

PRICING AND HEDGING LOOKBACK OPTIONS USING BLACK-SCHOLES IN  
BORSA ISTANBUL

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BLACK-SCHOLES IN BORSA ISTANBUL**

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**I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.**

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## ABSTRACT

### PRICING AND HEDGING LOOKBACK OPTIONS USING BLACK-SCHOLES IN BORSA ISTANBUL

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The lookback option is a path dependent option that looks at the behaviour of the underlying asset for a specified time frame known as the lookback period. The maximum (minimum) attained during the lookback period is used to determine the option's payoff. In this thesis, the floating strike lookback option, which uses the maximum to determine the strike price for the put options will be examined and will be applied to the assets appearing in the BIST30 index. We estimate the historical volatility of these assets and compute the price of the floating strike lookback options written on these assets using the Black Scholes (BS) framework. We then apply the delta hedging algorithm given by Black Scholes to see its replication performance for these lookback options. We apply the algorithm in two different periods: October 2015 and January 2016.

*Keywords* : Lookback options, floating strike, Turkish Market, Black Scholes, model fitting, hedging, Borsa Istanbul





## ÖZ

### BIST30 INDEKSİNDEKİ HİSSE SENETLERİ ÜZERİNE YAZILI LOOKBACK OPSİYONLARIN BLACK-SCHOLES MODELİ İLE FİYATLANMASI VE REPLİKASYONU

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Geriye Bakışlı opsiyonlar, şimdiki fiyatı opsiyonun dayanak varlığının geçmiş fiyatlarına bağlı finansal araçlardır. Bu tezde BIST30 endeksini oluşturan stok fiyatları üzerine yazılı maksimum Geriye Bakışı opsiyonları Black-Scholes modeli kullanılarak fiyatlanıp ve replike (garanti) edilmiştir. Replikasyon ve fiyatlama algoritmaları iki ayrı dönemde uygulanmıştır: Ekim 2015 ve Ocak 2016. Black-Scholes modelinin uygulanması için gerekli olan volatilité parametresi dayanak varlığın tarihsel volatilitesi kullanılarak hesaplanmıştır.

*Anahtar Kelimeler:* Geriye Bakışlı opsiyonlar, sabit olmayan işlem fiyatı, Türkiye Piyasasında, Black Scholes, model uydurma, garantiye alma, Borsa İstanbul





*To My Mother Yolanda Paul.*



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## LIST OF ABBREVIATIONS

ABBREV	Abbreviation
$\mathbb{R}$	Set of Real Numbers
DPS	Dividend Per Share
EPS	Earning Per Share
$r$	Risk-Free Interest Rate
$\sigma$	Volatility
BS	Black-Scholes
i.e.	That is
e.g.	Example
Jan	January
Oct	October
wrt	With respect to
$E(t)$	Error at time $t$
$\mathbb{E}(X)$	Expectation of $X$
METU	Middle East Technical University
IAM	Institute of Applied Mathematics



# CHAPTER 1

## Introduction

The classical European option ‘calls’ and ‘puts’ give holders the right to buy or sell an asset at a pre-agreed price,  $K$ , at a predetermined maturity date  $T$ . Furthermore, the payoff for this type of option is dependent only on  $K$  and the asset price on the maturity date and is given as:

$$V_c(T) = [S(T) - K]^+, \quad V_p(T) = [K - S(T)]^+,$$

(for calls and puts respectively) where

$$[f(x)]^+ = \begin{cases} f(x), & \text{if } f(x) \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

The **Lookback Option**, unlike classical options, is an option where the movement of the underlying assets is observed for a specified duration called *the lookback period* in order to determine its payoff. In general, the payoff is determined by either the maximum or minimum observed during the lookback period. These types of options are also often referred to as *path dependent options*. See Figures 1.1, 1.2, and 1.3.

The lookback options have several varieties which include:

- The fixed strike lookback,
- The reverse fixed strike lookback, and
- The floating strike lookback.

Below these options are defined.

### 1.1 The Fixed Strike Lookback

With the fixed strike lookback, the strike price for a put (call) is determined before hand and the minimum (maximum) is used in place of  $S(T)$ . The payoff of a put (call)

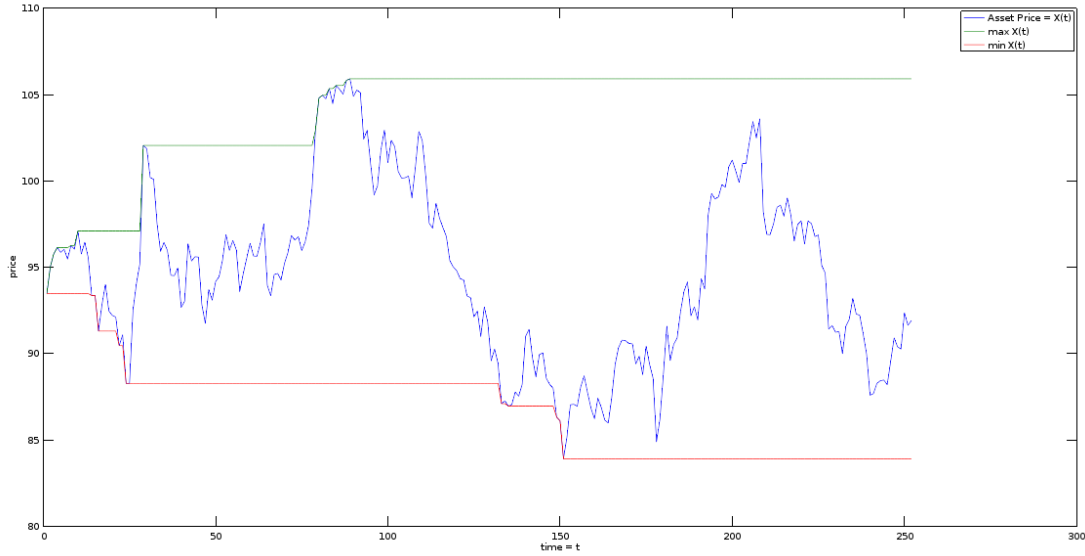


Figure 1.1: Example of a stock movement over a one year lookback period.

is therefore given as ([2] section 8.4, page 198):

$$V_p(T) = \left[ K - \min_{0 \leq t \leq T} S(t) \right]^+, \quad \left( V_c(T) = \left[ \max_{0 \leq t \leq T} S(t) - K \right]^+ \right).$$

**Example 1.1.** Let us assume we have a stock with starting price  $S(0) = 85$ ,  $K = 90$ , maturity date  $T$  and the lookback period,  $T - \text{start date} = 100$  days.  $S(T) = 43.5$ ,

$$m(t) = \min_{0 \leq t \leq T} S(t) = 20.89 \quad \text{and} \quad M(T) = \max_{0 \leq t \leq T} S(t) = 104.48,$$

then

$$V_p(T) = 90 - 20.89 = 69.11,$$

and

$$V_c(T) = 104.48 - 90 = 14.48.$$

## 1.2 The Reverse Fixed Strike Lookback Option

This is a reversal of the Fixed Strike option where, for a put (call) option the maximum (minimum) is used. Hence, the pay-off is given as ([2] section 8.4, page 198):

$$V_p(T) = \left[ K - \max_{0 \leq t \leq T} S(t) \right]^+, \quad \left( V_c(T) = \left[ \min_{0 \leq t \leq T} S(t) - K \right]^+ \right).$$



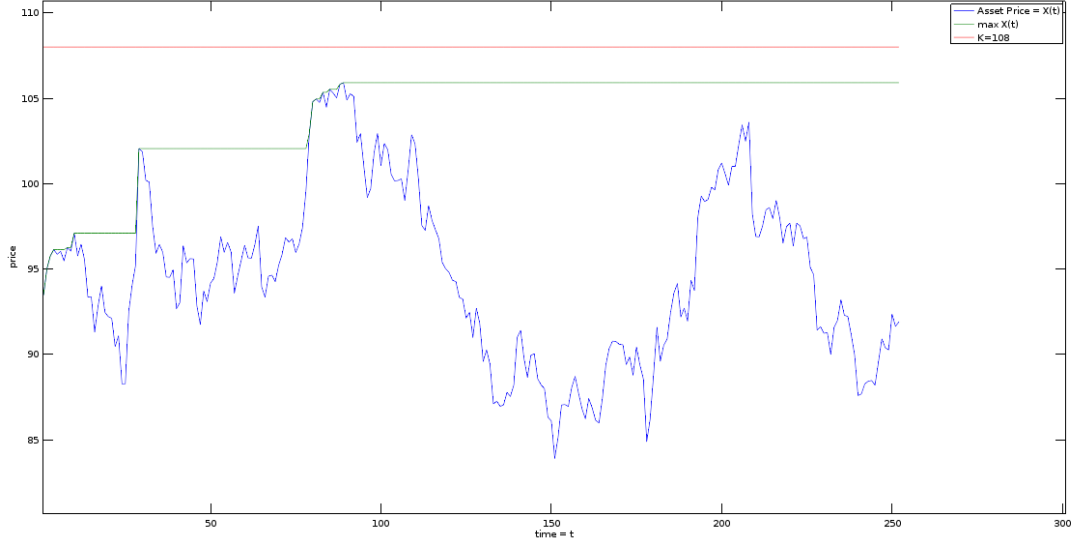


Figure 1.2: Reverse Strike Lookback Put Option.

With this type of option the investor is betting that the stock will perform bearishly and will remain below the strike price in the case of a put; whereas, in the case of a call, it will perform bullishly and will always be higher than the strike price. This is represented in Figures 1.2 and 1.3.

### 1.3 The Floating Strike Lookback

With this lookback option the maximum (minimum) attained over the lookback period is used as its strike price and  $S(T)$  is maintained in the payoff's computation. Therefore, its payoff is given by ([2], section 8.4, page 198):

$$V_p(T) = \left[ \max_{0 \leq t \leq T} S(t) - S(T) \right]^+, \quad \left( V_c(T) = \left[ S(T) - \min_{0 \leq t \leq T} S(t) \right]^+ \right).$$

This option is also referred to as “the no regret option” since the payoff is always greater than or equal to zero as can be seen in Figure 1.4.

For the purposes of this thesis we will focus on the last of these, i.e., “the floating strike lookback option.” Within the Black Scholes (BS) framework the pricing and hedging of this option (as well as those of the rest of the lookback options covered above) are well known and explicit formulas are already available in the literature, see, ‘Stochastic calculus for finance II: Continuous-time models’ [11, chapter 7] and ‘An introduction to exotic option pricing’ [2, chapter 8]; the derivations of these formulas are reviewed in the next chapter. From time to time we will use the term “lookback option” loosely to refer to the “floating-strike lookback option” unless otherwise specified.

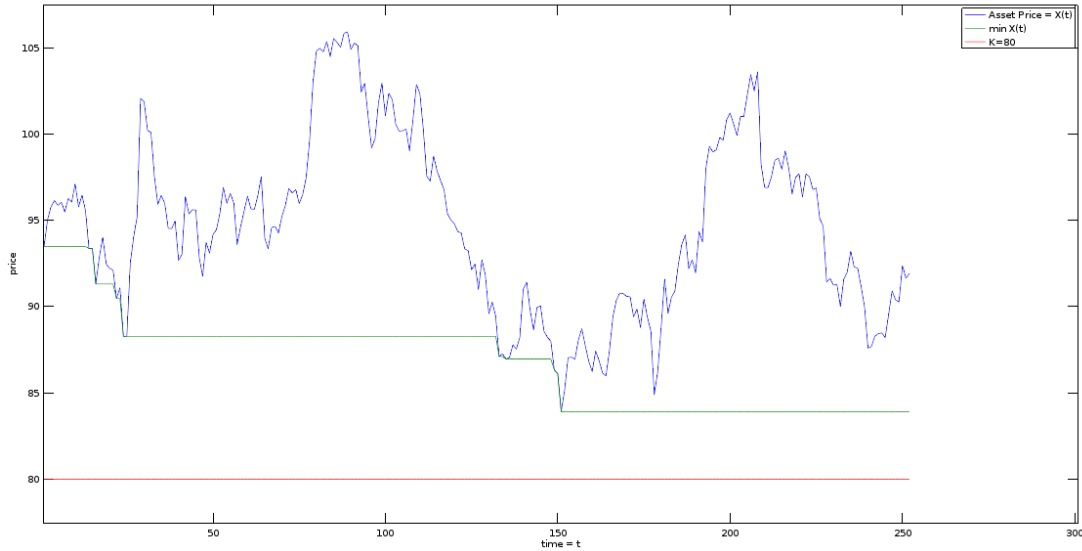


Figure 1.3: Reverse Strike Lookback Call Option.

The goal of this thesis is to see how well these pricing and hedging formulas perform for floating strike lookback options written on assets which form the BIST 30 index<sup>1</sup>. The details of this application are explained in chapter 3. Let us very briefly comment on the main aspects of our approach. Since the publicly available data is daily, we perform our fits and run the hedging algorithm daily, i.e., a hedging portfolio is updated at the end of each trading day. This discretization and model error imply that the hedge is not perfect and a hedging error is accumulated throughout the running of the hedging algorithm (precise definition is given in chapter 3). The average hedging error (normalized by the initial price of the lookback option) is the main performance measure by which we will assess the effectiveness of the BS formulas in pricing / hedging the lookback options. To fit the model to data, there are two parameters to be estimated: the constant volatility and the constant interest rate. The details are explained in chapter 3. We will see that as the lookback period increased the frequency of under-hedging [over-hedging] decreased [increased]. We also look at the effect of updating the volatility estimation using the latest price changes; our main observation here is that such updates have little influence on the performance of the algorithms.

In conclusion, we will see that, for the exception of a few stock used as underlying assets, the BS lookback pricing and delta hedging model performs relatively poorly in pricing and hedging lookback puts in BIST30, with an average error between -2%, and 33% for the options period tested and their standard deviation ranging between 16% and 30% (see Table 3.1.) We will also see that the BS model fails to capture shocks to underlying prices; we observe this in the sudden performance change of the hedging algorithm as it goes over the Turkish November 2015 elections and other political disturbance (see Figure 3.3 and Figure 3.15). In the future we would like to

<sup>1</sup> See <http://www.borsaistanbul.com/endeksler/bist-pay-endeksleri> for more on BIST30.

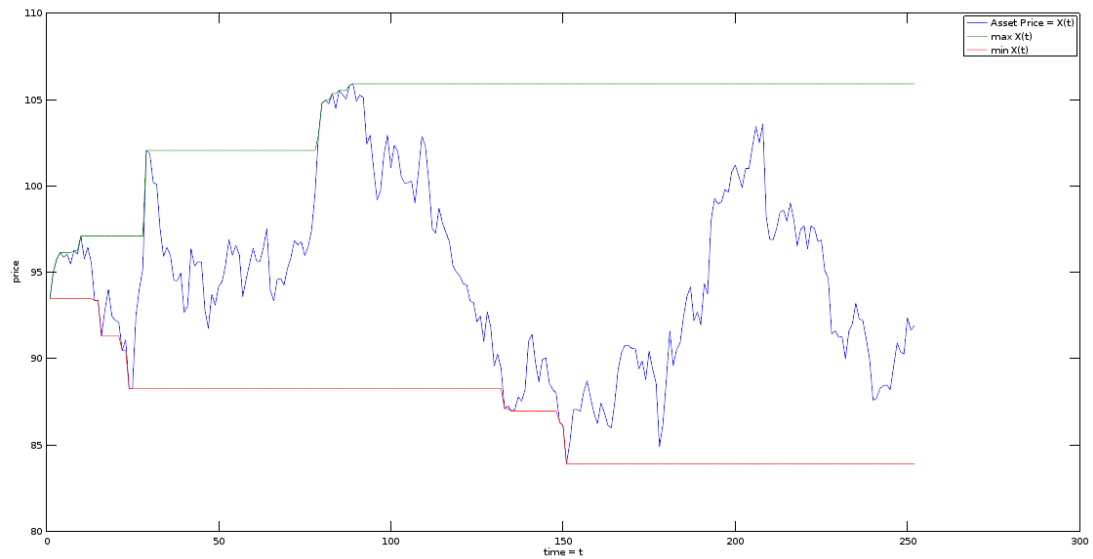


Figure 1.4: Stock Movement over a one year (250 days) Lookback Period for Floating Strike.

consider the use of the gamma hedge in the model instead of just the delta hedge as well looking at alternative pricing model such as pricing using stochastic volatility as is given by Jianwei Zhu in “Modular Pricing of Options: An Application of Fourier Analysis” [14, section 3.3].



## CHAPTER 2

### Pricing The LookBack Option

In this chapter we review the Black-Scholes pricing formulas for lookback options. Many authors, including Frans De Weert in “Exotic options trading,” [4, section 10.2], Peter Zhang in “A guide to second generation options,” [12, section 12.3], Marek Musiela, and Marek Rutkowski in Martingale methods in financial modelling ([9], section 6.7), and ([7], section 25.10) has given expositions on the lookback Option. Some brief, others in depth. Two well known expositions of this topic are also given in “Stochastic calculus for finance II: Continuous-time models” [11]<sup>1</sup> and in “An introduction to exotic option pricing” [2]<sup>2</sup>. This chapter reviews these two expositions and the methods of derivation they employ; we will refer to the first as “Shreve’s method” and to the second as “Buchen’s method.”

#### 2.1 Shreve’s Method

Since the payoff for this option is based on the maximum of an underlying asset over the lookback period or the remaining time thereof,  $[t, T]$ , and the value of the asset at time of maturity  $T$ , as per ([11] chapter 7), is computed as follows:

Given

$$S(t) = S(0)e^{\sigma\hat{W}(t)}, \quad (2.1)$$

$$\hat{W}(t) = \alpha t + \tilde{W}(t) \quad \text{and} \quad \alpha = \frac{1}{\sigma} \left( r - \frac{1}{2}\sigma^2 \right);$$

where  $S(t)$  is the price of the asset at some time  $t \in [0, T]$ ,  $\tilde{W}(t)$  is a Brownian motion,  $r$  is the risk free rate and  $\sigma$  is volatility of the underlying asset.

Defining

$$\hat{M}(t) = \max_{0 \leq u \leq t} \hat{W}(u), \quad (0 \leq t \leq T)$$

---

<sup>1</sup> The method of decomposition is used in its derivation. see Decomposition methods for differential equations: theory and applications [6] for a review of the method of decomposition

<sup>2</sup> The method of Images is used in its derivation. see(Introduction to electromagnetic and microwave engineering[8], section 6.7) for a review of the method of images

and

$$Y(t) = \max_{0 \leq u \leq t} S(u) = S(0)e^{\sigma \hat{M}(t)}; \quad (2.2)$$

the payoff is given as

$$V(T) = Y(T) - S(T), \quad (2.3)$$

while the risk neutral price is given as

$$V(t) = \tilde{E}[e^{-r\tau}(Y(T) - S(T)) | F(t)]. \quad (2.4)$$

Here  $\tau = T - t$ . Since (2.3) has the Markov property, see A.1 in the appendix, there exists a  $V(t, x, y)$  function such that

$$V(t) = V(t, S(t), Y(t)).$$

Now, for  $0 \leq t \leq T$  and  $\tau = T - t$ , using (2.2) we have

$$\begin{aligned} Y(T) &= S(0)e^{\sigma \hat{M}(t)} e^{\sigma(\hat{M}(T) - \hat{M}(t))} \\ &= Y(t)e^{\sigma(\hat{M}(T) - \hat{M}(t))}. \end{aligned}$$

If

$$\max_{t \leq u \leq T} \hat{W}(u) > \hat{M}(t),$$

then the max lies in the interval  $[t, T]$ . In that case

$$\hat{M}(T) - \hat{M}(t) = \max_{t \leq u \leq T} \hat{W}(u) - \hat{M}(t),$$

else

$$\hat{M}(T) - \hat{M}(t) = 0.$$

Therefore

$$\begin{aligned} \hat{M}(T) - \hat{M}(t) &= [\max_{t \leq u \leq T} \hat{W}(u) - \hat{M}(t)]^+ \\ &= [\max_{t \leq u \leq T} (\hat{W}(u) - \hat{W}(t)) - (\hat{M}(t) - \hat{W}(t))]^+. \end{aligned}$$

From (2.1) and (2.2) and multiplying by a factor of  $\sigma$  we have

$$\begin{aligned} \sigma(\hat{M}(T) - \hat{M}(t)) &= [\max_{t \leq u \leq T} \sigma(\hat{W}(u) - \hat{W}(t)) - \sigma(\hat{M}(t) - \hat{W}(t))]^+ \\ &= \left[ \max_{t \leq u \leq T} \sigma(\hat{W}(u) - \hat{W}(t)) - \log \left( \frac{Y(t)}{S(t)} \right) \right]^+. \end{aligned}$$

Now, with a substitution into equation 2.4 we derive

$$V(t) = e^{-rt} \tilde{E} \left( Y(t) \exp \left\{ \left[ \max_{t \leq u \leq T} \sigma(\hat{W}(u) - \hat{W}(t)) - \log \left( \frac{Y(t)}{S(t)} \right) \right]^+ \right\} \middle| F(t) \right) - e^{rt} \tilde{E}[e^{-rT} S(T) | F(t)].$$

Since,  $S(T)$  is a martingale under  $\tilde{P}$ ,  $Y(t)$  and  $S(t)$  are  $F(t)$  measurable and the function  $\max_{t \leq u \leq T} (\hat{W}(u) - \hat{W}(t))$  is independent of  $F(t)$  hence we obtain

$$V(t) = e^{-rt} Y(t) g(S(T), Y(T)) - S(t).$$

This can be expressed as,

$$V(t, x, y) = e^{-rt} y g(x, y) - x \quad (2.5)$$

where  $x = S(t)$ ,  $y = Y(t)$  and

$$\begin{aligned} g(x, y) &= \tilde{E} \left( \exp \left\{ \left[ \max_{t \leq u \leq T} \sigma(\hat{W}(u) - \hat{W}(t)) - \log \left( \frac{y}{x} \right) \right]^+ \right\} \right) \\ &= \tilde{E} \left( \exp \left\{ \left[ \sigma \hat{M}(\tau) - \log \left( \frac{y}{x} \right) \right]^+ \right\} \right) \\ &= \tilde{P} \left\{ \hat{M}(\tau) \leq \frac{1}{\sigma} \log \left( \frac{y}{x} \right) \right\} + \frac{x}{y} \tilde{E} \left[ e^{\sigma \hat{M}(\tau) \mathbf{I}_{\{\hat{M}(\tau) \geq \frac{1}{\sigma} \log(\frac{y}{x})\}}} \right]. \end{aligned} \quad (2.6)$$

Using the formula

$$\tilde{P}\{\hat{M}(T) \leq m\} = N \left( \frac{m - \alpha T}{\sqrt{T}} \right) - e^{2\alpha m} N \left( \frac{-m - \alpha T}{\sqrt{T}} \right); m \geq 0,$$

and substituting “ $m$ ” with “ $\frac{1}{\sigma} \log \left( \frac{y}{x} \right)$ ”, we derive

$$\begin{aligned} \tilde{P} \left\{ \hat{M}(T) \leq \frac{1}{\sigma} \log \left( \frac{y}{x} \right) \right\} &= N \left( -\delta_- \left( \tau, \frac{x}{y} \right) \right) \\ &\quad - \left( \frac{y}{x} \right)^{\frac{2r}{\sigma^2} - 1} N \left( -\delta_- \left( \tau, \frac{y}{x} \right) \right). \end{aligned} \quad (2.7)$$

Utilizing the density function

$$\tilde{f}_{\hat{M}(T)}(m) = \frac{2}{\sqrt{2\pi T}} e^{-\frac{1}{2T}(m - \alpha T)^2} - 2\alpha e^{2\alpha m} N \left( \frac{-m - \alpha T}{\sqrt{T}} \right); m > 0,$$

we have

$$\begin{aligned}
& \frac{x}{y} \tilde{E} \left[ e^{\sigma \tilde{M}(\tau)} \mathbf{I}_{\{\tilde{M}(\tau) \geq \frac{1}{\sigma} \log(\frac{y}{x})\}} \right] \\
&= \frac{x}{y} \int_{\frac{1}{\sigma} \log \frac{y}{x}}^{\infty} \frac{2}{\sqrt{2\pi\tau}} e^{\alpha m - \frac{1}{2\tau}(m-2\tau)^2} dm \\
&\quad - \frac{x}{y} \int_{\frac{1}{\sigma} \log \frac{y}{x}}^{\infty} 2\alpha e^{(\sigma+2\alpha)m} N\left(\frac{-m-\alpha\tau}{\sqrt{\tau}}\right) dm \\
&= 2e^{r\tau} N\left(\delta_+\left(\tau, \frac{x}{y}\right)\right) \left(\frac{x}{y}\right) \\
&\quad - \left(1 - \frac{\sigma^2}{2r}\right) e^{r\tau} N\left(\delta_+\left(\tau, \frac{x}{y}\right)\right) \left(\frac{x}{y}\right) \\
&\quad + \left(1 - \frac{\sigma^2}{2r}\right) e^{r\tau} N\left(-\delta_-\left(\tau, \frac{x}{y}\right)\right) \left(\frac{y}{x}\right)^{\frac{2r}{\sigma^2}-1}. \tag{2.8}
\end{aligned}$$

Then by substituting equations 2.8 and 2.7 into equation 2.6, and then making a final substitution into equation 2.5, we derive

$$\begin{aligned}
V(t, x, y) &= \left(1 + \frac{\sigma^2}{2r}\right) x N\left(\delta_+\left(\tau, \frac{x}{y}\right)\right) + e^{-r\tau} y N\left(-\delta_-\left(\tau, \frac{x}{y}\right)\right) \\
&\quad - \frac{\sigma^2}{2r} e^{-r\tau} \frac{x}{y} x^{\frac{2r}{\sigma^2}} N\left(-\delta_-\left(\tau, \frac{y}{x}\right)\right) - x,
\end{aligned}$$

where

$$\delta_{\pm}(\tau, k) = \frac{1}{\sigma\sqrt{\tau}} \left[ \log(k) + \left(r \pm \frac{1}{2}\sigma^2\right) \tau \right].$$

Since the function  $V(t, x, y)$  has a linear scaling property<sup>3</sup>, we can write the function as

$$V\left(t, \frac{x}{y}, 1\right) = U\left(t, \frac{x}{y}\right).$$

Thereby, deriving

$$\begin{aligned}
U\left(t, \frac{x}{y}\right) &= \left(1 + \frac{\sigma^2}{2r}\right) \frac{x}{y} N\left(\delta_+\left(\tau, \frac{x}{y}\right)\right) + e^{-r\tau} N\left(-\delta_-\left(\tau, \frac{x}{y}\right)\right) \\
&\quad - \frac{\sigma^2}{2r} e^{-r\tau} \frac{x}{y} x^{1-\frac{2r}{\sigma^2}} N\left(-\delta_-\left(\tau, \frac{y}{x}\right)\right) - \frac{x}{y}.
\end{aligned}$$

Making a final substitution of  $z = \frac{x}{y}$  we get

$$\begin{aligned}
U(t, z) &= \left(1 + \frac{\sigma^2}{2r}\right) z N\left(\delta_+\left(\tau, z\right)\right) + e^{-r\tau} N\left(-\delta_-\left(\tau, z\right)\right) \\
&\quad - \frac{\sigma^2}{2r} e^{-r\tau} z^{1-\frac{2r}{\sigma^2}} N\left(-\delta_-\left(\tau, z^{-1}\right)\right) - z.
\end{aligned}$$

---

<sup>3</sup> i.e., given  $\lambda \in \mathbb{R}$ ,  $\lambda V(t, x, y) = V(t, \lambda x, \lambda y)$



The graph of this payoff function is depicted below.

From the graph in Figure 2.1 we see that when  $Z$  is close to one, as time to maturity,  $\tau = T - t$ , decreases so does the price of the option. On the other hand, when  $z$  is close to zero as the price increases  $\tau$  decreases. This is in line with the behavior of a classical “at the money” and “deep in the money” put options respectively.

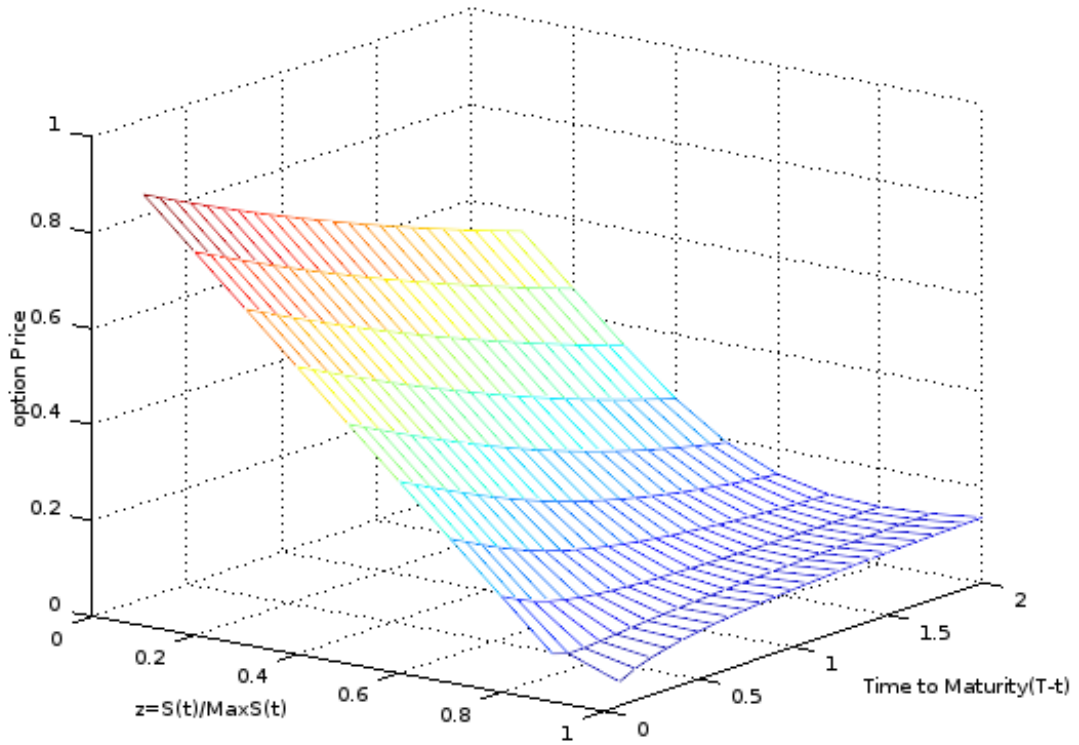


Figure 2.1: The floating lookback put option with constant volatility,  $U(t, z)$ .

### 2.1.1 Hedging

From the foregone equations we have

$$V(t, x, y) = yU(t, z).$$

By finding  $V_x(t, x, y)$  we will determine the delta of the option. Hence, our delta hedge is given as

$$\begin{aligned} V_x(t, x, y) &= yU_z(t, z) \frac{\partial}{\partial x} \left( \frac{x}{y} \right) = U_z(t, z) \\ &= \left( 1 + \frac{\sigma^2}{2r} \right) N(\delta_+(\tau, z)) \\ &\quad + \left( 1 - \frac{\sigma^2}{2r} \right) e^{-r\tau} z^{-\frac{2r}{\sigma^2}} N(-\delta_-(\tau, z^{-1})) - 1. \end{aligned}$$

This formula is given as an exercise in “*Stochastic calculus for finance II: Continuous-time models*” [11](section 7.8). We provide a solution to this in Appendix B.

The graph of the delta function is given in Figure 2.2.

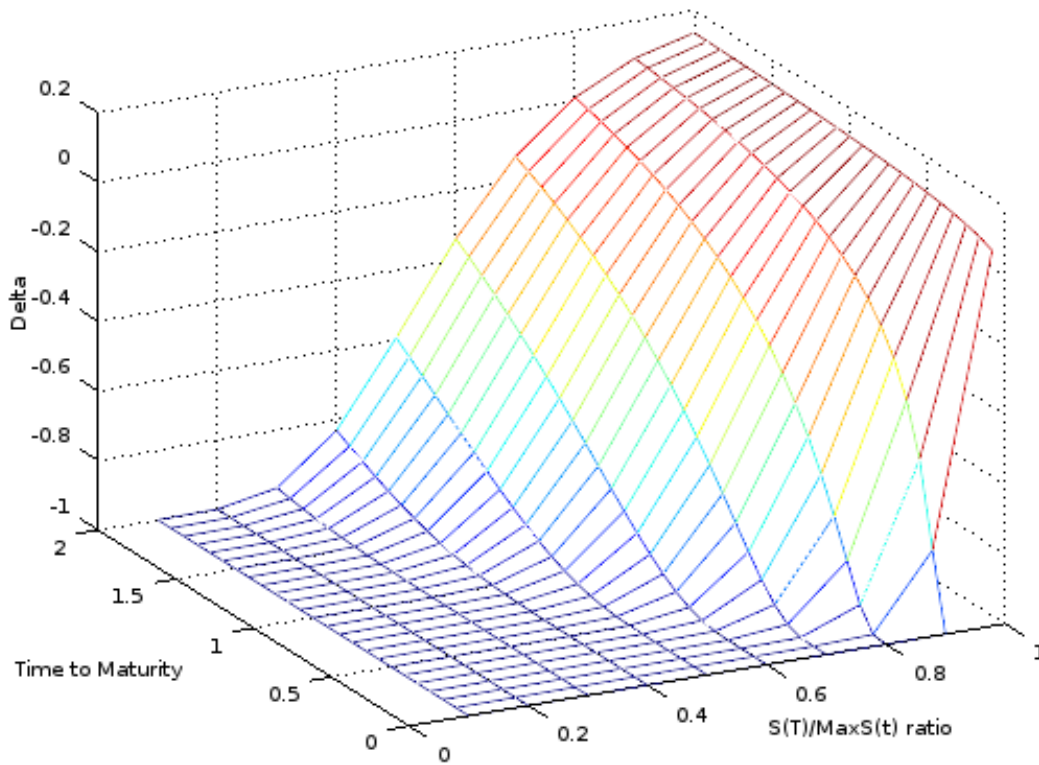


Figure 2.2: The Floating Lookback Put Option’s Delta Function,  $U_z(t, z)$ .

With the delta hedging of the lookback option it is notable that, unlike the standard option where  $\Delta \leq 0$ ,  $\Delta \in [-1, 1]$ , and in almost all cases when  $t = 0$ ,  $0 \leq \Delta$ .

The gamma hedge is derived from  $V_{xx}(t, z, y)$ , hence we have

$$\begin{aligned} V_{xx}(t, x, y) &= U_{zz}(t, z) \frac{\partial}{\partial x} \left( \frac{x}{y} \right) = \frac{1}{y} U_{zz}(t, z) \\ &= \frac{1}{y} \left( \left( 1 - \frac{2r}{\sigma^2} \right) e^{-r\tau} z^{-\frac{2r}{\sigma^2} - 1} N(-\delta_-(\tau, z^{-1})) \right. \\ &\quad \left. + \frac{2}{z\sigma\sqrt{\tau}} N'(\delta_+(\tau, z)) \right). \end{aligned}$$

This computation can also be found in Appendix B.

## 2.2 Buchen's Method

Using the the method of images to compute the BS lookback option price Buchen derives the price as an expression the classical European option price  $C_y(x, \tau)$  plus a premium  $L_p(x, y, \tau)$ . According to "An Introduction to Exotic Option Pricing" ([2] chapter 8), the price of this option is computed as follows.

To achieve the pricing formula, by letting the price of the underlying asset at time  $t$  be represented by  $S(t) = x$ , ( $0 \leq t \leq T$ ), where  $[0, T]$  is the lookback period, let  $F(x, y) = y = \max_{0 \leq t \leq T} S(t)$  denote a running maximum function.

**Theorem 2.1.** *For an option contract that pays the maximum the underlying asset attains over the  $0 \leq t \leq T$  the European option equivalent pay off is given as*

$$V(X_T, y, T) = F(x, y) \mathbb{I}_{(x < y)} + G(x, y) \mathbb{I}_{(x > y)},$$

where,

$$G(x, y) = F(x, x) + \int_y^x F_\xi^i(x, \xi) \partial \xi.$$

Note:

$$F^i(x) = \left( \frac{b}{x} \right)^\alpha F \left( \frac{b^2}{x} \right); \quad \alpha = \frac{2r}{\sigma^2} - 1$$

is the image of  $F(x)$  wrt  $x = b$  and the BS differential operator  $L$ ; and

$$F_\xi^i(x, \xi) = \mathcal{I}_\xi[F_\xi(x, \xi)] = \left( \frac{\xi}{x} \right)^\alpha F_\xi \left( \frac{\xi^2}{x}, \xi \right).$$

*Proof.* Since  $V(X_T, y, T)$  satisfies the Black-Scholes terminal boundary value problem, see Appendix A.2

we have:

$$\mathcal{L}V = 0, \quad V(x, y, T) = F(x, y), \quad \text{and} \quad V_y(x, y, t) = 0 \quad \text{at} \quad x = y,$$

in the domain  $[t, T]; x > y$ .

By making the substitution  $V_y(x, y, t) = U(x, y, t)$ , we have

$$\mathcal{L}U = 0, \quad U(x, y, T) = F_y(x, y), \quad \text{and} \quad U(x, y, t) = 0 \quad \text{at} \quad x = y,$$

on the above stated domain. Here  $y$  found in the equation itself. Therefore, we can use the payoff from the up and out barrier option, (see the appendix A.10), and we have

$$U(x, y, t) = F_y(x, y)\mathbb{I}_{(x < y)} - F_y^i(x, y)\mathbb{I}_{(x > y)}.$$

By integrating  $U(x, y, t)$  with respect to  $y$  we attain  $V(x, y, T)$ <sup>4</sup>.

As a result we have:

$$\begin{aligned} V(x, y, T) - V(x, 0, T) &= \int_0^y [F_\xi(x, \xi)\mathbb{I}_{(x < \xi)} - F_\xi^i(x, \xi)\mathbb{I}_{(x > \xi)}] \partial\xi \\ &= \int_0^{\max\{x, y\}} F_\xi(x, \xi) \partial\xi + \mathbb{I}_{(x > y)} \int_y^x F_\xi^i(x, \xi) \partial\xi \\ &= [F(x, \max\{x, y\}) - F(x, 0)] + \mathbb{I}_{(x > y)} \int_y^x F_\xi^i(x, \xi) \partial\xi \\ &= [F(x, y)\mathbb{I}_{(x < y)} + F(x, x)\mathbb{I}_{(x \leq y)} - F(x, 0)] \\ &\quad + \mathbb{I}_{(x > y)} \int_y^x F_\xi^i(x, \xi) \partial\xi \\ &= F(x, y)\mathbb{I}_{(x < y)} - F(x, 0) + G(x, y)\mathbb{I}_{(x > y)}. \end{aligned}$$

Since  $V(x, 0, T) = F(x, 0)$  we have  $V(x, y, T) = F(x, y)\mathbb{I}_{(x < y)} + G(x, y)\mathbb{I}_{(x > y)}$  as required.

□

Now, as per ([2] section 8.3, pages 197-198), by using  $M(x, y, t)$  to denote the present value of the generic maximum contract, where the payoff at time T is then given by  $F(x, y) = y$ , the maximum price the asset attained. Here  $F(x, y)$  is independent of  $x$ . Hence,

$$F(x, \xi) = \xi, \quad F_\xi(x, \xi) = 1 \quad \text{and} \quad F_\xi^i(x, \xi) = \left(\frac{\xi}{x}\right)^\alpha.$$

---

<sup>4</sup> this was given as an exercise to the reader.

$$\begin{aligned}
M(X_T, y, T) &= y\mathbb{I}_{(x < y)} + \left[ x + \int_y^x \left( \frac{\xi}{x} \right)^\alpha d\xi \right] \mathbb{I}_{(x > y)} \\
&= y\mathbb{I}_{(x < y)} + x\mathbb{I}_{(x > y)} + \beta \left[ x - y \left( \frac{y}{x} \right)^\alpha \right] \mathbb{I}_{(x > y)} \\
&= (1 + \beta)x\mathbb{I}_{(x > y)} + y[\mathbb{I}_{(x < y)} - \beta\mathbb{I}^i(x < y)],
\end{aligned}$$

where  $\beta = \frac{1}{\alpha+1} = \frac{\sigma^2}{2r}$  for non-dividend paying assets and  $\beta = \frac{\sigma^2}{2(r-q)}$  for dividend paying assets. Here  $q = \frac{DPS}{EPS}$  is the constant dividend yield (see [1] for a definition of “dividend.”).

Using assets and binary notation(see A.7), we have

$$M(x, y, t) = (1 + \beta)A_y^+(x, \tau) + y[B_y^-(x, \tau) - \beta B_y^{i-}(x, \tau)]. \quad (2.9)$$

The floating strike lookback put option’s strike price is set at  $k = y_T$ , the maximum asset price over  $[0, T]$ . Thus the holder gets to sell the asset at the highest price attained over the life of the option. The payoff at expiry is  $F(x, y) = (y - x)^+ = (y - x) = (k - x)$ , so the current value is

$$V_p(x, y, t) = M(x, y, t) - x.$$

Utilizing equation 2.9 we have

$$\begin{aligned}
V_p(x, y, t) &= (1 + \beta)A_y^+(x, \tau) + y[B_y^-(x, \tau) - \beta B_y^{i-}(x, \tau)] - x \\
&= [yB_y^-(x, \tau) - A_y^-(x, \tau)] + \beta[A_y^+(x, \tau) - yB_y^{i-}(x, \tau)] \\
&= [ye^{-rt}N(-d_\xi^1(x, t)) - xN(d_\xi(x, t))] \quad (*) \\
&\quad + \beta \left[ xN(d_\xi(x, t)) - y \left( \frac{\tau^2}{x} \right)^\alpha e^{-rt}N \left( -d_\xi^1 \left( \frac{t^2}{x}, t \right) \right) \right] \\
&= C_y(x, \tau) + L_p(x, y, \tau).
\end{aligned}$$

where (\*) follows from the binary notions given in Appendices A.8 and A.9.



## CHAPTER 3

### Scope, Methodology and Findings

#### 3.1 Scope and Methodology

The main contribution of this thesis is an application of the Black Scholes formulas for the pricing and hedging of the Floating Strike lookback option to asset prices observed in Borsa Istanbul and observe their performance. We measure performance in terms of the hedging error, which is defined below.

The underlying assets to which we apply the formulas are the stocks of the 30 companies listed on the BIST30 along with the BIST30 index. While collecting these data from Yahoo Finance it is noted that KARDEMIR KARABUK DEMIR CELIK SANAYI VE TICARET A.S. had three different stock listings, all are used in our study. Hence our application covers 33 underlying assets.

The risk free interest rates of 11.6% and 11.1% at Oct 1, 2015 and Jan 4, 2016 respectively is used as our  $r$  in the computation of prices and hedges. These are taken from Gosterge Faiz OraniGelişmeleri-Bloomberg HT <sup>1</sup>

For the volatility  $\sigma$ , we use the historical volatility as discussed by John C. Hull in “Options, futures, and other derivatives”([7]section 14.4 and 21.3) and by Don Chance and Roberts Brooks in the book “Introduction to derivatives and risk management” ([3] chapter 5.) For this, we use the closing prices from May 1, 2015 up to the starting date of the contract.

As a basis of control, a lookback period of 30-market-trading-days on the closing price, starting from Oct 1, 2015 is used. The rest of the results are centered around this for comparison.

A second 30-market-trading-days on the closing price, starting from Jan 4, 2016 is used to see how its hedging varied from the Oct 1, 2015 contract.

The 30-market-trading-days on the closing price, starting on Oct 1, 2015 is modified by changing the volatility at the end of each trading day. The day’s closing result is included in the computation of the volatility, which is then used in the pricing of the contract for the remaining days of the lookback period.

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<sup>1</sup> See <http://www.bloomberght.com/tahvil/gosterge-faiz>.

A 60 and a 90-market-trading-days contracts on the closing price, starting on Oct 1, 2015, is also used to assess how the hedging performed as the lookback period varied in length.

### 3.1.1 Hedging error computation

We used the standard delta hedge algorithm given by the BS framework ([7] section 18.4, pages 380-387)

The hedging error is computed by first creating a hedged portfolio with the proceeds from the option sale. This portfolio,  $\Pi$ , comprised of *Delta* Stocks and the excess or insufficient proceeds vested in the risk free bonds market. Hence our bond value is given as

$$V_b(t) = V_p(x, y, t) - \Delta(t, z)S(t)$$

Therefore, our hedged portfolio value is

$$\Pi(t) = V_b(t) + \Delta(t, z)S(t) = V_p(x, y, t)$$

where  $\Delta(t, z) = (V_p)_x(x, y, t)$ .

As indicated in the introduction discretize the hedging algorithm, taking one day as a step size, leads to hedging error defined as follows:

$$E(t) = V_b(t-1)e^r + \Delta(t-1)S(t) - \Pi(t) + E(t-1)e^r \text{ for } t \in (0, T], \text{ and } E(0) = 0.$$

This is the actual value of our previous day's portfolio at time t minus the value of our new portfolio plus the NPV of the previous day's hedging error.

We will express the error as a fraction of the initial value of the option, i.e.,  $\frac{E(t)}{V(x, y, 0)}100$ ,  $t \in [0, T]$ . This will be our main performance measure in evaluating the hedging performance. An ideal hedge will be one that is equal zero or is almost zero.

## 3.2 Findings

Table 3.1 gives a summary of the results from the tests performed.

From our control, October 1 30-market-trading-days lookback period the results' summary is given in Figures 3.1, 3.2 and Table 3.1. The following are observed:

The hedging on the various assets lookback options are scattered. Twenty (20) of the underlying assets' hedging errors fall within a plus or minus twenty percent ( $\pm 20\%$ ) range of the initial option price, whereas, twelve (12) fall within a plus or minus ten percent ( $\pm 10\%$ ), of which six (6) are within a plus or minus five percent ( $\pm 5\%$ ).



Table 3.1: Summary of Findings.

Detail	Oct 30-days	Jan 30-days	Oct 30-day with updated $\sigma$	Oct 60-days	Oct 90-days
Average	-2%	5%	-2%	33%	19%
Standard deviation	30%	16%	30%	19%	27%
Max over-hedging	70%	33%	72%	75%	61%
Max under-hedging	70%	51%	69%	5%	69%
Population size	33	33	33	33	33
No. under-hedged	19	11	18	2	6
No. over-hedged	13	22	13	31	26
Errors close to 0%	1	-	2	-	1
No of $E(T) \in \pm 5\%$	6	11	6	4	3
No of $E(T) \in \pm 10\%$	12	16	12	4	6
No of $E(T) \in \pm 20\%$	20	29	20	8	14

Furthermore, the average error is 2% under-hedged with a standard deviation of 30%. Moreover, nineteen (19) of the options are under-hedged using the delta hedging formula, while thirteen (13) are over-hedged and one perfectly hedged option. The results of these contracts are somewhat symmetric about -7%.

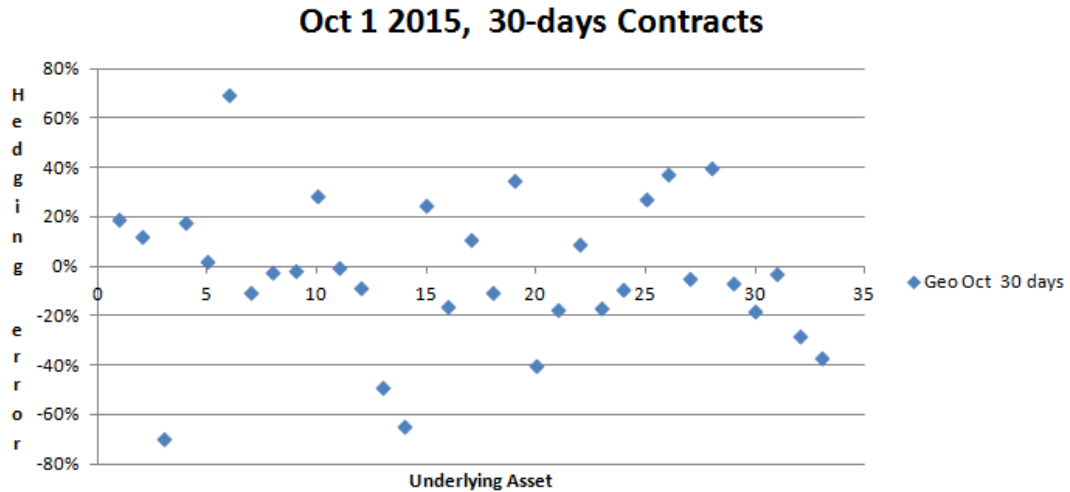


Figure 3.1: October 30 trading days lookback period delta hedging error.

Figure 3.3 gives the trajectory for four randomly selected runs of the hedging algorithm. This shows that immediately after the Nov 2015 elections, the hedging moved from a path of being over-hedged to becoming under-hedged.

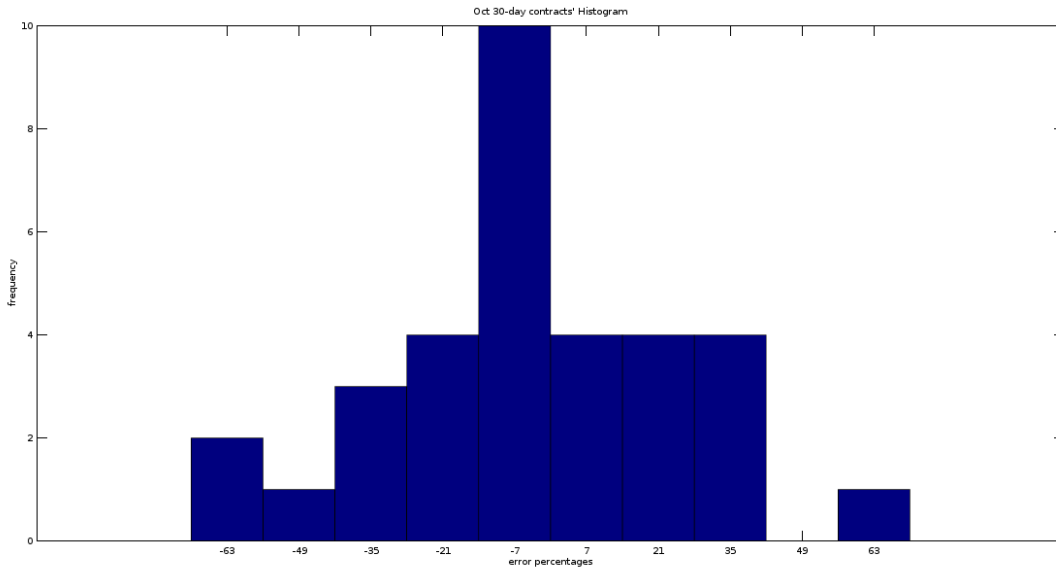
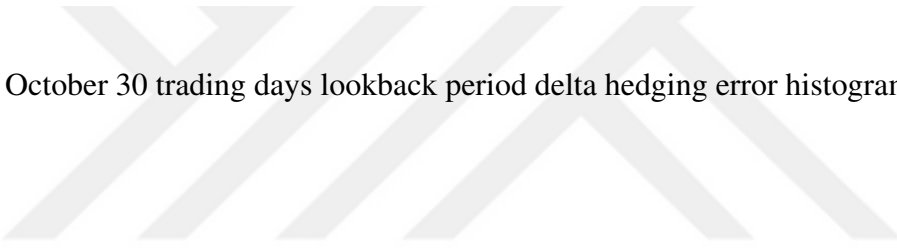


Figure 3.2: October 30 trading days lookback period delta hedging error histogram.



Four Companies' Hedging Error Progressions on the Oct 1st, 30 days Lookback

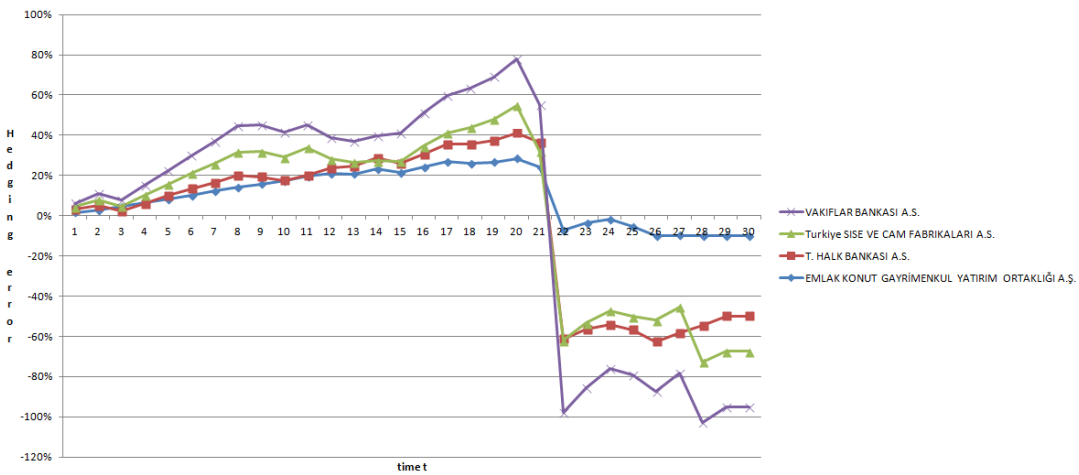


Figure 3.3: October 30 trading days lookback period delta hedging error trajectory for 4 underlying.

As is summarized in Figures 3.4, 3.5 and Table 3.1, the 30-trading-days lookback period contract, starting January 4, 2016, on the other hand, reflects a much more compact hedging performance. The average error was 5% over-hedged with a 16% deviation. Twenty-nine (29) of the underlying hedging is found within a plus or minus twenty percent ( $\pm 20\%$ ) and sixteen (16) within a plus or minus 10 percent ( $\pm 10\%$ ) of which eleven (11) fall between plus or minus five percent ( $\pm 5\%$ .)

While this contract type records no perfect hedge eleven (11) are under-hedged, whereas twenty-two (22) over-hedged. The errors for this contract type are left skewed, leaning towards being over-hedged.

The January 4, 2016 contracts record more over-hedging than the October 1, 2015 contracts.

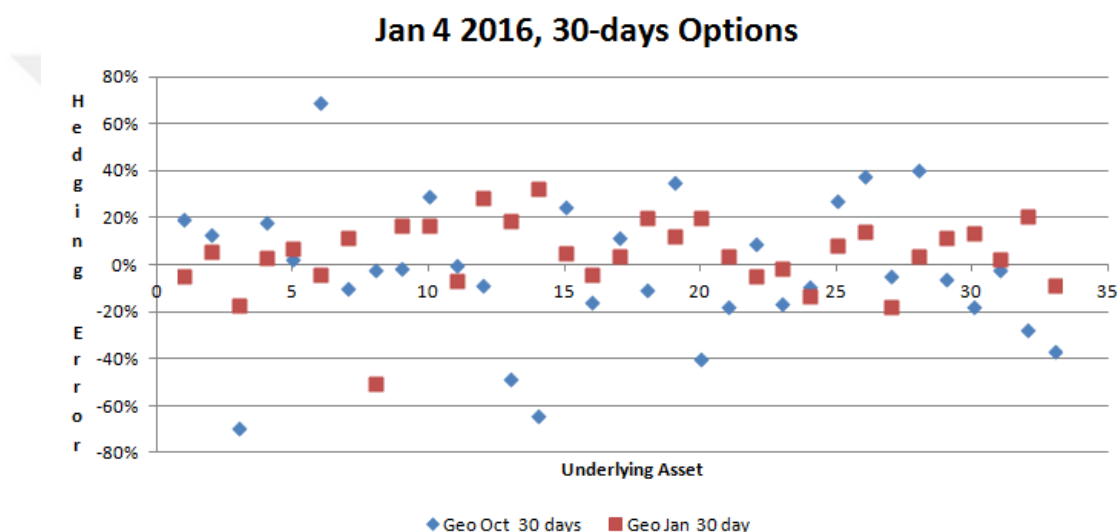


Figure 3.4: October and January 30 trading days lookback period delta hedging error.

Figure 3.6 gives the trajectory path of four randomly selected underlying. The price of the Arçelik stock significantly increases towards the end of the contract period. This results in its option moving from being over-hedged to being under-hedged.

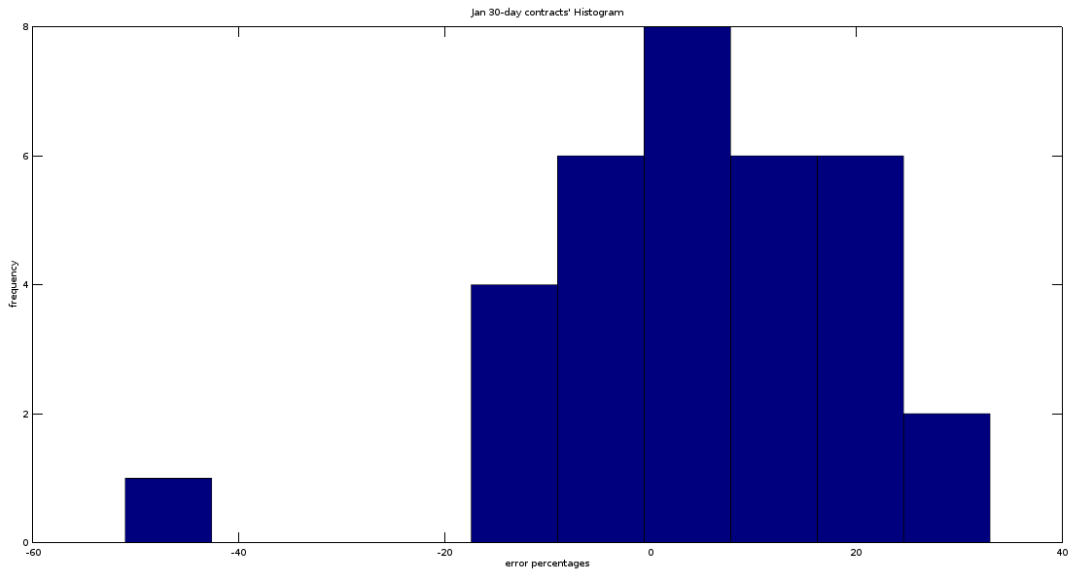


Figure 3.5: January 30 trading days lookback period delta hedging error histogram.

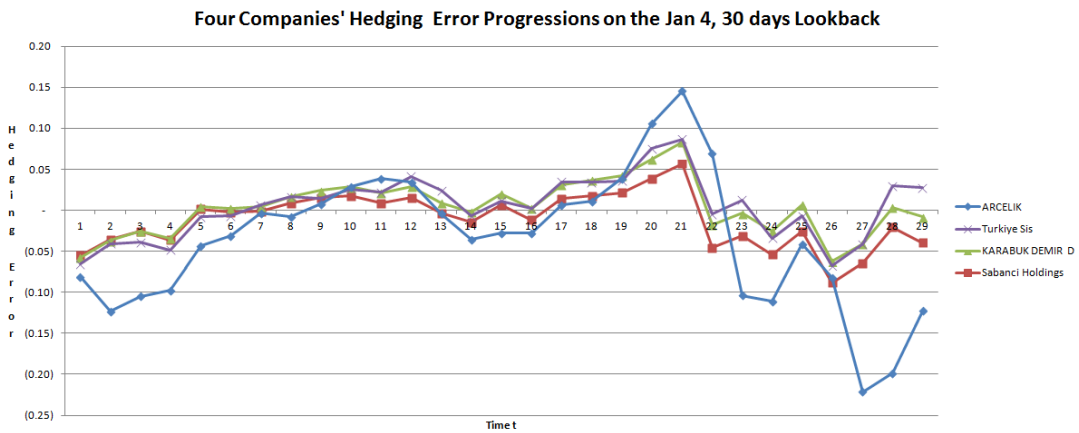


Figure 3.6: January 30 trading days lookback period delta hedging error trajectory for four companies.

The 30-market-trading-days on the closing price, starting on Oct 1, 2015 is modified by changing the volatility at the end of each trading day. The day's closing result is included in the computation of the volatility, which is then used in the pricing of the contract. The results from the updated contracts are summarized in Figures 3.7,3.8 and Table 3.1.

By updating the historical volatility daily to include  $(t - 1)$ 's data for our October 1, 2015 contracts, it is observed that eleven (11) contracts fall between a plus or minus ten percent( $\pm 10\%$ ), whereas there are twelve (12) for the non-recomputing volatility contract type.

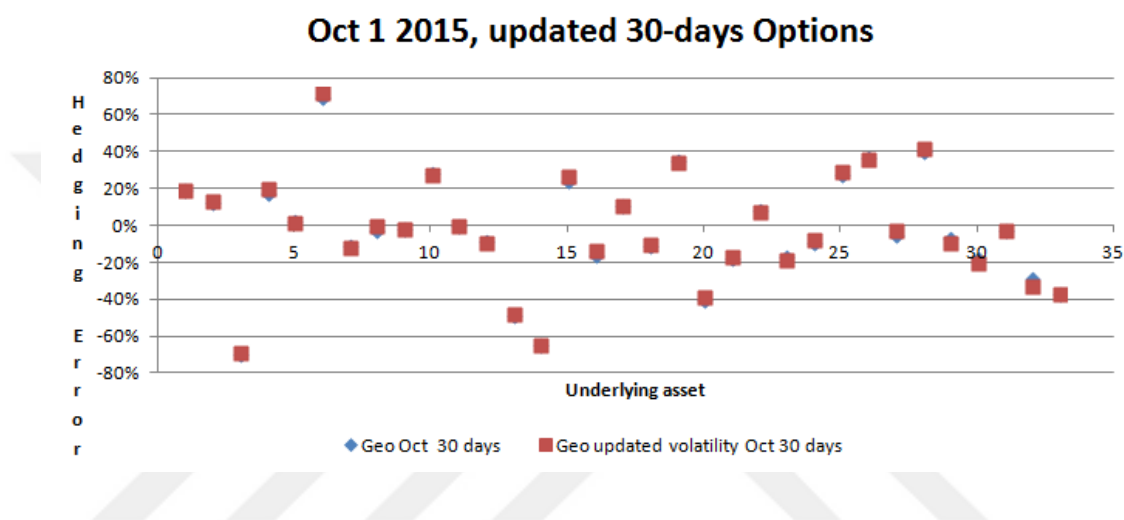


Figure 3.7: October 30 days lookback period with updated volatility hedging error.

The six (6) contracts that fall in the a plus or minus five percent( $\pm 5\%$ ) range for the non-updated contracts fall within the a plus or minus three percent( $\pm 3\%$ ) when the volatility is adjusted; of these, two are perfectly hedged.

Some contracts show slight improvement, while other show a worsened performance. The average is 2% under-hedged with a standard deviation of 30% which is identical with non-updated hedge.

By updating the volatility, the contracts' hedging performance show mixed results. However, in general, the performance is little to no better than when the volatility was not updated. Its skewness, as well, is not much varied from the non-updated volatility, which tends to be more under-hedged.

Moreover, Figure 3.9 displays the trajectory of four randomly selected assets' errors when the volatility is adjusted daily. The directions of these trajectories is not uniform where some are greater impacted by the Nov 2015 Turkish Elections than others.

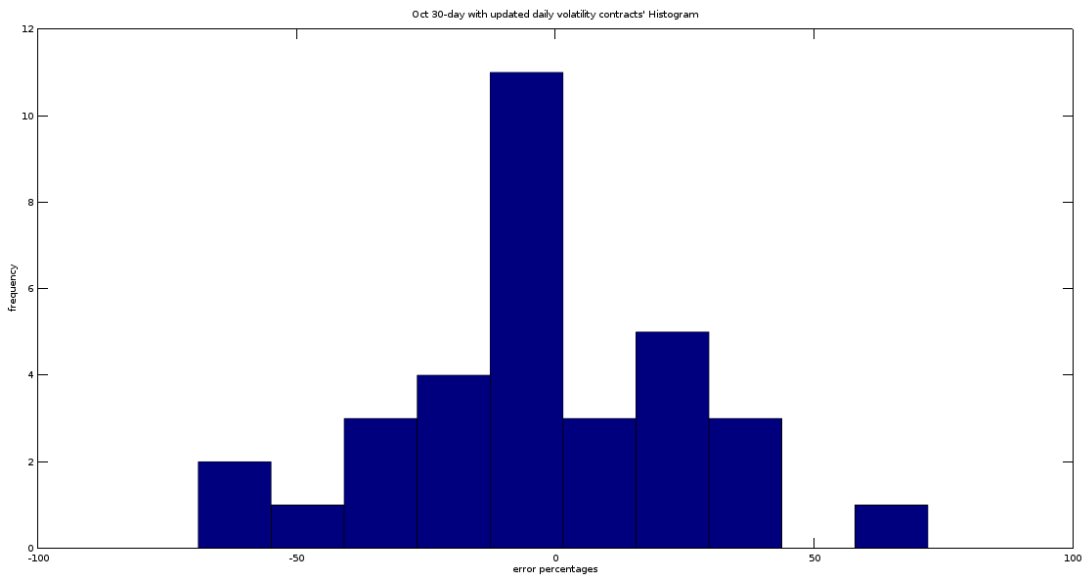


Figure 3.8: October 30 days lookback period with updated volatility hedging error histogram.

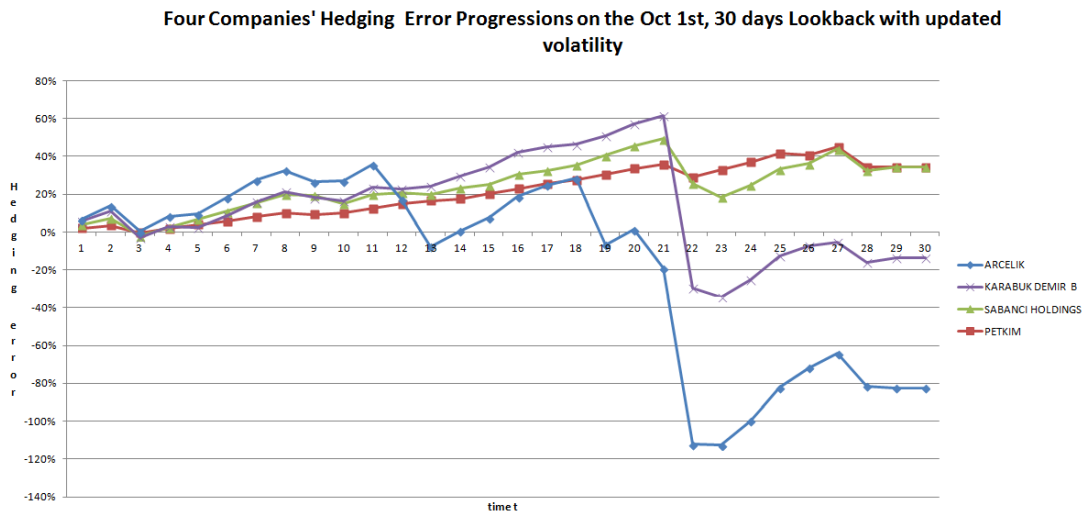


Figure 3.9: October 30 days lookback period with updated volatility hedging error underlying assets trajectory.

As per the summary provided in Figures 3.10, 3.11 and Table 3.1, the 60-days contract, on the other hand, had only two contracts under-hedged the remaining thirty-one(31) are over-hedged. There are four contracts with errors within a plus or minus five percent ( $\pm 5\%$ ) range, the remaining twenty-nine(29) rest in the interval of fifteen percent and seventy-five percent [15%, 75%].

Moreover, the average is a 33% over-hedging with a deviation of 0.19 and the errors are left skewed with over 50% of the population found between 20-43 percent.

Figure 3.12 gives the trajectory of four different underlying contracts. The trajectory shows a general upward trend with the errors of this contract type. The downward pull from the November 1st elections can be seen on step 21 to 22, however, since the contract period was longer, it soon continued on an its upward trajectory. The second dip in the trajectory between just around step 53 which corresponds to Dec 11 2015, another politically significant day [10].

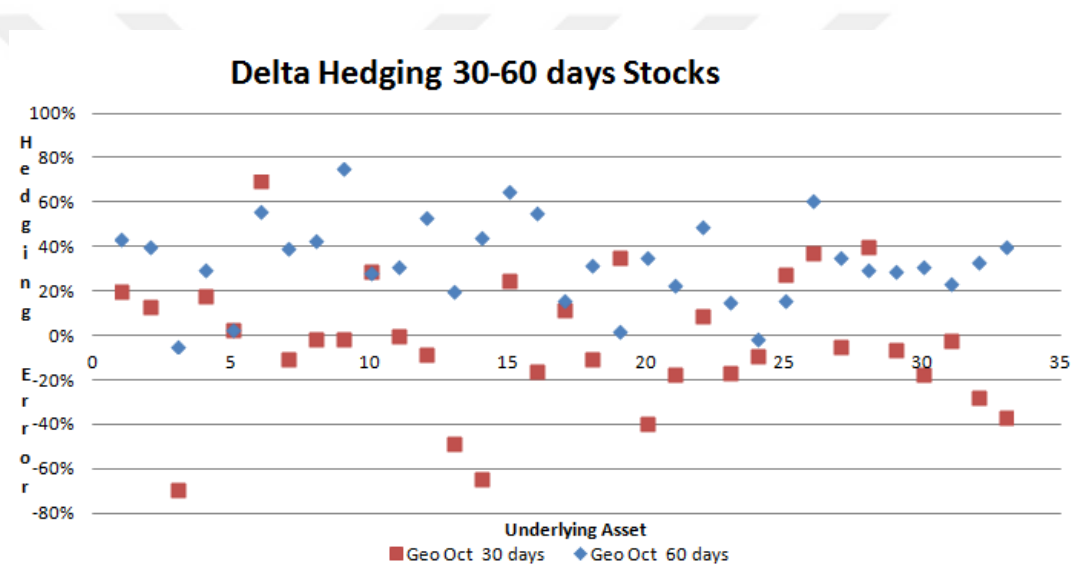


Figure 3.10: October 30 and 60 days lookback period hedging error.

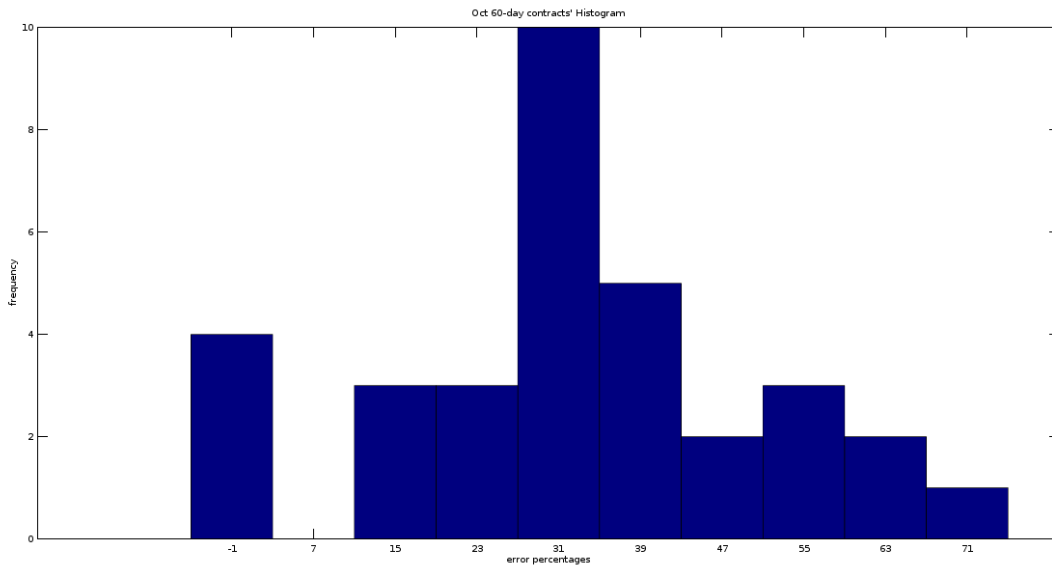


Figure 3.11: October 60 days lookback period hedging error Histogram.

Four Companies' Hedging Error Progressions on the Oct 1st, 60 days Lookback

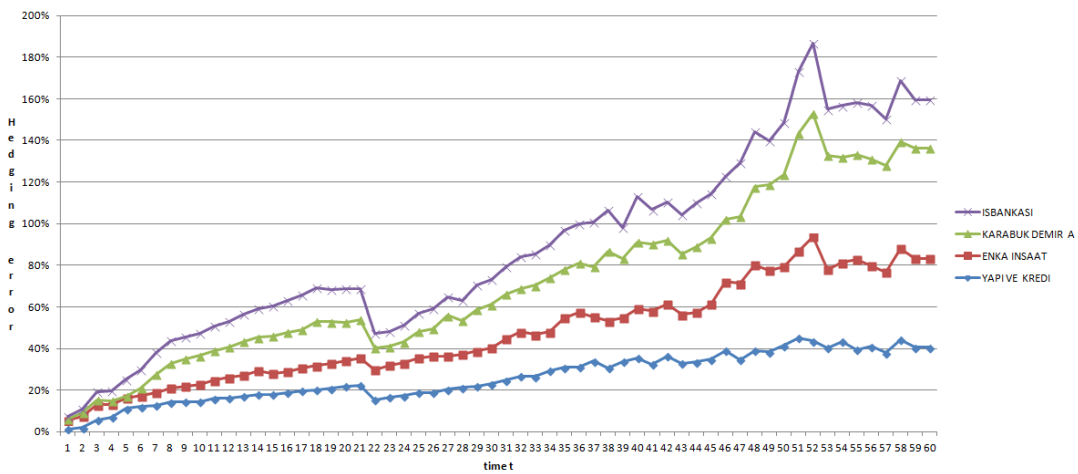


Figure 3.12: October 60 days lookback period hedging error trajectory for 4 underlying.



In Figures 3.13, 3.14 and Table 3.1 the results are summarized for the October 90 days lookback period. This, the other hand, had one perfect hedge, while twenty-six (26) are over-hedged, with only fourteen (14) hedging errors fall within the plus or minus twenty percent ( $\pm 20\%$ ) range and six (6) are between the plus or minus ten percent ( $\pm 10\%$ ) interval. The hedging error for this type of contracts is left skewed with an average error of 19% over-hedged and a standard deviation of 0.27.

From Figure 3.15 we see that until the middle of the lookback period there is a general upward trend. However, this trend goes in the opposite direction around step 53 which corresponds to Dec 11 2015,[10].

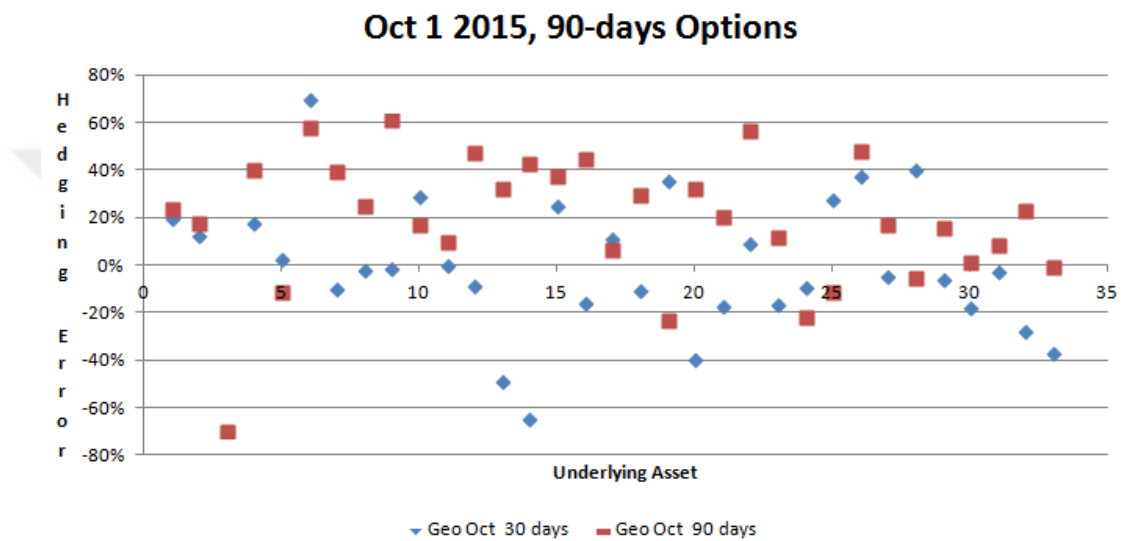


Figure 3.13: October 30 and 90 days lookback period hedging error.

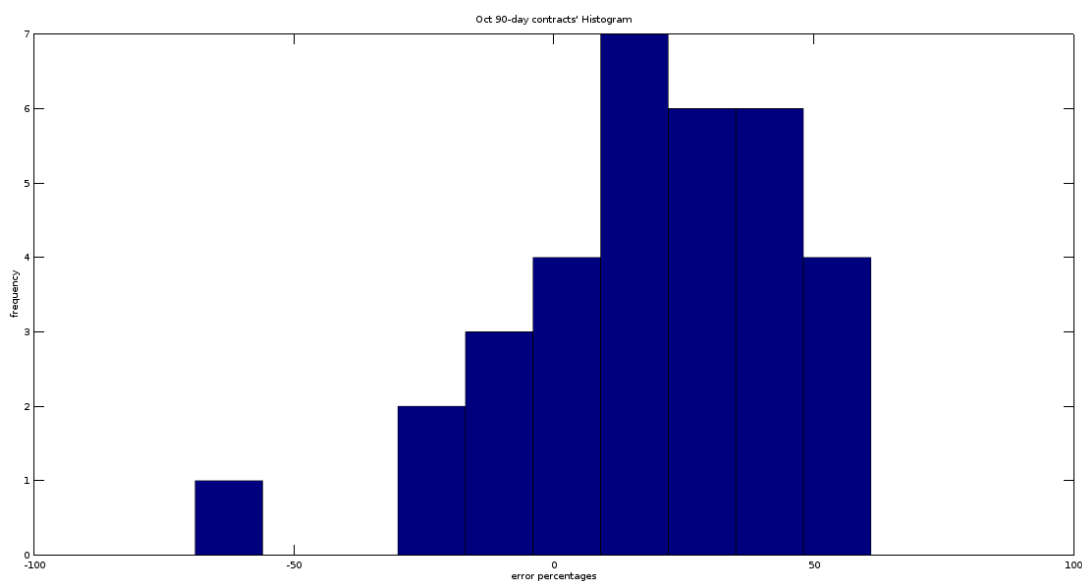


Figure 3.14: October 90 days lookback period hedging error histogram.

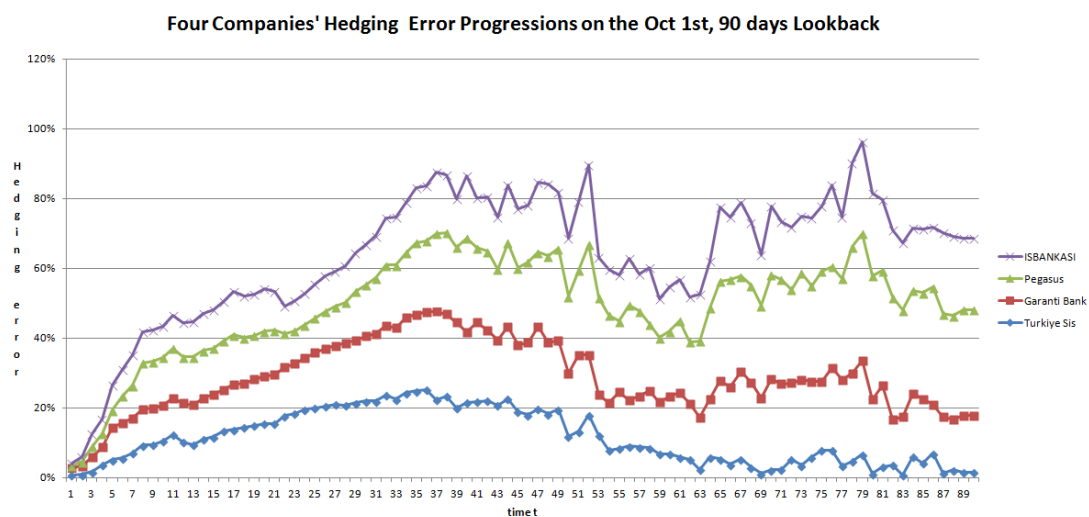


Figure 3.15: four underlying assets 90 days lookback period hedging error trajectory.

With the 60-days and 90-days contracts, the contracts are generally over-hedged. This may lead to unnecessary cash being tied up in the portfolio hedge, whereas it could have been used gainfully in other investments. The 90-days contracts' average was lower than the 60-days contracts' in part to the impact of the Dec 11 market shock.

Figure 3.16 gives a picture of how the contracts performs as the lookback period increases.

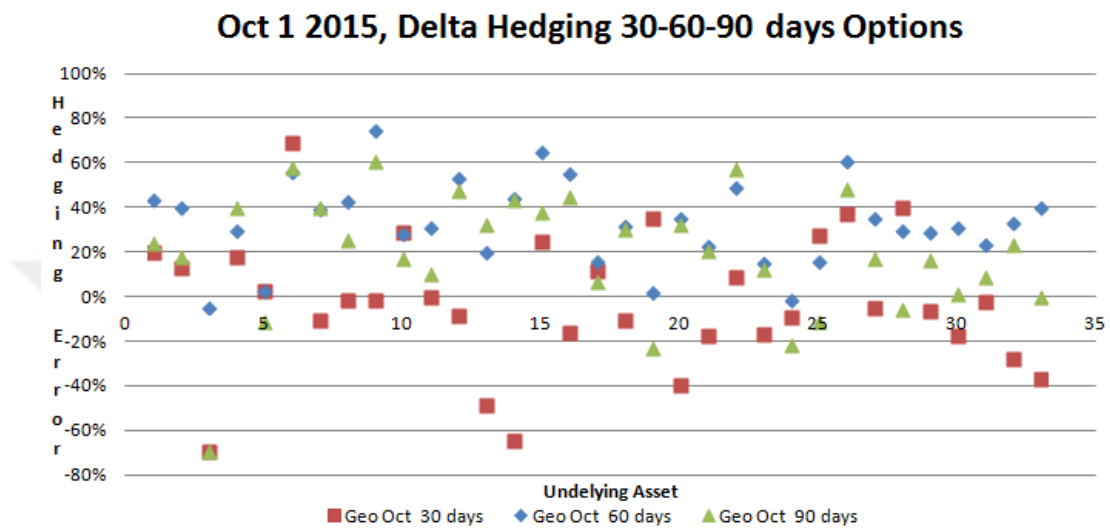


Figure 3.16: October 30, 60 and 90 days lookback period hedging error.

Figure 3.17 summarizes the performance of delta hedges on all contracts at different time period.

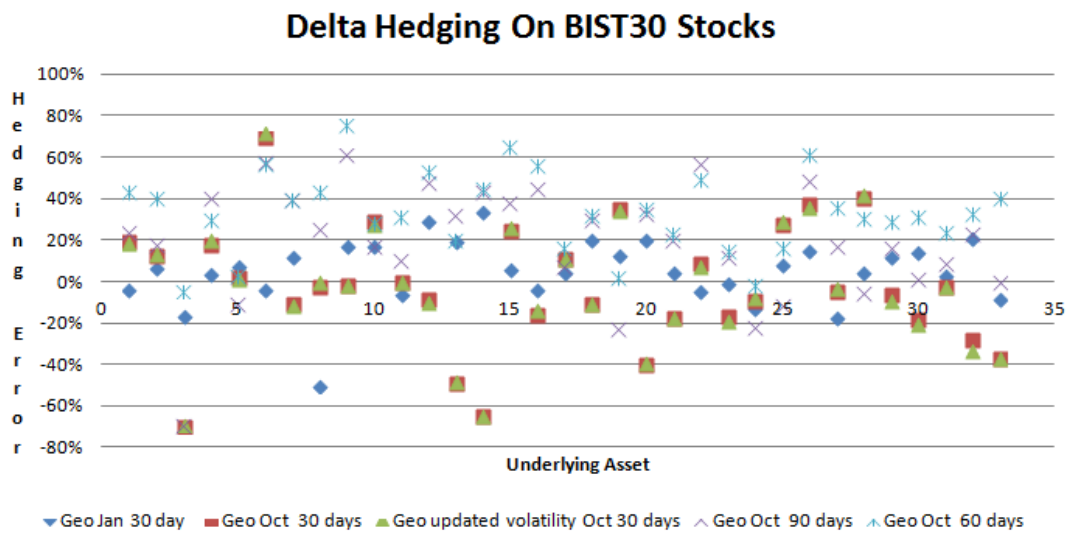


Figure 3.17: lookback period delta hedging error.

From the control of Oct 1, 2015, 30-trading-days hedging the top 10 companies results are summarized in Table 3.2 below:

Table 3.2: Summary of the BIST100 index and the top 10 assets with respect to the control results.

Company	Oct 30 days	Jan 30 days	Oct 30-day & a updated $\sigma$	Oct 60 days	Oct 90 days
TOFAS TURK OTOMOBIL FABRIKASI A.Ş.	-9%	-13%	-8%	-2%	-22%
KARDEMİR KARABUK DEMİR ÇELİK SANAYİ VE TİCARET A.Ş. "A"	-8%	29%	-10%	53%	48%
TÜRKİYE GARANTİ BANKASI A.Ş.	-6%	12%	-9%	29%	16%
TÜRK TELEKOMÜNİKASYON A.Ş.	-5%	-17%	-3%	35%	18%
ÜLKER BİSKÜVİ	-2%	2%	-2%	23%	9%
ENKA İNŞAAT VE SANAYİ A.Ş.	-2%	-51%	0%	43%	25%
EREĞLİ DEMİR ÇELİK FABRİKALARI A.Ş.	-1%	17%	-2%	75%	61%
HACI ÖMER SABANCI HOLDİNG A.Ş.	0%	-7%	0%	31%	10%
COCA COLA İÇECEK A.Ş.	3%	7%	1%	3%	-11%
TAV HAVALİMANLARI HOLDİNG A.Ş.	9%	-5%	7%	49%	57%
BIST100	20%	-4%	19%	43%	24%

From the results, delta hedging seem not to be ideal for hedging the lookback put option in the Turkish Market; more so for longer lookback periods. However, for individual stocks as underlying assets, there are some favorable results. Case in point, contracts written on Coca Cola İçecek A.Ş. (CCA) and Ülker Bisküvi (ÜB) stocks performed relatively well for the various contract period. ÜB performed within a plus or minus ten percent for all condition excepting for the 60-days contract where the error is twenty-three percent (23%). CCA on the other hand, performed within a one and seven percent range except in the case of the 90-days contract where it is under-hedged by 11%. Other stocks like Tav Havalimanları Holdings and Sabancı Holdings perform relatively well with the 30 contracts but have errors over 10% for the 60 and 90 days contract. On the other hand, others like Turkcell İletişim Hizmetleri A.Ş., Yapı Ve Kredi Bankası A.Ş., Türkiye Sise Ve Cam Fabrikaları A.Ş., and Otokar Otobüs Karoseri San. A.Ş have errors less than 10% for the 90-days contract but perform relatively poorly in the short term 30 days and even the 60-days contract. Likewise, Arçelik A.Ş., Tofaş Turk Otomobil Fabrikası A.Ş., and Petkim Petrokimya Holding A.Ş. are in the under 10% error range for the 60-day options but for the other options

their results were very poor relatively speaking.



## CHAPTER 4

### Conclusion and future works

From the results in chapter 3, standard delta hedging performs relatively poorly, with averages ranging between -2% and 33% and standard deviations between 16% and 30%, when hedging the lookback put option. In some cases  $E(T)$  was as high as 75% and as low as -70%. Furthermore, as is the case with the October-30-days lookback option and the Turkish November Elections, the hedging performs even worse when there are shocks in the market (see Figure 3.3 and Figure 3.9), since the model ignores such phenomena. In spite of this, a few contracts like Coca Cola İçecek A.Ş. and Ülker Bisküvi stocks performed relatively well for the various contract period. Contracts written on Tav Havalimanları Holding A.Ş. and Hacı Ömer Sabancı Holding stocks performed better with the 30-days lookback contracts than with the 60 and 90-days contracts. Whereas, contracts written on underlying like Tofaş Türk Otomobil Fabrikası, did better in the 60-days contract than with the 30 and 90 days contract.

#### 4.1 Future works

In future work, one may extend the tests done in this work (by studying different lookback option products, time periods, maturities, underlyings, markets, etc.) to improve our understanding of the practical performance of the BS pricing and hedging algorithm.

Additionally, we would like to explore the hedging performance of other hedging methods such as gamma hedging.

Another direction of research is to consider the use of implied volatility or more involved volatility methods as is discussed in “The volatility surface: a practitioner’s guide” [5], instead of historical volatility.

Finally, alternative pricing models may also be considered. This may include price with stochastic volatility methods as is discussed in “Modular pricing of options”[14] and in “Applications of Fourier transform to smile modeling” [13] and models that consider market shocks.





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## APPENDIX A

### Some Definitions and Theorem

#### Definitions

**Definition A.1** (From Stochastic calculus for finance II: Continuous-time models[11] page 74). Given a probability space  $(\Omega, F, \mathbb{P})$ ,  $0 \leq T$  a constant, and a filtration  $F(t)$ ,  $t \in [0, T]$ . Let  $X(t)$ ,  $t \in [0, T]$ . If for every non-negative, Borel-measurable function  $f$  and for  $\forall s \leq t$ ,  $s \in [0, T]$ , there is a Borel-measurable function  $g$  such that

$$\mathbb{E}[f(X(t))|F(s)] = g(X(s)),$$

then  $X$  is said to be a Markov process.

**Lemma**[From Stochastic calculus for finance II: Continuous-time models[11] page 73] Given a probability space  $(\Omega, F, \mathbb{P})$ , and  $G \subseteq F$  a  $\sigma$ -algebra. given  $G$ -measurable random variables  $X_1, \dots, X_K$  and let  $Y_1, \dots, Y_L$  be independent of  $G$ . Letting  $f(x_1, \dots, x_K, y_1, \dots, y_L)$  be a function with pseudo variables  $x_1, \dots, x_K$  and  $y_1, \dots, y_L$  and

$$g(x_1, \dots, x_k) = \mathbb{E}f(x_1, \dots, x_K, Y_1, \dots, Y_L),$$

then

$$\mathbb{E}[f(X_1, \dots, X_K, Y_1, \dots, Y_L)|G] = g(X_1, \dots, X_K).$$

From the previous lemma we have

$$\mathbb{E}[f(S(T), Y(T))|F(t)] = g(S(t), Y(t)). \quad (\text{A.1})$$

#### Definition A.2. Itô-Doebelin Formula for Multiple Processes

**Theorem A.1. Two-dimensional Itô-Doebelin formula**[from Stochastic calculus for finance II: Continuous-time models[11] page 141] Let  $f(t, x, y)$  be a function whose partial derivatives  $f_t, f_x, f_y, f_{xx}, f_{xy}, f_{yx}$  and  $f_{yy}$  are defined and are continuous. Let  $X(t)$  and  $Y(t)$  be Itô processes. The two-dimensional Itô-Doebelin formula in differen-

tial form is

$$\begin{aligned}
df(t, X(t), Y(t)) &= f_t(t, X(t), Y(t))dt + f_x(t, X(t), Y(t))dX(t) \\
&\quad + f_y(t, X(t), Y(t))dY(t) \\
&\quad + \frac{1}{2}f_{xx}(t, X(t), Y(t))dX(t)dX(t) \\
&\quad + f_{xy}(t, X(t), Y(t))dX(t)dY(t) \\
&\quad + \frac{1}{2}f_{yy}(t, X(t), Y(t))dY(t)dY(t). \tag{A.2}
\end{aligned}$$

**Theorem A.2** (From Stochastic calculus for finance II: Continuous-time models[11] page 313). *Let  $v(t, x, y)$  denote the price at time  $t$  of the floating strike lookback option under the assumption that  $S(t) = x$  and  $Y(t) = y$ . Then the pay off function  $V(t, x, y)$  satisfies the Black-Scholes-Merton partial differential equation*

$$V_t(t, x, y) + rxV_x(t, x, y) + \frac{1}{2}\sigma^2x^2V_{xx}(t, x, y) = rv(t, x, y). \tag{A.3}$$

in the region  $\{(t, x, y); 0 \leq t < T, 0 \leq x \leq y\}$  and satisfies the boundary conditions

$$V(t, 0, y) = e - r(T - t)y, 0 \leq t \leq T, y \geq 0, \tag{A.4}$$

$$Vy(t, y, y) = 0, 0 \leq t \leq T, y > 0, \tag{A.5}$$

$$V(T, x, y) = y - x, 0 \leq x \leq y. \tag{A.6}$$

*Proof.* Using equation A.2 and equating  $f(t, x, y)$  to  $e^{-rt}V(t, x, y)$ , we have

$$\begin{aligned}
df(t, X(t), Y(t)) &= e^{-rt}[-rv(t, X(t), Y(t))dt + v_t(t, X(t), Y(t))dt \\
&\quad + V_x(t, X(t), Y(t))dX(t) + V_y(t, X(t), Y(t))dY(t) \\
&\quad + \frac{1}{2}V_{xx}(t, X(t), Y(t))dX(t)dX(t) \\
&\quad + V_{xy}(t, X(t), Y(t))dX(t)dY(t) \\
&\quad + \frac{1}{2}V_{yy}(t, X(t), Y(t))dY(t)dY(t)].
\end{aligned}$$

However, the cross variation  $dY(t)dY(t) = 0$  and  $dX(t)dY(t)$  likewise. resulting in

$$\begin{aligned}
df(t, X(t), Y(t)) &= e^{-rt}[-rv(t, X(t), Y(t))dt + v_t(t, X(t), Y(t))dt \\
&\quad + V_x(t, X(t), Y(t))dX(t) + V_y(t, X(t), Y(t))dY(t) \\
&\quad + \frac{1}{2}V_{xx}(t, X(t), Y(t))dX(t)dX(t)] \\
&= e^{-rt}[-rV(t, X(t), Y(t)) + V_t(t, X(t), Y(t)) \\
&\quad + rX(t)V_x(t, X(t), Y(t)) \\
&\quad + \frac{1}{2}\sigma^2X^2(t)V_{xx}(t, X(t), Y(t))]dt \\
&\quad + e^{-rt}\sigma X(t)V_x(t, S(t), Y(t))d\tilde{W}(t) \\
&\quad + e^{-rt}V_y(t, X(t), Y(t))dY(t).
\end{aligned}$$

By definition, for  $f(t, x, y)$  to a martingale the  $dt$  part of the equation needs to be equal to zero, hence

$$\begin{aligned} & -rV(t, X(t), Y(t)) + V_t(t, X(t), Y(t)) \\ & + rX(t)V_x(t, X(t), Y(t)) + \frac{1}{2}\sigma^2 X^2(t)V_{xx}(t, X(t), Y(t)) = 0, \end{aligned}$$

giving us

$$\begin{aligned} rV(t, X(t), Y(t)) = & V_t(t, X(t), Y(t)) + rX(t)V_x(t, X(t), Y(t)) \\ & + \frac{1}{2}\sigma^2 X^2(t)V_{xx}(t, X(t), Y(t)). \end{aligned}$$

Alternatively

$$rV(t, x, y) = V_t(t, x, y) + rxV_x(t, x, y) + \frac{1}{2}\sigma^2 X^2(t)V_{xx}(t, x, y).$$

□

## Definitions

1.

**Definition A.3.** A “barrier option” is an exotic option whose pays off is dependent on whether the underlying asset’s price stays within a specified barrier condition or not.

2.

**Definition A.4.** An ‘up and out barrier option’ is an option that makes the option payoff only if the underlying assets stays below the specified barrier  $b \leq 0$ .

3.

**Definition A.5.** Assets and binary notation is given as:

$$\begin{aligned} A_\xi^+(x, \xi) &= x\mathbb{I}_{(x>\xi)} && \text{an up-type asset binary;} \\ A_\xi^-(x, \xi) &= x\mathbb{I}_{(x<\xi)} && \text{a down-type asset binary;} \\ B_\xi^+(x, \xi) &= \mathbb{I}_{(x>\xi)} && \text{an up-type bond binary;} \\ \text{and } B_\xi^-(x, \xi) &= \mathbb{I}_{(x<\xi)} && \text{a down-type bond binary;} \end{aligned} \quad (\text{A.7})$$

According to ‘An introduction to exotic option pricing’[[2]chapter 4 pages 83-85] these contracts payments are given as

$$A_\xi^\pm(x, t) = xN(\pm d_\xi(x, \tau)); \quad \text{and } B_\xi^\pm(x, t) = e^{-r\tau}N(\pm d_\xi^1(x, \tau)) \quad (\text{A.8})$$

$$d_\xi(x, \tau) = \frac{\log(\frac{x}{\xi}) + (r + \frac{\sigma^2}{\tau})\tau}{\sigma\sqrt{\tau}} \quad \text{and } d_\xi^1(x, \tau) = \frac{\log(\frac{x}{\xi}) + (r - \frac{\sigma^2}{\tau})\tau}{\sigma\sqrt{\tau}} \quad (\text{A.9})$$

Here,  $\tau = T - t$ , the assets binary pays the assets price  $x$  and the bond pays 1 monetary unit if the contracts’ conditions are met.

**Theorem A.3** (Taken from An introduction to exotic option pricing[2]). *The European equivalent payoff for the down and out barrier option is given as:*

$$V(x, y) = f(x)\mathbb{I}_{(x < b)} - f^i(x)\mathbb{I}_{(x > b)}. \quad (\text{A.10})$$



## APPENDIX B

### Computation of the Greeks

In this Section we compute the Greeks of the Lookback Option. These are given as exercises in "Stochastic calculus for finance II: Continuous-time models" [11] section 7.8.

#### Questions:

Given the following equations solve the questions that follow.

1.

$$U(t, z) = \left(1 + \frac{\sigma^2}{2r}\right) zN(\delta_+(\tau, z)) + e^{-r\tau}N(-\delta_-(\tau, z)) - \frac{\sigma^2}{2r}e^{-r\tau}z^{1-\frac{2r}{\sigma^2}}N(-\delta_-(\tau, z^{-1})) - z, \quad (\text{B.1})$$

2.

$$\frac{\partial}{\partial t}\delta_{\pm}(\tau, s) = -\frac{1}{2\tau}\delta_{\pm}\left(\tau, \frac{1}{s}\right), \quad (\text{B.2})$$

3.

$$e^{-r\tau}N'(\delta_-(\tau, s)) = sN'(\delta_+(\tau, s)), \quad (\text{B.3})$$

4.

$$N'(\delta_{\pm}(\tau, s^{-1})) = s^{\frac{2r}{\sigma^2}\pm 1}N'(\delta_{\pm}(\tau, s)), \quad \text{and} \quad (\text{B.4})$$

5. given  $c > 0$

$$\frac{\partial}{\partial t}\delta_{\pm}\left(\tau, \frac{x}{c}\right) = \frac{1}{x\sigma\sqrt{\tau}}, \quad \text{and} \quad \frac{\partial}{\partial t}\delta_{\pm}\left(\tau, \frac{c}{x}\right) = \frac{1}{x\sigma\sqrt{\tau}}. \quad (\text{B.5})$$

1. Show that:

$$\begin{aligned}
u_t(t, z) &= re^{-r\tau} N(-\delta_-(\tau, z)) - \frac{\sigma^2}{2} e^{-r\tau} z^{1-\frac{2r}{\sigma^2}} N(-\delta_-(\tau, z^{-1})) \\
&\quad - \frac{\sigma z}{\sqrt{\tau}} N'(\delta_+(\tau, z)). \tag{B.6}
\end{aligned}$$

Solution:

$$\begin{aligned}
U_t(t, z) &= \left(1 + \frac{\sigma^2}{2r}\right) z \left[\frac{d}{dt} \delta_+(\tau, z)\right] N'(\delta_+(\tau, z)) + re^{-r\tau} N(-\delta_-(\tau, z)) \\
&\quad + e^{-r\tau} \left[\frac{d}{dt} \delta_-(\tau, z)\right] N'(-\delta_-(\tau, z)) \\
&\quad - \frac{\sigma^2}{2} e^{-r\tau} z^{1-\frac{2r}{\sigma^2}} N(-\delta_-(\tau, z^{-1})) \\
&\quad - \frac{\sigma^2}{2r} e^{-r\tau} z^{1-\frac{2r}{\sigma^2}} \left[\frac{d}{dt} \delta_-(\tau, z^{-1})\right] N'(-\delta_-(\tau, z^{-1})).
\end{aligned}$$

Now using equation B.2, we get:

$$\begin{aligned}
U_t(t, z) &= \left(1 + \frac{\sigma^2}{2r}\right) z \left[-\frac{1}{2\tau} \delta_+\left(\tau, \frac{1}{z}\right)\right] N'(\delta_+(\tau, z)) \\
&\quad + re^{-r\tau} N(-\delta_-(\tau, z)) \\
&\quad + e^{-r\tau} \left[-\frac{1}{2\tau} \delta_-\left(\tau, \frac{1}{z}\right)\right] N'(-\delta_-(\tau, z)) \\
&\quad - \frac{\sigma^2}{2} e^{-r\tau} z^{1-\frac{2r}{\sigma^2}} N(-\delta_-(\tau, z^{-1})) \\
&\quad - \frac{\sigma^2}{2r} e^{-r\tau} z^{1-\frac{2r}{\sigma^2}} \left[-\frac{1}{2\tau} \delta_-(\tau, z)\right] N'(-\delta_-(\tau, z^{-1})) \\
&= re^{-r\tau} N(-\delta_-(\tau, z)) - \frac{\sigma^2}{2} e^{-r\tau} z^{1-\frac{2r}{\sigma^2}} N(-\delta_-(\tau, z^{-1})) \\
&\quad + \mathcal{K}(\tau, z), \tag{B.7}
\end{aligned}$$

where  $\mathcal{K}(\tau, z)$  is defined as:

$$\begin{aligned}
\mathcal{K}(\tau, z) &= -\left(1 + \frac{\sigma^2}{2r}\right) z \frac{1}{2\tau} \delta_+(\tau, z^{-1}) N'(\delta_+(\tau, z)) \\
&\quad - e^{-r\tau} \frac{1}{2\tau} \delta_-(\tau, z^{-1}) N'(-\delta_-(\tau, z)) \\
&\quad - \frac{\sigma^2}{2r} e^{-r\tau} z^{1-\frac{2r}{\sigma^2}} \frac{1}{2\tau} \delta_-(\tau, z) N'(-\delta_-(\tau, z^{-1})). \tag{B.8}
\end{aligned}$$



Since  $N'$  is an odd function, and with the use of equations (B.3) and (B.4), we then derive:

$$\begin{aligned}\mathcal{K}(\tau, z) &= -\left(1 + \frac{\sigma^2}{2r}\right) z \frac{1}{2\tau} \delta_+(\tau, z^{-1}) N'(\delta_+(\tau, z)) \\ &\quad + z \frac{1}{2\tau} \delta_-(\tau, z^{-1}) N'(\delta_+(\tau, z)) \\ &\quad - z \frac{\sigma^2}{2r} z^{1-\frac{2r}{\sigma^2}} \frac{1}{2\tau} \delta_-(\tau, z) z^{\frac{2r}{\sigma^2}-1} N'(\delta_+(\tau, z)).\end{aligned}$$

Now simplifying, we have:

$$\begin{aligned}\mathcal{K}(\tau, z) &= \frac{z}{2\tau} N'(\delta_+(\tau, z)) \left( -\left(1 + \frac{\sigma^2}{2r}\right) \delta_+ \sigma^2 2r \delta_-(\tau, z) \right) \\ &= \frac{z}{2\tau} N'(\delta_+(\tau, z)) F(\tau, z).\end{aligned}\tag{B.9}$$

Now solving for  $F(\tau, z)$ , we then attain:

$$\begin{aligned}F(\tau, z) &= \delta_-(\tau, z^{-1}) - \delta_+(\tau, z^{-1}) + \delta_-(\tau, z^{-1}) - (\tau, z^{-1}) \\ &\quad - \frac{\sigma^2}{2r} \delta_+(\tau, z^{-1}) - \frac{\sigma^2}{2r} \delta_-(\tau, z) \\ &= \frac{1}{\sigma\sqrt{\tau}} \left[ \log(z^{-1}) - \log(z^{-1}) - \frac{\sigma^2}{2r} (\log(z^{-1}) + \log(z)) \right. \\ &\quad \left. + \tau \left( r - \frac{\sigma^2}{2} \right) - \left( r + \frac{\sigma^2}{2} \right) - \frac{\sigma^2}{2r} \left( r - \frac{\sigma^2}{2} \right) \right. \\ &\quad \left. - \left( r + \frac{\sigma^2}{2} \right) \right] \\ &= \frac{1}{\sigma\sqrt{\tau}} (-2\tau\sigma^2) \\ &= -2\sigma\sqrt{\tau}.\end{aligned}$$

Substituting  $F(\tau, z)$  into equation (B.9),

$$\begin{aligned}\mathcal{K}(\tau, z) &= \frac{z}{2\tau} N'(\delta_+(\tau, z)) (-2\sigma\sqrt{\tau}) \\ &= -\frac{z\sigma}{\sqrt{\tau}} N'(\delta_+(\tau, z)),\end{aligned}$$

is derived. Further, substituting  $\mathcal{K}(\tau, z)$  into equation (B.7), we finally derive:

$$\begin{aligned}u_t(t, z) &= re^{-r\tau} N(-\delta_-(\tau, zt)) - \frac{\sigma^2}{2} e^{-r\tau} z^{1-\frac{2r}{\sigma^2}} N(-\delta_-(\tau, z^{-1})) \\ &\quad - \frac{\sigma z}{\sqrt{\tau}} N'(\delta_+(\tau, z)),\end{aligned}$$

as required.

2. Show that

$$\begin{aligned}
U_z(\tau, z) &= \left(1 + \frac{\sigma^2}{2r}\right) N(\delta_+(\tau, z)) \\
&\quad + \left(1 - \frac{\sigma^2}{2r}\right) e^{-r\tau} z^{-\frac{2r}{\sigma^2}} N(-\delta_-(\tau, z^{-1})) - 1. \tag{B.10}
\end{aligned}$$

Solution:

$$\begin{aligned}
U_z(\tau, z) &= \left(1 + \frac{\sigma^2}{2r}\right) N(\delta_+(\tau, z)) + \left(1 + \frac{\sigma^2}{2r}\right) z N'(\delta_+(\tau, z)) \left(\frac{d}{dz}\delta_+(\tau, z)\right) \\
&\quad + e^{-r\tau} N'(-\delta_-(\tau, z)) \left(\frac{d}{dz}\delta_-(\tau, z)\right) \\
&\quad - \frac{\sigma^2}{2r} e^{r\tau} z^{-\frac{2r}{\sigma^2}} N(-\delta_-(\tau, z^{-1})) \left(1 - \frac{2r}{\sigma^2}\right) \\
&\quad - \frac{\sigma^2}{2r} e^{r\tau} z^{-\frac{2r}{\sigma^2}} N'(-\delta_-(\tau, z^{-1})) \left(\frac{d}{dz}\delta_+(\tau, z)\right) - 1 \\
&= \left(1 + \frac{\sigma^2}{2r}\right) N(\delta_+(\tau, z)) + \left(1 - \frac{\sigma^2}{2r}\right) e^{-r\tau} z^{-\frac{2r}{\sigma^2}} N(-\delta_-(\tau, z^{-1})) - 1 \\
&\quad + \left(1 + \frac{\sigma^2}{2r}\right) z N'(\delta_+(\tau, z)) \left(\frac{d}{dz}\delta_+(\tau, z)\right) \\
&\quad + e^{-r\tau} N'(-\delta_-(\tau, z)) \left(\frac{d}{dz}\delta_-(\tau, z)\right) \\
&\quad - \frac{\sigma^2}{2r} e^{r\tau} z^{-\frac{2r}{\sigma^2}} N'(-\delta_-(\tau, z^{-1})) \left(\frac{d}{dz}\delta_+(\tau, z)\right) \\
&= \left(1 + \frac{\sigma^2}{2r}\right) N(\delta_+(\tau, z)) \\
&\quad + \left(1 - \frac{\sigma^2}{2r}\right) e^{-r\tau} z^{-\frac{2r}{\sigma^2}} N(-\delta_-(\tau, z^{-1})) - 1 + G(\tau, z). \tag{B.11}
\end{aligned}$$

Now, let us see that  $G(\tau, z) = 0$ .

From equation (B.11) it is derived that:

$$\begin{aligned}
G(\tau, z) &= \left(1 + \frac{\sigma^2}{2r}\right) z N'(\delta_+(\tau, z)) \left(\frac{d}{dz}\delta_+(\tau, z)\right) \\
&\quad + e^{-r\tau} N'(-\delta_-(\tau, z)) \left(\frac{d}{dz}\delta_-(\tau, z)\right) \\
&\quad - \frac{\sigma^2}{2r} e^{-r\tau} z^{1-\frac{2r}{\sigma^2}} N'(-\delta_-(\tau, z^{-1})) \left(\frac{d}{dz}\delta_+(\tau, z)\right).
\end{aligned}$$

Using equation (B.2), we now have:

$$\begin{aligned} G(\tau, z) &= \left(1 + \frac{\sigma^2}{2r}\right) z N'(\delta_+(\tau, z)) \left(\frac{1}{z\sigma\sqrt{\tau}}\right) \\ &\quad + e^{-r\tau} N'(-\delta_-(\tau, z)) \left(\frac{1}{z\sigma\sqrt{\tau}}\right) \\ &\quad + \frac{\sigma^2}{2r} e^{-r\tau} z^{1-\frac{2r}{\sigma^2}} N'(-\delta_-(\tau, z^{-1})) \left(\frac{1}{z\sigma\sqrt{\tau}}\right). \end{aligned}$$

The fact that  $N'$  is an odd function and using equation (B.3) and simplifying, we then have:

$$G(\tau, z) = \frac{\sigma^2}{2r} N'(\delta_+(\tau, z)) \left(\frac{1}{\sigma\sqrt{\tau}}\right) - \frac{\sigma^2}{2r} z^{1-\frac{2r}{\sigma^2}} N'(\delta_+(\tau, z^{-1})) \left(\frac{1}{\sigma\sqrt{\tau}}\right).$$

Finally using equation (B.4), we see:

$$\begin{aligned} G(\tau, z) &= \frac{\sigma^2}{2r} N'(\delta_+(\tau, z)) \left(\frac{1}{\sigma\sqrt{\tau}}\right) - \frac{\sigma^2}{2r} z^{1-\frac{2r}{\sigma^2}} z^{\frac{2r}{\sigma^2}-1} N'(\delta_+(\tau, z)) \left(\frac{1}{\sigma\sqrt{\tau}}\right) \\ &= 0. \end{aligned}$$

It therefore follows that:

$$U_z(\tau, z) = \left(1 + \frac{\sigma^2}{2r}\right) N(\delta_+(\tau, z)) + \left(1 - \frac{\sigma^2}{2r}\right) e^{-r\tau} z^{-\frac{2r}{\sigma^2}} N(-\delta_-(\tau, z^{-1})) - 1.$$

3. Show that:

$$U_{zz} = \left(1 - \frac{2r}{\sigma^2}\right) e^{-r\tau} z^{-\frac{2r}{\sigma^2}-1} N(-\delta_-(\tau, z^{-1})) + \frac{2}{z\sigma\sqrt{\tau}} N'(\delta_+(\tau, z)). \quad (\text{B.12})$$

By taking the partial derivative of equation (B.11) with respect to  $z$  and using equation (B.2), we have:

$$\begin{aligned} U_{zz}(\tau, z) &= \left(1 + \frac{\sigma^2}{2r}\right) N'(\delta_+(\tau, z)) \frac{1}{z\sigma\sqrt{\tau}} \\ &\quad - \frac{2r}{\sigma^2} \left(1 - \frac{\sigma^2}{2r}\right) e^{r\tau} z^{-\frac{2r}{\sigma^2}-1} N(-\delta_-(\tau, z^{-1})) \\ &\quad - \left(1 - \frac{\sigma^2}{2r}\right) e^{r\tau} z^{-\frac{2r}{\sigma^2}} N'(-\delta_-(\tau, z^{-1})) \frac{1}{z\sigma\sqrt{\tau}}. \end{aligned}$$

By applying equation (B.3) and simplifying, we then have:

$$\begin{aligned}
U_{zz}(\tau, z) &= \left(1 - \frac{2r}{\sigma^2}\right) e^{-r\tau} z^{-\frac{2r}{\sigma^2}-1} N(-\delta_-(\tau, z^{-1})) \\
&\quad + \left(1 + \frac{\sigma^2}{2r}\right) N'(\delta_+(\tau, z)) \frac{1}{z\sigma\sqrt{\tau}} \\
&\quad + \left(1 - \frac{\sigma^2}{2r}\right) z^{1-\frac{2r}{\sigma^2}} N'(\delta_+(\tau, z^{-1})) \frac{1}{z\sigma\sqrt{\tau}}.
\end{aligned}$$

Now applying equation (B.4), we have:

$$\begin{aligned}
U_{zz}(\tau, z) &= \left(1 - \frac{2r}{\sigma^2}\right) e^{-r\tau} z^{-\frac{2r}{\sigma^2}-1} N(-\delta_-(\tau, z^{-1})) \\
&\quad + \left(1 + \frac{\sigma^2}{2r}\right) N'(\delta_+(\tau, z)) \frac{1}{z\sigma\sqrt{\tau}} \\
&\quad + \left(1 - \frac{\sigma^2}{2r}\right) N'(\delta_+(\tau, z)) \frac{1}{z\sigma\sqrt{\tau}} \\
&= \left(1 - \frac{2r}{\sigma^2}\right) e^{-r\tau} z^{-\frac{2r}{\sigma^2}-1} N(-\delta_-(\tau, z^{-1})) \\
&\quad + N'(\delta_+(\tau, z)) \frac{1}{z\sigma\sqrt{\tau}} \left[ \left(1 + \frac{\sigma^2}{2r}\right) + \left(1 - \frac{\sigma^2}{2r}\right) \right] \\
&= \left(1 - \frac{2r}{\sigma^2}\right) e^{-r\tau} z^{-\frac{2r}{\sigma^2}-1} N(-\delta_-(\tau, z^{-1})) + \frac{2}{z\sigma\sqrt{\tau}} N'(\delta_+(\tau, z)).
\end{aligned}$$

4. Verify that the Black-Scholes-Merton Equation holds, i.e.,

$$U_t(\tau, z) + rzU_z(\tau, z) + \frac{1}{2}\sigma^2 z^2 U_{zz}(\tau, z) = rU(\tau, z).$$

with the use of equations (B.6),(B.10) and (B.12), we have:

$$\begin{aligned}
& U_t(\tau, z) + rzU_z(\tau, z) + \frac{1}{2}\sigma^2 z^2 U_{zz}(\tau, z) \\
&= re^{-r\tau} N(-\delta_-(\tau, z)) - \frac{\sigma^2}{2} e^{-r\tau} z^{1-\frac{2r}{\sigma^2}} N(-\delta_-(\tau, z^{-1})) \\
&\quad - \frac{\sigma z}{\sqrt{\tau}} N'(\delta_+(\tau, z)) \\
&\quad + rz \left[ \left(1 + \frac{\sigma^2}{2r}\right) N(\delta_+(\tau, z)) \right] \\
&\quad + rz \left[ \left(1 - \frac{\sigma^2}{2r}\right) e^{-r\tau} z^{-\frac{2r}{\sigma^2}} N(-\delta_-(\tau, z^{-1})) - 1 \right] \\
&\quad + \frac{1}{2}\sigma^2 z^2 \left[ \left(1 - \frac{2r}{\sigma^2}\right) e^{-r\tau} z^{-\frac{2r}{\sigma^2}-1} N(-\delta_-(\tau, z^{-1})) \right. \\
&\quad \left. + \frac{2}{z\sigma\sqrt{\tau}} N'(\delta_+(\tau, z)) \right] \\
&= re^{-r\tau} N(-\delta_-(\tau, z)) - \frac{\sigma^2}{2} e^{-r\tau} z^{1-\frac{2r}{\sigma^2}} N(-\delta_-(\tau, z^{-1})) \\
&\quad + rz \left(1 + \frac{\sigma^2}{2r}\right) N(\delta_+(\tau, z)) \\
&\quad + rz \left(1 - \frac{\sigma^2}{2r}\right) e^{-r\tau} z^{-\frac{2r}{\sigma^2}} N(-\delta_-(\tau, z^{-1})) - rz \\
&\quad + \frac{1}{2}\sigma^2 z^2 \left(1 - \frac{2r}{\sigma^2}\right) e^{-r\tau} z^{-\frac{2r}{\sigma^2}-1} N(-\delta_-(\tau, z^{-1})) \\
&= r \left[ e^{-r\tau} N(-\delta_-(\tau, z)) - \frac{\sigma^2}{2r} e^{-r\tau} z^{1-\frac{2r}{\sigma^2}} N(-\delta_-(\tau, z^{-1})) \right. \\
&\quad + z \left(1 + \frac{\sigma^2}{2r}\right) N(\delta_+(\tau, z)) \\
&\quad + z \left(1 - \frac{\sigma^2}{2r}\right) e^{-r\tau} z^{-\frac{2r}{\sigma^2}} N(-\delta_-(\tau, z^{-1})) - z \\
&\quad \left. + \frac{1}{2}\sigma^2 z^2 \left(1 - \frac{2r}{\sigma^2}\right) e^{-r\tau} z^{-\frac{2r}{\sigma^2}-1} N(-\delta_-(\tau, z^{-1})) \right] \\
&= rU(\tau, z).
\end{aligned}$$

5. Show that:

$$U(t, 1) = U_z(t, 1).$$

Substituting  $z=1$  in equation(B.1), we have:

$$\begin{aligned} U(t, 1) &= \left(1 + \frac{\sigma^2}{2r}\right) N(\delta_+(\tau, 1)) + e^{-r\tau} N(-\delta_-\tau, 1) \\ &\quad - \frac{\sigma^2}{2r} e^{-r\tau} N(-\delta_-(\tau, 1)) - 1 \\ &= \left(1 + \frac{\sigma^2}{2r}\right) N(\delta_+(\tau, 1)) + \left(1 - \frac{\sigma^2}{2r}\right) e^{-r\tau} N(-\delta_-(\tau, 1)) - 1 \\ &= U_z(t, 1). \end{aligned}$$

