APPLICATIONS OF THE HESTON MODEL ON BIST30 WARRANTS : HEDGING AND PRICING

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ABSTRACT

APPLICATIONS OF THE HESTON MODEL ON BIST30 WARRANTS :
HEDGING AND PRICING

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The Heston model is one of the first and known stochastic volatility models. The aim of this work is to study the performance of the Heston Model on pricing and hedging the warrants written on BIST30 and the compatibility between the observation of the Heston Model in the literature and BIST30 data.

Keywords : The Heston model, BIST30, warrants, pricing, hedging, fitting, comparison, Black-Scholes

ÖZ

HESTON MODELİNİN BIST30 VARANTLARI ÜZERİDEKİ UYGULAMALARI:
ÜRETME VE FİYATLAMA

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Heston modeli ilk ve en bilinen stokastik volatilite modellerinden biridir. Bu çalışmanın amacı Heston modelinin BIST30 üzerine yazılmış varantlar üzerindeki fiyat ve üretme (pricing and replication) performansını ve Heston modeli ile ilgili literatürde yapılmış gözlemlerin BIST30 verisiyle uyumunu incelemektir.

Anahtar Kelimeler: Heston modeli, BIST30, varant, fiyatlama, üretme, yerleştirme, karşılaştırma, Black-Scholes

To My Family

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LIST OF ABBREVIATIONS

$\mathbb{P}$	The Probability Measure
$\mathbb{Q}$	The Risk-Neutral Measure
S_t	The Stock Price at time t
X_t	The Log Stock Price at time t
W	The Brownian Motion
κ	The Mean Reversion Speed
θ	The Mean Reversion Level
σ	The Volatility of the Variance
ρ	The Correlation between the Brownian motions
v_0	Initial Variance
$(SDE)_S$	Stochastic Differential Equations

CHAPTER 1

INTRODUCTION

The Black-Scholes model to price and hedge options [6] gives a simple formula to price a given European option in terms the price of the underlying and its assumed constant volatility and the interest rate (again assumed constant). The assumption of constant volatility has long been known to be unrealistic and many models allowing stochastic volatility have been developed over the years, see, Scott [22], Hull and White [13], and Wiggins [23]. Melino and Turnbull have reported that the performance of the Black-Scholes model is worse on foreign currency options when it is compared to the models allowing stochastic volatility in [18, 19]. However, these models do not have closed-form solutions and extensive numerical techniques are needed to solve the two dimensional partial differential equations in these models. Following these developments, Steven L. Heston proposed [12] a stochastic volatility model having a closed-form solution for European call option prices when there is a correlation between the underlying security and the volatility. This gives the Heston model a computational efficiency in the valuation of European options, which is important in fitting the model to known option prices.

The Heston model is a stochastic volatility model in which two correlated Brownian motions drive the stock price and the volatility. There are a number of studies in the literature on the applications of the Heston model on actual markets, see, e.g., [9], [24], [14], and [4]. The goal of the current thesis is to study the applicability of the Heston model to standard call and put options written on the BIST30 index, which tracks 30 of the most traded stocks in the Borsa Istanbul and compare its performance to the standard Black Scholes model (BIST30 is one of the main indices tracking Borsa Istanbul, for more on it we refer the reader to [1]). To the best of our knowledge, the present thesis is a first attempt in this direction.

Chapter 2 is a review of the Heston model: its definition, the class of risk neutral measures under the Heston model, the derivation of the partial differential equation (PDE) satisfied by European options in the Heston model and a solution of the PDE. The resulting PDE is linear and can be treated using Fourier analysis (i.e., computation of the characteristic functions of the distribution of the process at fixed times). The computed Fourier transforms are inverted to compute European option prices. This chapter mainly follows [10, 21], giving further details of some of the arguments present in these works.

Chapter 3 fits the Heston model to BIST30 European call warrants traded in Borsa Istanbul (BIST). This involves the estimation of the Heston parameters $\kappa, \theta, \sigma, v_0, \rho$ from the warrant prices. For this purpose we use the minimization of loss functions which is defined as the difference between the market implied volatilities and the model implied volatilities generated by the Heston model; the bisection algorithm is used to extract the model implied volatilities from the model prices. This fitting algorithm is explained in [21] and we use the implementation provided in [21]. For the fit, we used the call warrants with strikes $95, 100, 105$ and maturities 29.04.2016 and 30.06.2016 as input; the fit is done across the time interval (01.03.2016-31.03.2016). To test the performance of the fitting process and see how it varies, we repeat the fit across another time interval: (15.01.2016-03.02.2016). For the second fit we use the call warrants with strikes $90, 95, 100$ and maturities 29.02.2016 and 29.04.2016. Further comments on the fitting can be found in Chapter 3 and Chapter 4.

Chapter 4 applies the hedging algorithm implied by the Heston model to the hedging of BIST30 warrants. For this, we first review the hedging algorithm to be applied and define the hedging error process, denoted $(PL)_t$. This will be our main measure of the performance of the Heston hedging algorithm. In an ideal hedging process the the hedging portfolio must be self financing and the hedging error must be zero. Due to discretization and model errors the hedging error will be nonzero in practice. The hedging error is computed both for the Heston model and the Black Scholes model and these are compared. Since the Heston model includes one more Brownian motion driving volatility, to hedge an option, the volatility needs to be hedged by including an additional option in the hedging portfolio. We applied the hedging algorithm to warrants having a short time and a long time to the maturity separately. Two warrants with strike 95 and with maturities 29.04.2016 and 30.06.2016 are hedged along in the period (01.03.2016-31.03.2016) using the market prices and the Heston model. To compare the performance of the hedging using the Heston model, we also hedge these warrants using the Black-Scholes model. To see how the hedge performance changes with time, we run the hedging algorithm in the time period (15.01.2016-03.02.2016) and compare it with the hedging implied by Black-Scholes; in this time period we use the strike 90 and maturities 29.02.2016 and 29.04.2016. All of these hedges are performed on warrants already traded in the market. As a final application, the last example is the hedging of a call warrant untraded in the market with strike 96 and maturity 29.04.2016 in the period (01.03.2016-31.03.2016). Comments on how the strike and maturity choices have been made can be found in the introduction of Chapter 4. The main finding of this chapter is this: in all of the hedging examples we studied, the use of the Heston model to model and hedge for stochastic volatility greatly reduces the hedging error from that of Black Scholes. A detailed commentary on our results is given in Chapter 4.

In all of the computations the interest rate is taken to be constant and is estimated from an average of the benchmark interest rate [1] over a single month preceding the hedge period.

Conclusion comments on future work.

[1] see, e.g, `http://www.bloomberght.com/tahvil/gosterge-faiz`

CHAPTER 2

The Heston Model

In this chapter, we review the Heston model (following mostly [12] and [21]): the derivation of changes of measure to risk-neutral probabilities, the Heston partial differential equation (PDE) satisfied by the price of a European call option. The solution of this PDE is reduced to the computation of in-the-money probabilities that are derived from Heston characteristic functions using the inverse Fourier transformation. Finally, we use the price of the European call option to obtain the European put option's price using the put-call parity. To finish the chapter, we give the option prices under the case that the stock pays dividend.

2.1 The Model

In the Heston Model [12], the underlying stock price S_t follows the Black-Scholes process with a stochastic variance v_t following Cox-Ingersoll-Ross process [8] which is actually an example of a square root process. Hence, the model is the bivariate system of stochastic differential equations (SDEs) [21]

$$dS_t = \mu S_t dt + \sqrt{v_t} S_t dW_t^{(1)}, \tag{2.1a}$$

$$dv_t = \kappa(\theta - v_t)dt + \sigma\sqrt{v_t} dW_t^{(2)}, \tag{2.1b}$$

where μ the drift of the process for the stock, $\kappa > 0$ the mean reversion speed for the variance, $\theta > 0$ the mean reversion level for the variance, and $\sigma > 0$ the volatility of the variance. Moreover, $W^{(1)}$ and $W^{(2)}$ are two Brownian motions such that $E^{\mathbb{P}}[dW_t^{(1)} dW_t^{(2)}] = \rho dt$. In other words, $\rho \in [-1, 1]$ represents the correlation between $W^{(1)}$ and $W^{(2)}$.

In the Heston Model, the volatility is modeled by using the variance v_t. The variance process derives from the Orstein-Uhlenbeck process for the volatility $h_t = \sqrt{v_t}$ given by

$$dh_t = -\beta h_t dt + \delta dW_t^{(2)}. \tag{2.2}$$

Applying Ito's lemma on the function $f(h_t) = h_t^2 = v_t$,

$$v_t = h_t^2 = h_0 + \int_0^t 2h_s dh_s + \frac{1}{2}\int_0^t 2d\langle h\rangle_s.$$

Taking differentiation with respect to t and using Equation 2.2,

$$\begin{aligned} dv_t &= 2h_t dh_t d\langle h\rangle_t \\ &= 2h_t[-\beta h_t dt + \delta dW_t^2] + \delta^2 dt \\ &= (-2\beta v_t + \delta^2)dt + 2\delta\sqrt{v_t}dW_t^{(2)}. \end{aligned} \tag{2.3}$$

Letting $\kappa = 2\beta$, $\theta = \delta^2/(2\beta)$, and $\sigma = 2\delta$ in Equation 2.3, Equation 2.1b is obtained.

2.2 The risk neutral measure

The stock price and variance processes in Equation 2.1a and Equation 2.1b are under the probability measure $\mathbb{P}$. For pricing, the measure $\mathbb{P}$ needs to be changed into a risk neutral measure $\mathbb{Q}$. In this sense, applying Girsanov's theorem for Equation 2.1a and Equation 2.1b separately leads to the risk-neutral process for the stock price which is the following:

$$dS_t = rS_t dt + \sqrt{v_t}S_t d\widetilde{W_t^{(1)}}, \tag{2.4}$$

where $\widetilde{W_t^{(1)}} = W_t^{(1)} + \int_0^t \left(\frac{\mu - r}{\sqrt{v_s}}\right) ds$ and r is the risk-free rate.

Sometimes, using log price is needed. For the log price process, using Ito's lemma on $\log(S_t)$ gives

$$\log(S_t) = \log(S_0) + \int_0^t \frac{1}{S_s}dS_s + \frac{1}{2}\int_0^t \frac{-1}{S_s^2}d\langle S\rangle_s.$$

Taking the derivation of the both sides with respect to t and using the fact that $d\langle S\rangle_s = v_s S_s^2 ds$,

$$d(\log(S_t)) = \frac{dS_t}{S_t} - \frac{1}{2S_t^2}v_t S_t^2 dt = \left(\mu - \frac{v_t}{2}\right)dt + \sqrt{v_t}dW_t^{(1)}.$$

The log price process under risk neutral measure is

$$d(\log(S_t)) = \left(r - \frac{v_t}{2}\right) dt + \sqrt{v_t} d\widetilde{W_t^{(1)}}. \tag{2.5}$$

In the case that the stock pays a continuous dividend yield, which is denoted by q, r is replaced by $r - q$ in the equations Equation 2.4 and Equation 2.5.

The variance process under the risk neutral measure is obtained by subtracting a function $\lambda(S_t, v_t, t)$ from the drift of dv_t in Equation 2.1b. Thus,

$$dv_t = [\kappa(\theta - v_t) - \lambda(S_t, v_t, t)]dt + \sigma\sqrt{v_t} d\widetilde{W_t^{(2)}}, \tag{2.6}$$

where $\widetilde{W_t^{(2)}} = W_t^{(2)} + \int_0^t \left(\frac{\lambda(S_s, v_s, s)}{\sigma\sqrt{v_s}}\right) ds$ and $\lambda(S, v, t)$ is called the volatility risk premium or the market price of the volatility risk. As in [7], $\lambda(S_t, v_t, t) = \lambda t$. We will assume that $\lambda = 0$ using [10].

Letting $\lambda(S, v, t) = \lambda v_t$ in Equation 2.6,

$$dv_t = \kappa^\star(\theta^\star - v_t)dt + \sigma\sqrt{v_t} d\widetilde{W_t^{(2)}}, \tag{2.7}$$

where $\kappa^\star = \kappa + \lambda$ and $\theta^\star = \kappa\theta/(\kappa + \lambda)$.

$\kappa^\star$ and $\theta^\star$ are called the risk neutral parameters for the variance process.

Since $\lambda = 0$ using our assumption above, $\kappa^\star = \kappa$ and $\theta^\star = \theta$.

To sum up, the Heston model under the risk neutral measure $\mathbb{Q}$ is

$$dS_t = rS_t dt + \sqrt{v_t} S_t d\widetilde{W_t^{(1)}}, \tag{2.8a}$$

$$dv_t = \kappa^\star(\theta^\star - v_t)dt + \sigma\sqrt{v_t} d\widetilde{W_t^{(2)}}, \tag{2.8b}$$

where $E^{\mathbb{Q}}[d\widetilde{W_t^{(1)}} d\widetilde{W_t^{(2)}}] = \rho dt$.

2.3 The derivation of the Heston PDE

Due to the presence of a second random process driving the volatility, the Heston model is incomplete and for a perfect hedge a second asset is needed driven by the same underlying processes (this is in contrast to the Black Scholes model, which is complete). Thus, we will be using a hedging portfolio consisting of the underlying asset, riskless bond and a second derivative asset written on the underlying; added to this portfolio is one unit of the option to be hedged. We would like the resulting portfolio to be riskless. The value of $V = V(S, v, t)$ of such a portfolio, consisting of one unit of the option to be hedged, Δ units of the stock, and ψ units of another option $U = U(S, v, t)$ for the volatility as in [10] is

$$\Pi = V + \Delta S + \psi U,$$

where $\Pi_t = \Pi$, $V_t = V$, and $U_t = U$ for simplicity.

Suppose that the portfolio is self-financing. So the change in the value of the portfolio is

$$d\Pi = dV + \Delta dS + \psi dU. \tag{2.9}$$

Applying Ito's lemma on V,

$$\begin{aligned} V = V_0 &+ \int_0^t \frac{\partial V}{\partial s} ds + \int_0^t \frac{\partial V}{\partial S} dS + \int_0^t \frac{\partial V}{\partial v} dv \\ &+ \frac{1}{2}\int_0^t \frac{\partial^2 V}{\partial S^2} d\langle S\rangle_s + \frac{1}{2}\int_0^t \frac{\partial^2 V}{\partial v^2} d\langle v\rangle_s \\ &+ \frac{1}{2}\int_0^t \frac{\partial^2 V}{\partial S \partial v} d\langle S, v\rangle_s. \end{aligned} \tag{2.10}$$

Using the fact that $d\langle S\rangle_t = vS^2 dt$, $d\langle v\rangle_t = \sigma^2 v dt$, and $d\langle S, v\rangle_t = \rho v \sigma S dt$ and differentiating the Equation 2.10 with respect to t,

$$\begin{aligned} dV &= \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} dS + \frac{\partial V}{\partial v} dv \\ &\quad + \frac{1}{2}\frac{\partial^2 V}{\partial S^2} d\langle S\rangle_t + \frac{1}{2}\frac{\partial^2 V}{\partial v^2} d\langle v\rangle_t + \frac{1}{2}\frac{\partial^2 V}{\partial S \partial v} d\langle S, v\rangle_t \\ &= \left[\frac{\partial V}{\partial t} + \frac{1}{2} v S^2 \frac{\partial^2 V}{\partial S^2} + \frac{1}{2}\sigma^2 v \frac{\partial^2 V}{\partial v^2} + \frac{1}{2}\rho v \sigma S \frac{\partial^2 V}{\partial S \partial v}\right] dt \\ &\quad + \frac{\partial V}{\partial S} dS + \frac{\partial V}{\partial v} dv. \end{aligned} \tag{2.11}$$

Using again Ito's lemma on U, dU is obtained by the same argument with the procedure obtaining dV. For this, V terms in Equation 2.11 are replaced by U terms. Therefore,

$$dU = \left[\frac{\partial U}{\partial t} + \frac{1}{2}vS^2\frac{\partial^2 U}{\partial S^2} + \frac{1}{2}\sigma^2 v\frac{\partial^2 U}{\partial v^2} + \frac{1}{2}\rho v\sigma S\frac{\partial^2 U}{\partial S\partial v}\right]dt + \frac{\partial U}{\partial S}dS + \frac{\partial U}{\partial v}dv. \tag{2.12}$$

Substituting dV and dU terms in equations Equation 2.11 and Equation 2.12, respectively, into Equation 2.9,

$$\begin{aligned} d\Pi &= dV + \Delta dS + \psi dU \\ &= \left[\frac{\partial V}{\partial t} + \frac{1}{2}vS^2\frac{\partial^2 V}{\partial S^2} + \frac{1}{2}\sigma^2 v\frac{\partial^2 V}{\partial v^2} + \frac{1}{2}\rho v\sigma S\frac{\partial^2 V}{\partial S\partial v}\right]dt \\ &+ \psi\left[\frac{\partial U}{\partial t} + \frac{1}{2}vS^2\frac{\partial^2 U}{\partial S^2} + \frac{1}{2}\sigma^2 v\frac{\partial^2 U}{\partial v^2} + \frac{1}{2}\rho v\sigma S\frac{\partial^2 U}{\partial S\partial v}\right]dt \\ &+ \left[\frac{\partial V}{\partial S} + \psi\frac{\partial U}{\partial S} + \Delta\right]dS + \left[\frac{\partial V}{\partial v} + \psi\frac{\partial U}{\partial v}\right]dv. \end{aligned} \tag{2.13}$$

To get the hedged portfolio against the movements in the stock and the volatility, the terms $\left[\frac{\partial V}{\partial S} + \psi\frac{\partial U}{\partial S} + \Delta\right]dS$ and $\left[\frac{\partial V}{\partial v} + \psi\frac{\partial U}{\partial v}\right]dv$ in Equation 2.13 must be zero. Thus,

$$\psi = -\frac{\partial V}{\partial v}\Big/\frac{\partial U}{\partial v}, \tag{2.14a}$$

$$\Delta = -\psi\frac{\partial U}{\partial S} - \frac{\partial V}{\partial S} = \frac{\frac{\partial V}{\partial v}\frac{\partial U}{\partial S} - \frac{\partial V}{\partial S}\frac{\partial U}{\partial v}}{\frac{\partial U}{\partial v}}. \tag{2.14b}$$

In this light, ψ and Δ are called the hedge parameters. Moreover, the change in hedged portfolio comes from equating the last two terms of Equation 2.13 to zero. Therefore,

$$\begin{aligned} d\Pi &= \left[\frac{\partial V}{\partial t} + \frac{1}{2}vS^2\frac{\partial^2 V}{\partial S^2} + \frac{1}{2}\sigma^2 v\frac{\partial^2 V}{\partial v^2} + \frac{1}{2}\rho v\sigma S\frac{\partial^2 V}{\partial S\partial v}\right]dt \\ &+ \psi\left[\frac{\partial U}{\partial t} + \frac{1}{2}vS^2\frac{\partial^2 U}{\partial S^2} + \frac{1}{2}\sigma^2 v\frac{\partial^2 U}{\partial v^2} + \frac{1}{2}\rho v\sigma S\frac{\partial^2 U}{\partial S\partial v}\right]dt. \end{aligned} \tag{2.15}$$

Since the portfolio is riskless, the earning on this portfolio must be the risk-free rate, r. This is to say that $d\pi = r\Pi dt$. Thus, Equation 2.9 becomes

$$d\Pi = r(V + \Delta S + \psi U)dt. \tag{2.16}$$

$d\Pi$ is in common in both Equation 2.15 and Equation 2.16. So equating these equations and substituting the hedge parameters ψ and Δ represented explicitly in Equation 2.14 to obtain

$$\frac{\frac{\partial U}{\partial v}\left[\frac{\partial V}{\partial t} + \frac{1}{2}vS^2\frac{\partial^2 V}{\partial S^2} + \frac{1}{2}\sigma^2 v\frac{\partial^2 V}{\partial v^2} + \frac{1}{2}\rho v\sigma S\frac{\partial^2 V}{\partial S\partial v}\right] - \frac{\partial V}{\partial v}\left[\frac{\partial U}{\partial t} + \frac{1}{2}vS^2\frac{\partial^2 U}{\partial S^2} + \frac{1}{2}\sigma^2 v\frac{\partial^2 U}{\partial v^2} + \frac{1}{2}\rho v\sigma S\frac{\partial^2 U}{\partial S\partial v}\right]}{\frac{\partial U}{\partial v}}$$
$$= \frac{r\left(\frac{\partial U}{\partial v} + \left[\frac{\partial V}{\partial v}\frac{\partial U}{\partial S} - \frac{\partial V}{\partial S}\frac{\partial U}{\partial v}\right]S - \frac{\partial V}{\partial v}U\right)}{\frac{\partial U}{\partial v}}.$$

Cancelling out $\frac{\partial U}{\partial v}$ from both sides and gathering V terms in left side as a function of V only and U terms in right side as a function of U only,

$$\begin{aligned} &\frac{\left[\frac{\partial V}{\partial t} + \frac{1}{2}vS^2\frac{\partial^2 V}{\partial S^2} + \frac{1}{2}\sigma^2 v\frac{\partial^2 V}{\partial v^2} + \frac{1}{2}\rho v\sigma S\frac{\partial^2 V}{\partial S\partial v}\right] - rV + \frac{\partial V}{\partial S}rS}{\frac{\partial V}{\partial v}} \\ &= \frac{\left[\frac{\partial U}{\partial t} + \frac{1}{2}vS^2\frac{\partial^2 U}{\partial S^2} + \frac{1}{2}\sigma^2 v\frac{\partial^2 U}{\partial v^2} + \frac{1}{2}\rho v\sigma S\frac{\partial^2 U}{\partial S\partial v}\right] - rU + \frac{\partial U}{\partial S}rS}{\frac{\partial U}{\partial v}}. \end{aligned} \tag{2.17}$$

The left side and the right side of Equation 2.17 are functions of V and U, respectively. This leads to the fact that both side can be written as a function $f(S, v, t)$. In the Heston model, this function is taken as [12]

$$f(S, v, t) = -\kappa(\theta - v) + \lambda(S, v, t),$$

where $\lambda(S, v, t)$ is the market price of volatility risk. This justifies the appearance of $\lambda(S, v, t)$ in Equation 2.16. As mentioned before, the market of the volatility risk is a linear function of volatility, i.e. $\lambda(S, v, t) = \lambda t$, where λ is constant [7]. Since our assumption is that $\lambda = 0$ depending on [10], without loss of generality, putting $f(S, v, t) = -\kappa(\theta - v)$ instead of the left side in Equation 2.17,

$$-\kappa(\theta - v) = \frac{\left[\frac{\partial U}{\partial t} + \frac{1}{2}vS^2\frac{\partial^2 U}{\partial S^2} + \frac{1}{2}\sigma^2 v\frac{\partial^2 U}{\partial v^2} + \frac{1}{2}\rho v\sigma S\frac{\partial^2 U}{\partial S\partial v}\right] - rU + \frac{\partial U}{\partial S}rS}{\frac{\partial U}{\partial v}}. \tag{2.18}$$

Rearranging Equation 2.18 gives the Heston's partial differential equation (PDE) which is

$$\left[\frac{\partial U}{\partial t} + \frac{1}{2}vS^2\frac{\partial^2 U}{\partial S^2} + \frac{1}{2}\sigma^2 v\frac{\partial^2 U}{\partial v^2} + \frac{1}{2}\rho v\sigma S\frac{\partial^2 U}{\partial S\partial v}\right] - rU + rS\frac{\partial U}{\partial S} + \kappa(\theta - v)\frac{\partial U}{\partial v} = 0. \tag{2.19}$$

Whether $U(S, v, t)$ is an European call or put option determines the boundary conditions for the Heston PDE. In this approach, let $U(S, v, t)$ be an European call option with strike K and maturity T.

The value of the call at the maturity is $(S_T - K)_+$. If the stock price is zero, the value is also zero. The change in the value of the call option U with respect to the stock price goes to 1 since if the stock price is large enough, the strike K has no effect on the value of the call option U. If the volatility goes to infinity, the stock price grow to large enough so that the strike K cannot affect the value of the option. Therefore, stock price S is the only factor left determining the value.

These arguments lead to the following boundary conditions for the Heston PDE:

$$U(S, v, T) = (S_T - K)_+, \tag{2.20a}$$

$$U(0, v, t) = 0, \tag{2.20b}$$

$$\frac{\partial U}{\partial S}(\infty, v, t) = 1, \tag{2.20c}$$

$$U(S, \infty, t) = S. \tag{2.20d}$$

Moreover, Equation 2.19 can be written as

$$\frac{\partial U}{\partial t} + \mathcal{A}U - rU = 0, \tag{2.21}$$

where

$$\mathcal{A} = rS\frac{\partial}{\partial S} + \frac{1}{2}vS^2\frac{\partial^2}{\partial S^2} + \kappa(\theta - v)\frac{\partial}{\partial v} + \frac{1}{2}\sigma^2 v\frac{\partial^2}{\partial v^2} + \frac{1}{2}\rho v\sigma S\frac{\partial^2}{\partial S\partial v}$$

is the generator of the Heston model.

It is known that $rS\frac{\partial}{\partial S} + \frac{1}{2}vS^2\frac{\partial^2}{\partial S^2}$ is the generator of the Black-Scholes model [17] , with $v = \sqrt{\sigma_{BS}}$, where σ_{BS} is the Black-Scholes volatility.

In addition to this, $\kappa(\theta - v)\frac{\partial}{\partial v} + \frac{1}{2}\sigma^2 v\frac{\partial^2}{\partial v^2} + \frac{1}{2}\rho v\sigma S\frac{\partial^2}{\partial S\partial v}$ is the part of the PDE, which is related to the stochastic volatility.

As mentioned earlier, let the log price $X_t = \log(S_t)$, $t \geq 0$. Taking $X = \log(S)$ for simplification, the followings can be obtained:

$$\frac{\partial U}{\partial S} = \frac{\partial U}{\partial X}\frac{\partial X}{\partial S} = \frac{\partial U}{\partial X}\frac{1}{S}, \tag{2.22a}$$

$$\frac{\partial^2 U}{\partial v\partial S} = \frac{\partial}{\partial v}\left(\frac{\partial U}{\partial X}\frac{1}{S}\right) = \frac{1}{S}\frac{\partial^2 U}{\partial v\partial X}, \tag{2.22b}$$

$$\frac{\partial^2 U}{\partial S^2} = \frac{\partial}{\partial S}\left(\frac{\partial U}{\partial X}\frac{1}{S}\right) = \frac{1}{S^2}\frac{\partial^2 U}{\partial X^2} - \frac{1}{S^2}\frac{\partial U}{\partial X}. \tag{2.22c}$$

Substituting expressions in Equation 2.22 into the Heston PDE in Equation 2.19 gives

$$\begin{aligned}\frac{\partial U}{\partial t} + \frac{1}{2}vS^2\left(\frac{1}{S^2}\left(\frac{\partial^2 U}{\partial X^2} - \frac{\partial U}{\partial X}\right)\right) + \frac{1}{2}\rho v\sigma S\left(\frac{1}{S}\frac{\partial^2 U}{\partial X\partial v}\right) + \frac{1}{2}\sigma^2 v\frac{\partial^2 U}{\partial v^2} \\ + -rU + rS\left(\frac{1}{S}\frac{\partial U}{\partial X}\right) + \kappa(\theta - v)\frac{\partial U}{\partial v} = 0.\end{aligned} \tag{2.23}$$

Rearranging Equation 2.23,

$$\begin{aligned}\frac{\partial U}{\partial t} + \frac{1}{2}v\frac{\partial^2 U}{\partial X^2} + \left(r - \frac{v}{2}\right)\frac{\partial U}{\partial X} + \frac{1}{2}\sigma^2 v\frac{\partial^2 U}{\partial v^2} + \frac{1}{2}\rho v\sigma\frac{\partial^2 U}{\partial X\partial v} - rU \\ +\kappa(\theta - v)\frac{\partial U}{\partial v} = 0.\end{aligned} \tag{2.24}$$

2.3.1 The Heston PDE for a European call

Let $C_t(K)$ be a European call price at time t where the option is written on the underlying asset S with maturity T and strike K.

The European call price at time t is the discounted expected value of the payoff at the maturity under the risk neutral measure $\mathbb{Q}$. That is

$$\begin{aligned}C_t(K) &= e^{-r(T-t)}E^{\mathbb{Q}}[(S_T - K)^+] \\ &= e^{-r(T-t)}E^{\mathbb{Q}}[(S_T - K)\mathbb{1}_{\{S_T>K\}}] \\ &= e^{-r(T-t)}E^{\mathbb{Q}}[S_T\mathbb{1}_{\{S_T>K\}}] - Ke^{-r(T-t)}E^{\mathbb{Q}}[\mathbb{1}_{\{S_T>K\}}] \\ &= S_tE^{\mathbb{Q}}[\mathbb{1}_{\{S_T>K\}}] - Ke^{-r(T-t)}\mathbb{Q}(S_T > K).\end{aligned} \tag{2.25}$$

To evaluate $e^{-r(T-t)}E^{\mathbb{Q}}[S_t\mathbb{1}_{\{S_T>K\}}]$ in Equation 2.25, one can chance the measure $\mathbb{Q}$ as $\mathbb{Q}^S$ by defining the following Rodon-Nikodym derivative:

$$\frac{d\mathbb{Q}}{d\mathbb{Q}^S} = \frac{S_t}{B_t}\Big/\frac{S_T}{B_T}, \tag{2.26}$$

where $B_t = e^{rt}$,$t \geq 0$.

Thus,

$$\frac{d\mathbb{Q}}{d\mathbb{Q}^S}\frac{S_T/S_t}{B_T/B_t} = 1.$$

Therefore,

$$\begin{aligned} e^{-r(T-t)}E^{\mathbb{Q}}[S_T\mathbb{1}_{\{S_T>K\}}] &= S_tE^{\mathbb{Q}}\left[\mathbb{1}_{\{S_T>K\}}\frac{S_T/S_t}{B_T/B_t}\frac{d\mathbb{Q}}{d\mathbb{Q}^S}\right] = S_tE^{\mathbb{Q}^S}[\mathbb{1}_{\{S_T>K\}}] \\ &= S_t\mathbb{Q}^S(S_T > K) = S_t\mathbb{Q}^S(\log(S_T) > \log(K)). \end{aligned} \tag{2.27}$$

Moreover,

$$\mathbb{Q}(S_T > K) = \mathbb{Q}(\log(S_T) > \log(K)).$$

Letting $\mathbb{Q}^S(S_T > K) = P_1$ and $\mathbb{Q}(S_T > K) = P_2$ and using the log price process $\log(S_t) = X_t$, Equation 2.25 can be written as following:

$$C_t(K) = S_tP_1 - Ke^{-r(T-t)}P_2, \tag{2.28a}$$

$$C_t(K) = e^{X_t}P_1 - Ke^{-r(T-t)}P_2. \tag{2.28b}$$

Furthermore, P_1 and P_2 can be considered as the probabilities of the call expiring in-the-money, conditional on the value $S_t = e^{X_t}$ of the stock and the value v_t on the variance at time t, respectively.

To get more information on the measures $\mathbb{Q}$ and $\mathbb{Q}^{\mathbb{S}}$, one can consider that the measure $\mathbb{Q}$ uses B_t as numeraire, while the measure $\mathbb{Q}^{\mathbb{S}}$ uses S_t as numeraire. Using [5], this change of measure method is valid for many models including the Heston model and the Black-Scholes model. That is why there are resembles between the call price in the Heston model and the Black-Scholes model. For example, using the same change of measure technique, one can find $\mathbb{Q}^{\mathbb{S}}(S_T > K) = \Phi(d_1)$ and $\mathbb{Q}(S_T > K) = \Phi(d_2)$, which are used to calculate the call price in the Black-Scholes model.

Since the European call satisfies the Heston PDE in Equation 2.24, accepting $X_t = X$ and using Equation 2.28b, the Heston PDE can be written in terms of the call option with price $C_t(K)$ as follows:

$$\frac{\partial C}{\partial t} + \frac{1}{2}v\frac{\partial^2 C}{\partial X^2} + \left(r - \frac{v}{2}\right)\frac{\partial C}{\partial X} + \frac{1}{2}\sigma^2 v\frac{\partial^2 C}{\partial v^2} + \frac{1}{2}\rho\sigma v\frac{\partial^2 C}{\partial X \partial v} - rC + \kappa(\theta - v)\frac{\partial C}{\partial v} = 0. \tag{2.29}$$

$\frac{\partial C}{\partial t}$, $\frac{\partial C}{\partial X}$, $\frac{\partial^2 C}{\partial X^2}$, $\frac{\partial^2 C}{\partial X \partial v}$, $\frac{\partial C}{\partial v}$, and $\frac{\partial^2 C}{\partial v^2}$ need to be found to extend Equation 2.29. Thus,

$$\frac{\partial C}{\partial t} = e^X\frac{\partial P_1}{\partial t} - Ke^{-r(T-t)}\left[\frac{\partial P_2}{\partial t} + rP_2\right], \tag{2.30a}$$

$$\frac{\partial C}{\partial X} = e^X\left[\frac{\partial P_1}{\partial X} + P_1\right] - Ke^{-r(T-t)}\frac{\partial P_2}{\partial X}, \tag{2.30b}$$

$$\frac{\partial^2 C}{\partial X^2} = e^X\left[\frac{\partial^2 P_1}{\partial X^2} + 2\frac{\partial P_1}{\partial X} + P_1\right] - Ke^{-r(T-t)}\frac{\partial^2 P_2}{\partial X^2}, \tag{2.30c}$$

$$\frac{\partial^2 C}{\partial X \partial v} = e^X\left[\frac{\partial^2 P_1}{\partial X \partial v} + \frac{\partial P_1}{\partial v}\right] - Ke^{-r(T-t)}\frac{\partial^2 P_2}{\partial X \partial v}, \tag{2.30d}$$

$$\frac{\partial C}{\partial v} = e^X\frac{\partial P_1}{\partial v} - Ke^{-r(T-t)}\frac{\partial P_2}{\partial v}, \tag{2.30e}$$

$$\frac{\partial^2 C}{\partial v^2} = e^X\frac{\partial^2 P_1}{\partial v^2} - Ke^{-r(T-t)}\frac{\partial^2 P_2}{\partial v^2}. \tag{2.30f}$$

Plugging the expressions in Equation 2.30 into Equation 2.29,

$$\begin{aligned}
\frac{\partial C}{\partial t} &= e^X\frac{\partial P_1}{\partial t} - Ke^{-r(T-t)}\left[\frac{\partial P_2}{\partial t} + rP_2\right] \\
&+ \frac{1}{2}v\left[e^X\left[\frac{\partial^2 P_1}{\partial X^2} + 2\frac{\partial P_1}{\partial X} + P_1\right] - Ke^{-r(T-t)}\frac{\partial^2 P_2}{\partial X^2}\right] \\
&+ \left(r - \frac{v}{2}\right)\left[e^X\frac{\partial P_1}{\partial t} - Ke^{-r(T-t)}\left[\frac{\partial P_2}{\partial t} + rP_2\right]\right] \\
&+ \frac{1}{2}\sigma^2 v\left[e^X\frac{\partial^2 P_1}{\partial v^2} - Ke^{-r(T-t)}\frac{\partial^2 P_2}{\partial v^2}\right] \\
&+ \rho\sigma v\left[e^X\left[\frac{\partial^2 P_1}{\partial X \partial v} + \frac{\partial P_1}{\partial v}\right] - Ke^{-r(T-t)}\frac{\partial^2 P_2}{\partial X \partial v}\right] \\
&- r\left[e^{X_t}P_1 - Ke^{-r(T-t)}P_2\right] \\
&+ \kappa(\theta - v)\left[e^X\frac{\partial P_1}{\partial v} - Ke^{-r(T-t)}\frac{\partial P_2}{\partial v}\right] = 0.
\end{aligned} \tag{2.31}$$

Rearranging Equation 2.31,

$$
\begin{aligned}
&e^X \frac{\partial P_1}{\partial t} - Ke^{-r(T-t)} \frac{\partial P_2}{\partial t} + \left(r + \frac{v}{2}\right) e^X \frac{\partial P_1}{\partial X} - \left(r - \frac{v}{2}\right) Ke^{-r(T-t)} \frac{\partial P_2}{\partial X} \\
&+ \frac{1}{2} v e^X \frac{\partial^2 P_1}{\partial X^2} - \frac{1}{2} v K e^{-r(T-t)} \frac{\partial^2 P_2}{\partial X^2} + \rho\sigma v e^X \frac{\partial^2 P_1}{\partial X \partial v} \\
&- \rho\sigma v K e^{-r(T-t)} \frac{\partial^2 P_2}{\partial X \partial v} + \left(\frac{1}{2}\rho\sigma v + \kappa(\theta - v)\right) e^X \frac{\partial P_1}{\partial v} \\
&- \kappa(\theta - v) K e^{-r(T-t)} \frac{\partial P_2}{\partial v} \\
&+ \frac{1}{2}\sigma^2 v e^X \frac{\partial^2 P_1}{\partial v^2} - \frac{1}{2}\sigma^2 K e^{-r(T-t)} \frac{\partial^2 P_2}{\partial v^2} = 0.
\end{aligned}
\tag{2.32}
$$

Since $C_t(K)$ satisfies the Heston PDE in Equation 2.29 for non-negative stock prices, strikes and risk-free rate, $S \geq 0$, $K \geq 0$ and $r \geq 0$, taking $S = 1$, $K = 0$ and $S = 0$, $K = 1$ in Equation 2.28a results the prices of the options as P_1 and $-P_2$, respectively. The options obtained as above with price P_1 and P_2 follow the Heston PDE. Since one can simply multiply the Heston PDE in Equation 2.29 by -1, the option with price P_2 also follows the Heston PDE.

Since P_1 satisfies the Heston PDE, it can be assumed that $P_2 = 0$ to write the PDE in terms of P_1. Thus, Equation 2.32 changes to the following equation:

$$
\begin{aligned}
e^x \Bigg[\frac{\partial P_1}{\partial t} + \left(r + \frac{v}{2}\right) \frac{\partial P_1}{\partial X} + \frac{1}{2} v \frac{\partial^2 P_1}{\partial X^2} + \rho\sigma v \frac{\partial^2 P_1}{\partial X \partial v} \\
+ \left(\rho\sigma v + \kappa(\theta - v)\right) \frac{\partial P_1}{\partial v} + \frac{1}{2}\sigma^2 v \frac{\partial^2 P_1}{\partial v^2}\Bigg] = 0.
\end{aligned}
\tag{2.33}
$$

Since e^X is not zero if X is not $-\infty$ or S is not 0 and since P_1 is obtained by letting $S = 1$ and $K = 0$, the term e^X in Equation 2.33 is not zero. Thus, Equation 2.33 simplifies to

$$
\begin{aligned}
\frac{\partial P_1}{\partial t} + \left(r + \frac{v}{2}\right) \frac{\partial P_1}{\partial X} + \frac{1}{2} v \frac{\partial^2 P_1}{\partial X^2} + \rho\sigma v \frac{\partial^2 P_1}{\partial X \partial v} \\
+ \left(\rho\sigma v + \kappa(\theta - v)\right) \frac{\partial P_1}{\partial v} + \frac{1}{2}\sigma^2 v \frac{\partial^2 P_1}{\partial v^2} = 0.
\end{aligned}
\tag{2.34}
$$

Since P_2 satisfies the Heston PDE as well, it can be assumed that $P_2 = 0$ to write the PDE in terms of P_2. Thus, Equation 2.32 changes to the following equation:

$$\frac{\partial P_2}{\partial t} + \left(r - \frac{v}{2}\right)\frac{\partial P_2}{\partial X} + \frac{1}{2}v\frac{\partial^2 P_2}{\partial X^2} + \rho\sigma v\frac{\partial^2 P_2}{\partial X \partial v}$$
$$\kappa(\theta - v)\frac{\partial P_2}{\partial v} + \frac{1}{2}\sigma^2 v\frac{\partial^2 P_2}{\partial v^2} = 0. \tag{2.35}$$

Letting $u_1 = \frac{1}{2}$, $u_2 = -\frac{1}{2}$, $b_1 = \kappa - \rho\sigma$, $b_2 = \kappa$ and $a = \kappa\theta$ unifies Equation 2.34 and Equation 2.35 as follows:

$$\frac{\partial P_j}{\partial t} + (r + u_j v)\frac{\partial P_j}{\partial X} + \frac{1}{2}v\frac{\partial^2 P_j}{\partial X^2} + \rho\sigma v\frac{\partial^2 P_j}{\partial X \partial v}$$
$$+ (a - b_j)\frac{\partial P_j}{\partial v} + \frac{1}{2}\sigma^2 v\frac{\partial^2 P_j}{\partial v^2} = 0. \tag{2.36}$$

2.3.2 The Heston Characteristic Functions

When $S_T > K$ at expiration, the probabilities of the call being in the money are stated as $P_j = \mathbb{1}_{\{X_T > \log K\}}$, $j = 1, 2$. In this sense, the initial guess [12] for the characteristic functions $f_j(\phi; x_t, v_t)$ for the log of the terminal stock price, $X_T = \log(S_T)$, is

$$f_j(\phi; X_t, v_t) = exp(C_j(\tau, \phi) + D_j(\tau, \phi)v_t + i\phi X_t),$$

where $i = \sqrt{-1}$, C_j and D_j are coefficients and τ is the time to maturity.

Using Feynman-Kac theorem which states that if a function $g(X_t, t)$ of the Heston bivariate system of SDE's $X_t = (X_t, v_t) = (\log(S_t), v_t)$ satisfies the PDE $\frac{\partial g}{\partial t} - rg + \mathcal{A}g = 0$, where $\mathcal{A}$ is mentioned earlier as the Heston generator , $\mathcal{A} = rS\frac{\partial}{\partial S} + \frac{1}{2}vS^2\frac{\partial^2}{\partial S^2} + \kappa(\theta - v)\frac{\partial}{\partial v} + \frac{1}{2}\sigma^2 v\frac{\partial^2}{\partial v^2} + \frac{1}{2}\rho v\sigma S\frac{\partial^2}{\partial S \partial v}$, then

$$g(X_t, t) = E[g(X_T, T)|\mathcal{F}_t] = E[g(X_T, T)|X_t, v_t].$$

Using $g(X_t, t) = exp(i\phi \log(S_t))$,

$$g(X_t, t) = E[e^{i\phi X_T}|\mathcal{F}_t] = E[e^{i\phi X_T}|X_t, v_t].$$

Since f_j's are characteristic functions for the terminal condition $X_T = \log(S_T)$ on in the money call probabilities P_j's for $j = 1, 2$, f_j satisfies the PDE $\frac{\partial f}{\partial t} - rf + \mathcal{A}f = 0$ above. Therefore, with risk neutral probabilities $\mathbb{Q}^s$ and $\mathbb{Q}$, which use the stock and the bond as numeraire, respectively,

$$f_1(\phi; X_t, v_t) = E^{\mathbb{Q}^s}[e^{i\phi X_T}], f_2(\phi; X_t, v_t) = E^{\mathbb{Q}}[e^{i\phi X_T}].$$

Since f_j satisfy the PDE, it can be written as follows:

$$\begin{aligned} &\frac{\partial f_j}{\partial t} + (r + u_j v)\frac{\partial f_j}{\partial X} + \frac{1}{2}v\frac{\partial^2 f_j}{\partial X^2} + \rho\sigma v\frac{\partial^2 f_j}{\partial X \partial v} \\ &\quad + (a - b_j)\frac{\partial f_j}{\partial v} + \frac{1}{2}\sigma^2 v\frac{\partial^2 f_j}{\partial v^2} = 0. \end{aligned} \tag{2.37}$$

To write Equation 2.37 in terms of $\tau = T - t$, one can simply use $\frac{\partial f}{\partial t} = -\frac{\partial f}{\partial \tau}$. Hence,

$$\begin{aligned} &-\frac{\partial f_j}{\partial \tau} + (r + u_j v)\frac{\partial f_j}{\partial X} + \frac{1}{2}v\frac{\partial^2 f_j}{\partial X^2} + \rho\sigma v\frac{\partial^2 f_j}{\partial X \partial v} \\ &\quad + (a - b_j)\frac{\partial f_j}{\partial v} + \frac{1}{2}\sigma^2 v\frac{\partial^2 f_j}{\partial v^2} = 0. \end{aligned} \tag{2.38}$$

Since the initial guess of the characteristic functions are $f_j(\phi; X_t, v_t) = exp(C_j(\tau, \phi) + D_j(\tau, \phi)v_t + i\phi X_t)$, $j = 1, 2$,

$$\frac{\partial f_j}{\partial \tau} = f_j\left[\frac{\partial C_j}{\partial \tau} + \frac{\partial D_j}{\partial \tau}v\right], \tag{2.39a}$$

$$\frac{\partial f_j}{\partial X} = (i\phi)f_j, \tag{2.39b}$$

$$\frac{\partial^2 f_j}{\partial X^2} = -\phi^2 f_j, \tag{2.39c}$$

$$\frac{\partial^2 f_j}{\partial X \partial v} = \frac{\partial (f_j D_j)}{\partial X} = (i\phi)f_j D_j, \tag{2.39d}$$

$$\frac{\partial f_j}{\partial v} = f_j D_j, \tag{2.39e}$$

$$\frac{\partial^2 f_j}{\partial v^2} = f_j D_j^2. \tag{2.39f}$$

Plugging the expressions in Equation 2.39 into Equation 2.38, $j = 1, 2$,

$$\begin{aligned} &f_j\left[-\left(\frac{\partial C_j}{\partial \tau} + \frac{\partial D_j}{\partial \tau}v\right) + \rho\sigma v i\phi D_j - \frac{1}{2}v\phi^2 \right. \\ &\quad \left. + \frac{1}{2}\sigma^2 v D_j^2 + (r + u_j v)i\phi + (a - b_j v)D_j\right] = 0. \end{aligned} \tag{2.40}$$

Since f_j is nonnegative for $j = 1,2$, Equation 2.40 turns out to be the following equation:

$$
\begin{aligned}
&-\left(\frac{\partial C_j}{\partial \tau} + \frac{\partial D_j}{\partial \tau} v\right) + \rho\sigma v i\phi D_j - \frac{1}{2} v\phi^2 \\
&\quad + \frac{1}{2}\sigma^2 v D_j^2 + (r + u_j v) i\phi + (a - b_j v) D_j = 0.
\end{aligned} \tag{2.41}
$$

Equation 2.41 can be rewritten as follows:

$$
\begin{aligned}
&v\left[-\frac{\partial D_j}{\partial \tau} + \rho\sigma i\phi D_j - \frac{1}{2}\phi^2 + \frac{1}{2}\sigma^2 D_j^2 + u_j i\phi - b_j D_j\right] \\
&-\frac{\partial C_j}{\partial \tau} + ri\phi + aD_j = 0.
\end{aligned} \tag{2.42}
$$

Since the variance v is non-negative, Equation 2.42 can be divided into two differential equations as

$$
-\frac{\partial D_j}{\partial \tau} + \rho\sigma i\phi D_j - \frac{1}{2}\phi^2 + \frac{1}{2}\sigma^2 D_j^2 + u_j i\phi - b_j D_j = 0, \tag{2.43a}
$$

$$
-\frac{\partial C_j}{\partial \tau} + ri\phi + aD_j = 0. \tag{2.43b}
$$

If one solves the differential equations above, the characteristic functions f_j is obtained for $j = 1,2$. In fact Equation 2.43a is a Ricatti equation in D_j. Once D_j is found explicitly, by integrating Equation 2.43b, C_j can be found as well.

Moreover, since $f_j(\phi; X_t, v_t) = exp(C_j(\tau,\phi) + D_j(\tau,\phi)v_t + i\phi X_t)$, $j = 1,2$, when $C_j(\tau,\phi)$ and $D_j(\tau,\phi)$ are known explicitly, the characteristic functions for the log of the terminal stock price are obtained explicitly.

2.3.2.1 Obtaining the Coefficients C_j and D_j

In Equation 2.43a, letting $M_j = u_j i\phi - \frac{1}{2}\phi^2$, $Q_j = b_j - \rho\sigma i\phi$, $j = 1,2$, and $R = \frac{1}{2}\sigma^2$ converts the equation Equation 2.43a into the following:

$$
-\frac{\partial D_j}{\partial \tau} - Q_j D_j + R D_j^2 + M_j = 0. \tag{2.44}
$$

Assuming $D_j = -\frac{1}{R}\left(\frac{\partial w}{\partial \tau}\Big/ w\right) = -\frac{1}{R}\frac{w'}{w}$ and putting this into Equation 2.44, $j = 1,2,$

$$-\frac{1}{R}\frac{w''}{w}-\frac{1}{R}\left(\frac{w'}{w}\right)^2+M_j+Q_j\frac{1}{R}\left(\frac{w'}{w}\right)+\frac{1}{R}\left(\frac{w'}{w}\right)^2=0. \tag{2.45}$$

Rearranging and multiplying both sides of Equation 2.45 by Rw,

$$w''+Q_jw'+RM_jw=0. \tag{2.46}$$

To solve the second order ODE in (Equation 2.46), we use its characteristic roots:

$$\alpha_j=\frac{-Q_j+\sqrt{Q_j^2-4M_jR}}{2}, \tag{2.47a}$$

$$\beta_j=\frac{-Q_j-\sqrt{Q_j-4M_jR}}{2}. \tag{2.47b}$$

To simplify Equation 2.47a and Equation 2.47b,
let $d_j=\sqrt{Q_j^2-4M_jR}=\sqrt{(b_j-\rho\sigma i\phi)^2+\sigma^2(\phi^2-2u_ji\phi)}$.

Thus, $\alpha_j=\frac{-Q_j+d_j}{2}$ and $\beta_j=\frac{-Q_j-d_j}{2}$.

The well known solution of the ODE in Equation 2.46 is

$$w=Ye^{\alpha_j\tau}+Ze^{\beta_j\tau},$$

where Y and Z are constants.

Since $D_j=-\frac{1}{R}\frac{w'}{w}$,

$$D_j=-\frac{1}{R}\left[\frac{Y\alpha_je^{\alpha_j\tau}+Z\beta_je^{\beta_j\tau}}{Ye^{\alpha_j\tau}+Ze^{\beta_j\tau}}\right]. \tag{2.48}$$

Let $N=\frac{Y}{Z}$. Equation 2.48 becomes

$$D_j=-\frac{1}{R}\left[\frac{N\alpha_je^{\alpha_j\tau}+\beta_je^{\beta_j\tau}}{Ne^{\alpha_j\tau}+e^{\beta_j\tau}}\right]. \tag{2.49}$$

Using the initial condition $D_j(0,\phi)=0$ and Equation 2.49, $N\alpha_j+\beta_j=0$, from which $N=-\beta_j\big/\alpha_j$. This changes Equation 2.49 into

$$D_j = -\frac{\beta_j}{R}\left[\frac{-e^{\alpha_j\tau} + e^{\beta_j\tau}}{(\frac{-\beta_j}{\alpha_j})e^{\alpha_j\tau} + e^{\beta_j\tau}}\right]. \tag{2.50}$$

Dividing numerator and denominator of the right side in Equation 2.50 by $e^{\beta_j\tau}$,

$$D_j = -\frac{\beta_j}{R}\left[\frac{1 - e^{(\alpha_j-\beta_j)\tau}}{1 - (\frac{\beta_j}{\alpha_j})e^{(\alpha_j-\beta_j)\tau}}\right]. \tag{2.51}$$

Moreover, note that $\alpha_j - \beta_j = d_j$.

Let $g_j = \frac{\beta_j}{\alpha_j} = \frac{Q_j+d_j}{Q_j-d_j}$. Thus, Equation 2.51 simplifies to the following equation:

$$D_j = \frac{Q_j + d_j}{2R}\left[\frac{1 - e^{d_j\tau}}{1 - g_j e^{d_j\tau}}\right]. \tag{2.52}$$

Putting $Q_j = b_j - \rho\sigma i\phi$ and $R = \frac{1}{2}\sigma^2$ into Equation 2.52, the coefficient $D_j(\tau, \phi)$, $j = 1, 2$, can be written explicitly as

$$D_j(\tau, \phi) = \frac{b_j - \rho\sigma i\phi + d_j}{\sigma^2}\left[\frac{1 - e^{d_j\tau}}{1 - g_j e^{d_j\tau}}\right]. \tag{2.53}$$

After finding $D_j(\tau, \phi)$ as above, integrating the equation Equation 2.43b, the coefficient $C_j(\tau, \phi)$ can be written as

$$C_j = \int_0^\tau \left[ri\phi + a\left(\frac{b_j - \rho\sigma i\phi + d_j}{\sigma^2}\right)\left(\frac{1 - e^{d_j y}}{1 - g_j e^{d_j y}}\right)\right] dy + N_1, \tag{2.54}$$

where N_1 is a constant.

Rearranging Equation 2.54,

$$C_j = \int_0^\tau ri\phi dy + a\left(\frac{b_j - \rho\sigma i\phi + d_j}{\sigma^2}\right)\int_0^\tau \left(\frac{1 - e^{d_j y}}{1 - g_j e^{d_j y}}\right) dy + N_1. \tag{2.55}$$

Using the initial condition $C_j(0, \phi) = 0$ and Equation 2.55, $N_1 = 0$.

To find the coefficient C_j, two integrals in Equation 2.55 need to be found. The first integral equals

$$\int_0^\tau ri\phi dy = ri\phi\tau. \tag{2.56}$$

To evaluate the second integral, suppose $z = e^{d_j y}$. Differentiating both side with respect to τ gives $dz = e^{djy} d_j dy$, from which $dy = dz \big/ zd_j$. Thus, substituting $z = e^{d_j y}$ into the second integral, we get

$$\int_1^{e^{d_j\tau}} \left(\frac{1-z}{1-g_j z}\right)\left(\frac{1}{zd_j}\right) dz = \frac{1}{d_j}\int_1^{e^{d_j\tau}} \left(\frac{1-z}{z(1-g_j z)}\right) dz. \tag{2.57}$$

To evaluate the integral in Equation 2.57, a fractional expansion can be used as follows:

$$\frac{1-z}{z(1-g_j z)} = \frac{1}{z} - \frac{1-g_j}{1-g_j z}.$$

Thus, Equation 2.57 becomes

$$\begin{aligned}
\frac{1}{d_j}\int_1^{e^{d_j\tau}} \left[\frac{1}{z} - \frac{1-g_j}{1-g_j z}\right] dz &= \frac{1}{d_j}\left[\int_1^{e^{d_j\tau}} \frac{1}{z} dz - \int_1^{e^{d_j\tau}} \left(\frac{1-g_j}{1-g_j z}\right) dz\right] \\
&= \frac{1}{d_j}\left[\log(z) + \left(\frac{1-g_j}{g_j}\right)\log(1-g_j z)\right]_{z=1}^{z=e^{d_j\tau}} \\
&= \tau + \left(\frac{1-g_j}{d_j g_j}\right)\log\left(\frac{1-g_j e^{d_j\tau}}{1-g_j}\right).
\end{aligned} \tag{2.58}$$

Putting $g_j = \frac{b_j - \rho\sigma i\phi + d_j}{b_j - \rho\sigma i\phi - d_j}$ in Equation 2.58 gives the solution of the second integral as:

$$\tau - \frac{2}{b_j - \rho\sigma i\phi + d_j}\log\left(\frac{1-g_j e^{d_j\tau}}{1-g_j}\right).$$

Substituting the value of the integrals evaluated above into Equation 2.55, the coefficient $C(\tau,\phi)$ is found as

$$C(\tau,\phi) = ri\phi\tau + \left(\frac{a}{\sigma^2}\right)\left[(b_j - \rho\sigma i\phi + d_j)\tau - 2\log\left(\frac{1-g_j e^{d_j\tau}}{1-g_j}\right)\right]. \tag{2.59}$$

2.4 Obtaining in-the-money Probabilities

The characteristic functions are known. They can be inverted to get the desired in-the-money probabilities P_1 and P_2 by using inversion theorem [11] as, for $j = 1, 2$,

$$\begin{aligned} P_j = Pr(\log(S_t) > \log(k)) &= \frac{1}{2} + \frac{1}{2\pi}\int_{-\infty}^{\infty} Re\left[\frac{e^{-i\phi\log(K)} f_j(\phi; x, v)}{i\phi}\right] d\phi \\ &= \frac{1}{2} + \frac{1}{\pi}\int_{0}^{\infty} Re\left[\frac{e^{-i\phi\log(K)} f_j(\phi; x, v)}{i\phi}\right] d\phi. \end{aligned} \tag{2.60}$$

Additionally, P_1 and P_2 are Arrow-Debreu prices for the binary options with $(S = 1, K = 0)$ and $(S = 0, K = 1)$, respectively.

2.5 Pricing Options

Since P_1 and P_2 are known, option prices can be found by using these probabilities.

2.5.1 Pricing a European Call Option

To price a European option price, implementing in-the-money probabilities P_1 and P_2 into Equation 2.28a which is

$$C_t(K) = S_t P_1 - K e^{-r(T-t)} P_2,$$

the call option price is obtained.

Thus,

$$\begin{aligned} C_t(K) = S_t &\left[\frac{1}{2} + \frac{1}{\pi}\int_{0}^{\infty} Re\left[\frac{e^{-i\phi\log(K)} f_1(\phi; x, v)}{i\phi}\right] d\phi\right] \\ - K e^{-r(T-t)} &\left[\frac{1}{2} + \frac{1}{\pi}\int_{0}^{\infty} Re\left[\frac{e^{-i\phi\log(K)} f_2(\phi; x, v)}{i\phi}\right] d\phi\right]. \end{aligned} \tag{2.61}$$

Rearranging Equation 2.61, the call price at time t can be found as below:

$$C_t(K) = \frac{1}{2}S_t - \frac{1}{2}K e^{-r(T-t)} + \frac{1}{\pi}\int_{0}^{\infty} Re\left[\frac{e^{-i\phi\log(K)}\left(S_t f_1(\phi; x, v) - K e^{-r(T-t)} f_2(\phi; x, v)\right)}{i\phi}\right] d\phi. \tag{2.62}$$

2.5.2 Pricing a European Put Option

In order to protect arbitrage free market, the put-call parity must hold [16], which is

$$P_t(K) = C_t(K) + Ke^{-r(T-t)} - S_t$$

The European call option price at time t is already known explicitly. So using the put-call parity, to price the put option at time t, it is just needed to put the call price in the parity. Thus,

$$P_t(K) = \frac{1}{2}Ke^{-r(T-t)} - \frac{1}{2}S_t + \frac{1}{\pi}\int_0^\infty Re\left[\frac{e^{-i\phi\log(K)}\left(S_t f_1(\phi;x,v) - Ke^{-r(T-t)}f_2(\phi;x,v)\right)}{i\phi}\right]d\phi. \tag{2.63}$$

In addition to this, the other method for pricing put options to use in-the money probabilities.

In-the-money probabilities for the put option are the complement of those which are P_1 and $P2$ [25].

Let these probabilities be $P_1^\star$ and $P_2^\star$. Thus, these probabilities can be written as below:

For $j = 1, 2$,

$$P_j^\star = Pr(\log(S_T) < \log(K)) = 1 - P_j = \frac{1}{2} - \frac{1}{\pi}\int_0^\infty Re\left[\frac{e^{-i\phi\log(K)}f_1(\phi;x,v)}{i\phi}\right]. \tag{2.64}$$

Moreover, Equation 2.28a needs to be modified for the put price as following:

$$P_t(K) = Ke^{-r(T-t)}P_2^\star - S_t P_1^\star. \tag{2.65}$$

When Equation 2.64 is inserted in Equation 2.65, the result is exactly the same with the put option price at time t, which is obtained in Equation 2.63.

2.5.3 Pricing an Option with the Stock Paying Dividend

To insert the dividend, q, into the model which is demonstrated in risk-neutral probabilities in Equation 2.8a, the risk free rate r needs to be replaced by $(r - q)$ so that the stochastic process for the stock price can written as following:

$$dS_t = (r - q)S_t dt + \sqrt{v_t} S_t d\widetilde{W_t^{(1)}}. \tag{2.66}$$

Furthermore, the call option price and the put option price at time t needs to be written with compatible to the stock paying dividend q. Thus, Equation 2.28a and Equation 2.65 are changed to the following equations:

$$C_t(K) = S_t e^{-q(T-t)} P_1 - K e^{-r(T-t)} P_2, \tag{2.67a}$$

$$P_t(K) = K e^{-r(T-t)} P_2^{\star} - S_t e^{-q(T-t)} P_1^{\star}. \tag{2.67b}$$

Putting the probabilities P_j and $P_j^{\star}$ into Equation 2.67a and Equation 2.67b establishes the call and put option price at time t with the stocks paying the dividend, respectively.

CHAPTER 3

Fitting the Heston Model to BIST30 Warrants

In this chapter, we will fit the Heston model to the warrants written on BIST30; this fit is done in section 3.3. For the fit we use the method of loss functions, where the goal is to minimize the difference between model prices and the observed market prices. An excellent reference which explains a number of fitting algorithms is [21]. The first section below reviews the one based on loss functions as covered in [21]. For our actual computations, we also use the implementation of this algorithm as given by [21]. The effects of the Heston parameters on implied volatilities is briefly reviewed in section 3.2; these will be useful in interpreting our fit results.

3.1 The Heston Parameters

To fit the model to the BIST30 warrants, the parameters in the model needs to be found. Some of these parameters are mentioned before in Equation 2.8a which we copy below for reader's convenience:

$$dv_t = \kappa(\theta - v_t)dt + \sigma\sqrt{v_t}d\widetilde{W_t^{(2)}}$$

As seen above, the parameters are κ, θ, σ which are the mean reversion speed for the variance, the mean reversion level for the variance, the volatility of the variance.

In addition to these, the correlation ρ between the Brownian motions $\widetilde{W_t^{(1)}}$ and $\widetilde{W_t^{(2)}}$ and the initial variance v_0 are also among the Heston parameters.

In total, there are five many parameters that are $\kappa, \theta, \sigma, v_o, \rho$.

3.1.1 Obtaining the Heston Parameters with the Loss Function

The loss function estimating the parameters uses the error between the market prices and the model prices or between the market and the model implied volatility. To use the method above, we impose the following constraints on the parameters:

$$\kappa > 0, \theta > 0, \sigma > 0, v_0 > 0, \rho \in [-1, 1]$$

Since the loss functions use the market option prices or the implied volatilities obtained from them, this method estimates the risk-neutral parameters of the Heston model. That is the reason fitting the Heston's parameters to option prices generates the risk neutral measure. In other words, $\kappa = \kappa^\star$ and $\theta = \theta^\star$ under both measures $\mathbb{P}$ and $\mathbb{Q}$.

Let N_T be the number of maturities T_i $(i = 1, ..., N_T)$ and N_K be the number of strikes K_j $(j = 1, ..., N_K)$. $C(T_i, K_j) = C_{ij}$ represents the market price of the option which has T_i maturity and K_j strike. $C(T_i, K_j, \Theta) = C_{ij}^{\Theta}$ represents the price generated by the Heston model, which has again T_i maturity and K_j strike.

Let IV_{ij} and IV_{ij}^{Θ} be the market implied volatility derived from C_{ij} using the Black-Scholes model and the model implied volatility derived from C_{ij}^{Θ}, respectively. The aim is to minimize the error between the market and model implied volatility by using the loss function. Thus, the implied volatility mean error sum of squares (IVMSE) parameter estimates are based on the loss function,

$$\frac{1}{N} \sum_{i,j=1}^{N_T, N_K} w_{i,j} \left[IV_{ij} - IV_{ij}^{\Theta} \right]^2, \tag{3.1}$$

where $N = N_T N_K$ and w_{ij} is the weight of the option C_{ij} among the all options that are used. For more on these weights, please see [21].

C_{ij}^{Θ} needs to be calculated at first place to extract IV_{ij}^{Θ} in Equation 3.1. For this, the bisection algorithm which is a root finding algorithm is used.

3.2 Effects of the Heston Parameters on the Heston Implied Volatilities

The Heston parameters have some effects on the Heston implied volatilities derived from the option prices generated by the model. Whether these volatilities show smile or skew is determined by the parameters.

The correlation between Brownian motions ρ affects the direction of the skew. In other words, when $\rho < 0$ and $\rho > 0$, the slope of the skew is negative and positive, respectively. Additionally, the more the volatility of the variance σ increases, the more the curvature of the smile increases.

Moreover, the parameters the mean reversion speed for the variance, κ, the mean reversion level for the variance, θ, and the initial variance, v_0 , control the level of the curvature. The higher values in κ, θ and v_0 make the curvature flattening.

We will observe these effects in our estimation results below.

3.3 Fitting the Heston Model to BIST30 Warrants

Before we examine the results of our fits, let us give some information about the warrants and the data used for our fit.

Warrants and options both give the holder of that warrant or option the right to buy or sell a financial asset on specific terms and conditions (maturity, exercise price etc.) but not the obligation. Thus, an investor can not lose more money than that he/she pays for a warrant. Warrants differ from options by the fact that they represent certificated option rights so that all warrants have an International Securities Identification Number ("ISIN") . [3]

To fit the model, warrants that are written by Deutsche Bank and traded in BIST30 are used. We took the warrants from the date December 2015 since Borsa Istanbul passes single session era at the date 30.11.2015. [2]

Recall that we use the constant risk-free rate. To find it, we used the bond having two years maturity. We took the average of its yields in the previous month to obtain the risk-free rate of the month we worked on. For example, using the average of the daily returns in the month (December, 2015) and (February, 2016) , we get the risk-free rate 0.1081 and 0.1086 to use in (January, 2016) and (March, 2016), respectively. The reason of mentioning about the risk-free rates in (January, 2016) and (March, 2016) is that we use these months as examples of our applications in the remaining part of this thesis.

3.3.1 Results of the Fit

For the fit, we selected six call warrants with two different maturities ($29.04.2016$ and $30.06.2016$) and three different strikes ($95, 100, 105$) in the period (01.03.2016-31.03.2016).

We fit the model for each day in ($01.03.2016 - 31.03.2016$) and obtain the Heston parameters, Heston implied volatilities and the model prices of the call warrants.

To give the reader an idea about what the fits look like, we give several examples in the tables below.

Table 3.1, Table 3.2, Table 3.3, and Table 3.4 show the parameters and comparison between the market and the model prices of six call warrants.

Table 3.1: 1March2016-HestonParameters

kappa	**theta**	**sigma**	**v0**	**rho**	**IVMSE**
4.2693	0.0435	0.3006	0.0696	0.8798	1.15e-07

Table 3.2 and Table 3.4 show that the market prices and the model prices are close to each other. From this, it is clear that the fits work very well.

Table 3.2: Comparison between Market Prices and Model Prices of Call Warrants in 1March2016

DATE	MATURITIES	STRIKES	MARKET PRICES	Model PRICES
01.03.2016	29April2016	95	3.03	3.0284
		100	1.43	1.4351
		105	0.63	0.6259
	30June2016	95	4.78	4.7820
		100	2.93	2.9255
		105	1.73	1.7334

Table 3.3: 28March2016-HestonParameters

kappa	theta	sigma	v0	rho	IVMSE
19.8663	0.0645	0.3686	0.0645	0.9827	5.05e-06

Table 3.4: Comparison between Market Prices and Model Prices of Call Warrants in 28March2016

DATE	MATURITIES	STRIKES	MARKET PRICES	MODEL PRICES
28.03.2016	29April2016	95	6.33	6.3505
		100	3.03	3.022
		105	1.18	1.1745
	30June2016	95	8.50	8.5210
		100	5.50	5.4460
		105	3.23	3.2677

To see how the results of the fit vary, we change the period through which we fit the model, and the maturities and the strikes of the warrants. Again we take six warrants with maturities ($29.02.2016$ and $29.04.2016$) and strikes $90, 95, 100$ in the period $(15.01.2016 - 03.02.2016)$.

We fit the model for each day in $(15.01.2016 - 03.02.2016)$ and obtain the Heston parameters, the Heston implied volatilities and the model prices of the call warrants.

There is an example which shows the comparison between the market and the model prices of the call warrants selected as tables Table 3.5 Table 3.6:

Table 3.5: 20January2016-HestonParameters

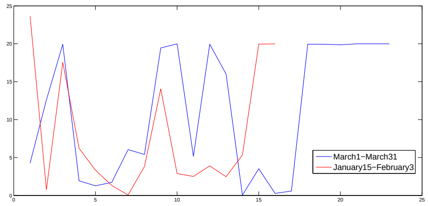
kappa	theta	sigma	v0	rho	IVMSE
6.2121	0.0807	1.1939	0.1372	-0.3329	5.53e-06

As seen in Table 3.6 above, although the period, strikes and maturities are changed, the fitting process gives the close model prices to the market prices.

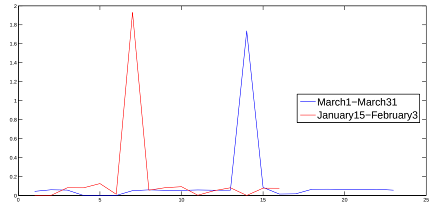
Additionally, since we fit the model along both periods mentioned above, we found the estimated parameters for each day. Therefore, we can observe the change in all parameters in both periods and compare them using Figure 3.1 .

Table 3.6: Comparison between Market Prices and Model Prices of Call Warrants in 20January2016

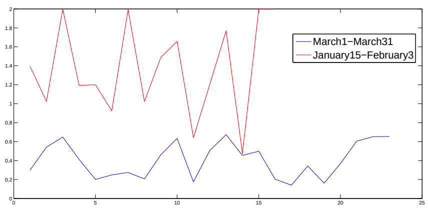
DATE	MATURITIES	STRIKES	MARKET PRICES	MODEL PRICES
20.01.2016	29Februaryt2016	90	1.58	1.5711
		95	0.53	0.5498
		100	0.18	0.1689
	29April2016	90	3.38	3.3848
		95	1.83	1.8237
		100	0.93	0.9302



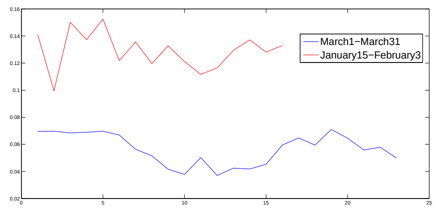
(a) The change in κ



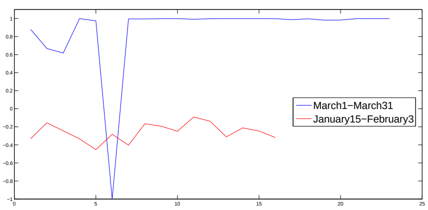
(b) The change in θ



(c) The change in σ



(d) The change in v_0



(e) The change in ρ

Figure 3.1: The change in the Heston parameters in the periods (01.03.2016-31.03.2016) and (15.01.2016-03.02.2016)

As seen in the Figure 3.1, it can be observed that σ and v_0 values during 15January-3February is bigger than σ and $v0$ values during 1March - 31March. Moreover, while ρ values are negative during 15January-3February, ρ values are positive during 1March-31March.

To evaluate the comparison between κ and θ values, the mean of all parameters can be checked by using Table 3.7 below:

Table 3.7 shows that the mean of parameters σ, v_0 and ρ are the same with the obser-

Table 3.7: Mean of Parameters

	Mean of Params. in 15.01.2016-03.02.2016	**Mean of Params. in 01.03.2016-31.03.2016**
κ	11.2126	6.6562
θ	0.1209	0.1718
σ	0.4073	1.4238
v_0	0.0565	0.1292
ρ	0.8719	-0.2578

vation made using the Figure 3.1. Furthermore , the mean of κ all along 15January-3February is bigger than the mean of κ all along 1March-31March.Although the mean of θ during 15January-3February is bigger than the mean of θ during 1March-31March, since the means are close to each other, this does not affect more than the other parameters' effect on the curvature of the smile of the Heston implied volatilities.

Using Section 3.2 and the observation on the comparisons of the parameters in the periods $(15.01.2016 - 03.02.2016)$ and $(01.03.2016 - 31.03.2016)$, the model implied volatilities's curvature for the most of the days in $(01.03.2016 - 31.03.2016)$ are more flattening that the curvature of the days in $(15.01.2016 - 03.02.2016)$.

To see this, since the parameters of 28 March 2016 and 20 January 2016 are known from Table 3.3 and Table 3.5, which enables to compare the parameters of these two days as well, the figures of the model implied volatilities in these days are used. Additionally, the figures gives the comparisons between the market implied volatilities and the model implied volatilities simultaneously are:

As seen in Table 3.6 above, although the period, strikes and maturities are changed, the fitting process gives the close model prices to the market prices.

Additionally, since we fit the model along both periods mentioned above, we found the estimated parameters for each day. Therefore, we can observe the change in all parameters in both periods and compare them using Figure 3.1 .

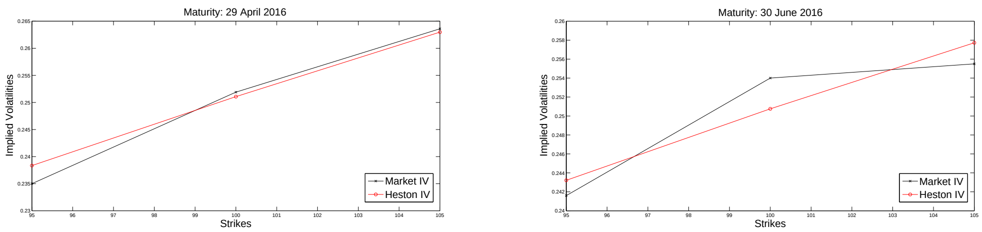


Figure 3.2: The comparison between the market and the model implied volatilities at 28.03.2016

When we compare the parameters in 28.03.2016 and 20.01.2016, the comparison of the parameters of these days are convenient to the observations made above for the comparison of the parameters in. Thus, we can make an initial opinion about that the model implied volatilities' curvature in 28.03.2016 are more flattening than the curvature in 20.01.2016. Figure 3.2 and Figure 3.3 show that this opinion is true.

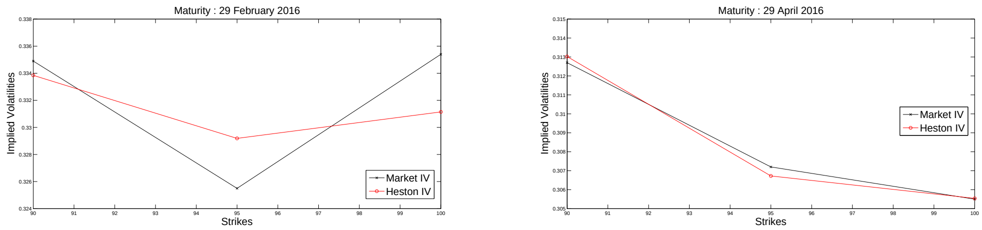


Figure 3.3: The comparison between the market and the model implied volatilities at 20.01.2016

Therefore, the parameters' effects on the model implied volatilities observed in the literature are in the same way with BIST30 warrants.

CHAPTER 4

Hedging a BIST30 Call Warrant with the Heston Model

In this chapter, we review how one hedges and replicates an option in the Heston model. Then we compare the hedging performance of the Heston model and the Black-Scholes model on a collection of call warrants written on the BIST30 index traded in BIST. Finally, we perform the hedging of a call warrant with a strike that was not being traded during the period covered.

4.1 Hedging an Option Using the Heston Model

4.1.1 Hedging an Option

As we have already indicated in Section 2.3, to get a riskless portfolio and to hedge the random volatility, an additional asset (other than the underlying index) driven by the same Brownian motions is needed in the hedging portfolio. The value of the riskless portfolio Π, consisting of one option with price $V = V(S, v, t)$, Δ units of the underlying asset (in our case it is the BIST30 index), and ψ units of another option $U = U(S, v, t)$ for the volatility is

$$\Pi = V + \Delta S + \psi U. \tag{4.1}$$

Using Equation 4.1, one obtains Δ and ψ given in Equation 2.14 of Section 2.3, for reader's convenience we repeat them here:

$$\psi = -\frac{\partial V}{\partial v} \Big/ \frac{\partial U}{\partial v}, \tag{4.2a}$$

$$\Delta = -\psi \frac{\partial U}{\partial S} - \frac{\partial V}{\partial S}. \tag{4.2b}$$

Because we are looking at the problem from the point of the seller of the option, the V position in Equation 4.1 needs to be changed to a short position [15]. Thus, the riskless portfolio, Π, including a short position on the call to be hedged will be:

$$\Pi = -V + \Delta S + \psi U \tag{4.3}$$

Equation 4.3 changes Equation 4.2a and Equation 4.2b and we obtain Δ and ψ for the hedging portfolio as

$$\psi = \frac{\partial V}{\partial v} \Big/ \frac{\partial U}{\partial v}, \tag{4.4a}$$

$$\Delta = -\psi \frac{\partial U}{\partial S} + \frac{\partial V}{\partial S}. \tag{4.4b}$$

In Equation 4.3, V is the price of the option hedged , U is the price of another option used to hedge V and S is the stock price. Since Π represents the value of the riskless portfolio, and because the market is no arbitrage it must be that Π equals the payoff of a riskless bond. Thus, how much bond must be present in the hedging portfolio at time t, can be calculated via

$$B_t = V_t - (\Delta_t S_t + \psi_t U_t). \tag{4.5}$$

For the purposes of this work the financial data available to us have been daily prices. Thus, to apply the above continuous time hedging algorithm, we discretize time and we perform the hedge at the end of each trading day. This leads to a hedging error H_t for day t and an accumulated hedge error (denoted $(PL)_t$) up to day t, computed as follows:

1. At $t = 0$ the hedging portfolio is constructed for the first time and $H_0 = 0$,
2. The hedging portfolio at t has long positions in Δ_t units of stock, ψ_t units of the other option with price U and B_t units of the bond; these are computed using the formulas above.
3. This hedging portfolio, (Δ_t, ψ_t, B_t) is updated to $(\Delta_{t+1}, \psi_{t+1}, B_{t+1})$ at time $t+1$; due to discretization and model errors this update will not be self financing and the portfolio will accrue the following hedging error

   $$H_{t+1} = V_{t+1} - (B_t e^{r/365} + \Delta_t S_{t+1} + \psi U_{t+1}), \tag{4.6}$$

   where $B_t e^{r/365}$ represents the bond's value at $t + 1$.
4. The total hedging error up to time $t + 1$ then will be

   $$(PL)_{t+1} = (PL)_t e^{r/365} - H_{t+1}, \tag{4.7}$$

   where $(PL)_t$ is the accumulated hedging error up to day t. Thus, the total hedging error up to day $t + 1$ is the sum of the future value of the total hedging error up to day t plus the hedging error stemming from the transactions of day $t + 1$. Note that a positive value of $(PL)_t$ means that the hedger makes a profit from the transaction and a negative value corresponds to a loss. But, since we are looking at this problem from the point of view of hedging, closer (PL) is to 0 better the hedging algorithm works (if the algorithm worked perfectly (PL) would have been 0 identically).

The total hedge error as a percentage of the current value of the option is defined as

$$(HE)_t = \frac{(PL)_t}{V_t}. \tag{4.8}$$

We will be using this value to measure how well the hedging algorithm works.

4.2 Results

In this section, we report on the performance of hedging based on the Heston model using the steps given in the previous section. In subsection 4.2.1 we apply the hedging call warrants on the BIST30. Two periods and four options are covered:

1. (01.03.2016-31.03.2016) (24 working days); we hedged two warrants in this period: a call option with strike $K = 95$ with maturity 29.04.2016 and a call option with the same strike $K = 95$ and maturity 30.06.2016. The parameter estimation of the Heston model for these periods have been carried out in Chapter 3; for the parameter values of the Heston model to be used in the present chapter we use the results of Chapter 3.

2. (15.01.2016-03.02.2016) (14 working days), the warrants are call options with strike $K = 90$ and maturity 29.02.2016 and 29.04.2016.

In choosing these we have used the following guidelines:

1. we wanted to cover at least one option with a short term maturity and one with a longer term.

2. In choosing the strikes: we have chosen one of the most liquid strikes traded in the period covered and used in model fitting (the greatest number of strikes used in any of the model fits we performed was three, so in choosing our example strikes we did not have many options to begin with).

3. the time period was chosen to be between 1.12.2015 (the date when BIST switched to a single trading session) and 31.3.2016 (the date around which we were in the middle of carrying out the work that is the subject of the present thesis). We wanted to cover at least two different time periods to get an idea about how much the results obtained depend on the period covered.

In the final subsection of the chapter, we apply the hedging algorithm to a warrant which was not traded in the market during the period covered. For this example, we have chosen the period between 1.03.2016 and 31.03.2016, a call option warrant with strike $K = 96$ and maturity 29.04.2016.

4.2.1 Hedging BIST30 warrants already traded in the market

To measure the hedging performance of the Heston model on BIST30 warrants, we compare its performance with the hedging performance of the Black-Scholes model.

We use the delta-hedging of the Black-Scholes model [20] and apply the Black-Scholes delta-hedging for each time period one day as in Section 4.1.1 .

4.2.1.1 Hedging a warrant having a shorter time to maturity

Using the method in Section 4.1.1, we hedged a warrant with strike 95 and maturity 29 April 2016 both using the Heston and the Black-Scholes models for each day in the period (01.03.2016-31.03.2016). Let us denote the warrant being hedged with C .

Furthermore, to hedge a warrant using the Heston model, we need to use another BIST30 warrant. For this, we choose a warrant with strike 100 and maturity 29 April 2016. Let us denote this warrant by $C^{\star}$.

In our hedging computations, we used the market prices of C and $C^{\star}$ in both the Heston and the Black-Scholes model (in the case of hedging with the BS model, this corresponds to using the implied volatility in the computations, rather than historical volatility). The trajectory of the BIST30 index during the period covered is given in Figure 4.1 .

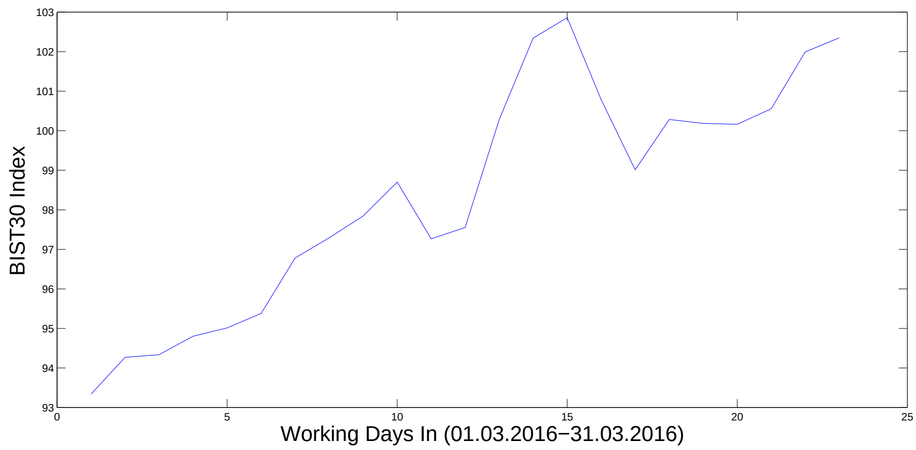


Figure 4.1: BIST30 index change in MARCH 2016

The hedging errors of the Heston model and the Black-Scholes models are given in Table 4.1.

When we compare the models' hedge performance listed in Table 4.1, we see that the percentage errors of the Heston model are much closer to zero than the errors of the Black-Scholes model, i.e., the hedge performance of the Heston model is better than the performance of the Black-Scholes model. The hedge percentage errors of both algorithms are also depicted in Figure 4.2. It can also be observed from this figure that the Heston model's hedge errors are closer to zero while the Black-Scholes hedge error can fluctuate wildly.

Table 4.1: The comparison of the hedge performance between the Heston and the Black-Scholes models on a warrant with short time to maturity in (01.03.2016-31.03.2016)

#	**Heston Model Hedging**		**BS Model Hedging**	
DATE	Profit/Loss	**Hedge Error(%)**	Profit/Loss	**Hedge Error(%)**
1March2016	0	0	0	0
2March2016	0.0391	1.40	-0.1279	-3.73
3March2016	-0.0268	-0.80	-0.0072	-0.22
4March2016	-0.0719	-1.98	-0.0867	-2.39
7March2016	-0.0813	-2.18	-0.0946	-2.54
8March2016	-0.0407	-1.10	0.1029	2.78
9March2016	-0.0171	-0.39	0.0939	2.14
10March2016	-0.0537	-1.16	0.1412	3.10
11March2016	0.0516	1.08	0.3396	7.07
14March2016	0.1162	2.21	0.5792	11.03
15March2016	-0.0841	-2.89	0.1772	3.98
16March2016	-0.3945	-0.90	0.4486	10.20
17March2016	0.0466	0.73	0.8043	12.57
18March2016	0.3802	4.87	1.3581	17.41
21March2016	0.2245	2.66	1.1944	14.14
22March2016	-0.2023	-2.94	0.5809	8.32
23March2016	-0.3246	-4.76	0.1796	3.19
24March2016	-0.3093	-4.73	0.5130	7.85
25March2016	-0.3552	-4.50	0.3851	5.90
28Mart2016	-0.2952	-2.29	0.5367	8.48
29March2016	-0.1837	-2.85	0.7981	12.41
30March2016	0.1155	-1.86	1.0228	13.41
31March2016	0.1422	1.52	1.3991	18.46

Hedging in the time period 15.01 to 03.02 To further understand the performance of the hedging algorithm given by the Heston algorithm we conducted another hedge covering the time period January 15 to February 2 2016 (approximately 45 days before the period studied above). We now consider the warrant with strike 90 and maturity 29 February 2016 in both models. Let C_1 denote the warrant hedged.

To hedge C_1 using the Heston model , another BIST30 warrant is needed. We choose this warrant with strike 95 and maturity 29 February 2016, which is called $C_1^\star$.

The trajectory of BIST30 in the period (15.01.2016-03.02.2016) can be observed in Figure 4.3.

The comparison of the hedging performance between the Heston model and the Black-Scholes model can be observed using the following Table 4.2

From Table 4.2 , the profits or the losses of the hedging process of the warrant C_1 for both models in the columns of profit/loss can be observed. The percentage of the hedge

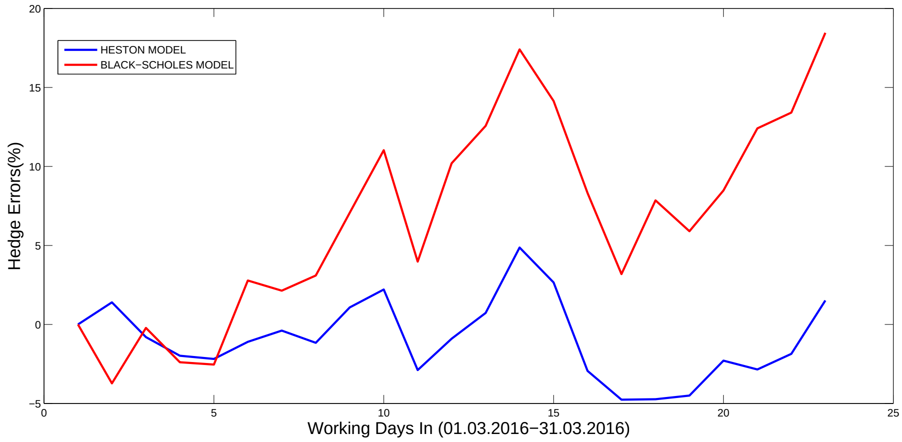


Figure 4.2: The Hedge Errors of the Heston and The Black-Scholes Model in (01.03.2016-31.03.2016)

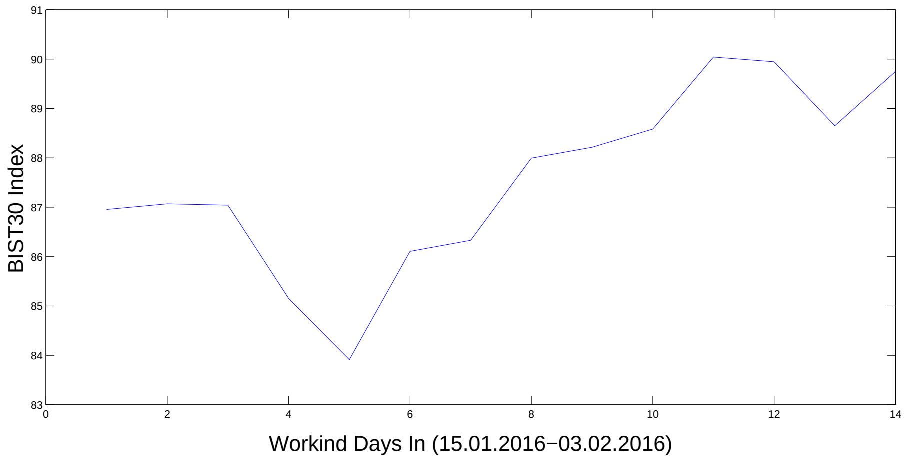


Figure 4.3: BIST30 index change in (15.01.2016-03.02.2016)

errors of both models can be obtained in the hedge error columns; see also Figure 4.4, for a graphical representation of the hedge errors of the models.

As in the previous period we see once again that, in the period (15.01.2016-03.02.2016) the percentage errors of the Heston model are much closer to zero than the errors of the Black-Scholes model.

4.2.1.2 Hedging a warrant having a longer time to maturity

In Section 4.2.1.1, we hedged a warrant having a short time to maturity by using the Heston model and the Black-Scholes model and compared the performance of these with each other.

Table 4.2: The comparison of the hedge performance between the Heston and the Black-Scholes models on a warrant with short time to maturity in (15.01.2016-03.02.2016)

#	**Heston Model Hedging**		**BS Model Hedging**	
DATE	Profit/Loss	**Hedge Error(%)**	Profit/Loss	**Hedge Error(%)**
15January2016	0	0	0	0
18January2016	-0.0033	-0.13	0.0151	0.62
19January2016	0.0375	1.68	0.1975	8.86
2OJanuary2016	-0.0127	-0.81	0.1887	11.94
21January2016	0.0657	5.81	0.5221	46.20
22January2016	0.0315	1.82	0.0985	5.70
25January2016	0.0542	3.13	0.1362	7.87
26January2016	0.1061	4.55	-0.1960	-8.41
27January2016	0.0871	3.65	-0.1793	-7.53
28January2016	0.0948	3.90	-0.1122	-4.62
29January2016	0.1445	4.69	-0.3149	-10.22
1February2016	0.0994	3.33	-0.3089	-10.36
2February2016	0.0160	0.68	-0.0135	-0.58
3February2016	0.1041	3.78	-0.1400	-5.09

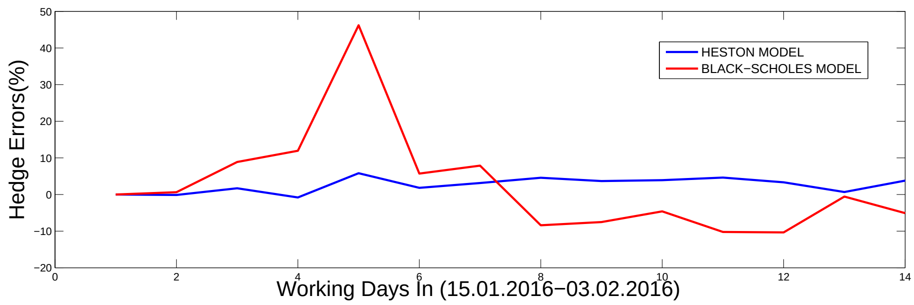


Figure 4.4: The Hedge Errors of the Heston and The Black-Scholes Model in (15.01.2016-03.02.2016)

To see these models' performance on a warrant with a longer time to maturity, we will know perform the hedge of a warrant $\bar{C}$ with strike 95 and maturity 30.06.2016 in the time period 01.03.2016 to 31.03.2016 using both models (in the previous subsection the maturity was 29.04.2016; thus the warrants in the present subsection have two additional months to maturity). For the Heston model hedge we will use the additional warrant with strike 100 and maturity 30.06.2016; we will denote this last warrant by $\bar{C}^{\star}$.

Table 4.3 lists the hedging errors of the models, Figure 4.5 shows the same errors graphically. We see from both of these that the hedging results for the 30.06 maturity warrants are similar to those of the warrants with 29.04 maturity. The hedge errors of the Heston model are much closer to zero compared to the hedging errors of the Black Scholes model.

Table 4.3: The comparison of the hedge performance between the Heston and the Black-Scholes models on a warrant with long time to maturity in (01.03.2016-31.03.2016)

#	Heston Model Hedging		BS Model Hedging	
DATE	Profit/Loss	**Hedge Error(%)**	Profit/Loss	**Hedge Error(%)**
1March2016	0	0	0	0
2March2016	-0.0800	-1.50	-0.2974	-5.56
3March2016	-0.0252	-0.48	-0.1262	-2.43
4March2016	0.0276	0.50	-0.2051	-3.73
7March2016	0.0433	0.78	-0.1625	-2.93
8March2016	-0.0531	-0.94	-0.0445	-0.79
9March2016	0.0060	0.10	-0.0730	-1.16
10March2016	-0.0513	-0.77	-0.1022	-1.54
11March2016	0.0062	0.09	0.0438	0.64
14March2016	0.0328	0.45	0.2839	3.89
15March2016	-0.0914	-1.41	-0.1176	-1.81
16March2016	-0.0284	-0.44	0.1043	1.60
17March2016	0.1007	1.21	0.6606	7.96
18March2016	0.2581	2.67	1.2649	13.11
21March2016	0.2243	2.19	1.1518	11.24
22March2016	-0.0748	-0.84	0.4187	4.70
23March2016	-0.1024	-1.33	-0.1321	-1.72
24March2016	-0.0896	-1.04	0.1518	1.76
25March2016	-0.1534	-1.76	-0.0255	-0.29
28March2016	-0.1513	-1.78	0.1265	1.49
29March2016	-0.1034	-1.20	0.3886	4.52
30March2016	-0.0959	-0.99	0.7138	7.36
31March2016	0.0415	0.43	1.0936	11.30

Moreover, using Figure 4.5, it can be observed that the Heston model's hedge error is more close. Thus, the hedge performance of the Heston model on $\bar{C}$ is better than the Black-Scholes' one in (01.03.2016-31.03.2016).

Finally, let us change the time period to (15.01.2016-03.02.2016) and hedge a warrant (denoted $\bar{C}_1$) with strike $K = 90$ with maturity 29.04 (in the previous section we covered the same strike in the same time period with maturity 29.02). For the Heston hedge we will use the additional warrant with strike 95 and the maturity 29.04.2016 (denoted $\bar{C}_1^{\star}$). Table 4.4 and Figure 4.6 shows that the Heston hedge errors for this setup. Once again, we get results to those we have observed in the previous cases: The Heston hedge error is closer to zero compared to the hedging errors of the Black-Scholes model.

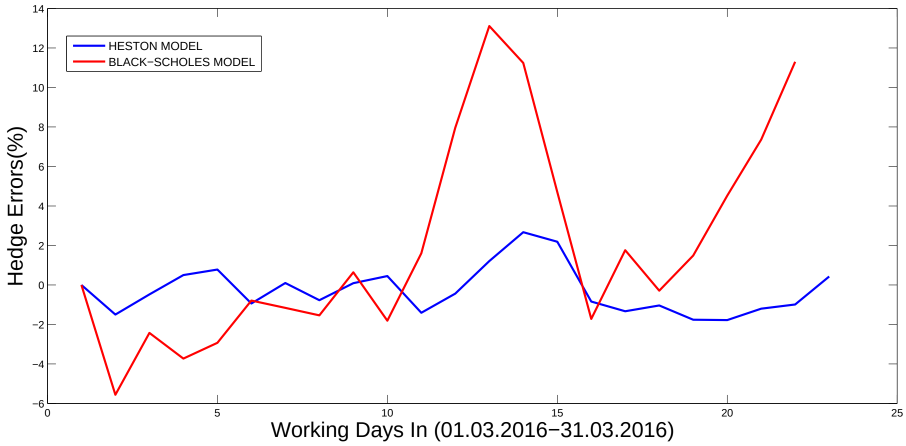


Figure 4.5: The Hedge Errors of the Heston and The Black-Scholes Model in (01.03.2016-31.03.2016)

Table 4.4: The comparison of the hedge performance between the Heston and the Black-Scholes models on a warrant with short time to maturity in (15.01.2016-03.02.2016)

#	**Heston Model Hedging**		**BS Model Hedging**	
DATE	Profit/Loss	**Hedge Error(%)**	Profit/Loss	**Hedge Error(%)**
15January2016	0	0	0	0
18January2016	0.0146	0.34	0.0156	0.37
19January2016	0.0384	0.94	0.1486	3.64
20January2016	-0.0285	-0.84	0.1903	5.63
21January2016	-0.0969	-3.42	0.6242	22.06
22January2016	0.0528	1.45	0.0012	0.03
25January2016	0.0193	5.30	0.0394	1.08
26January2016	0.2302	5.25	-0.4422	-10.10
27January2016	0.2151	4.80	-0.4750	-10.60
28January2016	0.1683	3.67	-0.4574	-9.99
29January2016	0.3140	5.98	-0.6795	-12.64
1February2016	0.2921	5.67	-0.6729	-13.07
2February2016	0.1719	3.82	-0.3771	-8.38
3February2016	0.2184	4.33	-0.6330	-12.03

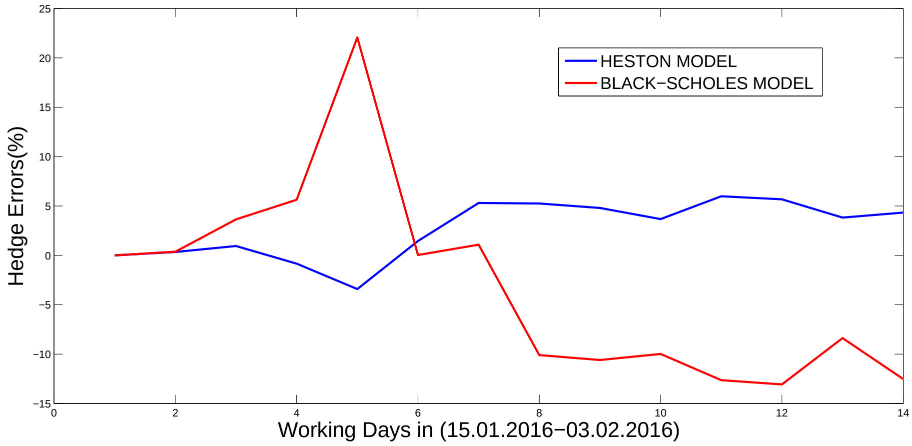


Figure 4.6: The Hedge Errors of the Heston and The Black-Scholes Model in (15.01.2016-03.02.2016)

4.2.2 Hedging a warrant not traded in the market

In the previous sections we have hedged 4 different warrants in two different time periods, all of which were traded in the time periods covered. In this section we would like to apply the Heston hedging algorithm to a warrant with a strike not available in the market. For our warrant we have chosen a call warrant with strike $K = 96$ and maturity 29.04.2016; we will denote this warrant with C_3. As before, for the Heston hedge an additional warrant is required: for this, we will use the warrant with strike $K = 100$ with the same maturity 29.04.2016 as that of C_3. To maintain the comparison with the hedging previous hedging performance mentioned in Section 4.2.1.1 and Section 4.2.1.2, we observe the hedge performance of C_3 during the period (01.03.2016-31.03.2016) .

Since C_3 is not traded, the hedging algorithm must use the model prices (as opposed to the market prices, which are not available for $K = 96$). The model prices of C_3 and the hedge errors are shown in Table 4.5. the hedging error is also depicted in Figure 4.7.

When we compare these results with those of the previous section, we see that the hedging performance of the Heston model for the untraded C_3 is slightly worse than those for the traded options. In particular, we see that the hedging error grows up to 8% whereas in the previous hedges the maximum hedging error was around 4%.

Table 4.5: The Model Prices and the Hedge Performance of The Heston Model on the Untraded Warrant

DATE	The Model Prices of C3	Profit/Loss	Hedge Errors(%)
1March2016	2.6274	0	0
2March2016	2.9949	0.0291	0.97
3March2016	2.8784	-0.0227	-0.79
4March2016	3.0802	0.0101	0.33
7March2016	3.2203	-0.0481	-1.49
8March2016	3.2579	-0.0930	-2.86
9March2016	3.8012	-0.0096	-0.25
10March2016	4.0301	-0.0360	-0.89
11March2016	4.1751	0.0743	1.78
14March2016	4.5950	0.1652	3.59
15March2016	3.8285	-0.1018	-2.66
16March2016	3.7694	-0.0283	-0.75
17March2016	5.6354	0.0310	0.55
18March2016	7.3127	-0.0392	-0.54
21March2016	7.8037	-0.0403	-0.52
22March2016	6.2163	-0.3376	-5.43
23March2016	4.9151	-0.4108	-7.35
24March2016	5.7542	-0.4042	-7.02
25March2016	5.7656	-0.4495	-7.80
28March2016	5.5685	-0.3980	-7.14
29March2016	5.6810	-0.3032	-5.33
30March2016	6.8552	-0.3105	-4.53
31March2016	7.0531	-0.3506	-5.08

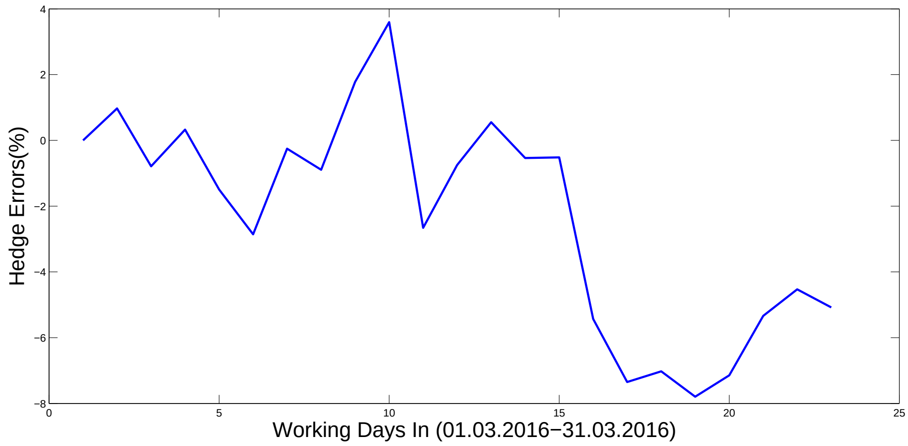


Figure 4.7: The Hedge Errors of the Heston Model on the untraded warrant in (01.03.2016-31.03.2016)

CHAPTER 5

Conclusion

In this chapter we comment further on our results and suggest several directions for future research.

As seen in Chapter 3, Figure 3.1, there is a great change in the Heston estimated parameters between the period (01.03.2016-31.03.2016) and (15.01.2016-03.02.2016). This can be interpreted that the Heston model is not perfectly compatible with the market (BIST30 warrants). However, with the appropriate choices of parameters, the Heston model generates the model prices close to the known market prices as seen in Table 3.2, Table 3.4 and Table 3.6.

We see in all of the examples that the hedge performance of the Heston model is superior to that of the Black-Scholes model; this is the case as we change the strike, the maturity and the period of hedge. This seems to be clearly related with the fact that the Heston model uses an additional option to hedge volatility.

For future work, the following goals seem relevant: a better model for the interest rate, allowing jumps in the Heston model, extending the analysis to earlier periods (not just the period after November 2015, to which we have confined ourselves in this work) and to more systematically cover the option parameters.

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