THE COMPARISON OF RISK MEASURES ON CLAIM DISTRIBUTIONS: TURKISH MOTOR INSURANCE CASE

A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF APPLIED MATHEMATICS
OF
MIDDLE EAST TECHNICAL UNIVERSITY

BY

CANSU TELKES

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR
THE DEGREE OF MASTER OF SCIENCE
IN
ACTUARIAL SCIENCES

APRIL 2018

Approval of the thesis:

THE COMPARISON OF RISK MEASURES ON CLAIM DISTRIBUTIONS: TURKISH MOTOR INSURANCE CASE

submitted by **CANSU TELKES** in partial fulfillment of the requirements for the degree of **Master of Science in Department of Actuarial Sciences, Middle East Technical University** by,

Prof. Dr. Ömür UĞUR
Director, Graduate School of **Applied Mathematics** _______________

Prof. Dr. A. Sevtap KESTEL
Head of Department, **Actuarial Sciences** _______________

Prof. Dr. A. Sevtap KESTEL
Supervisor, **Actuarial Sciences, METU** _______________

Prof. Dr. Fatih TANK
Co-supervisor, **Department of Insurance and Actuarial Sciences, Ankara University** _______________

Examining Committee Members:

Prof. Dr. Fatih TANK
Department of Insurance and Actuarial Sciences, Ankara University _______________

Prof. Dr. A. Sevtap KESTEL
Actuarial Sciences, METU _______________

Assoc. Prof. Dr. B. Burçak BAŞBUĞ ERKAN
Department of Statistics, METU _______________

Assoc. Prof. Dr. Ceylan YOZGATLIGİL
Department of Statistics, METU _______________

Assist. Prof. Dr. Sibel AÇIK KEMALOĞLU
Department of Statistics, Ankara University _______________

Date: _______________

I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

Name, Last Name: CANSU TELKES

Signature :

ABSTRACT

THE COMPARISON OF RISK MEASURES ON CLAIM DISTRIBUTIONS:
TURKISH MOTOR INSURANCE CASE

Telkes, Cansu
M.S., Department of Actuarial Sciences
Supervisor : Prof. Dr. A. Sevtap KESTEL
Co-Supervisor : Prof. Dr. Fatih TANK

April 2018, 93 pages

In this thesis, the impact of various risk measures on pricing methodology of automobile insurance product by using the historical claim data which is obtained from one of the most reputable insurance company in Turkey is investigated. To model the distribution of claim experience for pricing methodology, four right skewed distributions are chosen, namely Gamma, Weibull, Lognormal and Pareto. Two classical methods, which are methods of moment estimation and maximum likelihood estimation, are used to estimate the parameters from the data. Lognormal distribution explains the data ideally. After estimating the parameters of loss distribution, premium and capital based risk measures are calculated. A comparison is done with the help of the coherency criteria for the result of risk measures as a favorable price of the automobile insurance products.

Keywords : Risk Measures, Coherent Risk Measure, Loss Distribution, Method of Moment Estimation, Method of Maximum Likelihood Estimation

ÖZ

HASAR DAĞILIMLARINDAKİ RİSK ÖLÇÜMLERİNİN KARŞILAŞTIRILMASI: TÜRKİYE KASKO SİGORTASI ÖRNEĞİ

Telkes, Cansu
Yüksek Lisans, Aktüerya Bilimleri
Tez Yöneticisi : Prof. Dr. A. Sevtap KESTEL
Ortak Tez Yöneticisi : Prof. Dr. Fatih TANK

Nisan 2018, 93 sayfa

Bu çalışmada, Türkiye'nin ünlü sigorta şirketlerinden birine ait geçmiş hasar verisi kullanılarak, çeşitli risk ölçümlerinin kasko sigortası ürününün fiyatlandırılması üzerindeki etkisi incelenmiştir. Fiyatlandırmada kullanılacak olan hasar verisinin dağılımını modellemek için Gamma, Weibull, Lognormal ve Pareto olmak üzere dört adet sağa dayalı dağılım seçilmiştir. Ayrıca, bu dağılımların parametrelerini tahmin etmek üzere iki klasik yöntem olan, moment ve maksimum olasılık tahmini yöntemleri kullanılmıştır. Bu yöntemlerin sonucuna göre veriyi en iyi şekilde temsil eden dağılımın Lognormal dağılım olduğu görülmüştür; sonrasında ise risk ölçümleri, tahmin edilen bu parametrelere göre hesaplanmıştır. Kasko sigortası ürününde en uygun fiyatın belirlenmesi için hesaplanan risk ölçüm sonuçları, tutarlı risk ölçümü olma özelliğine göre karşılaştırılmıştır.

Anahtar Kelimeler: Risk Ölçümleri, Tutarlı Risk Ölçümleri, Hasar Dağılımları, Moment Tahmini Yöntemi, Maksimum Olasılık Tahmini

To My Beloved Family

ACKNOWLEDGMENTS

After a long and intensive period, I would like to express the deepest appreciation to my thesis supervisor Prof. Dr. A. Sevtap KESTEL and my thesis co-supervisor Prof. Dr. Fatih TANK for their patient guidance, enthusiastic encouragement and valuable advices during all stages of this thesis. Their willingness to give their time and to share their experiences has brightened my path.

Besides my advisors, I would like to thank to the members of my thesis examining committee for their encouragement, insightful comments and suggestions, constructive criticism and hard questions.

Furthermore, I am thankful to all institute members and friends for providing such a friendly working environment. I always felt at home with their lovely and warm behavior.

Many thanks to my lovely and dear friends Duygu Taşfiliz, Ezgi Aladağlı, Meral Şimşek and my lovely homemate Selin Tekten. My friends have helped me mentally through this difficult period. Their support and care helped me overcome setbacks and stay focused on my graduate study. I greatly value their friendship and I deeply appreciate their belief in me.

My sincere thanks also goes to my colleagues of the Actuarial Department at AvivaSA Emeklilik ve Hayat A.Ş. Their supports have always enlightened my path to complete my thesis successfully. I am especially grateful to my manager, Gregory Jones, for all his help in finishing my thesis successfully.

I am thankful to my beloved boyfriend Ahmet Çağlayan, who makes this thesis possible. Without his trust, support, encouragement, smile and love, I would not have been anywhere close to where I am now.

Most importantly, none of this would have been possible without the love and patience of my family. My family to whom this thesis is dedicated to, has been a constant source of love, care, support and strength all these time. I would like to express my heart-felt gratitude to my family. Especially, to my mother İkbal Işıklı, who has always helped me with everything.

TABLE OF CONTENTS

ABSTRACT vii

ÖZ ix

ACKNOWLEDGMENTS xiii

TABLE OF CONTENTS xv

LIST OF FIGURES xix

LIST OF TABLES xxi

LIST OF ABBREVIATIONS xxiii

CHAPTERS

1 INTRODUCTION 1

1.1 Literature Survey 2

1.2 Aim of The Study 3

2 RISK MEASURES 5

2.1 Premium Based Risk Measures 5

2.1.1 Net Premium Principle 6

2.1.2 Expected Value Premium Principle 6

2.1.3 Variance Premium Principle 6

2.1.4 Standard Deviation Premium Principle 6

2.1.5 Exponential Premium Principle 6

2.1.6 Esscher Premium Principle 7

2.2 Capital Based Risk Measures 7

2.2.1 Value at Risk 7

2.2.2 Conditional Value at Risk 9

2.3 Coherency Criteria of Risk Measures 10

2.3.1 Derivations for Coherency Criteria of Risk Measures 10

3 LOSS DISTRIBUTIONS AND ESTIMATION METHODS 21

3.1 Loss Distributions 21

3.1.1 Gamma Distribution 22

3.1.2 Weibull Distribution 24

3.1.3 Lognormal Distribution 25

3.1.4 Pareto Distribution 26

3.2 Parameter Estimation Methods 27

3.2.1 Method of Moments Estimation 28

3.2.2 Method of Maximum Likelihood Estimation 28

4 CASE STUDY: MOTOR INSURANCE 31

4.1 Descriptive Statistics 31

4.2 Homogeneity Test 37

4.3 Parameter Estimation 40

4.3.1 Method of Moments Estimation 41

4.3.2 Method of Maximum Likelihood Estimation 42

4.4 Risk Measures Calculations 44

5 CONCLUSION 47

REFERENCES 49

APPENDICES

A Derivations of Mean, Variance, MGF, MME and MLE of Selected Distributions 51

A.1 Gamma Distribution 51

A.2 Weibull Distribution 55

A.3 Lognormal Distribution 58

A.4 Pareto Distribution 62

B Graphical Representation of Fitting Distributions 65

B.1 Methods of Moment Estimation 65

B.2 Methods of Maximum Likelihood Estimation 72

C Mathematical Representation of Risk Premiums 81

C.1 Based on Methods of Moments Estimation 81

C.2 Based on Methods of Maximum Likelihood Estimation . . . 87

LIST OF FIGURES

Figure 2.1 Graphical definition of VaR based on 99.5% confidence level (Jorion, 1997, p.52 [12]) 8

Figure 2.2 Graphical definition of CVaR based on 99.5% confidence level (Jorion, 1997, p.52 [12]) 9

Figure 3.1 Probability Density Function of Gamma (α, 1) (Tse, 2009, p.52 [25]) 23

Figure 3.2 Probability Density Function of Weibull (α, 3) (Tse, 2009, p.52 [25]) 24

Figure 3.3 Probability Density Function of Lognormal (0, σ^2) (Tse, 2009, p.52 [25]) 26

Figure 3.4 Probability Density Function of Pareto (a,1) (Tse, 2009, p.52 [25]) . 27

Figure 4.1 Histogram of Claim Amounts: a) Istanbul, b) Ankara 33

Figure 4.2 Histogram of Claim Amounts Based on Gender and Selected Cities 34

Figure 4.3 Histogram of Claim Amounts Based on Discount Rates for Istanbul 36

Figure 4.4 Histogram of Claim Amounts Based on Discount Rates for Ankara 36

Figure 4.5 Claim Amounts Based on Gender and Discount Rates: a) Istanbul, b) Ankara 40

Figure B.1 Gamma Distribution Graphs of Istanbul 65

Figure B.2 Weibull Distribution Graphs of Istanbul 66

Figure B.3 Lognormal Distribution Graphs of Istanbul 67

Figure B.4 Pareto Distribution Graphs of Istanbul 68

Figure B.5 Gamma Distribution Graphs of Ankara 69

Figure B.6 Weibull Distribution Graphs of Ankara 70

Figure B.7 Lognormal Distribution Graphs of Ankara 71

Figure B.8 Pareto Distribution Graphs of Ankara 72

Figure B.9 Gamma Distribution Graphs of Istanbul 73
Figure B.10 Weibull Distribution Graphs of Istanbul 74
Figure B.11 Lognormal Distribution Graphs of Istanbul 75
Figure B.12 Pareto Distribution Graphs of Istanbul 76
Figure B.13 Gamma Distribution Graphs of Ankara 77
Figure B.14 Weibull Distribution Graphs of Ankara 78
Figure B.15 Lognormal Distribution Graphs of Ankara 79
Figure B.16 Pareto Distribution Graphs of Ankara 80

LIST OF TABLES

Table 2.1 Summary of Coherency Criteria of Risk Measures 20

Table 3.1 Summary of Parameter Estimation for Loss Distributions 30

Table 4.1 Summary Statistics of Claim Amounts Based on City 32

Table 4.2 Summary Statistics of Claim Amounts Based on Gender and City . . 34

Table 4.3 Coding of Bonus Categorization for Discount Rates 35

Table 4.4 Summary Statistics of Claim Amounts Based on Discount Rates and City 35

Table 4.5 Summary Statistics for Gender and Discount Rates of Selected Cities 37

Table 4.6 Result of Two Sample t-test for Selected Cities Based on Gender . . 38

Table 4.7 Results of Two Sample t-test for Selected Cities Based on Discount Rates 38

Table 4.8 Contingency Table for Gender and Discount Rates of Selected Cities 39

Table 4.9 Result of Chi Square Test for Homogeneity for Selected Cities . . . 39

Table 4.10 Summary of MME Results for Selected Cities 41

Table 4.11 Summary Statistics of Goodness of Fit Test for Selected Cities . . . 42

Table 4.12 Summary Results of Goodness of Fit Test for Selected Cities 42

Table 4.13 Summary of MLE Results for Selected Cities 43

Table 4.14 Summary Statistics of Goodness of Fit Tests for Selected Cities . . . 44

Table 4.15 Summary Results of Goodness of Fit Tests for Selected Cities 44

Table 4.16 Risk Measures Results For Selected Cities 45

LIST OF ABBREVIATIONS

AIC	Akaike's Information Criteria
BIC	Bayesian Information Criteria
CV	Coefficient of Variation
CVaR	Conditional Value at Risk
c	Constant
k	Count
CDF	Cumulative Distribution Functions
C	Cumulative Generating Function
DR	Discount Rates
E	Expectation
ESPP	Esscher Premium Principle
EVPP	Expected Value Premium Principle
EXPP	Exponential Premium Principle
Q1	First Quantile
inf	Infimum
m_4	Kurtosis
L(θ)	Likelihood Function
X	Loss Amount 1
Y	Loss Amount 2
max	Maximum
MLE	Maximum Likelihood Estimation
μ	Mean
MME	Method of Moments Estimation
min	Minimum
M	Moment Generating Function
M	Monotonicity
NPP	Net Premium Principle
$\mathcal{H}$	Premium Principle Function
P&C	Property and Casualty Insurance
$\mathcal{R}$	Real Numbers

$\mathcal{X}$	Risk of Loss
α	Safety Loading
SI	Scale Invariance
Q2	Second Quantile
m_3	Skewness
SD	Standard Deviation
σ	Standard Deviation
SDPP	Standard Deviation Premium Principle
SA	Subadditivity
Σ	Sum
Q3	Third Quantile
TI	Translation Invariance
VaR	Value at Risk
Var	Variance
VPP	Variance Premium Principle

CHAPTER 1

INTRODUCTION

Every moment we live, we experience various risks and it is not possible to completely eliminate these risks in advance. As a consequence, risk management methods have been developed not to be exposed to the negative effects of these risks. In other words, management of risk is the process of identifying, accepting or mitigating the probability of the occurrence of the threats [17] because the sense of security is the main purpose of the mankind after food and shelter in the world.

Insurance companies provide people with feeling of confidence with various insurance products. Therefore, individuals seek insurance products to manage their own risks and also companies try to improve their risk management mechanisms to better serve individuals with confidence.

Risk management mechanisms mainly depend on the measurement of risk, which is the heart of the process, and risk classification, which are primarily market risk, credit risk and operational risk. The main risk classification of an insurance company is the operational risks because the occurrence of losses are the main issue from insurance policies for the company [25]. According to operational risks, insurance companies have to protect the policyholders against their losses and also sustain the economic conditions of the companies against the bankruptcy by using their risk measures processes.

Risk measures help the insurance companies to contribute to profitability by providing the optimal premiums for insurance products in a competitive environment while covering the incoming claims and expenses of the company. As a result, the insurance companies selling their products with a favorable premiums so that we can feel more secure.

According to July 2017 report of the Insurance Association of Turkey, there are a total of 60 life and non-life insurance companies in Turkey. 38 of them are non-life companies and 22 of them are life companies.

Depend on the distribution of premium production of these companies, it is seen that automobile insurance has the biggest share in Turkey. In addition to premium production share, according to Turkish Statistical Institute; there are approximately 20 million registered vehicles in Turkey and 1.3 million of these vehicles are involved in an accident within a year. The vast majority of these property damage accidents take place in Istanbul and Ankara. As a result, individuals buy automobile insurance products to protect themselves because automobile insurance is usually used to cover the financial loss of insured motor vehicle when the insured motor vehicle has an accident so it is important to take an automobile insurance product for these losses in Turkey.

In automobile insurance, we have some problems for charging adequate premiums to meet the future claims [28] and the size of claims is usually not known for cities in advance. However, we can comment on the adequate premiums for automobile insurance by looking the claim-premium ratio. The ratio of claim to premium for automobile insures is 70% in accordance with Insurance Association of Turkey. Therefore, insurance companies are required to analyze historical claims data in order to protect their assets and financial statute and also determine adequate premium amounts for their new policies in such a competitive environment.

1.1 Literature Survey

In actuarial applications, risk measures are part of the necessary management procedures for reducing the exposure of adverse event and Bühlmann (1970) ([6]) took them into account for the first-time. Bühlmann (1970) stated that premium principles were defined as assigning a number to claim distributions. As a risk measure, he made premium principle calculations based on the assumption of known claim distributions thus expected value premium principle, variance premium principle, standard deviation premium principle and zero utility premium principle were proposed by Bühlmann. Moreover, he discussed the usage of them. For instance, the expected value premium principle is always used in life insurance instead of non-life due to the lack of homogeneity in non-life insurance area. In non-life insurance, the standard deviation premium principle is more commonly used because it is more sensitive to the distribution of claims. On the contrary, the variance premium principle is not as popular as the standard deviation premium principle because the variance premium principle does not have delicate characteristics for changing the claim experience. In addition to these, the zero utility premium principle is also not chosen frequently due to the difficulty of choosing the right utility function for calculations[6].

In addition to Bühlmann (1970), Artzner et al. (1999) ([2]) stated four desirable properties for premium principles, and called these measures which satisfy these four properties as coherent. Firstly, they define risk as a future value of a position given a time period. Secondly, they stated the risk measures, which are associated with the acceptance set. In other words, they ensure that the risk measures have the same unit in the future value. Therefore; they obtain a bounded and finite set of states of nature. Thirdly, four axioms are stated for being coherent risk measures. These axioms are *Translation Invariane*, *Subadditivity*, *Scale Invariance* and *Monotonicity*. In addition to the coherent risk measure, they demonstrate that Value at Risk is not coherent due to the fact that Value at Risk does not satisfy the subadditivity property and they conclude that Value at Risk does not behave nicely with respect to the addition of these risks. However, they noted that Value at Risk is coherent under specific conditions.

According to Rockafellar and Uryasev (2002) ([23]), Value at Risk is a popular risk measure all around the world even though it does not satisfy the property of subadditivity. Therefore; they study Conditional Value at Risk as an alternative risk measure because Conditional Value at Risk satisfies the properties of a coherent risk measure for portfolio optimization. Moreover, Inui and Kijima (2004) show that Conditional Value at Risk calculates the loss behind the Value at Risk [10].

In order to offer suitable premiums in Property and Casualty (P&C) insurance, the number of claims within a given time period and the sizes of those claims are the main issues. According to Gray and Pitts (2012) ([8]) claim sizes are more important than the claim frequencies for sufficient premiums of insurance coverage in P&C insurance because insurers have to stay solvent against those claims. Mostly right skewed distributions are used for modeling the loss data because small claim amounts occur more frequently than large claim amounts. Selection of the right skewed distribution also is an important issue for modeling the claim sizes. According to Mikosch (2006) ([18]), modeling the claim size with exponential family distributions gives more adequate results than other distribution families.

Furthermore, Wüthrich and Merz (2008) ([26]) stated that the estimation of claim size and claim probability with algorithmic or simple stochastic techniques (specially Chain-Ladder and Bornhuetter-Ferguson methods) give poor results. Moreover, Jorgensen and Souza (1994) ([11]) suggested Tweedie random variables to estimate the claim size of a risk. On the other hand, Gray and Pitts (2012) ([8]), Achieng (2010) ([1]), Packova and Brebara (2015) ([22]) and most writers used classical estimation methods, which are Methods of Moments Estimation and Methods of Maximum Likelihood Estimation, for claim size distributions. All these studies showed that the classical estimation methods give consistent and efficient results for claim size distributions. In addition to classical estimation approaches, Mazviona and Chiduza ([16]) used Bayesian estimation methods and compared the results with classical approach and based on their results, classical and Bayesian estimation methods gave same results.

1.2 Aim of The Study

The aim of the present study is to calculate affordable premiums of automobile insurance in a competitive insurance environment in Turkey by estimating the claim amount distributions with the help of historical claims data . Well-known risk measures in the insurance field are selected. To compare the sensitivity of risk measures, automobile insurance data, which are obtained from one of the most reputable insurance company in Turkey, are used. The data set contains the claim amounts information of a one-year period automobile insurance based on gender, discount rates and selected two city. Additionally, heavy-tailed distributions are selected to estimate the parameters of claim distributions by using Methods of Moment Estimation (MME) and Maximum Likelihood Estimation (MLE). The claim experience, moreover, is used as a loading factor for premium principle calculations.

This thesis consists of five chapters. The first chapter has introduced the purpose of the study and given a brief literature review. In the second chapter, mathematical definitions of risk measures are defined based on premium and capital category. Furthermore, the coherency of risk measures are discussed. In the third chapter, a brief review of the loss distribution is given. In addition to loss distribution, MME and MLE methodologies are presented. In the fourth chapter, empirical results of risk measures are calculated based on estimated loss distributions, which are applied on automobile insurance data with the help of R programming . The last chapter discusses results of the study and some more comments for future studies.

CHAPTER 2

RISK MEASURES

In this chapter, existing approaches for measuring risk in the insurance business will be discussed. Two main categories of risk measures will be examined. The first one is the premium based risk measures, which are important for the determination of optimal insurance premium. The second one is capital based risk measures, which are important for the determination of economic capital and also optimal insurance premiums.

2.1 Premium Based Risk Measures

The first use of premium based risk measures in actuarial science was the development of premium principle, which is also called traditional premium principle. It is defined as a functional, which is a function of functions, between insured and insurer that determines an appropriate insurance premium to charge for transferring the risk of loss [14].

In the premium principle, the random variable X is defined as the loss amount and $\mathcal{X}$ is described as risk of loss and it is a set of non-negative random variables on a probability space (Ω, $\mathcal{F}$, P). Moreover; $\mathcal{H}$ denotes the premium principle function as a risk measure. The mathematical definition of premium calculation function is $\mathcal{H} : \mathcal{X} \to \mathbb{R}$, which assigns a nonnegative value to a loss variable X. In short; it is defined as a mapping from $\mathcal{X}$ to $\mathcal{R}$.

The marginal distribution function of the random variable is also important for calculating sufficient insurance premium for the transferring the risk of loss within the premium principles. Furthermore, the assumption of bounded claims are made because unbounded claims will be resulted in an infinite premium. In other words, unbounded claims reveal situations which are not insurable by an insurance company [13].

In this section, we examined the six most popular premium principles by using mathematical definition of risk measures $\mathcal{H}$ and the random variable X as a risk of loss.

2.1.1 Net Premium Principle

Net Premium Principle states that

$$\mathcal{H}(X) = E[X] \tag{2.1}$$

It does not load for risk and it is only sufficient for a risk neutral insurer, which means that the risk is basically non-existent [13]. The biggest problem of Net Premium principle is that it is insufficient pricing methods for an insurance company to remain solvent.

2.1.2 Expected Value Premium Principle

Expected Value Premium Principle is defined by following equation:

$$\mathcal{H}(X) = (1 + \alpha)E[X] \tag{2.2}$$

where α is a loading factor and $\alpha \geq 0$. It is the most widely used premium principle within the insurance business because it is easy to understand by the users. However, it is not sensitive to the fluctuations in the large claim amounts.

2.1.3 Variance Premium Principle

Variance Premium Principle is described by following equation:

$$\mathcal{H}(X) = E[X] + \alpha Var[X] \tag{2.3}$$

where α is a loading factor and $\alpha \geq 0$. Furthermore, it is sensitive to the change in the extreme claim amounts.

2.1.4 Standard Deviation Premium Principle

Standard Deviation Premium Principle states that

$$\mathcal{H}(X) = E[X] + \alpha\sqrt{Var[X]} \tag{2.4}$$

where α is a loading factor and $\alpha \geq 0$. Furthermore, it is sensitive to the change of claim amounts.

2.1.5 Exponential Premium Principle

Exponential Premium Principle is defined as

$$\mathcal{H}(X) = \frac{1}{\alpha} \ln E[e^{\alpha X}] \tag{2.5}$$

where α is a loading factor and $\alpha \geq 0$. However, it is not practical to apply in insurance pricing methodology because it requires so many assumptions to perform the mathematical calculations.

2.1.6 Esscher Premium Principle

Esscher Premium Principle is expressed as

$$\mathcal{H}(X) = \frac{E[Xe^{X}]}{E[e^{\alpha X}]} \tag{2.6}$$

where α is a loading factor and $\alpha \geq 0$. Furthermore, it is sensitive to the change in the extreme claim amounts.

2.2 Capital Based Risk Measures

Risk measures, depending on the usage of the percentile principle, are called capital based risk measures and these are also used for constructing an appropriate economic capital for insurance company. The definition of economic capital is the amount of assets that a financial firm needs to hold in order to remain solvent at a given time period [15]. We can say that it is a buffer limit against for unexpected losses and it's important for the premium calculations. In this section, we investigate Value at Risk (VaR) and Conditional Value at Risk (CVaR), which are the most widely used capital based risk measures.

2.2.1 Value at Risk

The usage of Value at Risk is increased as a standard risk measure in the financial and actuarial fields in order to evaluate the exposure of risk. The definition of VaR is described as a measurement of the worst expected loss over a given horizon under the assumption of normal market conditions with a given level of confidence [12]. More simply, it is just the quantile of distribution over the target horizon. Moreover, it captures the effect of volatility and the risk exposure. VaR summarizes the risk in a single number, which is the greatest advantage of it.

In the financial sector, companies take precautions against for unexpected risks in order to keep their solvency level steady by using VaR. Therefore, company-specific level of capital is decided with VaR and we define it as the lower quantile value of the their loss distributions and the risks. Losses are occurred by exceeding this quantile value. In other words, losses will only occur with a probability of $1-\delta$ and δ is the quantile value of the loss distribution. In mathematical term, VaR can be written as

$$VaR_{\delta}(X) = F_{X}^{-1}(\delta) = x_{\delta} \tag{2.7}$$

where X is the random variable as a loss and F^{-1} is the inverse cumulative distribution functions (inverse CDF).

On the other hand, sometimes X is not a continuous random variable, so that it creates some calculation problems. The more general definition of VaR, hence, is given by

$$VaR_{\delta}(X) = \text{inf}\ x \in [0, \infty) : F_X(x) \geq \delta \tag{2.8}$$

where inf is defined as the greatest lower bound over the defined range and F(x) is the function of cumulative distribution (CDF) over the random variable X. Figure 2.1 shows the visual definition of VaR;

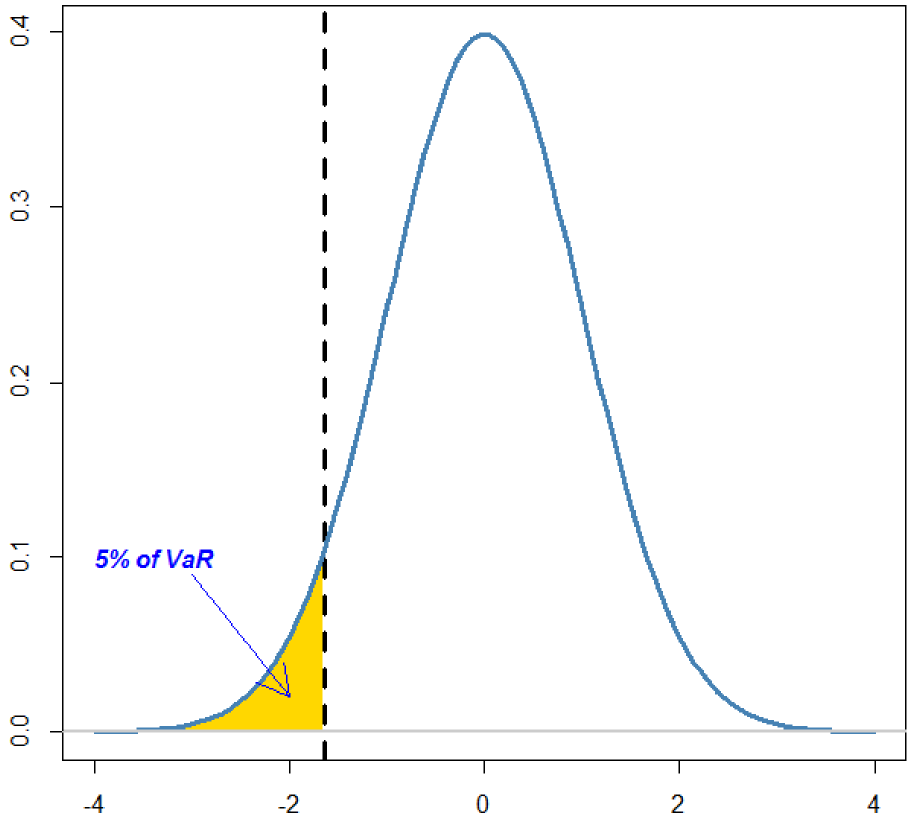

Figure 2.1: Graphical definition of VaR based on 99.5$\%$ confidence level (Jorion, 1997, p.52 [12])

From the Equation 2.8 and the Figure 2.1, we can say that the calculation of VaR is simple and it is easy to understand for users because it gives a single number in order to compare the risk. Therefore, it has a wide usage all over the world [5]. However, it has some important deficiencies that should be considered because it measures the total risk but there are different risks in the market. VaR, hence, does not warn about the severity of losses, which occur in the remaining $1-\delta$ probability. In other words, it does not sensitive in the tail so that the effect of diversification risk will be ignored.

2.2.2 Conditional Value at Risk

Due to the drawback of Value at Risk, as an alternative measure, Conditional Value at Risk (CVaR) was developed because it is more sensitive to the loss distribution at the tails. CVaR has many names such as Tail Value at Risk (TVaR), Tail Conditional Expectation (TCE) and Expected Shortfall (ES) due to the lack of consistent terminology in the literature. In this study, CVaR is basically defined as an expectation value of the VaR over the percentiles from 0 to α and its' mathematical definition is;

$$CVaR_{\delta}(X) = E[X|X > VaR_{\delta}(X)] \tag{2.9}$$

The Figure 2.2 shows an illustration of CVaR by using the comparison between the threshold of VaR and CVaR;

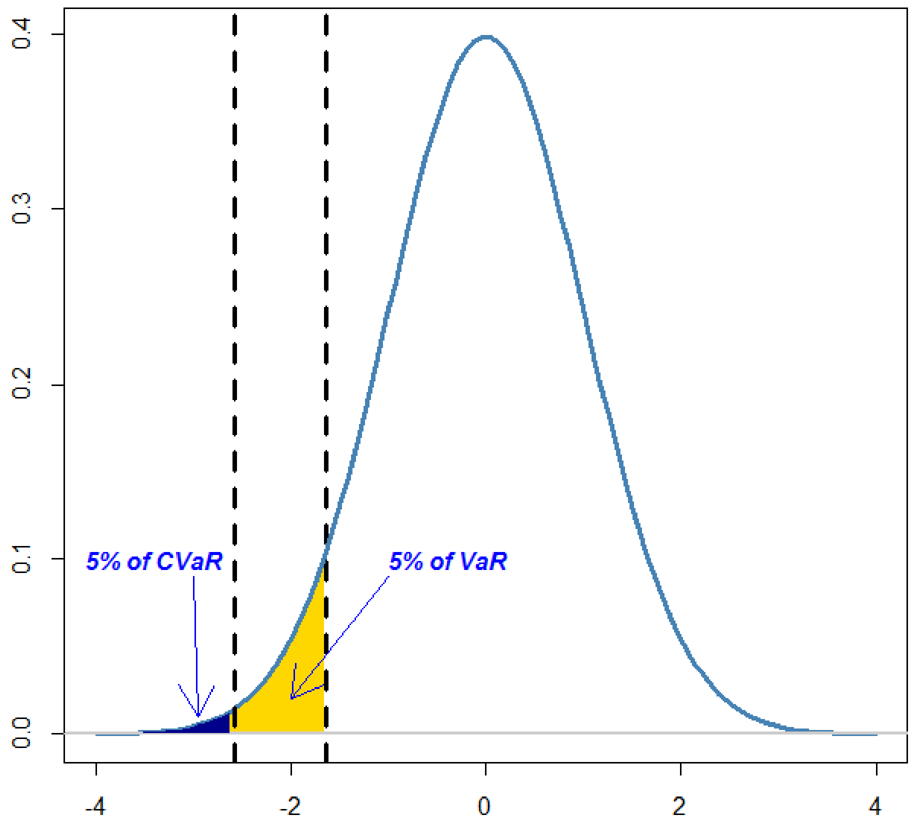

Figure 2.2: Graphical definition of CVaR based on 99.5% confidence level (Jorion, 1997, p.52 [12])

According to the Figure 2.2, value of CVaR is always greater than value of VaR because CVaR is calculated from the arithmetic average of VaR. Moreover, CVaR has an advantage of more significant estimation than VaR because it behaves more variably and sensitively to the risks in the tails and it also adequately assesses the risk in the market. Additionally, CVaR overcomes the drawback of VaR by satisfying the properties of coherency so that CVaR is a natural extension of VaR.

2.3 Coherency Criteria of Risk Measures

In order to obtain an optimal risk measure, coherency criteria have to be fulfilled with the following four axioms, which are proposed by Artzner (1999) [2]. Coherency refers to criterion which give economically rational contributions to the risk and efficient allocation of the premium. In other words, coherency describes the acceptance position of the regulators. Therefore, any risk measure, which satisfy the coherency properties, will be considered as an appropriate and optimal risk management tool. In addition to risk management tool, consistent and competitive premiums will be produced after satisfying the all criteria so that insurance companies manage the risk of loss in an effective way.

(i) *Translation Invariance*: If a principle satisfies the condition $\mathcal{H}(X+c) = \mathcal{H}(X) + c$ for all $X \in \mathcal{X}$ and $c \geq 0$, then the translation invariance property is satisfied. In simple terms, the translation invariance is defined as follows: increasing/decreasing the loss, increases/decreases the risk by the same amount. In other word; It explains the result of loss probability.

(ii) *Scale Invariance*: If a principle satisfies the condition $\mathcal{H}(cX) = c\mathcal{H}(X)$ for all $X \in \mathcal{X}$ and $c \geq 0$, then the scale invariance property is satisfied. As a small note, the scale invariance is also known as *positive homogeneity* in economic literature. We simply define the scale invariance as follows: increasing losses, increases the risk. In other word; it point out the level of risk.

(iii) *Subadditivity*: If a principle satisfies the following: $\mathcal{H}(X+Y) \leq \mathcal{H}(X) + \mathcal{H}(Y)$ for all $X, Y \in \mathcal{X}$, then it is called a subadditivity property and simply, the subadditivity property measures the diversification effect of a portfolio, which is decreasing the risk.

(iv) *Monotonicity*: If a principle satisfies the condition $\mathcal{H}(X) \leq \mathcal{H}(Y)$ when $X \leq Y$ for all $X, Y \in \mathcal{X}$, then the monotonicity property is satisfied and it is defined as enlarging the size of a portfolio results with higher risks. In other word; monotonicity shows the severity of risk.

2.3.1 Derivations for Coherency Criteria of Risk Measures

In this section, we investigate the properties of coherent risk measure for both premium and capital based risk measures. We express the four desirable properties for premium based risk measures with respect to definitions of six well known premium principles and for capital based risk measures with respect to definitions of VaR and CVaR .

In order to prove these criteria, we need to define the expected value of discrete and continuous distributions since we assume that the distribution of risk is known.

$\sum_{i=0}^{\infty} x f_X(x)$, if the distribituion of risk is discrete

$\int_{-\infty}^{\infty} x f(X) dx$, if the distribituion of risk is continuous

1. **Net Premium Principle:** It does not load for risk and it is only sufficient for a risk neutral insurer, which means that the risk is basically non-existent [13]. The biggest problem of Net Premium principle is that it is insufficient pricing methods for an insurance company to remain solvent. However, it satisfies the coherency criteria of risk measures by satisfying following four axioms.

   (a) **Translation Invariance**: Let's consider X is loss and c is a nonnegative constant, where is $c > 0$, then

   $$\begin{aligned} \mathcal{H}(X+c) &= E[X+c] \\ &= E[X]+c \\ &= \mathcal{H}(X)+c \end{aligned}$$

   We can thus express that the net premium principle satisfies the translation invariance property.

   (b) **Scale Invariance**: Let's consider X is loss and c is a constant, where is $c \geq 0$, then

   $$\begin{aligned} \mathcal{H}(cX) &= E[cX] \\ &= cE[X] \\ &= c\mathcal{H}(X) \end{aligned}$$

   We thus have shown that the net premium principle satisfies the scale invariance property.

   (c) **Subadditivity**: Let's consider X and Y are losses, then

   $$\begin{aligned} \mathcal{H}(X+Y) &\leq E[X+Y] \\ &\leq E[X]+E[Y] \\ &\leq \mathcal{H}(X)+\mathcal{H}(Y) \end{aligned}$$

   We therefore have shown that the net premium principle satisfies the subadditivity property.

   (d) **Monotonicity**: Let's consider X and Y are losses and $X \geq Y$ implies $E[X] \geq E[Y]$, then

   $$\mathcal{H}(X) = E[X] \geq E[Y] = \mathcal{H}(Y)$$

   According to the above equations, the net premium principle satisfies the property of monotonicity.

2. **Expected Value Premium Principle:** It is obtained by adding the proportional risk loading on the Net Premium Principle. Moreover, the usage of it is extremely wide in the insurance business because it is easy to understand. Following four axioms are investigated based on Expected Value Premium Principle.

   (a) **Translation Invariance**: Let's consider a loss X and a nonnegative constant c, where is $c > 0$, then

   $$\begin{aligned} \mathcal{H}(X+c) &= (1+\theta)E[X+c] \\ &= (1+\theta)(E[X]+c) \\ &= (1+\theta)E[X]+(1+\theta)c \\ &= \mathcal{H}(X)+(1+\theta)c \\ &> \mathcal{H}(X)+c \end{aligned}$$

   We showed that the expected value premium principle **does not** satisfy the translation invariance property.

   (b) **Scale Invariance**: Let's consider a loss X and a constant c, where is $c \geq 0$, then

   $$\begin{aligned} \mathcal{H}(cX) &= (1+\theta)E[cX] \\ &= (1+\theta)cE[X] \\ &= c\mathcal{H}(X) \end{aligned}$$

   We satisfy the property of scale invariance for expected value premium principle.

   (c) **Subadditivity**: Let's consider losses X and Y, then

   $$\begin{aligned} \mathcal{H}(X+Y) &\leq (1+\theta)E[X+Y] \\ &\leq (1+\theta)(E[X]+E[Y]) \\ &\leq (1+\theta)E[X]+(1+\theta)E[Y] \\ &\leq \mathcal{H}(X)+\mathcal{H}(Y) \end{aligned}$$

   Thus, the expected value premium principle satisfies the subadditivity property.

   (d) **Monotonicity**: Let's consider losses X and Y and $X \geq Y$ implies $E[X] \geq E[Y]$, then

   $$\mathcal{H}(X) = (1+\theta)E[X] \geq (1+\theta)E[Y] = \mathcal{H}(Y)$$

   Expected value premium principle satisfies the monotonicity property according to above equation.

3. **Variance Premium Principle:** According to Young [27], this premium principle also obtain from the Net Premium Principle with the additional risk loading which is proportional to the variance of the risk. Following four axioms are investigated based on the definition of Variance Premium Principle.

(a) **Translation Invariance**: Let's consider $Y = X + c$, where X is loss and c is a nonnegative constant, $c > 0$ and Y is a function of X, then

$$\begin{aligned}\mathcal{H}(Y) &= E[Y] + \theta Var[Y] \\ &= E[X+c] + \theta Var[X+c] \\ &= c + E[X] + \theta Var[X] \\ &= \mathcal{H}(X) + c\end{aligned}$$

Thus the variance premium principle satisfies the translation invariance property.

(b) **Scale Invariance**: Let's consider $Y = cX$, where X is loss and c is a constant, $c \geq 0$ and Y is a function of X, then

$$\begin{aligned}\mathcal{H}(Y) &= E[Y] + \theta Var(Y) \\ &= E[cX] + \theta Var(cX) \\ &= cE[X] + \theta c^2 Var(X) \\ &= c(E(X) + \theta c Var(X)) \neq c\mathcal{H}(X)\end{aligned}$$

The above expression shows that variance premium principle **does not** satisfy the scale invariance property.

(c) **Subadditivity**: Let's consider X and Y are losses, then

$$\begin{aligned}\mathcal{H}(X+Y) &\leq E[X+Y] + \theta Var(X+Y) \\ &\nleq (E[X] + E[Y]) + \theta(Var(X) + Var(Y) + 2Cov(X,Y))\end{aligned}$$

According to above equation, the variance premium principle **does not** satisfy the subadditivity property unless both loss variables, X and Y, are independent.

(d) **Monotonicity**: Let's consider X and Y are losses and $X \geq Y$ implies $E[X] \geq E[Y]$, then

$$\mathcal{H}(X) = E[X] + \theta Var(X) \ngeq E[Y] + \theta Var(Y) = \mathcal{H}(Y)$$

We stated that variance premium principle **does not** satisfy the monotonicity property due to the fact that it is not necessary to meet X has to be higher or equal to Y.

4. **Standard Deviation Premium Principle:** According to Young [27], this premium principle also obtain from the Net Premium Principle with the additional risk loading which is proportional to the standard deviation of the risk. Following four axioms are investigated based on the definition of Standard Deviation Premium Principle.

(a) **Translation Invariance**: Let's consider $Y = X + c$, where X is loss and c is a nonnegative constant, $c > 0$ and Y is a function of X, then

$$\begin{aligned}\mathcal{H}(Y) &= E[Y] + \theta\sqrt{Var[Y]} \\ &= E[X+c] + \theta\sqrt{Var[X+c]} \\ &= E[X] + c + \theta\sqrt{Var[X]} \\ &= \mathcal{H}(X) + c\end{aligned}$$

We have shown that the standard deviation premium principle satisfies the translation invariance property.

(b) **Scale Invariance**: Let's consider $Y = cX$, where X is loss and c is a constant, $c \geq 0$ and Y is a function of X, then

$$\begin{aligned}\mathcal{H}(Y) &= E[Y] + \theta\sqrt{Var(Y)} \\ &= E[cX] + \theta\sqrt{Var(cX)} \\ &= cE[X] + c\theta\sqrt{Var(X)} \\ &= c(E(X) + \theta\sqrt{Var(X)}) = c\mathcal{H}(X)\end{aligned}$$

We have expressed that the standard deviation premium principle satisfies the scale invariance property.

(c) **Subadditivity**: Let's consider losses X and Y, then

$$\begin{aligned}\mathcal{H}(X+Y) &\leq E[X+Y] + \theta\sqrt{Var(X+Y)} \\ &\nleq (E[X] + E[Y]) + \theta\sqrt{Var(X) + Var(Y) + 2Cov(X,Y)}\end{aligned}$$

We have shown that the standard deviation premium principle **does not** satisfy the subadditivity property unless X and Y are independent.

(d) **Monotonicity**: Let's consider losses X and Y and $X \geq Y$ implies $E[X] \geq E[Y]$ then

$$\mathcal{H}(X) = E[X] + \theta\sqrt{Var(X)} \ngeq E[Y] + \theta\sqrt{Var(Y)} = \mathcal{H}(Y)$$

Thus the standard deviation premium principle **does not** satisfy the monotonicity property due to the fact that the condition that X has to be higher or equal to Y is insufficient.

5. **Exponential Premium Principle:** According to Young [27], this premium principle occurs the exponential utility function. Following four axioms are investigated based on the definition of Exponential Premium Principle.

(a) **Translation Invariance**: Let's consider $Y = X + c$, X is loss and c is a nonnegative constant, $c > 0$ and Y is a function of X, then

$$\begin{aligned}\mathcal{H}(Y) &= \frac{\ln[E(e^{\theta Y})]}{\theta} \\ \mathcal{H}(X+c) &= \frac{\ln[E(e^{\theta(X+c)})]}{\theta} \\ &= \frac{\ln[E(e^{\theta X+\theta c})]}{\theta} \\ &= \frac{\ln[E(e^{\theta X})e^{\theta c}]}{\theta} \\ &= \frac{\ln[E(e^{\theta X})] + \ln[e^{\theta c}]}{\theta} \\ &= \frac{\ln[E(e^{\theta X})] + \theta c}{\theta} \\ &= \frac{\ln[E(e^{\theta X})]}{\theta} + \frac{\theta c}{\theta} = \mathcal{H}(X) + c\end{aligned}$$

We have shown that the exponential premium principle satisfies the translation invariance property.

(b) **Scale Invariance**: Let's consider $Y = cX$, where X is loss and c is a constant, $c \geq 0$ and Y is a function of X, then

$$\begin{aligned} \mathcal{H}(Y) &= \frac{\ln[E(e^{\theta Y})]}{\theta} \\ \mathcal{H}(cX) &= \frac{\ln[E(e^{\theta cX})]}{\theta} \neq c\mathcal{H}(X) \end{aligned}$$

We have shown that the exponential premium principle **does not** satisfy the scale invariance property.

(c) **Subadditivity**: Let's consider losses X and Y, then

$$\begin{aligned} \mathcal{H}(X) &= \frac{\ln[E(e^{\theta X})]}{\theta} \\ \mathcal{H}(Y) &= \frac{\ln[E(e^{\theta Y})]}{\theta} \\ \mathcal{H}(X+Y) &= \frac{\ln[E(e^{\theta(X+Y)}]}{\theta} \\ &= \frac{\ln[E(e^{\theta(X)}e^{\theta(Y)})]}{\theta} \\ &\nleq \frac{\ln[E(e^{\theta X})]}{\theta} + \frac{\ln[E(e^{\theta Y})]}{\theta} \end{aligned}$$

The above expression shows that the exponential premium principle **does not** satisfy the subadditivity property unless both losses X and Y are independent.

(d) **Monotonicity**: Let's consider losses X and Y, verifying that $X \leq Y$, then

$$\begin{aligned} \mathcal{H}(X) &= \frac{\ln[E(e^{\theta X})]}{\theta} \\ \mathcal{H}(Y) &= \frac{\ln[E(e^{\theta Y})]}{\theta} \\ \mathcal{H}(X) &\leq \mathcal{H}(Y) \end{aligned}$$

We have shown that the exponential premium principle satisfies the monotonicity property.

6. **Esscher Premium Principle:** Bühlmann [6] derived this premium principle from the framework of utility theory and risk exchange. Moreover, the results of Esscher Premium Principle depend on mainly the selection of loss function. Following four axioms are investigated based on only the mathematical definition of Esscher Premium Principle.

(a) **Translation Invariance**: Let's consider $Y = X + c$, where X is loss and c is a nonnegative constant, $c > 0$ and Y is a function of X, then

$$\begin{aligned}
\mathcal{H}(Y) &= \frac{E[Ye^{\theta Y}]}{E[e^{\theta Y}]} \\
\mathcal{H}(X+c) &= \frac{E[(X+c)e^{\theta(X+c)}]}{E[e^{\theta(X+c)}]} \\
&= \frac{E[(X+c)e^{\theta X}e^{\theta c}]}{E[e^{\theta X}e^{\theta c}]} \\
&= \frac{e^{\theta c}E[(X+c)e^{\theta X}]}{e^{\theta c}E[e^{\theta X}]} \\
&= \frac{e^{\theta c}E[Xe^{\theta X} + ce^{\theta X}]}{e^{\theta c}E[e^{\theta X}]} \\
&= \frac{E[Xe^{\theta X}] + E[ce^{\theta X}]}{E[e^{\theta X}]} \\
&= \frac{E[Xe^{\theta X}]}{E[e^{\theta X}]} + \frac{cE[e^{\theta X}]}{E[e^{\theta X}]} \\
\mathcal{H}(X) &= \mathcal{H}(X) + c
\end{aligned}$$

The above expressions have proved that the Esscher premium principle satisfies the translation invariance property.

(b) **Scale Invariance**: Let's consider $Y = cX$, where X is loss and c is a constant, $c \geq 0$ and Y is a function of X, then

$$\begin{aligned}
\mathcal{H}(Y) &= \frac{E[Ye^{\theta Y}]}{E[e^{\theta Y}]} \\
\mathcal{H}(cX) &= \frac{E[cXe^{\theta cX}]}{E[e^{\theta cX}]} \\
&= \frac{cE[Xe^{\theta cX}]}{E[e^{\theta cX}]} \neq \frac{cE[Xe^{\theta X}]}{E[e^{\theta X}]} = c\mathcal{H}(X)
\end{aligned}$$

The above statement shows that the Esscher premium principle **does not** satisfy the scale invariance property.

(c) **Subadditivity**: Let's consider losses X and Y, then

$$\begin{aligned}
\mathcal{H}(X) &= \frac{E[Xe^{\theta X}]}{E[e^{\theta X}]} \\
\mathcal{H}(Y) &= \frac{E[Ye^{\theta Y}]}{E[e^{\theta Y}]} \\
\mathcal{H}(X+Y) &= \frac{E[(X+Y)e^{\theta(X+Y)}]}{E[e^{\theta(X+Y)}]} \\
&= \frac{E[(Xe^{\theta(X+Y)}) + (Ye^{\theta(X+Y)})]}{E[e^{\theta(X+Y)}]} \\
&= \frac{E[(Xe^{\theta X}e^{\theta Y}) + (Ye^{\theta X}e^{\theta Y})]}{E[e^{\theta X}e^{\theta Y}]} \\
&\nleq \frac{E[Xe^{\theta X}]}{E[e^{\theta X}]} + \frac{E[Ye^{\theta Y}]}{E[e^{\theta Y}]} = \mathcal{H}(X) + \mathcal{H}(Y)
\end{aligned}$$

We have shown that the Esscher premium principle **does not** satisfy the subadditivity property.

(d) **Monotonicity**: Let's consider losses X and Y, assuming that $X \leq Y$, then

$$\begin{aligned}
\mathcal{H}(X) &= \frac{E[Xe^{\theta X}]}{E[e^{\theta X}]} \\
\mathcal{H}(Y) &= \frac{E[Ye^{\theta Y}]}{E[e^{\theta Y}]} \\
\mathcal{H}(X) &\nleq \mathcal{H}(Y)
\end{aligned}$$

We have stated that the Esscher premium principle **does not** satisfy the monotonicity property as despite the fact that the risk X is lower or equal to the risk Y, the above statement does not have to be always true.

7. **Value at Risk:** It is the most used risk measure in the field of finance and insurance. However, Rockafellar and Uryasev [23] showed that VaR does not satisfy the coherency criteria. Following axioms are investigated based on the satisfaction level of coherency criteria.

(a) **Translation Invariance**: Let's consider a loss X and a nonnegative constant c, where $c > 0$, and Y=X+c, then

$$\begin{aligned}
VaR_\delta(Y) &= VaR_\delta(X+c) \\
&= \inf\{y : P(Y \leq y) \geq \delta\} \\
&= \inf\{x+c : P(X+c \leq x+c) \geq \delta\} \\
&= c + \inf\{x : P(X+c \leq x+c) \geq \delta\} \\
&= c + \inf\{x : P(X \leq x) \geq \delta\} \\
&= c + VaR_\delta(X)
\end{aligned}$$

We have shown that VaR satisfies the translation invariance property by using the definition of VaR.

(b) **Scale Invariance**: Let's assume X is loss and c is a constant, where $c \geq 0$, and Y=cX, then

$$\begin{aligned} VaR_\delta(Y) &= VaR_\delta(cX) \\ &= \text{inf}\,\{y : P(Y \leq y) \geq \delta\} \\ &= \text{inf}\,\{cx : P(cX \leq cx) \geq \delta\} \\ &= c * \text{inf}\,\{x : P(cX \leq cx) \geq \delta\} \\ &= c * \text{inf}\,\{x : P(X \leq x) \geq \delta\} \\ &= c * VaR_\delta(X) \end{aligned}$$

We have shown that VaR satisfies the scale invariance property.

(c) **Subadditivity**: Let's assume X and Y are losses, then

$$VaR_\delta(X+Y) \leq VaR_\delta(X) + VaR_\delta(Y)$$

Let's define the each equation one by one.

$$\begin{aligned} VaR_\delta(X+Y) &= \text{inf}\,\{x+y : P((X+Y) \leq (x+y)) \geq \delta\} \\ &= \text{inf}\,\{x : P((X+Y) \leq (x+y)) \geq \delta\} \\ &+ \text{inf}\,\{y : P((X+Y) \leq (x+y)) \geq \delta\} \end{aligned}$$

$$VaR_\delta(X) = \text{inf}\,\{x : P((X) \leq (x)) \geq \delta\}$$

$$VaR_\delta(Y) = \text{inf}\,\{y : P((Y) \leq (y)) \geq \delta\}$$

Then, we can write the subadditivity property of VaR as follows;

$$\begin{aligned} A &\leq B + C \\ \text{inf}\,\{x : P((X+Y) \leq (x+y)) \geq \delta\} &+ \text{inf}\,\{y : P((X+Y) \leq (x+y)) \geq \delta\} \\ &\leq \text{inf}\,\{x : P((X) \leq (x)) \geq \delta\} \\ &+ \text{inf}\,\{y : P((Y) \leq (y)) \geq \delta\} \end{aligned}$$

We, thus, showed that VaR **does** not satisfies the subadditivity property.

(d) **Monotonicity**: Let's assume X and Y are losses and $X \leq Y$ under probability space (Ω, $\mathcal{F}$, P) then

$$\begin{aligned} VaR_\delta(X) &= \text{inf}\,\{x : P(X \leq x) \geq \delta\} \\ &\leq \\ VaR_\delta(Y) &= \text{inf}\,\{y : P(Y \leq y) \geq \delta\} \end{aligned}$$

According to above equation, VaR satisfies the property of monotonicity.

8. **Conditional Value at Risk:** Following axioms are examined based on the definition of CVaR to see the satisfaction level of coherency criteria of CVaR.

(a) **Translation Invariance**: Let's assume X is loss and c is a nonnegative constant, where $c > 0$, and Y=X+c, then

$$\begin{aligned} CVaR_\delta(Y) &= CVaR_\delta(X+c) \\ &= E[Y|Y > VaR_\delta(Y)] \\ &= E[X+c|X+c > VaR_\delta(X+c)] \\ &= E[X+c|X > VaR_\delta(X)] \\ &= c + E[X|X > VaR_\delta(X)] \\ &= c + CVaR_\delta(X) \end{aligned}$$

We have shown that CVaR satisfies the translation invariance property by using the definition of VaR.

(b) **Scale Invariance**: Let's assume X is loss and c is a constant, where $c \geq 0$, and Y=cX, then

$$\begin{aligned} CVaR_\delta(Y) &= CVaR_\delta(cX) \\ &= E[Y|Y > VaR_\delta(Y)] \\ &= E[cX|cX > VaR_\delta(cX)] \\ &= E[cX|X > VaR_\delta(X)] \\ &= c * E[X|X > VaR_\delta(X)] \\ &= c * CVaR_\delta(X) \end{aligned}$$

We have stated that CVaR satisfies the scale invariance property.

(c) **Subadditivity**: Let's assume X and Y are losses and Z=X+Y, then

$$\begin{aligned} CVaR_\delta(Z) &= CVaR_\delta(X+Y) \\ &= E[X|X+Y > VaR_\delta(X+Y)] \\ &+ E[Y|X+Y > VaR_\delta(X+Y)] \\ &\leq E[X|X > VaR_\delta(X)] \\ &+ E[Y|Y > VaR_\delta(Y)] \\ &= CVaR_\delta(X) + CVaR_\delta(Y) \end{aligned}$$

We have shown that CVaR satisfies the subadditivity property as long as losses have a continuous distribution.

(d) **Monotonicity**: Let's assume X and Y are losses and $X \leq Y$ under probability space (Ω, $\mathcal{F}$, P) then

$$\begin{aligned} CVaR_\delta(X) &= E[X|X > VaR_\delta(X)] \\ &\leq \\ CVaR_\delta(Y) &= E[Y|Y > VaR_\delta(Y)] \end{aligned}$$

According to above equations, CVaR satisfies the property of monotonicity.

According to the proofs of coherency level of risk measures, they are summarized based on the coherency satisfaction level in Table 2.1. In the table, the '✓' shows that

Table 2.1: Summary of Coherency Criteria of Risk Measures

Risk Measures	TI	SI	SA	M
Net Premium Principle	✓	✓	✓	✓
Expected Value Premium Principle	x	✓	✓	✓
Variance Premium Principle	✓	x	x	x
Standard Deviation Premium Principle	✓	✓	x	x
Exponential Premium Principle	✓	x	x	✓
Esscher Premium Principle	✓	x	x	x
Value at Risk	✓	✓	x	✓
Conditional Value at Risk	✓	✓	✓	✓

the premium principle satisfies the coherency property and the 'x' indicates that the premium principle does not satisfy the coherency property.

Based on Table 2.1, we see that Net Risk Premium and Conditional Value at Risk are only called as coherent risk measure by satisfying the four properties of coherency.

CHAPTER 3

LOSS DISTRIBUTIONS AND ESTIMATION METHODS

In this chapter, the characteristics and the usage of loss distributions in insurance and finance field will be discussed because they are important for pricing processes of an insurance product. Continuous loss distributions will be mainly examined based on their usage in insurance and finance field. In addition to loss distributions, the classical methods of parameter estimation will be expressed. These methods are Moment Estimation and Maximum Likelihood Estimation.

3.1 Loss Distributions

Insurance is defined as a transfer of risk from the insured to the insurer so that loss experience has a significant role in the pricing processes of insurance contracts because in order to offer a reasonable premium for individuals, the estimation of loss events has to be done accurately in advance.

(Ω, $\mathcal{F}$, P) is the probability space and X is a random variable defined on Ω. Random variables can be categorized as *discrete* or *continuous*. Discrete random variables have a countable set of distinct possible values, which compose a countable range space. On the other hand, continuous random variables have any value within a specified interval range space[3]. Moreover, random variable X is usually expressed by a function, which is called probability distribution, defined on all of the real numbers, $\mathcal{R}$ [4]. If a countable range space is used for defining the random variable X, it will be called a *Discrete Probability Distribution*. Otherwise, it will be called a *Continuous Probability Distribution*.

Measurement of risk needs the examination and modeling of two independent stochastic processes, which are loss frequency and loss severity, because there is a lack of predictability of these two stochastic processes in advance in the insurance field [28].

Discrete distributions are used for modeling the frequency of loss, whereas continuous distributions are used for modeling the loss severity because they are basically the probabilistic representations of the magnitude of loss. In other words, discrete distributions enable us to calculate the probability of occurrence of the loss and continuous distributions enable us to calculate the probability of the size of the claim, which will be not greater than a certain size.

In this study, the severity of loss is only modeled for examining the claim distribution so that continuous probability distributions are used. In insurance and finance field, losses can be not only non-negative, but also very high [8] in many cases. When looking at the past claims of an insurance company for automobile insurance products, small scale damage occurs very frequently. On the contrary, large scale damage occurs less frequently compared to small scale damages. In order to deal with these situations, heavy or fat tailed distributions are applied to the estimation of the distribution of claim severity. One of the main characteristic of these distributions is being right skewed. Weibull, Exponential, Pearson, Lognormal, Pareto and Gamma are the most suitable distributions for estimating the claim severity in the insurance field due to the fact that all of them have a right skewed feature.

Four right skewed distribution are chosen for estimating the claim severity by using claim amounts [9]. This choice is made based on the thickness level of distributions' tail. These four distributions are Gamma, Weibull, Lognormal and Pareto distributions; where Gamma distribution has the most thin tail and Pareto distribution has the most thick tail.

3.1.1 Gamma Distribution

The Gamma family has two positive parameters (α, β). The probability density function is defined as;

$$f(x) = \begin{cases} 0, & \text{if} -\infty < x \leq 0 \\ \frac{1}{\beta^{\alpha}\Gamma(\alpha)} x^{\alpha-1} e^{-\frac{x}{\beta}}, & \text{if } 0 < x < \infty \end{cases}$$

where $\alpha > 0, \beta > 0$ and the Gamma function is defined as

$$\Gamma(x) = \int_0^{\infty} u^{x-1} e^{-u} du, 0 < x < \infty$$

A special notation for the Gamma distribution is $X \sim \text{Gamma}(\alpha, \beta)$, where β is defined the scale parameter of distribution and α is the shape parameter of distribution.

Figure 3.1 shows the shape of the Gamma distribution based on different shape parameters.

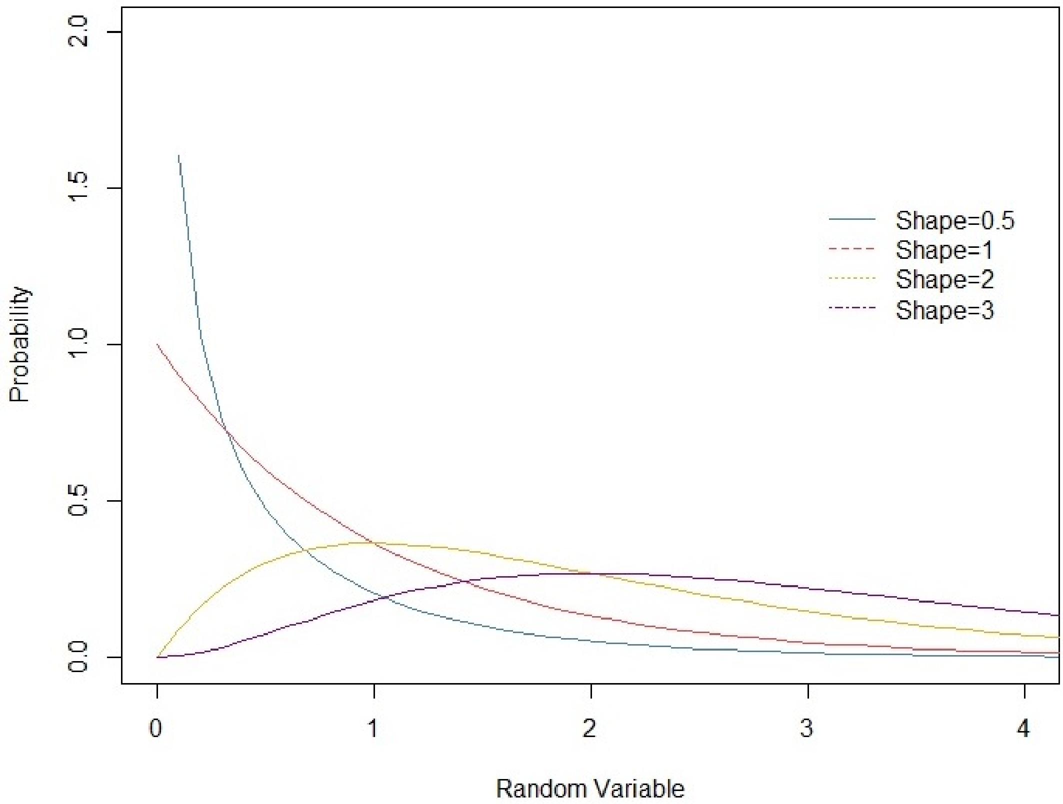

Figure 3.1: Probability Density Function of Gamma (α, 1) (Tse, 2009, p.52 [25])

According to Figure 3.1; shape parameter of Gamma distribution determines the shape of the distribution basically [4]. Depend on the value of shape parameter, there are three basic shape for Gamma distribution.

- $\alpha < 1$; The distribution decreases monotonically.
- $\alpha = 1$; Gamma distribution become Exponential distribution and the distribution decreases monotonically. It is a special case for Gamma distribution.
- $\alpha > 1$; The distribution increases monotonically from 0 to mode value of the distribution and then, it starts to decrease but it has skewed shape.

Mean, variance and moment generating function of the Gamma distribution are defined as;

$$
\begin{aligned}
E[X] &= \alpha\beta \\
V[X] &= \alpha\beta^2 \\
M(t) &= (1-\beta t)^{-\alpha}
\end{aligned}
$$

and the proof of mean, variance and moment generating function can be found in Appendix A.

3.1.2 Weibull Distribution

The Weibull distribution has two positive parameters (α, β) and it is described as $X \sim$ Wei(α, β). The probability density function is defined as;

$$f(x) = \begin{cases} 0, & \text{if } -\infty < x \leq 0 \\ \frac{\beta}{\alpha^\beta} x^{\beta-1} e^{-(\frac{x}{\alpha})^\beta} & \text{if } 0 < x < \infty \end{cases}$$

where the parameter $\alpha > 0$ is the scale parameter and $\beta > 0$ is the shape parameter. Shape parameter of Weibull is affected the behavior of the distribution because shape parameter act like a slope of the line [3]. Therefore, we only deal with the scale parameter of Weibull distribution.

In the following Figure 3.2, by increasing the scale parameter of the Weibull distribution, we spread out the dispersion of the distribution. In other words, the peakedness of the distribution is decreasing because the area of under the curve is holding constant.

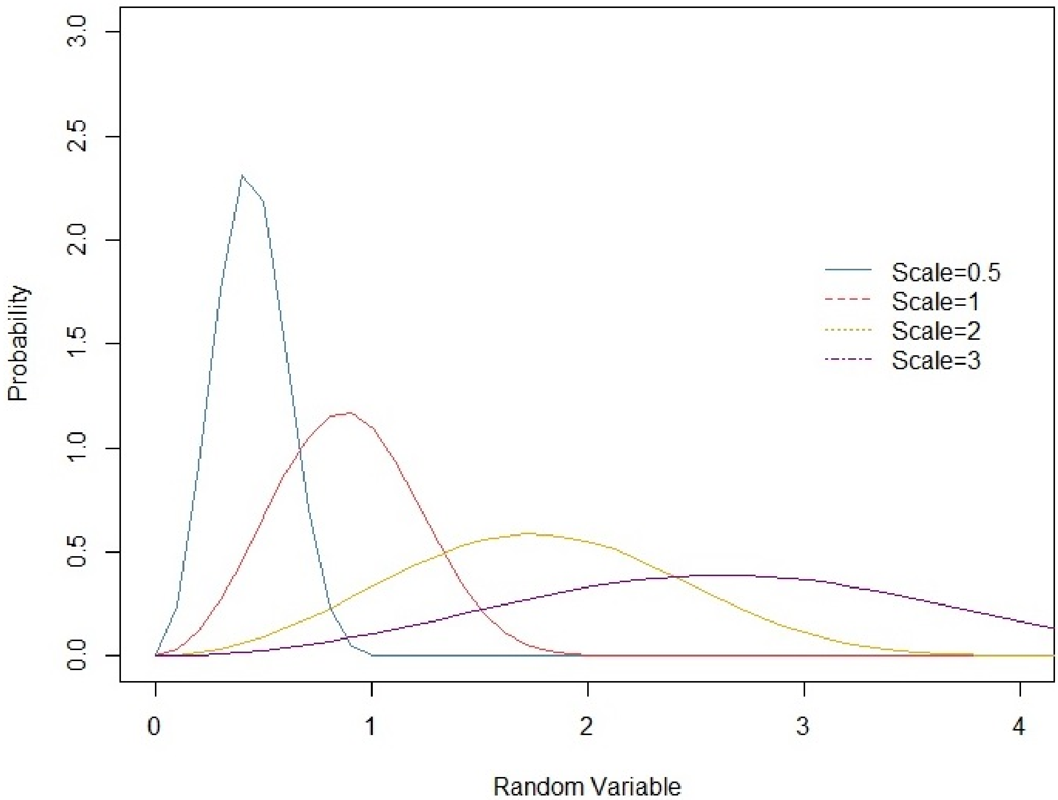

Figure 3.2: Probability Density Function of Weibull (α, 3) (Tse, 2009, p.52 [25])

Mean, variance and moment generating function of the Weibull distribution are described as;

$$
\begin{aligned}
E[X] &= \alpha\Gamma(1+\frac{1}{\beta}) \\
V[X] &= \alpha^2\big[\Gamma(1+\frac{2}{\beta}) - \Gamma^2(1+\frac{1}{\beta})\big] \\
M(t) &= \int_0^\infty e^{tx}\frac{\beta}{\alpha}(\frac{x}{\alpha})^{\beta-1}e^{-(\frac{x}{\alpha})^\beta}
\end{aligned}
$$

and the proof of mean, variance and moment generating function also can be found in Appendix A.

3.1.3 Lognormal Distribution

The Lognormal family has two parameters, which are μ and σ^2. It is also described as $X \sim \text{LN}(\mu, \sigma^2)$. The probability density function is given by

$$
f(x) = \begin{cases} 0, & \text{if} - \infty < x \leq 0 \\ \frac{1}{\sigma\sqrt{2\pi}}\frac{1}{x}e^{-\frac{1}{2}(\frac{\log x - \mu}{\sigma})^2}, & \text{if } 0 < x < \infty \end{cases}
$$

where $\mu \in \mathcal{R}$ is the location parameter and $\sigma^2 \in (0, \infty)$ is the scale parameter.

Moreover, the following relation can be defined between Normal and Lognormal distributions [8];

$$
X \sim \text{LN}(\mu, \sigma^2) \Leftrightarrow Y = \ln X \sim N(\mu, \sigma^2)
$$

Furthermore, in Figure 3.3, the effect of different scale parameters can be seen in depth. By increasing the scale parameter, tails of the Lognormal distribution are more stretched. In other words, the effect of the scale parameter of Lognormal distribution with value greater than 1 is stretching the distribution. On the contrary, the distribution is compressing with the value of scale parameter is less than 1.

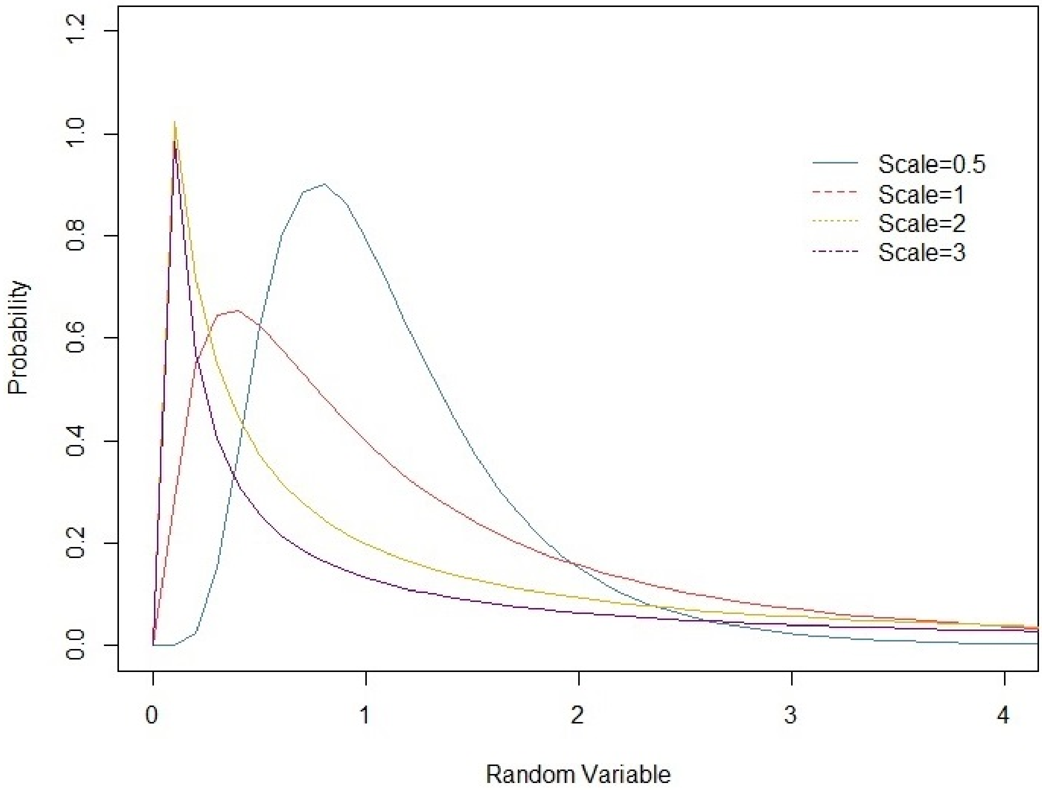

Figure 3.3: Probability Density Function of Lognormal (0, σ^2) (Tse, 2009, p.52 [25])

Mean, variance and moment generating function of the Lognormal distribution are defined as;

$$\begin{aligned} E[X] &= e^{(\mu+\frac{\sigma^2}{2})} \\ V[X] &= e^{(2\mu+\sigma^2)}(e^{\sigma^2}-1) \\ M(t) &= e^{t\mu+\frac{1}{2}t^2\sigma^2} \end{aligned}$$

and the proof of mean, variance and moment generating function also can be found in Appendix A.

3.1.4 Pareto Distribution

The Pareto family has two positive parameters (a, λ) and the notation of Pareto distribution is $X \sim \text{Pareto}(a, \lambda)$. The probability density function is given by

$$f(x) = \begin{cases} 0, & \text{if } -\infty < x \leq 0 \\ \frac{a\lambda^a}{x^{a+1}}, & \text{if } \lambda < x < \infty \end{cases}$$

where the shape parameter $a > 0$ and the scale parameter $\lambda > 0$.

As seen in Figure 3.4, the shape of the distribution is getting more clear with the larger shape parameters. In other words, the skewness of the Pareto distribution is increased by incrementing the shape parameter.

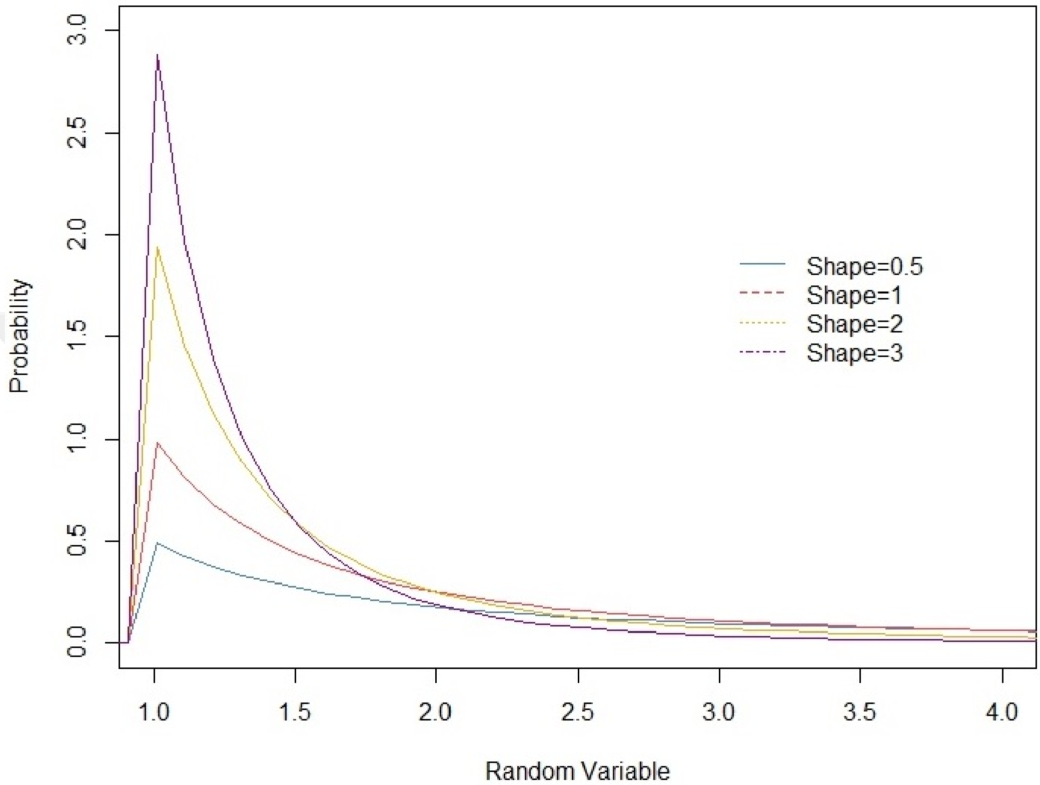

Figure 3.4: Probability Density Function of Pareto (a,1) (Tse, 2009, p.52 [25])

Mean, variance and moment generating function of the Pareto distribution are defined as;

$$
\begin{aligned}
E[X] &= \frac{a\lambda}{a-1} \\
V[X] &= \frac{a\lambda^2}{(a-1)^2(a-2)} \\
M(t) &= \text{Does not exist.}
\end{aligned}
$$

and the proof of mean and variance also can be found in Appendix A.

3.2 Parameter Estimation Methods

In order to fit the model of loss distributions of an insurance company products, the estimation of unknown parameters of loss distributions have to be calculated by using the classical parameter estimation methods. These two methods should give unbiased, consistent and efficient estimators. This section presents two of the most popular classical methods of estimation, which are Methods of Moments Estimation (MME) and Methods of Maximum Likelihood Estimation (MLE).

3.2.1 Method of Moments Estimation

Method of Moments Estimation (MME) is one of the oldest and simplest estimation methods. Although MME estimators are unbiased and consistent, all of the relevant information in the sample may not be taken into account with this method [4]. In other words, it does not always provide the efficiency feature of the estimators. In statistics, MME is defined as a way for estimating population parameters by using sample moments.

Let's consider a population probability density function $f(x;\theta_1;\theta_2;......;\theta_k)$, which depends on *k* unknown parameters and it is also independent. We also assume that identical random variables of X_1, X_2, ..., X_n are chosen from a population distribution.

The k^{th} population moment of a random variable X is showed as;

$$\mu_k' = E[X^k] \text{ where k =1,2,....} \tag{3.1}$$

The k^{th} sample moment of a sample X_1, X_2,......, X_n is showed as;

$$m_k' = \frac{1}{n}\Sigma_{i=1}^{n} X_i^k \text{ where k =1,2,....} \tag{3.2}$$

By replacing the population moment (3.1) with sample moment (3.2), then the following equation system gives the formula for the MME of $(\hat{\theta_1};\hat{\theta_2};......;\hat{\theta_k})$,

$$\mu_k' = m_k' \tag{3.3}$$

$$E[X^k] = \frac{1}{n}\Sigma_{i=1}^{n} X_i^k \tag{3.4}$$

We thus have obtain the estimator of method of moments by solving the Equation 3.4.

3.2.2 Method of Maximum Likelihood Estimation

Maximum Likelihood Estimation (MLE) method is the most preferred estimation approach for parameters in the world due to the fact that MLE has the properties of consistency, efficiency, asymptotic normality and invariance [7].

In statistics, MLE is defined as a method of finding the value of parameters, which maximizes the probability of obtaining the observed data [19]. Therefore; MLE procedure begins with getting the Likelihood function (L(θ)). The definition of L(θ) is given the joint density function of n random variables X_1, X_2,......, X_n, which are assessed at x_1, x_2, ..., x_3 respectively.

$$\begin{aligned} L(\theta) &= f(x_1, x_2, ..., x_n; \theta) && (3.5) \\ L(\theta) &= f(x_1;\theta)f(x_2;\theta).....f(x_n;\theta) && (3.6) \\ L(\theta) &= \prod_{i=1}^{n} f(x_i;\theta) && (3.7) \end{aligned}$$

By differentiating Equation 3.7 with respect to θ, the maximum likelihood equation is obtained.

$$MLE = \frac{dL(\theta)}{d\theta} \tag{3.8}$$

By solving Equation 3.8 with respect to zero, the value of the estimated parameter is obtained.

$$\frac{dL(\theta)}{d\theta} = 0 \tag{3.9}$$

However, it should be verified that this value maximizes the L(θ). In order to get verified results, a second derivative of L(θ) is taken and it has to be shown that the result of this equation is less than zero.

$$MLE = \frac{d^2L(\theta)}{d\theta^2} \leq 0 \tag{3.10}$$

If Equation 3.10 is less than zero, then it can be said that the value of Equation 3.9 maximizes the likelihood function L(θ). Therefore; we can assume that the result of Equation 3.9 gives the estimated parameter based on the maximum likelihood estimation method.

Following Table 3.1 shows the summary of parameter estimators based on two classical estimation methods for selected loss distributions. The proof of estimation of parameters based on two methods can be found in Appendix A.

Table 3.1: Summary of Parameter Estimation for Loss Distributions

Distribution	Parameters	MME	MLE
Gamma	α	$\frac{n\bar{x}^2}{\sum_{i=1}^{n}(x_i-\bar{x})^2}$	$n(ln(\beta) - \frac{\Gamma'(\alpha)}{\Gamma(\alpha)}) + nln(\bar{X})$
	β	$\frac{1}{n\bar{x}}\sum_{i=1}^{n}(x_i - \bar{x})^2$	$\frac{\bar{X}}{\hat{\alpha}}$
Weibull	α	$\frac{\bar{x}}{\Gamma(\frac{1}{\hat{\beta}}+1)}$	$(\frac{1}{n}\sum_{i=1}^{n} x_i^{\beta})^{\frac{1}{\beta}}$
	β	Not have simple and closed expression	Not have simple and closed expression
Lognormal	μ	$-\frac{ln(\sum_{i=1}^{n} X_i^2)}{2} + 2ln(\sum_{i=1}^{n} X_i) - \frac{3}{2}ln(n)$	$\frac{\sum_{i=1}^{n} ln(X_i)}{n}$
	σ	$ln(\sum_{i=1}^{n} X_i^2) - 2ln(\sum_{i=1}^{n} X_i) + ln(n)$	$\frac{\sum_{i=1}^{n}(ln(X_i) - \frac{\sum_{i=1}^{n} ln(X_i)}{n})^2}{n}$
Pareto	a	$1 + \sqrt{1 + \frac{\frac{n}{n-1}\bar{X}}{S^2}}$	$\frac{n}{\sum_{i=1}^{n} ln(X_i) - nln(\hat{\lambda})}$
	λ	$\frac{\bar{X}\sqrt{1+\frac{\frac{n}{n-1}\bar{X}}{S^2}}}{1+\sqrt{1+\frac{\frac{n}{n-1}\bar{X}}{S^2}}}$	$\min x_i$

CHAPTER 4

CASE STUDY: MOTOR INSURANCE

In this chapter, automobile insurance data, which belong to policy year of 2008, are used to test for an appropriate statistical distribution for claim amounts and to evaluate the sensitivity of risk measures based on these distributions. Data are obtained from one of the most reputable insurance company in Turkey. Data contain the information about motor insurance policy for one-year period. In 2008, there was an increase in the frequency of claim due to excessive rainfall and flood. This situation might have affected the data but there is no effect of regulation on the data set. Claim amounts of this data are used as random variables to estimate the claim distribution and this estimation process is performed with R programming. Moreover, the following assumptions and specifications are made on the analyses and data set:

(i) Claim amounts are independent.

(ii) Gamma, Weibull, Lognormal and Pareto distributions are considered to describe the behavior of claim amount.

(iii) Two highly populated cities in Turkey, Istanbul and Ankara, are chosen.

(iv) Among the other types of vehicles, automobiles whose model years are 2004-2008 are selected.

(v) The brand of the vehicle is taken as insignificant on claim distribution.

(vi) Deductible limits are not considered as the information per policy.

(vii) Reinsurance limits are not available.

4.1 Descriptive Statistics

The aim of descriptive statistics is to point out the outstanding features of the analyzed data by using numerical and graphical representations. In this data, which includes 2008 policy year, total 29,017 vehicles are registered to insurance company by buying an automobile insurance policy. 10,255 vehicles of them belong to Istanbul and these vehicles cover approximately 35% of the data. Moreover, 3,614 vehicles of this data belong to Ankara, which is approximately equal to 12% of the data. This change between registered vehicles is directly proportional to the populations of the selected cities and higher population result with the high traffic claims penetration in selected cities. Therefore, drivers' gender, city, bonus categorization for discount rates and claim amounts are the main components of the data.

Claim amounts are the most important component for the insurance companies because the estimation of claim amounts is a major factor in pricing methodology for an automobile insurance product. Claim amounts over a one-year policy period of the two selected cities are summarized in the Table 4.1 primarily.

Table 4.1: Summary Statistics of Claim Amounts Based on City

	Istanbul	**Ankara**
k	2,387	697
μ	3,360.35 TL	2,322.15 TL
σ	9,162.84 TL	4,228.47 TL
Σ	8,021,164.59 TL	1,618,538.98 TL
$\min$	0.05 TL	0.19 TL
$\max$	189,758.52 TL	46,745.96 TL
Q1	495.04 TL	434.83 TL
Q2	1,137.00 TL	906.00 TL
Q3	2,657.32 TL	2,447.00 TL
m_3	10.45 TL	5.12 TL
m_4	160.62 TL	40.39 TL
CV	2.7268	1.8209

According to Table 4.1, Istanbul is exposed to more claims than Ankara because there is a huge difference between the numbers of claim, which is indicated by the letter "k" on the table. In other words; based on the claim frequency, the probability of having a car accident is more common in Istanbul than in Ankara.

In addition to claim frequency, the severity of each claim is more major in Istanbul than in Ankara because "μ" stands for the average claim amount and the average claim amount of Istanbul is approximately 1.5 times higher than the claim amounts of Ankara. However, outliers, which are the values of min and max, may have caused this large difference in claim severity for Istanbul because the maximum claim amount is 189,758.52 TL. Moreover, the variation of claim amount in Istanbul is much higher than in Ankara because the coefficient of variation for Istanbul is 1.5 times higher than the coefficient of variation for Ankara. Sum of claim amounts of Istanbul is also greater than the sum of claim amounts of Ankara so that simply, claims are more likely to take place in Istanbul with higher variability than in Ankara proportional to population of these cities.

Based on the skewness statistics (m_3), the shape of claim amounts of both cities are right skewed because the value of skewness statistics are positive, which is the characteristic of right skewed data. Positive kurtosis value (m_4) indicates that the claim amounts of both cities have heavier tails. In addition to skewness and kurtosis, the median (Q_2) of the claim amounts is smaller than the mean of the claim amounts shows that the shape of the claim amounts is right skewed for both selected cities.

In addition to the numerical representation of the claim amounts data, the following figures show the graphical representations of claim amounts based on the selected two cities. Figure 4.1 shows that the dispersion of claim amounts in both Istanbul and Ankara are right skewed because most of the claims take place in small amounts.

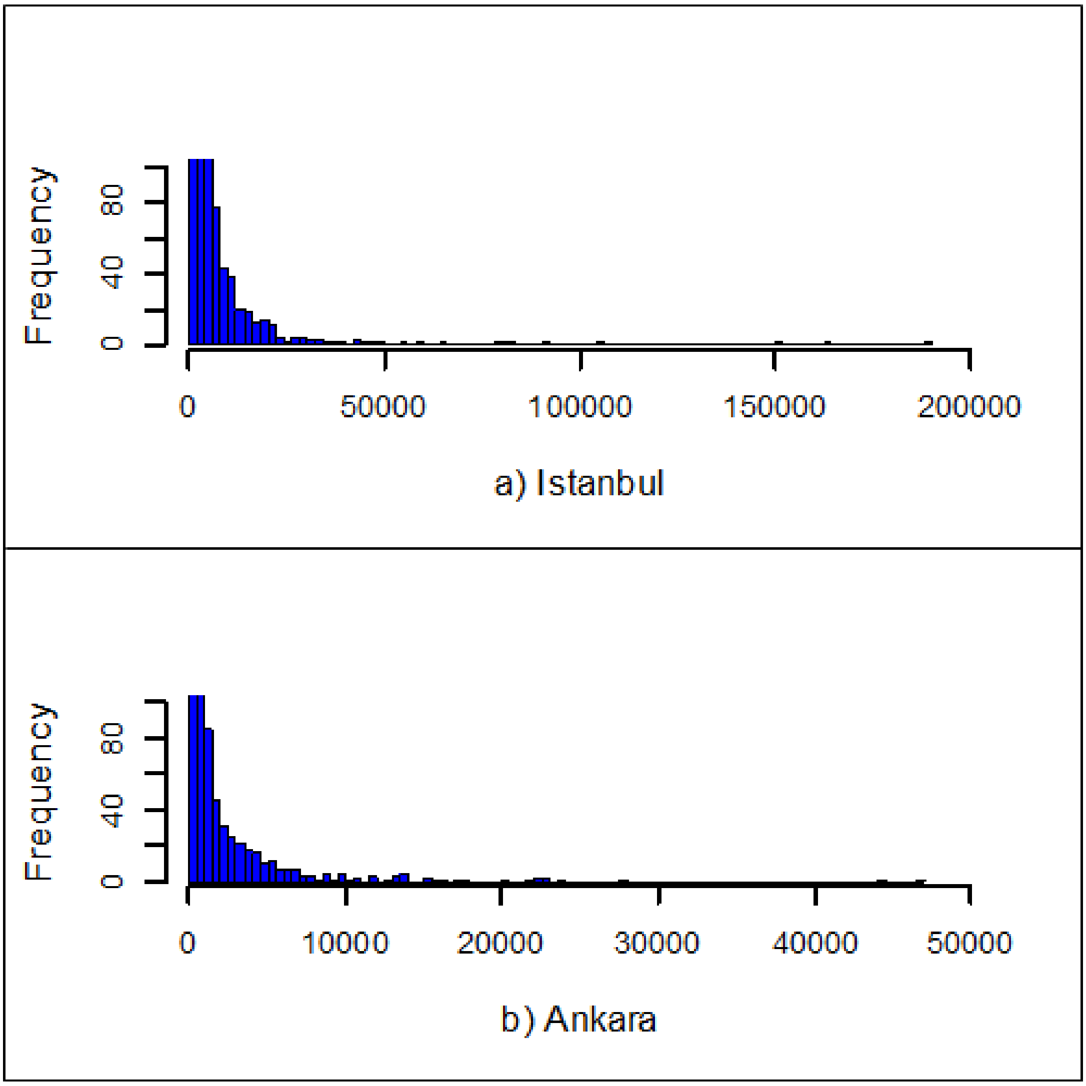

Figure 4.1: Histogram of Claim Amounts: a) Istanbul, b) Ankara

The costs for covering the accident, morbidity and mortality risks are different for gender in the pricing process of an automobile insurance [21] so that claim amounts are examined based on genders secondly. In Table 4.2; the descriptive statistics of claim amounts are shown with respect to gender and city.

According to gender classification in Table 4.2, female drivers are less likely to get involved a car accident than male drivers for both cities. Even though female drivers

Table 4.2: Summary Statistics of Claim Amounts Based on Gender and City

	Istanbul		Ankara	
	F	**M**	**F**	**M**
k	628	1,759	203	494
μ	3,580.22 TL	3,281.64 TL	2,855.06 TL	2,103.16 TL
σ	8,186.21 TL	9,488.14 TL	5,069.40 TL	3,814.13 TL
Σ	2,248,756.02 TL	5,772,408.57 TL	579,576.38 TL	1,038,962.60 TL
CV	2.2865	2.8913	1.7756	1.8135

have less claim numbers, the severity of these claims are higher than male drivers so that the average claim amounts are higher for female drivers in Istanbul and also in Ankara. The variation of claim severity in Istanbul, furthermore, is almost the same for female and male drivers, but female drivers have more volatile claim amounts than male drivers in Ankara. In brief, female drivers have fewer claims but those claims have larger amounts for both selected cities.

Figure 4.2 meanwhile, shows the distribution of claim amounts based on gender classification in both selected cities. From the histograms, we can say that the shape of claim amounts distribution is right skewed for each gender and city.

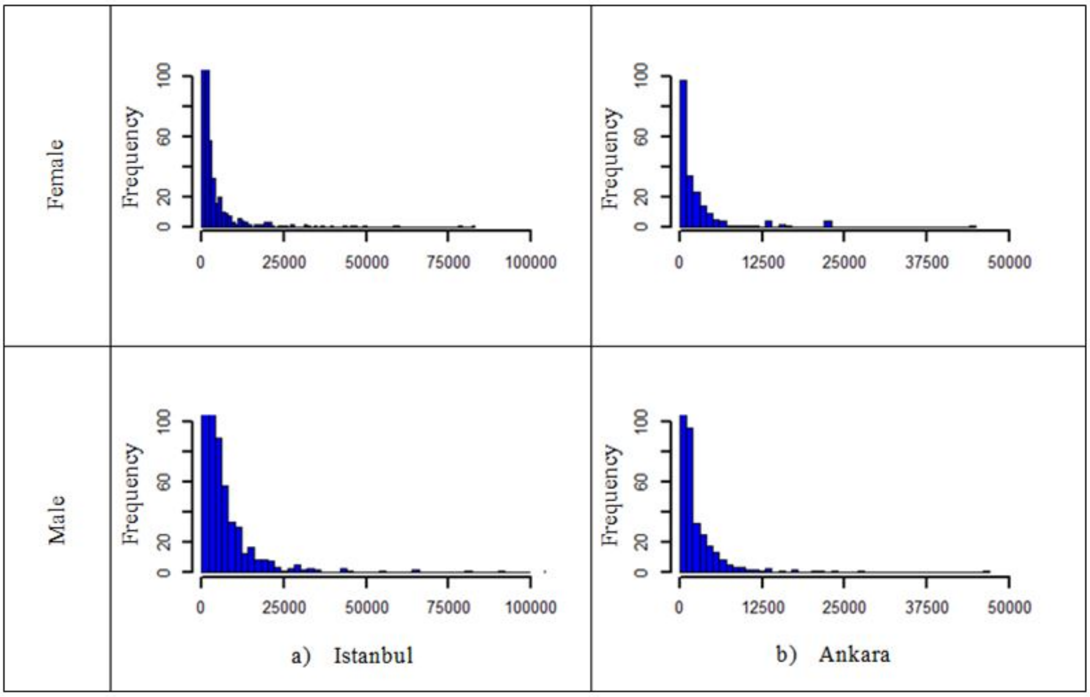

Figure 4.2: Histogram of Claim Amounts Based on Gender and Selected Cities

Thirdly, we examine the claim amounts depending on bonus categorization for discount rates and city. Bonus categorization is defined as a reduction in the premium with a determined rate, which is given on the renewal of policy with the condition that there have been no claims within the last policy year [7]. The coding of bonus categorization for DR is defined in the Table 4.3 and their descriptive statistics are represented in the Table 4.4 based on selected cities.

Table 4.3: Coding of Bonus Categorization for Discount Rates

Discount Rates in Premium (%)	**Coding**
0	0
30	1
40	2
50	3
55	4
60	5

According to Table 4.3, a discount rate of 0% means that the policyholder is new to the related insurance company or that the policyholder had an accident within the previous policy year. A 30% discount rate means that the policyholder has had no accident within the last policy year with the related insurance company. A 40% discount rate means that the policyholder has made no accident claim within the last two policy years with the related insurance company. A 50% discount rate means that the policyholder has had no accident within the last three policy years with the related insurance company. A discount rate of 55% means that the policyholder has made no accident claim within the last four policy years with the related insurance company and lastly, a 60% discount rate means that the policyholder has had no accident within the last five policy years with the related insurance company.

Table 4.4: Summary Statistics of Claim Amounts Based on Discount Rates and City

	Istanbul			**Ankara**		
DR (%)	**k**	μ (TL)	σ (TL)	**k**	μ (TL)	σ (TL)
0	1469	4,082.9	11,107.61	408	2,817.82	5,025.37
30	581	2,385.41	4,447.98	146	1,896.8	3,039.31
40	192	2,318.93	5,193.4	74	1,584.8	2,355.8
50	71	1,144.81	1,464.45	39	1,287	1,715.52
55	27	1,304.4	1,442.39	10	781.5	778.97
60	47	1,611.73	2,273.99	20	832.2	1,198.47

According to the definition of discount rates and Table 4.4, most of the claims are occur with new customers or policyholders who have had an accident in the previous year. Also, the average amount and severity of these claims varies greatly. Based on the discount rates, claim numbers, severity, average and dispersion of claims decrease year by year except for the 60% discount rate. The reason for this increase at the sixth year may depend on the behavior of drivers because among the defects causing accidents, driver defects has the biggest share based on the 2017 report of Turkish Statistical Institute.

Figures 4.3 and 4.4 show the distribution of claim amounts based on the discount rates for selected cities. In the figures, each discount rate has a right skewed shape of claim amount distribution.

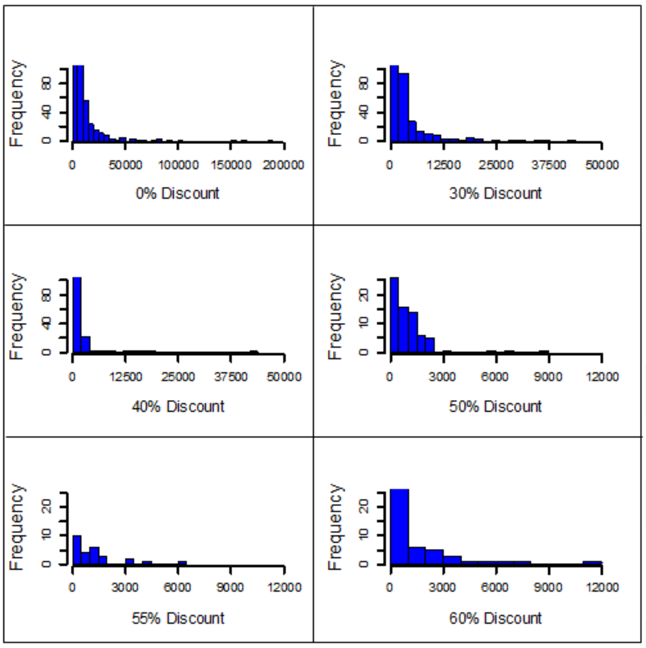

Figure 4.3: Histogram of Claim Amounts Based on Discount Rates for Istanbul

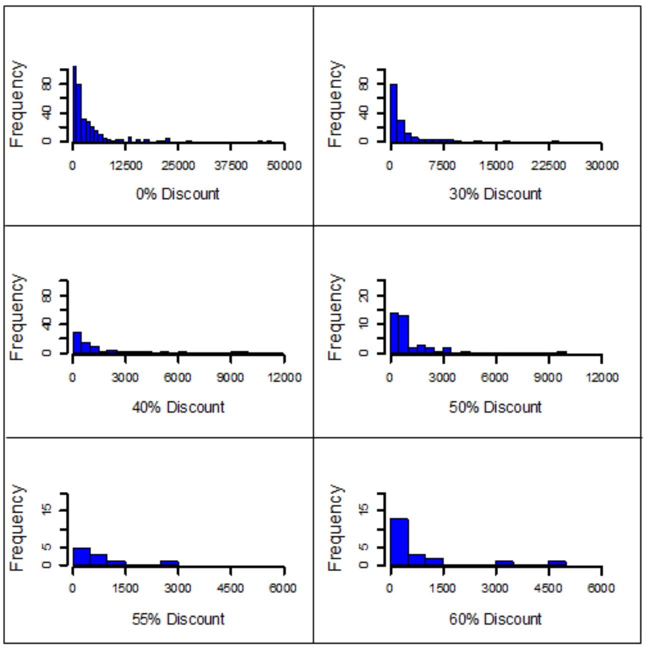

Figure 4.4: Histogram of Claim Amounts Based on Discount Rates for Ankara

In addition, we also examine the claim amounts of the discount rates by adding the gender level to see the effect of gender and discount rates on claim amounts.

Table 4.5: Summary Statistics for Gender and Discount Rates of Selected Cities

Istanbul	Female			Male		
DR (%)	**k**	μ (TL)	σ (TL)	**k**	μ (TL)	σ (TL)
0	382	4,255.55	9,810.84	1,087	4,022.2	11,532.35
30	155	2,862.57	4,918.09	426	2,211.8	4,257.03
40	59	2,417.15	4,269.95	133	2,275.36	5,568.6
50	18	896.6	546.11	53	1,229.1	1,661.6
55	7	1,345.76	1,034.83	20	1,289.88	1,583.65
60	7	1,609.35	2,082.34	40	1,612.14	2,330.68

Ankara	Female			Male		
DR (%)	**k**	μ (TL)	σ (TL)	**k**	μ (TL)	σ (TL)
0	118	3,744.1	6,228.63	290	2,440.92	4,400.77
30	48	1,718.96	2,712.51	98	1,983.98	3,196.88
40	20	1,763.2	1,652.29	54	1,518.73	2,578.55
50	12	1,223.25	1,358.45	27	1,315.51	1,875.6
55	2	574	377.6	8	833.38	862.79
60	3	1,390.67	1,744.23	17	733.69	1,120.97

Based on Table 4.5, both gender drivers are less likely to get involved a car accident as the discount rates get higher except the sixth policy year. This situation is same as the results of Table 4.4.

4.2 Homogeneity Test

In this section; firstly, we check whether there is an effect of gender on claim amounts so that we apply t-test on the claim amounts of gender with the assumption that the variances of gender are not equal. The distribution of claim amounts by gender can be found in Table 4.2.

According to Table 4.6, we conclude that the gender does not affect the claim amounts distribution by comparing the p-values of t test with the significance level (0.05).

In addition to the gender effect, we also analyze the effect of discount rates on claim amounts by applying t-test on the claim amounts of discount rates with the assumption

Table 4.6: Result of Two Sample t-test for Selected Cities Based on Gender

Two Sample t-test			
Istanbul	t = 0.75293 df = 1268.6 p value = 0.4516	Ankara	t=1.9034 df = 300.25 p value = 0.05794

of unequal variances. The distribution of claim amounts by discount rates can be found in Table 4.4.

Table 4.7: Results of Two Sample t-test for Selected Cities Based on Discount Rates

Two Sample t-test					
Istanbul	0 %	p value = 0.703	Ankara	0 %	p value = 0.0398
	30 %	p value = 0.1455		30 %	p value = 0.6026
	40 %	p value = 0.8476		40 %	p value = 0.6333
	50 %	p value = 0.2087		50 %	p value = 0.864
	55 %	p value = 0.9169		55 %	p value = 0.555
	60 %	p value = 0.9975		60 %	p value = 0.5857

Based on Table 4.7, by comparing the p-values with the significance level, we conclude that gender and discount rates do not affect the distribution of claim amounts except the claim amount of 0 % discount rate of Ankara but we will ignore this situation in the sense that the analyzes to be done are consistent.

Secondly, we check the homogeneity of data to see whether it is necessary to cluster the claim amounts for fitting the distribution or not. Therefore, we apply the Chi-Square test on the selected categorical variable of data with the assumption that the distribution of claim amounts of discount rates for female drivers are the same as the distribution of claim amounts of discount rates for male drivers for selected two cities.

Claim amounts data have two categorical variables, which are the gender and discount rates. According to them, Table 4.8 represents the number of claims based on gender and discount rates for both two cities. Summary statistics of the contingency table can be found in Table 4.5. Then, Figure 4.5 also shows the distribution of claim numbers with respect to gender and discount rates. Based on the Table 4.8 and Figure 4.5, most of the claims occur with a 0% discount rate, which represents the new policyholders or old policyholders, who have had an accident in the previous policy year. Moreover, male drivers produce claims more often than female drivers. Table 4.9 shows the result of the Chi-Square test. Based on the p values of the result of the Chi-Square test, discount rates and gender have the same distribution for both selected cities. In conclusion, the distribution of claim amounts for both cities is homogeneous, so that there is no need to cluster the data for the process of fitting distribution.

Table 4.8: Contingency Table for Gender and Discount Rates of Selected Cities

Number of Claims		Discount Rates %					
		0	**30**	**40**	**50**	**55**	**60**
Istanbul	**Female**	382	155	59	18	7	7
	Male	1087	426	133	53	20	40
Ankara	**Female**	118	48	20	12	2	3
	Male	290	98	54	27	8	17

Table 4.9: Result of Chi Square Test for Homogeneity for Selected Cities

Pearson's Chi Squared Test	
Istanbul	χ^2 = 5.2408
	df = 5
	p value = 0.3872
Ankara	χ^2 =3.5490
	df = 5
	p value = 0.616

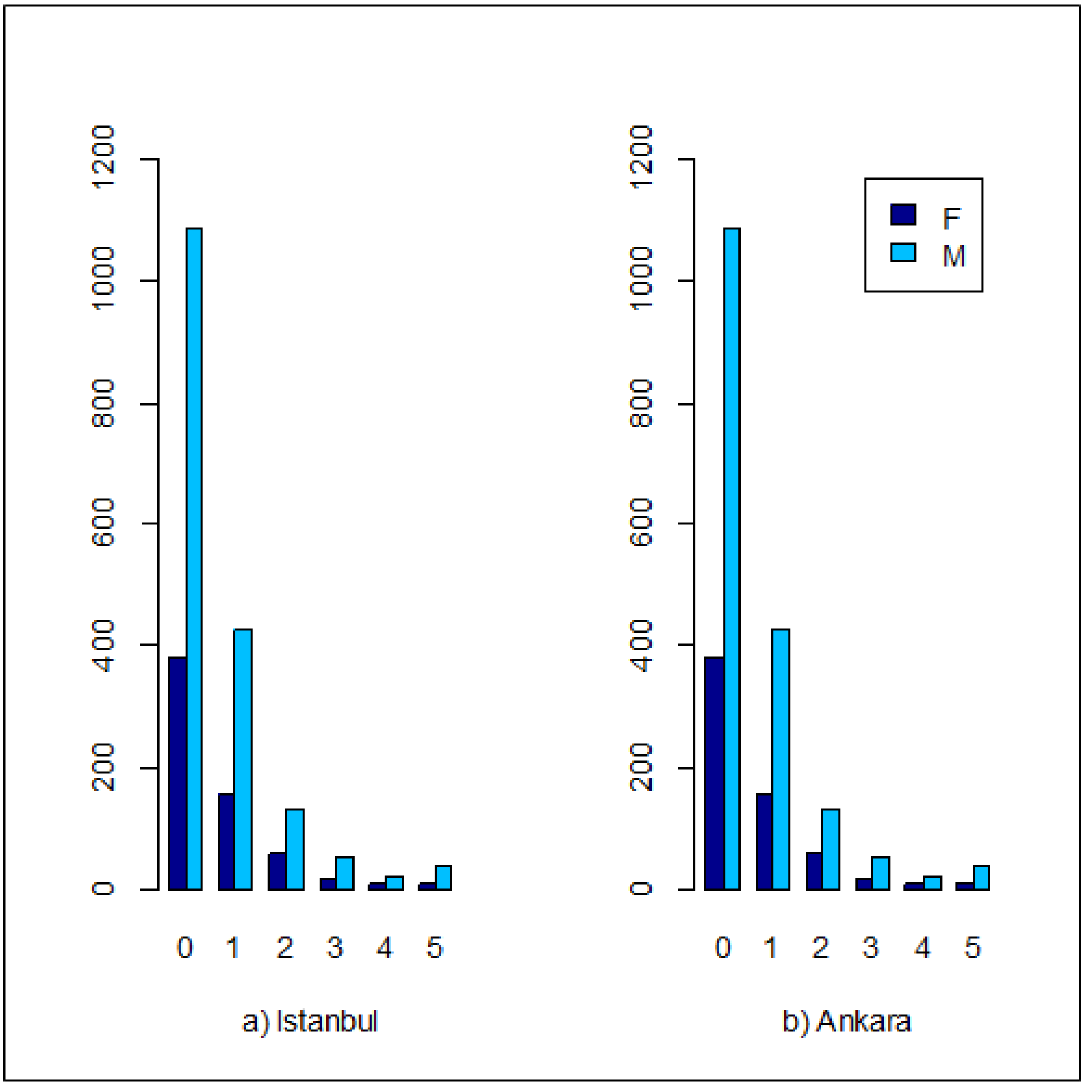

Figure 4.5: Claim Amounts Based on Gender and Discount Rates: a) Istanbul, b) Ankara

4.3 Parameter Estimation

In order to fit a distribution to claim data for an automobile insurance, the estimation of parameters for the selected distribution has to be done. We choose the Moments Estimation (MME) and Maximum Likelihood Estimator (MLE) methods for estimating the parameters due to the fact that MME and MLE always result in unbiased, consistent and efficient estimators [4]. Based on the homogeneity test results, the data, which are used for parameter estimation, consist only of claim amounts and city information because the gender and discount rates do not significantly affect the distribution of claim amounts of this data.

4.3.1 Method of Moments Estimation

By using the MME, estimated parameters are obtained for selecting the best fitted distribution of claim amounts. Table 4.10 gives the parameters of four selected distributions having been fit to the claims data based on the selected cities. In the Table, *Shape* is the first parameter value and *Scale* is the second parameter value of the distribution. In addition to the value of parameters, *Loglikelihood*, *AIC* and *BIC* show the estimation results of the fitted distribution for claims data. The highest Loglikelihood represents the best results for fitting distribution.

Table 4.10: Summary of MME Results for Selected Cities

Istanbul	Shape	Scale	Loglikelihood	AIC	BIC
Gamma	0.2160	0.0007	-21,970.47	43,944.95	43,956.48
Weibull	0.6081	2,521.0197	-21,141.21	42,286.41	42,297.95
Lognormal	**7.1874**	**0.0615**	-20,789.87	41,583.73	41,595.27
Pareto	2.0000	5.3209	-40,902.57	81,809.14	81,820.67
Ankara	**Shape**	**Scale**	**Loglikelihood**	**AIC**	**BIC**
Gamma	0.3060	0.0001	-6,162.823	12,329.65	12,338.72
Weibull	0.6840	2,074.2757	-6,006.285	12,016.57	12,025.64
Lognormal	**7.0348**	**0.0505**	-5,927.634	11,859.27	11,868.34
Pareto	2.000	4.4589	-11,835.18	23,674.37	23,683.44

According to Table 4.10, the highest Loglikelihood for Istanbul belongs to the Lognormal distribution amongst the sampled distributions so a Lognormal distribution gives the best result of parameter estimation for claims data of Istanbul. Therefore; claim amounts of Istanbul can be modelled by using a Lognormal distribution with a mean of 7.1874 and a standard deviation of 0.0615.

Furthermore, the Lognormal distribution also gives the best estimated result for claim amounts of Ankara due to having the highest Loglikelihood value. Hence; claim amounts of Ankara can be modelled by using a Lognormal distribution with a mean of 7.0348 and a standard deviation of 0.0505.

However, values of Loglikelihood are not sufficient evidence to show that it is the right distribution for the claim amounts for selected cities. Therefore; we have to make an assessment to how well these distributions fit the claim amounts by applying the Goodness of Fit Test having a null hypothesis which states that the claim amounts of selected cities come from the specified distribution.

The following Tables 4.11 and 4.12 show the results of a Goodness of Fit Test for selected cities. The graphical representations of Goodness of Fit Test are presented in Appendix B.

According to Table 4.11, we see that the p-value of the Kolmogorov-Smirnov test is

Table 4.11: Summary Statistics of Goodness of Fit Test for Selected Cities

Istanbul	Gamma	Weibull	Lognormal	Pareto
Kolmogorov-Smirnov Statistic	0.3703	0.1297	0.0633	0.0984
p value	0.3006	0.1053	0.0514	0.0799
Ankara	**Gamma**	**Weibull**	**Lognormal**	**Pareto**
Kolmogorov-Smirnov Statistic	0.2887	0.1085	0.0780	0.9865
p value	0.2344	0.0881	0.0633	0.8010

Table 4.12: Summary Results of Goodness of Fit Test for Selected Cities

Istanbul	Gamma	Weibull	Lognormal	Pareto
AIC	43,944.95	42,286.41	41,583.73	81,809.14
BIC	43,956.48	42,297.95	41,595.27	81,820.67
Ankara	**Gamma**	**Weibull**	**Lognormal**	**Pareto**
AIC	12,329.65	12,016.57	11,859.27	23,674.37
BIC	12,338.72	12,025.64	11,868.34	23,683.44

greater than the significance level, which is 0.05, so we fail to reject the null hypothesis and the AIC values from Table 4.12 show that the Lognormal distribution has the smallest value of AIC against other distribution value of AIC. Therefore; we conclude that our claim amounts of Istanbul come from Lognormal distribution with 95% level of confidence and this means that the future claim amounts of Istanbul could be estimated by using Lognormal distribution in order to use as an claim assumption of future premiums in the pricing methodology of automobile insurance.

In addition to Istanbul, we see that the p-value of the Kolmogorov-Smirnov test for Ankara is greater than the confidence level, which is 0.05, so we fail to reject the null hypothesis and the AIC values of Ankara from Table 4.12 show that the Lognormal distribution has the smallest AIC value. Therefore; we conclude that the Lognormal distribution would be the best statistical distribution to the model the claim amounts of Ankara at a 95% level of confidence and this means that the future claim amounts of Ankara could be estimated by using Lognormal distribution to determine the future premiums of automobile insurance.

4.3.2 Method of Maximum Likelihood Estimation

By using the MLE method, estimated parameters are obtained for selecting the best distribution to be fitted with the help of R programming. Table 4.13 gives the parameters of four selected distributions having been fit to the claims data based on the selected cities. In the Table, *Shape* is the first parameter value and *Scale* is the second parameter value of the distribution. In addition to the value of parameters, *Loglikelihood*, *AIC* and *BIC* show the estimation results of the fitted distribution for claims data. Selection of the best fitted distribution is the main goal, hence the highest Loglikelihood represents the best results for fitting distribution.

Table 4.13: Summary of MLE Results for Selected Cities

Istanbul	Shape	Scale	Loglikelihood	AIC	BIC
Gamma	0.6477	0.0002	-21,203.36	42,410.73	42,422.26
Weibull	0.7238	2,392.3097	-21,064.15	42,132.31	42,143.84
Lognormal	**7.1082**	**0.0314**	-20,785.58	41,575.16	41,586.70
Pareto	1.7936	2,581.0196	-20,853.59	41,711.18	41,722.71
Ankara	**Shape**	**Scale**	**Loglikelihood**	**AIC**	**BIC**
Gamma	0.7291	0.0003	-6,018.89	12,041.79	12,050.86
Weibull	0.7767	1,949.1527	-5,993.05	11,990.10	11,999.18
Lognormal	**6.9361**	**0.0355**	-5,923.93	11,851.85	11,860.93
Pareto	2.1294	2,716.4880	-5,948.42	11,900.85	11,909.92

According to Table 4.13, the highest Loglikelihood value for Istanbul belongs to the Lognormal distribution amongst the sampled distributions so a Lognormal distribution gives the best result for claims data of Istanbul. Therefore; claim amounts of Istanbul can be modelled by using a Lognormal distribution with a mean of 7.1082 and a standard deviation of 0.0314.

Furthermore, the Lognormal distribution also gives the best estimated result for claim amounts of Ankara due to having the highest Loglikelihood value. Hence; claim amounts of Ankara can be modelled by using a Lognormal distribution with a mean of 6.9361 and a standard deviation of 0.0355.

As stated previously, high values of Loglikelihood are not sufficient evidence to prove that the correct distribution has been chosen. We must assess how well these distributions fit the claims data using the Goodness of Fit Test.

Tables 4.14 and 4.15 below show the results of Goodness of Fit Test for selected cities. The graphical representations of the Goodness of Fit Test are presented in Appendix B.

In Table 4.14, we see that the p-value of Kolmogorov-Smirnov test is greater than the significance level of 0.05, so we cannot reject the null hypothesis. The AIC values from Table 4.15 show that the Lognormal distribution has the smallest value amongst sampled distribution. Therefore; we state that the claim amounts for Istanbul come from a Lognormal distribution with a 95% level of confidence. Thus, future claim amounts of Istanbul can be estimated by using Lognormal distribution when modelling

Table 4.14: Summary Statistics of Goodness of Fit Tests for Selected Cities

Istanbul	Gamma	Weibull	Lognormal	Pareto
Kolmogorov-Smirnov Statistic	0.1495	0.1020	0.051	0.0618
p value	0.1469	0.1002	0.0501	0.0607
Ankara	**Gamma**	**Weibull**	**Lognormal**	**Pareto**
Kolmogorov-Smirnov Statistic	0.1320	0.0938	0.0521	0.0665
p value	0.1297	0.0921	0.0512	0.0653

Table 4.15: Summary Results of Goodness of Fit Tests for Selected Cities

Istanbul	Gamma	Weibull	Lognormal	Pareto
AIC	42,410.73	42,132.31	41,575.16	41,711.18
BIC	42,422.26	42,143.84	41,586.70	41,722.71
Ankara	**Gamma**	**Weibull**	**Lognormal**	**Pareto**
AIC	12,041.79	11,990.10	11,851.85	11,900.85
BIC	12,050.86	11,999.18	11,860.93	11,909.92

future premiums of an automobile insurance.

The p-value Kolmogorov-Smirnov test for Ankara is also greater than 0.05, so again we fail to reject the null hypothesis. The AIC values of Ankara from Table 4.15 show that the Lognormal distribution has the smallest value. Therefore; we conclude that the Lognormal distribution would be the best statistical distribution to the model the claim amounts of Ankara with a 95% level of confidence. Future claim amounts of Ankara can therefore be estimated by using Lognormal distribution when pricing an automobile insurance.

4.4 Risk Measures Calculations

After estimating the sample parameters of the best fitted distribution, we can now compare the risk premiums based on these estimated parameters, which are found by using MME and MLE.

According to estimated parameters of claim amount distributions for selected cities, Table 4.16 shows the results of risk measures based on the estimation methods. All mathematical calculations of risk measures can be found in Appendix C.

According to risk measures results of both MME and MLE for selected cities, NPP

Table 4.16: Risk Measures Results For Selected Cities

Istanbul	NPP	EVPP	VPP	SDPP	EXPP	ESPP	VaR	CVaR
MME	1,325.16	1,590.20	2,656.05	1,341.48	1,990.61	2,656.05	1,463.47	1,510.69
MLE	1,222.55	1,467.06	1,517.42	1,230.23	1,369.99	1,517.42	1,286.72	1,308.13
Ankara	NPP	EVPP	VPP	SDPP	EXPP	ESPP	VaR	CVaR
MME	1,136.92	1,364.30	1,797.04	1,148.41	1,466.98	1,797.04	1,233.82	1,271.07
MLE	1,029.40	1,235.28	1,296.66	1,036.71	1,163.03	1,296.66	1,090.62	1,105.54

has the smallest premium charge for an automobile insurance. However, as it is known from ruin theory, using NPP is insufficient for an insurance company to be solvent [24] so the result of NPP is not the right premium charge for transferring the risk of policyholders and it is also not appropriate for companys' reserving purposes.

EVPP is safer side NPP due to the usage of a claim experience rate as a safety loading in the pricing methodology. However, both NPP and EVPP are not sensitive to the change in the amounts of claim thus most insurance companies do not prefer to use NPP nor EVPP as a pricing methodology of an automobile insurance.

In contrast to NPP and EVPP, VPP and ESPP are sensitive to the change of claim amounts yielding the highest premium for an automobile insurance. Higher premiums, which are calculated based on VPP and ESPP, show that there are more risky losses to be protected by customers but trying to sell automobile insurance with such high premiums in a competitive environment puts the sustainability of the insurance company at risk. SDPP is also sensitive the fluctuations of claim amounts but premiums are relatively lower than VPP and ESPP premiums.

EXPP has the one of the largest premium charges for an automobile insurance compared to the other premium principles. Actually this premium principle is sensitive to the risk aversion level of the insureds so while increasing the level of risk aversion of the insured, the premium is quate large. Also, it is not practical to apply in insurance pricing methodology because it requires so many assumption to perform the mathematical calculations.

Beside the premium based risk measures, VaR and CVaR produce a more competitive premium for the automobile. In addition to the lowest premiums, they are sensitive to the fluctuations of large claim amounts.

All these risk measures generate a premium which is higher than the expected loss. It is good for the company's sustainability but the coherency property is required for risk measure calculations thus insurance companies prefer CVaR as a pricing methodology of an automobile insurance.

In this study, we conclude that CVaR gives the most affordable premium, which is calculated by using the estimators of MLE for Lognormal distribution, because CVaR is sensitive to the extreme values on the tail of Lognormal distributions and also it satisfies the coherency conditions which makes a risk measure a candidate for an appropriate and optimal risk management tool for the insurance company. Also, MLE gives more unbiased, consistent and efficient estimators than MME [4]. Hence, the comparison of studied risk measures show that applying CVaR for both selected cities keep automobile insurance companies on the safer side. In other words, insurance companies can stay solvent using the this risk measure in the their pricing methodology of automobile insurance.

CHAPTER 5

CONCLUSION

Since people want to feel safe while they live, one of humankind's main purposes is to achieve a sense of security. They can achieve this emotion by buying an insurance product from the various insurance companies providing proper prices so that both the insureds and the insurers manage their own risk.

Companies determine the suitable premiums for their customers by using risk measure methods. The definition of risk measure method is a single number to summarize the future random losses.

In this thesis, the selection of appropriate risk measures is analyzed using the claim amounts of an automobile insurance, which are obtained from one of the most reputable general insurance company to determine efficient and competitive premiums in the insurance market by using claim amount distributions.

This thesis shows that, in automobile insurance, the estimation of claim amount distribution is independent from the categorical variables, which are the gender of the driver and the no-claims discount. Homogeneity tests show that the claim amount distribution is not dependent on these categorical variables.

The claims are separated based on the selected cities. After the separation of claims, the estimation of parameters of the claim distributions is performed by applying the Methods of Moments Estimation and Maximum Likelihood Estimation Method with the help of R programming. For both cities, the Lognormal distribution is fitted as the distribution of claim amounts with two estimation methods. Furthermore, the Goodness of Fit test is applied to test whether the estimators of selected cities fit the claim distributions adequately or not. According to the Goodness of Fit test results, the Lognormal distribution is well suited to the claims data.

Then, selected risk measures are calculated with respect to the Lognormal distribution by using estimated parameters of MME and MLE. The comparison of studied risk measures show that applying Conditional Value at Risk for both selected cities by using two estimation methods give affordable insurance premiums for both the insurer and insured because CVaR is sensitive to the extreme values of claim amounts and also CVaR satisfies the coherency criteria, which makes a risk measure a candidate for the risk management tool of the insurance company. Furthermore, insureds feel more secure with the help of well-priced insurance products and the insurers can continue to remain solvent with this well-priced products.

In future studies, in addition to the classical parameter estimation methods, Bayesian parameter estimation methods or parametric interval estimation methods may be considered for estimating the parameter of loss distributions and also, other variables affecting the claim amounts (including brand of the automobile, age of the driver, etc) could be added for the calculation of well-priced premiums in the competitive insurance market.

REFERENCES

[1] O. M. Achieng and I. No, Actuarial modeling for insurance claim severity in motor comprehensive policy using industrial statistical distributions, in *International Congress of Actuaries, Cape Town*, volume 712, 2010.

[2] P. Artzner, F. Delbaen, J.-M. Eber, and D. Heath, Coherent measures of risk, Mathematical Finance, 9(3), pp. 203–228, 1999.

[3] D. Bahnemann, Distributions for actuaries, CAS Monograph Series, (2), 2015.

[4] L. J. Bain and M. Engelhardt, *Introduction to Probability and Mathematical statistics*, Brooks/Cole, 1987.

[5] V. Bignozzi, *Contributions to Solvency Risk Measurement*, Ph.D. thesis, City University London, 2012.

[6] H. Bühlmann, *Mathematical Methods in Risk Theory*, Springer, 1970.

[7] M. Denuit, X. Marechal, S. Pitrebois, and J.-F. Walhin, *Actuarial Modelling of Claim Counts: Risk Classification, Credibility and Bonus-Malus Systems*, John Wiley & Sons, 2007.

[8] R. J. Gray and S. M. Pitts, *Risk Modelling in General Insurance: From Principles to Practice*, Cambridge University Press, 2012.

[9] R. V. Hogg and S. A. Klugman, *Loss Distributions*, volume 249, John Wiley & Sons, 2009.

[10] K. Inui and M. Kijima, On the significance of expected shortfall as a coherent risk measure, Journal of Banking & Finance, 29(4), pp. 853–864, 2005.

[11] B. Jørgensen and M. C. Paes De Souza, Fitting tweedie's compound poisson model to insurance claims data, Scandinavian Actuarial Journal, 1994(1), pp. 69–93, 1994.

[12] P. Jorion, *Value at Risk*, McGraw-Hill, New York, 1997.

[13] R. Kaas, M. Goovaerts, J. Dhaene, and M. Denuit, *Modern Actuarial Risk Theory: Using R*, volume 128, Springer Science & Business Media, 2008.

[14] Z. Landsman and M. Sherris, Risk measures and insurance premium principles, Insurance: Mathematics and Economics, 29(1), pp. 103–115, 2001.

[15] P. Liebwein, Risk models for capital adequacy: Applications in the context of solvency ii and beyond, The Geneva Papers on Risk and Insurance-Issues and Practice, 31(3), pp. 528–550, 2006.

[16] B. W. Mazviona and T. Chiduza, The use of statistical distributions to model claims in motor insurance, Journal of Business, Economics and Law, 3, pp. 44–57, 2013.

[17] A. J. McNeil, R. Frey, and P. Embrechts, *Quantitative Risk Management: Concepts, Techniques and Tools*, Princeton University Press, 2015.

[18] T. Mikosch, *Non-life Insurance Mathematics: An Introduction With The Poisson Process*, Springer Science & Business Media, 2009.

[19] I. J. Myung, Tutorial on maximum likelihood estimation, Journal of Mathematical Psychology, 47(1), pp. 90–100, 2003.

[20] M. A. Nielsen, *Parameter Estimation for the two-parameter Weibull Distribution*, Master's thesis, Brigham Young University-Provo, 2011.

[21] Oxera, The use of gender in insurance pricing, Association of British Insurers Research Paper, (24), 2010.

[22] V. Packová and D. Brebera, Loss distributions in insurance risk management, Business Administration, p. 17, 2015.

[23] R. T. Rockafellar and S. Uryasev, Optimization of conditional value-at-risk, Journal of Risk, 2, pp. 21–42, 2000.

[24] E. Straub and S. A. of Actuaries (Zürich), *Non-life insurance mathematics*, 517/S91n, Springer, 1988.

[25] Y.-K. Tse, *Nonlife Actuarial Models: Theory, Methods and Evaluation*, Cambridge University Press, 2009.

[26] M. V. Wüthrich and M. Merz, *Stochastic Claims Reserving Methods in Insurance*, volume 435, John Wiley & Sons, 2008.

[27] V. R. Young, Premium principles, Encyclopedia of Actuarial Science, 2004.

[28] Y. Ziai, *Statistical Models of Claim Amount Distributions in General Insurance*, Ph.D. thesis, City University London, 1970.

APPENDIX A

Derivations of Mean, Variance, MGF, MME and MLE of Selected Distributions

A.1 Gamma Distribution

The mean of Gamma distribution is obtained as follows;

$$\begin{aligned}
E[X] &= \int_0^\infty x \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-\frac{x}{\beta}} dx \\
&= \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^\infty x^{(\alpha+1)-1} e^{-\frac{x}{\beta}} dx \\
&= \frac{\beta^{1+\alpha}\Gamma(1+\alpha)}{\beta^\alpha \Gamma(\alpha)} \int_0^\infty \frac{1}{\beta^{1+\alpha}\Gamma(1+\alpha)} x^{(1+\alpha)-1} e^{-\frac{x}{\beta}} dx \\
&= \frac{\beta^{1+\alpha}\Gamma(1+\alpha)}{\beta^\alpha \Gamma(\alpha)} \\
&= \frac{\beta\alpha\Gamma(\alpha)}{\Gamma(\alpha)} \\
&= \alpha\beta
\end{aligned}$$

In order to find variance of Gamma distribution, $E[X^2]$ has to be found firstly.

$$\begin{aligned}
E[X^2] &= \int_0^\infty x^2 \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-\frac{x}{\beta}} dx \\
&= \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^\infty x^{(2+\alpha)-1} e^{-\frac{x}{\beta}} dx \\
&= \frac{\beta^{2+\alpha}\Gamma(2+\alpha)}{\beta^\alpha \Gamma(\alpha)} \int_0^\infty \frac{1}{\beta^{2+\alpha}\Gamma(2+\alpha)} x^{(2+\alpha)-1} e^{-\frac{x}{\beta}} dx \\
&= \frac{\beta^{2+\alpha}\Gamma(2+\alpha)}{\beta^\alpha \Gamma(\alpha)} \\
&= \beta^2 \frac{(\alpha+1)!}{(\alpha-1)!} \\
&= \beta^2 \frac{(\alpha+1)(\alpha)(\alpha-1)!}{(\alpha-1)!} \\
&= \beta^2(\alpha+1)(\alpha)
\end{aligned}$$

Thus, the variance of Gamma distribution is described as follows;

$$\begin{aligned}
Var[X] &= E[X^2] - E[X]^2 \\
&= \beta^2(\alpha+1)(\alpha) - (\alpha\beta)^2 \\
&= \alpha^2\beta^2 + \alpha\beta^2 - \alpha^2\beta^2 \\
&= \alpha\beta^2
\end{aligned}$$

Moment generating function of Gamma distribution is obtained as follows;

$$\begin{aligned}
M(t) &= E[e^{tx}] \\
&= \int_0^\infty e^{tx} \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-\frac{x}{\beta}} dx \\
&= \frac{1}{\Gamma(\alpha)\beta^\alpha} \int_0^\infty x^{\alpha-1} e^{-x(\frac{1}{\beta}-t)} dx
\end{aligned}$$

By applying the transformation $\frac{1}{\beta} - t = \frac{1}{u}$ and then, we obtain the probability distribution of Gamma (α, u) so that the inside of the integral is equal to 1.

$$
\begin{aligned}
&= \frac{1}{\Gamma(\alpha)\beta^{\alpha}} \int_0^{\infty} x^{\alpha-1} e^{-\frac{x}{u}} \\
&= \frac{1}{\Gamma(\alpha)\beta^{\alpha}} \Gamma(\alpha) u^{\alpha} \\
&= (\frac{u}{\beta})^{\alpha}
\end{aligned}
$$

Now, $\frac{u}{\beta} = \frac{\beta}{1-\beta t} - \frac{1}{\beta} = \frac{1}{1-\beta t}$, then we can write the moment generating function of Gamma distribution as follows;

$$
\begin{aligned}
M(t) &= (\frac{u}{\beta})^{\alpha} \\
&= (\frac{1}{1-\beta t})^{\alpha} \\
&= (1-\beta t)^{-\alpha}
\end{aligned}
$$

Method of Moment Estimator of Gamma distribution is described as follows; but firstly, we need to definition of the mean and the variance of Gamma distribution based on the sample moments.

$$E[X] = \alpha\beta = \frac{1}{n}\sum_{i=1}^{n} x_i \tag{A.1}$$

$$Var[X] = \alpha\beta^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2 \tag{A.2}$$

By solving equation A.1, we obtain the parameter α as;

$$\alpha = \frac{\bar{x}}{\beta} \tag{A.3}$$

Now, substituting A.3 into the equation A.2, we obtain the parameter β as;

$$
\begin{aligned}
\alpha\beta^2 &= (\frac{\bar{x}}{\beta})\beta^2 && \text{(A.4)} \\
&= \bar{x}\beta && \text{(A.5)} \\
&= \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2 && \text{(A.6)}
\end{aligned}
$$

From equation A.6, we thus obtain $\hat{\beta}_{MM}$ as $\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2$. Now, we can find the estimator of α by substituting the $\hat{\beta}_{MM}$ into the equation A.1.

$$\hat{\alpha}_{MM} = (\frac{\bar{x}}{\hat{\beta}_{MM}}) \tag{A.7}$$

$$= \frac{\bar{x}}{(\frac{1}{n\bar{x}}\sum_{i=1}^{n}(x_i - \bar{x})^2)} \tag{A.8}$$

$$= \frac{n\bar{x}^2}{\sum_{i=1}^{n}(x_i - \bar{x})^2)} \tag{A.9}$$

Maximum Likelihood Estimator of Gamma distribution is described as;

$$\begin{aligned} L(\alpha, \beta|X) &= \prod_{i=1}^{n} f(X_i) \\ &= \prod_{i=1}^{n} \frac{1}{\beta^{\alpha}\Gamma(\alpha)} x_i^{\alpha-1} e^{-\frac{x_i}{\beta}} \end{aligned}$$

Now, the natural logarithm of likelihood function has to be taken:

$$\begin{aligned} ln(L(X)) &= (\alpha - 1)\sum_{i=1}^{n} ln(X_i) - nln(\Gamma(\alpha)) - n\alpha ln(\beta) - \frac{1}{\beta}\sum_{i=1}^{n} X_i \\ &= n(\alpha - 1)ln(\bar{X}) - nln(\Gamma(\alpha)) - n\alpha ln(\beta) - \frac{n\bar{X}}{\beta} \end{aligned}$$

Now, the derivatives of likelihood function has to be taken with respect to parameters.

$$\begin{aligned} \frac{dln(L(X))}{\partial\alpha} &= nln(\bar{X}) - \frac{n\Gamma'(\alpha)}{\Gamma(\alpha)} - nln(\beta) \\ \hat{\alpha} &= n(ln(\beta) - \frac{\Gamma'(\alpha)}{\Gamma(\alpha)}) + nln(\bar{X}) \\ \frac{dln(L(X))}{\partial\beta} &= -\frac{n\alpha}{\beta} + \frac{n\bar{X}}{\beta^2} \\ \hat{\beta} &= \frac{\bar{X}}{\hat{\alpha}} \end{aligned}$$

Maximum likelihood values of α and β must satisfy the two equations of $\hat{\alpha}$ and $\hat{\beta}$ because the equation of $\hat{\alpha}$ cannot be solved in closed form so that an iterative method for finding the roots has to be done.

A.2 Weibull Distribution

The k^{th} moment of Weibull distribution is obtained as follows;

$$\begin{aligned} E[X^k] &= \int_0^\infty x^k f(x)dx \\ &= \int_0^\infty x^k(\frac{\beta}{\alpha})(\frac{x}{\alpha})^{\beta-1}e^{-(\frac{x}{\alpha})^\beta}dx \\ &= \int_0^\infty (\frac{\beta}{\alpha})(\frac{1}{\alpha})^{\beta-1}x^{k+\beta-1}e^{-(\frac{x}{\alpha})^\beta}dx \end{aligned}$$

By applying the transformation $(\frac{x}{\alpha})^\beta = t$ and $(\frac{\beta}{\alpha})(\frac{x}{\alpha})^{\beta-1}dx = dt$

$$\begin{aligned} &= \int_0^\infty (\frac{\beta}{\alpha})(\frac{1}{\alpha})^{\beta-1}(\alpha t^{\frac{1}{\beta}})^{k+\beta-1}e^{-t}(\frac{\alpha}{\beta})t^{(\frac{1}{\beta}-1)}dt \\ &= \int_0^\infty \frac{\alpha^{k+\beta-1}}{\alpha^{\beta-1}}t^{\frac{k}{\beta}+1-\frac{1}{\beta}}t^{\frac{1}{\beta}-1}e^{-t}dt \\ &= \int_0^\infty \alpha^k t^{(\frac{k}{\beta}+1)-1}e^{-t}dt \\ &= \alpha^k\int_0^\infty t^{(\frac{k}{\beta}+1)-1}e^{-t}dt \end{aligned}$$

Inside of the integral is equal to the Gamma function with parameter $(\frac{k}{\beta}+1)$, so that k^{th} moment of the Weibull distribution is written as;

$$E[X^k] = \alpha^k\Gamma(\frac{k}{\beta}+1)$$

The mean of Weibull distribution is equal to 1^{st} moment and it is written as;

$$E[X] = \alpha\Gamma(\frac{1}{\beta}+1)$$

In order to find variance of Weibull distribution, $E[X^2]$ has to be found firstly. It is equal the 2^{nd} moment of the Weibull distribution and it is written as;

$$E[X^2] = \alpha^2\Gamma(\frac{2}{\beta}+1)$$

Hence, the variance of Weibull distribution is described as;

$$
\begin{aligned}
Var[X] &= E[X^2] - E[X]^2 \\
&= \alpha^2\Gamma(\frac{2}{\beta}+1) - (\alpha\Gamma(\frac{1}{\beta}+1))^2 \\
&= \alpha^2[\Gamma(\frac{2}{\beta}+1) - \Gamma^2(\frac{1}{\beta}+1)]
\end{aligned}
$$

The moment generating function of Weibull distribution is described as;

$$
\begin{aligned}
M(t) &= E[e^{tx}] \\
&= \int_0^\infty e^{tx}\frac{\beta}{\alpha}(\frac{x}{\alpha})^{\beta-1}e^{-(\frac{x}{\alpha})^\beta} \\
&= \int_0^\infty \frac{\beta}{\alpha}\frac{1}{\alpha^{\beta-1}}x^{\beta-1}e^{tx-(\frac{x}{\alpha})^\beta}
\end{aligned}
$$

Weibull distribution does not have a simple and closed form of moment generating function, so that we left it in the form of definition of moment generating function.

Method of Moment Estimator of Weibull distribution is described as follows;

$$
\begin{aligned}
m_1 &= E[X] = \alpha\Gamma(\frac{1}{\beta}+1) \\
m_2 &= E[X^2] = \alpha^2\Gamma(\frac{2}{\beta}+1)
\end{aligned}
$$

From the above equations, we thus obtain $\hat{\alpha}_{MM}$ as $\frac{\frac{1}{n}\sum_{i=1}^n x_i}{\Gamma(\frac{1}{\beta}+1)}$. Now, we can find the estimator of shape parameter of Weibull distribution.

By taking the second sample moment divided by the square of the first sample moment, a function of the theoretical moments is

$$
\frac{m_2}{m_1^2} = \frac{\alpha^2\Gamma(\frac{2}{\beta}+1)}{\alpha^2\Gamma^2(\frac{1}{\beta}+1)}
$$

By using the root finding techniques, we can solve the equation. In this study, I used the technique of Nielsen (2011) for obtaining the moment estimator of shape parameter [20].

Maximum Likelihood Estimator of Weibull distribution is described as;

$$
\begin{aligned}
L(\alpha, \beta|X) &= \prod_{i=1}^{n} f(X_i) \\
&= \prod_{i=1}^{n} \frac{\beta}{\alpha}(\frac{X_i}{\alpha})^{\beta-1} e^{-(\frac{X_i}{\alpha})^{\beta}} \\
&= \frac{\beta^n}{\alpha^{n\beta}} e^{-\sum_{i=1}^{n}(\frac{X_i}{\alpha})^{\beta}} \prod_{i=1}^{n} X_i^{\beta-1}
\end{aligned}
$$

Firstly, the natural logarithm of likelihood function has to be taken:

$$
ln(L(X)) = nln(\beta) - n\beta ln(\alpha) - \sum_{i=1}^{n}(\frac{X_i}{\alpha})^{\beta} + (\beta - 1)\sum_{i=1}^{n} ln(X_i)
$$

Secondly, the first derivative of likelihood function has to be taken with respect to parameters.

$$
\begin{aligned}
\frac{dln(L(X))}{\partial\alpha} &= \frac{n\beta}{\alpha^{\beta+1}}[\frac{1}{n}\sum_{i=1}^{n} x_i^{\beta} - \alpha^{\beta}] \\
\hat{\alpha} &= (\frac{1}{n}\sum_{i=1}^{n} x_i^{\beta})^{\frac{1}{\beta}} \\
\frac{dln(L(X))}{\partial\beta} &= n(\frac{1}{n}\sum_{i=1}^{n} ln(x_i) + \frac{1}{\beta} - \frac{\sum_{i=1}^{n}(x_i^{\beta} ln(x_i))}{\sum_{i=1}^{n}(x_i^{\beta})}) \\
\hat{\beta} &= \text{No simple and closed form}
\end{aligned}
$$

Once the likelihood function has been obtained, the likelihood function is optimized to find its minimum value. This has to be done iteratively by using an optimization function, such as Newton-Raphson method because Weibull distribution does not have a simple and closed form of maximum likelihood estimator for shape parameter.

To verify these solutions are the maximum likelihood estimators of Weibull distribution, second derivative has to be taken to maximize the loglikelihood function.

$$
\begin{aligned}
\frac{d^2ln(L(X))}{\partial\alpha^2} &= \frac{\beta}{\alpha^2}[n - \frac{\sum_{i=1}^{n} x_i^{\beta}}{\alpha^{\beta}}] \\
\frac{d^2ln(L(X))}{\partial\beta^2} &= -\frac{n}{\beta^2} - \frac{\sum_{i=1}^{n}(x_i^{\beta} ln(x_i))}{\sum_{i=1}^{n}(x_i^{\beta})}
\end{aligned}
$$

Second derivative of both parameters are less than zero, so that we have been proved that these parameters are the maximum likelihood parameters of Weibull distribution.

A.3 Lognormal Distribution

Let us assume that Y=e^X and X is Normally distributed with mean μ and variance σ^2; ie; N(μ;σ^2). Then, the mean of Lognormal distribution is obtained as follows;

$$\begin{aligned} E[Y] = E[e^X] &= \int_{-\infty}^{\infty} e^x f(x)dx \\ &= \int_{-\infty}^{\infty} e^x \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx \end{aligned}$$

By applying the transformation $y = x - \mu$ and $dy = dx$

$$\begin{aligned} &= \int_{-\infty}^{\infty} e^{\mu+y} \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(y)^2}{2\sigma^2}} dy \\ &= e^{\mu} \int_{-\infty}^{\infty} e^{y} \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(y)^2}{2\sigma^2}} dy \\ &= e^{\mu} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{2y\sigma^2-y^2}{2\sigma^2}} dy \\ &= e^{\mu} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(y-\sigma^2)^2+\sigma^4}{2\sigma^2}} dy \\ &= e^{\mu} e^{\frac{1}{2}\sigma^2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(y-\sigma^2)^2}{2\sigma^2}} dy \\ &= e^{\mu} e^{\frac{1}{2}\sigma^2} \\ &= e^{\mu+\frac{1}{2}\sigma^2} \end{aligned}$$

In order to find variance of Lognormal distribution, $E[Y^2]$ has to be found firstly.

$$\begin{aligned} E[Y^2] = E[e^{2X}] &= \int_{-\infty}^{\infty} e^{2x} f(x)dx \\ &= \int_{-\infty}^{\infty} e^{2x} \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx \end{aligned}$$

By applying the transformation $y = x - \mu$ and $dy = dx$

$$\begin{aligned}
&= \int_{-\infty}^{\infty} e^{2(\mu+y)} \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(y)^2}{2\sigma^2}} dy \\
&= e^{2\mu} \int_{-\infty}^{\infty} e^{2y} \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(y)^2}{2\sigma^2}} dy \\
&= e^{2\mu} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{4y\sigma^2 - y^2}{2\sigma^2}} dy \\
&= e^{2\mu} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(y-2\sigma^2)^2 + 4\sigma^4}{2\sigma^2}} dy \\
&= e^{2\mu} e^{2\sigma^2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(y-2\sigma^2)^2}{2\sigma^2}} dy \\
&= e^{2\mu} e^{2\sigma^2} \\
&= e^{2\mu+2\sigma^2}
\end{aligned}$$

Thus, the variance of Lognormal distribution is described as follows;

$$\begin{aligned}
Var[Y] &= E[Y^2] - E[Y]^2 \\
&= e^{2\mu+2\sigma^2} - (e^{\mu+\frac{1}{2}\sigma^2})^2 \\
&= e^{2\mu+2\sigma^2} - e^{2\mu+\sigma^2} \\
&= e^{2\mu+\sigma^2}(e^{\sigma^2} - 1)
\end{aligned}$$

Moment generating function of Lognormal distribution is described as;

$$\begin{aligned}
M_X(t) &= E[e^{tx}] = M_Y(t) \\
&= e^{t\mu+\frac{1}{2}t^2\sigma^2}
\end{aligned}$$

Method of Moment Estimator of Lognormal distribution is described as follows;

$$\begin{aligned}
m_1 &= E[X] = exp(\mu + \frac{\sigma^2}{2}) \\
m_2 &= E[X^2] = exp(2\mu + 2\sigma^2)
\end{aligned}$$

By solving equations, first moment of Lognormal distribution is $\hat{\mu} = ln(\sum_{i=1}^{n} X_i) - ln(n) - \frac{\hat{\sigma}^2}{2}$ and second moment of Lognormal distribution is $\hat{\mu} = \frac{ln(\sum_{i=1}^{n} X_i^2)}{2} - \frac{ln(n)}{2} - \hat{\sigma}^2$.

By solving these two equations for $\hat{\sigma}^2$:

$$
\begin{aligned}
ln(\sum_{i=1}^{n} X_i) - ln(n) - \frac{\hat{\sigma}^2}{2} &= \frac{ln(\sum_{i=1}^{n} X_i^2)}{2} - \frac{ln(n)}{2} - \hat{\sigma}^2 \\
2ln(\sum_{i=1}^{n} X_i) - 2ln(n) - \hat{\sigma}^2 &= ln(\sum_{i=1}^{n} X_i^2) - ln(n) - 2\hat{\sigma}^2 \\
\hat{\sigma}^2 &= ln(\sum_{i=1}^{n} X_i^2) - 2ln(\sum_{i=1}^{n} X_i) + ln(n)
\end{aligned}
$$

Now, by inserting the value of $\hat{\sigma}^2$ into the first or second moment of Lognormal distribution, we can find the moment estimator of location parameter.

$$
\begin{aligned}
\hat{\mu} &= ln(\sum_{i=1}^{n} X_i) - ln(n) - \frac{\hat{\sigma}^2}{2} \\
&= ln(\sum_{i=1}^{n} X_i) - ln(n) - \frac{1}{2}[ln(\sum_{i=1}^{n} X_i^2) - 2ln(\sum_{i=1}^{n} X_i) + ln(n)] \\
&= 2ln(\sum_{i=1}^{n} X_i) - \frac{3}{2}ln(n) - \frac{ln(\sum_{i=1}^{n} X_i^2)}{2}
\end{aligned}
$$

Maximum Likelihood Estimator of Lognormal distribution is described as;

$$
\begin{aligned}
L(\mu, \sigma^2|X) &= \prod_{i=1}^{n} f(X_i) \\
&= \prod_{i=1}^{n}((2\pi\sigma^2)^{-\frac{1}{2}} X_i^{-1} exp[\frac{-(ln(X_i) - \mu)^2}{2\sigma^2}]) \\
&= (2\pi\sigma^2)^{-\frac{n}{2}} \prod_{i=1}^{n} X_i^{-1} exp[\sum_{i=1}^{n} \frac{-(ln(X_i) - \mu)^2}{2\sigma^2}]
\end{aligned}
$$

Now, the natural logarithm of likelihood function has to be taken:

$$
\begin{aligned}
ln(L(X)) &= ln((2\pi\sigma^2)^{-\frac{n}{2}}\prod_{i=1}^{n} X_i^{-1} exp[\sum_{i=1}^{n}\frac{-(ln(X_i)-\mu)^2}{2\sigma^2}]) \\
&= -\frac{n}{2}ln(2\pi\sigma^2) - \sum_{i=1}^{n} ln(X_i) - \sum_{i=1}^{n}\frac{(ln(X_i)-\mu)^2}{2\sigma^2} \\
&= -\frac{n}{2}ln(2\pi\sigma^2) - \sum_{i=1}^{n} ln(X_i) - \frac{\sum_{1}^{i=n} ln(X_i)^2 - 2ln(X_i)\mu + \mu^2}{2\sigma^2} \\
&= -\frac{n}{2}ln(2\pi\sigma^2) - \sum_{i=1}^{n} ln(X_i) - \frac{\sum_{i=1}^{n} ln(X_i)^2}{2\sigma^2} + \frac{\sum_{i=1}^{n} ln(X_i)\mu}{\sigma^2} - \frac{n\mu^2}{2\sigma^2}
\end{aligned}
$$

Now, the first derivative of likelihood function has to be taken with respect to parameters.

$$
\begin{aligned}
\frac{dln(L(X))}{\partial\mu} &= \frac{\sum_{i=1}^{n} ln(X_i)}{\hat{\sigma}^2} - \frac{2n\hat{\mu}}{2\hat{\sigma}^2} \\
\hat{\mu} &= \frac{\sum_{i=1}^{n} ln(X_i)}{n} \\
\frac{dln(L(X))}{\partial\sigma^2} &= \frac{-n}{2\hat{\sigma}^2} - \frac{\sum_{i=1}^{n}(ln(X_i)-\hat{\mu})^2}{2}(-\hat{\sigma}^2)^{-2} \\
\hat{\sigma}^2 &= \frac{\sum_{i=1}^{n}(ln(X_i) - \frac{\sum_{i=1}^{n} ln(X_i)}{n})^2}{n}
\end{aligned}
$$

To verify these solutions are the maximum likelihood estimators of Lognormal distribution, second derivative has to be taken to maximize the loglikelihood function.

$$
\begin{aligned}
\frac{d^2ln(L(X))}{\partial\mu^2} &= \frac{-n}{\hat{\sigma}^2} \\
\frac{d^2ln(L(X))}{\partial(\sigma^2)^2} &= \frac{1}{2(\hat{\sigma}^2)^3}[-\sum_{i=1}^{n}(ln(X_i)-\hat{\mu})^2]
\end{aligned}
$$

Second derivative of both parameters are less than zero, so that we prove that these parameters are the maximum likelihood parameters of Lognormal distribution.

A.4 Pareto Distribution

The k^{th} moment of Pareto distribution is obtained as follows;

$$\begin{aligned} E[X^k] &= \int_{\lambda}^{\infty} x^k f(x)dx \\ &= \int_{\lambda}^{\infty} x^k (\frac{a\lambda^a}{x^{a+1}})dx \\ &= a\lambda^a \int_{\lambda}^{\infty} (\frac{1}{x^{a+1-k}})dx \end{aligned}$$

By multiplying and dividing the equation with $(a-k)\lambda^{a-k}$, we obtain the following equation;

$$= \frac{a\lambda^a}{(a-k)\lambda^{a-k}} \int_{\lambda}^{\infty} \frac{(a-k)\lambda^{a-k}}{x^{a-k+1}} dx$$

The integral in equation is equal to the Pareto distribution with parameters $(a-k, \lambda)$, so it is equal to 1 whenever $a - k > 0$ thus k^{th} moment of the Weibull distribution is written as;

$$E[X^k] = \frac{a\lambda^k}{(a-k)}$$

The mean of Pareto distribution is equal to 1^{st} moment and it is written as;

$$E[X] = \frac{a\lambda}{(a-1)}$$

In order to find variance of Pareto distribution, $E[X^2]$ has to be found firstly. It is equal the 2^{nd} moment of the Pareto distribution and it is written as;

$$E[X^2] = \frac{a\lambda^2}{(a-2)}$$

Hence, the variance of Pareto distribution is described as;

$$
\begin{aligned}
Var[X] &= E[X^2] - E[X]^2 \\
&= \frac{a\lambda^2}{(a-2)} - (\frac{a\lambda}{(a-1)})^2 \\
&= \frac{a\lambda^2}{(a-2)} - \frac{a^2\lambda^2}{(a-1)^2} \\
&= \frac{a\lambda^2(a-1)^2 - a^2\lambda^2(a-2)}{(a-1)^2(a-2)} \\
&= \frac{a\lambda^2[(a-1)^2 - a(a-2)]}{(a-1)^2(a-2)} \\
&= \frac{a\lambda^2[a^2 - 2a + 1 - a^2 + 2a]}{(a-1)^2(a-2)} \\
&= \frac{a\lambda^2}{(a-1)^2(a-2)}
\end{aligned}
$$

Method of Moment Estimator of Pareto distribution is described as follows;

$$
\begin{aligned}
m_1 &= E[X] = \frac{a\lambda}{a-1} \\
m_2 &= E[X^2] = \frac{a\lambda^2}{a-2}
\end{aligned}
$$

By taking the second sample moment divided by the squared of the first sample moment, a function of the theoretical moments is obtained as follows;

$$
\frac{\hat{\mu_2}}{\hat{\mu_1}^2} = \frac{(a-1)^2}{a(a-2)}
$$

We know that $\frac{(a-1)^2}{a(a-2)} - 1 = \frac{1}{a(a-2)}$ so that the equation is equal to $a(a-2) = \frac{\hat{\mu_1}^2}{(\hat{\mu_2} - \hat{\mu_1}^2)}$. There is a unique solution as long as $a > 2$, then the moment estimator of shape parameter is as follows;

$$
\hat{a}_{MM} = 1 + \sqrt{1 + \frac{\frac{n}{n-1}\bar{X}}{S^2}}
$$

Now, we can calculate the moment estimator of scale parameter by using $\hat{a}_{MM}$.

$$
\hat{\lambda}_{MM} = \frac{\bar{X}\sqrt{1 + \frac{\frac{n}{n-1}\bar{X}}{S^2}}}{1 + \sqrt{1 + \frac{\frac{n}{n-1}\bar{X}}{S^2}}}
$$

Maximum Likelihood Estimator of Pareto distribution is described as;

$$\begin{aligned} L(a,\lambda|X) &= \prod_{i=1}^{n} f(X_i) \\ &= \prod_{i=1}^{n}(\frac{a\lambda^a}{X_i^{a+1}}) \end{aligned}$$

Now, the natural logarithm of likelihood function has to be taken:

$$\begin{aligned} ln(L(X)) &= ln(\prod_{i=1}^{n}(\frac{a\lambda^a}{X_i^{a+1}})) \\ &= nln(a) + naln(\lambda) - (a+1)\sum_{i=1}^{n} ln(X_i) \end{aligned}$$

Since a higher scale parameter (λ) will always result in a higher likelihood ($\ln(\lambda)$ is monotonically increasing), we maximize the likelihood by setting $\hat{\lambda}$ as high as possible. Since $\lambda < x_i$ for all i, we maximize the likelihood by setting $\hat{\lambda} = \min x_i$ with the smallest x_i in the sample.

Now, the derivative of likelihood function has to be taken with respect to shape parameter.

$$\begin{aligned} \frac{dln(L(X))}{\partial a} &= \frac{n}{a} + nln(\lambda) - \sum_{i=1}^{n} ln(X_i) \\ \hat{a} &= \frac{n}{\sum_{i=1}^{n} ln(X_i) - nln(\lambda)} \end{aligned}$$

To verify these solutions are the maximum likelihood estimators of Pareto distribution, second derivative has to be taken to maximize the loglikelihood function of Pareto distribution.

$$\frac{d^2ln(L(X))}{\partial a^2} = \frac{-n}{a^2}$$

Second derivative of the shape parameter is less than zero, so that we have been proved that the shape parameter is the maximum likelihood parameter of Pareto distribution.

APPENDIX B

Graphical Representation of Fitting Distributions

In this part, graphical results of the distribution fitting of automobile claim amounts data for selected two cities are shown with respect to methods of moment estimation and methods of maximum likelihood estimation.

B.1 Methods of Moment Estimation

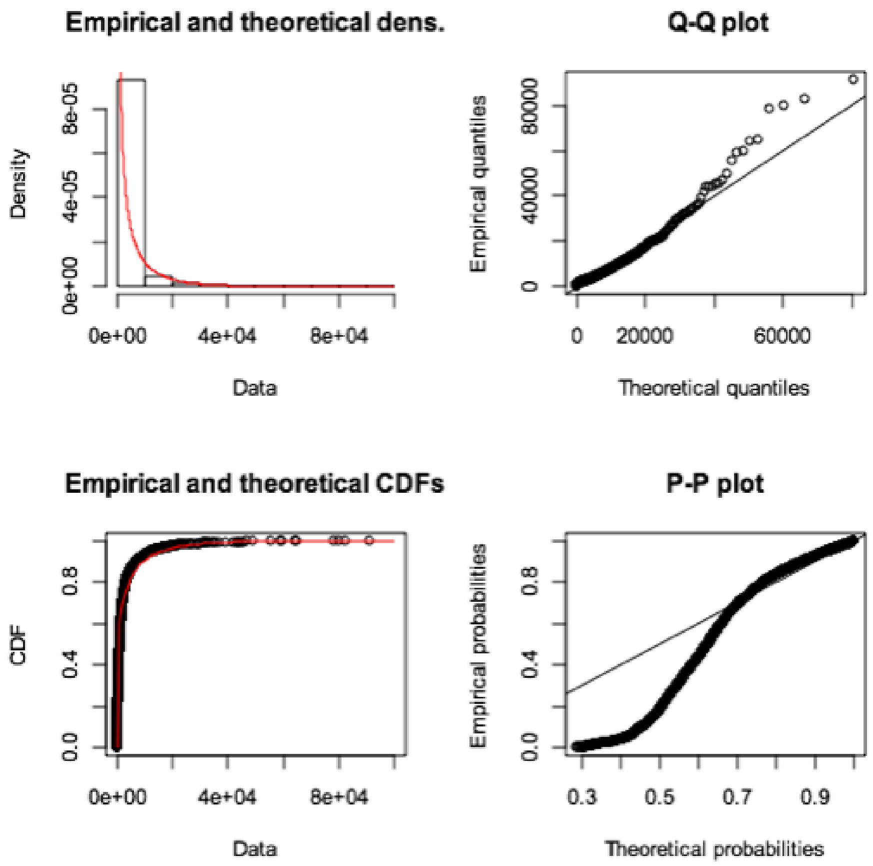

Figure B.1: Gamma Distribution Graphs of Istanbul

According to Figure B.1; probability density function graph shows that distribution of our claim amounts have right skewed feature and the red line shows the characteristic of Gamma distribution. Therefore; claim amounts of Istanbul have the same shape as Gamma distribution. Both of them are distributed with positively skewed. Moreover, cumulative distribution function graph shows that claim amounts of Istanbul come from the family of Gamma distribution. However, Q-Q and P-P plots show that our claim amounts do not fit Gamma distribution because data of claim amounts are not on the reference line at the tails.

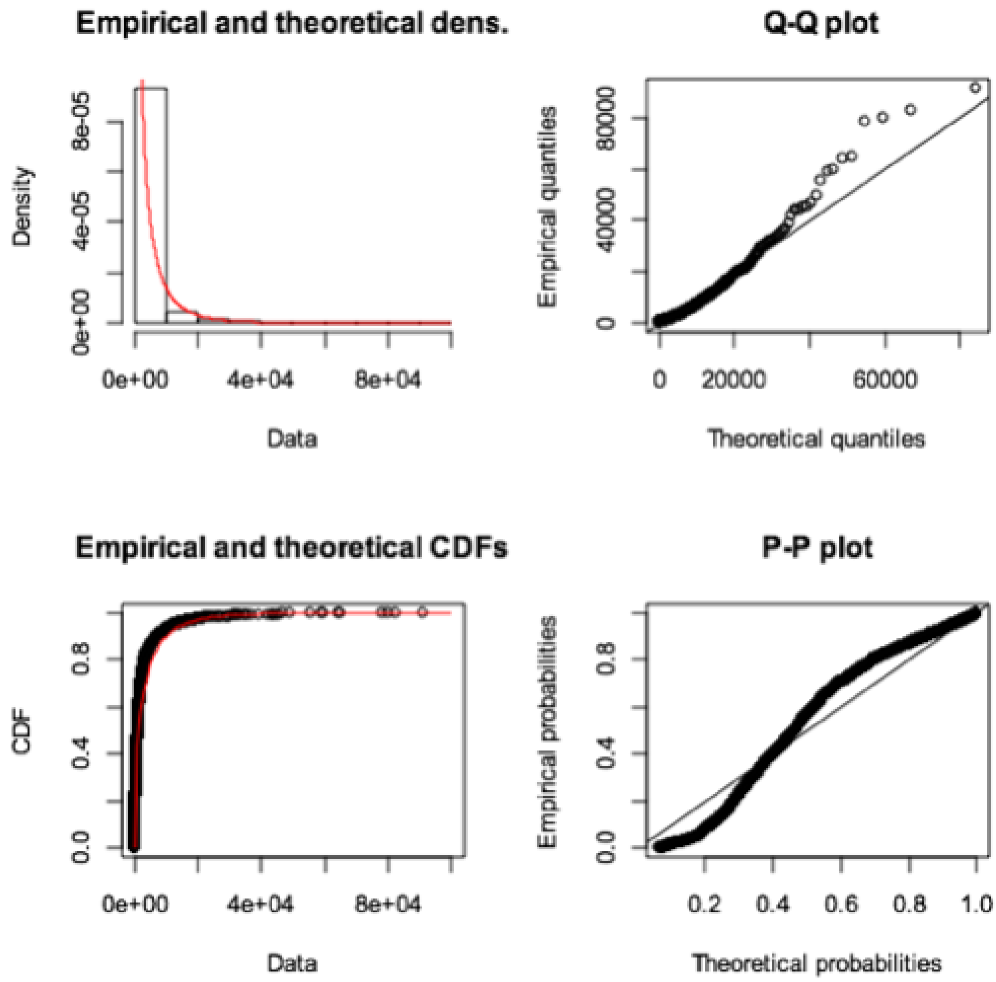

Figure B.2: Weibull Distribution Graphs of Istanbul

According to Figure B.2; probability density function graph shows that distribution of our claim amounts comes from Weibull distribution because claim amounts of Istanbul have the same shape as Weibull distribution. Both of them are distributed with right skewed. Moreover, cumulative distribution function graph shows that claim amounts of Istanbul come from the family of Weibull distribution. However, Q-Q and P-P plots show that our claim amounts do not fit Weibull distribution because data of claim amounts are not on the reference line perfectly.

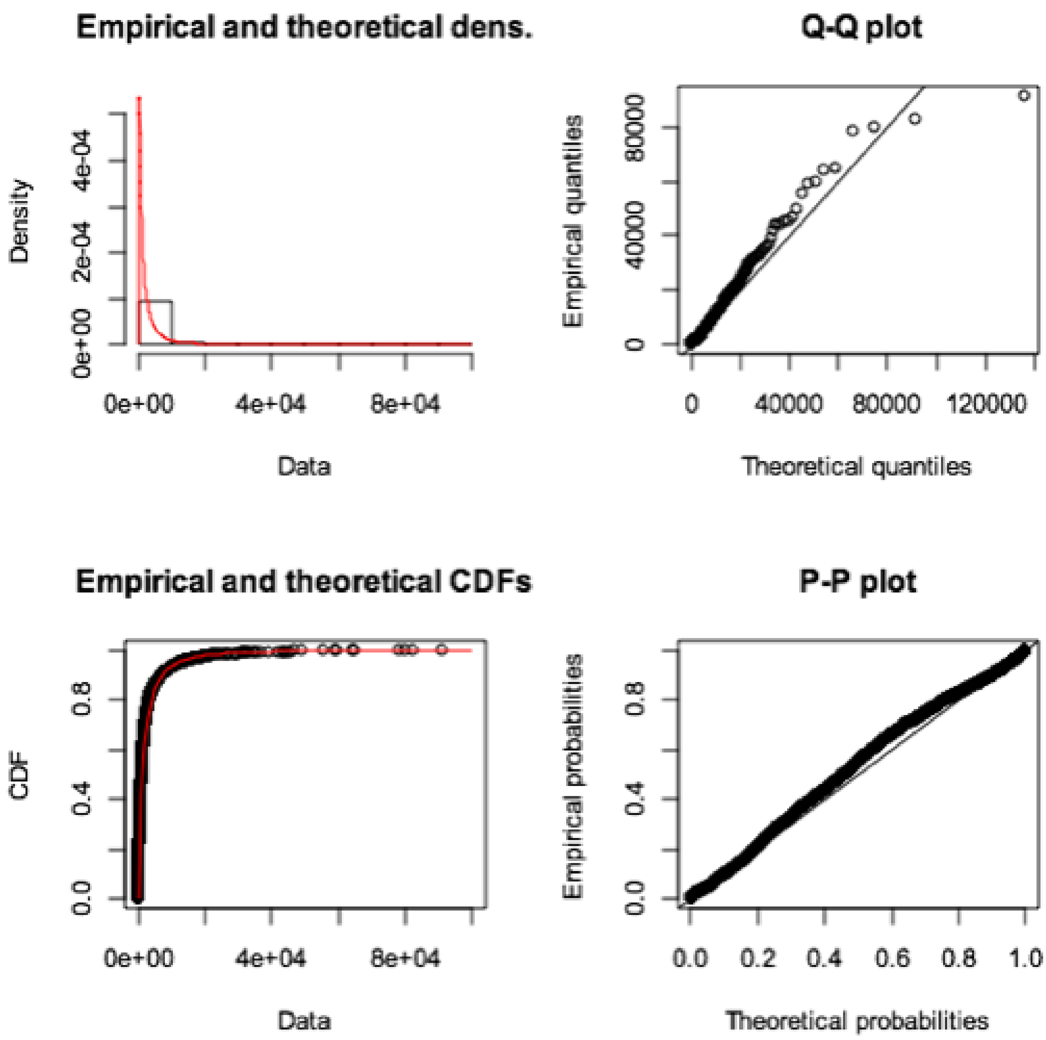

Figure B.3: Lognormal Distribution Graphs of Istanbul

According to Figure B.3; probability density function graph shows that distribution of our claim amounts comes from Lognormal distribution because claim amounts of Istanbul have the same shape as Lognormal distribution. Both of them are distributed with right skewed. Moreover, cumulative distribution function graph shows that claim amounts of Istanbul come from the family of Weibull distribution. Furthermore, P-P plot shows that our claim amounts fit Lognormal distribution because data of claim amounts are perfectly match on the reference line. Most of the quantiles of our claim amounts are on the line of Q-Q plot but they are few outliers at the high end, which are out of the range. Therefore; from Q-Q plot, the claim amounts have heavy tails on the right ends.

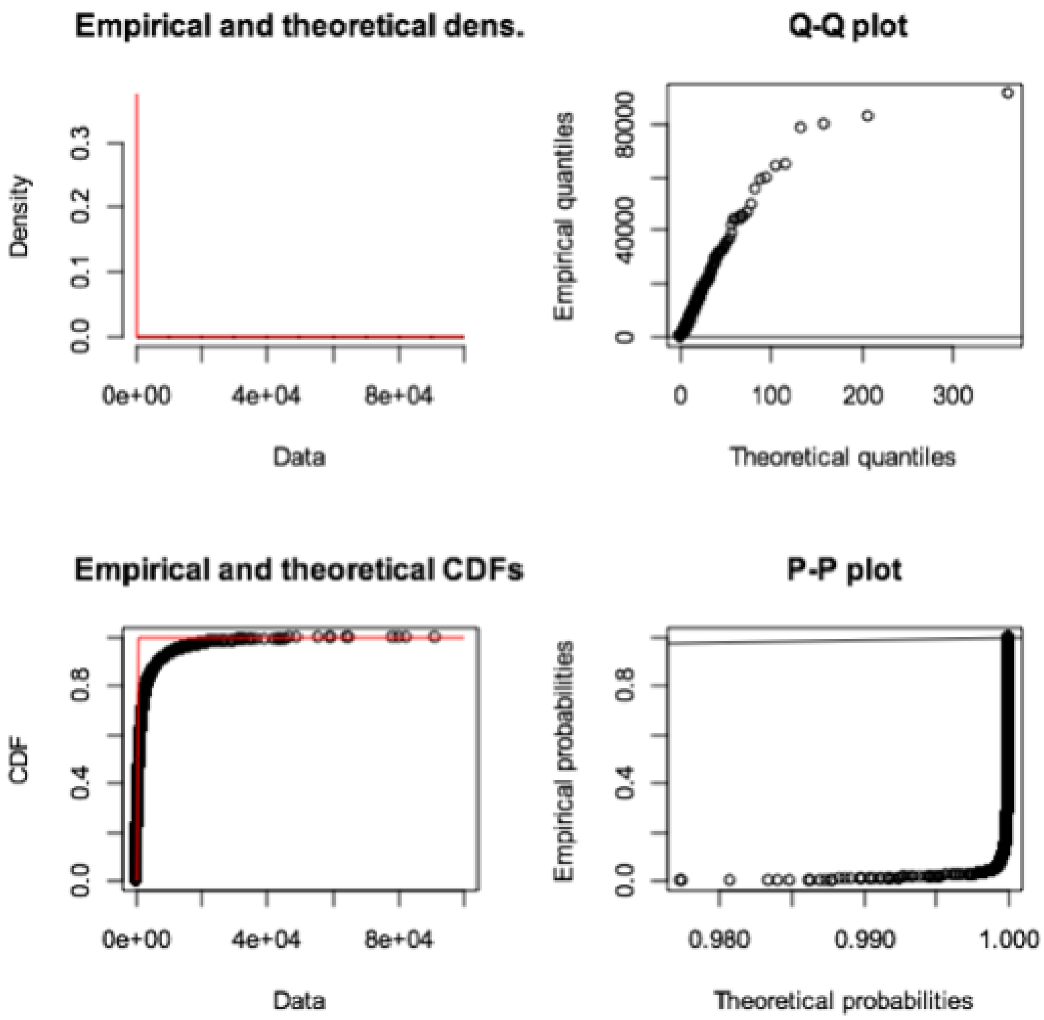

Figure B.4: Pareto Distribution Graphs of Istanbul

According to Figure B.4; probability density function graph shows that distribution of our claim amounts comes from Pareto distribution because claim amounts of Istanbul have the same shape as Pareto distribution. Both of them are distributed with positively skewed. Moreover, cumulative distribution function graph shows that claim amounts of Istanbul come from the family of Pareto distribution. Furthermore, P-P plot shows that our claim amounts fit Pareto distribution because data of claim amounts are on the straight line. Quantiles of our claim amounts are not on the line of Q-Q plot at right tail but they are as close as the theoretical value of Pareto distribution except few outliers at the high end of the range. Hence, we say that Pareto distribution also fit the claim amounts.

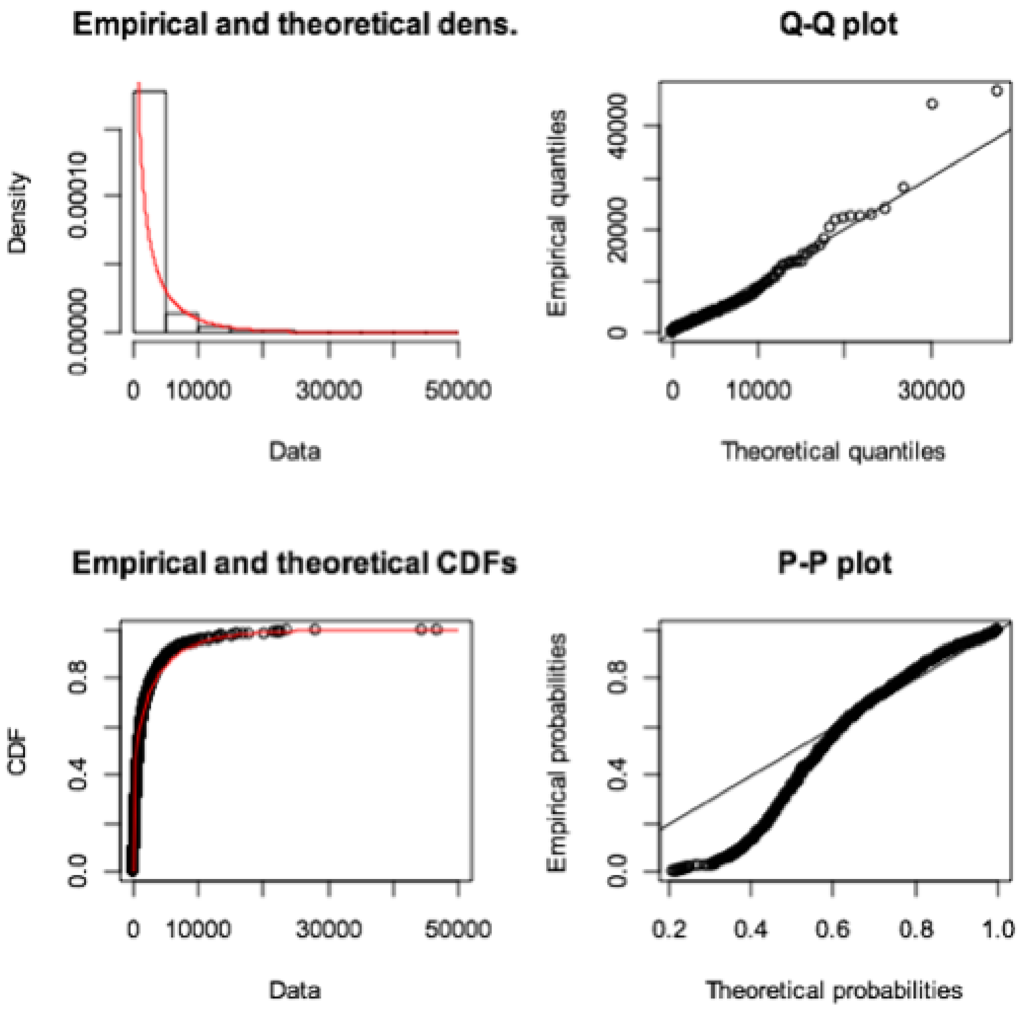

Figure B.5: Gamma Distribution Graphs of Ankara

According to Figure B.5; probability density function graph shows that distribution of our claim amounts have right skewed feature and the red line shows the characteristic of Gamma distribution. Therefore; claim amounts of Ankara have the same shape as Gamma distribution. Both of them are distributed with positively skewed. Moreover, cumulative distribution function graph shows that claim amounts of Ankara come from the family of Gamma distribution. However, Q-Q and P-P plots show that our claim amounts do not fit Gamma distribution because data of claim amounts are far away from the reference line at the tails.

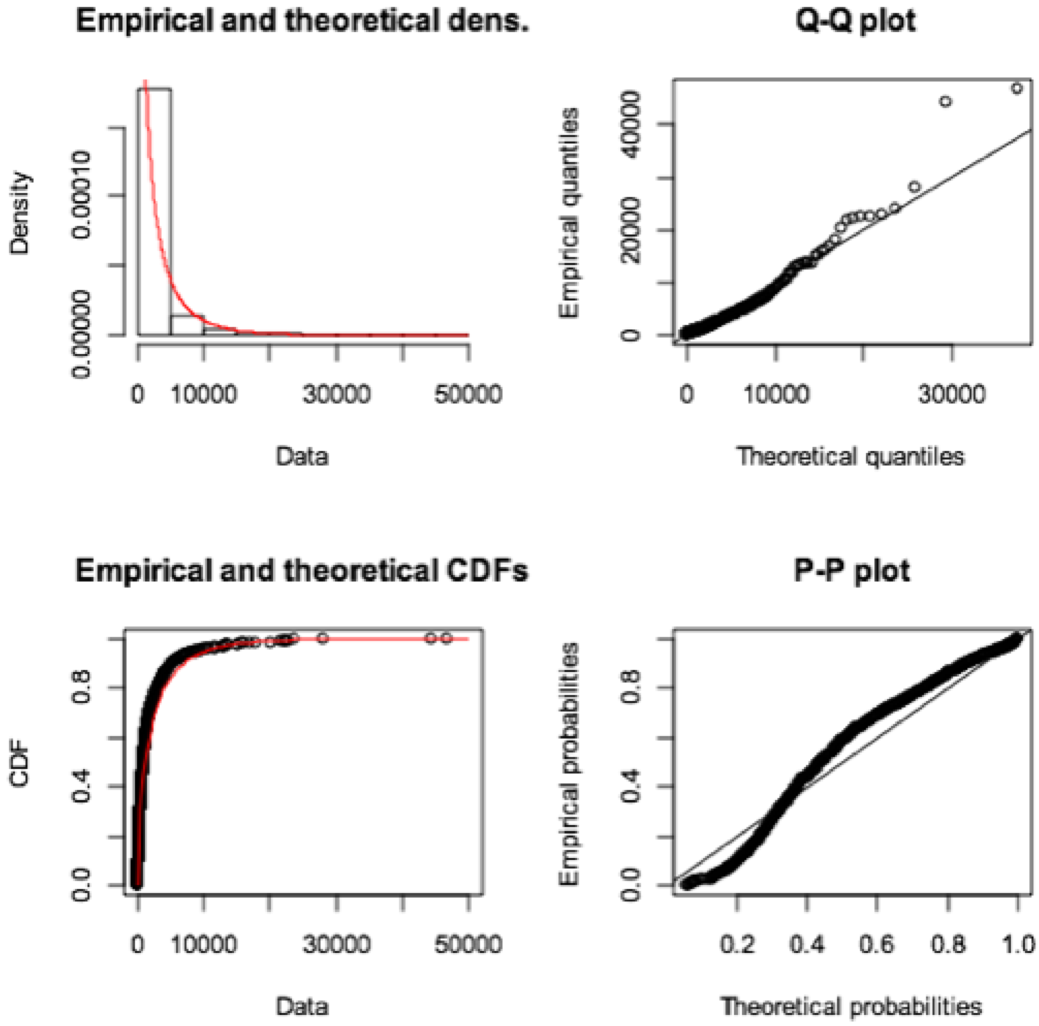

Figure B.6: Weibull Distribution Graphs of Ankara

According to Figure B.6; probability density function graph shows that distribution of our claim amounts comes from Weibull distribution because claim amounts of Ankara have the same shape as Weibull distribution. Both of them are distributed with right skewed. Moreover, cumulative distribution function graph shows that claim amounts of Ankara come from the family of Weibull distribution. However, Q-Q and P-P plots show that our claim amounts do not fit Weibull distribution because data of claim amounts are not on the reference line perfectly.

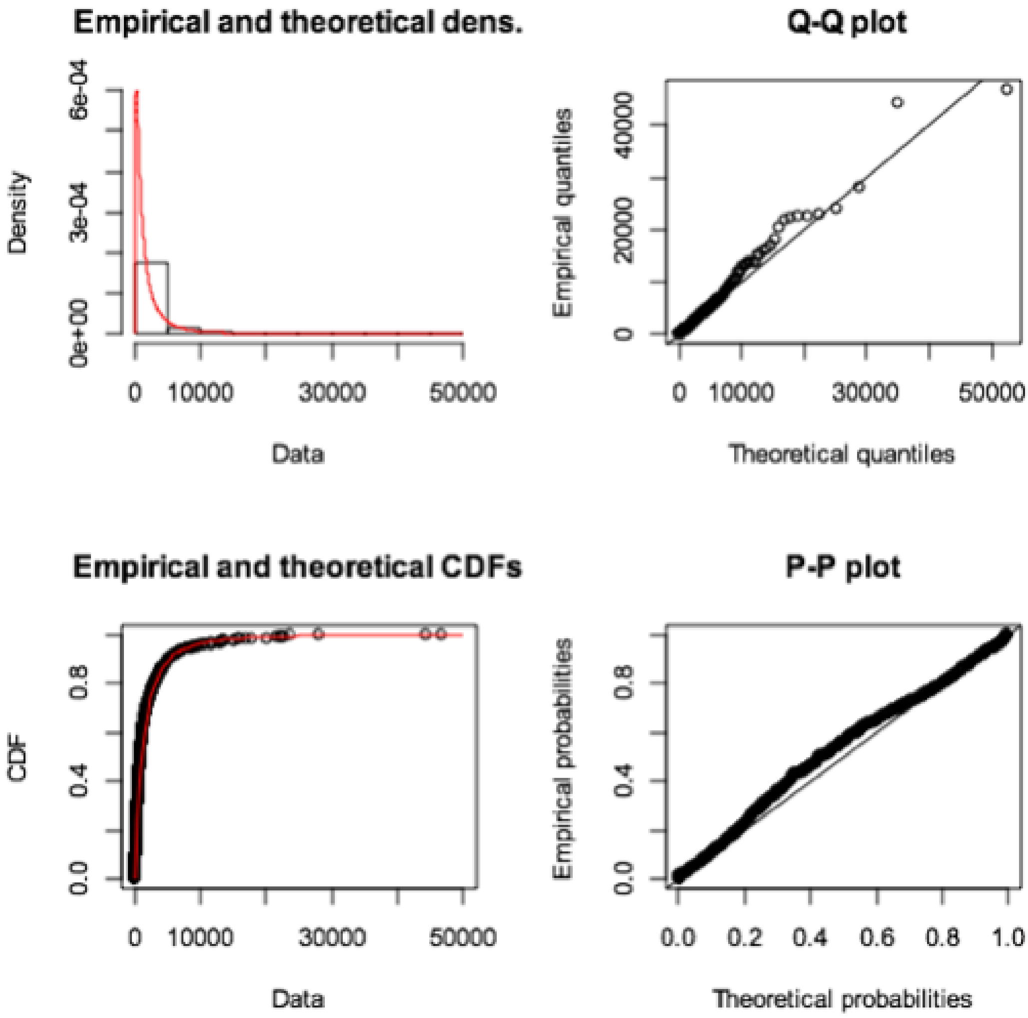

Figure B.7: Lognormal Distribution Graphs of Ankara

According to Figure B.7; probability density function graph shows that distribution of our claim amounts comes from Lognormal distribution because claim amounts of Ankara have the same shape as Lognormal distribution. Both of them are distributed with right skewed. Moreover, cumulative distribution function graph shows that claim amounts of Ankara come from the family of Weibull distribution. Furthermore, P-P plot shows that our claim amounts fit Lognormal distribution because data of claim amounts are perfectly match on the reference line. Most of the quantiles of our claim amounts are on the line of Q-Q plot but they are few outliers at the high end, which are out of the range. Therefore; from Q-Q plot, the claim amounts have heavy tails on the right ends.

According to Figure B.8; probability density function graph shows that distribution of our claim amounts comes from Pareto distribution because claim amounts of Ankara have the same shape as Pareto distribution. Both of them are distributed with positively

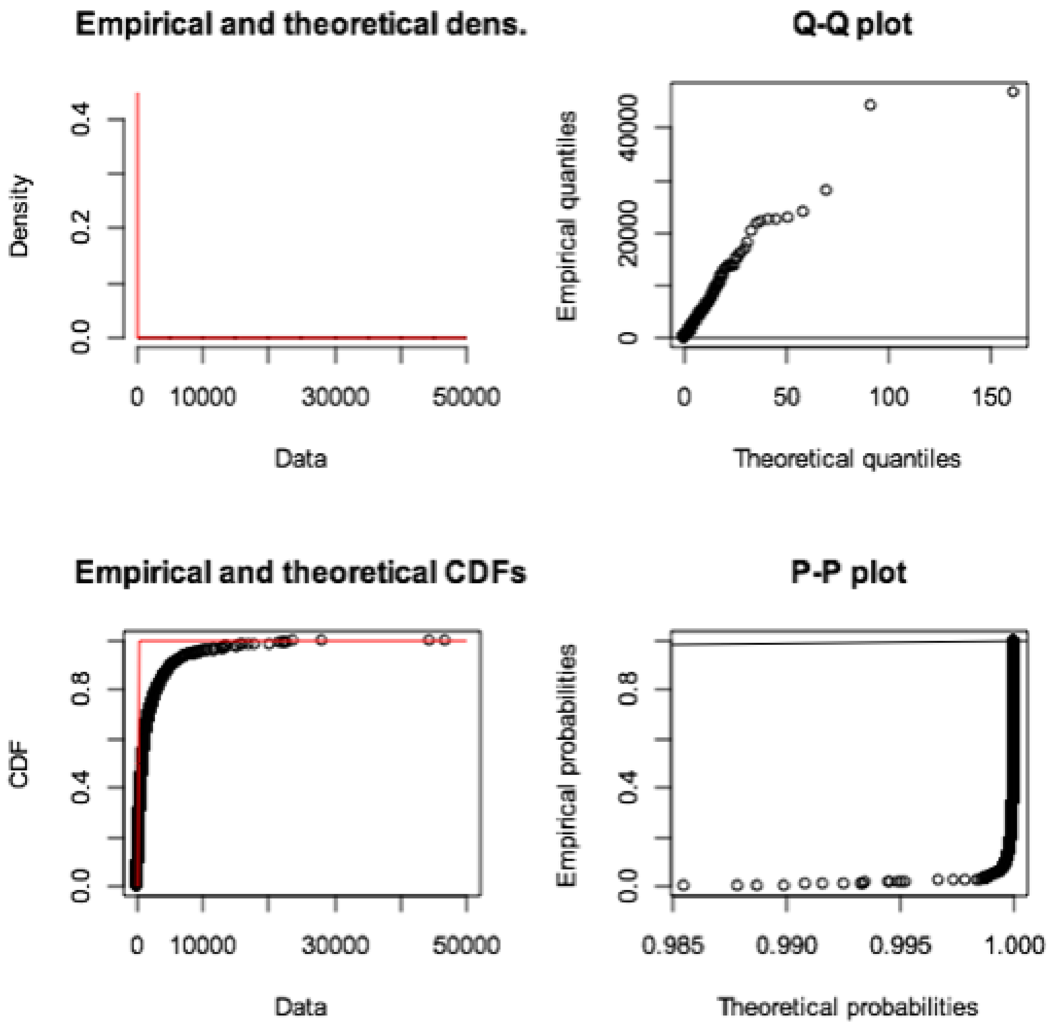

Figure B.8: Pareto Distribution Graphs of Ankara

skewed. Moreover, cumulative distribution function graph shows that claim amounts of Ankara come from the family of Pareto distribution. Furthermore, P-P plot shows that our claim amounts fit Pareto distribution because data of claim amounts are on the straight line. Quantiles of our claim amounts are not on the line of Q-Q plot at right tail but they are as close as the theoretical value of Pareto distribution except few outliers at the high end of the range. Hence, we say that Pareto distribution also fit the claim amounts.

B.2 Methods of Maximum Likelihood Estimation

According to Figure B.9; probability density function graph shows that distribution of our claim amounts have right skewed feature and the red line shows the characteristic

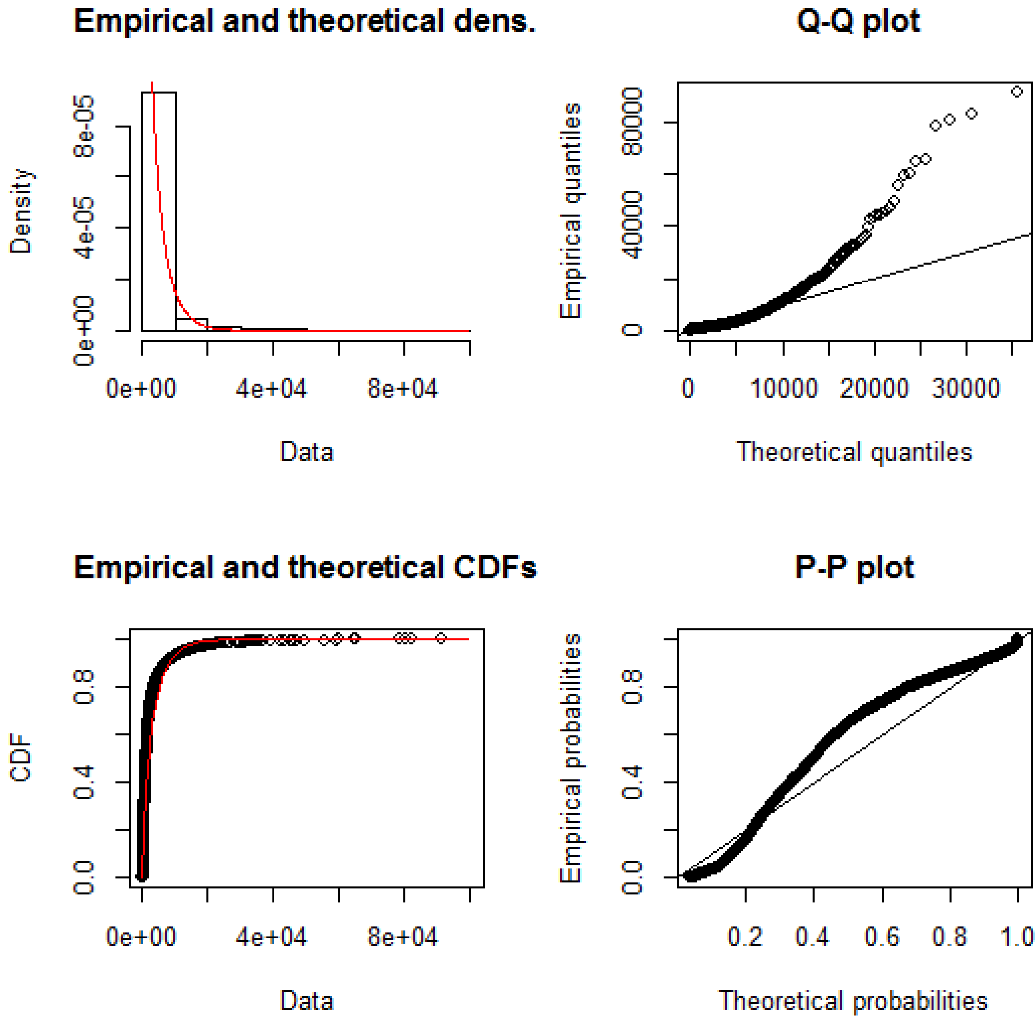

Figure B.9: Gamma Distribution Graphs of Istanbul

of Gamma distribution. Therefore; claim amounts of Istanbul have the same shape as Gamma distribution. Both of them are distributed with positively skewed. Moreover, cumulative distribution function graph shows that claim amounts of Istanbul come from the family of Gamma distribution. However, Q-Q and P-P plots show that our claim amounts do not fit Gamma distribution because data of claim amounts are not on the reference line at the tails.

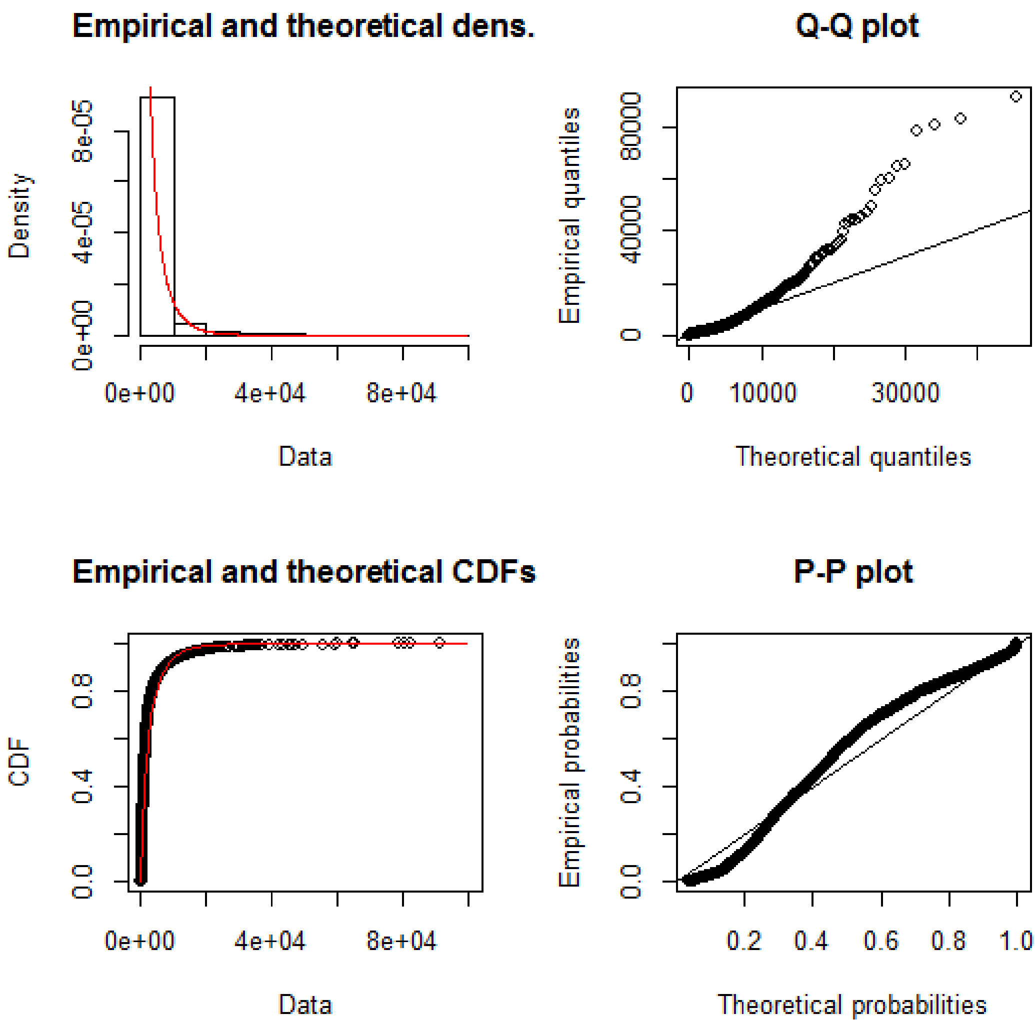

Figure B.10: Weibull Distribution Graphs of Istanbul

According to Figure B.10; probability density function graph shows that distribution of our claim amounts comes from Weibull distribution because claim amounts of Istanbul have the same shape as Weibull distribution. Both of them are distributed with right skewed. Moreover, cumulative distribution function graph shows that claim amounts of Istanbul come from the family of Weibull distribution. However, Q-Q and P-P plots show that our claim amounts do not fit Weibull distribution because data of claim amounts are not on the reference line perfectly.

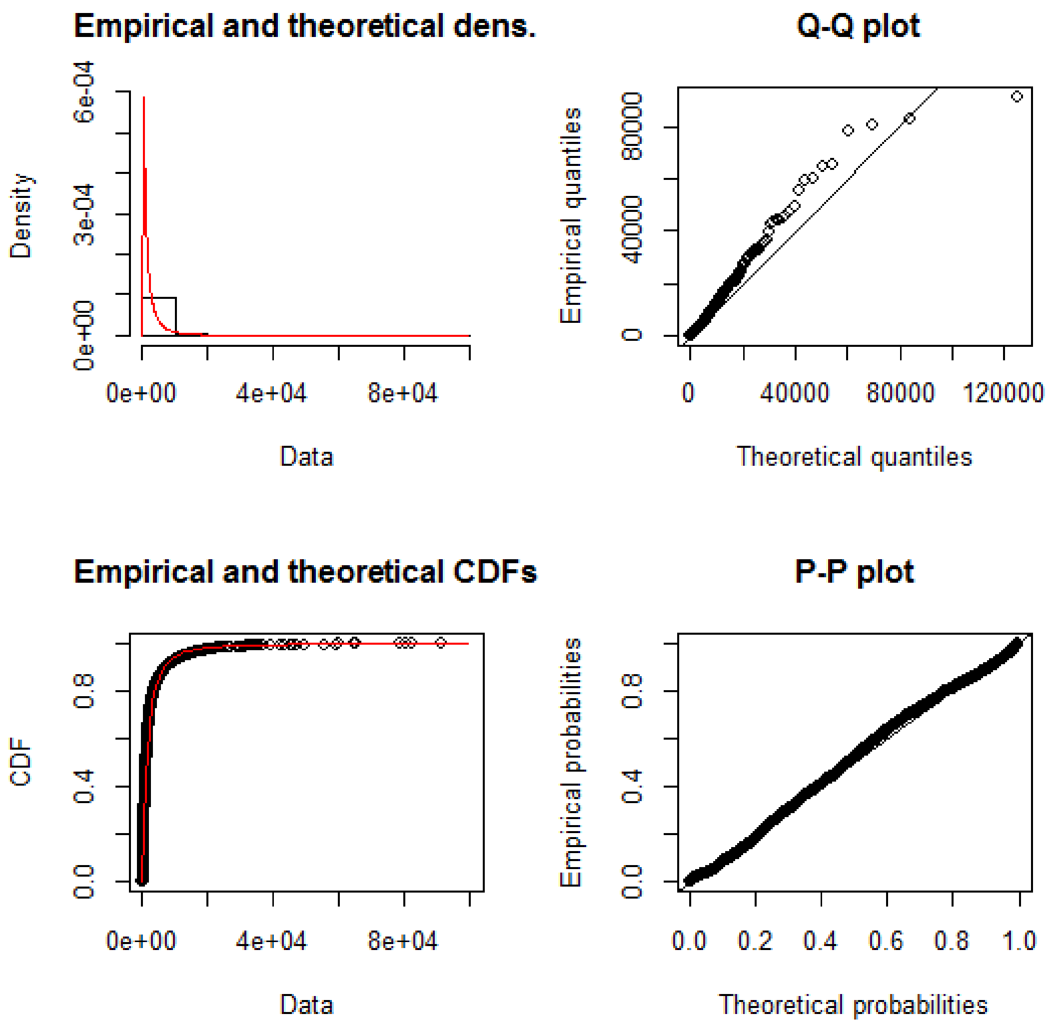

Figure B.11: Lognormal Distribution Graphs of Istanbul

According to Figure B.11; probability density function graph shows that distribution of our claim amounts comes from Lognormal distribution because claim amounts of Istanbul have the same shape as Lognormal distribution. Both of them are distributed with right skewed. Moreover, cumulative distribution function graph shows that claim amounts of Istanbul come from the family of Weibull distribution. Furthermore, P-P plot shows that our claim amounts fit Lognormal distribution because data of claim amounts are perfectly match on the reference line. Most of the quantiles of our claim amounts are on the line of Q-Q plot but they are few outliers at the high end, which are out of the range. Therefore; from Q-Q plot, the claim amounts have heavy tails on the right ends.

According to Figure B.12; probability density function graph shows that distribution of our claim amounts comes from Pareto distribution because claim amounts of Istanbul have the same shape as Pareto distribution. Both of them are distributed with positively

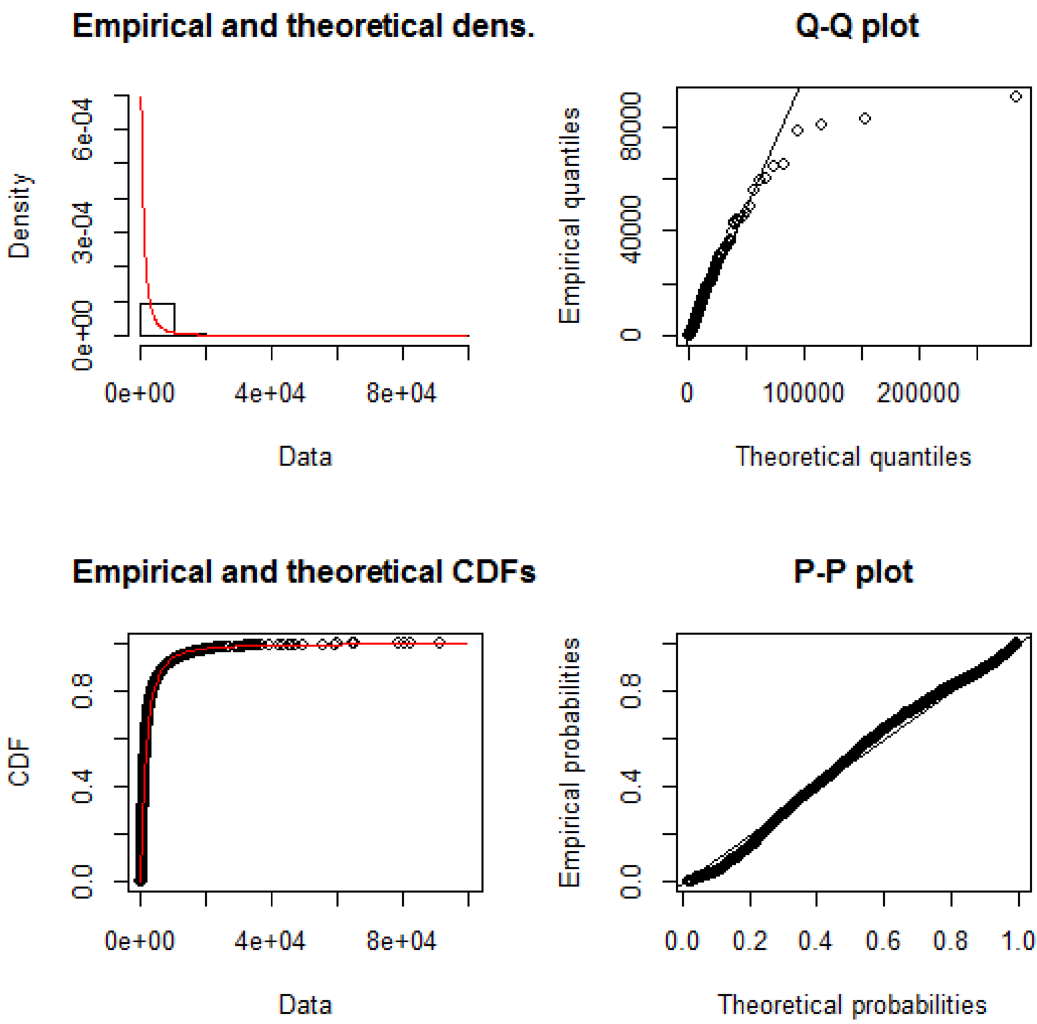

Figure B.12: Pareto Distribution Graphs of Istanbul

skewed. Moreover, cumulative distribution function graph shows that claim amounts of Istanbul come from the family of Pareto distribution. Furthermore, P-P plot shows that our claim amounts fit Pareto distribution because data of claim amounts are on the straight line. Quantiles of our claim amounts are not on the line of Q-Q plot at right tail but they are as close as the theoretical value of Pareto distribution except few outliers at the high end of the range. Hence, we say that Pareto distribution also fit the claim amounts.

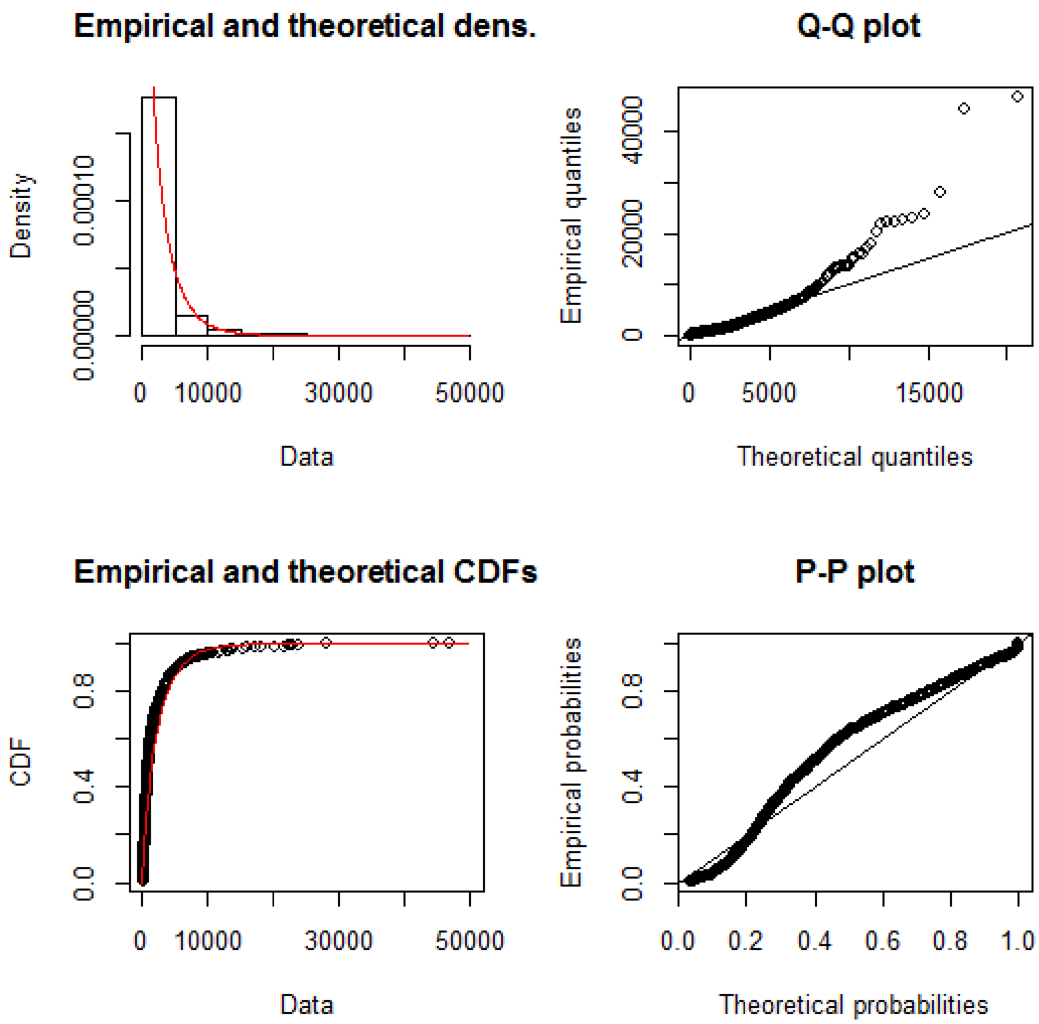

Figure B.13: Gamma Distribution Graphs of Ankara

According to Figure B.13; probability density function graph shows that distribution of our claim amounts have right skewed feature and the red line shows the characteristic of Gamma distribution. Therefore; claim amounts of Ankara have the same shape as Gamma distribution. Both of them are distributed with positively skewed. Moreover, cumulative distribution function graph shows that claim amounts of Ankara come from the family of Gamma distribution. However, Q-Q and P-P plots show that our claim amounts do not fit Gamma distribution because data of claim amounts are far away from the reference line at the tails.

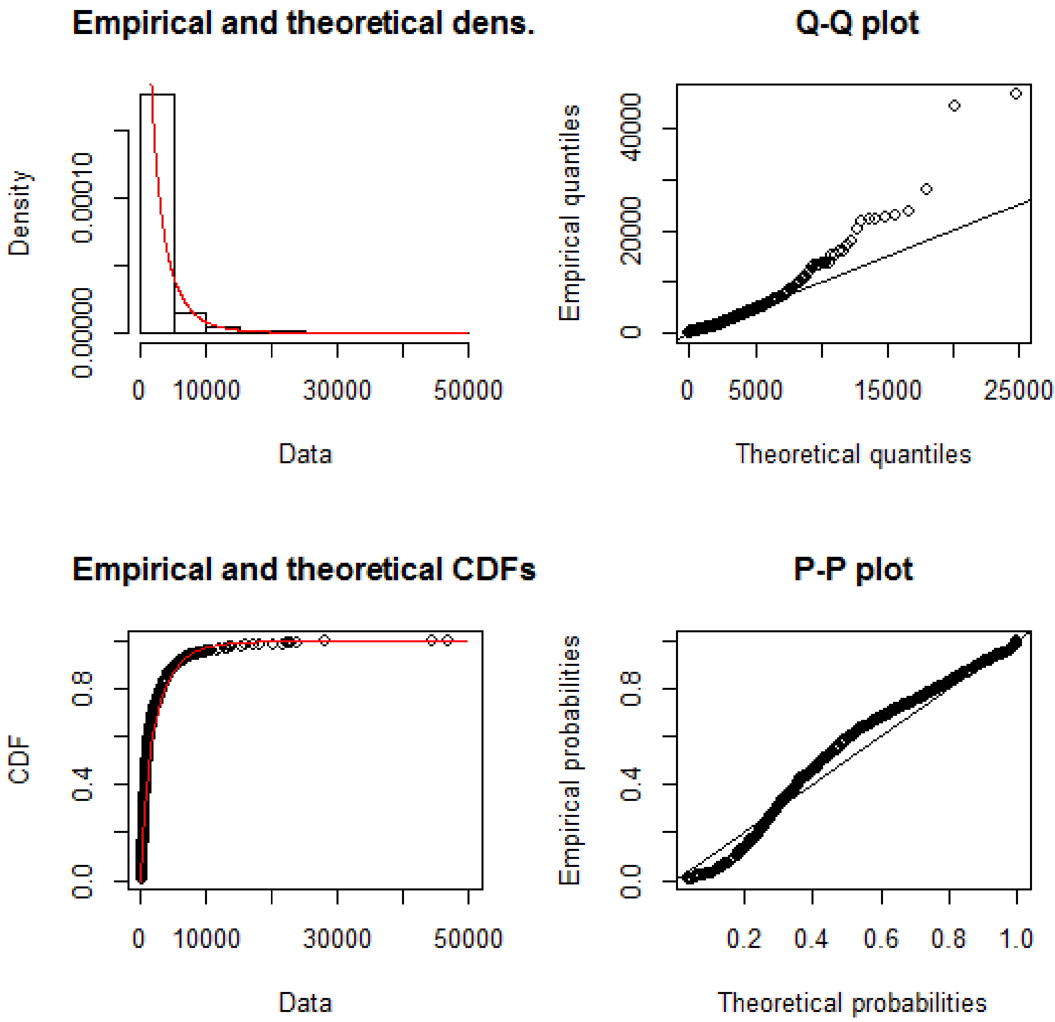

Figure B.14: Weibull Distribution Graphs of Ankara

According to Figure B.14; probability density function graph shows that distribution of our claim amounts comes from Weibull distribution because claim amounts of Ankara have the same shape as Weibull distribution. Both of them are distributed with right skewed. Moreover, cumulative distribution function graph shows that claim amounts of Ankara come from the family of Weibull distribution. However, Q-Q and P-P plots show that our claim amounts do not fit Weibull distribution because data of claim amounts are not on the reference line perfectly.

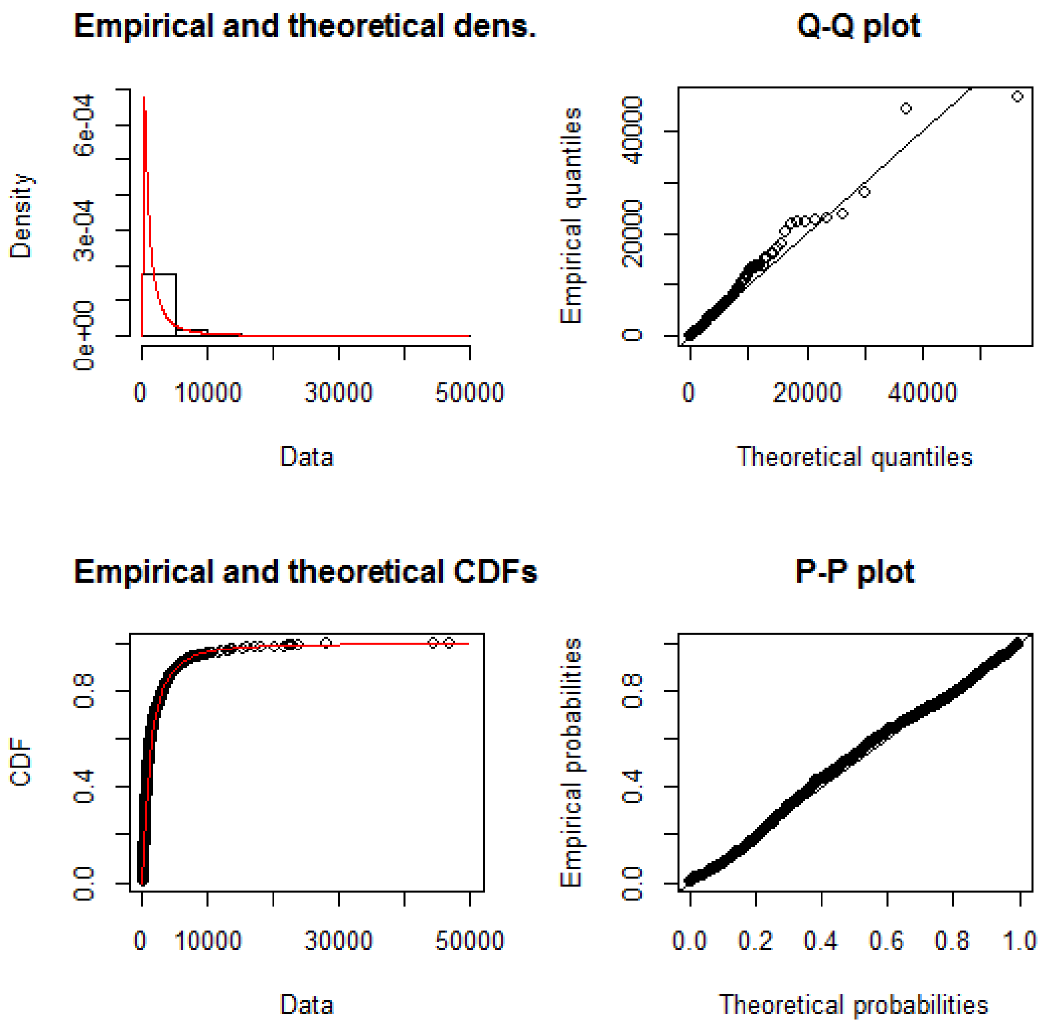

Figure B.15: Lognormal Distribution Graphs of Ankara

According to Figure B.15; probability density function graph shows that distribution of our claim amounts comes from Lognormal distribution because claim amounts of Ankara have the same shape as Lognormal distribution. Both of them are distributed with right skewed. Moreover, cumulative distribution function graph shows that claim amounts of Ankara come from the family of Weibull distribution. Furthermore, P-P plot shows that our claim amounts fit Lognormal distribution because data of claim amounts are perfectly match on the reference line. Most of the quantiles of our claim amounts are on the line of Q-Q plot but they are few outliers at the high end, which are out of the range. Therefore; from Q-Q plot, the claim amounts have heavy tails on the right ends.

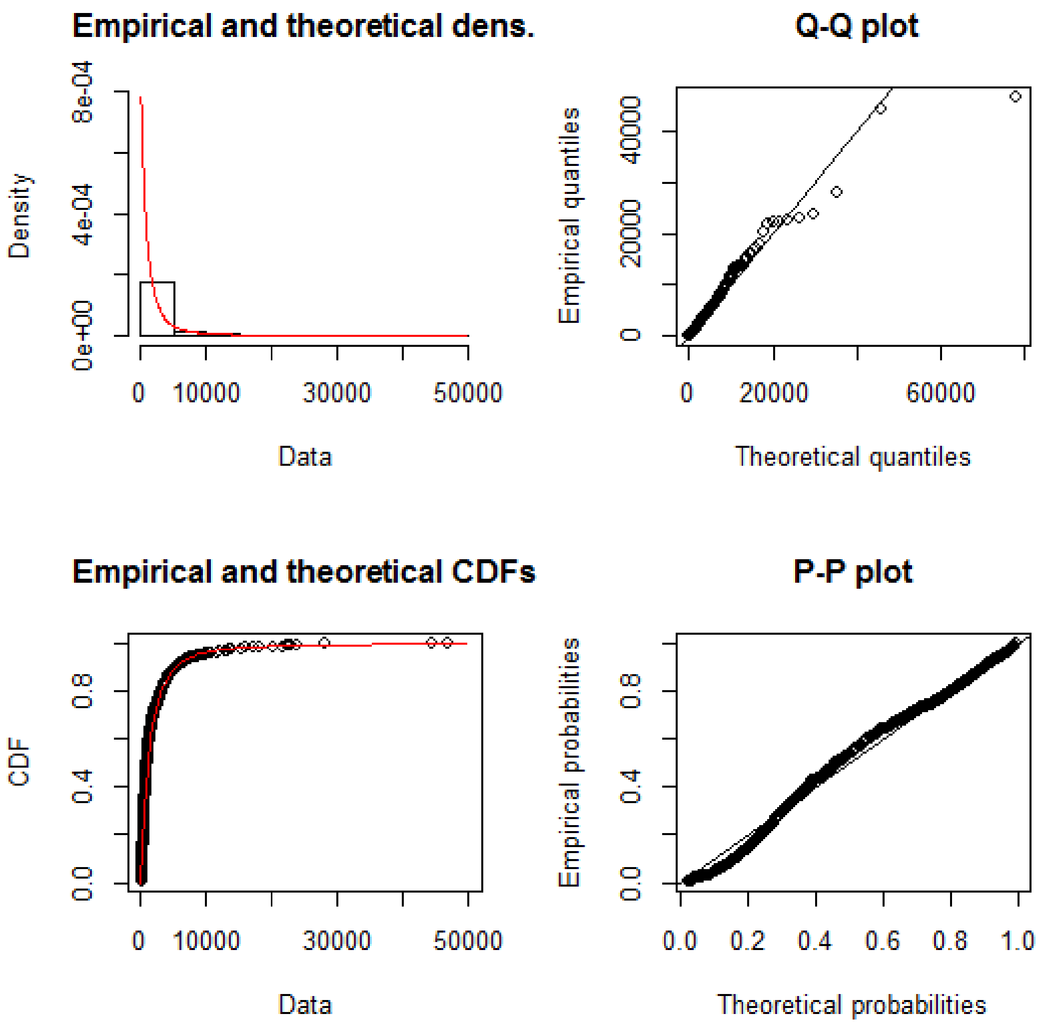

Figure B.16: Pareto Distribution Graphs of Ankara

According to Figure B.16; probability density function graph shows that distribution of our claim amounts comes from Pareto distribution because claim amounts of Ankara have the same shape as Pareto distribution. Both of them are distributed with positively skewed. Moreover, cumulative distribution function graph shows that claim amounts of Ankara come from the family of Pareto distribution. Furthermore, P-P plot shows that our claim amounts fit Pareto distribution because data of claim amounts are on the straight line. Quantiles of our claim amounts are not on the line of Q-Q plot at right tail but they are as close as the theoretical value of Pareto distribution except few outliers at the high end of the range. Hence, we say that Pareto distribution also fit the claim amounts.

APPENDIX C

Mathematical Representation of Risk Premiums

C.1 Based on Methods of Moments Estimation

Parameter estimation results of methods of moments give us the mean and variance of the selected cities based on Lognormal distribution so that we can calculate the risk premiums based on the their definitions.

The mean and the variance of Lognormal Distribution of Istanbul as follows;

$$\begin{aligned} E[X] &= e^{\mu+\frac{1}{2}\sigma^2} \\ &= e^{7.1874+\frac{1}{2}(0.0615)^2} \\ &= 1,325.163 \end{aligned}$$

$$\begin{aligned} Var[X] &= e^{2\mu+\sigma^2}(e^{\sigma^2}-1) \\ &= e^{2*7.1874+(0.0615)^2}(e^{(0.0615)^2}-1) \\ &= 6,654.428 \end{aligned}$$

1. **Net Risk Premium:**

$$\mathcal{H}(X) = E[X] = 1,325.163$$

2. **Expected Value Risk Premium:**

$$\mathcal{H}(X) = (1+\alpha)E[X] = (1+0.20)*1,325.163 = 1,590.196$$

3. **Variance Risk Premium:**

$$\mathcal{H}(X) = E[X] + \alpha Var[X] = 1,325.163 + (0.20*6,654.428) = 2,656.049$$

4. **Standard Deviation Risk Premium:**

$$\mathcal{H}(X) = E[X] + \alpha\sqrt{Var[X]} = 1,325.163 + (0.20*\sqrt{6,654.428}) = 1,341.478$$

5. **Exponential Risk Premium**
   We can write the Exponential Risk Premium as follow;

$$\mathcal{H}(X) = \frac{1}{\alpha} \ln E[e^{\alpha x}] = \frac{1}{\alpha} \ln M_x(\alpha)$$

   The moment Generating Function of Lognormal Distribuion is that

$$M_x(\alpha) = e^{\alpha\mu + \frac{1}{2}\alpha^2\sigma^2}$$

   Then, Exponential Risk Premium is

$$\begin{aligned}
\mathcal{H}(X) &= \frac{1}{\alpha} \ln E[e^{\alpha x}] = \frac{1}{\alpha} \ln M_x(\alpha) \\
&= \frac{1}{\alpha} \ln(e^{\alpha\mu + \frac{1}{2}\alpha^2\sigma^2}) \\
&= \frac{1}{\alpha}(\alpha\mu + \frac{1}{2}\alpha^2\sigma^2) \\
&= \mu + \frac{1}{2}\alpha\sigma^2 \\
&= 1,325.163 + \frac{1}{2} * 0.20 * 6,654.428 \\
&= 1,990.606
\end{aligned}$$

6. **Esscher Risk Premium** We know that Esscher Risk Premium can be represented by using the first derivative of Cumulative Generating Function.

$$C(t) = \ln M_x(t)$$

   Then, the first derivative of the Cumulative Generating function is the following equation;

$$C'(t) = \frac{M'_x(t)}{M_x(t)} = \frac{E[Xe^{tX}]}{E[e^{tX}]}$$

   The moment Generating Function and it's first derivative of Lognormal Distribuion are that

$$M_x(\alpha) = e^{\alpha\mu + \frac{1}{2}\alpha^2\sigma^2}$$

$$M'_x(\alpha) = (\mu + \alpha\sigma^2) * e^{\alpha\mu + \frac{1}{2}\alpha^2\sigma^2}$$

   Then, Esscher Risk Premium is

$$\begin{aligned}
\mathcal{H}(X) &= \frac{E[Xe^{\alpha X}]}{E[e^{\alpha X}]} \\
&= \frac{M'_x(t)}{M_x(t)} \\
&= \frac{(\mu + \alpha\sigma^2) * e^{\alpha\mu + \frac{1}{2}\alpha^2\sigma^2}}{e^{\alpha\mu + \frac{1}{2}\alpha^2\sigma^2}} \\
&= \mu + \alpha\sigma^2 \\
&= 1,325.163 + (0.20 * 6,654.428) \\
&= 2,656.049
\end{aligned}$$

7. **Value at Risk**
Let's assume that X is distributed with lognormal with meanμ and variance σ^2; ie; LN(μ,σ^2) and we also know the cumulative and inverse cumulative function of Lognormal distribution as follows;

$$\begin{aligned} F[X] &= \Phi(\frac{\ln x - \mu}{\sigma}) \\ F^{-1}[\delta] &= exp[\mu + \sigma\Phi^{-1}(\delta)] \end{aligned}$$

By the definition of VaR, we obtained the VaR of Lognormal distribution is as follows;

$$\begin{aligned} VaR_{0.95}[X] &= F^{-1}[\delta] \\ VaR_{0.95}[X] &= exp(\mu + \sigma\Phi^{-1}(0.95)) \end{aligned}$$

where Φ denotes the standard normal cumulative distribution function. Hence, VaR for Lognormal distribution is

$$\begin{aligned} VaR_{0.95}[X] &= e^{\mu+\sigma\Phi^{-1}(0.95)} \\ &= e^{7.1874+(0.0615*1.645)} \\ &= e^{7.2886} \\ &= 1,463.473 \end{aligned}$$

8. **Conditional Value at Risk**

$$\begin{aligned} CVaR_{0.95}(X) &= E[X|X > VaR_{0.95}(X)] \\ &= \frac{\int_{x_{0.95}}^{\infty} xf(x)dx}{1 - 0.95} \end{aligned}$$

The denominator of above equation is

$$\int_{x_{0.95}}^{\infty} xf(x)dx = \int_{x_{0.95}}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(\ln x-\mu)^2}{2\sigma^2}} dx$$

By applying the transformation $z = \frac{\ln x - \mu}{\sigma} - \sigma$ and $dx = \sigma e^{\mu+\sigma^2+\sigma z} dz$

$$\begin{aligned} &= e^{\mu+\frac{\sigma^2}{2}} * \int_{z^*}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{z^2}{2}} dz \\ &= e^{\mu+\frac{\sigma^2}{2}} [1 - \Phi(z^*)] \end{aligned}$$

where Φ is the cumulative distribution function of standard normal distribution and $z^* = \frac{\ln x_{0.95} - \mu}{\sigma} - \sigma = \frac{7.2886 - 7.1874}{0.0615} - 0.0615 = 1.5835$. Hence, CVaR for

Lognormal distribution is

$$
\begin{aligned}
CVaR_{0.95}(X) &= \frac{e^{\mu+\frac{\sigma^2}{2}}[1-\Phi(z^*)]}{1-0.95} \\
&= \frac{e^{7.1874+\frac{(0.0615)^2}{2}}[1-\Phi(1.5835)]}{0.05} \\
&= \frac{e^{7.1893}[1-0.9430]}{0.05} \\
&= \frac{1,325.163 * 0.057}{0.05} \\
&= 1,510.686
\end{aligned}
$$

Secondly; the following calculations represent the risk premiums results of Ankara based on methods of moments.

$$
\begin{aligned}
E[X] &= e^{\mu+\frac{1}{2}\sigma^2} \\
&= e^{7.0348+\frac{1}{2}(0.0505)^2} \\
&= 1,136.917
\end{aligned}
$$

$$
\begin{aligned}
Var[X] &= e^{2\mu+\sigma^2}(e^{\sigma^2}-1) \\
&= e^{7.0348+(0.0505)^2}(e^{(0.0505)^2}-1) \\
&= 3,300.607
\end{aligned}
$$

1. **Net Risk Premium:**

$$\mathcal{H}(X) = E[X] = 1,136.917$$

2. **Expected Value Risk Premium:**

$$\mathcal{H}(X) = (1+\alpha)E[X] = (1+0.20) * 1,136.917 = 1,364.3$$

3. **Variance Risk Premium:**

$$\mathcal{H}(X) = E[X] + \alpha Var[X] = 1,136.917 + (0.20 * 3,300.607) = 1,797.038$$

4. **Standard Deviation Risk Premium:**

$$\mathcal{H}(X) = E[X] + \alpha\sqrt{Var[X]} = 1,136.917 + (0.20 * \sqrt{3,300.607}) = 1,148.407$$

5. **Exponential Risk Premium**
We can write the Exponential Risk Premium as follow;
$$\mathcal{H}(X) = \frac{1}{\alpha}\ln E[e^{\alpha x}] = \frac{1}{\alpha}\ln M_x(\alpha)$$
The moment Generating Function of Lognormal Distribuion is that
$$M_x(\alpha) = e^{\alpha\mu + \frac{1}{2}\alpha^2\sigma^2}$$
Then, Exponential Risk Premium is
$$\begin{aligned}
\mathcal{H}(X) &= \frac{1}{\alpha}\ln E[e^{\alpha x}] = \frac{1}{\alpha}\ln M_x(\alpha) \\
&= \frac{1}{\alpha}\ln(e^{\alpha\mu + \frac{1}{2}\alpha^2\sigma^2}) \\
&= \frac{1}{\alpha}(\alpha\mu + \frac{1}{2}\alpha^2\sigma^2) \\
&= \mu + \frac{1}{2}\alpha\sigma^2 \\
&= 1,136.917 + \frac{1}{2} * 0.20 * 3,300.607 \\
&= 1,466.977
\end{aligned}$$

6. **Esscher Risk Premium**
We know that Esscher Risk Premium can be represented by using the first derivative of Cumulative Generating Function.
$$C(t) = \ln M_x(t)$$
Then, the first derivative of the Cumulative Generating function is the following equation;
$$C'(t) = \frac{M'_x(t)}{M_x(t)} = \frac{E[Xe^{tX}]}{E[e^{tX}]}$$
The moment Generating Function and it's first derivative of Lognormal Distribuion are that
$$M_x(\alpha) = e^{\alpha\mu + \frac{1}{2}\alpha^2\sigma^2}$$
$$M'_x(\alpha) = (\mu + \alpha\sigma^2) * e^{\alpha\mu + \frac{1}{2}\alpha^2\sigma^2}$$
Then, Esscher Risk Premium is
$$\begin{aligned}
\mathcal{H}(X) &= \frac{E[Xe^{\alpha X}]}{E[e^{\alpha X}]} \\
&= \frac{M'_x(t)}{M_x(t)} \\
&= \frac{(\mu + \alpha\sigma^2) * e^{\alpha\mu + \frac{1}{2}\alpha^2\sigma^2}}{e^{\alpha\mu + \frac{1}{2}\alpha^2\sigma^2}} \\
&= \mu + \alpha\sigma^2 \\
&= 1,136.917 + (0.20 * 3,300.607) \\
&= 1,797.038
\end{aligned}$$

7. **Value at Risk**
Let's assume that X is distributed with lognormal with meanμ and variance σ^2; ie; LN(μ,σ^2) and we also know the cumulative and inverse cumulative function of Lognormal distribution as follows;

$$\begin{aligned} F[X] &= \Phi(\frac{\ln x - \mu}{\sigma}) \\ F^{-1}[\delta] &= exp[\mu + \sigma\Phi^{-1}(\delta)] \end{aligned}$$

By the definition of VaR, we obtained the VaR of Lognormal distribution is as follows;

$$\begin{aligned} VaR_{0.95}[X] &= F^{-1}[\delta] \\ VaR_{0.95}[X] &= exp(\mu + \sigma\Phi^{-1}(0.95)) \end{aligned}$$

where Φ denotes the standard normal cumulative distribution function. Hence, VaR for Lognormal distribution is

$$\begin{aligned} VaR_{0.95}[X] &= e^{\mu+\sigma\Phi^{-1}(0.95)} \\ &= e^{7.0348+(0.0505*1.645)} \\ &= e^{7.1179} \\ &= 1,233.823 \end{aligned}$$

8. **Conditional Value at Risk**

$$\begin{aligned} CVaR_{0.95}(X) &= E[X|X > VaR_{0.95}(X)] \\ &= \frac{\int_{x_{0.95}}^{\infty} xf(x)dx}{1 - 0.95} \end{aligned}$$

The denominator of above equation is

$$\int_{x_{0.95}}^{\infty} xf(x)dx = \int_{x_{0.95}}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(\ln x-\mu)^2}{2\sigma^2}} dx$$

By applying the transformation $z = \frac{\ln x - \mu}{\sigma} - \sigma$ and $dx = \sigma e^{\mu+\sigma^2+\sigma z} dz$

$$\begin{aligned} &= e^{\mu+\frac{\sigma^2}{2}} * \int_{z^*}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{z^2}{2}} dz \\ &= e^{\mu+\frac{\sigma^2}{2}} [1 - \Phi(z^*)] \end{aligned}$$

where Φ is the cumulative distribution function of standard normal distribution and $z^* = \frac{\ln x_{0.95} - \mu}{\sigma} - \sigma = \frac{7.1179 - 7.0348}{0.0505} - 0.0505 = 1.5950$. Hence, CVaR for

Lognormal distribution is

$$\begin{aligned} CVaR_{0.95}(X) &= \frac{e^{\mu+\frac{\sigma^2}{2}}[1-\Phi(z^*)]}{1-0.95} \\ &= \frac{e^{7.0348+\frac{(0.0505)^2}{2}}[1-\Phi(1.5950)]}{0.05} \\ &= \frac{e^{7.0361}[1-0.9441]}{0.05} \\ &= \frac{1,136.917 * 0.0559}{0.05} \\ &= 1,271.073 \end{aligned}$$

C.2 Based on Methods of Maximum Likelihood Estimation

Parameter estimation results of maximum likelihood estimation give us the mean and variance of the selected cities based on Lognormal distribution so that we can calculate the risk premiums based on the their definitions.

The mean and the variance of Lognormal Distribution of Istanbul as follows;

$$\begin{aligned} E[X] &= e^{\mu+\frac{1}{2}\sigma^2} \\ &= e^{7.1082+\frac{1}{2}(0.0314)^2} \\ &= 1,222.549 \end{aligned}$$

$$\begin{aligned} Var[X] &= e^{2\mu+\sigma^2}(e^{\sigma^2}-1) \\ &= e^{2*7.1082+(0.0314)^2}(e^{(0.0314)^2}-1) \\ &= 1,474.367 \end{aligned}$$

1. **Net Risk Premium:**
$$\mathcal{H}(X) = E[X] = 1,222.55$$
2. **Expected Value Risk Premium:**
$$\mathcal{H}(X) = (1+\alpha)E[X] = (1+0.20) * 1,222.549 = 1,467.059$$
3. **Variance Risk Premium:**
$$\mathcal{H}(X) = E[X] + \alpha Var[X] = 1,222.549 + (0.20 * 1,467.367) = 1,517.421$$
4. **Standard Deviation Risk Premium:**
$$\mathcal{H}(X) = E[X] + \alpha\sqrt{Var[X]} = 1,222.549 + (0.20 * \sqrt{1,467.367}) = 1,230.231$$

5. **Exponential Risk Premium**
   We can write the Exponential Risk Premium as follow;

$$\mathcal{H}(X) = \frac{1}{\alpha} \ln E[e^{\alpha x}] = \frac{1}{\alpha} \ln M_x(\alpha)$$

   The moment Generating Function of Lognormal Distribuion is that

$$M_x(\alpha) = e^{\alpha\mu + \frac{1}{2}\alpha^2\sigma^2}$$

   Then, Exponential Risk Premium is

$$\begin{aligned}
\mathcal{H}(X) &= \frac{1}{\alpha} \ln E[e^{\alpha x}] = \frac{1}{\alpha} \ln M_x(\alpha) \\
&= \frac{1}{\alpha} \ln(e^{\alpha\mu + \frac{1}{2}\alpha^2\sigma^2}) \\
&= \frac{1}{\alpha}(\alpha\mu + \frac{1}{2}\alpha^2\sigma^2) \\
&= \mu + \frac{1}{2}\alpha\sigma^2 \\
&= 1,222.549 + \frac{1}{2} * 0.20 * 1,467.367 \\
&= 1,369.987
\end{aligned}$$

6. **Esscher Risk Premium** We know that Esscher Risk Premium can be represented by using the first derivative of Cumulative Generating Function.

$$C(t) = \ln M_x(t)$$

   Then, the first derivative of the Cumulative Generating function is the following equation;

$$C'(t) = \frac{M_x'(t)}{M_x(t)} = \frac{E[Xe^{tX}]}{E[e^{tX}]}$$

   The moment Generating Function and it's first derivative of Lognormal Distribuion are that

$$M_x(\alpha) = e^{\alpha\mu + \frac{1}{2}\alpha^2\sigma^2}$$
$$M_x'(\alpha) = (\mu + \alpha\sigma^2) * e^{\alpha\mu + \frac{1}{2}\alpha^2\sigma^2}$$

   Then, Esscher Risk Premium is

$$\begin{aligned}
\mathcal{H}(X) &= \frac{E[Xe^{\alpha X}]}{E[e^{\alpha X}]} \\
&= \frac{M_x'(t)}{M_x(t)} \\
&= \frac{(\mu + \alpha\sigma^2) * e^{\alpha\mu + \frac{1}{2}\alpha^2\sigma^2}}{e^{\alpha\mu + \frac{1}{2}\alpha^2\sigma^2}} \\
&= \mu + \alpha\sigma^2 \\
&= 1,222.549 + (0.20 * 1,467.367) \\
&= 1,517.421
\end{aligned}$$

7. **Value at Risk**
   Let's assume that X is distributed with lognormal with meanμ and variance σ^2; ie; LN(μ,σ^2) and we also know the cumulative and inverse cumulative function of Lognormal distribution as follows;

$$
\begin{aligned}
F[X] &= \Phi(\frac{\ln x - \mu}{\sigma}) \\
F^{-1}[\delta] &= exp[\mu + \sigma\Phi^{-1}(\delta)]
\end{aligned}
$$

   By the definition of VaR, we obtained the VaR of Lognormal distribution is as follows;

$$
\begin{aligned}
VaR_{0.95}[X] &= F^{-1}[\delta] \\
VaR_{0.95}[X] &= exp(\mu + \sigma\Phi^{-1}(0.95))
\end{aligned}
$$

   where Φ denotes the standard normal cumulative distribution function. Hence, VaR for Lognormal distribution is

$$
\begin{aligned}
VaR_{0.95}[X] &= e^{\mu+\sigma\Phi^{-1}(0.95)} \\
&= e^{7.1082+(0.0314*1.645)} \\
&= e^{7.1599} \\
&= 1,286.722
\end{aligned}
$$

8. **Conditional Value at Risk**

$$
\begin{aligned}
CVaR_{0.95}(X) &= E[X|X > VaR_{0.95}(X)] \\
&= \frac{\int_{x_{0.95}}^{\infty} xf(x)dx}{1-0.95}
\end{aligned}
$$

   The denominator of above equation is

$$
\int_{x_{0.95}}^{\infty} xf(x)dx = \int_{x_{0.95}}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(\ln x-\mu)^2}{2\sigma^2}} dx
$$

   By applying the transformation $z = \frac{\ln x - \mu}{\sigma} - \sigma$ and $dx = \sigma e^{\mu+\sigma^2+\sigma z} dz$

$$
\begin{aligned}
&= e^{\mu+\frac{\sigma^2}{2}} * \int_{z^*}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{z^2}{2}} dz \\
&= e^{\mu+\frac{\sigma^2}{2}} [1 - \Phi(z^*)]
\end{aligned}
$$

   where Φ is the cumulative distribution function of standard normal distribution and $z^* = \frac{\ln x_{0.95} - \mu}{\sigma} - \sigma = \frac{7.1599-7.1082}{0.0314} - 0.0314 = 1.6136$. Hence, CVaR for

Lognormal distribution is

$$\begin{aligned}
CVaR_{0.95}(X) &= \frac{e^{\mu+\frac{\sigma^2}{2}}[1-\Phi(z^*)]}{1-0.95} \\
&= \frac{e^{7.1082+\frac{(0.0314)^2}{2}}[1-\Phi(1.6136)]}{0.05} \\
&= \frac{e^{7.1087}[1-0.9465]}{0.05} \\
&= \frac{1,222.549 * 0.0535}{0.05} \\
&= 1,308.133
\end{aligned}$$

Secondly; the following calculations represent the risk premiums results of Ankara based on methods of maximum likelihood estimation.

$$\begin{aligned}
E[X] &= e^{\mu+\frac{1}{2}\sigma^2} \\
&= e^{6.9361+\frac{1}{2}(0.0355)^2} \\
&= 1,029.403
\end{aligned}$$

$$\begin{aligned}
Var[X] &= e^{2\mu+\sigma^2}(e^{\sigma^2}-1) \\
&= e^{2*6.9361+(0.0355)^2}(e^{(0.0355)^2}-1) \\
&= 1,336.280
\end{aligned}$$

1. **Net Risk Premium:**

$$\mathcal{H}(X) = E[X] = 1,029.403$$

2. **Expected Value Risk Premium:**

$$\mathcal{H}(X) = (1+\alpha)E[X] = (1+0.20) * 1,029.403 = 1,235.284$$

3. **Variance Risk Premium:**

$$\mathcal{H}(X) = E[X] + \alpha Var[X] = 1,029.403 + (0.20 * 1,336.280) = 1,296.659$$

4. **Standard Deviation Risk Premium:**

$$\mathcal{H}(X) = E[X] + \alpha\sqrt{Var[X]} = 1,029.403 + (0.20 * \sqrt{1,336.280}) = 1,036.714$$

5. **Exponential Risk Premium**
We can write the Exponential Risk Premium as follow;

$$\mathcal{H}(X) = \frac{1}{\alpha} \ln E[e^{\alpha x}] = \frac{1}{\alpha} \ln M_x(\alpha)$$

The moment Generating Function of Lognormal Distribuion is that

$$M_x(\alpha) = e^{\alpha\mu + \frac{1}{2}\alpha^2\sigma^2}$$

Then, Exponential Risk Premium is

$$\begin{aligned}
\mathcal{H}(X) &= \frac{1}{\alpha} \ln E[e^{\alpha x}] = \frac{1}{\alpha} \ln M_x(\alpha) \\
&= \frac{1}{\alpha} \ln(e^{\alpha\mu + \frac{1}{2}\alpha^2\sigma^2}) \\
&= \frac{1}{\alpha}(\alpha\mu + \frac{1}{2}\alpha^2\sigma^2) \\
&= \mu + \frac{1}{2}\alpha\sigma^2 \\
&= 1,029.403 + \frac{1}{2} * 0.20 * 1,336.280 \\
&= 1,163.031
\end{aligned}$$

6. **Esscher Risk Premium**
We know that Esscher Risk Premium can be represented by using the first derivative of Cumulative Generating Function.

$$C(t) = \ln M_x(t)$$

Then, the first derivative of the Cumulative Generating function is the following equation;

$$C'(t) = \frac{M_x'(t)}{M_x(t)} = \frac{E[Xe^{tX}]}{E[e^{tX}]}$$

The moment Generating Function and it's first derivative of Lognormal Distribuion are that

$$M_x(\alpha) = e^{\alpha\mu + \frac{1}{2}\alpha^2\sigma^2}$$

$$M_x'(\alpha) = (\mu + \alpha\sigma^2) * e^{\alpha\mu + \frac{1}{2}\alpha^2\sigma^2}$$

Then, Esscher Risk Premium is

$$\begin{aligned}
\mathcal{H}(X) &= \frac{E[Xe^{\alpha X}]}{E[e^{\alpha X}]} \\
&= \frac{M_x'(t)}{M_x(t)} \\
&= \frac{(\mu + \alpha\sigma^2) * e^{\alpha\mu + \frac{1}{2}\alpha^2\sigma^2}}{e^{\alpha\mu + \frac{1}{2}\alpha^2\sigma^2}} \\
&= \mu + \alpha\sigma^2 \\
&= 1,029.403 + (0.20 * 1,336.280) \\
&= 1,296.659
\end{aligned}$$

7. **Value at Risk**
   Let's assume that X is distributed with lognormal with meanμ and variance σ^2; ie; LN(μ,σ^2) and we also know the cumulative and inverse cumulative function of Lognormal distribution as follows;

$$\begin{aligned} F[X] &= \Phi(\frac{\ln x - \mu}{\sigma}) \\ F^{-1}[\delta] &= exp[\mu + \sigma\Phi^{-1}(\delta)] \end{aligned}$$

By the definition of VaR, we obtained the VaR of Lognormal distribution is as follows;

$$\begin{aligned} VaR_{0.95}[X] &= F^{-1}[\delta] \\ VaR_{0.95}[X] &= exp(\mu + \sigma\Phi^{-1}(0.95)) \end{aligned}$$

where Φ denotes the standard normal cumulative distribution function. Hence, VaR for Lognormal distribution is

$$\begin{aligned} VaR_{0.95}[X] &= e^{\mu+\sigma\Phi^{-1}(0.95)} \\ &= e^{6.9361+(0.0355*1.645)} \\ &= e^{6.9945} \\ &= 1,090.616 \end{aligned}$$

8. **Conditional Value at Risk**

$$\begin{aligned} CVaR_{0.95}(X) &= E[X|X > VaR_{0.95}(X)] \\ &= \frac{\int_{x_{0.95}}^{\infty} xf(x)dx}{1 - 0.95} \end{aligned}$$

The denominator of above equation is

$$\int_{x_{0.95}}^{\infty} xf(x)dx = \int_{x_{0.95}}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(\ln x-\mu)^2}{2\sigma^2}} dx$$

By applying the transformation $z = \frac{\ln x - \mu}{\sigma} - \sigma$ and $dx = \sigma e^{\mu+\sigma^2+\sigma z} dz$

$$\begin{aligned} &= e^{\mu+\frac{\sigma^2}{2}} * \int_{z^*}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{z^2}{2}} dz \\ &= e^{\mu+\frac{\sigma^2}{2}} [1 - \Phi(z^*)] \end{aligned}$$

where Φ is the cumulative distribution function of standard normal distribution and $z^* = \frac{\ln x_{0.95} - \mu}{\sigma} - \sigma = \frac{6.9945 - 6.9361}{0.0355} - 0.0355 = 1.6095$. Hence, CVaR for

Lognormal distribution is

$$\begin{aligned}
CVaR_{0.95}(X) &= \frac{e^{\mu+\frac{\sigma^2}{2}}[1-\Phi(z^*)]}{1-0.95} \\
&= \frac{e^{6.9361+\frac{(0.0355)^2}{2}}[1-\Phi(1.6095)]}{0.05} \\
&= \frac{e^{6.9367}[1-0.9463]}{0.05} \\
&= \frac{1,029.368 * 0.0537}{0.05} \\
&= 1,105.541
\end{aligned}$$