CPPI STRATEGY ON DEFINED CONTRIBUTION PENSION SCHEME UNDER CUSHION OPTION AND DISCRETE TIME TRADING SETTING

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ABSTRACT

CPPI STRATEGY ON DEFINED CONTRIBUTION PENSION SCHEME UNDER CUSHION OPTION AND DISCRETE TIME TRADING SETTING

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A Defined Contribution (DC) pension plan requires periodic contributions for a certain time in order to aggregate funds for the old ages. It enables the participant to pay a monthly/yearly premium whose valuation is done in financial markets. The choice of the market products (assets, bonds etc.) is processed by a centralized system at which a pension company/government has authority to manage the fund. Therefore, a risk management on the fund has to be taken into account. Such long term investment schemes are sensitive and prone to market risks. Additionally, a long-term planned funding might be distorted when premature terminations occur. For such cases, it is benefical for both parties (insurer and insured) to secure the fund in such a way that its return remains within the expected margins. We consider Constant Proportion Portfolio Insurance (CPPI) strategy in a DC pension plan. We examine a model that the price dynamics of a risky asset and labor income process are defined by a continuous-time stochastic process and trading is restricted to discrete time scheme. An exotic option is proposed as cushion option to reduce the risk that the portfolio value comes under the floor which is known as gap risk in the literature. We analyze the effectiveness of derived cushion option on CPPI strategy in a DC pension plan by measuring its sensivity with respect to the parameters through Monte Carlo Simulation.

Keywords : CPPI strategy, Defined Contribution Pension Plan, Exotic (cushion) option

ÖZ

KESİKLİ İŞLEM TARİHLERİ VE YASTIK OPSİYON VARSAYIMI ALTINDA BELİRLENMİŞ KATKI PAYI ESASLI EMEKLİLİK PLANLARININ SABİT ORANLI PORTFÖY SİGORTASI METODU İLE YÖNETİMİ

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Yaşlılık dönemi için fonlar toplamak için belirlenmiş katkı payı esaslı emeklilik planı belirli zamanda periyodik katkılar gerektirir. Bu katılımcıya finansal piyasalarda değerlenen aylık/yıllık prim ödemeleri olanağı tanır. Piyasa ürünlerinin seçmi (varlıklar, tahviller v.b.) emeklilik şirketinin /hükümetin fon yönetiminde yetkisi olduğu merkezileştirilmiş sistem tarafindan işlem görür. Bu yüzden, fona ilişkin risk yönetimi dikkate alınmalıdır. Böyle uzun dönem yatırım planları hassas ve piyasa risklerine eğilimlidir. Buna ek olarak, erken ayrılma gerçekleştiği zaman uzun dönem planlanmış fonlama tahrif edilebilir. Böyle durumlarda, fonun gelirinin beklenen marj içinde kalmasını sağlayan bir yolla fonu güvenceye almak her iki taraf (sigortalı ve sigortacı) içinde yararlı olacaktır. Belirlenmiş katkı payı esaslı emeklilik planlarında Sabit Oranlı Portföy Sigortası (SOPS) stratejisi ele alınmıştır. Emek gelirinin ve riskli varlığın fiyat dinamiklerinin sürekli-zaman stokastik süreç ile tanımlanmış ve alım-satımın kesikli-zaman çerçevesinde kısıtlandırılmış olan model incelenmiştir. Literatürde boşluk riski olarak bilinen portföy değerinin zeminin altına geçme riskini azaltmak için yastık opsiyon olarak bir egzotik opsiyon önerilmiştir. Belirlenmiş katkı payı esaslı emeklilik planlarında SOPS stratejisinde türetilmiş yastık opsiyonunun etkinliği Monte Carlo simulasyonları ile parametrelere göre onun hassaslığı ölçülerek analiz edilmiştir.

Anahtar Kelimeler : Sabit Oranlı Portföy Sigortası, Belirlenmiş Katkı Payı Esaslı Emeklilik Planları, Egzotik (Yastık) Opsiyon

To My Family

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CHAPTER 1

Introduction

Pension planning might be the longest and important financial decisions of a lifetime for individuals. Contribution and investment decision during this accumulation phase are the important issues in pension plan due to the fact that induviduals want to guarantee to maintain their standard of living at least after the retirement. Since contributions are invested in financial markets the pension system faces many risks especially, the financial market risk. Therefore, the downside protection strategies become outstanding topics in the literature. This thesis investigates a well-konwn downside protection strategy called Constant Proportion Portfolio Insurance (CPPI) in Defined Contribution (DC) pension fund modeling. Under discrete time trading, this strategy faces risk of portfolio value hits the floor. In this thesis, we examine the way that deals with so called gap risk.

A pension plan which has significant influence on maintaining indivuduals' standard of living after the retirement, become prominent issue in the literature due to the life expectancy which has raised over last decades.

Basically, there are two types of pension plans: Defined Benefit (DB) and Defined Contribution (DC). The benefits received by participant at retirement are fixed in advance in DB pension plans. Benefits are computed in relation to factors such as final salary, the length of pensionable service and the age of the member. Participants' contributions initially are set, but subsequently adjusted in order to accure the amount of the defined benefit at retirement [2]. On the contrary, participants' contributions are defined in advance and retirement benefits of a participant is a function of the contributions made to his account and fund's investment return during, and at the end of the period in DC pension scheme. The participant in a DC plan bears the investment risk in the sense that the amount of pension might be high or low depending on success of the fund management [10].

Defined Benefit pension scheme has been more prominent historically, but nowadays pension plans mostly is based on defined contribution scheme which transfers equity market risk to participants because of increasing life expectancy and structure of equity market [16]. Since all participants in both DB and DC pension scheme need to contribute a similar part of labor income, participants in DC plan will be more concerned about the performance of their pension fund investments. In related literature, Blake et al [11] examines the optimal dynamic asset allocation strategy for DC plan taking into account stochastic process for labor income including a non-hedgable risk component. Considering stochastic behaviour of labor income and stochastic inflation rate, closed form solution is given by Battochio et al [2] for the optimal portfolio in complete financial market. In the discrete-time trading framework, Haberman et al [18] and [19] investigate optimal investment strategy in DC plan by using dynamic programming techniques.

The main risk in DC pension plan for participants is the investment risk during accumulation phase in which their pension wealth have been built up. To moderate investment risk, a minimum guarantee is introduced as a lower bound for pension wealth that will be paid out to the participants in retirement. With respect to this, Bouiler et al [12] study the optimal management of a defined contribution plan with deterministic contribution and the guarantee in retirement depending on the level of stochastic interest rate which follows Vasicek [33] model. In their setup, the guarantee has a form that is annuity paid out from retirement time till the date of death which is also deterministic. Deelstra et al [15] investigates optimal investment problem for DC plan that allows for minimum guarantee with stochtastic contribution and assumes that interest rate dynamics are given as Duffie et al [17] in its one-dimensional version. It includes CIR [13] and Vasicek [33] models. Although authors make various assumptions, their aim is to provide the minimum guarantee. Appropriate investment strategies providing minimum guarantee are so-called portfolio insurance strategies.

Portfolio insurance is a trading strategy designed to protect portfolio value. A trading strategy allows for the guarantee of an acceptable terminal portfolio value in falling markets while taking a potential gain on the upwards market move. The Constant Proportion Portfolio Insurance (CPPI) strategy and the Option-Based Portfolio Insurance (OBPI) strategy are prominent examples of portfolio insurance strategies.

Bertnard and Prigent [5] compares the performances of CPPI and OBPI strategy when the volatility of the risky asset is stochastic. They also examine both strategies under first-order stochastic dominance criteria that is related to increasing utility functions [6]. Considering various stochastic dominance criteria up to third order Zagst and Kraus [34] compare the two portfolio insurance strategies. They conclude that CPPI strategy is likely to dominate OBPI strategy at third order. Péizer et al [26] show that CPPI strategy is superior to OBPI strategy under discrete trading and asset prices that might jump.

An outstanding example of the portfolio insurance strategies is Constant Proportion Portfolio Insurance (CPPI). The concept of CPPI strategy is introduced by Perold [25] for fixed-income instruments and by Black et al [8] for equity instruments. In CPPI strategy, the investor initially sets a floor which is the lowest acceptable portfolio value. The cushion is calculated as the excess value of current wealth over the floor. Cushion multiplied by pre-determined multiplier, defined as exposure, is allocated to risky asset. Remaining funds are invested in the riskless asset. The properties of CPPI strategies in continous time are studied by Black et al [9]. Assuming HARA utiliy funciton they show that CPPI strategy can maximize expected utility. This study is extended by Horsky [21] with the interest rate that follows Vasiček model and stock modeled by Heston process. Temoçin [32] studies CPPI strategy in DC pension fund with different floor assumption under continous time and discrete time trading. The gap risk and cash-lock risk which is the risk that portfolio wealth ends up fully invested in the risk-free asset without recovering are analyzed for discrete time trading.

In Black and Scholes framework which is the basis for most academic studies on CPPI, there is no gap risk due to the fact that probability of portfolio value above the floor at any time equals to one in CPPI settings. However, Balder et al [1], Cont and Tankov [14] , Paulot and Laucraze [23] show that portfolio value crashes through the floor in the incomplete market in which asset price jumps may occur or in which portfolio may only be rebalanced at a finite number of trading days. Balder et al [1] investigates CPPI strategy under trading restriction. The multiplier can not be changed between instant trading dates. So that, investor will not have an opportunity to rebalance the portfolio, which then crashes through the floor between trading dates on the downwards market move. Cont and Tankov [14] quantify the gap risk that comes from instantaneous price jumps analytically for CPPI strategy under continuous-time trading. Paulot and Laucraze [23] show that if the underlying has independent increments, the dynamics of the portfolio at trading dates is described by a discrete-time Markov process in a single variable. Extreme value approach is also used by Bernard et al [4] to estimate gap risk of CPPI strategy.

1.1 Aim of Thesis

The insufficiency of the CPPI strategy before the end of the termination of the policy (such as withdrawal and retirement) remains still as a risk. To reduce such risk the investor may improve the course of the asset growth by considering a financial tool, such as a put option. In other words, considering that CPPI strategy is applied in DC plan under discrete time trading setting, investors encounter the risk of the portfolio value hits the floor value since investors can not trade instantly against stock changes in the discrete trading scenario. If portfolio value drops below floor value, entire portfolio value is invested in the riskless asset. Hence, the potential gain of the risky asset is missed on the upwards market move and portfolio value may not meet the guarantee at the termination date.

For such case, we propose an exotic option to cope with this risk called ”gap risk”. When there is no arbitrage opportunity we can price this exotic option uniquely as an expectation of discounted claim with its boundary under the equivalent martingale measure in a complete market. However, trading restriction causes incomplete market. Due to that, there is a class of equivalent martingale measure to price the exotic option. Applying Schweizer[31]'s variance-optimal criterion we attain explicit price of that exotic option based on minimal martingale measure. That exotic option shall be henceforth called as cushion option. Through sensitivity analyses, we examine the effectiveness of exotic option on CPPI strategy in DC pension plan to deal with gap risk. We also analyse the optimal Strike price interval of the exotic option under predetermined market parameter.

The plan of this thesis is as follows. In Chapter 2, the basic concepts from finance and classical CPPI strategy are presented. Chapter 3 gives an overview of variance-optimal pricing methodologies. The results presented here are taken from research papers: [31],[28] and [24]. CPPI strategy for DC pension plan is presented in Chapter 4. The cushion option is also proposed and CPPI strategy for DC pension scheme under cushion option is derived. In Chapter 5, a graphic illustration of the evolution of wealth in pension fund with and without cushion option till given withdrawal time. The effectiveness of cushion insurance for CPPI strategy is also discussed. We conclude with a brief discussion of the observations and suggestions for further research in Chapter 6.

CHAPTER 2

Preliminaries

In this chapter, we introduce some basic concepts from finance, providing relevant definitions and theorems that are used in following chapters. We refer to [22], [27] and [7]. CPPI strategy is also introduced as described in [14].

2.1 Concepts from Finance

Definition 2.1. (Self-financing portfolio) A portfolio is called self-financing when all the changes in the portfolio are due to realized gains from investment. In a sense, no fund are borrowed or withdrawn from the portfolio at any time

$$V_t = V_0 + \sum_{i=1}^{t} (a_i \bigtriangleup S_i + b_i \bigtriangleup B_i) \tag{2.1}$$

where $t = 1, 2, .., T$ and trading strategy (a_t, b_t) denotes the number of shares of stock and bond units held during $[t-1, t)$, respectively.

Definition 2.2. (Admissible strategy) A strategy π is called H-admissible if $V_T^\pi = H$ $\mathbb{P}$-almost surely (as).

Definition 2.3. (Arbitrage opportunity) An arbitrage opportunity is an admissible trading strategy such that initial portfolio value $V_0 = 0$, but

$$E[V_T] > 0.$$

The market is said to be *arbitrage-free* if there are no arbitrage opportunities.

Definition 2.4. A claim X is called attainable if there exists self financing strategy π replicating the claim. In fact, V_t satisfies (2.1) , $V_t \geq 0$ and

$$V_T^\pi = X.$$

The market is called *complete* if every claim is attainable.

Definition 2.5. Q is equivalent to $\mathbb{P}$ if they have same null sets, i.e $Q(A) = 0 \Leftrightarrow \mathbb{P}(A) = 0$.

Definition 2.6. A probability measure Q on $(\Omega, \mathcal{F})$ equivalent to $\mathbb{P}$ is called a martingale measure for *S* if the process *S* follows a Q-martingale with respect to filtration $\mathcal{F}$.

Theorem 2.1. *The market is arbitrage-free if and only if there exists a probability measure Q equivalent to $\mathbb{P}$ under which the discounted d-dimensional asset price process S is a Q-martingale.*

Proposition 2.2. *The price process of any attainable claim X is given by*

$$\Pi_X(t) = \beta_t^{-1} E_Q[X\beta_T|\mathcal{F}_t] \qquad \forall t = 0, 1, .., T.$$

Definition 2.7. (Ito process) Let W_t, $t \geq 0$, be a Brownian motion, and let $\mathcal{F}_t$, $t \geq 0$, be an associated filtration. An Ito process is a stochastic process of the form

$$X_t = X_0 + \int_0^t \triangle(u) dW_u + \int_0^t \Theta(u) du, \tag{2.2}$$

where X_0 is constant and $\triangle(u)$ and $\Theta(u)$ are adapted stochastic processes.

Definition 2.8. (Ito formula) Let X_t, $t \geq 0$, be an Ito process as described in Definition 2.7, and let $f(t, x)$ be a function for which the partial derivatives $f_t(t, x)$, $f_x(t, x)$ and $f_{xx}(t, x)$ exist and are continuous. Then, for every $T \geq 0$,

$$f(T, X_T) = f(0, X_0) + \int_0^T f_t(t, X_t) dt + \int_0^T f_x(t, X_t) \triangle (t) dX_t$$
$$+ \int_0^T f_x(t, X_t)\Theta(t) dt + \frac{1}{2} \int_0^T f_{xx}(t, X_t) \triangle^2 (t) dX_t.$$

Theorem 2.3. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let Z be an almost surely positive random variable with $E[Z] = 1$. Define*

$$\mathbb{Q}(A) = E_{\mathbb{Q}}[\mathbb{I}_{\mathbb{A}} Z] = \int_A Z(w) d\mathbb{P}(w) \qquad \forall A \in \mathcal{F},$$

then $\mathbb{Q}$ is a probability measure.

Moreover, if X is a positive random variable, then

$$E_{\mathbb{Q}}[X] = E[XZ].$$

Definition 2.9. (Radon-Nikodym derivative) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $\mathbb{Q}$ be another probability measure on the space $(\Omega, \mathcal{F})$ which is equivalent to $\mathbb{P}$ and let Z be an almost surely nonnegative random variable as Theorem 2.3. Then Z is called the Radon-Nikodym derivative of $\mathbb{Q}$ with respect to $\mathbb{P}$, and is denoted as

$$Z = \frac{d\mathbb{Q}}{d\mathbb{P}}.$$

Suppose that Z satisfies Definition 2.9 with a filtration $\mathcal{F}_t$, defined for $0 \leq t \leq T$, where T is fixed final time. Then, we can define the Radon-Nikodym derivative process as

$$Z_t = \mathbb{E}[Z|\mathcal{F}_t], \qquad 0 \leq t \leq T.$$

Lemma 2.4. *Given $t, 0 \leq t \leq T$, let X be an $\mathcal{F}_t - measurable$ random variable. Then*

$$E_{\mathbb{Q}}[Y] = \mathbb{E}[YZ_t].$$

Theorem 2.5. *(Girsanov Theorem) Let W_t, $0 \leq t \leq T$, be a Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $\mathcal{F}_t$, $0 \leq t \leq T$, be a filtration for this Brownian motion.*

Let $\Theta(t)$, $0 \leq t \leq T$, be an adapted process. Define

$$Z_t = exp\left\{ -\int_0^t \Theta(u) dW_u - \frac{1}{2}\int_0^t \Theta^2(u) du \right\},$$

$$W_t^Q = W_t + \int_0^t \Theta(u) du$$

and assume that

$$\mathbb{E}\int_0^T \Theta^2(u) Z^2(u) du < \infty.$$

Set $Z = Z_T$, then $\mathbb{E}[Z] = 1$ and under the probability measure Q given by

$$Q(A) = \int_A Z(w) dP(w) \qquad \forall A \in \mathcal{F},$$

the process W_t^Q, $0 \leq t \leq T$, is a Brownian motion.

2.2 CPPI Strategy

The CPPI strategy is a self financing strategy whose aim is to take potential gain on upward market move while guaranteeing at least an specified fixed amount of money at maturity time T. In classic Black-Scholes market, where two basic assets are traded continuously in time during time horizon $[0, T]$, the fund manager invests into two assets: riskless asset (money market account), B_t, and risky asset (stock or stock index), S_t, with price dynamics given by

$$dB_t = rB_t dt, \quad B_0 = b, \tag{2.3}$$

$$\frac{dS_t}{S_t} = \mu_s d_t + \sigma_s dW_t, \quad S_0 = s, \tag{2.4}$$

where b, s, μ_s and σ_s are constant and W_t is a Brownian process defined on complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Here, $\mathbb{P}$ is real world probability measure and the filtration $\mathcal{F} = (\mathcal{F}_t)_{[t\geq 0]}$ represents the history of the Brownian motion up to time t.

The basic idea of CPPI approach is that terminal portfolio value, V_T, at the end of the investment horizon T stays above an investor-defined floor given as a percentage $\varphi \geq 0$ of initial value V_0, i.e.

$$F_T = \varphi_T V_0.$$

Since the market is arbitrage-free, it is impossible to find an investment that returns more than the risk-free rate of return, r, with no risk. Hence, the maximum guaranteed portfolio value at maturity time, T, is limited by

$$\varphi_T \leq e^{rT}.$$

The floor F_t, $0 \leq t \leq T$, denotes the present value of guarantee. By discounting with respect to deterministic interest rate r

$$F_t = \varphi_t V_0, \qquad \varphi_t = \varphi_T e^{-r(T-t)}.$$

Hence, the floor has following dynamics

$$dF_t = rF_t dt.$$

The surplus of current portfolio value V_t above the floor F_t is called cushion, denoted by C_t and its price at any time $t \in [0, T]$ is given as

$$C_t = max\{V_t - F_t, 0\}.$$

At any time t,

(1) if $V_t > F_t$, wealth invested into the risky asset called exposure is given by

$$e_t = mC_t = m(V_t - F_t),$$

where $m > 1$ is a constant multiplier and the remaining part of portfolio

$$R_t = V_t - e_t$$

is invested in the riskless asset.

(2) if $V_t \leq F_t$, the entire portfolio is invested into the riskless asset.

Proposition 2.6. *The t-value of the cushion C_t in time period $[0, T]$ is*

$$C_t = C_0 exp\{rt + m(\int_0^t \mu_S ds - rt) - \frac{1}{2}m^2 \int_0^t \sigma_S^2 d_S + m \int_0^t \sigma_S dW_S\}.$$

Proof. The value at time t of the CPPI portfolio is given as [6]

$$dV_t = e_t \frac{dS_t}{S_t} + (V_t - e_t)\frac{dB_t}{B_t}.$$

As we can see from definition of strategy, the cushion must satisfy the Black-Scholes stochastic differential equation

$$\frac{dC_t}{C_t} = d(V_t - F_t).$$
$$\frac{dC_t}{C_t} = (V_t - e_t)\frac{dB_t}{B_t} + e_t\frac{dS_t}{S_t} - \frac{dF_t}{F_t}$$
$$\frac{dC_t}{C_t} = (C_t + F_t - mC_t)\frac{dB_t}{B_t} + mC_t\frac{dS_t}{S_t} - \frac{dF_t}{F_t}$$
$$\frac{dC_t}{C_t} = (C_t - mC_t)\frac{dB_t}{B_t} + mC_t\frac{dS_t}{S_t}$$
$$\frac{dC_t}{C_t} = (m\mu_S + (1-m)r)dt + m\sigma_S dW_t$$

Thus, the process C_t is the $Doléans - Dade$ exponential of the process Z_t defined by

$$Z_t = \left(rt + m(\int_0^t \mu_S ds - rt) + m \int_0^t \sigma_S dW_S \right).$$

By using Ito formula, we obtain

$$C_t = C_0 exp\{rt + m(\int_0^t \mu_S ds - rt) - \frac{1}{2}m^2 \int_0^t \sigma_S^2 d_S + m \int_0^t \sigma_S dW_S\}.$$

$\square$

CHAPTER 3

Option Valuation at Fixed Trading Dates

The contingent claim is perfectly replicated in well-known Black Scholes model under the assumption that market is complete and arbitrage free. The claim is priced by taking expectation of discounted contingent claim under equivalent martingale measure which is unique in Black Scholes market. However, the finite number of trading dates causes incomplete market. The price or hedging strategy of contingent claim is not unique in incomplete market in the sense that there is a set of equivalent martingale measure. According to different optimality criteria, choice of an equivalent martingale measure called 'pricing measure' differs. An outstanding approach is Schweizer [31] (or Schäl[28])'s variance-optimal criteration among the existing theories for pricing and hedging in incomplete markets. The idea of this approach is to seek an initial capital and a strategy which minimize the variance of hedging cost related to discretization of the hedging strategy for a given claim. We refer to [28], [29], [30] and [31] for the exhaustive explanation of this theory.

3.1 Basic Definitions and Assumptions

We consider that risky asset (stock) and riskless asset (bond) are traded in financial market. Given finite time horizon $T \in \mathbb{N}$, the set of possible trading dates is represented by $\tau = \{t_0 = 0 < t_1 < ... < t_{M-1} < t_M = T\}$ where $M \in \mathbb{N}$ is a fixed natural number and a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\mathcal{F}_t(t = t_0, t_1, ..., t_{M-1}, t_M)$. The discounted price process S of risky asset is assummed to be $\mathcal{F}_t$-adapted square-integrable with respect to the objective measure $\mathbb{P}$:

$$S_t \in \mathcal{L}^2(\Omega, \mathcal{F}_t, \mathbb{P}) \qquad \forall\, t,$$

and we define

$$\Delta S_{t_k} = S_{t_k} - S_{t_{k-1}} \qquad \text{with} \quad S_{t_{-1}} = 0 \quad \text{for} \quad k = 0, ..., M-1.$$

We set money market account process equals to one. This setting does not cause any loss of generality. The results obtained still hold after a change of numeraire [20]. In order to change a numeraire, all the prices in the formula can be substituted by their discounted values.

Definition 3.1. A trading strategy π is a pair of two process (η, ξ) such that $\eta = (\eta_t)_{t\in\tau}$ is adapted, and such that $\xi = (\xi_t)_{t\in\tau}$ is a predictable process.

It satisfies the following:

$(i)\ \xi_{t_0} = 0; \xi_{t_k}\Delta S_{t_k} \in \mathcal{L}^2$

$(ii)\xi_{t_k}S_{t_k} + \eta_{t_k} \in \mathcal{L}^2$

ξ_{t_k} is interpreted as the number of share held during the time interval $(t_{k-1}, t_k]$ while η_{t_k} denotes the amount invested in bond during time interval $(t_k, t_{k+1}]$.

The value process V of $\pi(\eta, \xi)$ is defined as

$$V_{t_0} = \eta_{t_0} \qquad \text{and} \qquad V_{t_k} := \eta_{t_k} + \xi_{t_k}X_{t_k} \qquad \text{for} \quad k = 0, 1, ..., M.$$

The gain process associated with a trading strategy π :

$$G_{t_k}(\pi) = \Sigma_{j=1}^{k}\xi_{t_j}\Delta S_{t_j} \qquad \text{for} \quad k = 0, 1, ..., M$$

where $G_{t_k}(\pi) = 0$.

Furthermore, we define the cost process associated with a trading strategy π :

$$C_{t_k}(\pi) = V_{t_k}(\pi) - G_{t_k}(\pi) = V_{t_k}(\pi) - \Sigma_{j=1}^{k}\xi_{t_j}\Delta S_{t_j} \qquad \text{for} \quad k = 0, 1, ..., M.$$

Let $H \in \mathcal{L}^2$ be a square integrable contingent claim representing payoff of option at maturity time. We can not find a self financing strategy replicating the claim, H, perfectly by definition of incomplete market. The aim is to find an initial endowment $x \in \mathbb{R}$ and self financing strategy π, such that $V_T(\pi) = H$ $\mathbb{P}$-as,
satisfying

$$min\mathbb{E}[(H - x - G_T(\pi))^2] \qquad \text{over all } \ x \in \mathbb{R} \ \text{ and } \ \xi \in \pi. \tag{3.1}$$

Definition 3.2. Given a trading strategy π we define [24] :

(i) The process conditional remaining risk $R(\pi)$ by:

$$R_{t_k}(\pi) := \mathbb{E}\left[(C_{t_M}(\pi) - C_{t_k}(\pi))^2 | \mathcal{F}_{t_k}\right] \qquad k = 0, ..., M-1$$

(ii) The process local risk $r(\pi)$ by:

$$r_{t_k}(\pi) := \mathbb{E}\left[(C_{t_{k+1}}(\pi) - C_{t_k}(\pi))^2 | \mathcal{F}_{t_k}\right] \qquad k = 1, ..., M-1$$

$$= \mathbb{E}\left[(V_{t_{k+1}}(\pi) - \xi_{t_k}\Delta S_{t_k} - V_{t_k}(\pi)^2 | \mathcal{F}_{t_k}\right]$$

For the purpose of minimizing hedging error, we consider following problems related to claim H [24] :

(i) Minimize the local risk $r_{t_k}(\pi)$ by choosing $V_{t_k}(\pi)$ and $\xi_{t_{k+1}}$ for each $k = 0, ..., M-1$,

(ii) Minimize the remaining risk $R_{t_k}(\pi)$ for each $k = 0, ..., M-1$, over all π, such that $V_T(\pi) = H$ $\mathbb{P}$-as,

(iii) (a) Minimize $\mathbb{E}[(H - x - G_T(\pi))^2]$ over all $\xi \in \pi$ with x a given real number

(iii) (b) Minimize $\mathbb{E}[(H - x - G_T(\pi))^2]$ over all $\xi \in \pi$ and $x \in \mathbb{R}$

To solve local risk minimizing problem, set $V_M = H$ as a starting point and choose $\xi_{t_{k+1}}$ and V_{t_k}, recursively for $k = 0, 1, ..., M-1$ to minimize local risk $r_{t_k}(\pi)$. Then the admissible trading strategy $\pi^L(\xi^L, \eta^L)$ is given by

$$\xi^L_{t_k} = \frac{Cov\left[H - \sum_{j=k+1}^{M} \xi^L_{t_j}\Delta S_{t_j}, \Delta S_{t_k} | \mathcal{F}_{t_{k-1}}\right]}{Var\left[\Delta S_{t_k} / \mathcal{F}_{t_{k-1}}\right]} \tag{3.2}$$

and

$$\eta^L_{t_k} = \mathbb{E}\left[H - \sum_{j=k+1}^{M} \xi^L_{t_j}\Delta S_{t_j} | \mathcal{F}_{t_{k-1}}\right] - \xi^L_{t_k} S_{t_k}. \tag{3.3}$$

The problem of minimizing the remaining risk has no obtainable solution in general settings. If $M = 1$, then this problem resembles the local risk minimizing problem and the solution exists. The problem (iii)(a) and (iii)(b) have a solution if S has a bounded mean-variance trade off defined by [31]

$$\frac{\left(\mathbb{E}\left[\Delta S_{t_k} \mid \mathcal{F}_{t_{k-1}}\right]\right)^2}{Var\left[\Delta S_{t_k} \mid \mathcal{F}_{t_{k-1}}\right]}. \tag{3.4}$$

In the case that mean-variance trade off is deterministic, the explicit solutions of the problem (iii)(a) and (iii)(b) are given by ([28] or [31])

$$\xi_{t_k} = \xi_{t_k}^{(0,\hat{x}_{t_0})} := \xi_{t_k}^L + \alpha_{t_k}\left(\hat{\mathbb{E}}(H|\mathcal{F}_{t_{k-1}}) - x - \sum_{j=1}^{k-1}\xi_{t_j}^L \Delta S_{t_j}\right) \qquad k = 1, ..., M \tag{3.5}$$

and

$$\begin{aligned} &\xi_k = \xi_k^{(0,\hat{x}_{t_0})} \qquad k = 1, ..., M, \\ &x = \hat{x}_{t_0} := \hat{\mathbb{E}}(H), \end{aligned} \tag{3.6}$$

respectively. Here, $\hat{\mathbb{E}}$ is the expectation under the minimal martingale measure $\hat{\mathbb{P}}$ defined by

$$\frac{d\hat{\mathbb{P}}}{d\mathbb{P}} := \hat{Z}_M = \prod_{j=1}^{k} \frac{1 - \alpha_{t_j}\Delta S_{t_j}}{1 - \alpha_{t_j}\mathbb{E}\left[\Delta S_{t_j} \mid \mathcal{F}_{t_{j-1}}\right]} \tag{3.7}$$

where

$$\alpha_{t_k} = \frac{\mathbb{E}\left[\Delta S_{t_k} \mid \mathcal{F}_{t_{k-1}}\right]}{\mathbb{E}\left[\Delta S_{t_k}^2 \mid \mathcal{F}_{t_{k-1}}\right]}.$$

Let discounted price S_t of the stock be described by the geometric Brownian motion

$$S_t = Se^{(\mu - \frac{\sigma^2}{2})t + \sigma W_t}, \qquad S_0 = S \tag{3.8}$$

and the observed Stock price follows

$$S_{t_k} = S_{t_{k-1}} e^{(\mu - \frac{\sigma^2}{2})\triangle t_k + \sigma \triangle W_{t_k}}, \qquad S_0 = S \qquad k = 1, .., M \tag{3.9}$$

where

$\triangle t_k = t_k - t_{k-1}$ and $\triangle W_{t_k} = W_{t_k} - W_{t_{k-1}}$.

Proposition 3.1. *The explicit formula for the fair value of the claim is [24]*

$$\hat{x} = \hat{\mathbb{E}}\left[h(S_{t_1}, ..., S_{t_M})\right]$$

$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h(S_{t_1}, ..., S_{t_M}) \prod_{i=1}^{k} A(x_i) f(x_i) dx_1 \cdots dx_M \tag{3.10}$$

where $\forall i = 1, ..., M$

$$h(S_{t_1}, ..., S_{t_M}) := h(Se^{(\mu - \frac{\sigma^2}{2})\Delta t_1 + \sigma x_1 \sqrt{\Delta t_1}}, ..., Se^{(\mu - \frac{\sigma^2}{2})T + \sigma \sum_{i=1}^{M} x_i \sqrt{\Delta t_i}})$$

$$A(x_i) := \frac{(1 - e^{\mu \Delta t_i}) e^{-\frac{\sigma^2}{2}\Delta t_i + \sigma x_i \sqrt{\Delta t_i}} + e^{(\mu + \sigma^2)\Delta t_i} - 1}{e^{\mu \Delta t_i}\left(e^{\sigma^2 \Delta t_i - 1}\right)}$$

and

$$f(x_i) := \frac{1}{\sqrt{2\pi}} e^{-\frac{x_i^2}{2}}.$$

CHAPTER 4

CPPI Strategy in DC Pension Plan under Discrete Time Trading Setting

In this chapter, we use the similar discrete time setting as suggested in [32]. Let τ be set of equidistant trading dates along time period $[0,T]$,i.e.

$\tau = \{t_0 = 0 < t_1 < ... < t_{n-1} < t_n = T\}$,

where t_0 is the inception date of DC pension plan, T is retirement date and

$t_{k+1} - t_k = T/n := \triangle t$ for $k = 0, 1, .., n-1$.

4.1 Market Model

For simplicity, we consider two assets in the financial market for investing pension contributions: risky asset (stock or stock index) and riskless asset (money market account). Money market account, denoted by B_t, grows with constant interest rate r and its price dynamic is given in equation 2.3

$$dB_t = rB_t dt, \qquad B_0 = b,$$

where b is constant.

Price dynamics of risky asset, denoted by S_t, is defined as in equation 2.4

$$dS_t = S_t(\mu_s d_t + \sigma_s dW_t), \qquad S_0 = s,$$

where s, μ_s and σ_s are positive constants and W_t is a Brownian process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Here, $\mathbb{P}$ is real world probability measure and the filtration $\mathcal{F} = (\mathcal{F}_t)_{[0 \leq t \leq T]}$ represents the history of the Brownian motion up to time t.

Since trading takes place at equidistant trading dates, the observed stock price follows as

$$S_{t_{k+1}} = S_{t_k} e^{(\mu_S - \frac{\sigma_S^2}{2})\triangle t + \sigma_S \triangle W_{t_k}} \qquad k = 0, ..., n-1$$

where $S_{t_0} = s$ and $\triangle W_{t_k} = W_{t_{k+1}} - W_{t_k}$ for $k = 0, ..., n-1$.

4.2 Defined Contribution Fund Modeling

DC fund modeling has two key aspects that there is no consumption before retiremet date and labor income plays a central role in the wealth accumulated phase. Labor income is hard to model realistically due to its stochastic components such as financial and political crisis, disability and mortality. Here, labor income, denoted by L_t, is assumed to have stochastic process reflecting the risk of financial market.

The participants contribute into the pension fund every equidistant trading dates as a certain proportion, γ , of labor income. Here, we define the labor income dynamics as a stochastic differential equation as follows

$$dL_t = L_t(\mu_L dt + \sigma_L dW_t) \ , \quad L_0 = l \tag{4.1}$$

with constant l, μ_L and σ_L and W_t is a Brownian process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ as described in Section 4.1.

The observed labor income becomes

$$L_{t_{k+1}} = L_{t_k} e^{(\mu_L - \frac{\sigma_L^2}{2})\triangle t + \sigma_L \triangle W_{t_k}} \qquad k = 0, ..., n-1 \tag{4.2}$$

where $L_{t_0} = l$ and $\triangle W_{t_k} = W_{t_{k+1}} - W_{t_k}$ for $k = 0, ..., n-1$.

Therefore, defined contribution with respect to labor process and the value of proportion, γ, is defined as

$$\gamma_{t_k} = \gamma L_{t_k} \tag{4.3}$$

for all $k = 0, 1, .., n$ with dynamics

$$d\gamma_t = \gamma dL_t. \tag{4.4}$$

4.2.1 Replicating Strategy

We assume that there is a portfolio which has same dynamics as labor income so labor income can be replicated perfectly between equidistant trading dates by using assets in the market. In order to prove this, we need to show the relevant replicating portfolio.

When we consider self financing replicating strategy $\pi = (\pi_B, \pi_S, \pi_L)$ and choose $\pi_L = -1$, we set the fund

$$dV_t^{\pi} = \pi_B dB_t + \pi_S dS_t - dL_t \tag{4.5}$$

$$dV_t^{\pi} = \pi_B (rB_t dt) + \pi_S (S_t \mu_s dt + S_t \sigma_s dW_t) - L_t(\mu_L dt + \sigma_L dW_t) \tag{4.6}$$

$$dV_t^{\pi} = (\pi_B r B_t + \pi_S S_t \mu_s - L_t \mu_L)dt + (\pi_S S_t \sigma_s - L_t \sigma_L) dW_t \tag{4.7}$$

Equating the diffusion terms and the drift terms to zero, we obtain the number of assets as

$$\pi_S(t) = \frac{L\sigma_L}{S\sigma_S} \qquad \text{and} \qquad \pi_B(t) = \frac{L}{rB\sigma_S}(\sigma_S \mu_L - \sigma_L \mu_S).$$

4.3 CPPI strategy with random-growth floor

We consider CPPI strategy with random-growth floor on DC pension plan. Unlike the floor in classical CPPI strategy, floor evolves not only with interest rate but also with the portion of each contribution. In such case, floor has stochastic process due to the stochastic labor income process.

Floor at time t is defined as [32]

$$F_t = \begin{cases} \sum_{i=0}^{k} e^{r(t-t_i)} c\gamma_{t_i}, & t\epsilon(t_k, t_{k+1}) \qquad k = 0, 1, ..., n-1 \\ \\ F^{-}_{t_{k+1}} + c\gamma_{t_{k+1}}, & t = t_{k+1} \end{cases} \tag{4.8}$$

where $0 < c < 1$ is constant and $F_0 = c\gamma_{t_0}$.

The wealth process for all $k = 0, 1, .., n-1$ is given as [32]

$$V_{t_{k+1}} = \begin{cases} (V_{t_k} - mC_{t_k})e^{r\triangle t} + mC_{t_k}\frac{S_{t_{k+1}}}{S_{t_k}} + \gamma(t_{k+1}), & C_{t_k} > 0 \\ V_{t_k} e^{r\triangle t} + \gamma(t_{k+1}), & C_{t_k} \leq 0 \end{cases} \tag{4.9}$$

and for $t\epsilon(t_k, t_{k+1})$

$$V_t = \begin{cases} (V_{t_k} - mC_{t_k})\frac{B_t}{B_{t_k}} + mC_{t_k}\frac{S_t}{S_{t_k}}, & C_{t_k} > 0 \\ V_{t_k}\frac{B_t}{B_{t_k}}, & C_{t_k} \leq 0 \end{cases} \tag{4.10}$$

As cushion is defined by $C_t = V_t - F_t$ for all *t*, the cushion process can be written as [32]

$$C_{t_{k+1}} = \begin{cases} C_{t_k}(m\frac{S_{t_{k+1}}}{S_{t_k}} + (1-m)e^{r\triangle t}) + (1-c)\gamma(t_{k+1}), & C_{t_k} > 0 \\ C_{t_k}e^{r\triangle t} + (1-c)\gamma(t_{k+1}), & C_{t_k} \leq 0 \end{cases} \tag{4.11}$$

for all $k = 0, 1.., n-1$ and

$$C_t = \begin{cases} C_{t_k}(m\frac{S_t}{S_{t_k}} + (1-m)\frac{B_t}{B_{t_k}}), & C_{t_k} > 0 \\ C_{t_k}\frac{B_t}{B_{t_k}}, & C_{t_k} \leq 0 \end{cases}$$

for $t\epsilon(t_k, t_{k+1})$.

After describing CPPI-DC design we introduce the implementation of a cushion insurance by employing an exotic option.

4.3.1 Cushion option

There is a risk of cushion becoming negative between trading dates due to the fact that rebalancements are done only at trading dates. Considering cushion dynamics we can define probability of gap risk between trading dates just before contribution payment as follows:

$$P_{gap} = P\left(C^-_{t_{k+1}} < 0 \mid C_{t_k} > 0\right)$$

$$= P\left(C_{t_k}(m\frac{S_t}{S_{t_k}} + (1-m)e^{r\triangle t}) < 0 \mid C_{t_k} > 0\right)$$

$$= P\left(\frac{S_t}{S_{t_k}} < \frac{(m-1)}{m}e^{r\triangle t}\right).$$

For the purpose of dealing with gap risk, we propose an exotic option with payoff defined by

$$(K - (1-c)\gamma L_{t_{k+1}})^+ \mathbb{I}_{\left\{\frac{S_{t_{k+1}}}{S_{t_k}} < (1-\frac{1}{m})e^{r\triangle t}\right\}}.$$

We name the exotic option as ”cushion option” from now on, in regard to structure of our setup. The main aim of this custom-made option is to generate additional gains in case of a sudden decrease in the portfolio value. As it will be illustrated in Section 5.1, the option will especially benefit when the cushion becomes negative and will function as a guarantee in the event of an early withdrawal. Here, the strike price K represents the fixed part of the payoff and is determined by the seller. Since this is a theoretical derivative, our primary assumption is that there are parties in the market who sell these options with various strike prices. Then, based on the position of the pension fund the beneficiary can select and buy the relevant option. To be able provide an intuitive idea on the selection of K, we conduct a sensitivity analysis and present the results in Section 5.1.

In order to obtain fair value of the cushion option, we refer to Proposition 3.1.

Recall that

$$S_{t_{k+1}} = S_{t_k} exp\left((\mu_S - \frac{\sigma_S^2}{2})\triangle t + \sigma_S (W_{t_{k+1}} - W_{t_k}) \right)$$

$$(W_{t_{k+1}} - W_{t_k}) = \frac{ln\frac{S_{t_{k+1}}}{S_{t_k}} - (\mu_S - \frac{\sigma_S^2}{2})\triangle t}{\sigma_S} \tag{4.12}$$

Substituting $(W_{t_{k+1}} - W_{t_k})$ into the expression for $L_{t_{k+1}}$ leads to:

$$L_{t_{k+1}} = L_{t_k} exp\left((\mu_L - \frac{\sigma_L^2}{2})\triangle t + \sigma_L \frac{ln\frac{S_{t_{k+1}}}{S_{t_k}} - (\mu_S - \frac{\sigma_S^2}{2})\triangle t}{\sigma_S} \right)$$

$$L_{t_{k+1}} = L_{t_k} \left(\frac{S_{t_{k+1}}}{S_{t_k}} \right)^{\frac{\sigma_L}{\sigma_S}} exp\left((\mu_L - \frac{\sigma_L^2}{2} - \frac{\mu_S \sigma_L}{\sigma_S} + \frac{\sigma_L \sigma_S}{2})\triangle t \right) \tag{4.13}$$

$$(K - (1-c)\gamma L_{t_{k+1}})^+ \mathbb{I}_{\left\{ \frac{S_{t_{k+1}}}{S_{t_k}} < (1-\frac{1}{m})e^{r\triangle t} \right\}} =$$

$$\left(K - (1-c)\gamma L_{t_k} \left(\frac{S_{t_{k+1}}}{S_{t_k}} \right)^{\frac{\sigma_L}{\sigma_S}} exp\left((\mu_L - \frac{\sigma_L^2}{2} - \frac{\mu_S \sigma_L}{\sigma_S} + \frac{\sigma_L \sigma_S}{2})\triangle t \right) \right)^+ \mathbb{I}_{\left\{ \frac{S_{t_{k+1}}}{S_{t_k}} < (1-\frac{1}{m})e^{r\triangle t} \right\}}$$

$$(K - (1-c)\gamma L_{t_{k+1}})^+ \mathbb{I}_{\{ \frac{S_{t_{k+1}}}{S_{t_k}} < (1-\frac{1}{m})e^{r\triangle t} \}} = \zeta (K^* - S_{t_{k+1}}^{\frac{\sigma_L}{\sigma_S}})^+ \mathbb{I}_{\{ \frac{S_{t_{k+1}}}{S_{t_k}} < (1-\frac{1}{m})e^{r\triangle t} \}} \tag{4.14}$$

where

$$\zeta = \frac{(1-c)\gamma L_{t_k}}{S_{t_k}^{\frac{\sigma_L}{\sigma_S}}} exp\left(\left(\mu_L - \frac{\sigma_L^2}{2} - \frac{\mu_S \sigma_L}{\sigma_S} + \frac{\sigma_L \sigma_S}{2} \right) \triangle t \right)$$

and

$$K^* = \frac{K}{\zeta}.$$

Proposition 4.1. *The cushion option under CPPI-DC design with payoff defined by*

$$(K-(1-c)\gamma L_{t_{k+1}})^{+}\mathbb{I}_{\left\{\frac{S_{t_{k+1}}}{S_{t_k}}<(1-\frac{1}{m})e^{r\triangle t}\right\}}$$

can be priced, P_{t_k} *as*

$$P_{t_k}=\int_{-\infty}^{\infty}A(z)\zeta(K^{*}-S_{t_{k+1}}^{\frac{\sigma_L}{\sigma_S}})^{+}\mathbb{I}_{\left\{\frac{S_{t_{k+1}}}{S_{t_k}}<(1-\frac{1}{m})e^{r\triangle t}\right\}}\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz. \tag{4.15}$$

where

$$A(z):=\frac{(1-e^{\mu_S\triangle t})e^{-\frac{\sigma_S^2}{2}\triangle t+\sigma_S z\sqrt{\triangle t}}+e^{(\mu_S+\sigma_S^2)\triangle t}-1}{e^{\mu_S\triangle t}(e^{\sigma_S^2\triangle t}-1)}$$

Proof. see Proposition 3.1. □

Proposition 4.2. *The explicit price of cushion option is derived as*

$$P_{t_k}=\lambda\left[K^{*}\Phi\left(min(A,B)-\sigma_S\sqrt{\triangle t}\right)-S_{t_k}^{\frac{\sigma_L}{\sigma_S}}e^{\beta\triangle t}\Phi\left(min(A,B)-(\sigma_S+\sigma_L)\sqrt{\triangle t}\right)\right]$$

$$+\psi\left[K^{*}\Phi\left(min(A,B)\right)-S_{t_k}^{\frac{\sigma_L}{\sigma_S}}e^{(\beta-\frac{\sigma_L\sigma_S}{2})\triangle t}\Phi\left(min(A,B)-\sigma_L\sqrt{\triangle t}\right)\right] \tag{4.16}$$

where

$$\beta=\frac{\mu_S\sigma_L}{\sigma_S}+\frac{\sigma_L^2}{2}+\frac{\sigma_L\sigma_S}{2},$$

$$\lambda=\zeta\frac{(1-e^{\mu_S\triangle t})}{e^{\mu_S\triangle t}(e^{\sigma_S^2\triangle t}-1)},$$

$$\psi=\zeta\frac{e^{(\mu_S+\sigma_S^2)\triangle t}-1}{e^{\mu_S\triangle t}(e^{\sigma_S^2\triangle t}-1)},$$

$$A:=\frac{ln\left(\frac{K^{*}}{S_{t_k}^{\frac{\sigma_L}{\sigma_S}}}\right)-\frac{\sigma_L}{\sigma_S}(\mu_S-\frac{\sigma_S^2}{2})\triangle t}{\sigma_L\sqrt{\triangle t}},$$

$$B := \frac{ln(\frac{m-1}{m}) + (r - \mu_S + \sigma_S^2/2)\triangle t}{\sigma_S\sqrt{\triangle t}}$$

and $\Phi(.)$ denotes the cumulative distribution function of the standard normal distribution.

Proof. In order to get payoff, two conditions (A and B) need to be satisfied :

<u>Condition A :</u>

$$K^* - S_{t_{k+1}}^{\frac{\sigma_L}{\sigma_S}} > 0$$

$$K^* > S_{t_k}^{\frac{\sigma_L}{\sigma_S}} e^{\frac{\sigma_L}{\sigma_S}(\mu_S - \sigma_S^2/2)\triangle t + \sigma_L(W_{t_{k+1}} - W_{t_k})}$$

$$ln\left(\frac{K^*}{S_{t_k}^{\frac{\sigma_L}{\sigma_S}}}\right) > \frac{\sigma_L}{\sigma_S}(\mu_S - \frac{\sigma_S^2}{2})\triangle t + \sigma_L(W_{t_{k+1}} - W_{t_k})$$

$$\frac{Z_{\triangle t}}{\sqrt{\triangle t}} < \frac{ln\left(\frac{K^*}{S_{t_k}^{\frac{\sigma_L}{\sigma_S}}}\right) - \frac{\sigma_L}{\sigma_S}(\mu_S - \frac{\sigma_S^2}{2})\triangle t}{\sigma_L\sqrt{\triangle t}} := A \tag{4.17}$$

where $Z_{\triangle t} = W_{t_{k+1}} - W_{t_k}$ and $\frac{Z_{\triangle t}}{\sqrt{(\triangle t)}} = z \sim N(0,1)$.

<u>Condition B :</u>

$$\frac{S_{t_{k+1}}}{S_{t_k}} < (1 - \frac{1}{m})e^{r\triangle t}$$

$$\frac{S_{t_k} e^{(\mu_S - \sigma_S^2/2)\triangle t + \sigma_S(W_{t_{k+1}} - W_{t_k})}}{S_{t_k}} < (1 - \frac{1}{m})e^{r\triangle t}$$

$$(\mu_S - \sigma_S^2/2)\triangle t + \sigma_S(W_{t_{k+1}} - W_{t_k}) < ln(\frac{m-1}{m}) + r\triangle t$$

$$\frac{Z_{\triangle t}}{\sqrt{\triangle t}} < \frac{ln(\frac{m-1}{m}) + (r - \mu_S + \sigma_S^2/2)\triangle t}{\sigma_S\sqrt{\triangle t}} := B \tag{4.18}$$

where $Z_{\triangle t} = W_{t_{k+1}} - W_{t_k}$ and $\frac{Z_{\triangle t}}{\sqrt{(\triangle t)}} = z \sim N(0,1)$.

Under these conditions :

$$P_{t_k} = \int_{-\infty}^{B} \frac{(1-e^{\mu_S\triangle t})e^{-\frac{\sigma_S^2}{2}\triangle t+\sigma_S z\sqrt{\triangle t}} + e^{(\mu_S+\sigma_S^2)\triangle t} - 1}{e^{\mu_S\triangle t}(e^{\sigma_S^2\triangle t}-1)}\zeta(K^* - S_{t_{k+1}}^{\frac{\sigma_L}{\sigma_S}})^+ \frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz$$

$$= \int_{-\infty}^{min(A,B)} \frac{(1-e^{\mu_S\triangle t})e^{-\frac{\sigma_S^2}{2}\triangle t+\sigma_S z\sqrt{\triangle t}} + e^{(\mu_S+\sigma_S^2)\triangle t} - 1}{e^{\mu_S\triangle t}(e^{\sigma_S^2\triangle t}-1)}\zeta(K^* - S_{t_{k+1}}^{\frac{\sigma_L}{\sigma_S}}) \frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz$$

Right hand side of the equation can be decomposed into two part as follows:

$$P_{t_k} = \int_{-\infty}^{min(A,B)} \frac{(1-e^{\mu_S\triangle t})e^{-\frac{\sigma_S^2}{2}\triangle t+\sigma_S z\sqrt{\triangle t}}}{e^{\mu_S\triangle t}(e^{\sigma_S^2\triangle t}-1)}\zeta(K^* - S_{t_{k+1}}^{\frac{\sigma_L}{\sigma_S}}) \frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz \tag{4.19}$$

$$+ \int_{-\infty}^{min(A,B)} \frac{e^{(\mu_S+\sigma_S^2)\triangle t} - 1}{e^{\mu_S\triangle t}(e^{\sigma_S^2\triangle t}-1)}\zeta(K^* - S_{t_{k+1}}^{\frac{\sigma_L}{\sigma_S}}) \frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz \tag{4.20}$$

$$(4.19) = \frac{(1-e^{\mu_S\triangle t})\zeta}{e^{\mu_S\triangle t}(e^{\sigma_S^2\triangle t}-1)} \int_{-\infty}^{min(A,B)} (K^* - S_{t_{k+1}}^{\frac{\sigma_L}{\sigma_S}}) \frac{1}{\sqrt{2\pi}}e^{-\frac{\sigma_S^2}{2}\triangle t+\sigma_S z\sqrt{\triangle t}-\frac{z^2}{2}}dz$$

$$= \frac{(1-e^{\mu_S\triangle t})\zeta}{e^{\mu_S\triangle t}(e^{\sigma_S^2\triangle t}-1)} \int_{-\infty}^{min(A,B)} K^* \frac{1}{\sqrt{2\pi}}e^{-\frac{(z-\sigma_S\sqrt{\triangle t})^2}{2}}dz$$

$$-\frac{(1-e^{\mu_S\triangle t})\zeta}{e^{\mu_S\triangle t}(e^{\sigma_S^2\triangle t}-1)} \int_{-\infty}^{min(A,B)} S_{t_k}^{\frac{\sigma_L}{\sigma_S}} \frac{1}{\sqrt{2\pi}}e^{\frac{\sigma_L}{\sigma_S}(\mu_S-\frac{\sigma_S^2}{2})\triangle t+\sigma_L z\sqrt{\triangle t}-\frac{\sigma_S^2}{2}\triangle t+\sigma_S z\sqrt{\triangle t}-\frac{z^2}{2}}dz$$

$$= \lambda K^* \int_{-\infty}^{min(A,B)} \frac{1}{\sqrt{2\pi}} e^{-\frac{(z-\sigma_S\sqrt{\triangle t})^2}{2}} dz$$

$$- \lambda S_{t_k}^{\frac{\sigma_L}{\sigma_S}} e^{(\frac{\sigma_L \mu_S}{\sigma_S} - \frac{\sigma_L \sigma_S}{2} - \frac{\sigma_S^2}{2})\triangle t} \int_{-\infty}^{min(A,B)} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2} + (\sigma_S+\sigma_L) z \sqrt{\triangle t}} dz$$

where

$$\lambda = \frac{(1 - e^{\mu_S \triangle t})\zeta}{e^{\mu_S \triangle t}(e^{\sigma_S^2 \triangle t} - 1)}.$$

By change of variable $y = z - \sigma_S\sqrt{\triangle t}$, adding and substracting $\frac{(\sigma_S+\sigma_L)^2}{2}\triangle t$, we deduce

$$= \lambda K^* \Phi\left(min(A, B) - \sigma_S\sqrt{\triangle t}\right)$$

$$-\lambda S_{t_k}^{\frac{\sigma_L}{\sigma_S}} e^{(\frac{\sigma_L \mu_S}{\sigma_S} - \frac{\sigma_L \sigma_S}{2} - \frac{\sigma_S^2}{2} + \frac{(\sigma_S+\sigma_L)^2}{2})\triangle t} \int_{-\infty}^{min(A,B)} \frac{1}{\sqrt{2\pi}} e^{-\frac{(z-(\sigma_S+\sigma_L)\sqrt{\triangle t})^2}{2}} dz$$

By change of variable $h = z - (\sigma_S + \sigma_L)\sqrt{\triangle t}$,

$$= \lambda K^* \Phi\left(min(A, B) - \sigma_S\sqrt{\triangle t}\right) - \lambda S_{t_k}^{\frac{\sigma_L}{\sigma_S}} e^{\beta \triangle t} \Phi\left(min(A, B) - (\sigma_S + \sigma_L)\sqrt{\triangle t}\right)$$

where

$$\beta = \frac{\mu_S \sigma_L}{\sigma_S} + \frac{\sigma_L^2}{2} + \frac{\sigma_L \sigma_S}{2},$$

$$(4.20) = \frac{e^{(\mu_S+\sigma_S^2)\triangle t} - 1}{e^{\mu_S \triangle t}(e^{\sigma_S^2 \triangle t} - 1)} \int_{-\infty}^{min(A,B)} \zeta(K^* - S_{t_{k+1}}^{\frac{\sigma_L}{\sigma_S}}) \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz$$

$$= \frac{e^{(\mu_S+\sigma_S^2)\triangle t} - 1}{e^{\mu_S \triangle t}(e^{\sigma_S^2 \triangle t} - 1)} \zeta \int_{-\infty}^{min(A,B)} K^* \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz$$

$$- \frac{e^{(\mu_S+\sigma_S^2)\triangle t}-1}{e^{\mu_S \triangle t}(e^{\sigma_S^2 \triangle t}-1)}\zeta \int_{-\infty}^{min(A,B)} S_{t_{k+1}}^{\frac{\sigma_L}{\sigma_S}} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz$$

$$= \psi K^* \int_{-\infty}^{min(A,B)} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz - \psi \int_{-\infty}^{min(A,B)} S_{t_k}^{\frac{\sigma_L}{\sigma_S}} \frac{1}{\sqrt{2\pi}} e^{\frac{\sigma_L}{\sigma_S}(\mu_S - \frac{\sigma_S^2}{2})\triangle t + \sigma_L z \sqrt{\triangle t} - \frac{z^2}{2}} dz$$

where

$$\psi = \frac{e^{(\mu_S+\sigma_S^2)\triangle t}-1}{e^{\mu_S \triangle t}(e^{\sigma_S^2 \triangle t}-1)}\zeta,$$

which leads to

$$= \psi K^* \Phi\left(min(A,B)\right) - \psi S_{t_k}^{\frac{\sigma_L}{\sigma_S}} e^{(\frac{\sigma_L \mu_S}{\sigma_S} - \frac{\sigma_S \sigma_L}{2})\triangle t} \int_{-\infty}^{min(A,B)} \frac{1}{\sqrt{2\pi}} e^{\sigma_L z \sqrt{\triangle t} - \frac{z^2}{2}} d_z$$

By adding and substracting $\frac{\sigma_L^2}{2}\triangle t$ to the right side,

$$= \psi K^* \Phi\left(min(A,B)\right) - \psi S_{t_k}^{\frac{\sigma_L}{\sigma_S}} e^{(\frac{\sigma_L \mu_S}{\sigma_S} - \frac{\sigma_S \sigma_L}{2} + \frac{\sigma_L^2}{2})\triangle t} \int_{-\infty}^{min(A,B)} \frac{1}{\sqrt{2\pi}} e^{-\frac{\sigma_L^2}{2}\triangle t + \sigma_L z \sqrt{\triangle t} - \frac{z^2}{2}} dz$$

$$= \psi K^* \Phi\left(min(A,B)\right) - \psi S_{t_k}^{\frac{\sigma_L}{\sigma_S}} e^{(\beta - \sigma_L \sigma_S)\triangle t} \int_{-\infty}^{min(A,B)} \frac{1}{\sqrt{2\pi}} e^{-\frac{(z-\sigma_L\sqrt{\triangle t})^2}{2}} dz.$$

Using change of variable $u = z - \sigma_L\sqrt{\triangle t}$, we obtain

$$= \psi K^* \Phi\left(min(A,B)\right) - \psi S_{t_k}^{\frac{\sigma_L}{\sigma_S}} e^{(\beta - \sigma_L \sigma_S)\triangle t} \Phi\left(min(A,B) - \sigma_L\sqrt{\triangle t}\right)$$

Finally, we deduce the result which yields the price, P_{t_k} as follows:

$$P_{t_k} = \lambda K^* \Phi\left(min(A,B) - \sigma_S\sqrt{\triangle t}\right) - \lambda S_{t_k}^{\frac{\sigma_L}{\sigma_S}} e^{\beta \triangle t} \Phi\left(min(A,B) - (\sigma_S + \sigma_L)\sqrt{\triangle t}\right)$$

$$+ \psi K^* \Phi\left(min(A,B)\right) - \psi S_{t_k}^{\frac{\sigma_L}{\sigma_S}} e^{(\beta - \sigma_L \sigma_S)\triangle t} \Phi\left(min(A,B) - \sigma_L\sqrt{\triangle t}\right).$$

□

Cushion option is assumed to be bought at time t_k for all $k = 0, 1, ..., n-1$ if the cushion value at time t_k, C_{t_k} is higher than the price of the cushion option at time t_k, P_{t_k}. Since $C_{t_k} > P_{t_k}$, new portfolio value and cushion value at time t_k are given by

$$V^*_{t_k} = V_{t_k} - P_{t_k}$$

and

$$C^*_{t_k} = max\{V^*_{t_k} - F_{t_k}, 0\},$$

respectively. Therefore, the wealth process can be written as follows :

Since $C_{t_k} > 0$ for all $k = 0, 1, .., n-1$

$V_{t_{k+1}} =$

$$\left[(V^*_{t_k} - mC^*_{t_k})e^{r\triangle t} + mC^*_{t_k}\frac{S_{t_{k+1}}}{S_{t_k}} + \gamma L_{t_{k+1}} + (K - (1-c)\gamma L_{t_{k+1}})^+\right]\mathbb{I}_{\left\{\frac{S_{t_{k+1}}}{S_{t_k}} \leq (1-\frac{1}{m})e^{r\triangle t}, C_{t_k} > P_{t_k}\right\}}$$

$$+\left[(V^*_{t_k} - mC^*_{t_k})e^{r\triangle t} + mC^*_{t_k}\frac{S_{t_{k+1}}}{S_{t_k}} + \gamma L_{t_{k+1}}\right]\mathbb{I}_{\left\{\frac{S_{t_{k+1}}}{S_{t_k}} > (1-\frac{1}{m})e^{r\triangle t}, C_{t_k} > P_{t_k}\right\}}$$

$$+\left[(V_{t_k} - mC_{t_k})e^{r\triangle t} + mC_{t_k}\frac{S_{t_{k+1}}}{S_{t_k}} + \gamma L_{t_{k+1}}\right]\mathbb{I}_{\{C_{t_k} \leq P_{t_k}\}}$$

and since $C_{t_k} \leq 0$ for all $k = 0, 1, .., n-1$

$V_{t_{k+1}} = V_{t_k}e^{r\triangle t} + \gamma L_{t_{k+1}}$ yields the value of the pension fund at t_{k+1}.

For $C_{t_k} > 0$, the wealth process between equidistant trading dates is defined as

$$V_t = \left[(V^*_{t_k} - mC^*_{t_k})\frac{B_t}{B_{t_k}} + mC^*_{t_k}\frac{S_t}{S_{t_k}}\right]\mathbb{I}_{\{C_{t_k} > P_{t_k}\}} + \left[(V_{t_k} - mC_{t_k})\frac{B_t}{B_{t_k}} + mC_{t_k}\frac{S_t}{S_{t_k}}\right]\mathbb{I}_{\{C_{t_k} \leq P_{t_k}\}}$$

and since $C_{t_k} \leq 0$ for all $k = 0, 1, .., n-1$

$$V_t = V_{t_k}\frac{B_t}{B_{t_k}}$$

yields the value of the pension fund at $t\epsilon(t_k, t_{k+1})$.

CHAPTER 5

Implementation

In order to indicate the influence of exotic option on CPPI-DC plans under certain assumptions, we perform simulation analyses as the realization of real data requires long years and market conditions, plan and country-specify characteristics may change. By means of Monte Carlo simulation, CPPI strategy with cushion option and CPPI strategy without cushion option are simulated and the performance levels of both strategies are investigated under various market and DC plan assumptions. All analysis and runs are coded via MATLAB.

In more detail, the scenarios of the simulations depending on many factors are expected to expose sensivity analyses according to varying parameter values. As exotic options can be short-term leverage in financial markets, their contributions are evaluated in two basic terms :

(i) short-term which corresponds to the earliest time to terminate the pension plan (T=3 years)

(ii) long-term at which the participants fulfill the retirement conditions (T=20 years).

The comparisons with respect to terms, parameters are perfomed based on the value of the fund with exotic option (namely cushion option) and without cushion option.

Figure 5.1 and Figure 5.2 give us the outlook and the algorithm of proposed study, respectively. In the main setup we define contributing parameters with selected constant values except Strike price, K. We define two stochastic processes (Figure 5.1) to determine the labor income and stock price. Then the algorithm determines the price of the cushion option for different Strike prices, K at each fixed trading date. Next step considers a CPPI scheme for each fixed trading date without and with cushion option for each Strike price and finds the wealth for each case. Strike price, K which gives the highest final wealth is chosen for the last step. The process ends with a comparison of the setup with respect to cushion option and different values of contributing parameters.

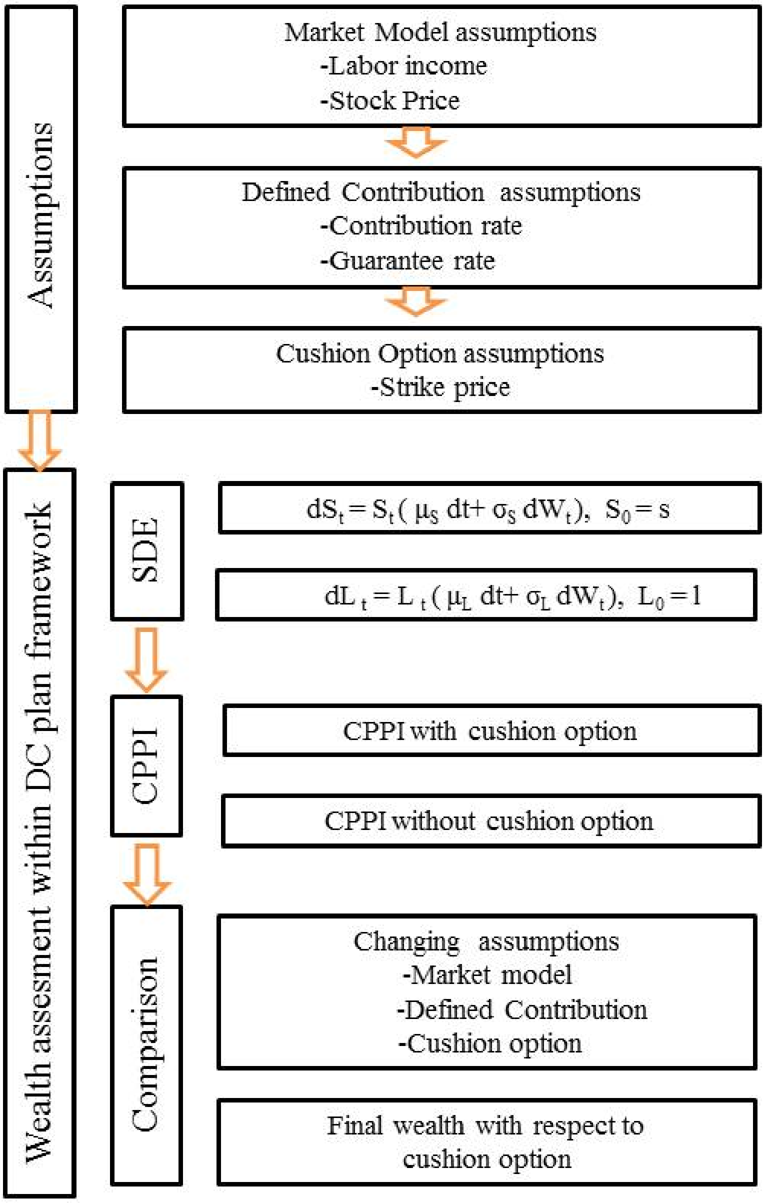


Figure 5.1: The framework of developed methodology

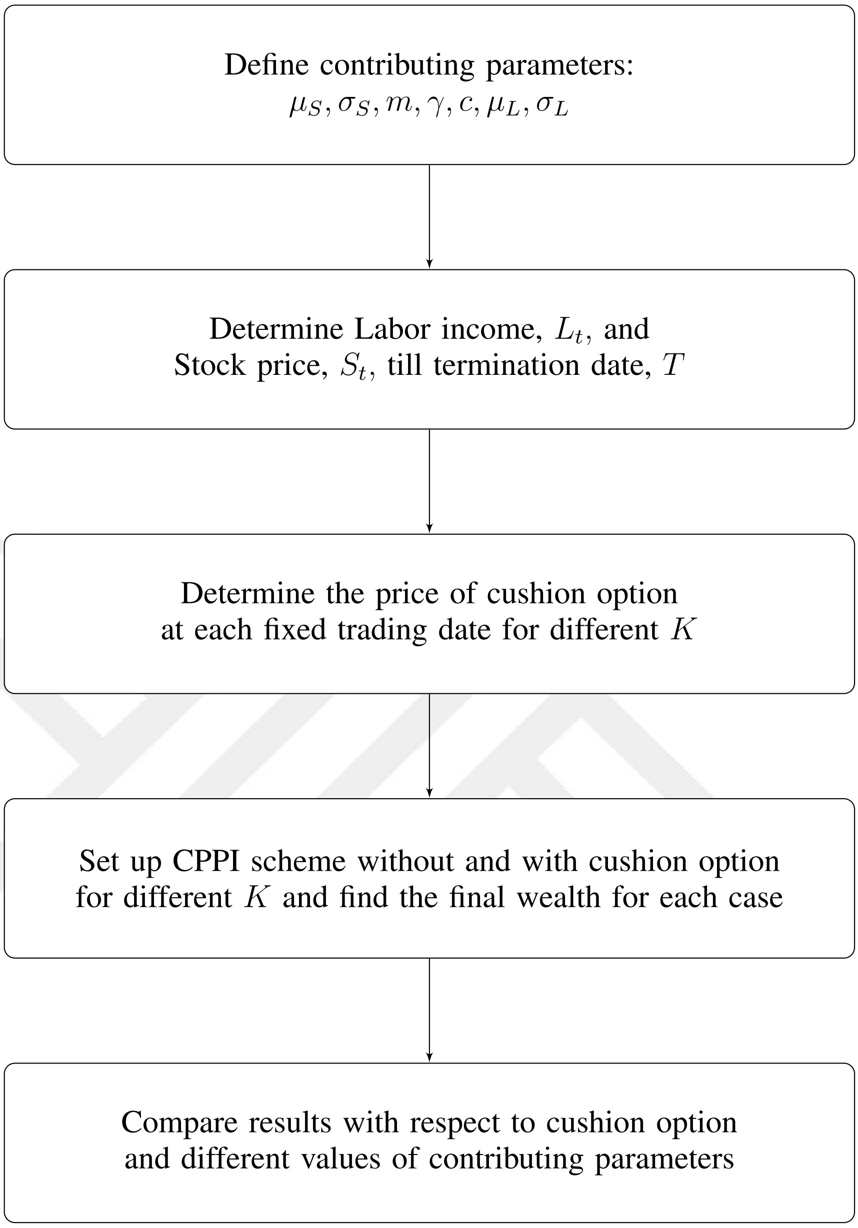


Figure 5.2: Alghoritm of the framework

5.1 Numerical Implementation and Results

The portfolio strategies are numerically simulated for the set of parameters whose values are summarized in Table 5.1. To keep the consistency with the study [32], we choose the same valuse of the parameters except some adjustments on the interest rate, r and market volatility, σ_S in order to be more realistic in cushion-option setup.

Table 5.1: Assumed values of parameters under discrete-time trading setting

	Value
Interest rate, r	0.03
Stock parameters	
Drift, μ_S	0.12
Volatility, σ_S	0.3
Labor income parameters	
Drift, μ_L	0.06
Volatility, σ_L	0.09
Strike price, K	28
Contribution rate, γ	0.1
Guarantee rate, c	0.8
Multiplier, m	8
Time horizon , T (year)	3

It is required to determine optimal Strike price, K to take the advantage of cushion option on CPPI strategy. In line with this requirement, Table 5.2 and 5.3 are obtained for long term period of T=20 and short term period of T=3, respectively. The choice of three years for short-term is made with respect to the most commonly official withdrawal duration set by the life and pension insurance companies. Those tables show the means and standard deviations of final wealth for CPPI strategy without cushion option and CPPI strategy with cushion option under the various Strike prices, K for a three-year period and a twenty-year period. According to Table 5.2, cushion option is not a profitable choice for $T = 20$. Mean, $E(V_T)$ and standard deviation, $\sigma(V_T)$ of final wealth for CPPI strategy are always higher than the mean, $E(V_T^c)$ and standard deviation, $\sigma(V_T^c)$ of final wealth for CPPI strategy under cushion option for each Strike price, K. Since CPPI strategy provides a downside protection, another protection for shortfall risk is not effective for a long period such as 20-year.

At first glance, it is noticeable in the Table 5.3 that the means, as well as the standard deviations, of CPPI portfolio under cushion option exceed the mean and standard deviation of CPPI portfolio. Overall, the mean of terminal value has an increasing trend till Strike price, K equals to 26 and there is no significant difference on the mean of terminal value between $K = 26$ and $K = 28$ and it decreases afterwards. This states that the terminal value for short-term withdrawal profits the implementation of cushion option for a certain strike price increment which reverses its impact after a certain break-even point.

Coefficient of variation (CV) of CPPI portfolio value under cushion option has the highest value in the case of $K = 28$. Considering downside protection is provided by CPPI strategy, higher standard deviation indicates a longer right tail of terminal wealth distribution in spite of general consideration as a negative indicator of performance. Therefore, we choose $K = 28$ for following analyses as a break-even point. It should be noted that the CPPI-DC under cushion option set up performs well for short term period. For this reason, the sensitivity to change in parameters is done for the case when T=3 years.

Table 5.4 presents the sensitivity on strategy basis for T=3. Higher market volatility, σ_S decreases the mean and standard deviation of terminal wealth under cushion option but it has a positive effect on moments of terminal wealth for CPPI strategy. An increase on the market drift, μ_S raise the terminal wealth for CPPI strategies. However, we can not observe accurate information about its effect on the terminal wealth for CPPI strategy under cushion option. A rise on labor income drift, μ_L, as well as labor income volatility, σ_L, increases the moments of terminal wealth for CPPI strategy. Also, σ_L enhances standard deviation of CPPI portfolio under cushion option. Increasing on guarantee rate, c resulting in the higher floor has a negative effect on moments of both strategies in the sense that small amount of wealth is invested in the risky asset and take fewer potential gains on the upwards market move. By contrast, a raise on contribution rate, γ enhances moments of both strategies. When the more amount of money is invested in the risky asset, you can take more gains from increasing market. Higher multiplier, m goes up the moments of CPPI portfolio but on the other hand higher m decreases moments of CPPI strategy under cushion option. The gap risk increases in the case of the high multiplier, m and therefore the value of cushion option depending on gap risk raises. As expected, the high price of cushion option lower the profits. Inversely, smaller multiplier, m decreases the gap risk in line with the price of cushion option. For a small multiplier such as $m = 4$, the cushion option has no significant effect on CPPI strategy.

It should be pointed out that the sensivity to parameters for long-run term (T=20) is not presented here, as the scheme under with and without cushion options does not show a significant improvement in the performance as mentioned before.

Table 5.2: The final wealth of CPPI strategies for T=20 years

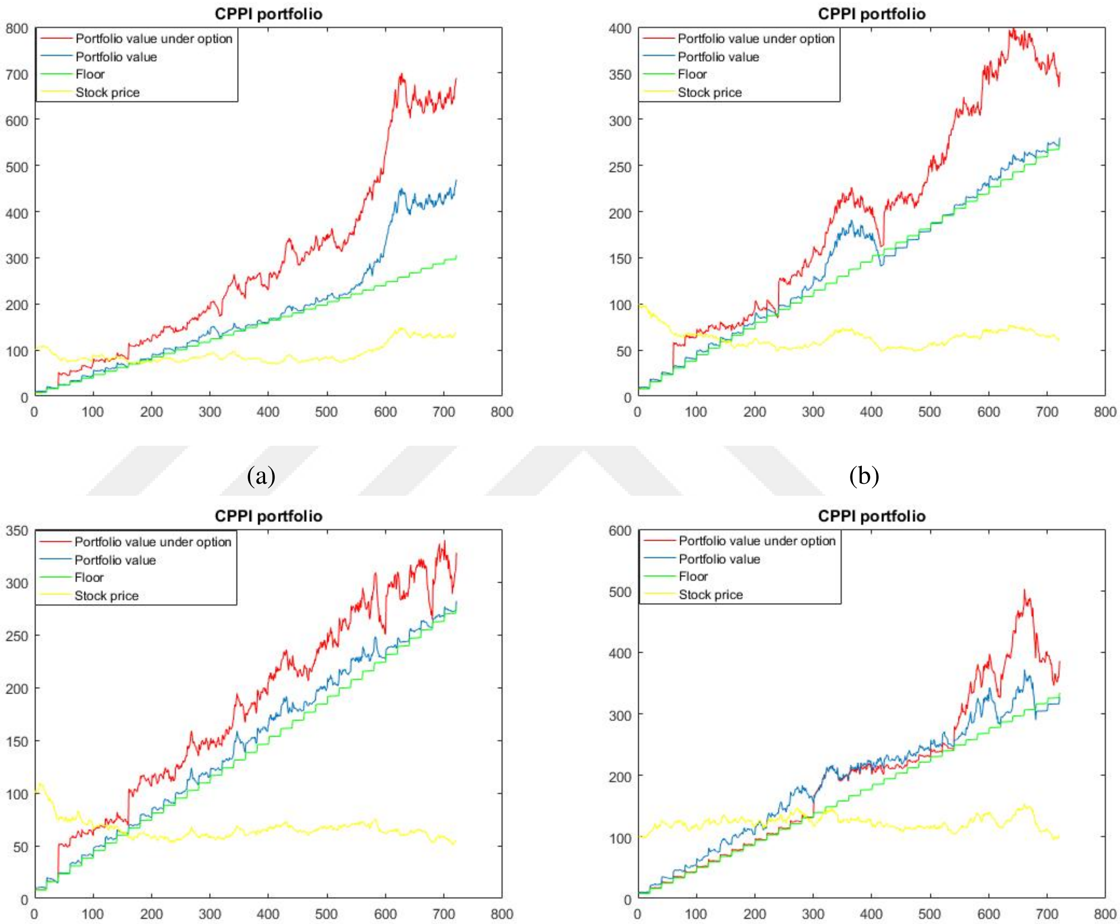
	K	6	8	10	12	14	16	18	20	22	24
Cushion	$E(V_T^c)$	3867.303	3852.725	3846.102	3845.839	3837.233	3826.053	3804.156	3773.730	3739.180	3682.215
option	$\sigma(V_T^c)$	4241.488	4186.104	4214.666	4213.391	4195.289	4155.835	4146.438	4092.478	4078.161	3989.882
	$CV(V_T^c)$	1.097	1.087	1.096	1.096	1.093	1.086	1.090	1.085	1.091	1.084
No cushion	$E(V_T)$	3889.607	3889.607	3889.607	3889.607	3889.607	3889.607	3889.607	3889.607	3889.607	3889.607
option	$\sigma(V_T)$	4285.938	4285.938	4285.938	4285.938	4285.938	4285.938	4285.938	4285.938	4285.938	4285.938
	$CV(V_T)$	1.102	1.102	1.102	1.102	1.102	1.102	1.102	1.102	1.102	1.102

Table 5.3: The final wealth of CPPI strategies for T=3 years

	K	16	18	20	22	24	26	28	30	32	34	36
Cushion	$E(V_T^c)$	443.009	446.205	449.495	452.592	454.889	456.283	455.896	433.976	325.25	319.712	317.329
option	$\sigma(V_T^c)$	155.566	156.854	158.336	159.959	162.056	165.097	168.667	167.594	55,243	42.744	35.805
	$CV(V_T^c)$	0.351	0.352	0.352	0.353	0.356	0.362	0.370	0.386	0.170	0.134	0.113
No cushion	$E(V_T)$	419.447	419.447	419.447	419.447	419.447	419.447	419.447	419.447	419.447	419.447	419.447
option	$\sigma(V_T)$	145.204	145.204	145.204	145.204	145.204	145.204	145.204	145.204	145.204	145.204	145.204
	$CV(V_T)$	0.346	0.346	0.346	0.346	0.346	0.346	0.346	0.346	0.346	0.346	0.346

Table 5.4: Distributional properties and sensitivity to parameters of CPPI portfolio with and without cushion option under certain parameter values (T=3 years)

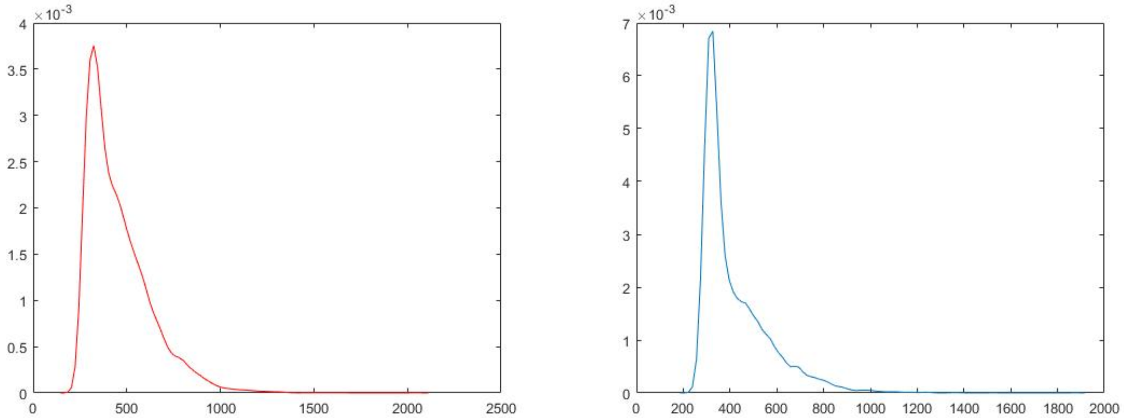
Parameter	Value	CPPI with cushion option		CPPI without cushion option	
		$E(V_T^c)$	$\sigma(V_T^c)$	$E(V_T)$	$\sigma(V_T)$
σ_S	0.2	431.508	113.726	412.741	104.552
	0.3	433.976	167.594	419.447	145.204
	0.4	316.021	28.818	429.064	193.162
μ_S	0.06	440,022	160,056	409,316	137,258
	0.12	433.976	167.594	419.447	145.204
	0.24	492.873	185.398	443.149	162.276
σ_L	0.05	460.311	153.237	418.453	133.031
	0.09	433.976	167.594	419.447	145.204
	0.18	446.127	202.540	424.447	178.757
μ_L	0.03	448.795	167.489	413.301	143.476
	0.06	433.976	167.594	419.447	145.204
	0.1	465.646	170.197	427.774	147.566
c	0.7	464.508	168.321	423.167	154.487
	0.8	433.976	167.594	419.447	145.204
	0.9	356.907	32.644	413.288	119.968
γ	0.05	157.605	14.019	209.724	72.602
	0.1	433.976	167.594	419.447	145.204
	0.2	879.259	308.290	838.894	290.409
m	4	416.976	132.652	417.534	132.995
	6	430.250	154.930	419.265	142.132
	8	433.976	167.594	419.447	145.204
	10	315.689	28.112	419.926	146.977
	12	316.249	28.527	420.126	148.673



(a) (b) (c) (d)

Figure 5.3: Illustrative trajectories of CPPI portfolio-scheme

Figure 5.3 presents trajectories of each strategy to illustrate the evolution of the portfolio for both strategies. Taking a glance at graphs, portfolio value under cushion option initially stands below the portfolio value of CPPI strategy due to hedging cost (a, b, c, d). When the portfolio value drops below the floor, it recovers quickly by means of cushion option and make benefits from increasing in the market (a, b, c). However, portfolio value for CPPI strategy needs time to recover and can not make use of an advantage of potential gains on the upwards move in the market (b, d).



(a) (b)

Figure 5.4: Short-term estimated kernel densities of final wealth for CPPI (a) under cushion option; (b) under no cushion option

Figure 5.4 illustrates the tail properties and the distributional behaviour of terminal portfolio value for both strategies. As clearly seen, CPPI strategy under cushion option has longer right tail and higher frequency of higher terminal wealth value than CPPI strategy. This illustrative distributional behaviour of both strategies supports the results depicted on Table 5.4.

CHAPTER 6

Conclusion

Fund management for DC pension plan is crucial topics for individuals and social security systems. With respect to this, portfolio insurance is studied in the literature to provide a downside protection for portfolio value. CPPI strategy with the random floor is introduced by [32] is applied to DC pension plan under discrete-time trading setting. Considering trading only takes place at predefined dates, the portfolio value can drop below the floor between trading dates. Additionally, contribution at trading dates may not be enough to push up the floor. Therefore, we propose a cushion option to reduce this gap risk in this thesis. By means of Monte Carlo simulation, the effectiveness of the cushion option on CPPI strategy is tested by comparing portfolio performance for CPPI strategy with and without cushion option. To support the effectiveness of cushion option, kernel densities are estimated for terminal wealth for both strategies.

This thesis has following contributions:

- The price of proposed cushion option is derived explicitly under discrete-time setting with respect to the variance-optimal criterion.
- Numerical results obtained by Monte Carlo simulation indicates that CPPI strategy under cushion option outperforms CPPI strategy in a short period such as 3-year in the sense that the moments of terminal wealth for CPPI strategy under cushion option are higher than the moments of terminal wealth for CPPI strategy.

Even though CPPI strategy provides downside protection, a sudden drop in the market may cause the portfolio value to drop below the floor. If remaining time before termination date is not enough for the portfolio value to recover, the investor faces the risk that the portfolio value stays under the floor representing the acceptable minimum portfolio value. Cushion option which is introduced in this thesis is shown to be a profitable choice on DC pension plan such a long period for an investor who considers early withdrawal.

As a future study our setup can be applied to hybrid pension scheme for reducing funding shortfall which is prominent risk on calculation of reserve with respect to Solvency II framework. Another research development would study how guarantee rate, c may become optimal.

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