STOCHASTIC SURPLUS PROCESSES WITH VaR AND CVaR SIMULATIONS IN ACTUARIAL APPLICATIONS

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ABSTRACT

STOCHASTIC SURPLUS PROCESSES WITH VaR AND CVaR SIMULATIONS IN ACTUARIAL APPLICATIONS

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The theory of ruin is a substantial study for those who are interested in financial survival probability based on the patterns imposed by the surplus process, which determines the insurer's capital balance at a given time. In other words, fluctuations in aggregate claims as well as premiums in such processes can be secured by a certain capital. In this study, we simulate various surplus processes under different claim size-distribution assumptions and extend the analyses by adding perturbation of a Brownian motion in order to capture the possible uncertainty on aggregate claims as well as premiums. The capital, which is required to prevent an insurance company from possible losses, is achieved by using the capital-based risk measures,such as the Value-at-Risk (VaR) and the Conditional Value-at-Risk (CVaR) associated to the surplus process. Findings of the thesis fill a gap in the related literature, especially for the claim size distributions whose closed-form expressions for the ruin probabilities cannot be obtained. This study sheds light on practitioners who allocate capital by means of VaR and CVaR when ruin is considered.

Keywords : ruin theory, risk measure, value-at-risk, conditional value-at-risk

ÖZ

AKTÜERYA UYGULAMALARINDA STOKASTİK REZERV SÜREÇLERİ İÇİN VaR VE CVaR SİMULASYONU

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İflas teorisi bir sigorta şirketinin finansal varlığını sürdürmekle ilgilenenler için büyük önem arzeden bir konudur. Şirketin sermaye dengesine bu teorinin temel konusu olarak incelenen rezerv süreçlerinin belirlenen zaman içindeki örüntüsü karar verir. Başka bir deyişle, bu tip stokastik süreçlerde toplam hasar yada primlerdeki dalgalanmalar şirketin likit olarak elinde bulundurduğu sermaye ile kontrol altında tutulabilir. Bu çalışmada, farklı hasar dağılımları varsayımıyla simüle edilen rezerv süreci ve olası belirsizlikleri yansıtabilmesi için bu stokastik sürece Brown hareketi difüzyonu eklenerek elde edilen model çalışılmıştır. Her iki stokastik modelden yola çıkarak, oluşabilecek risklerden korunmak için ayırılan sermaye değeri Riske Maruz Değer (VaR) ve koşullu Riske Maruz Değer (CVaR) gibi risk ölçümleri kullanılarak erişilebilir. Bu çalışmada bulunan sonuçlar, litaratürde birçok hasar dağılımı için kapalı formülü bulunamayan iflas olasılıklarını hesaplamada literatürdeki boşluğun doldurulmasına yardımcı ve stokastik rezerv süreçleri kullanarak sermaye hesabı yapan sigorta sektör çalışanlarına yol gösterici olacağı düşünülmektedir.

Anahtar Kelimeler : iflas teorisi, risk ölçümü, riske maruz değer (VaR), koşullu riske maruz değer (CVaR)

To my peerless Grandma,
and the times both we are young,

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TABLE OF CONTENTS

ABSTRACT vii

ÖZ ix

ACKNOWLEDGMENTS xiii

TABLE OF CONTENTS xv

LIST OF FIGURES xvii

LIST OF TABLES xix

LIST OF ABBREVIATIONS xxi

CHAPTERS

1 INTRODUCTION 1

2 RISK MEASURE 7

2.1 Premium-Based Risk Measures 8

2.2 Coherent Risk Measures 9

2.3 Capital-Based Risk Measures 10

2.3.1 Value-at-Risk 11

2.3.2 Conditional Value-at-Risk 12

3 BUILDING RISK MODELS and RUIN THEORY 15

3.1 Risk Models 15

3.1.1 Models for the Claim Number Process 16

3.1.1.1 Poisson Process 17
3.1.1.2 Renewal Process 18
3.1.2 Modeling Total Claim Size 19
3.1.3 Claim Size Distributions 20
3.1.4 Distribution of Total Claim Size 25
3.2 Ruin Theory 28
3.2.1 Continuous Time Surplus Model 29
3.2.2 Bounds for Ruin Probability 32
3.2.3 Discrete Time Surplus Model 34
4 IMPLEMENTATION 37
5 CONCLUSION 51
REFERENCES 53
APPENDICES
A Further Simulation Figures 59
B Further Simulation Tables 65
B.1 Tables when claim sizes are distributed as Gamma(3, 1) . . . 65
B.2 Tables when claim sizes are distributed as Weibull(1.5, 0.5) . 67
B.3 Tables when claim sizes are distributed as Lognormal(1, 0.4441) 69

LIST OF FIGURES

Figure 1.1 The state of solvency. MCR: Mininumum Capital Requirement, SCR: Solvency Capital Requirement, ASM: Available Solvency Margin. (Adopted from: Sandström, 2010, p. 5) 3

Figure 1.2 Relation of capital with other business instruments. (Adopted from:IAA (2004) [34], p. 25.) 4

Figure 2.1 The graph shows that CVaR considers risk beyond VaR. 13

Figure 3.1 Poisson process with different intensities. 18

Figure 3.2 Q–Q plot of distributions with different characteristic. 22

Figure 3.3 Shape of different mean excess functions for different distributions. 24

Figure 3.4 Examples of mean excess plot for specified distributions. 24

Figure 3.5 Paths of simulated surplus processes for compound Poisson claim and the histogram of final values of these surplus processes. 30

Figure 4.1 Different simulation paths of two models. 39

Figure 4.2 Simulated risk components with claim distribution Exponential(3), and $\sigma_B = 0$. 42

Figure 4.3 Simulated risk components with claim distribution Exponential(3), and $\sigma_B = 1$. 43

Figure 4.4 Change in ruin probabilities when $T = 1, 2, \ldots, 10$, and $\sigma_B = 0$. . 45

Figure 4.5 Change in ruin probabilities of Exponentially distributed claim when $dt = 1$ (daily), 30 (monthly), 120 (quarterly), 180 (semi-annually) and $\sigma_B = 0$. 45

Figure 4.6 Change in ruin probabilities of Exponentially distributed claim size when σ_B increases. 46

Figure 4.7 Simulated risk components with claim distribution Pareto(1, 2), $\sigma_B = 0$. 46

Figure 4.8 Simulated risk components with claim distribution Pareto(1, 2), $\sigma_B = 1$. 47

Figure A.1 Simulated risk components with claim distribution Gamma(3, 1), $\sigma_B = 0$. 59

Figure A.2 Simulated risk components with claim distribution Weibull(1.5, 0.5), $\sigma_B = 0$. 60

Figure A.3 Simulated risk components with claim distribution Lognormal(1, 0.441), $\sigma_B = 0$. 61

Figure A.4 Simulated risk components with claim distribution Gamma(3, 1), $\sigma_B = 1$. 62

Figure A.5 Simulated risk components with claim distribution Weibull(1.5, 0.5), $\sigma_B = 1$. 63

Figure A.6 Simulated risk components with claim distribution Lognormal(1, 0.4441), $\sigma_B = 1$. 64

LIST OF TABLES

Table 4.1 $\text{VaR}_{0.95}$ and $\text{CVaR}_{0.95}$ estimates and the ruin probabilities with claim size process Exponential(3) and $\sigma_B = 0$. 42

Table 4.2 $\text{VaR}_{0.95}$ and $\text{CVaR}_{0.95}$ estimates and the ruin probabilities with claim size process Exponential(3) and $\sigma_B = 1$. 42

Table 4.3 $\text{VaR}_{0.995}$ and $\text{CVaR}_{0.995}$ estimates and the ruin probabilities with claim size process Exponential(3) and $\sigma_B = 0$. 44

Table 4.4 $\text{VaR}_{0.995}$ and $\text{CVaR}_{0.995}$ estimates and the ruin probabilities with claim size process Exponential(3) and $\sigma_B = 1$. 44

Table 4.5 $\text{VaR}_{0.95}$ and $\text{CVaR}_{0.95}$ estimates and the ruin probabilities when claim size process follows Pareto(1, 2) and $\sigma_B = 0$. 48

Table 4.6 $\text{VaR}_{0.95}$ and $\text{CVaR}_{0.95}$ estimates and the ruin probabilities with claim size process Pareto(1, 2) and $\sigma_B = 1$. 48

Table 4.7 $\text{VaR}_{0.995}$ and $\text{CVaR}_{0.995}$ estimates and the ruin probabilities with claim size process Pareto(1, 2) and $\sigma_B = 0$. 48

Table 4.8 $\text{VaR}_{0.995}$ and $\text{CVaR}_{0.995}$ estimates and the ruin probabilities with claim size process Pareto(1, 2) and $\sigma_B = 1$. 49

Table B.1 $\text{VaR}_{0.95}$ and $\text{CVaR}_{0.95}$ estimates and the ruin probabilities with claim size process Gamma(3, 1) and $\sigma_B = 0$. 65

Table B.2 $\text{VaR}_{0.95}$ and $\text{CVaR}_{0.95}$ estimates and the ruin probabilities with claim size process Gamma(3, 1) and $\sigma_B = 1$. 66

Table B.3 $\text{VaR}_{0.995}$ and $\text{CVaR}_{0.995}$ estimates and the ruin probabilities with claim size process Gamma(3, 1) and $\sigma_B = 0$. 66

Table B.4 $\text{VaR}_{0.995}$ and $\text{CVaR}_{0.995}$ estimates and the ruin probabilities with claim size process Gamma(3, 1) and $\sigma_B = 1$. 66

Table B.5 $\text{VaR}_{0.95}$ and $\text{CVaR}_{0.95}$ estimates and the ruin probabilities with claim size process Weibull(1.5, 0.5) and $\sigma_B = 0$. 67

Table B.6 $\text{VaR}_{0.95}$ and $\text{CVaR}_{0.95}$ estimates and the ruin probabilities with claim size process Weibull(1.5, 0.5) and $\sigma_B = 1$. 67

Table B.7 $\text{VaR}_{0.995}$ and $\text{CVaR}_{0.995}$ estimates and the ruin probabilities with claim size process Weibull(1.5, 0.5) and $\sigma_B = 0$. 67

Table B.8 $\text{VaR}_{0.995}$ and $\text{CVaR}_{0.995}$ estimates and the ruin probabilities with claim size process Weibull(1.5, 0.5) and $\sigma_B = 1$. 68

Table B.9 $\text{VaR}_{0.95}$ and $\text{CVaR}_{0.95}$ estimates and the ruin probabilities with claim size process Lognormal(1, 0.4441) and $\sigma_B = 0$. 69

Table B.10 $\text{VaR}_{0.95}$ and $\text{CVaR}_{0.95}$ estimates and the ruin probabilities with claim size process Lognormal(1, 0.4441) and $\sigma_B = 1$. 69

Table B.11 $\text{VaR}_{0.995}$ and $\text{CVaR}_{0.995}$ estimates and the ruin probabilities with claim size process Lognormal(1, 0.4441) and $\sigma_B = 0$. 69

Table B.12 $\text{VaR}_{0.995}$ and $\text{CVaR}_{0.995}$ estimates and the ruin probabilities with claim size process Lognormal(1, 0.4441) and $\sigma_B = 1$. 70

LIST OF ABBREVIATIONS

$\mathbb{R}$	Real Numbers
$\mathbb{N}$	Natural Numbers
$F_X(x)$	distribution of X
$\mathbb{P}$	probability
$\mathbb{E}$	expectation
Var	variance
$\phi_X(x)$	characteristic function of X
$M_X(t)$	moment generating function of X
Φ	distribution function of $\mathcal{N}(0,1)$
$\pi(X)$	premium principle of X
$\mathrm{VaR}_\alpha(X)$	Value-at-Risk of X at α confidence level
$\mathrm{CVaR}_\alpha(X)$	conditional Value-at-Risk of X at α confidence level
i.i.d.	independent identically distributed
m.g.f.	moment generating function
a.s.	almost surely
NPC	net profit condition
$\tilde{R}$	adjustment coefficient
δ	safety loading

CHAPTER 1

INTRODUCTION

Risk is a fragment of our daily lives. As a dictionary definition risk is an ‘exposure to the chance of injury or loss; a hazard or dangerous chance’. However, only a single definition is not entirely satisfactory since financial economists, behavioral scientists, risk theorists, statisticians, and actuaries adjust their concepts in the definition of risk. Generally, it is connected with *uncertainty*. Therefore, a popular answer to ‘what is risk?’ can be uncertainty concerning an occurrence of a loss [28]. Although mostly risk is seen as downside, rarely upside, potential of gain.

Since risk is inherent in everything that we act, essence of risk management shows up. Kloman (1990) stated the risk management as [38]:

> To many analysts, politicians, and academics it is the management of environmental and nuclear risks, those technology generated macro-risks that appear to threaten our existence. To bankers and financial officers it is the sophisticated use of such techniques as currency hedging and interest-rate swaps. To insurance buyers or sellers it is coordination of insurable risks and the reduction of insurance cost. To hospital administrators it may mean ‘quality assurance’. To safety professionals it is reducing accidents and injuries. In summary, risk management is *a discipline for living with the possibility that future events may cause adverse effects.*

As it is understood from the quotation, risk management protects our lives to some extent via different implementations. Risk management is main business of especially banks and insurance companies. Risk can be undertaken by an insurance which encapsulates from designing, pricing and marketing the insurance products, the underwriting procedures, the calculation of liabilities, technical provisions and asset backing these provisions, to the overall claims and risk management [57]. Indeed, insurance provides an insured pooling losses, payment of fortuitous losses, transferring the risk, and indemnification. Those insurance contracts are signed on the condition that losses are insurable risk. Ideally, an insurable risk has different characteristics which are having a large number of exposure units, the loss must be accidental or unintentional and measurable, the chance of loss must be calculable, and finally premium for risk must be feasible [28].

In the risk management, no matter the sector operate in, financial institutions are affected by three types of core risk: *credit*, *market*, and *operational risks* [56]. Besides, *liquidity risk* can be added, too. An insurance company encounters with *underwriting risk* as well. Credit risk is the default risk and, on the company investment portfolio, it is the change in the quality of issuers of security, counter-parties (e.g. reinsurance, derivative contracts), intermediaries, etc. This risk resembles also direct default, spread, sovereign, and concentration risk. Market risk is related to the level or volatility of market prices on assets and considers the movements in the level of stock prices, interest rates, exchange rates or commodity rates. It consists of interest-rate risk, equity and property risk, currency risk, etc. Operational risk arises from inadequacy of business plan or falling system such as fraud and external events. It is also named as residual risks. Liquidity risk comes up due to insufficient liquid assets. For instance, early termination of insurance contracts poses liquidity risk of which company should be aware. Finally, underwriting risk (or insurance risk) is associated with insurance contracts on that misleading issues cannot be covered. Subclasses of such risk are pricing, reserving, policyholder behavior, claim and net retention risk.

Road to regulation for an insurance company in terms of risk management is drawn by *Solvency* framework that is inspired by the Basel Accord for banking system. This regulation is used for protection of the policyholder by enforcing a law, enough capitalization for the insurance company, hereby providing it financial stability. Same as the Basel Accord, Solvency is based on three pillar system, where the first pillar is interested in quantitative requirements, the second pillar involves governance and supervision review of the process, and the third pillar concerned with market reporting to compare risk profiles of companies easier. Pillar I rules on the amount of the capital in order to ensure an insurance company from probability of insolvency. In fact, it focuses on sustaining the appropriate technical provisions (policy liabilities such as premiums, not funded claims and claim reserves from unearned premiums), compensating the obligations with appropriate assets, and the primary interest required capital. On the purpose of calculating the capital requirements, company is free to use *internal models*, or *standard formula approach*. Pillar II beholds competing way with risk of an insurance company i.e. supervisory or designed qualified party reviews, independent opinions about the determination of risk assessment for solvency purposes. This is especially required for the determination of internal models in Pillar I. Moreover, Pillar II not only ensures having sufficient amount required capital, but also supports insurers to improve better risk management techniques. Pillar III assists the disclosure of information to public to strengthen market discipline. Thereby, business of an insurance company can easily be judged by policyholders, analysts, investors and supervisors.

Figure 1.1 demonstrates different states of solvency. Generally, when Available Solvency Margin (ASM), which equals to the difference between assets and liabilities, is less than zero, insolvency arises. Solvency Capital Requirements (SCR) is the target requirement for an insurance company. Regulatory capital requirements account both Minimum Capital Requirements (MCR) and SCR. The value lower than MCR is considered to be that the company is not in the regularity state solvency [57]. In an ideal situation,

$$\mathrm{MCR} \leq \mathrm{SCR} \leq \mathrm{ASM}.$$

The maintenance of financial stability is satisfied by the economic capital view. It is a

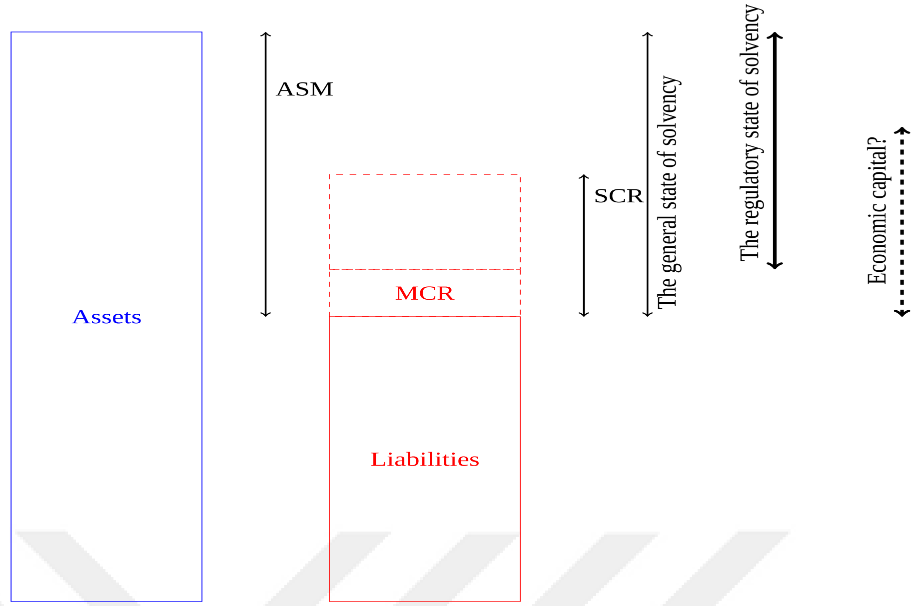


Figure 1.1: The state of solvency. MCR: Mininumum Capital Requirement, SCR: Solvency Capital Requirement, ASM: Available Solvency Margin. (Adopted from: Sandström, 2010, p. 5)

more complex situation than satisfying regularity state of solvency. Financial stability can be guaranteed with the use of *Own Risk and Solvency Assessment (ORSA)* that is proposed in [35, 36].

Business capital has central role in operations of insurer environment, as illustrated in Figure 1.2, by holding clear vision on the handling purposes of each elements such as risks and design of it, pricing, liabilities, assets and their managements, experience rating, profit and solvency. A report of International Actuarial Association [34] (2004) remarks that capital requirement purpose is a rainy day fund which covers bad situations effects, a way of risk management to decrease level of risk, an aid for avoiding undesirable level of risk from a policyholder perspective, a function of actual economic risk, a tool for supervisors in order to control financial failure of the company, and a signal of emerging trends on the market.

Risk based capital regulation is the way of measuring the minimum amount of capital in order to maintain overall business operations according to company's size and its risk profile. Indeed, as indicated, capital allocation is seen as a buffer to the company against insolvency [50]. Under risk based capital regulations, financial regulators should choose the correct approach of measuring the risk. These measures should give intervention before the insolvency gets large, at least they should help minimize the risk.

Pentikäinen (2004) [52] points out that solvency can be evaluated by using risk theoretical techniques, for example, ruin theory which plays important role in addressing the issue of investigating business activities of an insurance company by means of its surplus process. Besides pursuing inflows and outflows, other ingredient of this

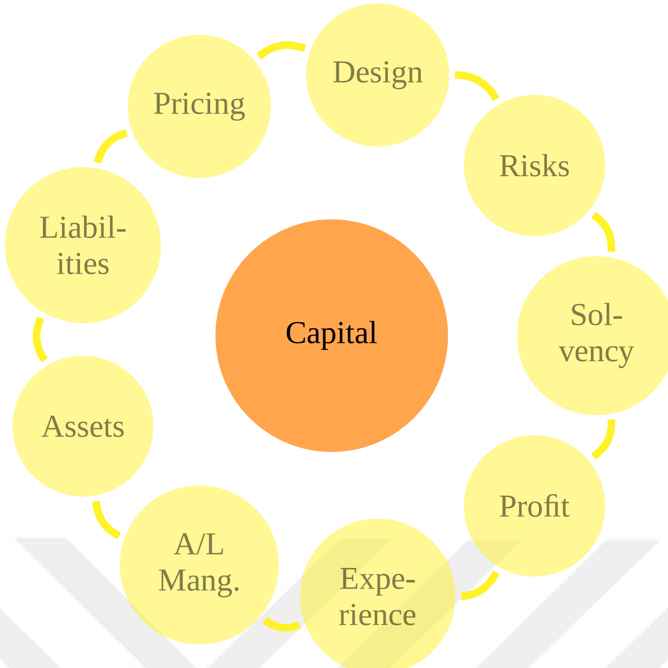


Figure 1.2: Relation of capital with other business instruments. (Adopted from:IAA (2004) [34], p. 25.)

theory is the ruin probabilities which specify the amount of required initial capital for a certain probability of solvency. As a matter of fact, Sandström (2010) [57] reported that ruin probability is the risk measure of insurer having an initial ASM; if ASM greater or equal to SCR, it means insolvent during the given time interval. Over the decades, prominent studies, for instance, by Asmussen & Albrecher (2010) [5], Bühlman (1970) [8], Dickson (2005) [18], Embrechts (2000) [24], Mikosch (2006) [47], Rolski et al. (2009) [55], have developed the ruin theory to a certain level. In fact, the first studies in ruin are traced back Lundberg and Cramér about collective risk models.

In this study, we aim to find capital requirement for an insurance company via considering its solvency by attaining the ruin. Therefore, we shape the current paper in the light of the ruin theory, in which the modeling of the surplus process plays an important role. It figures out business act of an insurance company in terms of cash flows, premium inflows and claim outflows, with setting up initial capital. Premiums are collected at a constant rate from each contingent insurance policy within a given unit of time. Although premium payments are received at the fixed rate, loss claims are random and unpredictable. Financial survival of the company primarily relies upon both estimating loss exposure, of which real values cannot be well-predicted, and balancing claims with premiums. We focus on estimating aggregate claims under various distributions and, we extend the classical model by adding perturbation of a Brownian motion in the surplus process to handle the uncertainty in premium payment, and resolve the problem of uncertainties of aggregate claim distribution [5, 21].

Preliminary work on extension of classical surplus model by adding a diffusion term was investigated by Gerber and Dufresne (1991) [21] for the sake of expressing an additional uncertainty of aggregate claims and premium income. They presented defective renewal equation for extended surplus process, thereby standard techniques of renewal theory for this model has become applicable for finding ruin probability especially in combinations of exponential distribution. Then, Veraverbeke (1993) [63] moved perturbed risk models, concerning to asymptotic estimates of ruin probability and they proved that asymptotically this is equal to integrated tail of claim size distributions. Alter on, Schlegel (1998) [59] obtained asymptotic ruin behavior of perturbed risk model by allowing claim arrival process. Yang and Zhang (2001) [67] used perturbed risk process which included not only compound Poisson but also Gamma process. They obtained joint distribution of ruin time and the first recovery time. Lin (2009) [44] derived optimal investment strategy that minimizes ruin probability of risk process, perturbed by a diffusion, and discussed the relation between ruin and investment by investigating adjustment coefficient and diffusion volatility parameter, risk free rate and correlation coefficient.

Studies of risk measures in actuarial context were generally connected with calculation of the insurance premium; among those are the works of Bühlmann (1970) [8], Gerber (1970) [29], Goovaerts (1984) [30], and Denuit et al. (2006) [15], Kaas et al. (2008) [37]. Risk measures such as VaR or CVaR are started to use in 1990s in actuarial by National Association of Insurance Commissioners (NAIC) establishment as an early warning system for insurance regulators [50].

Due to its ease in calculations and clarity in interpretation, Value-at-Risk (VaR) has become one of the popular capital-based risk measures. It describes maximum probable loss in the fixed time horizon at a specific confidence level. Despite the fact that it is a widely used financial risk measure, it has some drawbacks such as concealing any idea about the value of the actual loss above confidence level; further it does not not satisfy the subadditivity property of coherent risk measures. These drawbacks motivate us to use Conditional Value-at-Risk (CVaR) [1, 54]. The use of CVaR leads us to calculate the risk beyond VaR. In addition, it satisfies all axioms of coherency and, it provides mathematical superior properties than those of VaR. Since CVaR turns out to be a convex risk measure, it can be invoked in well-established convex programming problems of risk management. A number of different variations of CVaR come out such as Expected Shortfall, Tail Value at Risk, Tail Conditional or Average Value at Risk provided that underlying distribution is continuous.

In this study, regarding the theoretical framework of ruin, to ensure required capital allocation, we implement capital-based risk measures: VaR and CVaR for both classical and its extended (alternative) surplus processes.

Over the years, there has been highly appreciated studies made public in the related literature. Here, we emphasize some of these: Dhaene et al. (2003) [16] examine risk measures based on exponential premiums from the Cramér-Lundberg upper bound and for the fixed initial capital, they find annual premium for a give ruin probability level. Cheridito et al. (2004) [12] give example about measuring risk of an insurance company by employing VaR and Average Value-at-Risk (AVaR) of ruin probability function which is limited at a certain confidence level. They bound these measures for risk component with Cramér-Lundberg upper bound. In the studies of Trufin et al. (2011) [61], on the other hand, directly ruin probability for risk measures VaR and TVaR are used; these are the so-called ruin-consistent Value-at-Risk and ruin-consistent Tail Value-at-Risk. Assuming the stationarity of insurance business, these risk measures indicate the smallest amount of capital to provide the ruin probability to be below a specific confidence level. Moreover, Gatto et al. (2014) [27] define the value at ruin as well as the tail value at ruin of surplus process with diffusion. They also propose efficient computation of these risk measures by using saddle point approximation and Fast Fourier Transform methods. Recent work on combination of risk measure and ruin theory is attributed to Mitric et al. (2015) [48]. They suggest a risk measure which consists of expected deficit at ruin, as well as ruin-consistent VaR and TVaR. They conclude their work with a closed-form expression of this risk measure, despite the fact that obtaining closed-from solution of probability might not be possible apart from some specific distributions such as exponential.

Therefore, implementation of these formulations on practical use might be difficult under different distributional assumptions on the claim sizes. With various scenarios for the claim distributions, this thesis study focusses on obtaining VaR and CVaR for the risk components to allocate a necessary amount of initial capital as well as the corresponding ruin probabilities associated to those (initial) capitals.

The thesis is structured as follows: Chapter 2 and Chapter 3 presents theoretical dimension of this study which mainly focuses on risk measures and ruin theory. Starting with definition risk measure, we discuss coherency concept, concordantly popular risk measures, VaR and CVaR are explained in Chapter 2. Further theme of the Chapter 3 is ruin theory queries step by step constructing surplus process elements and the model. Chapter 4 explains proposed approach for capital allocation with risk measures and shows the results via tables and graphical tools. Finally, Chapter 5 concludes the study by harmonizing the theoretical facts and simulation results handled on application part.

CHAPTER 2

RISK MEASURE

Risk measurement is one of the predominant issues of actuaries and practitioners in insurance business. Especially, risk measures are used in determination of capital or reserve, premiums, reinsurance, deductible threshold etc. In this chapter, we start with the definition of risk measure and their usage, then we continue with premium calculation principles which are introduced as risk measure in actuarial context: that is why it is called a premium-based risk measure. We argue the coherency axioms which ideal risk measure should have and we finally give the properties of two important capital-based risk measures: VaR and CVaR.

A risk measure explains the overall risk exposure with a single number. However, economically, a risk measure represents the preferences of the decision maker in the economic situation [43]; mathematically speaking that a risk measure is a real-valued mapping on the space of random variables: $\rho : X \mapsto \rho(X) \in \mathbb{R}$ [4]. The risk measure is an important tool for establishing internal and external models, insurance premium and economic capital allocation [57, 62], and so on.

Internal and external model. In order to monitor solvency of a company, regulatory committees agree on some standard reporting system like Basel Accord for banking and Solvency for insurance companies. Since external reporting system calculations have some deficiencies of setting companies' targets on different business level, companies have right to develop their own internal models. These models should based on certain risk measures.

Insurance premium. Along with marketing concerns, computation of insurance premium is compromised of expected loss, cost of insurance capital, and risk loading which are calculated via appropriate risk measures.

Economic capital allocation. Economic capital is considered as a cushion for unexpected losses. The capital, symbolizes the level of credit standing and tolerance level for the probability of insolvency that the company can possibly rule out.

Common examples of risk measures in actuarial studies are based on the premium principles for pricing insurance, ruin probability given initial capital for controlling solvency, in order to determine capital requirements to run the business.

2.1 Premium-Based Risk Measures

Targeting the limited ruin probability, a risk characteristic or risk measure for calculating premiums including safety loading should be considered [31]. Such a risk measure is called premium-based risk measure which gives the minimum amount premium that the insurer collects to compensate its obligations. In this section, widely used premium-based risk measures, such as, expected-value premium principle, variance premium principle, standard deviation, zero-utility premium principle are introduced.

Here, X is a non-negative random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ that represents the risk, and $\pi(X)$ is called as *premium principle* which is a functional on the space of risks X. Below are the most popular examples of the premium-based risk measures on actuarial applications.

(i) Expected-value premium principle:

$$\pi(X) = (1+\delta)\mathbb{E}\left[X\right], \tag{2.1}$$

where $\delta > 0$ is the safety loading which is utilized to protect the company losses higher than anticipated. In the special case of $\delta = 0$ and, $\pi(X) = \mathbb{E}\left[X\right]$, the risk measure is called *pure premium*.

Generally, expected-value premium principle is used in life insurance, however, it is rarely used in property and casualty insurance. Nevertheless, such an average value calculation is not enough to explain loss exposure especially in extreme event modeling. Apparently, risks with identical means might have different dispersion, and hence, they should be priced with different premiums [8].

(ii) Variance premium principle:

$$\pi(X) = \mathbb{E}\left[X\right] + \delta \text{Var}\left[X\right], \quad \delta \geq 0,$$

where δ is a constant which refers to non-negative loading.

(iii) Standard deviation premium principle:

$$\pi(X) = \mathbb{E}\left[X\right] + \delta\sqrt{\text{Var}\left[X\right]}, \quad \delta \geq 0,$$

where δ is again a constant which refers to non-negative loading.

Under variance and standard deviation premium principle, loss distribution with higher dispersion has higher risk, hereby it has a higher premium. But, as noted in [41], in standard deviation principle negative variation and positive variation yields the same effect. This problem can be resolved by using *semi-standard deviation principle* is given by

$$\pi(X) = \mathbb{E}\left[X\right] + \delta\sqrt{\mathbb{E}\left[\max(0, X - \mathbb{E}\left[X\right])^2\right]}, \quad 0 \leq \delta \leq 1.$$

(iv) Zero-utility premium principle:

$$U(c) = \mathbb{E}\left[U(c - X + \pi(X))\right],$$

where c is constant wealth of the insurer. Here, U is a given utility function being concave and strictly increasing ($U' > 0$ and $U'' < 0$); that is, *risk averse*. Under zero-utility premium principle is charged according to insurer's wealth. Special case of zero utility principle is when utility function is exponential:

$$U(x) = \frac{1}{\beta}\left(1 - e^{-\beta x}\right)$$

for some fixed $\beta > 0$. In this case, premium principle becomes

$$\pi(X) = \frac{1}{\beta}\ln\left[\mathbb{E}\left[e^{\beta x}\right]\right].$$

This premium calculation is attractive since it is based on the moment generating function of the loss distribution and it is called exponential utility function premium principle.

2.2 Coherent Risk Measures

Many different kinds of risk measures have been introduced in literature, showing which one is reasonable or right is passed on stating axiomatic characterizations of these risk measures. For detailed explanations of axioms considered, we refer to see [64]. Indeed, deciding whether a risk measure is reliable or not is, in general, based on the idea of *coherent* risk measures. Artzner (1999) [2] stated that in order to regulate or manage risk effectively, axioms of coherence should be fulfilled for any risk measure. Economically, coherence of a risk measure, for example, has a meaning of consistency. Artzner et al. also point out that describing the risk with a single number may cause a great loss of information. However, choosing the correct risk measure may help decrease the effect of information loss. Furthermore, Bouwer, in [14], claims that using incoherent risk measure for reducing risk causes extreme risk taking, opposite diversification, and, it prompts blindness for investment cost etc.

Thus a coherent risk measure ρ should satisfy the following properties; for given two financial positions X and Y,

(i) $\rho(X + a) = \rho(X) - a$, for all $a \in \mathbb{R}$, (translation invariance)

(ii) $\rho(X) \leq \rho(Y)$ for $X \geq Y$, (monotonicity)

(iii) $\rho(aX) = a\rho(X)$ for all $a \geq 0$, (positive homogeneity)

(iv) $\rho(X + Y) \leq \rho(X) + \rho(Y)$, (subadditivity)

Translation invariance axiom implies that adding a constant amount of a, the risk decreases with that amount of a. Also, adding/subtracting $\rho(X)$ instead of a as a capital annihilates the risk:

$$\rho(X - \rho(X)) = \rho(X) - \rho(X) = 0.$$

Monotonicity axiom means that among two financial position, less risky financial position requires less money. *Positive homogeneity* is related to the linear utility which signifies that position of a risk linearly depends on its size. Seemingly, this property is not really desirable in insurance context since it calculates risk as a linear function of the scale. Another criticism of the positive homogeneity property in literature is the independence of currency, yet Denuit et al. [15] warn that it is the wrong interpretation. *Subadditivity* indicates that the risk of combined financial position is less than the separate risk of these financial positions, reflecting the diversification effect. In other words, as Artzner et al. [2, 3] state that 'a merger does not create additional risk'. Intuitively, it is possible to reduce economic capital required or the appropriate premium for a risk by pooling it [33]. Nevertheless, together with positive homogeneity, subadditivity guarantees the *convexity axiom*:

$$\rho[aX + (1-a)Y] \leq a\rho(X) + a\rho(Y) \quad \text{for} \quad a \geq 0.$$

Premium-based risk measures are described previously, are not coherent risk measures see [49] for detailed proofs. However, it is worth mentioning that the axioms that a risk measure satisfies rely on the conditions of economic environment. Those axioms above should be viewed as a typical set of rules or guidelines [17]. Having defined what is meant by coherent risk measure, now move on to discuss capital-based risk measures.

2.3 Capital-Based Risk Measures

Both premium-based and capital-based risk measures are governed to judge the risk. Nevertheless, premium-based risk measures are generally attached to pricing of the insurance. Moreover, Goovaerts et al. compare these two risk measures and indicate that mathematically both concepts are functionals, mapping random variables to a single real number, but the justifications and derivations are different [32].

This section is mainly interested in the use of risk measure as a determination of economic capital that is interpreted as a buffer for unexpected losses. Föllmer and Schied highlight that a risk measure might be seen as a capital requirement which helps the position be acceptable when added to position and then invested [26].

Below, the most common examples of capital-based risk measures in literature and financial institutions are going to be reviewed. Among those are Value-at-Risk (VaR) and its extension Conditional Value-at-Risk (CVaR) will be presented. While VaR is not a coherent measure, CVaR is; and, it is considered to be superior than VaR in many applications in finance.

2.3.1 Value-at-Risk

In actuarial risk theory, actuaries already have been using quantile as a risk measure for years. However, since 1990's using quantiles as risk measure has gained popularity in financial, particularly, in actuarial applications. First attempts start with the release of Risk Metrics in order to set up a standard in the market [45].

VaR can be regarded a statistical summary of all possible losses in a portfolio. In fact, it describes the maximum potential loss of a specific portfolio within a given time horizon and a confidence level α; for $X \sim F$, VaR_α is defined by the quantile function

$$\text{VaR}_\alpha(X) = F_X^{-1}(\alpha) = \inf\{x \in [0, \infty] : F_X(x) \geq \alpha\}.$$

VaR attracts many regulators by virtue of the following [19, 20, 58]:

- It is easy to calculate since it is basically statistical quantile function.
- It gives a single number representing all risk; this helps regulators understand and interpret the value of VaR easily in order to react accordingly.
- It is a probabilistic measure, that presents to risk managers informations associated with the amount of loss. Some of the traditional measures such as durations, Greeks do not indicate loss likelihood.
- It is a stable estimate as it neglects the tail of the underlying distributions.
- It measures maximum amount of loss likely to lose; this helps to determine the capital and the risk targets of the company.
- It ensures a more consistent and integrated approach to the management of different risks and provides greater risk transparency and disclosure.

These facilities make VaR popular among capital-based risk measures. However, VaR has also unfavorable properties:

- It is inefficient risk measure for skewed distributions.
- It may give conflicting results, because calculation of VaR depends on the specified confidence level.
- It cannot give any other information beyond the quantile.
- It is not a coherent risk measure since it does not satisfy the subadditivity property. Although this is necessary for diversification of portfolios, failing subadditivity sometimes may not be so important to reject VaR [33].
- It is non-linear, non-convex, and non-smooth; it has multiple local extrema which keep VaR off from optimization problems [42].

In order to avoid the drawbacks of VaR stated above, use of a coherent risk measure CVaR is proposed by Uryasev (2000) [3, 4]. We briefly discuss it in the following section.

2.3.2 Conditional Value-at-Risk

A VaR at a given confidence level α does not provide any further information about the loss beyond the quantile. In practice, this weakness might cause to give a wrong decision, as regulators do not only consider frequency of default but also they consider the severity of default. Similarly, they often want to know 'how bad is?', when judging the risk. Therefore, CVaR is proposed to quantify the risk beyond VaR and further, it is also a coherent risk measure [3, 4]. Roughly speaking, CVaR is a downside risk measure which determines the average loss (in the tail of the loss distribution) with a certain confidence level. Similar to VaR, for a given confidence level α, CVaR is defined:

$$\text{CVaR}_{\alpha}(X) = \mathbb{E}\left[X | X \geq \text{VaR}_{\alpha}(X)\right].$$

CVaR has superior properties, as given in [42, 58], than those of VaR; here are a couple of those:

- It leads to find the value beyond the VaR, which ensures the estimation of extreme tail losses, see Figure 2.1.
- It is a coherent risk measure satisfying all axioms in Section 2.2.
- It fits the loss distribution without distinction of skewness.
- It is convex and smooth; hence, it proves useful in related optimization problems of finance and actuarial sciences.

CVaR has also some criticism stated in [58]:

- CVaR is more sensitive to estimation error than VaR since accuracy of CVaR is extensively affected by the accuracy of the modeling of the tail. As a result, VaR may be regarded more robust than CVaR.
- In order to establish a reliable estimate, CVaR generally needs a large number of (realizations) observations. Even then, it may fail to estimate the most extreme potential losses.

Nonetheless, an important question still persists, in literature, for the choice of the risk measure [58]: which one should be preferred? Answer depends on the preferences and the objective in general;

- If high uncontrolled risk is needed, VaR can offer better results since it is more unrestricted than CVaR.
- For distributions with light tails, such as, normal or elliptical distributions which might be regarded an extension of multivariate normal distribution, CVaR may not perform well. Also, one should remember that under normal distribution CVaR converges to VaR.

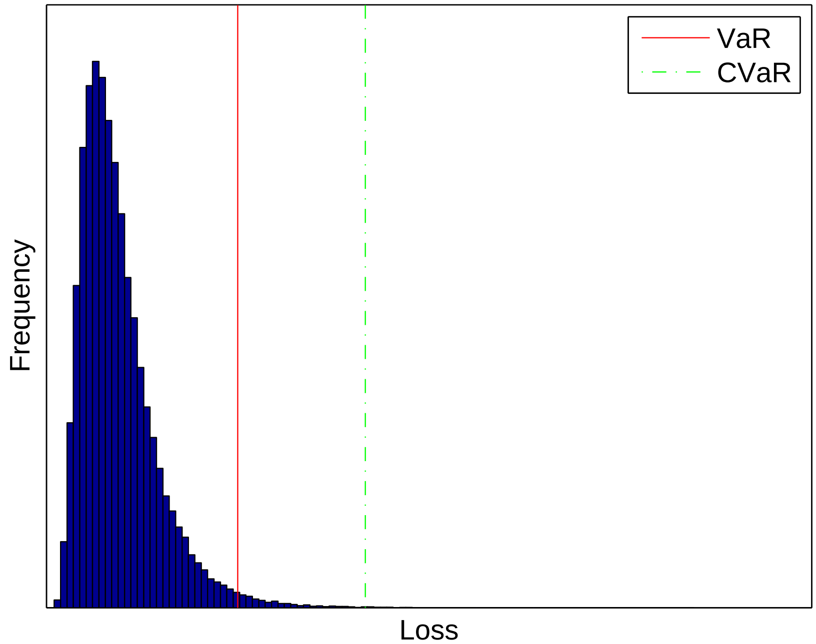


Figure 2.1: The graph shows that CVaR considers risk beyond VaR.

- CVaR needs a large number of observation, otherwise it does not give a consistent result.
- VaR is more stable (in terms of robustness) than CVaR in estimation problems, since VaR does rarely consider the tail of the distribution.

CHAPTER 3

BUILDING RISK MODELS and RUIN THEORY

In this chapter, step by step we construct different collective risk model in order to obtain stochastic surplus processes. After, we discuss the key elements of ruin theory framework.

3.1 Risk Models

In actuarial modeling of risk, two major approaches are conducted for aggregate loss: individual and collective models. Individual model considers n independent policies which may or may not have losses. Consequently, for a certain time period, it has to be constructed as two sources of variability: either loss occurs or not, and if so, the size of the loss.

Let X_i denote the claim size of contract i in the insurer's portfolio, the aggregate loss S_n is

$$S_n = \sum_{i=1}^{n} X_i. \tag{3.1}$$

It is important to emphasize that the amount of claim may be zero under individual policy which implies that individual model has probability mass at zero. It should also be noted that the mean and variance of S_n are given by

$$\mathbb{E}\left[S_n\right] = n\mathbb{E}\left[X\right] \quad \text{and} \quad \text{Var}\left[S_n\right] = n\text{Var}\left[X\right].$$

On the other hand, collective risk model examines the compound distribution of the aggregate loss which is inferred by counting the number of claims from the insurance portfolio, not from the individual policies. Instead of analyzing each policy separately, as in individual model, each claim is analyzed separately. It, therefore, consists of a stochastic $N(t)$ for the number of claims between the time intervals (to be discussed in the next section in detail). In the collective risk model, non-negative X_i refers to independent claims and

$$S(t) = \sum_{i=1}^{N(t)} X_i \tag{3.2}$$

is the aggregate loss. Here, we should impose the basic assumptions that

i. the number of claims is independent of the claims;

ii. individual claims is independent and identically distributed (i.i.d.).

These assumptions provide us the derivation of some distributional inferences for the total claim size. Herewith, we can reach the mean $\mathbb{E}\left[S(t)\right]$ an the variance Var $\left[S(t)\right]$ of aggregate claims, respectively,

$$\mathbb{E}\left[S(t)\right] = \mathbb{E}\left[N(t)\right]\mathbb{E}\left[X\right], \tag{3.3}$$

and

$$\text{Var}\left[S(t)\right] = \mathbb{E}\left[N(t)\right]\text{Var}\left[X\right] + \text{Var}\left[N\right]\left[\mathbb{E}\left[X\right]\right]^2. \tag{3.4}$$

Using collective model approach, namely, modeling distribution of claim numbers and claim sizes separately has various advantages [39]:

1. The expected number of claims changes according to number of insurance policy, which helps insurer to control growth in business volume for forecasting the number of claims in future using past year data.

2. Economic inflation and additional claims inflation which affects the losses incurred and these claims are paid back to insureds. Such inflation effects generally conceal when insurance policies have deductibles or limits which are independent of inflation. Thereby, aggregate results are used.

3. Influence of changes in deductibles and policy limits can easily be applied by changing the specifications of the distribution claim size. Also, influence of changes in deductibles can be understood better with respect to the claim number.

4. Non-covered losses, claim cost for insurers and claim cost of reinsurer can mutually be consistent which results in shifting losses to a reinsurer easier.

5. Understanding of the relative distribution of total claim is important to implement modified policy details. In collective model, distribution of total claims is defined as a combination of the number of claims and claim size distributions.

3.1.1 Models for the Claim Number Process

In this section, one of the benchmark in counting process, so is called Poisson process is covered. Since it has appealing theoretical properties, it is commonly used in applied probability and theory of stochastic processes. Then, we mention renewal process. For a more general version of Poisson process, we will briefly explain the renewal process and give its asymptotic properties.

3.1.1.1 Poisson Process

We start with recalling Poisson distribution with intensity $\lambda > 0$,

$$\mathbb{P}\left\{X = k\right\} = \frac{e^{-\lambda}\lambda^k}{k!}.$$

Also, we know that $\mathbb{E}\left[X\right] = \text{Var}\left[X\right] = \lambda$.

Definition 3.1. (Poisson Process [47]): A stochastic process $N(t)$ with intensity $\lambda > 0$, is called as a *Poisson process* if

(1) $N(0) = 0$,

(2) The process has independent increments with $N(t) - N(s)$ or $N(s,t]$ has a Poisson distribution with intensity $\lambda(t-s)$ for $t > s > 0$, and $\lambda > 0$,

(3) The process $N(t)$ is right-continuous and has left limit (*càdlàg process*)

For simplicity, we can also write a Poisson process as

$$N(t) = \#\{n \geq 1 : T_n \leq t\}, \quad t \geq 0,$$

where inter-arrival times the sequence of W_i, which are i.i.d. Exponential random variables with mean λ. The arrival times T_i's are such that $T_0 = 0$, $T_n = W_1 + \cdots + W_n, \quad n \geq 1$.

A Poisson process has some additional properties that we should mention.

(i) Since $N(0) = 0$ a.s., it follows that

$$N(t) - N(0) = N(0,t] \sim \textit{Poisson}\left(\lambda t\right).$$

(ii) The independent increment property facilitates to work with finite-dimensional distribution of N.

(iii) Cádlág process indicates that the value of the jump of the Poisson process added to the process already [40].

Figure 3.1 shows sample paths of Poisson processes with different intensities: $\lambda = 1$, $\lambda = 3$, $\lambda = 5$. For $\lambda = 1$ jumps occur less frequently, while for $\lambda = 5$ they occur more frequently.

Furthermore, Poisson process is divided into two parts as *homogeneous Poisson process* and *inhomogeneous Poisson process* which are closely related. Homogeneous Poisson process has a linear intensity function which has an intuitive meaning that claims arrive uniformly over time since it evolves linearly. In other words, this process adds to condition (2) in Definition 3.1 stationarity property. Moreover, homogeneous Poisson process is a primitive example of Lévy processes that satisfy the conditions: having stationary and independent increments with $N(0) = 0$, and being a càdlàg process. On the other hand, inhomogeneous Poisson process can be constructed non-constant intensity function that time slows down or speed up according to magnitude of $\lambda(t)$.

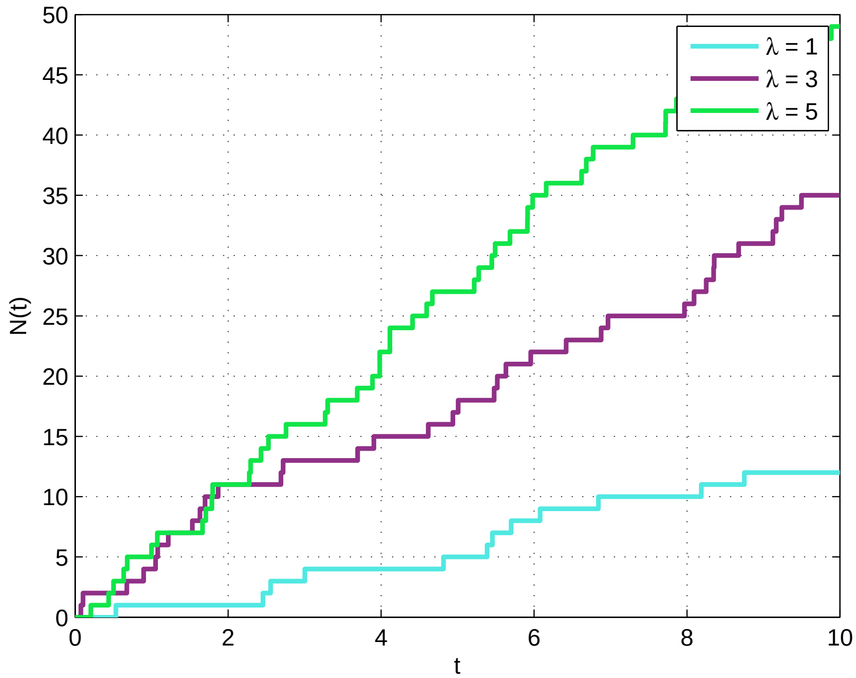


Figure 3.1: Poisson process with different intensities.

3.1.1.2 Renewal Process

When there is a large interval between the arrival times, Poisson process moves away from a being realistic model to describe arrivals. Then, for such situations, modeling the inter-arrival times via a distribution is necessary. In this respect, renewal process models occurrences at random times at which inter-arrival times are i.i.d. distributed random variables [53].

Definition 3.2. (Renewal Process [47]): A renewal sequence can be given

$$T_0 = 0, \quad T_n = W_1 + \cdots + W_n \quad n \geq 1,$$

where the variables W_i's are i.i.d. sequence of almost surely positive random variables, which refer to inter-arrival times and T_i's are the arrival times. The (renewal) counting process then can be written as

$$N(t) = \#\{n \geq 1 : T_n \leq t\}, \quad t \geq 0.$$

A Poisson process is a special case of renewal process when inter-arrival times are distributed as i.i.d. Exponential. Although, renewal processes are more preferable than Poisson processes, in the cases of large gaps between the arrival times and models for long-time period; homogeneous Poisson and renewal processes both have numerous asymptotic properties in common.

Theorem 3.1 (Strong Law of Large Numbers for the Renewal Process [47]). *If expectation of inter-arrival times $\mathbb{E}[W_1] = \lambda^{-1}$, is finite, then number of claim process satisfies the strong law of large numbers:*

$$\lim_{t\to\infty} \frac{N(t)}{t} = \lambda \quad a.s.$$

For homogeneous Poisson process, it is readily known that the exact value of expected claim number process is $\mathbb{E}[N(t)] = \lambda t$, whereas for a general renewal process, expectation of the renewal process is asymptotically $\mathbb{E}[N(t)] = \lambda t$ by Theorem 3.1.

Theorem 3.2 (Elementary Renewal Theorem [47]). *If the expectation of inter-arrival times $\mathbb{E}[W_1] = \lambda^{-1}$ exist, then*

$$\lim_{t\to\infty} \frac{\mathbb{E}[N(t)]}{t} = \lambda.$$

Proposition 3.3 (Asymptotic Behavior of the Variance of Renewal Model [47]). *Suppose that variance of the inter-arrival time exists, that is Var $[W_1] < \infty$. Then,*

$$\lim_{t\to\infty} \frac{\mathit{Var}[N(t)]}{t} = \frac{\mathit{Var}[N(t)]}{[\mathbb{E}[W_1]]^3}.$$

Theorem 3.4 (Central Limit Theorem for the Renewal Process [47]). *Suppose that variance of the inter-arrival time exists ($\mathit{Var}[W_1] < \infty$). Then, by the Central Limit Theorem*

$$\frac{N(t) - \lambda t}{\sqrt{\mathit{Var}[N(t)]\,[\mathbb{E}[W_1]]^{-3}}} \xrightarrow{d} Y \sim \mathcal{N}(0,1),$$

as $t \to \infty$.

3.1.2 Modeling Total Claim Size

The total claim size modeling, given in Eq. 3.2, there are many different approaches. Specifically, if the claim number follows a homogeneous Poisson process, the model is called Cramér-Lundberg model; on the other hand, if the claim number is a renewal process, the model Eq. 3.2 is called renewal or Sparre-Anderson model.

Broadly, finding analytical distribution of model Eq. 3.2 is hard to derive for an arbitrarily given distribution for the claim numbers or the claim sizes. For this reason, asymptotic properties are suggested to use in order to find how much premium should be charged for a given time period to avoid insolvency or avoid ruin. As shown in Section 2.1, premium calculation basically depends on identifying expectation and variance based on strong law of large numbers (Theorem 3.1), elementary renewal theorem (Theorem 3.2) and the Central Limit Theorem (Theorem 3.4).

Note that the expected value of $S(t)$ previously given Eq. 3.3 now changes to

$$\mathbb{E}[S(t)] = \lambda t \mathbb{E}[X_t],$$

under Cramér-Lundberg model. On the other hand, for the general renewal model such a compact formula does not exist. However, by using Theorem 3.2, if the expectation of inter-arrival times exist and equals to λ^{-1}, then we have $\dfrac{\mathbb{E}\left[N(t)\right]}{t} \rightarrow \lambda$ a.s. $t \rightarrow \infty$. So, the expectation of renewal model can be computed as

$$\mathbb{E}\left[S(t)\right] = \lambda t \mathbb{E}\left[X_1\right](1 + o(1)), \quad t \rightarrow \infty.$$

Furthermore, using the variance of total claim size in Eq. 3.4, the variance of Cramér-Lundberg model becomes $\mathbb{E}\left[N(t)\right] = \text{Var}\left[N(t)\right]$. Hence,

$$\begin{aligned}\text{Var}\left[S(t)\right] &= \lambda t[\text{Var}\left[X_1\right] + (\mathbb{E}\left[X_1\right]^2] \\ &= \lambda t \mathbb{E}\left[{X_1}^2\right].\end{aligned}$$

For the renewal model, again by using Theorem 3.2 and Proposition 3.3, the variance can be expressed as

$$\begin{aligned}\text{Var}\left[S(t)\right] &= [\lambda t \text{Var}\left[X_1\right] + \text{Var}\left[W_1\right]\lambda^3 t \mathbb{E}\left[X_1\right]^2](1 + o(1)) \\ &= \lambda t[\text{Var}\left[X_1\right] + \text{Var}\left[W_1\right]\lambda^2 \mathbb{E}\left[X_1\right]^2](1 + o(1)).\end{aligned}$$

Apart from the expectation and variance, asymptotic behavior of the renewal model can be stated by invoking Theorem 3.1 and Theorem 3.4 as follows:

(1) if the expectation of inter-arrival times W_i and claim sizes X_i exist, $S(t)$ satisfies the *strong law of large numbers*

$$\lim_{t \rightarrow \infty} \frac{S(t)}{t} = \lambda \mathbb{E}\left[X_1\right] \quad a.s. \tag{3.5}$$

(2) if the variance of inter-arrival times W_i and the claim sizes X_i exist, $S(t)$ satisfies the Central Limit Theorem

$$\sup_{x \in \mathbb{R}} \left| \mathbb{P}\left\{ \frac{S(t) - \mathbb{E}\left[S(t)\right]}{\sqrt{\text{Var}\left[S(t)\right]}} \leq x \right\} - \Phi(x) \right| \longrightarrow 0, \tag{3.6}$$

where Φ is the distribution function of the Standard Normal distribution.

3.1.3 Claim Size Distributions

Here, in this section, we introduce commonly used class of distributions in insurance business practice. These distributions are studied by some explanatory statistical tools such as *quantile–quantile (Q–Q) plot* which gives the best fit to the real life data, and another graphical tool: *mean excess plot* which helps discriminate the tail and decide whether the distribution is heavy-tailed or light-tailed.

Quantile–Quantile Plot (Q–Q plot). Q–Q plot is a scatter plot for identifying, at-a-glance, which distribution can give better fit to the insurance data. Firstly, it takes a set of observation i.e. empirical distribution, sort them in the ascending order, then plot them versus quantiles of reference distribution which is calculated as quantile of p_i where $p_i = (i + 1/2)/(n + 1)$.

If one see roughly linear line proceeding by points between two quantiles, it can be said that the data is distributed as presumed distributions. Contrary to common misunderstanding, the claimed distribution should not be Normal distribution necessarily.

In Figure 3.2, Q–Q plots of Generalized Pareto (top), Exponential (middle) and Normal distribution (bottom) are given for illustration. Since the aim of Figure 3.2 is to show whether the distribution is Exponential or not, simulated data quantiles are calculated versus simulated Exponential distribution quantile. In the Q–Q plot, if the data points nearly spread around the linear line, it means that input distribution is as the same as the assumed distribution. On the top figure, since data comes from the Generalized Pareto distribution the data points move away from the linear line on the right, whereas on the bottom figure, data are generated from Normal distribution, it can be seen that both left and right end of the linear line, there is a curve down shape. Also, one can reach an interpretation about the outliers of the distribution using Q–Q plots. For example, although data points almost fit the linear line in the middle figure, one or two points seem to be extreme. Likewise, on the top and bottom figures extreme points are clearly distinguishable.

Q–Q plot allows us to understand the following properties of a distribution stated by Chambers [11]: It

(1) ensures comparison of distribution by looking at almost linear relationship,

(2) finds outliers of the distribution which are the data values move away from the linear line appearance extremely, whereas the other data values scattered near the linear line,

(3) determines location and scale of the distribution which might be estimated by graphically through intercept and slope,

(4) enables to make inference for the shape of the distribution, e.g., if the data follows one of the heavy-tailed and right-skewed, the Q–Q plot is seen as nearly curving down shape at right-up of the linear line,

Mean Excess Plot. There is no precise way to decide whether claim size distribution is heavy-tailed or light-tailed. Yet, the intuitive approach is that, a distribution is a heavy-tailed if 20% of the claims account for more than 80% of the total claims [5]. Also, by accepting Exponential distribution as a benchmark, and denoting the right tail of distribution by $\overline{F}(x) = 1 - F(x)$ for $x > 0$, if,

$$\limsup_{x\to\infty} \frac{\overline{F}(x)}{e^{-\lambda x}} < \infty \quad \text{for some} \quad \lambda > 0,$$

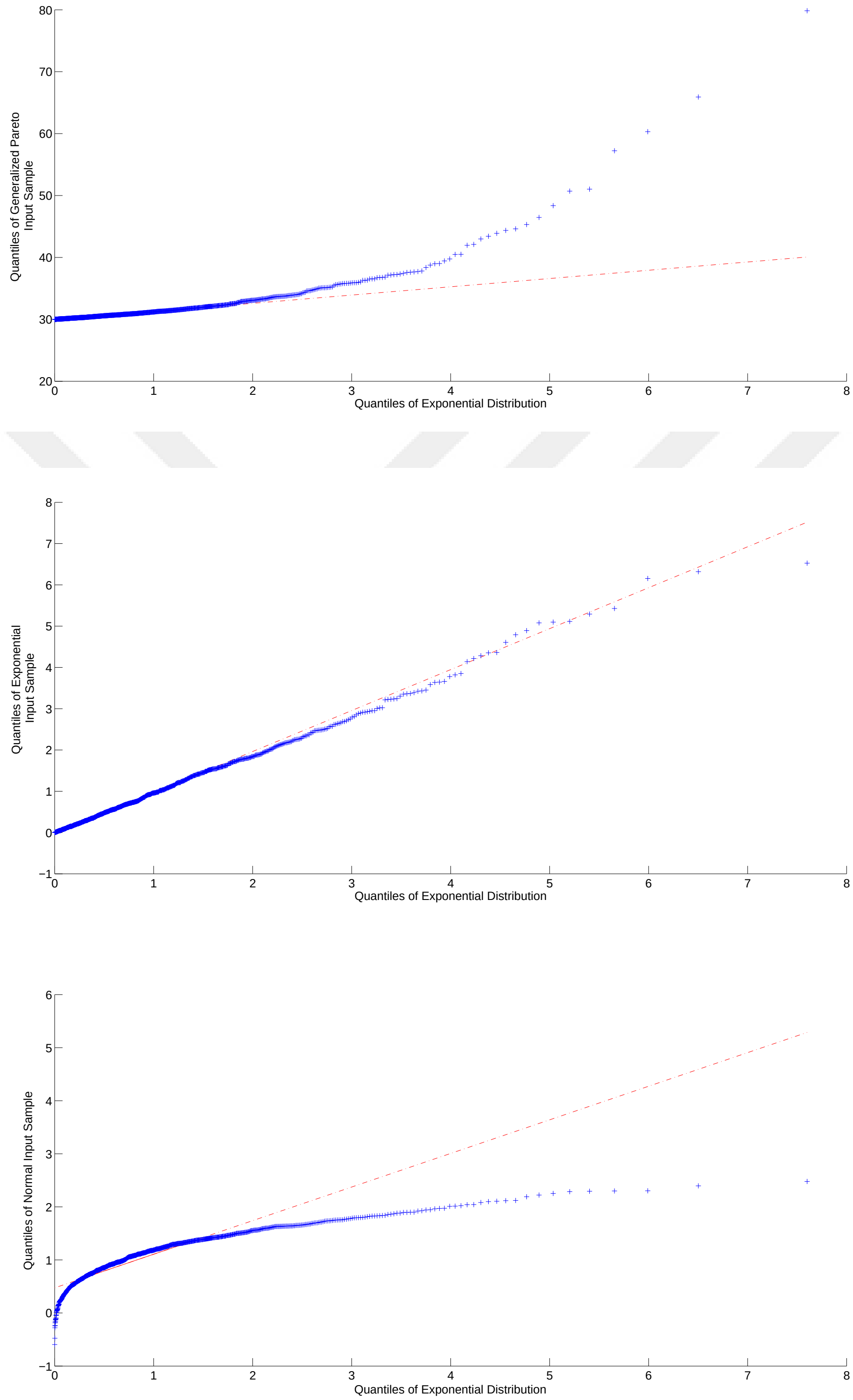


Figure 3.2: Q–Q plot of distributions with different characteristic.

then F is considered a light-tailed distribution. If, on the other hand,

$$\liminf_{x\to\infty} \frac{\overline{F}(x)}{e^{-\lambda x}} > 0 \quad \text{for all} \quad \lambda > 0,$$

then, F is a heavy-tailed one.

These two results help us interpret the tail of the distribution. While in graphical interpretation, mean excess plot deduces a comparison about thickness of the tail of the distribution.

Definition 3.3. (Mean Excess Function [25].) Given a non-negative random variable Y with finite mean, cumulative distribution F, and y being the right end point. Mean excess function (mean residual life function) is

$$e_F(u) = \mathbb{E}\left[Y - u \middle| Y > u\right], \quad 0 \le u \le y.$$

Mean excess function can also be written as

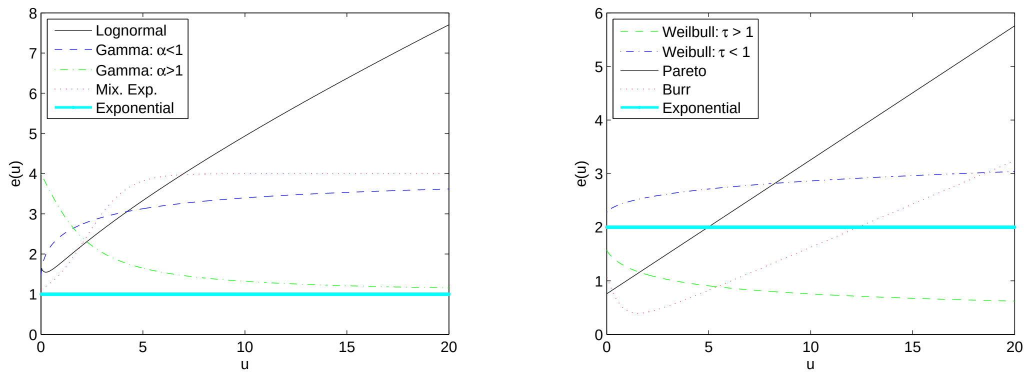
$$e_F(u) = \frac{1}{\overline{F}(u)} \int_u^\infty \overline{F}(y)dy, \quad u > 0.$$

If $e_F(u)$ converges to infinity as $u \to \infty$, then F is called heavy-tailed, otherwise if $e_F(u)$ converges to a finite constant as $u \to \infty$, F is called light-tailed. In insurance business practice, unlimited growth of mean excess function indicates the danger of the underlying distribution F in its right-tail. This means, given claim sizes X_i excess the high threshold u.

Mean excess functions for some distributions are illustrated in Figure 3.3, (for their mean excess functions, see [9]). As mentioned before, mean excess function approach exponential distribution is a benchmark since it has memoryless property. In other words, the expected value of $Y - u$ in Definition 3.3 does not change, whether that is conditioned on $Y > u$ or not. If the distribution has heavier tail than exponential distribution, $e(u)$ ultimately increases, if it has lighter tail, $e(u)$ deceases ultimately. Figure 3.3a shows shapes of Lognormal, Gamma with $\alpha < 1$ and $\alpha > 1$, mixture of Exponentials, and Exponential. In this panel of the figure, continuous increase in mean excess function in Lognormal, for example, proves that distribution has heavier tail, while mean excess function of Gamma with $\alpha > 1$ decreases which means that distribution has lighter tail. Figure 3.3b demonstrates mean excess function of Weibull with $\tau > 1$ and $\tau < 1$, Pareto, Burr and Exponential distributions. In this panel of the figure, for example, Pareto and Burr distributions have heavier tails according to Exponential, while Weibull with $\tau > 1$ has lighter tail.

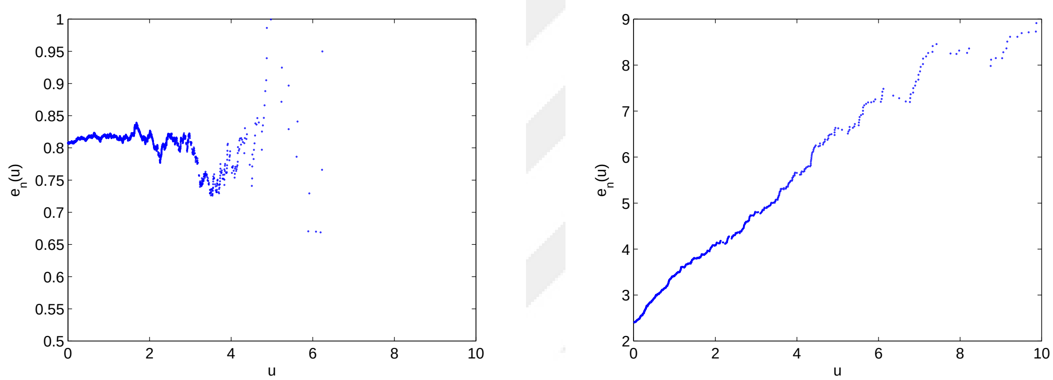
Graphical method is based on empirical mean excess function $e_{F_n}(u)$ considered in $u \in (X_{(1)}, X_{(n)})$, where $X_{(k)}$ represents kth order statistics. Given F_n empirical distribution has bounded support and by the strong law of large numbers, $e_{F_n}(u) \to e_F(u)$, a.s. as $n \to \infty$. The mean excess plot consists of the set

$$\left\{(X_{(k)}, e_{F_n}(X_{(k)})) : \quad k = 1, 2, \ldots, n-1\right\}.$$



(a) (b)

Figure 3.3: Shape of different mean excess functions for different distributions.



(a) Exponential(0.8) (b) Lognormal(0, $\log(4)$)

Figure 3.4: Examples of mean excess plot for specified distributions.

Figure 3.4 is drawn to depict graphical method of mean excess function. Figure 3.4a contains simulations of mean excess function having Exponential with mean 0.8, coinciding with Figure 3.3 for Exponential distribution, mean excess plot shows straight line around nearly mean 0.8, whereas in Figure 3.4b, we see a linear line shape which supports the Figure 3.3a for Lognormal mean excess function.

A note of caution is due here since the data might have sparsity, and hence the plot mean excess function calculates the larger threshold u, which may cause misleading information about heaviness of the tail. For this reason, one might consider different excess functions which are not affected by sparsity of the data, e.g., median excess function and respectively median excess plot [47].

Some Realistic Claim Size Distributions. After some statistical inference property about claim size distribution, some of the most popular distributions in non-life insurance modeling are mentioned below by classifying them as light-tailed and heavy-tailed.

a. Light-tailed distributions: such as Exponential, Gamma, Erlang, Phase-type distributions are in this category. As well as in many applied probability, Exponential distribution is a benchmark in risk theory. Especially, it is widely used in ruin theory framework, since compound Poisson process with Exponential claim sizes is employed to find closed-form solution for the ruin probability. Most prominent property of Exponential distribution is its *lack of memory*. The distribution which has favorable property is Gamma, due to being one of the infinitely divisible distributions which allows computational tractability in the calculation of the distribution of the total claim.

b. Heavy-tailed distributions: such as Weibull, Lognormal, Pareto, Loggamma, distributions with regularly varying tails and subexponential class of distributions are the examples of such distributions. Lognormal has heavier tail than Weibull distribution, and also it has the property that $\log X \sim \mathcal{N}(\mu, \sigma)$, Pareto or Generalized Pareto is particularly used for large claims modeling, and they settle a (large claim) threshold, say $\theta > 0$, then one can consider the claims above that threshold.

3.1.4 Distribution of Total Claim Size

Thus far, we focus on constructing the total claim size model with different assumptions and distributions of claim number and size. As much as constructing a model, detection of the distribution is significant for practitioners in the field. One might observe the distribution of the total claim size $S(t)$ in Eq. 3.2 by using characteristic function, by decomposing it as claim size state and time of a compound Poisson into independent compound Poisson processes. If these are not applicable, some numerical methods might be used such as Panjer recursion, fast Fourier transform (FFT), and some approximation methods, for instance, Central Limit Theorem (CLT), and Monte Carlo method.

One of the ways of finding distribution of total claim amount $S(t)$ is to use *characteristic function* and moment generating function (m.g.f.) of $S(t)$ which can be helpful especially compound Poisson and compound geometric cases. Assuming the independence of $N(t)$ and X_i, where $i = 1, 2, \ldots, N(t)$, the characteristic function of $S(t)$ can be found as

$$\begin{aligned}\phi_S(s) &= \mathbb{E}\left[\mathbb{E}\left[e^{is(X_1+X_2+\cdots+X_N)}\middle|N\right]\right] \\ &= \mathbb{E}\left[\mathbb{E}\left[(e^{isX_1})^N\right]\right] \\ &= \mathbb{E}\left[(\phi_{X_1}(s))^N\right] = M_N(\log \phi_{X_1}(s)).\end{aligned} \tag{3.7}$$

Moreover, if the total amount process can be written as a *mixture distribution* such as

$$G(x) = p_1F_1(x) + \cdots + p_nF_n, \quad x \in \mathbb{R},$$

where p_i is probabilities and F_i is distribution function of real-valued random variables when $i = 1, \ldots, n$, then, characteristic function of mixture distribution becomes

$$\phi(s) = p_1\phi_1(s) + \cdots + p_n\phi_n(s). \tag{3.8}$$

Likewise, with Eq. 3.8, one might reach a conclusion that sum of independent compound Poisson variables are again compound Poisson.

In addition to these, compound Poisson process can be decomposed into independent compound Poisson processes by presenting a disjoint partition on the time and claim size spaces. The detailed explanations and examples are presented in [47].

To find exact distribution of $S(t)$, one may use a *numerical method* so-called *Panjer recursion*. However, to use this recursion formula is based on a condition that the claim size should be expressed on a lattice form. In fact, it is good to emphasize that every continuous claim size distributions can be approximated by a lattice distribution and in real life models, for example, claim sizes can be explained by a lattice form in terms of monetary amounts.

Theorem 3.5. *(Panjer Recursion Formula [7])*

A compound random variable S, satisfying the condition

$$\frac{\mathbb{P}\{N=n\}}{\mathbb{P}\{N=n-1\}} = \alpha + \frac{\beta}{n} \quad \text{for} \quad n = 1, 2, \ldots$$

has the recursive equation for the distribution of claim size with the initial condition $f_s(0) = \mathbb{P}\{N=0\}$ as follows:

$$f_S(x) = \frac{1}{1-\alpha f_x(0)} \sum_{k=1}^{x} \left(\alpha + \frac{\beta k}{x}\right) f_x(k) f_s(x-k),$$

Although Panjer recursion method is the most extensively used technique to find distribution of total claim size, *fast Fourier transform (FFT)* method attracts many practitioners since it provides an easy and fast alternative with controllable error; and it neglects aliasing error. See [23, 37, 60].

A comparison of Panjer recursion and FFT, other numerical methods in order to find exact distribution of $S(t)$, one might refer to [23, 60].

Finally, as it is not easy to find the exact distribution of total claim $S(t)$, *approximation techniques* based on the Central Limit Theorem approximation and Monte Carlo method may be preferred.

Central Limit Theorem approximation fundamentally relys on the asymptotic behavior of renewal process. As in many statistical applications, asymptotic confidence interval for large t is found by using the Central Limit Theorem for $S(t)$,

$$\mathbb{P}\{S(t) \leq z\} \approx \Phi\left(\frac{z - \mathbb{E}[S(t)]}{\sqrt{\text{Var}[S(t)]}}\right),$$

where Φ is the distribution function of $\mathcal{N}(0,1)$ distribution. Then, for example, 95% confidence interval for average claim size can be found

$$\mathbb{P}\left\{S(t) \in \left[\mathbb{E}[S(t)] \mp 1.96\sqrt{\text{Var}[S(t)]}\right]\right\} \approx 0.95.$$

An insurance portfolio with high claim numbers, conditional on $N(t) = n(t)$, Central Limit Theorem provides good approximations. Given the finite mean and variance of total claim size $S(t)$, the probability becomes [47]

$$\mathbb{P}\left\{S(t) > z | N(t) = n(t)\right\} = \mathbb{P}\left\{\frac{S(t) - n(t)\mathbb{E}\left[X_1\right]}{\sqrt{n(t)\text{Var}\left[X_1\right]}} > \frac{z - n(t)\mathbb{E}\left[X_1\right]}{\sqrt{n(t)\text{Var}\left[X_1\right]}}\right\}$$

for large z. However, it is suggested that normal approximation should be avoided. Since the number of claims is random, if extreme event probabilities are of the interest, instead saddle point approximation can be more favorable to normal approximation.

One of the easiest way of computing the approximate distribution of $S(t)$ is *Monte Carlo technique*. If we know the distribution of claim number $N(t)$, and claim sizes X_i, where $i = 1, 2, \ldots, N(t)$, one can generate the i.i.d. samples for total claim size. Let

$$X_1^{(1)}, \ldots, X_{N_1}^{(1)}, \ldots, X_1^{(m)}, \ldots, X_{N_m}^{(m)},$$

where $X_i^{(j)}$ is the j^{th} generated variate from the selected distribution of i^{th} claim. Therefore,

$$S_1 = \sum_{i=1}^{N_1} X_i^{(1)}, \ldots, S_m = \sum_{i=1}^{N_m} X_i^{(m)}$$

Then the probability of $\mathbb{P}\left\{S(t) \in A\right\}$ for some Borel set A by the strong law of large numbers it follows that

$$\hat{p}_m = \frac{1}{m}\sum_{i=1}^{m} \mathbb{1}_A(S_i) \xrightarrow{a.s.} \mathbb{P}\left\{S(t) \in A\right\} = p = 1 - q \quad \text{as} \quad m \to \infty.$$

Note that $m\hat{p}_m \sim \mathcal{B}\left(m, p\right)$. The relative frequencies of $\hat{p}_m$ of the event A is p, which is called *crude Monte-Carlo simulation*. If one use larger samples or replications in Monte Carlo method, it is sure that better convergence in estimation is obtained. Also, using the Central Limit Theorem, asymptotic 95% confidence interval is

$$[\hat{p}_m \pm 1.96\sqrt{pq/m}].$$

Here, generic algorithm for simulating of a collective model is given in Algorithm 1. This algorithm lists the steps on generating total claim sizes conditional to random number of claims in an insurance portfolio.

The main pitfall of the crude Monte Carlo method is slow convergence. That can be settled up variance reduction techniques due to standard deviation error only decreases as a square root in terms of the required number of simulations [40]. Therefore, decreased variance speeds the computations with desired accuracy and less simulation runs. One of the variance reduction techniques is *importance sampling* which finds a distribution for the underlying random variables assigning a high probability to those values are important. Detailed explanations or examples can be found in [6, 40].

Another way to approximate $S(t)$ is *bootstrap technique* which is applicable with small sample size. Contrary to other approximation techniques, it does not require any information about distribution of X_i's, instead it uses information reached from the data.

Algorithm 1 Monte Carlo Simulation for Collective Models

for $m = 1, 2, \dots, k$ **do**

Simulate $N^{(m)}$ from a claim size distributions, e.g., Poisson, Geometric, Negative Binomial etc.

Simulate $X_1^{(m)}, X_2^{(m)}, \dots, X_N^{(m)}$ from a claim size distribution among the light-tailed or the heavy tailed distributions.

Calculate $S^{(m)} = \sum_{i=1}^{N^{(m)}} X_i^{(m)}$ (using Eq. 3.2).

end for

The bootstrap technique develops the idea that replacing the quantity based on unknown distribution F with the known empirical distribution F_n, then it simulates i.i.d. random variables from pseudo-samples of empirical distribution function of F_n. For detailed explanations, one can look at [22]. In insurance context, bootstrap approach ensures to approximate distribution of the aggregated claim sizes. Mikosch [47] urges that naive bootstrap does not work properly when one uses heavy tailed distributions and also bootstrap does not help to solve the probability of rare events.

This section has reviewed the constructing collective risk model with describing the key instruments of that model. Moreover, exact distribution of total claim size and approximations are discussed. In the next section, we will explain the ruin theory framework which uses collective risk model.

3.2 Ruin Theory

The amount of aggregate claims has vital impact on companies financial stability. A catastrophic claim size may result in insolvency of the insurance company. For this reason, it is vital to estimate the robustness of the firm under stressed conditions. Ruin probability is one of the important indicator to detect such situations.

Ruin theory considers stochastic behavior of capital of the insurance company. Pioneering work on ruin theory roots back to study of Lundberg (1903) on collective risk model. He modeled surplus process of an insurance company with compound Poisson process intuitively [46]. However, precise and critical works are carried out by Cramér [65] in order to implant foundations into mathematical context.

One of the main ingredient of the ruin theory is the surplus process: it explains the excess of initial capital raised by the constant rate premiums collected over claims. Aspects of insurance business in ruin theory leans on whether this process falls below zero or not.

We first launch continuous time surplus process where an insurance company collects premiums at constant rate, while losses may occur at any time at any size. The formulation of the ruin probability, its bound and asymptotic results are given. We drag ruin considerations to discrete time set-up as well, since in some applications yearly grid might be preferable.

3.2.1 Continuous Time Surplus Model

If an insurance company collects the premiums continuously over losses occurring at any time, the surplus process can be defined as:

$$U(t) = u + p(t) - S(t), \quad t \geq 0, \tag{3.9}$$

where, in the framework of Cramér-Lundberg model,

$$p(t) = ct \quad \text{and} \quad S(t) = \sum_{i=1}^{N(t)} X_i, \quad \text{for} \quad t \geq 0.$$

Here, $u = U(0)$ is the initial capital, the equity of an insurance company at $t = 0$. Deterministic function $p(t)$ is the premium income that is gathered from each contingent portfolio per unit of time and c is the premium rate such that $c > 0$. $S(t)$ given in Eq. 3.2 is an aggregate claims up to given unit time t. Claim amounts are independent of claim arrivals sequence which is described as

$$T_0 = 0, \quad T_n = W_1 + \cdots + W_n, \quad n \geq 1,$$

where W_i's are called inter-arrival times and assumed to be i.i.d.. Then, the claim number process is defined as

$$N_t = \#\{n \geq 1 : T_n \leq t\}, \quad t \geq 0.$$

We assume that the $X_i$is an i.i.d. sequence of positive claim sizes with the underlying distribution function F. Thereby, it is assumed that $N(t)$ is a Poisson process with intensity λ so that $S(t)$ is a compound Poisson process. Under these assumptions, the surplus process

$$U(t) = u + ct - \sum_{i=1}^{N(t)} X_i, \quad t \geq 0, \tag{3.10}$$

is rather well understood.

An illustration of surplus process is performed by assuming an exponentially distributed claim sizes with mean 2, yielding a compound Poisson process having $\lambda = 5$. Given the initial amount, $u = 150$ with premium rate $c = 11$ and a time frame $T = 7$ days with daily basis, the surplus process is simulated for 1000 runs and presented in Figure 3.5. It can be seen that by time $U(t)$ decreases dramatically. Some of the runs fall below zero and the others do not keep the financial stability. Histogram in the Figure 3.5 demonstrates the frequency of the final values of the surplus processes. As seen, small part of tail of the histogram implies the surplus processes which are below zero.

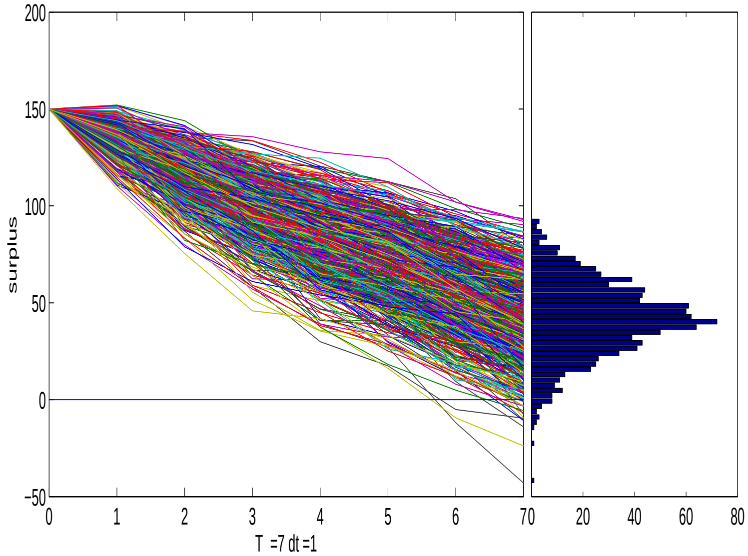


Figure 3.5: Paths of simulated surplus processes for compound Poisson claim and the histogram of final values of these surplus processes.

Therefore, having specified the surplus process $U(t)$, ruin event will occur when surplus process falls below zero, let us a give definitions related to ruin and ruin probability,

Definition 3.4. (Ruin, ruin time, infinite and finite ruin probability [47]) Ruin is an event defined as

$$\text{Ruin} = \{U(t) < 0, \quad \text{for some } t > 0\}.$$

The first time at which the ruin occurs is called the ruin time, which might implicitly depend on the initial capital:

$$\tau(u) = \inf\{t > 0 : U(t) < 0\}.$$

Hence, the probability of ruin for the initial capital $u > 0$ turns out to be

$$\psi(u) = \mathbb{P}\{\text{Ruin}|U(0) = u\} = \mathbb{P}\{\tau(u) < \infty\} \tag{3.11}$$

for the infinite time horizon. Likewise, for the finite time horizon T, as in the work [5] by Asmussen and Albrecher, the probability of ruin with the definition of Ruin $= \{U(t) < 0 : 0 < t < T\}$, and given the positive initial capital u is, therefore,

$$\psi(u, T) = \mathbb{P}\{\text{Ruin}|U(0) = u\} = \mathbb{P}\{\tau(u) < T\}; \tag{3.12}$$

it consequently may depend on the time horizon T.

As mentioned in [10], contrary to infinite time ruin probability, for exact ruin probabilities in finite time there is no such a method like Pollaczek-Khinchin formula, on the literature, it is possible to obtain partial integro-differential equation for probability of non-ruin and Asmussen [5] suggest that explicit formula of finite time ruin probability solely is known when the claims are Exponentially distributed, even for that numerical integration should be necessarily done.

Alternatively,

$$\text{Ruin} = \bigcup_{t\geq 0}\{U(t) < 0\} = \Big\{ \inf_{t\geq 0} U(t) < 0\Big\} = \{T < \infty\}$$

Ruin can occur only at the times $t = T_n$ for some $n \geq 1$, since $U(t)$ linearly increases in the interval $[T_n, T_{n+1})$.

We can also represent ruin in terms of the inter-arrival times W_n, the claim sizes X_n and the premium rate c,

$$\begin{aligned}
\text{Ruin} &= \Big\{ \inf_{t\geq 0} U(t) < 0\Big\} = \Big\{ \inf_{n\geq 1} U(T_n) < 0\Big\} \\
&= \Big\{ \inf_{n\geq 1}[u + p(T_n) - S(T_n)]\Big\} \\
&= \Big\{ \inf_{n\geq 1}[u + cT_n - \sum_{i=1}^{n} X_i]\Big\}.
\end{aligned}$$

Recalling the fact that

$$N(T_n) = \#\{i \geq 1 : T_i \leq T_n\} = n \quad \text{a.s.}$$

Since $W_j > 0$ a.s. is assumed for all $j \geq 1$, then, a random walk, S_n, can be characterized:

$$Z_n = X_n - cW_n, \quad S_n = Z_1 + \cdots + Z_n, \quad n \geq 1, \quad S_0 = 0.$$

Here, the sequence of W_i and X_i are mutually independent. As a result of this set-up, the following alternative expression for *ruin probability*, $\psi(u)$, with the initial capital u can be presented

$$\psi(u) = \mathbb{P}\left\{ \inf_{n\geq 1}(-S_n) < -u \right\} = \mathbb{P}\left\{\sup S_n > u\right\}.$$

Supposing that $\mathbb{E}[W_1]$ and $\mathbb{E}[X_1]$ exist, certain asymptotic results with strong law of large numbers,

$$\frac{S_n}{n} \xrightarrow{a.s} \mathbb{E}[Z_1] \quad \text{as} \quad n \to \infty.$$

allows

$$\mathbb{E}[Z_1] = \mathbb{E}[X_1] - c\mathbb{E}[W_1].$$

If $\mathbb{E}[Z_1] > 0$ is independent of the initial capital, ruin probability is one. In other words, ruin is inevitable which is stated also in Proposition 3.6.

Proposition 3.6. *(Ruin probability with one [47]) If $\mathbb{E}[W_1]$ and $\mathbb{E}[X_1]$ exist, and the condition*

$$\mathbb{E}[Z_1] = \mathbb{E}[X_1] - c\mathbb{E}[W_1] \geq 0,$$

then, the ruin occur with probability one for every $u > 0$.

Proof of the Proposition 3.6 is available in [66]. Proposition 3.6 expresses that if an insurance company avoids the ruin, it should choose the premium $p(t) = ct$ conditional to $\mathbb{E}[Z_1] < 0$. Therefore, *Net Profit Condition (NPC)* has to be stated.

Definition 3.5. (Net Profit Condition (NPC)) The surplus process satisfies the net profit condition if

$$c > \mathbb{E}[X]\,\mathbb{E}[N]\,. \tag{3.13}$$

Given the unit of time, NPC clarifies the business act of the insurance company with guaranteeing 'No Ruin' if the expected claim size is smaller than premium income. In other words, if the NPC is not satisfied, even for large values of initial capital, ruin occurs with probability one [13]. However, Mikosch [47] states that the ruin event is not going to be avoided completely, since the expectation might diverse due to the uncertain nature of the stochastic process. Therefore, in the calculation of constant premium rate, c, besides NPC, one should consider expected-value principle premium given in Eq. 2.1:

$$\begin{aligned} p(t) &= (1+\delta)\mathbb{E}[S(t)] \\ &= (1+\delta)\mathbb{E}[X]\,\mathbb{E}[N(t)] \\ &= (1+\delta)\frac{\mathbb{E}[X_1]}{\mathbb{E}[W_1]}t, \end{aligned}$$

where $\delta > 0$ is the safety loading. Consequently, the premium rate

$$c = (1+\delta)\frac{\mathbb{E}[X_1]}{\mathbb{E}[W_1]}$$

is obtained.

3.2.2 Bounds for Ruin Probability

In this section, upper bound for the ruin probability is presented. Assume that the renewal model with the NPC in Eq. 3.13, and a small claim condition, if m.g.f. of the claim size distributions exist. By the Markov's Inequality, we have

$$\mathbb{P}\{X_1 > x\} \le e^{-rx} M_{X_1}(r) \quad \text{for all} \quad x > 0.$$

As a result, one can infer that $\mathbb{P}\{X_1 > x\}$ decays to zero exponentially. Note that this inequality contradicts real life claim size examples which expose generally heavier tails, and non-existing m.g.f..

Definition 3.6. (Adjustment or Lundberg coefficient) Let m.g.f. of Z_1 exist in a neighborhood of the origin. The unique positive solution of the equation

$$M_{Z_1}(r) = \mathbb{E}\left[e^{r(X_1 - cW_1)}\right] = 1$$

exists, and the solution $r = \tilde{R}$ then $\tilde{R}$ is called *adjustment coefficient or Lundberg coefficient.*

Theorem 3.7 (Lundberg Inequality)**.** *Given the basic assumption is the renewal model with NPC, and the adjustment coefficient $\tilde{R}$ exists, then*

$$\psi(u) \leq e^{-\tilde{R}u}$$

holds for all $u > 0$.

As it is understood from Theorem 3.7, if the initial capital large enough, the probability of ruin becomes very small. As well as the initial capital, ruin probability depends upon the magnitude of the adjustment coefficient. Indeed smaller $\tilde{R}$ means getting a more riskier portfolio.

Exact Asymptotics for the Ruin Probability: the Small and Large Claim Sizes

The bounds for ruin probability are derived for small and large claim sizes. The asymptotics alter with respect to the magnitude of the claim size.

a. **Small Claim Size Case**

Theorem 3.8 (Cramér's ruin bound [47])**.** *For the Cramér-Lundberg model satisfying the NPC in Eq. 3.13, assume that claim size distribution function F_{X_1} has a density and m.g.f. of X_1 exists in an interval neighborhood of origin, at which adjustment coefficient $\tilde{R}$ exist. Then, there exists $C > 0$ such that*

$$\lim_{u\to\infty} e^{\tilde{R}u}\psi(u) = C,$$

where

$$C = \left[\frac{\tilde{R}}{\delta\mathbb{E}\left[X_1\right]}\int_0^\infty x e^{\tilde{R}x}\overline{F}_{X_1}(x)dx\right]^{-1}.$$

Here, δ denotes the safety loading, and tail of distribution function is

$$\overline{F}_{X_1} = 1 - F_{X_1}.$$

Renaming probability of no-ruin as 'non-ruin' probability, $\varphi(u)$, is denoted by

$$\varphi(u) = 1 - \psi(u),$$

as the expression in case of small claims which is presented in Lemma 3.9.

Lemma 3.9 (Fundamental integration for non-ruin probability [47])**.** *For the Cramér-Lundberg model satisfying the NPC in Eq. 3.13 and $\mathbb{E}\left[X_1\right]$ exists, assume that claim size distribution function F_{X_1} has a density. Then, the non-ruin probability is written as:*

$$\varphi(u) = \varphi(0) + \frac{1}{(1+\delta)\mathbb{E}\left[X_1\right]}\int_0^u \overline{F}_{X_1}(y)\varphi(u-y)dy. \quad (3.14)$$

Remark 3.1. $\varphi(0)$ can be evaluated as $\varphi(0) = \dfrac{\delta}{1-\delta}$ when $\varphi(u) \to 1$ as $u \to \infty$.

The proof is available in [47].

Analogously, we assume Cramér-Lundberg model with NPC condition. The Cramér bound for ruin probability for small claim size is defined by virtue of Theorem 3.8:

$$\varphi(u) = Ce^{-\tilde{R}u}(1 + o(1)) \quad \text{where} \quad u \to \infty.$$

Following theorem states the ruin probability for the large claim sizes.

b. **Large Claim Size Case**

Theorem 3.10 (Ruin probability when the integrated claim size distribution is subexponential claims [47])**.** *Suppose that the Cramér-Lundberg model with NPC and* $\mathbb{E}[X_1]$ *exist, also the claim sizes* X_i *have a density with integrated claim size distribution* $F_{X_i,\mathbb{1}}$ *is subexponential. Then the asymptotic relationship of ruin is probability*

$$\lim_{u\to\infty} \frac{\psi(u)}{\overline{F}_{X_i,\mathbb{1}}} = \delta^{-1}.$$

This theorem emphasizes that fundamentally the probability $\psi(u)$ is of the same order as $\overline{F}_{X_i,\mathbb{1}}$ which should not be neglected even if the initial capital u is large enough.

3.2.3 Discrete Time Surplus Model

Up to now, different ruin probabilities and their properties are found in the continuous time setting. As stated in [65], insurance companies and regulators cannot follow the surplus process continuously, since balance sheet estimates for an insurance company can be on regular basis e.g. quarterly, monthly, weekly, etc. If the companies use the discrete time surplus process, they have the opportunity of a slightly delay in claim payments. Moreover, computation of discrete time surplus process resulting approximation can be handled easier [5]. However, a drawback of this approach is that, generally, influence of model parameters on the final results cannot be traced and respectively, the qualitative behavior of ruin probabilities cannot be well recognized [5].

Again the same representation of initial capital u, and imposing the NPC with $\mathbb{E}[X_1] > 1$, the surplus process $U_d(u)$ for the discrete time setting is given by [5]

$$U_d(u) = u + n - \sum_{i=1}^{n} X_n, \quad n \in \mathbb{N}, \tag{3.15}$$

where claim size in a time unit i is X_i.

The ruin time for the discrete surplus model in Eq. 3.15 is

$$\tau_d(u) = \min\{n \geq 1 : U_d(u) \leq 0\},$$

and the ruin probability is defined as [5]

$$\psi_d(u) = \mathbb{P}\left\{\tau_d(u) < \infty\right\} = \mathbb{P}\left\{\min_{n\geq 1} U_d(n) \leq 0\right\}.$$

Moreover, on the discrete time set-up, adjustment coefficient $r = \tilde{R}_d$ is positive unique root of the equation

$$\mathbb{E}\left[e^{r(X_1-1)}\right] = 1,$$

if it exists.

Proposition 3.11. *Suppose that the adjustment coefficient exists, then probability ruin in discrete time is given by*

$$\psi_d(u) = \frac{e^{-\tilde{R}_d u}}{\mathbb{E}\left[\exp\{-\tilde{R}_d U_d(u)\}|\tau_d(u) < \infty\right]}.$$

Notably, the Lundberg inequality for the discrete ruin probability is $\psi_d(u) \leq e^{-\tilde{R}_d u}$.

Proof of Proposition 3.11 can be found in [5].

CHAPTER 4

IMPLEMENTATION

Development of financial engineering in insurance applications based on securitization of insurance products growing use of risk measures in regularity capital, and solvency requirement conduce increasing trend in popularity of risk measures [20]. Indeed, risk measures are highly crucial for capital allocation and evaluation of the performance of a company.

Ruin theory prospects the probability that a company bears for a specific time horizon with initial capital and collected premiums. Main ingredient of the underlying theory is based on the surplus process which resembles a business stream of the cash inflows and outflows of the company.

This chapter aims to find required capital for an insurance company in order to maintain its business activities and be aware of the early warning signal due to breakdowns. Works on risk measures derived from ruin theory framework such as [12, 16, 27, 48, 61], suggest the theoretical applicability of that the concept, although closed-form solution of ruin probability except for a few specific distributions like exponential, mixture exponential etc. cannot be derived. Therefore, results in these works are difficult to put into real life applications for practitioners in insurance sector. In the following section, we mention procedures that we use with all aspects.

An amount of initial capital which is a buffer against the insurance risk when premium income cannot compensate the future claim payments. We consider the risk measures stated accordingly for the surplus process in the framework of ruin theory.

We consider the surplus process as in Eq. 3.10,

$$U(t) = u + ct - \sum_{i=1}^{N(t)} X_i, \quad t \geq 0, \tag{4.1}$$

in which premiums are collected at the fixed rate c and realization of aggregate claims $S(t)$ are drawn from a given distribution. In principle, such a simple surplus process gives rise to skepticism on the modeling and applicability in real life problems. Rolski et al. [55], for instance, express that even though premium is not random for a given time, their calculation should include stochastic elements within the portfolio, based on the economic environment. Simple justification of this view might be attributed to either increase or decrease in the number of customers, events trigger unexpected large claims, or even, investment of the surplus on (domestic or foreign) financial (risky or volatile) markets.

In construction of the mathematical model for the surplus process, the effects of interest, or possibly inflation, should also be taken into account. Consequently, extension of the Cramér-Lundberg model by adding a perturbation of a Brownian motion to Eq. 4.1 is inevitable. Such an additional term may help model the (underlying) stochastic nature of the process; it may also explain other criticism of the classical model in literature.

Hence, we consider the extended surplus process at time t as

$$U_B(t) = u + ct + \sigma_B B(t) - \sum_{i=1}^{N(t)} X_i, \quad t \geq 0, \tag{4.2}$$

where $B(t)$ is a standard Brownian motion and independent of $S(t)$. Inclusion of the Brownian motion may also be regarded as a perturbation of the classical (unperturbed) Cramér-Lundberg model [55]. Indeed, such a model was first introduced by Gerber (1970) to capture additional uncertainties of the aggregate claims as well as the stochastic fluctuations in the premium income [21]. Since then, perturbed surplus models has been used by other researchers whose main concerns have been either the investment of the surplus [21] or the asymptotic behavior of the ruin or, simply, approximation of the Cramér-Lundberg model [59, 63].

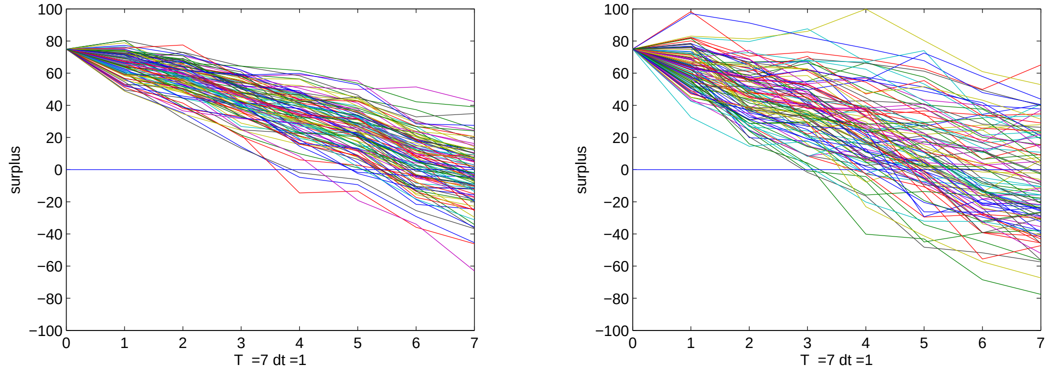
As a matter of fact, the probability of ruin is one of the main desired quantities of interest in such processes. Here, within the framework of the proposed surplus process, a ruin event may occur either by random oscillation of the process itself or, purely, by the jumps realized due to the aggregate claims.

We depict in Figure 4.1a simulated 7 days realizations of different surplus processes $U(t)$ given in Eq. 4.1 for Exponential claim sizes with mean 2 and a Poisson process with $\lambda = 5$ with a predefined initial capital 75. Alternatively, in Figure 4.1b perturbed model of Eq. 4.2 is depicted with diffusion coefficient $\sigma_B = 10$ using Eq. 4.2. It is clear that Figure 4.1b reveals marked oscillations in simulation paths which proves the occurrence of ruin incurred by both jumps due to aggregate claims and oscillation of the process.

Since the cardinal goal is to seek capital requirement in a monetary unit, configuration of the surplus model and its extension should be made. Thereby, given the surplus process as in Eq. 4.1, risk component, $S(t) - p(t)$ is redefined as

$$C(t) = \sum_{i=1}^{N(t)} X_i - ct \quad t \geq 0, \tag{4.3}$$

where the fixed premium rate $c > 0$. Recall that aggregate claim $S(t)$ is a compound Poisson process fulfilling the assumptions that $N(t)$ is homogeneous Poisson process with intensity $\lambda > 0$, the X_i are i.i.d. with mean $\mathbb{E}\left[X_i\right] = \mu > \infty$ and $N(t)$ and X_i are mutually independent.



(a) Surplus processes

(b) Extended surplus processes

Figure 4.1: Different simulation paths of two models.

Respectively, we use extended surplus process in Eq. 4.2 which captures uncertainties in premium, and aggregate claim. The risk component for the extended model under the same assumptions is as follows:

$$C_B(t) = \sum_{i=1}^{N(t)} X_i - ct - \sigma_B B(t), \quad t \geq 0. \tag{4.4}$$

Considering the aggregate claim under different distribution assumptions, we simulate paths $C(t)$ or $C_B(t)$ using Algorithm 2.

Algorithm 2 Simulating Risk Components

for $m = 1, 2, \ldots, k$ **do**
  Simulate $N^{(m)}(t)$ from Poisson distibution with the intensity λt where $t \in [0, T]$
  Simulate $X_1^{(m)}, X_2^{(m)}, \ldots, X_N^{(m)}$ from any given claim size distribution
  Calculate $S^{(m)} = \sum_{i=1}^{N^{(m)}} X_i^{(m)}$,
  Calculate $C^{(m)}$
  (or $C_B^{(m)} = C^{(m)} - \sigma_B \sqrt{t} Z^{(m)}$
  where $Z^{(m)}$ is a standart normally distributed random variable for mth run)
end for

Having defined the risk component for both classical surplus and extended model, for a defined time interval we determine the value which address the required capital by using capital-based risk measures: VaR and CVaR. So, at α confidence level, given the distribution F, VaR and CVaR for any risk component C are defined, respectively,

$$\text{VaR}_\alpha(C) = F_C^{-1}(\alpha),$$

and

$$\text{CVaR}_\alpha(C) = \mathbb{E}\left[\, C \mid C \geq \text{VaR}_\alpha(C)\right].$$

For the estimation of VaR and CVaR of risk components, there are couple of methods proposed in literature. Basically, VaR can be obtained by parametric, non-parametric or Monte Carlo simulation methods. Among these Monte Carlo simulation technique is more powerful and flexible than others, since randomly generated loss distribution takes into account nearly all degree of complexity [20]. It carries on large number of trials which promises a good approximation to unknown distribution we want to know. Moreover, this method is considered as an appropriate way to cope with complicating factors such as valuation problems, badly behaved distribution, nonlinearity, parameter and model risk, long horizons, etc. Indeed, based on the given distribution for the claim sizes, having an idea about the distribution of the risk components $C(t)$ and $C_B(t)$ may be quite difficult to obtain analytically. Thus, simulation, as in the Algorithm 3 might be necessary.

Algorithm 3 Simulating VaR and CVaR Estimates of Risk Components

---

for $m = 1, 2, \ldots, k$ **do**

  Simulate $N^{(m)}(t)$ from Poisson distibution with the intensity λt where $t \in [0, T]$.

  Simulate $X_1^{(m)}, X_2^{(m)}, \ldots, X_N^{(m)}$ from any suitable claim size distributions.

  Calculate $S^{(m)} = \sum_{i=1}^{N^{(m)}} X_i^{(m)}$,

  Calculate $C^{(m)}$, (or $C_B^{(m)} = C^{(m)} - \sigma_B \sqrt{t} Z^{(m)}$,

  where $Z^{(m)}$ is a standart normally distributed random variable for mth run.)

end for

Calculate VaR(C), CVaR(C) (or VaR(C_B), CVaR(C_B))

Having constructed the risk components, for simulations, one more step remains: the choice of distribution and parameter. As mentioned in Subsection 3.1.3 claim size distribution, in modeling non-life insurance commonly analyzed via light-tailed distributions. Hence we use Exponential and Gamma. To represent heavy-tailed distributions we take Weibull, Lognormal and Pareto, which are used generally in catastrophic rare events. For comparison, all claim size distributions satisfy the same mean, say an arbitrary chosen value 3 That is Exponential distribution with mean 3, Gamma distribution with shape 3 and scale 1, Weibull distribution with shape 0.5, scale 1.5, Lognormal distribution with log mean 1, and log standard deviation 0.4441. Instead of Pareto distribution, for easiness we use MATLAB random generator. For Generalized Pareto with shape 0.5, scale 0.5, threshold 2 is used which is equivalent to Pareto distribution with shape 1, scale 2.

We recall that one of the stated model assumptions is that claim numbers are generated by a Poisson distribution with a given intensity. For simplicity, we choose arbitrarily a Poisson process with intensity $\lambda = 10$. Also, a vital requisite is that the premium rate should be greater than 30 due to NPC in Eq. 3.13 so that we select the premium rate to be $c = 31$.

With all parameters and distributions, simulations for risk components are attained for $k = 10000$ times, and estimations of VaR and CVaR at 95% confidence level for different time horizons (1-year, 2-year, 5-year, 10-year) in daily basis are obtained. One should note that this time basis infers to checking the risk components for both model in that range: if the company want to check their risk components weekly for a year, it would be appropriate to choose take one year as 52 times 7 days, thereby time basis is taken as 7.

Furthermore, we simulate ruin probabilities for each risk components under these distributions and parameters in specified time horizons by using Monte Carlo techniques. Indeed, we take VaR and CVaR estimations that are obtained via simulations and we use them as initial capital, then in selected time horizons with defined basis, we estimate the ruin probabilities for the given claim size distributions.

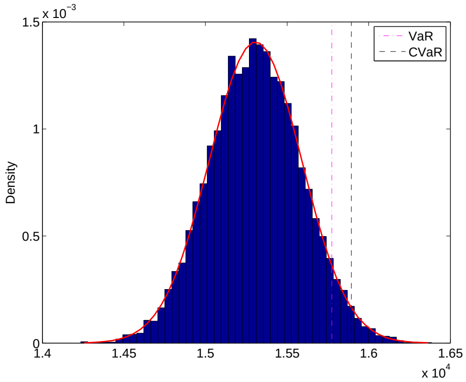
For the comparison of light-tailed and heavy tailed distributions for the claim size, we make use of the Exponential and Pareto distributions, for the other simulation results when claim size distribution is Gamma, Weibull and Lognormal whose results are given in Appendix A and Appendix B.

Figure 4.2 and Figure 4.3 summarize the risk component behavior when claim sizes are distributed as Exponential with mean 3, with and without perturbation of a Brownian motion, respectively, with $\sigma_B = 0$ and $\sigma_B = 1$. The paths generated on the 1-year, 2-year, 5-year, 10-year in daily basis time horizons. As can be easily be inferred from the histograms CVaR considers tail better than VaR. When time horizon increases, gradually fit better to a normal distribution.

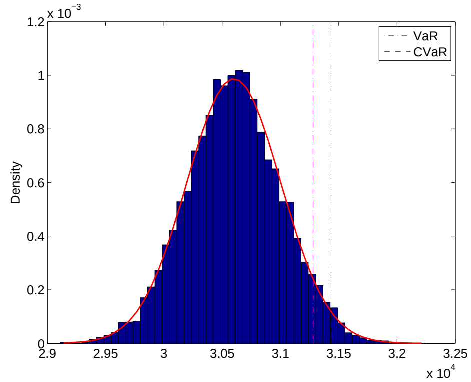
To go deep inside the figures, we tabulate the results in Table 4.1 and Table 4.2. VaR and CVaR estimates and their corresponding ruin probabilities are presented. First, as expected, when the time horizon increased from 1-year to 10-year increase in VaR and CVaR estimates are obvious. Accordingly, these tables when we allocate the initial capital by using CVaR estimates instead of using VaR estimates, we can clearly catch up the difference in ruin probabilities. It is apparent to deduce that even in slight increase between VaR and CVaR values, ruin probabilities decline nearly half in all cases. These decreasing ruin probabilities suggest that a risk averse insurance company should fancy CVaR in allocation of required capital; this considerably minimizes its insurance risk.

In order to detect the effects of this perturbed model, the values in Table 4.2 are compared to those in Table 4.1. Along with the specified time horizon, there is a small difference among the estimates of both VaR and CVaR. Consequently, unsteady increase or decrease in ruin probabilities of these estimates appear. However, this should not be considered bad, for the fact that perturbed models is used to reflect uncertainties in premiums, aggregate claims and market conditions.

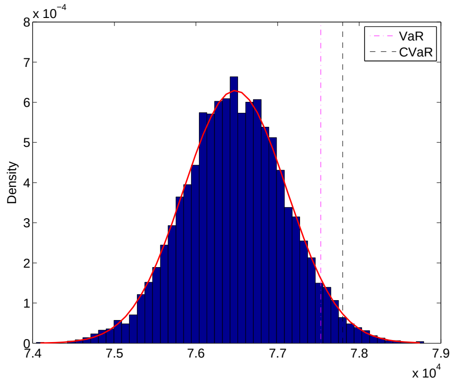
Panjer [51] stated that lower percentage may reflect the inter-unit diversification that exists. In this study, all simulations are investigated in 95% confidence level; however, it is known that solvency regulation uses a confidence level of 99.5% for entire enterprise. For that reason, we simulate surplus processes in order to find capital requirement for 99.5% confidence level, and find their ruin probabilities associated to those capitals. When we compare Table 4.3 and Table 4.4 with Table 4.1 and Table 4.2, as it is expected, all VaR and CVaR estimates rise, consequently, corresponding ruin



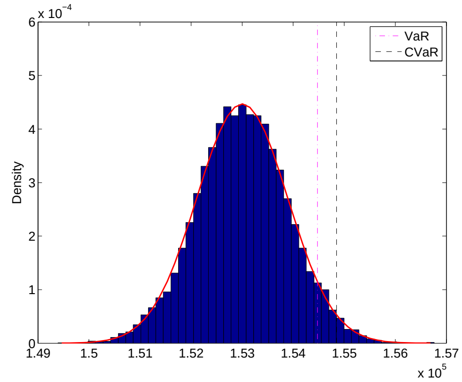
(a) 1-year with daily basis.



(b) 2-year with daily basis.



(c) 5-year with daily basis.



(d) 10-year with daily basis.

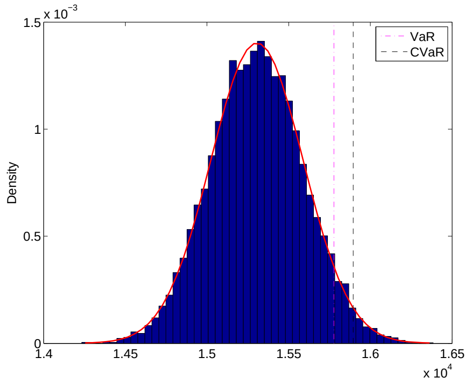
Figure 4.2: Simulated risk components with claim distribution Exponential(3), and $\sigma_B = 0$.

Table 4.1: $\text{VaR}_{0.95}$ and $\text{CVaR}_{0.95}$ estimates and the ruin probabilities with claim size process Exponential(3) and $\sigma_B = 0$.

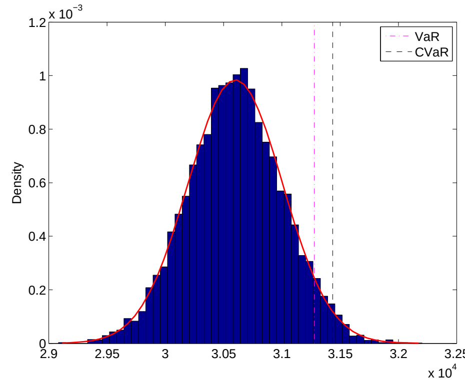
Time	VaR	CVaR	ψ(VaR)	ψ(CVaR)
dt = 1; T = 365	15774.5542	15893.5989	0.0430	0.0170
dt = 1; T = 2×365	31281.7036	31435.7473	0.0537	0.0212
dt = 1; T = 5×365	77530.3213	77798.0859	0.0517	0.0192
dt = 1; T = 10×365	154475.6833	154850.5623	0.0131	0.0049

Table 4.2: $\text{VaR}_{0.95}$ and $\text{CVaR}_{0.95}$ estimates and the ruin probabilities with claim size process Exponential(3) and $\sigma_B = 1$.

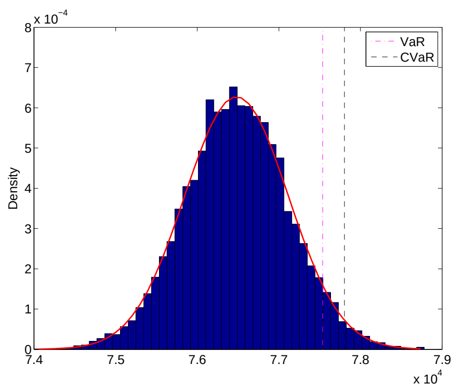
Time	VaR	CVaR	ψ(VaR)	ψ(CVaR)
dt = 1; T = 365	15776.5397	15895.2196	0.0422	0.0173
dt = 1; T = 2×365	31278.6609	31437.8074	0.0563	0.0218
dt = 1; T = 5×365	77536.6294	77804.3774	0.0503	0.0192
dt = 1; T = 10×365	154487.2658	154857.3203	0.0124	0.0047



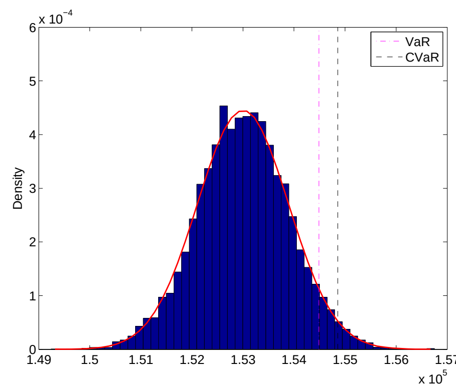
(a) 1-year with daily basis.



(b) 2-year with daily basis.



(c) 5-year with daily basis.



(d) 10-year with daily basis.

Figure 4.3: Simulated risk components with claim distribution Exponential(3), and $\sigma_B = 1$.

Table 4.3: $\text{VaR}_{0.995}$ and $\text{CVaR}_{0.995}$ estimates and the ruin probabilities with claim size process Exponential(3) and $\sigma_B = 0$.

Time	VaR	CVaR	ψ(VaR)	ψ(CVaR)
dt = 1; T = 365	16040.6107	16124.9383	0.0048	0.0020
dt = 1; T = 2×365	31625.3920	31768.0860	0.0053	0.0018
dt = 1; T = 5×365	78133.2822	78335.3219	0.0059	0.0028
dt = 1; T = 10×365	155288.3805	155581.6487	0.0013	0.0003

Table 4.4: $\text{VaR}_{0.995}$ and $\text{CVaR}_{0.995}$ estimates and the ruin probabilities with claim size process Exponential(3) and $\sigma_B = 1$.

Time	VaR	CVaR	ψ(VaR)	ψ(CVaR)
dt = 1; T = 365	16044.9878	16129.8398	0.0044	0.0021
dt = 1; T = 2×365	31635.5772	31769.2284	0.0056	0.0017
dt = 1; T = 5×365	78131.5395	78341.2831	0.0058	0.0027
dt = 1; T = 10×365	155302.7512	155578.5398	0.0010	0.0004

probabilities when those values used as initial capitals are markedly decrease.

As Figure 4.4 presents changes in ruin probabilities, when time increases from 1-year to 10-year using daily basis. It is striking that in 10-year analysis, with the help of fitted line through the points, ruin probabilities for both VaR and CVaR drop. This is clearly explained with rising VaR and CVaR estimates given in Table 4.1.

On the other hand, we consider the influence of time basis on the ruin probabilities, since an insurance company checks its surplus process in different time basis referring to a specific time discretization. Here, we simulate models when surplus of an insurance company is controlled daily, monthly, quarterly and semi-annually within 1-year. In Figure 4.5 depicts the effect of such time discretization on the ruin probabilities; clearly, as time discretization increases, ruin probabilities increase.

Furthermore, in order to see the effects of the diffusion coefficient in the model with perturbation of Brownian motion in ruin probabilities, we check over ruin probabilities of VaR and CVaR by increasing the diffusion coefficient σ_B, $0 \leq \sigma_B \leq 100$. We infer from Figure 4.6 that adding diffusion coefficient does not provide a stable ruin probability according to situations which cause uncertainties in premiums and aggregate claims.

Similarly, Figure 4.7 and Figure 4.8 show demonstration of a heavy-tailed claim size distributed risk component behavior for both unperturbed and perturbed models. As stated all claim size mean is 3; that is, Pareto(1, 2) is generated. At a first glance, Figure 4.7 and Figure 4.8 seems to be different from Figure 4.2 and Figure 4.3. That is, due to the fact that Pareto distribution is a model for low frequency and high severity situations, namely, for extreme events. In such situations it is natural to expect higher estimates of VaR and CVaR.

Table 4.5 and Table 4.6 are constructed with the estimates of VaR and CVaR when

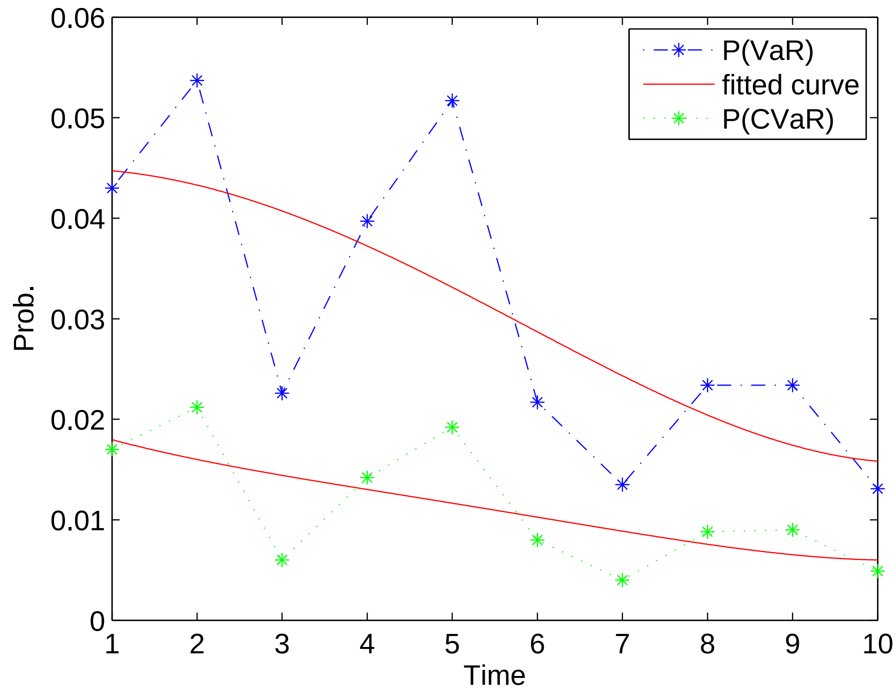


Figure 4.4: Change in ruin probabilities when $T = 1, 2, \ldots, 10$, and $\sigma_B = 0$.

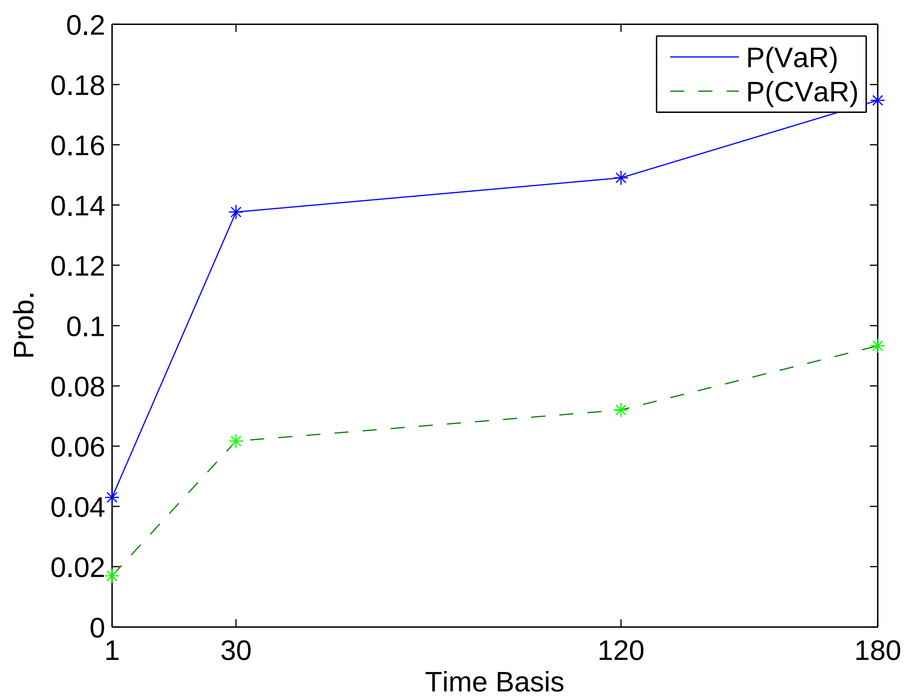


Figure 4.5: Change in ruin probabilities of Exponentially distributed claim when $dt = 1$ (daily), 30 (monthly), 120 (quarterly), 180 (semi-annually) and $\sigma_B = 0$.

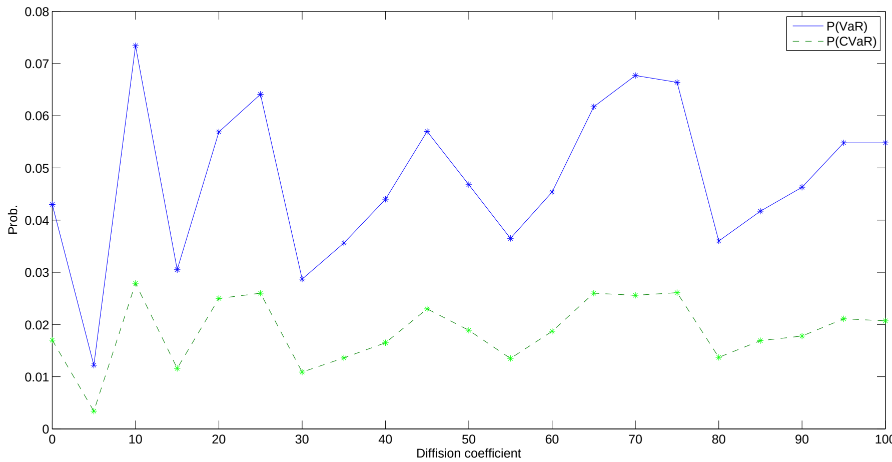
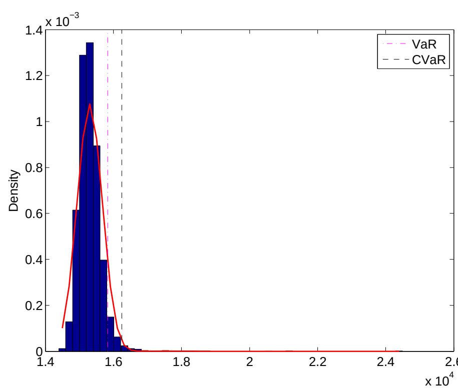
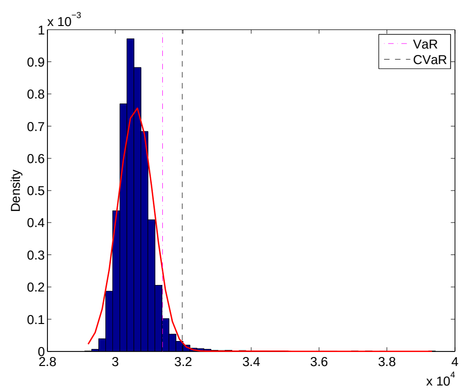


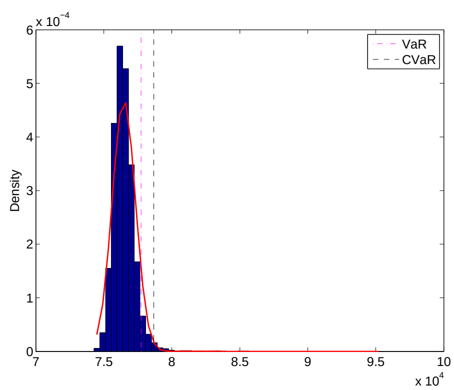
Figure 4.6: Change in ruin probabilities of Exponentially distributed claim size when σ_B increases.



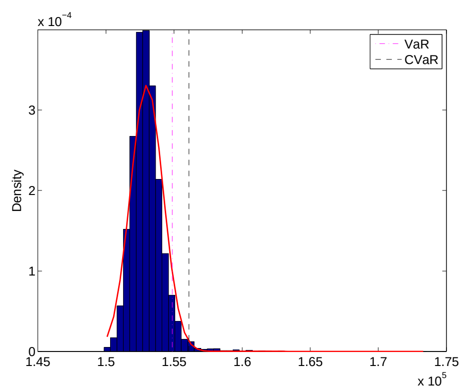
(a) 1-year with daily basis.



(b) 2-year with daily basis.

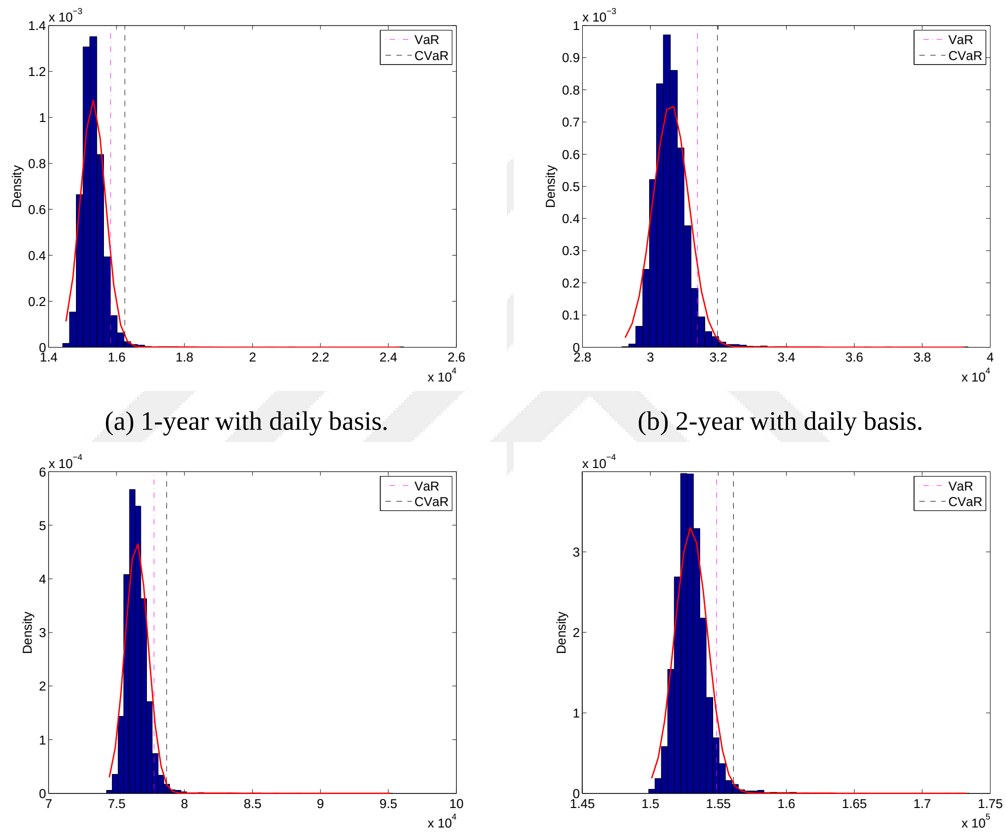


(c) 5-year with daily basis.



(d) 10-year with daily basis.

Figure 4.7: Simulated risk components with claim distribution Pareto(1, 2), $\sigma_B = 0$.



(a) 1-year with daily basis.

(b) 2-year with daily basis.

(c) 5-year with daily basis.

(d) 10-year with daily basis.

Figure 4.8: Simulated risk components with claim distribution Pareto(1, 2), $\sigma_B = 1$.

Table 4.5: $\text{VaR}_{0.95}$ and $\text{CVaR}_{0.95}$ estimates and the ruin probabilities when claim size process follows Pareto(1, 2) and $\sigma_B = 0$.

Time	VaR	CVaR	ψ(VaR)	ψ(CVaR)
dt = 1; T = 365	15832.8755	16243.2002	0.0514	0.0127
dt = 1; T = 2×365	31391.3823	31973.7217	0.0477	0.0113
dt = 1; T = 5×365	77744.3984	78683.1634	0.0553	0.0129
dt = 1; T = 10×365	154868.4314	156097.6582	0.0279	0.0082

Table 4.6: $\text{VaR}_{0.95}$ and $\text{CVaR}_{0.95}$ estimates and the ruin probabilities with claim size process Pareto(1, 2) and $\sigma_B = 1$.

Time	VaR	CVaR	ψ(VaR)	ψ(CVaR)
dt = 1; T = 365	15830.0355	16244.3285	0.0517	0.0129
dt = 1; T = 2×365	31389.4173	31973.8131	0.0481	0.0112
dt = 1; T = 5×365	77762.0244	78685.2456	0.0541	0.0129
dt = 1; T = 10×365	154870.2301	156103.4810	0.0284	0.0080

claim size are distributed as Pareto(1, 2): not surprisingly, increasing estimate results are faced again. Furthermore, when diffusion coefficient σ_B is choosen to be 1, rise and decline of the estimates of VaR and CVaR induces unsteady increase and decrease in ruin probabilities, as seen in Table 4.6.

Finally, in order to indicate differences between choosing 95% and 99.5% for a heavy-tailed distribution such as Pareto, VaR and CVaR estimates and their corresponding ruin probabilities are displayed in Table 4.7 and Table 4.8. Interestingly, in these tables significant increase of VaR and CVaR estimates are observed. That can be reasoned by these estimates of heavy-tailed distributions at 99.5% confidence level move further beyond the tail. Inversely, depending on these increased estimates, corresponding ruin probabilities decrease.

For the other distributions that considered in this study the same analyses are done and summarized in Appendix B for 99.5% degree of certainty.

Table 4.7: $\text{VaR}_{0.995}$ and $\text{CVaR}_{0.995}$ estimates and the ruin probabilities with claim size process Pareto(1, 2) and $\sigma_B = 0$.

Time	VaR	CVaR	ψ(VaR)	ψ(CVaR)
dt = 1; T = 365	16689.4173	17935.5261	0.0057	0.0014
dt = 1; T = 2×365	32727.8489	34227.7038	0.0037	0.0016
dt = 1; T = 5×365	79782.1542	82267.8557	0.0045	0.0014
dt = 1; T = 10×365	158066.2547	160527.9438	0.0028	0.0016

Table 4.8: $\text{VaR}_{0.995}$ and $\text{CVaR}_{0.995}$ estimates and the ruin probabilities with claim size process Pareto(1, 2) and $\sigma_B = 1$.

Time	VaR	CVaR	$\psi(\text{VaR})$	$\psi(\text{CVaR})$
dt = 1; T = 365	16690.9627	17934.7743	0.0056	0.0014
dt = 1; T = 2×365	32725.0879	34223.5661	0.0037	0.0016
dt = 1; T = 5×365	79774.8693	82268.4054	0.0046	0.0014
dt = 1; T = 10×365	158061.7920	160528.5048	0.0028	0.0016

CHAPTER 5

CONCLUSION

In this study, we bring into focus of determining initial capital that any insurance company should allocate to recompense all possible contingent claims. Indeed, required capital ensures the insurance company from insolvency. We base upon ruin theory framework and risk measures in calculation of the required capital. First, we take basic surplus model, and and then, we extend it by model with adding a perturbation of Brownian motion to explain uncertainties in aggregate claims better. This help us understand the uncertainties in premiums that might be due to economic conditions, for instance, insurance market volatility and underwriting factors such as factors that cause large claims, and increase or decrease in policyholder. Then, we constitute risk components via removing initial capital, and taking a negative position in the surplus processes. After simulating these risk components under different claim size distributions, namely, Exponential, Gamma, Weibull, Lognormal, Pareto, we track down VaR and CVaR estimates for various time units. Accordingly, taking VaR and CVaR as the proposed initial capital, we calculate the associated ruin probabilities.

Findings of this study suggest that in all models, not surprisingly if the claim size follows heavy-tailed distributions like Weibull and Pareto, VaR and CVaR estimates are high, when compared to those distributions with light tails. Their ruin probabilities by writing VaR and CVaR estimates as an initial capital at different times $T = 1$, $T = 2$, $T = 5$, and $T = 10$ with daily basis are generally higher than light-tailed distributions.

Furthermore, in this study, we observe that ruin probabilities obtained by using CVaR estimate are always halfed in all simulations. That yields us to comment that depending on risk appetite of the company, it should decide which risk measure they choose: VaR or CVaR. For instance, risk averse companies should choose CVaR in their risk management regulations since it indicates conservative as proven in this study.

Another result we can infer from these simulations is that adding perturbation of Brownian motion generally demonstrates their effect by increasing and decreasing the VaR and CVaR estimates. However, unstable behavior in ruin probabilities of the estimates VaR and CVaR does not make a precise inference between perturbed and unperturbed models, however perturbed model might be thought of a more realistic model since it fulfills the stochastic nature of the process. Nevertheless, a decrease in the estimates of VaR or CVaR supports the idea that if the insurance companies modeling their surplus process with Brownian motion, need less initial capital to run the business, and vice versa.

Another crucial aspects of this study shows that time discretization is important in calculation of the ruin probability. In order to avoid ruin, it is better to use small time basis to investigate the surplus.

In addition to simulations built on at 95% confidence level, relevant with degree of certainty for risk measures on Solvency II regulation, same simulations is repeated for 99.5% confidence level. Comparison of those reveals that, not strikingly, VaR and CVaR estimates are always higher while ruin probabilities associated with those estimates are lower. As expected, VaR and CVaR estimates of simulations made for heavy-tailed claim sizes are even more higher since the difference on the tails of such distributions at 95% to 99.5% goes beyond more. Yet, 99.5% confidence level might be generally skeptical for CVaR estimates which is already conservative risk measure.

To sum up, for a better risk management of an insurance company, regulators or actuaries can use proposed approach in order to pursue company cash flow with classical and extended surplus models, and by employing VaR and CVaR associated to those surplus processes, the required capital should be attained without neglecting their ruin probabilities.

Further in this topic, the optimization of the proposed risk measures for precise capital allocation can be carried out. With this motivation, further study of constrained optimization problems involving the surplus process, VaR and CVaR would be worthwhile since quantifying economic capital, and allocating it from an insurance portfolio optimally has a pivotal role in Asset/Liability Management (ALM). Leading study in optimizing portfolio under VaR and CVaR constrained based on Rockafeller and Uryasev [54], and the collected studies are summarized in [58]. As implied also in these studies, CVaR gains more importance since constrained optimization with that risk measure presents promising results. To such studies, more that one additional constraint, the ruin probability,would certainly be helpful since our proposed approach focuses on ruin theory.

Undoubtedly, the reinsurance is mutual agreement with insurance companies and others specialized reinsurance products such as Swiss, Munich Re, Lloyd’s, in order to reduce the risk in a portfolio. For that reason, observing optimal capital allocation under ruin consideration both provides healthier reinsurance agreements and ensures the success in fair investments of the company.

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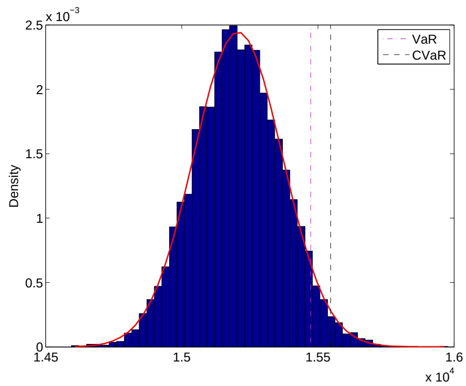
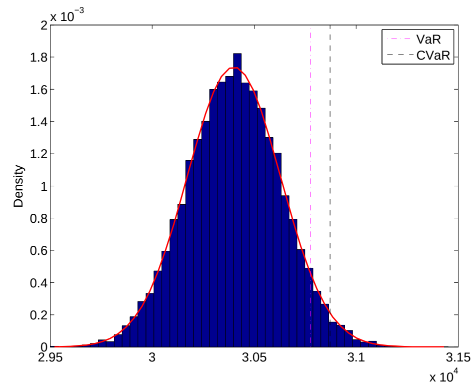
APPENDIX A

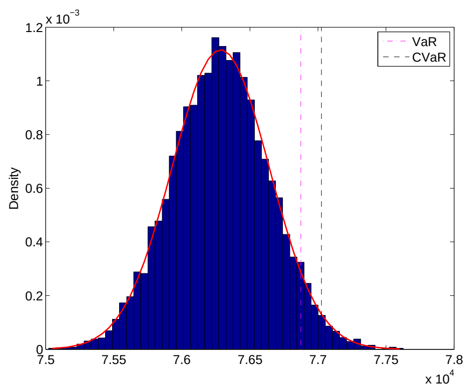
Further Simulation Figures



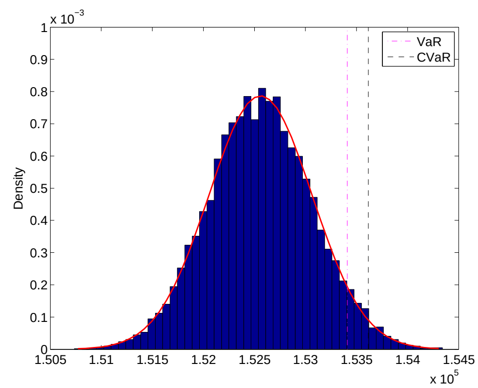
(a) 1-year with daily basis.



(b) 2-year with daily basis.

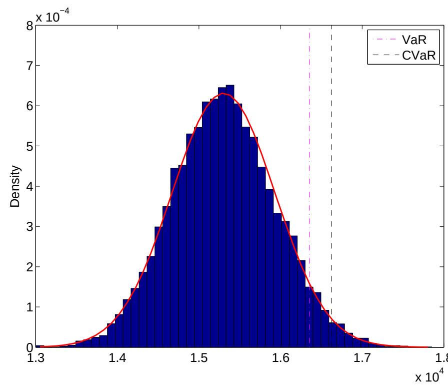


(c) 5-year with daily basis.

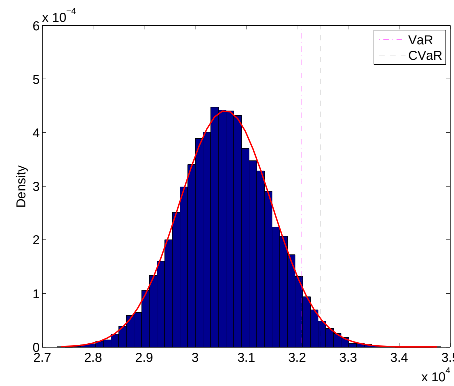


(d) 10-year with daily basis.

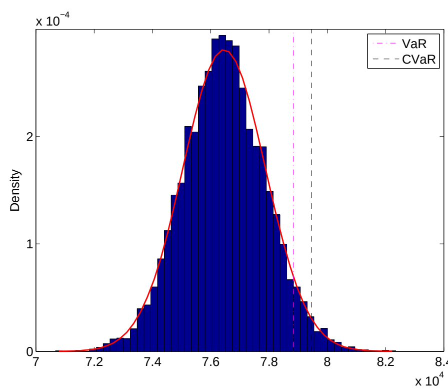
Figure A.1: Simulated risk components with claim distribution Gamma(3, 1), $\sigma_B = 0$.



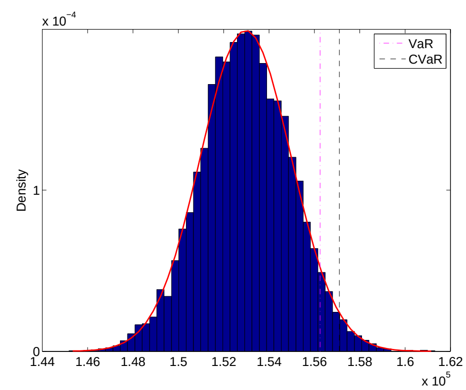
(a) 1-year with daily basis.



(b) 2-year with daily basis.

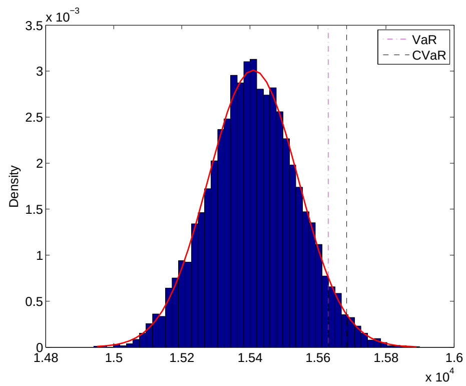


(c) 5-year with daily basis.

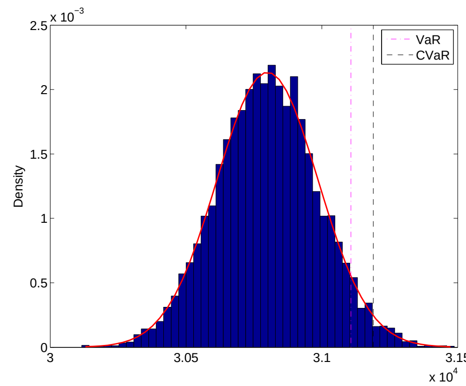


(d) 10-year with daily basis.

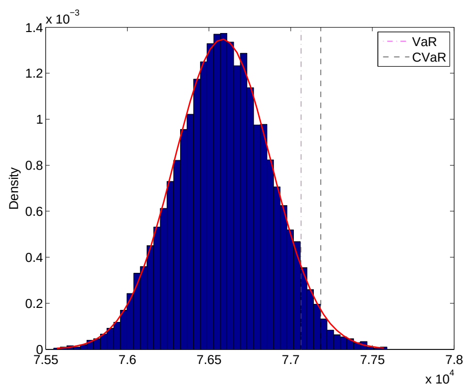
Figure A.2: Simulated risk components with claim distribution Weibull(1.5, 0.5), $\sigma_B = 0$.



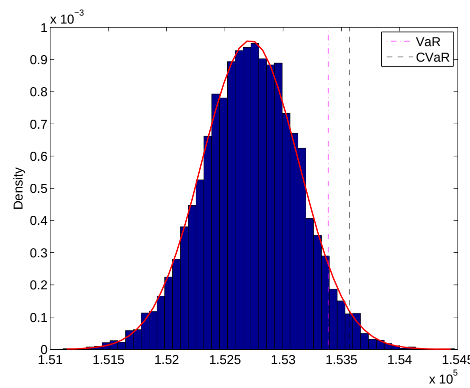
(a) 1-year with daily basis.



(b) 2-year with daily basis.

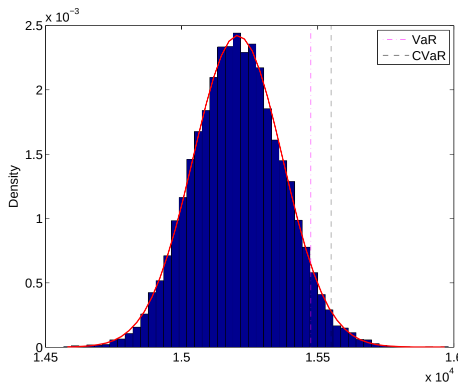


(c) 5-year with daily basis.

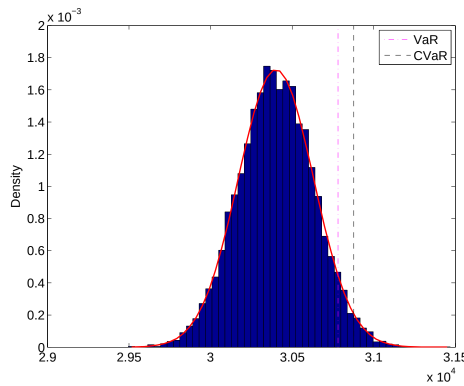


(d) 10-year with daily basis.

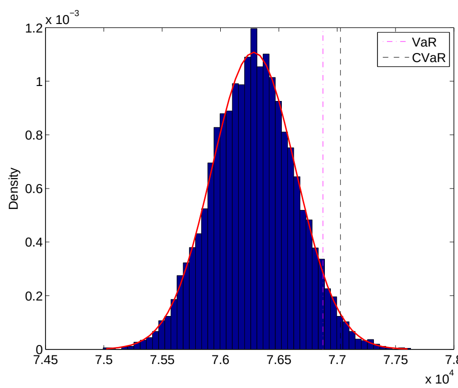
Figure A.3: Simulated risk components with claim distribution Lognormal(1, 0.441), $\sigma_B = 0$.



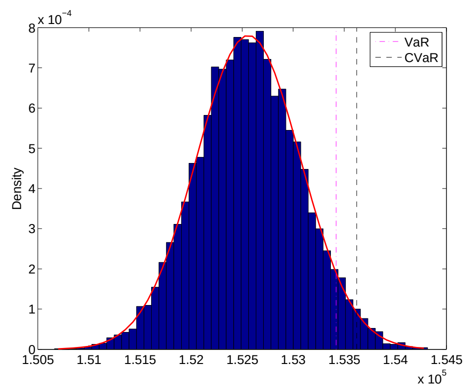
(a) 1-year with daily basis.



(b) 2-year with daily basis.

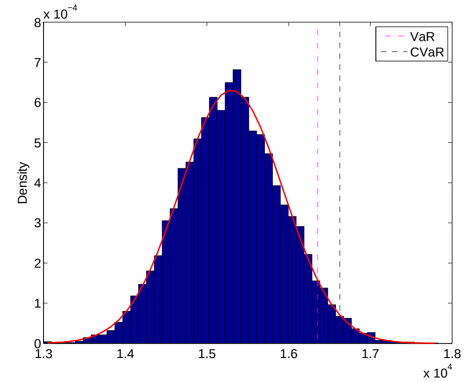


(c) 5-year with daily basis.

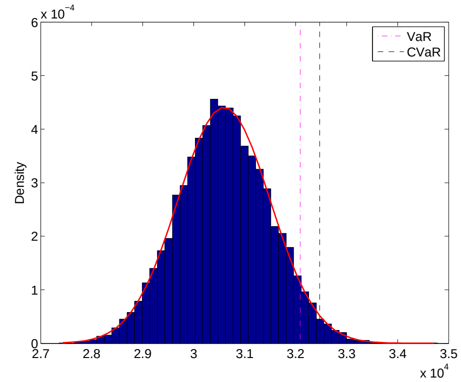


(d) 10-year with daily basis.

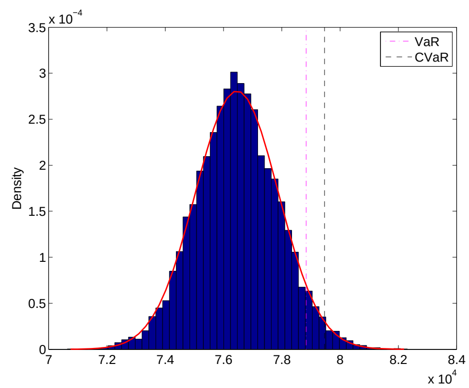
Figure A.4: Simulated risk components with claim distribution Gamma(3, 1), $\sigma_B = 1$.



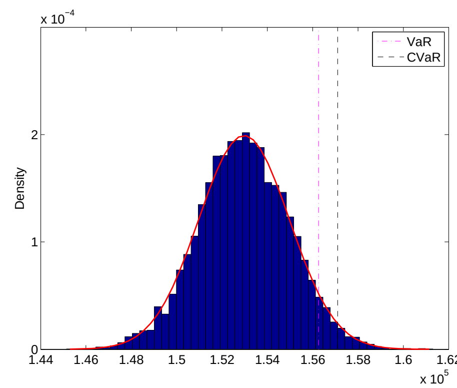
(a) 1-year with daily basis.



(b) 2-year with daily basis.

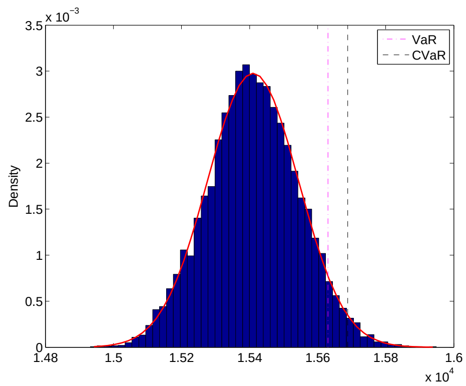


(c) 5-year with daily basis.

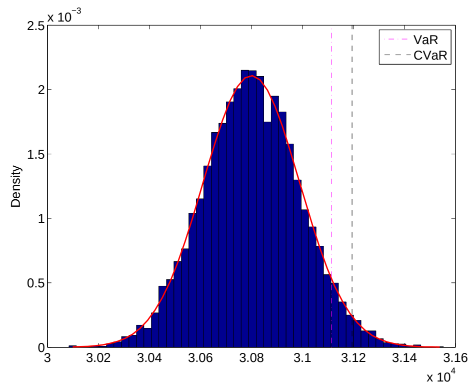


(d) 10-year with daily basis.

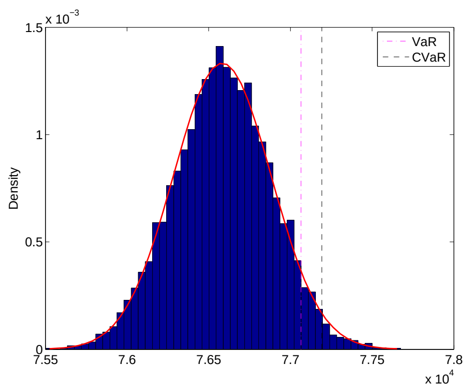
Figure A.5: Simulated risk components with claim distribution Weibull(1.5, 0.5), $\sigma_B = 1$.



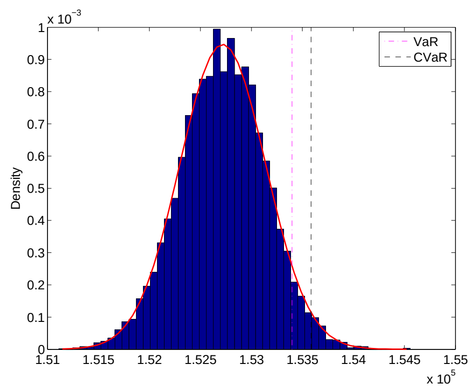
(a) 1-year with daily basis.



(b) 2-year with daily basis.



(c) 5-year with daily basis.



(d) 10-year with daily basis.

Figure A.6: Simulated risk components with claim distribution Lognormal(1, 0.4441), $\sigma_B = 1$.

APPENDIX B

Further Simulation Tables

In this part of this thesis, we present the comparison of light-tailed distributions with Exponential, and further heavy-tailed distribution with Pareto distribution.

B.1 Tables when claim sizes are distributed as Gamma(3, 1)

Gamma distribution with shape parameter 3 and scale paramater 1 is a distribution which has lighter tail than Exponential distribution. (see Figure 3.3a). Therefore, when Table B.1, Table B.2, Table B.3, and Table B.3 are compared to with those results from the use of Exponential distribution, it is clear that the use of Gamma(3, 1) yields lower VaR and CVaR estimates. However, the ruin probabilities with initial capital slightly changes.

Table B.1: $\text{VaR}_{0.95}$ and $\text{CVaR}_{0.95}$ estimates and the ruin probabilities with claim size process Gamma(3, 1) and $\sigma_B = 0$.

Time	VaR	CVaR	ψ(VaR)	ψ(CVaR)
dt = 1; T = 365	15473.3609	15546.4560	0.0421	0.0155
dt = 1; T = 2×365	30776.2071	30872.4523	0.2689	0.1455
dt = 1; T = 5×365	76875.3575	77026.5844	0.0501	0.0194
dt = 1; T = 10×365	153409.5737	153616.6194	0.2037	0.1102

Table B.2: $\text{VaR}_{0.95}$ and $\text{CVaR}_{0.95}$ estimates and the ruin probabilities with claim size process Gamma(3, 1) and $\sigma_B = 1$.

Time	VaR	CVaR	ψ(VaR)	ψ(CVaR)
dt = 1; T = 365	15475.4770	15549.5473	0.0409	0.0162
dt = 1; T = 2×365	30780.7116	30877.1559	0.2628	0.1438
dt = 1; T = 5×365	76878.1751	77029.1300	0.0504	0.0212
dt = 1; T = 10×365	153420.3815	153623.4969	0.1980	0.1101

Table B.3: $\text{VaR}_{0.995}$ and $\text{CVaR}_{0.995}$ estimates and the ruin probabilities with claim size process Gamma(3, 1) and $\sigma_B = 0$.

Time	VaR	CVaR	ψ(VaR)	ψ(CVaR)
dt = 1; T = 365	15638.0865	15692.9458	0.0049	0.0017
dt = 1; T = 2×365	30990.5156	31062.8770	0.0582	0.0301
dt = 1; T = 5×365	77223.7490	77341.9016	0.0044	0.0016
dt = 1; T = 10×365	153854.6805	154007.4651	0.0465	0.0242

Table B.4: $\text{VaR}_{0.995}$ and $\text{CVaR}_{0.995}$ estimates and the ruin probabilities with claim size process Gamma(3, 1) and $\sigma_B = 1$.

Time	VaR	CVaR	ψ(VaR)	ψ(CVaR)
dt = 1; T = 365	15641.1363	15698.2739	0.0048	0.0014
dt = 1; T = 2×365	30986.9138	31070.1879	0.0622	0.0298
dt = 1; T = 5×365	77235.5288	77338.8302	0.0039	0.0018
dt = 1; T = 10×365	153863.8518	154014.6297	0.0449	0.0247

Table B.5: $\text{VaR}_{0.95}$ and $\text{CVaR}_{0.95}$ estimates and the ruin probabilities with claim size process Weibull(1.5, 0.5) and $\sigma_B = 0$.

Time	VaR	CVaR	ψ(VaR)	ψ(CVaR)
dt = 1; T = 365	16352.1492	16624.4890	0.0483	0.0185
dt = 1; T = 2×365	32091.3380	32468.4802	0.0495	0.0196
dt = 1; T = 5×365	78834.5941	79465.3718	0.0479	0.0175
dt = 1; T = 10×365	156248.9798	157101.3741	0.0324	0.0120

Table B.6: $\text{VaR}_{0.95}$ and $\text{CVaR}_{0.95}$ estimates and the ruin probabilities with claim size process Weibull(1.5, 0.5) and $\sigma_B = 1$.

Time	VaR	CVaR	ψ(VaR)	ψ(CVaR)
dt = 1; T = 365	16355.8537	16625.1636	0.0473	0.0188
dt = 1; T = 2×365	32090.8261	32469.8726	0.0497	0.0200
dt = 1; T = 5×365	78841.7191	79469.4481	0.0477	0.0170
dt = 1; T = 10×365	156267.0462	157105.4636	0.0311	0.0121

B.2 Tables when claim sizes are distributed as Weibull(1.5, 0.5)

Weibull distribution with shape 0.5, scale 1.5 has a heavy tail. Table B.5, Table B.6 Table B.7, and Table B.8 with the results when claim size is distributed as Pareto(1, 2), it proves that VaR and CVaR estimates are obtained by using Weibull(1.5, 0.5). Moreover, if the results for heavy-tailed distributions are reviewed, surprisingly VaR and CVaR estimates when claim size is considered as Weibull are always higher. That may be reasoned by parameter choice of Weibull. Although the VaR and CVaR estimates higher when claim size distributed as Weibull, corresponding ruin probabilities is generally higher than the other heavy-tailed distributions. For instance, when Table B.7 and Table B.8 compared to Table B.9 and Table B.10, ruin probabilities for 99.5% confidence level cannot reach to the ruin probabilities on 95% level of confidence.

Table B.7: $\text{VaR}_{0.995}$ and $\text{CVaR}_{0.995}$ estimates and the ruin probabilities with claim size process Weibull(1.5, 0.5) and $\sigma_B = 0$.

Time	VaR	CVaR	ψ(VaR)	ψ(CVaR)
dt = 1; T = 365	16955.7903	17148.8771	0.0054	0.0021
dt = 1; T = 2×365	32918.3447	33250.2423	0.0059	0.0022
dt = 1; T = 5×365	80254.1607	80744.4581	0.0041	0.0018
dt = 1; T = 10×365	158140.0671	158835.0510	0.0026	0.0012

Table B.8: $\text{VaR}_{0.995}$ and $\text{CVaR}_{0.995}$ estimates and the ruin probabilities with claim size process Weibull(1.5, 0.5) and $\sigma_B = 1$.

Time	VaR	CVaR	$\psi(\text{VaR})$	$\psi(\text{CVaR})$
dt = 1; T = 365	16965.1108	17146.8439	0.0056	0.0021
dt = 1; T = 2×365	32915.1052	33250.0659	0.0058	0.0021
dt = 1; T = 5×365	80264.1818	80745.1410	0.0042	0.0018
dt = 1; T = 10×365	158176.3777	158831.0642	0.0026	0.0012

Table B.9: $\text{VaR}_{0.95}$ and $\text{CVaR}_{0.95}$ estimates and the ruin probabilities with claim size process Lognormal(1, 0.4441) and $\sigma_B = 0$.

Time	VaR	CVaR	ψ(VaR)	ψ(CVaR)
dt = 1; T = 365	15630.2244	15684.8001	0.0378	0.0154
dt = 1; T = 2×365	31107.4199	31189.2957	0.0003	0.0000
dt = 1; T = 5×365	77063.7144	77184.1861	0.0008	0.0001
dt = 1; T = 10×365	153388.6887	153573.0642	0.0037	0.0011

Table B.10: $\text{VaR}_{0.95}$ and $\text{CVaR}_{0.95}$ estimates and the ruin probabilities with claim size process Lognormal(1, 0.4441) and $\sigma_B = 1$.

Time	VaR	CVaR	ψ(VaR)	ψ(CVaR)
dt = 1; T = 365	15630.6779	15688.1694	0.0381	0.0145
dt = 1; T = 2×365	31114.2279	31194.7622	0.0003	0.0000
dt = 1; T = 5×365	77065.4204	77192.9527	0.0010	0.0002
dt = 1; T = 10×365	153398.9478	153584.4237	0.0030	0.0007

B.3 Tables when claim sizes are distributed as Lognormal(1, 0.4441)

Lognormal distribution with log mean of 1, and log standard deviation of 0.4441 is one of the examples of heavy-tailed distributions. Here, the results are compared to Pareto distributed claim size estimates and their ruin probabilities. In Table B.9, Table B.10, Table B.11 and Table B.12, VaR and CVaR estimates are generally are found to be low. Despite the lower estimates in VaR and CVaR, ruin probabilities are mostly lower than the results constructed for Pareto claim size. This situation can be due to the fact that Pareto distribution is predominantly used the modeling extreme events for that reason the estimates of VaR and CVaR may not be enough to reduce the ruin probabilities like those obtained using Lognormal distribution situation.

Table B.11: $\text{VaR}_{0.995}$ and $\text{CVaR}_{0.995}$ estimates and the ruin probabilities with claim size process Lognormal(1, 0.4441) and $\sigma_B = 0$.

Time	VaR	CVaR	ψ(VaR)	ψ(CVaR)
dt = 1; T = 365	15755.0479	15791.8451	0.0040	0.0016
dt = 1; T = 2×365	31285.2752	31342.2580	0.0000	0.0000
dt = 1; T = 5×365	77353.2821	77432.2375	0.0000	0.0000
dt = 1; T = 10×365	153796.1481	153942.3861	0.0002	0.0000

Table B.12: $\text{VaR}_{0.995}$ and $\text{CVaR}_{0.995}$ estimates and the ruin probabilities with claim size process Lognormal(1, 0.4441) and $\sigma_B = 1$.

Time	VaR	CVaR	ψ(VaR)	ψ(CVaR)
dt = 1; T = 365	16965.1108	17146.8439	0.0056	0.0021
dt = 1; T = 2×365	32915.1052	33250.0659	0.0058	0.0021
dt = 1; T = 5×365	80264.1818	80745.1410	0.0042	0.0018
dt = 1; T = 10×365	158176.3777	158831.0642	0.0026	0.0012