# A COMPARISON OF CONSTANT AND STOCHASTIC VOLATILITY IN MERTON'S PORTFOLIO OPTIMIZATION PROBLEM

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submitted by **OZAN ÖZTÜRK** in partial fulfillment of the requirements for the degree of **Master of Science in Department of Financial Mathematics, Middle East Technical University** by,

Prof. Dr. Ömür Uğur
Director, Graduate School of **Applied Mathematics** _____

Prof. Dr. A. Sevtap Selçuk-Kestel
Head of Department, **Financial Mathematics** _____

Assoc. Prof. Dr. Ali Devin Sezer
Supervisor, **Financial Mathematics, METU** _____

**Examining Committee Members:**

Prof. Dr. A. Sevtap Selçuk-Kestel
Financial Mathematics, METU _____

Assoc. Prof. Dr. Ali Devin Sezer
Financial Mathematics, METU _____

Asst. Prof. Dr. Çağın Ararat
Industrial Engineering, Bilkent University _____

**Date:** _____

**I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.**

Name, Last Name: OZAN ÖZTÜRK

Signature :

# ABSTRACT

A COMPARISON OF CONSTANT AND STOCHASTIC VOLATILITY IN MERTON'S PORTFOLIO OPTIMIZATION PROBLEM

Öztürk, Ozan

M.Sc., Department of Financial Mathematics

Supervisor : Assoc. Prof. Dr. Ali Devin Sezer

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Merton's Portfolio Problem is a dynamic portfolio choice problem, which assumes asset returns and covariances are constant. There is well documented evidence that, stock returns and volatilities are correlated. Therefore, stochastic volatility models in dynamic portfolio problems can give better results. The work [J. Liu, Portfolio selection in stochastic environments, Review of Financial Studies, 20(1), 2007] developed a general dynamic portfolio model that allows the parameters of the model to depend on an external process $X$; this general model includes Merton's portfolio problem with Heston stochastic volatility (Merton H) and constant volatility as special cases. Liu's solution involves substituting solutions of a specific form into the Hamilton Jacobi Bellman (HJB) equation associated with the problem and reducing it first to a simpler Partial Differential Equation (PDE), and then reducing this PDE into a sequence of Ordinary Differential Equations (ODE). In this thesis we give the details of these reductions. We then use the explicit solutions provided by Liu for the Merton H model to see the effect of replacing stochastic volatility with constant volatility in Merton's problem. We find that, a ratio(sensitivity to stochastic volatility ratio) depending on mean reversion rate, risk aversion and Sharpe ratio is the most important parameter in this respect. When the value of this ratio is small, incorporating stochastic volatility into the model has little effect on the optimal portfolio. When it is large (when Sharpe ratio is high and the investor has low risk aversion) taking stochastic volatility into consideration is meaningful.

*Keywords* : Dynamic portfolio choice, Merton's portfolio problem, Stochastic volatility, Heston model, Stochastic control.

# ÖZ

MERTON'UN PORTFÖY PROBLEMİNİN, SABİT VOLATİLİTE İLE STOKASTİK VOLATİLİTE OLDUĞU DURUMLARDA KARŞILAŞTIRILMASI

Öztürk, Ozan
Yüksek Lisans, Finansal Matematik Bölümü
Tez Yöneticisi : Doç. Dr. Ali Devin Sezer

Ağustos 2018, 44 sayfa

Merton'un portföy problemi, varlık getirilerinin ve volatilitelerinin sabit olduğunu varsayan bir dinamik portföy seçim problemidir. Varlık getirileri ve volatiliteleri arasinda korelasyon olduğuna dair önemli bulgular mevcuttur. Bu durum göz önününde bulundurulduğunda, stokastik volatilite kullanılan dinamik portföy seçimlerinin daha iyi sonuç vermesi öngorülmektedir. Bu bağlamda, Liu, model parametrelerinin, X ile ifade edilen, dışsal süreçlere bağlı olduğu genel bir portföy seçim modeli geliştirmiştir[J. Liu, Portfolio selection in stochastic environments, Review of Financial Studies, 20(1), 2007]. Merton portföy probleminde, Heston stokastik volatilitesi olduğu durum (Merton H) ve volatilitenin sabit olduğu durum, farklı durumlar olarak bu genel modelin içeriği dahilindedir. Liu'nun çözümü, belli bir sonuç formunun, ilgili Hamilton Jacobi Bellman (HJB) denkleminde değiştirilmesini, bunun sonucunda bu denklemin kısmi diferansiyel denkleme (PDE) dönüşümünü ve daha sonra da bu PDE'nin bir takim adi diferansiyel denklemlere (ODE) indirgenmesini içermektedir. Bu tezde bu işlemlerin detayları verilmiştir. Daha sonra Liu'nun Merton H modeli için verdiği eksplisit çözüm kullanılarak, Merton probleminde, stokastik volatilitenin sabit volatilite ile değitirilmesinin etkisi incelenmiştir. Bunun sonucunda ortalamaya dönüş hızı, Sharpe oranı, risk iştahı parametrelerine bağlı olan bir oranın (stokastik volatiliteye hassasiyet oranı) önemli olduğu tespit edilmiştir. Bu oran küçük iken stokastik volatilitenin modele dahil edilmesinin optimal portföy seçimi üzerindeki etkisi azdır. Bu oran büyük iken (Sharpe oranının yüksek olduğu ve yatırımcıların risk iştahı yüksek iken), stokastik volatilitenin hesaba katılması portföy seçimini etkilemektedir.

*Anahtar Kelimeler*: Dinamik sistemlerde portföy seçimi, Merton'un portföy problemi, Stokastik volatilite, Heston modeli, Stokastik kontrol.

*To My Family*

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# CHAPTER 1

# Introduction

Portfolio selection is one of the popular topics in finance literature for a long time. Investors always looked for an answer for how to manage their wealths more efficiently; whether to invest in stock market or buy government bonds, which stocks to buy, how to allocate wealth between different stocks, etc. In this sense, the first modern approach to answer these questions is advanced by Harry Markowitz in 1952[12]. He developed a model which optimizes portfolios according to their expected returns and variances (mean-variance optimization). Markowitz's model is based on a static framework which only includes one period of time and excludes future possible events which means the asset returns and correlations stay constant in the investment period. As a result, Markowitz's model fall short due to its limitation in adapting to changes in the market.

A dynamic model which can take into account fluctuations in the market would be more realistic and more accurate for investors, especially for long term investments. However, compared to static models, with the dynamic models it is more difficult to obtain solutions. For this reason, dynamical models are developed many years later. In 1969 Nobel laureate Robert Merton solved a dynamic portfolio problem, where an investor seeks to invest some part of its wealth and consume remaining part in a limited market setting[15]. In this setting, the investor either buy a risky asset such as a stock and/or a risk free asset, such as a treasury bond. In order to find both optimal asset allocation and optimal consumption, Merton used dynamic programming principles in continuous time setting. His seminal work is considered as the starting point of dynamic portfolio choice which is also subject of Stochastic Control theory.

The Merton problem can be explicitly solved by the help of a series of simplifying assumptions such as; risky assets follow geometric Brownian motion, mean and covariance of asset returns stay constant until the end of investment horizon, there are no transaction costs, assets are traded continuously in time, the market is perfectly liquid, investor has constant relative risk aversion(CRRA) etc. These assumptions contradict with dynamics of the stock market. However, Merton's solution to the problem gives a good understanding of dynamic portfolio problem. Thus, Merton's portfolio problem became a well established approach in the literature. But since the assumptions made were not very realistic, many researchers attempted to integrate more realistic assumptions to the model.

The assumption of constant mean and covariance of asset returns, is one of the major unrealistic assumptions in Merton's portfolio problem. There is a strong empirical evidence in the literature that stock market returns is not constant over time. In the works of French et al.[8], Campbell and Hentschel [3], fluctuation in volatility is observed and correlation between asset returns and volatility is documented. Further to this, the volatility is not constant as can be inferred in volatility smiles in option prices. For pricing risky assets, taking asset return mean and covariance as time varying processes fit market dynamics better. In literature, stochastic volatility models such as Heston [10], SABR [9], can explain smile curves. Considering this, stochastic volatility models in a dynamic portfolio choice may give better results than a classical Merton's portfolio model.

In this context, Liu [11] developed a general dynamical portfolio model for a CRRA type investor in incomplete markets. Liu characterized asset returns by a series of differential equations which he calls 'quadratic returns' and by the help of these differential equation he obtained a general solution; the model studied by Liu includes optimal investment under Heston stochastic volatility model as a special case. This extends Merton's classical framework and gives explicit solution of dynamic portfolio choice problem for a CRRA investor in a setting where the stock price follows the stochastic volatility model of Heston for incomplete markets.

In this thesis, an attempt has been made to see the difference between the constant volatility and stochastic volatility in dynamic portfolio choice and tried to find an answer to in which conditions it is relevant to integrate stochastic volatility to the dynamic portfolio choice problems. Liu's model of Merton's portfolio problem is the framework of this thesis. Based on this model, a comparison is made and a conclusion is driven. This thesis is organized as follows; the second chapter gives a brief background information about stock price movement and stochastic control theory. The third chapter deals with the Merton Portfolio Problem in its classical form. Liu's dynamic portfolio model is the subject matter of the fourth chapter. In fifth chapter, we give Liu's reduction of his general model to the problem of dynamic portfolio choice under stochastic volatility. We also show how one can reduce this solution to the constant volatility case. Finally we compare the two models of volatility in the sixth chapter.

## 1.1 Literature Review

Portfolio choice is a subject of considerable attention. Even though there are numerous studies on the subject, there are a few studies on the dynamic portfolio choice which incorporates stochastic volatility of the risky asset's return. These studies are differentiated from each other with respect to their assumptions on asset returns, utility settings and market conditions. Schroder and Skiadas [17], for example, studied stochastic volatility for homothetic generalized recursive utility and driven closed-form solutions for the optimal consumption for incomplete markets as a special case of complete markets. Chacko and Viceira [4], studied dynamic portfolio choice in incomplete markets for one-factor stochastic volatility model (with constant expected return and

time varying precision–the reciprocal of volatility) with recursive utility preferences and gave a closed-form approximate analytic solution in their study. They found out that the optimal portfolio demand for stocks includes an inter-temporal hedging component. They implemented their model to US stock data and found that the stock return volatility did not appear to be variable and persistent enough to generate large inter-temporal hedging demands. Liu[11] focused on dynamic portfolio choice problems when the asset returns were quadratic and the investor had a constant relative risk aversion utility function. Liu, gave exact explicit solution, while other studies gave approximate solutions to the problem. Liu applied his method to portfolio choice problem for incomplete markets when the stock return follows Heston's [10] stochastic volatility model.

Following Liu's work, Buraschi et al.[2] developed a new multivariate inter-temporal portfolio choice which allows volatility and correlation risk to have separate roles and in that context, they derived an optimal portfolio implications for economies in which the degree of correlation across industries, countries, or asset classes is stochastic. They found that the optimal hedging demand can be significantly different from the one implied by more common models with constant correlations or single-factor stochastic volatility. Later, Faria et al.[7], extending the work of Chacko and Viceira[4], took ambiguity into account in the dynamic portfolio choice and test, how ambiguity about the stochastic processes that generate the return and volatility of the risky asset impacts portfolio choice. Faria, found that the ambiguity affects portfolio choice, through a specific correlation structure, that also induces ambiguity about the return process. Another example of stochastic volatility in dynamic portfolio choice is the study of Escobar et al.[6]. This study incorporated multi-factor stochastic volatility model (which is introduced by Christoffersen et al.[5] in option pricing) into the portfolio choice problem. In this respect, a model for multivariate inter-temporal portfolio choice in complete and incomplete markets with multi-factor stochastic covariance matrix of asset returns is developed and its performance on investors welfare is analyzed. Escobar et al. concluded that investors who invest myopically, ignore derivative assets, model volatility by one factor and ignore stochastic covariance between asset returns can incur significant welfare losses.

# CHAPTER 2

# Background - Stochastic Control Theory

In this chapter we will give some brief information about models of stock price movements. We give the definition of the geometric Brownian motion and we look at the stock price movements given by the Heston stochastic volatility model. Then we will introduce the concepts of Stochastic Control Theory.

## 2.1 Stock price model

### 2.1.1 Geometric Brownian Motion

The standard Brownian motion $B$ is often considered the simplest possible continuous time continuous stochastic process. It can be considered to be a continuous time random walk; rigorously, it is defined through the following properties:

1. $B(0) = 0$ and $B_t$ is almost surely continuous.
2. $B_t$ has $t \geqslant 0$ has stationary independent increments. if $r \leq s \leq t \leq u$ then $B_u - B_t$ and $B_s - B_r$ are independent stochastic variables
3. $B_t - B_s \sim \mathcal{N}(0, t-s)$ for $0 \leq s \leq t$

The simplest and earliest model for stock prices assumes that the underlying risky assets follow a geometric Brownian motion, which is represented by the following stochastic process:

$$dS_t = \mu S_t dt + \sigma S_t dB_t, \tag{2.1}$$

where $\mu$ and $\sigma$ are constants and $dB_t$ denotes integration with respect to the standard Brownian Motion. The constant $\mu$ is the drift coefficient and the $\mu S_t dt$ component of $dS_t$ corresponds to the drift process which represents the trend in the movement of the price. In this model, the random fluctuations in the price increment depends linearly on the current price itself (this is the $\sigma S_t dB_t$ term; the constant $\sigma$ (volatility) is the coefficient of this linear dependence.

A geometric Brownian motion model dictates that asset returns are assumed to be log normally distributed, and the standard deviation of the returns (i.e. the volatility) is assumed to be constant.

### 2.1.2 Heston model

A geometric Brownian motion is based on the assumption that the volatility of the risky asset is constant. In Heston's stochastic volatility model, Heston relaxed this assumption and defined the volatility as a stochastic process which affects the dynamics of stock prices. In Heston model the risky asset follows the following stochastic differential equation [10];

$$dS_t = \mu S_t dt + \sqrt{V_t} S_t dB_t, \tag{2.2}$$

$$dV_t = K(\theta - V_t)dt + \sigma_v \sqrt{V_t} dB_t^V, \tag{2.3}$$

Where $S_t$ denotes stock price and $V_t$ denotes variance, where $K, \theta, \sigma_v \geqslant 0$. $B_t$ and $B_t^v$ are Brownian motions. $K$ (mean-reversian rate )is the rate at which $V_t$ returns to the long-term volatility mean $\theta$. $\sigma_v$ is the volatility of the volatility. $B_t$ and $B_t^V$ are correlated Brownian motions:

$$dB_t dB_t^v = \rho dt, \tag{2.4}$$

where $\rho \in (-1, 1)$ is the correlation coefficient.

## 2.2 Basic Structure of Stochastic Control Problems

We now give a brief summary of concepts from stochastic control theory following Pham [16]. A stochastic control problem formed by three main features: state of the system, control and performance criterion. Brief description of each is given below.

### 2.2.1 State of the System

Stochastic Control Theory problems take place in a dynamic system where the system is characterized by its states and evolves in an uncertain environment. The system represented by a filtered probability space $(\Omega, F, \mathbb{F} = \{F_t, t \in [0, T]\}, \mathbb{P})$ . State of the system is described by some set of quantitative variables(state variables) which formalized as $X = X_t$, which is a $\mathbb{F}$ adapted stochastic process that shows the evolution of the variables describing the dynamic system through a stochastic differential equation that maps $t \to X_t(w)$ for all $\omega \in \Omega$.

For controlled diffusion process in $\mathbb{R}^n$ our state of the system follows the following stochastic differential equation

$$dX_s = b(X_s, \alpha_s)ds + \sigma(X_s, \alpha_s)dB_s, \tag{2.5}$$

where $B$ denotes d-dimensional Brownian motion on the filtered probability space.

In order to obtain a unique solution, one imposes regularity and growth conditions on the coefficients $a$ and $\sigma$. $b : \mathbb{R}^n \times A \to \mathbb{R}^n$ and $\sigma : \mathbb{R}^n \times A \to \mathbb{R}^{n\times d}$ satisfy following Lipschitz condition in $A$ for some $K$:

$$\sup_{a\in\mathcal{A}}|b(x,a) - b(y,a)| + |\sigma(x,a) - \sigma(y,a)| \leq K|x-y|. \tag{2.6}$$

### 2.2.2 Control

Control is a process, which influences the state variable $X$ in order to achieve a desired goal. If this goal is to optimize some payoff function then this concept is defined as optimal control. Control is denoted as $\alpha = \alpha_t$ whose value is decided at time t in function of the available information $F_t$. We want our control to be admissible. The set of all admissible control is represented by $A$

For controlled diffusion process in $\mathbb{R}^n$, our control needs to satisfy the following conditions in order to obtain unique and strong solutions for $A$

$$E\left[\int_0^T |b(0,\alpha_t)|^2 + |\sigma(0,\alpha_t)|^2 dt\right] < \infty, \tag{2.7}$$

$$E[\sup_{t\leqslant s\leqslant T}|X_s^{t,x}|^2] < \infty, \tag{2.8}$$

where the existence and uniqueness of a strong solution to the SDE (2.5) starting from $x$ at $s = t$ is denoted by $X_s^{t,x}$, $t \leq s \leq T$.

### 2.2.3 Performance Criterion

In stochastic control theory we want to optimize all admissible controls over a functional $J(t, x\alpha)$. Our performance criterion for controlled diffusion process is in the form;

$$J(t,x,\alpha) = E\bigg(\int_t^T f(s, X_s^{t,x}, \alpha_s)ds + g(X_s^{t,x})\bigg)(t,x) \in [0,T]\times\mathbb{R}^n,$$

on a finite horizon $[0, T]$

where $f$ represents a running profit function and $g$ represents a terminal reward function. Both $f$ and $g$ are utility functions which measures the happiness of the investor

by taking into account of risk aversion of the investor. Utility functions are strictly concave functions. $f',\ g' > 0\ f",\ g" < 0$

The maximum value attained from the performance criterion is defined as value function and denoted by $v$. The main purpose in stochastic control to find an optimal control that attains the value function.

$$v(t,x) = \sup_{\alpha \in A} J(t,x,\alpha). \tag{2.9}$$

$\hat{\alpha} \in A$ is an optimal control if $v(t,x) = J(t,x,\hat{\alpha})$

The process $\alpha$ is in the form $\alpha_s = (s, X_s^{t,x})$ for some measurable function $\alpha$ from $[0,T] \times \mathbb{R}^n$ into $A$ is called Markovian control.

## 2.3 Solution of the Stochastic Control Problem

A popular method for the solution of the stochastic control problem formulated above is Bellman's Dynamic Programing principle. This principle (see Bellman[1]) can be described as follows: if one knows the expected minimal cost for each of the possible states in the next step, the optimal control for the current state is one that optimizes the sum of the cost for the control chosen for the present step and the minimal cost of the state resulting from this choice. This gives a recursive description of the value function and the optimal control. In the continuous time framework, if the value function is differentiable enough, this recursion can be expressed as the HJB equation. Further details of these ideas are given below.

### 2.3.1 Dynamic Programming and HJB equation

In dynamic programing (DP), optimization problem was broken down to smaller subproblems by varying initial state values. Some relation among the value functions are sought. Let us see how this process leads to nonlinear Hamilton-Jacobi-Bellman (HJB) partial differential equations.

Breaking up the problem into two pieces (before and after a stopping time $\theta$ gives the following recursion of $v$:

$$v(t,x) = \sup_{\alpha \in A} \mathbb{E}\left[ \int_t^\theta f(X_s^{t,x}, \alpha_s) ds + v(\theta, X_\theta^{t,x}) \right], \tag{2.10}$$

for any stopping time $\theta \in \mathcal{T}_{t,T}$ (the set of stopping times valued in $[t,T]$)

When Ito's formula applied to $v(s, X_s^{(t,s)})$ between $s = t$ and $s = t + h$, the following

equation is obtained:

$$v(t+h, X_{t+h}^{t,x}) = v(t,x) + \int_t^{t+h} ((\frac{\partial t}{\partial v}) + L^{\alpha_s} v)(s, X_s^{t,x}) ds + \text{(local) martingale}, \quad (2.11)$$

where for $\alpha \in A$, $L^a$ is the second order operator associated to the diffusion $X$ with constant control $\alpha$

$$L^a v = b(x,a) D_x v + \frac{1}{2} \text{tr}(\sigma(x,\alpha)\sigma'(x,\alpha) D_x^2 v), \quad (2.12)$$

if we substitute this into the DP equation we will obtain

$$\sup_{\alpha \in A} \mathbb{E}\left[ \int_t^{t+h} ((\frac{\partial t}{\partial v}) + L^{\alpha_s} v)(s, X_s^{t,x}) ds + f(X_s^t, x, \alpha_s) ds \right] = 0, \quad (2.13)$$

divide by $h$ send $h$ to zero and obtain by mean value theorem. Then HJB equation is obtained

$$\frac{\partial v}{\partial t} + \sup_{\alpha \in A} [L^a v + f(x,a)] = 0, \quad (t,x) \in [0,T) \times \mathbb{R}^n, \quad (2.14)$$

with terminal condition

$$v(T^-, x) = v(T,x) = g(x), \quad x \in \mathbb{R}^n$$

Classical approach to stochastic control is to show the existence of a smooth solution and if possible obtain an explicit solution from HJB equation, if we can prove the existence of a smooth solution we can solve stochastic control problem by verification method

### 2.3.2 Verification argument

A verification argument consists of the following steps: through a study of the HJB equation or otherwise, produce a guess for the value function. Then use the HJB equation itself, Ito's formula and perhaps other tools from probability theory and the theory of stochastic processes to prove that your guess is indeed correct. The work Liu [11] and Merton's solution of the original optimal investment problem use this type of an argument. The argument starts with the assumption that the value function has a specific form and then it follows the consequences of this assumption, which eventually lead to a pair of ODE which are solved explicitly. Once the ODE are solved, the value function can be defined in terms of the solutions. Then these candidate solutions are rigorously verified to satisfy the HJB equations. Their smoothness, along with

Ito's formula imply that they are indeed the value functions of the associated control problems. Some of these ideas and reductions are given in detail in the following chapters.

# CHAPTER 3

## Merton's Portfolio Problem

Following the background information given, we will now look at Merton's portfolio problem. The Merton's portfolio problem is set in a limited market which consists of only two kind of securities. A riskless asset $S_0$ and a risky asset $S_1$. There is an investor who wants to allocate its wealth between these assets, and additionally, the investor intends to consume some amount of the wealth. Merton[15] looked for an optimal investment and consumption strategy for an constant relative risk aversion type investor in a setting where assets are traded continuously in time without any transaction cost and indivisibility. Also mean and covariance of asset returns assumed to stay constant until the end of the investment horizon

The price process of risky and riskless assets are given by the following equations Merton[14]

$$dS_{0t} = rS_{0t}dt, \tag{3.1}$$
$$dS_{1t} = \mu S_{1t}dt + \sigma S_{1t}dB_t, \tag{3.2}$$

where $B_t$ is a Brownian motion and $\mu, r, \sigma$ are constants with conditions;
$\mu \geqslant r \geqslant 0$ and $\sigma \geqslant 0$

Under these settings, investors wealth process evolves according to following equation

$$dW_t = (1-\phi_t)W_t r dt + \phi_t W_t(\mu dt + \sigma dB_t) - C_t dt, \tag{3.3}$$

where $\phi$ is the fraction of the wealth in risky asset and $C_t$ is the consumption rate. This process continues until some terminal time , $T$.

The agent will maximize the following expected utility:

$$\max E\left[\int_0^T e^{-\beta t} f[C_t]dt + f(W,T)\right], \tag{3.4}$$

subject to the budget constraints

$$C_t \geqslant 0, \quad W_t \geqslant 0, \quad W_0 > 0,$$

where $f$ is a strictly concave utility function $f'(t) > 0, f''(t) < 0$. Remembering dynamic programing equation we can state this equation in the form,

$$J(T, W, X) = \max_{\phi_{t=0}^{T}, c_{t=0}^{T}} E\left[\int_t^T e^{-\beta t} f(C_t)dt + f(W_t)\right], \tag{3.5}$$

where $J(T, W, X) = f(W_T)$.

If we take $T$ as $t + h$ and apply Ito's formula, we will obtain the following equation;

$$0 = \max_{\phi_t, C_t}\left[e^{-\beta t} f(C_t) + \frac{\partial J_t}{\partial t} + \frac{\partial J_t}{\partial W}[(\phi(\mu - r) + r)W_t - C_t] + \frac{\partial^2 J_t}{2\partial W^2}\sigma^2\phi_t^2 W_t^2\right]. \tag{3.6}$$

If we look for the first order conditions of the derivation, we will obtain

$$e^{-\beta t} f'(\hat{C}) - \frac{\partial J_t}{\partial W} = 0, \tag{3.7}$$

$$(\mu - r)\frac{\partial J_t}{\partial W} + \frac{\partial^2 J_t}{\partial W^2}\hat{\phi} W\sigma^2 = 0. \tag{3.8}$$

Nonlinear partial differential equations are difficult to solve, In order to make these nonlinear equation solvable, CRRA utility functions are selected. With CRRA utility functions we can obtain explicit solution.

Let $f(C)$ of the form $\dfrac{C^\gamma}{\gamma}$ where $0 < \gamma < 1$

If we rewrite our equation using the CRRA utility function we will obtain following equation;

$$0 = \frac{1-\gamma}{\gamma}\left[\frac{\partial J_t}{\partial W}\right]^{\frac{1-\gamma}{\gamma}} \exp\left(\frac{-\beta t}{1-\gamma}\right) + \frac{\partial J_t}{\partial t} + \frac{\partial J_t}{\partial W} rW - \frac{(\mu - r)^2[\partial J_t/\partial W]^2}{2\sigma^2 \partial^2 J_t/\partial W^2}, \tag{3.9}$$

$$C^* = \left[e^{\beta} t \frac{\partial J_t}{\partial W}\right]^{\frac{1}{\gamma - 1}}. \tag{3.10}$$

$$\phi^* = \frac{-(\mu - r)\partial J_t/\partial W}{\sigma^2 W \partial^2 J_t/\partial W^2}. \tag{3.11}$$

Remembering Stochastic Control Theory, we need a candidate solution which attain the value function. Let conjecture the solution in the form

$$J(W_t, t) = K\frac{e^{-\beta t}}{\gamma}[W(t)]^\gamma. \tag{3.12}$$

If we put our candidate solution into (3.9), K must satisfy the following ordinary differential equation;

$$K(t) = \alpha K(t) - (1-\gamma)[K(t)]^{-\frac{\gamma}{1-\gamma}},$$

where $\alpha \equiv \beta - \gamma\left[\frac{(\mu-r)^2}{2\sigma^2(2-\gamma)} + r\right]$ subject to $K(t) = e^{1-\gamma}$

Solution of this differential equation is ;

$$K(t) = \left[\frac{1 + (ve-1)e^{v(t-T)}}{v}\right]^{1-\gamma},$$

$$\text{where} \quad v = \frac{\mu}{1-\gamma}.$$

So our optimal control policy becomes

$$C^*(t) = [K(t)]^{\frac{1}{1-\gamma}} W(t), \tag{3.13}$$

$$w^*(t) = \frac{\mu - r}{\sigma^2(1-\gamma)}. \tag{3.14}$$

which suggests leaving a certain fraction of the wealth on risky asset. This fraction is directly proportional with risk premium. As the award of taking risk increases, stock weight increases as well. On the other hand, as variance increases, a riskier environment evolves, thus amount of stock weight decreases. As risk aversion increases, investors willingness to avoid risk increases, so an investor is unwilling to hold risky asset thus stock weight decreases.

# CHAPTER 4

## Liu's General Explicit Solution

In this chapter, we will look at Liu's general solution for dynamic portfolio choice. Let us consider a market, consist of a riskless asset ($P_0$) and M risky assets ($P_i$) whose price process is given by:

$$\frac{dP_{0t}}{P_{0t}} = r(X_t)dt, \tag{4.1}$$

$$\frac{dP_{it}}{P_{it}} = \mu_i(X_t)dt + \Sigma_i(X_t)dB_t, i = 1, ..., N. \tag{4.2}$$

where $B_t$ is m dimensional Brownian motion. The state of the system evolves according to following process.

$$dX_t = \mu^X dt + \Sigma^X dB_t^X. \tag{4.3}$$

Here, $r(X_t), \mu_i(X_t)$ and $\Sigma_i(X_t)$ represent the functions of an $N$dimensional state variable vector of $X_t$, $\mu^X$ is the drift coefficient and $\Sigma^X$ is the diffusion coefficient.

It is assumed that the agent is a price taker, assets are traded contentiously in time without transaction costs or asset indivisibility and short sales are allowed.

Under these settings, the following expected utility function will be maximized

$$\max_{\phi_t{}_{t=0}^T, c_t{}_{t=0}^T} E_0\left[\int_0^T \alpha e^{-\beta t}\frac{C_t^{1-\gamma}}{1-\gamma}dt + (1-\alpha)e^{-\beta T}\frac{W_T^{1-\gamma}}{1-\gamma}\right], \tag{4.4}$$

where $\phi_t$ represents the portfolio weights of the risky assets which is an M dimensional vector. $c_t$ is the consumption rate and $W_t$ denotes the value of trading strategy which finances $c_{t\,t=0}^T$ at time $T$.

$$dW_t = \Big(W_t[\phi_t^T(\mu - r) + r] - C_t\Big)dt + W_t\phi_t^T\Sigma dB_t. \tag{4.5}$$

$\gamma$ is the risk aversion coefficient. $\beta$ is the discount rate and $\alpha$ determines the importance rate of inter-temporal consumption. When $\alpha$ equals to zero then, the problem turns into an asset allocation problem with zero inter-temporal consumption.

Stochastic control theory approach leads to following Hamilton Jacobi Bellman equation for $J$ where $J(t, W, X)$ is an indirect utility function.

$$\max_{\phi,C}\Big\{\frac{\partial J}{\partial t}+\frac{1}{2}W^2\phi^T\Sigma\Sigma^T\phi J_{WW}+W[\phi^T(\mu-r)+r]J_W-CJ_W \\ +W\phi^T\Sigma\rho^T\Sigma^{XT}J_{WX}+\frac{1}{2}Tr(\Sigma^X\Sigma^{XT}J_{XX^T}) \\ +\mu^{XT}J_x+\alpha e^{-\beta t}\frac{C^{1-\gamma}}{1-\gamma}\Big\}=0, \tag{4.6}$$

with boundary condition

$$J(T,W,X)=(1-\alpha)e^{-\beta t}\frac{W^{1-\gamma}}{1-\gamma}.$$

Similar to the solution of the Merton Model, Liu searches a solution to (4.6) of the form:

$$J(t,w,x)=e^{-\beta t}\frac{w^{1-\gamma}}{1-\gamma}f(x,t)^\gamma. \tag{4.7}$$

The next proposition derives the equation that $f$ must satisfy so that $J$ above solves (4.6); this equation is given as (7) in [11]. Reduction of (4.6) to (4.8) is given in the appendix of [11]; the following proof is a detailed version of this reduction.

**Proposition 4.1.** *For $J$ of the form* (4.7) *the equation* (4.6) *reduces to*

$$f_t+0.5tr\left(\Sigma^X\Sigma^{XT}f_{xx}\right)+\left(\mu^X+\frac{1-\gamma}{\gamma}\Sigma^X\rho\Sigma^{-1}(\mu-r)\right)^T f_x \\ +\frac{1}{2f}(\gamma-1)f_x^T((\Sigma^X\Sigma^X)^T-\Sigma^X\rho\rho^T(\Sigma)^T)f_x \\ +\left(\frac{1-\gamma}{2\gamma^2}(\mu-r)^TA^{-1}(\mu-r)+\frac{1-\gamma}{\gamma}r-\frac{\beta}{\gamma}\right)f+\alpha^{1/\gamma}=0. \tag{4.8}$$

For two vectors $a$ and $b$ let $a\otimes b$ denote the matrix with entries $(a\otimes b)_{i,j}=a_ib_j$.

*Proof.* The derivatives of $J$ are

$$J_t=-\beta J+\gamma\frac{J}{f}f_t,\quad J_w=\frac{1-\gamma}{w}J,\quad J_{ww}=\frac{(\gamma-1)\gamma}{w^2}J \\ J_x=\frac{\gamma}{f}Jf_x,\quad J_{xw}=\frac{(1-\gamma)\gamma}{wf}Jf_x,\quad J_{xx}=\frac{\gamma(\gamma-1)}{f^2}Jf_x\otimes f_x+\frac{\gamma}{f}Jf_{xx}.$$

Substituting these in (4.6) and canceling out the $W$'s wherever possible give

$$\max_{\phi,C}\Bigg\{-\beta J+\gamma\frac{J}{f}f_t+\phi^T\frac{1}{2}A\phi(\gamma-1)\gamma J+(\phi^T(\mu-r)+r)(1-\gamma)J-C\frac{1-\gamma}{w}J$$
$$+\phi^T\Sigma\rho^T\Sigma^{XT}\frac{(1-\gamma)(\gamma)}{f}Jf_x+0.5\text{tr}\left(\Sigma^X\Sigma^{XT}\left(\frac{\gamma(\gamma-1)}{f^2}Jf_x\otimes f_x+\frac{\gamma}{f}Jf_{xx}\right)\right)$$
$$+\mu^X\frac{\gamma}{f}Jf_x+\alpha e^{-\beta t}\frac{C^{1-\gamma}}{1-\gamma}\Bigg\}=0,$$

Multiplying both side of the last equation by $f/J$ reduces the last display to

$$\max_{\phi,C}\Bigg\{-\beta f+\gamma f_t+\phi^T\frac{1}{2}A\phi(\gamma-1)\gamma f+(\phi^T(\mu-r)+r)(1-\gamma)f-C\frac{1-\gamma}{w}f$$
$$+\phi^T\Sigma\rho^T\Sigma^{XT}(1-\gamma)(\gamma)f_x+0.5\text{tr}\left(\Sigma^X\Sigma^{XT}\left(\frac{\gamma(\gamma-1)}{f}f_x\otimes f_x+\gamma f_{xx}\right)\right)$$
$$+\mu^X\gamma f_x+\alpha e^{-\beta t}\frac{f}{J}\frac{C^{1-\gamma}}{1-\gamma}\Bigg\}=0.$$

Taking those terms not depending on $\phi$ and $C$ out of the bracket reduce the last display to

$$0.5\text{tr}\left(\Sigma^X\Sigma^{XT}\left(\frac{\gamma(\gamma-1)}{f}f_x\otimes f_x+\gamma f_{xx}\right)\right)+\mu^X\gamma f_x-\beta f+\gamma f_t+r(1-\gamma)f$$
$$\max_{\phi,C}\Bigg\{-C\frac{1-\gamma}{w}f+\phi^T\Sigma\rho^T\Sigma^{XT}(1-\gamma)(\gamma)f_x+\phi^T\frac{1}{2}A\phi(\gamma-1)\gamma f$$
$$+\phi^T(\mu-r)(1-\gamma)f+\alpha e^{-\beta t}\frac{f}{J}\frac{C^{1-\gamma}}{1-\gamma}\Bigg\}=0. \tag{4.9}$$

Denote the expression inside the last brackets by $E(\phi,C)$. $E$ is concave with gradient

$$E_{(\phi,C)}=\Big((\gamma-1)(\gamma)fA\phi+(1-\gamma)(\mu-r)f+\Sigma\rho^T\Sigma^{XT}(1-\gamma)(\gamma)f_x\ ,$$
$$-\frac{1-\gamma}{w}f+\alpha e^{-\beta t}C^{-\gamma}\frac{f}{J}\Big). \tag{4.10}$$

The concavity of $E$ implies that the optimizer of the above problem $(\phi^*,C^*)$ is the solution of

$$E_{(\phi,C)}=0.$$

Substituting the formula (4.10) for the gradient gives the following equation for $C$

$$-\frac{1-\gamma}{w}f+\alpha e^{-\beta t}C^{-\gamma}\frac{f}{J}=0. \tag{4.11}$$

and the following for $\phi$

$$\gamma A\phi - (\mu - r) - \Sigma\rho^T\Sigma^{XT}\gamma\frac{f_x}{f} = 0, \tag{4.12}$$

where we have divided both sides by $(\gamma - 1)f$. Let us solve (4.11):

$$-\frac{1-\gamma}{w}f + \alpha e^{-\beta t}C^{-\gamma}\frac{f}{J} = 0 \Rightarrow -\frac{1-\gamma}{w} + \alpha e^{-\beta t}C^{-\gamma}\frac{1}{J} = 0$$

Substitute the formula (4.7) for $J$:

$$\begin{aligned} -\frac{1-\gamma}{w} + \alpha e^{-\beta t}C^{-\gamma}\frac{1-\gamma}{e^{-\beta t}w^{1-\gamma}f^{\gamma}} = 0 \Rightarrow -1 + \alpha C^{-\gamma}\frac{1}{w^{-\gamma}f^{\gamma}} = 0 \\ C^{-\gamma} = \alpha^{-1}w^{-\gamma}f^{\gamma} \\ C^* = \alpha^{\frac{1}{\gamma}}wf^{-1}. \end{aligned}$$

(4.12) is linear in $\phi$, whose solution is obtained by inverting $A$:

$$\phi^* = \frac{1}{\gamma}A^{-1}\left(\mu - r + \gamma\Sigma\rho^T\Sigma^{XT}\frac{f_x}{f}\right). \tag{4.13}$$

Now let us substitute $C^*$ in (4.9) to simplify that equation:

$$\begin{aligned} &0.5\text{tr}\left(\Sigma^X\Sigma^{XT}\left(\frac{\gamma(\gamma-1)}{f}f_x \otimes f_x + \gamma f_{xx}\right)\right) + \mu^X\gamma f_x - \beta f + \gamma f_t + r(1-\gamma)f \\ &- \alpha^{\frac{1}{\gamma}}wf^{-1}\frac{1-\gamma}{w}f + \phi^*\Sigma\rho^T\Sigma^{XT}(1-\gamma)(\gamma)f_x + \phi^*\frac{1}{2}A\phi(\gamma-1)\gamma f \\ &\quad + \phi^*(\mu - r)(1-\gamma)f + \alpha e^{-\beta t}\frac{f}{J}\frac{\left(\alpha^{\frac{1}{\gamma}}wf^{-1}\right)^{1-\gamma}}{1-\gamma} = 0. \end{aligned} \tag{4.14}$$

Cancel the $w$ and $f$ terms in the first term of the second line and substitute the formula for $J$ in the last line to get:

$$\begin{aligned} &0.5\text{tr}\left(\Sigma^X\Sigma^{XT}\left(\frac{\gamma(\gamma-1)}{f}f_x \otimes f_x + \gamma f_{xx}\right)\right) + \mu^X\gamma f_x - \beta f + \gamma f_t + r(1-\gamma)f \\ &- \alpha^{\frac{1}{\gamma}}(1-\gamma) + \phi^*\Sigma\rho^T\Sigma^{XT}(1-\gamma)(\gamma)f_x + \phi^*\frac{1}{2}A\phi(\gamma-1)\gamma f \\ &\quad + \phi^*(\mu - r)(1-\gamma)f + \alpha^{\frac{1}{\gamma}} = 0. \end{aligned}$$

Combining the first term on the second line and the last term on the third line reduces the last equation to

$$\begin{aligned} &0.5\text{tr}\left(\Sigma^X\Sigma^{XT}\left(\frac{\gamma(\gamma-1)}{f}f_x \otimes f_x + \gamma f_{xx}\right)\right) + \mu^X\gamma f_x - \beta f + \gamma f_t + r(1-\gamma)f \\ &+ \phi^*\Sigma\rho^T\Sigma^{XT}(1-\gamma)(\gamma)f_x + \phi^*\frac{1}{2}A\phi(\gamma-1)\gamma f + \phi^*(\mu - r)(1-\gamma)f + \gamma\alpha^{\frac{1}{\gamma}} = 0. \end{aligned} \tag{4.15}$$

Let us now expand the terms $\phi^*\Sigma\rho^T\Sigma^{XT}(1-\gamma)(\gamma)f_x$, $\phi^*\frac{1}{2}A\phi(\gamma-1)\gamma f$ and $\phi^*(\mu-r)(1-\gamma)f$ by substituting (4.13) for $\phi^*$; we begin with the last one:

$$
\begin{aligned}
(\phi^*)^T(\mu-r)(1-\gamma)f &= \left(\frac{1}{\gamma}A^{-1}\left(\mu-r+\gamma\Sigma\rho^T\Sigma^{XT}\frac{f_x}{f}\right)\right)^T(\mu-r)(1-\gamma)f \\
&= \frac{1-\gamma}{\gamma}(\mu-r)^T(A^{-1})(\mu-r)f + (1-\gamma)\left(A^{-1}\Sigma\rho^T(\Sigma^X)^Tf_x\right)^T(\mu-r), \quad (4.16)
\end{aligned}
$$

where we have used $A^{-1}=(A^{-1})^T$. Now remember that $A=\Sigma\Sigma^T$. Therefore, $A^{-1}=(\Sigma^T)^{-1}\Sigma^{-1}$. Substituting this in the last expression gives:

$$
\begin{aligned}
(1-\gamma)\left(A^{-1}\Sigma\rho^T(\Sigma^X)^Tf_x\right)^T &= (1-\gamma)\left((\Sigma^T)^{-1}\rho^T(\Sigma^X)^Tf_x\right)(\mu-r) \\
&= (1-\gamma)\left(\Sigma^X\rho\Sigma^{-1}(\mu-r)\right)^T f_X, \quad (4.17)
\end{aligned}
$$

where we have used $(Mx)^Ty=(M^Ty)^Tx$. Substitute (4.13) for the first $\phi^*$ in $[(\phi^*)^T\frac{1}{2}A\phi(\gamma-1)\gamma f\,]$

$$
\begin{aligned}
&(\phi^*)^T\frac{1}{2}A\phi^*(\gamma-1)\gamma f \\
&= \left(\frac{1}{\gamma}A^{-1}\left(\mu-r+\gamma\Sigma\rho^T\Sigma^{XT}\frac{f_x}{f}\right)\right)^T\frac{1}{2}A\phi^*(\gamma-1)\gamma f \\
&= \left(A^{-1}\left(\mu-r+\gamma\Sigma\rho^T\Sigma^{XT}\frac{f_x}{f}\right)\right)^T\frac{1}{2}A\phi^*(\gamma-1)f \\
&= \frac{1}{2}(\gamma-1)(\mu-r)^T\phi^*f + \frac{1}{2}\gamma(\gamma-1)\left(\Sigma\rho^T(\Sigma^X)^Tf_x\right)^T\phi^* \quad (4.18)
\end{aligned}
$$

Substitute again (4.13) for the remaining $\phi^*$, the first term:

$$
\begin{aligned}
&\frac{1}{2}(\gamma-1)(\mu-r)^T\phi^*f \\
&= f\frac{1}{2}(\gamma-1)(\mu-r)^T\frac{1}{\gamma}A^{-1}\left(\mu-r+\gamma\Sigma\rho^T\Sigma^{XT}\frac{f_x}{f}\right) \\
&= f\frac{1}{2}(\gamma-1)\frac{1}{\gamma}(\mu-r)^TA^{-1}(\mu-r) + \frac{1}{2}(\gamma-1)(\mu-r)^T\left(A^{-1}\left(\Sigma\rho^T\Sigma^{XT}f_x\right)\right) \\
&= f\frac{1}{2}\frac{(\gamma-1)}{\gamma}(\mu-r)^TA^{-1}(\mu-r) + \frac{1}{2}(\gamma-1)(\mu-r)^T\left((\Sigma^T)^{-1}\rho^T\Sigma^{XT}f_x\right) \\
&= f\frac{1}{2}\frac{(\gamma-1)}{\gamma}(\mu-r)^TA^{-1}(\mu-r) + \frac{1}{2}(\gamma-1)\left(\Sigma^X\rho\Sigma^{-1}(\mu-r)\right)^Tf_x; \quad (4.19)
\end{aligned}
$$

and the second term:

$$\begin{aligned}
&\frac{1}{2}(\gamma-1)\gamma(\gamma-1)\left(\Sigma\rho^T(\Sigma^X)^T f_x\right)^T\phi^* \\
&=\frac{1}{2}(\gamma-1)\left(\Sigma\rho^T(\Sigma^X)^T f_x\right)^T A^{-1}\left(\mu-r+\gamma\Sigma\rho^T\Sigma^{XT}\frac{f_x}{f}\right) \\
&=\frac{1}{2}(\gamma-1)\left(\Sigma\rho^T(\Sigma^X)^T f_x\right)^T A^{-1}(\mu-r)+\frac{1}{2}(\gamma-1)\left(\Sigma\rho^T(\Sigma^X)^T f_x\right)^T A^{-1}\gamma\Sigma\rho^T\Sigma^{XT}\frac{f_x}{f} \\
&=\frac{1}{2}(\gamma-1)\left(\Sigma^X\rho\Sigma^{-1}(\mu-r)\right)^T f_x+\frac{1}{2}(\gamma-1)\left(\Sigma\rho^T(\Sigma^X)^T f_x\right)^T A^{-1}\gamma\Sigma\rho^T\Sigma^{XT}\frac{f_x}{f}
\end{aligned} \tag{4.20}$$

The last term of this expression is computed as follows:

$$\begin{aligned}
(\Sigma\rho^T(\Sigma^X)^T f_x)^T A^{-1}\Sigma\rho^T\Sigma^{XT}f_x &= f_x^T\Sigma^X\rho\Sigma^T(\Sigma^T)^{-1}\Sigma^{-1}\Sigma\rho^T\Sigma^X T f_x \\
&= f_x^T\Sigma^X\rho\rho^T\Sigma^{XT}f_x,
\end{aligned} \tag{4.21}$$

where we have used, once again, $A^{-1}=(\Sigma^T)^{-1}\Sigma^{-1}$. It remains to expand $(\phi^*)^T\Sigma\rho^T\Sigma^{XT}(1-\gamma)(\gamma)f_x$:

$$\begin{aligned}
&(\phi^*)^T\Sigma\rho^T\Sigma^{XT}(1-\gamma)(\gamma)f_x \\
&=\left(\frac{1}{\gamma}A^{-1}\left(\mu-r+\gamma\Sigma\rho^T\Sigma^{XT}\frac{f_x}{f}\right)\right)^T\Sigma\rho^T\Sigma^{XT}(1-\gamma)(\gamma)f_x \\
&=\left(A^{-1}\left(\mu-r+\gamma\Sigma\rho^T\Sigma^{XT}\frac{f_x}{f}\right)\right)^T\Sigma\rho^T\Sigma^{XT}(1-\gamma)f_x \\
&=(\mu-r)^T(\Sigma^T)^{-1}\rho^T(\Sigma^X)^T(1-\gamma)f_x+\gamma(1-\gamma)\frac{1}{f}f_x^T\Sigma^X\rho\rho^T(\Sigma^X)^T f_x \\
&=(1-\gamma)(\Sigma^X\rho(\Sigma)^{-1}(\mu-r))^T f_x+\gamma(1-\gamma)\frac{1}{f}f_x^T\Sigma^X\rho\rho^T(\Sigma^X)^T f_x.
\end{aligned} \tag{4.22}$$

Before we substitute these terms in (4.15) let us note one more fact; it follows from the involved definitions that for a vector $v$ and a matrix $M$, $\text{tr}(Ma\otimes a)=a^TMa$. Therefore, $\text{tr}(\Sigma^X(\Sigma^X)^T f_x\otimes f_x)=f_x^T\Sigma^X(\Sigma^X)^T f_x$. Then we can rewrite (4.15) as

$$\begin{aligned}
&0.5\gamma\text{tr}\left(\Sigma^X\Sigma^{XT}f_{xx}\right)+0.5\frac{\gamma(\gamma-1)}{f}f_x^T\Sigma^X(\Sigma^X)^T f_x+\mu^X\gamma f_x-\beta f+\gamma f_t+r(1-\gamma)f \\
&+\phi^*\Sigma\rho^T\Sigma^{XT}(1-\gamma)(\gamma)f_x+\phi^*\frac{1}{2}A\phi(\gamma-1)\gamma f+\phi^*(\mu-r)(1-\gamma)f+\gamma\alpha^{\frac{1}{\gamma}}=0.
\end{aligned} \tag{4.23}$$

Now substituting (4.16), (4.17), (4.18), (4.19) , (4.20), (4.21) and (4.22) reduce the last equation to

$$\begin{aligned}
&\gamma f_t+0.5\gamma\text{tr}\left(\Sigma^X\Sigma^{XT}f_{xx}\right)+\left(\gamma\mu^X+(1-\gamma)\Sigma^X\rho\Sigma^{-1}(\mu-r)\right)^T f_x \\
&+\frac{1}{2f}(\gamma-1)\gamma f_x^T((\Sigma^X\Sigma^X)^T-\Sigma^X\rho\rho^T(\Sigma)^T)f_x \\
&+\left(\frac{1-\gamma}{2\gamma}(\mu-r)^TA^{-1}(\mu-r)+(1-\gamma)r-\beta\right)f+\gamma\alpha^{1/\gamma}=0.
\end{aligned}$$

Finally we divide both sides by $\gamma$ to obtain

$$f_t + 0.5\text{tr}\left(\Sigma^X \Sigma^{XT} f_{xx}\right) + \left(\mu^X + \frac{1-\gamma}{\gamma}\Sigma^X \rho \Sigma^{-1}(\mu - r)\right)^T f_x$$
$$+ \frac{1}{2f}(\gamma - 1) f_x^T ((\Sigma^X \Sigma^X)^T - \Sigma^X \rho \rho^T (\Sigma)^T) f_x$$
$$+ \left(\frac{1-\gamma}{2\gamma^2}(\mu - r)^T A^{-1}(\mu - r) + \frac{1-\gamma}{\gamma} r - \frac{\beta}{\gamma}\right) f + \alpha^{1/\gamma} = 0. \tag{4.24}$$

□

HJB equation now reduced to a simpler equation. Only the last term $\alpha^{1/\gamma}$ creates nonlinearity in this equation, we need to remove this non linear term in order to solve this equation. One approach is to take $\alpha = 0$ which reduces it to

$$\hat{f}_t + 0.5\text{tr}\left(\Sigma^X \Sigma^{XT} \hat{f}_{xx}\right) + \left(\mu^X + \frac{1-\gamma}{\gamma}\Sigma^X \rho \Sigma^{-1}(\mu - r)\right)^T \hat{f}_x$$
$$+ \frac{1}{2\hat{f}}(\gamma - 1) \hat{f}_x^T ((\Sigma^X \Sigma^X)^T - \Sigma^X \rho \rho^T (\Sigma)^T) \hat{f}_x$$
$$+ \left(\frac{1-\gamma}{2\gamma^2}(\mu - r)^T A^{-1}(\mu - r) + \frac{1-\gamma}{\gamma} r - \frac{\beta}{\gamma}\right) \hat{f} = 0. \tag{4.25}$$

The second approach to deal with the nonlinearity in (4.8) given in [11] is to focus on the "complete markets" case which assumes $X$ and the asset prices to be driven by the same Brownian motion; in this case (4.8) can again be reduced to (4.25). The Heston model application studied in later chapters is an incomplete model, for this reason we focus on the incomplete case and assume $\alpha = 0$; for the details of the complete case we refer the reader to [11].

To be able to solve and obtain analytic solution of equation (4.25), some restrictions are made on both dynamic state variable vector $X$ and asset returns. Mainly we assume that state variable $X$ has a drift and diffusion coefficient which are quadratic functions of itself and asset returns are quadratic functions as well.

The first assumption which is made is the following;

$$\mu^x = k - KX + \frac{1}{2} X^T \eta^T . K_2 . \eta X, \tag{4.26}$$

$$\Sigma^X \Sigma^{XT} = h_0 + h_1 . X + X^T \eta^T . h_2 . \eta X, \tag{4.27}$$

additional restrictions on parameters are applied

$$K^T \eta^T = \eta^T \hat{K}, \quad K_2 \eta^T = 0, \quad h_1 \eta^T = 0, \quad h_2 \eta^T = 0 \tag{4.28}$$

In these equations $k$ represents a $N \times 1$ constant vector. $K$ and $h_0$ are $N \times N$ constant matrices, $K_2 = K_{2k}^{ij}, i, j = 1, ..., N_1, k = 1, ..., N$ denotes a constant tensor(generalization of vector and matrix) with three indices (two upper one lower) and

$h_1 = h^i_{1jk}, i, j, k = 1, ..., N$ is constant tensor with three indices (one upper two lower)
$h_2 = h^{ij}_{2kl}, i, j = 1, ..., N_1, k = 1, ..., N$ is constant tensor with four indices (two upper two lower)

Secondly, it is stated that the asset returns should satisfy following conditions;

$$r = \delta_0 + \delta_1 X + \frac{1}{2} X^T \eta^T \delta_2 \eta X, \tag{4.29}$$

$$(\mu - r)^T (\Sigma\Sigma^T)^{-1} (\mu - r) = H_0 + H_1 X + \frac{1}{2} X^T \eta^T H_2 \eta X, \tag{4.30}$$

$$\Sigma^x \rho \Sigma^{-1} (\mu - r) = g_0 + g_1 X + \frac{1}{2} X^T \eta^T . g_2 . 4 \eta X, \tag{4.31}$$

$$\Sigma^X \rho \rho^T \Sigma^{XT} - \Sigma^X \Sigma^{XT} = l_0 + l_1 X + X^T \eta^T . l_2 . \eta X. \tag{4.32}$$

with restrictions

$$g_1^T \eta^T = \eta^T \hat{g}_1, \quad g_2^T \eta^T = 0, \quad l_1 \eta^T = 0, \quad l_2 \eta^T = 0. \tag{4.33}$$

In these equations $\delta_0$ is a constant, $\delta_1$ is a constant N dimension vector, $\delta_2$ is a constant $N_1 \times N_1$ matrix. $H_0$ is a constant, $H_1$ is a constant vector of dimension $N$, $H_2$ is a constant matrix of dimension $N_1 \times N_1$, $g_0$ is a constant vector of dimension $N$, $g_1$ is a constant matrix of dimension $N \times N$, $g_2$ is a constant tensor with three indices (two upper indices running from 1 to $N_1$ and one lower index running from 1 to $N$), $\hat{g}_1$is a constant matrix of dimension $N1 \times N1$, $l_0$ is an $N \times N$constant matrix, $l_1$is a constant tensor with three indices (with one upper index and two lower indices all running from 1 to N), and $l_2$is a constant tensor with four indices (with two upper indices running from 1 to $N_1$and two lower indices running from 1 to $N$).

These assumptions imply that all the coefficients of the (4.25) are quadratic in $\eta X$ and linear in $X$. Under these assumptions [11] we seek a solution of (4.25) of the form:

$$\hat{f} = e^{c(t) + d(t)^T X + 0.5 X^T \eta^T Q(t) \eta X}, \tag{4.34}$$

where $c(t)$ is a scalar function, $d(t)$ is a $N$ dimensional vector function and $Q(t)$ is a $N_1 \times N_1$ matrix function. Since the equation holds for all $X$, the coefficients of these terms have to be zero, which leads to ordinary differential equations for $c(t)$, $d(t)$, and $Q(t)$. The next proposition reduces (4.25) to equations satisfied by these functions; these equations are given as (18),(19) and (20) in [11] ; the derivation of these equations are given in [11]; the following proof is a detailed version of this derivation.

**Proposition 4.2.** *For $\hat{f}$ of the form* (4.34) *the equation* (4.25) *reduces to*

$$\frac{d}{dt}c + (k + \frac{1-\gamma}{\gamma}g_0)^T d + \frac{1}{2}d^T[h_0 + (1-\gamma)l_0]d$$
$$+ \frac{1}{2}tr(h_0\eta^T Q\eta) + +\frac{1-\gamma}{2\gamma^2}H_0 + \frac{1-\gamma}{\gamma}\delta_0 - \frac{\beta}{\gamma} = 0,$$

$$\frac{d}{dt}d + \big[-K + \frac{1-\gamma}{\gamma}(g_1)\big]^T d$$
$$+ \frac{1}{2}d^T[h_1 + (1-\gamma)l_1] + \eta^T Q\eta[h_0 + (1-\gamma)l_0]d$$
$$+ \eta^T Q\eta(k + \frac{1-\gamma}{\gamma}g_0) + \frac{1-\gamma}{2\gamma^2}H_1 + \frac{1-\gamma}{\gamma}\delta_1 = 0,$$

*and*

$$\frac{d}{dt}Q + \big(-\hat{K} + \frac{1-\gamma}{\gamma}\hat{g}_1\big)^T Q$$
$$+ Q\big(-\hat{K} + \frac{1-\gamma}{\gamma}\hat{g}_1\big) + Q\eta^T[h_0 + (1-\gamma)l_0]\eta Q$$
$$+ dt[h_2 + (1-\gamma)l_2]d + \frac{1-\gamma}{\gamma}g_2^T d + \frac{1-\gamma}{2\gamma^2}H_2 + \frac{1-\gamma}{\gamma}\delta_2 = 0,$$

*for $c$, $d$ and $Q$.*

*Proof.* The proof consists of substituting the form of the solution, the assumptions and the restrictions into equation(4.25):

$$\hat{f}_t + \underbrace{\frac{1}{2}\text{tr}\left(\Sigma^X\Sigma^{XT}\hat{f}_{xx}\right)}_{1} + \underbrace{\left[(\mu^X + \frac{1-\gamma}{\gamma}\Sigma^X\rho\Sigma^{-1}(\mu - r)\right]^T\hat{f}_x}_{2}$$
$$+ \underbrace{\frac{1}{2\hat{f}}(\gamma-1)\hat{f}_x^T((\Sigma^X\Sigma^X)^T - \Sigma^X\rho\rho^T(\Sigma)^T)\hat{f}_x}_{3}$$
$$+ \underbrace{\frac{1-\gamma}{2\gamma^2}(\mu - r)^T A^{-1}(\mu - r)}_{4} + \frac{1-\gamma}{\gamma}r - \frac{\beta}{\gamma}\ \hat{f} = 0. \tag{4.35}$$

$\hat{f}$ and derivative of $\hat{f}$ are;

$$\hat{f} = \epsilon^{c(t)+d(t)^T X + \frac{1}{2}X^T\eta^T Q(t)\eta X^T}, \tag{4.36}$$
$$\hat{f}_t = (\frac{d}{dt}c + \frac{d}{dt}d^T X + \frac{1}{2}X^T\eta^T\frac{d}{dt}Q\eta X)\hat{f}, \tag{4.37}$$
$$\hat{f}_x = (d + \eta^T Q\eta X)\hat{f}, \tag{4.38}$$
$$\hat{f}_{xx^T} = [\eta^T Q\eta + (d + \eta^T Q\eta X)(d + \eta^T Q\eta X)^T]\hat{f}. \tag{4.39}$$

If we look at the terms shown with 1 in (4.35) and replace (4.39), (4.26)

$$\frac{1}{2}\text{tr}\left(\Sigma^X\Sigma^{XT}\hat{f}_{xx}\right) = \frac{1}{2}\text{tr}\left[(h_0 + h_1X + X^T\eta^T.h_2.\eta X).[\eta^TQ\eta + (d + \eta^TQ\eta X)(d + \eta^TQ\eta X)^T]\hat{f}\right].$$

Since $\hat{f}$ is a scalar we can take it out. By using basic operation rules we will have;

$$= \frac{1}{2}\hat{f}\Big[\text{tr}(h_0\eta^TQ\eta) + \text{tr}(h_0dd^T) + \text{tr}(h_0dX^T\eta^TQ\eta) + \text{tr}(h_0\eta^TQ\eta Xd^T) + \text{tr}(h_0\eta^TQ\eta XX^T\eta^TQ^T\eta) + \text{tr}(h_1Xdd^T) + \text{tr}(X^T\eta^Th_2\eta Xdd^T)\Big].$$

other terms will cancel due to $h_1\eta^T = 0$ and $h_2\eta^T = 0$ restrictions

We can take $dd^T$ as a common factor which will reduce the equation to;

$$= \frac{1}{2}\hat{f}\Big[\text{tr}(h_0\eta^TQ\eta) + \text{tr}[h_0 + h_1X + X^T\eta^Th_2\eta X](dd^T) + \text{tr}(h_0dX^T\eta^TQ^T\eta) + \text{tr}(h_0\eta^TQ\eta Xd^T) + \text{tr}(h_0\eta^TQ\eta XX^T\eta^TQ\eta)\Big].$$

Remembering $\text{tr}(Ma \otimes a) = a^TMa$ property, we can represent the equation as;

$$= \frac{1}{2}\hat{f}\Big[\text{tr}(h_0\eta^TQ\eta) + d^T[h_0 + h_1X + x^T\eta^Th_2\eta X]d + (X^T\eta^TQ^T\eta h_0d) + (d^Th_0\eta^TQ\eta X) + (X^T\eta^TQ^T\eta h_0\eta^TQ\eta X)\Big]. \tag{4.40}$$

For any matrix $A$ and a vector $d$ $X^tAd$ is equal to $d^TA^TX$. Using this property we can reduce this equation to;

$$= \frac{1}{2}\hat{f}\Big[\text{tr}(h_0\eta^TQ\eta) + d^T[h_0 + h_1X + x^T\eta^Th_2\eta X]d + 2(X^T\eta^TQ^T\eta h_0d) + (X^T\eta^TQ^T\eta h_0\eta^TQ\eta X)\Big]. \tag{4.41}$$

If we look at the term shown 2 in (4.35) and replace (4.26), (4.31),(4.38);

$$(\mu^X + \frac{1-\gamma}{\gamma}\Sigma^X\rho\Sigma^{-1}(\mu - r)^T\hat{f}_x = \Big[(k - KX + \frac{1}{2}X^T\eta^T.K_2.\eta X) + \frac{1-\gamma}{\gamma}(g_0 + g_1X + \frac{1}{2}X^T\eta^T.g_2.\eta X)\Big]^T(d + \eta^TQ\eta X)\hat{f}.$$

Multiplying the equation above gives;

$$\underbrace{= \hat{f}\left[(k - KX + \frac{1}{2}X^T\eta^T.K_2.\eta X) + \frac{1-\gamma}{\gamma}(g_0 + g_1X + \frac{1}{2}X^{^T}\eta^{^T}.g_2.\eta X)\right]^T d +}_{i}$$

$$\underbrace{\hat{f}\left[(k - KX + \frac{1}{2}X^T\eta^T.K_2.\eta X) + \frac{1-\gamma}{\gamma}(g_0 + g_1X + \frac{1}{2}X^{^T}\eta^{^T}.g_2.\eta X)\right]^T (\eta^T Q\eta X)}_{ii}.$$

(4.42)

After the operations of $ii$ in (4.42), we will obtain;

$$\begin{aligned}\hat{f}\Bigg[(k - KX + \frac{1}{2}X^T\eta^T.K_2.\eta X) + \frac{1-\gamma}{\gamma}(g_0 + g_1X + \frac{1}{2}X^{^T}\eta^{^T}.g_2.\eta X)\Bigg]^T (\eta^T Q\eta X)\\ = \hat{f}\Bigg[(k + \frac{1-\gamma}{\gamma}g_0))^T(\eta^T Q\eta X) + (-KX + \frac{1-\gamma}{\gamma}g_1X)^T(\eta^T Q\eta X)\\ + \left(\frac{1}{2}X^T\eta^T.K_2.\eta X + \frac{1-\gamma}{2\gamma}X^{^T}\eta^{^T}.g_2.\eta X\right)^T(\eta^T Q\eta X)\Bigg]\end{aligned}$$

$g_2^T\eta^T = 0$ and $K_2\eta^T = 0$ restrictions cancels out the other remaining terms which makes this equation;

$$\hat{f}\left[(k + \frac{1-\gamma}{\gamma}g_0))^T(\eta^T Q\eta X) + (-X^T K^T\eta^T Q\eta X) + (\frac{1-\gamma}{\gamma}X^T g_1^T\eta^T Q\eta X)\right].$$

Additionally, if we apply $K\eta^T = \eta^T\hat{K}$ and $g_1^T\eta^T = \eta^T\hat{g}_1$ our equation will become;

$$\hat{f}\left[(k + \frac{1-\gamma}{\gamma}g_0)^T(\eta^T Q\eta X) + (-X^T\eta^T\hat{K}Q\eta X) + (\frac{1-\gamma}{\gamma}X^T\eta^T\hat{g}_1^T Q\eta X)\right].$$

If we place the terms $i$ and $ii$ together, this gives;

$$\begin{aligned}= \Bigg[(k - KX + \frac{1}{2}X^T\eta^T.K_2.\eta X) + \frac{1-\gamma}{\gamma}(g_0 + g_1X + \frac{1}{2}X^{^T}\eta^{^T}.g_2.\eta X)\Bigg]^T (d\hat{f}) +\\ (k + \frac{1-\gamma}{\gamma}g_0)^T(\eta^T Q\eta X\hat{f}) + (-X^T\eta^T\hat{K}Q\eta X\hat{f}) + (\frac{1-\gamma}{\gamma}X^T\eta^T\hat{g}_1^T Q\eta X\hat{f}).\end{aligned}$$

If we also apply $g_2^T\eta^T = 0$ and $K_2\eta^T = 0$ restrictions, they cancel $\frac{1}{2}X^T\eta^T.K_2.\eta X$, $\frac{1}{2}X^{^T}\eta^{^T}.g_2.\eta X$ terms in $i$ and then we obtain;

$$\begin{aligned}= \Bigg[(k - KX) + \frac{1-\gamma}{\gamma}(g_0 + g_1X + \frac{1}{2}X^{^T}\eta^{^T}.g_2.\eta X)\Bigg]^T (d\hat{f})\\ +(k + \frac{1-\gamma}{\gamma}g_0)^T(\eta^T Q\eta X\hat{f}) + (-X^T\eta^T\hat{K}Q\eta X\hat{f})\\ +(\frac{1-\gamma}{\gamma}X^T\eta^T\hat{g}_1^T Q\eta X\hat{f}).\end{aligned} \tag{4.43}$$

If we look at the term 3 in (4.35) and replace (4.38), (4.32)

$$\frac{1}{2\hat{f}}(\gamma-1)f_x^T((\Sigma^X\Sigma^X)^T-\Sigma^X\rho\rho^T(\Sigma)^T)f_x$$
$$=\frac{1}{2\hat{f}}(\gamma-1)[(d+\eta^TQ\eta X)\hat{f}]^T[-l_0-l_1X-X^T\eta^T.l_2.\eta X](d+\eta^TQ\eta X)\hat{f}$$
$$=\frac{\hat{f}}{2}(1-\gamma)(d^T+X^T\eta^TQ^T\eta)l_0(d+\eta^TQ\eta X)$$
$$+\frac{\hat{f}}{2}(1-\gamma)(d^T+X^T\eta^TQ^T\eta)(l_1X)d+$$
$$+\frac{\hat{f}}{2}(1-\gamma)(d^T+X^T\eta^TQ^T\eta)(X^T\eta^Tl_2\eta X)d,$$

other terms will cancel out because of $l_1\eta^T=0$ and $l_2\eta^T=0$ restrictions. If we multiply and use $X^tAd$ is equal to $d^TA^TX$ property, this would lead to final of;

$$\frac{\hat{f}}{2}(1-\gamma)\Big[d^Tl_0d+2(X^T\eta^TQ^T\eta)l_0d+d^Tl_1Xd+$$
$$(X^T\eta^TQ^T\eta)l_0(\eta^TQ\eta X)++d^TX^T\eta^Tl_2\eta Xd\Big]. \tag{4.44}$$

If we look at the terms 4 in equations (4.35) and replace (4.30);

$$\frac{1-\gamma}{2\gamma^2}(\mu-r)^TA^{-1}(\mu-r)+\frac{1-\gamma}{\gamma}r-\frac{\beta}{\gamma}\hat{f}$$
$$=\hat{f}\Big[\frac{1-\gamma}{2\gamma^2}(H_0+H_1X+\frac{1}{2}X^T\eta^TH_2\eta X)$$
$$+\frac{1-\gamma}{\gamma}(\delta_0+\delta_1X+\frac{1}{2}X^T\eta^T.\delta_2.\eta X)-\frac{\beta}{\gamma}\Big]. \tag{4.45}$$

Lastly, we plug (4.41),(4.43),(4.44),(4.45) into (4.35), the equation becomes

$$\Big(\frac{d}{dt}c+\frac{d}{dt}d^TX+\frac{1}{2}X^T\eta^T\frac{d}{dt}Q\eta X\hat{f}\Big)+\frac{1}{2}\hat{f}\Big[\text{tr}(h_0\eta^TQ\eta)$$
$$+d^T[h_0+h_1X+X^T\eta^Th_2\eta X]d+2X^T\eta^TQ\eta h_0d+X^T\eta^TQ\eta h_0\eta^TQ\eta X\Big]$$
$$+\Big[(k-KX)+\frac{1-\gamma}{\gamma}(g_0+g_1X+\frac{1}{2}X^T\eta^Tg_2\eta X)\Big]^T(d\hat{f})+$$
$$(k+\frac{1-\gamma}{\gamma}g_0)^T(\eta^TQ\eta X\hat{f})+(-X^T\eta^T\hat{K}Q\eta X\hat{f})+(\frac{1-\gamma}{\gamma}X^T\eta^T\hat{g}_1^TQ\eta X\hat{f})$$
$$+\frac{\hat{f}}{2}(1-\gamma)\Big[d^Tl_0d+2(X^T\eta^TQ^T\eta)l_0d+d^Tl_1Xd+(X^T\eta^TQ^T\eta)l_0(\eta^TQ\eta X)$$
$$+d^TX^T\eta^Tl_2\eta Xd\Big]+\hat{f}\Big[\frac{1-\gamma}{2\gamma^2}(H_0+H_1X+\frac{1}{2}X^T\eta^TH_2\eta X)$$
$$+\frac{1-\gamma}{\gamma}(\delta_0+\delta_1X+\frac{1}{2}X^T\eta^T.\delta_2.\eta X)-\frac{\beta}{\gamma}\Big]=0.$$

This equation consists of three classes of terms; terms that are independent of $X$, terms linear in $X$ and quadratic in $\eta X$. In order for the equation to hold for all $X$, all coefficients of these terms must be zero. This would lead to three ordinary differential equations;

if we separate scalar terms we obtain:

$$\begin{aligned}\frac{d}{dt}c + \frac{1}{2}\text{tr}(h_0\eta^T Q\eta) + \frac{1}{2}d^T h_0 d + (k + \frac{1-\gamma}{\gamma}g_0)^T d \\ + \frac{1}{2}(1-\gamma)(d^T l_0 d) + \frac{1-\gamma}{2\gamma^2}H_0 + \frac{1-\gamma}{\gamma}\delta_0 - \frac{\beta}{\gamma} \quad = 0.\end{aligned}$$

After reshaping and taking into $d^T.d$ parenthesis we obtain the equation for the $c$ function:

$$\begin{aligned}\frac{d}{dt}c + (k + \frac{1-\gamma}{\gamma}g_0)^T d + \frac{1}{2}d^T[h_0 + (1-\gamma)l_0]d \\ +\frac{1}{2}\text{tr}(h_0\eta^T Q\eta) + +\frac{1-\gamma}{2\gamma^2}H_0 + \frac{1-\gamma}{\gamma}\delta_0 - \frac{\beta}{\gamma} \quad = 0.\end{aligned} \tag{4.46}$$

If we separate linear terms we obtain:

$$\begin{aligned}&\frac{d}{dt}d^T X + \frac{1}{2}\Big[d^T h_1 X d + 2(X^T\eta^T Q\eta h_0 d)\Big] \\ &+ \big[-KX + \frac{1-\gamma}{\gamma}(g_1 X)\big]^T d + (k + \frac{1-\gamma}{\gamma}g_0)^T(\eta^T Q\eta X) \\ &+ \frac{1}{2}(\gamma-1)\Big[2(X^T\eta^T Q^T\eta)l_0 d + d^T l_1 X d\Big] \\ &+ \frac{1-\gamma}{2\gamma^2}H_1 X + \frac{1-\gamma}{\gamma}\delta_1 X.\end{aligned}$$

All coefficient of these terms must be zero which leads to following equation for $d$:

$$\begin{aligned}&\frac{d}{dt}d + \frac{1}{2}\Big[d^T h_1 d + (2\eta^T Q\eta h_0 d)\Big] + \big[-K + \frac{1-\gamma}{\gamma}(g_1)\big]^T d \\ &+ (k + \frac{1-\gamma}{\gamma}g_0)^T(\eta^T Q\eta) + \frac{1}{2}(\gamma-1)\Big[(2\eta^T Q^T\eta)l_0 d + d^T l_1 d\Big] \\ &+ \frac{1-\gamma}{2\gamma^2}H_1 + \frac{1-\gamma}{\gamma}\delta_1 \quad = 0.\end{aligned}$$

After reshaping and taking into $d^T.d$ and $\eta^T Q\eta$ parenthesis, we obtain the final equation for $d$:

$$\begin{aligned}&\frac{d}{dt}d + \big[-K + \frac{1-\gamma}{\gamma}(g_1)\big]^T d + \frac{1}{2}d^T[h_1 + (1-\gamma)l_1] + \eta^T Q\eta \\ &[h_0 + (1-\gamma)l_0]d + \eta^T Q\eta(k + \frac{1-\gamma}{\gamma}g_0) + \frac{1-\gamma}{2\gamma^2}H_1 + \frac{1-\gamma}{\gamma}\delta_1 = 0.\end{aligned} \tag{4.47}$$

Finally, collecting the quadratic terms together we obtain the equation for $Q$:

$$\begin{aligned}
&\frac{1}{2}X^T\eta^T\frac{d}{dt}Q\eta X + \frac{1}{2}\big[d^TX^T\eta^Th_2\eta Xd + X^T\eta^TQ\eta h_0\eta^TQ\eta X\big] \\
&-(X^T\eta^T\hat{K}Q\eta X) + \frac{1-\gamma}{\gamma}\hat{f}(\frac{1}{2}X^T\eta^Tg_2\eta X)^Td + (\frac{1-\gamma}{\gamma}X^T\eta^T\hat{g}_1^TQ\eta X\hat{f}) \\
&\quad + \frac{1}{2}(\gamma-1)\Big[(X^T\eta^TQ^T\eta)l_0(\eta^TQ\eta X) + d^TX^T\eta^Tl_2\eta Xd \\
&\quad + (\eta^TQ\eta X)X^T\eta^Tl_2\eta Xd)\Big] + \frac{1-\gamma}{4\gamma^2}X^T\eta^TH_2\eta X + \frac{1-\gamma}{2\gamma}X^T\eta^T.\delta_2.\eta X = 0.
\end{aligned}$$

To solve this equation all coefficients must be zero if we take into $\frac{1}{2}X^T\eta^T..\eta X$ parenthesis, we obtain:

$$\begin{aligned}
&\frac{d}{dt}Q + d^Th_2d + Q\eta h_0\eta^TQ - \hat{K}Q + \frac{1-\gamma}{\gamma}\hat{g}_2d + (\frac{1-\gamma}{\gamma}\hat{g}_1Q) \\
&+ (1-\gamma)[(Q\eta^T)l_0(\eta Q) + d^Tl_2d] + \frac{1-\gamma}{2\gamma^2}H_2 + \frac{1-\gamma}{\gamma}\delta_2 = 0.
\end{aligned}$$

After reshaping and taking into $d.d^T$, $Q\eta^T.\eta Q$ parenthesis we get the equation for $Q$:

$$\begin{aligned}
&\frac{d}{dt}Q + \big(-\hat{K} + \frac{1-\gamma}{\gamma}\hat{g}_1\big)^TQ + Q\big(-\hat{K} + \frac{1-\gamma}{\gamma}\hat{g}_1\big) + Q\eta^T[h_0 + (1-\gamma)l_0]\eta Q \\
&+ dt[h_2 + (1-\gamma)l_2]d + \frac{1-\gamma}{\gamma}g_2^Td + \frac{1-\gamma}{2\gamma^2}H_2 + \frac{1-\gamma}{\gamma}\delta_2 = 0. \qquad (4.48)
\end{aligned}$$

□

Substituting the various terms arising from the form of the value function (4.34) in (4.13) gives the following formula for the optimal control:

$$\begin{aligned}
\phi^* =& \frac{1}{\gamma}(\Sigma\Sigma^T)^{-1}\left(\mu - r + \gamma\Sigma\rho^T\Sigma^{XT}\frac{f_x}{f}\right), \\
\phi^* =& \frac{1}{\gamma}(\Sigma\Sigma^T)^{-1}\big(\mu - r + \gamma\Sigma\rho^T\Sigma^{XT}(d + Q\eta X)\big), \qquad (4.49)
\end{aligned}$$

in terms of $d$ and $Q$. The ordinary equations derived in the above proposition for $d$ and $Q$ can be solved to obtain optimal controls through the above formula. An application is given in the following chapter.

# CHAPTER 5

# Dynamic Portfolio Choice Problem when Stocks have Stochastic Volatility

In this chapter, we will first review the application of Liu's framework to Merton's problem with Heston stochastic volatility model. This application is given in Section 2.2 in reference [11]. Then we reduce the model to a setting where volatility is constant as in the classical Merton's model. The next chapter will compare these two models.

## 5.1 Stochastic Volatility Case

In stochastic volatility case, risky assets are assumed to follow;

$$\begin{aligned} \frac{dP_s^t}{P_t^s} =& (r + \lambda_s V_t)dt + \sqrt{V_t}dBt, \\ dV_t =& (k_v - K_v V_t)dt + \sigma_v \sqrt{V_t} dB_t^v. \end{aligned} \tag{5.1}$$

which can also be expressed as

$$dV_t = K_v(\frac{k_v}{K_v} - V_t)dt + \sigma_v \sqrt{V_t} dB_t^v,$$

where $\frac{k_v}{K_v}$ is the long mean of variance, as t goes to infinity, the expected value of $V_t$ tends to $\frac{k_v}{K_v}$. $K$ is the rate at which $V_t$ reverts to $\frac{k_v}{K_v}$, $\sigma_v$ is the variance of the volatility which shows the variance of $V_t$.

When asset returns follow (5.1) , variables can be expressed as follows in terms of the notation of Liu general solution;

$$\mu - r = \lambda_s V, \quad \Sigma^X = \sigma_v \sqrt{V}, \tag{5.2}$$

$$\Sigma = \sqrt{V}, \quad \rho = \rho_v, \tag{5.3}$$

$$r = \delta_0. \tag{5.4}$$

these specifications imply;

$$(\mu - r)^T(\Sigma\Sigma^T)^{-1}(\mu - r) = \lambda_s^2 V, \tag{5.5}$$

$$\Sigma^X \rho \Sigma^{-1}(\mu - r) = \rho_v \sigma_v \lambda_s V, \tag{5.6}$$

$$\Sigma^X(\rho\rho^T - I)\Sigma^{XT} = -(1 - \rho_v^2)\sigma_v^2 V. \tag{5.7}$$

In the previous chapter, we showed that, the optimal asset allocation rule is given by, $\phi^* = \frac{1}{\gamma}(\Sigma\Sigma^T)^{-1}\left(\mu - r + \gamma\Sigma\rho^T\Sigma^{XT}(d + Q\eta X)\right)$ (4.49). As it can be seen from the equation optimal allocation rule does not depend on $c(t)$ and moreover, when asset returns follow Heston's model (5.1) then there is no quadratic relation between assets which makes $Q$ as 0. So in order to find optimal weight for stochastic volatility, only solving ordinary differential equation for $d(t)$ is sufficient.

$d(t)$ is given in the previous chapter(4.47). If we change variables according to Heston's model we will obtain the following equation;

$$\frac{d}{dt}d_v - \left[K_v - \frac{1-\gamma}{\gamma}\lambda_s\sigma_v\rho_v\right] + \frac{\sigma_v^2}{2}[1 - (1-\gamma)(1-\rho_v^2)]d_v^2 + \frac{1-\gamma}{2\gamma^2}\lambda_s^2 = 0, \tag{5.8}$$

with terminal condition $d_v(T) = 0$

Solution of this differential equation is;

$$d_v(t) = \begin{array}{rl} -\dfrac{2[\exp(\xi_v\tau) - 1]}{(\hat{\kappa}_v + \xi_v)[\exp(\xi_v\tau) - 1] + 2\xi_v}\delta_v & \text{if} \quad \xi_v^2 \geq 0, \\ -\dfrac{2}{\hat{\kappa}_v + \zeta\frac{cos(\zeta_v\tau/2)}{sin(\zeta_v\tau/2)}}\delta_v & \text{if} \quad \zeta_v^2 \geq 0, \end{array} \tag{5.9}$$

where

$$\begin{aligned} \delta_v &= \frac{1-\gamma}{2\gamma^2}\lambda_s^2, \\ \hat{\kappa}_v &= K_v - \frac{1-\gamma}{\gamma}\lambda_s\sigma_v\rho_v, \\ \xi_v &= \sqrt{\hat{\kappa}_v^2 + 2\delta_v[\rho_v^2 + \gamma(1-\rho_v^2)]\sigma_v^2}, \\ \zeta_v &= -i\xi_v \quad \text{and} \quad \tau = (T-t). \end{aligned}$$

and the optimal portfolio allocation rule is

$$\boxed{\phi_s^* = \frac{1}{\gamma}\lambda_s \ + \ \rho_v\sigma_v d_v}. \tag{5.10}$$

The first part of the solution is the myopic component. It represents mean-variance optimization of the optimal allocation under the short time horizon. Second part is the inter-temporal hedging demand component which shows the agents behavior according to the changing market situations. Merton's optimal portfolio allocation rule for constant volatility case is equal only to the myopic component of stochastic volatility case.

Interestingly, optimal stock portfolio weight does not depend on volatility. In a stochastic volatility model it is expected that stock portfolio weight should be related to volatility. However, according to Liu's solution, $\mu - r$ is given as $\lambda_s V$ which indicates that, risk premium is directly proportional to volatility. When volatility is high, return is equivalently high. When volatility is low the return is also equivalently low. Thus, through risk premium's effect, the volatility impact on stock portfolio weight is neutralized, which makes volatility independent from optimal portfolio rule.

### 5.2 Constant Volatility Case

To be able to compare portfolio choice problems, in case of stochastic volatility and constant volatility, it is necessary to make two models related with each other. In this case, if we can remove stochastic features of volatility in Liu's portfolio problem, then portfolio problem would downgrade to a solution for constant volatility case, which would be very similar to Merton's classical model. So in this part, we solve Merton's portfolio problem in Liu's framework as a special case.

If $\sigma_v$ is taken as zero, the stochastic component of volatility would be canceled. Additionally, if volatility, $V_t$, is taken as $\frac{k_v}{K_v}$, the volatility is set to its long term mean. Under these circumstances, the optimization problem with stochastic volatility should downgrade to a constant volatility case.

Making $\sigma_v = 0$ reduces optimal stock weight allocation rule to;

$$\phi_s^* = \frac{1}{\gamma}\lambda_s, \tag{5.11}$$

which is the same as the Merton's classical model, only a myopic component represents the optimal allocation rule.

To correctly downgrade stochastic volatility to constant volatility case, it is necessary to check the value function as well. After making volatility a constant process, the value function should also satisfy the characteristic of classical Merton's Portfolio Problem. The new value function should not change as long as the volatility stays constant.

In Liu's general solution, the value function is given as

$$J(t,w,x) = e^{-\beta t}\frac{W^{1-\gamma}}{1-\gamma}\left[e^{c(t)+d(t)x+0.5X^T\eta^T Q(t)\eta X}\right]^{\gamma}. \tag{5.12}$$

If asset returns follow Heston's model, then Q becomes zero

$$J(t,w,x) = e^{-\beta t}\frac{W^{1-\gamma}}{1-\gamma}\left[e^{c(t)+d(t)x}\right]^{\gamma}. \tag{5.13}$$

Only $c(t) + d(t)x$ affects the value function which might have stochastic features regarding volatility. Hence, instead of looking at the whole value function we will examine only constant volatility cases of functions $c(t)$ and $d(t)$

If we apply $\sigma_v$ = 0 rule to $d_v$ (5.8), $d_v$ reduces to constant volatility version which is denoted by $d_{cv}$.

$$\frac{d}{dt}d_v - \Big[K_v - \frac{1-\gamma}{\gamma}\lambda_s\sigma_v\rho_v\Big] + \frac{\sigma_v^2}{2}[1-(1-\gamma)(1-\rho_v^2)]d_v^2 + \frac{1-\gamma}{2\gamma^2}\lambda_s^2 = 0 \quad (5.14)$$

Therefore, in passing from Heston's stochastic volatility to constant volatility, the ODE satisfied by $d$ reduces to:

$$\frac{d}{dt}d_{cv} - K_v d_{cv} + \frac{1-\gamma}{2\gamma^2}\lambda_s^2 = 0, \quad (5.15)$$

with terminal condition $d_{cv}(T) = 0$.

**Proposition 5.1.** *Solution of* (5.15) *is given by*

$$d_{cv}(t) = \frac{(1-\gamma)\lambda_s^2}{2\gamma^2 K_v}(1 - e^{K_v(t-T)}). \quad (5.16)$$

*Proof.* Equation(5.15) is a first order constant coefficient ODE with a constant non homogeneous term. A well known method to solve such ODE is to use an integrating factor.

$$\underbrace{\frac{d}{dt}d_{cv}}_{y'} - \underbrace{K_v d_{cv}}_{ay} + \underbrace{\frac{1-\gamma}{2\gamma^2}\lambda_s^2}_{b} = 0 \quad (5.17)$$

$$\begin{aligned}
y' - ay &= -b,\\
I &= e^{-at},\\
e^{-at}y' + -ae^{-at}y &= -be^{-at},\\
(e^{-at}y)' &= -be^{-at},\\
e^{-at}y &= \frac{b^{-at}}{a} + C_1,\\
y &= \frac{b}{a} + C_1 e^{at}, \quad (5.18)
\end{aligned}$$

with terminal condition $y(T) = 0$.

$$\begin{aligned}
\frac{b}{a} + C_1 e^{aT} &= 0,\\
C_1 &= -\frac{b}{a}e^{-aT}.
\end{aligned}$$

If we replace $C_1$ in (5.18)

$$\begin{aligned}
y &= \frac{b}{a} + (-\frac{b}{a}\,e^{-aT})\,e^{at}\\
&= \frac{b}{a}[1 - e^{a(t-T)}]. \quad (5.19)
\end{aligned}$$

After changing a b and y we obtain the solution for $d_{cv}(t)$;

$$d_{cv}(t) = \frac{(1-\gamma)\lambda_s^2}{2\gamma^2 K_v}(1 - e^{K_v(t-T)}) \tag{5.20}$$

□

c(t) is given in the previous chapter. If we reduce the equation of c(t) to Heston's model and change $d(t)$ with $d_{cv}(t)$ our equation becomes;

$$\frac{d}{dt}c + k_v d_{cv} + \left[\frac{1-\gamma}{\gamma}\delta_0 + \frac{\beta}{\gamma}\right] = 0. \tag{5.21}$$

It is observed that, the difference in $c(t)$ arises from the function $d(t)$.

$$c_{cv}(t) = \frac{k_v b}{a^2}\big[e^{(t-T)} - 1\big] + (T-t)\big[\frac{k_v b}{a} + \alpha_1\big]. \tag{5.22}$$

For value function to be constant with volatility we need to check the outcome of

$$c(t) + d(t) \times V. \tag{5.23}$$

If we replace $c(t)$, $d(t)$ and $V$ with $c_{cv}$, $d_{cv}$ and $\frac{k_v}{K_v}$ ;

$$\begin{aligned}(c(t) + d(t) \times V) =& \frac{k_v b}{K_v^2}\big[e^{(t-T)} - 1\big] + (t-T)\big[\frac{k_v b}{K_v} + \alpha_1\big] \\ &+ \left[\frac{b}{K_v}[1 - e^{K_v(t-T)}] * \big(\frac{k_v}{K_v}\big)\right] \\ =& (t-T)\big[\frac{k_v b}{K_v} + \alpha_1\big].\end{aligned} \tag{5.24}$$

As we can see, the solution only depends on $\frac{k_v}{K_v}$. If this value is fixed, stochastic volatility case can be downgraded to constant volatility case. To test this, a graph is plotted for different terms of $k_v$ and $K_v$ when $\gamma$, $\lambda$ $\beta$, $\delta$ is set to $0.5$, $0.2$, $0.3$, $0.1$ respectively. This is given in Figure 5.1;

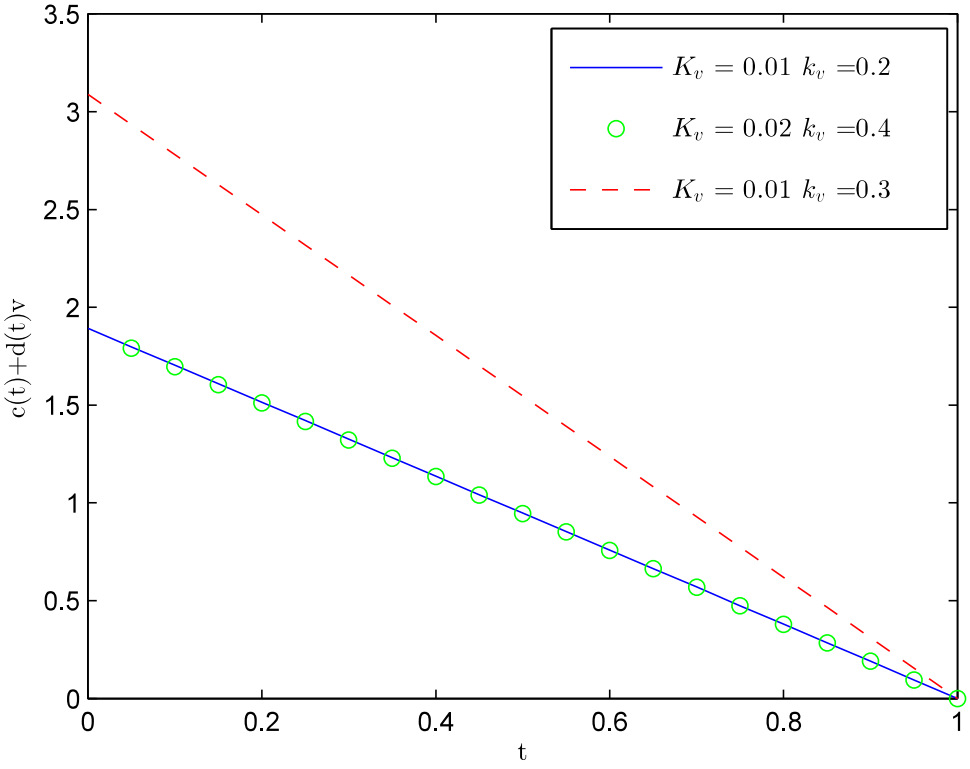


Figure 5.1: Effects of $K_v$ and $k_v$ on value function

In Figure 5.1, three sets of values were used. These were $K_v = 0.01$ , $k_v = 0.2$, $K_v = 0.02$ , $k_v = 0.4$, $K_v = 0.01$ , $k_v = 0.3$. It should be noted that in the former two sets $\frac{k_v}{K_v}$ has the same value and they fall on the same line meaning that their value functions are the same. In the third set, $\frac{k_v}{K_v}$ has a different value and thus it gives a different value function. This implies that, when volatility is fixed at its long term mean, then value function behaves the same as Merton's constant volatility model.

Considering both optimal risky asset weight rule and the value function, we can come to the conclusion that when $\sigma_v$ is fixed at 0 and $V$ is taken as $\frac{k_v}{K_v}$ the stochastic volatility model of dynamic portfolio choice would change to constant volatility case. Thus a comparison between stochastic volatility and constant volatility can be made by relaxing and applying these restrictions and additionally, the difference between value functions is only arose from the function $d$.

# CHAPTER 6

## Comparison of the Two Models

In the previous chapter, we showed that under restriction of $\sigma_v$ and $V = \frac{k_v}{K_v}$, the stochastic volatility case downgrades to the constant volatility case. In this chapter we will make numerical analysis to show how stochastic volatility affects the dynamic portfolio choice.

Optimal risky asset weight in stochastic volatility case is given as,

$$\phi_s^* = \frac{1}{\gamma}\lambda_s \ + \ \rho_v \sigma_v d_v, \tag{6.1}$$

the second term $\rho_v \sigma_v d_v$, is the only term that differs compared to (5.11) and shows the increase of proportion of the risky asset when there is stochastic volatility in the system.

Optimal stock weight is dependent on variables $K_v$ , $\gamma$, $\lambda$, $\rho_v$, $\sigma_v$. Since there are too many variables, it is difficult to see and interpret the effect of each variable on portfolio choice. But since the variables $\rho_v$ and $\sigma_v$ are valued between $(-1, 1)$ and $(0, 1)$ respectively, their multiplication would be between $(-1, 1)$. Only in extreme cases these values would be close to 1 or -1, so only for large values of $d_v$ this equation would make a substantial difference as compare to constant volatility case. Therefore we will examine the value of $d_v$ as a way of comparison. Moreover, the difference between value functions of two cases depends only on the functions $d_v$ and $d_{cv}$. Thus, comparing $d_{cv}$ and $d_v$ will indicate the difference in the value functions. So by only comparing $d$ functions we can see how two models differentiate from each other.

The solution of $d_v$ is more complex to work with as compared to $d_{cv}$. When $K_v$ , $\gamma$, $\lambda$ is fixed, the solutions of $d_v$ is bounded by $d_{cv}$ for different values of $\rho_v$ and $\sigma_v$. Therefore, we can examine $d_{cv}$ instead of $d_v$.

$$d_{cv}(t) = \frac{(1-\gamma)\lambda_s^2}{2\gamma^2 K_v}(1 - e^{K_v(t-T)}).$$

$K_v$ and $\frac{(1-\gamma)\lambda_s^2}{2\gamma^2 K_v}$ directly affect the value of $d_{cv}$. $K_v$ represents the mean reversion rate($K_v$), and the ratio $\frac{(1-\gamma)\lambda_s^2}{2\gamma^2 K_v}$ is referred to as SSVR (sensitivity to stochastic volatility ratio) that combines mean reversion rate($K_v$), risk aversion rate of the investor ($\gamma$) and Sharpe ratio($\lambda_s$). Here, Sharpe ratio defines the relation between risk premium per volatility.

To see the effect of stochastic volatility in Merton's problem, we will pursue the following methodology. First, we will look at how the SSVR and $K_v$ affect $d_{cv}$ and $d_v$. Then, we will look at how SSVR is affected by $\gamma$, $\lambda_s$

In order to see the effect of SSVR on $d_{cv}$ and $d_v$, we will set SSVR to the values $[0.2,\ 0.8,\ 1.6]$ and $K_v$ to the values of $[0.2,\ 0.7,\ 1.2]$. We will fix $\rho_v$ to $0.5$ and let $\sigma_v$ to take values $0.1$ to $1$. We then plot the graph of $d_{cv}$ and $d_v$ with all combinations of SSVR and $K_v$ for different values of $\sigma_v$ as time $t$ goes form $0$ to $1$, Figure 6.1.

It should be noted that, in Figure 6.1, blue lines represent the values of $d_v$ when $\sigma_v$ varies and red line represents $d_{cv}$. SSVR increases from top to bottom. It is clearly seen that the values of $d_{cv}$ and $d_v$ are determined by SSVR. When SSVR is high (figures at the bottom) the corresponding $d$ values ($d_{cv}$ , $d_v$) also high, which supports the argument that, $\frac{(1-\gamma)\lambda_s^2}{2\gamma^2 K_v}$ ratio measures the sensitivity of stochastic volatility to the optimal portfolio choice as a whole.

At the same time, the value of $K_v$ increases from left to right in Figure 6.1 . The figure, confirms the argument that, beyond its direct effect to SSVR, the parameter $K_v$ influences the shape of the curves i.e. it determines how fast $d_v$ decays with time.

Further, it can be seen from the Figure 6.1, $d_v$ values become large enough to affect portfolio weight, when $\frac{(1-\gamma)\lambda_s^2}{2\gamma^2 K_v} = 1.6$. When $\frac{(1-\gamma)\lambda_s^2}{2\gamma^2 K_v}$ is equals to $0.8$ the values of $d_v$ varies from $0.15$ to $0.6$ for different values of of given $K_v$. When $\frac{(1-\gamma)\lambda_s^2}{2\gamma^2 K_v} = 0.2$ the values of $d_v$ are not large enough to affect the equation (6.1).

Having evaluated how SSVR and $K_v$ affects $d_{cv}$ and $d_v$, we now look at how SSVR is affected by $\gamma$ and $\lambda_s$. In order to see the direct effect of $\lambda_s$ and $\gamma$ on $\frac{(1-\gamma)\lambda_s^2}{2\gamma^2 K_v}$ (SSVR), a table is prepared for each variables under the conditions where $\gamma$ and $\lambda$ take up values $[0.2,\ 0.5,\ 0.8]$ and $[0.1,\ 0.4,\ 0.7]$ respectively. $K_v$ is taken as $1$. As seen from the Table $6.1$, SSVR varies form $0.01$ to $4.9$ depending on the values $\gamma$ and $\lambda$. In case of a risk loving investor (low values of $\gamma$) in a market with very high Sharpe ratio (high values of $\lambda_s$) would yield values of SSVR high enough to affect $d_v$. By looking at the values, we can observe that $\frac{(1-\gamma)\lambda_s^2}{2\gamma^2 K_v}$ is most likely will have values lower than $0.2$. In order to better observe the outcome of these parameters an application to real data is made in the following section.

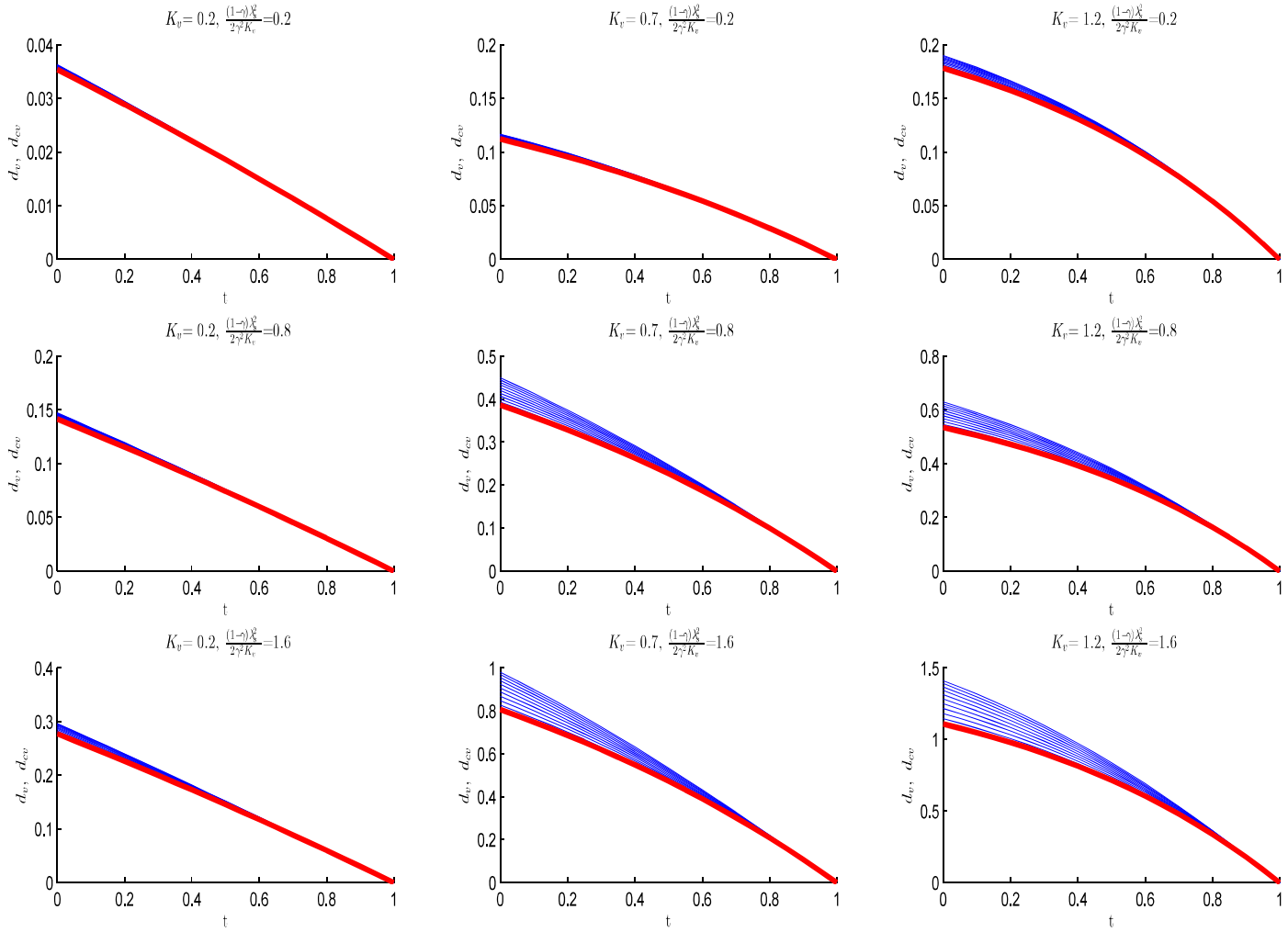


Figure 6.1: Comparison of stochastic volatility (blue) and constant volatility (red) based on differential equation $d$. Note that, plots are obtained at discrete values of $K_v$ and SSRV

Table 6.1: Values of SSVR for different values of $\gamma$ and $\lambda_s$

| $\gamma$ | $\lambda_s$ | SSVR |
|---|---|---|
| 0.2 | 0.1 | 0.100 |
| 0.5 | 0.1 | 0.010 |
| 0.8 | 0.1 | 0.001 |
| 0.2 | 0.4 | 1.600 |
| 0.5 | 0.4 | 0.160 |
| 0.8 | 0.4 | 0.025 |
| 0.2 | 0.7 | 4.900 |
| 0.5 | 0.7 | 0.490 |
| 0.8 | 0.7 | 0.076 |

## 6.1 An Application of Dynamic Portfolio Choice with Stochastic Volatility and Constant Volatility to BIST30 Stock Exchange

In this section, we will test how stochastic volatility affect the dynamic portfolio choice when it is implemented with real data. Thus, an application was made using BIST30 stock exchange data. In a previous study, Mert[13] priced BIST30 European call warrants (a special derivative product, which traded in Borsa Istanbul) using Heston Stochastic volatility model. This study includes calibration of Heston's parameters across two time intervals (01.03.2016-31.03.2016) and (15.01.2016-03.02.2016), which are given below;

Table 6.2: Heston's parameters for BIST30 for two time intervals [13]

| Mean of Parameters | 15.01.2016-03.02.2016 | 01.03.2016-31.03.2016 |
|---|---|---|
| $K_v$ | 11.2126 | 6.6562 |
| $\frac{k_v}{K_v}$ | 0.1209 | 0.1718 |
| $\sigma_v$ | 0.4073 | 1.4238 |
| $v_0$ | 0.0565 | 0.1292 |
| $\rho$ | 0.8719 | -0.2578 |
| $r$ | 0.1081 | 0.1086 |

Using these values, it is possible to make an dynamic portfolio application. In order to implement Liu's model, only $\mu$ value is missing. To obtain the $\mu$ value, BIST30 index stock prices are extracted for the given time periods. Their daily log returns are calculated and annualized to yearly data $(a.\mu)$ to be compatible with the Heston parameters in [13]. And, then values of $\lambda_s$, SSVR and $d_v$ are calculated. These processes yield the following results;

Table 6.3: Values of $\mu$, $\mu - r$, $\lambda_s$, SSVR, $d_v(0)$ calculated for BIST30 data in two time intervals

| Mean of Parameters | 15.01.2016-03.02.2016 | 01.03.2016-31.03.2016 |
|---|---|---|
| $\mu$ | 0.0024 | -0.0028 |
| $a.\mu$ | 0.8208 | -0.5039 |
| $\mu - r$ | 0.7127 | -0.6125 |
| $\lambda_s$ | 0.9870 | -2.1601 |
| $\frac{(1-\gamma)\lambda_s^2}{2\gamma^2 K_v}$ | 0.0869 | 0.7001 |
| $d_v(0)$ | 0.0896 | 1.2267 |

In order to compare stochastic volatility with constant volatility, relevant values of $d_{cv}$ and $d_v$ are plotted in Figure 6.2. The red line indicate the effect of constant volatility on value function while blue line indicates the effect of stochastic volatility on the same. Moreover, the values of SSVR are also given in the figure.

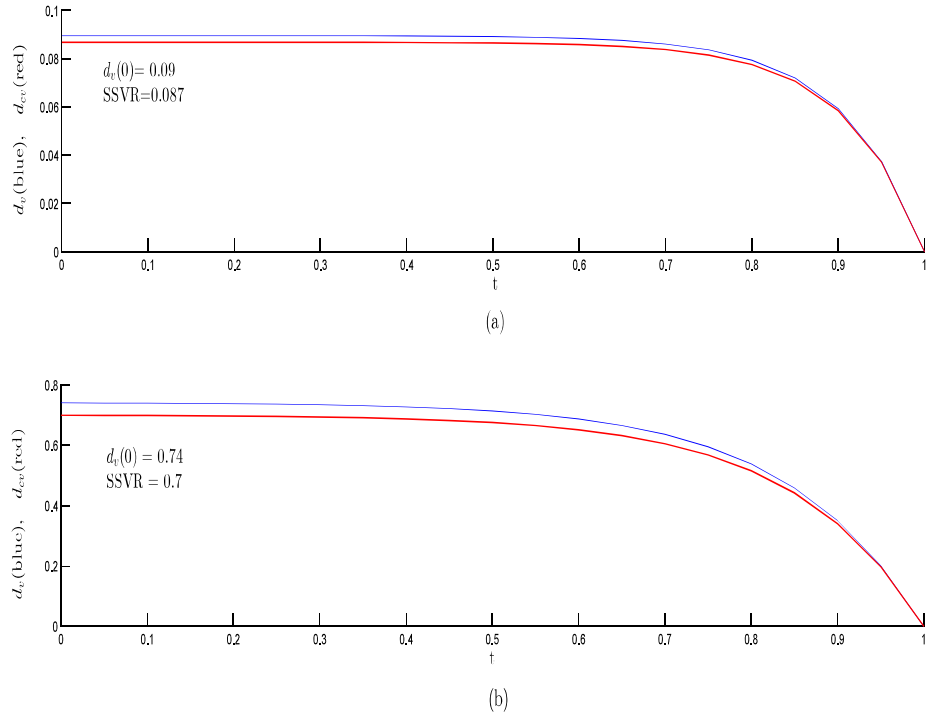


Figure 6.2: Values of $d_{cv}$ and $d_v$ calculated from BIST30 data for a) time period 15.01.2016-03.02.2016, b) time period 01.03.2016-31.03.2016

The graph at Figure 6.2(a) shows the values of $d_{cv}$ and $d_v$ for time period (15.01.2016-03.02.2016) for given parameters in Table 6.2 and Table 6.3. The corresponding value of SSVR equals to 0.089, which is a small value, consequently the value of $d_v$ is equivalently small. This correspondence is consistent with the findings of this thesis that the value of $d_v$ is determined by SSVR. For such low values of $d_v$, stochastic volatility would result in very small change in portfolio choice that can be ignorable regarding the rule for optimal hedging demand $\rho_v \sigma_v d_v$.

The graph at Figure 6.2(b) refers to parameters for time period (01.03.2016-31.03.2016). This gives SSVR = $0.7$ and $d_v$ = $0.74$. The value of $d_v$ is much greater compared to that in the previous time period, which indicates that the impact of stochastic volatility is now much greater. This is reflected by the difference between blue line (stochastic volatility) and the red line (constant volatility) which is larger in the second time period. Considering value of the $d_v$ the incorporating of stochastic volatility become relevant as it would lead to more significant differences in portfolio choice.

# CHAPTER 7

## Conclusion and Future Work

In this thesis, a comparative study was carried out between stochastic volatility and constant volatility in Merton's portfolio optimization problem within the framework of Liu's model. Liu's solution involves substituting solutions of a specific form into the Hamilton Jacobi Bellman (HJB) equation associated with the problem and reducing it first to a simpler Partial Differential Equation (PDE), and then reducing this PDE into a sequence of Ordinary Differential Equations (ODE). In this thesis we give the details of these reductions. We then used explicit solutions provided by Liu for the Merton H model to see the effect of replacing stochastic volatility with constant volatility. We drived a ratio(SSVR) which measures the sensitivity of dynamic portfolio choice to stochastic volatility. Values taken by this ratio and how they affect the portfolio choice were examined through graphs by making comparisons. Finally an application on BIST30 was made which verified the findings of the current work that when the value of SSVR is small, incorporating stochastic volatility into the model has little effect on the optimal portfolio. When this ratio is large (when Sharpe ratio is high and the investor has low risk aversion) taking stochastic volatility into consideration is meaningful.

As a future work, the current application can be extended; BIST30 data can be analyzed over a wider range and similarly the application can be implemented to other markets data. This would lead to a better empirical interpretation of stochastic volatility on dynamical portfolio choice.

# REFERENCES


[1] R. Bellman, On the theory of dynamic programming, Proceedings of the National Academy of Sciences, 38(8), pp. 716–719, 1952, ISSN 0027-8424.

[2] A. Buraschi, P. Porchia, and F. Trojani, Correlation risk and optimal portfolio choice, The Journal of Finance, 65(1), pp. 393–420, 2010.

[3] J. Y. Campbell and L. Hentschel, No news is good news: An asymmetric model of changing volatility in stock returns, Journal of Financial Economics, 31(3), pp. 281 – 318, 1992, ISSN 0304-405X.

[4] G. Chacko and L. Viceira, Dynamic consumption and portfolio choice with stochastic volatility in incomplete markets, Review of Financial Studies, 18(4), pp. 1369–1402, 2005.

[5] P. Christoffersen, S. Heston, and K. Jacobs, The shape and term structure of the index option smirk: Why multifactor stochastic volatility models work so well, Management Science, 55(12), pp. 1914–1932, 2009.

[6] M. Escobar, S. Ferrando, and A. Rubtsov, Optimal investment under multi-factor stochastic volatility, Quantitative Finance, 17(2), pp. 241–260, 2017.

[7] G. Faria and J. C. da Silva, Is stochastic volatility relevant for dynamic portfolio choice under ambiguity?, The European Journal of Finance, 22(7), pp. 601–626, 2016.

[8] K. R. French, G. Schwert, and R. F. Stambaugh, Expected stock returns and volatility, Journal of Financial Economics, 19(1), pp. 3 – 29, 1987, ISSN 0304-405X.

[9] P. Hagan, D. Kumar, A. Lesniewski, and D. E. Woodward, Managing smile risk, 1, pp. 84–108, 2002.

[10] S. L. Heston, A closed-form solution for options with stochastic volatility with applications to bond and currency options, The Review of Financial Studies, 6(2), pp. 327–343, 1993.

[11] J. Liu, Portfolio selection in stochastic environments, Review of Financial Studies, 20(1), pp. 1–39, 2007.

[12] H. Markowitz, Portfolio selection*, The Journal of Finance, 7(1), pp. 77–91, 1952.

[13] O. M. Mert, *Applications of the Heston Model on BIST30 Warrants: Hedging and Pricing*, Master's thesis, Middle East Technical University, 2016.

[14] R. C. Merton, Lifetime portfolio selection under uncertainty: The continuous-time case, The Review of Economics and Statistics, 51(3), pp. 247–257, 1969, ISSN 00346535, 15309142.

[15] R. C. Merton, *Continuous-Time Finance*, Macroeconomics and Finance Series, Wiley, 1992, ISBN 9780631185086.

[16] H. Pham, *Continuous-time Stochastic Control and Optimization with Financial Applications*, Springer Publishing Company, Incorporated, 1st edition, 2009, ISBN 3540894993, 9783540894995.

[17] M. D. Schroder and C. Skiadas, Lifetime consumption-portfolio choice under trading constraints, recursive preferences and nontradeable income, 2003, kellogg School of Management, Finance Working Paper No. 324.