \frac{1}{2} \left( (\Delta W_n)^2 - \Delta_n \right) \tag{3.9}$$

and

$$I_{(1,1,1),\Delta_n} = \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} \int_{t_n}^{u} dW_v dW_u dW_s = \frac{1}{3!} \left( (\Delta W_n)^2 - 3\Delta_n \right) \Delta W_n. \tag{3.10}$$

After constructing the Itô-Taylor schemes by benefiting from above equations (3.5)–(3.10), we try to measure the accuracy of the approximation methods by looking that how much the approximation process is close to the analytic solution of the stochastic differential equation. Therefore, we state strong and weak convergence criteria of the stochastic processes, in advance.

**Definition 3.1.** [27] Let $Y_t$ be an approximation process to the true solution $X_t$ of a stochastic integral equation (3.1) at the discretization times. We say that the process $Y_t$ *strongly converges* to the solution process $X_t$ if there exist a positive constant $c_1$ and the maximum step size $\Delta_n$ of the interval $[t_0, T]$ satisfying

$$E(|X_T - Y_T|) \leq c_1 \Delta_n^p \tag{3.11}$$

where $p$ is the strong order of convergence.

Now, we use the multi-indices and hierarchical sets we mentioned before in Chapter 2. Let us define a new set $\Lambda_p$ in order to define the strong Taylor approximation schemes according to a value of $p$ which shows the order of the scheme [26].

$$\Lambda_p := \left\{ \alpha \in \mathcal{M} : l(\alpha) + n(\alpha) \leq 2p \text{ or } l(\alpha) = n(\alpha) = p + \frac{1}{2} \right\}. \tag{3.12}$$

By increasing values of $p$ for the set $\Lambda_p$, we get additional multi-indices which are used in Itô-Taylor expansion (2.34) so as to create the strong Itô-Taylor schemes. For different $p$ values, we can obtain strong approximation schemes with different orders.

We remark that the summation of $l(\alpha)$ and $n(\alpha)$ which is less than or equal to $2p$, defined in the set $\Lambda_p$, should be integer because both $l(\alpha)$ and $n(\alpha)$ are always integers. Therefore, we can choose $p$ from $\mathbb{Z}/2$ to determine the order of a strong Taylor approximation scheme.

**Definition 3.2.** [27] Let $Y_t$ be an approximation process to the true solution $X_t$ of a stochastic differential equation (3.1) at the discretization times. We say that the process $Y_t$ *weakly converges* to the solution process $X_t$ if there exist a positive constant $c_2$ and the maximum step size $\Delta_n$ of the interval $[t_0, T]$ satisfying

$$|E(X_T) - E(Y_T)| \leq c_2 \Delta_n^q \tag{3.13}$$

where $q$ is the order of weak convergence.

In order to find the weak Taylor approximation schemes according to a value of $q$, we again benefit from the hierarchical sets stated in Chapter 2 and introduce a new set $\Gamma_q$ which will be used for the definitions of the weak Itô-Taylor schemes as follows:

$$\Gamma_q := \{\alpha \in \mathcal{M} : l(\alpha) \leq q\}\,. \tag{3.14}$$

We obtain weak Itô-Taylor schemes by trying different values for $q$ in the set $\Gamma_q$ and finding the corresponding multi-indices to use in Itô-Taylor expansion (2.34). This process will be more clear when we explain the stochastic Taylor schemes in the following chapter.

Also, since $l(\alpha)$ is defined as an integer, choices for $q$ for the set $\Gamma_q$ should be also integer. Therefore, we note that we always take integer numbers for the orders of the weak Itô-Taylor approximation schemes.

### 3.2 Euler-Maruyama Scheme

We now start to give different values to $p$ and to $q$ for the strong and weak orders of the Itô-Taylor approximation schemes, respectively. Firstly, we try the smallest value $1/2$ for the strong order $p$ and we see that the set $\Lambda_p$ (3.12) has the multi-indices $\alpha$ such as

$$\begin{aligned} \Lambda_{1/2} &= \{\alpha : l(\alpha) + n(\alpha) \leq 1 \text{ or } l(\alpha) = n(\alpha) = 1\} \\ &= \{\alpha : l(\alpha) = 1 \text{ and } n(\alpha) = 0 \text{ or } l(\alpha) = n(\alpha) = 1\} \\ &= \{\alpha = (1) \text{ or } \alpha = (0)\}\,. \end{aligned} \tag{3.15}$$

We also try the smallest value of $q = 1$ for the set $\Gamma_q$ (3.14) which has the same multi-indice $\alpha$:

$$\begin{aligned} \Gamma_1 &= \{\alpha : l(\alpha) \leq 1\} \\ &= \{\alpha : l(\alpha) = 1\} \\ &= \{\alpha = (0) \text{ or } \alpha = (1)\}\,. \end{aligned} \tag{3.16}$$

It can be seen that the sets $\Lambda_{1/2}$ and $\Gamma_1$ have the same multi-indices $\alpha = (0)$ and $\alpha = (1)$ in the equations (3.15) and (3.16). This means the approximation scheme has both strong order and weak order for the convergence criteria. When

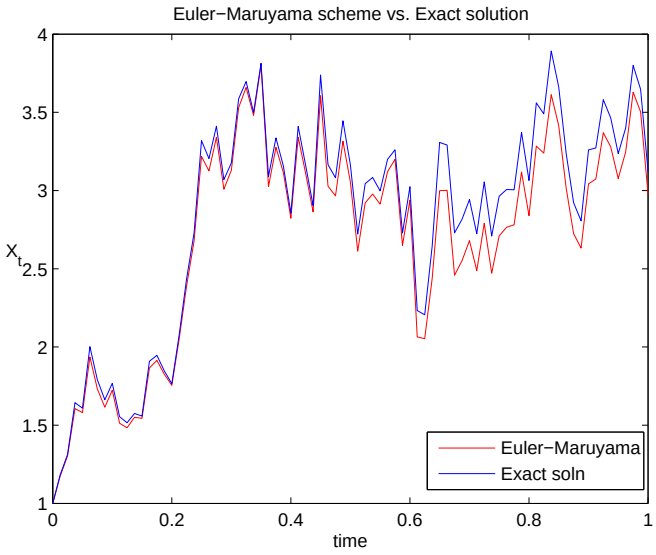


Figure 3.1: Comparison the Euler-Maruyama scheme with the exact solution where $a$=3, $b$=1 and at $N$=80 discretization time steps.

we use these multi-indices in the Itô-Taylor expansion (2.34) to construct the first approximation scheme, namely Euler-Maruyama scheme, the approximation process $Y_t$ will be written with the following stochastic differential equation

$$\begin{aligned} Y_{n+1} &= Y_n + I_{(0)} f_{(0)} + I_{(1)} f_{(1)} \\ &= Y_n + a(t_n, X_{t_n}) \int_{t_n}^{t_{n+1}} dt + b(t_n, X_{t_n}) \int_{t_n}^{t_{n+1}} dW_t \\ &= Y_n + a(t_n, X_{t_n}) \left(t_{n+1} - t_n\right) + b(t_n, X_{t_n}) \left(W_{t_{n+1}} - W_{t_n}\right) \\ &= Y_n + a(t_n, X_{t_n}) \Delta_n + b(t_n, X_{t_n}) \Delta W_n, \end{aligned}$$

which is called the one-dimensional *Euler-Maruyama scheme*. Therefore, regarding the $p$ and $q$ values, we say that Euler-Maruyama scheme has the strong order $p = 1/2$ and weak order $q = 1$ satisfying the convergence criteria (3.11) and (3.13), respectively.

The Euler-Maruyama scheme is the most known and commonly used approximation method which is derived by using the first derivations of Itô-Taylor expansion of the stochastic differential equation. However, it is not an enough accurate approximation always; therefore, this situation emerges the need for better methods that have higher $p$ and $q$ values for the sets $\Lambda_p$ and $\Gamma_q$, respectively. See Figure 3.1 for a numerical performance of Euler-Maruyama scheme applied to a simple stochastic differential equation.

### 3.3 Milstein Scheme

When we choose $p = 1$ for the strong order Itô-Taylor approximation scheme, the multi-indices $\alpha$ can get only these values:

$$\begin{aligned}
\Lambda_1 &= \{\alpha : l(\alpha) + n(\alpha) \leq 2 \text{ or } l(\alpha) = n(\alpha) = 3/2\} \\
&= \{\alpha : l(\alpha) = 1 \text{ and } n(\alpha) = 0 \text{ or } l(\alpha) = 1 \text{ and } n(\alpha) = 1 \text{ or} \\
&\quad l(\alpha) = 2 \text{ and } n(\alpha) = 0\} \\
&= \{\alpha = (1) \text{ or } \alpha = (0) \text{ or } \alpha = (1,1)\}. \qquad (3.17)
\end{aligned}$$

We realized that $\Lambda_1$ has one more multi-index $\alpha = (1,1)$ than $\Lambda_{1/2}$ which emerges an extra term to the Euler-Maruyama scheme when we solve the Itô-Taylor expansion (2.34) with the multi-indices found in the above equation (3.17). Then, we can rewrite the stochastic integral equation (3.1) for the set $\Lambda_1$ as follows:

$$\begin{aligned}
Y_{n+1} &= Y_n + I_{(0)}f_{(0)} + I_{(1)}f_{(1)} + I_{(1,1)}f_{(1,1)} \\
&= Y_n + a(t_n, X_{t_n}) \int_{t_n}^{t_{n+1}} dt + b(t_n, X_{t_n}) \int_{t_n}^{t_{n+1}} dW_t \\
&\quad + b(t_n, X_{t_n}) \frac{\partial}{\partial x} b(t_n, X_{t_n}) \int_{t_n}^{t_{n+1}} \int_{t_n}^{t} dW_s dW_t \\
&= Y_n + a(t_n, X_{t_n}) (t_{n+1} - t_n) + b(t_n, X_{t_n}) \left(W_{t_{n+1}} - W_{t_n}\right) \\
&\quad + \frac{1}{2} b(t_n, X_{t_n}) \frac{\partial}{\partial x} b(t_n, X_{t_n}) \left((W_{t_{n+1}} - W_{t_n})^2 - (t_{n+1} - t_n)\right) \\
&= Y_n + a(t_n, X_{t_n})\Delta_n + b(t_n, X_{t_n})\Delta W_n \\
&\quad + \frac{1}{2} b(t_n, X_{t_n}) b'(t_n, X_{t_n}) \left((\Delta W_n)^2 - \Delta_n\right),
\end{aligned}$$

which is actually the one-dimensional *Milstein scheme* [19, 20]. Therefore, we can deduce that the Milstein scheme has the strong order $p = 1$.

We note that Milstein scheme has not corresponding weak order Itô-Taylor approximation scheme because of the choices for the value of $q$ and the structure of the set $\Gamma_q$. To be more precise, it is impossible to add the extra term to the Euler-Maruyama scheme to get the Milstein scheme for the weak order Itô-Taylor scheme due to the restriction of $l(\alpha) \leq q$ for the set $\Gamma_q$.

It can be seen that Milstein scheme has an extra term when compared to Euler-Maruyama scheme. This extra term,

$$\frac{1}{2} b(t_n, X_{t_n}) b'(t_n, X_{t_n}) \left((\Delta W_n)^2 - \Delta_n\right),$$

brings a chance to obtain more satisfactory approximation to the Milstein method when comparing to the Euler-Maruyama method. Moreover, it can be seen that Milstein scheme is closer to the exact solution in the Figure **??** when comparing the closeness of the Euler-Maruyama scheme in the Figure **??** with the same exact

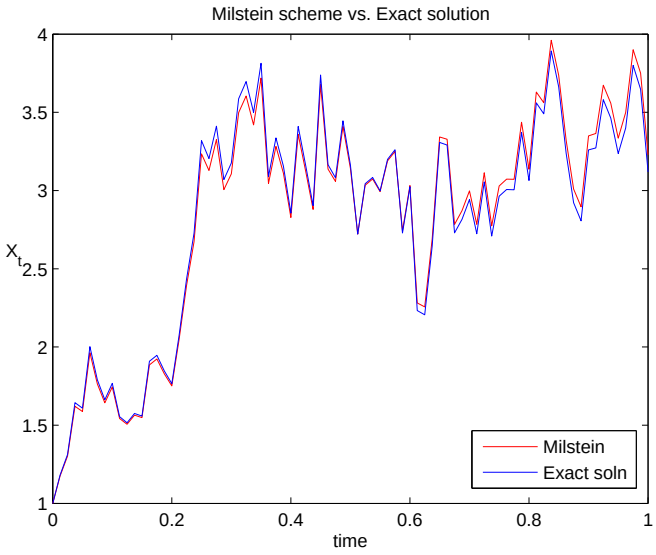


Figure 3.2: Comparison the Milstein scheme with the exact solution where $a$=3, $b$=1 and at $N$=80 discretization time steps.

solution of the same stochastic differential equation with the variables $a = 3$ and $b = 1$. See Figure 3.2.

We may carry on trying higher values for $p$ and $q$ in search of obtaining more efficient stochastic Itô-Taylor approximation methods in the next sections.

### 3.4 Weak Taylor Scheme of Order 2

The another choice for the value of $q$ is equal to 2. Then, by the properties of the set $\Gamma_q$ (3.14), the multi-indices can get these values such as

$$\begin{aligned} \Gamma_2 &= \{\alpha : l(\alpha) \leq 2\} \\ &= \{\alpha : l(\alpha) = 1 \text{ or } l(\alpha) = 2\} \\ &= \{\alpha = (0) \text{ or } \alpha = (1) \text{ or } \alpha = (0,0) \text{ or } \alpha = (0,1) \\ &\quad \text{or } \alpha = (1,0) \text{ or } \alpha = (1,1)\}. \end{aligned} \tag{3.18}$$

$\Gamma_2$ includes six different multi-indices $\alpha$ while $\Gamma_1$ in the section of Euler-Maruyama includes only two of them. These extra multi-indices should increase the accuracy of the approximation scheme to the exact solution. By using the Itô-Taylor expansion (2.34) with the set of multi-indices found in (3.18), we can write the following stochastic integral equation

$$\begin{aligned} Y_{n+1} &= Y_n + I_{(0)}f_{(0)} + I_{(1)}f_{(1)} + I_{(0,0)}f_{(0,0)} + I_{(0,1)}f_{(0,1)} \\ &\quad + I_{(1,0)}f_{(1,0)} + I_{(1,1)}f_{(1,1)}. \end{aligned}$$

Now, we need the calculations of corresponding multiple Itô integrals $I_{(1,0)}$ and $I_{(0,1)}$ given in (3.7) and (3.8), respectively. By reorganizing the above equation, we have

$$\begin{aligned}
Y_{n+1} &= Y_n + a(t_n, X_{t_n}) \int_{t_n}^{t_{n+1}} dt + b(t_n, X_{t_n}) \int_{t_n}^{t_{n+1}} dW_t \\
&\quad + \left( a(t_n, X_{t_n}) a'(t_n, X_{t_n}) + \frac{1}{2} b^2(t_n, X_{t_n}) a''(t_n, X_{t_n}) \right) \int_{t_n}^{t_{n+1}} \int_{t_n}^{t} ds dt \\
&\quad + \left( a(t_n, X_{t_n}) b'(t_n, X_{t_n}) + \frac{1}{2} b^2(t_n, X_{t_n}) b''(t_n, X_{t_n}) \right) \int_{t_n}^{t_{n+1}} \int_{t_n}^{t} ds dW_t \\
&\quad + a'(t_n, X_{t_n}) b(t_n, X_{t_n}) \int_{t_n}^{t_{n+1}} \int_{t_n}^{t} dW_s dt \\
&\quad + b(t_n, X_{t_n}) b'(t_n, X_{t_n}) \int_{t_n}^{t_{n+1}} \int_{t_n}^{t} dW_s dW_t,
\end{aligned}$$

and carrying out further calculations,

$$\begin{aligned}
Y_{n+1} &= Y_n + a(t_n, X_{t_n}) (t_{n+1} - t_n) + b(t_n, X_{t_n}) \left( W_{t_{n+1}} - W_{t_n} \right) \\
&\quad + \frac{1}{2} \left( a(t_n, X_{t_n}) a'(t_n, X_{t_n}) + \frac{1}{2} b^2(t_n, X_{t_n}) a''(t_n, X_{t_n}) \right) (t_{n+1} - t_n)^2 \\
&\quad + \left( a(t_n, X_{t_n}) b'(t_n, X_{t_n}) + \frac{1}{2} b^2(t_n, X_{t_n}) b''(t_n, X_{t_n}) \right) \\
&\qquad \times \left( \left( W_{t_{n+1}} - W_{t_n} \right) (t_{n+1} - t_n) - \int_{t_n}^{t_{n+1}} \int_{t_n}^{t} dW_s dt \right) \\
&\quad + a'(t_n, X_{t_n}) b(t_n, X_{t_n}) \int_{t_n}^{t_{n+1}} \int_{t_n}^{t} dW_s dt \\
&\quad + \frac{1}{2} b(t_n, X_{t_n}) b'(t_n, X_{t_n}) \left( \left( W_{t_{n+1}} - W_{t_n} \right)^2 - (t_{n+1} - t_n) \right).
\end{aligned}$$

Further, rearranging the terms, we have

$$\begin{aligned}
Y_{n+1} &= Y_n + a(t_n, X_{t_n}) \Delta_n + b(t_n, X_{t_n}) \Delta W_n \\
&\quad + \frac{1}{2} \left( a(t_n, X_{t_n}) a'(t_n, X_{t_n}) + \frac{1}{2} b^2(t_n, X_{t_n}) a''(t_n, X_{t_n}) \right) \Delta_n^2 \\
&\quad + \left( a(t_n, X_{t_n}) b'(t_n, X_{t_n}) + \frac{1}{2} b^2(t_n, X_{t_n}) b''(t_n, X_{t_n}) \right) (\Delta W_n \Delta_n - \Delta Z_n) \\
&\quad + a'(t_n, X_{t_n}) b(t_n, X_{t_n}) \Delta Z_n \\
&\quad + \frac{1}{2} b(t_n, X_{t_n}) b'(t_n, X_{t_n}) \left( (\Delta W_n)^2 - \Delta_n \right),
\end{aligned}$$

which is called one-dimensional Itô-Taylor approximation scheme with weak order $p = 2$. This scheme has additional terms to the Euler-Maruyama scheme and the Milstein scheme because of the multi-indices $\alpha = (0, 0)$, $\alpha = (0, 1)$ and $\alpha = (1, 0)$. Therefore, we may preassume that weak Itô-Taylor scheme of order 2 creates

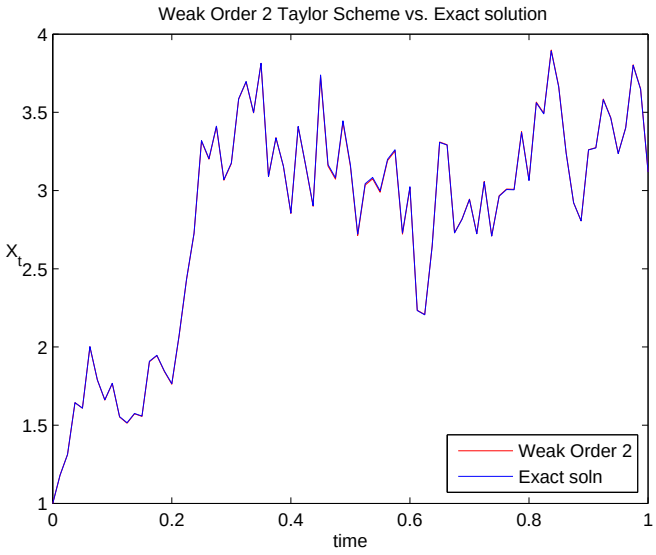


Figure 3.3: Comparison the weak order of 2 Taylor scheme with the exact solution where $a$=3, $b$=1 and at $N$=80 time steps.

better approximation process to the exact solution. Numerical performance of the method can be seen in Figure 3.3: the approximation process is very close to the exact solution when compared to the figures of the previous schemes.

However, the terms of the approximation process are getting more complex to calculate while the order of the scheme increases. In fact, constructing higher order schemes from Itô-Taylor expansion is getting expensive because of the need of the calculation of higher differentiations of the terms in the stochastic Taylor expansion.

### 3.5 Strong Taylor Scheme of Order 3/2

Let us try $p = 3/2$ for the set $\Lambda_p$ in the equation (3.12) to construct better approximation methods. Then, the multi-indices found in $\Lambda_{3/2}$ is given as

$$\begin{aligned}
\Lambda_{3/2} &= \{\alpha : l(\alpha) + n(\alpha) \leq 3 \text{ or } l(\alpha) = n(\alpha) = 2\} \\
&= \{\alpha : l(\alpha) = 1 \text{ and } n(\alpha) = 0 \text{ or } l(\alpha) = 1 \text{ and } n(\alpha) = 1 \text{ or} \\
&\quad l(\alpha) = 2 \text{ and } n(\alpha) = 0 \text{ or } l(\alpha) = 2 \text{ and } n(\alpha) = 1 \text{ or} \\
&\quad l(\alpha) = 3 \text{ and } n(\alpha) = 0 \text{ or } l(\alpha) = 2 \text{ and } n(\alpha) = 2\} \\
&= \{\alpha = (0) \text{ or } \alpha = (1) \text{ or } \alpha = (1,1) \text{ or } \alpha = (1,0) \text{ or} \\
&\quad \alpha = (0,1) \text{ or } \alpha = (1,1,1) \text{ or } \alpha = (0,0)\}. \qquad (3.19)
\end{aligned}$$

We have one more multi-index $\alpha = (1,1,1)$ different from the set $\Gamma_2$ (3.18) for

the weak Taylor scheme of order 2. For this set of multi-indices (3.19), we can write the stochastic differential equation by using the Itô-Taylor expansion (2.34) as follows:

$$\begin{aligned} Y_{n+1} &= Y_n + I_{(0)}f_{(0)} + I_{(1)}f_{(1)} + I_{(1,1)}f_{(1,1)} + I_{(1,0)}f_{(1,0)} + I_{(0,1)}f_{(0,1)} \\ &\quad + I_{(0,0)}f_{(0,0)} + I_{(1,1,1)}f_{(1,1,1)}. \end{aligned}$$

We again need the equations of corresponding multiple Itô integrals $I_{(1,0)}, I_{(0,1)}$ and $I_{(1,1,1)}$. These terms are calculated in the introduction part of this chapter, see equations (3.7), (3.8) and (3.10), respectively. By rearranging the terms with this information, we have

$$\begin{aligned} Y_{n+1} &= Y_n + a(t_n, X_{t_n}) \int_{t_n}^{t_{n+1}} dt + b(t_n, X_{t_n}) \int_{t_n}^{t_{n+1}} dW_t \\ &\quad + b(t_n, X_{t_n}) b'(t_n, X_{t_n}) \int_{t_n}^{t_{n+1}} \int_{t_n}^{t} dW_s dW_t \\ &\quad + a'(t_n, X_{t_n}) b(t_n, X_{t_n}) \int_{t_n}^{t_{n+1}} \int_{t_n}^{t} dW_s dt \\ &\quad + \left( a(t_n, X_{t_n}) b'(t_n, X_{t_n}) + \frac{1}{2} b^2(t_n, X_{t_n}) b''(t_n, X_{t_n}) \right) \int_{t_n}^{t_{n+1}} \int_{t_n}^{t} ds dW_t \\ &\quad + \left( a(t_n, X_{t_n}) a'(t_n, X_{t_n}) + \frac{1}{2} b^2(t_n, X_{t_n}) a''(t_n, X_{t_n}) \right) \int_{t_n}^{t_{n+1}} \int_{t_n}^{t} ds dt \\ &\quad + \left( \frac{1}{2} b^2(t_n, X_{t_n}) b''(t_n, X_{t_n}) + \frac{1}{2} b^2(t_n, X_{t_n}) b''(t_n, X_{t_n}) \right) \\ &\qquad \times \int_{t_n}^{t_{n+1}} \int_{t_n}^{t} \int_{t_n}^{s} dW_u dW_s dW_t, \end{aligned}$$

and, even further,

$$\begin{aligned} Y_{n+1} &= Y_n + a(t_n, X_{t_n}) (t_{n+1} - t_n) + b(t_n, X_{t_n}) \left( W_{t_{n+1}} - W_{t_n} \right) \\ &\quad + \frac{1}{2} b(t_n, X_{t_n}) b'(t_n, X_{t_n}) \left( \left( W_{t_{n+1}} - W_{t_n} \right)^2 - (t_{n+1} - t_n) \right) \\ &\quad + a'(t_n, X_{t_n}) b(t_n, X_{t_n}) \int_{t_n}^{t_{n+1}} \int_{t_n}^{t} dW_s dt \\ &\quad + \left( a(t_n, X_{t_n}) b'(t_n, X_{t_n}) + \frac{1}{2} b^2(t_n, X_{t_n}) b''(t_n, X_{t_n}) \right) \\ &\qquad \times \left( (W_{t_{n+1}} - W_{t_n})(t_{n+1} - t_n) - \int_{t_n}^{t_{n+1}} \int_{t_n}^{t} dW_s dt \right) \\ &\quad + \frac{1}{2} \left( a(t_n, X_{t_n}) a'(t_n, X_{t_n}) + \frac{1}{2} b^2(t_n, X_{t_n}) a''(t_n, X_{t_n}) \right) (t_{n+1} - t_n)^2 \\ &\quad + \frac{1}{2} \left( b^2(t_n, X_{t_n}) b''(t_n, X_{t_n}) + b(t_n, X_{t_n}) b'(t_n, X_{t_n})^2 \right) \\ &\qquad \times \left( \frac{1}{3} (W_{t_{n+1}} - W_{t_n})^2 - (t_{n+1} - t_n) \right) \left( W_{t_{n+1}} - W_{t_n} \right). \end{aligned}$$

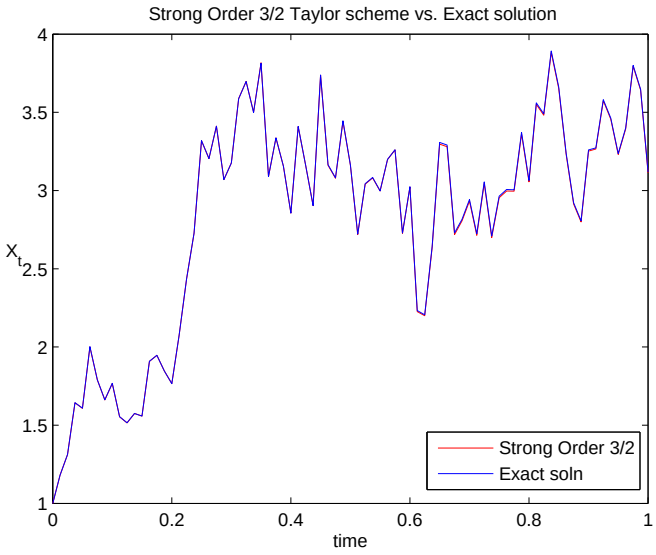


Figure 3.4: Comparison the strong order of 3/2 Taylor scheme with the exact solution where $a$=3, $b$=1 and at $N$=80 time steps.

Having rearranged the terms, we have,

$$\begin{aligned} Y_{n+1} &= Y_n + a(t_n, X_{t_n})\Delta_n + b(t_n, X_{t_n})\Delta W_n \\ &\quad + \frac{1}{2}b(t_n, X_{t_n})b'(t_n, X_{t_n})\left((\Delta W_n)^2 - \Delta_n\right) + a'(t_n, X_{t_n})b(t_n, X_{t_n})\Delta Z_n \\ &\quad + \left(a(t_n, X_{t_n})b'(t_n, X_{t_n}) + \frac{1}{2}b^2(t_n, X_{t_n})b''(t_n, X_{t_n})\right)(\Delta W_n\Delta_n - \Delta Z_n) \\ &\quad + \frac{1}{2}\left(a(t_n, X_{t_n})a'(t_n, X_{t_n}) + \frac{1}{2}b^2(t_n, X_{t_n})a''(t_n, X_{t_n})\right)\Delta_n^2 \\ &\quad + \frac{1}{2}\left(b^2(t_n, X_{t_n})b''(t_n, X_{t_n}) + b(t_n, X_{t_n})b'(t_n, X_{t_n})^2\right) \\ &\quad \times \left(\frac{1}{3}(\Delta W_n)^2 - \Delta_n\right)\Delta W_n. \end{aligned}$$

This equation is called the *order 3/2 strong Taylor scheme* for the one-dimensional case. In this scheme, we have greater order and extra terms different than the other Itô-Taylor methods. However, we should obtain more accurate approximation process than previous methods due to the contribution of the extra terms. Figure 3.4 depicts the approximation obtained and the exact solution for a stocahstic process: both seems to be identical.

The improvement in the graphics can be seen evidently when comparing the methods. As we stated before in the section of the weak Taylor scheme of order 2, it is hard to see the difference between the approximation method and the exact solution process in Figure 3.4.

By adding more terms from the stochastic Taylor expansion to the approximation schemes we have covered, we can get greater orders for the strong Itô-Taylor approximation schemes. Yet, it is not necessarily required to examine the higher order strong Taylor schemes, because the calculations of higher differentiation of the terms are getting harder in the stochastic Taylor expansion.

Although it is also possible to obtain higher orders Itô-Taylor schemes by adding more terms to the existing approximation schemes from the stochastic Taylor expansion and also by taking greater $p$ and $q$ values for the strong orders and the weak orders, respectively, we face with more computational difficulties and time consuming when we use greater $p$ and $q$ values. This situation shows the necessity of some other Taylor approximation schemes which avoid from the higher differentiations of the terms in the stochastic Taylor scheme. Thus, based on the application in practice, one needs a trade-off between the orders of the methods to be applied.

## 3.6 Runge-Kutta Schemes

In this section, we will deal with the method of Runge-Kutta both in ordinary and stochastic cases. This type of method is used for higher order schemes of the approximation processes. Firstly, we review the deterministic Runge-Kutta methods in order to understand the stochastic version of Runge-Kutta methods. The latter we will be covered later in the sequel.

The distinctive property of Runge-Kutta scheme from other approximation methods is that we do not have to calculate the derivatives of the drift and the diffusion terms in constructing the method although Runge-Kutta scheme is also derived from the Taylor expansions like the others. Calculating the derivatives of the terms in the stochastic differential equations is computationally costly and this situation emerges the need of approximation schemes that are free from derivatives. Runge-Kutta scheme avoids to compute the derivatives and, as a result, it becomes an important and valuable type of stochastic approximation methods.

### 3.6.1 Ordinary Runge-Kutta Scheme

In this subsection, we are interested in only deterministic part of the Itô stochastic differential equation (2.10) which is, in fact, the ordinary differential equation

$$dX_t = a(t, X_t)dt, \quad X_{t_0} = x_0 \in \mathbb{R}, \tag{3.20}$$

where the function $a(t, X_t)$ is a continuously differentiable $\mathbb{R}$-valued function in the time interval $I = [t_0, T]$. We should consider the partial derivatives of the

function $a(t, X_t)$ with respect to the time variable $t$ [11] as follows:

$$\begin{aligned}
\frac{dX_t}{dt} &= a(t, X_t), && (3.21)\\
\frac{d^2X_t}{dt^2} &= a_t(t, X_t) + a_x(t, X_t)a(t, X_t), && (3.22)\\
\frac{d^3X_t}{dt^3} &= a_{tt}(t, X_t) + 2a_{tx}(t, X_t)a(t, X_t) + a_{xx}(t, X_t)a^2(t, X_t) \\
&\quad + a_t(t, X_t)a_x(t, X_t) + a_x^2(t, X_t)a(t, X_t) && (3.23)
\end{aligned}$$

and so on. These partial derivatives help us to derive Runge-Kutta schemes by defining the Taylor expansion of $X_t$. Recall the discretization, defined in the equation (3.2), of the time interval $I = [t_0, T]$ ;

$$0 \leq t_0 < t_1 < \cdots < t_n < \cdots < t_N = T \tag{3.24}$$

with $n = 0, 1, \ldots, N$. Let $\Delta_n$ be sufficiently small step size such that $\Delta_n = t_{n+1} - t_n$ as in the equation (3.5). And, let again $Y_n = Y(t_n)$ be an approximation process to the analytic solution $X_{t_n}$ of the initial value problem.

Then, the Taylor expansion of the solution $X_t$ of an ordinary differential equation (3.20) is given by

$$X_{t+\Delta_n} = X_{t_{n+1}} = X_{t_n} + \Delta_n \frac{dX_t}{dt} + \frac{\Delta_n^2}{2!}\frac{d^2X_t}{dt^2} + \frac{\Delta_n^3}{3!}\frac{d^3X_t}{dt^3} + \cdots . \tag{3.25}$$

By using the Taylor expansion of $X_t$ (3.25), we will define the Runge-Kutta schemes with different stages [6]. For example, we can show how to construct the *second-order Runge-Kutta scheme* step by step as follows:

Firstly, we take the third-order Taylor expansion of the solution $X_t$ by substituting the partial derivatives of the function $a(t, X_t)$, defined in (3.21)–(3.23), into (3.25). Then, we have the following:

$$X_{t_{n+1}} = X_{t_n} + \Delta_n a + \frac{1}{2}\Delta_n^2 (a_t + a_x a) + \mathcal{O}(\Delta_n^3). \tag{3.26}$$

Let us now state a remark which will help us to simplify (3.26):

*Remark* 3.1. We assume that the function $\phi(t, X_t)$ satisfies the following property:

$$\phi(t + ck, X_t + dl) = a(t, X_t) + cka_t(t, X_t) + dla_x(t, X_t) + \mathcal{O}(k^2),$$

where $c, d, k, l$ are constants.

By this remark, we give some values to the variables $c = 1$, $k = \Delta_n$, $d = \Delta_n$, $l = a(t_n, X_{t_n})$ in order to get the following representation

$$a(t_n + \Delta_n, X_{t_n} + \Delta_n a(t_n, X_{t_n})) = a + \Delta_n a_t + \Delta_n a_x a + \mathcal{O}(\Delta_n^2). \tag{3.27}$$

We realize that this function in (3.27) is also appearing in (3.26), so we can rewrite the process $X_{t_{n+1}}$ as follows:

$$\begin{aligned} X_{t_{n+1}} = X_{t_n} &+ \frac{1}{2}\Delta_n a + \frac{1}{2}\Delta_n \left(a + \Delta_n(a_t + a_x a)\right) + \mathcal{O}(\Delta_n^3) \\ = X_{t_n} &+ \frac{1}{2}\Delta_n a(t_n, X_{t_n}) \\ &+ \frac{1}{2}\Delta_n a\left(t_n + \Delta_n, X_{t_n} + \Delta_n a(t_n, X_{t_n})\right) + \mathcal{O}(\Delta_n^3). \end{aligned} \tag{3.28}$$

As a result, reorganizing (3.28) leads to the *second-order ordinary Runge-Kutta scheme* defined as

$$Y_{n+1} = Y_n + \frac{\Delta_n}{2}(\Phi_1 + \Phi_2), \tag{3.29}$$

where the functions $\Phi_1$ and $\Phi_2$ are given by

$$\Phi_1 = a(t_n, X_{t_n})$$

and

$$\Phi_2 = a(t_n + \Delta_n, X_{t_n} + \Delta_n \Phi_1).$$

We can also state the fourth-order Runge-Kutta scheme by following similar steps which lead us to construct the second-order Runge-Kutta scheme above. When we study on the fifth-order Taylor expansion (3.25) which has the terms until $\mathcal{O}(\Delta_n^5)$, we add two extra terms into the expansion. After substituting the derivatives of the function $a(t, X_t)$ into the Taylor expansion and doing similar simplifying steps together with *Remark* 3.1, we can reach the *fourth-order ordinary Runge-Kutta* scheme:

$$Y_{n+1} = Y_n + \frac{\Delta_n}{6}(\Phi_1 + 2\Phi_2 + 2\Phi_3 + \Phi_4), \tag{3.30}$$

where the functions $\Phi_i$ for $i = 1, 2, 3, 4$ are defined to be

$$\begin{aligned} \Phi_1 &= a(t_n, X_{t_n}), \\ \Phi_2 &= a(t_n + \frac{\Delta_n}{2}, X_{t_n} + \frac{\Delta_n}{2}\Phi_1), \\ \Phi_3 &= a(t_n + \frac{\Delta_n}{2}, X_{t_n} + \frac{\Delta_n}{2}\Phi_2), \\ \Phi_4 &= a(t_n + \Delta_n, X_{t_n} + \Delta_n \Phi_3). \end{aligned}$$

It can be seen that the fourth-order Runge-Kutta scheme has only two more terms than the second-order Runge-Kutta scheme. Hence, we understand that adding more terms from the Taylor expansion in (3.25) leads us to get more accurate approximation with higher orders. Furthermore, we may generalize the *ordinary Runge-Kutta scheme of order s* as follows:

$$Y_{n+1} = Y_n + \Delta_n \sum_{i=1}^{s} \mu_i \Phi_i, \quad n \geq 0, \tag{3.31}$$

where the functions $\Phi_i$'s are defined as

$$\Phi_i = a(t_n + c_i\Delta_n, X_{t_n} + \Delta_n \sum_{j=1}^{s} \beta_{ij}\Phi_j)$$

for $i = 1, 2, \ldots, s$ and the coefficients $\beta_{ij}$, $c_i$ and $\mu_i$ are constants. The *Butcher array* is defined for the coefficients to demonstrate them in a table like environment as in [32]:

$$\begin{array}{c|ccccc} 0 & & & & & \\ c_2 & \beta_{21} & & & & \\ c_3 & \beta_{31} & \beta_{32} & & & \\ \vdots & \vdots & \vdots & \ddots & & \\ c_s & \beta_{s1} & \beta_{s2} & \ldots & \beta_{s,s-1} & \\ \hline & \mu_1 & \mu_2 & \ldots & \mu_{s-1} & \mu_s \end{array}$$

with $\beta_{ij} = 0$ for $j \geq i$ and

$$c_i = \sum_{j=1}^{i-1} \beta_{ij} \quad \text{and} \quad \sum_{i=1}^{s} \mu_i = 1.$$

We are interested in the case of $\beta_{ij} = 0$ for $j \geq i$ which means that the Runge-Kutta scheme is in the *explicit form*. If the coefficients satisfy $\beta_{ij} \neq 0$ for $j \geq i$, then the method becomes in *implicit form*; we do not cover the implicit case of the Runge-Kutta schemes, here, in this thesis. However, interested readers can refer to the book [26] by Andreas Rößler for a detailed study of Runge-Kutta schemes.

We note that the function $\Phi_1$ is always equal to the function $a(t_n, X_{t_n})$ since $c_1 = 0$ and $\beta_{1j} = 0$ for all $j \geq 1$ for the explicit form of the Runge-Kutta scheme.

For illustration purposes, we can write the Butcher arrays for the second-order ordinary Runge-Kutta scheme (3.29) as

$$\begin{array}{c|cc} 0 & & \\ 1 & 1 & \\ \hline & 1/2 & 1/2 \end{array}$$

and the fourth-order ordinary Runge-Kutta scheme (3.30) as

$$\begin{array}{c|cccc} 0 & & & & \\ 1/2 & 1/2 & & & \\ 1/2 & 0 & 1/2 & & \\ 1 & 0 & 0 & 1 & \\ \hline & 1/6 & 1/3 & 1/3 & 1/6 \end{array}$$

As we have covered the basics of Runge-Kutta methods in the ordinary case, we can generalize this knowledge to the stochastic case. In the next section, we give the general forms of some stochastic Runge-Kutta schemes with different stages.

### 3.6.2 Stochastic Runge-Kutta Scheme

In this subsection, we are moving on the stochastic case of Runge-Kutta schemes [32], thus we recall the Itô stochastic differential equation (2.10):

$$dX_t = a(t, X_t)dt + b(t, X_t)dW_t, \quad X_{t_0} = x_0, \tag{3.32}$$

or, in the integral form

$$X_t = X_{t_0} + \int_{t_0}^{t} a(s, X_s)ds + \int_{t_0}^{t} b(s, X_s)dW_s \tag{3.33}$$

where $W_t$ is a one-dimensional Brownian motion in the interval $I = [t_0, T]$.

Recursive formula for successive approximations has the form

$$Y_{n+1} = Y_n + \int_{t_n}^{t_{n+1}} a(s, X_s)ds + \int_{t_n}^{t_{n+1}} b(s, X_s)dW_s,$$

where $Y_n$ converges to a solution $X_{t_n}$ in the interval $[t_0, T]$ accordingly the definition of the partition (3.24) in the previous subsection. We have known from Chapter 2 that we can obtain the Itô-Taylor expansion recursively by applying the Itô formula (2.1) to the solution process $X_t$ (3.33).

By using similar steps as in the deterministic case, we can introduce the stochastic Runge-Kutta schemes from the Itô-Taylor expansions by applying *Remark* 3.1 in order to simplify the scheme.

In the light of these information, the general form of the stochastic Runge-Kutta scheme with $s$-stage, with $s \geq 1$, is defined by [18],

$$Y_{n+1} = Y_n + \sum_{i=1}^{s} \mu_i \Phi_i \Delta_n + \sum_{i=1}^{s} \nu_i \Psi_i \Delta W_n, \tag{3.34}$$

where

$$\begin{aligned} \Phi_i &= a(t_n + c_i\Delta_n, Y_n^{(i)}), \qquad i = 1, 2, \ldots, s, \\ \Psi_i &= b(t_n + c_i\Delta_n, Y_n^{(i)}), \qquad i = 1, 2, \ldots, s, \end{aligned}$$

and

$$Y_n^{(i)} = Y_n + \sum_{j=1}^{i-1} \beta_{ij} \Phi_j \Delta_n + \sum_{j=1}^{i-1} \gamma_{ij} \Psi_j \Delta W_n.$$

Coefficients $\mu_i, \nu_i, c_i, \beta_{ij}, \gamma_{ij}$ are constant which are selected in similar way as in the deterministic case; however,

$$\sum_{i=1}^{s} \mu_i = \sum_{i=1}^{s} \nu_i = 1$$

must be fulfilled.

The Butcher array for the stochastic sense of the Runge-Kutta scheme is given by

$$\begin{array}{c|ccccc|ccccc}
0 & & & & & & & & & & \\
c_2 & \beta_{21} & & & & & \gamma_{21} & & & & \\
c_3 & \beta_{31} & \beta_{32} & & & & \gamma_{31} & \gamma_{32} & & & \\
\vdots & \vdots & \vdots & \ddots & & & \vdots & \vdots & \ddots & & \\
c_s & \beta_{s1} & \beta_{s2} & \cdots & \beta_{s,s-1} & & \gamma_{s1} & \gamma_{s2} & \cdots & \gamma_{s,s-1} & \\
\hline
 & \mu_1 & \mu_2 & \cdots & \mu_{s-1} & \mu_s & \nu_1 & \nu_2 & \cdots & \nu_{s-1} & \nu_s
\end{array}$$

The additional part of the Butcher array, the rightest column, shows the coefficients of the diffusion part of the stochastic differential equation. The above table is given for the stochastic Runge-Kutta scheme including one-dimensional Brownian motion. If we have multi-dimensional Brownian motion in the scheme (3.34), then we have additional columns in the Butcher array for each Brownian motion in the stochastic part. For example, the general structure of the stochastic Runge-Kutta scheme including 2-dimensional Brownian motion is given such as

$$Y_{n+1} = Y_n + \sum_{i=1}^{s} \mu_i \Phi_i \Delta_n + \sum_{i=1}^{s} \nu_i \Psi_i \Delta W_n^1 + \sum_{i=1}^{s} \nu_i \Psi_i \Delta W_n^2 \tag{3.35}$$

where

$$\begin{aligned}
\Phi_i &= a(t_n + c_i\Delta_n, Y_n^{(i)}), \qquad i = 1, 2, \ldots, s, \\
\Psi_i &= b(t_n + c_i\Delta_n, Y_n^{(i)}), \qquad i = 1, 2, \ldots, s,
\end{aligned}$$

and

$$Y_n^{(i)} = Y_n + \sum_{j=1}^{i-1} \beta_{ij} \Phi_j \Delta_n + \sum_{j=1}^{i-1} \gamma_{ij} \Psi_j \Delta W_n^1 + \sum_{j=1}^{i-1} \gamma_{ij} \Psi_j \Delta W_n^2$$

subject to the conditions

$$\sum_{i=1}^{s} \mu_i = \sum_{i=1}^{s} \nu_i = 1.$$

Coefficients $\mu_i, \nu_i, c_i, \beta_{ij}, \gamma_{ij}$ are constant. Also, the corresponding Butcher array for the above scheme (3.35) is defined as

$$\begin{array}{c|ccccc|ccccc|ccccc}
0 & & & & & & & & & & & & & & & \\
c_2 & \beta_{21} & & & & & \gamma_{21}^1 & & & & & \gamma_{21}^2 & & & & \\
c_3 & \beta_{31} & \beta_{32} & & & & \gamma_{31}^1 & \gamma_{32}^1 & & & & \gamma_{31}^2 & \gamma_{32}^2 & & & \\
\vdots & \vdots & \vdots & \ddots & & & \vdots & \vdots & \ddots & & & \vdots & \vdots & \ddots & & \\
c_s & \beta_{s1} & \beta_{s2} & \cdots & \beta_{s,s-1} & & \gamma_{s1}^1 & \gamma_{s2}^1 & \cdots & \gamma_{s,s-1}^1 & & \gamma_{s1}^2 & \gamma_{s2}^2 & \cdots & \gamma_{s,s-1}^2 & \\
\hline
 & \mu_1 & \mu_2 & \cdots & \mu_{s-1} & \mu_s & \nu_1^1 & \nu_2^1 & \cdots & \nu_{s-1}^1 & \nu_s^1 & \nu_1^2 & \nu_2^2 & \cdots & \nu_{s-1}^2 & \nu_s^2
\end{array}$$

With the analogous properties as in the deterministic case, for the explicit form of the stochastic Runge-Kutta schemes, the coefficients satisfy $\beta_{ij} = \gamma_{ij} = 0$ for $j \geq i$ and

$$\sum_{j=i}^{s} \mu_i = \sum_{i=1}^{s} \nu_i = 1.$$

Moreover, the functions satisfy $\Phi_1 = a(t_n, Y_n^{(i)})$ and $\Psi_1 = b(t_n, Y_n^{(i)})$ since $c_1 = \beta_{1j} = \gamma_{1j} = 0$ for all $j \geq 1$ and $i = 1, 2, \ldots, s$ in the explicit form.

Let us define some stochastic Runge-Kutta schemes with different stages for given Buthcer arrays. Firstly, for the Butcher array of the *second-stage stochastic Runge-Kutta scheme* [21] is given as follows:

| | | | | |
|---|---|---|---|---|
| 0 | 0 | | 0 | |
| 1 | 1 | 0 | 1 | 0 |
| | 1/2 | 1/2 | 1/2 | 1/2 |

and the corresponding approximation scheme has the form of

$$\begin{aligned} Y_{n+1} = Y_n &+ \frac{1}{2}\left(a(t_n, Y_n^{(1)}) + a(t_{n+1}, Y_n^{(2)})\right)\Delta_n \\ &+ \frac{1}{2}\left(b(t_n, Y_n^{(1)}) + b(t_{n+1}, Y_n^{(2)})\right)\Delta W_n \end{aligned} \tag{3.36}$$

where the process $Y_n^{(i)}$ with $i = 1, 2$ satisfies

$$\begin{aligned} Y_n^{(1)} &= Y_n, \\ Y_n^{(2)} &= Y_n + a(t_n, Y_n^{(1)})\Delta_n + b(t_n, Y_n^{(1)})\Delta W_n. \end{aligned}$$

We show a comparison of the approximation process and the exact solution in Figure 3.5. The paths of the processes indeed get closer to the true solution when the number of stages is increased.

The third-stage explicit stochastic Runge-Kutta scheme [33, 35] is similarly expressed as

$$\begin{aligned} Y_{n+1} = Y_n &+ \left(\frac{1}{2}a(t_n, Y_n^{(1)}) + \frac{3}{4}a(t_n + \frac{1}{2}\Delta_n, Y_n^{(2)}) - \frac{1}{4}a(t_{n+1}, Y_n^{(3)})\right)\Delta_n \\ &+ \left(\frac{1}{2}b(t_n, Y_n^{(1)}) + \frac{3}{4}b(t_n + \frac{1}{2}\Delta_n, Y_n^{(2)}) - \frac{1}{4}b(t_{n+1}, Y_n^{(3)})\right)\Delta W_n, \end{aligned}$$

where the functions $Y_n^{(i)}$ with $i = 1, 2, 3$ satisfy

$$\begin{aligned} Y_n^{(1)} &= Y_n, \\ Y_n^{(2)} &= Y_n + \frac{2}{3}a(t_n, Y_n^{(1)})\Delta_n + \frac{2}{3}b(t_n, Y_n^{(1)})\Delta W_n, \\ Y_n^{(3)} &= Y_n + \left(a(t_n, Y_n^{(1)}) - a(t_n, Y_n^{(2)})\right)\Delta_n + \left(b(t_n, Y_n^{(1)}) - b(t_n, Y_n^{(2)})\right)\Delta W_n. \end{aligned}$$

The Butcher array of the scheme is

| | | | | | | |
|---|---|---|---|---|---|---|
| 0 | 0 | | | 0 | | |
| 1/2 | 2/3 | 0 | | 2/3 | 0 | |
| 1 | 1 | -1 | 0 | 1 | -1 | 0 |
| | 1/2 | 3/4 | −1/4 | 1/2 | 3/4 | −1/4 |

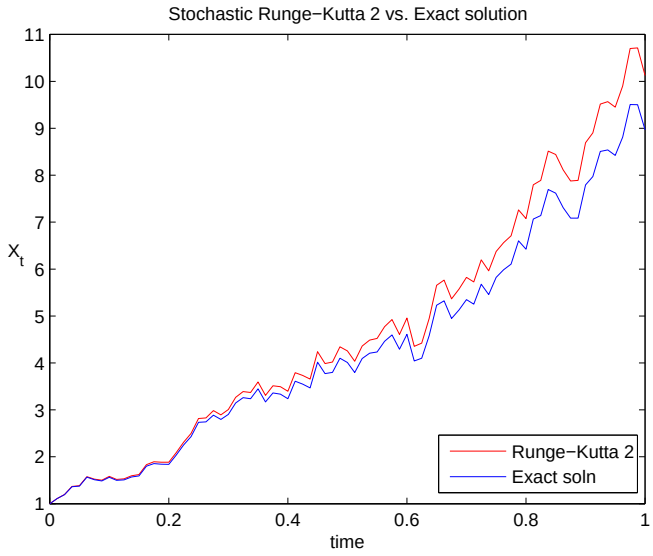


Figure 3.5: Comparison the stochastic Runge-Kutta scheme of stage 2 with the exact solution where $a$=3, $b$=0.5 and at $N$=80 time steps.

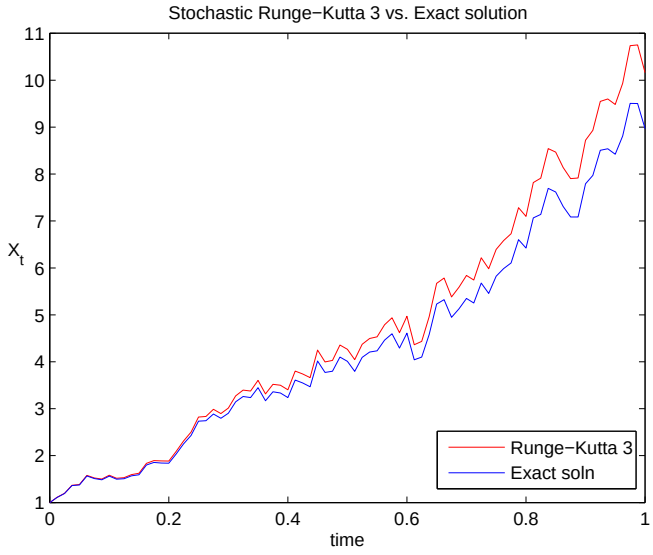


Figure 3.6: Comparison the stochastic Runge-Kutta scheme of stage 3 with the exact solution where $a$=3, $b$=0.5 and at $N$=80 time steps.

Figure 3.6 shows the accuracy and numerical performance of the approximation process to the exact solution for the third-stage stochastic Runge-Kutta scheme.

For the fourth-stage stochastic Runge-Kutta scheme, we have the below Butcher array with three column [35] which means that we have 2-dimensional Brownian motion in the scheme and the coefficients in the first column are given for the deterministic part while the coefficients in the second and the third column are given for the stochastic part of the approximation scheme.

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | | | | 0 | | | | 0 | | | |
| 1/2 | 1/2 | 0 | | | -0.72 | 0 | | | 2.70 | 0 | | |
| 1/2 | 0 | 1/2 | 0 | | 0.42 | -0.20 | 0 | | 1.76 | 0 | 0 | |
| 1 | 0 | 0 | 1 | 0 | -1.58 | 0.84 | 1.74 | 0 | -2.92 | 0 | 0 | 0 |
| | 1/6 | 1/3 | 1/3 | 1/6 | -0.78 | 0.07 | 1.49 | 0.22 | 1.69 | 1.64 | -3.02 | -0.31 |

Therefore, the fourth-stage explicit stochastic Runge-Kutta scheme including two independent Brownian motions is defined by using the coefficients of the above Butcher array in the following:

$$\begin{aligned} Y_{n+1} = Y_n \;&+\; \Big(\frac{1}{6}a(t_n, Y_n^{(1)}) + \frac{1}{3}a(t_n + \frac{1}{2}\Delta_n, Y_n^{(2)}) \\ &-\; \frac{1}{3}a(t_n + \frac{1}{2}\Delta_n, Y_n^{(3)}) + \frac{1}{6}a(t_{n+1}, Y_n^{(4)})\Big)\Delta_n \\ &-\; \Big(0.78\; b(t_n, Y_n^{(1)}) + 0.07\; b(t_n + \frac{1}{2}\Delta_n, Y_n^{(2)}) \\ &+\; 1.49\; b(t_n + \frac{1}{2}\Delta_n, Y_n^{(3)}) + 0.22\; b(t_{n+1}, Y_n^{(4)})\Big)\Delta W_n^1 \\ &+\; \Big(1.69\; b(t_n, Y_n^{(1)}) + 1.64\; b(t_n + \frac{1}{2}\Delta_n, Y_n^{(2)}) \\ &-\; 3.02\; b(t_n + \frac{1}{2}\Delta_n, Y_n^{(3)}) - 0.31\; b(t_{n+1}, Y_n^{(4)})\Big)\Delta W_n^2, \end{aligned}$$

where the functions $Y_n^{(i)}$ with $i = 1, 2, 3, 4$ satisfy

$$\begin{aligned} Y_n^{(1)} &= Y_n, \\ Y_n^{(2)} &= Y_n + \frac{1}{2}a(t_n, Y_n^{(1)})\Delta_n - 0.72\; b(t_n, Y_n^{(1)})\Delta W_n^1 + 2.7\; b(t_n, Y_n^{(1)})\Delta W_n^2, \\ Y_n^{(3)} &= Y_n + \frac{1}{2}a(t_n, Y_n^{(2)})\Delta_n + \big(0.42\; b(t_n, Y_n^{(1)}) - 0.20\; b(t_n, Y_n^{(2)})\big)\,\Delta W_n^1 \\ &\quad +1.76\; b(t_n, Y_n^{(1)})\Delta W_n^2, \\ Y_n^{(4)} &= Y_n + a(t_n, Y_n^{(3)})\Delta_n - \big(1.58\; b(t_n, Y_n^{(1)}) - 0.84\; b(t_n, Y_n^{(2)}) \\ &\quad -1.74\; b(t_n, Y_n^{(3)})\big)\Delta W_n^1 - 2.92\; b(t_n, Y_n^{(1)})\Delta W_n^2. \end{aligned}$$

It can be seen in Figure 3.7 that the distance of the approximation process and the exact solution is smaller than the distance of the second-stage and the third-stage stochastic Runge-Kutta schemes: compare Figures 3.5 and 3.6.

In this chapter, we stated different approximation schemes with different orders regarding the convergence criteria (3.11) and (3.13). However, we observe the

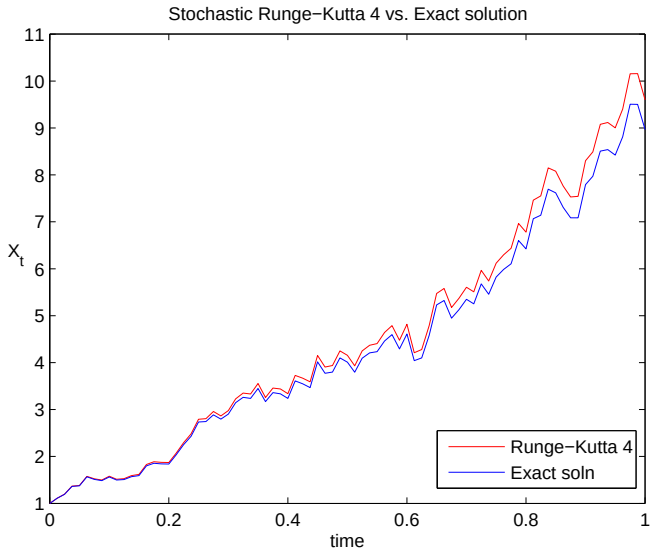


Figure 3.7: Comparison the stochastic Runge-Kutta scheme of stage 4 with the exact solution where $a$=3, $b$=0.5 and at $N$=80 the discretization times.

accuracy of the schemes by comparing the associated paths. We need better processes to compare the stochastic Itô-Taylor scheme with each other. Therefore, in the next chapter, we calculate some error terms for each of the schemes and apply the methods for option pricing models via Monte Carlo simulations.

# CHAPTER 4

# APPLICATION

In this chapter, we try to decide which Itô-Taylor scheme supplies the best approximation process to the exact solution of a given stochastic differential equation

$$dX_t = a(t, X_t)dt + b(t, X_t)dW_t \tag{4.1}$$

with the initial condition $X_{t_0} = x_0 \in \mathbb{R}$ [10, 13].

Firstly, we handle the error analysis of the approximation methods covered in Chapter 3 by creating and comparing the error functions of each Itô-Taylor scheme. Secondly, we examine some volatility models to see how the approximation methods are proceeding in the case of option pricing by using *Monte Carlo method*. The price of a European option is derived by Black & Scholes pricing model on the risk-neutral probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{Q})$ where $\mathbb{Q}$ is the risk-neutral probability measure [17]. The use of standard Brownian motion is an advantage for the opportunity to apply Black & Scholes model to the stochastic differential equations because of the martingale property.

## 4.1 Introduction

In this section, we give some basic definitions and helpful preliminaries before moving on the applications of the Itô-Taylor schemes. This section includes meanings of local and global error as well as European option and Black & Scholes model.

For analyzing the error functions of the stochastic processes which are constructed by Itô-Taylor approximation schemes, we need to give the definition of local error. We recall the partition $0 \leq t_0 < t_1 < \cdots < t_N = T$ of the time interval $I = [t_0, T]$. In addition, $Y_t$ is the approximation process to the analytic solution $X_t$ of a stochastic differential equation at the discretization times $t_n$ for $n = 0, 1, \ldots, N$, which is defined before in (3.3) at the beginning of the Chapter 3. We recall the approximation process as follows:

$$Y_t = Y_{t_0} + \int_{t_0}^{t} a(s, X_s)ds + \int_{t_0}^{t} b(s, X_s)dW_s, \quad t \in [t_0, T]. \tag{4.2}$$

The approximation method is the best one when the approximation process $Y_t$ is the closest to the exact form solution $X_t$ at the discretization points. The exact solution can generally be found by applying the Itô formula (2.19) to the stochastic differential equation.

The errors of the approximation schemes will help us to judge the accuracy of the methods. We will examine the errors at the time steps of the partition. The error is generally defined as the difference between the exact solution and the approximate solution at the discretization times. The *local truncation error* of any approximation process $Y_t$ to the true solution $X_t$ in the time interval $[t_0, T]$ is given by

$$\text{local error} = \epsilon = X_{t_n} - Y_n \tag{4.3}$$

where $Y_n = Y(t_n)$ with $n = 0, 1, \ldots, N$. The numerical method is *consistent* if the local truncation error is $\mathcal{O}(\Delta_n)$ where $\Delta_n$ is the increment of the subinterval $[t_n, t_{n+1}]$ defined in the equation (3.5). Furthermore, we say that the numerical method has order $p$ if the local truncation error is $\mathcal{O}(\Delta_n^{p+1})$ for any solution of the initial value problem.

On the other hand, the fact that the local truncation error can be accumulated over all of the iterations gives the *global error* with the assumption of the perfect knowledge of the exact form of the solution at the initial time step. The numerical method is said to be *convergent* if global truncation error goes to zero as the step size goes to zero.

For the rest of this application chapter, we cover the option pricing models and analyze them in Monte Carlo simulations. Therefore, we need to know what the option means and how to calculate their values with the Black & Scholes model. We can define a European option as the right to buy or sell an underlying stock at an agreed price $K$ in the certain time $T$. In other words, it is a contract and you have compromised to a strike price $K$ at the beginning of the agreement assuming that the stock price $X_t$ will go up or down in the future and you will be free to use or not to use this option at the maturity time $T$. You can buy or sell the stock at the market price or the price you have decided on at the beginning of the agreement.

There are two types of option, calls and puts. While the call option gives a person the right to buy the stock at a certain price at the maturity, the put option gives the person the right to sell the stock at an agreed price at the maturity. Therefore, the person who buys a call option expects that the stock price will be higher and the person who buys a put option hopes that the stock price will decrease at the maturity. The payoff functions of the options are introduced as follows:

$$\Pi_C(t, X_t) = \max(X_T - K, 0), \quad \text{for a call option,} \tag{4.4}$$

$$\Pi_P(t, X_t) = \max(K - X_T, 0), \quad \text{for a put option.} \tag{4.5}$$

We use the Black & Scholes formula to find the theoretical option prices under the assumption that the market is arbitrage free, so we work on the risk-neutral

probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{Q})$, where $\mathbb{Q}$ is the risk-neutral probability measurebsmodel, lamber. Regarding that the option is a call or a put, we take a hedge position on the market in the process of the Black & Scholes model. We apply the Black & Scholes formula to the stock price $X_t$ where the formula describes the value $V(t, X_t)$ of the option over time and is defined as

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 X_t^2 \frac{\partial^2 V}{\partial X^2} + rX_t \frac{\partial V}{\partial X} - rX_t = 0. \tag{4.6}$$

By the Black & Scholes model, the value of a call option $V_C(t, X_t)$ and the value of a put option $V_P(t, X_t)$ are respectively given by

$$V_C(t, X_t) = X_t \mathcal{N}(d_1) - Ke^{-r(T-t)}\mathcal{N}(d_2) \tag{4.7}$$

and

$$V_P(t, X_t) = Ke^{-r(T-t)}\mathcal{N}(-d_2) - X_t \mathcal{N}(-d_1) \tag{4.8}$$

with

$$\begin{aligned} d_1 &= \frac{1}{\sigma\sqrt{T-t}}\left(\ln\left(\frac{X_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)\right), \\ d_2 &= \frac{1}{\sigma\sqrt{T-t}}\left(\ln\left(\frac{X_t}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)(T-t)\right) = d_1 - \sigma\sqrt{T-t}, \end{aligned}$$

where $T-t$ is the time to maturity, $X_t$ is the spot price of the underlying stock, $K$ is the strike price, $r$ is the risk-free interest rate, $\sigma$ is the volatility of the returns of the stock and the $\mathcal{N}(d)$ is the cumulative distribution function of standard normal distribution:

$$\mathcal{N}(d) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d} e^{\frac{-x^2}{2}} dx.$$

More information about Black & Scholes model can be found in [17].

Now, we are ready to proceed the applications of the Itô-Taylor schemes with some examples by analyzing the error functions and option pricing models.

## 4.2 Geometric Brownian Motion

In this part, we introduce the process of Geometric Brownian Motion and try to decide which method, covered in the previous chapters, creates the best approximation to the stochastic process by examining the error function.

Geometric Brownian Motion is a continuous-time stochastic process and it can be used to model stock prices in the Black & Scholes model. General structure of the Geometric Brownian Motion satisfies the following stochastic differential equation

$$dX_t = rX_t dt + \sigma X_t dW_t, \quad X_0 = x_0 \in \mathbb{R}, \tag{4.9}$$

where $r$ and $\sigma$ are non-negative constants and $W_t$ is a Brownian Motion with $t \in [0, T]$. Here, by comparing the stochastic differential equation (4.1), the drift term is $a(t, X_t) = rX_t$ and the diffusion term is $b(t, X_t) = \sigma X_t$. Drift term determines the trend of the stochastic process while the diffusion term controls the effect of the randomness of the Brownian motion $W_t$.

In finance, we can think that the process $X_t$ behaves like a stock price where $r$ and $\sigma$ represent the risk-free rate and the volatility of the stock price, respectively. Therefore, we can determine that a stock price $X_t$ goes up or goes down in the time interval $t \in [0, T]$ by modeling with Geometric Brownian Motion (4.9) with respect to the risk-free rate $r$ and the volatility $\sigma$.

We can reach the exact (closed) form of the solution of Geometric Brownian Motion in the following steps: firstly, we reorganize the the stochastic differential equation (4.9) by

$$\frac{1}{X_t} dX_t = r dt + \sigma dW_t.$$

By using Itô formula (2.19) to the function $f(t, X_t) = \ln X_t$ it follows that

$$\begin{aligned} d(\ln X_t) &= \frac{1}{X_t} dX_t + \frac{1}{2}\sigma^2 X_t^2 \left(-\frac{1}{X_t^2}\right) dt \\ &= r dt + \sigma dW_t - \frac{1}{2}\sigma^2 dt \\ &= \left(r - \frac{1}{2}\sigma^2\right) dt + \sigma dW_t. \end{aligned}$$

As a result, we get the general form of the exact solution of Geometric Brownian Motion as

$$\ln \frac{X_t}{X_0} = \left(r - \frac{\sigma^2}{2}\right) t + \sigma W_t,$$

or, more explicitly,

$$X_t = X_0 \exp\left(\left(r - \frac{\sigma^2}{2}\right) t + \sigma W_t\right). \tag{4.10}$$

Now, we start to compare the approximation schemes with each other by examining the local error functions with respect to the exact solution process of the Geometric Brownian Motion (4.10). As we mentioned before, the local error functions are calculated by the differences between the approximation process and the closed form process at the discretization times by using the equation (4.3). We use MATLAB to construct the approximation and exact solution processes and some codes with some explanations are given in the appendix.

In Figure 4.1, we have demonstrated that the best approximation methods for given Geometric Brownian Motion process are the higher order Itô-Taylor schemes: weak Taylor scheme of order 2 and strong Taylor scheme of order 3/2. The error functions for those two are around zero line and it can be seen that the stochastic processes modeled with stochastic Runge-Kutta methods are getting greater

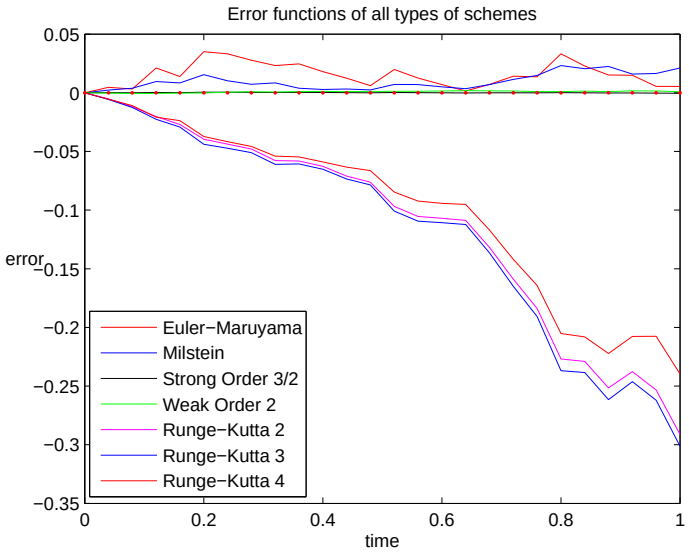


Figure 4.1: Error functions of all methods for Geometric Brownian Motion with the variables $X_0 = 1,\ T = 1,\ N = 25,\ r = 0.5$ and $\sigma = 0.5$.

error exponentially. Moreover, the error measures can be found for the first 5 steps in Table 4.1. It is acceptable that the higher order Itô-Taylor schemes give better results than the Euler-Maruyama and Milstein schemes. However, the stochastic Runge-Kutta schemes do not have good approximation processes for the Geometric Brownian Motion process surprisingly.

We continue with another stochastic process in the next section and determine the best approximation method by the error functions, similarly.

Table 4.1: The errors $\epsilon$ at the first 5 steps calculated as $\epsilon = X_{t_n} - Y_n$ for $n = 1, 2, \ldots, 5$.

| | Error Calculations | | | | | | |
|---|---|---|---|---|---|---|---|
| $N$ | E-M | Mils | SO3/2 | WO2 | SRK2 | SRK3 | SRK4 |
| 1 | 0.0046 | 0.0023 | 0.0001 | -0.0003 | -0.0053 | -0.0057 | -0.0054 |
| 2 | 0.0032 | 0.0039 | 0.0000 | -0.0007 | -0.0118 | -0.0125 | -0.0110 |
| 3 | 0.0212 | 0.0096 | 0.0003 | -0.0005 | -0.0203 | -0.0226 | -0.0208 |
| 4 | 0.0139 | 0.0084 | 0.0001 | -0.0003 | -0.0270 | -0.0292 | -0.0238 |
| 5 | 0.0350 | 0.0154 | 0.0005 | -0.0001 | -0.0396 | -0.0439 | -0.0373 |

## 4.3 Orsntein-Uhlenbeck Process

In this section, we state another type of the stochastic differential process, namely *Orsntein-Uhlenbeck process*. This process is also used for modeling the stock prices or interest rates in finance. General form of Orsntein-Uhlenbeck process is given in the following stochastic differential equation:

$$dX_t = \theta(r - X_t)dt + \sigma dW_t, \quad \theta > 0, \tag{4.11}$$

where $\theta, r$ and $\sigma$ are nonnegative parameters and $W_t$ is a Brownian Motion with $t \in [0, T]$. Again, when comparing the stochastic differential equation (4.1), the drift term is $a(t, X_t) = \theta(r - X_t)$ and the diffusion term is $b(t, X_t) = \sigma$.

If we are modeling a stock price $X_t$ with Orsntein-Uhlenbeck process, then $\theta$ gives the speed of turning into the long-run level of the process $X_t$, $r$ represents the long-run mean of the process $X_t$, and $\sigma$ shows the volatility of the process $X_t$. The closed form solution of the stochastic process (4.11) is calculated by applying Itô formula (2.19) to the function $f(t, X_t) = X_t e^{\theta t}$ as follows:

$$\begin{aligned} d(X_t e^{\theta t}) &= \theta X_t e^{\theta t} dt + e^{\theta t} dX_t \\ &= \theta X_t e^{\theta t} dt + e^{\theta t} \left(\theta(r - X_t)dt + \sigma dW_t\right) \\ &= e^{\theta t}\theta r dt + \sigma e^{\theta t} dW_t. \end{aligned}$$

Then, in the integration form, we have

$$X_t e^{\theta t} = X_0 + \int_0^t e^{\theta s}\theta r ds + \int_0^t \sigma e^{\theta s} dW_s.$$

Therefore, we obtain the exact solution of the Orsntein-Uhlenbeck process as

$$X_t = X_0 e^{-\theta t} + r(1 - e^{-\theta t}) + \int_0^t \sigma e^{\theta(s-t)} dW_s \tag{4.12}$$

with $X_0 = x_0 \in \mathbb{R}$.

Now, we can construct and plot the analytic solution process of the Orsntein-Uhlenbeck and the approximation processes we have covered before. Related MATLAB codes are also given in the appendix at the end of the thesis.

In Figure 4.2, we have showed that the best approximation methods for the Orsntein-Uhlenbeck process is the stochastic Runge-Kutta schemes with different stages. Of course, the best one is the fourth-stage stochastic Runge-Kutta scheme as seen obviously in the figure. However, we plot only four lines in the graphic because the diffusion term $b(t, X_t)$ is equal to a constant $\sigma$ and the derivatives of the diffusion term with respect to space will be zero. So, Euler-Maruyama scheme gives the same results with Milstein scheme; strong Taylor scheme of order 3/2 has the same results with weak Taylor scheme of order 2 at the discretization points for the Orsntein-Uhlenbeck process.

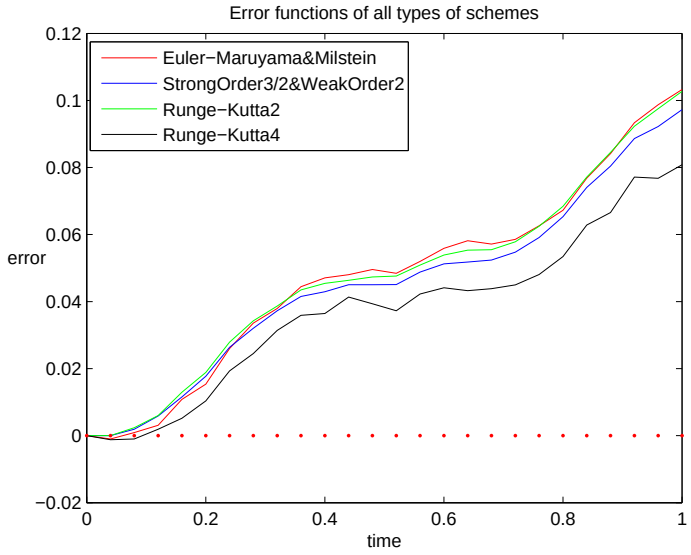


Figure 4.2: Error functions of all methods for Orsntein-Uhlenbeck process with the variables $X_0 = 1,\ T = 1,\ N = 25,\ \theta = 0.5,\ \sigma = 0.5$ and long run mean $r = 0$.

Table 4.2: The errors $\epsilon$ at the first 6 steps calculated as $\epsilon = X_{t_n} - Y_n$ for $n = 1, 2, \ldots, 6$.

| | Error Calculations | | | | |
|---|---|---|---|---|---|
| $N$ | EM & Mils | SO32 & WO2 | SRK2 | SRK3 | SRK4 |
| 1 | -0.0010 | 0.0000 | 0.0000 | 0.0000 | -0.0012 |
| 2 | 0.0009 | 0.0019 | 0.0024 | 0.0024 | -0.0010 |
| 3 | 0.0031 | 0.0058 | 0.0060 | 0.0060 | 0.0019 |
| 4 | 0.0108 | 0.0116 | 0.0130 | 0.0129 | 0.0052 |
| 5 | 0.0153 | 0.0177 | 0.0189 | 0.0188 | 0.0104 |
| 6 | 0.0260 | 0.0263 | 0.0279 | 0.0279 | 0.0193 |

Moreover, the local errors of the approximation processes by comparing the closed form of the solution of the Orsntein-Uhlenbeck (4.12) process are given in Table 4.2 for only the first 6 steps.

In addition to the Geometric Brownian Motion and the Orsntein-Uhlenbeck process, we can think of the volatility models. In above stochastic processes, we always take the drift and the diffusion terms as constant; however, we can take the time-dependent drift and diffusion terms in the volatility models. Later in the sequel, we mention this perspective of stochastic differential models.

## 4.4 Barrier Down-And-Out Option

Up to now, we have covered different stochastic differential processes and done some stock price modeling. However, we need to be more clear about the accuracy of approximation methods. We decide to study on European options after modeling the stochastic processes and we believe that studying on option pricing models will be the best way to measure the success of the approximation methods. In this section, we apply *Monte Carlo simulation* since we obtain possible outcomes obtained from possible paths of the stock prices, and then taking the average of them gives us the expected outcome in this simulation.

The best way to get the approximate option price is to make Monte-Carlo simulation. In such a simulation, we firstly determine how the stock prices (paths) behaves in the time interval $[t_0, T]$ with an approximation method we mentioned in Chapter 3, then we calculate the value $V(T, X_T)$ of the option at time $T$ which is also equal to the payoff $\Pi(t, X_t)$ of the option at the maturity time $T$ for the simulated path of the underlying stock prices. When we discount this price to the present time, we obtain the option price at time $t_0$.

To set an example, we can give a down-and-out barrier option which means that if the stock price $X_t$ crushes the barrier line $B$ at any time until the maturity time, then the value of the option $V(t, X_t)$ will be zero; otherwise, the option price will be calculated. Shortly, we can think that we never want the stock price to be lower than a fixed value $B$ which is called the barrier line. Moreover, the closed form solution of a down-and-out barrier call option for $B < K$ is given by

$$V_C^B(t, X_t) = V_C(t, X_t) - \left(\frac{X_t}{B}\right)^{1-\frac{2r}{\sigma^2}} V_C\left(t, \frac{B^2}{X_t}\right), \tag{4.13}$$

where $V_C(t, X_t)$ is the value of the European call option with the strike price $K$, $B$ is the barrier, $r$ is the risk-free interest rate and $\sigma$ is the volatility of the stock price.

In Table 4.3, we can understand that when we increase the number of paths keeping the discretization at the same level, we get better option values. Similarly, if we look at the error results of different methods for the same iteration and discretization level, we can say that higher order schemes have lower errors

Table 4.3: The results of Monte Carlo simulation of a down-and-out call option, where $X_0 = 50$, $K = 50$, $B = 45$, $r = 0.03$ and $\sigma = 0.1$ in the time interval $[0, 1]$. (The price of the option found by the closed form solution (4.13) is 2.7543.)

| | | Error Calculations | | | | | |
|---|---|---|---|---|---|---|---|
| # Paths | N | E-M | errorEM | SRK2 | errorSRK2 | SRK4 | errorSRK4 |
| $10^2$ | 20 | 2.9761 | -0.2218 | 3.1511 | -0.3968 | 2.6910 | 0.0633 |
| $10^3$ | 20 | 2.6202 | 0.1341 | 2.7762 | -0.0219 | 2.8145 | -0.0602 |
| $10^4$ | 20 | 2.7132 | 0.0411 | 2.8739 | -0.1196 | 2.9048 | -0.1505 |
| $10^5$ | 20 | 2.7791 | -0.0248 | 2.9412 | -0.1869 | 2.9438 | -0.1895 |
| $10^6$ | 20 | 2.7843 | -0.0300 | 2.9472 | -0.1929 | 2.9359 | -0.1816 |
| $10^2$ | 50 | 2.6295 | 0.1248 | 2.7811 | -0.0268 | 2.3844 | 0.3699 |
| $10^3$ | 50 | 2.8342 | -0.1799 | 2.9983 | -0.2440 | 2.9808 | -0.2265 |
| $10^4$ | 50 | 2.7819 | -0.0276 | 2.9450 | -0.1907 | 2.9753 | -0.2210 |
| $10^5$ | 50 | 2.7694 | -0.0151 | 2.9310 | -0.1767 | 2.9696 | -0.2153 |
| $10^6$ | 50 | 2.7684 | -0.0141 | 2.9301 | -0.1758 | 2.9473 | -0.1930 |

absolutely by giving better approximate values to the exact value of the option as we have expected. As a result, when we analyze all results in Table 4.3, we examine that the fourth-stage of stochastic Runge-Kutta scheme, which has the highest approximation order among the schemes in the table, gives the smallest values for the errors comparing the closed form solution (4.13).

For another example, we state a volatility model in the next section. The difference of the this model is that the volatility in the diffusion part of the stochastic differential equation is also a stochastic process, thus modeling the volatility as well.

## 4.5 Heston Model

In Black & Scholes model, the only variable we cannot forecast is the *volatility* $\sigma$, so to reach the perfect hedging is not possible. Therefore, we move on stochastic volatility models to get rid of the uncertain variable, volatility. In this section, we study on one of the stochastic volatility models: *Heston model*. In this model, the stochastic process has a time-dependent diffusion term instead of a constant diffusion term as before. The diffusion part is initially modeled by the methods we have stated before, then the stochastic process is modeled depending on this diffusion part.

In Heston model, the stochastic process $X_t$ "*seems*" to be created by Geometric Brownian Motion (4.9); however, the volatility in the diffusion part of the stochastic differential equation is constructed with correlated Brownian motion.

The general structure of Heston model can be defined as in the following equation

$$dX_t = rX_t dt + \sqrt{\sigma_t} X_t dW_t^X \tag{4.14}$$

with

$$d\sigma_t = \kappa(\theta - \sigma_t)dt + \xi\sqrt{\sigma_t} dW_t^\sigma \tag{4.15}$$

where the Brownian motions are correlated and they satisfy

$$dW_t^X dW_t^\sigma = \rho dt, \tag{4.16}$$

and $r$ is the risk-free interest rate, $\theta$ is the long variance, $\kappa$ is the speed of the reverting $\sigma_t$ to $\theta$, and $\xi$ is the volatility of the volatility process $\sigma_t$ [28].

To be more understandable, we can rewrite (4.14) with the correlation $dW_t^X = \rho dW_t^\sigma + \sqrt{1-\rho^2} dZ_t$ where $Z_t$ is a standard Brownian motion as follows:

$$dX_t = rX_t dt + \sqrt{\sigma_t} X_t \left(\rho dW_t^\sigma + \sqrt{1-\rho^2} dZ_t\right). \tag{4.17}$$

Moreover, the closed form solution of Heston model is given as in the following:

$$X_t = X_0 \exp\left(rt + \frac{\rho}{\xi}(\sigma_t - \kappa\theta t) + \left(\frac{\kappa\rho}{\xi} - \frac{1}{2}\right)\sigma_t + \sqrt{1-\rho^2}\int_0^t \sqrt{\sigma_s} dW_s'\right), \tag{4.18}$$

where $dW_t'$ is standard Brownian motion independent of $W_t^X$ and $W_t^\sigma$.

While plotting a path of a stochastic process constructed by Heston model, we follow the steps below:

- Firstly, we model the process $\sigma_t$ with its own Brownian motion process $dW_t^\sigma$.
- Secondly, we determine the Brownian motion $dW_t^X$ for the process $X_t$ by regarding the correlation $\rho$ between Brownian motions in the equation (4.16).
- Finally, we construct the process $X_t$ with the risk-free interest rate $r$, stochastic volatility process $\sigma_t$ and the Brownian motion $dW_t^X$.

The effect of the stochastic volatility is clearly seen in Figure 4.3. The blue path is plotted by Heston model with a stochastic volatility $\sigma_t$ by taking care of the equation (4.15) and the red path is obtained by Geometric Brownian Motion with a constant volatility $\sigma = 0.2$. Although the processes go down and up at the same time steps, the oscillation of the blue path is higher than the oscillation of the red one due to the effect of the stochastic volatility.

Lastly, we study numerically approximating the price of a European option which is constructed by Heston model. We model the stochastic volatility in different ways, and then, we construct the stock price $X_t$ depending on this stochastic volatility. We make a decision about which approximation method is the best one by examining the values given in Monte Carlo simulation. The price $V(t, X_t)$

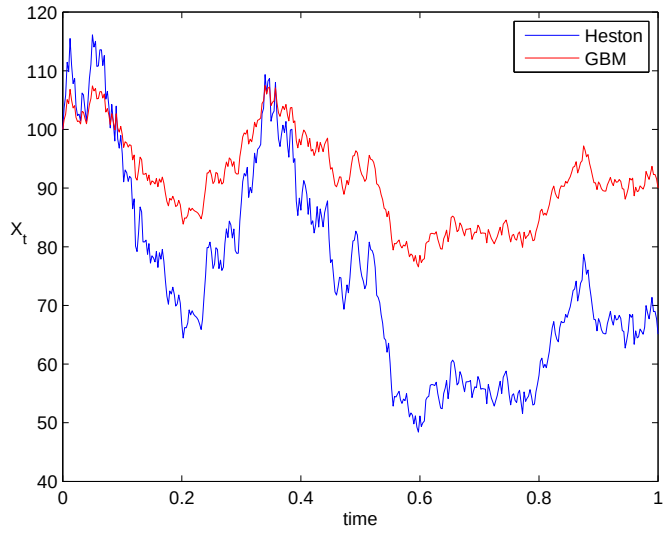


Figure 4.3: The process $X_t$ constructed by Heston model with the variables $X_0 = 100,\ \sigma_0 = 0.2,\ T = 1,\ N = 400,\ \kappa = 0.001,\ \theta = 0.15,\ \xi = 0.001,\ r = 0.05,\ \rho = 0.5$.

of an option, with the underlying $X_t$, governed by the Heston model can be obtained by solving the following partial differential equation,

$$\begin{aligned} \frac{\partial V}{\partial t} &+ rX_t\frac{\partial V}{\partial x} + \kappa(\theta-\sigma)\frac{\partial V}{\partial \sigma} + \frac{1}{2}\sigma X_t^2\frac{\partial^2 V}{\partial x^2} \\ &+ \rho\xi\sigma X_t\frac{\partial^2 V}{\partial X_t \partial \sigma} + \frac{1}{2}\xi^2\sigma\frac{\partial^2 V}{\partial \sigma^2} - rV = 0. \end{aligned} \tag{4.19}$$

Figure 4.4 shows the behaviors of the stock prices $X_t$ modeled by stochastic Taylor schemes, and one can observe that Euler-Maruyama scheme constructs the farthest solution process (path) to the exact, true solution. In addition, the stochastic Runge-Kutta schemes give the best approximation processes to the closed form solution.

In Table 4.4, indeed, we observe that when the number of paths increases while keeping the number of discretization the same, we get better option values regarding the error. Similarly, comparing the results of different methods at the same number of paths and discretization, we observe that the fourth-stage of stochastic Runge-Kutta scheme has the smallest error values which means that this method gives the most closest approximation to the exact option price.

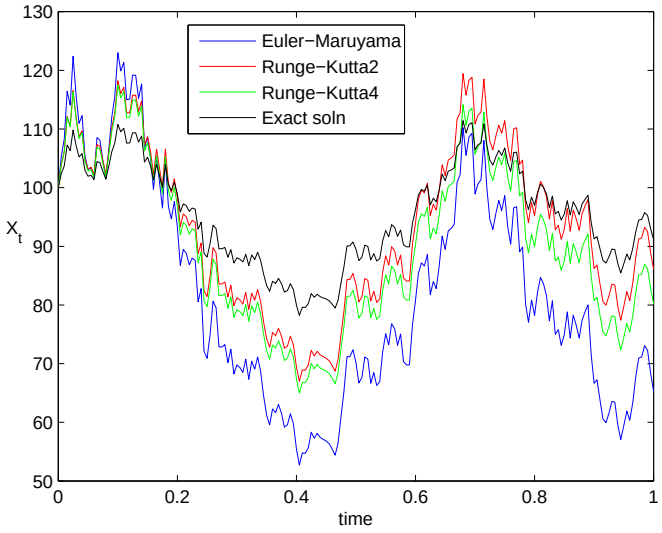


Figure 4.4: The stochastic process $X_t$ constructed by Heston model with different Itô-Taylor schemes.

Table 4.4: The results of Monte Carlo simulation of the value of an option defined by Heston model, where $X_0 = 100$, $K = 100$, $\sigma_0 = 0.04$ $r = 0.04$, $\xi = 0.2$, $\kappa = 0.6$, $\theta = 0.04$ and the correlation $\rho = -0.15$ between Brownian motions $W_t^X$ and $W_t^\sigma$ in the time interval $[0, 1/2]$. (Exact price of the option is 6.5473 given in [15].)

| | | Error Calculations | | | | | |
|---|---|---|---|---|---|---|---|
| # Paths | N | E-M | errorEM | SRK2 | errorSRK2 | SRK4 | errorSRK4 |
| $10^2$ | 20 | 6.1287 | 0.4186 | 6.8577 | -0.3104 | 6.1101 | 0.4372 |
| $10^3$ | 20 | 6.3649 | 0.1824 | 7.1078 | -0.5605 | 6.8022 | -0.2549 |
| $10^4$ | 20 | 6.4830 | 0.0643 | 7.2478 | -0.7005 | 6.7563 | -0.2090 |
| $10^5$ | 20 | 6.5649 | -0.0176 | 7.3411 | -0.7938 | 6.5710 | -0.0237 |
| $10^6$ | 20 | 6.5449 | 0.0024 | 7.3195 | -0.7722 | 6.5780 | -0.0307 |

# CHAPTER 5

# CONCLUSION

In this work, we studied the stochastic Itô-Taylor schemes with the aim of investigating the best approximation method to a given stochastic differential equation.

In the preliminaries part of the study, Itô calculus and Stratonovich calculus were stated in detail and their relationships with each other were given properly. The stochastic Taylor expansions of two types of calculus were discussed at the end of this part. Then, we moved on the stochastic approximation schemes by interesting in only one type of calculus, Itô calculus. Using only Itô process was not a problem because we had explained how to convert Itô calculus to Stratonovich calculus clearly by giving examples.

In the next chapter, the main part of the study, the stochastic Taylor schemes were given one by one in each section. Both strong and weak convergence criterias were defined first, then the different types of the approximation schemes were discussed by regarding the order of convergence. The basic type of them was the Euler-Maruyama scheme with strong order of 1/2 and weak order of 1. This scheme was constructed by using the first derivatives of the Itô-Taylor expansion of the stochastic differential equation. After that, we continued with the Milstein scheme by adding one more term to the Euler-Maruyama scheme from the stochastic Itô-Taylor expansion. It was shown that the Milstein scheme had strong order of 1. Weak Taylor scheme of order 2 and strong Taylor scheme of order 3/2 were found by analyzing more derivatives of the Itô-Taylor expansion. Furthermore, we examined the Runge-Kutta schemes with different stages. In order to derive the stochastic Runge-Kutta schemes, we prefered to understood the ordinary Runge-Kutta scheme first. The opportunity of not using the derivatives of the stochastic Itô-Taylor expansion gives the importance and the utilizability to the Runge-Kutta schemes instead of the other types of the stochastic Itô-Taylor schemes. We obtained higher order approximation schemes by testing different stages of Runge-Kutta schemes and recognized that they were more useful and computational simplicity when comparing the other types of the stochastic Itô-Taylor schemes. Therefore, analyzing the Itô-Taylor schemes, especially the stochastic Runge-Kutta schemes, with their general structure and paths are plotted.

Later in the application part, we justified our assumption that higher order

stochastic Taylor schemes is the more preferable approximate process to the exact solution of a given stochastic differential equation. Although the improvement of the approximation schemes could be seen in the figures of the stochastic Itô-Taylor schemes visually in the previous chapter, we numerically analyzed the accuracy of each approximation scheme by comparing the exact solution processes and using the error analysis for some given well-known stochastic processes and option pricing models. We started with Geometric Brownian Motion whose closed form solution process was found by applying the Itô formula (2.19) to the function $f(t, X_t) = \ln X_t$. When we constructed and computed the stochastic Taylor schemes for the Geometric Brownian Motion process, we got the best approximation methods as weak Taylor scheme of order 2 and strong Taylor scheme of order 3/2. This can be seen in Figure 4.1 where the error process calculated by subtracting the approximate process from the exact solution process at the discretization points. Then, we examined Orsntein-Uhlenbeck process and observed that the stochastic Runge-Kutta schemes, especially the fourth-stage stochastic Runge-Kutta scheme, gave the best approximation processes to the analytic solution process which was obtained by applying the Itô formula (2.19) to the function $f(t, X_t) = X_t e^{\theta t}$.

After we had different results from the approximation methods for these two stochastic processes, we decided to study on Monte Carlo simulations for option pricing models in order to make more reliable and consistent decisions about the success of the approximation methods in more realistic situations in finance: we looked into the barrier down-and-out option. In Monte Carlo simulation, we repeatedly got the value of the options by changing the number of paths or the number of the discretizations of the stochastic process. By comparing the Monte Carlo results with the closed form solution of the barrier down-and-out option given in (4.13), we obtained the best results from the approximation method modeled by the fourth-stage stochastic Runge-Kutta scheme. In the last section of that chapter, we tried to improve our calculations by analyzing one of the stochastic volatility model, namely Heston model, since letting volatility be stochastic instead of constant helps us to obtain better approximations. We again got better outcomes from the fourth-stage stochastic Runge-Kutta scheme when applying Monte Carlo simulation for an option which was created by the Heston model with stochastic volatility.

As a result, we introduced various types of the stochastic Taylor approximation methods and tried to decide the best one throughout the study. At the end, it was seen that we had the best results from the fourth-stage stochastic Runge-Kutta scheme while doing the error analysis and Monte Carlo simulation. This method was not only easy to implement, but it also avoids the calculation of the derivatives of the stochastic Taylor expansion. Furthermore, it also gives the best approximate solutions when compared to the exact solution processes.

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# APPENDIX A

## Some MATLAB Codes

Euler-Maruyama Method

```
% Function of Euler-Maruyama approximation

% For the stochastic process dXt = a(Xt,t)dt + b(Xt,t)dWt
% with the initial condition X(t0=0) = X0 and t0<t<T
% the Euler-Maruyama approximation is calculated by
% X(t+1) = X(t) + a*dt + b*dWt

function [X, t, dt] = EulerMaruyama (X0, T, N, a, b)

randn('state', 13);

X = zeros(1, N+1); X(1)=X0;     % initial value of Xt
t = linspace(0,T,N+1)';     % discretization
dt = diff(t);       % step size
dW = sqrt(dt).*randn(N,1);  % Brownian motion

for j = 1:N
    X(j+1) = X(j) + a(t(j),X(j)).*dt(j) + b(t(j),X(j)).*dW(j);
end
```

Milstein Method

```
% Function of Milstein approximation

% For the stochastic process dXt = a(Xt,t)dt + b(Xt,t)dWt
% with the initial condition X(t0=0) = X0 and t0<t<T
% the Milstein approximation is calculated by
% X(t+1) = X(t) + a*dt + b*dWt + (1/2)*b*b'*(dWt^2-dt)

function [X, t, dt] = Milstein (X0, T, N, a, b, diff_b)

randn('state', 13);

X = zeros(1, N+1); X(1)=X0;     % initial value of Xt
t = linspace(0,T,N+1)';     % discretization
dt = diff(t);           % step size
dW = sqrt(dt).*randn(N,1); % Brownian motion
```

```
for j = 1:N
    X(j+1) = X(j) + a(t(j),X(j)).*dt(j) + b(t(j),X(j)).*dW(j) + ...
        (1/2).*b(t(j),X(j)).*diff_b(t(j),X(j)).*((dW(j)^2)-dt(j));
end
```

Strong Taylor Method of Order 3/2

```
% Strong Taylor Approximation of Order 3/2

% For the stochastic process dXt = a(Xt,t)dt + b(Xt,t)dWt
% with the initial condition X(t0=0) = X0 and t0<t<T
% the strong Taylor approximation of order 3/2 is calculated by
% X(t+1) = X(t) + a*dt + b*dWt + (1/2)*b*b'*(dWt^2-dt) + a'*b*dZt +
        %(1/2)*(a*a'+(1/2)*b^2*a'')*dt^2 +
        %(a*b'+(1/2)*b^2*b'')*(dWt*dt-dZt) +
        %(1/2)*(b^2*b''+b*(b')^2)*((1/3)*dWt^2-dt)*dWt

function [X, t, dt] = StrongOrder32 (X0, T, N, a, b, diff_a, diff_b, ...
                                        diff_aa, diff_bb)
randn('state', 13);

X = zeros(1, N+1); X(1)=X0;     % initial value of Xt
t = linspace(0,T,N+1)';     % discretization
dt = diff(t);           % step size
dW = sqrt(dt).*randn(N,1);  % Brownian motion (BM)
dZ = (1/2)*dt.*(dW + sqrt(dt).*randn(N,1)/sqrt(3)); % dependent BM with dW

for j = 1:N
    X(j+1) = X(j) + a(t(j),X(j)).*dt(j) + b(t(j),X(j)).*dW(j) + ...
        (1/2).*b(t(j),X(j)).*diff_b(t(j),X(j)).*((dW(j)^2)-dt(j)) + ...
        diff_a(t(j),X(j)).*b(t(j),X(j)).*dZ(j) +  ...
        (1/2).*(a(t(j),X(j)).*diff_a(t(j),X(j)) + ...
        (1/2).*diff_aa(t(j),X(j)).*(b(t(j),X(j)))^2).*((dt(j))^2) + ...
        (a(t(j),X(j)).*diff_b(t(j),X(j)) + ...
        (1/2).*diff_bb(t(j),X(j)).*(b(t(j),X(j)))^2).*(dW(j).*dt(j) - ...
        dZ(j)) + (1/2).*(diff_bb(t(j),X(j)).*(b(t(j),X(j)))^2 + ...
        b(t(j),X(j)).*((diff_b(t(j),X(j)))^2)).*((1/3).*(dW(j))^2 - ...
        dt(j)).*dW(j);
end
```

Weak Taylor Method of Order 2

```
% Weak Taylor Approximation of Order 2

% For the stochastic process dXt = a(Xt,t)dt + b(Xt,t)dWt
% with the initial condition X(t0=0) = X0 and t0<t<T
% the weak Taylor approximation of order 2 is calculated by
% X(t+1) = X(t) + a*dt + b*dWt + (1/2)*b*b'*(dWt^2-dt) + a'*b*dZt +
        %(1/2)*(a*a'+(1/2)*b^2*a'')*dt^2 +
        %(a*b'+(1/2)*b^2*b'')*(dWt*dt-dZt)

function [X, t, dt] = WeakOrder2 (X0, T, N, a, b, diff_a, diff_b, ...
                                    diff_aa, diff_bb)
```

```
randn('state', 13);

X = zeros(1, N+1); X(1)=X0;     % initial value of Xt
t = linspace(0,T,N+1)';     % discretization
dt = diff(t);           % step size
dW = sqrt(dt).*randn(N,1);      % Brownian motion (BM)
dZ = (1/2)*dt.*(dW + sqrt(dt).*randn(N,1)/sqrt(3)); % dependent BM with dW

for j = 1:N
    X(j+1) = X(j) + a(t(j),X(j)).*dt(j) + b(t(j),X(j)).*dW(j) + ...
        (1/2).*b(t(j),X(j)).*diff_b(t(j),X(j)).*((dW(j)^2)-dt(j)) + ...
        diff_a(t(j),X(j)).*b(t(j),X(j)).*dZ(j) +  ...
        (1/2).*(a(t(j),X(j)).*diff_a(t(j),X(j)) + ...
        (1/2).*diff_aa(t(j),X(j)).*(b(t(j),X(j)))^2).*((dt(j))^2) + ...
        (a(t(j),X(j)).*diff_b(t(j),X(j)) + ...
        (1/2).*diff_bb(t(j),X(j)).*(b(t(j),X(j)))^2).*(dW(j).*dt(j)-dZ(j));
end
```

Second-Stage Runge-Kutta Method

```
% Second-stage Stochastic Runge-Kutta Method

% For the stochastic process dXt = a(Xt,t)dt + b(Xt,t)dWt
% with the initial condition X(t0=0) = X0 and t0<t<T
% the second-stage Runge-Kutta method is calculated by
% X(t+1) = X(t) + (1/2)*(a(t,Xt)+a(t+1,Yt))*dt + ...
%           (1/2)*(b(t,Xt)+b(t+1,Yt))*dWt
% where
% Y(t+1) = Y(t) + a(t,Xt)*dt + b(t,Xt)*dWt (constructed by the function Xhat)

function [X, t] = SRK2(X0, T, N, a, b)

randn('state', 13);

X(1) = X0;      % initial value of Xt
t = linspace(0,T,N+1)';     % discretization
dt = diff(t);       % step size
dW = sqrt(dt).*randn(N,1);      % Brownian motion

for j = 1:N
    Y = Xhat(t(j), X(j), dt(j), dW(j), a, b);
    X(j+1) = X(j) + (1/2)*(a(t(j),X(j)) + a(t(j+1),Y)).*dt(j) + ...
        (1/2)*(b(t(j),X(j)) + b(t(j+1),Y)).*dW(j);
end
```

The Function of $Y_t$ in Runge-Kutta Method

```
function X_hat = Xhat(t, X, dt, dW, a, b)

X_hat = X + a(t,X).*dt + b(t,X).*dW;
```

## Fourth-Stage Runge-Kutta Method

```
% Fourth-stage Stochastic Runge-Kutta Method

% For the stochastic process dXt = a(Xt,t)dt + b(Xt,t)dWt
% with the initial condition X(t0=0) = X0 and t0<t<T
% the fourth-stage Runge-Kutta method is calculated by
% X(t+1) = X(t) + sum(mu_i*a(t+c_i*dt,Yt_i)*dt) + ...
%            sum(nu_i*b(t+c_i*dt,Yt_i)*dWt)
% where i = 1,2,3,4 and the function
% Yt_i = Y(t) + sum(beta_j*a(t+c_j*dt,Yt_j)*dt) + ...
%            sum(gamma_j*b(t+c_j*dt,Yt_j)*dWt)
% where j=1,2,...,i-1 with the coefficients c, mu, nu, beta, gamma defined
% in Buthcer array.

function [X, t] = SRK4(X0, T, N, a, b)

rand('state',13); randn('state',13);

X(1) = X0;      % initial value of Xt
t = linspace(0,T,N+1)';     % discretization
dt = diff(t);       % step size
dW1 = sqrt(dt).*randn(N,1);     % Brownian motion
dW2 = sqrt(dt).*randn(N,1);     % independent Brownian motion

for j = 1:N
    Y2 = Xhat2(t(j), X(j), dt(j), dW1(j), dW2(j), a, b);
    Y3 = Xhat3(t(j), X(j), dt(j), dW1(j), dW2(j), a, b, Y2);
    Y4 = Xhat4(t(j), X(j), dt(j), dW1(j), dW2(j), a, b, Y2, Y3);

    X(j+1) = X(j) + ((1/6)*a(t(j),X(j)) + (1/3)*a(t(j)+(1/2)*dt(j),Y2) + ...
        (1/3)*a(t(j)+(1/2)*dt(j),Y3) + (1/6)*a(t(j+1),Y4)).*dt(j) + ...
        + ((-0.78)*b(t(j),X(j)) + (0.07)*b(t(j)+(1/2)*dt(j), Y2) + ...
        (1.49)*b(t(j)+(1/2)*dt(j),Y3) + (0.22)*b(t(j+1), Y4)).*dW1(j) + ...
        ((1.69)*b(t(j),X(j)) + (1.64)*b(t(j)+(1/2)*dt(j), Y2) + ...
        (-3.02)*b(t(j)+(1/2)*dt(j),Y3) + (-0.31)*b(t(j+1), Y4)).*dW2(j);
end
```

## Exact Solution of Geometric Brownian Motion

```
% Geometric Brownian Motion

% The stochastic process dXt = a(t,Xt)dt + b(t,Xt)dWt is given
% with the initial condition X(t0=0) = X0 and t0<t<T.
% The Geometric Brownian model is constructed when the functions satisfy
    % a(t,Xt)=mu*Xt   and   b(t,Xt)=sigma*Xt.
% Then, the exact solution Xt will be as follows
    % Xt = X0*exp((mu-0.5*sigma^2)*t + sigma*Wt)

function [X, t, dt] = Exact_GBM (X0, T, N, rate, sigma)

randn('state', 13);
```

```
X = zeros(1, N+1);      % solution process Xt
t = linspace(0,T,N+1)';     % discretization
dt = diff(t);       % time step
dW = sqrt(dt).*randn(N,1); W = cumsum([0;dW]);      % Brownian motion

X = X0*exp(rate.*t - (1/2)*sigma^2.*t + sigma.*W);  % exact solution
```

### Exact Solution of Orsntein-Uhlenbeck Process

```
% Orsntein-Uhlenbeck Process

% The stochastic process dXt = a(t,Xt)dt + b(t,Xt)dWt is given
% with the initial condition X(t0=0) = X0 and t0<t<T.
% The Orsntein-Uhlenbeck process is constructed when the functions satisfy
    % a(t,Xt)=theta*(r-Xt)   and   b(t,Xt)=sigma
% Then, the exact solution Xt will be as follows
    % Xt = X0*exp(-theta*t) + r(1-exp(-theta*t)) + ...
    %        int(sigma*exp(theta*(s-t))*dWs

function [X, t, dt] = Exact_OUP(X0, T, N, rate, mu, sigma)

randn('state', 13);

X = zeros(1, N+1);      % solution process Xt
t = linspace(0,T,N+1)';     % discretization
dt = diff(t);       % step size
dW = sqrt((1-exp((-2)*rate.*dt))/(2*rate)).*randn(N,1); % Brownian motion
W = cumsum([0;dW]);

X = X0*exp((-1)*rate.*t) + mu*(1 - exp((-1)*rate.*t)) + sigma.*W; %exact solution
```

### Closed Form Solution of Down-and-Out Barrier Call Option

```
% Closed form solution of a barrier down-and-out call option is given by
%   VB(t,Xt) = V(t,Xt) - V(t,B^2/Xt)*(Xt/B)^(1-(2*r/sigma^2)
% where V(t,Xt) is the European call option price, B is the barrier, r is
% the risk-free interest rate and sigma is the volatility of the process.

function value = Exact_BarrierDO (S0, B, K, rate, T, sigma)

randn('state', 13);

A = (B^2)./S0;
V1 = blsprice(S0,K,rate,T,sigma); % option pricing with Black&Scholes model
V2 = blsprice(A,K,rate,T,sigma); % option pricing with Black&Scholes model

value = V1 - ((S0/B)^(1-2*rate/(sigma^2))).*V2;
```

### Monte Carlo Simulation of a Down-and-Out Barrier Option

```
% Monte Carlo Simulation for Barrier Down-and-Out European Call Option

clear all, close all,
randn('state', 13)
rand('state', 13)

X0=50; T=1; N=200; M=100; rate=0.03; sigma=0.1; K = 50; B=45;   % variables

X = zeros(1, N+1); X(1)=X0;     % solution process Xt
V = zeros(1, M);    % value of the option
V_sum = 0;
t = linspace(0,T,N+1)';     % discretization
dt = diff(t);       % step size

for i=1:M
    for j=1:N
        dW = randn(N,1);    % Brownian motion
        X(j+1) = X(j)*exp(rate.*dt(j) - (1/2)*sigma^2.*dt(j) + ...
                    sigma.*sqrt(dt(j)).*dW(j));
    end
    X_min = min(X);
    if X_min < B        % B is the barrier
        V(i)=0;
    else
        V(i) = exp((-1)*rate*T)*max(X(end)-K,0);    % discounted payoff
    end
    V_sum = V_sum + V(i);
end
V_GBM = (1/M)*V_sum     % value of the option
```

### Monte Carlo Simulation for Heston Model

```
% Monte Carlo simulation for Heston model

randn('state', 13); rand('state',13);

sigma0=0.04; X0=100; K=100; T=0.5;
rho=-0.15; kappa=0.6; theta=0.04; xi=0.2; rate=0.04;
N=50; M=50;

exact_value=6.5473;     % exact solution of the model with these variables

sigma = zeros(1, N+1); sigma(1) = sigma0; % initial value of volatility
X = zeros(1, N+1); X(1) = X0;   % initial value of X_t
t = linspace(0,T,N+1)';     % discretization
dt = diff(t);       % step size
V = zeros(1, M);    % value of the option
V_sum = 0;      % initial value of iterative sum

for i=1:M
    for j=1:N
```

```
        dZ = randn(N,1); dY = randn(N,1);   % standard Brownian motions
        dW = rho*dZ + sqrt(1-rho^2)*dY;     % correlated Brownian motion

        sigma(j+1) = sigma(j) + kappa.*(theta-sigma(j)).*dt(j) + ...
            xi.*sqrt(abs(sigma(j))).*sqrt(dt(j)).*dW(j);

        X(j+1) = X(j).*exp((rate - (1/2)*sigma(j)).*dt(j) + ...
            sqrt(abs(sigma(j))).*sqrt(dt(j)).*dZ(j));
    end
    V(i) = exp((-1)*rate*T)*max(X(end)-K,0); % discounted payoff
    V_sum = V_sum + V(i);
end
V_EM = (1/M)*V_sum      % value of the option
error = exact_value - V_EM      % difference btwn analytic and calculated value
```