# STOCHASTIC OPTIMAL CONTROL THEORY: NEW APPLICATIONS TO FINANCE AND INSURANCE

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# ABSTRACT

STOCHASTIC OPTIMAL CONTROL THEORY: NEW APPLICATIONS TO FINANCE AND INSURANCE

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In this study, the literature, recent developments and new achievements in stochastic optimal control theory are studied. Stochastic optimal control theory is an important direction of mathematical optimization for deriving control policies subject to time-dependent processes whose dynamics follow stochastic differential equations. In this study, this methodology is used to deal with those infinite-dimensional optimization programs for problems from finance and insurance that are indeed motivated by the real life. Stochastic optimal control problems can be further treated and solved along different avenues, two of the most important ones of being (i) Pontryagin's maximum principle together with stochastic adjoint equations (within both necessary and sufficient optimality conditions), and (ii) Dynamic Programming principle together with Hamilton-Jacobi-Bellman (HJB) equations (within necessary and sufficient versions, e.g., a verification analysis). Here we introduce the needed instruments from economics and from Ito calculus, such as the theory of jump-diffusion and Lévy processes. We then present Dynamic Programing Principle, HJB Equations, Verification Theorem, Sufficient Maximum Principle for stochastic optimal control of diffusions and jump diffusions, and we state some connections and differences between Maximum Principle and the Dynamic Programing Principle. As financial applications, we investigate mean-variance portfolio selection problem and Merton optimal portfolio and consumption problem. From actuarial sciences, we study the optimal investment and liability ratio problem for an insurer and the problem of purchase of optimal life-insurance, optimal investment and consumption of a wage-earner within a market of several life-insurance providers, respectively. In our examples, we shall refer to various utility functions such as exponential, power and logarithmic ones, and to different parameters of risk averseness. We provide some graphical representations of the optimal solutions to illustrate the theoretical results. The thesis ends with a conclusion and an outlook to future studies, addressing elements of information, memory and stochastic robust optimal control problems.

*Keywords*: Dynamic Programming Principle, Life-Insurance, Maximum Principle, Optimal Investment Strategy, Utility Maximization

# ÖZ

STOKASTİK OPTİMAL KONTROL TEORİ: FİNANS VE SİGORTACILIKTA YENİ UYGULAMALAR

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Bu tezde, stokastik optimal kontrol teorisinin literatürü, ve bu teori üzerindeki son gelişmeler ve yeni edinimler üzerinde durulmuştur. Stokastic optimal kontrol teorisi, dinamikleri stokastik diferansiyel denklemleri takip eden zamana bağlı süreçlere tabi tutulan en uygun kontrol politikalarının türetilmesi için kullanılmaktadır. Bu çalışmada, bu metodoloji, gerçek hayattan finans ve sigorta problemleri için sonsuz boyutlu optimizasyon programlarını çözmek için kullanılmaktadır. Stochastic optimal kontrol problemleri, (i) Pontryagin'in maksimum prensibi ile birlikte stokastik adjoint denklemleri (hem gerekli hem de yeterli optimallik koşulları dahilinde) ve (ii) Hamilton-Jacobi-Bellman (HJB) denklemleri ile birlikte Dinamik Programlama prensibi (gerekli ve yeterli şartlar içinde, örneğin bir doğrulama analizi) olmak üzere çözülebilir. Bu tezde Dinamik Programlama Prensibi, HJB denklemleri, doğrulama teoremi, sıçramalı difüzyonların stokastik optimal kontrolü için Yeterli Maksimum Prensip ve Maksimum Prensip ile Dinamik Programlama Prensibi arasındaki bağlantıları ve farklılıkları açıklayacağız. Finansal uygulamalar kısmında sırasıyla bir sigortacının ortalama-varyans portföy seçimi problemi ve Merton optimal portföy ve tüketim problemini inceleyeceğiz. Aktüerya biliminden ise bir sigorta şirketinin optimal yatırım ve yükümlülük oranı problemini ve bir gündelikçi için en iyi hayat sigortası seçimi ve satın alımı, en uygun tüketim ve yatırım oranlarını bulma problemini inceleyeceğiz. Örneklerimizde, üstel, güç ve logaritmik gibi çeşitli fayda fonksiyonları ve risk farklılığının farklı parametrelerini inceleyeceğiz. Bu örneklerden optimal çözümlerin bazı grafiksel sonuçlarını sunacağız. Çalışmamızı sonuş ve gelecekteki çalışmalar kısmı ile bitireceğiz.

*Anahtar Kelimeler* : Dinamik Programlama Prensibi, Fayda Maksimizasyonu, Hayat Sigortası, Maksimum Prensibi, Optimal Yatırım Stratejisi

*To My Mother*

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| | |
|---|---|
| CRRA | Constant Relative Risk Aversion |
| $\mathbb{N}$ | Set of natural numbers |
| $\mathbb{R}$ | Set of real numbers |
| $\mathbb{R}^+$ | Set of nonnegative real numbers |
| $\mathbb{R}^d$ | $d$-dimensional Euclidean space |
| $\mathbb{R}^{d\times n}$ | Real-valued $d \times n$ vectors |
| ODE | Ordinary Differential Equations |
| PDE | Partial Differential Equations |
| PIDE | Partial Integro Differential Equations |
| SDE | Stochastic Differential Equations |
| DPP | Dynamic Programming Principle |
| MP | Maximum Principle |
| RCLL | Right-Continuous with Left Limits |
| BSDE | Backward Stochastic Differential Equations |
| FBSDE | Forward-Backward Stochastic Differential Equations |
| a.s. | almost surely |
| a.e. | almost everywhere |

# CHAPTER 1

# INTRODUCTION

Optimal control theory which is an extension of the calculus of variations is a modern technique to solve Dynamic Optimization problems. Calculus of variations has some limitations, because it relies on differentiability and deals with interior solutions. Optimal control theory, a contemporary mathematical optimization method, is not being constrained by interior solutions, but it still relies on differentiability. In optimal control, the objective is to derive control policies which optimize the performance functional for a given system. Once the optimal control variables are found, the optimal paths for the the given state variables are derived.

The parameters to be optimized of the control problems may be taken as constant or random. Stochastic optimal control theory is a subfield of optimal control theory and it deals with mathematical models which contain randomness. To choose the best path (or best parameter values) among all choices under uncertainty is the goal of stochastic optimal control. In stochastic optimal control, controlled systems are described by stochastis differential equations (SDEs) and the controlled system involves a state process, control process, and a performance functional. In this thesis, we consider the systems which are dynamic and described by SDEs.

Recently, stochastic optimal control has been of great interest by many researchers, and is used with its several applications in many fields such as physics, economics, finance, biology, ecology, medicine, engineering and economics. Since Merton optimal consumption and portfolio problem [15], portfolio optimization problems occupy an important place in finance. In the literature, portfolio optimization problems can be solved by Maximum Principle, Dynamic Programming Principle, and the Convex Duality Martingale method. For the Convex Duality Martingale method, we refer the reader to see [21] and, in this thesis, we will look more closely the Maximum Principle (MP) and Dynamic Programming Principle (DPP). Interestingly, MP and DPP, the two main and most commonly used approaches, were introduced simultaneously, but separately. Maximum Principle is based on necessary optimality conditions for controls and leads to forward-backward stochastic differential equations (FBSDE). We call the optimal control problems as stochastic recursive optimal control problems if their state equations are described by the solution of a FBSDE. Maximum Principle was introduced by Pontryagin and his group for deterministic problems. The inspiring idea was coming from the classical calculus of variations. The maximum principle for diffusions was studied by Kushner [13], Bismut [3], Bensoussan [2], Haussmann [12], Peng [20], Young and Zhou [30]. To handle stochastic optimal control problems, Bismut [3] introduced the linear backward stochastic differential equations (BSDEs). Pardoux and Peng [19] introduced the nonlinear BSDEs. Peng [20] first examined the stochastic recursive optimal control problems and derived a stochastic maximum principle for the convex domains. For the non-convex domain, Xu [28] derived a maximum principle. Tang and Li [27] extended Peng's study to the jump-diffusion processes. Zhou [33] proved that the study of Peng is enough when the certain convexity conditions are satisfied. A sufficient Maximum Principle for general jump-diffusion processes was formulated by Framstad et al. [9]. In Chapter 3, we will review the study of Framstad et al. [9] in detail and explain the methodology for general jump-diffusion processes.

In the early 1950s, the other important approach Dynamic Programming Principle was introduced by Richard Bellman. This principle leads to Hamilton Jacobi Bellman (HJB) equation, a nonlinear second-order partial differential equation (PDE) in continuous-time finance for Markov processes. Instead of solving the entire problem, it is enough to solve the HJB equation, and if the HJB equation is solvable, then the optimal values are obtained. Moreover, the HJB equation is satisfied by Verification Theorem in the DPP. When the HJB equation has an explicit smooth solution, the verification theorem says that this solution is indeed the value function of the problem. However, this case is not general. Here, a convenient framework, namely, the viscosity solutions, introduced by Crandal and Lions [6], provides to go beyond the classical verification theorem by relaxing the smoothness. In this thesis, it is not our purpose to study viscosity solutions and we refer [8], [30] for viscosity solutions and for more literature review related to DPP.

The purpose of this thesis is to review the stochastic optimal control problems by using the two main approaches, namely, DPP and MP, with their applications to finance and insurance. The thesis is structured as follows: Chapter 2 presents some preliminaries that will be used in this thesis. In Chapter 3, we introduce the formulation of stochastic optimal control problems. Then, we proceed with the study of Framstad et al. [9], which is a sufficient maximum principle for general jump-diffusion processes. We will give a brief exposition of the MP without proofs, introduce the Hamiltonian systems, and discuss mean-variance portfolio selection problem taken from Framstad et al. [9] as a financial application of the MP. Chapter 3 also contains the Dynamic Programming methodology for controlled systems. We will derive the HJB equation, Verification Theorem, and examine the Merton's optimal consumption-portfolio problem for diffusion and jump-diffusion processes [15]. Finally, the relation between MP and DPP will be established in this chapter. In Chapter 4, some applications of stochastic optimal control problems in actuarial sciences are presented. In this chapter, firstly, we will investigate the submitted study of Özalp et al. [31] which is entitled with optimal investment strategy and liability ratio for insurer with Lévy risk processes. In this application, the aim is to obtain the optimal liability ratio and investment policy which maximizes the expected utility of an insurer at terminal time via Maximum Principle. We obtained the same results as obtained in Özalp et al. [31] for the logarithmic utility function. Then, secondly, we will investigate the study of Mousa et al. [16] which is selection and purchase of an optimal life-insurance contract from a market which contains many insurance companies. This application is a wage-earner's problem whose lifetime is uncertain and confronted with a problem of to find the optimal rates for his consumption, investment and the premium amont which he pays for a life-insurance contract. In this application, as an investment strategy the wage-eaner may buy riskless asset and a fixed number of risky assets, and selects life-insurance contracts from insurance companies which offer different contracts. The aim of the wage-earner is to optimize the joint expected benefit from his expenditures, from his wealth at retirement time or the legacy in the event of early death before his retirement age. To solve this control problem, DPP is used to get an explicit optimal solutions for the discounted constant relative risk aversion (CRRA) utilities. Finally, we developed the numerical results of Duarte et al. [7] with the author's helps and visualized these optimal results using Matlab. We analysed the optimal results with respect to different parameters. In the last Chapter, we conclude and propose some interesting and promising research projects for the future.

# CHAPTER 2

# MATHEMATICAL FOUNDATIONS

As for prerequisites, the reader is expected to be familiar with a basic knowledge of probability theory, measure theory and stochastic calculus. In this chapter, we recall the relevant material, some basic definitions and theorems (without proofs) of stochastic calculus, that will be needed to solve the stochastic control problems from finance and insurance. This chapter is rather very short and for a treatment of a more detailed theory we refer the reader to Cont [5], Kyprianou [14], Papapantoleon [18], and Øksendal and Sulem [17]. Throughout this thesis we work with a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, where $\Omega$ denotes a sample space, $\mathcal{F}$ is a $\sigma-$algebra, $(\mathcal{F}_t)_{t\geq 0}$ is a filtration, and $\mathbb{P}$ is a probability measure.

As it is well known, Brownian motion is a substantially important process which appears in the most financial models. It is an example of a diffusion process which is a solution to a stochastic differential equation. A diffusion process is a Markov process that has continuous paths, namely, it has no jumps, and it models a "standardized" random fluctuation. Diffusion models are beneficial for mathematical finance in practice, but they cannot generate sudden discontinuities. However, in the real world, empirical observations indicate us that price movements have jumps. Therefore, we need to consider models which involve sudden discontinuities for describing the observed reality of financial markets. In this thesis, financial models with jumps and no jumps are studied. Since the jump-diffusion models contain the diffusion models, we proceed with the study of models with jumps. We can classify these models into two, namely, jump-diffusion models and infinite activity models.

## 2.1 Jump-Diffusion Models

The first category consists of the *jump-diffusion* models which contain a Brownian motion component and jumps at random times. That is to say the process jumps at some times and has a continuous random path between jumps. Here, in every finite time interval there are only finitely many jumps, jumps are rarely appearing and represented by a compound Poisson process. In jump-diffusion models, since the distribution of jump sizes is known, they carry out quite well for a realistic description of price dynamics and market risks; moreover, jump-diffusion models are easy to simulate. In jump-diffusion models, characteristic functions of random variables have great importance, because while the densities are not known in closed form, the characteristic function is known explicitly. As an example of jump diffusion models, we can give the Merton model with the stock price $(S(t)) = (S(0)\exp\{X(t)\})_{\{t\geq 0\}}$ and Gaussian distributed jumps.

A jump-diffusion process is described in the following form:

$$X(t) = X(0) + \int_0^t \mu(u)du + \int_0^t \sigma(u)dW(u) + J(t), \tag{2.1}$$

where $J(t)$ is a right continuous and adapted pure jump process.

A *pure jump process* begins at zero, is constant between jump times and has finitely many jumps in each finite time interval. The fundamental pure jump process is the Poisson process.

**Definition 2.1. (Poisson Process)**

Let $\{\tau_j\}_{j\in\mathbb{N}}$ be a sequence of independent exponentially distributed random variables with parameter $\lambda$, i.e., with cumulative distribution function $\mathbb{P}\{\tau_j \leq x\} = 1 - e^{-\lambda x}$ and $S(n) = \sum_{k=1}^{n} \tau_k$. Then, the process

$$N(t) = \sum_{n\geq 1} 1_{\{t\geq S(n)\}}$$

is called the *Poisson process* with intensity $\lambda$.

*Remark* 2.1. The Poisson process $(N(t) : t \geq 0)$ counts the number of jumps that occur at or before time $t$ because all jumps of a Poisson process are of size one. The random variables $\{\tau_k\}$, $k = 1, 2, \ldots, n$, are called the inter arrival times and they are exponentially distributed.

The arrival times are defined by

$$S(n) = \sum_{k=1}^{n} \tau_k, \tag{2.2}$$

i.e., $S(n)$ is the time of the $n^{\text{th}}$ jump.

Since the expected jump time between jumps is $\frac{1}{\lambda}$, the jumps are arriving at an average rate of $\lambda$ per unit time.

**Proposition 2.1.** *(Cont, [5])*

*Let* $\{N(t)\}_{t\geq 0}$ *be a Poisson process.*

*1. For any* $\omega$*, the sample path* $t \mapsto N(t)$ *is right continuous with left limit (RCLL, cádlág) piecewise constant.*

*2. The Poisson process* $N(t)$ *with intensity* $\lambda$ *has the distribution*

$$\mathbb{P}\{N(t)=k\}=\frac{(\lambda t)^k}{k!}e^{-\lambda t}.$$

*3. The characteristic function of a Poisson process* $N(t)$ *is given by*

$$\mathbb{E}\left[e^{ixN(t)}\right]=\exp\{\lambda t(e^{ix}-1)\}.$$

*4. The Poisson process* $N(t)$ *has independent increments:*

*If* $t \geq s$, $N(t)-N(s)$ *is independent of the* $\sigma$*-algebra* $\mathcal{F}_s$.

*5. The Poisson process* $N(t)$ *has stationary increments:*

*If* $t \geq s \geq 0$, *then* $N(t)-N(s)$ *and* $N(t-s)-N(0)$ *have the same law.*

*6. The Poisson process* $N(t)$ *has the Markov property,*

*i.e.,* $\mathbb{E}[N(t)|\mathcal{F}_s]=N(s), \forall t \geq s \geq 0$.

**Corollary 2.2.** *(Shreve, [25])*

*A Poisson process* $N(t)$ *with intensity* $\lambda$ *satisfies that* $\mathbb{E}[N(t)]=\lambda t$ *and* $Var\,[N(t)]=\lambda t$.

**Definition 2.2. (Compensated Poisson Process)**

Let $N(t)$ be a Poisson process as in Definition (2.1). Then $M(t)=N(t)-\lambda t$ is called a compensated Poisson process where $\lambda$ is the parameter of the Poisson process.

**Theorem 2.3.** *(Shreve, [25])*

*The compensated Poisson process* $M(t)=N(t)-\lambda t$ *is a martingale.*

**Definition 2.3. (Compound Poisson Process)**

A compound Poisson process is a stochastic process with intensity $\lambda$ and jump size distribution $F$ defined as

$$Q(t)=\sum_{j=1}^{N(t)} Y_j, \quad t \geq 0,$$

where $N(t)$ is Poisson process with intensity $\lambda$, and the jump sizes $Y_j$ are independent of one another and also independent of $N(t)$ with distribution $F$.

*Remark* 2.2. A compound Poisson process can be considered as a Poisson process with random jump sizes.

**Proposition 2.4.** *(Cont, [5])*

*Let Q(t) be a compound Poisson process. Then, the following conditions are fulfilled:*

*1. For any $\omega$, the sample path $t \mapsto Q(t)$ is RCLL (cádlág) piecewise constant.*

*2. The characteristic function of a compound Poisson process $Q(t)$ is given by*

$$\mathbb{E}[e^{ixQ(t)}] = \exp\{\lambda t \int_{\mathbb{R}} (e^{ix} - 1)F(dx)\}.$$

*3. The compound Poisson process $Q(t)$ has independent increments: If $t \geq s \geq 0$, then $Q(t) - Q(s)$ is independent of the $\sigma$-algebra $\mathcal{F}_s$.*

*4. The compound Poisson process $Q(t)$ has stationary increments: If $t \geq s \geq 0$, then $Q(t) - Q(s)$ and $Q(t - s) - Q(0)$ have the same law.*

*5. The jump sizes $(Y_j)_{j \geq 1}$ are independent and identically distributed (i.i.d.) random variables with law F and the same mean $\mu = \mathbb{E}[Y_j]$.*

**Corollary 2.5.** *(Shreve, [25])*

*A compound Poisson process $Q(t)$ with intensity $\lambda$ satisfies the equation $\mathbb{E}[Q(t)] = \mu\lambda t$.*

**Theorem 2.6.** *(Shreve, [25])*

*The compensated compound Poisson process $\tilde{Q}(t) = Q(t) - \mu\lambda t$ is a martingale.*

**Theorem 2.7.** ***(Itô-Doeblin formula for jump-diffusion processes)*** *(Shreve, 2004, [25])*

*Let $f \in C^2(\mathbb{R})$ and $X(t)$ be a jump-diffusion process given in Eqn. (2.1). Then, we have*

$$\begin{aligned} f(X(t)) =& f(X(0)) + \int_0^t f^{'}(X(s))dX^C(s) + \frac{1}{2}\int_0^t f^{''}(X(s))d[X^C, X^C](s) \\ &+ \sum_{0<s\leq t} [f(X(s)) - f(X(s-))], \end{aligned} \tag{2.3}$$

*where $X^C(t) = X(0) + \int_0^t \mu(s)ds + \int_0^t \sigma(s)dW(s)$ represents the continuous part of the process according to Eqn. (2.1).*

## 2.2 Infinite Activity Models

In jump-diffusion models, jumps are rare events and in every finite time interval finitely many jumps occur. In infinite activity Lévy processes, in every interval there are infinitely many jumps that most of them are very small and there is only a finite number of jumps with absolute value greater than a given number. These models do not necessarily involve a Brownian motion and move actually by jumps. As compared with jump-diffusion models, infinite activity models can be constructed by Brownian subordination which gives them additional tractability. Some examples of Lévy processes are linear drift (simplest Lévy process), Brownian motion (the only non-deterministic continuous Lévy process), Poisson process, compound Poisson process, Gamma process (an increasing Lévy process, also called as subordinator).

**Definition 2.4. (Lévy Process)**

An adapted, $c\acute{a}dl\acute{a}g$, real valued stochastic process $(\eta(t))_{t\geq 0}$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$ is called a **Lévy process** if it satisfies the followings are satisfied:

1. $\mathbb{P}\left(\eta(0) = 0\right)$.

2. Independent increments: For all $0 \leq s \leq t$, $\eta(t) - \eta(s)$ is independent of $\mathcal{F}_s$.

3. Stationary increments: For all $0 \leq s \leq t$, $\eta(t) - \eta(s)$ is equal in distribution to $\eta(t-s)$.

4. Stochastic continuity: $\forall \epsilon > 0, \quad \lim_{h\to 0} P\left(|\eta(t+h) - \eta(t)| \geq \epsilon\right) = 0.$

**Definition 2.5. (Lévy Measure)**

Let $\eta(t)$ be a Lévy process on $\mathbb{R}^d$ and $\mathcal{B}(\mathbb{R}^d)$ is the Borel $\sigma$-algebra of $\mathbb{R}^d$. The measure

$$\nu(A) = \mathbb{E}[M(1, A)] = \mathbb{E}\left[\#\{t \in [0,1] : \Delta\eta(t) \neq 0, \Delta\eta(t) \in A\}\right], A \in \mathcal{B}(\mathbb{R}^d),$$

on $\mathbb{R}^d$ is called as Lévy measure of $\eta$.

This means that, $\nu(A)$ is the expected number, per unit time, of jumps whose size is in $A$. Furthermore, $M([0,t], A)$, called as jump measure of $\eta$, counts the number of jumps of $\eta$ up to time $t$ with jump size in the set $A$, and $M(dt, dx)$ is the differential notation of the $M([0,t], A)$. The compensated jump measure of $\eta$ is defined by $\tilde{M}(dt, dx) = M(dt, dx) - \nu(dx)$.

**Definition 2.6.** (**Poisson Random Measure**) (Cont, [5])

Suppose that $\mathcal{E}$ be a $\sigma$-algebra of subsets of $E \subseteq \mathbb{R}$, $(E, \mathcal{E})$ be a measurable space and $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A Poisson random measure $M$ on $E$ with intensity measure $\lambda$ (a given positive Radon measure on $(E, \mathcal{E})$) is an integer-valued random measure such that

$$\begin{aligned} M : \Omega \times \mathcal{E} &\to \mathbb{N} \\ (\omega, A) &\mapsto M(\omega, A), \end{aligned}$$

which satisfies the following conditions:

(i) For (almost all) $\omega \in \Omega$, $M(w, \cdot)$ is an integer-valued Radon measure on $E$.

(ii) For each measurable set $A \subseteq E$, $M(\cdot, A) := M(A) < \infty$ is a Poisson random variable with parameter $\lambda(A)$; for all $k \in \mathbb{N}$,

$$\mathbb{P}(M(A) = k) = \frac{(\mu(A))^k}{k!} e^{\lambda(A)}.$$

(iii) The variables $M(A_1), ...., M(A_n)$ are independent when $A_1, ...., A_n \in \mathcal{E}$ are disjoint sets.

**Proposition 2.8.** *(Jump Measure of a Compound Poisson Process) (Cont, [5])*

*The jump measure $M_X$ of a compound Poisson process $(X(t))_{t\geq 0}$ is a Poisson random measure on $\mathbb{R}^n \times [0, \infty)$ with intensity measure $\mu(dx, dt) = \nu(dx)dt = \lambda F(dx)dt$, where $\lambda$ is the intensity and $F$ is the jump size distribution of $(X(t))_{t\geq 0}$.*

According to above proposition, every compound Poisson process $X(t)$ can also be written as

$$X(t) = \sum_{s\in[0,t]} \Delta X(s) = \int_{[0,t]\times\mathbb{R}^n} x M_X(ds, dx),$$

where $M_X$ is a Poisson random measure with intensity measure $\nu(dx)dt$.

There is a strong, intimate relation between the Lévy processes and the infinite divisibility. To see this relation, we now give the definition of infinite divisibility and Lévy-Khintchine Formula.

**Definition 2.7. (Infinite Divisibility)**

A real-valued random variable $X$ has an *infinitely divisible distribution* if for all $n \in \mathbb{N}$ there exist a sequence of i.i.d. random variables $X_1^{(\frac{1}{n})}, X_2^{(\frac{1}{n})}, ...., X_n^{(\frac{1}{n})}$ such that

$$X \stackrel{d}{=} X_1^{(\frac{1}{n})} + X_2^{(\frac{1}{n})} + ... + X_n^{(\frac{1}{n})}.$$

Alternatively, in terms of probability distributions, the probability distribution $F$ of a random variable $X$ is *infinitely divisible* if for all $n \in \mathbb{N}$ there exists another law $F_{X^{(\frac{1}{n})}}$ of a random variable $X^{(\frac{1}{n})}$ such that

$$F_X = F_{X^{(\frac{1}{n})}} * F_{X^{(\frac{1}{n})}} * .... * F_{X^{(\frac{1}{n})}},$$

where $F_{X^{(\frac{1}{n})}} * F_{X^{(\frac{1}{n})}} * .... * F_{X^{(\frac{1}{n})}}$ is the $n$-th convolution of $F_{X^{(\frac{1}{n})}}$. For instance, the Normal, Poisson, Gamma, negative binomial, geometric, Cauchy, Gaussian, Dirac delta, stable distributions are infinitely divisible. For more details, see Papapantoleon [18].

**Proposition 2.9.** *(Papapantoleon, [18])*

*If $(\eta(t))_{t\geq 0}$ is a Lévy process, then $\eta(t)$ is infinitely divisible for each $t > 0$.*

*Proof.* For all $t \geq 0$ and all $n \in \mathbb{N}$, we have

$$\eta(t) = \eta(t/n) + (\eta(2t/n) - \eta(t/n)) + .... + (\eta(t) - \eta((n-1)t/n)\,.$$

By the stationary and the independent increment properties of a Lévy process, we conclude that $\eta(t)$ is infinitely divisible. $\square$

**Theorem 2.10 (Lévy-Khintchine Formula).** *(Papapantoleon, [18])*

*The probability distribution $F_X$ of a random variable $X$ is infinitely divisible with characteristic exponent*

$$\psi(u) = iub - \frac{u^2\sigma}{2} + \int_{\mathbb{R}} (e^{iux} - 1 - iux1_{|x|<1})\nu(dx),$$

*if and only if there exists a triplet $(b, \sigma, \nu)$ with $\sigma \in \mathbb{R}$, $b \in \mathbb{R}$ and $\nu$ is a measure satisfying $\nu(0) = 0$ and $\int_{\mathbb{R}}(1 \wedge |x|^2)\nu(dx) < \infty$, such that*

$$\mathbb{E}[e^{iuX}] = e^{\psi(u)}.$$

Here, the triplet $(b, \sigma, \nu)$ is called the characteristic or *Lévy triplet* and the characteristic exponent $\psi(u)$ is called the Lévy exponent, $b \in \mathbb{R}$ is drift, $\sigma \in \mathbb{R}^+$ is the diffusion coefficient and $\nu$ is the Lévy measure.

**Theorem 2.11 (Lévy-Khintchine Formula for Lévy processes).** *(Papapantoleon, [18])*

*Let $b \in \mathbb{R}$, $\sigma \geq 0$ and $\nu$ is a measure satisfying $\nu(0) = 0$ and $\int_{\mathbb{R}}(1 \wedge |x|^2)\nu(dx) < \infty$. Define characteristic exponent*

$$\psi(u) = iub - \frac{u^2\sigma}{2} + \int_{\mathbb{R}} (e^{iux} - 1 - iux1_{|x|<1})\nu(dx).$$

*Then, there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which a Lévy process with characteristic exponent $\psi(u)$ is defined.*

**Theorem 2.12 (Lévy-Itô Decomposition).** *(Papapantoleon, [18])*

*Suppose that $(b, \sigma, \nu)$ is a triplet with $\sigma \in \mathbb{R}$, $b \in \mathbb{R}$ and $\nu$ is a measure satisfying $\nu(0) = 0$ and $\int_{\mathbb{R}}(1 \wedge |x|^2)\nu(dx) < \infty$. Then there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which three independent Lévy processes, $\eta^{(1)}, \eta^{(2)}, \eta^{(3)}$ exist, where $\eta^{(1)}$ is a Brownian motion with drift, $\eta^{(2)}$ is a compound Poisson process, and $\eta^{(3)}$ is a square-integrable martingale with an a.s. countable number of jumps with magnitude less than unity on every finite interval. By taking $\eta = \eta^{(1)} + \eta^{(2)} + \eta^{(3)}$, there exists a probability space on which a Lévy process $\eta$ is defined for all $u \in \mathbb{R}$ with Lévy exponent*

$$\psi(u) = iub - \frac{u^2\sigma}{2} + \int_{\mathbb{R}} (e^{iux} - 1 - iux1_{|x|<1})\nu(dx).$$

**Proposition 2.13.** *(Papapantoleon, [18])*

*Let $\eta(t)$ be a square integrable Lévy process with Lévy measure $\nu$. Then, there exist $a$ and $b \in \mathbb{R}$ such that*

$$\begin{aligned}
\eta(t) &= at + bW(t) + \int_0^t \int_{|x|\geq 1} xM(ds,dx) + \int_0^t \int_{|x|<1} x\tilde{M}(ds,dx) \\
&= at + bW(t) + \int_0^t \int_{|x|\geq 1} x(\tilde{M}(ds,dx) + \nu(dx)ds) + \int_0^t \int_{|x|<1} x\tilde{M}(ds,dx) \\
&= at + bW(t) + \int_0^t \int_{\mathbb{R}} x(\tilde{M}(ds,dx) + \nu(dx)dt) + \int_0^t \int_{|x|\geq 1} x\nu(dx)ds \\
&= \tilde{a}t + bW(t) + \int_0^t \int_{\mathbb{R}} x(\tilde{M}(ds,dx),
\end{aligned}$$

*where $\tilde{a} = a + t\int_{|x|\geq 1} x\nu(dx)$.*

This theorem indicates that every square integrable Lévy process is a combination of a Brownian motion with drift and infinite sum of independent compound Poisson processes.

In brief, if $\eta(t)$ is a Lévy process, then for every $t$, $\eta(t)$ has an infinitely divisible distribution. Beside, if $F$ is an infinitely divisible distribution then we can construct a Lévy process $\eta(t)$, $t \geq 0$, such that the distribution of $\eta(1)$ is given by $F$.

**Proposition 2.14.** ***(Itô Formula for multidimensional Lévy processes)*** *(Cont, [5])*

*Let $X(t) = \left(X^1(t), ..., X^d(t)\right)$ be a multidimensional (d-dimensional) Lévy process with characteristic triplet $(b, \sigma, \nu)$. Then, for any $f \in C^{1,2}(\mathbb{R})$ such that $f : [0,T] \times \mathbb{R}^d \to \mathbb{R}$, we have*

$$\begin{aligned}
f\left(t, X(t)\right) =& f(0, X(0)) + \int_0^t \sum_{i=1}^d \frac{\partial f}{\partial x_i}(s, X(s-))dX^i(s) + \int_0^t \frac{\partial f}{\partial s}(s, X(s))ds \\
&+ \frac{1}{2}\int_0^t \sum_{i,j=1}^d \sigma_{ij} \frac{\partial^2 f}{\partial x_i x_j}(s, X(s))ds \\
&+ \sum_{0\leq s\leq t}^{\Delta X(s)\neq 0} [f(s, X(s)) - f(s, X(s-)) - \sum_{i=1}^d \Delta X^i(s) f^{'}(s, X(s-))],
\end{aligned} \tag{2.4}$$

*where $\Delta X^i(s) = X^i(s) - X^i(s-)$.*

**Definition 2.8.** (***Infinitesimal Generator***)

Let $(X(t))_{t\geq 0}$ be a jump-diffusion process. The infinitesimal generator $\mathcal{L}$ of $(X(t))_{t\geq 0}$ on function $f : \mathbb{R}^n \to \mathbb{R}$ is defined as

$$(\mathcal{L}f)(x) = \lim_{t\to 0} \frac{\mathbb{E}\left[f(X(t))\right] - f(x)}{t},$$

if the limit exists.

**Proposition 2.15 (Infinitesimal generator of a Lévy Process).** *(Cont, [5])*

*Let $\eta(t)$ be a Lévy process on $\mathbb{R}^n$ with Lévy triplet $(b, \sigma, \nu)$ and $f \in C_0^2(\mathbb{R})$, where $C_0^2(\mathbb{R})$ is the set of twice differentiably functions;*

$$
\begin{aligned}
d\eta(t) = & b\left(t, \eta(t), u(t)\right)\ dt + \sigma\left(t, \eta(t), u(t)\right)\ dW(t) \\
& + \int_{\mathbb{R}^n} h\left(t, \eta(t-), u(t-), z\right)\ \tilde{M}(dt, dz),
\end{aligned} \tag{2.5}
$$

*where $b : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{U} \to \mathbb{R}^n$, $\sigma : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{U} \to \mathbb{R}^{n\times d}$, and $h : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{U} \times \mathbb{R}^n \to \mathbb{R}^{n\times l}$ are given functions, $W(t) = W$ is a $d$-dimensional standard Brownian Motion, and*

$$
\begin{aligned}
\tilde{M}(dt, dz) &= \left(\tilde{M}_1(dt, dz), \ldots.., \tilde{M}_l(dt, dz)\right)^T \\
&= \left(M_1(dt, dz) - \nu_1(dz_1)dt, \ldots., M_l(dt, dz) - \nu_l(dz_l)dt\right)^T
\end{aligned}
$$

*is a compensated Poisson process where $\{M_i\}$ are independent $\mathbb{R}^{l\times 1}$-valued Poisson random measures with Lévy measures $\nu_i$ on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ for $i = 1, ..., l$.*

*Then, the infinitesimal generator $\mathcal{L}f(x)$ of $\eta$ is defined as follows:*

$$
\begin{aligned}
\mathcal{L}f(x) = & \sum_{j=1}^{n} b_j(x)\frac{\partial f}{\partial x_j}(x) + \frac{1}{2}\sum_{j,i=1}^{n}(\sigma\sigma^T)_{ij}(x)\frac{\partial^2 f}{\partial x_j \partial x_i}(x) \\
& + \int_{\mathbb{R}^n}\left(f(x + h(x, z)) - f(x) - \sum_{k=1}^{n}\nabla f(x)\cdot h(x, z)\right)\nu(dz),
\end{aligned}
$$

*where $T$ represents the transpoze and $\nabla$ represents the gradient operator.*

# CHAPTER 3

# STOCHASTIC OPTIMAL CONTROL PROBLEMS

## 3.1 Introduction

Optimal Control theory is a mathematical optimization methodology. It aims to find control policies for a given system which give the optimal results. Optimal control problems can be either deterministic or stochastic. In this thesis, we study the dynamic systems which evolve over time and are described by stochastic differential equations. In stochastic optimal control problems, the goal is to reach the best expected result, and for this purpose the decision makers must select an optimal decision over all possible decisions. The decision has to be non-anticipative, that is to say, the decision or control must be based only on the past and present information. Moreover, the decisions which are made based on the most updated information and no any future information must also be dynamic.

An optimal control problem consists of a state process $X \in \mathbb{R}^n$, a control process $u = u(t, w) \in \mathbb{U}$, $w \in \Omega$ for a given set $\mathbb{U} \subset \mathbb{R}^n$, and a performance functional $J(u)$.

Suppose the state of a stochastic process $X(t) = X^u(t)$ at time $t$ with an initial value $x$ is governed by an SDE:

$$\begin{aligned} dX(t) =& b\left(t, X(t), u(t)\right)\ dt + \sigma\left(t, X(t), u(t)\right)\ dW(t) \\ &+ \int_{\mathbb{R}^n} h\left(t, X(t-), u(t-), z\right)\ \tilde{M}(dt, dz), \end{aligned} \tag{3.1}$$

where $b : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{U} \to \mathbb{R}^n$, $\sigma : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{U} \to \mathbb{R}^{n\times d}$, and $h : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{U} \times \mathbb{R}^n \to \mathbb{R}^{n\times l}$ are given functions, $W(t) = W$ is a $d$-dimensional standard Brownian Motion, and

$$\begin{aligned} \tilde{M}(dt, dz) &= \left(\tilde{M}_1(dt, dz), \ldots.., \tilde{M}_l(dt, dz)\right)^T \\ &= \left(M_1(dt, dz) - \nu_1(dz_1)dt, \ldots., M_l(dt, dz) - \nu_l(dz_l)dt\right)^T \end{aligned}$$

is a compensated Poisson process where $\{M_i\}$ are independent $\mathbb{R}^{l\times 1}$-valued Poisson random measures with Lévy measures $\nu_i$ on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ for $i = 1, ..., l$.

Here, $u(t)$ is our *control process* which represents the value of the control at time $t$, and we assume that it is RCLL ($c\acute{a}dl\acute{a}g$) and adapted, i.e., progressively measurable-valued in $\mathbb{U}$. From now on, $u$ stands for the control variable, and we call $X(t) = X^u(t)$, $t \in [0, T]$, as a controlled stochastic process.

We define a performance criterion which is called as cost functional in minimization problems, and gain functional in maximization problems as follows

$$J(t, x, u) = \mathbb{E}\left[\int_t^T f\left(s, X^u(s), u(s)\right)\, ds + g\left(X^u(T), u(T)\right)\right], \tag{3.2}$$

where $T$ is the terminal time, $f : [0, T] \times \mathbb{R}^n \times \mathbb{U} \to \mathbb{R}$ is a continuous function, and $g : \mathbb{R}^n \to \mathbb{R}$ is a function which is lower bounded and satisfying quadratic growth condition.

We call the control process $u$ as *admissible process* if Eqn. (3.1) has a unique, strong solution, and the condition below is satisfied:

$$\mathbb{E}\left[\int_0^T |f(t, X(t), u(t))| dt + \max\{0, g(X(T))\}\right] < \infty. \tag{3.3}$$

We will denote by $\mathbb{A}$ for the set of all admissible control processes.

In stochastic optimization problems, the aim is to maximize the performance criterion $J(t, x, u)$ given in Eqn. (3.2) over all admissible controls. Therefore, a stochastic optimization problem can be typically written in the form of

$$V(t, x) = \sup_{u \in \mathbb{A}} \mathbb{E}\left[\int_t^T f\left(s, X^u(s), u(s)\right)\, ds + g\left(X^u(T), u(T)\right)\right],$$

where $V$ is called *value function*. The value function is obtained by choosing the best control $\tilde{u}$ among all controls, and $\tilde{u}$ is called as an *optimal control process*.

In other words, we can consider the stochastic optimal control problem as to find $\tilde{u} \in \mathbb{A}$ such that

$$J(t, x, \tilde{u}) = \sup_{u \in \mathbb{A}} J(t, x, u) = V(t, x).$$

Here, the optimization problem is a maximization problem where $f$ and $g$ are utility functions that give a measure of satisfaction. Furthermore, $f$ represents the total utility between time $[t, T]$ and is called as running gain while $g$ represents the remaining utility by time $T$ and is called as terminal gain.

If we consider a minimization problem instead of a maximization problem, we call the performance functional as a cost functional, $f$ represents running cost, while $g$ represents terminal cost, and the stochastic optimal control problem is written in the form of

$$V(t, x) = \inf_{u \in \mathbb{A}} \mathbb{E}\left[\int_t^T f(s, X^u(s), u(s))\, ds + g(X^u(T), u(T))\right].$$

An optimal control problem is called as Bolza-type problem if $f \neq 0$ and $g \neq 0$. If $f = 0$, the problem is called as Mayer problem, and if the $g = 0$, we call the problem as Lagrange type.

This chapter is organized as follows. In the subsequent two sections, two principal and most commonly used methods in solving stochastic optimal control problems, namely, Pontryagin's Maximum Principle and Bellman's Dynamic Programming Principle, will be introduced with their applications to finance. In the last section, the relationship between these two approaches will be discussed.

### 3.2 Maximum Principle

In this section, we will study how to solve a stochastic optimal control problem by the Maximum Principle approach. In the 1950s, the Maximum Principle for deterministic problems was first derived by Pontryagin and his group. Then, Kushner [13] introduced the Necessary Stochastic Maximum Principle for diffusions. Following Kushner's studies, necessary conditions for stochastic Maximum Principle were developed by Bismut [3], Bensoussan [2], Haussmann [12], Peng [20], and Young and Zhou [30]. A Necessary Maximum Principle for the jump-diffusions was given by Tang and Li [27]. The sufficient conditions for the Stochastic Maximum Principle was first introduced by Bismut in 1978, and developed by Zhou [32]. A Sufficient Maximum Principle for general jump-diffusion processes was formulated by Framstad et al. [9].

We present here the sufficient Maximum Principle for jump-diffusion processes by following closely Framstad et al. [9]. We introduce the notion of a stochastic Hamiltonian system that consists of two backward stochastic differential equations (which can also be called as adjoint equations) and one forward stochastic differential equation (the original state equation) along with a maximum condition. The Maximum Principle says that any optimal control must solve the Hamiltonian system, and that is the importance of Maximum Principle because optimizing the Hamiltonian is much more easy than the original control problem which is infinite dimensional. Moreover, we will see that the Dynamic Programming techniques are applicable only if the system is Markovian. The advantage of using Maximum Principle lies in the fact that the Maximum Principle techniques is also applicable for the non-Markovian systems.

We introduce here a Verification Theorem (the Sufficient Maximum Principle) which says when a stochastic control satisfies the optimality conditions, then it is optimal. In general jump-diffusion problems, a Verification Theorem based on Dynamic Programming Principle involves a Partial-Integro Differential Equation (PIDE) in the HJB equation which is challenging to solve. Here, the principle significance of the sufficient Maximum Principle is that it is a useful alternative to the verification theorem based on DPP.

#### 3.2.1 Sufficient Maximum Principle

Let $X(t) = X^u(t)$ be a controlled jump-diffusion process on $\mathbb{R}^n$ given in Eqn. (3.1), $u(t) = u(t,w) : [0,T] \times \Omega \to \mathbb{U}$ is the control process which is predictable and $c\acute{a}dl\acute{a}g$.

Consider the performance functional $J(u)$ of the form

$$J(u) = \mathbb{E}\left[\int_0^T f\left(t, X(t), u(t)\right)\, dt + g\left(X(T)\right)\right],$$

where $u \in \mathbb{A}$, $T > 0$ is a fixed constant, $f : [0,T] \times \mathbb{R}^n \times \mathbb{U} \to \mathbb{R}$ is continuous and $g : \mathbb{R}^n \to \mathbb{R}$ is concave.

Recall that the objective is to maximize the value function $J$ over all admissible controls. Therefore, the problem is to find $u^* \in \mathbb{A}$ which satisfies the following equation

$$J(u^*) = \sup_{u\in\mathbb{A}} J(u).$$

Set the **Hamiltonian function**, $H : [0,T] \times \mathbb{R}^n \times \mathbb{U} \times \mathbb{R}^n \times \mathbb{R}^{n\times m} \times R \to \mathbb{R}$, by

$$\begin{aligned} H(t,x,u,q_1,q_2,q_3) =& f(t,x,u) + b^T(t,x,u)q_1 + tr\left(\sigma^T(t,x,u)q_2\right) \\ &+ \sum_{i=1}^{l}\sum_{j=1}^{n}\int_{\mathbb{R}^n} h_{ij}(t,x,u,z_j)q_{3ij}(t,z)\nu_j(dz_j). \end{aligned} \tag{3.4}$$

The *adjoint equation* in the adapted adjoint processes $q_1, q_2, q_3$ is defined as

$$\begin{aligned} dq_1(t) =& -\nabla_x H\left(t, X(t), u(t), q_1(t), q_2(t), q_3(t,\cdot)\right)\, dt + q_2(t)dW(t) \\ &+ \int_{\mathbb{R}^n} q_3(t-,z)\, \tilde{M}(dt,dz) \end{aligned} \tag{3.5}$$

with boundary condition

$$q_1(T) = \nabla g\left(X(T)\right). \tag{3.6}$$

The adjoint equation above is also called as backward stochastic differential equation since we know the terminal value.

**Theorem 3.1** (**Sufficient Maximum Principle**). *(Framstad et al., [9])*

Let $\left(\tilde{u}(t), X^{\tilde{u}}(t)\right)$ be an admissible pair with corresponding solutions $\tilde{q}_1(t), \tilde{q}_2(t), \tilde{q}_3(t)$, $\tilde{q}_4(t,z)$ of the corresponding adjoint equation, and assume that the growth condition is satisfied, $g$ is a concave function of $x$ and that

$$\begin{aligned} &\tilde{H}\left(t, \tilde{X}(t), \tilde{u}(t), \tilde{q}_1(t), \tilde{q}_2(t), \tilde{q}_3(t), \tilde{q}_4(t,z)\right) \\ &\quad = \max_{u\in\mathbb{A}} H\left(t, X(t), u, \tilde{q}_1(t), \tilde{q}_2(t), \tilde{q}_3(t), \tilde{q}_4(t,z)\right) \end{aligned} \tag{3.7}$$

exists and is concave. Moreover, suppose that

$$\begin{aligned} &\tilde{H}\left(t, \tilde{X}(t), \tilde{u}(t), \tilde{q}_1(t), \tilde{q}_2(t), \tilde{q}_3(t), \tilde{q}_4(t,z)\right) \\ &\quad = \sup_{u\in A} H\left(t, \tilde{X}(t, u, \tilde{q}_1(t), \tilde{q}_2(t), \tilde{q}_3(t), \tilde{q}_4(t,z)\right) \end{aligned} \tag{3.8}$$

for all $t \in [0,T]$. Then $\tilde{u}$ is an optimal control.

*Proof.* See Framstad et al. [9] for the details of the proof. □

### 3.2.2 Applications to Finance

Now, we will apply Maximum Principle approach to Mean-Variance portfolio selection problem taken from Framstad et al. [9].

This problem is an application of the stochastic optimization problems to finance. We consider a financial market which consists of a risk-free asset and a risky asset, where the price dynamics at time $t$ are given by, respectively:

$$dS_0(t) = r(t)S_0(t)\,dt, \quad S_0(0) = s_0 > 0,$$

$$dS(t) = \mu(t)S(t)\,dt + \sigma(t)S(t)\,dW(t) + S(t-)\int_{\mathbb{R}} \gamma(t,z)\,\tilde{M}(dt,dz), \quad S_1(0) = s_1 > 0,$$

where $\mu(t) > r(t) > 0$, $\mu(t) \neq 0$ (mean rate of return), $\sigma(t) \neq 0$, and $h(t,z) > -1$ are locally bounded deterministic functions, $\tilde{M}$ is a compensated random measure with the assumption that $t \mapsto \int_{\mathbb{R}} h^2(t,z)\nu(dz)$ is a locally bounded function.

We also consider a predictable and *cádlág* portfolio such as $\theta(t) = (\theta_0(t), \theta_1(t))$, where $\theta_0(t)$ and $\theta_1(t)$ represent the number of units for the risk-free and the risky asset at time $t$, respectively.

We call this portfolio as self-financing if

$$dX(t) = \theta_0(t)dS_0(t) + \theta_1(t)dS_1(t). \tag{3.9}$$

Let $\pi(t) := \theta_1(t)S_1(t)$ denote the amount of the risky-asset at time $t$, therefore, we can express the amount of the risk-free asset at time $t$ as $X(t) - \pi(t)$. Then, we can write the wealth process in Eqn. (3.9) as

$$\begin{aligned} dX(t) =& \{r(t)X(t) + (\mu(t) - r(t))\pi(t)\}\,dt + \sigma(t)\pi(t)\,dW(t) \\ &+ \pi(t-)\int_{\mathbb{R}} h(t,z)\,\tilde{M}(dt,dz). \end{aligned} \tag{3.10}$$

Here, $u(t) = \pi(t)$ is our control process, and we call $u(t)$ admissible, i.e., $u(t) \in \mathbb{A}$, if Eqn. (3.9) has a unique solution with the assumption that $\mathbb{E}[(X^u(T))^2] < \infty$.

**Stochastic Optimal Control Problem:**

In this mean-variance portfolio selection problem, the goal is to find an admissible control $u(t)$ which minimizes the variance

$$\text{Var}[X(T)] = \mathbb{E}\left[(X(T) - \mathbb{E}[X(T)])^2\right]$$

providing that

$$\mathbb{E}[X(T)] = A,$$

where $A$ is a given constant.

**Proposition 3.2.** *(Framstad et al., [9]) Consider the wealth process in Eqn. (3.10). The optimal control policy which minimizes the variance is given by*

$$\tilde{u}(t) = \frac{(r(t) - \mu(t))(m(t)x + n(t)}{m(t)\gamma(t)}.$$

*Proof.* Using the Lagrange multiplier method, this problem is written as minimize

$$\mathbb{E}[(X(T) - a)^2],$$

for a given real number $a \in \mathbb{R}$, without any constraint. This is because

$$\mathbb{E}\left[(X(T) - A)^2 - \lambda\left(\mathbb{E}[X(T)] - A\right)\right] = \mathbb{E}\left[\left(X(T) - \left(A + \frac{\lambda}{2}\right)\right)^2\right] - \frac{\lambda^2}{4}, \tag{3.11}$$

where $\lambda \in \mathbb{R}$ is a constant and called as the Lagrange multiplier.

Therefore, instead of (3.11), we can consider the following equivalent optimization problem

$$\sup_{u \in \mathbb{A}} \mathbb{E}\left[-\frac{1}{2}\left(X^u(T) - a\right)^2\right].$$

Combining Eqns. (3.4) and (3.10), we can write the corresponding Hamiltonian function as

$$H(t, x, u, q_1, q_2, q_3) = \{r(t)x + (\mu(t) - r(t))u\}q_1 + \sigma(t)uq_2 + u\int_{\mathbb{R}} h(t, z)q_3(t, z)\nu(dz). \tag{3.12}$$

Besides, combining Eqns. (3.5) and (3.20), the corresponding adjoint equations are

$$\begin{aligned} dq_1(t) &= -\, r(t)q_1(t)\, dt + q_2(t)\, dW(t) + \int_{\mathbb{R}} q_3(t, z)\, \tilde{M}(dt, dz), \\ q_1(T) &= -\, (X(T) - a) = -X(T) + a. \end{aligned} \tag{3.13}$$

Now, we make a guess for $q_1(t)$ :

$$q_1(t) = m(t)X(t) + n(t), \tag{3.14}$$

where $m(t)$ and $n(t)$ are deterministic and differentiable funtions.

Now, we differentiate Eqn. (3.14) with respect to $t$, and get the result

$$dq_1(t) = m(t)\, dX(t) + m'(t)X(t)\, dt + n'(t)\, dt. \tag{3.15}$$

Combining Eqns. (3.15) with (3.10), we obtain

$$dq_1(t) = m(t)\Big[\{r(t)X(t) + (\mu(t) - r(t))\pi(t)\}\, dt + \sigma(t)\pi(t)\, dW(t)$$

$$+ \pi(t-) \int_{\mathbb{R}} h(t,z)\, \tilde{M}(dt,dz)\Big] + m'(t)X(t)\, dt + n'(t)\, dt$$

$$= \Big[m(t)r(t)X(t) + m(t)(\mu(t) - r(t)u(t) + X(t)m'(t) + n'(t)\Big]\, dt$$

$$+ m(t)\sigma(t)u(t)\, dW(t) + m(t)u(t-) \int_{\mathbb{R}} h(t,z)\, \tilde{M}(dt,dz). \tag{3.16}$$

Comparing Eqn. (3.16) with Eqn. (3.13), we get

$$\begin{aligned} dq_1(t) &= -r(t)q_1(t) = -r(t)\,(m(t)X(t) + n(t)) \\ &= m(t)r(t)X(t) + m(t)(\mu(t) - r(t)u(t) + X(t)m'(t) + n'(t), \end{aligned} \tag{3.17}$$

$$q_2(t) = m(t)\sigma(t)u(t), \tag{3.18}$$

$$q_3(t,z) = m(t)u(t)h(t,z). \tag{3.19}$$

With the assumption $\tilde{u} \in \mathbb{A}$ be an optimal control with corresponding wealth $\tilde{X}$, and corresponding adjoint variables $\tilde{q}_1, \tilde{q}_2, \tilde{q}_3$ , we have

$$\begin{aligned} &H\left(t, \tilde{X}(t), u, \tilde{q}_1(t), \tilde{q}_2(t), \tilde{q}_3(t,\cdot)\right) = r(t)\tilde{X}(t)\tilde{q}_1(t) \\ &\quad + u\left[(\mu(t) - r(t))\,\tilde{q}_1(t) + \sigma(t)\tilde{q}_2(t) + \int_{\mathbb{R}} h(t,z)\tilde{q}_3(t,z)\nu(dz)\right]. \end{aligned} \tag{3.20}$$

Then, from first-order conditions we have

$$\frac{\partial \tilde{H}}{\partial \tilde{u}(t)} = (\mu(t) - r(t))\,\tilde{q}_1(t) + \sigma(t)\tilde{q}_2(t) + \int_{\mathbb{R}} h(t,z)\tilde{q}_3(t,z)\nu(dz) = 0. \tag{3.21}$$

Substituting Eqns. (3.18) and (3.19) into Eqn. (3.21) we can write it as

$$\tilde{u}(t) = \frac{(r(t) - \mu(t))\tilde{q}_1(t)}{m(t)\gamma(t)}, \tag{3.22}$$

where

$$\gamma(t) = \sigma^2(t) + \int_{\mathbb{R}} h^2(t,z)\nu(dz). \tag{3.23}$$

Besides, from Eqn. (3.17) we have

$$\tilde{u}(t) = \frac{(m(t)r(t) + m'(t))\tilde{X}(t) + r(t)(m(t)\tilde{X}(t) + n(t)) + n'(t)}{m(t)(r(t) - \mu(t)}. \tag{3.24}$$

Connecting Eqn. (3.22) and Eqn. (3.24) yields the following equations:

$$(r(t)-\mu(t))^2\, m(t) - \left[2r(t)m(t) + m'(t)\right]\gamma(t) = 0, \quad m(T) = -1,$$

$$(r(t)-\mu(t))^2\, n(t) - \left[r(t)n(t) + n'(t)\right]\gamma(t) = 0, \quad n(T) = a.$$

If we solve these equations, we get

$$m(t) = -\exp\left(\int_t^T \frac{(r(s)-\mu(s))^2}{\gamma(s)} - 2r(s)\, ds\right), \quad 0 \le t \le T, \tag{3.25}$$

$$n(t) = a\exp\left(\int_t^T \frac{(r(s)-\mu(s))^2}{\gamma(s)} - r(s)\, ds\right), \quad 0 \le t \le T. \tag{3.26}$$

Subtracting (3.25) and (3.26) to Eqns. (3.17), (3.18) and (3.19) , the adjoint processes solves the Eqn. (3.16), and all conditions of Theorem 3.1 are satisfied. Therefore,

$$\tilde{u}(t) = \frac{(r(t)-\mu(t))(m(t)x + n(t)}{m(t)\gamma(t)} \tag{3.27}$$

is an optimal control. □

### 3.3 Dynamic Programming Principle and Hamilton-Jacobi-Bellman Equations

In this section, we review the theory of Dynamic Programming Principle which is another fundamental methodology to solve the stochastic optimal control problems. Dynamic Programming Principle was initiated by Richard Bellman in the 1950s, and this methodology results in a necessary condition and as well as a sufficient condition for optimality. For discrete-time optimization problems Bellman equation refers to a Dynamic Programming equation, while for continuous-time optimization problems it refers to a nonlinear and second-order PDE, the so-called Hamilton-Jacobi-Bellman (HJB) equation.

In this section, we will first derive the HJB equation in a heuristic manner for diffusion processes, and then for jump-diffusion processes. When the HJB equation is solvable, optimality of the candidate solution, namely, the value function that satisfies the HJB, is proved with the Verification Theorem. In the Verification Theorem, it is required that the solution of the HJB equation must be smooth enough which is not the case in general, and this is the main drawback of Dynamic Programming principle. To overcome this problem, viscosity solutions are used. In this thesis, we will not cover the viscosity solutions and we refer to Pham [21], Yong and Zhou [30], and Fleming and Soner [8] for details of viscosity solutions. In the applications, we will solve the Merton's portfolio problem for optimal consumption first under a diffusion process and then under a jump-diffusion process, for a logarithmic utility function. The aim of starting with a diffusion process is to see the essential differences with the jump-diffusion process.

Consider a control system which is driven by following SDE:

$$dX(t) = b\left(t, X(t), u(t)\right)\; dt + \sigma(t, X(t), u(t))\, dW(t). \tag{3.28}$$

Here $W$ is a $d$-dimensional Brownian motion on $(\Omega, \mathbb{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, $t \in [0,T]$, where $T > 0$ is constant, $b : [0,T] \times \mathbb{R}^n \times \mathbb{U} \to \mathbb{R}^n$ and $\sigma : [0,T] \times \mathbb{R}^n \times \mathbb{U} \to \mathbb{R}^{n\times d}$ are given deterministic and continuous functions satisfying Lipschitz continuity and linear growth conditions; hence, a unique $L^2-$solution to Eqn. (3.28) exists.

Here, $X(t) \in \mathbb{R}^n$ is the state process that represents the wealth at time $t$, and Eqn. (3.28) will be the given constraint of optimization problem. Moreover, $X(t)$ is controlled by a stochastic process $u(t)$ as mentioned in the introduction of this chapter. We assume that $u(t)$ is $c\acute{a}dl\acute{a}g$ and predictable, that means, the optimal control at time $t$ depends on the information at time $t$.

**Definition 3.1. (Markovian Control)**

Let $X^{s,x}$ be the state process with initial value $X(s) = x$. A control process $u(t)$, $t \in [s,T]$, is called *Markovian control* if $u(t) = a(t, X^{s,x}(t))$ for some measurable function
$a : [0,T] \times \mathbb{R}^n \to \mathbb{A}$.

In the remainder of this section we only consider Markovian controls.

**Theorem 3.3.** ***(Dynamic Programming Principle)***

Let $(t,x) \in [0,T] \times \mathbb{R}^n$. Then, for $\theta \in [t,T]$, we have

$$V(t,x) = \sup_{u\in\mathbb{A},\theta\in[t,T]} \mathbb{E}\left[\int_t^\theta f(s, X^{t,x}(s), u(s))ds + V\left(\theta, X^{t,x}(\theta)\right)\right]. \tag{3.29}$$

*Proof.* By definition of the value function, for any $\theta \in [t,T]$, we have

$$J(t,x,u) = \mathbb{E}\left[\int_t^\theta f(s, X^{t,x}(s), u(s))ds + J\left(\theta, X^{t,x}(\theta), u\right)\right].$$

Then,

$$J(t,x,u) \leq \mathbb{E}\left[\int_t^\theta f(s, X^{t,x}(s), u(s))ds + V\left(\theta, X^{t,x}(\theta), u\right)\right].$$

By taking the supremum at both sides,

$$V(t,x) \leq \sup_{u\in\mathbb{A},\theta\in[t,T]} \mathbb{E}\left[\int_t^\theta f(s, X^{t,x}(s), u(s))ds + V\left(\theta, X^{t,x}(\theta), u\right)\right]. \tag{3.30}$$

For the other side of the proof, we define the process

$$\hat{u}(s,w) = \begin{cases} u(s,w), s \in [t,\theta], \\ \tilde{u}(s,w), s \in [\theta, T], \end{cases}$$

where $\tilde{u}(s,w)$ is the optimal control. Then, we have

$$V(t,x) = J(t,x,\tilde{u}) \geq J(t,x,\hat{u}) = \mathbb{E}\left[\int_t^T f(s, X^{t,x}(s), \hat{u}(s))ds + J\left(T, X^{t,x}(T), \hat{u}\right)\right]$$

$$=\mathbb{E}\left[\int_t^\theta f(s,X^{t,x}(s),\hat{u}(s))ds\right]+\mathbb{E}\left[\int_\theta^T f(s,X^{t,x}(s),\hat{u}(s))ds+J\left(T,X^{t,x}(T),\hat{u}\right)\right]$$
$$=\mathbb{E}\left[\int_t^\theta f(s,X^{t,x}(s),\hat{u}(s))ds\right]+V\left(\theta,X^{t,x}(\theta),u\right),$$

which implies

$$V(t,x)\geq \sup_{u\in\mathbb{A},\theta\in[t,T]}\mathbb{E}\left[\int_t^\theta f(s,X^{t,x}(s),u(s))\,ds+V\left(\theta,X^{t,x}(\theta),u\right)\right]. \qquad (3.31)$$

Thus, from Eqn. (3.30) and Eqn. (3.31) the desired result is obtained. □

If we investigate the local behaviour of the value function, when $\theta\to t$ in Theorem 3.2 leads to the HJB equation which is the infinitesimal version of the Dynamic Programming Principle.

**Theorem 3.4.** ***(Hamilton-Jacobi-Bellman equation)***

Assume that $V\in C^{1,2}$ and there exists an optimal control $\tilde{u}$ such that for any $(t,x)\in[0,T]\times\mathbb{R}^n$,

$$J\left(t,x,\tilde{u}(\cdot)\right)=V(t,x).$$

Then, the value function $V$ satisfies the HJB equation

$$\frac{\partial V}{\partial t}(t,x)+\sup_{u\in\mathbb{A}}\left[\mathcal{L}^uV(t,x)+f(t,x,u)\right]=0, \qquad (3.32)$$

with terminal condition

$$V(T,x)=g(x),$$

where

$$\mathcal{L}^uV(t,x)=b(x,u)\,\frac{\partial V}{\partial x}+\frac{1}{2}tr(\sigma\sigma^T)(x,u)\,\frac{\partial^2 V}{\partial x^2}$$

is the infinitesimal generator of a diffusion process.

Furthermore, for each $(t,x)\in[0,T]\times\mathbb{R}^n$, the supremum in the HJB equation of Eqn. (3.32) is attained by the optimal control $\tilde{u}(t,x)$.

*Proof.* Let us choose $\theta$ in Theorem 3.2 as $\theta=t+\delta t$, where $\delta t$ is a small time increment and $t+\delta t<T$.

By assuming that $V$ is smooth enough and applying Itô Formula to $V$ between $t$ and

$t + \delta t$, we get

$$V\left(t+\delta t, X^{t,x}(t+\delta t)\right) = V(t,x) + \int_t^{t+\delta t} \frac{\partial V}{\partial t}(s, X^{t,x}(s))\, ds$$
$$+ \int_t^{t+\delta t} \frac{\partial V}{\partial x}(s, X^{t,x}(s))\, dX(s)$$
$$+ \frac{1}{2}\int_t^{t+\delta t} \frac{\partial^2 V}{\partial x^2}(s, X^{t,x}(s))\, [X,X](s),$$
$$= V(t,x) + \int_t^{t+\delta t} \frac{\partial V}{\partial t}(s, X^{t,x}(s))\, ds$$
$$+ b\int_t^{t+\delta t} \frac{\partial V}{\partial x}(s, X^{t,x}(s))\, ds$$
$$+ \sigma\int_t^{t+\delta t} \frac{\partial V}{\partial x}(s, X^{t,x}(s))\, dW(s)$$
$$+ \frac{1}{2}\int_t^{t+\delta t} \frac{\partial^2 V}{\partial x^2}(s, X^{t,x}(s))\, ds.$$

Then, we obtain

$$V\left(t+\delta t, X^{t,x}(t+\delta t)\right) = V(t,x) + \int_t^{t+\delta t} \frac{\partial V}{\partial t}(s, X^{t,x}(s))\, ds$$
$$+ \int_t^{t+\delta t} \mathcal{L}^u V(s, X^{t,x}(s))\, ds$$
$$+ \sigma\int_t^{t+\delta t} \frac{\partial V}{\partial x}(s, X^{t,x}(s))\, dW(s). \tag{3.33}$$

We already know that

$$V(t,x) \geq \mathbb{E}\left[\int_t^{t+\delta t} f\left(s, X^{t,x}(s), u(s)\right)\, ds + V\left(t+\delta t, X^{t,x}(t+\delta t)\right)\right]. \tag{3.34}$$

Additionally, since the expected value of a Brownian Motion is 0, we have

$$\mathbb{E}\left[\sigma\int_t^{t+\delta t} \frac{\partial V}{\partial x}(s, X^{t,x}(s))\, dWs\right] = 0. \tag{3.35}$$

By taking expectation of Eqn. (3.33) and combining it with Eqn. (3.34) and Eqn. (3.35), we get

$$\mathbb{E}\left[\int_t^{t+\delta t}\left(f(s, X^{t,x}(s), u) + \frac{\partial V}{\partial t}(s, X^{t,x}(s)) + \mathcal{L}^u V(s, X^{t,x}(s))\right) ds\right] \leq 0. \tag{3.36}$$

Dividing Eqn. (3.36) by $t + \delta t$ and letting $t + \delta t \to 0$, finally we obtain by the mean value theorem that

$$f(t,x,u) + \frac{\partial V}{\partial t}(t,x) + \mathcal{L}^u V(t,x) \leq 0. \tag{3.37}$$

Since Eqn. (3.37) holds for any control process $u$, we have

$$\sup_{u \in \mathbb{A}} \left[ f(s, X^{t,x}(s), u) + \mathcal{L}^u V(s, X^{t,x}(s)) \right] + \frac{\partial V}{\partial t}(s, X^{t,x}(s)) \leq 0. \tag{3.38}$$

By assumption, we know that

$$J\left(t, x, \tilde{u}(\cdot)\right) = V(t, x) = \mathbb{E}\left[ \int_t^{t+\delta t} f\left(s, \tilde{X}^{t,x}(s), \tilde{u}(s)\right) \, ds + V\left(t + \delta t, \tilde{X}^{t,x}(t + \delta t)\right) \right].$$

Applying the same arguments as above, for an optimal control $\tilde{u}$ we have

$$f(t, x, \tilde{u}) + \frac{\partial V}{\partial t}(t, x) + \mathcal{L}^{\tilde{u}} V(t, x) = 0. \tag{3.39}$$

Thus, if we combine Eqn. (3.38) and Eqn. (3.39), then it is seen that the supremum in Eqn. (3.32) is attained by the optimal control $\tilde{u}(t, x)$ and $V$ satisfies

$$\sup_{u \in \mathbb{A}} \left[ f(s, X^{t,x}(s), u) + \mathcal{L}^u V(s, X^{t,x}(s)) \right] + \frac{\partial V}{\partial t}(s, X^{t,x}(s)) = 0.$$

□

Interpretation of the HJB equation is that if $V$ is the value function and the optimal control $\tilde{u}$ exists, then $V$ satisfies the HJB equation. Moreover, the supremum in the HJB equation is attained by the optimal control $\tilde{u}$. Indeed, this means that the theorem is a necessary condition for optimality.

On the other hand, the HJB equation is also provided in a form of sufficient condition. This means that if a smooth solution to the HJB equation is given, indeed, the solution is equal to the optimal solution. This validates the optimality of the given solution and is known as the Verification Theorem. Now, we will state the Verification theorem, and then prove it.

**Theorem 3.5.** ***(Verification Theorem)***

Let $H(t, x)$, $t \in [0, T]$, $x \in \mathbb{R}$, be a function such that $H \in C^{1,2}$ satisfies quadratic growth condition and solve the HJB equation

$$\frac{\partial H}{\partial t}(t, x) + \sup_{u \in \mathbb{A}} \left[ L^u H(t, x) + f(t, x, u) \right] = 0 \tag{3.40}$$

with boundary condition

$$H(T, x) = g(x).$$

Let the supremum in Eqn. (3.40) be attained by an admissible control process $\hat{u}$.

Then, there exists an optimal control $\tilde{u}$ such that $\tilde{u} = \hat{u}$, and the function $H$ is equal to the optimal value function, i.e.,

$$H(t, x) = V(t, x).$$

*Proof.* We know that $\hat{u} \in \mathbb{A}$ and the supremum in Eqn. (3.40) is attained by $\hat{u}$. For any control process $u$, choose a point $(t,x)$ and apply Itô Formula to $H(T, X^{t,x}(T))$.

Then, we have

$$\begin{aligned} H\left(T, X^{t,x}(T)\right) =& H(t,x) + \int_t^T \frac{\partial H}{\partial t}(s, X^{t,x}(s))\, ds + \int_t^T \frac{\partial H}{\partial x}(s, X^{t,x}(s))\, dX(s) \\ &+ \frac{1}{2}\int_t^T \frac{\partial^2 H}{\partial x^2}(s, X^{t,x}(s))\, [X,X]\,(s), \\ =& H(t,x) + \int_t^T \frac{\partial H}{\partial t}(s, X^{t,x}(s))\, ds + b\int_t^T \frac{\partial H}{\partial x}(s, X^{t,x}(s))\, ds \\ &+ \sigma \int_t^T \frac{\partial H}{\partial x}(s, X^{t,x}(s))\, dW(s) + \frac{1}{2}\int_t^T \frac{\partial^2 H}{\partial x^2}(s, X^{t,x}(s))\, ds, \end{aligned}$$

which yields

$$\begin{aligned} H\left(T, X^{t,x}(T)\right) =& H(t,x) + \int_t^T \frac{\partial H}{\partial t}(s, X^{t,x}(s))\, ds + \int_t^T \mathcal{L}^u H(s, X^{t,x}(s))\, ds \\ &+ \sigma \int_t^T \frac{\partial H}{\partial x}(s, X^{t,x}(s))\, dW(s). \end{aligned} \tag{3.41}$$

Since $H$ solves the HJB equation (3.40), for any feasible control process $u$, we also know that

$$\frac{\partial H}{\partial t}(t,x) + \mathcal{L}^u H(t,x) + f(t,x,u) \leq 0. \tag{3.42}$$

Eqn. (3.42) implies that

$$\frac{\partial H}{\partial t}(t,x) + \mathcal{L}^u H(t,x) \leq -f(t,x,u), \tag{3.43}$$

and associating Eqn. (3.41) and Eqn. (3.43), we obtain

$$H\left(T, X^{t,x}(T)\right) = H(t,x) + \int_t^T -f(t,x,u)\, ds + \sigma\int_t^T \frac{\partial H}{\partial x}(s, X^{t,x}(s))\, dW(s). \tag{3.44}$$

We have $H(T, X(T)) = g(X(T))$ from the boundary condition. Moreover, since the expected value of Brownian Motion is 0, we have

$$\mathbb{E}\left[\int_t^T \frac{\partial H}{\partial x}(s, X^{t,x}(s))\sigma dW(s)\right] = 0.$$

Finally, we obtain

$$H(t,x) \geq \mathbb{E}\left[\int_t^T f(t,x,u) + g(X(T))\right] = J(t,x,u).$$

Hence,

$$H(t,x) \geq \sup_{u\in\mathbb{A}} J(t,x,u) = V(t,x). \tag{3.45}$$

The proof will be completed by showing that $H(t,x) \leq V(t,x)$.

By assumption, for the control process $\hat{u}$ we have

$$\frac{\partial H}{\partial t}(t,x) + \sup_{u\in\mathbb{A}}[\mathcal{L}^u H(t,x) + f(t,x,u)] = \frac{\partial H}{\partial t}(t,x) + \mathcal{L}^{\hat{u}} H(t,x) + f(t,x,\hat{u}) = 0$$

$$\Rightarrow \frac{\partial H}{\partial t}(t,x) + \mathcal{L}^{\hat{u}} H(t,x) = -f(t,x,\hat{u}). \tag{3.46}$$

Applying Itô Formula to $H\left(T, X^{t,x}(T)\right)$ and from similar calculations, the desired result will be obtained.

Similarly, we have

$$\begin{aligned} H\left(T, X^{t,x}(T)\right) =& H(t,x) + \int_t^T \frac{\partial H}{\partial t}(s, X^{t,x}(s))\, ds + \int_t^T \mathcal{L}^{\hat{u}} H(s, X^{t,x}(s))\, ds \\ &+ \sigma \int_t^T \frac{\partial H}{\partial x}(s, X^{t,x}(s))\, dW(s). \end{aligned} \tag{3.47}$$

By connecting Eqn. (3.46) and Eqn. (3.47), we receive

$$\begin{aligned} H\left(T, X^{t,x}(T)\right) = g(X(T)) = H(t,x) + \int_t^T -f(t,x,\hat{u})\, ds \\ + \sigma \int_t^T \frac{\partial H}{\partial x}(s, X^{t,x}(s))\, dW(s). \end{aligned}$$

Since the expected value of Brownian component is equal to 0, taking expectation of both sides yields that

$$H(t,x) = \mathbb{E}\left[\int_t^T f(t,x,\hat{u}) + g(X_T)\right] = J(t,x,\hat{u})$$

$$\Rightarrow H(t,x) = J(t,x,\hat{u}) \leq V(t,x). \tag{3.48}$$

Therefore, by Eqn. (3.45) and Eqn. (3.48) we get

$$H(t,x) = V(t,x),$$

and $\tilde{u}$ is the optimal control process which is the desired conclusion. □

Now, we extend the results of the Verification Theorem 3.5 to the jump-diffusion case considering the wealth process of Eqn. (3.1).

**Theorem 3.6 (HJB for Optimal Control of Jump Diffusions).** *(Øksendal and Sulem, [17])*

Suppose $H \in C^2(\mathbb{R})$ satisfies the following:

$(i)$ $\mathcal{L}^u H(t,x) + f(t,x,u) \leq 0$, for all controls $u \in \mathbb{A}$, where $\mathcal{L}$ is the infinitesimal generator of a Lévy process as in Proposition 2.15.

$(ii)$ $\lim_{t \to T} H(X(t)) = g(X(T))$ a.s., for all $u \in \mathbb{A}$.

$(iii)$ $\mathbb{E}_x \left[ |H(X(T))| + \int_t^T |\mathcal{L}H(X(t))| \; dt \right] < \infty$, for all $u \in \mathbb{A}$.

$(iv)$ $(H(X(t)))_{t \leq T}$ is uniformly integrable for all $u \in \mathbb{A}$.

Then,

$$H(t,x) \geq V(t,x). \tag{3.49}$$

Moreover, suppose that there exists $u = \tilde{u}(t,x)$ such that

$$\mathcal{L}^{\tilde{u}} H(t,x) + f(t,x,\tilde{u}) = 0,$$

and $\tilde{u}$ is a Markov control. Then $\tilde{u}$ is an optimal control and

$$H(t,x) = V(t,x). \tag{3.50}$$

*Proof.* By assumption, we know that the growth condition $(iii)$ is satisfied and $(H(X(t)))_{t \leq T}$ is uniformly integrable. Then we can use the Dynkin Formula taking $f = H$. It follows that

$$\mathbb{E}_x \left[ H(X(T)) \right] = H(x) + \mathbb{E}_x \left[ \int_t^T \mathcal{L}H(X(t)) \, dt \right]. \tag{3.51}$$

Using the condition $(i)$ which is $\mathcal{L}^u H(t,x) + f(t,x,u) \leq 0$, we can rewrite the Eqn. (3.51) as

$$\mathbb{E}_x \left[ H(X(T)) \right] \leq H(x) - \mathbb{E}_x \left[ \int_t^T f(t, X(t), u(t)) \, dt \right]$$

and, consequently, we have

$$H(t,x) \geq \mathbb{E}_x \left[ \int_t^T f(t, X(t), u(t)) \, dt + g\left(X(T)\right) \right] = J(t,x,u).$$

Hence, we obtain

$$H(t,x) \geq \sup_{u \in \mathbb{A}} J(t,x,u) = V(t,x), \tag{3.52}$$

which proves the first part of the theorem.

The rest of the proof runs as before. If we apply the same argument in the first part of the proof, to $\mathcal{L}^{\tilde{u}} H(t,x) + f(t,x,\tilde{u}) = 0$, we obtain

$$H(t,x) = J(t,x,\tilde{u}) \leq V(t,x). \tag{3.53}$$

Combining the inequalities of Eqns. (3.52) and (3.53), we can assert that

$$H(t, x) = V(t, x),$$

and $\tilde{u}$ is an optimal control. $\square$

### 3.3.1 Applications to Finance

Now, we will apply Dynamic Programming Principle approach to Merton optimal investment and consumption problem under diffusion processes and jump-diffusion processes, respectively.

**Example 3.1.** (Merton Portfolio Problem for Optimal Consumption) [15]

In this application, we consider an optimal portfolio-consumption problem of an investor. Let $X(t) \geq 0$ represents the wealth of the investor at time $t$ with an initial wealth $x \geq 0$ at time $t$. He is allowed to consume for his utility and invests his savings in a financial market with two possibilities: one is riskless asset (bond) and the other one is risky asset (stock) whose price dynamics are governed by, respectively:

$$dS_0(t) = rS_0(t)dt, \quad S_0(0) = s_0 > 0,$$

$$dS_1(t) = \mu S_1(t)dt + \sigma S_1(t)dW(t), \quad S_1(0) = s_1 > 0,$$

where $r > 0$, the interest rate of the bank, $\mu > 0$, the mean rate of return with the assumption $\mu > r$, and $\sigma \in \mathbb{R}$, the volatility of the stock, are constants. Finally, $(W(t))$ is a Brownian motion on $(\Omega, \mathbb{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$.

In this problem $c(t) \geq 0$ is the consumption rate at time $t$, and it is one of the control variables. We also assume that the portfolio is self-financing, short selling is allowed and there is no transaction cost between money transfers from one asset to another one.

Let $\pi(t) \cdot X(t)$ and $(1 - \pi(t)) \cdot X(t)$ be the amounts of the risky asset and risk-free asset, respectively. Here, $\pi(t)$ is another control variable for this problem.

Therefore, we can write the wealth process as

$$\begin{aligned} dX(t) &= \pi(t)\frac{X(t)}{S_1(t)}\, dS_1(t) + r\,(1 - \pi(t))\, X(t)\, dt - c(t)\, dt \\ &= (\mu\pi(t)X(t) + r(1 - \pi(t))X(t) - c(t))\; dt + \sigma\pi(t)X(t)\, dW(t). \end{aligned} \tag{3.54}$$

The goal of this optimization problem is to find the value function $V(t, x)$ and the optimal control $\tilde{u}(t) = (\tilde{\pi}(t), \tilde{c}(t)) \in \mathbb{A}$ which maximizes the discounted utility for some constant $\rho > 0$.

So, the objective function is defined as

$$J(t, x; u) = E\left[\int_0^\infty e^{-\rho t} U(c(t))dt\right].$$

**Stochastic Optimal Control Problem:**

$$V(t,x) = \max_{u\in A} E\left[\int_0^\infty e^{-\rho t}U(c(t))dt\right] = J(t,x;\tilde{u}), \tag{3.55}$$

where $V(\cdot)$ is the value function. Here, $U(c)$ is chosen as the logarithmic utility which is a differentiable, strictly increasing and concave utility function, implying that the investor is risk averse.

**Theorem 3.7.**

Given the wealth process as in Eqn. (3.54) and the utility function $U(c) = \log c$, the optimal strategy is given by

$$\tilde{\pi} = \frac{\mu - r}{\sigma^2} \quad \text{and} \quad \tilde{c} = \rho X(t), \tag{3.56}$$

over the period $0 \leq t < \infty$.

Furthermore, the maximum utility over the period $0 \leq t < \infty$ is given by $a \log X(0) + b$, where

$$a = \frac{1}{\rho}, \quad b = \frac{1}{\rho}\left(\log\rho + \frac{r}{\rho} + \frac{(\mu-r)^2}{2\rho\sigma^2} - 1\right).$$

*Proof.* The HJB equation for this problem is

$$\begin{aligned}\frac{\partial V}{\partial t}(t,x) + \max_{u\in\mathbb{A}}[\{\pi x(\mu - r) + rx - c\}\frac{\partial V}{\partial x}(t,x) \\ + \frac{1}{2}\sigma^2\pi^2x^2\frac{\partial^2 V}{\partial x^2}(t,x) + e^{-\rho t}\log c] = 0.\end{aligned} \tag{3.57}$$

It follows from the first-order conditions that

$$\begin{aligned}e^{-\rho t}\frac{1}{c} - \frac{\partial V}{\partial x} &= 0, \\ x(\mu - r)\frac{\partial V}{\partial x} + \sigma^2\pi x^2\frac{\partial^2 V}{\partial x^2} &= 0.\end{aligned}$$

Therefore, we obtain the optimal control variables as

$$\tilde{c} = e^{-\rho t}\left(\frac{\partial V}{\partial x}\right)^{-1}, \tag{3.58}$$

$$\tilde{\pi} = \frac{\mu - r}{\sigma^2}\left(-\frac{\partial V}{\partial x}\right)\left(x\frac{\partial^2 V}{\partial x^2}\right)^{-1}. \tag{3.59}$$

We are now looking for a candidate solution $V$ of the ansatz form

$$V(x,t) = e^{-\rho t}(a\log x + b).$$

Then, we receive the following derivatives

$$\begin{aligned}\frac{\partial V}{\partial t} &= -\rho e^{-\rho t}(a\log x + b),\\ \frac{\partial V}{\partial x} &= \frac{a}{x}e^{-\rho t},\\ \frac{\partial^2 V}{\partial x^2} &= -\frac{a}{x^2}e^{-\rho t}.\end{aligned} \tag{3.60}$$

Hence, by inserting partial derivatives into Eqn. (3.58) and Eqn. (3.59), we have

$$\tilde{c} = \frac{x}{a}, \quad \tilde{\pi} = \frac{\mu - r}{\sigma^2}(-\frac{a}{x}e^{-\rho t})(-\frac{a}{x}e^{-\rho t})^{-1} = \frac{\mu - r}{\sigma^2}. \tag{3.61}$$

Now, we substitute the results in Eqn. (3.61) into the HJB equation of Eqn. (3.57) to gradually find $a$ and $b$:

$$\frac{\partial V}{\partial t} + \{\pi x(\mu - r) + rx - c\}\frac{\partial V}{\partial x} + \frac{1}{2}\sigma^2\pi^2x^2\frac{\partial^2 V}{\partial x^2} + e^{-\rho t}\log c = 0,$$

hence, by Eqns. in (3.60),

$$\begin{aligned}&-\rho e^{-\rho t}(a\log x + b) + e^{-\rho t}(-\log a + \log x) + \{x\frac{(\mu - r)^2}{\sigma^2} + rx - \frac{x}{a}\}\frac{a}{x}e^{-\rho t}\\ &-\frac{1}{2}\sigma^2\frac{(\mu - r)^2}{\sigma^4}x^2\frac{a}{x^2}e^{-\rho t} = 0,\end{aligned}$$

thus,

$$-\rho e^{-\rho t}(a\log x + b) + e^{-\rho t}(-\log a + \log x) + \left(\frac{1}{2}\frac{(\mu - r)^2}{\sigma^2} + r - \frac{1}{a}\right)ae^{-\rho t} = 0.$$

Then, we divide by $e^{-\rho t}$ and obtain

$$-\rho(a\log x + b) + (-\log a + \log x) + \left(\frac{1}{2}\frac{(\mu - r)^2}{\sigma^2} + (r - \frac{1}{a}\right)a = 0.$$

Finally, comparison of the coefficients yields the following result

$$a = \frac{1}{\rho}, \quad b = \frac{1}{\rho}\left(\log\rho + \frac{r}{\rho} + \frac{(\mu - r)^2}{2\rho\sigma^2} - 1\right).$$

□

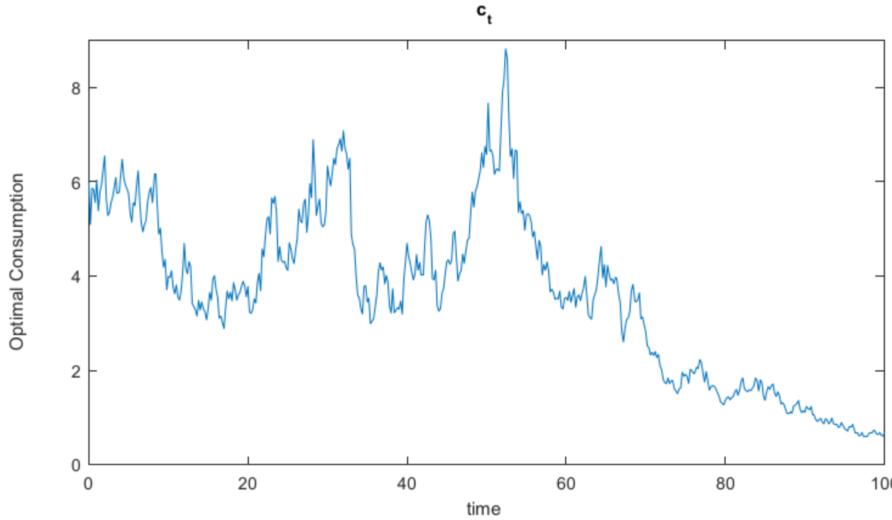


Figure 3.1: Optimal consumption for logarithmic utility.

In Figures (3.1) and (3.2) we plot the sample paths of $X$ with initial value $X(0) = 100$. We choose the parameters as $\mu = 0.1$, $r = 0.05$, $\sigma = 0.3$, $\rho = 0.06$, and $T = 100$. For these parameters, $\pi = 0.5553$ which is a constant rate proportional to $\mu - r$. The interpretation of this result is that the investor has to invest more in the risky asset for larger values of $\mu$ and more in the risk-free asset for higher interest rate $r$ and for larger volatility $\sigma$.

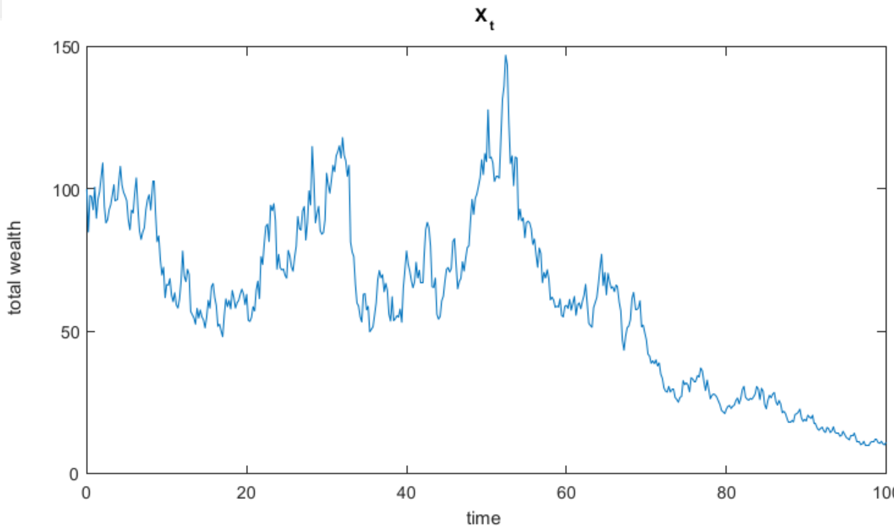


Figure 3.2: Wealth process with logarithmic utility.

Now, we will consider the same problem under jump-diffusion processes.

**Example 3.2.** (Merton Portfolio Problem for Optimal Consumption under Jump-Diffusion Process)

As we said earlier, since sudden changes in price movements can not be explained by diffusion models, jump-diffusion processes are more realistic for description of price movements, and now we will solve the above application under a jump-diffusion process. In this problem, again an investor has two investment opportunities which are risk-free and risky assets. The price dynamics of risk-free and risky assets are given below, respectively:

$$dS_0(t) = rS_0(t)\,dt, \quad S_0(0) = s_0 > 0, \tag{3.62}$$

$$dS(t) = \mu S(t)\,dt + \sigma S(t)\,dW(t) + S(t)\int_{\mathbb{R}} \gamma(t,z)\,\tilde{M}(dt,dz), \quad S_1(0) = s_1 > 0, \tag{3.63}$$

where $r > 0$, the interest rate of the bank, $\mu > 0$, the mean rate of return with the assumption $\mu > r$, and $\sigma \in \mathbb{R}$, the volatility of the stock, are constants. Eventually, $W(t)$ is a Brownian motion on $(\Omega, \mathbb{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. We assume that $\gamma > -1$ which implies that $X(t)$ can never jump to 0 or a negative value.

In this problem, $c(t) \geq 0$ is one of the control variables representing the consumption rate at time $t$. The assumptions of the previous example that the portfolio is self-financing, short selling is allowed and there is no transaction cost between money transfers from one asset to another one is still valid.

Let $\pi(t) \cdot X(t)$ and $(1-\pi(t)) \cdot X(t)$ be the amounts of the risky and the risk-free assets, respectively. Here, $\pi(t)$ is another control variable for our problem.

Therefore, we can write the wealth process as

$$\begin{aligned} dX(t) =& \pi(t)\frac{X(t)}{S_1(t)}\,dS_1(t) + r\,(1-\pi(t))\,X(t)\,dt - c(t)\,dt \\ & + \pi(t)X(t)\int_{\mathbb{R}} h(t,z)\,\tilde{M}(dt,dz) \qquad (3.64) \\ =& \left[\mu\pi(t)X(t) + r\,(1-\pi(t))\,X(t) - c(t)\right]dt + \sigma\pi(t)X(t)\,dW(t) \\ & + \pi(t)X(t)\int_{\mathbb{R}} h(t,z)\,\tilde{M}(dt,dz). \qquad (3.65) \end{aligned}$$

The goal of this optimization problem is to find the value function $V(t,x)$ and an optimal control $\tilde{u}(t) = (\tilde{\pi(t)}, \tilde{c(t)}) \in \mathbb{A}$ which maximizes the discounted utility for some constant $\rho > 0$.

The objective function is defined as

$$J(t, x; u) = E\left[\int_0^{\tau_s} e^{-\rho t} U(c(t))dt\right].$$

**Stochastic Optimal Control Problem:**

$$V(t, x) = \max_{u \in A} E\left[\int_0^{\tau_s} e^{-\rho t} U(c(t))dt\right] = J(t, x; \tilde{u}), \tag{3.66}$$

where $V(\cdot)$ is the value function. Here, we choose $U(c)$ as the logarithmic utility as in the previous example.

**Theorem 3.8.**

Given the wealth process as in Eqn. (3.64) and utility function $U(c) = \log c$, the optimal consumption is given by

$$\tilde{c} = \rho X(t), \tag{3.67}$$

and optimal amount of the risky asset is the solution of the equation

$$\tilde{\pi}\sigma^2 + \tilde{\pi} \int_{\mathbb{R}} \frac{h^2(t, z)\nu(dz)}{1 + \tilde{\pi}(t)h(t, z)} = \mu - r. \tag{3.68}$$

Moreover, the maximum utility is given by $a \log X(0) + b$, where

$$a = \frac{1}{\rho},$$

$$b = \frac{1}{\rho^2}\left(\rho \log \rho + (\mu - r)\pi + r - \rho^2 - \frac{\sigma^2\pi^2}{2} + \int_{\mathbb{R}} \{\log(1 + \pi h) - \pi h\}\nu(dz)\right).$$

*Proof.* The HJB equation for this problem is

$$\frac{\partial V}{\partial t}(t, x) + \sup_{u \in \mathbb{A}} [e^{-\rho t} \log c + \{\pi x(\mu - r) + rx - c\} \frac{\partial V}{\partial x}(t, x) + \frac{1}{2}\sigma^2\pi^2 x^2 \frac{\partial^2 V}{\partial x^2}(t, x)$$

$$+ \int_{\mathbb{R}} \{V(t, x + x\pi h) - V(t, x) - \frac{\partial V(t, x)}{\partial x} x\pi h\}\nu(dz)] = 0. \tag{3.69}$$

It follows from the first-order conditions that

$$e^{-\rho t}\frac{1}{c} - \frac{\partial V}{\partial x} = 0,$$

$$\frac{\partial}{\partial \pi}\left(\int_{\mathbb{R}} \{V(t, x + x\pi h) - V(t, x) - \frac{\partial V(t, x)}{\partial x} x\pi h\}\nu(dz)\nu(dz)\right)$$

$$+ x(\mu - r)\frac{\partial V}{\partial x} + \sigma^2\pi x^2 \frac{\partial^2 V}{\partial x^2} = 0. \tag{3.70}$$

If we choose a candidate solution $V$ in the form

$$V(x,t) = e^{-\rho t}(a \log x + b),$$

with partial derivatives

$$\frac{\partial V}{\partial t} = -\rho e^{-\rho t}(a \log x + b), \tag{3.71}$$

$$\frac{\partial V}{\partial x} = \frac{a}{x} e^{-\rho t}, \tag{3.72}$$

$$\frac{\partial^2 V}{\partial x^2} = -\frac{a}{x^2} e^{-\rho t}, \tag{3.73}$$

Eqn. (3.70) becomes

$$\frac{\partial}{\partial \pi}\left(\int_{\mathbb{R}} \{\{e^{-\rho t}\left((a\log(x + x\pi h) + b)\right) - e^{-\rho t}(a \log x + b) - e^{-\rho t}\frac{a}{x} x\pi h\}\right)\nu(dz)$$
$$+ \frac{a}{x} e^{-\rho t} x(\mu - r) - \frac{a}{x^2} e^{-\rho t}\sigma^2 \pi x^2 = 0. \tag{3.74}$$

Dividing Eqn. (3.74) by $e^{-\rho t}$, we get

$$a(\mu - r) - a\sigma^2\pi + \frac{\partial}{\partial \pi}\left(\int_{\mathbb{R}} a\{\log\left(\frac{x + x\pi h}{x}\right) - \pi h\}\right)\nu(dz) = 0,$$

hence,

$$(\mu - r) - \sigma^2\pi + \int_{\mathbb{R}} \left(\frac{-\pi h^2}{1 + \pi h}\right)\nu(dz) = 0.$$

Then, we have

$$\tilde{\pi}\sigma^2 + \tilde{\pi}\int_{\mathbb{R}} \frac{h^2}{1 + \tilde{\pi} h}\nu(dz) = \mu - r, \tag{3.75}$$

and we find $\tilde{c} = \frac{x}{a}$. Inserting $\tilde{c}$ and partial derivatives from Eqns. (3.71)-(3.73), Eqn. (3.69) is equal to

$$- \rho e^{-\rho t}(a \log x + b) + e^{-\rho t} \log\left(\frac{x}{a}\right) + \{\pi x(\mu - r) + rx - c\}e^{-\rho t}\frac{a}{x}$$

$$- e^{-\rho t}\frac{1}{2}\sigma^2\pi^2 x^2 \frac{a}{x^2} + \int_{\mathbb{R}} \{e^{-\rho t}\left((a\log(x + x\pi h) + b)\right) - e^{-\rho t}(a \log x + b)$$

$$- e^{-\rho t}\frac{a}{x} x\pi h\}\nu(dz) = 0,$$

thus,

$$- \rho(a \log x + b) + \log x - \log a + \pi a(\mu - r) + ra - 1 - \frac{1}{2}\sigma^2\pi^2 a$$
$$+ a\int_{\mathbb{R}} \{\log(x + x\pi h) - \log x - \pi h\}\nu(dz) = 0. \tag{3.76}$$

Therefore, we have

$$(1-\rho a)\log x-\rho b-\log a+\pi a(\mu-r)+ra-1-\frac{1}{2}\sigma^2\pi^2 a$$
$$+a\int_{\mathbb{R}}\{\log(1+\pi h)-\pi h\}\nu(dz)=0, \tag{3.77}$$

where

$$a=\frac{1}{\rho},$$

$$b=\frac{1}{\rho^2}\left(\rho\log\rho+(\mu-r)\pi+r-\rho^2-\frac{\sigma^2\pi^2}{2}+\int_{\mathbb{R}}\{\log(1+\pi h)-\pi h\}\nu(dz)\right).$$

□

Note that when $\nu=0$, we obtain the same results with Merton's Portfolio-Consumption Problem in the no-jump case.

## 3.4 The Relationship Between the Maximum Principle and the Dynamic Programming Principle

In this chapter, we examined the theory of Maximum Principle and Dynamic Programing Principle. The relationship between these two fundamental methodology is first studied by [4] and [1]. Yong and Zhou [30] discussed this topic for the stochastic case, and Framstad et al. [9] extended this to the jump-diffusion processes. Now, following Framstad et al. [9], we will briefly establish the relationship between these commonly used approaches in solving the stochastic optimal control problems. As we mentioned earlier, these two methods have been developed simultaneously, but independently and separately.

The relationship between Maximum Principle and Dynamic Programming Principle is fundamentally the relationship among ODEs, PDEs and SDEs. In fact, the Hamiltonians in the Maximum Principle are an ordinary differential equation in the deterministic case, whereas a stochastic differential equation in the stochastic case. On the other hand, the HJB equations in the Dynamic Programming Principle are nonlinear PDEs, of first order in the deterministic case and of second order in the stochastic case. That is the reason why we establish relationship between ODEs, PDEs, and SDEs with these two fundamental principles.

In addition to that, in the diffusion case, the relation between Maximum Principle and Dynamic Programming Principle is that the adjoint processes of the Maximum Principle ($q_1, q_2, q_3$, in Section 3.2) can be expressed as

$$q_1(t)=\frac{\partial V}{\partial x}(t,x),$$

$$q_2(t) = \frac{\partial^2 V}{\partial x^2}(t,x),$$

where $V(t,x)$ is the value function, $x$ is the initial value of the state process.

Furthermore, for the jump-diffusion case, the relation between these two approaches are given by

$$\begin{aligned}
q_1^{(i)}(t) &= \frac{\partial V}{\partial x_i}(t,\tilde{X}(t)),\\
q_2^{(ik)}(t) &= \sum_{j=1}^{n} \sigma_{jk}(t,\tilde{X}(t),\tilde{u}(t))\frac{\partial^2 V}{\partial x_i \partial x_j}(t,\tilde{X}(t)),\\
q_3^{(ij)}(t,z) &= \frac{\partial V}{\partial x_i}\left(t,\tilde{X}(t) + h^{(j)}(t,\tilde{X}(t),\tilde{u}(t),z) - \frac{\partial V}{\partial x_i}(t,\tilde{X}(t))\right),
\end{aligned}$$

for all $i = 1, ...., n$; $j = 1, ....., l$; $k = 1, ....., m$, where $\tilde{X}$ is an optimal solution and $\tilde{u}$ is an optimal control.

Therefore, we see that the relationship between these two methods is substantially equal to the relationship between the derivatives of the value function and the solutions of the adjoint equations of the Maximum Principle.

# CHAPTER 4

# APPLICATIONS TO INSURANCE

## 4.1 Introduction

Stochastic control has been a new research area in insurance, and it has been of great interest. In the previous chapter, we review the theory of stochastic optimal control theory with applications to finance. In this chapter, we will examine two applications of stochastic optimal control to insurance. The first application is about to find optimal control policies of an insurer, optimal investment decision and optimal liability ratio, which maximizes the expected utility of an insurer at terminal time. This application is studied by Özalp et al. [31] under controlled Lévy risk processes and solved by Maximum Principle. Then, investigating the paper of Mousa et al. [16], we analyze an insurance problem from the perspective of a wage-earner who wants to buy a life-insurance contract. This problem is solved by Dynamic Programming Principle and for the diffusion processes. Optimal strategies for constant relative risk aversion utilities are given explicitly. Finally, we will demonstrate some numerical results.

## 4.2 Optimal Investment Strategy and Liability Ratio for Insurer with Lévy Risk Processes

In this example, we investigate the study of Özalp et al. [31] which is the optimal investment and liability problem of an insurer with the wealth process controlled by a Lévy process. In this optimization problem, the goal is to find the optimal investment strategy that will maximize the expected utility of terminal wealth of an insurer for various utility functions such as exponential, power, and logarithmic.

In this study, the risk process of the insurer is controlled by Lévy process, and the control variables are the investment strategy under the risk-free and risky assets and the liability ratio. By using the Maximum Principle approach, a closed form solution is obtained for the optimal investment strategy and the liability ratio.

A financial market consisting of one risk-free asset (bond) and one risky-asset (stock), whose price-dynamics are given as below, respectively,

$$dS_0(t) = r(t)S(t)dt,\ S_0(0) = s_0,$$

$$dS_1(t) = \mu S_1(t)dt + \sigma_1(t)S_1(t)dW^1(t),\ S_1(0) = s_1,$$

is considered, where $r$ is the interest rate of the bank, $\mu$ is the mean rate of the return and $\sigma$ is the volatility of the stock. Here, $r$, $\mu$, and $\sigma$ are positive bounded deterministic functions and $W^1$ is a standard Brownian motion.

The risk process of the insurer is modeled by a Lévy process and given as

$$dP(t) = \bar{b}dt + \sigma_2 d\bar{W}(t) + \int_{\mathbb{R}} h(t,z)\tilde{M}(dt,dz),$$

where $\bar{b} = b + \int_{h(t,z)\geq 1} h(t,z)\nu(dz)$ and $\bar{W}(t)$ is a standard Brownian motion.

According to studies of Stein [26] on the financial crises of 2007-2008, liability of the insurer and return of the risky assets are negatively correlated and, hence, $\bar{W}(t)$ is defined as

$$\bar{W}(t) = \rho W^1(t) + \sqrt{1-\rho^2}W^2(t),$$

where $W^1(t)$ and $W^2(t)$ are independent standard Brownian motions and $\rho \in [-1, 0]$ is a correlation coefficient.

In this study, the premium is considered as constant, and we assume a constant ratio of insurer's liability, denoted by $p$. Then, the premium at time $t$ is calculated by $pL(t)$, where $L(t)$ is the total liability at time $t$ and one of the control variables in this optimization problem. In addition, expected premium income must be greater than or equal to expected losses and expenses. Otherwise, it is meaningless for the insurer. Therefore, the premium has a lower bound such as

$$p \geq \bar{b} = b + \int_{h(t,z)\geq 1} h(t,z)\nu(dz).$$

In this problem, another control variable is the amount of the risky asset at time $t$, which is denoted by $\pi(t)$. Let us call $X(t)$ the total wealth of the insurer at time $t$ with initial condition $X(t) = x$; then, automatically, $X(t) - \pi(t)$ is the amount of risk-free asset at time $t$.

Insurer's wealth process is affected by the stochastic cash flow which is a consequence of investment and insurance operations, and we formulate it as:

**Wealth = Initial Wealth + Premium Income + Financial Gain - Claim Payments.**

Mathematically speaking, referring to incremental changes:

$$dX(t) = \pi(t)\frac{dS_1(t)}{S_1(t)} + \{X(t) - \pi(t)\}\frac{dS_0(t)}{S_0(t)} + L(t)[pdt - dP(t)].$$

Therefore, the wealth process $X(t)$ is equal to, in differential form,

$$
\begin{aligned}
dX(t) = &\left[r(t)X(t) + (\mu(t) - r(t))\pi(t) + (p - \bar{b})L(t)\right] dt \\
&+ (\sigma_1(t)\pi(t) - \sigma_2\rho L(t))dW^1(t) - \sigma_2 L(t)\sqrt{1-\rho^2}dW^2(t) \\
&- \int_{\mathbb{R}} L(t)h(t,z)\tilde{M}(dt,dz). \qquad (4.1)
\end{aligned}
$$

Specifying $L(t)$ with $L(t) = X(t) \cdot K(t)$ enables us to write the wealth process in Eqn. (3.8) as

$$
\begin{aligned}
\frac{dX^{\tilde{u}}(t)}{X^{\tilde{u}}(t)} = &\left[r(t) + (\mu(t) - r(t))\pi(t) + (p - \bar{b})K(t)\right] dt \\
&+ (\sigma_1(t)\pi(t) - \sigma_2\rho K(t))dW^1(t) - \sigma_2 K(t)\sqrt{1-\rho^2}dW^2(t) \\
&- \int_{\mathbb{R}} K(t)\gamma(t,z)\tilde{M}(dt,dz), \qquad (4.2)
\end{aligned}
$$

where $u(t) = (\pi(t), K(t))$ is an admissable control process, i.e., $u(t) \in \mathbb{A}$, with the following conditions:

$$
\int_0^t \pi(s)ds < \infty \quad \text{and} \quad \int_0^t K(s)ds < \infty.
$$

Furthermore, for $u$ to be in $\mathbb{A}$ as we mentioned in Eqn. (3.2), it is required that

$$
\mathbb{E}\left[\max_{u\in\mathbb{A}} U(X(T))\right] < \infty.
$$

In this study, the objective is to choose the optimal control $\tilde{u}(t) = (\tilde{\pi}(t), \tilde{K}(t))$ which maximizes the expected utility of the insurer's terminal wealth. The objective function is defined as

$$
J(t,x;u) = E\left[U(X^u(T)) \,\middle|\, X^u(t) = x\right],
$$

where $T > 0$ is a fixed constant and terminal time.

**The Stochastic Optimal Control Problem:**

$$
V(t,x) = \sup_{u\in\mathbb{A}} E\left[U(X^{\tilde{u}}(T)) \,\middle|\, X^{\tilde{u}}(t) = x\right] = J(t,x;\tilde{u}),
$$

where $V(\cdot)$ is the value function, $U(\cdot)$ is a differentiable, strictly increasing and concave utility function which implies that the insurer is risk averse, and we look for an optimal control

$$
\tilde{u}(t) = (\tilde{\pi}(t), \tilde{K}(t)) \in \mathbb{A}.
$$

In this optimization problem, there are two Brownian components and the wealth process is given in the form

$$dX(t) = b\left(t, X(t), u(t)\right)\ dt + \sigma^1\left(t, X(t), u(t)\right)\ dW^1(t) + \sigma^2\left(t, X(t), u(t)\right)\ dW^2(t) + \int_{\mathbb{R}} h(t, X(t-), u(t-), z)\ \tilde{M}(dt, dz). \tag{4.3}$$

The corresponding Hamiltonian function is defined in the form of

$$H(t, x, u, q_1, q_2, q_3, q_4) = b^T\left(t, x, u\right)\ q_1 + \sigma^1\left(t, x, u\right)\ q_2 + \sigma^2\left(t, x, u\right)\ q_3 + \int_{\mathbb{R}} h\left(t, x, u, z\right)\ q_4(t, z)\nu(dz). \tag{4.4}$$

Furthermore, the corresponding adjoint equation is defined as

$$dq_1(t) = -\nabla_x H\left(t, X(t), u(t), q_1(t), q_2(t), q_3(t), q_4(t, z)\right)\ dt + q_2(t)\ dW^1(t) + q_3(t)\ dW^2(t) + \int_{\mathbb{R}^n} q_4(t-, z)\ \tilde{M}(dt, dz) \tag{4.5}$$

with terminal condition

$$q_1(T) = \nabla U\left(X(T)\right).$$

After defining Hamiltonian function and adjoint equation, now we will solve this optimization problem for various utility functions such as exponential, power and logarithmic utility functions which maximizes the expected utility of terminal wealth of the insurer. In this thesis, we will give the proof for logarithmic utility function. For the proofs in the cases of exponential and power utility functions, see Özalp et al. [31]

**Proposition 3.1.** (Özalp et al. [31])

Suppose that the utility function is given by $U(x) = \ln(x), x > 0$. Then, the optimal investment strategy is such that

$$\tilde{\pi}(t) = \frac{\mu(t) - r(t)}{\sigma^2(t)} + \frac{\rho\sigma_2}{\sigma_1(t)}\tilde{K}(t).$$

The optimal liability ratio satisfies the following equation:

$$\Lambda(\tilde{K}(t)) = -\left(p - \bar{b}\right) - \left[-\rho\sigma_2\sigma_1(t)\tilde{\pi}(t) + \sigma_2^2\tilde{K}(t)\right] - \int_{\mathbb{R}} \left[\frac{\gamma(t, z)}{1 + \gamma(t, z)\tilde{K}(t)} - 1\right]\nu(dz) = 0.$$

*Proof.* The proof is based on Theorem 3.1. By using the wealth process given in Eqn.

(4.3), the Hamiltonian function can be written as

$$H(t,x,\tilde{\pi}(t),L(t),q_1,q_2,q_3,q_4) = \left[xr(t + (\mu(t)-r(t))\,\tilde{\pi}(t) + (p-(\bar{b}))L(t)\right] q_1(t)$$
$$+ (\sigma_1(t)\tilde{\pi}(t) - \sigma_2\rho L(t))q_2(t)$$
$$+ (-\sigma_2 L(t)\sqrt{1-\rho^2})q_3(t)$$
$$+ \int_{\mathbb{R}} h(t,x)L(t)q_4(t-,z)\nu(dz),$$

and the adjoint equation can be written as

$$\begin{aligned} dq_1(t) =& -\nabla_x H(t,X(t),u(t),q_1(t),q_2(t),q_3(t),q_4(t,z))dt \\ &+ q_2(t)dW^1(t) + q_3(t)dW^2(t) + \int_{\mathbb{R}^n} q_4(t-,z)\tilde{M}(dt,dz) \\ =& -r(t)q_1(t) + q_2(t)dW^1(t) + q_3(t)dW^2(t) + \int_{\mathbb{R}^n} q_4(t-,z)\tilde{M}(dt,dz) \end{aligned} \quad (4.6)$$

with terminal condition

$$q_1(T) = \frac{1}{X(T)}. \quad (4.7)$$

Then, we make a guess for $q_1(t)$:

$$\tilde{q_1}(t) = \frac{\phi(t)}{X(t)}, \quad (4.8)$$

where $\phi \in C^1$ with $\phi(T) = 1$.

Applying Itô Formula to the unknown adjoint variable $\tilde{q_1}(t)$, we have

$$\begin{aligned} d\tilde{q_1}(t) =& \frac{\phi'(t)}{X(t)} - \frac{\phi(t)}{(X(t))^2}\Big[\{xr(t) + (\mu(t)-r(t))\tilde{\pi(t)} + (p-\bar{b})L(t)\}dt \\ &+ (\sigma_1(t)\tilde{\pi}(t) - \sigma_2\rho L(t))dW^1(t) + (-\sigma_2 L(t)\sqrt{1-\rho^2})dW^2(t)\Big] \\ &+ \frac{1}{2}\left(((\sigma_1(t))^2(\tilde{\pi}(t))^2 - 2\sigma_1(t)\tilde{\pi}(t)\sigma_2 L(t) + (\sigma_2)^2L^2(t)\right)\cdot\frac{2\phi(t)}{(X(t)^3)}dt \\ &+ \int_{\mathbb{R}}\left[\frac{\phi(t)}{X(t)-h(t,z)L(t)} - \frac{\phi(t)}{X(t)} - \frac{-\phi(t)}{(X(t))^2}h(t,z)1_{\epsilon\le\gamma<1}\right]\nu(dz) \\ &+ \int_{\mathbb{R}}\left[\frac{\phi(t)}{X(t)-h(t,z)L(t)} - \frac{\phi(t)}{X(t)}\right]\tilde{M}(dt,dz). \end{aligned} \quad (4.9)$$

Then, comparing the adjoint equation $dq_1(t)$ (cf. Eqn. (3.14)) with differentiation of the unknown adjoint variable defined in Eqn. (3.16), i.e., Eqn. (3.17), we obtain the following solutions:

$$\begin{aligned}\tilde{q_2}(t) &= -\frac{\phi(t)}{(X(t))^2}(\sigma_1(t)\tilde{\pi}(t) - \sigma_2\rho L(t)) \\ &= -\frac{\phi(t)}{(X(t))}(\sigma_1(t)\tilde{\pi}(t) - \sigma_2\rho K(t)), \qquad (4.10)\\ \tilde{q_3}(t) &= -\frac{\phi(t)}{(X(t))^2}\sigma_2 L(t)\sqrt{1-\rho^2} \\ &= -\frac{\phi(t)}{(X(t))}\sigma_2 K(t)\sqrt{1-\rho^2}, \qquad (4.11)\\ \tilde{q_4}(t,z-) &= \frac{\phi(t)}{X(t) - h(t,z)L(t)} - \frac{\phi(t)}{X(t)}. \qquad (4.12)\end{aligned}$$

Then, from the first-order conditions it is easily seen that

$$\begin{aligned}\frac{\partial \tilde{H}}{\partial \tilde{\pi}(t)} &= (\mu(t) - r(t))\tilde{q_1}(t) + \sigma_1(t)\tilde{q_2}(t) = 0 \\ &= (\mu(t) - r(t))\frac{\phi(t)}{X(t)}(t) - \sigma_1(t)\frac{\phi(t)}{(X(t))}(\sigma_1(t)\tilde{\pi}(t) - \sigma_2\rho K(t)) = 0;\end{aligned}$$

hence, the optimal investment strategy is obtained as

$$\tilde{\pi(t)} = \frac{(\mu(t) - r(t))}{(\sigma_1)^2(t)} + \frac{\sigma_2\rho\tilde{K(t)}}{\sigma_1(t)}.$$

Similarly, we have

$$\frac{\partial \tilde{H}}{\partial \tilde{L}(t)} = (p - \bar{b})\tilde{q_1}(t) - \sigma_2\rho\tilde{q_2}(t) - \sigma_2\sqrt{1-\rho^2}\tilde{q_3}(t) - \int_{\mathbb{R}} h(t,z)\tilde{q_4}(t,z-)\nu(dz) = 0.$$

Thus, the optimal liability ratio satisfies the following equation as claimed:

$$\begin{aligned}\Lambda(\tilde{K}(t)) = &-(p - \bar{b}) - [-\rho\sigma_2\sigma_1(t)\tilde{\pi}(t) + \sigma_2^2\tilde{K}(t)] \\ &- \int_{\mathbb{R}}\left[\frac{\gamma(t,z)}{1+\gamma(t,z)\tilde{K}(t)} - 1\right]\nu(dz) = 0.\end{aligned}$$

□

**Proposition 3.2.** (Özalp et al. [31])

Suppose that the utility function is given by $U(x) = -\frac{1}{\alpha}e^{-\alpha x}, \alpha > 0$. Then, the optimal investment strategy is

$$\tilde{\pi}(t) = e^{-r(T-t)} \cdot \left[\frac{\mu(t) - r(t)}{\alpha x \sigma_1^2(t)}\right] + \frac{\rho\sigma_2}{\sigma_1(t)}\tilde{K}(t).$$

Moreover, the optimal liability ratio satisfies the following equation:

$$\begin{aligned}\Lambda(\tilde{K}(t)) = &- (p - \bar{b}) + (-\rho\sigma_2\sigma_1(t)\tilde{\pi}(t) + \sigma_2^2\tilde{K}(t))\alpha x e^{r(T-t)} \\ &+ \int_{\mathbb{R}} \gamma(t,z)[\exp(\alpha e^{r(T-t)}\gamma(t,z)\tilde{K}(t)x) - 1]\nu(dz).\end{aligned}$$

*Proof.* Özalp et al. [31].

**Proposition 3.3.** (Özalp et al. [31])

Suppose that the utility function is given by $U(x) = \frac{1}{\alpha}x^{\alpha}$, with $\alpha \neq 0, \alpha \neq 1$.

Then, the optimal investment strategy is such that

$$\tilde{\pi}(t) = \frac{\mu(t) - r(t)}{(\alpha - 1)\sigma_1^2(t)} + \frac{\rho\sigma_2}{\sigma_1(t)}\tilde{K}(t).$$

Furthermore, the optimal liability ratio satisfies the following equation

$$\begin{aligned}\Lambda\left(\tilde{K}(t)\right) = &- (p - \bar{b}) + \left[-\rho\sigma_2\sigma_1(t)\tilde{\pi}(t) + \sigma_2^2\tilde{K}(t)\right](\alpha - 1) \\ &- \int_{\mathbb{R}} \gamma(t,z)[(1 - \gamma(t,z)\tilde{K}(t))^{\alpha-1} - 1]\nu(dz) = 0.\end{aligned}$$

*Proof.* See Özalp et al. [31].

For more details, analysis and numerical results about this application we refer the reader to Özalp et al. [31].

### 4.3 Selection and Purchase of an Optimal Life-Insurance contract among Several Life-Insurance Companies

In 1965, Yaari [29] introduced an optimal consumption problem from the point of an individual with uncertain lifetime under a pure deterministic setup, and Hakansson [11] included risky assets to this study and extended it to the discrete case. In the previous chapter, we investigated Merton's continuous-time optimal portfolio and consumption problem. In 1975, Richard [23] extended this problem including life-insurance purchase using Yaari's setting. In 2007, Pliska and Ye [22] studied the optimal portfolio consumption and life-insurance problem under an unbounded random time interval, and developed a new numerical method which is Markov Chain Approximation with logarithmic transformation. Duarte et al. [7] extended the study of Pliska and Ye [22] where a wage-earner invests his savings in an incomplete financial market with multi-dimensional diffusive terms and purchases a life-insurance contract from a single insurance company with a random time horizon. In 2014, Shen and Wei [24] considered the same problem in a complete market with random unbounded parameters such as stochastic income, stochastic hazard rate and stochastic appreciation rate. In 2015, Guambe and Kufakunesu [10] extended the study of Shen and Wei [24] under jump-diffusion processes. In 2016, Mousa et al. [16] extended Duarte et al. [7] with $K$ insurance companies, and now, we will look more closely at this study.

In this application, we examine the study of Mousa et al. [16]. It is on a problem of a wage-earner whose lifetime is uncertain, investing his savings on riskless and risky assets; the wage-earner has to decide concerning consumption and select a life-insurance contract. The wage-earner's lifetime is uncertain, during the random interval $[0, \min\{\tau, T\}]$, his objective is to maximize his total expected utility obtained from consumption, the legacy in the situation of a premature death and the investor's terminal wealth at time $T$ if he lives that long. Here, $\tau$ is a positive and continuous random variable representing the wage-earner's eventual time of death and $T$ is a fixed constant representing the retirement date of the wage-earner. Since his lifetime is random, we have a random time horizon problem and it is the distinctive feature of this problem. Moreover, it is assumed that there is a life-insurance market composed by $K$ life-insurance companies in which he can buy a life-insurance contract from the $k^{\text{th}}$ company by paying a premium insurance rate $p_k(t)$, where $k = 1, 2, ..., K$.

In the event of the wage-earner's instantaneous death at time $\tau \leq T$, the $k^{\text{th}}$ insurance company will pay his family the amount

$$Z_k(\tau) = \frac{p_k(\tau)}{\eta_k(\tau)}, \tag{4.13}$$

where $\eta_k$ is the premium-payout ratio of the $k^{\text{th}}$ insurance company.

Here, $\eta_k : [0, T] \to \mathbb{R}^+$ is a continuous, deterministic and positive function, and, the assumption, the insurance companies offer different contracts for Lebesgue a.e., that is, $\eta_{k_1} \neq n_{k_2}$ for every $k_1 \neq k_2$, will be needed throughout the paper. In the case of premature death, thanks to $\eta_k$ the payout of the life-insurance contract is fixed.

Suppose that $W(t) = (W_1(t), ...., W_M(t))^T$ is a $M$-dimensional Brownian motion on $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ which attains its values in $\mathbb{R}^M$. Here, $\mathcal{F}_t$ represents the information available at time $t$.

We consider a financial market consisting of one risk-free asset and a specified number $(N)$ of risky-assets, whose price-dynamics are, respectively, given as follows:

$$dS_0(t) = r(t)S_0(t)dt, \quad S_0 = s_0,$$

$$dS_n(t) = \mu_n(t)S_n(t)dt + S_n(t)\sum_{m=1}^{M}\sigma_{nm}(t)dW_m(t), \quad S_n = s_n,$$

where $n = 1, 2, ...., N$, $r(t)$ is the interest rate of the bank, $\mu(t) = (\mu_1(t), ...., \mu_N(t))^T$ is the random vector of mean rate of returns with values in $\mathbb{R}^N$, and

$\sigma(t) = (\sigma_{nm}(t))_{1\leq n\leq N, 1\leq m\leq M}$ is the random $N \times M$ matrix of volatilities. It is also assumed that $\mu(t), r(t)$, and $\sigma(t)$ are continuous and deterministic functions. Here, we define the appreciation rate as $\alpha = (\mu_1(t) - r(t), ...., \mu_N(t) - r(t))^T$.

Another assumption is that the wage-earner is alive at time $t = 0$ and the wage-earner's remaining lifetime is a nonnegative random variable $\tau$, defined on $(\Omega, \mathcal{F}, \mathbb{P})$, with a probability density function (pdf) $f$ and distribution function (cdf) $F$, such as

$$F(t) := \mathbb{P}(\tau < t) = \int_0^t f(s)\, ds.$$

Furthermore, it is known that the *survival function* is defined as the probability that the lifetime $\tau$ is greater than or equal to $t$; i.e.,

$$\hat{F}(t) := \mathbb{P}(\tau \geq t) = 1 - F(t).$$

The hazard rate function, which is also called as the instantaneous force of mortality, is the instantaneous death rate for an individiul who has survived to time $t$, and defined by

$$\begin{aligned}
\lambda(t) :&= \lim_{\delta t \to 0^+} \frac{\mathbb{P}(t \leq \tau < t + \delta t \mid \tau \geq t)}{\delta t} \\
&= \lim_{\delta t \to 0^+} \frac{\mathbb{P}(t \leq \tau < t + \delta t\ )}{\delta t \cdot \mathbb{P}(\tau \geq t)} \\
&= \lim_{\delta t \to 0^+} \frac{F(t+\delta t) - F(t)}{\delta t} \frac{1}{\hat{F}(t)}.
\end{aligned}$$

Then, we have

$$\lambda(t) = \frac{f(t)}{\hat{F}(t)} = -\frac{d}{dt}(\ln \hat{F}(t)), \tag{4.14}$$

and *survival function*

$$\hat{F}(t) = \mathbb{P}(\tau > t) = \exp\{-\int_0^t \lambda(s)\, ds\}. \tag{4.15}$$

From Eqn. (4.14) we know that there is a relation between hazard rate function and the pdf of $\tau$:

$$f(t) = \lambda(t) \exp\{-\int_0^t \lambda(s)\, ds\}. \tag{4.16}$$

In the remainder of this application, we assume $\lambda(\cdot) : [0, \infty] \to \mathbb{R}^+$ is a continuous and deterministic function with the condition

$$\int_0^\infty \lambda(t)\, dt = \infty.$$

For every $0 \leq t \leq s$, suppose that $f(s,t)$ denotes the conditional probability density for the wage-earner be death at time $s$ conditional upon being alive at time $t \leq s$.

Combination of Eqn. (4.15) and Eqn. (4.16) gives us

$$f(s,t) := \frac{f(s)}{\hat{F}(t)} = \lambda(s)\exp\{-\int_t^s \lambda(u)\, du\}. \tag{4.17}$$

Furthermore, let $\hat{F}(s,t)$ denote the conditional probability for the wage-earner to be alive at time $s$ conditional upon being alive at time $t \leq s$,

$$\hat{F}(s,t) := \frac{\hat{F}(s)}{\hat{F}(t)} = \exp\{-\int_t^s \lambda(u)\, du\}. \tag{4.18}$$

Moreover, every contract ends at time $t = \min\{\tau, T\}$, namely, when the wage-earner dies or reaches the retirement, which ever comes first. Hence, in the event of a premature death at time $\tau \leq T$ the wage-earner's total legacy is given by

$$Z(\tau) = X(\tau) + \sum_{k=1}^{K} \frac{p_k(\tau)}{\eta_k(\tau)}, \tag{4.19}$$

where $X(\tau)$ is the wage-earner's wealth at time $\tau$.

From now on, we make the following assumptions:

**(A1)** The wage earner has a revenue $i(t)$ which will be terminated by his death or his retirement, which ever happens first. Moreover, the income function $i : [0,T] \to \mathbb{R}^+$ is a deterministic function which satisfies:

$$\int_0^T i(t)\, dt < \infty.$$

**(A2)** The nonnegative consumption process $(c(t))_{0 \leq t \leq T}$ is $\mathcal{F}_t$-measurable which satisfies:

$$\int_0^T c(t)\, dt < \infty \quad \text{a.s.}$$

**(A3)** We will denote by $\pi_n(t)$ the proportion of the wealth invested in the security $S_n$ at time $t$. Then, it is assumed that $\pi_n(t) \in \mathcal{F}_t$ and given by $\pi(t) = (\pi_0(t), \pi_1(t), ...., \pi_N(t))^T$ with the condition

$$\int_0^T \left|\left|\pi(t)^2\right|\right| dt < \infty \quad \text{a.s.},$$

where $||\cdot||$ denotes Euclidean norm.

Note that, we have

$$\sum_{n=0}^{N} \pi_n(t) = 1, \quad 0 \leq t \leq T.$$

Now, we define the wealth process of the wage-earner such as

$$X(t) = X(0) + \int_0^t \left( i(s) - c(s) - \sum_{k=1}^{K} p_k(s) \right) ds + \sum_{n=0}^{N} \int_0^t \frac{\pi_n(s) X(s)}{S_n(s)} \, dS_n(s),$$

where $t \in [0, \min\{\tau, T\}]$ and the initial wealth is $X(0) = x$.

We write the wealth process $X(t)$ in differential form by

$$\begin{aligned} dX(t) = &\left[ i(t) - c(t) - \sum_{k=1}^{K} p_k(t) + (\pi_0(t) r(t) \sum_{n=1}^{N} \pi_n(t) \mu_n(t)) X(t) \right] dt \\ &+ \sum_{n=1}^{N} \pi_n(t) X(t) \sum_{m=1}^{M} \sigma_{nm}(t) \, dW_m(t). \end{aligned} \tag{4.20}$$

To solve this stochastic optimal control problem, we will use Dynamic Programming approach. In this problem, the control variables are the control process $(c(t))$, the fraction of the wealth in the assets $(\pi(t))$, and the premium insurance rate process $(p_k(t))$. Then, our control process is $u = (c(\cdot), \pi(\cdot), p(\cdot)) \in \mathbb{A}$.

Therefore, we may rewrite the stochastic optimal control problem as: Find an admissable strategy $u$ that maximizes the joint expected utility. We define the objective function subject to the state process $X(t)$ given by Eqn. (4.20) and with initial value $X(0) = x$ as

$$\begin{aligned} J(t, x; u) = E_{t,x} \Bigg[ \int_t^{\tau \wedge T} & U_1(s, c(s)) ds + U_2(\tau, Z(\tau)) I_{[0,T]}(\tau) \\ & + U_3(X_{t,x}^{\nu}(T) I_{(T,+\infty)}(\tau)) \Bigg], \end{aligned} \tag{4.21}$$

where $I$ is an indicator function, and $U_1(t, \cdot)$, $U_2(\cdot)$, $U_3(t, \cdot)$ are utility functions describing the wage-earner's family preference, for the size of the wage-earner's legacy and for the terminal wealth of retirement date, respectively.

We assume also that the utility functions $U_1$ and $U_2$ are twice differentiable, strictly increasing and strictly concave functions on their second variable, and $U_3$ is a twice differentiable, strictly increasing and strictly concave function.

Here, the problem is defined on a random time horizon. Since we will solve the problem using the Dynamic Programming Principle, we transform this problem to a fixed-planning horizon thanks to Eqn. (4.17), on the conditional survival probability of the wage-earner, and Eqn. (4.18), on the conditional probability density of the death of the wage-earner.

The following lemma provides our key transformation (from a random terminal time to a fixed terminal time) of the optimal control problem to an equivalent one which is placed a fixed planning horizon.

**Lemma 4.1.**

Let $\tau$, independent from $\mathcal{F}_t$, is a random variable. The objective function at any time $t \in [0, T]$ looks as follows:

$$J(t, x; u) = E_{t,x}\bigg[\int_t^T \hat{F}(s,t)U_1(s, c(s))ds + f(s,t)U_2(s, Z(s))(\tau) \\ + \hat{F}(T,t)U_3(X(T))|F_t\bigg]. \tag{4.22}$$

*Proof.* See Ye and Pliska [22] for the details of the proof. □

So, we can restate our control problem under fixed plan horizon as

**Stochastic Optimal Control Problem:**

Find

$$V(t, x) = \sup_{u \in \mathbb{A}} J(t, x; u) = J(t, x; \tilde{u}),$$

where $V$ is the value function, and $\tilde{u}(t) = (\tilde{c}(t), \tilde{\pi}(t), \tilde{p}(t)) \in \mathbb{A}$ is an optimal control.

**Lemma 4.2** (*Dynamic Programming Principle*, [22])

For $0 \leq t < s < T$, we have

$$V(t, x) = \sup_{u \in \mathbb{A}} \mathbb{E}\bigg[\exp\left(-\int_t^s \lambda(u)\,du\right) V(s, X^u(s)) \\ + \int_t^s \left(\hat{F}(u,t)U_1(u, c(u)) + f(u,t)U_2(u, Z^u)\right)\,du|\mathcal{F}_t\bigg].$$

*Proof.* See [22]. □

**Theorem 4.1** ***(Hamilton-Jacobi-Bellman Equation)***, [22]

Assume that there exists an optimal control process $u^*$ and the value function $V$ is of class $C^{1,2}$. Then $V$ satisfies the HJB equation

$$V_t(t, x) - \lambda(t)V(t, x) + \sup_{(c,\pi,p) \in \mathbb{R}^{N+1} \times (\mathbb{R}_0^+)^K} H(t, x; c, \pi, p) = 0, \tag{4.23}$$

with terminal condition

$$V(T, x) = U_3(x),$$

where the Hamiltonian function $H$ is given by

$$H(t,x;u) = \left( i(t) - c(t) - \sum_{k=1}^{K} p_k + \left( r(t) + \sum_{n=1}^{N} \pi_n(\mu_n(t) - r(t)) \right) x \right) V_x(t,x)$$
$$+ \frac{x^2}{2} \sum_{m=1}^{M} \left( \sum_{n=1}^{N} \pi_n \sigma_{nm}(t) \right) V_{xx}(t,x) + U_1(t,c)$$
$$+ \lambda(t) U_2 \left( t, x + \sum_{k=1}^{K} \frac{p_k}{\eta_k(t)} \right).$$

Moreover, the supremum in the HJB equation is attained by $u = \tilde{u}(\tilde{c}(\cdot), \tilde{\pi}(\cdot), \tilde{p}(\cdot)) \in \mathbb{A}$ with $V_t(s, \tilde{X}(s)) - \lambda(s)V(s, \tilde{X}(s)) + H(s, \tilde{X}(s); \tilde{u}) = 0$.

*Proof.* See Ye and Pliska [22] for the details of the proof. □

**Theorem 4.2** (Mousa et al., [16])

Suppose that the value function $V$ is of class $C^{1,2}(\mathbb{R})$ and the Hamiltonian function $H$ given in Theorem 4.1 has a unique maximum $u = \tilde{u}(\tilde{c}(\cdot), \tilde{\pi}(\cdot), \tilde{p}(\cdot)) \in \mathbb{A}$.

Moreover, the optimal strategies are

$$\tilde{c}(t,x) = \mathbb{I}_1(t, \frac{\partial V}{\partial x}(t,x)),$$
$$\tilde{\pi}(t,x) = -\frac{\frac{\partial V}{\partial x}(t,x)}{x\,\frac{\partial^2 V}{\partial x^2}(t,x)} \cdot \mathcal{E} \cdot \alpha(\sqcup),$$

and, for each $k \in \{1, 2, \dots, K\}$,

$$\tilde{p}_k(t,x) = \begin{cases} \max\left\{0, \left[\mathbb{I}_2(t, \eta_k(t)\left(\lambda(t))^{-1}\frac{\partial V}{\partial x}(t,x)\right) - x\right] \eta_k(t)\right\}, & \text{if} \quad k = \tilde{k}(t), \\ 0, & \text{otherwise,} \end{cases}$$

where

$$\tilde{k}(t) = \arg \min_{k \in \{1,2,\dots K\}} \{\eta_k(t)\}. \tag{4.24}$$

Here, $\mathcal{E}$ denotes the non-singular square matrix given by $(\sigma\sigma^T)^{-1}$, and

$$\alpha(t) = (\mu_1(t) - r(t), \dots, \mu_N(t) - r(t))^T \in \mathbb{R}^N$$

which consists of the appreciation rates for the wage-earner, and $\mathbb{I}_i : [0,T] \times \mathbb{R}^+ \to \mathbb{R}^+$, $i = 1, 2$, are continuous inverse functions such that

$$\mathbb{I}_1\left(t, \frac{\partial U_1}{\partial x}(t,x)\right) = x \quad \text{and} \quad \frac{\partial U_1}{\partial x}(t, \mathbb{I}_1(t,x)) = x, \tag{4.25}$$

and

$$\mathbb{I}_2\left(t, \frac{\partial U_2}{\partial x}\right) = x \quad \text{and} \quad \frac{\partial U_2}{\partial x}\left(t, \mathbb{I}_2(t,x)\right) = x. \tag{4.26}$$

*Proof.* By Theorem 4.1, we know that the supremum in the HJB equation of Eqn. (4.23) is attained by an optimal control $\tilde{u}$. We have partitioned the supremum in the HJB equation of Eqn. (4.23) into three separate suprema:

$$\begin{aligned}
&\sup_{(c,\pi,p)\in\mathbb{R}^{N+1}\times(\mathbb{R}_0^+)^K} H(t,x;c,\pi,p) = \sup_{c\in\mathbb{R}}\{U_1(t,c) - c\frac{\partial V}{\partial x}(t,x)\} \\
&+ \sup_{\pi\in\mathbb{R}^n}\{\frac{x^2}{2}\sum_{m=1}^{M}\left(\sum_{n=1}^{N}\pi_n\sigma_{nm}(t)\right)^2 V_{xx}(t,x) + \sum_{n=1}^{N}\pi_n(\mu_n(t)-r(t))x\frac{\partial V}{\partial x}(t,x)\} \\
&+ \sup_{p\in(\mathbb{R}_0^+)^K}\{\lambda(t)U_2\left(t, x+\sum_{k=1}^{K}\frac{p_k}{n_k(t)}\right) - \frac{\partial V}{\partial x}(t,x)\sum_{k=1}^{K}p_k\} \\
&+ r(t)x\frac{\partial V}{\partial x}(t,x) + i(t)\frac{\partial V}{\partial x}(t,x).
\end{aligned} \tag{4.27}$$

We apply unconstrained optimization rules for $c$ and $\pi$. From the first-order necessary conditions, we obtain the following equations:

$$\begin{aligned}
-\frac{\partial V}{\partial x}(t,x) + U_1(x)(t,\tilde{c}) &= 0 \\
x^2 V_{xx}(t,x)\sigma\sigma^T\tilde{\pi} + x\frac{\partial V}{\partial x}(t,x)\alpha &= 0_{\mathbb{R}^N},
\end{aligned} \tag{4.28}$$

where $0_{\mathbb{R}^N}$ is origin of $\mathbb{R}^N$. Combining these equations with inverse functions, Eqn. (4.25) and Eqn. (4.26), implies that

$$\tilde{c}(t,x) = I_1(t, \frac{\partial V}{\partial x}(t,x)),$$

$$\tilde{\pi}(t,x) = -\frac{\frac{\partial V}{\partial x}(t,x)}{xV_{xx}(t,x)}\alpha(t)\mathcal{E}.$$

Now, we apply constrained optimization rules for $p$. We can write the corresponding Kuhn-Tucker conditions (which are necessary but not sufficient optimality conditions) as:

$$\begin{aligned}
\frac{\lambda(t)}{n_k(t)}\frac{\partial U_2}{\partial x}\left(t, x+\sum_{k=1}^{K}\frac{p_k}{n_k(t)}\right) - \frac{\partial V}{\partial x}(t,x) &= -\mu_k, \\
p_k &\geq 0, \\
\mu_k &\geq 0, \\
p_k\mu_k &= 0,
\end{aligned} \tag{4.29}$$

where $k = 1, 2, \ldots, K$.

Note that when $k_1 \neq k_2$ for some $(t, x) \in [0, T] \times \mathbb{R}$, from Eqn. (4.29) equality $\mu_{k_1} = \mu_{k_2}$ implies that $p_{k_1} = p_{k_2}$ which contradicts the assumption that all insurance companies offer pairwise distinct contracts. So, we have for every $k_1$, $k_2 \in \{1, 2, ..., K\}$ that when $k_1 \neq k_2$, $\mu_{k_1}(t, x) \neq \mu_{k_2}(t, x)$ for every for some $(t, x) \in [0, T] \times \mathbb{R}$ for Lebesgue a.e., which yields there is at most one $k \in \{1, 2, \ldots, K\}$ such that $\mu_k = 0$. Hence, this implies that there is at most one $p_k(t, x) \neq 0$ for Lebesgue a.e.

From the first condition of Eqn. (4.29), we have

$$\lambda(t)\frac{\partial U_2}{\partial x}\left(t, x + \sum_{k=1}^{K}\frac{p_k}{\eta_k(t)}\right) = \eta_k(t)\left(\frac{\partial V}{\partial x}(t, x) - \mu_k\right)$$

$$\Rightarrow \eta_{k_1}(t)\left(\frac{\partial V}{\partial x}(t, x) - \mu_{k_1}\right) = \eta_{k_2}\left(\frac{\partial V}{\partial x}(t, x) - \mu_{k_2}\right). \tag{4.30}$$

From Eqn. (4.30), it follows that for $(t, x) \in [0, T] \times \mathbb{R}$, if $\mu_{k_1}(t, x) > \mu_{k_2}(t, x)$, then $\eta_{k_1}(t) > \eta_{k_2}(t)$. Now, if for some $t \in [0, T]$ it holds $\mu_{k_1}(t, x) = 0$, then $\eta_{k_1}(t) < \eta_{k_2}(t)$ for every $k = 1, ...., K$, and $k_1 \neq k_2$. In what follows, if we select $\tilde{k}(t)$ as given in Eqn. (4.24), then $p_k(t, x) = 0$ for every $k = 1, ...., K$, or $\tilde{p}_k(t, x) > 0$ is a solution of

$$\frac{\lambda(t)}{\eta_{\tilde{k}}(t)}\frac{\partial U_2}{\partial x}\left(t, x + \sum_{k=1}^{K}\frac{\tilde{p}_k}{\eta_{\tilde{k}}(t)}\right) = \frac{\partial V}{\partial x}(t, x). \tag{4.31}$$

We can rewrite Eqn. (4.31) as

$$\frac{\partial U_2}{\partial x}\left(t, x + \sum_{k=1}^{K}\frac{\tilde{p}_k}{\eta_{\tilde{k}}(t)}\right) = \frac{\partial V}{\partial x}(t, x)\frac{\eta_{\tilde{k}}(t)}{\lambda(t)}.$$

Therefore, by our inverse functions we get the following result:

$$\tilde{p}_k(t, x) = \begin{cases} \max\left\{0, \left[\mathbb{I}_2(t, \eta_k(t)\Big(\lambda(t))^{-1}V_x(t, x)\Big) - x\right]\eta_k(t)\right\} & \text{, if} \quad k = \tilde{k}(t), \\ 0 & \text{, otherwise.} \end{cases}$$

To check the optimality of $\tilde{c}$ and $\tilde{p}$, we shall look to the second-order derivatives of the Hamiltonian $H$:

$$H_{cc}(t, x; \tilde{u}) = \frac{\partial^2 U_1}{\partial c^2}(t, \tilde{c}),$$

$$H_{p_{k_1}p_{k_2}}(t, x; \tilde{u}) = \frac{\lambda(t)}{\eta_{k_1}(t)\eta_{k_2}(t)}\frac{\partial^2 U_2}{\partial x^2}\left(t, x + \sum_{k=1}^{K}\frac{\tilde{p}_k}{\eta_{\tilde{k}}(t)}\right), \quad k_1, k_2 = 1, \ldots, K,$$

$$H_{\pi\pi}(t, x; \tilde{u}) = x^2\frac{\partial^2 V}{\partial x^2}(t, x)\sigma\sigma^T.$$

The strict concavity of the utility functions $U_1$ and $U_2$ with respect to their second variables guarantees the optimality of $\tilde{c}$ and $\tilde{p}$, and it also makes the first-order and Kuhn-Tucker conditions sufficient conditions.

For the optimality of $\pi$, we show that $H_{\pi\pi}(t, x; \tilde{u})$ is negative definite. Since $\sigma\sigma^T$ is assumed to be nonsingular, thus, positive definite, and $\frac{\partial^2 V}{\partial x^2}(t, x)$ must be negative (otherwise, $H$ would not be bounded from above which yields that either $\frac{\partial V}{\partial t}(t, x)$ or $V(t, x)$ would have to be infinity, which is a contradiction with the smoothness of $V$), $H_{\pi\pi}$ is negative definite. Therefore, we can say that $H$ has a regular interior maximum, which completes the proof. □

Now, in the following proposition we obtain optimal strategies for the discounted CRRA utility functions, which are given by

$$U_1(t, c) = e^{-\rho t}\frac{c^\gamma}{\gamma}, \quad U_2(t, Z) = e^{-\rho t}\frac{Z^\gamma}{\gamma}, \quad \text{and} \quad U_3(X) = e^{-\rho t}\frac{X^\gamma}{\gamma}, \tag{4.32}$$

where $\gamma < 1$, $\gamma \neq 0$, is the risk-aversion parameter of power utility functions, and $\rho > 0$ is the discount rate for the utility functions.

**Proposition 4.1.** ***(Optimal Strategies for Discounted Power Utility)*** *(Mousa et al., [16])*

Under the assumptions made, the optimal strategies for the same discounted CRRA utilities which are given by Eqn. (4.32) are such that

$$\begin{aligned}
\tilde{c}(t, x) &= \frac{1}{\xi(t)}(x + A(t)),\\
\tilde{\pi}(t, x) &= \frac{1}{1 - x}\frac{x + A(t)}{x}(\mu(t) - r(t))\mathcal{E},\\
\tilde{p}_k(t, x) &= \begin{cases} \max\left\{0, n_k(t)\Big((B(t) - 1)x + A(t)B(t)\Big)\right\} & \text{, if} \quad k = \tilde{k}(t),\\ 0 & \text{, otherwise,}\end{cases}
\end{aligned}$$

where $\mathcal{E}$ is the non-singular square matrix given by $(\sigma\sigma^T)^{-1}$, and

$$
\begin{aligned}
A(t) &= \int_t^T i(s) \exp\left(-\int_t^s (r(u) + \eta_{\tilde{k}}(u))\, du\right) ds,\\
B(t) &= \frac{1}{\xi(t)} \left(\frac{\lambda(t)}{\eta_{\tilde{k}}(t)}\right)^{\dfrac{-1}{\gamma-1}},\\
\xi(t) &= \exp\left(-\int_t^T D(u)\, du\right) + \int_t^T \exp\left(-\int_t^s D(u)\, du\right) E(s)\, ds,\\
D(t) &= \frac{\lambda(t)+\rho}{1-\gamma} - \frac{\gamma}{1-\gamma}\left(r(t) + \eta_{\tilde{k}}(t)\right) - \frac{\gamma}{(1-\gamma)^2} H(t),\\
E(t) &= 1 + \left(\frac{\lambda(t)}{(\eta_{\tilde{k}}(t))^{\gamma}}\right)^{\dfrac{-1}{\gamma-1}},\\
H(t) &= \alpha^T(t)\alpha(t)\mathcal{E} - \frac{1}{2}\|\sigma^T\alpha(t)\mathcal{E}\|^{\epsilon}.
\end{aligned}
$$

*Proof.* By Theorem 4.2, we have the subsequent results for optimal strategies:

$$
\tilde{c}(t,x) = \left(e^{\rho t}\frac{\partial V}{\partial x}(t,x)\right)^{\dfrac{1}{\gamma-1}}, \tag{4.33}
$$

$$
\tilde{\pi}(t,x) = -\frac{\dfrac{\partial V}{\partial x}(t,x)}{x\,\dfrac{\partial^2 V}{\partial x^2}(t,x)} \cdot \mathcal{E} \cdot \alpha(t), \tag{4.34}
$$

$$
\tilde{p}_k(t,x) = \begin{cases} \max\left\{0, \left(\left(\dfrac{e^{\rho t}\,\eta_k(t)\,\dfrac{\partial V}{\partial x}(t,x)}{\lambda(t)}\right) - x\right)\eta_k(t)\right\} & ,\text{if} \quad k = \tilde{k}(t),\\ 0 & ,\text{otherwise,} \end{cases} \tag{4.35}
$$

for each $k \in \{1, 2, \ldots, K\}$.

Substituting these results into the HJB equation in Eqn. (4.23), we obtain the following PDE:

$$
\begin{aligned}
&\frac{\partial V}{\partial t}(t,x) - \lambda(t)V(t,x) + \left(i(t) + \left(r(t) + n_{\tilde{k}}(t)\right)x\right)\frac{\partial V}{\partial x}(t,x)\\
&-H(t)\frac{\left(\dfrac{\partial V}{\partial x}(t,x)\right)^2}{\dfrac{\partial^2 V}{\partial x^2}(t,x)} + \frac{1-\gamma}{\gamma}e^{\rho t/(\gamma-1)}E(t)\left(\frac{\partial V}{\partial x}(t,x)\right)^{\gamma/(\gamma-1)} = 0,
\end{aligned} \tag{4.36}
$$

with the terminal condition

$$
V(T,x) = U_3(x).
$$

Now, we make a guess for the value function $V$ such as

$$V(t,x) = M(t)\frac{(x+A(t))^{\gamma}}{\gamma}. \tag{4.37}$$

Then, we get the first-order and the second-order partial derivatives such that

$$\frac{\partial V}{\partial t} = \frac{1}{\gamma}\ (x+A(t))^{\gamma}\ \frac{dM(t)}{dt} + M(t)\ (x+A(t))^{\gamma-1}\ \frac{dA(t)}{dt},$$

$$\frac{\partial V}{\partial x} = M(t)\ (x+A(t))^{\gamma-1},$$

$$\frac{\partial^2 V}{\partial x^2} = M(t)\ (\gamma-1)(x+A(t))^{\gamma-2}.$$

Substituting these partial derivatives into the PDE Eqn. (4.36), we can rewrite Eqn. (4.36) as

$$\begin{aligned}\frac{1}{\gamma}\ \frac{dM(t)}{dt} + \frac{M(t)}{x+A(t)}\frac{dA(t)}{dt} - \frac{\lambda(t)}{\gamma}\ M(t) + \frac{(i(t)+(r(t)+\eta_{\tilde{k}}(t))\,x)}{x+A(t)}M(t)\\ +H(t)\frac{M(t)}{1-\gamma} + e^{\rho t/(\gamma-1)}\ \frac{1-\gamma}{\gamma}E(t)\,(M(t))^{\gamma/(\gamma-1)} = 0,\end{aligned}$$

which is equal to

$$\begin{aligned}\frac{1}{\gamma}\ \frac{dM(t)}{dt} - \frac{\lambda(t)}{\gamma}\ M(t) + H(t)\frac{M(t)}{1-\gamma} + e^{\rho t/(\gamma-1)}\ \frac{1-\gamma}{\gamma}E(t)\,(M(t))^{\gamma/(\gamma-1)}\\ +\frac{M(t)}{x+A(t)}\left(\frac{dA(t)}{dt} + (i(t)+(r(t)+\eta_{k^*}(t))\,x)\right) = 0,\end{aligned}$$

i.e.,

$$\begin{aligned}\frac{1}{\gamma}\ \frac{dM(t)}{dt} - \frac{\lambda(t)}{\gamma}\ M(t) + H(t)\frac{M(t)}{1-\gamma} + e^{\rho t/(\gamma-1)}\ \frac{1-\gamma}{\gamma}E(t)\,(M(t))^{\gamma/(\gamma-1)}\\ +r(t)M(t) + \eta_{\tilde{k}}(t)M(t) - r(t)M(t) - \eta_{\tilde{k}}(t)M(t)\\ +\frac{M(t)}{x+A(t)}\frac{dA(t)}{dt} + \frac{M(t)}{x+A(t)}i(t) + \frac{M(t)}{x+A(t)}r(t)x + \frac{M(t)}{x+A(t)}\eta_{\tilde{k}}(t)x = 0.\end{aligned}$$

Consequently, we have

$$\frac{1}{\gamma}\,\frac{dM(t)}{dt} - \frac{\lambda(t)}{\gamma}\,M(t) + H(t)\frac{M(t)}{1-\gamma} + e^{\rho t/(\gamma-1)}\,\frac{1-\gamma}{\gamma}E(t)\left(M(t)\right)^{\gamma/(\gamma-1)}$$

$$+ r(t)M(t) + \eta_{\tilde{k}}(t)M(t) - r(t)M(t) - \eta_{\tilde{k}}(t)M(t) + \frac{M(t)}{x+A(t)}\frac{dA(t)}{dt}$$

$$+ \frac{M(t)}{x+A(t)}i(t) + \frac{M(t)}{x+A(t)}r(t)x + \frac{M(t)}{x+A(t)}\eta_{\tilde{k}}(t)x = 0.$$

Hence, this gives us the following ODEs

$$\frac{1}{\gamma}\,\frac{dM(t)}{dt} + M(t)\left(-\frac{\lambda(t)}{\gamma} + \frac{H(t)}{1-\gamma} + r(t) + n_{\tilde{k}}(t)\right)$$

$$+e^{\rho t/(\gamma-1)}\,\frac{1-\gamma}{\gamma}E(t)\left(M(t)\right)^{\gamma/(\gamma-1)} = 0 \qquad (4.38)$$

$$M(T) = e^{-\rho T},$$

and

$$\frac{dA(t)}{dt} - \left(r(t) + \eta_{\tilde{k}}(t)\right)A(t) + i(t) = 0, \qquad (4.39)$$

$$A(T) = 0.$$

We now suppose $M(t)$ in Eqn. (4.38) as

$$M(t) = e^{-\rho t}\left(\xi(t)\right)^{1-\gamma},$$

and define

$$D(t) = \frac{\lambda(t)+\rho}{1-\gamma} - \frac{\gamma}{1-\gamma}\left(r(t) + \eta_{\tilde{k}}(t)\right) - \frac{\gamma}{(1-\gamma)^2}H(t),$$

where

$$H(t) = a^T(t)\mathcal{E}a(t) - \frac{1}{2}\|\sigma^T\mathcal{E}a(t)\|^2.$$

Thus, we obtain a new boundary value problem, i.e., a linear, first-order ODE,

$$\frac{d\xi(t)}{dt} - D(t)\xi(t) + H(t) = 0,$$

$$\xi(T) = 1. \qquad (4.40)$$

We can solve Eqn. (4.40) by using an integrating factor, and obtain the result

$$\xi(t) = \exp\left(-\int_t^T D(s)\,ds\right) + \int_t^T \exp\left(-\int_t^s D(s)\,ds\right)H(s)\,ds,$$

which gives

$$M(t) = e^{-\rho t}\left[\exp\left(-\int_t^T D(s)\,ds\right) + \int_t^T \exp\left(-\int_t^s D(s)\,ds\right) H(s)\,ds\right]^{1-\gamma}. \tag{4.41}$$

To find the solution of Eqn. (4.39), we again apply the same argument to obtain the following result

$$A(t) = \int_t^T i(s)\exp\left(-\int_t^s (r(u) + \eta_{\tilde{k}}(u))\,du\right) ds. \tag{4.42}$$

Evaluating Eqns. (4.33), (4.34) and (4.35), with (4.37), (4.41) and (4.42), we obtain the desired results. □

A remarkable result of Proposition 4.1 is that for a wage-earner who has a large amount of wealth and is close to retirement age, to buy no life-insurance is the optimal choice. Because, when $t$ approaches to $T$, $\xi(t)$ goes to 1, and $A(t)$ goes to 0. Moreover, provided that $\lambda(t) < \eta_{\tilde{k}}$, we have $B(t) < 1$. Therefore, the premium becomes negative. Since a negative life-insurance purchase is not more realistic, the wage earner who has a large amount of wealth and is close to retirement age, usually buys no life-insurance.

Now, we will show the optimal results with graphics obtained from Matlab. We consider a wage-earner who starts to work at age 25, and retires 40 years later, at age 65. At the beginning, his initial wealth is $\$50,000$, with growing parameter $\rho = 0.03$. His total wealth is denoted by $x + A(t)$, where $A(t)$ is called as *future income* of the wage-earner and calculated as in Proposition 4.1:

$$A(t) = \int_t^T i(s)\exp\left(-\int_t^s (r(u) + \eta_{\tilde{k}}(u))\,du\right) ds.$$

The other parameters were taken as $r = 0.04$, $\gamma = -3$, $\mu_1 = 0.09$, $\mu_2 = 0.1$, $\sigma_{11} = 0.2$, $\sigma_{12} = 0.18$, $\sigma_{21} = 0.18$ and $\sigma_{22} = 0.25$.

The hazard rate function of the model were taken as $\lambda(t) = \lambda_c(\lambda_1 + \exp(\lambda_2 + \lambda_3 t))$, where $\lambda_c = 1$, $\lambda_1 = 0.001$, $\lambda_2 = -9.5$ and $\lambda_3 = 0.1$. We know that the premium is fair when $\lambda(t) \leq \eta(t)$. So, to make profit, insurance company must establish his premium-payout ratio as $\lambda(t) \leq \eta(t)$. To prevent this, premium-payout ratio was taken as $\eta(t) = 1.05\lambda(t)$.

Using these parameters and the results of Proposition 4.1 with utility functions

$$U_1(t,c) = e^{-\rho t}\frac{c^\gamma}{\gamma}, \quad U_2(t,Z) = e^{-\rho t}\frac{Z^\gamma}{\gamma}, \quad \text{and} \quad U_3(X) = e^{-\rho t}\frac{X^\gamma}{\gamma}, \tag{4.43}$$

Figures 4.1-4.2 are obtained.

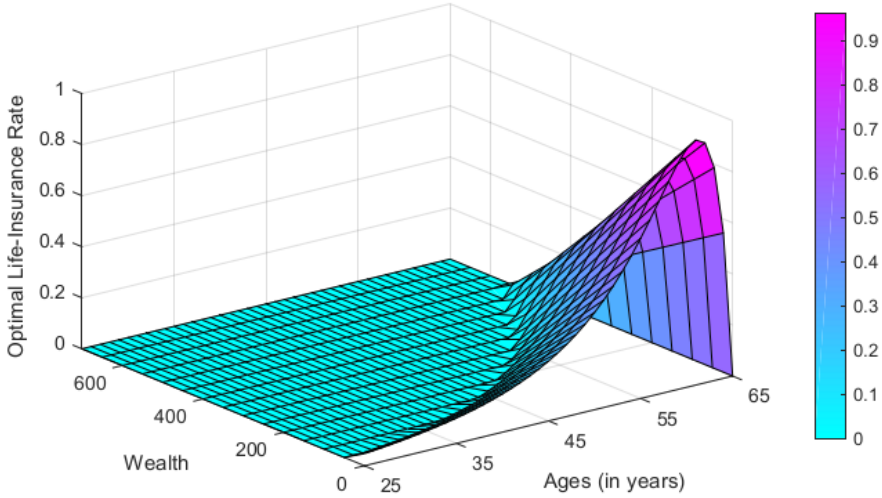


Figure 4.1: A wage-earner's optimal ptimal life-insurance purchase with respect to his age and total wealth.

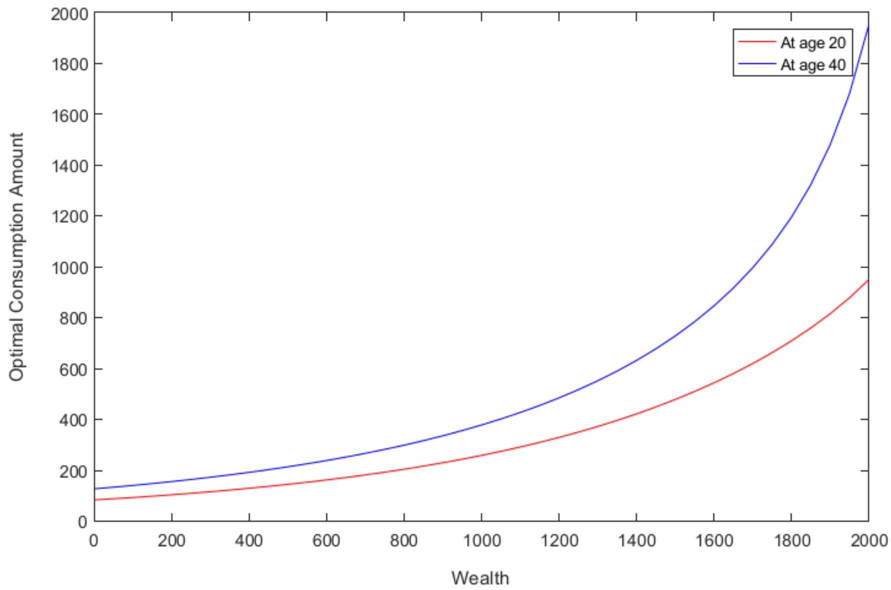


Figure 4.2: A wage-earner's optimal consumption amount with respect to his total wealth at ages 20 and 40.

In the Figure 4.1, we see that the optimal life-insurance purchase rate of an insurer is at higher rates if the wage-earnes has not sufficiently large wealth his age is close to his retirement. According to same figure, if he has large amount of wealth, his optimal decision is to buy no life insurancecontract from any insurance company. In the Figure 4.2, we compare the consumption amounts of the wage-earner at age 20 and 40 according to the different level of initial wealths. We see that as the initial wealth increases the consumption level increases. Moreover, we see that the consumption level increases as the retirement time approaches.

# CHAPTER 5

# CONCLUSION AND OUTLOOK

In this thesis, we investigate two fundamental methods of stochastic optimal control, namely Maximum principle and Dynamic Programming Principle, and indicate how these techniques are applied to finance and insurance problems. As a beginning, we set up notation and terminology of probability theory and stochastic calculus, following closely Cont [5], Papapanteleon [18], Øksendal and Sulem [17]. Then, we give a detailed summary of stochastic optimal control problems. In the following, we summarized the Maximum Principle methodology for general jump-diffusion processes without proofs, following closely Framstad et al. [9]. The Hamiltonian systems and adjoint equations were sketched and mean-variance portfolio selection problem was solved by MP. We also studied the DPP which reduces the stochastic optimal control problem to the problem of solving a second-order nonlinear PDE, called as Hamilton-Jacobi-Bellman Equation. Firstly, we derived the HJB equation for diffusion processes in a heuristic manner, and then we developed it for jump-diffusion processes using the similar arguments. The Verification Theorem, a sufficient condition for optimization problem, is also derived. The Verification Theorem says that if there exists a solution for the HJB equation, indeed it is an optimal solution and called as the value function. By using DPP, a financial application, Merton Portfolio Problem for Optimal Consumption, for both diffusion and jump-diffusion processes was studied. Moreover, the relationship between MP and DPP was briefly discussed.

Chapter 4 was devoted to applications of stochastic optimal control to the insurance sector. We examined two problems from insurance. In the first problem, the purpose was to find an optimal investment strategy and liability ratio for an insurer where the insurer's aggregate claims are represented by Lévy processes. By using MP, closed-form solutions for logarithmic utility function were obtained. In the second problem, the objective was to find an optimal life-insurance purchase, investment and consumption of a wage-earner, where the financial market consists of a fixed number of risky assets, and the insurance market comprises of multiple life-insurance providers. Using DPP, the optimal strategies were determined for discounted CRRA utility functions. As a result, buying a life-insurance from the insurance company that offers the smallest premium-payout ratio, or buying no life-insurance makes sense for the wage-earner. Another result obtained from this problem was that for a wage-earner who is close to retirement time and has a sufficiently large wealth, to buy no life-insurance is an optimal choice. Finally, using Maple and Matlab, we graphed the optimal results of life-insurance purchase problem as a contribution.

As a future work, we would like to solve the optimal investment, consumption and life-insurance purchase problem of Mousa et al. [16] from the perspective of insurance companies by investigating the optimal premium, loading factor and optimal premium-payout ratio. We also would like to study the same problem by developing the hazard rate models. This problem can also be extended to hybrid or regime-switch models, e.g., the models in which the mean rate of return can be considered in three different economical states. Another possible extension is to study the robust optimal control problems.

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