INVESTIGATION OF FRACTIONAL BLACK SCHOLES OPTION PRICING APPROACHES AND THEIR IMPLEMENTATIONS

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ABSTRACT

INVESTIGATION OF FRACTIONAL BLACK SCHOLES OPTION PRICING APPROACHES AND THEIR IMPLEMENTATIONS

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One of the fundamental research areas in the financial mathematics is option pricing. With the emergence of Black-Scholes model, the partial differential equations (PDE) for option pricing have started to be used widely. PDEs are adopted for both finding numerical and analytical solutions and developing new models for option pricing. One of the significant PDE is fractional Black-Scholes PDE. Essentially, a PDE can become non-local with fractionalization and this non-localization enables to expand the time frame of that equation. Several fractional Black Scholes equations are proposed in literature. The ones relevant to the topic of this thesis are summarized. The main contribution of this thesis is the development of new fractional Black-Scholes PDE through fractional heat equation and fractional Brownian motion. The new models are evaluated for particular cases and correspondence with Black Scholes PDE is noticed. Moreover, because the valuation of option is as necessary as the derivation of an option valuation model, the explicit method is expanded to a fractional explicit method. The new method is to find a numerical solution. The Fractional Black Scholes PDE is solved by the proposed fractional explicit method and the solutions are compared with the classical ones.

Keywords : Fractional Calculus, Fractional Brownian Motion, Fractional Black Scholes PDE, Fractional Explicit Method

ÖZ

KESİRLİ BLACK SCHOLES OPSİYON FİYATLANDIRMA YAKLAŞIMLARININ İNCELENMESİ VE UYGULAMALARI

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Opsiyon fiyatlama, finansal matematikteki en temel araştırma konularındann birisidir. Black Scholes modelinden sonra kısmi diferansiyel denklemlerle opsiyon fiyatlamak daha yaygın hale gelmiştir. Kısmi diferansiyel denklemler hem nümerik ve analitik çözümle bulunmasına hem de yeni modeller geliştirilmesine olanak sağlamaktadır. En önemli kısmi diferansiyel denklemlerden biri kesirli Black Scholes denklemidir. Bir kısmi diferansiyel denklemi zamana göre kesirli hale getirmek onu zaman sınırlamasından çıkartır; bu da denklemin kısıtlı zaman aralığını genişletir. Literatürde, çeşitli metotlar kullanılarak çok sayıda kesirli Black Scholes denklemi önerilmiştir. Bu önerilerden tez konusuyla ilgili olanları incelenmiş ve özetlenmiştir. Bu tezin literatüre katkısı kesirli ısı denklemi ve kesirli Brown hareketinden elde edilen iki yeni kesirli Black Scholes denklemidir. Önerilen bu modeller, belirli durumlar için Black Scholes denklemine karşılık gelmiştir. Diğer yandan, opsiyonun değerini bulmak, opsiyon fiyatlama modeli elde etmek kadar önemli olduğu için, ileri doğru farklar metodu kesirli metoda genişletilmiştir. Bu yeni metodun amacı nümerik çözümler bulmaktır. Kesirli Black Scholes denklemi, önerilen bu kesirli ileri doğru farklar metodu ile çözülmüş ve çözümler, ileri doğru farklar metodu ile elde edilen çözümlerle karşılaştırılmıştır.

Anahtar Kelimeler : Kesirli Analiz,Kesirli Brown Hareketi, Kesirli Black Scholes Kısmi Diferansiyel Denklemi, Kesirli İleri Doğru Farklar Metodu

To My Family

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CHAPTER 1

INTRODUCTION

In finance, a derivative is a contract whose value is determined according to an underlying asset. The derivatives are widely used especially for hedging which is basically the insurance of the price movements. Options are one of the fundamental derivatives which are commonly traded in the derivatives market. An option is a contract that its buyer may buy or sell the underlying asset at an explicit price on or before the explicit date. According to the expectations of the buyer, call and put options are traded. For example, if the buyer has an expectation that the stock will go up, the call options are traded for the right to buy at a specified price. If the buyer has an expectation that the stock will go down, the put options are traded for the right to sell at a specified price. According to the expiration date of the option, European and American options are traded. A European option may be exercised only at the expiration date of the option. An American option may be exercised at any time before the expiration date.

Option valuation is a topic of ongoing research in the academic and practical finance. Although the option valuation has been studied since 19th century, the contemporary approach is still based on the Black Scholes model, which was first published in 1973 and awarded the 1977 Nobel Prize in economics. [6] The Black Scholes model is used to calculate the theoretical price of European put and call options where the underlying stock price follows a geometric Brownian motion.

The Black Scholes equation is a second order partial differential equation in financial mathematics which is fulfilled by the price of the European option.
The Black Scholes PDE for the European call or put on an underlying stock without paying dividends is:

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

where $V(S,t)$ is the price of European option as a function of stock price S and time t, r is the risk-free interest rate, σ is the volatility of the stock.

For the European call option $C(S,t)$ for $S \in (0,\infty)$ and $t \in (0,T)$, the initial and boundary conditions of Black Scholes PDE are $C(S,T) = \max(S-E, 0)$, $C(0,t) = 0$

and $C(S,t) \approx S$ as $S \to \infty$ where $C(S,T)$ is the value of option at T when the option matures.

In this thesis, fractional calculus is used to derive and analyze the Black Scholes PDE. The fractional calculus is a branch of mathematics which is applicable for non-integer powers of the differentiation operator. The arbitrary order derivatives are called differintegrals. The non-integer order of differential operator was first presented by Leibniz[19]. Later, Abel[1], Fourier[12], Lioville[20] and Riemann[30] made important contributions to the literature. They defined and developed fractional integral and differentiation. Especially, the integer order derivatives and integrals are widely used for physical and geometric interpretations. But, an acceptable interpretation for differintegrals is missing in literature. Podlubny [29] shows that the geometric interpretation of fractional integration is “shadows on the walls” and its physical interpretation is “shadows of the past.” [4]

Several authors proposed new approaches for derivation of fractional Black Scholes PDE. In literature, Jafari, Khan, Kmar, Sayevand, Wei, Yıldırım [15], Jumarie [16], Wyss [33] derived fractional Black Scholes PDE.

However, the question is why the fractional derivative is non-local? The fractional order derivatives of a function are based on the values of the function over the entire range but not on the value of a single point. The space fractional derivate is non-local and it does not have a local meaning. Therefore, the boundary conditions are necessary to find a valid interpretation. For time fractional derivatives, non-locality represents that the properties of the curve should be taken into consideration over a large extend in time. Hence, the system has a long term memory and the evaluation at a point depends on the past values of the function. The fractional derivatives can be used for different physical systems such as, diffusion equations, food engineering, robotics, control theory and econophysics etc.[10]

Fractional calculus is recently used for the finance and stock market analysis. Because, the historical information about the market can be included in the analysis. Moreover, the stochastic calculus for fractional Brownian motion (fBm) has been widely used to develop financial models for the same reason. fBm is an extension of the classical Brownian motion.

In brief, the fractional order model is based on the historical data of the system. The financial variables such as stock market prices need more long-term memory to forecast future fluctuations better based on the past fluctuations. In financial markets, the main aim is making profit by trading through the right estimations. The motivation of this research is to build a robust financial model to make the right estimations by employing long-term memory of the fractional calculus efficiently.

Chapter 2 gives a detailed introduction to fractional calculus. The definitions and properties of differintegrals are presented in the first two sections. In the third section, Mittag-Leffler function is discussed, which is a necessary tool to be able to analyze differintegrals. In the last section, Brownian motion is for following chapters to be understood in the right context.

Chapter 3 presents five different approaches to derive fractional Black Scholes partial differential equations. The preliminaries of Black Scholes PDE are given in the first section, In between five approaches, we propose two new derivations by using fractional heat equation and using fractional Brownian motion in the last two sections.

Chapter 4 provides the finite difference method. The explicit method is the most useful one to find numerical solutions for Black Scholes PDE. The explicit method is given in the first section. The consistency, convergence and stability of the explicit method are discussed. In the last section, a fractional explicit method is proposed. This method is applied to fractional Black Scholes PDE. Five different examples with the different values of the variables in PDE are presented and the results are compared.

Chapter 5 concludes the research and gives an outlook of the future work.

CHAPTER 2

PRELIMINARIES

Fractional calculus is a branch of mathematics which studies differintegrals. The subject of fractional calculus has gained popularity especially in the past decade because mathematics is needed for engineering and scientific applications. In other words, differintegrals can find practical solutions for many modern problems. In this chapter, different definitions of differintegrals and their properties are given.

In addition, Mittag-Leffler function is needed to understand the usage of differintegrals. In this chapter, Mittag-Leffler function is discussed.

Fractional Brownian motion is given at the end of chapter in order to make a fractional and financial basis for the following chapters.

2.1 Definitions for Differintegrals

In this section, the most commonly used definitions of differintegrals are given which are Grünwald-Letnikov definition, Riemann-Liouville definition, Riemann definition, Laplace transform for differintegral, Caputo definition of fractional derivative and Riesz definition. For an extensive discussion of differintegrals, see Bayın [5] and Oldham and Spanier [26].

2.1.1 Notation for Differintegrals

For n is integer, the common notations for n-th order derivative of a function $f(x)$ at x are as follows:

$$\frac{d^n f(x)}{dx^n} = f^{(n)}(x) = D_x^n f(x).$$

Similarly, when the integral considered as inverse of derivative, the notations for n-th integral of a function $f(x)$ at x are as follows:

$$\frac{d^{-n} f(x)}{dx^{-n}} = f^{(-n)}(x) = D_x^{-n} f(x).$$

It is common to use q when the power n is real or complex number. Therefore, combining derivative and integral definitions for arbitrary q gives

$$\frac{d^q f(x)}{dx^q} = f^{(q)}(x) = D_x^q f(x).$$

Remark 2.1. Note that when a is lower limit for $a < x$, the notation for differintegral is as follows:

$${}_aD_x^q f(x) = \frac{d^q f}{d(x-a)^q}$$

2.1.2 Grünwald-Letnikov Definition of Differintegrals

Grünwald and Letnikov defined fractional differintegral in 1868 as limit of a sum which is generalized form of definition of differentiation and successive integration for arbitrary q numbers.

Definition 2.1. [5] Grünwald-Letnikov q-th order differintegral for a continuous function $f(x)$ is given

$${}_aD_x^q f(x) = \frac{d^q f}{d(x-a)^q} = \lim_{N\to\infty} \left\{ \frac{\left(\frac{x-a}{N}\right)^{-q}}{\Gamma(-q)} \sum_{j=0}^{N-1} \frac{\Gamma(j-q)}{\Gamma(j+1)} f\left(x - j\left(\frac{x-a}{N}\right)\right) \right\}. \tag{2.1}$$

Here, the expression a is lower limit for $a < x$ and N represents the number of segments which the interval $(x-a)$ divided into.

It is worth noting that the number q in the above definition can takes all values. Furthermore, the main property of this definition is consisting only of the values of the function. There is no need for derivatives and integrals.

Proposition 2.1. *The Grünwald-Letnikov definition of differintegral is obtained by extending integer n to arbitrary q in definitions of derivative and integral.*

Proof. Let N be the number of segments which the interval $(x-a)$ divided into by definition. Then let us define $\delta_N x$ as follows:

$$\delta_N x = \frac{x-a}{N}, \quad \text{for } N = 1, 2, 3, ...$$

The following definition of $\frac{d^n f}{d(x-a)^n}$ is derived from the definition of a derivative and for the coefficients, binomial expansion is considered as:

$$\frac{d^n f}{d(x-a)^n} = \lim_{N\to\infty} \left\{ (\delta_N x)^{-n} \sum_{j=0}^{N-1} (-1)^j \binom{n}{j} f(x - j(\delta_N x)) \right\}.$$

The following definition of $\frac{d^{-n}f}{d(x-a)^{-n}}$ is derived from the expression for n successive integrals and Riemann sum as:

$$\frac{d^{-n}f}{d(x-a)^{-n}} = \lim_{N\to\infty}\left\{(\delta_N x)^n \sum_{j=0}^{N-1}\binom{j+n-1}{j} f(x-j(\delta_N x))\right\}.$$

The Binomial expansions in the above definitions can be expressed as follows using Gamma functions

$$(-1)^j\binom{n}{j} = \binom{j+n-1}{j} = \frac{\Gamma(j-n)}{\Gamma(-n)\Gamma(j+1)}. \tag{2.2}$$

Note that Equation (2.2) is valid for integer numbers n and noninteger numbers q and the proof of the equation is in Appendix B.

It follows easily that a unified definition is possible for both positive and negative integers. The Grünwald-Letnikov definition can be obtained by extending integer n to real or even complex numbers q:

$$\frac{d^q f}{d(x-a)^q} = \lim_{N\to\infty}\left\{\frac{(\delta_N x)^{-q}}{\Gamma(-q)}\sum_{j=0}^{N-1}\frac{\Gamma(j-q)}{\Gamma(j+1)} f(x-j\delta_N x)\right\}.$$

□

2.1.3 Riemann-Liouville Definition of Differintegral

Riemann and Liouville defined fractional differintegral in 1832 from an integral definition.

Definition 2.2. [5] Riemann-Liouville definition of q-th order differintegral for a continuous function $f(x)$ is expressed as follows:

for $\forall q < 0$:

$${}_aD_x^q f(x) = \frac{d^q f}{d(x-a)^q} = \frac{1}{\Gamma(-q)}\int_a^x (x-\xi)^{-q-1} f(\xi)d\xi,$$

for $q - n < 0$ and $\forall q > 0$:

$${}_aD_x^q f(x) = \frac{d^q f}{d(x-a)^q} = \frac{d^n f}{d(x-a)^n}\left(\frac{1}{\Gamma(n-q)}\int_a^x (x-\xi)^{-(q-n)-1} f(\xi)d\xi\right).$$

where the expression a is lower limit for $a < x$ and n is an integer number.

Remark 2.2. Riemann-Liouville definition of differintegral can be derived by extending Cauchy's integral formula for n-th order

$$f^{(-n)}(x) = \int_a^x \int_a^{x_n} ... \int_a^{x_3} \int_a^{x_2} f(x_1)dx_1...dx_{n-1}dx_n = \frac{1}{(n-1)!}\int_a^x (x-\xi)^{n-1} f(\xi)d\xi.$$

Theorem 2.2. *Riemann-Liouville definition and Grünwald-Letnikov definition of differintegrals are equal*

$$\left\{\frac{d^q f}{d(x-a)^q}\right\}_{Riemann-Liouville} = \left\{\frac{d^q f}{d(x-a)^q}\right\}_{Grunwald-Letnikov}.$$

Proof. See [5].

It is worth to note that Grünwald-Letnikov and Riemann-Liouville definitions are the most common and basic definitions in literature. Besides, there are several other definitions which are stated in following sections.

2.1.4 Riemann Definition of Differintegral

Definition 2.3. Riemann definition of q^{th} order differintegral for functions as $f(x) = x^p$ and $p > -1$ is

$$\frac{d^q x^p}{dx^q} = \frac{\Gamma(p+1)}{\Gamma(p-q+1)} x^{p-q},$$

where q takes all the values.

Remark 2.3. The Riemann definition of differintegrals is a generalization of the formula for positive integer m and n

$$\frac{d^n x^m}{dx^n} = \frac{(m)!}{(m-n)!} x^{m-n}.$$

2.1.5 Laplace Transform for Differintegral

Definition 2.4. The q^{th} order differintegral of a function $f(x)$ at a point x can be defined by using Laplace Transform as:

for $\forall \ q < 0$:

$$\frac{d^q f}{dx^q} = \mathcal{L}^{-1}(s^q \tilde{f}(s)),$$

for $\forall\ \ q > 0$ and $n-1 < q < n$:

$$\frac{d^q f}{dx^q} = \mathcal{L}^{-1}\left(s^q \tilde{f}(s) - \sum_{k=0}^{n-1} s^k \frac{d^{q-1-k} f}{dx^{q-1-k}}(0)\right),$$

where $\tilde{f}(s)$ is Laplace transform of $f(x)$.

Theorem 2.3. *Riemann-Liouville definition and definition of differintegral by Laplace Transform are equal*

$$\left\{\frac{d^q f}{d(x-a)^q}\right\}_{Riemann-Liouville} = \left\{\frac{d^q f}{d(x-a)^q}\right\}_{Laplace}$$

Proof. See [5].

2.1.6 Caputo Definition of Fractional Derivative

Caputo defined fractional derivative in 1960s, using Laplace transform. This definition is widely used, especially for viscoelasticity problems.

Definition 2.5. Caputo definition of q^{th} order differintegral of a function $f(x)$ at a point x for $0 < q < 1$:

$$\frac{d^q f}{dx^q} = \frac{1}{\Gamma(1-q)} \int_0^x (x - \acute{x})^{-q} \left(\frac{df(\acute{x})}{d\acute{x}}\right) d\acute{x}$$

Note that since it is valid for $0 < q < 1$, this definition is given only for derivative.

Theorem 2.4. *The relationship between Riemann-Liouville definition and Caputo definition for* $0 < q < 1$ *is as follows:*

$$\left\{\frac{d^q f}{d(x-a)^q}\right\}_{R-L} - \frac{x^{-q}}{\Gamma(1-q)} f(0) = \left\{\frac{d^q f}{d(x-a)^q}\right\}_{Caputo}.$$

Proof. See [5].

2.1.7 Riesz Definition of Differintegral

Riesz definition of differintegrals is commonly used in applications which is derived using Fourier transform.

2.1.7.1 Riesz Definition of Fractional Integral

Definition 2.6. Riesz Definition of q^{th} order fractional integral for a function $f(x)$ at a point x for $q > 0$ and $q \neq 1, 3, 5, ...$:

$$\frac{d^{-q}f}{dx^{-q}} = \frac{{}_{-\infty}D_x^q +_{\infty} D_x^q}{2\cos\left(\frac{q\pi}{2}\right)} f(x),$$

where ${}_{-\infty}D_x^q$ and ${}_{\infty}D_x^q$ are Riemann-Liouville differintegrals for lower limits $-\infty$ and ∞

2.1.7.2 Riesz Definition of Fractional Derivative

Definition 2.7. Riesz definition of q^{th} order fractional derivative for a function $f(x)$ at a point x for $0 < q \leq 2$ and $q \neq 1$:

$$\frac{d^q f}{dx^q} = -\frac{{}_{-\infty}D_x^q +_{\infty} D_x^q}{2\cos\left(\frac{q\pi}{2}\right)} f(x),$$

where ${}_{-\infty}D_x^q$ and ${}_{\infty}D_x^q$ are Riemann-Liouville differintegrals for lower limits $-\infty$ and ∞

Theorem 2.5. *[5] If $D_x^k f(0) = 0$ for $k = 0, 1, 2, ..., n-1$ then definitions of Riemann-Liouville, Grünwald-Letnikov, Riemann, Laplace, Caputo and Riesz are all agree.*

2.2 Properties of Differintegrals

Linearity, homogeneity and scale transformation of differintegrals, differintegral of a series and composition of differintegrals are the most useful properties and listed below. Leibniz Rule is a product rule for differintegrals. These properties are widely used for derivation of new differintegrals.

- Linearity of q-th order differintegrals can be expressed as:

$$\frac{d^q(f_1 + f_2)}{dx^q} = \frac{d^q f_1}{dx^q} + \frac{d^q f_2}{dx^q}.$$

- Homogeneity of q-th order differintegrals can be expressed as:

$$\frac{d^q(C_0 f)}{dx^q} = C_0 \frac{d^q f}{dx^q}.$$

- Scale transformation of a function for q-th order differintegrals can be expressed as:

$$\frac{d^q f(\gamma x)}{dx^q} = \gamma^q \frac{d^q f(\gamma x)}{d(\gamma x)^q}.$$

- In order to differintegrate q-th order of functions with power series, Riemann definition can be used as:

$$\frac{d^q}{d(x-a)^q}\sum_{j=0}^{\infty} a_j(x-a)^{p+\frac{j}{n}} = \sum_{k=0}^{\infty} a_j \frac{\Gamma(p+\frac{j}{n}+1)}{\Gamma(p+\frac{j}{n}-q+1)}(x-a)^{p+\frac{j}{n}-q}.$$

- Composition of q-th order differintegrals is as follows:

$$\frac{d^q}{d(x-a)^q}\left(\frac{d^Q f}{d(x-a)^Q}\right) = \frac{d^{q+Q}}{d(x-a)^{q+Q}}.$$

However, it is valid under certain condition that is

$$f - \frac{d^{-Q}}{d(x-a)^{-Q}}\left(\frac{d^Q f}{d(x-a)^Q}\right) = 0.$$

Since the general rule for composition of differintegrals for noninteger q and Q is

$$\begin{aligned}\frac{d^q}{d(x-a)^q}\left(\frac{d^Q f}{d(x-a)^Q}\right) &= \frac{d^{q+Q}}{d(x-a)^{q+Q}} \\ &\quad - \frac{d^{q+Q}}{d(x-a)^{q+Q}}\left\{f - \frac{d^{-Q}}{d(x-a)^{-Q}}\left(\frac{d^Q f}{d(x-a)^Q}\right)\right\}.\end{aligned}$$

- Leibniz Rule is to differintegrate of the q^{th} order of the multiplication of two functions as:

$$\frac{d^q(f.g)}{d(x-a)^q} = \sum_{j=0}^{\infty}\binom{q}{j}\frac{d^{q-j}(f)}{d(x-a)^{q-j}}\frac{d^j(g)}{d(x-a)^j}.$$

where $\binom{q}{j}$ can be calculated by gamma functions as:

$$\binom{q}{j} = \frac{\Gamma(q+1)}{\Gamma(q-j+1)\Gamma(j+1)}. \tag{2.3}$$

2.3 The Mittag-Leffler Function

The Mittag-Leffler function plays an important role for representing solutions of fractional order partial differintegral equations and fractional integral equations. Moreover, Mittag-Leffler function found widespread usage in applications in many branches of science.

Definition 2.8. [21] Mittag-Leffler function is defined for $\alpha > 0$ as follows:

$$E_\alpha(x) = \sum_{k=0}^{\infty}\frac{x^k}{\Gamma(\alpha k+1)}.$$

Remark 2.4. The Mittag-Leffler function turns out to be exponential function for $\alpha = 1$. So the exponential function is a specific form of an infinite series

$$E_1(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x.$$

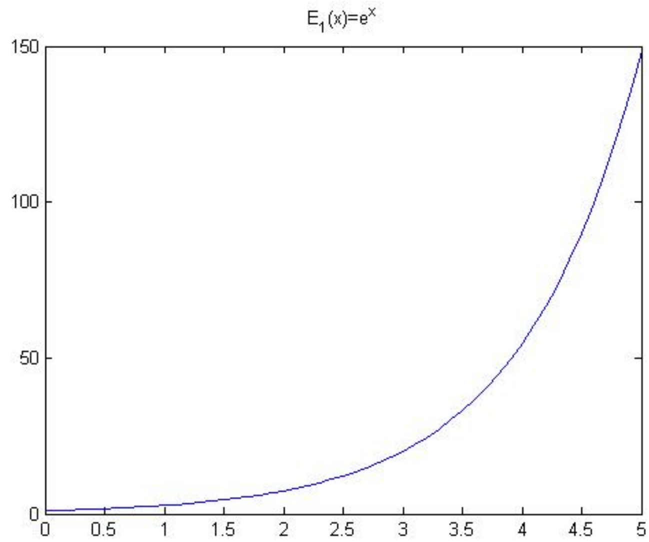

Figure 2.1: Mittag-Leffler function for $\alpha = 1$

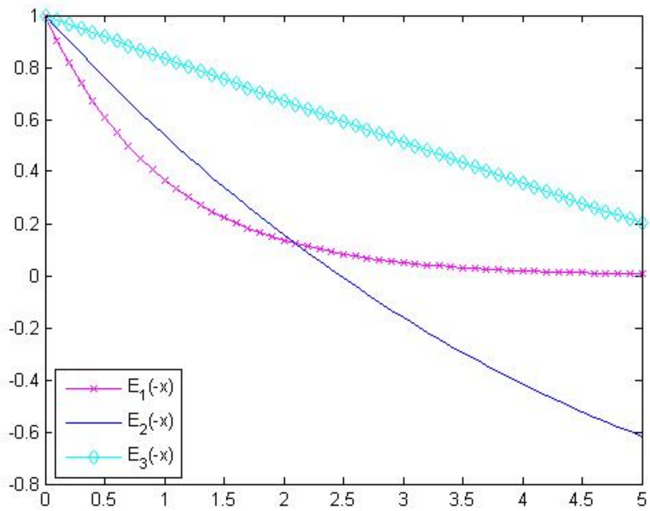

Figure 2.2: Mittag-Leffler function of $(-x)$ for $\alpha = 1, 2$ and 3.

Remark 2.5. Extraordinary differential equations (Fractional order differintegral equations) can be given as follows:

$$\frac{d^q f(x)}{dx^q} = F(x),$$

where q is any number, $F(x)$ is a given function and $f(x)$ is unknown function.

Proposition 2.6. *For constant λ and positive integer n the differential equation is*

$$\frac{d^n x(t)}{dt^n} = \lambda^n x(t), \tag{2.4}$$

and has the solution

$$x(t) = x_0 E_n((\lambda t)^n),$$

where (E_n) is the Mittag-Leffler function.

For constant λ and arbitrary number q extraordinary differential equation is

$$\frac{d^q x(t)}{dt^q} = \lambda^q x(t) \tag{2.5}$$

and has the solution

$$x(t) = x_0 E_q((\lambda t)^q).$$

Note that Equation (2.5) is fractional extension of Equation (2.4).

2.4 Brownian Motion

In order to construct a basis for the fractional Brownian motion and Itô process, the following section will be useful. Since, not only fractional calculus but also stochastic calculus for the fractional Brownian motion is used for derivation of the new fractional financial option pricing models, fractional Brownian motion is needed.

2.4.1 Standard Brownian Motion

Weiner process also called standard Brownian motion plays an important role in mathematics, economics, and applied mathematics especially in stochastic calculus.

Definition 2.9. [18] A real valued one dimensional Brownian motion $(W_t)_{t\geq 0}$ is a continuous time stochastic process with the following properties

- $W_0 = 0$,

- W_t has independent increments

$$W_t - W_s \approx N(0, t-s),$$

where $N(\mu, \sigma^2)$ denotes the normal distribution with expected value μ and variance σ^2,

- $t \to W_t$ has a continuous path.

Remark 2.6. Standard Brownian motion is a centered Gaussian process. For W_t is a standard Brownian motion expectation and variance of $(W_t)_{t\geq 0}$ are

$$\mathbb{E}(W_t) = 0,$$
$$Var(W_t) = t.$$

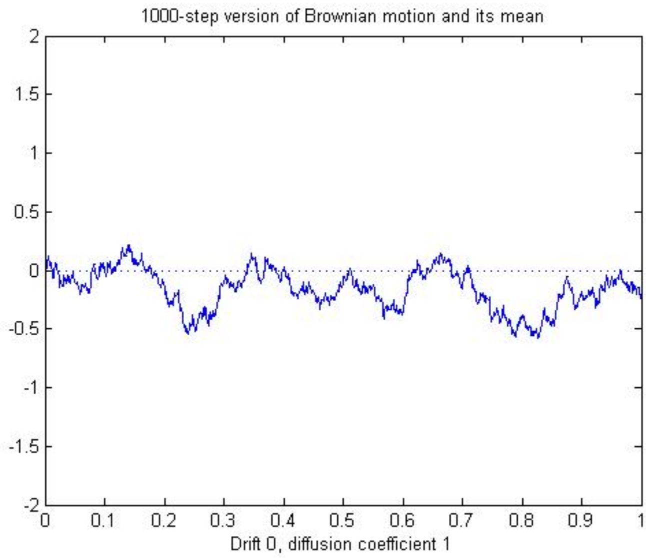

Figure 2.3: Standard Brownian Motion for drift 0 and diffusion coefficient 1

Stochastic differential equation is a differential equation which has stochastic process. Also it has a solution which is itself a stochastic process.

Definition 2.10. Let W_t be a standard Brownian motion, X_t is an $\mathbb{R}$-valued Itô process and $a(X_t, t)$ is drift coefficient and $b(X_t, t)$ is the diffusion term.

Then the definition of stochastic differential equation (SDE) is

$$dX_t = a(X_t, t)dt + b(X_t, t)dW_t.$$

Definition 2.11. (Geometric Brownian Motion)[18] A stochastic differential equation under risky probability measure $\mathbb{P}$ of the form

$$dS_t = \mu S_t dt + \sigma S_t dW_t. \tag{2.6}$$

is called geometric Brownian motion where S_t is stock price and μ and σ are constants.

Remark 2.7. The solution of geometric Brownian motion is

$$S_t = S_0 \exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right\},$$

where S_0 is the initial value of the asset price.

Remark 2.8. The expectation of a geometric Brownian motion S_t is

$$\mathbb{E}(S_t) = S_0 e^{\mu t}.$$

The variance of a geometric Brownian motion S_t is

$$Var(S_t) = {S_0}^2 e^{2\mu t}(e^{\sigma^2 t} - 1).$$

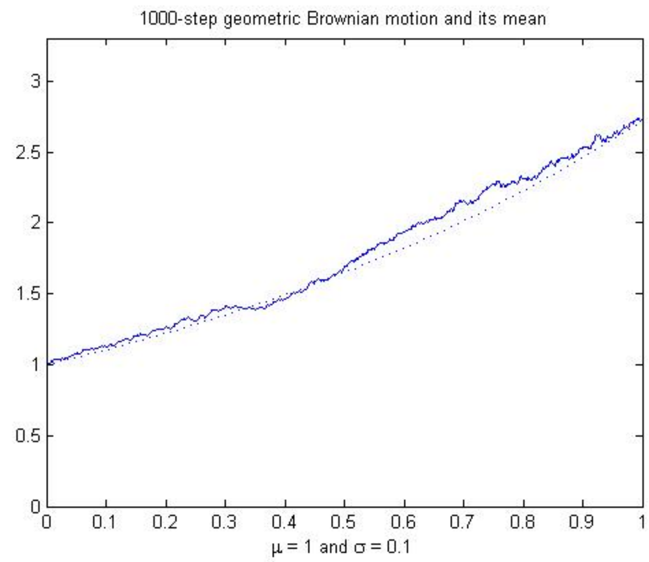

Figure 2.4: Geometric Brownian Motion for $\mu = 1$ and $\sigma = 0.1$

Brownian motion $\tilde{W}_t$ under risk neutral probability measure $\tilde{\mathbb{P}}$, where r is the risk free rate, is as follows:

$$\tilde{W}_t = \frac{\mu - r}{\sigma}t + W_t.$$

Therefore, Equation (2.6) under $\tilde{\mathbb{P}}$ is

$$dS_t = rS_t dt + \sigma S_t d\tilde{W}_t.$$

Lemma 2.7. *(Itô Lemma) [18] Let X_t be an Itô process where $X_{t_0} = X_0$*

$$X_t = X_0 + \int_{t_0}^{t} K_s ds + \int_{t_0}^{t} H_s dW_s,$$

and let f be a twice continuously differentiable function. Then the Itô Lemma is stated as follows:

$$f(X_t,t) = f(X_0,0) + \int_0^t f_s^{'}(X_s,s)ds + \int_0^t f_x^{'}(X_s,s)dX_s \tag{2.7}$$

$$+ \frac{1}{2}\int_0^t f_{xx}^{''}(X_s,s)d < X,X >_s,$$

where quadratic variation is

$$d < X,X >_s = H_s^2 ds.$$

2.4.2 Fractional Brownian Motion

Fractional Brownian motion (fBm) is a generalization of Brownian motion which has following definition

Definition 2.12. [14] fBm; $B^{(H)}(t)$, for $t \geq 0$ and with Hurst index $H \in (0,1)$ is a centered Gaussian process, and has the following covariance function

$$\mathbb{E}[B^H(t)B^H(s)] = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} + |t-s|^{2H}\right).$$

Remark 2.9. [27] For different values of H, properties of fBm are as follows:

1. If $H = \frac{1}{2}$, fBm is a standard Brownian Motion.

2. If $H > \frac{1}{2}$, then the increments of the process are positively correlated. In other words, the times series has persistent behavior. Also, fBm has long-range dependence, implying that it has long memory.

3. If $H < \frac{1}{2}$, then the increments of the process are negatively correlated. In other words, the times series has antipersistent behavior.

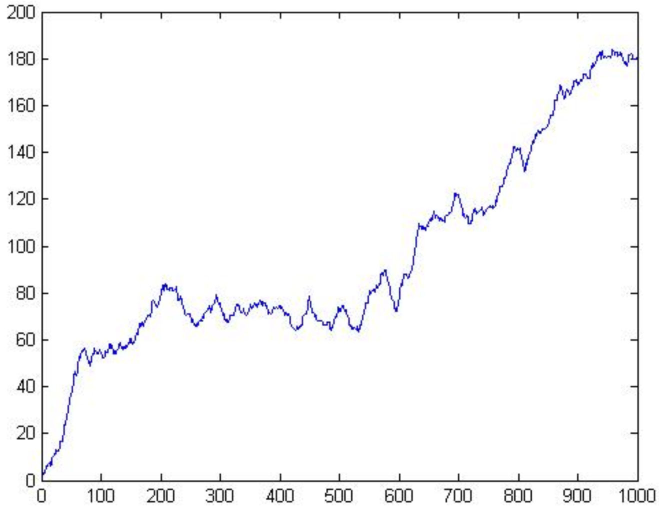

Figure 2.5: Fractional Brownian Motion for $H = 0.7$

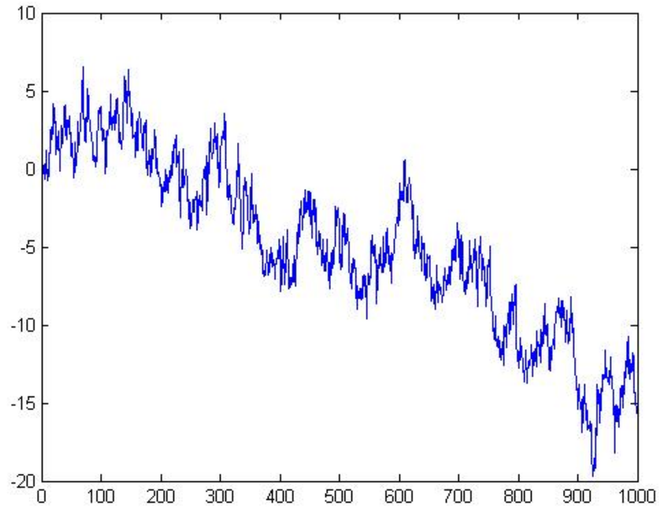

Figure 2.6: Fractional Brownian Motion for $H = 0.3$

Properties of fBm

- $B^H(0) = 0$.
- The expectation of fBm is

$$\mathbb{E}[B^H(t)] = 0 \quad \forall t > 0.$$

- The variance of fBm is

$$Var[B^H(t)] = t^{2H}.$$

- If $H \neq \frac{1}{2}$, then fBm is non-Markovian and is not a semimartingale.
- The process is self-similar

$$B^H(\alpha t) \approx |\alpha|^H B^H(t).$$

- fBm has stationary increments

$$B^H(t) - B^H(s) \approx B^H(t-s).$$

Proposition 2.8. *[27] Fractional stochastic differential equation under $\mathbb{P}$ for constant; x, drift μ and volatility σ is given as with the initial condition $S(0) = x > 0$*

$$dS(t) = \mu S(t)dt + \sigma S(t)dB^H(t). \tag{2.8}$$

Then, the solution of Equation (2.8) is given by

$$S(t) = x \exp\left(\sigma B^H(t) + \mu t - \frac{1}{2}\sigma^2 t^{2H}\right):$$

Remark 2.10. Fractional Brownian motion $\tilde{B}^H(t)$ under risk neutral probability measure $\tilde{\mathbb{P}}$ where r is the risk free rate is as follows:

$$\tilde{B}^H(t) = \frac{\mu - r}{\sigma}t + B^H(t).$$

The fractional stochastic differential equation (2.8) under $\tilde{\mathbb{P}}$ is

$$dS_t = rS_t dt + \sigma S_t d\tilde{B}^H(t).$$

Theorem 2.9. *(Fractional Itô formula)[14] Let $H \in (0,1)$. Let $f(S,x): \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ belongs to $\mathcal{C}^{1,2}(\mathbb{R}\times\mathbb{R})$ and $f(t, B^H(t))$, $\int_0^t \frac{\partial f}{\partial s}(s, B^H(s))ds$ and $\int_0^t \frac{\partial^2 f}{\partial x^2}(s, B^H(s))s^{2H-1}ds$ belong to $L^2(\mathbb{P})$.*

Then fractional Itô formula is presented as follows:

$$f(t, B^H(t)) = f(0,0) + \int_0^t \frac{\partial f}{\partial s}(s, B^H(s))ds + \int_0^t \frac{\partial f}{\partial x}(s, B^H(s))dB^H(s)$$

$$+H\int_0^t \frac{\partial^2 f}{\partial x^2}(s, B^H(s))s^{2H-1}ds.$$

Remark 2.11. For following stochastic differential equation

$$dS_t = \mu S_t dt + \sigma S_t dB^H(t),$$

[14] The quadratic variation of S_t is given by:

$$d < S, S >_t = S_t^2 2H\sigma^2 t^{2H-1} dt.$$

Note that Brownian motion is presented in detailed in this section. For further details on the properties of fBm, see [14].

CHAPTER 3

FRACTIONAL BLACK SCHOLES APPROACHES

3.1 Preliminaries

Black Scholes equation is a second order partial differential equation in financial mathematics which estimates the price of the European option under the Black Scholes Model (1973). The model is awarded Nobel Prize in Economics in 1997 and widely used in option pricing and risk elimination since then.

Definition 3.1. For European call or put options on an underlying stock paying no dividends, Black Scholes PDE is stated as follows:

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0, \tag{3.1}$$

where $V(S,t)$ is the price of European option as a function of stock price S and time t, and r is the risk-free interest rate, σ is the volatility of the stock.

The main point of the Black Scholes formula is that there is one price of the option. One can eliminate risk by hedging by buying or selling the underlying asset.

Remark 3.1. Derivation of Black Scholes PDE by using Itô formula in Equation (2.7) and geometric Brownian motion (2.6) is on Appendix A

Remark 3.2. For European call option initial and boundary condition of Black Scholes PDE for $S \in (0,\infty)$ and $t \in (0,T)$ is

$$\begin{aligned} C(S,T) &= \max(S-E,0), \\ C(0,t) &= 0, \\ C(S,t) &\approx S \text{ as } S \to \infty, \end{aligned}$$

where E is the strike price, $C(S,T)$ is value of the option at T when the option matures.

Proposition 3.1. *The solution of Black Scholes PDE in Equation (3.1) for European call option $C(S,t)$ with the initial and boundary conditions in Remark 3.2 is*

$$C(S,t) = SN(d_1) - Ee^{-r(T-t)}N(d_2),$$

where

$$d_1 = \frac{\ln\left(\frac{S}{E}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}},$$

$$d_2 = \frac{\ln\left(\frac{S}{E}\right) + \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}},$$

and

$$N(d) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d} e^{-\frac{1}{2}x^2} dx.$$

Example 3.1. For a European call option with an exercise price of 95. The option has the underlying stock at price 100$ which pays no dividends, and has a volatility of 50% and the risk-free rate is 10%. For $T = 0.25$. By using MATLAB code in Appendix C, we obtain the call option price as 13.6953$.

Example 3.2. For a European put option with an exercise price of 95. The option has the underlying stock at price 100$ which pays no dividends, and has a volatility of 50% and the risk-free rate is 10%. For $T = 0.25$. By using MATLAB code in Appendix C, we obtain the put option price as 6.3497$.

In this chapter, different approaches to derive fractional Black Scholes PDE are given. First of all, definitions of fractional order differintegral were given in previous chapter. Then in order to fractionalize Black Scholes PDE, the term with time derivative can be extended to non-integer order q.

Thus one can simply write time fractional Black Scholes PDE as follows

$$\frac{\partial^q V}{\partial t^q} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0.$$

Several authors proposed new approaches for derivation of fractional Black Scholes PDE. In this chapter, three approaches for derivation of fractional Black Scholes PDE are given. First approach [33] is using the equation of evolution for derivation of PDE. In second approach [15], Laplace transform and homotopy perturbation method is used to derive fractional Black Scholes PDE. The third approach [16] use fractional Taylor series method .

Furthermore, we propose two new derivation of the Black Scholes PDE. First the derivation of the PDE is by using time fractional heat equation. Second one is derivation of Black Scholes PDE using fractional Brownian motion and It$\hat{o}$ formula.

3.2 Equation of Evolution Approach

In [33], fractional Black Scholes PDE is derived using equation of evolution. Moreover, the relation between the solutions of the classical and fractional equations is proposed.

The equation of evolution of $U(x,t)$ with initial condition $U(x,0) = f(x)$ is as follows:

$$\frac{\partial U(x,t)}{\partial t} = (L[x]U)(x,t), \tag{3.2}$$

where $L(x)$ is an operator of x.

We first take the derivative of Equation (3.2) and then restate the equation as;

$$U(x,t) = U(x,0) + \int_0^t (L[x]U)(x,\tau)d\tau. \tag{3.3}$$

Equation (3.3) is derived from the classical equation of evolution. Fractional extension of Equation (3.3) is proposed in [33] as:

$$U_q(x,t) = U(x,0) + \frac{1}{\Gamma(q)} \int_0^t (t-\tau)^{q-1} (L[x]U_q)(x,\tau)d\tau. \tag{3.4}$$

Theorem 3.2. *[33] For $A(S,\tau) = C(S,t)$ where $C(S,t)$ is European call option, τ is a variable depending on T, σ^2 and t, $L(x)$ is an operator and c. Therefore, q^{th} order time fractional Black Scholes PDE is*

$$\frac{\partial^q A}{\partial \tau^q} = \frac{\partial^q A(S,0)}{\partial \tau^q} + \frac{\partial^q}{\partial \tau^q} \left(\int_0^\tau (\tau - z)^{q-1} \left(S^2 \frac{\partial^2 A_q}{\partial S^2} + \lambda_0 S \frac{\partial A_q}{\partial S} - \lambda_0 A_q \right) (S,z) \right) dz.$$

Proof. The original Black Scholes equation (3.1) for European call option is restated with the boundary conditions $C(S,T) = \max(S-E,0)$:

$$\frac{\partial C}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} - rC = 0.$$

With the transformation $t = T - \frac{2}{\sigma^2}\tau$, $C(S,t) = A(S,\tau)$ and $\lambda_0 = \frac{2r}{\sigma^2}$ the equation of evolution is obtained with boundary conditions $A(S,0) = \max(S-E,0)$:

$$\frac{\partial A}{\partial \tau} = S^2 \frac{\partial^2 A}{\partial S^2} + \lambda_0 S \frac{\partial A}{\partial S} - \lambda_0 A = (L[S]A)(S,\tau). \tag{3.5}$$

According to Equation (3.3) $A(S,\tau)$ can be stated as:

$$A(S,\tau) = A(S,0) + \int_0^\tau (L[S]A)(S,z)dz \tag{3.6}$$

$$= A(S,0) + \int_0^\tau \left(S^2 \frac{\partial^2 A}{\partial S^2} + \lambda_0 S \frac{\partial A}{\partial S} - \lambda_0 A \right) (S,z)dz.$$

According to Equation (3.4) $A_q(S,\tau)$ is the fractional extension of $A(S,\tau)$ which can be stated as

$$A_q(S,\tau) = A(S,0) + \int_0^\tau (\tau - z)^{q-1}(L[S]A_q)(S,z)dz \tag{3.7}$$

$$= A(S,0) + \frac{1}{\Gamma(q)} \int_0^\tau (\tau - z)^{q-1} \left(S^2 \frac{\partial^2 A_q}{\partial S^2} + \lambda_0 S \frac{\partial A_q}{\partial S} - \lambda_0 A_q \right)(S,z)dz.$$

Finally, by taking the fractional derivative of Equation(3.7) we get the fractional Black Scholes PDE

$$\frac{\partial^q A}{\partial \tau^q} = \frac{\partial^q A(S,0)}{\partial \tau^q} + \frac{\partial^q}{\partial \tau^q} \left(\int_0^\tau (\tau - z)^{q-1}(L[S]A_q)(S,z) \right) dz$$

$$= \frac{\partial^q A(S,0)}{\partial \tau^q} + \frac{\partial^q}{\partial \tau^q} \left(\int_0^\tau (\tau - z)^{q-1} \left(S^2 \frac{\partial^2 A_q}{\partial S^2} + \lambda_0 S \frac{\partial A_q}{\partial S} - \lambda_0 A_q \right)(S,z) \right) dz.$$

□

Corollary 3.3. *[33] $U(x,t)$ is solution of Equation (3.3) which is equation of evolution and $U_q(x,t)$ is solution of its fractional extension in Equation (3.4). The relation between $U(x,t)$ and $U_q(x,t)$ is given by*

$$U_q(x,t) = t^{-q} \int_0^\infty f_q(t^{-q}z)U(x,z)dz,$$

where $f_q(z)$ is an entire function and can be represented as follows

$$f_q(z) = \sum_{k=0}^{\infty} (-1)^k \frac{1}{\Gamma(1-q-qk)} \frac{z^k}{k!},$$

where $0 < q < 1$ and $z \in \mathbb{R}^+$.

Remark 3.3. The relationship between $A(S,\tau)$ in Equation (3.6) and $A_q(S,\tau)$ in equation (3.7) from Corollary 3.3 is

$$A_q(S,\tau) = \tau^{-q} \int_0^\infty f_q(\lambda^{-q}\tau)A(S,\tau)d\tau.$$

Proposition 3.4. *The solution of Equation (3.5) is $A(S,\tau)$. Since $A(S,\tau)$ also satisfies the Black Scholes PDE the solution can be obtained by same transformation as:*

$$A(S,\tau) = SN(d_1) - Ee^{-\lambda_0 \tau}N(d_2),$$

where

$$d_1 = (2\tau)^{-\frac{1}{2}} \left[\ln\left(\frac{S}{E}\right) + (\lambda_0 + 1) \right],$$

$$d_2 = (2\tau)^{-\frac{1}{2}} \left[\ln\left(\frac{S}{E}\right) + (\lambda_0 - 1) \right],$$

and

$$N(d) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d} e^{-\frac{1}{2}x^2} dx.$$

3.3 Laplace Homotopy Perturbation Approach

In [15], analytical solution of the fractional Black Scholes equation is calculated via Laplace homotopy perturbation method, which is combined form of the Laplace transform and the homotopy perturbation method.

Theorem 3.5. *[15] Fractional Black Scholes PDE for* $0 < q \leq 1$ *for European option* $V(x,t)$ *is considered as:*

$$\frac{\partial^q V}{\partial t^q} = \frac{\partial^2 V}{\partial x^2} + (k-1)\frac{\partial V}{\partial x} - kV, \tag{3.8}$$

where $V(x,0) = \max(e^x - 1, 0)$ *and* $k = \frac{2r}{\sigma^2}$

Proposition 3.6. *[15] The analytical solution of the Equation (3.8) is found using Laplace transform and homotopy perturbation method as follows:*

$$V(x,t) = \lim_{p \to 1} \sum_{i=0}^{\infty} p^i V_i(x,t) = \max(e^x - 1, 0) E_q(-kt^q) + \max(e^x, 0)(1 - E_q(-kt^q)),$$

where $E_q(z)$ *is the Mittag-Leffler function.*

Remark 3.4. The solution in Proposition 3.6 is closed form solution and for $q = 1$ we get the exact solution of the Black Scholes formula of Equation (3.8)

$$V(x,t) = \max(e^x - 1, 0)e^{-kt} + \max(e^x, 0)(1 - e^{-kt}) \tag{3.9}$$

3.4 Fractional Taylor's Series Method

The Black Scholes equation is derived and it is claimed that in order to get the suitable Black Scholes equation it is not sufficient to extend the time derivative to fractional case [16]. Moreover, the solutions of these Black Scholes equations are obtained using Lagrange technique.

Theorem 3.7. *[16] Fractional Black Scholes PDE is derived using new fractional Taylor's series of fractional order and Riemann-Liouville fractional derivative definitions. The fractional Black Scholes PDE for European call option is*

$$\frac{\partial^q C}{\partial t^q} = \left(rC - rS\frac{\partial C}{\partial S} \right) \frac{t^{1-q}}{(1-q)!} - \frac{q!}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2}, \tag{3.10}$$

with standard boundary conditions $C(S,T) = \max(S-E,0)$.

Proposition 3.8. *The solution of the fractional Black Scholes equation (3.10)*

$$C(x,t) = \int_{-\infty}^{\infty} \Psi(x-v, T-t) C(v,T) dv, \tag{3.11}$$

$$\textit{where} \quad \Psi(x, T-t) = \int_{-\infty}^{\infty} e^{i\xi x} E_q(\xi^2 (T-t)^q) d\xi,$$

for $E_q(x)$ *is Mittag-Leffler function.*

Proof. In order to find the solution of Equation (3.10) first fractional heat equation is obtained from fractional Black Scholes PDE

$$C_t{}^q(x,t) = -\rho C_{xx}(x,t), \tag{3.12}$$

where $\rho^2 = (q!)\frac{\sigma^2}{2}$ and $C(x,T) = E(e^x - 1)$ for $E(x)$ is Mittag-Leffler function. Equation (3.11) is solution of Equation (3.12).

For detailed proof, see [16]. □

Note that, the theorems in the previous sections are approaches for derivation of fractional Black Scholes PDE in literature. In the following sections, we propose new fractional Black Scholes PDEs.

3.5 Heat Equation Approach

Derivation of fractional Black Scholes PDE using fractional heat equation is proposed in this section. The definitions, properties and rules in previous chapter will be used. First, heat equation is transformed to Black Scholes PDE by change of variables then fractional Black Scholes PDE is obtained using same transformations.

3.5.1 Classical Heat Equation

The heat equation is a partial differential equation that is used to determine the change in a function over time where the function is of space and time.

Theorem 3.9. *[28] Black Scholes PDE can be derived from heat equation using suitable transformations as:*

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0,$$

where $V(S,t)$ is European option price, r is risk free rate and σ is volatility of the stock.

Proof. The main idea of the proof is transformation of the heat equation to Black Scholes equation. Thus, let us first state the heat equation for function $u(x,\tau)$

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}.$$

At first, the transformation is applied to the function $u(x,\tau)$ as $u(x,\tau) = e^{-(\alpha x + \beta \tau)} v(x,\tau)$. Then we get

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} - 2\alpha \frac{\partial v}{\partial x} + (\alpha^2 + \beta) v.$$

Second change of variables are for coefficients as $\alpha = \frac{1-k}{2}$, $\beta = -\frac{(1+k)^2}{4}$ and $k = \frac{2r}{\sigma^2}$. Then we obtain

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + \left(\frac{2r}{\sigma^2} - 1\right) \frac{\partial v}{\partial x} - \frac{2r}{\sigma^2} v.$$

Last transformations are for both the functions and variables $v(x,\tau) = \frac{V(S,t)}{E}$, $\tau = \frac{\sigma^2}{2}(T-t)$ and $x = ln\left(\frac{S}{E}\right)$. Then we get

$$\begin{aligned}
\frac{\partial v}{\partial \tau} &= \frac{1}{E}\frac{\partial V}{\partial t}\left(-\frac{2}{\sigma^2}\right), \\
\frac{\partial v}{\partial x} &= \frac{S}{E}\frac{\partial V}{\partial S}, \\
\frac{\partial^2 v}{\partial x^2} &= \frac{S}{E}\frac{\partial V}{\partial S} + \frac{S^2}{E}\frac{\partial^2 V}{\partial S^2}.
\end{aligned}$$

Finally, by substitutions and suitable regulations Black Scholes PDE is obtained as in Equation(3.1)

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0.$$

□

Theorem 3.10. *Initial and boundary conditions of Black Scholes PDE for European call option in Remark 3.2 can be obtained from the initial and boundary conditions of heat equation.*

Proof. The initial and boundary conditions of heat equation for $\tau \in \left(0, \frac{T\sigma^2}{2}\right)$ and $x \in (-\infty, \infty)$ are

$$\begin{aligned}
&u(x,0) = \max\left(e^{\frac{k+1}{2}x} - e^{\frac{k-1}{2}x}, 0\right),\\
&u(x,\tau) \to 0 \text{ as } x \to -\infty,\\
&u(x,\tau) \approx e^{\frac{k+1}{2}\left(x+\frac{k+1}{2}\tau\right)} \text{ as } x \to \infty.
\end{aligned}$$

As in the proof of Theorem 3.9 the first change of variables are $u(x,\tau) = e^{-(\alpha x+\beta\tau)}v(x,\tau)$, $\alpha = \frac{1-k}{2}$, $\beta = -\frac{(1+k)^2}{4}$ and $k = \frac{2r}{\sigma^2}$. Then, we obtain for $x \in (-\infty, \infty)$, $\tau \in \left(0, \frac{T\sigma^2}{2}\right)$:

$$\begin{aligned}
&v(x,0) = \max(e^x - 1, 0),\\
&v(x,\tau) \to 0 \text{ as } x \to -\infty,\\
&u(x,\tau) \approx e^x \text{ as } x \to \infty.
\end{aligned}$$

Second change of variables are $v(x,\tau) = \frac{C(S,t)}{E}$, $\tau = \frac{\sigma^2}{2}(T-t)$ and $x = \ln\left(\frac{S}{E}\right)$.

Then, we get initial and boundary conditions of European call option for $S \in (0,\infty)$ and $t \in (0,T)$ as in Remark 3.2

$$\begin{aligned}
&C(S,T) = \max(S-E, 0),\\
&C(0,t) = 0,\\
&C(S,t) \approx S \text{ as } S \to \infty,
\end{aligned}$$

where $C(S,T)$ is value of the option at T when option matures. □

3.5.2 Fractional Heat Equation

Fractional Black Scholes PDE can be derived from fractional heat equation by using Riemann definition of differintegral, Liebniz rule and linearity and homogeneity properties of differintegrals. Especially, this derivation is inspired from the original derivation of Black Scholes PDE as in Theorem 3.9 since same transformations are used.

Theorem 3.11. *Time fractional Black Scholes PDE for European option price $V(S,t)$ which is derived from time fractional heat equation as:*

$$-\sum_{\substack{j=0\\ j\neq q}}^{\infty} \binom{q}{j} \frac{t+T(j-q)}{\Gamma(2-q+j)} (T-t)^{j-q} \left(\frac{\sigma^2}{2}\right)^{j-q+1} \left(\frac{\sigma^2+2r}{2\sigma^2}\right)^{\frac{2j}{q}} \frac{\partial^{q-j}V}{\partial t^{q-j}} \tag{3.13}$$

$$+\frac{\sigma^2}{2}S^2\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0.$$

Proof. Time fractional heat equation is basically extending the time derivative to q^{th} order differintegral

$$\frac{\partial^q u}{\partial \tau^q} = \frac{\partial^2 u}{\partial x^2}.$$

At first, the transformation is applied to function u as before $u(x,\tau) = e^{-(\alpha x+\beta\tau)}v(x,\tau)$. Then we get:

$$\frac{\partial^q}{\partial \tau^q}\left(e^{-(\alpha x+\beta\tau)}v(x,\tau)\right) = \frac{\partial^2}{\partial x^2}\left(e^{-(\alpha x+\beta\tau)}v(x,\tau)\right).$$

Since, there is a product of functions at the left hand side of the above equation, Liebniz Rule (2.3) should be used in order to differintegrate. Then we obtain

$$\sum_{j=0}^{\infty}\binom{q}{j}\frac{\partial^{q-j}v}{\partial \tau^{q-j}}\frac{\partial^j}{\partial \tau^j}\left(e^{-\alpha x-\beta\tau}\right) = e^{-\alpha x-\beta\tau}\left(\alpha^2 v - 2\alpha\frac{\partial v}{\partial x} + \frac{\partial^2 v}{\partial x^2}\right)$$

$$\sum_{j=0}^{\infty}\binom{q}{j}\frac{\partial^{q-j}v}{\partial \tau^{q-j}}(-\beta)^j = \alpha^2 v - 2\alpha\frac{\partial v}{\partial x} + \frac{\partial^2 v}{\partial x^2}$$

$$\sum_{\substack{j=0\\ j\neq q}}^{\infty}\binom{q}{j}\frac{\partial^{q-j}v}{\partial \tau^{q-j}}(-\beta)^j + \sum_{j=q}^{\infty}\binom{q}{j}\frac{\partial^{q-j}v}{\partial \tau^{q-j}}(-\beta)^j = \alpha^2 v - 2\alpha\frac{\partial v}{\partial x} + \frac{\partial^2 v}{\partial x^2}$$

$$\sum_{\substack{j=0\\ j\neq q}}^{\infty}\binom{q}{j}\frac{\partial^{q-j}v}{\partial \tau^{q-j}}(-\beta)^j + v(-\beta)^q = \alpha^2 v - 2\alpha\frac{\partial v}{\partial x} + \frac{\partial^2 v}{\partial x^2}$$

$$\sum_{\substack{j=0\\ j\neq q}}^{\infty}\binom{q}{j}\frac{\partial^{q-j}v}{\partial \tau^{q-j}}(-\beta)^j = \left(\alpha^2 - (-\beta)^q\right)v - 2\alpha\frac{\partial v}{\partial x} + \frac{\partial^2 v}{\partial x^2}.$$

Second change of variables are for coefficients such that $-2\alpha = k-1$, $k = \frac{2r}{\sigma^2}$ and $\alpha^2 - (-\beta)^q = -k$. Then, we get

$$\sum_{\substack{j=0\\ j\neq q}}^{\infty}\binom{q}{j}\frac{\partial^{q-j}v}{\partial \tau^{q-j}}\left(\frac{1}{2}+\frac{r}{\sigma^2}\right)^{\frac{2j}{q}} = \frac{\partial^2 v}{\partial x^2} + \left(\frac{2r}{\sigma^2}-1\right)\frac{\partial v}{\partial x} - \frac{2r}{\sigma^2}v. \tag{3.14}$$

Last transformations are for both functions and variables;
$v(x,\tau) = \frac{V(S,t)}{E}$, $\tau = \frac{\sigma^2}{2}(T-t)$ and $x = \ln\left(\frac{S}{E}\right)$ which can be also written as $S = Ee^x$.

In order to simplify the calculations let us start with the term $\frac{\partial^{q-j} v}{\partial \tau^{q-j}}$ on the left hand side of Equation (3.14)

$$\frac{\partial^{q-j} v}{\partial \tau^{q-j}} = \frac{1}{E} \frac{\partial^{q-j} V}{\partial t^{q-j}} \frac{\partial^{q-j} t}{\partial \tau^{q-j}}. \tag{3.15}$$

Then, we need to find the term $\frac{\partial^{q-j} t}{\partial \tau^{q-j}}$ on the right hand side of Equation (3.15). Since $t = T - \frac{2\tau}{\sigma^2}$ we have

$$\begin{aligned}
\frac{\partial^{q-j} t}{\partial \tau^{q-j}} &= \frac{\partial^{q-j}}{\partial \tau^{q-j}} \left(T - \frac{2\tau}{\sigma^2} \right) \\
&= \frac{\partial^{q-j}}{\partial \tau^{q-j}} (T) + \frac{\partial^{q-j}}{\partial \tau^{q-j}} \left(-\frac{2\tau}{\sigma^2} \right) \quad \text{from linearity} \\
&= T \frac{\partial^{q-j}}{\partial \tau^{q-j}} (\tau^0) - \frac{2\tau}{\sigma^2} \frac{\partial^{q-j}}{\partial \tau^{q-j}} (\tau^1) \quad \text{from homogeneity} \\
&= T \frac{\Gamma(0+1)\tau^{0-q+j}}{\Gamma(0-q+j+1)} - \frac{2}{\sigma^2} \frac{\Gamma(1+1)\tau^{1-q+j}}{\Gamma(1-q+j+1)} \quad \text{from Riemann formula} \\
&= T \frac{\tau^{j-q}}{\Gamma(1-q+j)} - \frac{2}{\sigma^2} \frac{\tau^{1-q+j}}{\Gamma(2-q+j)} \\
&= \frac{T}{\Gamma(1-q+j)} \left(\frac{\sigma^2}{2}(T-t) \right)^{j-q} - \frac{2}{\sigma^2} \frac{\left(\frac{\sigma^2}{2}(T-t) \right)^{1-q+j}}{\Gamma(2-q+j)} \\
&= \frac{1}{\Gamma(1-q+j)} \left(\frac{\sigma^2}{2}(T-t) \right)^{j-q} \left(T - \frac{T-t}{1-q+j} \right).
\end{aligned}$$

Therefore, Equation (3.15) is equal to the following equation

$$\frac{\partial^{q-j} v}{\partial \tau^{q-j}} = \frac{1}{E} \frac{\partial^{q-j} V}{\partial t^{q-j}} \left(\frac{\sigma^2}{2}(T-t) \right)^{j-q} \frac{1}{\Gamma(1-q+j)} \left(\frac{t - Tq + Tj}{1-q+j} \right).$$

Since we obtain the left hand side of Equation (3.14) we focus on the right hand side which is same as the one in original Black Scholes PDE.

$$\frac{\partial v}{\partial \tau} = \frac{1}{E}\frac{\partial V}{\partial t}\left(-\frac{2}{\sigma^2}\right),$$
$$\frac{\partial v}{\partial x} = \frac{S}{E}\frac{\partial V}{\partial S},$$
$$\frac{\partial^2 v}{\partial x^2} = \frac{S}{E}\frac{\partial V}{\partial S} + \frac{S^2}{E}\frac{\partial^2 V}{\partial S^2}.$$

Then, we get Equation (3.14) as follows

$$\sum_{\substack{j=0\\ j\neq q}}^{\infty}\binom{q}{j}\left[\frac{\partial^{q-j}V}{\partial t^{q-j}}\left(\frac{\sigma^2}{2}(T-t)\right)^{j-q}\frac{t-Tq+Tj}{\Gamma(2-q+j)}\left(\frac{1}{2}+\frac{r}{\sigma^2}\right)^{\frac{2j}{q}}\right]$$
$$= S^2\frac{\partial^2 V}{\partial S^2} + \frac{2r}{\sigma^2}S\frac{\partial V}{\partial S} - \frac{2r}{\sigma^2}V.$$

Finally, by multiplying both sides by $-\frac{\sigma^2}{2}$ we obtain the proposed fractional Black Scholes PDE as:

$$-\sum_{\substack{j=0\\ j\neq q}}^{\infty}\binom{q}{j}\frac{t+T(j-q)}{\Gamma(2-q+j)}(T-t)^{j-q}\left(\frac{\sigma^2}{2}\right)^{j-q+1}\left(\frac{\sigma^2+2r}{2\sigma^2}\right)^{\frac{2j}{q}}\frac{\partial^{q-j}V}{\partial t^{q-j}}$$
$$+\frac{\sigma^2}{2}S^2\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0.$$

□

Remark 3.5. Fractional Black Scholes PDE of Equation (3.13) can be stated using Gamma functions in Equation (2.3) as

$$-\sum_{\substack{j=0\\ j\neq q}}^{\infty}\frac{\Gamma(q+1)}{\Gamma(q-j+1)\Gamma(j+1)}\frac{t+T(j-q)}{\Gamma(2-q+j)}(T-t)^{j-q}\left(\frac{\sigma^2}{2}\right)^{j-q+1}\left(\frac{\sigma^2+2r}{2\sigma^2}\right)^{\frac{2j}{q}}\frac{\partial^{q-j}V}{\partial t^{q-j}}$$
$$+\frac{\sigma^2}{2}S^2\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0.$$

Then for different values q we obtain the following equations;

- For $q = 1$ we have original Black Scholes PDE of Equation (3.1)

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2}S^2\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0.$$

- For $q = 2$ we have:

$$\left(1+\frac{2r}{\sigma^2}\right)\frac{\partial V}{\partial t}+\frac{\sigma^2}{2}S^2\frac{\partial^2 V}{\partial S^2}+rS\frac{\partial V}{\partial S}-rV=0.$$

$$\underbrace{\frac{2r}{\sigma^2}\frac{\partial V}{\partial t}}_{\text{Extra Term}}+\underbrace{\frac{\partial V}{\partial t}+\frac{\sigma^2}{2}S^2\frac{\partial^2 V}{\partial S^2}+rS\frac{\partial V}{\partial S}-rV=0.}_{\text{Black Scholes PDE}}$$

- For $q = \frac{1}{2}$ we have:

$$-\sum_{j=0}^{\infty}\frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2}-j\right)\Gamma(j+1)}\frac{t+T\left(j-\frac{1}{2}\right)}{\Gamma\left(\frac{3}{2}+j\right)}(T-t)^{j-\frac{1}{2}}\left(\frac{\sigma^2}{2}\right)^{j-\frac{1}{2}}\left(\frac{\sigma^2+2r}{2\sigma^2}\right)^{4j}\frac{\partial^{\frac{1}{2}-j}V}{\partial t^{\frac{1}{2}-j}}$$

$$+\frac{\sigma^2}{2}S^2\frac{\partial^2 V}{\partial S^2}+rS\frac{\partial V}{\partial S}-rV=0.$$

Remark 3.6. Initial and boundary conditions of fractional Black Scholes PDE for European call option are as in Theorem 3.10

$$\begin{aligned}C(S,T)&=\max(S-E,0),\\ C(0,t)&=0,\\ C(S,t)&\approx S \text{ as } S\to\infty.\end{aligned}$$

3.6 Fractional Brownian Motion and It$\hat{o}$ Formula Approach

Fractional Black Scholes equation is derived by fractional Brownian motion and It$\hat{o}$ formula with the help of the preliminaries in previous chapter.

Theorem 3.12. *Fractional Black Scholes PDE for European option price $V(S,t)$ using fractional Brownian motion $B^{(H)}(t)$ with Hurst index $H\in(0,1)$ and It$\hat{o}$ formula with quadratic variation containing H, as follows:*

$$-rV+\frac{\partial V}{\partial t}+S_t r\frac{\partial V}{\partial S}+H\sigma^2 t^{2H-1}S_t^2\frac{\partial^2 V}{\partial S^2}=0,$$

where r is the risk free rate and σ is the volatility of the stock.

Proof. Initialy, we start derivation by using It$\hat{o}$ formula of Equation (2.7) for $V(S,t)$

$$V(S_t,t)=V(S_0,0)+\int_0^t\frac{\partial V}{\partial u}(S_u,u)du+\int_0^t\frac{\partial V}{\partial S}(S_u,u)dS_u \tag{3.16}$$
$$+\frac{1}{2}\int_0^t\frac{\partial^2 V}{\partial S^2}(S_u,u)d<S,S>_u .$$

For quadratic variation in Equation (3.16) the equality $d < S, S >_t = S_t^2 2H\sigma^2 t^{2H-1} dt$ in Remark 2.11 is used

$$V = V_0 + \int_0^t \left(\frac{\partial V}{\partial u} + \frac{1}{2} S_u^2 2H\sigma^2 u^{2H-1} \frac{\partial^2 V}{\partial S^2} \right) du + \int_0^t \frac{\partial V}{\partial S}(S_u, u) dS_u. \tag{3.17}$$

Fractional stochastic differential equation $dS_t = rS_t dt + \sigma S_t d\tilde{B}^H(t)$ can be substituted in Equation (3.17) as

$$V = V_0 + \int_0^t \left(\frac{\partial V}{\partial u} + S_u r \frac{\partial V}{\partial S} + H\sigma^2 u^{2H-1} S_u^2 \frac{\partial^2 V}{\partial S^2} \right) du + \int_0^t \frac{\partial V}{\partial S} S_u \sigma d\tilde{B}^H(u).$$

After taking the derivative we obtain

$$dV = \left(\frac{\partial V}{\partial t} + S_t r \frac{\partial V}{\partial S} + H\sigma^2 t^{2H-1} S_t^2 \frac{\partial^2 V}{\partial S^2} \right) dt + \left(\frac{\partial V}{\partial S} S_t \sigma \right) d\tilde{B}^H(t). \tag{3.18}$$

Since, $\tilde{V}$ is the notation for option value for discounted asset price such that

$$\begin{aligned} \tilde{V} &= e^{-rt} V, \\ d\tilde{V} &= -rte^{-rt} V + e^{-rt} dV. \end{aligned}$$

By substitution of these variables in (3.18) we get

$$d\tilde{V} = e^{-rt} \left(-rV + \frac{\partial V}{\partial t} + S_t r \frac{\partial V}{\partial S} + H\sigma^2 t^{2H-1} S_t^2 \frac{\partial^2 V}{\partial S^2} \right) dt + \left(e^{-rt} \frac{\partial V}{\partial S} S_t \sigma \right) d\tilde{B}^H(t).$$

Note that $\tilde{V}$ is the martingale transform of discounted asset prices. Since $\tilde{S}$ is a martingale under $\tilde{\mathbb{P}}$, then $\tilde{V}$ is also martingale under measure $\tilde{\mathbb{P}}$. Hence, by martingale representation theorem [18], the Fractional Black Scholes PDE is

$$-rV + \frac{\partial V}{\partial t} + S_t r \frac{\partial V}{\partial S} + H\sigma^2 t^{2H-1} S_t^2 \frac{\partial^2 V}{\partial S^2} = 0.$$

□

Remark 3.7. For $H = \frac{1}{2}$ we have classical Black Scholes PDE of Equation(3.1)

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2} + rS_t \frac{\partial V}{\partial S} - rV = 0.$$

Proposition 3.13. *The solution of fractional stochastic differential equation under risk neutral measure $\tilde{\mathbb{P}}$ is*

$$S_t = S_0 \exp\left(rt - \frac{1}{2}\sigma^2 t^{2H} + \sigma \tilde{B}^H(t) \right),$$

where $S(t)$ is price of a stock at time t.

Therefore for $0 \le t \le T$ the solution is

$$S(T) = S(t) \exp\left(r(T-t) - \frac{1}{2}\sigma^2 (T^{2H} - t^{2H}) + \sigma(\tilde{B}^H(T) - \tilde{B}^H(t)) \right).$$

Proof. Fractional stochastic differential equation for $H \in (0,1)$ under measure $\tilde{\mathbb{P}}$ is

$$dS_t = rS_t dt + \sigma S_t d\tilde{B}^H(t).$$

Itô formula of Equation (2.7) is applied for $f(x) = \ln(x)$

$$\begin{aligned}
\ln(S(t)) &= \ln(S(0)) + \int_0^t \frac{1}{S(u)} dS(u) - \frac{1}{2}\int_0^t \frac{1}{S^2(u)} d < S, S >_u \\
&= \ln(S(0)) + \int_0^t r du + \sigma d\tilde{B}^H(u) - \frac{1}{2}\int_0^t \frac{1}{S^2(u)} \sigma^2 2Hu^{2H-1} du \\
&= \ln(S(0)) + \int_0^t \left(r - \sigma^2 Hu^{2H-1}\right) du + \int_0^t \sigma d\tilde{B}^H(u) \\
&= \ln(S(0)) + rt - \frac{1}{2}\sigma^2 t^{2H} + \sigma\tilde{B}^H(t)
\end{aligned}$$

$$\ln\left(\frac{S(t)}{S(0)}\right) = rt - \frac{1}{2}\sigma^2 t^{2H} + \sigma\tilde{B}^H(t).$$

Therefore we obtain the solution of fractional stochastic differential equation as:

$$S(t) = S(0)\exp\left(rt - \frac{1}{2}\sigma^2 t^{2H} + \sigma\tilde{B}^H(t)\right).$$

Furthermore for $0 \le t \le T$ the solution is

$$S(T) = S(t)\exp\left(r(T-t) - \frac{1}{2}\sigma^2(T^{2H} - t^{2H}) + \sigma(\tilde{B}^H(T) - \tilde{B}^H(t))\right).$$

□

Theorem 3.14. *For $C(t, S(t))$ is price of a European call option at time t*

$$C(t, S(t)) = S(t)N(d_1) - Ee^{-r(T-t)}N(d_2),$$

where E is strike price, T is maturity and d_1 and d_2 are as follows:

$$d_1 = \frac{\ln(\frac{S}{E}) + r(T-t) + \frac{1}{2}\sigma^2(T^{2H} - t^{2H})}{\sigma\sqrt{T^{2H} - t^{2H}}},$$

$$\begin{aligned}
d_2 &= d_1 - \sigma\sqrt{T^{2H} - t^{2H}} \\
&= \frac{\ln(\frac{S}{E}) + r(T-t) - \frac{1}{2}\sigma^2(T^{2H} - t^{2H})}{\sigma\sqrt{T^{2H} - t^{2H}}},
\end{aligned}$$

where

$$N(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^t \exp\left(-\frac{x^2}{2}\right) dx.$$

Proof. See [14].

Remark 3.8. For $H = \frac{1}{2}$ the solution becomes the one in Proposition 3.1:

$$d_1 = \frac{\ln\left(\frac{S}{E}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}},$$
$$d_2 = \frac{\ln\left(\frac{S}{E}\right) + \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}.$$

Example 3.3. As in Example 3.1 for a European call option; $K = 95\$$,$S = 100\$$, $div = 0$, $\sigma = 0.5$, $r = 0.1$ and $T = 0.25$. When we have the Hurst parameter as $H = 0.5$, by using MATLAB code in Appendix C, we obtain the same call option price as 13.6953$.

Example 3.4. As in Example 3.3 for a European call option; $K = 95\$$,$S = 100\$$, $div = 0$, $\sigma = 0.5$, $r = 0.1$ and $T = 0.25$. When we have the Hurst parameter as $H = 0.7$, by using MATLAB code in Appendix C, we obtain the call option price as 11.5116$. Or when we have the Hurst parameter as $H = 0.3$, we obtain the call option price as 16.6061$.

Example 3.5. As in Example 3.2 for a European put option; $K = 95\$$,$S = 100\$$, $div = 0$, $\sigma = 0.5$, $r = 0.1$ and $T = 0.25$. When we have the Hurst parameter as $H = 0.5$, by using MATLAB code in Appendix C, we obtain the same put option price as 6.3497$.

Example 3.6. As in Example 3.3 for a European put option; $K = 95\$$,$S = 100\$$, $div = 0$, $\sigma = 0.5$, $r = 0.1$ and $T = 0.25$. When we have the Hurst parameter as $H = 0.7$, by using MATLAB code in Appendix C, we obtain the put option price as 4.1661$. Or when we have the Hurst parameter as $H = 0.3$, we obtain the put option price as 9.2606$.

Table 3.1: Comparison of European call and European put option prices for different H values

	$H = 0.3$	$H = 0.5$	$H = 0.7$
European Call	16.6061	13.6953	11.5116
European Put	9.2606	6.3497	4.1661

CHAPTER 4

FINITE DIFFERENCE METHOD

Finite difference methods for derivatives are used to solve differential equations with the approximation principle. In 1768, L. Euler studied finite difference methods in one dimension of space, and in 1908, C. Runge extended it to dimension two. In 1950s, since computers had become useful tools for complex problems, numerical application of finite difference methods had developed. The principle of the methods is using approximations of differential operators instead of differential quotients [13].

In this chapter, the most common finite difference method which is called the explicit method is presented in order to solve the Black Scholes equation [3].

The approximations of differential operator based on Taylor series expansions of derivatives. Suppose that the function f is C^2 continuous in the neighborhood of x. For $h > 0$ the Taylor series expansion is

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \mathcal{O}(h^3) \tag{4.1}$$

Putting in different order the expansion of Equation (4.1) gives the forward difference since the difference is in the forward direction

$$f'(x) = \frac{f(x+h) - f(x)}{h} + \mathcal{O}(h^2).$$

Therefore, the forward difference approximation is

$$\frac{\partial f(x)}{\partial x} \approx \frac{f(x+h) - f(x)}{h}.$$

For $h > 0$, the Taylor series expansion is also written as:

$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2}f''(x) - \mathcal{O}(h^3) \tag{4.2}$$

Putting in different order the expansion of Equation (4.2) gives the backward difference since the difference is in the backward direction

$$f'(x) = \frac{f(x) - f(x-h)}{h} + \mathcal{O}(h^2).$$

Therefore, the backward difference approximation is

$$\frac{\partial f(x)}{\partial x} \approx \frac{f(x) - f(x-h)}{h}.$$

In addition, by combining the forward and the backward difference approximations, the central difference formula is obtained as

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} + \mathcal{O}(h^2).$$

Therefore, the central difference approximation is

$$\frac{\partial f(x)}{\partial x} \approx \frac{f(x+h) - f(x-h)}{2h}.$$

Then, Taylor series expansion is used also for higher-order central difference approximations. Making some rearrangements in (4.1) and (4.2) the central difference approximation for second order derivative is

$$\frac{\partial^2 f(x)}{\partial x^2} \approx \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}.$$

Moreover, for functions of two variables as $f(x, y)$ for $h = \Delta x$, [28] the forward difference approximation for partial derivative $f_x(x, y)$ is

$$\frac{\partial f(x,y)}{\partial x} \approx \frac{f(x+h,y) - f(x,y)}{h}.$$

The backward difference approximation for partial derivative $f_x(x, y)$ is

$$\frac{\partial f(x,y)}{\partial x} \approx \frac{f(x,y) - f(x-h,y)}{h}.$$

The central difference approximation for partial derivative $f_x(x, y)$ is the combination of forward and backward difference approximations

$$\frac{\partial f(x,y)}{\partial x} \approx \frac{f(x+h,y) - f(x-h,y)}{2h}.$$

The central difference approximation for second-order partial derivative $f_{xx}(x, y)$ is

$$\frac{\partial^2 f(x,y)}{\partial x^2} \approx \frac{f(x+h,y) - 2f(x,y) + f(x-h,y)}{h^2}.$$

Finally, the central difference approximation for second-order partial derivative $f_{xy}(x, y)$ is [3]

$$\frac{\partial^2 f(x,y)}{\partial x \partial y} \approx \frac{f(x+h,y+k) - f(x,y+k) + f(x+h,y) + f(x,y)}{hk}.$$

where $h = \Delta x$ and $k = \Delta y$.

4.1 Explicit Method

This section mainly proposed to find the solution of Black Scholes PDE using finite difference method. Explicit method is the most popular one within finite difference methods. For derivation of explicit method backward difference approximation and central difference approximation are used [3, 25, 28].

As we have stated, Black Scholes PDE is

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0, \tag{4.3}$$

where $V(S,t)$ is the price of European option as a function of stock price S and time t, and r is the risk-free interest rate, σ is the volatility of the stock.

The boundary conditions for European call option can be given as

$$\begin{aligned} C(S,T) &= \max{(S-E,0)}, \\ C(S_{min},t) &= 0, \\ C(S_{max},t) &= S_{max} - Ee^{-r(T-t)}, \end{aligned}$$

where E is a strike price and S_{min} and S_{max} represents the minimum and maximum values of stock price.

For derivation of explicit method, first domain of S and t will be discretized. The intervals $[S_{min}, S_{max}]$ and $[t_0, T]$ will be divided into M and N parts.

$$\Delta t = \frac{T - t_0}{M} \quad \text{for} \quad t_0 \le t \le T,$$

$$\Delta S = \frac{S_{max} - S_{min}}{N} \quad \text{for} \quad S_{min} \le S \le S_{max}.$$

Then, making some rearrangements the general notation for S and t are obtained as

$$t_i = t_0 + i\Delta t \quad \text{for} \quad i = 0, 1, ..., M,$$

$$S_k = S_{min} + k\Delta S \quad \text{for} \quad k = 0, 1, ..., N.$$

From now on, for the simplicity of the notation for points (S_k, t_i), we denote the approximation of option price as

$$V(S_k, t_i) \approx w_{k,i},$$

The initial and boundary conditions for European call option in terms of $w_{k,i}$, M and N are

$$\begin{aligned} w_{k,M} &\approx \max(S_k - E, 0), \\ w_{0,i} &\approx 0, \\ w_{N,i} &\approx S_N - Ee^{-r(t_M - t_i)}. \end{aligned}$$

Then, the backward and central difference approximations for the partial derivatives $\frac{\partial V(S_k,t_i)}{\partial t_i}$, $\frac{\partial V(S_k,t_i)}{\partial S_k}$ and the second-order derivative $\frac{\partial^2 V(S_k,t_i)}{\partial {S_k}^2}$ in Black Scholes PDE can be written in terms of $w_{k,i}$, Δt and ΔS as follows

$$\frac{\partial V(S_k, t_i)}{\partial t_i} \approx \frac{w_{k,i} - w_{k,i-1}}{\Delta t}, \tag{4.4}$$

$$\frac{\partial V(S_k, t_i)}{\partial S_k} \approx \frac{w_{k+1,i} - w_{k-1,i}}{2\Delta S}, \tag{4.5}$$

$$\frac{\partial^2 V(S_k, t_i)}{\partial {S_k}^2} \approx \frac{w_{k+1,i} - 2w_{k,i} + w_{k-1,i}}{(\Delta S)^2}. \tag{4.6}$$

When we put backward and central difference approximations of partial derivatives of Equations (4.4), (4.5) and (4.6) in Black Scholes PDE (4.3) we get

$$\frac{w_{k,i} - w_{k,i-1}}{\Delta t} + \frac{\sigma^2}{2} {S_k}^2 \frac{w_{k+1,i} - 2w_{k,i} + w_{k-1,i}}{(\Delta S)^2} + rS_k \frac{w_{k+1,i} - w_{k-1,i}}{2\Delta S} - rw_{k,i} = 0.$$

The terms α_k, β_k and γ_k for the simplicity of the notations and the preceding equation can be written basically as

$$w_{k,i-1} = \alpha_k w_{k-1,i} + \beta_k w_{k,i} + \gamma_k w_{k+1,i}. \tag{4.7}$$

for $i = M, M-1, ..., 1$ and $k = 1, ..., N-1$ and where the terms α_k, β_k and γ_k are

$$\alpha_k = \frac{1}{2}\Delta t \left\{ \sigma^2 \left(\frac{S_k}{\Delta S} \right)^2 - r\frac{S_k}{\Delta S} \right\},$$

$$\beta_k = 1 - \Delta t \left\{ \sigma^2 \left(\frac{S_k}{\Delta S} \right)^2 + r \right\},$$

$$\gamma_k = \frac{1}{2}\Delta t \left\{ r\frac{S_k}{\Delta S} + \sigma^2 \left(\frac{S_k}{\Delta S} \right)^2 \right\}.$$

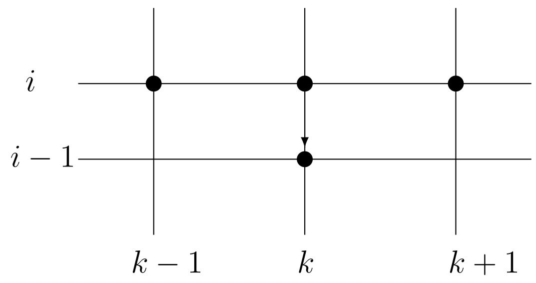

Figure 4.1: Molecules of explicit method

This figure shows the principle of the explicit method. The notation i is for time and k is for stock price. It explains Equation (4.7). Note that on explicit method the summation of the values of black dots on the i-th row with the coefficients α_k, β_k and γ_k is equal to the value of black dots on the $(i-1)$-th row and k-th column.

As in the vector form; $w_{1,i}$,$w_{2,i}$,...,$w_{N-1,i}$ can be written

$$w^{(i)} = \begin{pmatrix} w_{1,i} \\ \vdots \\ w_{N-1,i} \end{pmatrix}_{N-1\times 1} . \tag{4.8}$$

When the equation (4.7) has been written for each $i = M, M-1, ..., 1$, and $k = 1, ..., N-1$, $M \times N$ equations occur. These can be collected for each i and with the notation in the matrix of Equation(4.8) as follows:

$$w^{(i-1)} = Aw^{(i)} + y^{(i)},$$

where A is a $(N-1 \times N-1)$ matrix consists of the terms α_k, β_k and γ_k.

$$A = \begin{pmatrix} \beta_1 & \gamma_1 & 0 & \cdots & 0 & 0 \\ \alpha_2 & \beta_2 & \gamma_2 & \cdots & 0 & 0 \\ 0 & \alpha_3 & \beta_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \beta_{N-2} & \gamma_{N-2} \\ 0 & 0 & 0 & \cdots & \alpha_{N-1} & \beta_{N-1} \end{pmatrix}_{N-1\times N-1} ,$$

and the matrix $y^{(i)}$ is $(N-1 \times 1)$ matrix as:

$$y^{(i)} = \begin{pmatrix} \alpha_1 w_{0,i} \\ 0 \\ \vdots \\ 0 \\ \gamma_{N-1} w_{N,i} \end{pmatrix}_{N-1\times 1} .$$

Example 4.1. As in Example 3.1, a European call option has an exercise price of 95. The option has the underlying stock at price 100$ which pays no dividends and has a volatility of 50% and the risk-free rate is 10% and $T = 0.25$.

Explicit method is used to obtain the call option price. For valuation, the parameters should be chosen suitably as [28]:

$S_{min} = 0$,$S_{max} = 150$,$dS = 2$ and $dt = 1/1200$.

We obtain the call option price by using explicit method is 13.6982$ and closed form solution is as in example (3.1) is 13.6953$.

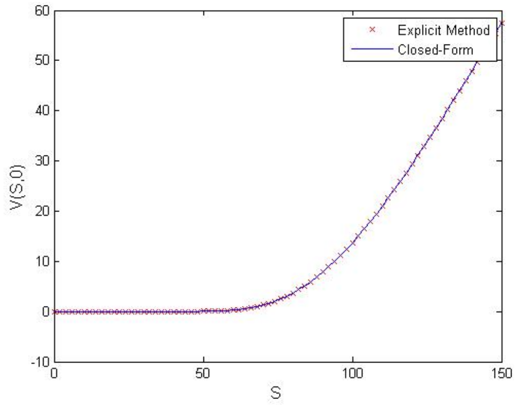

Figure 4.2: The exact solution and approximate solution in 2D

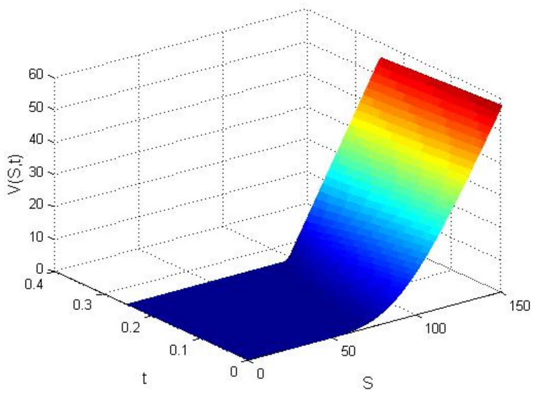

Figure 4.3: Explicit method solution for call option price $V(S,t)$ in 3D

4.1.1 Consistency, Convergence and Stability of Explicit Method

All finite difference approximation of partial differential equations have three properties

1. *Consistency* For a finite difference scheme $P_{\Delta x,\Delta t}v = f$ is consistent with a partial differential equation $P_u = f$ if for any smooth function $\phi(x,t)$

$$P\phi - P_{\Delta x,\Delta t}\phi \to 0 \quad \text{as} \;\; \Delta x, \Delta t \to 0,$$

2. *Convergence*

   If v_k^0 converges to $u_0(x)$ as $k\Delta x$ converges to x, then v_k^i converges to $u(x,t)$ as $(k\Delta x, i\Delta t)$ converges to (x,t) as $(\Delta x, \Delta t)$ converges to 0 where $u(x,t)$ is the solution of partial differential equation and v_k^i is the solution of finite difference scheme, then the scheme is convergent.

3. *Stability*

   If for a constant C and some positive integer N and M we have

$$||v_k^i|| \le C||v_k^0||_{\Delta x} \;\; \text{for} \;\; 0 \le \Delta x \le N \;\; \text{and} \;\; 0 \le \Delta t \le M,$$

   then finite difference scheme $P_{\Delta x,\Delta t}v_k^i = 0$ is stable.

Note that from now on we use the notation $w_{k,i}$ for v_k^i since $V(S_k, t_i) \approx w_{k,i}$ in Black Scholes equation.

Proving the stability is hard by using the definition for explicit method for Black Scholes PDE. Then we use Fourier analysis for evolution which is known as Von Neumann Analysis [3, 11, 24, 32]. the basis of the Fourier analysis is the following assumption for the solution of finite difference scheme as:

$$w_{k,i} = \lambda^i e^{zk\theta}, \tag{4.9}$$

where $z^2 = -1$, $Im(z) = 1$ and θ is arbitrary constant.

The condition for von Neumann criteria for stability is $|\lambda| \le 1$.

Substituting (4.9) into (4.7) we get

$$\lambda^{i-1}e^{zk\theta} = \alpha_k\lambda^i e^{z(k-1)\theta} + \beta_k\lambda^i e^{zk\theta} + \gamma_k\lambda^i e^{z(k+1)\theta}.$$

Therefore, using substitutions and rearrangements the sufficient condition for stability is [24]:

$$0 \le \frac{\Delta t}{(\Delta S)^2} \le \frac{1}{2}$$

4.2 Fractional Explicit Method

One of the main purposes of this thesis is finding a solution for fractional Black Scholes PDE. Let us restate the basic derivation of q-th order time fractional Black Scholes PDE as

$$\frac{\partial^q V}{\partial t^q} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0,$$

where q is arbitrary real or complex number $V(S,t)$ is the price of European option as a function of stock price S and time t, and r is the risk-free interest rate, σ is the volatility of the stock with the same boundary condition as in original one.

Since there are partial differences in fractional Black Scholes PDE, finite difference method can be used for solution. However, it is fundamental to derive a approximation for fractional derivative $\frac{\partial^q V}{\partial t^q}$.

Therefore, in this section, we propose a new technique for the solution of fractional Black Scholes PDE. For this purpose, first we derive fractional explicit method in order to approximate fractional derivatives then we find the Black Scholes equation in terms of $w_{k,i}$ as in previous section. Finally, the solutions for different values of q are presented.

Fractional explicit method consists of backward difference approximation, central difference approximation and approximation for fractional derivative which is presented in the following theorem.

Theorem 4.1. *[25] The approximation for q-th order time fractional derivative of $V(S_k, t_i)$ can be stated as the sum differences with the coefficients g_j as*

$$\frac{d^q V}{dt_i^q} \approx \frac{1}{(\Delta t)^q} \sum_{j=0}^{i} g_j w_{k,i-j}, \tag{4.10}$$

where g_j is the function of gamma functions of q and j,

$$g_j = \frac{\Gamma(j-q)}{\Gamma(-q)\Gamma(j+1)}. \tag{4.11}$$

Proof. The Grünwald-Letnikov definition of differintegral was stated in Equation (2.1)

$$\frac{d^q f}{d(x-a)^q} = \lim_{N\to\infty} \left\{ \frac{(\delta_N x)^{-q}}{\Gamma(-q)} \sum_{j=0}^{N-1} \frac{\Gamma(j-q)}{\Gamma(j+1)} f(x - j\delta_N x) \right\}.$$

Then, this definition of differintegral can be stated as approximation instead of limit definition and when for $a = 0$ and g_j is as in Equation (4.11):

$$\frac{d^q f}{dx^q} \approx \frac{1}{(\delta_N x)^q} \sum_{j=0}^{N-1} g_j f(x - j\delta_N x).$$

Then by changing notations for $f(x) = V(S_k, t_i)$ and then taking the q-th order time fractional derivative we have

$$\frac{d^q f}{dx^q} = \frac{d^q V}{dt_i^q}.$$

Moreover, for the right hand side of the equation we need following relations which are obtained from the change of notations $\delta_N x = \Delta t$ as follows:

$$f(x - j\delta_N x) = V(S_k, t_{i-j}) \approx w_{k,i-j}.$$

Finally, modifying the equation appropriate to explicit method for $\frac{d^q V}{dt_i^q}$ in terms of $w_{k,i}$, Δt and g_j we have

$$\frac{d^q V}{dt_i^q} \approx \frac{1}{(\Delta t)^q} \sum_{j=0}^{i} g_j w_{k,i-j}.$$

□

Proposition 4.2. *For $q = 1$ in equation (4.10) we have the backward difference in (4.4)*

$$\frac{dV}{dt_i} \approx \frac{1}{(\Delta t)} \sum_{j=0}^{i} \frac{\Gamma(j-1)}{\Gamma(-1)\Gamma(j+1)} w_{k,i-j}$$

$$\approx \frac{w_{k,i} - w_{k,i-1}}{(\Delta t)}.$$

Proof. Using Proposition B.1 in Appendix B, the equation (4.10) for $q = 1$ is as follows

$$\begin{aligned}
\sum_{j=0}^{i} \frac{\Gamma(j-1)}{\Gamma(-1)\Gamma(j+1)} w_{k,i-j} &= \sum_{j=0}^{i} (-1)^j \binom{1}{j} w_{k,i-j} \\
&= (-1)^0 \binom{1}{0} w_{k,i-0} + (-1)^1 \binom{1}{1} w_{k,i-1} \\
&= 1 w_{k,i} + (-1) w_{k,i-1} \\
&= w_{k,i} - w_{k,i-1}.
\end{aligned}$$

□

Remark 4.1. The properties of gamma function in Remark 4.2 are in Appendix B.

Therefore substituting the equations; backward difference approximations of Equation (4.5), central difference approximations of Equation (4.6) and approximation of fractional derivative in Equation (4.10) of partial derivatives in Black Scholes PDE (4.3) we get

$$\frac{1}{(\Delta t)^q}\sum_{j=0}^{i} g_j w_{k,i-j} + \frac{\sigma^2}{2}{S_k}^2\frac{w_{k+1,i}-2w_{k,i}+w_{k-1,i}}{(\Delta S)^2} + rS_k\frac{w_{k+1,i}-w_{k-1,i}}{2\Delta S} - rw_{k,i} = 0.$$

The terms $\tilde{\alpha_k}$, $\tilde{\beta_k}$ and $\tilde{\gamma_k}$ for the simplicity of the notations and the preceding equation can be written basically as:

$$\sum_{j=1}^{i} g_j w_{k,i-j} = \tilde{\alpha_k} w_{k-1,i} + \tilde{\beta_k} w_{k,i} + \tilde{\gamma_k} w_{k+1,i}, \tag{4.12}$$

for $i = M, M-1, ..., 1$ and $k = 1, ..., N-1$ and where the terms $\tilde{\alpha_k}$, $\tilde{\beta_k}$ and $\tilde{\gamma_k}$ are

$$\tilde{\alpha_k} = -\frac{1}{2}(\Delta t)^q\left\{\sigma^2\left(\frac{S_k}{\Delta S}\right)^2 - r\frac{S_k}{\Delta S}\right\},$$

$$\tilde{\beta_k} = (\Delta t)^q\left\{\sigma^2\left(\frac{S_k}{\Delta S}\right)^2 + r\right\} - 1,$$

$$\tilde{\gamma_k} = -\frac{1}{2}(\Delta t)^q\left\{r\frac{S_k}{\Delta S} + \sigma^2\left(\frac{S_k}{\Delta S}\right)^2\right\}.$$

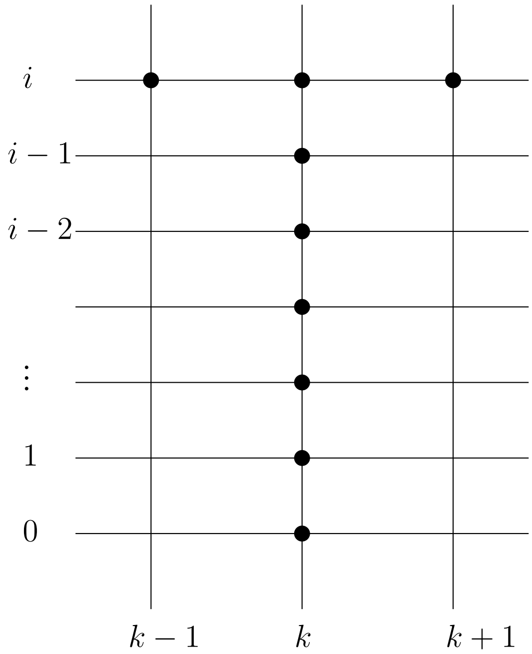

Figure 4.4: Molecules of fractional explicit method

This figure shows the principle of the fractional explicit method for Equation (4.12). The notation i is for time and k is for stock price as mentioned before. The summation of the values of black dots on the i-th row with the coefficients $\tilde{\alpha_k}$, $\tilde{\beta_k}$ and $\tilde{\gamma_k}$ is equal to the summation of the values of black dots on the $(i-1), (i-2), ..., 0$-th rows and k-th column with the coefficients g_i. Note that on explicit method the summation of the values of black dots on the i-th row with the coefficients α_k, β_k and γ_k is equal to the value of black dots on the $(i-1)$-th row and k-th column. As mentioned in the introduction chapter, fractionalization a partial differential equation makes the pde non-Markovian. In other words, the fractional Black Scholes PDE has memory. As it can be seen in the figure, the system has memory in other words the system takes into consideration the values at the entire time range. Since i is for time and in fractional case the summation includes $(i-1), (i-2), ..., 0$-th rows, the solution of the fractional Black Scholes PDE by fractional explicit method has affected by the values of historical data.

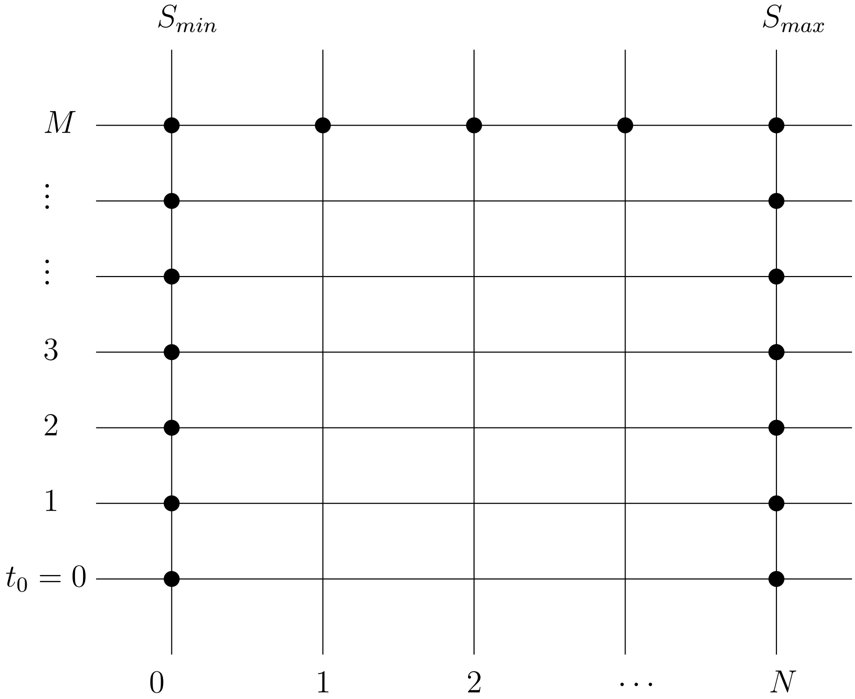

Figure 4.5: Fractional explicit method with boundaries

Note that the black dots shows the data for time T, and stock price S_{min} and S_{max} which are all given information. We aim to find the data when $t_0 = 0$ for $S = S_0$ by using the fractional explicit method. Then in order to solve the system we need matrices. When for each $i = M, M-1, ..., 1$ and $k = 1, ..., N-1$ the equation (4.12) is written, we need a matrix product to collect all the $M \times (N-1)$ equations. The matrix which consists of coefficients is a block matrix.

$$\underbrace{\begin{pmatrix} g_1 & g_2 & g_3 & | & 0 & 0 & 0 \\ -\tilde{\beta}_1 & g_1 & g_2 & | & -\tilde{\gamma}_1 & 0 & 0 \\ 0 & -\tilde{\beta}_1 & g_1 & | & 0 & -\tilde{\gamma}_1 & 0 \\ -- & -- & -- & | & -- & -- & -- \\ 0 & 0 & 0 & | & g_1 & g_2 & g_3 \\ -\tilde{\alpha}_2 & 0 & 0 & | & -\tilde{\beta}_2 & g_1 & g_2 \\ 0 & -\tilde{\alpha}_2 & 0 & | & 0 & -\tilde{\beta}_2 & g_1 \end{pmatrix}}_{G} \underbrace{\begin{pmatrix} w_{1,2} \\ w_{1,1} \\ w_{1,0} \\ -- \\ w_{2,2} \\ w_{2,1} \\ w_{2,0} \end{pmatrix}}_{W} = \underbrace{\begin{pmatrix} \tilde{\alpha}_1 w_{0,3} + \tilde{\beta}_1 w_{1,3} + \tilde{\gamma}_1 w_{2,3} \\ \tilde{\alpha}_1 w_{0,2} \\ \tilde{\alpha}_1 w_{0,1} \\ ------- \\ \tilde{\alpha}_2 w_{1,3} + \tilde{\beta}_2 w_{2,3} + \tilde{\gamma}_2 w_{3,3} \\ \tilde{\gamma}_2 w_{3,2} \\ \tilde{\gamma}_2 w_{3,1} \end{pmatrix}}_{B}$$

Here G is a matrix which contains the coefficients. W is a matrix which contains unknown values. In Figure W matrix represents the dots inside of black dots. Finally, B is a matrix which contains known values. In Figure B matrix represents the black dots.

Example 4.2. Let a European call option have an exercise price of 95. The option has the underlying stock at price 100$ which pays no dividends, and has a volatility of 50% and the risk-free rate is 10% and $T = 0.25$ as in Example 3.1.

Fractional explicit method is used to obtain the call option price. For valuation, the parameters should be chosen suitably as

$S_{min} = 0$, $S_{max} = 150$, $dS = 50$ and $dt = 0.0833$.

Then, we can find as $M = 3$ and $N = 3$ and we get following equations for each $i = 3, 2, 1$ and $k = 1, 2$:

$$\begin{aligned}
g_1 w_{1,2} + g_2 w_{1,1} + g_3 w_{1,0} &= \tilde{\alpha}_1 w_{0,3} + \tilde{\beta}_1 w_{1,3} + \tilde{\gamma}_1 w_{2,3}, \\
g_1 w_{1,1} + g_2 w_{1,0} &= \tilde{\alpha}_1 w_{0,2} + \tilde{\beta}_1 w_{1,2} + \tilde{\gamma}_1 w_{2,2}, \\
g_1 w_{1,0} &= \tilde{\alpha}_1 w_{0,1} + \tilde{\beta}_1 w_{1,1} + \tilde{\gamma}_1 w_{2,1}, \\
g_1 w_{2,2} + g_2 w_{2,1} + g_3 w_{2,0} &= \tilde{\alpha}_2 w_{1,3} + \tilde{\beta}_2 w_{2,3} + \tilde{\gamma}_2 w_{3,3}, \\
g_1 w_{2,1} + g_2 w_{2,0} &= \tilde{\alpha}_2 w_{1,2} + \tilde{\beta}_2 w_{2,2} + \tilde{\gamma}_2 w_{3,2}, \\
g_1 w_{2,0} &= \tilde{\alpha}_2 w_{1,1} + \tilde{\beta}_2 w_{2,1} + \tilde{\gamma}_2 w_{3,1}.
\end{aligned}$$

When we rewrite these equations as the unknown terms are on the left hand side and the known terms are on the right hand side we get:

$$\begin{aligned}
g_1 w_{1,2} + g_2 w_{1,1} + g_3 w_{1,0} &= \tilde{\alpha}_1 w_{0,3} + \tilde{\beta}_1 w_{1,3} + \tilde{\gamma}_1 w_{2,3}, \\
g_1 w_{1,1} + g_2 w_{1,0} - \tilde{\beta}_1 w_{1,2} - \tilde{\gamma}_1 w_{2,2} &= \tilde{\alpha}_1 w_{0,2}, \\
g_1 w_{1,0} - \tilde{\beta}_1 w_{1,1} - \tilde{\gamma}_1 w_{2,1} &= \tilde{\alpha}_1 w_{0,1}, \\
g_1 w_{2,2} + g_2 w_{2,1} + g_3 w_{2,0} &= \tilde{\alpha}_2 w_{1,3} + \tilde{\beta}_2 w_{2,3} + \tilde{\gamma}_2 w_{3,3}, \\
g_1 w_{2,1} + g_2 w_{2,0} - \tilde{\alpha}_2 w_{1,2} - \tilde{\beta}_2 w_{2,2} &= \tilde{\gamma}_2 w_{3,2}, \\
g_1 w_{2,0} - \tilde{\alpha}_2 w_{1,1} - \tilde{\beta}_2 w_{2,1} &= \tilde{\gamma}_2 w_{3,1}.
\end{aligned}$$

These equations can be written in matrix form where G and B matrices are known:

$$G \times W = B$$

where

$$G = \begin{pmatrix} g_1 & g_2 & g_3 & 0 & 0 & 0 \\ -\tilde{\beta}_1 & g_1 & g_2 & -\tilde{\gamma}_1 & 0 & 0 \\ 0 & -\tilde{\beta}_1 & g_1 & 0 & -\tilde{\gamma}_1 & 0 \\ 0 & 0 & 0 & g_1 & g_2 & g_3 \\ -\tilde{\alpha}_2 & 0 & 0 & -\tilde{\beta}_2 & g_1 & g_2 \\ 0 & -\tilde{\alpha}_2 & 0 & 0 & -\tilde{\beta}_2 & g_1 \end{pmatrix}, W = \begin{pmatrix} w_{1,2} \\ w_{1,1} \\ w_{1,0} \\ w_{2,2} \\ w_{2,1} \\ w_{2,0} \end{pmatrix}$$

and

$$B = \begin{pmatrix} \tilde{\alpha}_1 w_{0,3} + \tilde{\beta}_1 w_{1,3} + \tilde{\gamma}_1 w_{2,3} \\ \tilde{\alpha}_1 w_{0,2} \\ \tilde{\alpha}_1 w_{0,1} \\ \tilde{\alpha}_2 w_{1,3} + \beta_2 w_{2,3} + \gamma_2 w_{3,3} \\ \tilde{\gamma}_2 w_{3,2} \\ \tilde{\gamma}_2 w_{3,1} \end{pmatrix}$$

Then we can find the matrix W as $W = G^{-1} \times B$

Remark 4.2. Instead of using inverse of matrix G, one can apply LU decomposition not to have computational errors.

Remark 4.3. The purpose of this example is to find the matrix W which consists of unknown values. However, the solution is the value of w at $t = t_0$ and $S = S_0$. In this example, the solution is $w_{2,0}$ for $k = 2$ and $i = 0$ since

$$\begin{aligned} S &= S_{min} + k\Delta S = 0 + 2 \times 50 = 100 = S_0, \\ t &= t_0 + i\Delta t = 0 + 0 \times 0.0833 = 0 = t_0. \end{aligned}$$

All the values in the following table and figure are the values of $w_{2,0}$.

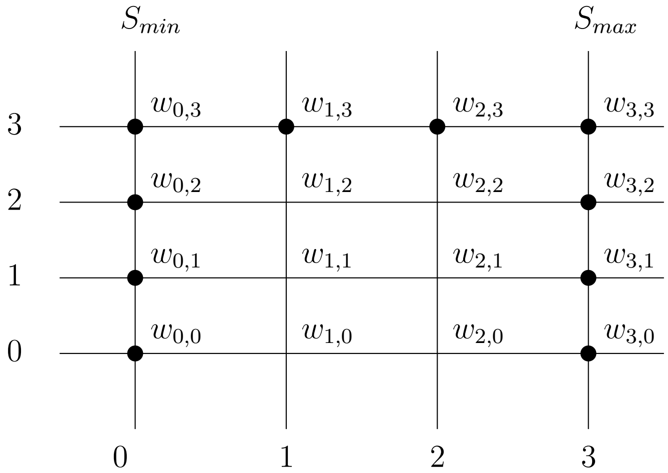

Figure 4.6: Fractional explicit method for $M = 3$ and $N = 3$

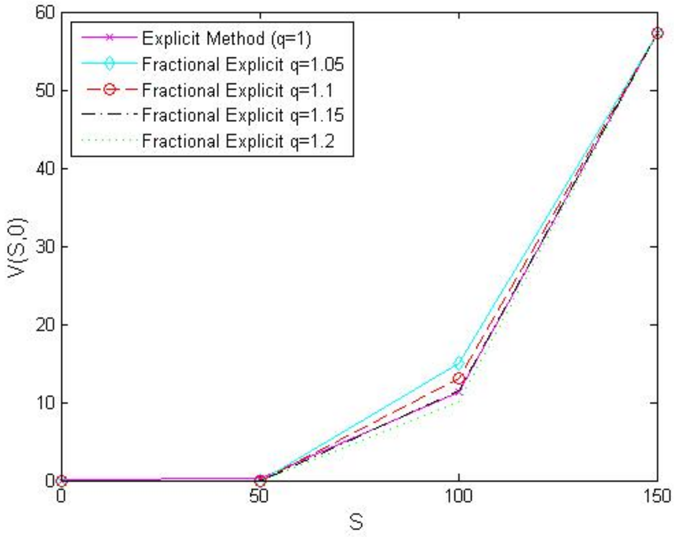

Figure 4.7: European call option prices for different values of q

This figure shows European Call Option prices which are found by explicit method for $q = 1$, fractional explicit method for $q = 1.05$, $q = 1.1$, $q = 1.15$, $q = 1.2$.

Table 4.1: Comparison of solutions for European call option prices for different values of q and the solution with explicit method and closed form solution for $M = 3$ and $N = 3$

$q = 1.05$	$q = 1.1$	$q = 1.15$	$q = 1.2$	PDE Solution	Closed-form
14.9293	13.0744	11.4792	10.1044	11.3842	13.6952

Example 4.3. Let a European call option have an exercise price of 95. The option has the underlying stock at price 100$ which pays no dividends, and has a volatility of 50% and the risk-free rate is 10% and $T = 0.25$ as in Example 3.1.

Fractional explicit method is used to obtain the call option price. For valuation, the parameters should be chosen suitably as

$S_{min} = 0$, $S_{max} = 150$, $dS = 37.5$ and $dt = 0.0833$.

Then, we can find as $M = 3$ and $N = 4$ and we get following equations for each $i = 3, 2, 1$ and $k = 1, 2, 3$:

$$
\begin{aligned}
g_1 w_{1,2} + g_2 w_{1,1} + g_3 w_{1,0} &= \tilde{\alpha}_1 w_{0,3} + \tilde{\beta}_1 w_{1,3} + \tilde{\gamma}_1 w_{2,3}, \\
g_1 w_{1,1} + g_2 w_{1,0} &= \tilde{\alpha}_1 w_{0,2} + \tilde{\beta}_1 w_{1,2} + \tilde{\gamma}_1 w_{2,2}, \\
g_1 w_{1,0} &= \tilde{\alpha}_1 w_{0,1} + \tilde{\beta}_1 w_{1,1} + \tilde{\gamma}_1 w_{2,1}, \\
g_1 w_{2,2} + g_2 w_{2,1} + g_3 w_{2,0} &= \tilde{\alpha}_2 w_{1,3} + \tilde{\beta}_2 w_{2,3} + \tilde{\gamma}_2 w_{3,3} \\
g_1 w_{2,1} + g_2 w_{2,0} &= \tilde{\alpha}_2 w_{1,2} + \tilde{\beta}_2 w_{2,2} + \tilde{\gamma}_2 w_{3,2}, \\
g_1 w_{2,0} &= \tilde{\alpha}_2 w_{1,1} + \tilde{\beta}_2 w_{2,1} + \tilde{\gamma}_2 w_{3,1}, \\
g_1 w_{3,2} + g_2 w_{3,1} + g_3 w_{3,0} &= \tilde{\alpha}_3 w_{2,3} + \tilde{\beta}_3 w_{3,3} + \tilde{\gamma}_3 w_{4,3} \\
g_1 w_{3,1} + g_2 w_{3,0} &= \tilde{\alpha}_3 w_{2,2} + \tilde{\beta}_3 w_{3,2} + \tilde{\gamma}_3 w_{4,2}, \\
g_1 w_{3,0} &= \tilde{\alpha}_3 w_{2,1} + \tilde{\beta}_3 w_{3,1} + \tilde{\gamma}_3 w_{4,1}.
\end{aligned}
$$

If we rewrite these equations as the unknown terms are on the left hand side and the known terms are on the right hand side we get:

$$
\begin{aligned}
g_1 w_{1,2} + g_2 w_{1,1} + g_3 w_{1,0} &= \tilde{\alpha}_1 w_{0,3} + \tilde{\beta}_1 w_{1,3} + \tilde{\gamma}_1 w_{2,3}, \\
g_1 w_{1,1} + g_2 w_{1,0} - \tilde{\beta}_1 w_{1,2} - \tilde{\gamma}_1 w_{2,2} &= \tilde{\alpha}_1 w_{0,2}, \\
g_1 w_{1,0} - \tilde{\beta}_1 w_{1,1} - \tilde{\gamma}_1 w_{2,1} &= \tilde{\alpha}_1 w_{0,1}, \\
g_1 w_{2,2} + g_2 w_{2,1} + g_3 w_{2,0} &= \tilde{\alpha}_2 w_{1,3} + \tilde{\beta}_2 w_{2,3} + \tilde{\gamma}_2 w_{3,3}, \\
g_1 w_{2,1} + g_2 w_{2,0} - \tilde{\alpha}_2 w_{1,2} - \tilde{\beta}_2 w_{2,2} - \tilde{\gamma}_2 w_{3,2} &= 0, \\
g_1 w_{2,0} - \tilde{\alpha}_2 w_{1,1} - \tilde{\beta}_2 w_{2,1} - \tilde{\gamma}_2 w_{3,1} &= 0, \\
g_1 w_{3,2} + g_2 w_{3,1} + g_3 w_{3,0} &= \tilde{\alpha}_3 w_{2,3} + \tilde{\beta}_3 w_{3,3} + \tilde{\gamma}_3 w_{4,3} \\
g_1 w_{3,1} + g_2 w_{3,0} - \tilde{\alpha}_3 w_{2,2} - \tilde{\beta}_3 w_{3,2} &= \tilde{\gamma}_3 w_{4,2}, \\
g_1 w_{3,0} - \tilde{\alpha}_3 w_{2,1} - \tilde{\beta}_3 w_{3,1} &= \tilde{\gamma}_3 w_{4,1}.
\end{aligned}
$$

These equations can be written in matrix form, where G and B matrices are known:

$$G \times W = B$$

where

$$
G = \begin{pmatrix}
g_1 & g_2 & g_3 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\tilde{\beta}_1 & g_1 & g_2 & -\tilde{\gamma}_1 & 0 & 0 & 0 & 0 & 0 \\
0 & -\tilde{\beta}_1 & g_1 & 0 & -\tilde{\gamma}_1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & g_1 & g_2 & g_3 & 0 & 0 & 0 \\
-\tilde{\alpha}_2 & 0 & 0 & -\tilde{\beta}_2 & g_1 & g_2 & -\tilde{\gamma}_2 & 0 & 0 \\
0 & -\tilde{\alpha}_2 & 0 & 0 & -\tilde{\beta}_2 & g_1 & 0 & -\tilde{\gamma}_2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & g_1 & g_2 & g_3 \\
0 & 0 & 0 & -\tilde{\alpha}_3 & 0 & 0 & -\tilde{\beta}_3 & g_1 & g_2 \\
0 & 0 & 0 & 0 & -\tilde{\alpha}_3 & 0 & 0 & -\tilde{\beta}_3 & g_1
\end{pmatrix}, W = \begin{pmatrix}
w_{1,2} \\ w_{1,1} \\ w_{1,0} \\ w_{2,2} \\ w_{2,1} \\ w_{2,0} \\ w_{3,2} \\ w_{3,1} \\ w_{3,0}
\end{pmatrix}
$$

and

$$B = \begin{pmatrix} \tilde{\alpha}_1 w_{0,3} + \tilde{\beta}_1 w_{1,3} + \tilde{\gamma}_1 w_{2,3} \\ \tilde{\alpha}_1 w_{0,2} \\ \tilde{\alpha}_1 w_{0,1} \\ \tilde{\alpha}_2 w_{1,3} + \tilde{\beta}_2 w_{2,3} + \tilde{\gamma}_2 w_{3,3} \\ 0 \\ 0 \\ \tilde{\alpha}_3 w_{2,3} + \tilde{\beta}_3 w_{3,3} + \tilde{\gamma}_3 w_{4,3} \\ \tilde{\gamma}_3 w_{4,2} \\ \tilde{\gamma}_3 w_{4,1} \end{pmatrix}.$$

Then we can find the matrix W as $W = G^{-1} \times B$

Remark 4.4. Instead of using inverse of matrix G, one can apply LU decomposition not to have computational errors.

Remark 4.5. The purpose of this example is to find the matrix W which consists of unknown values. However, the solution is the value of w at $t = t_0$ and $S = S_0$. In this example, the solution is for $i = 0$ as:

$$t = t_0 + i\Delta t = 0 + 0 \times 0.0833 = 0 = t_0.$$

However, there is not a suitable k value for $S = S_0$, thus linear interpolation method is used in order to find the option price for $S = S_0$. Linear interpolation method is commonly used for approximation between two values. In this example, linear interpolation method is applied to the values at $k = 2$ and $k = 3$.

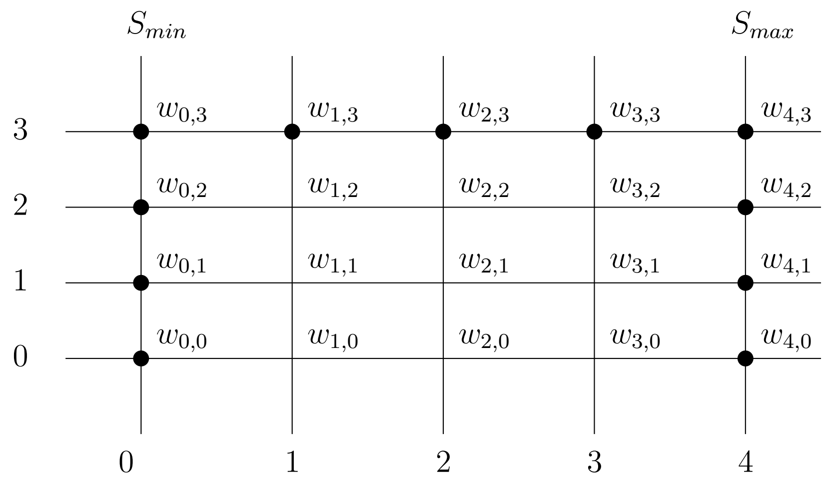

Figure 4.8: Fractional explicit method for $M = 3$ and $N = 4$

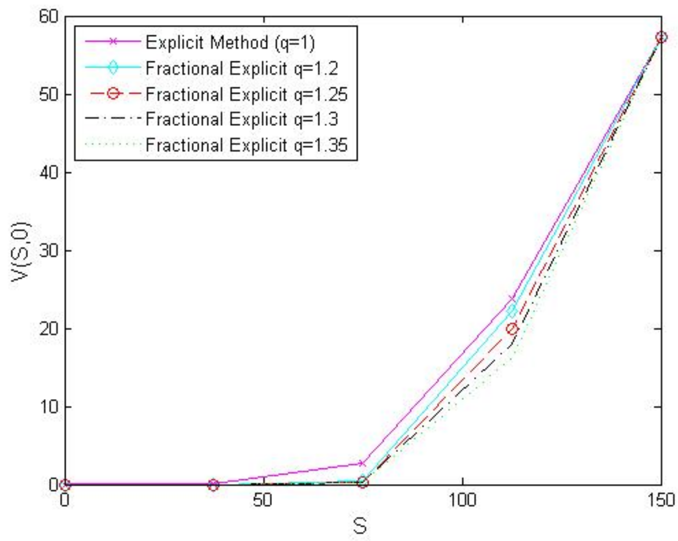

Figure 4.9: European call option prices for different values of q

This figure shows European Call Option prices which are found by explicit method for $q = 1$, fractional explicit method for $q = 1.2$, $q = 1.25$, $q = 1.3$, $q = 1.35$.

Table 4.2: Comparison of solutions for European call option prices for different values of q and the solution with explicit method and closed form solution for $M = 3$ and $N = 4$

$q = 1.2$	$q = 1.25$	$q = 1.3$	$q = 1.35$	PDE Solution	Closed-form
14.9154	13.3843	12.0430	10.8653	16.8140	13.6952

Example 4.4. Let a European call option have an exercise price of 95. The option has the underlying stock at price 100$ which pays no dividends, and has a volatility of 50% and the risk-free rate is 10% and $T = 0.25$ as in Example 3.1.

Fractional explicit method is used to obtain the call option price. For valuation, the parameters should be chosen suitably as:

$S_{min} = 0$, $S_{max} = 150$, $dS = 50$ and $dt = 0.0625$.

Then, we can find as $M = 4$ and $N = 3$ and we get following equations for each $i = 4, 3, 2, 1$ and $k = 1, 2$:

$$
\begin{aligned}
g_1 w_{1,3} + g_2 w_{1,2} + g_3 w_{1,1} + g_4 w_{1,0} &= \tilde{\alpha}_1 w_{0,4} + \tilde{\beta}_1 w_{1,4} + \tilde{\gamma}_1 w_{2,4},\\
g_1 w_{1,2} + g_2 w_{1,2} + g_3 w_{1,0} &= \tilde{\alpha}_1 w_{0,3} + \tilde{\beta}_1 w_{1,3} + \tilde{\gamma}_1 w_{2,3},\\
g_1 w_{1,1} + g_2 w_{1,0} &= \tilde{\alpha}_1 w_{0,2} + \tilde{\beta}_1 w_{1,2} + \tilde{\gamma}_1 w_{2,2},\\
g_1 w_{1,0} &= \tilde{\alpha}_1 w_{0,1} + \tilde{\beta}_1 w_{1,1} + \tilde{\gamma}_1 w_{2,1},\\
g_1 w_{2,3} + g_2 w_{2,2} + g_3 w_{2,1} + g_4 w_{2,0} &= \tilde{\alpha}_2 w_{1,4} + \tilde{\beta}_2 w_{2,4} + \tilde{\gamma}_2 w_{3,4},\\
g_1 w_{2,2} + g_2 w_{2,2} + g_3 w_{2,0} &= \tilde{\alpha}_2 w_{1,3} + \tilde{\beta}_2 w_{2,3} + \tilde{\gamma}_2 w_{3,3},\\
g_1 w_{2,1} + g_2 w_{2,0} &= \tilde{\alpha}_2 w_{1,2} + \tilde{\beta}_2 w_{2,2} + \tilde{\gamma}_2 w_{3,2},\\
g_1 w_{2,0} &= \tilde{\alpha}_2 w_{1,1} + \tilde{\beta}_2 w_{2,1} + \tilde{\gamma}_2 w_{3,1}.
\end{aligned}
$$

If we rewrite these equations as the unknown terms are on the left hand side and the known terms are on the right hand side we get:

$$
\begin{aligned}
g_1 w_{1,3} + g_2 w_{1,2} + g_3 w_{1,1} + g_4 w_{1,0} &= \tilde{\alpha}_1 w_{0,4} + \tilde{\beta}_1 w_{1,4} + \tilde{\gamma}_1 w_{2,4},\\
g_1 w_{1,2} + g_2 w_{1,2} + g_3 w_{1,0} - \tilde{\beta}_1 w_{1,3} - \tilde{\gamma}_1 w_{2,3} &= \tilde{\alpha}_1 w_{0,3},\\
g_1 w_{1,1} + g_2 w_{1,0} - \tilde{\beta}_1 w_{1,2} - \tilde{\gamma}_1 w_{2,2} &= \tilde{\alpha}_1 w_{0,2},\\
g_1 w_{1,0} - \tilde{\beta}_1 w_{1,1} - \tilde{\gamma}_1 w_{2,1} &= \tilde{\alpha}_1 w_{0,1},\\
g_1 w_{2,3} + g_2 w_{2,2} + g_3 w_{2,1} + g_4 w_{2,0} &= \tilde{\alpha}_2 w_{1,4} + \tilde{\beta}_2 w_{2,4} + \tilde{\gamma}_2 w_{3,4},\\
g_1 w_{2,2} + g_2 w_{2,2} + g_3 w_{2,0} - \tilde{\alpha}_2 w_{1,3} - \tilde{\beta}_2 w_{2,3} &= \tilde{\gamma}_2 w_{3,3},\\
g_1 w_{2,1} + g_2 w_{2,0} - \tilde{\alpha}_2 w_{1,2} - \tilde{\beta}_2 w_{2,2} &= \tilde{\gamma}_2 w_{3,2},\\
g_1 w_{2,0} - \tilde{\alpha}_2 w_{1,1} - \tilde{\beta}_2 w_{2,1} &= \tilde{\gamma}_2 w_{3,1}.
\end{aligned}
$$

These equations can be written in matrix form where G and B matrices are known:

$$G \times W = B$$

$$
G = \begin{pmatrix}
g_1 & g_2 & g_3 & g_4 & 0 & 0 & 0 & 0\\
-\tilde{\beta}_1 & g_1 & g_2 & g_3 & -\tilde{\gamma}_1 & 0 & 0 & 0\\
0 & -\tilde{\beta}_1 & g_1 & g_2 & 0 & -\tilde{\gamma}_1 & 0 & 0\\
0 & 0 & -\tilde{\beta}_1 & g_1 & 0 & 0 & -\tilde{\gamma}_1 & 0\\
0 & 0 & 0 & 0 & g_1 & g_2 & g_3 & g_4\\
-\tilde{\alpha}_2 & 0 & 0 & 0 & -\tilde{\beta}_2 & g_1 & g_2 & g_3\\
0 & -\tilde{\alpha}_2 & 0 & 0 & 0 & -\tilde{\beta}_2 & g_1 & g_2\\
0 & 0 & -\tilde{\alpha}_2 & 0 & 0 & 0 & -\tilde{\beta}_2 & g_1
\end{pmatrix}, W = \begin{pmatrix}
w_{1,3}\\ w_{1,2}\\ w_{1,1}\\ w_{1,0}\\ w_{2,3}\\ w_{2,2}\\ w_{2,1}\\ w_{2,0}
\end{pmatrix},
$$

and

$$
B = \begin{pmatrix}
\tilde{\alpha}_1 w_{0,4} + \tilde{\beta}_1 w_{1,4} + \tilde{\gamma}_1 w_{2,4} \\
\tilde{\alpha}_1 w_{0,3} \\
\tilde{\alpha}_1 w_{0,2} \\
\tilde{\alpha}_1 w_{0,1} \\
\tilde{\alpha}_2 w_{1,4} + \tilde{\beta}_2 w_{2,4} + \tilde{\gamma}_2 w_{3,4} \\
\tilde{\gamma}_2 w_{3,3} \\
\tilde{\gamma}_2 w_{3,2} \\
\tilde{\gamma}_2 w_{3,1}
\end{pmatrix}.
$$

Then we can find the matrix W as $W = G^{-1} \times B$

Remark 4.6. Instead of using inverse of matrix G, one can apply LU decomposition not to have computational errors.

Remark 4.7. The purpose of this example is to find the matrix W which consists of unknown values. However, the solution is the value of w at $t = t_0$ and $S = S_0$. In this example, the solution is $w_{2,0}$ for $k = 2$ and $i = 0$, since

$$
\begin{aligned}
S &= S_{min} + k\Delta S = 0 + 2 \times 50 = 100 = S_0, \\
t &= t_0 + i\Delta t = 0 + 0 \times 0.0625 = 0 = t_0.
\end{aligned}
$$

All the values in the following table and figure are the values of $w_{2,0}$.

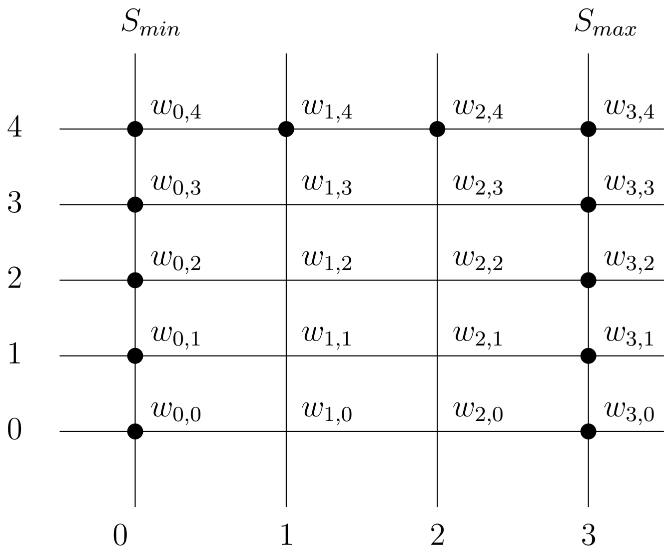

Figure 4.10: Fractional explicit method for $M = 4$ and $N = 3$

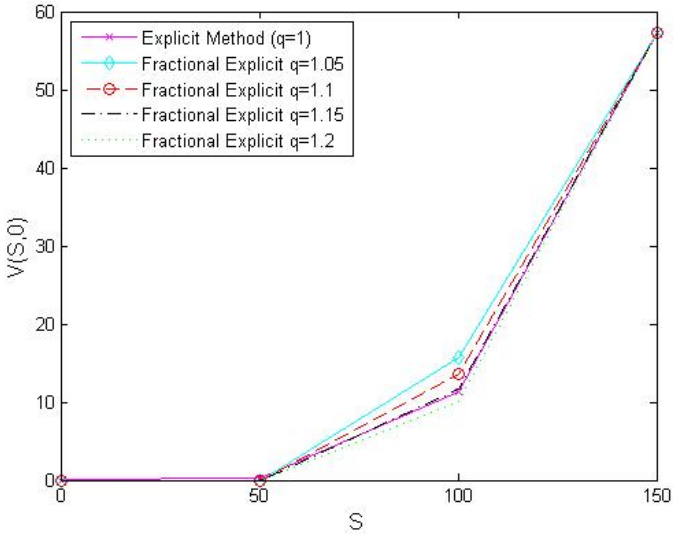

Figure 4.11: European call option prices for different values of q

This figure shows European Call Option prices which are found by explicit method for $q = 1$, fractional explicit method for $q = 1.05$, $q = 1.1$, $q = 1.15$, $q = 1.2$.

Table 4.3: Comparison of solutions for European call option prices for different values of q and the solution with explicit method and closed form solution for $M = 4$ and $N = 3$

$q = 1.05$	$q = 1.1$	$q = 1.15$	$q = 1.2$	PDE Solution	Closed-form
15.7167	13.5481	11.7092	10.1464	11.3341	13.6952

Example 4.5. Let a European call option have an exercise price of 95. The option has the underlying stock at price 100$ which pays no dividends, and has a volatility of 50% and the risk-free rate is 10% and $T = 0.25$ as in the preceding examples.

Fractional explicit method is used to obtain the call option price. In order to obtain accurate valuations, the parameters are chosen to have higher M and N values:

$S_{min} = 0$, $S_{max} = 150$, $dS = 10$ and $dt = 0.00625$.

Then we can find as $M = 40$ and $N = 15$.

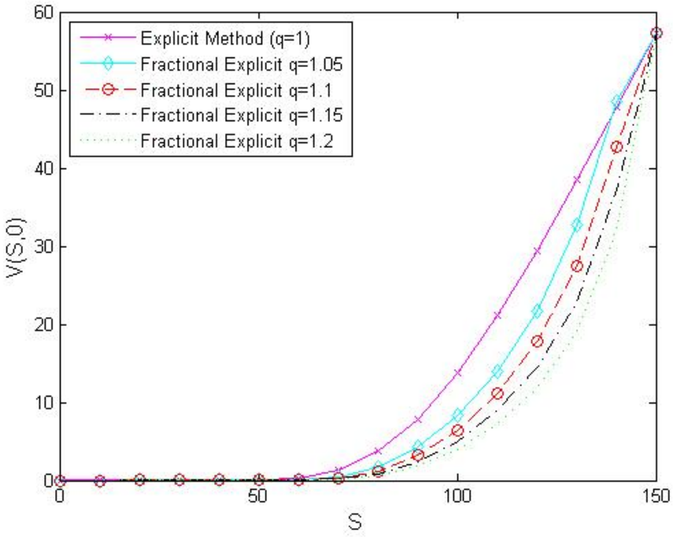

Figure 4.12: European call option prices for different values of q

This figure shows European Call Option prices which are found by explicit method for $q = 1$, fractional explicit method for $q = 1.05$, $q = 1.1$, $q = 1.15$, $q = 1.2$.

Table 4.4: Comparison of solutions for European call option prices for different values of q and the solution with explicit method and closed form solution for $M = 40$ and $N = 15$

$q = 1.001$	$q = 1.005$	$q = 1.008$	$q = 1.01$	PDE Solution	Closed-form
10.4853	10.2825	10.1330	10.0346	13.7353	13.6952

Example 4.6. Let a European call option have an exercise price of 95. The option has the underlying stock at price 100$ which pays no dividends, and has a volatility of 50% and the risk-free rate is 10% and $T = 0.25$ as in the preceding examples.

Fractional explicit method is used to obtain the call option price. In order to obtain accurate valuations, the parameters are chosen to have higher M and N values:

$S_{min} = 0$, $S_{max} = 150$, $dS = 5$ and $dt = 0.0025$.

Then we can find as $M = 100$ and $N = 30$.

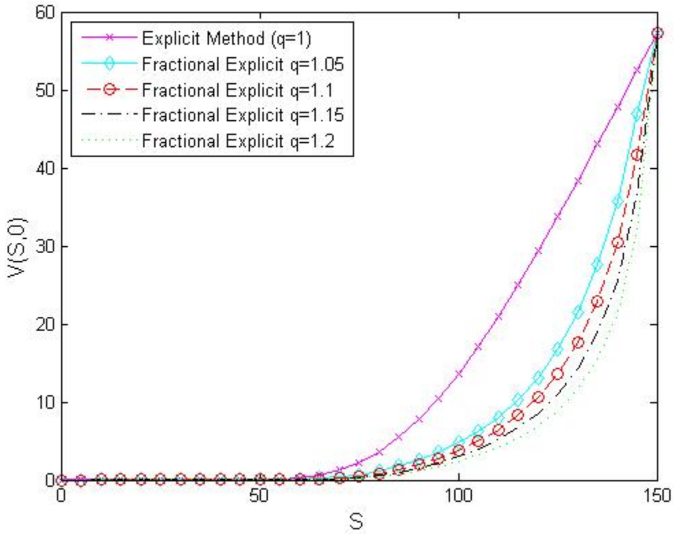

Figure 4.13: European call option prices for different values of q

This figure shows European Call Option prices which are found by explicit method for $q = 1$, fractional explicit method for $q = 1.05$, $q = 1.1$, $q = 1.15$, $q = 1.2$.

Table 4.5: Comparison of solutions for European call option prices for different values of q and the solution with explicit method and closed form solution for $M = 100$ and $N = 30$

$q = 0.86$	$q = 0.88$	$q = 0.9$	$q = 0.91$	PDE Solution	Closed-form
11.0186	10.1236	9.2781	8.8801	13.6581	13.6952

To summarize, the following table compare all the European call option values which are obtained by solution of q-th order time fractional Black Scholes PDE using fractional explicit method.

Table 4.6: Comparison of European call option prices for different values of M and N for different values of q

$q = 0.9$	$M = 3$	$M = 4$
$N = 3$	22.5812	24.9289
$N = 4$	30.3504	32.4841

$q = 0.95$	$M = 3$	$M = 4$
$N = 3$	19.6189	21.3183
$N = 4$	26.7554	28.2423

$q = 1.05$	$M = 3$	$M = 4$
$N = 3$	14.9293	15.7167
$N = 4$	20.9921	21.5463

$q = 1.1$	$M = 3$	$M = 4$
$N = 3$	13.0744	13.5481
$N = 4$	18.6784	18.9030

$q = 1.2$	$M = 3$	$M = 4$
$N = 3$	10.1044	10.1464
$N = 4$	14.9154	14.6748

$q = 1.3$	$M = 3$	$M = 4$
$N = 3$	7.8890	7.6794
$N = 4$	12.0430	11.521

According to the table it can be seen that when q increases, the value of the European option decreases. When q is in a close neighborhood of 1, European option price is considerably close to closed-form solution which is 13.6952.

In this chapter, finite difference method is restated and explicit method is reviewed with example. Fractional Black Scholes PDE is presented. In order to solve fractional Black Scholes PDE, fractional explicit method is proposed and applied to the PDE. Different examples are solved for different values of variables and the results are compared with tables and figures.

CHAPTER 5

CONCLUSION AND OUTLOOK

Investigation and application of fractional Black Scholes partial differential equation is mainly purposed in this thesis. At first, the problem of how to derive fractional Black Scholes PDE is considered. For this reason, in chapter 2 a detailed notations and definitions of fractional order derivatives and integrals, also called differintegrals, are investigated. Most of the definitions of differintegrals are derived by extending the definition of derivative or integral. It is also noticed that all the definitions; Grünwald-Letkinov, Riemann-Liouville, Riemann, Laplace Transform, Caputo and Riesz are all agree [5]. It is fundamental to use Mittag-Leffler function for calculations in differintegrals. Therefore, properties of Mittag-Leffler function are stated. Moreover, in a different point of view, stochastic calculus is also used to derive fractional Black Scholes PDE. Therefore, Brownian motion is reviewed in order to construct a basis for derivation.

Derivation of fractional Black Scholes PDE is investigated in chapter 3. For this reason, first Black-Scholes equation is given in detailed afterwards approaches for derivation fractional Black-Scholes PDE are stated. At the beginning, equation of evolution [33] approach is given which is derived by Wyss M. and Wyss W., then Laplace transform and homotopy perturbation method [15] is presented, then fractional Taylor series method [16] is stated. It is emphasized that there is a wide range of aspects for derivation of fractional PDEs. After revising approaches in literature, we proposed a new fractional Black-Scholes PDE using fractional heat equation. In order to derive the formula first derivation of Black Scholes PDE by classical heat equation is restated [17, 28]. Then, the definitions and properties of differintegrals are used for derivation. It is essential to notice that for $q = 1$ classical and fractional Black-Scholes equations are agree. Moreover, using stochastic calculus is another way to study fractional PDEs. First fractional Brownian motion [14] is considered and then by It$\hat{o}$ formula we derived fractional Black-Scholes equation.

On the other hand, we focus on finding the solution of fractional Black Scholes PDE. Therefore, numerical methods are considered and finite difference method is examined. One of the finite difference methods; the explicit method is stated which is commonly used to solve classical partial differential equations. Then, solution of classical Black Sholes PDE using explicit method is found. It is noticed that, closed form solution and the PDE solution are almost equal to each other. Additionally, consistency, convergence and stability analysis of explicit method are discussed. Furthermore, fractional explicit method is proposed for PDEs which contains fractional order differintegrals. The fundamental idea of fractional explicit method is adjusting the Grünwald-Letkinov definition for differintegrals. It is noticed that for $q = 1$ explicit method and fractional explicit method agree. Therefore, fractional explicit method is used to solve fractional Black-Scholes equation. Numerical solutions for fractional Black Scholes PDE are found for different M, N and q values. Moreover, the solutions of Black Scholes PDE via explicit method and fractional explicit method for different fractional order q is compared in tables and figures. It is an undeniable fact that fractional explicit method is derived using explicit method and Grünwald-Letkinov definition for differintegrals. It is noticed that for the suitable choice of variables, the solutions of Black Sholes PDE with the two different methods are found close to each other. However, we proposed that the solutions with fractional explicit method affected by the contribution of the previous data.

The future work may be on stability analysis of fractional explicit method and the choice of variables. Moreover, one can may take the fractional derivative not only for time but also stock price S. For Black Sholes PDE, one should also consider the fractional explicit method for higher order, since the PDE contains second order derivative with respect to stock price. Furthermore, fractional explicit method may be applied to other fractional PDEs and can be used to find solutions of other problems.

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APPENDIX A

DERIVATION OF BLACK SCHOLES PDE

Theorem A.1. *For European call or put options on an underlying stock paying no dividends, Black Scholes PDE is stated as follows;*

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0,$$

where $V(S,t)$ is the price of European option as a function of stock price S and time t, and r is the risk-free interest rate, σ is the volatility of the stock.

Proof. For derivation of Black Scholes PDE It$\hat{o}$ formula of Equation 2.7 is used for $V(S,t)$;

$$\begin{aligned} V(S_t,t) = V(S_0,0) + \int_0^t \frac{\partial V}{\partial u}(S_u,u)du + \int_0^t \frac{\partial V}{\partial S}(S_u,u)dS_u \\ + \frac{1}{2}\int_0^t \frac{\partial^2 V}{\partial S^2}(S_u,u)d < S,S >_u . \end{aligned}$$

The quadratic variation is $d < S,S >_t = S_t^2\sigma^2 dt$, then we have;

$$V = V_0 + \int_0^t \left(\frac{\partial V}{\partial u} + \frac{1}{2}S_u^2\sigma^2\frac{\partial^2 V}{\partial S^2} \right) du + \int_0^t \frac{\partial V}{\partial S}(S_u,u)dS_u.$$

Under risk neutral measure $\tilde{\mathbb{P}}$ geometric Brownian motion is $dS_t = rS_t dt + \sigma S_t dW_t$. Then by substitution we obtain;

$$V = V_0 + \int_0^t \left(\frac{\partial V}{\partial u} + S_u r\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S_u^2\frac{\partial^2 V}{\partial S^2} \right) du + \int_0^t \frac{\partial V}{\partial S} S_u \sigma dW_u.$$

Then by taking derivative with respect to t, the following equation is obtained;

$$dV = \left(\frac{\partial V}{\partial t} + S_t r\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 V}{\partial S^2} \right) dt + \left(\frac{\partial V}{\partial S} S_t \sigma \right) dW_t.$$

Since, $\tilde{V}$ is the notation for option value for discounted asset price such that;

$$\begin{aligned}\tilde{V} &= e^{-rt}V,\\ d\tilde{V} &= -rte^{-rt}V + e^{-rt}dV.\end{aligned}$$

By substitution we obtain $d\tilde{V}$ as;

$$d\tilde{V} = e^{-rt}\underbrace{\left(-rV + \frac{\partial V}{\partial t} + S_t r\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2}\right)}_{0} dt + \left(e^{-rt}\frac{\partial V}{\partial S}S_t\sigma\right) dW_t.$$

Note that, $\tilde{V}$ is martingale transform of discounted asset prices. Since $\tilde{S}$ is a martingale under risk neutral measure $\tilde{\mathbb{P}}$, the $\tilde{V}$ is also martingale under measure $\tilde{\mathbb{P}}$. Hence, by martingale representation theorem [17], Black Scholes PDE is;

$$-rV + \frac{\partial V}{\partial t} + S_t r\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2} = 0.$$

□

APPENDIX B

GAMMA FUNCTION

B.1 Properties of Gamma Function

Definition B.1. The Gamma function is defined for $z \in \mathbb{C}, z \neq 0, -1, -2, ...$ and for $x > 0$ to be;

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx. \tag{B.1}$$

Remark B.1. Properties of Gamma function functions;

1. One of the most important properties of Gamma function is valid for positive integer n;

   $$\Gamma(n+1) = n\Gamma(n) = n!. \tag{B.2}$$

   Note that, this property can be extended to noninteger values q as;

   $$\Gamma(q+1) = q\Gamma(q).$$

2. Using the property 1, we can obtain the following property for all integers. Note that, even though Gamma function gives infinity for the values $0, -1, -2, ...$ their ratios are finite [5].

   $$\frac{\Gamma(-n)}{\Gamma(-N)} = (-1)^{N-n} \frac{N!}{n!}. \tag{B.3}$$

3. Binomial coefficients lead us to obtain the following property for integer n;

   $$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{\Gamma(n+1)}{\Gamma(k+1)\Gamma(n-k+1)}. \tag{B.4}$$

   Note that, for noninteger values q this property can be extended as;

   $$\binom{q}{j} = \frac{\Gamma(q+1)}{\Gamma(j+1)\Gamma(q-j+1)} \tag{B.5}$$

For some n values, the values of gamma function are;

$$\Gamma(1) = 1 \;\; \Gamma(2) = 1 \;\; \Gamma(3) = 1,$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \;\; \Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\sqrt{\pi} \;\; \Gamma\left(\frac{5}{2}\right) = \frac{3}{4}\sqrt{\pi},$$

$$\Gamma(0) = \pm\infty \quad \Gamma(-1) = \pm\infty.$$

$$\Gamma\left(-\frac{1}{2}\right) = -2\sqrt{\pi} \;\; \Gamma\left(-\frac{3}{2}\right) = \frac{4}{3}\sqrt{\pi}$$

Proposition B.1. *Gamma function satisfies the following relation for integer n which is used to obtain Grünwald-Letnikov definition of differintegral of Equation (2.1).*

$$(-1)^j \binom{n}{j} = \binom{j+n-1}{j} = \frac{\Gamma(j-n)}{\Gamma(-n)\Gamma(j+1)}$$

Proof. Let us restate Equation (B.4) in property 3 as;

$$\binom{n}{k} = \frac{\Gamma(n+1)}{\Gamma(k+1)\Gamma(n-k+1)}.$$

Thus, first applying this relation to $\binom{j+n-1}{j}$, then using (B.3) in order to find the ratio of Gamma functions we get the result as;

$$\begin{aligned} \binom{j+n-1}{j} &= \frac{\Gamma(j-n)}{\Gamma(-n)\Gamma(j+1)} = \frac{\Gamma(-(n-j))}{\Gamma(-n)\Gamma(j+1)} \\ &= (-1)^{(n-j)-(n)} \frac{(n)!}{(n-j)!(j)!} \\ &= (-1)^j \binom{n}{j}. \end{aligned}$$

□

Remark B.2. This relation is valid not only for integers n but also non-integers q.

$$(-1)^j \binom{q}{j} = \binom{j+q-1}{j} = \frac{\Gamma(j-q)}{\Gamma(-q)\Gamma(j+1)}.$$

APPENDIX C

MATLAB CODES

1. MATLAB code for Figure 2.1. The plot of $E_1(x)$ [8];

```
function f = ml_func(aa,z,n,eps0)
aa=[aa,1,1,1]; a=aa(1); b=aa(2);c=aa(3); q=aa(4);
f=0; k=0; fa=1; aa=aa(1:4);
if nargin<4, eps0=eps; end
if nargin<3
  n=0
end
if n==0
  while norm(fa,1)>=eps0
   fa=gamma(k*q+c)/gamma(c)/gamma(k+1)/...
   gamma(a*k+b)*z.^k
   f=f+fa; k=k+1;
  end
else
  aa(2)=b+n*a; aa(3)=c+q*n;
  f=gamma(q+n*c)/gamma(c)*ml_func(aa,z,0,eps0)
end
end
```

```
t=0:0.1:5;
a1=ml_func(1,t)
plot(t,a1)
title('E_{1}(x)=e^x')
```

2. MATLAB code for Figure 2.2. The plot of $E_1(-x)$, $E_2(-x)$ and $E_3(-x)$ [8];

```
t=0:0.1:5;
b1=ml_func(1,-t);
b2=ml_func(2,-t);
b3=ml_func(3,-t);
plot(t,b1, 'm-x'), hold on,
plot(t,b2), hold on, plot(t,b3,'c-d'),hold on
legend('E_1(-x)','E_2(-x)', 'E_3(-x)','Location','southwest')
hold off
```

3. MATLAB code for Figure 2.3. The plot of Standard Brownian Motion[34];

```
function [B] = brownian(N,b,sigma,T)
t = (0:1:N)'/N;
W = [0; cumsum(randn(N,1))]/sqrt(N);
t = t*T;
W = W*sqrt(T);
B = b*t + sigma*W;
plot(t,B);
hold on
plot(t,b*t,':');
axis([0 T min(-sigma,(b-2*sigma)*T) ...
max(sigma,(b+2*sigma)*T)])
title([int2str(N) '-step version of ...
Brownian motion and its mean'])
xlabel(['Drift ' num2str(b) ',
diffusion coefficient ' num2str(sigma)])
hold off
```

```
brownian(1000,0,1,1)
```

4. MATLAB code for Figure 2.4. The plot of Geometric Brownian Motion[34];

```
function [X] = geometric_brownian(N,r,alpha,T)
t = (0:1:N)'/N;
W = [0; cumsum(randn(N,1))]/sqrt(N);
t = t*T;
W = W*sqrt(T);
Y = (r-(alpha^2)/2)*t + alpha * W;
X = exp(Y);
plot(t,X);
hold on
plot(t,exp(r*t),':');
axis([0 T 0 max(1,exp((r-(alpha^2)/2)*T+2*alpha))])
title([int2str(N) '-step geometric Brownian ...
motion and its mean'])
xlabel(['r = ' num2str(r) ' and ...
alpha = ' num2str(alpha)])
hold off
```

```
geometric_brownian(1000,1,0.1,1)
```

5. MATLAB code for Figure 2.5 and 2.6. The plot of Fractional Brownian Motion;

```
function [deltafBm,Sigma]=fBm(H,N)
Sigma=zeros(N,N);
for i=1:N
 for j=1:N
  Sigma(i,j)=0.5*(abs(i-j+1)^(2*H) +abs(i-j-1)^...
  (2*H)-2*abs(i-j)^(2*H));
 end
end
mu=zeros(N,1);
deltafBm=mvnrnd(mu,Sigma); fBm(1)=0;
for i=1:N-1
 fBm(i+1)=fBm(i)+deltafBm(i);
end
t=0:N-1;
plot(t,fBm)
```

```
fBm(0.7,1000)
fBm(0.3,1000)
```

6. • MATLAB code for Example 3.1, 3.2;

```
[Call, Put] = blsprice(100, 95, 0.1, 0.25, 0.5)
```

• A more detailed MATLAB code for Example 3.1, 3.2[28];

```
function [C, P] = CallPut_Delta(S,K,r,tau,...
sigma,div)% tau = time to expiry (T-t)
if nargin < 6
 div = 0.0;
end
if tau > 0
 d1 = (log(S/K) + (r + 0.5*sigma^2)*(tau)*...
 ones(size(S)))/(sigma*sqrt(tau));
 d2 = d1 - sigma*sqrt(tau);
 N1 = 0.5*(1+erf(d1/sqrt(2)));
 N2 = 0.5*(1+erf(d2/sqrt(2)));
 C = exp(-div*tau) * S.*N1-K*exp(-r*(tau))*N2;
 P = C + K*exp(-r*tau) - exp(-div*tau)*S;
else
 C = max(S-K,0);
 P = max(K-S,0);
end
```

```
[C, P]= CallPut_Delta(100, 95, 0.1, 0.25, 0.5)
```

7. MATLAB code for Examples 3.3, 3.4, 3.5 and 3.6;

```
function [C, P] = CallPut_Delta_Frac(S,K,r,...
tau,sigma,div,H)
% tau = time to expiry (T-t)
if nargin < 7
 div = 0.0;
end
if tau > 0
 d1 = (log(S/K) + (r + 0.5*sigma^2)*(tau^(2*H))*...
 ones(size(S)))/(sigma*sqrt(tau^(2*H)));
 d2 = d1 - sigma*sqrt(tau^(2*H));
 N1 = 0.5*(1+erf(d1/sqrt(2)));
 N2 = 0.5*(1+erf(d2/sqrt(2)));
 C = exp(-div*tau) * S.*N1-K*exp(-r*(tau))*N2;
 P = C + K*exp(-r*tau) - exp(-div*tau)*S;
else
 C = max(S-K,0);
 P = max(K-S,0);
end
```

```
[C, P]= CallPut_Delta_Frac(100, 95, 0.1, 0.25, 0.5, 0.5)\\
[C, P]= CallPut_Delta_Frac(100, 95, 0.1, 0.25, 0.5, 0.7)\\
[C, P]= CallPut_Delta_Frac(100, 95, 0.1, 0.25, 0.5, 0.3)
```

8. MATLAB code for the solution of explicit method and the plots of Explicit Method $2D$ and $3D$ in Figure 4.2 and 4.3 can be found in detailed in [28].

9. MATLAB code for Example 4.2 and Figure 4.6

Note that the first part of the MATLAB code is for calculating the solution of fractional Black Scholes price.

```
function[N,M,price,B]=BlackScholes_fExplicit(S0,...
K, r, div, sigma, T, f, alpha, beta, Smin, Smax,...
dS, dt ,q)

N = round((Smax-Smin)/dS);
M = round(T/dt);
Nv=0:N;
Mv=0:M;
S = Smin + dS*Nv;
t = dt*Mv;

Timet = feval(f,S,K); % t = T
StockSmin = (feval(alpha, t,T,r,K,Smin,div))';
% S = Smin
StockSmax = (feval(beta, t,T,r,K,Smax,div))';
% S = Smax

for j=1:M
g(j)=(gamma(j-q)/(gamma(-q)*gamma(j+1)));
end

mS = S/dS;
a = -0.5*(dt^q)*(mS.*(sigma^2*mS - r));
b = (dt^q)*( sigma^2*mS.^2 + r)-1;
c = -0.5*(dt^q)*(mS.*(sigma^2*mS + r));

p=M*(N-1);
G=zeros(p);
G=diag(g(1)*ones(1,p));

beta=zeros(1,p-1);
j=1;
 for i=1:M:(p-2)
     if(j-1)*M < i <j*M
           beta(i:i+(M-2))=-b(j)*ones(1,M-1);
            j=j+1;
     end
 end
G=G+diag(beta,-1);
```

```
alpha=zeros(1,p-M-1);
j=1;
 for i=1:M:p-(M+1)
     if(j-1)*M < i <j*M
           alpha(i:i+(M-2))=-a(j)*ones(1,M-1);
            j=j+1;
     end
  end
G=G+diag(alpha,-(M+1));

gamma2=g(M)*ones(1,p-M+1);
j=1;
 for i=2:M:(p-M+1)
     if(j-1)*M < i <j*M
           gamma2(i:i+(M-2))=-c(j)*ones(1,M-1);
           j=j+1;
     end
 end
G=G+diag(gamma2, M-1);

for k=2:M-1
gam=zeros(1,p-k+1);
j=1;
 for i=1:M:p-k+1
     if(j-1)*M < i <j*M
           gam(i:i+(M-k))=g(k)*ones(1,M-k+1);
           j=j+1;
     end
  end
G=G+diag(gam,k-1);
end

K=zeros(p,1);
j=1;
for i=1:M:p-M+1
 K(i)=a(j)*Timet(j)+b(j)*Timet(j+1)+c(j)*Timet(j+2);
 j=j+1;
end

j=0;
for i=2:M
    K(i)=a(1)*StockSmin(M-j);
    j=j+1;
end
```

```
j=0;
for i=p-M+2:p
    K(i)=c(end)*StockSmax(M-j);
    j=j+1;
end

[L U] = lu(G);
A = U \ ( L \ (K));

B=zeros(N+1,1);
B(1)=StockSmin(1);
B(end)=StockSmax(1);

j=M;
for i=1:N-1
    B(i+1)=A(j);
    j=j+M;
end

%linear interpolation
down = floor((S0-Smin)/dS); up = ceil((S0-Smin)/dS);
if (down == up)
   price = B(down+1);
else
   price = B(down+1)+(B(up+1) - B(down+1))*...
 (S0-Smin - down*dS)/dS;
end
```

Note that, the second part of the MATLAB code is for presenting and plotting the solutions for different q values. Therefore, except from the beginning part of the code, it is same for each example.

```
clear all, close all,
S0 = 100; K = 95; sigma = 0.5; r = 0.1; T = 0.25;
Smin = 0; Smax = 150; div=0;
dS = 50; dt = 0.0833;
q1=1.25; q2=1.35; q3=1.2; q4=1.5;

f = @BlackScholes_Payoff;
alpha = @BlackScholes_LeftBoundary;
beta = @BlackScholes_RightBoundary;
```

```
[a,b,c1,d1] = BlackScholes_fExplicit(S0, K, r,...
 div, sigma, T, f, alpha, beta, Smin, Smax,...
 dS, dt ,q1);
fprintf('Solution: %f\n for: %f\n',c1,q1);
[a,b,c2,d2] = BlackScholes_fExplicit(S0, K, r,...
 div, sigma, T,f, alpha, beta, Smin, Smax,...
 dS, dt ,q2 );
fprintf('Solution: %f\n for: %f\n',c2,q2);
[a,b,c3,d3] = BlackScholes_fExplicit(S0, K, r,...
 div, sigma, T, f, alpha, beta, Smin, Smax,...
 dS, dt ,q3 );
fprintf('Solution: %f\n for: %f\n',c3,q3);
[a,b,c4,d4] = BlackScholes_fExplicit(S0, K, r,...
 div, sigma, T, f, alpha, beta, Smin, Smax,...
 dS, dt ,q4);

fprintf('Solution: %f\n for: %f\n',c4,q4);
fprintf('N...........: %d\n', a);
fprintf('M...........: %d\n', b);

S = Smin + dS*[0:a];
[call, Cdelta, P, Pdelta] = ...
CallPut_Delta(S,K,r,sigma,T,div);
[w,p] = BlackScholes_Explicit(S0, K, r,...
 div, sigma, T, f, alpha, beta, Smin,...
 Smax, dS, dt);
[m,n] = size(w);
S = linspace(Smin,Smax,m);
t = linspace(0,T,n);

plot(S, w(:,1), 'm-x'), hold on,
plot(S, call),hold on, plot(S,d1,'c-d'),hold on,
plot(S,d2,'r--o'),hold on, plot(S,d3,'k-.'),
hold on, plot(S,d4,'g:'),hold on
xlabel('S','FontSize', 12),
ylabel('V(S,0)','FontSize', 12),

legend({'Explicit Method (q=1)',...
     'Fractional Explicit q=1.05',...
     'Fractional Explicit q=1.1',...
     'Fractional Explicit q=1.15'...
     'Fractional Explicit q=1.2'},'Location',...
     'northwest')
 hold off
```

10. MATLAB code for Example 4.3 and Figure 4.8

```
clear all, close all,
S0 = 100; K = 95; sigma = 0.5; r = 0.1; T = 0.25;
Smin = 0; Smax = 150; div=0;
dS = 37.5; dt = 0.0833;
q1=1.25; q2=1.35; q3=1.2; q4=1.5;
```

11. MATLAB code for Example 4.4 and Figure 4.10

```
clear all, close all,
S0 = 100; K = 95; sigma = 0.5; r = 0.1; T = 0.25;
Smin = 0; Smax = 150; div=0;
dS = 50; dt = 0.0625;
q1=0.9; q2=0.95; q3=1.05; q4=1.1;
```

12. MATLAB code for Example 4.5 and Figure 4.10

```
clear all, close all,
S0 = 100; K = 95; sigma = 0.5; r = 0.1; T = 0.25;
Smin = 0; Smax = 150; div=0;
dS = 10; dt = 0.00625;
q1=0.9; q2=0.95; q3=1.05; q4=1.1;
```

13. MATLAB code for Example 4.6 and Figure 4.10

```
clear all, close all,
S0 = 100; K = 95; sigma = 0.5; r = 0.1; T = 0.25;
Smin = 0; Smax = 150; div=0;
dS = 5; dt = 0.0025;
q1=0.9; q2=0.95; q3=1.05; q4=1.1;
```