STOCHASTIC DELAY DIFFERENTIAL EQUATIONS

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ABSTRACT

STOCHASTIC DELAY DIFFERENTIAL EQUATIONS

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In many areas of science like physics, ecology, biology, economics, engineering, financial mathematics etc. phenomenas do not show their effect immediately at the moment of their occurrence. Generally, they influence the future states. In order to understand the structure and quantitative behavior of such systems, stochastic delay differential equations (SDDEs) are constructed while inserting the information that are obtained from the past phenomena into the stochastic differential equations (SDEs). SDDEs become a new interest area due to the their potential to capture reality better. It can be said that SDDEs are in the infancy stage when we consider the SDEs. Some numerical approaches to SDDEs are constructed because obtaining closed form solutions by the help of stochastic calculus is very difficult most of the time and for some equations it is impossible. In recent years, scientist who are interest in economy and finance study option pricing formulation for systems that include time delay which can be stochastic or deterministic. The aim of this thesis is to understand general forms of SDDEs and their solution process for the deterministic time delay. Some examples are provided to see the exact solution process. Moreover, we examine numerical techniques to obtain approximate solution processes. In order to understand effect of delay term, these techniques are used to simulate the solution process for different choices of delay terms and coefficients. In the application part of the thesis, we investigate the stock returns and European call option price when the system is modeled with SDDEs.

Keywords : Stochastic delay differntial equations, stochastic differntial equations with memory, Euler Maruyama scheme for stochastic delay differntial equations, effect of delay term in the stochastic differentail equations via simulations, effect of delay term in stock returns , effect of delay term in European option pricing with simulations

ÖZ

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Fizik, ekoloji, biyoloji, ekonomi, mühendeslik, finansal matematik gibi bir çok bilim alanında, olayların etkisi hemen gerçekleştikleri anda olmaz. Etkiler genellikle ilerleyen zamanlarda ortaya çıkar. Bu tarz sistemlerin yapılarını ve hareketlerini anlamak için, geçmiş olayların bilgi ve verilerini stokastik diferansiyel denklemlere ekleyerek geciklemli stokastik diferansiyel denklemler elde edilmiştir. Bu denklemlerin gerçeği daha iyi yansıtacağı düşünüldüğü için yeni bir araştırma alanı oluşturmuştur. Stokatik diferansiyel denklemlerle karşılaştırıldığı zaman, gecikmeli stokastik diferansiyel denklemlerin gelişimlerinin çok daha başlarında olduğu söylenebilir. Kapalı çözüm bulmak zor olduğu için bu tür denklemler için nümerik analiz metodları geliştirilmiştir. Son yıllarda finans ve ekonomi alanlarında çalışan bir çok bilim insanı, zaman bağlılığı içeren modeller için opsiyonların fiyatlandırılması üzerine çalışıyor. Bizim bu tezde ki amacımız ise rastgele olmayan (deterministik) zaman gecikmeleriyle elde edilmiş olan geciklemli stokastik diferansiyel denklemleri anlamak. Bu denklemlerin nasıl çözüldüğünü anlam için bazı örnekleri ele alacağız. Zamandaki gecikmenin etkisini görmek için farklı simulasyonlar yapacağız. Tezin son bölümünde ise, bu denklemler yardımıyla elde edilen modellerde ki hisse senedi getirilerini ve Avrupa tarzında ki alım opsiyonlarının fiyatlarını inceleyeceğiz.

Anahtar Kelimeler: gecikmeli stokastik diferansiyel denklemler, gecikmeli stokastik diferansiyel süreçler, gecikmeli stokastik diferansiyel süreçler için Euler Maruyama tekniği

To My Family

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CHAPTER 1

INTRODUCTION

In many areas of science we need to construct models to understand the structure and behavior of systems. These models include some functions (parameters) and their derivatives where the parameters correspond to physical quantities. In such deterministic models, the parameters are completely known and they are called as ordinary differential equations (ODEs). However, we generally do not have sufficient information on the parameters because of noise. Therefore, stochastic models are constructed with the addition of that noise term into the deterministic model namely, stochastic (ordinary) differential equations (S(O)DEs). These type of modelings may offer a more sophisticated intuition about real-life phenomena than their deterministic counterparts do. Thus, SDEs have an important role in many application areas including biology, physics, engineering and finance. As Mao (2007) states, in many applications of SDEs, it is assumed that the system fulfills the principle of causality which means the future state of the system is determined only by the present i.e., it does not depend on the past, see [40] for further details. However, in many areas of science like medicine, physics, ecology, biology, economics, engineering, etc., phenomena do not show their effect at the moment of their occurrence. Consider the simplest example, "a patient shows symptoms of an illness, days (or even weeks) after the day in which he or she was infected" [41]. Hence, it can be said that SDEs, with principle of causality, give only an approximation to the real-life situations. Thus, some extra term, namely, time delay that is obtained from the past states of the system could be added in the model to create a more realistic one. Stochastic delay differential equations (SDDEs) give a mathematical formulation for such a system and in many areas of science, there is an increasing interest in the investigation of SDDEs. Moreover, SDDEs are actually a generalization of both deterministic delay differential equations (DDEs) and stochastic ordinary differential equations (SODEs). For the application of SDDEs, one could refer to [8, 9, 51] for applications in biology, [20, 38, 46] for applications in biophysic, [13, 42] for applications in physics, [19, 23] for applications in engineering and [2, 18, 27] for applications in finance and economy.

In this thesis, as a motivation of our study we consider an example from the financial market. In recent years, SDEs take an important role for valuation of financial assets. The advances in the theory of SDEs bring solutions to the many sophisticated pricing problems. In these models for asset prices, efficient market hypothesis is taken into consideration as a basic assumption. According to that hypothesis [40]:

- All available historical information is examined and already reflected in the present price of the stock and they give no information about future performance,
- Markets respond immediately to any new information about an asset, i.e., asset prices move randomly.

With these two assumptions, changes in the asset price define a Markov process. However, Scheinkman and LeBaron (1989) showed that stock returns depend on the past returns [**?**]. In the article [50], it is stated that the trader expects the stock price to follow a Black-Scholes diffusion process while the insider knows that both the drift and the volatility of the stock price process are influenced by certain events that happened before the trading period started. They wish to set a pricing model which includes that past information to evaluate present and to forecast future price. In that case, they can predict the market movement and make better investments. However, this is not possible via SDEs because of the efficient market hypothesis. By introducing that information as a delay term into the SDE, more realistic mathematical formulation, SDDE, is obtained for the evaluation of asset prices. With this new model, instead of assuming SDE as a model and the Markov property, it is actually assumed that the future asset price depends also on the historical states not only on the current state.

For the SDDEs, explicit solutions can hardly be obtained and in general they do not have a closed-form analytic solution. As a result, we need numerical techniques to produce approximate solution process and understand its quantitative behavior. These numerical analysis methods for SDDEs are actually based on the numerical analysis of DDEs and the numerical analysis of SDEs. For the numerical treatment of DDEs, one can see [3, 7, 22] . There is extensive work on the numerical treatment of SDEs, we can refer to [26, 33, 40, 47]. However, the numerical analysis of SDDEs only recently attracted attention and it is not a straight forward generalization of numerical analysis of DDEs and SDEs. Kühler and Platen (2000) [34] derived strong discrete time approximations of SDDEs. Baker and Buckwar (2005) [6] did a detailed convergence analysis for explicit one-step methods and a number of numerical stability results have been derived [37]. One can also see [5, 16, 24, 41]. The Euler schemes are stated in [5, 29, 34, 37, 52] and the Milstein schemes in [11, 28, 32]. Arriojas et al. (2007) in [2] provide a closed form formula for the fair price of European call options when the stock price follows nonlinear stochastic delay differential equations. The delay term in the model can be fix or variable. Meng et al. (2008) extend this formula in [43].

This thesis generally focuses on the mathematical foundations, constructions of SDDEs and numerical approximate solution technique namely Euler Maruyama scheme for SDDEs.

In the first chapter, we explain the need for and importance of SDDEs and give a short literature review.

In the second chapter, we discuss the existence and uniqueness theorem for the solution process of SDDEs and properties of them. To understand how the solution process is obtained if it exists, some examples are given. Moreover, we try to obtain a closed form solutions for these examples. In the last stage of this chapter, we give a comparison study between SDDEs and SDEs.

In the third chapter, some numerical methods for finding approximate solution of SDDEs are introduced and the properties of these methods are studied. With the help of Euler Maruyama scheme, solution processes of some SDDEs are derived to see the effect of the delay terms in the equations.

In the fourth chapter, we give two applications of SDDEs motivated by real-life examples in finance. In the first application, we model stock returns using an SDDE with the linear delay and we analyze the structure of this model. In the second one, we discuss the European option pricing model with SDDE and provide some numerical results to show the time delay effect.

In the last chapter, as a conclusion consequences of this study are provided with possible future works.

CHAPTER 2

STOCHASTIC DELAY DIFFERENTIAL EQUATIONS

The aim of this chapter is to give general information about stochastic delay differential equations (SDDEs). These equations have complicated characteristics; therefore, it is better to start with discussing the properties of stochastic differential equations (SDEs), which are the special case of SDDEs. Once the behavior of SDEs is understood, it is easier to follow the fundamental properties of SDDEs. For this reason we start with building notation and terminology on SDEs. Moreover, the necessity for their existence, the general definition of SDE, conditions to have a unique solution and the properties of that solution will be discussed. Also, we give some examples to make these concepts clear. For detailed information and proofs one can see [21, 35, 40, 45].

In the second part of this chapter, we introduce the notion of SDDEs. Firstly, we introduce vector valued SDDEs. For simplicity, we restrict our attention to real valued SDDEs. Secondly, the existence and properties of solutions for such kind of equations will be discussed. Finally, examples will be provided to make clear the concepts. For detailed information and proofs one can see : [5, 10, 39, 40, 44, 53].

In order to see the time delay effect, we conclude the chapter by comparing the solutions of the examples given for SDEs and SDDEs.

2.1 Stochastic Differential Equations

A mathematical equation that includes functions and their derivatives is called differential equation. In real-life applications, this functions generally corresponds to physical quantities while the derivatives represent their rates of change. With the help of differential equation, we can show relationship between them. Let us consider the simple population growth model as an example. Suppose $N(t)$ denotes the population size at time t, $\alpha(t)$ is the deterministic relative growth rate at time t, $\dfrac{dN}{dt}$ is the rate of change of the population size and N_0 is the given initial data value. Then, the corresponding differential equation is:

$$\frac{dN}{dt} = \alpha(t)N(t), \quad N(0) = N_0.$$

This equation means the rate of change of the population at time t is equal to multiplication of the growth rate and the population at that time . However, usually $\alpha(t)$ is not known completely and it is subject to some environmental effects. Thus, it can be written as

$$\alpha(t) = r(t) + \beta\, noise,$$

where $r(t)$ is deterministic term, β is a real valued constant number and noise corresponds to the random term (the behavior is not exactly known, only probability distribution is known). This noise term is generally taken as a white noise which is related with a Brownian motion, $W(t)$. Then the equation can be rewritten as

$$\begin{aligned} \frac{dN}{dt} &= (r(t) + \beta\, noise)N(t), \\ &= \big(r(t) + \beta\, W(t)\big)N(t); \end{aligned}$$

this implies that

$$dN(t) = r(t)N(t)dt \; + \; \beta N(t)dW(t).$$

Because of this noise term, we call that differential equation as stochastic differential equation (SDE). Generally, differential equations that include randomness in the coefficients are called SDEs. Actually, adding randomness leads to a model with more realistic form.

2.1.1 Existence and Uniqueness of Stochastic Differential Equations

After discussing the importance of SDEs, we are ready to give the definition of SDE. The definition will be followed by the existence and uniqueness theorem.

Definition 2.1. Assume that $(\Omega, \mathcal{F}, P)$ is a probability space with filtration $\{\mathcal{F}_t\}_{t\geq 0}$ and $W(t) = (W_1(t), W_2(t), ..., W_m(t))^T, t \geq 0$, be an m-dimensional Brownian motion on that given probability space. SDE with coefficient functions f and g is in the form of :

$$\left.\begin{array}{ll} dX(t) = f(t, X(t))dt + g(t, X(t))dW(t), & 0 \leq t \leq T, \\ X(0) = x_0, & \end{array}\right\} \tag{2.1}$$

where $T > 0,\ x_0$ is an n-dimensional random variable and the coefficient functions are in the form of $f : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n,$ and $g : [0, T] \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$.

Let us give some observations:

- That SDE can be equivalently written in the integral form as:

$$X(t) = x_0 + \int_0^t f(s, X(s))ds + \int_0^t g(s, X(s))dW(s)$$

- dX and dW terms in (2.1) are called stochastic differentials; because of that reason we call that differential equation as stochastic differential equation.
- An $\mathbb{R}^n$-valued stochastic process $X(t)$, satisfing equation (2.1), is called a solution of the SDE.

Now, let us state the conditions so that solution of equation (2.1) exists and properties of that solution.

Theorem 2.1. *Let $T > 0$ be a given final time and assume that the coefficient functions $f : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ and $g : [0, T] \times \mathbb{R}^n \to \mathbb{R}^{n \times m}$ are continuous. Moreover, there exists finite constant numbers K and L such that $\forall t \in [0, T]$ and for all $x, y \in \mathbb{R}^n$, the drift and diffusion terms satisfy*

$$||f(t, x) - f(t, y)|| + ||g(t, x) - g(t, y)|| \leq K||x - y||, \tag{2.2}$$

$$||f(t, x)|| + ||g(t, x)|| \leq L(1 + ||x||). \tag{2.3}$$

Suppose also that x_0 is any $\mathbb{R}^n$-valued random variable such that $E(||x_0||^2) < \infty$. Then the above stochastic differential equation has a unique solution X in the interval $[0, T]$. Moreover, it satisfies

$$E\left(\sup_{0 \leq t \leq T} ||X(t)||^2\right) < \infty$$

The detailed proof of the theorem can be found in [35] and [45].

- Condition in (2.2) means that f and g satisfy uniformly Lipschitz continuous with respect to second variable x while condition in (2.3) implies f and g satisfy linear growth condition.
- Assume that X and $\tilde{X}$ are two solutions to the same SDE with continuous sample paths. Since the solution is unique, they satisfy:

  $$P\left(X(t) = \tilde{X}(t), \quad \forall t \in [0, T]\right) = 1.$$

- If the coefficient functions f and g are in the form of

  $$f(t, x) := a(t) + b(t)x,$$
  $$g(t, x) := c(t) + d(t)x,$$

  then we say that equation (2.1) defines a linear SDE.
- If $a \equiv c \equiv 0$ for $0 \leq t \leq T$, then the linear SDE is called homogeneous.

Now, let us discuss some examples to understand solution strategy better.

2.1.2 Examples of Stochastic Differential Equations

The solution of SDEs can be obtained by Itô formula. Let us consider some examples and corresponding solutions to clarify the solution technique.

Example 2.1. (Geometric Brownian Motion) Assume that $S(t)$ denote the stock price at time $t \geq 0$ which changes randomly. The dynamics of the price of the stock is given as:

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma dW(t), \quad t \geq 0,$$

the $\dfrac{dS}{S}$ denotes the relative change of price, $\mu > 0$ is the drift term, σ corresponds to diffusion term (it can be considered as volatility) and $W(t)$ is a standard Brownian motion. In order to obtain a unique solution if the solution exists, we need an initial value. Thus, assume $S(0) = s_0$ is the given initial stock price. Then we actually have the following equalities:

$$\begin{aligned} dS(t) &= \mu S(t)dt + \sigma S(t)dW(t), \quad t \geq 0, \\ S(0) &= s_0, \end{aligned}$$

where $f(t,x) = \mu x$ and $g(t,x) = \sigma x$ according to general definition of SDE. Before finding the solution, let us check the conditions for the existence and uniqueness theorem:

$$\begin{aligned} |f(t,x) - f(t,y)| + |g(t,x) - g(t,y)| &= |\mu x - \mu y| + |\sigma x - \sigma y| \leq (|\mu| + |\sigma|)|x - y|, \\ |f(t,x)| + |g(t,x)| &= |x|(|\mu| + |\sigma|) \leq (1 + |x|)(|\mu| + |\sigma|). \end{aligned}$$

Therefore, this SDE with the given initial data has a unique solution. As we said before, we are going to use Itô formula to find this unique solution. Let us choose $F(x) = \ln x$ and apply Itô formula :

$$F(S(t)) = F(S(0)) + \int_0^t F'(S(u))dS(u) + \frac{1}{2}\int_0^t F''(S(u))d < S(u), S(u) >,$$

where F' and F'' are the derivatives of the function F of order one and two, respectively. This implies to:

$$\begin{aligned} \ln S(t) &= \ln S(0) + \int_0^t \frac{1}{S(u)} S(u)\left[\mu du + \sigma dW(u)\right] - \frac{1}{2}\int_0^t \frac{1}{S^2(u)}\sigma^2 S^2(u)du \\ &= \ln S(0) + \int_0^t \mu du + \int_0^t \sigma dW(u) - \frac{1}{2}\int_0^t \sigma^2 du \\ &= \ln S(0) + \mu t + \sigma W(t) - \frac{1}{2}\sigma^2 t. \end{aligned}$$

Therefore, the solution process is

$$S(t) = S(0)e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W(t)}.$$

Now let us find the expected value of the solution:

$$\begin{aligned} E(S(t)) &= E\Big(S(0)e^{(\mu-\frac{1}{2}\sigma^2)t+\sigma W(t)}\Big) \\ &= S(0)e^{(\mu-\frac{1}{2}\sigma^2)t}E\big(e^{\sigma W(t)}\big) \\ &= S(0)e^{(\mu-\frac{1}{2}\sigma^2)t}e^{\frac{1}{2}\sigma^2 t} \\ &= S(0)e^{\mu t}. \end{aligned}$$

Moreover, the variance of the solution is given by:

$$\begin{aligned} Var(S(t)) &= Var\Big(S(0)e^{(\mu-\frac{1}{2}\sigma^2)t+\sigma W(t)}\Big) \\ &= S^2(0)e^{2(\mu-\frac{1}{2}\sigma^2)t}Var\big(e^{\sigma W(t)}\big) \\ &= S^2(0)e^{2(\mu-\frac{1}{2}\sigma^2)t}\Big[E(e^{2\sigma W(t)}) - \big(E(e^{\sigma W(t)})\big)^2\Big] \\ &= S^2(0)e^{2(\mu-\frac{1}{2}\sigma^2)t}\big[e^{2\sigma^2 t} - e^{\sigma^2 t}\big] \\ &= S^2(0)e^{2\mu t}[e^{\sigma^2 t} - 1]. \end{aligned}$$

Remark 2.1. Since S has a Markovian property, it is also possible to write the solution as follows:

$$S(t) = S(u)e^{(\mu-\frac{1}{2}\sigma^2)(t-u)+\sigma(W(t)-W(u))}, \quad t \geq u.$$

Because of the same property, the conditional expectation and variance can be written, respectively, as

$$\begin{aligned} E(S(t)|\mathcal{F}_u) &= S(u)e^{\mu(t-u)}, \quad t \geq u, \\ Var(S(t)|\mathcal{F}_u) &= S^2(u)e^{2\mu(t-u)}[e^{\sigma^2(t-u)} - 1], \quad t \geq u. \end{aligned}$$

Example 2.2. Let us take the drift term in the Example 2.1 as zero and the initial time point as t_0 instead of 0. Then the equation becomes:

$$\begin{aligned} dS(t) &= \sigma S(t)dW(t), \quad t \geq t_0, \\ S(t_0) &= s_0. \end{aligned}$$

Applying Remark 2.1, we obtain the solution

$$S(t) = s_0 e^{\sigma(W(t)-W(t_0))-\frac{1}{2}\sigma^2(t-t_0)}.$$

Now let us take the conditional expectation of both sides to find the conditional mean of the solution:

$$\begin{aligned} E(S(t)|\mathcal{F}_{t_0}) &= E\Big(s_0 e^{\sigma(W(t)-W(t_0))-\frac{1}{2}\sigma^2(t-t_0)}|\mathcal{F}_{t_0}\Big) \\ &= s_0 e^{-\frac{1}{2}\sigma^2(t-t_0)}E\big(e^{\sigma(W(t)-W(t_0))}|S(t_0) = s_0\big) \\ &= s_0 e^{-\frac{1}{2}\sigma^2(t-t_0)}e^{\frac{1}{2}\sigma^2(t-t_0)} \\ &= s_0. \end{aligned}$$

Moreover, the conditional variance of the solution is obtained as follows:

$$\begin{aligned}
Var(S(t)|\mathcal{F}_{t_0}) &= Var\left(s_0 e^{\sigma(W(t)-W(t_0))-\frac{1}{2}\sigma^2(t-t_0)}|\mathcal{F}_{t_0}\right)\\
&= s_0^2 e^{-\sigma^2(t-t_0)} Var\left(e^{\sigma(W(t)-W(t_0))}|\mathcal{F}_{t_0}\right)\\
&= s_0^2 e^{-\sigma^2(t-t_0)}\left[E\big(e^{2\sigma W(t-t_0)}|S(t_0)=s_0\big) - E^2\big(e^{\sigma W(t-t_0)}|S(t_0)=s_0\big)\right]\\
&= s_0^2 e^{-\sigma^2(t-t_0)}\big[e^{2\sigma^2(t-t_0)} - e^{\sigma^2(t-t_0)}\big]\\
&= s_0^2\big[e^{\sigma^2(t-t_0)} - 1\big].
\end{aligned}$$

Example 2.3. Let us consider another example such that drift term is a function of random variable X while the diffusion term is a constant number, $\beta,$ with initial value x_0

$$\begin{aligned}
&dX(t) = X(t)dt + \beta dW(t), \quad t \geq t_0,\\
&X(t_0) = x_0.
\end{aligned}$$

Note that this is a kind of Ornstein-Uhlenbeck process. Let us firstly substitute $Y(t) = e^{-t}X(t)$ and take the derivative of both sides with respect to parameter t:

$$\begin{aligned}
dY(t) &= -e^{-t}X(t)dt + e^{-t}dX(t),\\
&= -e^{-t}X(t)dt + e^{-t}\big[X(t)dt + \beta dW(t)\big],\\
&= \beta e^{-t}dW(t).
\end{aligned}$$

According to the general definition of SDE, drift and diffusion terms corresponds to $f(t,x) = 0$ and $g(t,x) = \beta e^{-t}$, respectively. Now let us check whether they satisfy the Lipschitz condition and the linear growth condition:

$$\begin{aligned}
|f(t,x) - f(t,y)| + |g(t,x) - g(t,y)| &= 0 \leq |x-y|\\
|f(t,x)| + |g(t,x)| &= \beta e^{-t} \leq (1+|x|)\,\beta e^{-t}
\end{aligned}$$

Hence, this SDE has a unique solution. Now, we apply Itô formula for $F(x) = x$:

$$\begin{aligned}
Y(t) &= Y(t_0) + \int_{t_0}^{t} 1\, dY(u), \qquad (\text{since } F''(x) = 0 \text{ for all } x \in \mathbb{R})\\
&= Y(t_0) + \int_{t_0}^{t} \beta e^{-u} dW(u).
\end{aligned}$$

Now apply the back-substitution $Y(t) = X(t)e^{-t}$ to find solution $X(t)$:

$$\begin{aligned}
X(t)e^{-t} &= X(t_0)e^{-t_0} + \beta\int_{t_0}^{t} e^{-u}dW(u),\\
X(t) &= x_0 e^{t-t_0} + \beta\int_{t_0}^{t} e^{t-u}dW(u).
\end{aligned}$$

Moreover, the conditional expectation of the solution is given by:

$$\begin{aligned} E(X(t)|\mathcal{F}_{t_0}) &= E\bigg(x_0 e^{t-t_0} + \beta\int_{t_0}^{t} e^{t-u} dW(u)|\mathcal{F}_{t_0}\bigg) \\ &= x_0 e^{t-t_0} + \beta E\bigg(\int_{t_0}^{t} e^{t-u} dW(u)|X(t_0) = x_0\bigg) \\ &= x_0 e^{t-t_0}; \end{aligned}$$

since $\displaystyle\int_{t_0}^{t} e^{t-u} dW(u)$ is a martingale, it implies $E(\displaystyle\int_{t_0}^{t} e^{t-u} dW(u)) = 0$. Moreover, let us compute the variance of the solution using the fact that the variance of a constant is zero:

$$\begin{aligned} Var(X(t)|\mathcal{F}_{t_0}) &= Var\bigg(x_0 e^{t-t_0} + \beta\int_{t_0}^{t} e^{t-u} dW(u)|\mathcal{F}_{t_0}\bigg) \\ &= Var\bigg(\beta\int_{t_0}^{t} e^{t-u} dW(u)|\mathcal{F}_{t_0}\bigg) \\ &= \beta^2\bigg[E\bigg([\int_{t_0}^{t} e^{t-u} dW(u)]^2|\mathcal{F}_{t_0}\bigg) - \bigg(E(\int_{t_0}^{t} e^{t-u} dW(u)|\mathcal{F}_{t_0})\bigg)^2\bigg] \\ &= \beta^2 E\bigg([\int_{t_0}^{t} e^{t-u} dW(u)]^2|\mathcal{F}_{t_0}\bigg) \qquad \text{(since } E(\int_{t_0}^{t} e^{t-u} dW(u)) = 0) \\ &= \beta^2 E\bigg(\int_{t_0}^{t} e^{2(t-u)} d(u)|\mathcal{F}_{t_0}\bigg) \qquad \text{(by } It\hat{o} \text{ isometry)} \\ &= \beta^2 \int_{t_0}^{t} e^{2(t-u)} du \\ &= \frac{\beta^2}{2}\big[e^{2(t-t_0)} - 1\big]. \end{aligned}$$

Remark 2.2. Let us consider the following SDE:

$$\begin{aligned} dX(t) &= -cX(t)dt + \beta dW(t), \quad t \geq 0, \\ X(0) &= x_0, \end{aligned}$$

where c and β are any real numbers, x_0 is the given initial path and W is a standard Brownian motion. When we solve this equation, we get the solution

$$X(t) = x_0 e^{-ct} + \beta e^{-ct}\int_0^t e^{cs} dW(s).$$

This solution process is called an Ornstein-Uhlenbeck process and this is one of the most important and used process. The equation in Example 2.3 defines a Ornstein-Uhlenbeck process with $c = -1$.

2.2 Stochastic Delay Differential Equations

In the previous section, we mentioned about SDEs and stressed that they are imposed because of the weakness of ODEs. However, SDEs also include some weaknesses. For example, consider the finance sector; the traders wants to foresee the market movements (since the parameters in market behave randomly) and predict the risks before making their investments. Thus, they use past market informations that are available and make some statistical inferences. However, those historical data cannot be used in the SDE which is the model used for understanding the future market movements. We call this past information as *delay* or *memory* and it is generally denoted by τ. This delay term can be deterministic or stochastic. While adding this delay term in the model, we get a more realistic and better model. The equations with this additional term are called stochastic delay differential equations (SDDEs). Consider the stock price model in Example 2.1 again. The delay can be added only into the drift term, then equation becomes

$$dS(t) = f(t, S(t), S(t-\tau))dt + \sigma S(t)dW(t),$$

where $f : \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. Or, it can affect only the diffusion term and we obtain

$$dS(t) = \mu S(t)dt + g(t, S(t), S(t-\tau))dW(t),$$

where $g : \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. Adding the delay in both terms is also possible and our SDE becomes

$$dS(t) = f(t, S(t), S(t-\tau))dt + g(t, S(t), S(t-\tau))dW(t).$$

In these three cases, the new form of the equation defines an SDDE. Now let us see the general formulation of this kind of equations.

2.2.1 Existence and Uniqueness of Stochastic Delay Differential Equations

In this section, we first give the general formulation of stochastic delay differential equations. Second, the existence and uniqueness theorem for SDDEs will be discussed. Before giving this theorem, we introduce some definitions to understand the conditions clearly.

Definition 2.2. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space with filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions and $W(t) = (W_1(t), W_2(t), ..., W_m(t))^T, t \geq 0$, be an m-dimensional standard Brownian motion on that probability space $(\Omega, \mathcal{F}, P)$. The stochastic delay differential equations (SDDEs) with a fixed time horizon $T > 0$ are in the form of:

$$\left.\begin{array}{l} dX(t) = F(t, X(t), X(t-\tau))dt + G(t, X(t), X(t-\tau))dW(t),\ t \in [0, T] \\ X(t) = \varphi(t), \quad t \in [-\tau, 0], \end{array}\right\} \tag{2.4}$$

where the delay τ is fixed positive finite number and the initial path $\varphi(t) : [-\tau, 0] \to \mathbb{R}^n$ is assumed to be a continuous and $\mathcal{F}_0$-measurable random variable such that $\left[E\left(\sup_{t\in[-\tau,0]} |\varphi(t)|^p\right)\right]^{\frac{1}{p}} < \infty$. The drift and diffusion functions in the equation are given as $F : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and $G : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$, respectively.

- The filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions means that it is increasing and right continuous, and each $\mathcal{F}_t$ contains all *P*-null sets in $\mathcal{F}$ for all $t \geq 0$.
- SDDEs are actually a kind of stochastic functional differential equations (SFDEs). Stochastic delay differential equation, stochastic differential equation with delay, stochastic differential equation with memory all have the same meaning.
- $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^n$ and $||\cdot||$ denotes the corresponding induced matrix norm. $Z \in L_p(\Omega, \mathbb{R}^n)$ means that $E(|Z|^p) < \infty$. The L_p-norm of a random variable $Z \in L_p(\Omega, \mathbb{R}^n)$ will be denoted by $||Z||_p := (E(|Z|^p))^{\frac{1}{p}}$, where E is expectation with respect to probability measure P.
- If the function F and G do not depend on t explicitly, we call the above SDDE as autonomous SDDE.
- If the function g does not depend on X, we say that the above equation has an additive noise. If g depends on X , the equation has multiplicative noise.
- It is obvious that SDDE defines a SDE when τ is equal to 0. This means that SDEs are actually a kind of SDDE.
- If solution $X(t)$ of the equation (2.4) exists, it will be vector valued, i.e., $X(t) \in \mathbb{R}^n$.

In this thesis we restrict our attention on the real valued SDDEs, i.e, we are going to set $n = m = 1$. Now let us state the definition of the real valued SDDEs.

Definition 2.3. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space with filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions and $W(t), t \geq 0$, be a standard Brownian motion on the given probability space. Then SDDE is in the form of:

$$\left.\begin{array}{l} dX(t) = f(t, X(t), X(t-\tau))dt + g(t, X(t), X(t-\tau))dW(t),\ t \in [0, T] \\ X(t) = \varphi(t), \quad t \in [-\tau, 0], \end{array}\right\} \quad (2.5)$$

where the delay τ is fixed positive finite number and $\varphi(t) : [-\tau, 0] \to \mathbb{R}$ is initial path and it is assumed to be a continuous and $\mathcal{F}_0$-measurable random variable such that $\left(E\big(\sup_{t\in[-\tau,0]} |\varphi(t)|^p\big)\right)^{\frac{1}{p}} < \infty$. The drift and diffusion functions in the equation are given as $f : \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, respectively.

Remark 2.3. If real valued solution $X(t)$ of the SDDE exists, then the integral form of (2.5) can be written as:

$$X(t) = \varphi(0) + \int_0^t f(s, X(s), X(s-\tau))ds + \int_0^t g(s, X(s), X(s-\tau))dW(s).$$

This equation is a stochastic integral (because of the second integral in the equation) which is interpreted in the It$\hat{o}$ sense.

It is natural to ask the constraints to guarantee the existence and uniqueness of the solution of SDDEs. Now, our aim is to explain conditions imposed on the initial function $\varphi(t)$, the drift term f and diffusion term g to ensure that SDDE has a unique solution. These conditions are gathered under the existence and uniqueness theorem. Before stating the existence and uniqueness theorem, let us consider some concepts that make it easy to follow and understand this theorem.

Definition 2.4. If an $\mathbb{R}$-valued stochastic process $X(t) : [-\tau, T] \times \Omega \rightarrow \mathbb{R}$ is a measurable, sample-continuous process such that $X(t)$ is $(\mathcal{F}_t)_{0 \leq t \leq T}$ adapted satisfying (2.5) almost surely, and fulfills the initial condition $X(t) = \varphi(t)$ for $t \in [-\tau, 0]$, then it is called a strong solution for equation (2.5).

Remark 2.4. Two processes, namely, X and Y are said to be indistinguishable if there is an event $A \subseteq \mathcal{F}$ such that $P(A) = 1$ and these processes satisfy $X_t(\omega) = Y_t(\omega)$ for all $\omega \in A$ and all $t \geq 0$. This actually means that they almost surely (i.e., with probability one) have the same sample paths.

Definition 2.5. The functions f and g in equation (2.5) are said to fulfill the local Lipschitz condition, if there is a positive constant *K* satisfying:

$$|f(t, x_1, y_1) - f(t, x_2, y_2)| \vee |g(t, x_1, y_1) - g(t, x_2, y_2)| \leq K(|x_1 - x_2| + |y_1 - y_2|),$$

for any $x_1, x_2, y_1, y_2 \in \mathbb{R}$ and any $t \in \mathbb{R}^+$ where $|x| \vee |y| = \max\{|x|, |y|\}$. These constants *K* are called as the Lipschitz constants.

Definition 2.6. If there exists a positive constant *K* satisfying :

$$|f(t, x, y_1) - f(t, x, y_2)| \vee |g(t, x, y_1) - g(t, x, y_2)| \leq K|y_1 - y_2|,$$

for any y_1 and y_2 are $\in \mathbb{R}$ and any $(t, x) \in \mathbb{R}^+ \times \mathbb{R}$, then the functions f and g in equation (2.5) are said to satisfy the weakly local Lipschitz condition.

Definition 2.7. The functions f and g in equation (2.5) fulfill the linear growth condition, if there is a positive constant *L* satisfying:

$$|f(t, x, y)|^2 \vee |g(t, x, y)|^2 \leq L(1 + |x|^2 + |y|^2)$$

for all $(t, x, y) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}$.

Now we are ready to state the existence and uniqueness theorem for SDDEs.

Theorem 2.2. *If the functions f and g in equation* (2.5) *satisfy the local Lipschitz condition and linear growth condition, then there exists a path wise unique strong solution to equation* (2.5) *on $t \geq -\tau$. Moreover, the linear growth condition guarantees that the solution satisfies*

$$E\Big(\sup_{-\tau \leq t \leq T} |X(t)|^2\Big) < \infty, \quad \forall T > 0.$$

The detailed proof of this theorem can be found in [44]. That proof depends on the standard technique of Picard iterations. Note that, when $t \in [0, \tau]$, $X(t-\tau) = \varphi(t-\tau)$, which is the given initial path since $-\tau \leq t - \tau \leq 0$. Therefore, the SDDE in (2.5) can be written as :

$$dX(t) = f(t, X(t), \varphi(t-\tau))dt + g(t, X(t), \varphi(t-\tau))dW(t)$$

where the initial data is $X(0) = \varphi(0)$. Note that, this defines a SDE and it has a unique solution if the linear growth condition holds while $f(t, x, y)$ and $g(t, x, y)$ are locally Lipschitz continuous. Actually it is seen that a weakly local Lipschitz condition with respect to x for the drift and diffusion term, namely, $f(t, x, y)$ and $g(t, x, y)$ is enough instead of considering local Lipschitz condition. After the solution $X(t)$ on $[0, \tau]$ is found, we can proceed the iteration on the other intervals $[i\tau, (i+1)\tau]$ for all $i = 1, 2, \cdots$ to obtain the solution process on $[-\tau, \infty)$.

Theorem 2.3. *Under the assumption of weakly local Lipschitz condition and linear growth condition, given SDDE in* (2.5) *has a path-wise unique strong solution X(t) on* $t \geq -\tau$ *and it satisfies:*

$$E\bigg(\sup_{-\tau\leq t\leq T} |X(t)|^2\bigg) < \infty, \quad \forall T > 0.$$

For some particular cases of SDDE, it is not easy to check the linear growth condition. For this reason, let us consider a more general existence and uniqueness theorem.

Theorem 2.4. *[39] Assume that local Lipschitz condition holds and there exists a constant number* $K > 0$ *such that*

$$2x^T f(t, x, y) + |g(t, x, y)|^2 \leq K(1 + |x|^2 + |y|^2) \quad \forall (t, x, y) \in \mathbb{R}_t \times \mathbb{R}^n \times \mathbb{R}^n.$$

Then (2.5) *has a unique continuous solution X(t) on t* $\geq -\tau$*. Moreover, it satisfies*

$$E\bigg(\sup_{-\tau\leq t\leq T} |X(t)|^2\bigg) < \infty, \quad \forall T > 0.$$

Remark 2.5. [39] Consider the following particular SDDE,

$$dX(t) = \big[-X^3(t) + X(t-\tau)\big]dt + \bigg[\sin(X(t)) + \cos\big(X(t-\tau)\big)\bigg]dW(t), \quad t \geq 0,$$

with initial data $\varphi(t) = a$ for $t \in [-\tau, 0]$ where a is the real number and $\varphi(t) \in L^2_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R})$. Note that drift and diffusion terms satisfy local Lipschitz condition where $f(t, x, y) = -x^3 + y$ and $g(t, x, y) = \sin(x) + \cos(y)$, respectively. However, showing the linear growth condition is not easy. Thus, let us check the second condition in Theorem 2.4. We know that $|\sin(x)| \leq 1$ and $|\cos(x)| \leq 1$ for any $x \in \mathbb{R}$. Then, for any $(t,x,y) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}$,

$$\begin{aligned}
2xf(t,x,y) + |g(t,x,y)|^2 &= -2x^4 + 2xy + |\sin^2(x) + \cos^2(y) + 2\sin(x)\cos(y)| \\
&\leq -2x^4 + 2xy + |\sin^2(x)| + |\cos^2(y)| + 2|\sin(x)||\cos(y)| \\
&\leq 2xy + 1 + 1 + 2 \\
&\leq 2x^2 + 2y^2 + 4 \\
&\leq 4(x^2 + y^2 + 1).
\end{aligned}$$

So, according to Theorem 2.4, given equation has a path-wise unique solution.

2.2.2 Examples of Stochastic Delay Differential Equations

The procedure for finding the solution again depends on It$\hat{o}$ formula, but it is a little bit different than the way used in SDE because of the delay effect that we insert in the equation. We need to proceed step by step in the intervals with equal step-size τ starting from the initial point. Now, let us examine the following examples to understand the solution technique better and see the difference.

Example 2.4. Let us consider an example of SDDE such that the delay effects only the drift term and diffusion term is a constant real number β:

$$\begin{cases} dX(t) = X(t-\tau)dt + \beta dW(t), \quad t \geq 0, \\ X(t) = \varphi(t), \quad t \in [-\tau, 0]. \end{cases} \tag{2.6}$$

Assume that $\varphi(t)$ is a continuous function for $t \in [-\tau, 0]$, which satisfy the conditions in Theorem 2.2. We first check the necessary conditions for the existence and uniqueness theorem and then solve this SDDE with the given initial path. Note that $f(t,x,y) = y$ and $g(t,x,y) = \beta$ are drift and diffusion terms, respectively, according to the general definition of a SDDE. Then,

$$\begin{aligned}
|f(t,x_1,y_1) - f(t,x_2,y_2)| &= |y_1 - y_2| \leq |x_1 - x_2| + |y_1 - y_2|, \\
|g(t,x_1,y_1) - g(t,x_2,y_2)| &= |\beta - \beta| \leq |x_1 - x_2| + |y_1 - y_2|,
\end{aligned}$$

for any $x_1, x_2, y_1, y_2 \in \mathbb{R}$ and $t \in \mathbb{R}_+$. The above two inequalities show that the local Lipschitz condition is satisfied. Moreover, the linear growth condition is also satisfied for any $x, y \in \mathbb{R}$ and $t \in \mathbb{R}^+$ since:

$$\begin{aligned}
|f(t,x,y)|^2 &= |y|^2 \leq 1 + |x|^2 + |y|^2, \\
|g(t,x,y)|^2 &= |\beta|^2 \leq |\beta|^2(1 + |x|^2 + |y|^2).
\end{aligned}$$

As a result, given SDDE in the example has a path-wise unique strong solution and that solution $X(t)$ satisfies

$$E\left(\sup_{-\tau \leq t \leq T} |X(t)|^2\right) < \infty, \quad \text{for all} \quad T > 0.$$

After we show that the solution is unique, we are ready to solve it by It$\hat{o}$ formula. Define $\varphi(t) =: \varphi_1(t)$.

For $t \in [0, \tau]$: $t - \tau \in [-\tau, 0]$, which implies that $X(t-\tau) = \varphi_1(t-\tau)$ and our SDDE actually defines the following SDE:

$$dX(t) = \varphi_1(t-\tau)dt + \beta dW(t).$$

Applying It$\hat{o}$ formula for $F(x) = x$:

$$\begin{aligned} X(t) &= \varphi_1(0) + \int_0^t \varphi_1(u_1 - \tau)du_1 + \int_0^t \beta dW(u_1) \\ &= \varphi_1(0) + \int_0^t \varphi_1(u_1 - \tau)du_1 + \beta W(t) \\ &=: \varphi_2(t). \end{aligned}$$

For $t \in [\tau, 2\tau]$: $t - \tau \in [0, \tau]$. Hence, $X(t-\tau) = \varphi_2(t-\tau)$ and the equation becomes:

$$dX(t) = \varphi_2(t-\tau)dt + \beta dW(t).$$

Applying It$\hat{o}$ formula for $F(x) = x$ again:

$$\begin{aligned} X(t) &= \varphi_2(\tau) + \int_\tau^t \varphi_2(u_2 - \tau)du_2 + \int_\tau^t \beta dW(u_2) \\ &= \varphi_2(\tau) + \int_\tau^t \left[\varphi_1(0) + \int_0^{u_2-\tau} \varphi_1(u_1 - \tau)du_1 + \beta W(u_2 - \tau)\right] du_2 + \int_\tau^t \beta dW(u_2) \\ &= \varphi_2(\tau) + \varphi_1(0)(t-\tau) + \int_\tau^t \int_0^{u_2-\tau} \varphi_1(u_1 - \tau)du_1 du_2 \\ &\quad + \int_\tau^t \beta W(u_2 - \tau)du_2 + \beta(W(t) - W(\tau)) \\ &=: \varphi_3(t). \end{aligned}$$

For $t \in [2\tau, 3\tau]$: $t - \tau \in [\tau, 2\tau]$. Thus, $X(t-\tau) = \varphi_3(t-\tau)$ and the equation turns to

$$dX(t) = \varphi_3(t-\tau)dt + \beta dW(t).$$

Applying It$\hat{o}$ formula again:

$$\begin{aligned}
X(t) &= \varphi_3(2\tau) + \int_{2\tau}^{t} \varphi_3(u_3-\tau)du_3 + \int_{2\tau}^{t} \beta dW(u_3) \\
&= \varphi_3(2\tau) + \int_{2\tau}^{t} \Bigg[\varphi_2(\tau) + \varphi_1(0)(u_3-2\tau) + \int_{\tau}^{u_3-\tau}\int_{0}^{u_2-\tau} \varphi_1(u_1-\tau)du_1du_2 \\
&\quad + \int_{\tau}^{u_3-\tau} \beta W(u_2-\tau)du_2 + \beta\big(W(u_3-\tau)-W(\tau)\big)\Bigg] du_3 + \beta\big(W(t)-W(2\tau)\big) \\
&= \varphi_3(2\tau) + \varphi_2(\tau)(t-2\tau) + \int_{2\tau}^{t} \varphi_1(0)(u_3-2\tau)du_3 \\
&\quad + \int_{2\tau}^{t}\int_{\tau}^{u_3-\tau}\int_{0}^{u_2-\tau} \varphi_1(u_1-\tau)du_1du_2du_3 + \int_{2\tau}^{t}\int_{\tau}^{u_3-\tau} \beta W(u_2-\tau)du_2du_3 \\
&\quad + \int_{2\tau}^{t} \beta\big(W(u_3-\tau)-W(\tau)\big)du_3 + \beta\big(W(t)-W(2\tau)\big) \\
&=: \varphi_4(t).
\end{aligned}$$

We can repeat this procedure over the intervals $[i\tau, (i+1)\tau]$, for $i = 3, 4, \dots$ and construct the solution recursively for this SDDE. Up to now, we have computed:

$$X(t) = \begin{cases}
\varphi_1(t), \quad t \in [-\tau, 0] \\
\varphi_2(t) = \varphi_1(0) + \int_0^t \varphi_1(u_1-\tau)du_1 + \beta W(t), \quad t \in [0,\tau] \\
\varphi_3(t) = \varphi_2(\tau) + \varphi_1(0)(t-\tau) + \int_{\tau}^{t}\int_{0}^{u_2-\tau} \varphi_1(u_1-\tau)du_1du_2 \\
\qquad + \int_{\tau}^{t} \beta W(u_2-\tau)du_2 + \beta(W(t)-W(\tau)), \quad t \in [\tau, 2\tau] \\
\varphi_4(t) = \varphi_3(2\tau) + \varphi_2(\tau)(t-2\tau) + \int_{2\tau}^{t} \varphi_1(0)(u_3-2\tau)du_3 \\
\quad + \int_{2\tau}^{t}\int_{\tau}^{u_3-\tau}\int_{0}^{u_2-\tau} \varphi_1(u_1-\tau)du_1du_2du_3 + \int_{2\tau}^{t}\int_{\tau}^{u_3-\tau} \beta W(u_2-\tau)du_2du_3 \\
\quad + \int_{2\tau}^{t} \beta(W(u_3-\tau)-W(\tau))du_3 + \beta(W(t)-W(2\tau)), \quad t \in [2\tau, 3\tau].
\end{cases} \tag{2.7}$$

The recurrence relation for the solution $\varphi_n(t)$ for $t \in [(n-2)\tau, (n-1)\tau]$ can be written as follow:

$$\varphi_n(t) = \begin{cases}
\varphi_{n-1}((n-2)\tau) + \int_{(n-2)\tau}^{t} \varphi_{n-1}(s-\tau)ds + \beta(W(t) - W((n-2)\tau)), \; n = 2, 3, ..., \\
\varphi_1(t), \; n = 1.
\end{cases}$$

Proposition 2.5. *Suppose X(t) fulfills equation* (2.6)*. Then, the expected value of X(t) for any* $t \in [n\tau, (n+1)\tau]$, $n = 0, 1, 2, ...,$ *is given by*

$$E(X(t)) = y_n(n\tau) + \int_{n\tau}^{t} y_n(s-\tau)ds,$$

where y_n represents the expected value of the solution for $t \in [(n-1)\tau, n\tau]$.

Proof. Let us write equation (2.6) in the integral form as:

$$X(t) = X(0) + \int_0^t X(s-\tau)ds + \int_0^t \beta dW(s).$$

Set $E(X(t)) = m(t)$ and take the expectation of this stochastic integral while using the linearity property of expectation:

$$\begin{aligned} m(t) &= m(0) + \int_0^t m(s-\tau)ds + \boldsymbol{E}\Big(\int_0^t \beta dW(s)\Big) \\ &= m(0) + \int_0^t m(s-\tau)ds. \end{aligned}$$

Note that $\int_0^t \beta dW(s)$ is martingale and it implies to $E\big(\int_0^t \beta dW(s)\big) = 0$. Taking the derivative of both sides with respect to parameter t of the above equation, we get:

$$\begin{cases} m'(t) = m(t-\tau), \quad t \geq 0, \\ m(t) = E(\varphi(t)) := y_0(t), \quad t \in [-\tau, 0]. \end{cases}$$

Note that this defines ODE, solve this equation considering intervals with length τ.

For $t \in [0,\tau]$, $t-\tau \in [-\tau, 0]$, then our ODE becomes:

$$\begin{aligned} m'(t) &= y_0(t-\tau), \\ m(0) &= y_0(0). \end{aligned}$$

Thus, the corresponding solution is equal to $m(t) = y_0(0) + \int_0^t y_0(s-\tau)ds$. Call this solution as $y_1(t)$.

For $t \in [\tau, 2\tau]$, $t-\tau \in [0,\tau]$, then we get:

$$\begin{aligned} m'(t) &= y_1(t-\tau), \\ m(\tau) &= y_1(\tau) \end{aligned}$$

where the result is $m(t) = y_1(\tau) + \int_\tau^t y_1(s-\tau)ds$. Call this solution as $y_2(t)$.

For $t \in [2\tau, 3\tau]$, $t-\tau \in [\tau, 2\tau]$, then it becomes:

$$\begin{aligned} m'(t) &= y_2(t-\tau), \\ m(2\tau) &= y_2(2\tau). \end{aligned}$$

Hence, the corresponding solution is equal to $m(t) = y_2(2\tau) + \int_{2\tau}^{t} y_2(s-\tau)ds$. Call this solution as $y_3(t)$. We can continue to process and find the mean function over the intervals $[n\tau, (n+1)\tau]$ for $n = 3, 4, \ldots$. Up to now, we have computed:

$$E(X(t)) = \begin{cases} y_0(t), \qquad t \in [-\tau, 0], \\ y_1(t) = y_0(0) + \displaystyle\int_0^t y_0(s-\tau)ds, \qquad t \in [0, \tau], \\ y_2(t) = y_1(\tau) + \displaystyle\int_\tau^t y_1(s-\tau)ds, \qquad t \in [\tau, 2\tau] \\ y_3(t) = y_2(2\tau) + \displaystyle\int_{2\tau}^t y_2(s-\tau)ds, \qquad t \in [2\tau, 3\tau], \end{cases} \tag{2.8}$$

Then we can write $m(t)$ where $t \in [n\tau, (n+1)\tau], n = 0, 1, 2, ...,$ as:

$$m(t) = E(X(t)) = y_n(n\tau) + \int_{n\tau}^{t} y_n(s-\tau)ds,$$

where y_n represents the solution for $t \in [(n-1)\tau, n\tau]$. □

Example 2.5. In order to make the previous computations more understandable, let us consider Example 2.4 again with the assumption $\tau = 1$ and initial path $\varphi(t) = 1 + t$; in other words:

$$\begin{aligned} dX(t) &= X(t-1)dt + \beta dW(t), \quad 0 \le t \le T, \\ X(t) &= \varphi(t) = 1 + t, \quad t \in [-1, 0]. \end{aligned}$$

Using the general form of solution given in (2.7) or solving directly the SDDE, one can show

$$X(t) = \begin{cases} \varphi_1(t) = 1 + t, \qquad t \in [-1, 0] \\ \varphi_2(2) = 1 + \dfrac{t^2}{2} + \beta W(t), \qquad t \in [0, 1] \\ \varphi_3(t) = t + \dfrac{(t-1)^3}{6} + \dfrac{1}{2} + \displaystyle\int_1^t \beta W(s-1)ds + \beta W(t), \qquad t \in [1, 2] \\ \varphi_4(t) = \dfrac{16}{6} + \displaystyle\int_1^2 \beta W(s-1)ds + \beta W(2) + \int_2^t \big(s - \dfrac{1}{2} + \dfrac{(s-2)^3}{6}\big)ds \\ \quad + \displaystyle\int_2^t \int_1^{u-1} \beta W(s-1)dsdu + \int_2^t \beta W(s-1)ds + \int_2^t \beta dW(s), \qquad t \in [2, 3] \end{cases}$$

Similarly, expected value can be solved according to (2.8) and found

$$E(X(t)) = \begin{cases} 1 + t, \qquad t \in [-1, 0] \\ 1 + \dfrac{t^2}{2}, \qquad t \in [0, 1] \\ \dfrac{1}{3} + \dfrac{3t}{2} - \dfrac{t^2}{2} + \dfrac{t^3}{6}, \qquad t \in [1, 2] \\ \dfrac{10}{6} - \dfrac{t}{2} + \dfrac{t^2}{2} + \dfrac{(t-2)^4}{24}, \qquad t \in [2, 3] \end{cases}$$

For the complete solution one can see the Appendix A.

Example 2.6. In our third example in this section, let us consider an example in which the delay affects only the diffusion term. Moreover, assume that there is no drift term, in other words $f(t,x,y)=0$:

$$\begin{aligned} dX(t) &= \sigma X(t-\tau)dW(t), \qquad t\geq 0,\\ X(t) &= \varphi(t), \qquad t\in[-\tau,0], \end{aligned}$$

where σ is a constant real number and assume that initial data $\varphi(t)$ is an $\mathcal{F}_0$-measurable random variable with $E\left(\sup_{-\tau\leq t\leq 0}|\varphi(t)|^2\right)<\infty$ and $\varphi(.)$ is a continuous function on $[-\tau,0]$ and W is a standard Brownian motion.

Like in the previous SDDE examples, let us first check whether it satisfies the existence and uniqueness theorem or not. For this SDDE note that $f(t,x,y)=0$ and $g(t,x,y)=\sigma y$:

$$\begin{aligned} &|f(t,x_1,y_1)-f(t,x_2,y_2)| = 0\leq |x_1-x_2|+|y_1-y_2|,\\ &|g(t,x_1,y_1)-g(t,x_2,y_2)| = |\sigma y_1-\sigma y_2| = |\sigma||y_1-y_2|\leq|\sigma|(|x_1-x_2|+|y_1-y_2|). \end{aligned}$$

These inequalities imply that the local Lipschitz condition is satisfied. For the linear growth condition:

$$\begin{aligned} &|f(t,x,y)|^2 = 0\leq 1+|x|^2+|y|^2,\\ &|g(t,x,y)|^2 = |\sigma y|^2\leq|\sigma|^2(1+|x|^2+|y|^2). \end{aligned}$$

As a result the given SDDE in the example has a path-wise unique strong solution and that solution $X(t)$ satisfies

$$E\left(\sup_{-\tau\leq t\leq T}|X(t)|^2\right)<\infty, \quad \forall T>0.$$

Let us say $\varphi(t):=\varphi_1(t)$ and solve the equation.
For $t\in[0,\tau]$ $t-\tau\in[-\tau,0]$, so $X(t-\tau)=\varphi_1(t-\tau)$ so, we get

$$dX(t)=\sigma\varphi_1(t-\tau)dW(t),$$

i.e.,

$$\begin{aligned} X(t) &= \varphi_1(0)+\int_0^t \sigma\varphi_1(u_1-\tau)dW(u_1)\\ &=: \varphi_2(t). \end{aligned}$$

For $t\in[\tau,2\tau]$ $t-\tau\in[0,\tau]$, so $X(t-\tau)=\varphi_2(t-\tau)$ and we get

$$dX(t)=\sigma\varphi_2(t-\tau)dW(t).$$

Actually, it can be written as:

$$\begin{aligned}
X(t) &= \varphi_2(\tau) + \int_{\tau}^{t} \sigma\varphi_2(u_2-\tau)dW(u_2) \\
&= \varphi_2(\tau) + \sigma\int_{\tau}^{t}\left[\varphi_2(0) + \int_{0}^{u_2-\tau} \sigma\varphi_2(u_1-r)dW(u_1)\right]dW(u_2) \\
&= \varphi_2(\tau) + \sigma\int_{\tau}^{t}\varphi_1(0)dW(u_2) + \sigma^2\int_{\tau}^{t}\int_{0}^{u_2-\tau}\varphi_1(u_1-r)dW(u_1)dW(u_2) \\
&=: \varphi_3(t).
\end{aligned}$$

For $t \in [2\tau, 3\tau]$ $t-\tau \in [\tau, 2\tau]$, so $X(t-\tau) = \varphi_3(t-\tau)$, and equation turns to

$$dX(t) = \sigma\varphi_3(t-\tau)dW(t),$$

and the corresponding integral form is:

$$\begin{aligned}
X(t) &= \varphi_3(2\tau) + \int_{2\tau}^{t}\sigma\varphi_3(u_3-\tau)dW(u_3) \\
&= \varphi_3(2\tau) + \sigma\int_{2\tau}^{t}\left[\varphi_2(\tau) + \sigma\int_{\tau}^{u_3-\tau}\varphi_1(0)dW(u_2)\right. \\
&\qquad \left. +\sigma^2\int_{\tau}^{u_3-\tau}\int_{0}^{u_2-\tau}\varphi_1(u_1-\tau)dW(u_1)dW(u_2)\right]dW(u_3) \\
&= \varphi_3(2\tau) + \sigma\int_{2\tau}^{t}\varphi_2(\tau)dW(u_3) + \sigma^2\int_{2\tau}^{t}\int_{\tau}^{u_3-\tau}\varphi_1(0)dW(u_2)dW(u_3) \\
&\qquad + \sigma^3\int_{2\tau}^{t}\int_{\tau}^{u_3-\tau}\int_{0}^{u_2-\tau}\varphi_1(u_1-\tau)dW(u_1)dW(u_2)dW(u_3) \\
&=: \varphi_4(t).
\end{aligned}$$

We can continue the same procedure and compute recursively for the solutions defined in other intervals. Up to now, we have computed:

$$X(t) = \begin{cases}
\varphi_1(t), \qquad t \in [-\tau, 0] \\
\varphi_2(t) = \varphi_1(0) + \int_{0}^{t}\sigma\varphi_1(u_1-\tau)dW(u_1), \qquad t \in [0,\tau] \\
\varphi_3(t) = \varphi_2(\tau) + \sigma\int_{\tau}^{t}\varphi_1(0)dW(u_2) \\
\qquad\qquad +\sigma^2\int_{\tau}^{t}\int_{0}^{u_2-\tau}\varphi_1(u_1-r)dW(u_1)dW(u_2), \qquad t \in [\tau, 2\tau] \\
\varphi_4(t) = \varphi_3(2\tau) + \sigma\int_{2\tau}^{t}\varphi_2(\tau)dW(u_3) + \sigma^2\int_{2\tau}^{t}\int_{\tau}^{u_3-\tau}\varphi_1(0)dW(u_2)dW(u_3) \\
\quad +\sigma^3\int_{2\tau}^{t}\int_{\tau}^{u_3-\tau}\int_{0}^{u_2-\tau}\varphi_1(u_1-\tau)dW(u_1)dW(u_2)dW(u_3), \qquad t \in [2\tau, 3\tau]
\end{cases}$$

Then we can write $\varphi_n(t)$ where $t \in [(n-2)\tau, (n-1)\tau]$ as:

$$\varphi_n(t) = \begin{cases} \varphi_{n-1}((n-2)\tau) + \displaystyle\int_{(n-2)\tau}^{t} \varphi_{n-1}(s-\tau)dW(s), & n = 2, 3, ..., \\ \varphi_1(t), \qquad n = 1. \end{cases}$$

Proposition 2.6. *Assume that the solution process X of Example 2.6 exists. Then the expected value of $X(t)$ for any $t \in [n\tau, (n+1)\tau]$, $n = 0, 1, 2...,$ is given by*

$$E(X(t)) = E(\varphi_1(0)).$$

Proof. Let us compute the mean function of the solution process which is found in Example 2.6 in each interval.

For $t \in [0, \tau]$: our solution is $X(t) = \varphi_2(t) = \varphi_1(0) + \int_0^t \sigma\varphi_1(u_1 - \tau)dW(u_1)$, taking the expectation of both sides we obtain

$$E(X(t)) = E(\varphi_1(0)).$$

For $t \in [\tau, 2\tau]$, our solution is

$$X(t) = \varphi_3(t) = \varphi_2(\tau) + \sigma\int_\tau^t \varphi_1(0)dW(u_2) + \sigma^2\int_\tau^t\int_0^{u_2-\tau} \varphi_1(u_1 - r)dW(u_1)dW(u_2),$$

and the corresponding mean function is

$$E(X(t)) = E(\varphi_2(\tau)) = E(\varphi_1(0))$$

For $t \in [2\tau, 3\tau]$, our solution is

$$\begin{aligned} X(t) = \varphi_4(t) = \varphi_3(2\tau) + \sigma\int_{2\tau}^t \varphi_2(\tau)dW(u_3) + \sigma^2\int_{2\tau}^t\int_\tau^{u_3-\tau} \varphi_1(0)dW(u_2)dW(u_3) \\ + \sigma^3\int_{2\tau}^t\int_\tau^{u_3-\tau}\int_0^{u_2-\tau} \varphi_1(u_1-\tau)dW(u_1)dW(u_2)dW(u_3), \end{aligned}$$

and the corresponding mean function is

$$E(X(t)) = E(\varphi_3(2\tau)) = E(\varphi_1(0))$$

The process can be continued and the other mean functions in each interval can be computed. It is seen that

$$E(X(t)) = \begin{cases} E(\varphi_1(t)), & t \in [-\tau, 0], \\ E(\varphi_1(0)), & t \in [n\tau, (n+1)\tau], \end{cases}$$

where $n = 0, 1, 2, ...,$ it can be written as:

$$E(X(t)) = E(\varphi_{n+1}(n\tau)) = E(\varphi_1(0))$$

□

2.2.3 Comparison Study

Comparison of Example 2.3 and Example 2.4:
On $t \in [0, \tau]$:
Solution of SDE is

$$X(t) = x_0 e^t + \beta \int_0^t e^{t-u} dW(u),$$

where x_0 is the given initial data. Moreover, corresponding conditional mean function is

$$E(X(t)|\mathcal{F}_0) = E(X(t)|X(0) = x_0) = x_0 e^t.$$

Solution of SDDE is

$$X(t) = \varphi_1(0) + \int_0^t \varphi_1(u_1 - \tau) du_1 + \beta W(t),$$

where $\varphi_1(t)$ is the given initial path for $t \in [-\tau, 0]$ (this solution is called $\varphi_2(t)$). In addition to this result, corresponding conditional mean function is

$$E(X(t)|\mathcal{F}_0) = y_0(0) + \int_0^t y_0(s - \tau) ds,$$

where $y_0(t)$ represents the expected value of the solution for any $t \in [-\tau, 0]$.

On $t \in [\tau, 2\tau]$:
Solution of SDE is

$$X(t) = X(\tau) e^{t-\tau} + \beta \int_\tau^t e^{t-u} dW(u)$$

and the corresponding conditional mean function is

$$E(X(t)|\mathcal{F}_\tau) = E(X(t)|X(\tau)) = X(\tau) e^{t-\tau}.$$

Solution of SDDE is

$$\begin{aligned} X(t) = \varphi_2(\tau) + \varphi_1(0)(t - \tau) + \int_\tau^t \int_0^{u_2 - \tau} & \varphi_1(u_1 - \tau) du_1 du_2 \\ + \int_\tau^t \beta W(u_2 - \tau) du_2 & + \beta(W(t) - W(\tau)), \end{aligned}$$

where $\varphi_2(t)$ is the solution of the SDDE for any $t \in [0, \tau]$ and the corresponding conditional mean function is

$$E(X(t)|\mathcal{F}_\tau) = E(X(t)|X(\tau)) = y_1(\tau) + \int_\tau^t y_1(s - \tau) ds,$$

where $y_1(t)$ corresponds the expected value of the solution in the interval $[0, \tau]$.

Comparison of Example 2.2 and Example 2.6:

On $t \in [0, \tau]$:

Solution of SDE is

$$X(t) = x_0 \exp\{\sigma W(t) - \frac{1}{2}\sigma^2 t\},$$

and the corresponding conditional mean function is

$$E(X(t)|\mathcal{F}_0) = E(X(t)|X(0) = x_0) = x_0,$$

where $X(0) = x_0$ is the initial value.

Solution of SDDE is

$$X(t) = \varphi_1(0) + \int_0^t \sigma\varphi_1(u_1 - \tau)dW(u_1)$$

and the corresponding conditional mean function is

$$E(X(t)|\mathcal{F}_0) = \varphi_1(0),$$

where $\varphi_1(t)$ is the given initial path for $t \in [-\tau, 0]$.

On $t \in [\tau, 2\tau]$:

Solution of SDE is

$$X(t) = X(\tau)\exp\{\sigma(W(t) - W(\tau)) - \frac{1}{2}\sigma^2(t - \tau)\},$$

and the corresponding conditional mean function is

$$E(X(t)|\mathcal{F}_\tau) = E(X(t)|X(\tau)) = X(\tau).$$

Solution of SDDE is

$$\begin{aligned} X(t) &= \varphi_2(\tau) + \int_\tau^t \sigma\varphi_2(u_2 - \tau)dW(u_2), \\ &= \varphi_2(\tau) + \sigma\int_\tau^t \varphi_1(0)dW(u_2) + \sigma^2\int_\tau^t\int_0^{u_2-\tau} \varphi_1(u_1 - r)dW(u_1)dW(u_2), \end{aligned}$$

and the corresponding conditional mean function is

$$E(X(t)|\mathcal{F}_\tau) = E(X(t)|X(\tau)) = E(\varphi_2(\tau)) = \varphi_1(0),$$

where $\varphi_2(t)$ is the solution of given SDDE for $t \in [0, \tau]$.

We observe in this two comparison that after adding delay term in SDE, the solution of the equation change. Since solution alters, the expected value and variance of the result also change.

In this chapter, firstly SDEs are discussed. To understand these equations and their solutions better, some examples are provided and examined. Secondly, we consider SDDEs which are the equations obtained from SDEs while implementing delay term in SDEs. Contrary to the several examples given above which can be solved quite explicitly, in general, similar to most differential equation, one can rarely obtain closed form solutions of SDDEs. In the next chapter, our aim is to clarify numerical analysis technique for SDDEs which provides approximate solutions for them. In order to make this topic easily understandable, numerical analysis techniques for SDEs will be firstly discussed. After considering the techniques for SDDEs, we will give some graph sketching to see and understand time delay effect better.

CHAPTER 3

NUMERICAL METHODS

In the previous chapter we examined some examples of SDDEs. In order to find solution of SDDE, we need to proceed step by step. However, this is not easy to tackle.

In this chapter, our aim is to introduce some numerical methods for finding an approximate solution of SDDEs and to study the properties of these methods. Because of the complicated structure of SDDEs, it is better to understand numerical methods for SDEs. Hence, we firstly introduce and discuss numerical methods for SDEs. For the detailed information and proofs, one can see [1, 14, 15, 17, 25, 30, 48, 49].

After considering that, we focus on numerical methods for SDDEs. Some definitions and Euler Maruyama scheme for SDDEs are introduced. For the detailed information and proofs, one can see [4, 5, 10, 12, 16, 41, 53].

Before move on subsections let us remember and introduce some notations that are used in this chapter:

- $(\Omega, \mathcal{F}, P)$ represents a complete probability space where the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfies the usual conditions, namely, $\{\mathcal{F}_t\}_{t\geq 0}$ is increasing, right continuous and it contains all P-null sets for $t = 0$.
- $W(t), t \geq 0$, represents a standard Brownian motion on that complete probability space.
- $X \in L_p(\Omega, \mathbb{R}^n)$ for any finite positive number $p \geq 1$ if and only if $E(|X|^p) < \infty$, where E denotes the expectation with respect to the given probability function P.
- L_p norm of random variable X is defined as $||X||_p := (E(|X|^p))^{\frac{1}{p}}$.

Definition 3.1. Let $\{X_t\}_{t\geq 0}$ be a sequence of random variables defined on $L_p(\Omega, \mathcal{F}, P)$. We say that X_t converges to a random variable $X \in L_p(\Omega, \mathcal{F}, P)$ as $t \to t_0$ in the p-th mean (L_p convergence) if they satisfy

$$||X_t - X||_p \to 0 \quad as \quad t \to t_0$$

or, equivalently,

$$E(|X_t - X|^p) \to 0 \quad as \quad t \to t_0.$$

Remark 3.1. In the literature of numerical analysis for stochastic differential equations, it is usually considered either $p = 1$ or $p = 2$ that is namely convergence in the (absolute) mean or convergence in the mean square sense, respectively.

Let us provide some observations:

- Using Jensen's inequality, we get Lyapunov's inequality which states

  $$E\big(|Z|^q\big)^{\frac{1}{q}} \leq E\big(|Z|^p\big)^{\frac{1}{p}}$$

  for any $0 < q \leq p$. This inequality can be rewritten as $||Z||q \leq ||Z||p$.

  Proof of Lyapunov Inequality: Define $r = \frac{p}{q} > 1$ and $y = |Z|^q$. Now using equation (A.1) which is obtained from the application of Jensen's inequality in Appendix A, it can be written:

  $$E(|y|)^r \leq E(|y|^r).$$

  While applying back substitution in this inequality, we obtain:

  $$E(|Z|^q)^{\frac{p}{q}} \leq E(|Z|^p).$$

  After taking p-th root of previous inequality, we get:

  $$E(|Z|^q)^{\frac{1}{q}} \leq E(|Z|^p)^{\frac{1}{p}},$$

  which is the desired result.

- Taking q-th power of Lyapunov's inequality, we get $E(|Z|^q) \leq E(|Z|^p)^{\frac{q}{p}}$ where $0 < q < p$. This means if $Z \in L_p(\Omega, \mathcal{F}, P)$ and $p > q$, then $Z \in L_q(\Omega, \mathcal{F}, P)$.
- If X_t converges to X in the p-th mean and $p > q$ then X_t also converges in the q-th mean (one can prove this while setting $Z = X_t - X$ and using previous observations). In particular, convergence in the mean square (where $p = 2$) implies convergence in the mean (where $q = 1$).

Now, let us move our first subsection and introduce some definitions and methods for SDEs.

3.1 Numerical Methods for SDE

Let us remember our formulation for SDE in Section 2.1 which is given by:

$$\begin{aligned} dX(t) &= f(t, X(t))dt + g(t, X(t))dW(t), \quad 0 \leq t \leq T, \\ X(0) &= x_0. \end{aligned}$$

- Consider a partition of the time interval $[0, T]$, $0 = t_0 < t_1 < ... < t_N = T$ and define $\Delta t_{n+1} = t_{n+1} - t_n$ and $\Delta W_{n+1} = W(t_{n+1}) - W(t_n) = W(\Delta t_{n+1})$ for $n = 0, 1, 2, .., N - 1$. They are the step size for time and the increment of standard Brownian motion, respectively.
- We know that Brownian Motion $W(t)$ is a continuous time process which satisfies independent increment, continuous path and stationary increment properties. Moreover, $W(t)$ is normally distributed with mean 0 and variance t. Using Central Limit Theorem, we can write

  $$W_t - W_s = W_{t-s} \sim \sqrt{t-s}N(0,1)$$

  for some $0 \le s \le t$. Using that idea, we can rewrite increment function of the standard Brownian motion as:

  $$\begin{aligned} \Delta W_{n+1} &= \sqrt{t_{n+1} - t_n} Z_{n+1}, \\ \Delta W_{n+1} &= \sqrt{\Delta t_{n+1}} Z_{n+1}, \end{aligned}$$

  for some random variable $Z_{n+1} \sim N(0, 1)$.
- Note that uniform step size for time, h, means $h = T/N$ then $t_n = nh$ where $n = 0, 1, ..., N$. Moreover, the increment of time and a standard Brownian motion correspond to $\Delta t_{n+1} = h$ and $\Delta W_{n+1} = W(h) = \sqrt{h} Z_{n+1}$ for all $n = 0, 1, 2, ..., N - 1$ respectively.
- Suppose $\tilde{X}_n$ is an approximation of the strong solution to equation (2.1), using a stochastic one step method with an increment function ϕ.

  $$\left.\begin{aligned} &\tilde{X}_{n+1} = \tilde{X}_n + \phi(\Delta t_{n+1}, \tilde{X}_n, \Delta W_{n+1}) \quad n = 0, 1, 2, ..., N-1, \\ &\tilde{X}(0) = x_0 \end{aligned}\right\} \tag{3.1}$$

  where the increment function $\phi(\Delta t, x, \Delta W)$ is continuous in all three variables and satisfy local Lipschitz condition in x.
- We also use the following notations:
  $X(t_{n+1})$ denotes the value of the exact solution of equation (2.1) at the point t_{n+1},
  $\tilde{X}_{n+1}$ denotes the approximation of the strong solution using equation (3.1),
  $\tilde{X}(t_{n+1})$ denotes the locally approximate value obtained after just one step of equation (3.1), i.e.,

  $$\tilde{X}(t_{n+1}) = X(t_n) + \phi(\Delta t_{n+1}, X(t_n), \Delta W_{n+1}).$$

After these notations, let us provide some definitions.

Definition 3.2. The local error of $\{\tilde{X}(t_n)\}$ between two consecutive time t_n and t_{n-1} for any $n = 1, 2, ..., N$, is defined by

$$\delta_n = X(t_n) - \tilde{X}(t_n), \; n = 1, 2, .., N.$$

The numerical scheme $\tilde{X}(t_n)$ is called local of order α if

$$X(t_n) - \tilde{X}(t_n) = O(h^{\alpha+1}).$$

Definition 3.3. The global error of $\{\tilde{X}_n\}$ from the beginning point t_0 to the end point $t_N = T$ is defined by

$$\epsilon_n = X(t_n) - \tilde{X}_n, \quad n = 1, 2, .., N.$$

Similarly, the numerical scheme $\tilde{X}_n$ is called global of order β if

$$X(t_n) - \tilde{X}_n = O(h^{\beta+1}).$$

Now, we can consider the way of measuring the accuracy of a numerical solution of the SDE. The most used ones are strong convergence and weak convergence.

Definition 3.4. The time discretized approximation $\tilde{X}$ with step size h converges strongly to X at time T if

$$\lim_{h\to 0} E\big(|X(T) - \tilde{X}_N|\big) = 0.$$

$\tilde{X}$ is said to converge strongly to X with (global) order p if we have

$$E\big(|X(T) - \tilde{X}_N|\big) \leq C\, h^p,$$

for some $C > 0$ which does not depend on h.

Definition 3.5. The approximation $\tilde{X}$ with uniform step size h converges weakly to X at time T if the following condition is satisfied for any continuously differentiable function g

$$\lim_{h\to 0} \Big| E\big(g(X(T))\big) - E\big(g(\tilde{X}_N)\big) \Big| = 0.$$

$\tilde{X}$ converges weakly to X with order p means

$$\Big| E\big(g(X(T))\big) - E\big(g(\tilde{X}_N)\big) \Big| \leq C\, h^p,$$

for some positive constant number C which is independent of h.

Remark 3.2. Strong convergence measures mean of the error while weak convergence measures error of the means of solution and approximation with given any continuously differentiable function g.

After these definitions, we are ready to introduce two important numerical schemes, namely, Euler Maruyama method and Milstein method.

3.1.1 Euler Maruyama Method for SDE

Euler Maruyama scheme is one of the most well known and useful method that is used in stochastic calculus to find an approximate solution to the given SDE. Consider the general form of SDEs given in equation (2.1) and the partition of the time interval $[0, T]$, $0 = t_0 < t_1 < ... < t_N = T$ where the increment of time and a standard Brownian motion are $\Delta t_{n+1} = t_{n+1} - t_n$ and $\Delta W_{n+1} = W(t_{n+1}) - W(t_n) = W(\Delta t_{n+1})$, respectively. The function ϕ in equation (3.1) for the Euler Maruyama method is defined as:

$$\phi(\Delta t_{n+1}, \tilde{X}_n, \Delta W_{n+1}) = f(t_n, \tilde{X}_n)\Delta t_{n+1} + g(t_n, \tilde{X}_n)\Delta W_{n+1} \tag{3.2}$$

for all $n = 0, 1, ...N - 1$ where $\tilde{X}(t_0) = \tilde{X}(0) = x_0$. Then while implementing this increment function into equation (3.1), we get:

$$\begin{aligned}\tilde{X}(t_{n+1}) &= \tilde{X}(t_n) + f(t_n, \tilde{X}(t_n))\Delta t_{n+1} + g(t_n, \tilde{X}(t_n))\Delta W_{n+1} \\ &= \tilde{X}(t_n) + f(t_n, \tilde{X}(t_n))\Delta t_{n+1} + g(t_n, \tilde{X}(t_n))\sqrt{\Delta t_{n+1}} Z_{n+1},\end{aligned}$$

where Z_{n+1} is normally distributed random variable with mean 0 and variance 1 for all $0 \leq n \leq N - 1$. This equation is known as time discretized approximation of $X(t)$ by using Euler Maruyama Method. For the uniform step size h on that given interval, the equation can be rewritten as:

$$\tilde{X}(t_{n+1}) = \tilde{X}(t_n) + f(t_n, \tilde{X}(t_n))h + g(t_n, \tilde{X}(t_n))\sqrt{h} Z_{n+1}.$$

3.1.2 Milstein Method for SDE

Now, we are going to explain another most known method in order to find an approximate solution to the SDE namely, Milstein method. Again consider the same SDE and time interval partition. The increment function ϕ in equation (3.1) for the Milstein method is given by:

$$\begin{aligned}\phi(\Delta t_{n+1}, \tilde{X}_n, \Delta W_{n+1}) &= f(t_n, \tilde{X}(t_n))\Delta t_{n+1} + g(t_n, \tilde{X}(t_n))\Delta W_{n+1} \\ &\quad + \frac{1}{2} g(t_n, \tilde{X}(t_n)) g'(t_n, \tilde{X}(t_n))(\Delta W_{n+1}^2 - \Delta t_{n+1})\end{aligned}$$

where $g'(t_n, \tilde{X}(t_n))$ is the derivative of g with respect to $\tilde{X}$ for all $n = 0, 1, ...N - 1$. Moreover, define $\tilde{X}(t_0) = \tilde{X}(0) = x_0$ as an initial value. Then equation (3.1) can be rewritten while using this increment function as:

$$\begin{aligned}\tilde{X}(t_{n+1}) &= \tilde{X}(t_n) + f(t_n, \tilde{X}(t_n))\Delta t_{n+1} + g(t_n, \tilde{X}(t_n))\Delta W_{n+1} \\ &\quad + \frac{1}{2} g(t_n, \tilde{X}(t_n)) g'(t_n, \tilde{X}(t_n))(\Delta W_{n+1}^2 - \Delta t_{n+1})\end{aligned}$$

for all $n = 0, 1, ...N - 1$.
For mesh with uniform step $h$on the interval $[0, T]$, we can rewrite this equation as:

$$\begin{aligned}
\tilde{X}(t_{n+1}) &= \tilde{X}(t_n) + f(t_n, \tilde{X}(t_n))h + g(t_n, \tilde{X}(t_n))\Delta W_{n+1} \\
&\qquad\qquad + \frac{1}{2} g(t_n, \tilde{X}(t_n)) g'(t_n, \tilde{X}(t_n))(\Delta W_{n+1}^2 - h) \\
&= \tilde{X}(t_n) + f(t_n, \tilde{X}(t_n))h + g(t_n, \tilde{X}(t_n))\sqrt{h} Z_n \\
&\qquad\qquad + \frac{1}{2} g(t_n, \tilde{X}(t_n)) g'(t_n, \tilde{X}(t_n))(h Z_n^2 - h) \\
&= \tilde{X}(t_n) + f(t_n, \tilde{X}(t_n))h + g(t_n, \tilde{X}(t_n))\sqrt{h} Z_n \\
&\qquad\qquad + \frac{1}{2} g(t_n, \tilde{X}(t_n)) g'(t_n, \tilde{X}(t_n)) h (Z_n^2 - 1).
\end{aligned}$$

Remark 3.3. Note that Milstein Method and Euler Maruyama Method give same result whenever the derivative of g with respect to X namely $g'(t_n, \tilde{X}(t_n))$ is 0 .

Now let us consider the examples in Section 2.1 and use Euler Maruyama method to simulate the solution processes of them.

Example 3.1. Remember the geometric Brownian motion equation in Example 2.1,

$$\begin{aligned}
&dS(t) = \mu S(t)dt + \sigma S(t)dW(t), \quad t \geq 0, \\
&S(0) = s_0,
\end{aligned}$$

where μ and σ are some positive constant numbers. We found that the solution process of this SDE is

$$S(t) = S(0)e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W(t)}.$$

In Figure 3.1, $x(t)$ represents the approximate solution which is obtained from SDE by using Euler Maruyama method while $y(t)$ represents the approximate solution that is obtained by using the method on the exact solution. In the second graph of Figure 3.1, we measure the difference between these two approaches and we see that the difference is very small. Thus, we can apply Euler Maruyama method directly on given SDE in order to get solution process.

In Figure 3.2, the blue line represents the expectation of solution process which is obtained by using Euler Maruyama method on SDE while the red line represents the exact value of expectation. It is again seen that the method fits well.

In Figure 3.3, the different compositions of coefficients are used in order to see the effect of drift and diffusion terms to the solution process. When we increase the diffusion term (blue and green line), the volatility also increase as it is expected. However, change on the drift term does not affect the volatility too much. It affects the value of the process(red and blue line).

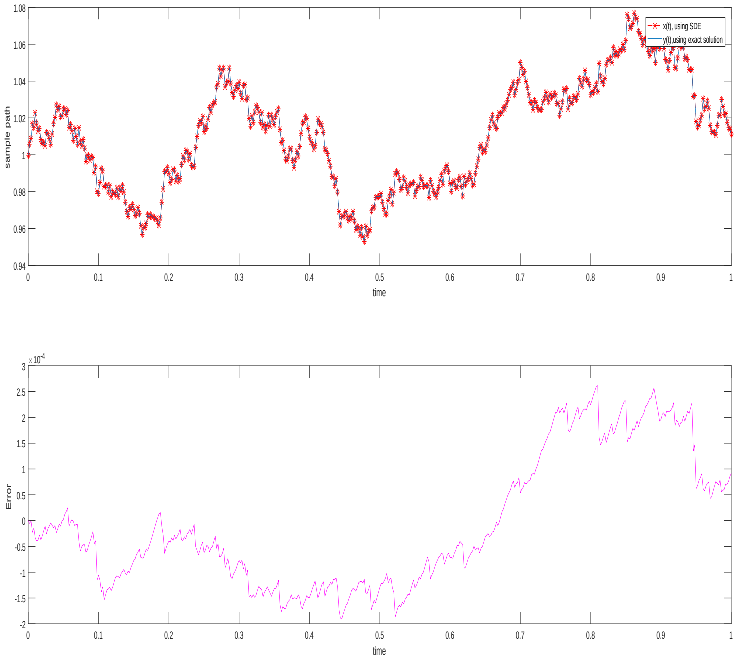

Figure 3.1: Solution process using Euler Maruyama method and exact solution and error between them

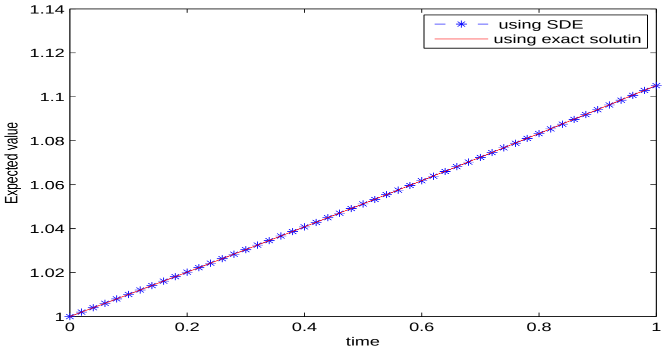

Figure 3.2: Expected value of the solution process

Example 3.2. Let us examine the kind of Ornstein-Uhlenbeck process given in Example 2.3,

$$\begin{aligned} &dX(t) = X(t)dt + \beta dW(t), \quad t \geq t_0, \\ &X(t_0) = x_0, \end{aligned}$$

where β is positive constant number and corresponding solution is

$$X(t) = x_0 e^{t-t_0} + \beta \int_{t_0}^{t} e^{t-u} dW(u).$$

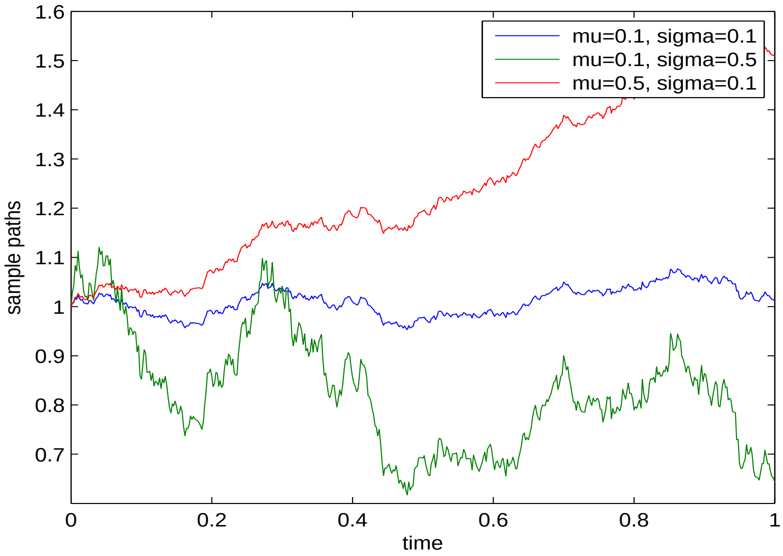

Figure 3.3: Sample path with different coefficients

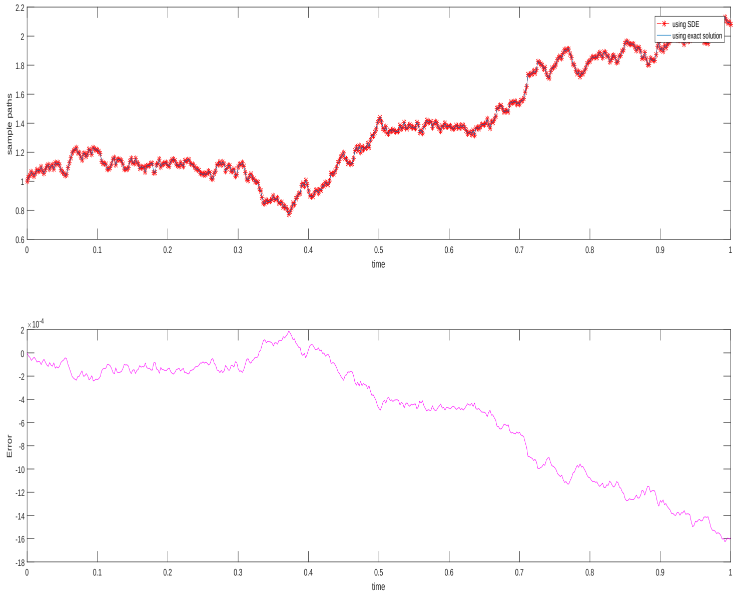

Figure 3.4: Solution process using Euler Maruyama method and exact solution and error between them.

In Figure 3.4, we see two sample path that are obtained from SDE and the exact solution of SDE with the help of Euler Maruyama method. In the second graph of Figure 3.4, the error between this two paths is given. Like in the previous example, the error is very small.

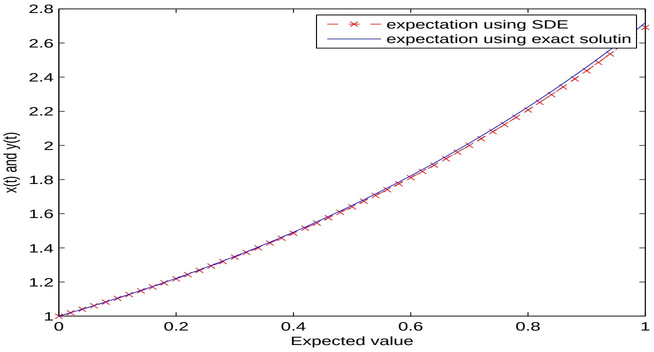

Figure 3.5: Expected value of the solution process.

In Figure 3.5, the red dashed line represents the expectation of solution process using Euler Maruyama method on SDE while the blue line represents the expectation of the exact solution. It is seen that the error between them is very small.

3.2 Numerical Methods for SDDE

We are going to consider our general SDDE in equation (2.5) in the autonomous form for simplicity, i.e., functions f and g do not depend explicitly on t:

$$\left.\begin{array}{l} dX(t) = f(X(t), X(t-\tau))dt + g(X(t), X(t-\tau))dW(t), \quad t \in [0,T], \\ X(t) = \varphi(t), \quad t \in [-\tau, 0]. \end{array}\right\} \tag{3.3}$$

- Consider a partition of the interval $[0,T]$, $0 = t_0 < t_1 < ... < t_N = T$ with uniform step size h then $h = T/N$ and $t_n = nh$ where $n = 0, 1, ..., N$. Moreover, we define a positive integer number N_τ such that $N_\tau h = \tau$.
- Define the increment of time and standard Brownian motion with a uniform step size h like in SDEs:

  $$\begin{aligned} \Delta t_{n+1} &= t_{n+1} - t_n = h, \\ \Delta W_{n+1} &= W(t_{n+1}) - W(t_n) = \Delta W(h) = \sqrt{h} Z_{n+1}, \end{aligned}$$

  for some random variable $Z_{n+1} \in N(0,1)$, where $0 \le n \le N-1$.
- Suppose $\tilde{X}_n$ is an approximation of the strong solution to equation (3.3), using a stochastic explicit one step method with an increment function ϕ:

  $$\left.\begin{array}{l} \tilde{X}_{n+1} = \tilde{X}_n + \phi(h, \tilde{X}_n, \tilde{X}_{n-N_\tau}, \Delta W_{n+1}), \quad 0 \le n \le N-1, \\ \tilde{X}_{n-N_\tau} = \varphi(t_n - \tau), \quad 0 \le n \le N_\tau. \end{array}\right\} \tag{3.4}$$

- Sometimes, we will assume that for any $x, x', y, y' \in \mathbb{R}$, the increment function

ϕ fulfills the following conditions:

$$\left.\begin{aligned} &\Big|E\big(\phi(h,x,y,\Delta W_{n+1})-\phi(h,x',y',\Delta W_{n+1})\big)\Big| \le C_1 h\big(|x-x'|+|y-y'|\big), \\ &E\Big(\big|\phi(h,x,y,\Delta W_{n+1})-\phi(h,x',y',\Delta W_{n+1})\big|^2\Big) \le C_2 h\big(|x-x'|^2+|y-y'|^2\big), \end{aligned}\right\} \tag{3.5}$$

where C_1 and C_2 are some positive constant numbers.

- The following notations are also used:
  $X(t_{n+1})$ denotes the value of the exact solution of equation (3.3) at the point t_{n+1},
  $\tilde{X}_{n+1}$ denotes the value of approximate solution using equation (3.4) and
  $\tilde{X}(t_{n+1})$ denotes the locally approximate value obtained after just one step of equation (3.4) i.e.,

$$\tilde{X}(t_{n+1}) = X(t_n) + \phi(h, X(t_n), X(t_{n-N_\tau}), \Delta W_{n+1}).$$

After these, we are going to provide some definitions that are related to the way of measuring the accuracy of a numerical approximate solution to SDDE.

Definition 3.6. The local error of $\{\tilde{X}(t_n)\}$ is the sequence of random variables:

$$\delta_n = X(t_n) - \tilde{X}(t_n), \quad n = 1, 2, .., N.$$

The local error measures the difference between the approximation and the exact solution on a subinterval of the integration.

Definition 3.7. The global error of $\{\tilde{X}_n\}$ is the sequence of random variables:

$$\epsilon_n = X(t_n) - \tilde{X}_n, \quad n = 1, 2, .., N.$$

The global error measures the difference between the approximation and the exact solution over the entire integration range.

Definition 3.8. If the explicit one step method defined in equation (3.4) satisfies the following conditions:

$$\max_{1\le n\le N} |E(\delta_n)| \le Ch^{p_1} \quad \text{as} \quad h \to 0,$$

$$\max_{1\le n\le N} (E|\delta_n|^2)^{\frac{1}{2}} \le Ch^{p_2} \quad \text{as} \quad h \to 0,$$

for some positive constants $p_2 \ge \frac{1}{2}$, $p_1 \ge p_2 + \frac{1}{2}$ and C which does not depend on h but may depend on the initial condition φ and T then it is called consistent with order p_1 in the mean and with order p_2 in the mean square sense.

Definition 3.9. The method in equation (3.4) is convergent in the mean with order p_1 and in the mean square with order p_2 if the following conditions are satisfied:

$$\max_{1\le n\le N} |E(\epsilon_n)| \le Ch^{p_1} \quad \text{as} \quad h \to 0,$$

$$\max_{1\le n\le N} (E|\epsilon_n|^2)^{1/2} \le Ch^{p_2} \quad \text{as} \quad h \to 0,$$

again the constant C is independent of h, but may depends on the initial function φ and T.

Remark 3.4. Note that the consistency of the method is about the local error while the convergence is related to the global error.

Theorem 3.1. *Assume that drift and diffusion terms namely functions f and g fulfill local Lipschitz condition and linear growth condition. Moreover, suppose the increment function ϕ in equation* (3.4) *satisfies conditions in equation* (3.5) *and the method in equation* (3.4) *is consistent with order p_1 in the mean and order p_2 in the mean square sense. Then approximation in equation* (3.4) *for the equation* (3.3) *is convergent in L_2 with order $p = p_2 - \frac{1}{2}$ which means that convergence occurs in the mean square sense and we can write*

$$\max_{1 \leq n \leq N} (E|\epsilon_n|^2)^{1/2} \leq Ch^p \quad as \quad h \to 0.$$

Proof. The detailed proof can be found in [10]. ☐

Now, we can state the most known numerical method namely Euler Maruyama for SDDEs to find approximate solution.

3.2.1 Euler Maruyama Method for SDDE

Consider approximation with uniform step size h on the interval $[0, T]$, i.e., $h = T/N$ and $t_n = nh$ where $n = 0, 1, ..., N$. Moreover define a positive integer number N_τ such that $N_\tau h = \tau$. The increment function ϕ in equation (3.4) for the Euler Maruyama method is defined as

$$\phi(h, \tilde{X}_n, \tilde{X}_{n-N_\tau}, \Delta W_{n+1}) = f(\tilde{X}_n, \tilde{X}_{n-N_\tau})h + g(\tilde{X}_n, \tilde{X}_{n-N_\tau})\Delta W_{n+1} \tag{3.6}$$

for $n = 0, 1, ...N - 1$. Then, equation (3.4) becomes

$$\begin{aligned} \tilde{X}_{n+1} &= \tilde{X}_n + f(\tilde{X}_n, \tilde{X}_{n-N_\tau})h + g(\tilde{X}_n, \tilde{X}_{n-N_\tau})\Delta W_{n+1}, \\ &= \tilde{X}_n + f(\tilde{X}_n, \tilde{X}_{n-N_\tau})h + g(\tilde{X}_n, \tilde{X}_{n-N_\tau})\sqrt{h} Z_{n+1} \end{aligned}$$

for all $n - N_\tau \geq 0$, where Z_{n+1} corresponds to normally distributed random variable with mean 0 and variance 1, and for all indices $n - N_\tau \leq 0$ we define $\tilde{X}_{n-N_\tau} := \psi(t_n - \tau)$.

Theorem 3.2. *Assume that the coefficient functions f and g in equation* (3.3) *satisfy the conditions of existence and uniqueness theorem, namely local Lipschitz and linear growth conditions. Then the Euler Maruyama scheme is consistent with order $p_1 = 2$ in the mean and order $p_2 = 1$ in the mean square sense.*

Proof. The complete proof can be found in [4]. ☐

Lemma 3.3. *If equation* (3.3) *has a unique strong solution, then the increment function* ϕ *in equation* (3.6) *satisfies the conditions in equation* (3.5).

Proof. Assume that we have a unique strong solution which means the coefficient functions f and g satisfy the local Lipschitz and linear growth conditions. Let us show for any $x, x', y, y' \in \mathbb{R}$, there exists constant numbers C_1 and C_2 so that the conditions in equation (3.5) hold:

$$
\begin{aligned}
&\big|E\big(\phi(h,x,y,\Delta W_{n+1}) - \phi(h,x',y',\Delta W_{n+1})\big)\big| \\
&\qquad = \big|E\big(f(x,y)h + g(x,y)\Delta W_{n+1} - f(x',y')h - g(x',y')\Delta W_{n+1}\big)\big| \\
&\qquad \le \big|E\big(f(x,y)h - f(x',y')h\big)\big| + \big|E\big(g(x,y)\Delta W_{n+1} - g(x',y')\Delta W_{n+1}\big)\big| \\
&\qquad \le h\big|f(x,y) - f(x',y')\big| + \big|g(x,y) - g(x',y')\big|\big|E(\Delta W_{n+1})\big| \\
&\qquad \le h\big|f(x,y) - f(x',y')\big| \qquad \text{(since } E(W(t)) = 0\text{)} \\
&\qquad \le C_1\, h(|x-x'| + |y-y'|), \qquad \text{(since } f \text{ satisfies local Lipschitz condition)}
\end{aligned}
$$

$$
\begin{aligned}
&E\big(\big|\phi(h,x,y,\Delta W_{n+1}) - \phi(h,x',y',\Delta W_{n+1})\big|^2\big) \\
&\quad = E\Big(\big|f(x,y)h + g(x,y)\Delta W_{n+1} - f(x',y')h - g(x',y')\Delta W_{n+1}\big|^2\Big) \\
&\quad = E\Big(\big|(f(x,y) - f(x',y'))h + (g(x,y) - g(x',y'))\Delta W_{n+1}\big|^2\Big) \\
&\quad \le E\Big(\Big(\big|(f(x,y) - f(x',y'))h\big| + \big|(g(x,y) - g(x',y'))\Delta W_{n+1}\big|\Big)^2\Big) \\
&\quad \le E\Big(2h^2|f(x,y) - f(x',y')|^2 + 2\Delta W_{n+1}^2|g(x,y) - g(x',y')|^2\Big) \\
&\quad \le 2h^2\big|f(x,y) - f(x',y')\big|^2 + 2\big|g(x,y) - g(x',y')\big|^2 E(\Delta W_{n+1}^2) \\
&\quad \le L_1 2h^2\big(|x-x'| + |y-y'|\big)^2 + L_2 2h\big(|x-x'| + |y-y'|\big)^2 \text{(since } (a+b)^2 \le 2(a^2+b^2)\text{)} \\
&\quad \le L_1 2h^2\big(2|x-x'|^2 + 2|y-y'|^2\big) + L_2 2h\big(2|x-x'|^2 + 2|y-y'|^2\big) \\
&\quad \le C_2 h\big(|x-x'|^2 + |y-y'|^2\big).
\end{aligned}
$$

□

Remark 3.5.

- According to Theorem 3.2 and Lemma 3.3, the Euler Maruyama method fulfills Theorem 3.1 with order of convergence $p = 1/2$ in the mean square sense and we can write:

$$\max_{1 \le n \le N} (E|\epsilon_n|^2)^{1/2} \le C h^{1/2} \quad \text{as} \quad h \to 0.$$

- If equation (3.3) has an additive noise (function g does not depend on X), the Euler Maruyama method is consistent with order $p_1 = 2$ in the mean and order

$p_2 = 3/2$ in the mean square sense. In this case, method is converge with order $p = 1$ in the mean square sense and we get:

$$\max_{1\leq n\leq N} |E(\epsilon_n)| \leq Ch \quad \text{as} \quad h \to 0.$$

Now, let us consider some examples to clarify the method.

Example 3.3. Now let us consider the SDDE in Example 2.4 while setting the coefficient of $X(t-\tau)$ as μ,

$$\begin{aligned} dX(t) &= \mu X(t-\tau)dt + \beta dW(t), \quad 0 \leq t \leq T, \\ X(t) &= \phi(t), \quad t \in [-\tau, 0]. \end{aligned}$$

Since the calculation of exact solution is not easy, we simulate the solution process using Euler Maruyama method on SDDE. In order to see the effect of the initial value on SDDE, we provide our simulations while setting $\phi(t) = e^{-t}$ and $\phi(t) = 1 + t$.

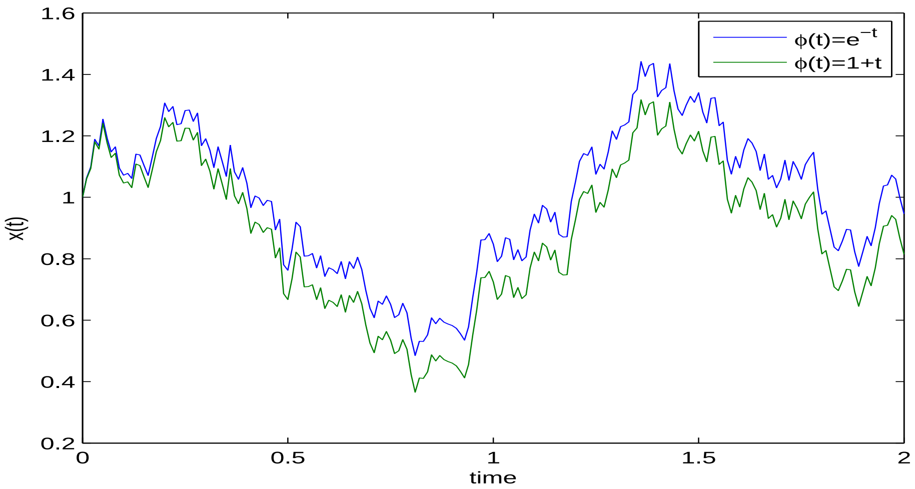

Figure 3.6: Sample path with different initial functions,

The Figure 3.6 shows two sample path for the different choice of initial values with $\tau = 1$, $T = 2$, $\mu = 0.1$ and $\beta = 0.5$. Up to time 0, the path with initial data $\phi(t) = e^{-t}$ decreases while path with initial data $\phi(t) = 1 + t$ increases. At time equal to 0, it is seen that both graphs take the same value, 1. After time 0, we observe that both graphs have the same structure and their values is always near to each other (difference of the values for the sample paths between $t = 0$ and $t = 2$ increases from 0 to 0.14).

In Figure 3.7, we see the effect of delay term on the mean function. When delay is getting smaller, graph of the mean function approaches to the non delayed one.

Figure 3.8 provides the information about the effects of coefficients for the choice of initial function $\phi(t) = 1+t$. From the first and second graphs, it is seen that increasing the diffusion term increases the volatility. From the second and third graphs, we realize that change in the drift term only affects the value of the solution process and structure is preserved.

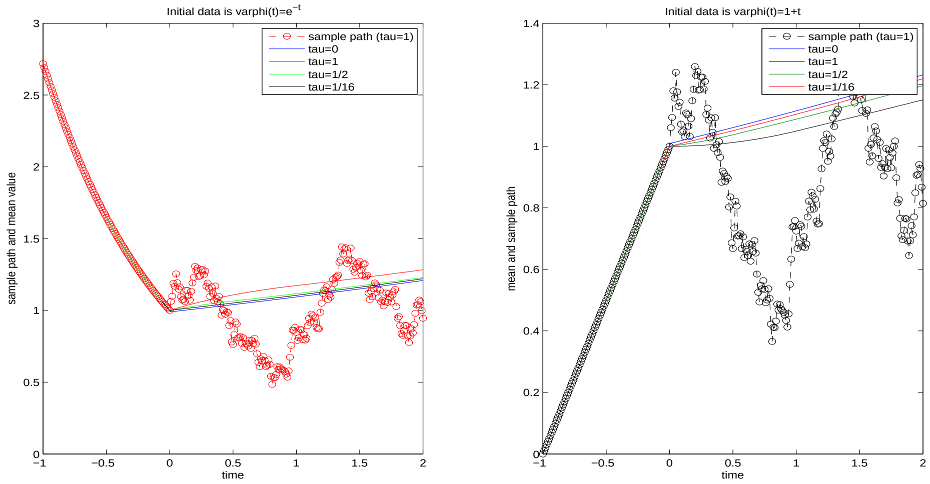

Figure 3.7: Sample path and mean function with different choice for initial functions and delay terms.

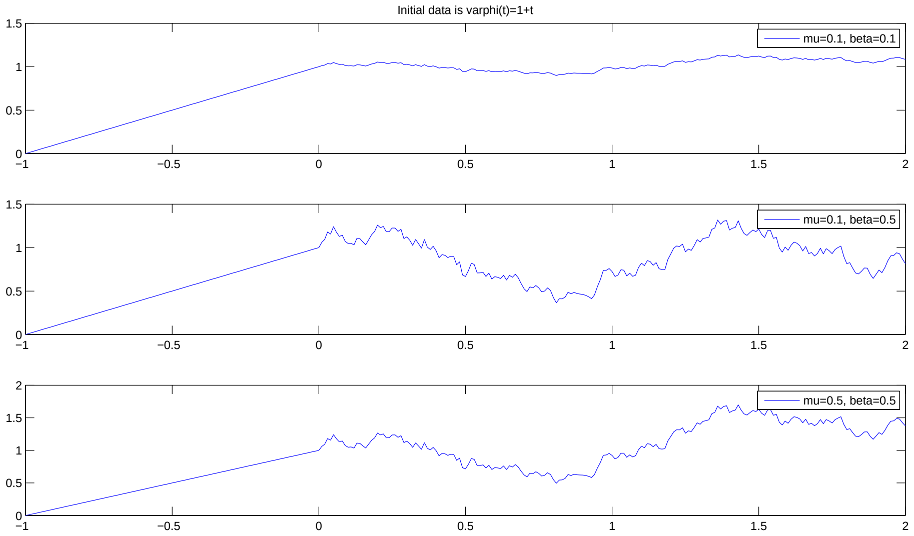

Figure 3.8: Sample path with different coefficients μ and σ.

Example 3.4. Now, let us consider SDDE in Example 2.6 again,

$$\begin{aligned} dX(t) &= \sigma X(t-\tau)dW(t), \qquad t \geq 0, \\ X(t) &= \phi(t), \qquad t \in [-\tau, 0], \end{aligned}$$

where σ is a constant real number to see the delay term effect in the diffusion. Like in the previous question, we take two different initial values.

In Figure 3.9, we see one sample path with $\tau = 1$, $T = 2$ and $\sigma = 1$ where initial values are $\phi(t) = e^{-t}$ and $\phi(t) = 1 + t$.

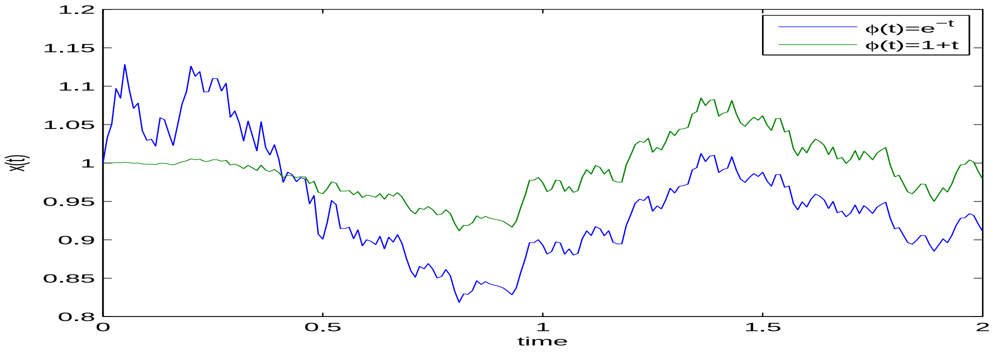

Figure 3.9: Sample path with different initial functions.

For the next simulation, we take initial value as $\phi(t) = 1 + t$.

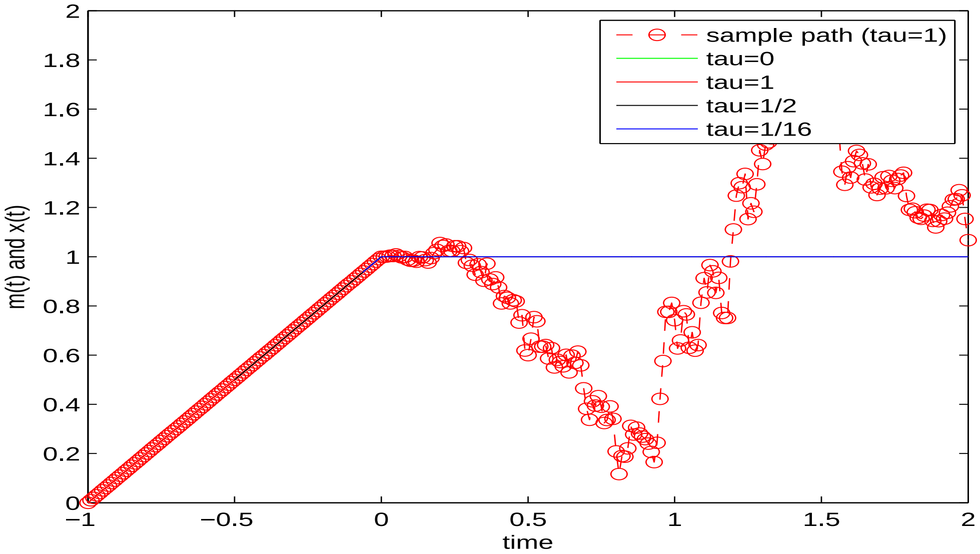

Figure 3.10: Sample path and corresponding mean function.

In Figure 3.10, we see one sample path with $\tau = 1$ and mean function with the different choice of τ. Since there is no drift term and mean of the martingale process is equal to the mean of the initial value, mean function is constant and equal to $E(\phi(0)) = 1$ for all $t \geq 0$ (we actually proved this theoretically in Proposition 2.6). Moreover, the choice of τ does not affect it.

In this chapter, we consider Euler Maruyama method for SDDEs and its convergence analysis. With the help of method, simulations of the examples in the previous chapter is done. In that simulations, the effect of initial function and length of delay term are considered. It is observed that they have an important effect on the evolution of the solution process and corresponding expected value in the future states. In the next chapter, applications of SDDEs in the financial market will be considered.

CHAPTER 4

APPLICATION

In this chapter we provide two applications of SDDEs in order to examine the concepts that are discussed in the previous chapters and see the effect of time delay. In the first application, we consider a market so that its stock returns depend on the historical information. Zheng (2015) provides a model for this system and examine for the choice of delay term as 1 in [53]. In this work as an our first application , the model is expressed as a function of delay term and the behavior of mean function in terms of delay term is examined. In the second application, firstly the value for a European call option under the delay effect is provided according to [2] (2007). Arriojas et al. handled options for finding fair price of them under the delay effect and provided a fair price formula. We understand the logic behind the model and then provide some numerical results to observe the effect of delay term. We consider the value of the option for the delay and no delay cases and give the results.

For more application of SDDEs in the financial market, one can see [31, 36, 43, 50].

4.1 Stock Return Equation with Time Delay

Let us construct our stock return model with time delay in the financial market. We assumed that trading occurs continuously over time. The stock returns react to the information that is gotten at the previous time point τ. In other words, the split of the trading asset could depend on the historical information at the time point τ. Under that assumption, this feedback process is modeled by an SDDE while implementing a linear delay into the most known stock return model, namely, geometric Brownian motion (GBM). The formulation of the model is:

$$\begin{aligned} dS(t) &= (xS(t) + yS(t-\tau) + a)dt + (zS(t) + wS(t-\tau) + b)dW(t), \quad 0 \leq t \leq T, \\ S(t) &= \varphi(t), \quad -\tau \leq t \leq 0, \end{aligned}$$

where the delay term τ is positive fixed number and the coefficients, namely, x, y, a, z, w and $b \in \mathbb{R}$. We assume that $\varphi(t) : [-\tau, 0] \to \mathbb{R}$ is a continuous initial function on its domain and $\mathcal{F}_0$-measurable random variable such that $E\big(\sup\limits_{t\in[-\tau,0]} |\varphi(t)|^2\big) < \infty$.

Now, we consider Example 3.4 given in [53] (2015). Our model is expressed as a function of delay instead of considering only $\tau = 1$ and the behavior of mean function with respect to the choice of the delay term is examined. Now, let us consider the linear SDDE:

$$\left.\begin{aligned} dS(t) &= (-3S(t) + 2e^{-1}S(t-\tau) + 3 - 2e^{-1})dt \\ &\qquad\qquad +(aS(t) + bS(t-\tau))dW(t), \quad 0 \leq t \leq T, \\ S(t) &= 1 + e^{-t}, \quad -\tau \leq t \leq 0, \end{aligned}\right\} \tag{4.1}$$

where a and b are some constants and $W(t)$ represents a standard Brownian motion. Let us first check whether this SDDE satisfies the existence and uniqueness theorem or not where $f(t,x,y) = -3x + 2e^{-1}y + 3 - 2e^{-1}$ and $g(t,x,y) = ax + by$, according to the general formulation given in equation (2.5). Now for any $x_1, x_2, y_1, y_2 \in \mathbb{R}$ and $t \in \mathbb{R}^+$, we have:

$$\begin{aligned} |f(t,x_1,y_1) - f(t,x_2,y_2)| &= |-3x_1 + 2e^{-1}y_1 + 3 - 2e^{-1} - (-3x_2 + 2e^{-1}y_2 + 3 - 2e^{-1})| \\ &= |-3(x_1 - x_2) + 2e^{-1}(y_1 - y_2)| \\ &\leq 3|x_1 - x_2| + 2e^{-1}|y_1 - y_2| \\ &\leq 3(|x_1 - x_2| + |y_1 - y_2|) \end{aligned}$$

and

$$\begin{aligned} |g(t,x_1,y_1) - g(t,x_2,y_2)| &= |ax_1 + by_1 - (ax_2 + by_2)| \\ &\leq |a||x_1 - x_2| + |b||y_1 - y_2| \\ &\leq (|a| \vee |b|)(|x_1 - x_2| + |y_1 - y_2|), \end{aligned}$$

where $|a| \vee |b| = \max\{|a|, |b|\}$. These two inequalities show that the local Lipschitz condition is satisfied. Moreover, the linear growth condition is also satisfied for any $x, y \in \mathbb{R}$ and $t \in \mathbb{R}^+$. As a result, given SDDE has a path-wise unique strong solution and that solution $S(t)$ satisfies

$$E\Big(\sup_{-\tau \leq t \leq T} |S(t)|^2\Big) < \infty, \quad \text{for all} \quad T > 0.$$

Proposition 4.1. *Suppose S(t) fulfills equation* (4.1) *where* $T = 2\tau$. *Then the expected value of S(t) namely* $m(t) = E(S(t))$ *is given by*

$$m(t) = \begin{cases} 1 + e^{-t}, \quad t \in [-\tau, 0] \\ e^{-3t}(1 - e^{\tau-1}) + e^{-t+\tau-1} + 1, \quad t \in [0, \tau] \\ 1 + e^{-t+2\tau-2} + 2te^{-3t}(e^{3\tau-1} - e^{4\tau-2}) \\ \quad +e^{-3t}(1 + e^{4\tau-2}(-1+2\tau) + e^{3\tau-1}(1-2\tau) - e^{\tau-1}), \quad t \in [\tau, T]. \end{cases} \tag{4.2}$$

Proof. Let us compute the mean function of the stochastic process $S(t)$. Consider (4.1) for $t \in [0, T]$ in the integral form:

$$S(t) = S(0) + \int_0^t (-3S(u) + 2e^{-1}S(u-\tau) + 3 - 2e^{-1})du + \int_0^t (aS(u) + bS(u-\tau))dW(u).$$

Now, take the expectation of both sides :

$$\begin{aligned} m(t) &= m(0) + \int_0^t (-3m(u) + 2e^{-1}m(u-\tau) + E(3-2e^{-1}))du \\ &\qquad + E(\int_0^t (aS(u) + bS(u-\tau))dW(u)) \\ &= m(0) + \int_0^t (-3m(u) + 2e^{-1}m(u-\tau) + 3 - 2e^{-1})du \end{aligned}$$

where $E(\int_0^t (aS(u) + bS(u-\tau))dW(u)) = 0$ since $\int_0^t (aS(u) + bS(u-\tau))dW(u)$ is a martingale. While taking the derivative of this equation with respect to t, we obtain:

$$\left.\begin{array}{l} m'(t) = -3m(t) + 2e^{-1}m(t-\tau) + 3 - 2e^{-1}, \quad t \in [0,T], \\ m(t) = 1 + e^{-t}, \quad t \in [-\tau, 0]. \end{array}\right\} \tag{4.3}$$

Now, the mean function in (4.3) is solved by iteration.
For $t \in [0,\tau]$: our equation becomes

$$\begin{aligned} &m'(t) = -3m(t) + 2e^{-t+\tau-1} + 3, \\ &m(0) = 2, \end{aligned}$$

and corresponding solution is

$$m(t) = e^{-3t}(1 - e^{\tau-1}) + e^{-t+\tau-1} + 1.$$

For $t \in [\tau, 2\tau]$: where $T = 2\tau$, equation can be written as

$$\begin{aligned} &m'(t) = -3m(t) + 2e^{-3t+3\tau-1} - 2e^{-3t+4\tau-2} + 2e^{-t+2\tau-2} + 3, \\ &m(\tau) = e^{-3\tau} - e^{-2\tau-1} + e^{-1} + 1, \end{aligned}$$

and the solution of this ODE is

$$m(t) = 1+e^{-t+2\tau-2}+2te^{-3t}(e^{3\tau-1}-e^{4\tau-2})+e^{-3t}(1+e^{4\tau-2}(-1+2\tau)+e^{3\tau-1}(1-2\tau)-e^{\tau-1}).$$

So, we get the result:

$$m(t) = \begin{cases} 1 + e^{-t}, \quad t \in [-\tau, 0], \\ e^{-3t}(1 - e^{\tau-1}) + e^{-t+\tau-1} + 1, \quad t \in [0,\tau], \\ 1 + e^{-t+2\tau-2} + 2te^{-3t}(e^{3\tau-1} - e^{4\tau-2}) \\ \quad +e^{-3t}(1 + e^{4\tau-2}(-1+2\tau) + e^{3\tau-1}(1-2\tau) - e^{\tau-1}), \quad t \in [\tau, 2\tau]. \end{cases}$$

Note that, we can also represent that mean function as $m(t,\tau)$ since τ is also a parameter. □

Remark 4.1. Let us set $T = 2$ and $\tau = 1$. According to equation (4.2), the mean function of this linear SDDE in (4.1) becomes

$$m(t) = 1 + e^{-t}, \quad -1 \leq t \leq 2.$$

4.1.1 Simulations

We will use Euler Maruyama method to simulate the sample path (since there is no closed form solution).

In the following figures, we set the maturity time as $T = 2$.

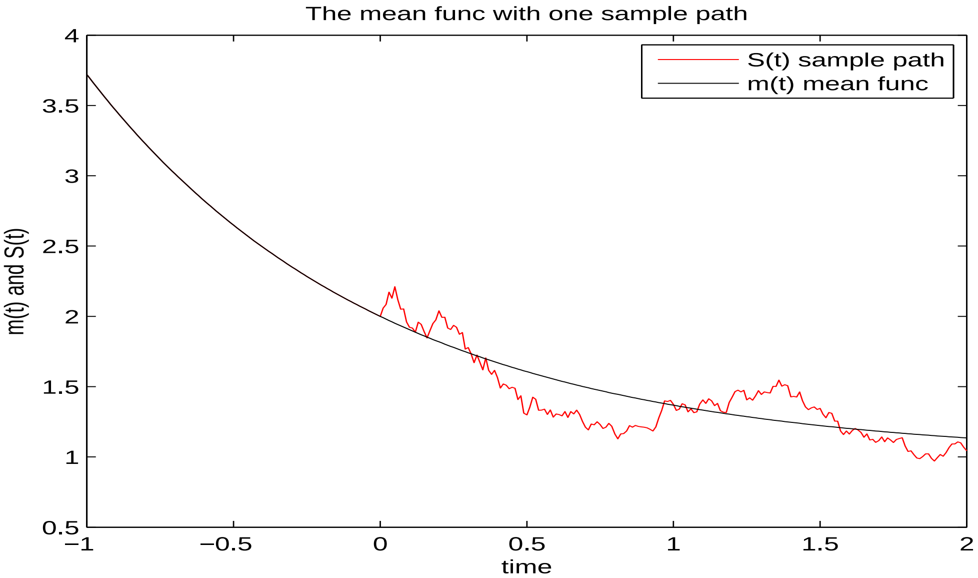

Figure 4.1: Solution process and mean function for $T = 2, \tau = 1, a = b = 0.1$.

Figure 4.1 shows one sample path and corresponding mean function of the equation (4.1) with the parameters $a = b = 0.1$ and $\tau = 1$. Since the closed form solution does not exist , we use Euler Maruyama method to simulate the sample path.

Let us examine the order of convergence in the mean for that equation where $a = b = 0.1$ and $\tau = 1$. We simulate 50000 sample paths with different Brownian motion over the time period [0, 2] for the different choice of mesh $h = 2^{-4}, 2^{-5}, 2^{-6}, 2^{-7}, 2^{-8}$, and calculate the average global error at the maturity $T = 2$. According to our remark case two in 3.5 , we expect that Euler Maruyama method converges in the mean with order 1, i.e.,

$$\max_{1\leq n\leq N} |E(\epsilon_n)| \leq C\,h \quad \text{as} \quad h \to 0$$

for some constant $C > 0$. Thus, we expect the plot of average global error, $E(\epsilon_n)$, versus mesh, h, to show a line pattern with slope C.
From Figure 4.2, we see that convergence in the mean is approximately order 1. The slope of the line is equal to 0.062, namely, C.

In order to show the effect of delay term on the process:

- Substitute different coefficients of delay term namely parameter b, and
- Substitute different lengths of delay term τ.

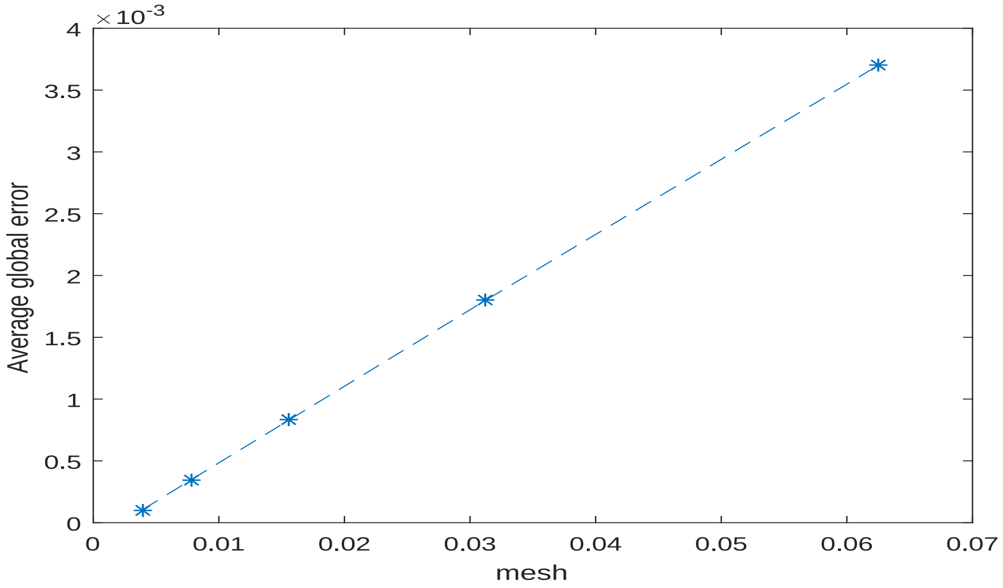

Figure 4.2: Convergence in the mean with order 1.

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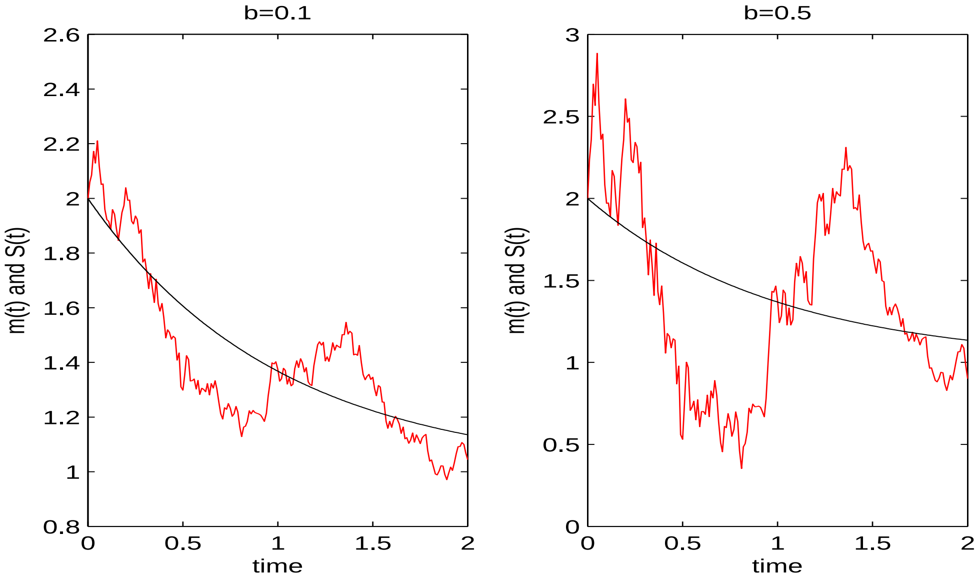

Figure 4.3: Impact of the delay term that is in the diffusion.

In Figure 4.3, we take $b = 0.1$ and $b = 0.5$ where $a = 0.1$ and $\tau = 1$ in both graphs. It is seen that the mean function is the same, since the choice of a and b does not affect the mean function. However, in the second graph volatility of the sample path increases with the increase in the term b. Actually, this is not a surprise, because increasing b means increasing the diffusion term which is directly related to volatility of the solution process.

Now, consider total coefficient of diffusion term is constant and equal to 0.5, i.e.,

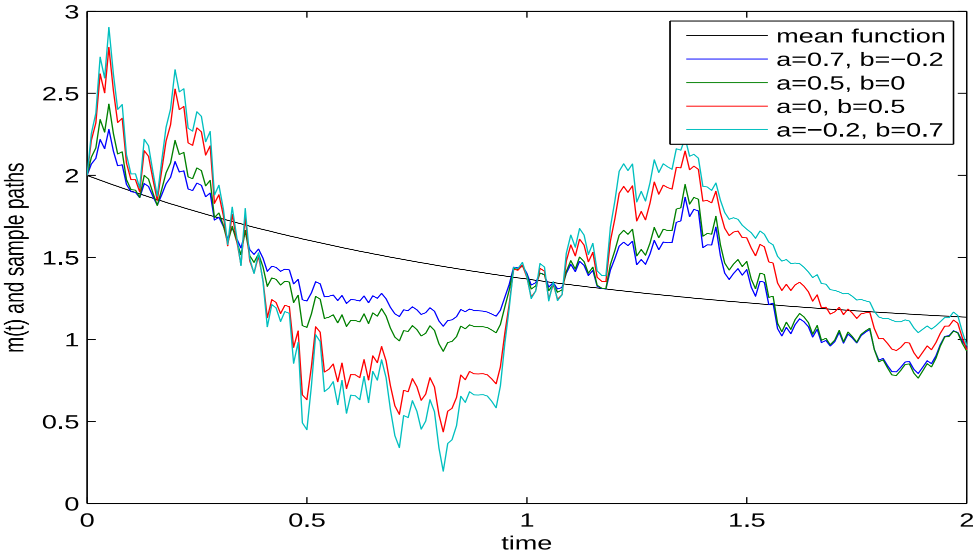

Figure 4.4: Different choice of coefficients a and b where $a + b = 0.5$.

$a + b = 0.5$. Let us set the different combinations of a and b. We observe in Figure 4.4 that although total coefficient of diffusion term is same, different combinations of a and b influence the volatility of the solution process (price process) for the underlying asset. Moreover, it is seen that the effect of term b is more than a.

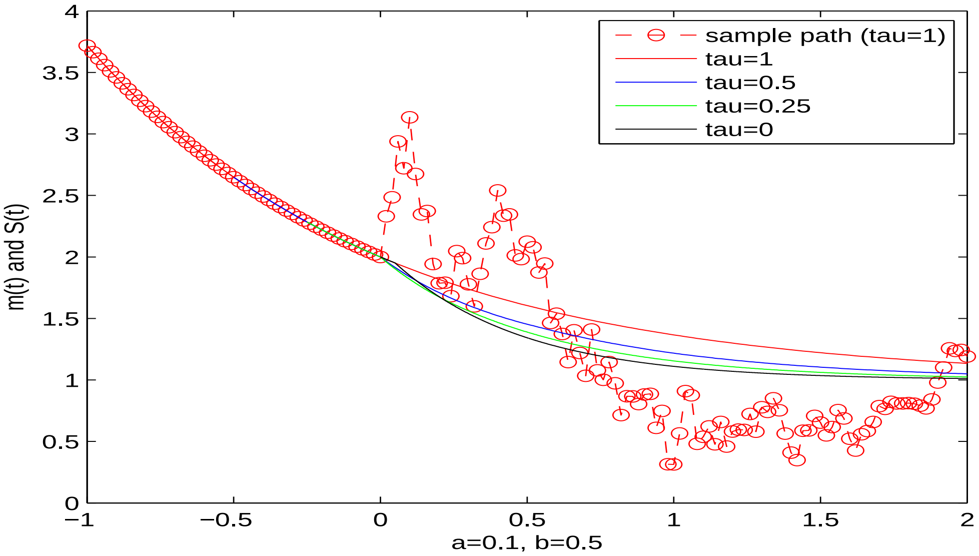

Figure 4.5: Effect of different choice of delay term on the mean function.

In Figure 4.5, we see the effect of delay term on the mean function for the different choice of $\tau = 1, 0.5, 0.25$ and $\tau = 0$, i.e., no delay. The black curve that lies at the bottom shows the mean function for no-delay case. Note that, it has a sharp change after the initial point (0,2), since we only know the information at $t = 0$. It is seen that when we make small perturbation in length of delay, $\tau = 0.25$, a small change in the mean function occurs. However, when τ is increased 0 to 1, we observe a big change in the mean. The red curve moves up and away from the black curve. According to these observations, we can say that the length of delay term has a significant impact on the mean function. From that choice of τ, it is realized that when we decrease the value of τ, our corresponding mean function value also decreases. In other words, the mean function is increasing with respect to τ when $\tau \in [0, 1]$. This is simply the case because

$$\frac{dm(t,\tau)}{d\tau} = \begin{cases} -e^{-3t+\tau-1} + e^{-t+\tau-1}, & t \in [0,\tau], \\ 2e^{-t+2\tau-2} + 2te^{-3t}(3e^{3\tau-1} - 4e^{4\tau-2}) & \\ \quad +e^{-3t}(2e^{4\tau-2}(-1+4\tau) + e^{3\tau-1}(1-6\tau) - e^{\tau-1}), & t \in [\tau, 2\tau], \end{cases}$$

according to equation (4.2), which shows us $\dfrac{dm(t,\tau)}{d\tau} > 0$ for any $\tau \in [0, 1]$.

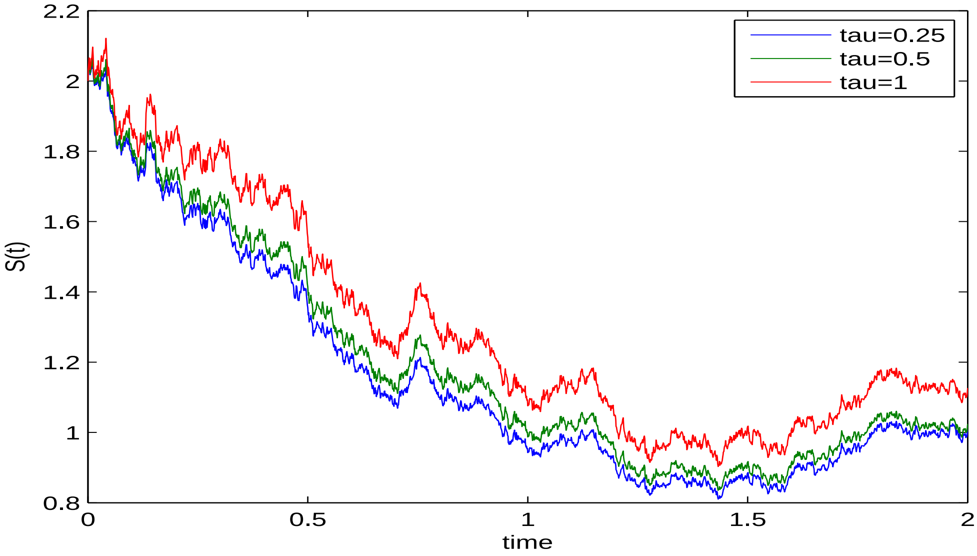

Figure 4.6: Effect of different choice of delay term on the sample path where $a = b = 0.1$.

In Figure 4.6, we see the effect of delay term on the solution process for the different choice of $\tau = 1, 0.5$ and 0.25. We realize that the value of the solution process increase with the increase of delay term.

4.2 European Option Pricing with Delay

In this section, we examine European call option pricing model when the stock price process satisfies nonlinear stochastic delay differential equations. We consider the delay term is fixed. In other words, we try to understand the influence of the history in the determination of the price of a European call option. Our second application is based on the study of Arriojas et al. (2007) [2]. In this study, the valuation of the European option is handled theoretically. There is no application and numerical result for the value of the option for the model with delay. We provide the followings numerical results:

- Effect of initial function, and
- Effect of delay term

on the evaluation of the fair price of the option where the stock price follows a SDDE. We provide exact value of the option by using formula given in [2] and approximate value obtained from Monte-Carlo simulation. Moreover, we use Black-Scholes formula to compute the value for no delay case while taking the same initial price.

Arriojas et al. derive the fair price formula of the European option for any time $t \leq T$. They consider a market consisting of a risk-less asset (a bond or bank account) and a single stock. They denote that risk-less asset by $B(t)$ where the rate of return is $r > 0$, i.e., $B(t) = e^{rt}$. Moreover, they denote the price process of that stock as $S(t)$ at time t and assume that it fulfills the following SDDE:

$$\left.\begin{array}{l} dS(t) = \mu S(t-a)S(t)dt + g(S(t-b))S(t)dW(t), \quad t \in [0,T], \\ S(t) = \varphi(t), \quad t \in [-L,0], \end{array}\right\} \tag{4.4}$$

on a complete probability space $(\Omega, \mathcal{F}, P)$ with filtration $\{\mathcal{F}_t\}_{0 \leq t \leq T}$ satisfying the usual conditions. The coefficient μ and maturity T are positive constant numbers, a and b are positive fixed delays with $L = \max\{a, b\}$ and $g : \mathbb{R} \to \mathbb{R}$ is a continuous function. Furthermore, $W(t)$ is a one-dimensional standard Brownian motion and $\varphi(t) : [-L, 0] \to C([-L,0], \mathbb{R})$ is $\mathcal{F}_0$-measurable random variable such that $\varphi(0) > 0$ a.s.

They consider an option written on the stock with the maturity T and an exercise price K under the assumption no transaction costs and no dividend payment. In this research, it is also proven that the model satisfies the no-arbitrage property and the completeness of the market. Then, the fair price formula for option which based on an equivalent local martingale measure in Theorem 3, [2] is provided. As a consequence of that theorem, they provide a Black-Scholes type formula for the value of a European call option where the stock price process satisfy (4.4) at any time $t < T$ in Theorem 4 which means the following.

Theorem 4.2. *[2] Assume that the stock price S satisfies equation* (4.4) *with conditions $\varphi(0) > 0$ and $g(u) \neq 0$ whenever $u \neq 0$. Let $r > 0$ be the rate of return for the risk free asset and Q be new probability measure where the discounted asset prices are local martingale under it. Let $V(t)$ be the pricing formula of a European call option which is written on the stock S with strike price K and maturity time T. Let ϕ denote the standard normal distribution function i.e.,*

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} du, \; x \in \mathbb{R}.$$

Then there exist two cases for the value of the option:

- *For all* $t \in [T-l, T]$*, where* $l = \min\{a, b\}$*:*

$$V(t) = S(t)\phi(\beta_1(t)) - Ke^{-r(T-t)}\phi(\beta_2(t)), \tag{4.5}$$

where

$$\beta_1 = \frac{\log(\frac{S(t)}{K}) + \int_t^T (r + \frac{1}{2}g(S(u-b))^2)du}{\sqrt{\int_t^T g(S(u-b))^2 du}},$$

$$\beta_2 = \frac{\log(\frac{S(t)}{K}) + \int_t^T (r - \frac{1}{2}g(S(u-b))^2)du}{\sqrt{\int_t^T g(S(u-b))^2 du}}.$$

- *For all* $T > l$ *and* $t < T - l$*:*

$$V(t) = e^{rt}E_Q\Big(H\Big(e^{-r(T-l)}S(T-l), -\frac{1}{2}\int_{T-l}^{T} g(S(u-b))^2 du, \int_{T-l}^{T} g(S(u-b))^2 du\Big)\Big|\mathcal{F}_t\Big), \tag{4.6}$$

where

$$H(x, m, \sigma^2) = xe^{m+\sigma^2/2}\phi(\alpha_1(x, m, \sigma)) - Ke^{-rT}\phi(\alpha_2(x, m, \sigma)),$$

and

$$\alpha_1(x, m, \sigma) = \frac{1}{\sigma}\Big[\log\big(\frac{x}{K}\big) + rT + m + \sigma^2\Big],$$

$$\alpha_2(x, m, \sigma) = \frac{1}{\sigma}\Big[\log\big(\frac{x}{K}\big) + rT + m\Big],$$

for $\sigma, x \in \mathbb{R}^+$*,* $m \in \mathbb{R}$*.*

The hedging strategy for $t \in [T-l, T]$ *is given by*

$$\pi_S(t) = \phi(\beta_1(t)),$$
$$\pi_B(t) = -Ke^{-r(T-t)}\phi(\beta_2(t)).$$

Remark 4.2. (i) Note that $-2m = \sigma^2$ in the above formula. When we can rewrite the function H as

$$H\Big(x, -\frac{\sigma^2}{2}, \sigma^2\Big) = x\phi\Big(\alpha_1\Big(x, -\frac{\sigma^2}{2}, \sigma\Big)\Big) - Ke^{-rT}\phi\Big(\alpha_2\Big(x, -\frac{\sigma^2}{2}, \sigma\Big)\Big).$$

(ii) If $g(x) = 1$ for all $x \in \mathbb{R}^+$, then equation (4.5) reduces to the classical Black-Scholes formula.

4.2.1 Numerical Treatment for European Option Pricing with Delay

Let us consider a European call option written on the stock S which satisfies equation (4.4) and its conditions for one fixed delay $a = b = \tau$. Assume $g(x) = \sigma x$ for some positive real number σ. Then, S fulfills the following equation:

$$dS(t) = \mu S(t-\tau)S(t)dt + \sigma S(t-\tau)S(t)dW(t), \quad t \in [0, T],$$
$$S(t) = \varphi(t), \quad t \in [-\tau, 0],$$

where μ, σ are positive real numbers and the delay $\tau > 0$ is fixed. The initial path $\varphi(t) : [-\tau, 0] \to C([-\tau, 0], \mathbb{R})$ is $\mathcal{F}_0$-measurable random variable and $\varphi(0) > 0$ a.s. This equation can be written under the new measure Q as:

$$\begin{aligned} dS(t) &= rS(t)dt + \sigma S(t-\tau)S(t)d\tilde{W}(t), \quad t \in [0, T], \\ S(t) &= \varphi(t), \quad t \in [-\tau, 0], \end{aligned}$$

where r is the risk free rate such that $r > 0$ and $\tilde{W}$ is a standard Brownian motion under Q.
Let us take the risk free rate as $r = 0.05$, $\sigma = 0.04$, $T = 2$ and mesh size $h = 0.01$ in the following computations.

When $t \in [T - \tau, T]$: Let us take $t = 1.8$.

- In order to see the effect of initial functions on the evolution of option price, we take two different initial values, namely, $\varphi_1(t) = 1 + t$ and $\varphi_2(t) = 0.01 + e^{-t}$. Note that φ_1 is increasing and φ_2 is decreasing on their domain $[-\tau, 0]$. We compute call option price for the delay case using formula in equation (4.5) and denote it in the table as "V". Moreover, we compute the call option price for the non-delay case using Black-Scholes formula while taking the same stock price at time $t = 1.8$ and denote it as "B.S.". Now under these settings, we provide two tables for the different choices of delay terms.
    - Let us take delay term as $\tau = 0.5$ then corresponding stock price at $t = 1.8$ are $S_{\varphi_1}(1.8) = 1.0587$ and $S_{\varphi_1}(1.8) = 1.0559$ for initial functions φ_1 and φ_2 respectively. We see the results in Table 4.1.

      Table 4.1: Effect of initial function, where $\tau = 0.5$.

		Exercise Price		
		$K = 1.04$	$K = 1.05$	$K = 1.07$
$\varphi_1(t) = 1 + t$	V	0.0297	0.0210	0.0078
	B.S.	0.0295	0.0206	0.0072
$\varphi_2(t) = 0.01 + e^{-t}$	V	0.0272	0.0187	0.0065
	B.S.	0.0269	0.0183	0.0060

      We realize that initial data has an important effect on the value of the option. When we check our results, the value of the option for the initial function φ_1 is greater than for the initial function φ_2, since S_{φ_1} at $t = 1.8$ is bigger than S_{φ_2}. Because of same reason, the value of the option for the initial function φ_1 in the no-delay case is greater than for φ_2.
    - Now, take delay term as $\tau = 0.25$, then the corresponding stock price at $t = 1.8$ are $S_{\varphi_1}(1.8) = 1.0547$ and $S_{\varphi_2}(1.8) = 1.0609$ for initial function φ_1 and φ_2 respectively. We see the results in Table 4.2. In this case, the value of the option for the initial data φ_2 is greater than for the initial data φ_1, since S_{φ_1} at $t = 1.8$ is less than S_{φ_2}.

Table 4.2: Effect of initial function, where $\tau = 0.25$.

		Exercise Price		
		$K = 1.04$	$K = 1.05$	$K = 1.07$
$\varphi_1(t) = 1 + t$	V	0.0260	0.0177	0.0058
	B.S.	0.0258	0.0174	0.0054
$\varphi_2(t) = 0.01 + e^{-t}$	V	0.0318	0.0228	0.0088
	B.S.	0.0316	0.0226	0.0084

From these two table, we notice that the value of the option is increase when the model contains past information i.e., delay term.

- In order to see the effect of delay term on the evolution of option price clearly, we set $\tau = 0.5$ and $\tau = 0.25$, where the initial process is $\varphi(t) = 1 + t$. Under this setting, corresponding stock price at $t = 1.8$ are $S_{\tau_1}(1.8) = 1.0587$ and $S_{\tau_2}(1.8) = 1.0547$ for $\tau_1 = 0.5$ and $\tau_2 = 0.25$, respectively. The results are provided in Table 4.3.

Table 4.3: Effect of delay term, where $\varphi(t) = 1 + t$.

		Exercise Price		
		$K = 1.04$	$K = 1.05$	$K = 1.07$
$\tau = 0.5$	V	0.0297	0.0210	0.0078
	B.S.	0.0295	0.0206	0.0072
$\tau = 0.25$	V	0.0260	0.0177	0.0058
	B.S.	0.0258	0.0174	0.0054

It seen that the values for τ_1 is greater than the τ_2, since when we increase τ, the corresponding stock price is also increases.

For $t \leq T - \tau$: Let us take $t = 0$ and $\tau = 0.5$. In the following tables, “V” corresponds to the call option value that is obtained by semi-analytic Monte-Carlo simulation using the formula in equation (4.6), “M.C.” corresponds to the call option value that is obtained by Monte-Carlo simulation on the given SDDE and “B.S.” corresponds to the call option value that is obtained from Black-Scholes model.

- When we take initial process as $\varphi(t) = 1+t$, our stock price at $t = 0$ is $S(0) = 1$. In Table 4.4, we see the corresponding results.

Table 4.4: Call option values at $t = 0$, where $\varphi(t) = 1 + t$.

	Exercise Price		
	$K = 0.95$	$K = 1$	$K = 1.05$
V	0.1404	0.0958	0.0549
M.C.	0.1403	0.0957	0.0547
B.S.	0.1405	0.0960	0.0554

Note that we realize that the values of the option price in "V" row is less than the values in "B.S." row. This actually shows us the effect of delay term. Although the initial stock price is same in both cases, the stock price in the delay case at any time $t \in [0, T]$ is less than in the no-delay case, see Figure 4.7.

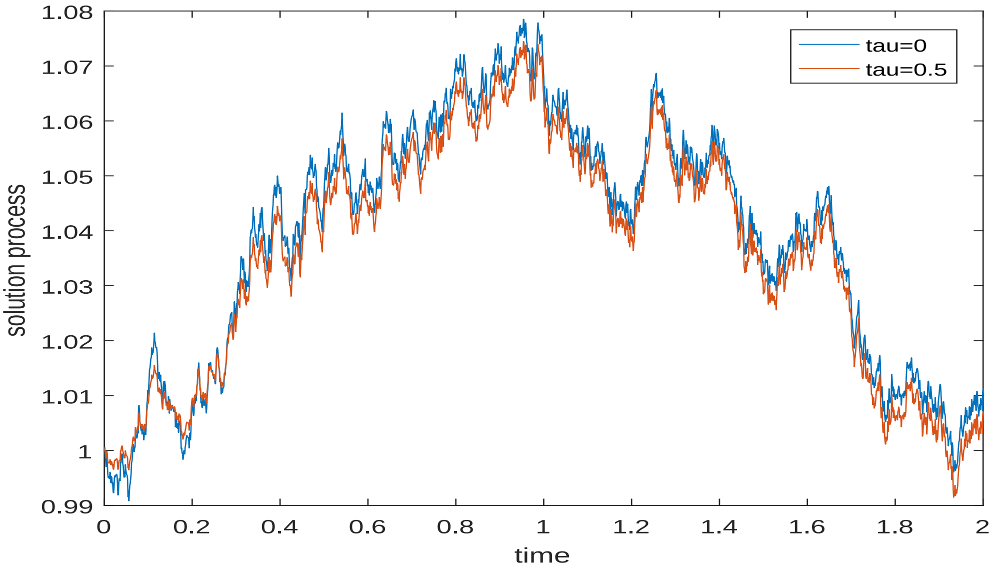

Figure 4.7: Sample paths of the stock price, where $\varphi(t) = 1 + t$ and $\tau = 0.5$.

- For the case of initial process as $\varphi(t) = e^{-t}$, our stock price at $t = 0$ is $S(0) = 1$. Table 4.5 provides the corresponding results.

Table 4.5: Call option values at $t = 0$, where $\varphi(t) = e^{t}$,

	Exercise Price		
	$K = 0.95$	$K = 1$	$K = 1.05$
V	0.1406	0.0965	0.0571
M.C.	0.1404	0.0963	0.0567
B.S.	0.1405	0.0960	0.0554

Note that we realize that the values of the option price in "V" row is greater than the values in "B.S." row because stock price in the delay case at any time $t \in [0, T]$ is greater than the no-delay cases, see Figure 4.8.

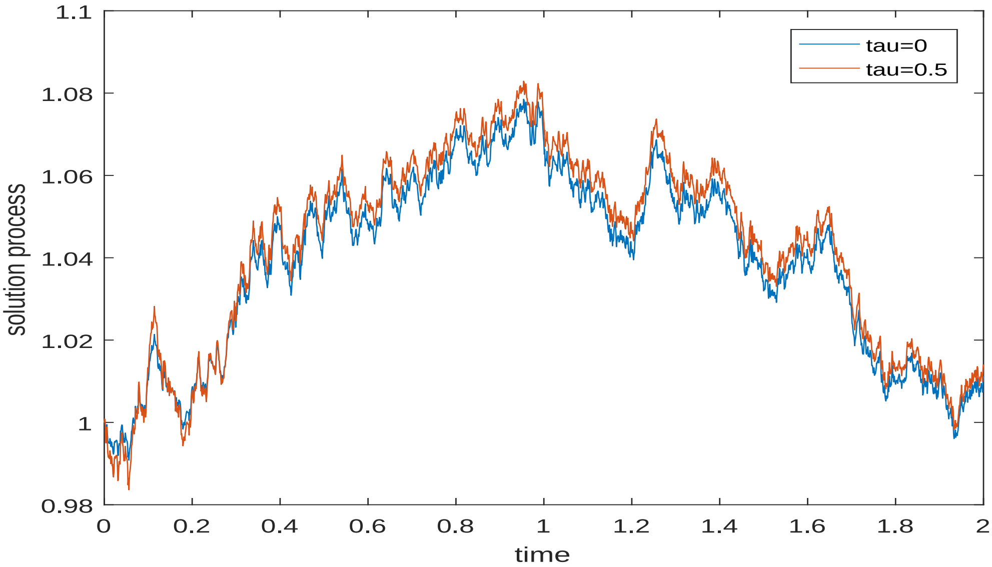

Figure 4.8: Sample paths of the stock price, where $\varphi(t) = e^{-t}$ and $\tau = 0.5$.

In this chapter, we consider two application of SDDEs in the financial markets. Stock returns are examined firstly where the system is modeled with SDDE. This example is actually based on [53] (2015). We consider the model in a more general case while setting τ as a parameter instead of taking directly $\tau = 1$. With this construction, the impact of delay term in the mean function is examined as a contribution to this work. In the second application, value of the European call option under the SDDE is handled. Arriojas et al. provide a formula for such a model in [2] (2007). They only consider is mathematically. As a contribution to that work, we provide numerical results for the values of the option with table.

CHAPTER 5

CONCLUSION

Stochastic delay differential equations (SDDEs) are become really important in many areas of science to understand the real world phenomena as well as to understand future behaviors of systems. They include both historical information and randomness. Thus, SDDEs provide a more realistic model for many systems than deterministic delay differential equations (DDEs) and stochastic differential equations (SDEs). SDDEs are actually a generalization of them.

In this thesis, stochastic delay differential equations (SDDEs) are handled together with definitions and their numerical approaches. The properties of SDEs are provided to make easy to follow concept for SDDEs because of the complicated characteristic of them. The existence and properties of the solution process for SDDEs are discussed and some examples of SDDEs are provided to make clear the concept in Chapter 2. These examples are obtained from examples of SDEs by adding a delay term into them. In order to solve them, iteration is used (the time interval is divided into pieces with a length of the delay term). We conclude our examples while giving general solutions of them in the iterative form and the corresponding expected values. While giving a comparison study between the examples of SDDE and SDE, chapter is completed.

Since, in general, finding closed form solution is not easy for a model with delay, numerical treatments are handled in Chapter 3. We consider Euler Maruyam method for SDDEs and its convergence analysis. With the help of method, simulations of the examples in the previous chapter is done. In these simulations, the effect of initial function and length of delay term are considered. It is observed that they have an important effect on the evolution of the solution process and corresponding expected value in the future states.

In Chapter 4, we consider two applications of SDDEs in finance. Stock returns are examined where the system is modeled with SDDE. This example is based on [53] (2015). We consider the model in a more general case while setting τ as a parameter instead of taking directly $\tau = 1$. With this construction, the impact of delay term in the mean function is examined. We show numerically that the Euler Maruyama method converges in the mean with order 1. In order to show the influence of delay, we substitute different coefficients of the delay term and lengths of the delay term τ. In the second application, value of the European call option under the SDDE is handled. Arriojas et al. provide a formula for such a model in [2] (2007). We provide numerical results for the values of the option. While keeping risk free rate, volatility and maturity as a constant, we create some tables. In table Table 4.1, Table 4.2 and Table 4.3, we provide the option value at the time in the last delay period. These values are obtained from the formula that is given in [2] for the SDDE case and using Black-Scholes formula for the no-delay case. The effect of initial data, length of delay term and strike price are analyzed in those tables. The value analysis of the option at any time less than the last delay period, i.e., $t \leq T - \tau$ (T is maturity, τ is length of delay), is considered in Table 4.4 and Table 4.5. We show that how the choice of initial function and strike price influence the value. In that tables, three kinds of value are provided. First ones are obtained from the value formula in Theorem 4.2 and second values are computed by using Monte-Carlo simulation. This two values are for the model with delay case. While using Black-Scholes formula, we compute the last values for the no-delay case.

For the future studies, the followings can be considered:

- System with random delay,
- When there is more than one delay term,
- Stochastic delay differential system with jump,
- The value of the option for the above cases,
- The value of the American option when stock price follows SDDE.

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APPENDIX A

Some Results

Definition A.1. A function $f : \mathbb{R} \rightarrow \mathbb{R}$ is called a convex function if it satisfies $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) \quad \forall x, y \in \mathbb{R}$ and $\lambda \in [0,1]$.

Definition A.2. For any convex function $f : \mathbb{R} \rightarrow \mathbb{R}$ and any random variable X we have $f(E(X)) \leq E(f(X))$. This inequality is called as Jensen's inequality.

Let us consider some examples of Jensen's inequality:

- Choose $f(x) = |x|$, we get $|E(x)| \leq E(|x|)$. Taking the t-th power of this inequality for any $t > 0$, we get $|E(x)|^t \leq E(|x|)^t$.
- Choose $f(x) = x^t$ for any $t > 0$, it implies $E(x)^t \leq E(x^t)$. Instead of x, take $|x|$. Then we get $E(|x|)^t \leq E(|x|^t)$.

Combining above examples, we get:

$$|E(x)|^t \leq E(|x|)^t \leq E(|x|^t). \tag{A.1}$$

Solution of Example 2.5: Let us solve the given SDDE in that example explicitly:

$$\begin{aligned} dX(t) &= X(t-1)dt + \beta dW(t), \quad 0 \leq t \leq T, \\ X(t) &= \varphi(t) = 1 + t, \quad t \in [-1, 0]. \end{aligned}$$

Define $\varphi(t) =: \varphi_1(t)$.

For $t \in [0,1]$: $t-1 \in [-1,0]$ which implies that $X(t-1) = \varphi_1(t-1)$ and our SDDE actually defines the following SDE:

$$\begin{aligned} dX(t) &= \varphi_1(t-1)dt + \beta dW(t), \\ dX(t) &= tdt + \beta dW(t). \end{aligned}$$

Now apply Itô formula for $F(x) = x$:

$$\begin{aligned}
X(t) &= X(0) + \int_0^t 1\, dX(s) \qquad \text{(since } F''(x) = 0\text{)} \\
&= 1 + \int_0^t \big[\varphi_1(s-1)ds + \beta dW(s)\big] \\
&= 1 + \int_0^t sds + \int_0^t \beta dW(s) \\
&= 1 + \frac{s^2}{2}\Big|_0^t + \beta W(s)\big|_0^t \\
&= 1 + \frac{t^2}{2} + \beta W(t) \\
&=: \varphi_2(t).
\end{aligned}$$

For $t \in [1,2]$: $t-1 \in [0,1]$ which implies that $X(t-1) = \varphi_2(t-1)$ and equation becomes:

$$\begin{aligned}
dX(t) &= \varphi_2(t-1)dt + \beta dW(t) \\
&= \left(1 + \frac{(t-1)^2}{2} + \beta W(t-1)\right) dt + \beta dW(t).
\end{aligned}$$

Now apply Itô formula for $F(x) = x$ again:

$$\begin{aligned}
X(t) &= X(1) + \int_1^t 1\, dX(s) \\
&= \frac{3}{2} + \beta W(1) + \int_1^t \left[1 + \frac{(s-1)^2}{2} + \beta W(s-1)\right] ds + \int_1^t \beta dW(s) \\
&= \frac{3}{2} + \beta W(1) + \left(s + \frac{(s-1)^3}{6}\right)\Big|_1^t + \int_1^t \beta W(s-1)ds + \beta\big(W(t) - W(1)\big) \\
&= t + \frac{(t-1)^3}{6} + \frac{1}{2} + \int_1^t \beta W(s-1)ds + \beta W(t) \\
&=: \varphi_3(t).
\end{aligned}$$

For $t \in [2,3]$: $t-1 \in [1,2]$, $X(t-1) = \varphi_3(t-1)$ and the equation turns to be:

$$\begin{aligned}
dX(t) &= \varphi_3(t-1)dt + \beta dW(t) \\
&= \left(t - 1 + \frac{(t-2)^3}{6} + \frac{1}{2} + \int_1^{t-1} \beta W(s-1)ds + \beta W(t-1)\right) dt + \beta dW(t).
\end{aligned}$$

Now again apply Itô formula for $F(x) = x$:

$$\begin{aligned}
X(t) &= X(2) + \int_2^t 1\, dX(s) \\
&= \frac{16}{6} + \int_1^2 \beta W(s-1)ds + \beta W(2) + \int_2^t \big(s - \frac{1}{2} + \frac{(s-2)^3}{6}\big)ds \\
&\qquad + \int_2^t \int_1^{u-1} \beta W(s-1)dsdu + \int_2^t \beta W(s-1)ds + \int_2^t \beta dW(s) \\
&=: \varphi_4(t).
\end{aligned}$$

It is possible to continue like this and compute recursively the solutions defined in the other intervals. Up to now, we compute:

$$X(t) = \begin{cases}
\varphi_1(t) = 1 + t, \qquad t \in [-1, 0], \\
\varphi_2(2) = 1 + \dfrac{t^2}{2} + \beta W(t), \qquad t \in [0, 1], \\
\varphi_3(t) = t + \dfrac{(t-1)^3}{6} + \dfrac{1}{2} + \displaystyle\int_1^t \beta W(s-1)ds + \beta W(t), \qquad t \in [1, 2], \\
\varphi_4(t) = \dfrac{16}{6} + \displaystyle\int_1^2 \beta W(s-1)ds + \beta W(2) + \int_2^t \big(s - \frac{1}{2} + \frac{(s-2)^3}{6}\big)ds \\
\quad + \displaystyle\int_2^t \int_1^{u-1} \beta W(s-1)dsdu + \int_2^t \beta W(s-1)ds + \int_2^t \beta dW(s), t \in [2, 3].
\end{cases}$$

Expected value of Example 2.5: Let us compute the mean function of $X(t)$ in $[-1, 3]$. Since we know solution in that interval we can take the expectation of them directly.
For $t \in [0, 1]$: our solution is $X(t) = 1 + \frac{t^2}{2} + \beta W(t)$, taking the expectation of both sides we get:

$$E(X(t)) = 1 + \frac{t^2}{2}.$$

For $t \in [1, 2]$: our solution is:

$$\begin{aligned}
X(t) &= t + \frac{(t-1)^3}{6} + \frac{1}{2} + \int_1^t \beta W(s-1)ds + \beta W(t), \\
&= \frac{t^3}{6} - \frac{t^2}{2} + \frac{3t}{2} + \frac{1}{3} + \int_1^t \beta W(s-1)ds + \beta W(t)
\end{aligned}$$

and the corresponding mean function is

$$E(X(t)) = \frac{1}{3} + \frac{3t}{2} - \frac{t^2}{2} + \frac{t^3}{6}.$$

For $t \in [2,3]$: our solution is:

$$
\begin{aligned}
X(t) = \frac{16}{6} + \int_1^2 \beta W(s-1)ds + \beta W(2) + \int_2^t \big(s - \frac{1}{2} + \frac{(s-2)^3}{6}\big)ds \\
+ \int_2^t \int_1^{u-1} \beta W(s-1)dsdu + \int_2^t \beta W(s-1)ds + \int_2^t \beta dW(s),
\end{aligned}
$$

and the corresponding mean function is

$$
E(X(t)) = \frac{10}{6} - \frac{t}{2} + \frac{t^2}{2} + \frac{(t-2)^4}{24}.
$$