EFFICIENT SIMULATION AND MODELLING OF COUNTERPARTY CREDIT RISK

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ABSTRACT

EFFICIENT SIMULATION AND MODELLING OF COUNTERPARTY CREDIT RISK

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After 2008-2009 crisis, measurement of Counterparty Credit risk has become an essential part of Basel-III regulations. The measurement involves a complex calculation, simulation and scenario generation process which involve a heavy computational cost. Moreover, the counterparty default calculation is an important part depending on scenario generation and state of the economy, state of the counterparty, liquidity as well as the bank itself.

In this thesis we develop flexible structural credit risk models and an efficient simulation framework for Counterparty Credit Risk calculations. The credit risk models are of Merton type, Black-Cox Barrier type and Stochastic Barrier type in Variance Gamma environment. We proceeded by modifying stochastic volatility models to be used for credit risk and default dependence. Moreover, we derive a liquidity adjusted option price for stochastic volatility models to measure indirect effect of liquidty on credit spreads. The models studied were all developed to include default dependence between counterparties using an affine factor framework.

Keywords: Counterparty Credit Risk, Efficient Simulation, Structural Credit Risk, Variance Gamma, Stochastic Volatility, Liquidity adjustment.

ÖZ

KARŞI TARAF KREDİ RİSKİ MODELLEMESİ VE ETKİN SİMÜLASYONU

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2008-2009 krizi sonrasında Karşıtaraf Kredi Riski ölçümü Basel-III düzenlemelerinin önemli bir parçası haline gelmniştir. Söz konusu riskin ölçümü yüksek hesaplama maliyeti doğuran karmaşık hesaplamalar, simülasyonlar ve senaryo üretim süreçlerini içermektedir. Ayrıca, Karşıtaraf temerrüt tahmin süreci, ekonomik ve finansal koşulları dikkate alan senaryo üretimine, karşıtarafın finansal durumu ve likidite koşulları ile bankanın durumuna da bağlı olarak değiştiğinden önem taşımaktadır.

Bu tezde, Karşıtaraf Kredi Riski ölçümü özelinde etkin bir simülasyon altyapısı ve esnek yapısal kredi riski modelleri geliştirilmiştir. Bu çerçevedeki kredi riski modelleri, Varyans Gama süreci çerçevesinde Merton yapısal özelliğni ve Black-Cox Bariyer-Stokastik bariyer özelliklerini taşıyacak şekilde tasarlanmıştır. Ayrıca söz konusu kredi riski modelleri, stokastik volatilite sürecini içerecek şekilde de genişletilmiştir. Ek olarak, stokastik volatilite modelleri, likidite düzeyinin opsiyon fiyatı üzerindeki etkisini gösterecek şekilde düzenlenmiştir. Tezdeki tüm modeller taraflar arası kredi riski bağımlılığını içerecek şekilde tasarlanmıştır.

Anahtar Kelimeler: Karşıtaraf Kredi Riski, Etkin Simülasyon, Yapısal Kredi Riski, Varyans Gama, Stokastik Volatilite, Likidite ayarlaması.

To Aliş, Sevgi and my parents

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BM	Brownian Motion
CCR	Counterparty Credit Risk
CDF	Cumulative Distribution Function
CDS	Credit Default Swap
CIR	Cox-Ingersoll-Ross
CVA	Credit Valuation Adjustment
DD	Distance-to-Default
DJS	Direct Jump to Simulation
DVA	Debt Valuation Adjustment
EAD	Exposure at Default
EL	Expected Loss
ENE	Expected Negative Exposure
EPE	Expected Positive Exposure
ES	Expected Shortfall
FV	Fair Value
FX	Foreign Exchange
GBM	Geometric Brownian Motion
GBMCIR	Geometric Brownian Motion Cox-Ingersoll-Ross
GBMSA	Geometric Brownian Motion Stochastic Arrival
GBMSAj	Geometric Brownian Motion Stochastic Arrival with Jump
GBMSAfj	Geometric Brownian Motion Affine Factor Stochastic Arrival with Jump
IRS	Interest Rate Swap
IAS	International Accounting Standards
LGD	Loss Given Default
MC	Monte Carlo Simulation
MLMC	Multi-level Monte Carlo
MSE	Mean Squared Error
MMA	Money Market Account

NIG	Normal Inverse Gaussian
PD	Probability of Default
PD	Probability of Default
PDE	Partial Differential Equation
PIDE	Partial Integro Differential Equation
PDF	Probability Density Function
RR	Recovery Rate
SABR	Stochastic Alpha Beta Rho
SDE	Stochastic Differential Equation
SV	Stochastic Volatility
UL	Unexpected Loss
VaR	Value at Risk
VG	Variance Gamma
VGCIR	Variance Gamma Cox-Ingersoll-Ross
VGSA	Variance Gamma Stochastic Arrival

CHAPTER 1

INTRODUCTION

1.1 Introduction

Aftermath of 2008 financial crisis the financial regulation environment had undergone a more strict phase in the context of financial derivatives [32]. These events elucidated the importance of measuring the risk concept Counterparty Credit Risk . Counterparty risk occurs when a party to an OTC (Over the Counter) derivatives contract may fail to perform on its contractual obligations, causing losses to the other party [22].

Especially after the aforementioned crisis, measurement of this risk has become one of the main challenges for regulators and banks. Hence, correct assessment and management of counterparty credit risk has become a major concern for financial market regulators. It contributed to the reconstructing the behavioural style of banks and their counterparties.

The regulatory environment has changed significantly with increasing speed aimed to reduce the risk of bank failures and to increase financial stability. Therefore, this regime shift helped constructing a more robust composition of capital via capital requirements, related to counterparty risk exposures. In that context, moving away from the Basel I regime which was introduced in 1988 with a first well established principle and method set on risk based regulation, to the Basel II (2005) and, even more, the Basel III (2010) regimes have pointed to the need and importance of an enhanced sensitivity of credit risk measurement. In Basel-III and complementaries, capital requirements have been linked to more sophisticated measures of counterparty credit risk such as the Credit Valuation Adjustment (CVA), Debt Value Adjustment (DVA).

More recently the potential volatility of the same (VaR of CVA) captured via VaR models inconjunction with stress testing under extreme market scenarios has been put in place [33].

In this thesis we wanted to construct first; flexible and realistic, calibratable to real data, credit risk models of different properties. This was aimed to fit the needs of Counterparty Credit Risk framework. The theoretical grounds of out credit risk models lies on subordinated Levy processes such as Variance Gamma , Normal Inverse Gaussian and stochastic volatility. Our modification of these models is to embed an affine decomposition structure [32, 6] to their architecture. Such a framework not only enables a more concrete Counterparty Credit Risk calculation but also enables right/wrong way risks which have become crucial concepts of Basel-III/IV framework particularly for Counterparty Credit Risk . It is important to note that, we extended Black-Cox type barrier models [8] to contain Levy process with the affine decomposition and a Monte Carlo (MC) Simulation procedure in that environment is derived.

Second, due to the computational burden elicited by the measurement of this risk, we study and develop some formulas to increase the efficiency of Counterparty Credit Risk simulation. The purpose is not to rely on hardware capacity, rather to find out analytical formulas for efficiency. In that context we developed formulas for optimal parameters in terms of MC simulation and these are used to calculate CCR metrics such as EPE, CVA and DVA etc. We find that it is possible to obtain some optimal levels for number of simulations and number of simulation paths for CVA calculation.

Third, the stress testing of Counterparty Credit Risk has not been implemented with a link economc state, so we propose Counterparty Credit Risk measurement allowing stress testing. We have constructed such framework based on Variance Gamma process. We see that this point is crucial, as the connection between macroeconomic and financial key factors with counterparty default has not been paid much importance in the calculation of Counterparty Credit Risk . The importance of this issue showed its role in the 2008 financial crisis where collapse of financial system emanated from macroeconomic breakdown as to what have observed.

Lastly, we derive an option formula embedded with a liquidity adjustment factor. It has structural feature, however it's a stochastic volatility model different from other classical models of that kind. The importance of this is to observe the effect of liquidity constraint, which is the first activity we see in markets during financial crisis, on the credit spreads.

1.2 Preliminaries

Credit risk occurs when one party of a financial agreement fails to satisfy financial obligations and causes losses to the other party due to its deterioration in credit quality [32]. According to Basel-II/III, CRD-IV of EU Banks are required to set aside a capital for the unexpected losses from that risk. There are two methods for calculation of required capital for banks:

1. **Standard Approach**: This method has no internal modelling component and all risk based components are predefined and preset by the Basel Banking Committee.

2. **Economic Capital Approach:** Economic Capital Approach mainly utilizes credit risk models to account for calculation of risk and adjustment of captial based on the loss obtained from the model. The main goal of this component is quantification of expected loss (EL) and unexpected loss (UL). The key components of this quantification are [33].

   A- Probability of Default (PD)

B- Loss Given Default (LGD)

C- Exposure at Default (EAD)

Thus, this method is mainly interested in modelling of those risk parameters.

As mentioned, Expected Loss (EL) calculation is done by the formula:

$$EL = PD \times LGD \times EAD$$

Then Unexpected Loss (UL) calculation, which is at the core of ratings based approaches in Basel II/III.

$$UL = VaR - PD \times LGD \times EAD$$

1.2.1 Recovery Ratio (RR)

The first exogenous input credit risk modelling is the recovery ratio [33]. The default event could be defined in many ways; it could be real inability to meet the obligations or missing payments. In the same way, loss on default can also vary in multiple ways.

The most common are: 1) Receive cash in proportion of market value, 2) Receive cash in proportion of principal value. Thus, the recovery is a ratio, $RR \in (0, 1)$.

A common assumption is to use $\%40$ which is also a predefined ratio in Basel-II/III regulations.

As a result, $\%60$ is loss given default, which is termed in practice LGD. In practice it is observed that [33]:

Recovery rate is a function of the seniority,

The distribution of recovery rate is asymmetrical and skewed at the right tail,

The data displays a remarkable dispersion around the mean for each debt class,

Bank debt has the highest recovery rate and is generally less volatile than the other debt classes.

CHAPTER 2

COUNTERPARTY CREDIT RISK ESTIMATION AND CHALLENGES

2.1 Definition

Counterparty Credit Risk, basically can be defined as credit risk or default risk, that counterparty will fail to satisfy the payment obligations due to another party. Consider a fixed rate loan without any subtelities in which the borrower makes annual fixed payments to the lender for ten years before paying back the principal of the loan [22]. In this case the borrower defaults if there is a failure to pay any of the interest payments or the principal of the loan during the life of the loan [22].

The same is true for bonds and other securities, however in this case the defaulting side will be the issuer of the security. Derivatives also expose counterparties to credit risk. Some of which the payments are bidirectional and the both counterparties incur default event such as interest rate swaps (IRS) [22].

Counterparty credit risk is present when one counterparty has an exposure to the other. The *exposure at default* (EAD) is the total amount owed by the defaulting party to the non-defaulting party. EAD could be written [22]:

$$EAD = \max(V, 0), \tag{2.1}$$

where V is the value of derivative.

There is no exposure if the non-defaulting party has liabilities towards the defaulting party and this gives the max function in equation 2.1. If the exposure above is positive and non-defaulting party has liabilities to the defaulting party, the net position will form the basis of the claim the creditor will make against the defaulter through bankruptcy proceedings [22].

The *Expected Positive Exposure* is the amount of exposure party A, the side that carries risky position due to the counterparty (party B), has at some future date given that the expectation or average taken over all simulated future outcomes on the date of interest as defined below [22]:

$$EPE(t) = \mathbb{E}\{\max(V, 0)|F_t\}. \tag{2.2}$$

Expected Exposure is simply the expected exposure at each specific point of time e.g payment dates. These exposure measures display an important role in the estimation of the important counterparty credit risk metric: *Credit Valuation Adjustment* (CVA) [22].

The *Expected Negative Exposure* (ENE) is the expected exposure that party B carries due to party A as seen from the perspective of A [22].

$$ENE(t) = \mathbb{E}\{\max(-V, 0)|F_t\}. \tag{2.3}$$

The expected negative exposure enters the calculation of debit and funding valuation adjustments (DVA and FVA) which are the extensions of CVA with multiple defaults are considered [22]. The unilateral and bilateral default cases are:

-
$$PD(t, T) = \mathbb{E}(\mathbb{1}_{t<\tau<T}) = P(\tau < T), \tag{2.4}$$

-
$$\begin{aligned} PD_{1,2}(t, T) = \mathbb{E}(\mathbb{1}_{\{t<\tau_{1,2}<T, \tau_{2,1}>T\}} &= P(\tau_{1,2} < T, \tau_{2,1} > T) \\ &= P(\tau_{1,2} < T|\tau_{2,1} > T)(1 - P(\tau_{2,1} < T)), \end{aligned} \tag{2.5}$$

respectively.

2.1.1 Credit Valuation Adjustment CVA

The major concept for the CCR modelling is Credit Valuation Adjustment (CVA). It is a metric to quantify CCR and measures the expected loss from missing payments of the derivative due to the default of counterparty. As we defined before, this is a unilateral assumption. If the bilateral assumption is used then it is called Debt Valuation Adjustment (DVA).

These have become an integral part of Basel III regulatory requirements and IAS39 accounting rules [33]. The CVA could be mathematically defined:

$$CVA(t) = \mathbb{E}\left(\mathbb{1}_{\{\tau \leq T\}}(1-R)(e^{(-\int_t^T r(s)ds)})V_+(T) \mid \mathcal{F}_t\right)$$

This can be calculated:

$$CVA(t) = \int_t^T \left((1-R)e^{-\int_s^t r(u)du}V_+(T)\right) dPD_s \tag{2.6}$$

$$= \mathbb{E}\left(\mathbb{1}_{\{\tau \leq T\}}|\mathcal{F}_t\right)\mathbb{E}\left((1-R) \mid \mathcal{F}_t\right)\mathbb{E}(\left(e^{(-\int_t^T r(u)du))}V_+(T) \mid \mathcal{F}_t\right))$$

2.2 CVA Estimation and Challenges

The major problem exists for CVA is the need for numerous simulations for the determination of derivative pricing components path and default risk on the path. The default risk to be the first and the others require scenarios for the simulation as well. It is affected by many macro and idiosynratic factors which are to be simulated.

As a result, a method which either by approximation of the path or a complexity / variance reduction will be very important. It is also possbile to find some semi-closed form solutions for same cases which alleviates the burden remarkably. These will significantly contibute to the calculation of CVA, DVA or other variants.

2.2.1 Scenario Generation and Path Simulation

The first step in calculating credit exposure is to generate potential market scenarios at a fixed set of simulation dates $\{t_k\}_{k=1}^N$ in the future. Market scenarios are specifications of a set of price factors that could impact the values of the trades in the portfolio severely. Some of the price/risk factors are foreign exchange (FX) rates, interest rates, equity prices, commodity prices and credit spreads.

The scenarios are usually specified via stochastic differential equations (SDE). Typically, these SDEs describe Markovian processes and are solvable in closed form [44]. For example, a popular choice for modelling FX rates and stock indices is the generalized geometric Brownian motion given by [44]:

$$dX(t) = \mu(t)X(t)dt + \sigma(t)dW_t. \tag{2.7}$$

The solution of this SDE is known as geometric Brownian Motion

$$X(t) = X(s)\exp\left((\int_s^t \mu(s) - 0.5\sigma(s)^2 ds) + \int_s^t \sigma(s)dW(s)\right). \tag{2.8}$$

For scenario generation there are two approaches namely Path-Dependent Simulation (PDS) and Direct Jump to Simulation Date (DJS) [44]. In PDS for each time interval on the path scenarios are generated whereas in DJS scenarios are directly generated or jumped from t=0 to scenario date t.

Above equations change for PDS:

$$X(t_k) = X(t_{k-1})\exp\left(\frac{\int_{t_{k-1}}^{t_k} \mu(s) - 0.5\sigma^2(s)ds}{t_{k-1} - t_k} + \frac{\int_{t_{k-1}}^{t_k} \sigma(s)dWs}{t_k - t_{k-1}}\right) \tag{2.9}$$

for DJS:

$$X(t_k) = X(t_0)\exp\left(\frac{\int_{t_0}^{t_k} \mu(s) - 0.5\sigma^2(s)ds}{t_k - t_0} + \frac{\int_{t_0}^{t_k} \sigma(s)dWs}{t_k - t_0}\right) \tag{2.10}$$

The price factor distribution at a given simulation date obtained using either PDS or DJS is identical. However, a PDS method may be more suitable for path-dependent, American/Bermudan and asset-settled derivatives [44].

Scenarios can be generated either under the real world probability measure or under the risk-neutral probability measure. Under the real world measure, both drifts and volatilities are calibrated to the historical data of price factors. Under the risk-neutral measure, drifts must be calibrated to ensure there is no arbitrage of traded securities on the price factors.

Additionally, volatilities must be calibrated to match market implied volatilities of options on the price factors.

2.2.2 Computational Complexity "An Illustration"

For CVA measurement the banks have to run simulations for different risk factors, economic scenarios for each counterparty so an efficient simulation with reduced computational burden is very beneficial.

For illustration, an example of a generic CVA calculation algorithm is given as follows [33]:

1. Simulate the Cox-Ingersoll-Ross (CIR) model,

   CIR Model has the SDE:

   $$dr(t) = \lambda(\mu - r(t))dt + \sigma\sqrt{r_t}dW_t$$

   Its solution is:

   $$r(t) = e^{-\lambda(t-s)}(r(s) - \mu) + \mu + \int_s^t e^{-\lambda(t-u)}\sigma\sqrt{r(t)}dW(u)$$

   The simulation will be based on discretization below:

   $$r^k(t_i) = r^k(t_{i-1}) + \lambda\left(\mu - r^k(t_{i-1})\right)\Delta t + \sigma_v\sqrt{r^k(t_{i-1})}\epsilon^k\sqrt{\Delta t}, \qquad (2.11)$$

   where $\epsilon \sim N(0,1)$.

   This simulation ensures the positive $r(t)$ if Feller condition ($2\lambda\mu \geq \sigma^2$) is satisfied.

2. Calculate discount rate process (could be Money market account):

   $$\text{MMA}(T) = \text{MMA}(t)e^{\int_t^T r(s)ds}$$

   The calculation starts with discretization,

   $$\text{MMA}(t_i) = \text{MMA}(t_{i-1})e^{\int_{t_{i-1}}^{t_i} r^{(k)}(s)ds}$$

where the integral in the exponential is approximated by (Simpson integration here but naturally more advanced technique could be used),

$$\text{MMA}(t_i) = \text{MMA}(t_{i-1}) \exp\left(\frac{h}{3}\left(r^{(k)}t_i - 4r^{(k)}t_{i-1} + r^k t_{i-2}\right)\right)$$

3. Use implied bond price of CIR model to simulate term structure,

$$B(t, t+\tau) = \exp\left(A(\tau) + B(\tau)r(t)\right)$$

After discretization of this equation we have:

$$B(s, t_i) = \exp\left(A(t_i - s) + B(t_i - s)r^{(k)}(s)\right)$$

4. Compute fair value of swap where $f(t)$ is fixed rate of the IRS:

$$FV^{(k)}(s) = 1 - P^{(k)}(s, T_n) + \frac{f(t)}{12}\sum_{i=1}^{n} P^{(k)}(s, t_i)$$

5. Calculate exposure value for simulation k:

$$E^{(k)} = \max(FV^{(k)}(s), 0)$$

6. Calculate CVA for simulation k and $t = t_i$:

$$\text{CVA}^{(k)} = (1 - R)PD(t_i, t_{i-1})E^{(k)}D(t_i)$$

7. Then calculate final CVA for N simulations:

$$\text{CVA} = \frac{1}{N}\sum_{k=1}^{N}\sum_{i=1}^{M}\text{CVA}^{(k)}(t_i) \qquad (2.12)$$

2.2.3 Default Probability Simulation with State of Economy (Stress Testing)

An overlooked but very important part of the CVA is the stress testing of credit risk component. Since CVA is averaged over marginal default probabilities, a default probability calculation with a strong link with state of the economy is very important. For practical calculations default probabilities are bootstrapped using CDS spreads [33]. However, there are not many credit risk models incorporated into CVA calculations directly.

CHAPTER 3

EFFICIENT SIMULATION FOR COUNTERPARTY CREDIT RISK

3.1 Introduction

There are four studies particularly directed at obtaining an efficient framework for Counterparty Credit Risk estimation [41, 19, 39, 9].

In the first study [41], the author obtains an approximation formula for CVA by applying asymptotic expansion method based on infinite dimensional analysis called the Watanabe-Yoshida theory and the Malliavin calculus. Author estimates CVA of the interest rate swap which has an underlying interest rate model of SABR.

This model is widely used by practitioners in the financial industry [50] and is popular, thus application of mentioned methodology is claimed to reduce the time it takes to calculate the CVA of the swap by Monte Carlo simulation. Using a new approach, he derives an approximate formula for the CVA of the swap.

This new formula is shown to enable CVA calculation much faster than Monte Carlo method. However, the method requires some tedious derivations and some Malliavin Calculus.

Definition 3.1. Asymptotic Expansion

Let's consider $\mathbb{R}^d$ valued diffusion process and X^ϵ to be the solution to the following SDEs [50]

$$dX^\epsilon = V_0(X^\epsilon, \epsilon)dt + \epsilon V(X^\epsilon)dW_t \tag{3.1}$$

$$X_0^\epsilon = x_0 \tag{3.2}$$

where $\epsilon \in [0, 1]$ is a known as perturbation parameter. The following therem is provided in [55].

Theorem: Suppose $V_0 : \mathbb{R} \times [0, 1] \to \mathbb{R}^d$ and $V : \mathbb{R}^d \times \mathbb{R}^m$ are smooth and derivatives of any order are bounded. Next, suppose that a function g: $\mathbb{R}^d \to \mathbb{R}$ to be smooth and all derivatives are of polynomial growth orders. Then for $\epsilon \downarrow 0, g(X_T^{(\epsilon)})$ has asymptotic expansion:

$$g(X_T^{(\epsilon)}) = g_{0T} + \epsilon g_{1T} + \epsilon^2 g_{2T} + \epsilon^3 g_{3T} + o(\epsilon^3) \tag{3.3}$$

The coefficients g_{0T}, g_{1T}, g_{2T} and g_{3T} could be obtained by Taylor's formula with the following Wiener-Ito integrals:

$$g_{0T} = g_{XT}^0$$

$$g(1T) = \sum_{j=1}^{d} \partial_j g(X_T^0) D_T^j$$

$$g(2T) = \sum_{i,j=1}^{d} \partial_i \partial_j g(X_T^0) D_T^i D_T^j + \frac{1}{2} \sum_{j=1}^{d} \partial_j E^j(T)$$

$$g(3T) = \sum_{i,j,k=1}^{d} \partial_i \partial_j \partial_k g(X_T^0) D_T^i D_T^j D_T^k + \frac{1}{2} \sum_{i,j=1}^{d} \partial_i \partial_j E^i(T) D^j(T) + \frac{1}{6} \sum_{i=1}^{d} \partial_i g(X_T^0) F_i^T$$

where $D_t = \frac{\partial X^{(\epsilon)}}{\partial \epsilon}|_{\epsilon=0}, D_t = \frac{\partial X^{(\epsilon)}}{\partial \epsilon}|_{\epsilon=0}, E_t = \frac{\partial^2 X^{(\epsilon)}}{\partial \epsilon^2}|_{\epsilon=0}, F_t = \frac{\partial^3 X^{(\epsilon)}}{\partial \epsilon^3}|_{\epsilon=0}$.

We can see that as long as precision of this approximation is good, there is no need for multiple Monte Carlo simulations for CVA calculation. Only single path simulation to calculate Wiener-Ito Integrals for conditional expectation is sufficient.

The study of [19] is the first one examine an efficient simulation framework for Counterparty credit risk measurement methods such as CVA. Their approach is based on comparison of PDS and DJS path simulation [44] in terms of Mean Squared Error (MSE). The PDS method as we know from (2.9),

$$X(t_k) = X(t_{k-1})e^{\left((\mu_{k,k-1}-0.5\sigma^2_{k,k-1})(t_k-t_{k-1})+\sigma_{k,k-1}\sqrt{(t_k-t_{k-1})}Z\right)} \tag{3.4}$$

The covariance between pathwise elements are:

$$\mathbb{E}(X_{t_k}X_{t_{k-1}}) = E(\exp(Z_k + Z_{k-1}))$$
$$= \exp\left(\mu_k + \mu_{k-1} + 0.5(\sigma_k^2 + \sigma_{k-1}^2 + 2Cov(Z_k, Z_{k-1})) - \mu_k - \mu_{k-1} - 0.5(\sigma_k^2 + \sigma_{k-1}^2)\right)$$
$$= \exp(Cov(Z_k, Z_{k-1}))$$

For DJS from (2.10) the solution is:

$$X(t_k) = X(t_0)\exp\left((\mu_{0,k-1} - 0.5\sigma^2_{0,k-1})(t_k - t_0) + \sigma_{0,k-1}\sqrt{(t_k-t_0)}\tilde{Z}\right) \tag{3.5}$$

where $Z \sim \mathcal{N}(0, \Sigma)$ and $\tilde{Z} \sim N(0, 1)$.

Here for DJS,

$$\mathbb{E}(X_{t_k}X_{t_{k-1}}) = \exp\left(Cov(\tilde{Z}_i, \tilde{Z}_{i-1})\right).$$

Since by construction $Cov(\tilde{Z}_i, \tilde{Z}_{i-1}) = 0$ then $E(X_{t_k}X_{t_k-1}) = 0$. We can see that PDS has a correlated path whereas DJS has uncorrelated path. This is important since CVA with no correlated path will have a smaller variance.

$$\text{CVA(T)} = \sum_{t=1}^{T} V_t dP_t$$

$$\text{Var(CVA(T))} = \sum_{t=1}^{T}\left[Var(V_t)\Delta P_t^2 + 2Cov(V_t, V_s)\Delta P_t \Delta P_s\right]$$

This is very favorable for DJS, however DJS has a computational cost $\left(N + \frac{N(N-1)}{2}\right)$ due to random number generations, PDS has N random number generations. However PDS has N^2 operations due to covariance terms whereas DJS has N operations. Given that Complexity $=$ Var$(replication) \times \mathbb{E}(C_t)$ we will have:

$$\frac{C_t(DJS)}{C_t(PDS)} = N$$

and

$$\frac{Var(DJS)}{Var(PDS)} = \frac{1}{N}$$

where C_t is the computation time. This shows that comparability is possible for both estimators since total complexity is approximately the same.

In the study [39] optimal quantization methodology was used to calculate Counterparty Credit Risk involving EPE and CVA. Let's first define quantization.

3.1.1 Quantization

Let X be a random vector on a probability space $(\Omega, F, \mathbb{P})$ taking values $\mathbb{R}^d$. Let $\mathbb{P}_x$ denote the image law in $\mathbb{R}^d$. Optimal quantization consists in the approximation of X by another random vector $q(X)$ and the (x_i), the optimal quantizer of X, are chosen[42] such that for integer $p \geq 1$,

$$\int_{\mathbb{R}^d} min(||x_i - v||^p) P_x(dv) = \min(\mathbb{E}||X - q(X)||^p). \tag{3.6}$$

Here $q(X) = \sum_{i=1}^{N} x_i \mathbb{1}_{C_i(x)}(X)$ and $C_i(x) = (y \in \mathbb{R}^d, ||y - x_i|| < ||y - x_j||, \forall i \neq j)$ is the i^{th} Voronoi of x. The quadratic quantization could be directly defined for $p = 2$. To make things clear on how to find out optimal Voronoi cells we can start with $\mathbb{R}^d$ where $d = 1$. For $p = 2$ we can write previous equation

$$Q = \int_{\mathbb{R}} min((x_i - v)^2) P_x(dv) = min(\mathbb{E}||X - q(X)||^2) \tag{3.7}$$

Then to find optimal value we can use the first and the second order conditions since quadratic quantization uses a convex function. Then, we proceed

$$\frac{\partial Q}{\partial x_i} = \int_{\mathbb{R}} 2(x_i - v) P_x(dv) = 0 \tag{3.8}$$

Then

$$x_i = \frac{\int_{x_{i-1}}^{x_{i+1}} v P_x(dv)}{\int_{x_{i-1}}^{x_{i+1}} P_x(dv)} \tag{3.9}$$

Using these optimal quantiles of the distributions it's possible to efficiently simulate and approximate the derivative price, EPE, CVA or any other parameter of interest. Hence, the authors of [39] used the methodolgy with slight extensions. As authors claim, results seem to be quite accurate and time efficient.

The concept of brownian local time and occupation time to approximate the value of brownian motion functional is used by [9] to estimate another Counterparty Credit Risk metric EPE where the authors claim high efficiency.

3.1.2 Brownian Local Time

Let $W(t)_{t\geq 0}$ be standard brownian motion defined over probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

The local time can be defined as the time it takes the brownian motion to stay close to level of $A \in \mathbb{R}$ [9]. The concept was first introduced by Paul Levy (1965). Levy has defined this random time in sound ground by [9]:

$$L_t(A) = \frac{1}{2\epsilon} \lim_{\epsilon \to 0} \mu\{0 \leq s \leq t, : |W(t) - \epsilon| \leq A\}.$$

Then it is possible to write the occupation time [9]:

$$\beta(B, \omega) = \int_0^t \mathbb{1}_B(W(s)ds) = \int_B L(t)(x, \omega)dx.$$

The benefit of this equality will be in writing Brownian Motion functionals that the following result will confirm,

$$\beta(B, \omega) = \int_0^t f(W(s)ds) = \int_B f(x)L(t)(x, \omega)dx,\ 0 \leq t \leq T.$$

The authors [9] exploit this result to calculate derivative prices which are naturally functionals of Brownian Motion and then to calculate EPE which they claim to be efficient in simulation.

3.2 Efficient Simulation of Derivative Price

Given that CCR is a computationally intensive task, reduction of variance and bias is of high importance. Hence, a possible issue as the target objective function is MSE. For that purpose we will be using Multilevel Monte Carlo Method introduced by [20].

3.2.1 Multilevel Monte Carlo Methodology

This methodology [20] defines a Monte Carlo estimator with telescopic sum which is an estimation with control variate.

$$\mathbb{E}(\tilde{P}_L) = \mathbb{E}(\tilde{P}_0) + \sum_{l=1}^{L} (\mathbb{E}(\tilde{P}_l) - \mathbb{E}(\tilde{P}_{l-1})) \tag{3.10}$$

Then, defining more levels, such as multi levels inside paths one, can obtain a better estimator:

$$\mathbb{E}(\tilde{P}_L) = \mathbb{E}(\tilde{P}_0) + \sum_{l=1}^{L} \frac{1}{N_l} \sum_{j=1}^{N_l} (\mathbb{E}(\tilde{P}_l^j) - \mathbb{E}(\tilde{P}_{l-1}^j))$$

Using multilevel estimator above we can find a prediction of exact $\mathbb{E}(P)$ by optimizing MSE:

$$\text{MSE(P)} = \mathbb{E}(P - \tilde{P}_L)^2$$
$$\text{MSE(P)} = (\mathbb{E}(P) - \mathbb{E}(\tilde{P}_L))^2 + \mathbb{E}(\mathbb{E}(\tilde{P}_L) - P))^2$$
$$\text{MSE(P)} = \text{Bias}^2 + \text{Var}$$

This could be done by optimizing number of N_l, L simulation number at each time step and number levels in the timeframe. Thus, our optimization problem will be given fixed computational budget:

$$\underset{N_l, L}{\text{argmax}}\, Lf(N_l, L) = \sum_{l=0}^{L} \frac{V_l}{N_l} + \lambda \Big[\sum_{l=0}^{L} N_l h_l - C \Big]$$

$$\frac{\partial Lf}{N_l} = \sum_{l=0}^{L} \left(\frac{-V_l}{N_l^2} \right) + \sum_{l=0}^{L} \frac{\lambda}{h_l} = 0$$

$$\frac{V_l}{N_l^2} = \frac{\lambda}{h_l}, \quad N_l = \sqrt{\lambda^{-1} \frac{V_l}{h_l}}$$

$$\sum_{l=0}^{L} \frac{V_l}{N_l} \le \frac{\epsilon^2}{2}, \quad \sum_{l=0}^{L} \sqrt{\frac{V_l}{h_l}} \lambda^{\frac{1}{2}} \le \frac{\epsilon^2}{2}, \quad \lambda^{(-\frac{1}{2})} \ge \sum_{l=0}^{L}$$

Using these we obtain an optimal value N_l as such:

$$N_l = \Big\lfloor \sqrt{\frac{V_l}{h_l}} \sum_{l=0}^{L} \sqrt{\frac{V_l}{h_l}} 2\epsilon^{-2} \Big\rfloor \tag{3.11}$$

Using the fact that Bias=Ch_l, $h_l = \frac{T}{2^L}$ and $Ch_l \le \frac{\epsilon}{\sqrt{2}}$ then an upper bound for L is obtained

$$L_{max} = \log \left(\frac{\sqrt{2} C T \epsilon^{-1}}{\log 2} \right)$$

3.2.2 MLMC Algorithm

The outline of the algorithm is as follows:

1. Begin with $L = 0$
2. Calculate the initial estimate of V_L using for example 100 samples
3. Determine optimal N_l using (3.11)
4. Generate additional samples as needed for new N_l
5. if $L < L_{max}$ set $L = L + 1$ and go to second step.

As suggested in [20] the L_{max} could be improved dramatically by evaluating bias.

3.3 An Optimal Simulation Framework for Counterparty Credit Risk

In this method the approach is to find optimal number of simulation amount N and M optimal number of simulation paths to calculate Counterparty credit risk measure of interest.

We take EPE in equation (2.2) first for illustrative purposes,

$$\text{EPE}_t = \theta_t = \int_0^t E[V_s]ds$$
$$\widehat{\text{EPE}_t} = \hat{\theta}_t = \sum_{i=1}^{M}\sum_{j=1}^{N}\frac{V_{ij}}{N}\Delta_i$$

and we can decompose bias and variance for PDS estimator as follows:

$$\text{Var}(\theta_t) = \sum_{i=1}^{M}\frac{\text{Var}(V_j)}{N}\Delta_i^2 + \frac{2}{N}\sum_{i,j=1}^{M}\text{Cov}(V_i, V_j)\Delta_i\Delta_j$$
$$\text{Bias}^2{}_t = (\text{EPE} - \widehat{\text{EPE}_t})^2$$
$$\text{Bias}^2{}_t = \left(\sum_{i=1}^{M}\sum_{j=1}^{N}\frac{V_{ij}}{N}\Delta_i - \text{EPE}_t\right)^2$$

The reason we set the framework over EPE rather than CVA is that it is more trivial to write it in terms of discrete time differences. However, we can extend the same to CVA assuming a disrcetized PD change of $O(\Delta t)$.

We can further write this in terms of order notation which will be the basis for our optimization problem and for this purpose we follow the construction of [19]. We can see that Δ_i term is $\{t_i - t_{i-1}\} = \frac{T}{M}$ which simply $O(\frac{1}{M})$. Then given the derivative price estimator $\hat{V}_t$ is of order $O(\frac{1}{N})$ and given Δ_i's order. We can write variance,

$$Var(\theta_t) = \sum_{i=1}^{M} \frac{Var(V_j)}{N} \Delta_i^2 + \frac{2}{N} \sum_{i,j=1}^{M} Cov(V_i, V_j) \Delta_i \Delta_j \tag{3.12}$$

Assume:

$$E(V_i^2) \leq L_1, E(V_i V_j) \leq L_2,$$

Then we write,

$$\begin{aligned} Var(\theta_t) &\leq \left(\frac{T}{M}\right)^2 \frac{L_1 M}{N} + \left(\frac{T}{M}\right)^2 \frac{2L_2 M}{N} \\ &= \frac{T^2(L_1 + 2L_2)}{MN} \cong O(\frac{1}{MN}) \end{aligned}$$

Therefore the variance is of $O(\frac{1}{MN}) + O(\frac{1}{N})$ and bias is of $O(\frac{1}{M^2})$.

Given that, we first start with the minimization problem defined by the authors [19] :

$$\begin{aligned} &\arg\min_{M,N} \frac{c_1}{MN} + \frac{c_2}{N} + \frac{c_3}{M^2} \\ &\text{subject to } \mathrm{C} = c_4 MN \end{aligned} \tag{3.13}$$

We must pay attention to the fact that the PDS based CCR estimator θ_t variance from equation (3.4) is dependent on derivative price estimator variance. This fact requires a resort to use of variance reduction techniques. Here a possible candidate is MLMC of Section 3.2.1 which we explained previously. Because at each time point of the derivative price path, it is possible to define levels and reduce variance by MLMC.

When we go back to our optimization problem, then we can see that it is straightforward to solve this problem via lagrange multiplier.

$$\begin{aligned} L(M, N, \lambda) &= \left(\frac{c_1}{MN} + \frac{c_2}{N} + \frac{c_3}{M^2}\right) + \lambda(c_4 MN - C) \\ \frac{\partial L}{\partial N} &= \frac{-c_1}{N^2 M} - \frac{-c_2}{N^2} + \lambda c_4 M = 0 \\ \frac{\partial L}{\partial M} &= \frac{-c_1}{NM^2} - \frac{2c_3}{N^2} + \lambda c_4 N = 0 \end{aligned} \tag{3.14}$$

Then we proceed by solving this set of equations:

$$\frac{c_1 + c_2 M}{c_4 N^2 M^2} = \frac{c_1 M + 2c_3 M}{c_4 N^2 M^3}$$
$$M = \frac{c_2 N^2}{2c_3}$$

Then the final solution yields:

$$M = (\frac{2Cc_3}{c_4 c_2})^{\frac{1}{3}} = C^{\frac{1}{3}}\tilde{C}$$
$$N = \frac{1}{2}(\frac{4c_2 C^2}{c_4^2 c_3}) = C^{\frac{2}{3}}\hat{C}$$
$$\hat{C} = (\frac{4c_2}{c_4^2 c_3})^{\frac{1}{3}}, \tilde{C} = (\frac{2c_3}{c_4 c_2})^{\frac{1}{3}}$$

We can change the problem above by reverse the case above such that, given an MSE level ϵ^2 what could the optimal budget be, then we see one-to-one relationship with the (3.14):

$$L(M, N, \lambda) = c_4 MN + \lambda(\frac{c_1}{MN} + \frac{c_2}{N} + \frac{c_3}{M^2} - \epsilon^2) \tag{3.15}$$
$$\frac{\partial L}{\partial N} = c_4 M + \frac{-c_1}{N^2 M} - \frac{-c_2}{N^2} = 0$$
$$\frac{\partial L}{\partial M} = c_4 N + \lambda \left(\frac{-c_1}{NM^2} - \frac{2c_3}{N^2} \right) = 0$$

We proceed by solving the equation set:

$$\frac{N^2}{c_1 + c_2 M} = \frac{M^3}{c_1 M + 2c_3 M}$$
$$N = \sqrt{\frac{2c_3 M}{c_2}}$$
$$\frac{1}{\sqrt{2n^3}} + \frac{3}{n} = \epsilon^2$$

Let $\sqrt{n} = x$ then we can proceed to solve a cubic equation below and find n such that $n = x^2$

$$x^3 - \frac{3x}{\epsilon^2} - \frac{\sqrt{2}}{2\epsilon^2} = 0 \tag{3.16}$$

If we compare equations (3.15) and (3.16) we see that they are the same and there is a primal-dual relationship. Thus, for a constraint MSE we can either solve nonlinear (3.14) using (3.16) and then find $C = NM$ complexity or we can solve (3.16) and find $C = NM$ directly.

We can see that optimal number of simulations, N, and optimal number of time paths, M, are obtained by previous solution. This solution could be used for calculating

EPE, CVA and other variant measures similarly. However, these estimators have time discretization bias since originally the measures are defined as an integral calculation whereas in practice the calculations are to be applied with rough Riemann sum. This naturally creates a time discretization bias. Another Bias source is the calculation of derivative price. Because some of the sophisticated derivatives and even the nature of derivative might require a discretized scheme which could be of an *Euler-Maruyama* or *Milstein* type.

In order to verify validity of the formulas we made an application using optimal time discretization M and optimal number of simulations N derived in (3.14). The application was implemented for various computational budgets regarding simple GBM model:

Table 3.1: Results of Optimal Simulation for GBM EPE

	MSEopt/MSE	MSE	N	M	Complexity
Coefs=1	119.02	0.00016	8963	134	12×10^5
Coefs=Opt	214.23	9.183-05	5646	213	12×10^5
Coefs=1	72.89	0.00027	41602	288	12×10^6
Coefs=Opt	356.88	5.592-05	26207	458	12×10^6

The table compares the MSE of plain Monte Carlo Simulations and the MSE of optimal parameter Monte Carlo Simulations regarding Geometric Brownian Motion (GBM). As per for tractability and as we have exact result for expectation of GBM, we use this simple model. We take total complexity budget, $N = 100.000, M = 12$ so $N \times M = 1.200.000$, in order to replicate industry convention of monthly cash flow schedule. First calculate the MSE with these parameters and the MSE with optimal N and M from (3.15) or (3.14). We can immediately see better results using optimal simulation parameters in terms of MSE compared to plain Monte Carlo Path simulation.

3.3.1 Removing Bias via Stratified Sampling

Our estimator $\theta_t = \sum_{i=1}^{M} E(V_{t_i})\Delta_i$ for EPE_t and $\theta_t = \sum_{i=1}^{M} E(V_{t_i})dF_{t_i}$ for CVA_t which can further write by conditioning on a new random variable τ,

$$\theta_t = \sum_{i=1}^{M} \mathbb{E}(V_\tau|\tau = t_i)p(\tau = t_i)$$

This is going to be random time for EPE and default time for CVA. This methodology is known as stratified sampling and reduces variance of the estimator by conditioning [46].

From law of total variance the variance of a random variable could be decomposed [46]:

$$Var(X) = Var(\mathbb{E}(X|Y)) + \mathbb{E}(Var(X|Y))$$

Thus, derivative price estimator $\mathbb{E}(V_\tau|\tau = t)$ has lower variance than $\mathbb{E}(V_\tau)$.

In addition, our new estimator has destroyed bias exposed during EPE and CVA calculation but not derivative price estimation, since the bias in this estimation has a different source which comes from path simulation of pricing SDE if exists. The stratified sampling opened us a way to further define more optimal and detailed simulation runs for each time point. This could be calculated again by another constrained optimization problem solution:

$$\begin{aligned} L(N_m, V_m, p_m) &= \sum_{m=1}^{M} \frac{V_m(p_m)^2}{N_m} + \lambda(\sum_{m=1}^{M} N_m - N) \\ \frac{\partial L}{\partial N_m} &= -\frac{V_m(p_m)^2}{(N_m)^2} + \lambda = 0 \\ \frac{\partial L}{\partial \lambda} &= \sum_{m=1}^{M} N_m - N = 0 \\ N_m &= \sqrt{\frac{V_m}{\lambda}} p_m \\ \sum_{m=1}^{M} \sqrt{\frac{V_m}{\lambda}} p_m = N, \quad N_m &= \frac{\sqrt{V_m} p_m N}{\sum_{m=1}^{M} \sqrt{V_m} p_m} \end{aligned} \tag{3.17}$$

This is very important for an additional efficiency gain. Given optimal total number of simulation runs N we will further have optimal number of simulations N_m for each time point m of total time path M. It is now possible to apply these formulas to calculate the aforementioned CCR measures for simple Geometric Brownian Motion based stock price which has an exact solution. This allows us to calculate MSE, variance and bias easily which is key to evaluate the efficiency of our methodology.

3.3.2 Finding Arbitrary Coefficients

The previous optimization problem is constructed by using convergence order and there are arbitrary coefficients involved. In applications it's not so easy to select these arbitrary coefficients. Because there is a huge search space which makes it difficult to optimize the coefficients. In this study, rather than finding the set coefficients over a large volume of coefficient pool, we can set a constraint on the sum of coefficients and narrow the search or optimization to this smaller space. In [19] the arbitrary coefficients are assumed to be one. Given this assumption the second constrained optimization problem will be defined as:

$$L(\lambda, c_1, c_2, c_3) = \left(\frac{4c_2C^2}{c_4^2c_3}\right)^{\frac{1}{3}} + \lambda(\sum_{j=2}^{3} c_j - K) \tag{3.18}$$

$$\frac{\partial L}{\partial c_2} = \frac{1}{3}\left(\frac{4C^2}{c_4^2c_3}\right)^{\frac{1}{3}} c_2^{-\frac{2}{3}} + \lambda$$

$$\frac{\partial L}{\partial c_3} = \frac{-1}{3}\left(\frac{4c_2C^2}{c_4^2}\right)^{\frac{1}{3}} c_3^{-\frac{4}{3}} + \lambda$$

$$\frac{\partial L}{\partial c_4} = \frac{-2}{3}\left(\frac{4c_2C^2}{c_3}\right)^{\frac{1}{3}} c_4^{-\frac{5}{3}} + \lambda$$

After some algebraic operations we find the optimal arbitrary coefficients given K are $c_2 = \frac{K}{4} = c_3, c_4 = \frac{K}{2}$

3.3.3 General Efficient Algorithm for EPE and CVA Simulation

1. Calculate M and N from 3.15

2. Calculate V_m for $p_m = \frac{1}{M}$ and $N_m = \frac{N}{M}$

3. Then set $N_m = \frac{\sqrt{V_m}p_mN}{\sum_{m=1}^{M}\sqrt{V_m}p_m}$

4. Then use MLMC to calculate path dependent derivative prices to reduce V_m

5. Recalculate EPE/CVA using

$$\hat{\theta}_t = \sum_{m=1}^{M}\sum_{j=1}^{N_m} \frac{V_{mj}}{N_m}\Delta_m(dF_m) \tag{3.19}$$

CHAPTER 4

CREDIT RISK MODELLING IN COUNTERPARTY CREDIT RISK

4.1 Introduction

The default modelling of Counterparty Credit Risk is a complicated task since there are many subtelities of this concept. As mentioned in Chapter 1, we should consider the cases of unilateral and multilateral defaults. In context of this modelling efforts two paradigms emerged over the years [14]. These are Reduced Form/Intensity/Hazard Rate models and Structural/Firm Value Models. In this chapter we examine these models, their properties and make some extensions through the inclusion of Levy Processes, particularly in structural credit risk models. We will start with intensity/hazard rate models.

4.2 Intensity Based Credit Risk Models

Intensity models (also called *reduced form* when convenient) defines the default through an exogenous jump process [14]. In the context of these models; the default time τ is the first jump time of a Poisson process, with deterministic or stochastic (Cox process) intensity. In these models the default is not triggered by well-known, easily observable market variables rather by an exogenous component, independent of all the default-free market information. Moreover, this modelling embraces the opinion that default-free market with its interest rates, exchange rates and so forth, does not give complete information on the default process, and there is not explicitly defined variables affecting the driving process behind the default [14].

This structure of modelling is not in the same line with firm value models (structural models) where default arrives, when the market value of assets of the firms hits a default barrier associated with the debt level. The benefit of this reduced form family of models is particularly its conformity with credit spread modelling and its basic formulation, which is easy to numerically calibrate to corporate bond data or Credit Default Swap (CDS) [14].

The explicit CDS calibration and CDS option formulas have been studied in [10] and [11]. Before moving into technical details intensity models, we can present their general properties and why these models are regarded particularly useful for CDS spread modelling.

In basic reduced form or intensity models, the default time τ is the first jump of a Poisson process. It is useful to point out that the first jump time of a (time-inhomogeneous) Poisson process obeys roughly the following:

- Given that the subject has not defaulted (we can say not jumped due to Poisson process) before *t*, (risk-neutral) probability of defaulting (jumping) in the next *dt* infinitesmal time interval is [14],

$$\mathbb{Q}\{\tau \in [t, t+dt] | \tau > t, \mathcal{F}_t\} = \lambda(t)dt.$$

There are well accepted models used in the context of CCR modelling as per CVA calculation. Most popular frequently used models contain Black-Cox intensity [8] and Vasicek [52] one factor model. Authors of the study [11] used JCIR++ model to extend Cox intensity model with exponential jump process. These models could be easily calibrated to CDS prices [14] JCIR++ model can be written [11]:

$$d\lambda_t = \kappa(\mu - \lambda_t)dt + \sigma\sqrt{\lambda_t}dW_t + ydJ_t(\alpha). \tag{4.1}$$

The model is of tractable affine jump diffiusion (AJD) class [14] and the survival probability can be defined by:

$$\mathbb{Q}(\tau > t|\mathcal{F}_t) = \mathbb{1}_{\{\tau>t\}}\tilde{\alpha}(t,T)e^{(-\tilde{\beta}(t,T))} = \mathbb{1}_{\{\tau>t\}}P^{\text{JCIR++}}(t,T,\lambda). \tag{4.2}$$

This has an exact solution below obtained by solving Riccati Equations [11]:

$$\tilde{\alpha}(t,T) = A(t,T)\left(\frac{2he^{\left(\frac{h+\kappa+2\gamma}{2}(T-t)\right)}}{2h+(\kappa+h+2\gamma)e^{h(T-t)}}\right).$$

$$\tilde{\beta}(t,T) = B(t,T).$$

4.3 Affine Factor Intensity Model

The general properties of intensity of models are given at the introduction. These models are comfortable to calibrate to CDS spreads and credit term structure, therefore, they are practical tools to generate market relevant default probabilities and default spreads. However, from a Counterparty Credit Risk context these models do not generate dependence between the counterparties to account for DVA or wrong/right way risks. In order to add dependence it's possible to introduce an affine structure to the intensity such as:

$$\lambda_t^j = \lambda_{1t}^j + c\lambda_{2t} \tag{4.3}$$

In this relation, the first term accounts for idiosyncratic component and second term is the systematic-common component. By doing so we decompose the intensity propagated by individual default and from systematic default effects. This enables us to setup a dependence between counterparties and the underlying asset.

In order to exploit the formula in [11] we should obtain similar process given in (4.1); a CIR process. This is possible if conditions explained at the Appendix C are satisfied. Hence, we setup the model to satisfy this structure. Given this setup we will be able to add a link between default intensities of different companies. Therefore, we will have the intensity models such that:

$$d\lambda_t = \kappa(\mu_1 + c\mu_2 - (\lambda_{1t} + c\lambda_{2t}))dt + \sigma\sqrt{\lambda_{1t} + c\lambda_{2t}}dW_t + ydJ_t(\alpha) \tag{4.4}$$

$$d\lambda_{1t} = \kappa(\mu_1 - \lambda_{1t})dt + \sigma\sqrt{\lambda_{1t}}dW_t + (1-c)ydJ_t(\alpha)$$

$$d\lambda_{2t} = \kappa(\mu_2 - \lambda_{2t})dt + \sigma\sqrt{\lambda_{2t}}dW_t + cydJ_t(\alpha)$$

4.4 Structural Credit Risk Models Based on Levy Processes

In the structural approach, there is an explicit assumption that the both firm's assets and capital dynamics follow a specific stochastic process and there are other particular assumpitons as well. Given these assumptions, the firm defaults if its value of assets falls below a specific threshold, mainly a liability barrier, which also named distress barrier. Based on those assumptions, a firm is assumed to default if its assets fall below some specified level at the maturity of the liabilities. This maturity is mostly supposed to be one year and accordingly liabilities of this maturity is taken into consideration. The other structural credit risk model is barrier structural model. This model percieves the default event as the knockout option where the option terminates when the asset value falls below a specific barrier at any time in the maturity of the option. Thus, the difference from Merton type structural model is the timing of default.

In our study we plan to calculate the credit risk regarding both the Merton type and Cox type (Barrier) [8] models in the context of some well-known Levy Processes as these models have attracted interest in financial literature. The reason for that is; they accommodate distributions with non zero higher moments (skewness and kurtosis) due to the presence of jumps, therefore allowing a more realistic representation of stylized features of market quantities such as assets returns [33].

Levy Process, could be described in general; it is a stochastic process endowed a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with independent and stationary increments which has an infinitely divisible distribution [51].

This process could be described by its characteristic function, which is a crucial result of well-known Levy-Khintchine representation. Charateristic function has a representation in that context:

$$\phi_L(u,t) = \mathbb{E}(e^{iuL(t)}) = e^{t\varphi_L(u)}, \quad u \in \mathbb{R} \tag{4.5}$$

which is basically Fourier transform of Levy process *L(t)* naturally *i* is imaginary part and $\varphi_L(\cdot)$ is log of chararteristic function or characteristic exponent. This further has decomposition of [51]

$$\varphi_L(u) = -\frac{1}{2}u \cdot Au + i\gamma \cdot u \int_{R^d} (e^{iux} - 1 - iu \cdot x\mathbb{1}_{\{|x| \le 1\}})\nu(dx) \tag{4.6}$$

For a scalar Levy process the formula above will be in the form

$$\varphi_L(u) = -\frac{1}{2}u^2 A + i\gamma u \int_{-\infty}^{\infty} (e^{iux} - 1 - iux\mathbb{1}_{\{|x| \le 1\}})\nu(dx) \tag{4.7}$$

where this is summarised by Levy triplet (A, γ, ν). The Levy processes frequently preferred in finance are Brownian Motion , Variance Gamma (VG) and Normal Inverse Gaussian (NIG) processes. In addition Levy stochastic volatility models are also used in order to employ volatility clustering into modelling process.

4.5 Variance Gamma Process

4.5.1 Introduction to Variance Gamma Model

The Variance Gamma (VG) model is used in the financial literature for several years. The model was introduced as a stochastic process more robust than the geometric Brownian motion and but is similar in representation. Yet, it is more realistic in terms of reflecting the structure of financial market returns, namely equity returns, to overcome the problems that the Black and Scholes model has in pricing of options, which are because of anti-symmetry and fat tail observed in the financial markets. To include these facts in the model, two additional parameters are introduced, compared with the Geometric Brownian Motion , which allow to control the skewness and the kurtosis of the distribution of equity price returns. The first presentation of the model was in Madan-Seneta (1990) [38]. The model presented in this paper is a symmetric variance gamma model where the only control parameter is for kurtosis, and there is no parameter to account for nonzero skewness observed of equity returns in financial markets. In a another paper by Seneta (2004) [48], the empirical properties of the model are studied.

In the paper by [37], the authors studied equilibrium option pricing for the symmetric variance gamma process using a representative agent model with a constant relative risk aversion utility function. The resulting risk neutral measure studied in the paper was a non-symmetric variance gamma process and the drift was negative to account for positive risk aversion. The complete study of this general variance gamma process was in Madan, Carr and Chang (1998) [36] where a closed form solution using this process as the underlying for European Vanilla options was presented.

4.5.2 Variance Gamma Process as Time Change

The Variance Gamma model is defined as a time changed Brownian Motion with and/or without drift where time change is a gamma process. The process is a pure jump process that accounts for high activity, in keeping with the normal distribution, by having an infinite number of jumps in any interval of time [24].

Unlike many other jump models, it is not necessary to introduce a diffusion component for the VG process, as the Black–Scholes model is a parametric special case already and high activity is already accounted for.

Unlike normal diffusion, the sum of absolute log price changes is finite for the VG process. Since VG has finite variation, it can be written as the difference of two increasing processes, the first of which accounts for the price increases, while the second explains the price decreases. In the case of the VG process, the two increasing processes that are subtracted to obtain the VG process are themselves gamma processes [24].

The model could be defined:

$$\begin{aligned} b(t,\sigma,\theta) &= \theta t + \sigma W_t \\ X(t;\sigma,\gamma,\theta) &= b(\gamma(t;1,\nu),\sigma,\theta) \\ &= \theta\gamma(t;1,\nu) + \sigma W(\gamma(t;1,\nu)) \end{aligned} \tag{4.8}$$

It is actually possible to write variance gamma process in terms of difference of two gamma processes. To start with, we consider a gamma process $Ga(t;\mu,\nu)$ where it has independent increments over non-overlapping intervals, $(t, t+h]$, and μ is the mean per unit and ν is variance per unit of the process.

Then we define $\gamma = Ga(t+h;\mu,\nu) - Ga(t;\mu,\nu)$ with $G > 0$ and given gamma function $\Gamma(\cdot)$, we can write density of the increment:

$$f_\gamma = \left(\frac{\mu}{\nu}\right)^{\frac{\mu^2 h}{\nu}} \frac{\gamma^{\frac{\mu^2 h}{\nu}} \exp(\frac{-\mu}{\gamma})}{\Gamma(\frac{\mu^2 h}{\nu})} \tag{4.9}$$

Using this process enables us to see each unit of calendar time can be seen as having a time length given by an independent random variable that has a gamma density with unit mean and positive variance [24], which we can write as $\gamma(t;1,\nu)$. Therefore we can understand this model as having correspondence for different levels of trading activity during different time periods [17].

The economic intuition [12] underlying the stochastic time change approach to stochastic volatility is due to the Brownian scaling property. This property connects changes in scale to changes in time and thus random changes in structure of volatility could be captured by random changes in time. Thus the stochastic time change of the variance gamma model enables us to represent stochastic volatility in a pure jump process[24].

Given the properties above, the probabilistic properties of variance gamma model could be written starting from density first by conditioning argument [24]:

$$\begin{aligned} F(x;\sigma,\theta,\nu) &= \int_0^\infty f(x;\sigma,\theta,\nu|\gamma = g) f(g) dg \qquad (4.10) \\ &= \int_0^\infty \phi(\theta g, \sigma\sqrt{g}) f_\gamma(\frac{t}{\nu},\nu) dg \end{aligned}$$

This proceeds,

$$= \int_0^\infty \frac{1}{\sigma\sqrt{2\pi g}} e^{\left(-\frac{(x-\theta g)^2}{2\sigma^2 g}\right)} \frac{g^{t/\nu - 1} e^{-g/\nu}}{\nu^{t/\nu}\Gamma(t/\nu)} dg$$

$$F(x;\sigma,\theta,\nu) = \int_0^\infty \Phi\left(\frac{x-\theta g}{\sigma\sqrt{g}}\right) \frac{g^{t/\nu - 1} e^{-g/\nu}}{\nu^{t/\nu}\Gamma(t/\nu)} dg$$

where $\Phi(x)$ is the cumulative standart normal distribution function. We start deriving characteristic function of variance-gamma model by writing:

$$\mathbb{E}(e^{iuX}) = \mathbb{E}(\mathbb{E}(e^{iu(\theta\gamma + \sigma W(\gamma))})|\gamma = g) \tag{4.11}$$

Then using the characteristic function of a $N(\mu, \sigma)$ random variable, we further continue deriving variance gamma characteristic function:

$$\mathbb{E}(e^{iu\theta g - 0.5\sigma^2 g}) = \mathbb{E}(e^{i(u\theta + i0.5u^2\sigma^2)g}) = \mathbb{E}(e^{izg}) = \Phi(g) \tag{4.12}$$
$$u\theta + i\frac{1}{2}u^2\sigma^2 = z$$

where $\Phi(g)$ is the characteristic function of gamma distributed random variable evaluated at scale parameter of $\frac{t}{\nu}$ and shape parameter of ν. Given this, the characteristic function is:

$$\Phi(u) = \left(\frac{\beta}{\beta - iu}\right)^{\alpha}$$

In our case $u = z$, $\alpha = \frac{t}{\nu}$ and $\beta = \nu$ which leads to:

$$\Phi(z) = \left(\frac{\frac{1}{\nu}}{\frac{1}{\nu} - iz}\right)^{\frac{t}{\nu}}$$

Then given 4.12 we will have final explicit formula for variance gamma characteristic function:

$$\Phi(u) = \left(\frac{1}{1 - iu\theta\nu + \frac{1}{2}u^2\sigma^2\nu)}\right)^{\frac{t}{\nu}} \tag{4.13}$$

4.5.3 Parameters of Variance Gamma Process

After observing equation (4.13) we can see that given the characteristic function above the moments of variance gamma process is written [17]:

$$\mathbb{E}(X(t)) = \theta$$

$$\mathbb{V}(X(t)) = \nu\theta^2 + \sigma^2$$

$$\mathbb{E}(X(t) - \theta)^3 = \theta\nu\frac{3\sigma^2 + 2\nu\theta^2}{(\sigma^2 + \nu\theta^2)^{\frac{3}{2}}}$$

$$\mathbb{E}(X(t) - \theta)^4 = 3(1 + 2\nu - \nu\sigma^4(\nu\theta^2 + \sigma^2)^{-2})$$

We begin by writing the model subordinated by γ conditional on the value of g:

$$X(t) = \theta g + \sigma\sqrt{g}z,$$

where z is a standart normally distributed random variable. Some useful results of gamma distribution are beneficiary. Assume a gamma density and reparametrize it for easyness such that $b = \frac{1}{\nu}$ and $a = \frac{t}{\nu}$

$$f(\gamma, a, b) = b^a x^{a-1} e^{-xb}$$

Corresponding moment generating function becomes:

$$M_x(u) = \mathbb{E}(e^{ux}) = \frac{\int_0^\infty b^a x^{a-1} e^{-x(b+u)}}{\Gamma(a)} = \left(\frac{b}{b+u}\right)^a = \left(\frac{1}{1+\frac{u}{b}}\right)^a$$

First and second derivatives of moment generating function at zero are:

$$\begin{aligned} M_x^{'}(0) &= \mathbb{E}[\gamma] \\ &= \frac{a}{b} \end{aligned}$$

$$\begin{aligned} M_x^{''}(0) &= \mathbb{E}[\gamma^2] \\ &= V(\gamma) + \mathbb{E}[\gamma] \\ &= \frac{a}{b^2} + (\frac{a}{b})^2 \end{aligned}$$

$$\begin{aligned} M_x^{'''}(0) &= \mathbb{E}[\gamma^3] \\ &= \frac{a(a+1)(a+2)}{b^3} \end{aligned}$$

$$\begin{aligned} M_x^{''''}(0) &= \mathbb{E}[\gamma^4] \\ &= \frac{a(a+1)(a+2)(a+3)}{b^4} \end{aligned}$$

Given that γ_t process which has a mean of t and variance of νt on the interval t and taking back reparametrization the formulas from above will change accordingly:

$$\mathbb{E}[\gamma] = \frac{a}{b} = t, \quad \nu = \frac{1}{b}$$

$$\mathbb{E}[\gamma^2] = \frac{a}{b^2} = \nu t + t^2$$

$$\mathbb{E}[\gamma^3] = t^3 + 3\nu t + 2\nu^2 t$$

$$\mathbb{E}[\gamma^4] = 6\nu^3 t + 11\nu^2 t^2 + 6\nu t^3 + t^4$$

Using these results from gamma process moments of variance gamma is derived as follows [17] :

First and Second Moments:

$$
\begin{aligned}
\mathbb{E}[X(t)] &= \mathbb{E}[\mathbb{E}[X(t)|\gamma = g]] \\
&= \theta\mathbb{E}[g] + \sqrt{g}\mathbb{E}[z] = \theta t
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}[X(t) - \mathbb{E}X(t)]^2 &= \mathbb{E}[X(t) - \theta t]^2 \\
&= \mathbb{E}[\theta(g-t) + \sigma\sqrt{g}z]^2 = \mathbb{E}[\theta(g-t)]^2 + 0 + \mathbb{E}[\sigma^2 g\mathbb{E}[z^2]] \\
&= \theta^2(\nu t - 2t^2 + t^2 + t^2) = \theta^2(\nu t - t^2 + t^2) + \sigma^2 t \\
&= (\theta^2\nu + \sigma^2)t
\end{aligned}
$$

Third Moment:

$$
\begin{aligned}
\mathbb{E}[X(t) - \mathbb{E}X(t)]^3 &= \mathbb{E}[\theta(g-t) + \sigma\sqrt{g}z]^3 \\
&= \mathbb{E}[\theta^3(g-t)^3 + 3\theta^2(g-t)^2\sigma\sqrt{g}z + 3\theta(g-t)\sigma^2 gz^2 + \sigma^3 g^{3/2}z^3] \\
&= \theta^3\mathbb{E}(g-t)^3 + 0 + 3\theta\sigma^2\mathbb{E}[(g-t)g] + 0 = \theta^3\mathbb{E}(g-t)^3 + 3\theta\sigma^2\nu t \\
&= (t^3 + 3\nu t^2 + 2\nu^2 t - 3\nu t^2 - 3t^3 + 3t^3 - t^3) + 3\theta\sigma^2\nu t \\
&= \theta^3\nu t(3t + 2\nu - 3t) + 3\theta\sigma^2\nu t = (2\theta^3\nu^2 + 3\theta\sigma^2\nu)t
\end{aligned}
$$

Fourth Moment:

$$
\begin{aligned}
\mathbb{E}[X(t) - \mathbb{E}X(t)]^4 &= \mathbb{E}[\theta(g-t) + \sigma\sqrt{g}z]^4 \\
&= \mathbb{E}[\theta^4(g-t)^4 + 4\theta^3(g-t)^3\sigma\sqrt{g}z + 6\theta^2(g-t)^2\sigma^2 gz^2 \\
&\qquad + 4\theta(g-t)\sigma^3 g^{\frac{3}{2}}z^3 + \sigma^4 g^2 z^4] \\
&= (3\sigma^4\nu + 12\sigma^2\theta^2\nu^2 + 6\theta^4\nu^3)t + (3\sigma^4 + 6\sigma^2\theta^2\nu + 3\theta^4\nu^2)t^2
\end{aligned}
$$

As we can see the variance gamma model has a higher kurtosis than normal distribution e.g. order of 3 and a non-zero skewness. Thus, is a possible candidate for modelling financial markets where empricially high kurtosis and skewed distributions are observed.

4.5.4 Variance Gamma Factor Construction

Variance gamma process can be decomposed into different variance gamma processes under some convolution restrictions. In our framework we will work with a multivariate variance gamma process in $\mathbb{R}^n$ with common components in each variance gamma process used in [33]. Hence, let $X(t) = (X_1(t), X_2(t), \ldots, X_n(t))$ be a $\mathbb{R}^n$ variance gamma with common components. Let's further assume that $Y(t) = (Y_1(t), Y_2(t), \ldots, Y_n(t))$ is another $\mathbb{R}^n$ variance gamma process with independent components.

Additionally, we assume last term, $Z(t)$, to be a multivariate variance gamma process where the commonality is imposed and it is independent from $Y(t)$. Finally, we have a coefficient $c_j \in \mathbb{R}$ for $j = 1, \ldots, n$. As a result, we can model dependence between $X_j(t)s$ via common term $Z(t)$ in an affine structure:

$$X(t) = Y(t) + cZ(t) \tag{4.14}$$

Here again $Y(t)$ and $Z(t)$ are of the form

$$Y(t) = \theta_Y \gamma + \sigma_Y W(\gamma)$$
$$Z(t) = \theta_Z \gamma + \sigma_Z W(\gamma)$$

The dependent variance gamma structure in the context of credit risk has been studied by [40] and in a general multivariate setting in [35]. The use of such a model has some advantages. This comes mainly in terms of calibration and setting up characteristic function and convoluted density. for example, in [17] the difficulty of calibration and complexity in copula based models is mentioned. Moreover, in [17] the affine model was used not for whole process but in the subordinator gamma part.

However, this does not entirely capture the dependence and negative dependence poses problems as the setting up connection via an increasing time change process causes numerical problems. Thus, use of an affine factor structure with a common component solves the problem of dependence and calibration with its flexible structure.

4.5.5 Variance Gamma Affine Factor Asset Value Model

Now given linear factor model we can define the related asset price process. It is possible to obtain an option price based variance gamma model using the linear factor structure above. We can define the asset price process as the Geometric Brownian Motion with gamma time change or Geometric Variance Gamma . Given a risk neutral probability measure we can set out the formal pricing equation:

$$\begin{aligned} S(t) =& S(0)e^{(r-\Phi(-i))t+X(t)} \\ =& S(0)e^{(r-\Phi(-i))t+Y(t)+cZ(t)} \end{aligned} \tag{4.15}$$

where $\Phi(u)$ is the characteristic function of a variance gamma process. Using this setup we can derive an option pricing formula that, in some way incorporate the effects of common systematic factor. Moreover, as a byproduct of this formula, we can obtain the Probability of Default (PD) to be used in credit risk estimations.

4.5.6 General European Option Price Formula for Variance Gamma Setting

A formula for variance gamma European call option has been derived in [37, 36]. Here, we derive a similar formula based on the characteristic functions but not via Fourier inversion. It's based on the application of the measure change through characteristic function. This methodolgy helps us to derive a similar formula more easily when underlying is an variance gamma affine factor model.

To begin with, we assume a European call option with strike K and with maturity T whose dynamics are governed by the risk neutral measure mentioned above. Define stock risk-adjusted probability measure P^S and its Radon-Nikodym derivative [18, 24]:

$$Z(t) = \frac{S(T)}{S(0)e^{rt}} = e^{-\phi(-i)t+X(t)}$$

Then we can write the price of such instrument:

$$\begin{aligned} C(K,r,T,\sigma) &= \mathbb{E}[e^{-rT}(S(T)-K)^+] \\ &= \mathbb{E}(S(0)e^{((-\Phi(-i))T+X(T))}\mathbb{1}_A) - \mathbb{E}(Ke^{-rT}\mathbb{1}_A) \\ &= S(0)\mathbb{E}^s(\mathbb{1}_A) - Ke^{-rT}\mathbb{E}(\mathbb{1}_A) \end{aligned}$$

This leads to the formula,

$$C(K,r,T,\sigma) = S(0)\mathbb{P}^s(S(T) > K) - Ke^{-rT}\mathbb{P}(S(T) > K) \tag{4.16}$$

where this general representation is given first in [5].

Let,

$$A := \left\{ \log\left(\frac{S(0)}{K}\right) + (r - \phi(-i))T > X(T) \right\}$$

and $\mathbb{E}^s(\mathbb{1}_A)$ is the expectation under the stock risk adjusted measure P^S induced by variance gamma asset price. Similar to Black-Scholes framework we can formulize well-known parameters d_1, d_2 as:

$$\begin{aligned} d_1 &= \frac{\log(\frac{S(0)}{K}) + (r - \phi(-i))t + \theta_s g}{\sigma\sqrt{g}} \\ d_2 &= \frac{\log(\frac{S(0)}{K}) + (r - \phi(-i))t + \theta g}{\sigma\sqrt{g}} \end{aligned} \tag{4.17}$$

Using the facts,

(i) $X(T) = \theta g + \sigma W_g$

(ii) $A := \left\{ \log\left(\frac{S(0)}{K}\right) + (r - \phi(-i))T + \theta g > -\sigma W_g \right\}$

After normalization we have,

$$A := \left\{ \frac{\log(\frac{S(0)}{K}) + (r - \phi(-i))T + \theta g}{\sigma\sqrt{g}} > \frac{-W_g}{\sigma\sqrt{g}} \right\} \tag{4.18}$$

Inserting A, we find that:

$$\begin{aligned}
\mathbb{E}(\mathbb{1}_A) &= \mathbb{P}(S(T) > K) \\
&= \mathbb{P}\left(\frac{\log(\frac{S(0)}{K}) + (r - \phi(-i))T + \theta g}{\sigma\sqrt{g}} > \frac{-W_g}{\sigma\sqrt{g}}\right) \\
&= \int_0^\infty N\left(\frac{\log(\frac{S(0)}{K}) + (r - \phi(-i))T + \theta g}{\sigma\sqrt{g}}\right)\frac{g^{t/\nu-1}e^{-g/\nu}}{\nu^{t/\nu}\Gamma(t/\nu)}dg
\end{aligned}$$

$$= \int_0^\infty N(d_2(g))\frac{g^{t/\nu-1}e^{-g/\nu}}{\nu^{t/\nu}\Gamma(t/\nu)}dg \tag{4.19}$$

Then we proceed,

$$\begin{aligned}
\mathbb{E}^s(\mathbb{1}_A) &= \mathbb{P}^s(S(T) > K) \\
&= \mathbb{P}\left(\frac{\log(\frac{S(0)}{K}) + (r - \phi(-i))T + \theta_s g}{\sigma\sqrt{g}} > \frac{-W_g}{\sigma\sqrt{g}}\right) \\
&= \int_0^\infty N\left(\frac{\log(\frac{S(0)}{K}) + (r - \phi(-i))T + \theta_s g}{\sigma\sqrt{g}}\right)\frac{g^{t/\nu-1}e^{-g/\nu_s}}{\nu_s^{t/\nu}\Gamma(t/\nu)}dg
\end{aligned}$$

This is compactly written,

$$\mathbb{P}^s(S(T) > K) = \int_0^\infty N(d_1(g))\frac{g^{t/\nu}e^{-g/\nu_s}}{\nu_s^{t/\nu}\Gamma(t/\nu)}dg \tag{4.20}$$

Using 4.19 and 4.20 we finally obtain Variance Gamma option price below:

$$C(K, r, T, \sigma) = S(0)P_1(T/\nu, \nu_s, g, \theta_s, \sigma) - Ke^{-rT}P_2(T/\nu, \nu, g, \theta, \sigma) \tag{4.21}$$

where the probabilities P_1 and P_2 are:

$$P_1 = F_{VG}(x; \sigma, \theta_s, t/\nu, \nu_s) = \int_0^\infty N\left(d_1(g)\right)\frac{g^{t/\nu-1}e^{-g/\nu_s}}{\nu_s^{t/\nu}\Gamma(t/\nu)}dg \tag{4.22}$$

$$P_2 = F_{VG}(x; \sigma, \theta, t/\nu, \nu) = \int_0^\infty N\left(d_2(g)\right)\frac{g^{t/\nu-1}e^{-g/\nu}}{\nu^{t/\nu}\Gamma(t/\nu)}dg \tag{4.23}$$

The difference between 4.22 and 4.23 comes from a stock risk adjusted probability measure P^S. These formulas are useful for derivations and understanding the model.

However, the density developed by [36] outlined below allevites the numerical computations in terms of both speed and precision. The cdf and pdf are as follows:

$$F_{VG}(x,\theta,\sigma,\nu,\mu) = \int_{-\infty}^{x} \frac{2e^{\frac{\theta y}{\sigma^2}}}{\nu^{\frac{\tau}{\nu}}\sqrt{2\pi}\sigma\Gamma(\frac{\tau}{\nu})} \left(\frac{(y-\mu)^2}{\frac{2\sigma^2}{\nu}+\theta^2}\right)^{\frac{2-\nu}{4\nu}} \times K_{\frac{\tau}{\nu}-\frac{1}{2}}\left(\frac{1}{\sigma^2}\sqrt{(y-\mu)^2\left(\frac{2\sigma^2}{\nu}+\theta^2\right)}\right) dy, \tag{4.24}$$

$$f_{VG}(x,\theta,\sigma,\nu,\mu) = \frac{2e^{\frac{\theta x}{\sigma^2}}}{\nu^{\frac{\tau}{\nu}}\sqrt{2\pi}\sigma\Gamma(\frac{\tau}{\nu})} \left(\frac{(x-\mu)^2}{\frac{2\sigma^2}{\nu}+\theta^2}\right)^{\frac{2-\nu}{4\nu}} \times K_{\frac{\tau}{\nu}-\frac{1}{2}}\left(\frac{1}{\sigma^2}\sqrt{(x-\mu)^2\left(\frac{2\sigma^2}{\nu}+\theta^2\right)}\right) \tag{4.25}$$

respectively.

Using Esscher Transform which is the method to obtain a measure change in Levy Processes, [51] and corresponding Radon-Nikodym derivative

$$\frac{dQ}{dP} = \frac{e^{\theta Y_t}}{\mathbb{E}(e^{\theta Y_t})}$$

for $\theta \in \mathbb{R}$ we define Theorem 4.1.

Theorem 4.1. *Let*

$$Z(t) = \frac{dQ}{dP} = e^{-\phi(-i)t+Y(t)}$$

be the Radon-Nikodym derivative, where $\theta = 1$ above and $Y(t)$ is a variance gamma random variable. Then under this new probability measure we will have a characteristic function:

$$\phi_y^s(u,t) = \mathbb{E}^s(e^{Y(t)}) = e^{t\psi^{(s)}}$$

$$\psi^{(s)} = \frac{1}{\nu}\log\left(\frac{1}{1 - iu\theta^s\nu^s + \frac{1}{2}(\sigma u)^2\nu^s}\right)$$

which again points to another variance gamma random variable with new parameters θ^s, ν^s and corresponding densities in (4.26) *and* (4.27).

$$f_s(x,g) = \frac{1}{\sqrt{2\pi\sigma^2 g}} e^{\left(\frac{x-\theta_s g}{\sigma\sqrt{g}}\right)^2} \frac{g^{1/\nu-1}e^{-g/\nu_s}}{\nu_s^{1/\nu}\Gamma(1/\nu)} dg \tag{4.26}$$

$$f(x,g) = \frac{1}{\sqrt{2\pi\sigma^2 g}} e^{\left(\frac{x-\theta g}{\sigma\sqrt{g}}\right)^2} \frac{g^{1/\nu-1}e^{-g/\nu}}{\nu^{1/\nu}\Gamma(1/\nu)} dg \tag{4.27}$$

Proof. Under new measure P^Y we can write characteristic function

$$\mathbb{E}^y(e^{Y(t)}) = \mathbb{E}(Z(t)e^{iuY(t)}) = \mathbb{E}(e^{Y(t)(iu+1)})e^{-\psi(-i)t} = \mathbb{E}(e^{Y(t)i(u-i)})e^{-\psi(-i)t}$$

Let $u - i = v$, then

$$\begin{aligned}
\mathbb{E}(e^{Y(t)iv})e^{-\psi(-i)t} &= \phi_x^y(v) \\
&= \frac{\phi(-i)}{\phi(v)} = \frac{1-\theta\nu - 0.5\sigma^2\nu}{1 - i(u-i)\theta\nu + 0.5(u-i)^2\sigma^2\nu} \\
&= \frac{1-\theta\nu - 0.5\sigma^2\nu}{1-\theta\nu-0.5\sigma^2\nu\left(1 - iu(\sigma^2+\theta)\frac{\nu}{1-\theta\nu-0.5\sigma^2\nu} + 0.5\sigma^2u^2\frac{\nu}{1-\theta\nu-0.5\sigma^2\nu}\right)}
\end{aligned}$$

Let $\kappa = (1-\theta\nu-0.5\sigma^2\nu)$ then we obtain

$$\begin{aligned}
\frac{\phi(-i)}{\phi(v)} &= \frac{\kappa}{\kappa\left(1 - iu(\sigma^2+\theta)\frac{\nu}{\kappa} + 0.5(\sigma u)^2\frac{\nu}{k}\right)} \\
\phi^y(u) &= \left(\frac{1}{1 - iu\theta^s\nu^s + 0.5(\sigma u)^2\nu_s}\right)^{\frac{1}{\nu}} \\
\theta^s &= \sigma^2 + \theta \\
\nu^s &= \frac{\nu}{\kappa}
\end{aligned}$$

Thus, using this new characteristic function and corresponding parameters the densities at (4.26) and (4.27) could be obtained. □

4.5.7 European Option Price Formula for affine factor Variance Gamma Model

Using the framework above we can obtain factor extended variance gamma option price for an affine factor model. We start by writing general pricing equation for a call option,

$$C(K, r, T, \sigma, c) = \mathbb{E}(e^{-rT}(S(T) - K)^+) \tag{4.28}$$

Then, using (4.15) we can write this expectation by law of iterated expectations,

$$\begin{aligned}
C(K, r, T, \sigma, c) &= \mathbb{E}(\mathbb{E}(S(0)e^{-\Phi_Z(-ci)T+cz}e^{-\Phi_Y(-i)T+Y(T)}\mathbb{1}_A)|Z(T)=z) \\
&\quad - \mathbb{E}(\mathbb{E}(Ke^{-rT}\mathbb{1}_A)|Z(T)=z) \\
&= S(0)\mathbb{E}\Big(e^{-\Phi_Z(-ci)T+cz}\mathbb{E}^y(\mathbb{1}_A|Z(T)=z)\Big) \\
&\quad - Ke^{-rT}\mathbb{E}\Big(\mathbb{E}(\mathbb{1}_A)|Z(T)=z\Big)
\end{aligned} \tag{4.29}$$

Then we define exponential Variance Gamma process $g(z)$,

$$g(z) = e^{-\Phi_Z(-ci)T + cz}$$

Then embedding $g(z)$ inside the $\log(S(T))$ we have cz in the resulting formula below:

$$\begin{aligned} C(K, r, T, \sigma, c, z) = S(0) \int_{-\infty}^{\infty} \mathbb{P}^y(S(T) > K) g(z) f_Z(z) dz \\ -Ke^{-rT} \int_{-\infty}^{\infty} \mathbb{P}(S(T) > K) f_Z(z) dz \end{aligned} \tag{4.30}$$

Here,

$$\begin{aligned} A :&= \{\log(S(T)) > \log(K)\} \\ &= \{\log(\frac{S(0)}{K}) + (r - \phi_x(-i))T + cz + \theta g > -Y(T)\} \end{aligned}$$

Furthermore:

$$A := \left\{ \frac{\log(\frac{S0}{K}) + (r - \phi_x(-i))T + cz + \theta g}{\sigma_Y \sqrt{g}} > \frac{-W_g}{\sigma_Y \sqrt{g}} \right\}$$

Then given the set A and using Theorem 4.1 we specify Radon-Nikodym derivative;

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = e^{-\phi_Y(-i)t + Y(t)}$$

Considering the general variance gamma option pricing formula (4.21) we obtain Black-Scholes type parameters d_1, d_2 for affine factor model below:

$$d_1 = \frac{\log(\frac{S(0)}{K}) + (r - q - \phi_X(-i))t + \theta_s^Y g + cz}{\sigma_Y \sqrt{g}} \tag{4.31}$$

$$d_2 = \frac{\log(\frac{S(0)}{K}) + (r - q - \phi_X(-i))t + \theta^Y g + cz}{\sigma_Y \sqrt{g}} \tag{4.32}$$

Using these parameters and adding the dividend we finally have variance gamma affine factor call option formula:

$$\begin{aligned} C(K, r, T, \sigma, c) = S(0)e^{-qT} \int_{-\infty}^{\infty} F_{VG}(d_1(z)) f_Z(z) dz \\ -Ke^{-rT} \int_{-\infty}^{\infty} F_{VG}(d_2(z)) f_Z(z) dz \end{aligned} \tag{4.33}$$

In conclusion; the model is quite fast to calculate using either numerical integral [53] with quadrature or semi-closed form distribution function (4.24). It's worth noting that the study of [53] has a simple yet very sucessful formulas to approximate normal cumulative distribution function (CDF) and probability density function (PDF). Using this study we present formulas below (4.34) and (4.35).

$$F(x) = \exp\left(-\frac{358x}{23} + 111\arctan\left(\frac{37x}{294}\right) + 1\right)^{-1}, \quad -\infty < x < \infty \tag{4.34}$$

$$g(x) = \frac{-358}{23} + 111\left(1 + \left(\frac{37}{294}x\right)^2\right)^{-1}\frac{37}{294}$$
$$f(x) = g(x)F(x)(F(x) - 1) \tag{4.35}$$

These two formulas above have enormously increased the speed of complex integrals involving normal cdf. For example any value on normal approximation took 300 times faster than regular cdf evaluation in python scipy package. The same timing took 343 times faster for pdf.

If we evaluate the model in terms of market compatibility, the model successfully satisfies the smile effect as we can see in Figure 4.1.

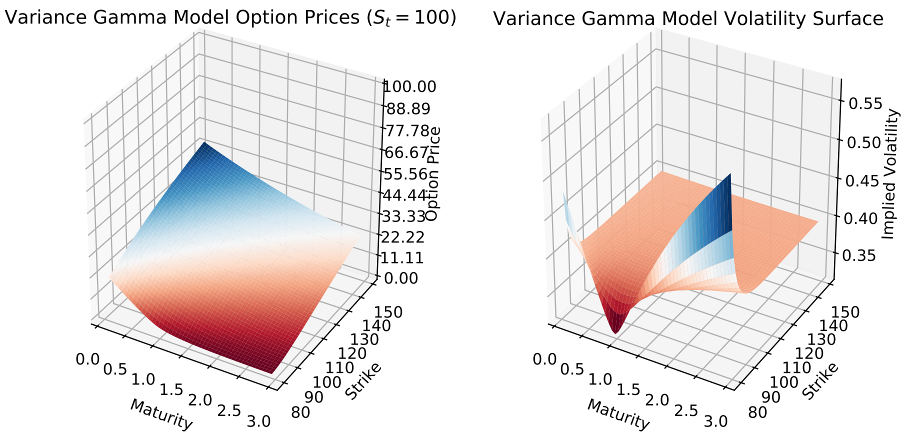


Figure 4.1: Variance Gamma European Option Model Volatility Surface

4.5.8 Barrier Model under Variance Gamma Process

In order to model the first passsage time of a variance gamma process, first we should write the joint distribution of Brownian Motion and its maximum/minimum separately as these formulas derived in detail by [49] and [29]:

$$\mathbb{P}(M_t \le \overline{m}) = \int_0^{\overline{m}} \int_w^{\overline{m}} \frac{2(2m-w)}{t\sqrt{2\pi t}} e^{\frac{-1}{2t}(2m-w)^2} dmdw + \int_{-\infty}^{0} \int_w^{\overline{m}} \frac{2(2m-w)}{t\sqrt{2\pi t}} e^{\frac{-1}{2t}(2m-w)^2} dmdw$$

$$\mathbb{P}(M_t \le \overline{m}, W_t \le w) = 1 - \int_{2\overline{m}-w}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz = \int_{-\infty}^{\overline{m}} \int_{-\infty}^{w} \frac{2(w-2m)}{t\sqrt{2\pi t}} e^{\frac{-1}{2t}(w-2m)^2} dmdw \tag{4.36}$$

For minimum:

$$\mathbb{P}(M_t \le \underline{m}) = \int_{-\infty}^{0} \int_{\underline{m}}^{w} \frac{2(w-2m)}{t\sqrt{2\pi t}} e^{\frac{-1}{2t}(w-2m)^2} dmdw + \int_{\underline{m}}^{\infty} \int_{-\infty}^{\underline{m}} \frac{2(w-2m)}{t\sqrt{2\pi t}} e^{\frac{-1}{2t}(w-2m)^2} dmdw$$

$$\mathbb{P}(M_t \le \underline{m}, W_t \ge w) = \int_{-\infty}^{w-2\underline{m}} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz$$

$$= \int_{-\infty}^{\underline{m}} \int_{w}^{\infty} \frac{2(w-2m)}{t\sqrt{2\pi t}} e^{\frac{-1}{2t}(w-2m)^2} dmdw \tag{4.37}$$

When we take the derivative of the distribution with respect to m we will get marginal of the maximum M_t:

$$f_{M_t} = \int_{\overline{m}}^{\infty} \frac{-2}{\sqrt{2\pi t}} e^{-\frac{(2m-w)^2}{2t}} dm$$

For minimum M_t:

$$f_{M_t} = \int_{-\infty}^{\underline{m}} \frac{2}{\sqrt{2\pi t}} e^{-\frac{(w-2m)^2}{2t}} dm$$

Given these, it is particularly appealing to write density of first passage time of the Variance Gamma after an immediate conditioning. However, this will be possible only under the fact that the passage time now depends on the time change of Brownian Motion where it is gamma process here. This notion is called first passage time of the second kind [26] and it is easier to derive formulas regarding to the maximum and minimum of Variance Gamma process as suggested in [26].

In [35] it is suggested that first passage time problems regarding variance gamma could be obtained after solving the Partial Integro Differential Equation (PIDE) implied by the asset price process and boundary conditions specific to the problem. Also, as stated in [51] barrier type problems regarding Levy Processes with jumps either requires the solution of Wiener Hopf factorization of characteristic exponent of Levy process or requires related PIDE solution. In [51] computationally expensive Monte Carlo simulation suggested as well. However, if we use First Passage Time of second kind for barrier type problems, the challenges and complexities are overcome.

As proved in [27] first passage time of the second kind converges to first passage time of the first kind with $\tau^1 \geq \tau^2$ [26]. When a continous time change (subordinator) is γ_t is used, these two stopping times coincide [26]. These concepts could be defined as:

(i) First Passage time of the first kind is the first time Variance Gamma passes a threshold.

$$\tau^1 = \inf \Big\{ t : W(\gamma_t) \leq L \Big\}, \quad 0 < t < T$$

- The τ^1 of Brownian Motion is defined as:

$$\tau^1_{W(t)} = \inf \Big\{ t : W(t) \leq L \Big\}, \quad 0 < t < T$$

(ii) First Passage Time of the second kind is the first time stochastic time change passes First Passage Time of Brownian Motion .

$$\tau^2 = \inf \Big\{ t : \tau^1_{W(t)} \leq \gamma_t \Big\}, \quad 0 < t < T$$

Thus, we use First Passage Time of the second kind for calculations regarding subordinated Levy processes since the subordinator becomes the key variable for Laplace transform of First Passage Time of the second kind and other calculations.

This overcomes the complexity of the First Passage Time of the first kind regarding subordinated Levy Processes. As a result after an intermediate conditioning on the subordinator [26]; density of the maximum of Variance Gamma process can be obtained as follows:

$$\begin{aligned}\mathbb{P}(M_{VG}(t) \leq \overline{m}) &= \int \mathbb{P}(M_{VG}(t) \leq \overline{m} | \gamma = g) f(g) dg \\ &= \int_0^{\overline{m}} \int_w^{\overline{m}} \frac{2(2m-w)}{g\sqrt{2\pi g}} e^{\frac{-(2m-w)^2}{2g}} dm dw \\ &+ \int_{-\infty}^0 \int_w^{\overline{m}} \frac{2(2m-w)}{g\sqrt{2\pi g}} e^{\frac{-(2m-w)^2}{2g}} dm dw\end{aligned}$$

$$\begin{aligned}\mathbb{P}(M_{VG}(t) \leq \overline{m}, W_t \leq w) &= \int_0^\infty \left(1 - \int_{2\overline{m}-w}^\infty \frac{1}{\sqrt{2\pi g}} e^{-\frac{z^2}{2g}} dz\right) \frac{g^{t/\nu-1} e^{-g/\nu}}{\nu^{1/\nu}\Gamma(1/\nu)} dg \\ &= \int_0^\infty \left(\int_{-\infty}^{\overline{m}} \int_{-\infty}^{w} \frac{2(w-2m)}{g\sqrt{2\pi g}} e^{\frac{-(w-2m)^2}{2g}} dm dw\right) \frac{g^{t/\nu-1} e^{-g/\nu}}{\nu^{t/\nu}\Gamma(t/\nu)} dg\end{aligned}$$

Similarly density of the minimum of Variance Gamma process can be obtained as follows:

$$\begin{aligned}\mathbb{P}(M_{VG}(t) \leq \underline{m}) &= \int \mathbb{P}(M_{VG}(t) \leq \underline{m} | \gamma = g) f(g) \\ &= \int_0^\infty \left(\int_{\underline{m}}^0 \int_w^{\underline{m}} \frac{2(w-2m)}{g\sqrt{2\pi g}} e^{\frac{-(w-2m)^2}{2g}} dm dw\right) \frac{g^{t/\nu-1} e^{-g/\nu}}{\nu^{t/\nu}\Gamma(t/\nu)} dg \\ &+ \int_0^\infty \int_{\underline{m}}^{w} \frac{2(w-2m)}{g\sqrt{2\pi g}} e^{\frac{-(w-2m)^2}{2g}} dm dw \frac{g^{t/\nu-1} e^{-g/\nu}}{\nu^{t/\nu}\Gamma(t/\nu)} dg \qquad (4.38)\end{aligned}$$

$$\begin{aligned}\mathbb{P}(M_{VG}(t) \leq \underline{m}, W_t \leq w) &= \int_0^\infty \left(\int_{-\infty}^{2w-\underline{m}} \frac{1}{\sqrt{2\pi g}} e^{-\frac{z^2}{2g}} dz\right) \frac{g^{t/\nu-1} e^{-g/\nu}}{\nu^{t/\nu}\Gamma(t/\nu)} dg \\ &= \int_0^\infty \left(\int_{-\infty}^{\overline{m}} \int_{w}^{\infty} \frac{2(w-2m)}{g\sqrt{2\pi g}} e^{\frac{-(w-2m)^2}{2g}} dm dw\right) \frac{g^{t/\nu-1} e^{-g/\nu}}{\nu^{t/\nu}\Gamma(t/\nu)} dg\end{aligned} \qquad (4.39)$$

4.6 Levy Process Stochastic Volatility Models

4.6.1 Introduction

The use stochastic volatility model in option pricing has been introduced by Heston (1993) which is a brand new approach to option price modelling through characteristic function of log-stock price [23]. He embedded stochastic volatility into SDE of asset price process which was previously modelled by geometric brownian motion of B-S framework. The model could be written [23]:

$$\begin{aligned} \frac{dS_t}{S_t} &= rdt + \sqrt{v_t} dW_t \\ dv_t &= \alpha(\mu - v_t)dt + \sigma_v \sqrt{v_t} dW_t^v \\ dW_t^S dW_t^v &= \rho dt \end{aligned} \tag{4.40}$$

Heston [23] states that this assumption of asset return process is successful to explain skew and smile facts in option prices where Black-Scholes model fails to satisfy. Black-Scholes model is successful in stock options although with such biases. Especially in currency options, the model's ability to fit is quite insufficient since the empirical asset price distributions are not generally normal [23]. Moreover, Heston [23] states that adding stochastic volatility solves smile whereas correlation between underlying asset return process and the stochastic volatility solves skew problem. Then conjecturing a structure similar to Black-Scholes he assumed option price of the form [23],

$$C(S, v, t) = SP_1 - KB(t, T)P_2 \tag{4.41}$$

where $B(t, T)$ is the discount bond maturing at T, $P_1 = \mathbb{P}^S(S(T) > K)$ and $P_2 = \mathbb{P}(S(T) > K)$.

Then writing $X = \log(S(t))$ and using characteristic function of this argument $f(S, v, T; u) = e^{iuX}$ with Feynman-Kac formula, he derives partial differential equation (PDE) of this characteristic function. Assuming a conjecture for this function satisfying the PDE which is [23],

$$f(S, v, T; u) = e^{A(t,T)+B(t,T)v(t)+iux} \tag{4.42}$$

he then finds a semi-closed form for (4.41). This conjecture is very useful and is used an extended version of this model for our later derivations introducing affine factor decomposition for stochastic volatility models. This stochastic volatility factor model carries all features of Heston model plus there is volatility decomposition. However, in order to progress we first start with time changed stochastic volatility models which are similar and mathematically more tractable.

4.6.2 Geometric Brownian Motion Stochastic Arrival Model (GBMSA-GBMCIR)

This model is most basic subordinated stochastic volatility models where the stochastic volatility comes from an Integrated CIR process. However, the subordinator ICIR and the subordinated brownian motion is uncorrelated. This is actually no correlated type of Heston model. This model is important to define, at least theoretically, in a sense that it is easier to generate slighly more advanced models equipped with stochastic vollatility through time changing.

We define the model as follows:

Definition 4.1. Let $\{W(t) : t \geq 0\}$ be a scaled Brownian Motion with drift θ. Let $\{X(t) : t \geq 0\}$ be a ICIR process with parameters α, β and σ_v. Then the Levy process,

$$Z(t) = W_{X(t)} + \theta X(t)$$

is called Geometric Brownian Motion stochastic arrival (GBMSA) process [24] $\theta, \alpha, \beta \in R$ and $\sigma_v \in R^+$

This model is actually time change representation of some type of Heston stochastic volatility model. The time change representation of Brownian Motion makes these models more flexible and very convenient for deriving charateristic function and moment generating function. Thus there is more room for easier model extension. The time change is a ICIR process since this allows mean reversion in stochastic volatility and volatility clustering.

Moreover, statistical properties of CIR and ICIR processes turns this model construction, mathematically tractable. We can illustrate the time change representation under risk neutral measure directly as follows:

$$\frac{dS_t}{S_t} = rdt + \sigma dW(V(t)) \tag{4.43}$$

where $V(t) = \int_0^t v(s)ds$ as defined in (4.44).

After recollecting all these Heston model SDE without correlation ($dW^S(t)dW^v(t) = 0$) is obtained

$$\frac{dS_t}{S_t} = rdt + \sqrt{v_t}dW_t$$

$$dv_t = \alpha(\mu - v_t)dt + \sigma_v\sqrt{v_t}dW_t^v \tag{4.44}$$

where $\sigma, \sigma_v \in R^+$ and $\alpha, \mu \in R$. Then further, Heston Model's jump extension, namely Bates Model, could be written below where $\ln(1+y) \sim \mathcal{N}(\ln(1+\hat{y}-\delta^2, \delta^2))$

$$\frac{dS_t}{S_t} = rdt + dW(V(t)) + \lambda y dJ_t \tag{4.45}$$

As mentioned before time change representation of Heston/Bates Model, which is named in [24] GBMSA and we use the same, is useful for deriving chararteristic function and makes the structure more flexible for different modelling specifications. The only caveat is that the correlation between the subordinator or stochastic volatility term and stock price is lost as stated earlier.

This is required because we use time change representation of *Bohner* type [12] and this does not admit correlation between the underlying Brownian Motion and the time change process. The reason we use this representation although the correlation is lost which accounts for the skew effect, is that we can define better models that can account skew as well with less effort.

Given this alternative time change representation for stochastic volatility models which have time change from integrated CIR process (ICIR) such that,

$$V(t) = \int_0^t \nu(s)ds, \tag{4.46}$$

$$d\nu_t = \kappa(\beta - \nu_t)dt + \sigma_\nu\sqrt{\nu}dW_t. \tag{4.47}$$

Then, we can write characteristic function for $\log(S_t)$ on the condition that $cor(dW_t, dW_t^v) = 0$ based on the method for writing characteristic function for subordinated processes [51] as:

$$\varphi_{X(V_t)}(u) = \varphi_{V_t}(i\phi_{X_t}(u)) \tag{4.48}$$

Here φ and ϕ denote characteristic function and the log-characteristic function respectively. Therefore, if we know characteristic functions of both subordinator and the original process we can easily derive characteristic function of time changed (subordinated) process. Using this method, we can then obtain characteristic function of the GBMSA [24] model. We can write the final characteristic function of (4.43) using (4.48) first by writing the characteristic function with subordination explicitly:

$$\varphi_{X(V_t)}(u) = \mathbb{E}(e^{iu(\theta V_t + \sigma W(V_t))}) = \mathbb{E}(\mathbb{E}(e^{iu(\theta V_t + \sigma W(V_t))})|V_t = v) \tag{4.49}$$

Then we can use the characteristic function of Geometric Brownian Motion to evaluate the expectation inside:

$$\begin{aligned} \mathbb{E}(\mathbb{E}(e^{iu(\theta V_t + \sigma W(V_t))})|V_t = v) &= \mathbb{E}(e^{iu\theta v - \frac{v\sigma^2 u^2}{2}}) \\ &= \mathbb{E}(e^{i(u\theta + i\frac{\sigma^2 u^2}{2})v}) = \mathbb{E}(e^{ibv}) \end{aligned} \tag{4.50}$$

Here, $b = u\theta + i\frac{\sigma^2 u^2}{2}$ which is our new pivot term in the characteristic function. As we know that V_t is an ICIR process and it has a characteristic function of the form [24, 49]:

$$\mathbb{E}(e^{iuV_\tau}|\mathcal{F}_t) = A(\tau, u)e^{B(\tau,u)v(t)} \tag{4.51}$$

$$A(\tau, u) = \frac{e^{\kappa^2 \frac{\beta\tau}{\sigma_v}^2}}{\left(\cosh(\frac{\gamma\tau}{2}) + \frac{\kappa}{\gamma}\sinh(\gamma\frac{\tau}{2})\right)^{\kappa^2 \frac{\beta\tau}{\sigma_v^2}}}$$

$$B(\tau, u) = \frac{2iu}{\kappa + \gamma\coth(\frac{\gamma\tau}{2})}.$$

and $\tau = T - t$. Using this general form for an ICIR process and (4.50), we can find the general form of characteristic function for GBMSA process by substiuting the pivot $b = u\theta + i\frac{\sigma^2 u^2}{2}$ term for u in (4.50).

Finally, we obtain,

$$\begin{aligned} \gamma_{GBMSA} &= \sqrt{\kappa^2 + \sigma_v^2(\sigma^2 u^2 - 2iu\theta)} \\ B_{GBMSA}(\tau, u) &= \frac{2iu\theta - u^2\sigma^2}{\kappa + \gamma\coth(\frac{\gamma\tau}{2})} \\ A_{GBMSA}(\tau, u) &= \frac{e^{\frac{\kappa^2\beta\tau}{\sigma_v^2}}}{\left(\cosh(\frac{\gamma\tau}{2}) + \frac{\kappa}{\gamma}\sinh(\gamma\frac{\tau}{2})\right)^{\frac{2\kappa\beta}{\sigma_v^2}}} \end{aligned}$$

At first, it looks the same in structure as the parameters above however, due to γ term all formulas change completely and $B(\tau, u)$ term also has different element in numerator. Finally for GBMSA process our characteristic function will be:

$$\mathbb{E}(e^{iu(\theta V_t+\sigma W(V_t))}) = A_{GBSA}(T-t,u)e^{B_{GBSA}(T-t,u)v(t)} \tag{4.52}$$

The jump augmented case as in Bates model above will be trivial to write since the filtrations generated by subordinated and jump components; $F_{W(V_t)} \bigcap F_J = \emptyset$ which means jumps and the subordinated processes are independent. Therefore, the jump augmented characteristic function of the model which we can call GBMSAj could be written as:

$$\mathbb{E}(e^{iu\left(\theta V_T+W(V_T)+\sum_{i=N(t)}^{N(T)}\log(y_i)\right)}|\mathcal{F}_t) = A_{GBMSA}(T-t,u)e^{B_{GBMSA}(T-t,u)v(t)} \times \mathbb{E}(e^{iu\sum_{i=N(t)}^{N(T)}\log(y_i)}) \tag{4.53}$$

When log-stock price is considered the complete formula adjusted for risk neutral measure will be:

$$\mathbb{E}(e^{iu\left(\log(S(0))+(r-\phi(-i))(T-t)+\theta V_T+W(V_T)+\sum_{i=N(t)}^{N(T)}\log(y_i)\right)}|\mathcal{F}_t) = e^{iu\left(\log(S(0))+(r-\phi(-i))(T-t)\right)} \times A_{GBMSA}(T-t,u)e^{B_{GBMSA}(T-t,u)v(t)}\mathbb{E}(e^{iu\sum_{i=N(t)}^{N(T)}\log(y_i)}) \tag{4.54}$$

Remark 4.1. As we can observe, the time change representation turns the derivation of stochastic volatility or brownian motion with ICIR time change into a very practically solvable case unless we want to integrate the correlation between the time change and the diffusion; $cor(dW^d, dW^V) \neq 0$.

4.6.3 Stochastic Volatility Option Price Model

This model uses GMBSA asset price process as the underlying. This is the similar to Heston Option Price Model with the only difference no correlation between subordinator $V(t)$ and stock price $S(t)$. To more general, we added other parameters of Brownian Motion ; drift and volatility.

As per used for Heston Model, the methodology for option pricing is based on characteristic function and obtaining cumulative distribution via Fourier inversion of this function.

This will enable us to calculate expected pay-off as presented below:

$$V(S_t, v_t, K, r, \tau, \sigma_v, \theta) = S_t F_s - e^{-r\tau} K F$$

$$F_s = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty Re\left(\frac{e^{-iu \log(K)} f_s(S_t, v_t, K, r, \tau, \sigma_v, \theta, u)}{iu} du \right) \tag{4.55}$$

$$F = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty Re\left(\frac{e^{-iu \log(K)} f(S_t, v_t, K, r, \tau, \sigma_v, \sigma, \theta, u)}{iu} du \right) \tag{4.56}$$

The F_s shows $\mathbb{P}^s(S(t) > K)$ is stock risk-adjusted probability measure [5] as we use in deriving Variance Gamma option price formula. The F shows $\mathbb{P}(S(t) > K)$ under risk neutral measure similar to Black-Scholes structure. As explained previously we obtain probabilities from inverse Fourier transform of GBMSA model charateristic function. However, the F_s requires a similar measure change process applied to (4.5.6). The characteristic functions used in (4.55) and (4.56) comes from (4.54). However, as mentioned, the measure change process slightly modifies the characteristic function required to find F_s, thus we obtain the characteristic function components $A(\tau, u)$ and $B(\tau, u)$ of (4.55), in the $\mathbb{P}^S$ measure, as follows:

$$\gamma^s_{GBSA} = \sqrt{\kappa^2 + \sigma_v^2((u^2 - 1) - 2(\theta + ui(1 + \theta)))}$$

$$B^s_{GBSA}(\tau, u) = \frac{2(iu(1 + \theta) + \theta) + (u^2 - 1)}{\kappa + \gamma^s \coth(\frac{\gamma^s \tau}{2})}$$

$$A^s_{GBSA}(\tau, u) = \frac{e^{\frac{\kappa^2 \beta \tau}{\sigma_v^2}}}{\left(\cosh(\frac{\gamma^s \tau}{2}) + \frac{\kappa}{\gamma^s} \sinh(\gamma^s \frac{\tau}{2})\right)^{\frac{2\kappa\beta}{\sigma_v^2}}}$$

4.6.4 A GBMSA-GBMCIR Model with Affine Factor Time Change

In this model we use and extend previous GBMSA model with an affine structure with two components similar to previous Levy process model Variance Gamma . However, the difference here is the affine combination takes place only in time change component not the overall Levy process.

$$W(X(t)) = W\Big(Y(t) + cZ(t)\Big) \tag{4.57}$$

where $Y(t)$ and $Z(t)$ are stochastic time change components.

We use the simple affine model (4.14) as the basis to capture the dependence between companies' asset value processes. The decomposition involves idiosyncratic $Y(t)$ and systematic $Z(t)$ as before and $Z(t)$ accounts for dependency since this is assumed to be common in all asset prices. This affine model could be used either in modelling dependency via time change or via direct modelling as in (4.14). Since our asset value process comes from GBMSA, we first start by constructing an affine model for time change to capture dependency at least in terms of business time, which is stochastic as mentioned before. Under this set-up log-asset value process is given in definition (4.2).

Definition 4.2. Let $\{W(t) : t \geq 0\}$ be a scaled Brownian Motion with drift θ. Let $\{X(t) : t \geq 0\}$ be a ICIR process with parameters α, β, σ_v. Then the Levy process,

$$W_{X(t)} + \theta X(t) = W_{Y(t)+cZ(t)} + \theta(Y(t) + cZ(t))$$

could be called Geometric Brownian Motion Stochastic Arrival Affine Factor (GBMSAf) process with $\theta, \alpha, \beta \in R$ and $\sigma_v \in R^+$ where $X(t)$ is as defined in (4.46).

However, this structure comes with an important constraint on the time change process. Because, both components of the time change have ICIR process. Thus, the linear combination should also have ICIR process. To achieve this, the process under the integral must be a CIR process. However, this could only be achieved after imposing some coefficient constraints on $Y(t)$ and $Z(t)$ otherwise we can not use the conjecture (4.51). Thus, after satisfying these constraints the sum $Y(t) + cZ(t)$ will be a ICIR process where we prove at the appendix (C) using moment generating function (MGF) of CIR process (non-central χ^2 distribution).

As a result, our final characteristic function for GBMSA model with affine ICIR time change will be:

$$\gamma_{GBMSAf} = \sqrt{\kappa^2 + \sigma_v^2(u^2 - 2iu\theta)}$$

$$B_{GBMSAf}(\tau, u) = \frac{2iu\theta - u^2}{\kappa + \gamma\coth(\frac{\gamma\tau}{2})} \tag{4.58}$$

$$A_{GBMSAf}(\tau, u) = \frac{e^{\frac{\kappa^2(\beta_1 + c\beta_2)\tau}{\sigma_v^2}}}{\left(\cosh(\frac{\gamma\tau}{2}) + \frac{\kappa}{\gamma}\sinh(\gamma\frac{\tau}{2})\right)^{\frac{2\kappa(\beta_1+c\beta_2)}{\sigma_v^2}}} \tag{4.59}$$

$$\varphi(u,\tau) = A_{GBMSAf}(u,\tau)e^{B_{GBSAf}(u,\tau)X(0)}$$
$$= A_{GBMSAf}(u,\tau)e^{B_{GBSAf}(u,\tau)[Y(0)+cZ(0)]} \quad (4.60)$$

The characteristic function of jump augmented GBMSAf model is GBMSAfJ and it is similar to (4.53).

$$\varphi(u,\tau) = A_{GBMSAf}(u,\tau)e^{B_{GBSAf}(u,\tau)X(0)}\mathbb{E}(e^{iuJ}) \quad (4.61)$$
$$= A_{GBMSAf}(u,\tau)e^{B_{GBSAf}(u,\tau)[Y(0)+cZ(0)]}$$
$$\times\mathbb{E}(e^{iu\sum_{i=0}^{N(t)}\log(y_i)})$$

where we now have idiosyncratic factors and systematic factor in the setup.

Moreover, the $Y(t)$ and $Z(t)$ are governed by,

$$Z(t) = \int_0^\infty z(s)ds$$
$$Y(t) = \int_0^\infty y(s)ds$$

The underlying CIR processes have the SDEs,

$$dy_t = \kappa(\beta_1 - y_t)dt + \sigma_v\sqrt{y(t)}dW_t^y,$$
$$dz_t = \kappa(\beta_2 - z(t))dt + \sigma_v\sqrt{z(t)}dW_t^z.$$

As we can see, the constraints are in the adjustment term κ and volatility of volatility term σ_v are the same in both processes. In order to account the assumption at the beginning, that $cor(Z,Y) = 0$ will still be valid here via $cor(dW_Y, dW_Z) = 0$.However, the long-run mean terms β_1, β_2 are different to distinguish the two process. As a result given these two constrained processes, we can still use general GBMSA and ICIR characteristic function structures for an affine ICIR time change process.

Lemma 4.2. *Given two ICIR processes $Y(t)$ and $Z(t)$ the affine combination, $Y(t) + cZ(t)$ and $c \in R$, is also an ICIR process unless they have different adjustment and volatility parameters*

Proof. Using MGF of CIR process, applying it to linear combination $Y(t) + cZ(t)$ and after some algebra, we can see that adjustment and volatility parameters have to be the same to keep the MGF of this combination still a valid CIR MGF. The details are given in Appendix C. □

4.6.5 Stochastic Volatility Affine Factor Option Pricing Model

This model is very similar to Heston model having the only difference that no correlation between subordinator and the subordinated process exists. However, we have added other parameters of Brownian Motion : drift and volatility to generalize the model.

As per used for Heston Model, the methodology for option pricing is based on characteristic function and obtaining cumulative distribution via Fourier inversion of this function. This will enable us to calculate expected pay-off. We define the option price,

$$C(S_t, v_t, K, r, \tau, \sigma_v, \theta, c, \kappa, \beta_1, \beta_2) = S_t F_s - e^{-r\tau} K F$$

$$F_s = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty Re\left(\frac{e^{-iu\log(K)} f_s(S_t, v_t, K, r, \tau, \sigma_v, \theta, c, \kappa, \beta_1, \beta_2, u)}{iu} du \right) \quad (4.62)$$

$$F = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty Re\left(\frac{e^{-iu\log(K)} f(S_t, v_t, K, r, \tau, \sigma_v, \theta, c, \kappa, \beta_1, \beta_2, u)}{iu} du \right) \quad (4.63)$$

The F_s shows $\mathbb{P}^s(S(t) > K)$ and F shows $\mathbb{P}(S(t) > K)$ as before. Again, we obtain probabilities from inverse Fourier transform of GBMSAj model characteristic function given by (4.61). The derivation of F_s requires a similar measure change process applied to (4.5.6). The characteristic functions used in (4.62) and (4.63) comes from (4.61).

As mentioned before, the measure change process slightly modifies the characteristic function required to find F_s. Thus we obtain the characteristic function components $A(\tau, u)$ and $B(\tau, u)$ of (4.62), in the $\mathbb{P}^S$ measure, as follows:

$$\gamma^s_{GBMSAf} = \sqrt{\kappa^2 + \sigma_v^2((u^2 - 1) - 2(\theta + ui(1 + \theta)))}$$

$$B^s_{GBMSAf}(\tau, u) = \frac{2(iu(1 + \theta) + \theta) + (u^2 - 1)}{\kappa + \gamma^s \coth(\frac{\gamma^s \tau}{2})}$$

$$A^s_{GBMSAf}(\tau, u) = \frac{e^{\frac{\kappa^2(\beta_1 + \beta_2 c)\tau}{\sigma_v^2}}}{\left(\cosh(\frac{\gamma^s \tau}{2}) + \frac{\kappa}{\gamma^s}\sinh(\gamma^s \frac{\tau}{2})\right)^{\frac{2\kappa(\beta_1 + \beta_2 c)}{\sigma_v^2}}}$$

We can then extend this affine factor setting to n factors case thanks to lemma (4.2) after satisfying the constraints about parameters. Then trivially the components of the characteristic function will be:

$$\gamma_{GBMSAnf} = \sqrt{\kappa^2 + \sigma_v^2(u^2 - 2iu\theta)}$$

$$B_{GBMSAf}(\tau, u) = \frac{2iu\theta - u^2}{\kappa + \gamma \coth(\frac{\gamma\tau}{2})} \tag{4.64}$$

$$A_{GBMSAnf}(\tau, u) = \frac{\exp\left(\frac{\kappa^2\left(\beta_0 + \sum_{j=1}^{n}(\beta_j c_j)\right)\tau}{\sigma_v^2}\right)}{\left(\cosh(\frac{\gamma\tau}{2}) + \frac{\kappa}{\gamma}\sinh(\gamma\frac{\tau}{2})\right)^{\frac{2\kappa\left(\beta_0 + \sum_{j=1}^{n}(\beta_j c_j)\right)}{\sigma_v^2}}} \tag{4.65}$$

For the $\mathbb{P}^S$ measure:

$$\gamma^s_{GBMSAnf} = \sqrt{\kappa^2 + \sigma_v^2((u^2 - 1) - 2(\theta + ui(1 + \theta)))}$$

$$B^s_{GBMSAnf}(\tau, u) = \frac{2(iu(1 + \theta) + \theta) + (u^2 - 1)}{\kappa + \gamma^s \coth(\frac{\gamma^s\tau}{2})}$$

$$A^s_{GBSAnf}(\tau, u) = \frac{\exp\left(\frac{\kappa^2\left(\beta_0 + \sum_{j=1}^{n}(\beta_j c_j)\right)\tau}{\sigma_v^2}\right)}{\left(\cosh(\frac{\gamma^s\tau}{2}) + \frac{\kappa}{\gamma^s}\sinh(\gamma^s\frac{\tau}{2})\right)^{\frac{2\kappa(\beta_0 + \sum_{j=1}^{n}(\beta_j c_j))}{\sigma_v^2}}}$$

Then complete form of characteristic function leads to the formula,

$$\begin{aligned}\varphi(u, \tau) &= A_{GBMSAf}(u, \tau)e^{B_{GBSAf}(u,\tau)X(0)} \\ &= A_{GBMSAnf}(u, \tau)e^{B_{GBSAnf}(u,\tau)\left[\sum_{j=0}^{n}(Y(0)+c_j Z_j(0))\right]}\end{aligned} \tag{4.66}$$

4.6.6 Variance Gamma with Stochastic Arrival (VGSA-VGCIR) Model

This model has been first introduced by [12]. For details [24] is suggested. In some studies, VGCIR name was used as well. The previous GMBSA model is a model to account for integrating volatility clustering into diffusion models via time change [24]. However, the original process Geometric Brownian Motion is still lacking the skewness inherent in log-returns of equities [32].

Therefore, it could be useful to employ a time change in a process which contains skewness already with a stochastic volatility coming from ICIR. For that reason, the model; Variance Gamma Stochastic Arrival (VGSA) could be a more complete in comparison to GBMSA and its variants. Similar to GBMSA model, the variance gamma model embraces stochastic volatility through stochastic time change.

There is a need for a stochastic time change even for the Variance Gamma which has a higher skewness and kurtosis value than pure Brownian Motion . The reason is that, the stochastic volatility in the VG model does not allow for volatility clustering, which is a feature of asset prices in financial markets in addition to skew. It is possible to implement volatility clustering in the model if random time changes persists, which requires that the rate of time change be mean reverting. The natural candidate for such a time change is the CIR process and is perfectly valid for modelling volatility if Feller conditions are satisfied. The importance of such a model in terms of option pricing is; this model will allow calibration of option prices to both maturity (τ) and strike (K) dimensions [24].

The model is defined:

$$\begin{aligned} dv_t &= \alpha(\beta - v_t)dt + \sigma\sqrt{v_t}dW_t \\ V(t) &= \int_0^t v(s)ds \\ X_{VGSA}(t) &= X_{VG}(V(t); \sigma, \alpha, \beta) \end{aligned}$$

As outlined (4.67) $Y(t)$ is a ICIR process. As we have written for GBMSA case it is better to write and define the model through its characteristic function below, where it is possible to write using (4.48).

$$\mathbb{E}((e^{iuX_{VGSA}(t)}) = \varphi_{V(t)}(i\phi_{VG}(u)) \tag{4.67}$$

From that equation we will have $\phi_{VG}(u)$ in the ICIR characteristic function instead of u term. We can write (4.67) using (4.51) in the explicit form:

$$A(\tau,u) = \frac{e^{\kappa^2 \frac{\beta\tau}{\sigma_v^2}}}{\left(\cosh(\frac{\gamma\tau}{2}) + \frac{\kappa}{\gamma}\sinh(\gamma\frac{\tau}{2})\right)^{\kappa\frac{\beta}{\sigma_v^2}}}$$
$$B(\tau,u) = \frac{2i\phi_{VG}(u)}{\kappa + \gamma\coth(\frac{\gamma\tau}{2})}$$
$$\gamma(u) = \sqrt{\kappa^2 - 2\sigma_v^2 i\phi_{VG}(u)}$$

4.6.7 An Affine Factor VGSA Model

Our result on convolution of ICIR process particular to affine decomposition model we have used throughout the study, could also be exploited here to still present the convoluted VGSA model in the form above. We will make use of the results involving GBMSAf model directly as we are still in ICIR environment after linear combination thanks to the 4.2.

Defining affine factor VGSA model below, we can proceed by exploiting the structure of (4.66).

$$X_{VGSA}(V(t)) = X_{VGSA}(Y(t) + cZ(t))$$

To have a general structure, we assume $Y(t)$ and $Z(t)$ are both $\in R^n$ as we defined for affine factor variance gamma case before.

Now, the only difference here compared to GBMSAf model is that, we work here with variance gamma characteristic function different from brownian motion characteristic function. The lemma 4.2 and VGSA characteristic function will lead to the following result for evaluating affine factor VGSA model.

$$\begin{aligned}\varphi(u,\tau) &= A_{VGSAf}(u,\tau)e^{B_{VGSAnf}(u,\tau)X(0)} \\ &= A_{VGSAnf}(u,\tau)e^{B_{VGSAf}(u,\tau)\left[\sum_{j=0}^{n}(Y(0)+c_j Z_j(0))\right]}\end{aligned} \tag{4.68}$$

where the components $A_{VGSAnf}(\tau, u)$ and $B_{VGSAf}(\tau, u)$ are

$$
\begin{aligned}
A_{VGSAnf}(\tau, u) &= \frac{e^{\kappa^2 \frac{\left[\beta_0 + \sum_{j=1}^{n}(\beta_j c_j)\right]^2}{\sigma_v} \tau}}{\left(\cosh(\frac{\gamma\tau}{2}) + \frac{\kappa}{\gamma}\sinh(\gamma\frac{\tau}{2})\right)^{\kappa^2 \frac{\beta_0 + \sum_{j=1}^{n}(\beta_j c_j)\tau}{\sigma_v^2}}} \\
B_{VGSAf}(\tau, u) &= \frac{2i\phi_{VG}(u)}{\kappa + \gamma\coth(\frac{\gamma\tau}{2})} \\
\gamma_{VGSA}(u) &= \sqrt{\kappa^2 - 2\sigma_v^2 i\phi_{VG}(u)}.
\end{aligned}
$$

In the next topic we proceed with a similar model where in this case, we allow correlation between subordinator and the stock price with jump as is in Bates model. Moreover, we now impose our linear decomposition structure over the subordinator. This model is naturally the correlated version of GBMSAfj model which also accounts the skew and smile effects as is taken into account VGSA model as well.

4.6.8 An Affine Factor Bates Model

In order to account for various non-gaussian features of return of asset prices, volatility smiles and more realistic structural credit risk modelling, Heston model and its jump added extension Bates model was produced. The stochastic volatility feature of these models enables volatility clustering and smile effect. However, it's possible to extend these models considering additional factors. During crisis or stress periods, the volatility and illiquidty may increase simultaneously in a way to switch dependence to a positive or negative tail structure. For these reasons and conditions it's possible to extend the asset price model of a company to a factor based structure. This could be in the form of decomposing time change or stochastic volatility into different components to account for systematic and unsystematic effects.

This will enable an environment in which dependencies between companies and even wrong-way risk is captured. We define the model as follows:

$$\frac{dS^L(t)}{S^L(t)} = rdt + W^S\sqrt{(V(t) + aL(t))} + ydN_t - \lambda\hat{y}dt$$

$$= rdt + \sqrt{(V_t + aL_t)}W(t)^S + ydN_t - \lambda\hat{y}dt$$

$$dL_t = \alpha(\beta - L(t))dt + \eta\sqrt{L_t}dW_t^L$$

$$dV_t = \kappa(\theta - V(t))dt + \sigma_v\sqrt{V(t)}dW^V(t)$$

$$L_t = \int_0^t L(s)ds, \quad V_t = \int_0^t V(s)ds$$

$$d\ln(S(t)) = \left(r - \lambda\hat{y} - \frac{V(t)}{2} - \frac{aL(t)}{2}\right)dt + \sqrt{(V(t) + aL(t))}dW_t^S + ydN_t$$

$$dW^SdW^L = \rho_{SL}dt, \quad dW^SdW^V = \rho_{SV}dt$$

The solution to the model above will be similar to Heston/Bates model which uses a charateristic function approach. Because it is trivial to see that the model is still a stochastic volatility model and the stochastic volatility is still in the same square root process form. The major difference is the addition a new term to volatility and so to the predefined functional structure of characteristic function.

Let $X_t^c = \log S_t$, then the characteristic function with the well-known form:

$$\mathbb{E}[e^{iuX_t^c}|X_t^c = x] = \int_{-\infty}^{\infty} e^{iux}f(x)dx \tag{4.69}$$

and using Feynman-Kac Theorem and applying Ito's Lemma over the function then setting dt terms to zero for martingale correction we obtain the PIDE below:

$$\mathbb{E}(e^{iuX_t^c}|X_t^c = x, V_t = \nu, L_t = \zeta) = f(t,\nu,x,\zeta)$$

$$= \frac{1}{2}f_{xx}(\nu + a\zeta) + \frac{1}{2}f_{\nu\nu}\sigma_v\nu + \frac{1}{2}f_{\zeta\zeta}\eta^2\zeta + f_{x\nu}\rho_{x\nu}\sigma_v\nu$$

$$+ f_{x\zeta}\rho_{x\zeta}\eta\zeta + f_x(r - \lambda\hat{y} - \frac{v}{2} - \frac{a\zeta}{2})$$

$$+ f_\nu[\kappa(\theta - \nu)] + f_\zeta[\alpha(\beta - \zeta)] + f_t$$

$$+ \lambda\mathbb{E}[f(t,\nu,x^J,\zeta) - f(t,\nu,x^c,\zeta)] = 0 \tag{4.70}$$

Then given at time $t = T$ characteristic function has terminal value $\mathbb{E}(e^{iux}|X_T = x)$ and $\tau = T - t$ we can write an Ansatz [51]:

$$f(t, \nu, x, \zeta) = \exp(A(\tau) + B(\tau)\nu + C(\tau)\zeta + iux)f^J \tag{4.71}$$

where f^J is the characteristic function of jump part, whose distribution is log-normal $\ln(1+y) \sim \mathcal{N}(\ln(1+\hat{y}) - \frac{\sigma_{jp}^2}{2}, \sigma_{jp}^2))$

We start by plugging the Ansatz above to the 4.70 then we obtain PDE with time dependent coefficients:

$$\begin{aligned}-u^2 f\frac{1}{2}(\nu + a\zeta) + B^2(\tau)\sigma_v^2 f + iuB(\tau)\rho_{x\nu}\sigma_v\nu f + iuC(\tau)\rho_{x\zeta}\eta\zeta f\\ +iu\left(r - \lambda\hat{y} - \frac{\nu}{2} - \frac{a\zeta}{2}\right) f + \left(A_\tau(\tau) + \nu B_\tau(\tau) + \zeta C_\tau(\tau)\right) f\\ +B(\tau)\left(\kappa(\theta - \nu)\right) f + C(\tau)\left(\alpha(\beta - \zeta)f\right) = 0\end{aligned}$$

Recollecting related to terms and setting each equal to zero we obtain three Ricatti type differential equations:

$$\nu\left(\frac{B(\tau)^2\sigma_v^2}{2} + B(\tau)[iu\rho_{x\nu}\sigma_v - \kappa] - B_\tau(\tau) - (\frac{u^2 + iu}{2})\right) = 0 \tag{4.72}$$

$$\zeta\left(\frac{C(\tau)^2\eta^2}{2} + C(\tau)[iu\rho_{x\zeta}\beta - \alpha] - C_\tau(\tau) - (\frac{a(u^2 + iu)}{2})\right) = 0 \tag{4.73}$$

$$A_\tau(\tau) + iu(r - \lambda\hat{y}) + B(\tau)\kappa\theta + C(\tau)\alpha\beta = 0, \tag{4.74}$$

subject to initial conditions:

$$B(0) = 0, \quad C(0) = 0, \quad A(0) = 0.$$

The solution of (4.72) requires some transformations to have a more solvable and familiar ODE form. For that purpose we have a transformation suggested by [49] to solve CIR characteristic function more practically. Let,

$$\mathrm{f_B}(\tau) = e^{\frac{-\sigma_v^2}{2}\int_0^\tau B(u)du} \tag{4.75}$$

$$\mathrm{f_C}(\tau) = e^{\frac{-\eta^2}{2}\int_0^\tau C(u)du}, \tag{4.76}$$

Then we proceed by writing $B(\tau)$ and $C(\tau)$ in terms of f_B and of f_C,

$$\mathrm{f}_{\mathrm{B}}^{'}(\tau) = \frac{-\sigma_v^2}{2}\mathrm{B}(\tau)\mathrm{f}_{\mathrm{B}}(\tau) \tag{4.77}$$

$$\mathrm{B}(\tau) = \frac{-2}{\sigma_v^2}\frac{\mathrm{f}_{\mathrm{B}}^{'}(\tau)}{\mathrm{f}_{\mathrm{B}}(\tau)} \tag{4.78}$$

$$\mathrm{B}^{'}(\tau) = \frac{-2}{\sigma_v^2}\left[\frac{\mathrm{f}_{\mathrm{B}}^{''}(\tau)}{\mathrm{f}_{\mathrm{B}}(\tau)} - \left(\frac{\mathrm{f}_{\mathrm{B}}^{'}(\tau)}{\mathrm{f}_{\mathrm{B}}(\tau)}\right)^2\right]. \tag{4.79}$$

We substitute back these results to (4.72) to simplify it and obtain a more familiar ODE form,

$$\begin{aligned}
\frac{2}{\theta^2}\left[\frac{\mathrm{f}_B^{''}(\tau)}{\mathrm{f}_B(\tau)} - \left(\frac{\mathrm{f}_B^{'}(\tau)}{\mathrm{f}_B(\tau)}\right)^2\right] - \frac{-2}{\theta^2}\frac{\mathrm{f}_B^{'}(\tau)}{\mathrm{f}_B(\tau)}A + \frac{4}{\theta^4}\left(\frac{\mathrm{f}_B^{'}(\tau)}{\mathrm{f}_B(\tau)}\right)^2\frac{\theta^2}{2} - C &= 0 \\
= \frac{2}{\theta^2}\left[\frac{\mathrm{f}_B^{''}(\tau)}{\mathrm{f}_B(\tau)} - A\frac{\mathrm{f}_B^{'}(\tau)}{\mathrm{f}_B(\tau)}\right] - C\frac{\mathrm{f}_B(\tau)}{\mathrm{f}_B(\tau)} &= 0 \\
= \mathrm{f}_B^{''}(\tau) - A\mathrm{f}_B^{'}(\tau) - \frac{C\theta^2}{2}\mathrm{f}_B(\tau) &= 0 \\
= \mathrm{f}_B^{''}(\tau) - A\mathrm{f}_B^{'}(\tau) - D\mathrm{f}_B(\tau) &= 0
\end{aligned}$$

$$A = (iu\rho\theta - \kappa), D = \frac{C\sigma_v^2}{2}, C = \frac{u^2 + iu}{2}$$

Finally for the Ricatti equations, (4.72) and (4.73) we obtain second order ODE below,

$$\mathrm{f}_B^{''}(\tau) - A\mathrm{f}_B^{'}(\tau) - \frac{C_B\theta^2}{2}\mathrm{f}_B(\tau) = 0 \tag{4.80}$$

and

$$\mathrm{f}_C^{''}(\tau) - A_C\mathrm{f}_C^{'}(\tau) - \frac{C_C\beta^2}{2}\mathrm{f}_C(\tau) = 0, \tag{4.81}$$

respectively. Since this is a general second order ODE form we can propose general solution,

$$f(x) = A_1e^{\lambda_1 x} + A_2e^{\lambda_2 x}$$

The boundary conditions of the new form will be:

$\mathrm{f}_B(\tau) = 1$ and $\mathrm{f}_B^{'}(\tau) = 0$.

Using these conditions we have the system of equations:

$$\mathrm{f}_B(0) = A_1 + A_2 = 1$$
$$\mathrm{f}_B^{'}(0) = A_1\lambda_1 + A_2\lambda_2 = 0$$

Computations for A_1, A_2 and $\mathrm{f}_B(\tau)$ yield,

$$A_1 = \frac{-\lambda_2}{\lambda_1 - \lambda_2}, \quad A_2 = \frac{\lambda_1}{\lambda_1 - \lambda_2}$$
$$\mathrm{f}_B(\tau) = \frac{\lambda_1}{\lambda_1 - \lambda_2} e^{\lambda_2 x} - \frac{\lambda_2}{\lambda_1 - \lambda_2} e^{\lambda_1 x} \tag{4.82}$$

Here λ_1 and λ_2 are the roots of the quadratic form obtained by plugging the proposed solution in the ODE. Using equation 4.80 we have,

$$\lambda^2 - \lambda A - D = 0$$
$$\lambda_{1,2} = \frac{A^2 \pm \sqrt{\Delta}}{2}$$
$$A = (iu\rho_{x\nu}\theta - \kappa)$$
$$D = \frac{C\sigma_v^2}{2} = \frac{\sigma_v^2(u^2 + iu)}{4}$$
$$\sqrt{\Delta} = \sqrt{(iu\rho_{x\nu} - \kappa)^2 + \sigma_v^2\,(u^2 + iu)} \tag{4.83}$$

Furthermore, using (4.77) and (4.79) we have,

$$B(\tau) = \frac{-2}{\theta^2}\left[\frac{\frac{\lambda_2\lambda_1 e^{\lambda_2\tau}}{\lambda_1-\lambda_2} - \frac{\lambda_1\lambda_2 e^{\lambda_1\tau}}{\lambda_1-\lambda_2}}{\frac{\lambda_1 e^{\lambda_2\tau}}{\lambda_1-\lambda_2} - \frac{\lambda_2 e^{\lambda_1\tau}}{\lambda_1-\lambda_2}}\right]$$

$$\lambda_1 \times \lambda_2 = \frac{A^2 - \Delta}{4} = \frac{-\sigma_v^2(u^2 + iu)}{4}$$

$$B(\tau) = \frac{\frac{(u^2+iu)\left[e^{\lambda_2\tau} - e^{\lambda_1\tau}\right]}{2}}{\lambda_2 e^{\lambda_2\tau} - \lambda_2 e^{\lambda_1\tau}} = \frac{\frac{(u^2+iu)\left[e^{(A-\sqrt{\Delta})\tau/2} - e^{(A+\sqrt{\Delta})\tau/2}\right]}{2}}{\frac{(A+\sqrt{\Delta})}{2}e^{(A-\sqrt{\Delta})\tau/2} - \frac{(A-\sqrt{\Delta})}{2}e^{(A+\sqrt{\Delta})\tau/2}}$$

$$= \frac{\frac{(u^2+iu)\left[e^{(-\sqrt{\Delta})\tau/2} - e^{(\sqrt{\Delta})\tau/2}\right]}{2}}{\frac{-A}{2}\left(e^{(\sqrt{\Delta})\tau/2} - e^{\sqrt{\Delta}\tau/2}\right) + \frac{\sqrt{\Delta}}{2}\left(e^{(\sqrt{\Delta})\tau/2} + e^{\sqrt{\Delta}\tau/2}\right)}$$

$$= \frac{-(u^2 + iu)}{-A + \sqrt{\Delta}\coth(\frac{\sqrt{\Delta}}{2})}$$

Finally, we have the solution as follows:

$$B(\tau) = \frac{-(u^2+iu)}{(\kappa - iu\rho_{x\nu}\theta) + \sqrt{\Delta}\coth(\frac{\sqrt{\Delta}\tau}{2})}. \tag{4.84}$$

Since $B(\tau)$ and $C(\tau)$ has similar ODE forms with mere difference in constant term we reach a similar solution for $C(\tau)$ by substituting the modified constant and the other terms of $C(\tau)$,

$$C(\tau) = \frac{-a(u^2+iu)}{(\alpha - iu\rho_{x\zeta}\beta) + \sqrt{\Delta_C}\coth(\frac{\sqrt{\Delta_C}\tau}{2})} \tag{4.85}$$

where the terms are obtained using (4.80).

$$\Delta_C = A_C^2 + 4D_C, A_C = (iu\rho_{x\zeta}\beta - \alpha), D_C = \frac{a(u^2+iu)\beta^2}{4}$$

$$\Delta_C = (iu\rho_{x\zeta}\beta - \alpha)^2 + \left[a(u^2+iu)\beta^2\right]$$

From these solutions for $C(\tau)$ and $B(\tau)$ we obtain the solution for $A(\tau)$ as:

$$A(\tau) = iu(r - \lambda\hat{y})\tau + \int_0^\tau [B(s)\kappa\theta + C(s)\alpha\beta ds] \tag{4.86}$$

The integral (4.86) can be evaluated as follows:

Write the solution of $B(\tau)$ or $C(\tau)$ in the form of below:

$$A_1 = u^2 + iu, \quad A_2 = \kappa - iu\rho_{x\nu}\theta, A_3 = \sqrt{\Delta}$$

$$\int B(s)ds = -\int \left[\frac{A_1 dx}{A_2 + A_3\coth(\frac{\tau}{2})}\right]$$

$$= -\left[\frac{A_1}{A_2 + A_3\frac{e^\tau+1}{e^x-1}}\right]$$

$$= -\int \left[\frac{A_1(e^x-1)dx}{(A_3+A_2)\left(e^x + \frac{A_3-A_2}{A_3+A_2}\right)}\right]$$

Letting $e^x = u$ the integral becomes:

$$\int B(s)ds = -\int \frac{(u-1)du}{u(A_3+A_2)(u + A_3 - A_2)}$$

Using partial fractions the result will be:

$$\frac{2}{A_3^2 - A_2^2}\ln(|A_2(e^x - 1) + A_3(e^x + 1)|) - \frac{x}{A_3 - A_2}$$

Using the substitutes for A_1, A_2, A_3 we will have:

$$\int B(s)\kappa\theta ds = \tau\kappa\theta\frac{(\kappa - iu\rho_{x\nu}\theta)}{\sigma_v^2} - \frac{2\kappa\theta}{\sigma_v^2}\ln\Big((\kappa - iu\rho_{x\nu}\theta)(e^{\frac{\sqrt{\Delta}\tau}{2}} - 1) + \sqrt{\Delta}(e^{\frac{\sqrt{\Delta}\tau}{2}} + 1)\Big) + K_B \quad (4.87)$$

Likewise the solution for $C(\tau)$ is going to be:

$$\int C(s)\alpha\beta ds = \tau\alpha\beta\frac{(\alpha - iu\rho_{x\zeta}\beta)}{\eta^2} - \frac{2\alpha\beta}{\eta^2}\ln\Big((\alpha - iu\rho_{x\zeta}\beta)(e^{\frac{\sqrt{\Delta_c}\tau}{2}} - 1) + \sqrt{\Delta_c}(e^{\frac{\sqrt{\Delta_c}\tau}{2}} + 1)\Big) + K_C \quad (4.88)$$

Using the boundary condition for $A(\tau)$; $A(0) = 0$ we find the integral constants K_B and K_C. This means using (4.87) and related boundary conditions:

$$\frac{-2\kappa\theta}{\sigma_v^2}\ln(2\sqrt{\Delta}) + K_B = 0$$

$$K_B = \frac{2\kappa\theta}{\sigma_v^2}\ln(2\sqrt{\Delta})$$

$$\frac{-2\kappa\theta}{\sigma_v^2}\ln(2\sqrt{\Delta}) + K_C = 0$$

$$K_C = \frac{2\alpha\beta}{\eta^2}\ln(2\sqrt{\Delta_C})$$

Substituting these into 4.87 and 4.88 we obtain:

$$\begin{aligned}\int B(s)\kappa\theta ds &= \tau\kappa\theta\frac{(\kappa - iu\rho_{x\nu}\theta)}{\sigma_v^2} \\ &- \frac{2\kappa\theta}{\sigma_v^2}\ln\Big((\kappa - iu\rho_{x\nu}\theta)(e^{\frac{\sqrt{\Delta}\tau}{2}} - 1) + \sqrt{\Delta}(e^{\frac{\sqrt{\Delta}\tau}{2}} + 1)\Big) + \frac{2\kappa\theta}{\sigma_v^2}\ln(2\sqrt{\Delta}) \\ &= \tau\kappa\theta\frac{(\kappa - iu\rho_{x\nu}\theta)}{\sigma_v^2} \\ &- \frac{2\kappa\theta}{\sigma_v^2}\ln\left((\kappa - iu\rho_{x\nu}\theta)\frac{(e^{\sqrt{\Delta}\tau} - 1)}{2\sqrt{\Delta}} + \frac{(e^{\sqrt{\Delta}\tau} + 1)}{2}\right)\end{aligned} \quad (4.89)$$

After writing exponential functions of above equation in the form of $\sinh$ and $\cosh$, we finally have

$$\int B(s)\kappa\theta ds = \tau\kappa\theta\frac{(\kappa - iu\rho_{x\nu}\sigma_v)}{\sigma_v^2} - \frac{2\kappa\theta}{\sigma_v^2}\ln\left(\frac{(\kappa - iu\rho_{x\nu}\sigma_v)}{\sqrt{\Delta}}\sinh(\frac{\sqrt{\Delta}\tau}{2}) + \cosh(\frac{\sqrt{\Delta}\tau}{2})\right) \tag{4.90}$$

and

$$\int C(s)\alpha\beta ds = \tau\alpha\beta\frac{(\alpha - iu\rho_{x\zeta}\nu)}{\eta^2} - \frac{2\alpha\beta}{\eta^2}\ln\left(\frac{(\alpha - iu\rho_{x\zeta}\eta)}{\sqrt{\Delta_C}}\sinh(\frac{\sqrt{\Delta_C}\tau}{2}) + \cosh(\frac{\sqrt{\Delta_C}\tau}{2})\right). \tag{4.91}$$

The formulas for Δ and Δ_C are

$$\sqrt{\Delta} = \sqrt{(iu\rho_{x\nu}\theta - \kappa)^2 + \sigma_v^2(u^2 + iu)}$$
$$\sqrt{\Delta_C} = \sqrt{(iu\rho_{x\zeta}\beta - \alpha)^2 + (a\beta^2(u^2 + iu))}$$

respectively.

Therefore, after collecting all the integral solutions above we finally obtain for $A(\tau)$:

$$A(\tau) = iu(r - \lambda\hat{y})\tau + \int_0^\tau [B(s)\kappa\theta + C(s)\alpha\beta ds]$$

The call option price implied by this affine factor structure for Heston/Bates Model will be:

$$V(S_t, v_t, \zeta_t, K, r, \tau, \rho_{x\zeta}, \rho_{xv}, \alpha, \beta, \theta, \kappa, a, \sigma_v, \eta) = S_t F_s - Ke^{-r\tau}F \tag{4.92}$$

$$F_s = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re\left(\frac{e^{-iu\ln(K)}f_s(\log S_t, v_t, \zeta_t, \tau, u)}{iu}du\right) \tag{4.93}$$

$$[10pt]F = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re\left(\frac{e^{-iu\ln(K)}f(\log S_t, v_t, \zeta_t, \tau, u)}{iu}du\right) \tag{4.94}$$

$$f^s = e^{A_s(\tau)+B_s(\tau)v_t+C_s(\tau)\zeta+iu\log S_t+\lambda\tau\left[(1+\mu)^{ui+1}\exp\left(\frac{\sigma_{jp}^2(ui-u^2)}{2}\right)-(1+\mu)\right]} \tag{4.95}$$

$$f = e^{A(\tau)+B(\tau)v_t+C(\tau)\zeta+iu\log S_t+\lambda\tau\left[(1+\mu)^{ui}\exp\left(\frac{-\sigma_{jp}^2(u^2+ui)}{2}\right)-1\right]} \tag{4.96}$$

The components of the characteristic functions are:

$$\Delta_s = ((1+ui)\rho_{xv}\theta - \kappa)^2 + \sigma_v^2(u^2 - ui)$$

$$\Delta_{cs} = ((1+ui)\rho_{x\zeta}\beta - \alpha)^2 + \eta^2(u^2 - ui)$$

$$B_s(\tau) = \frac{(-u^2+iu)}{\kappa - (1+ui)\rho_{xv}\theta + \sqrt{\Delta_s}\coth(\frac{\sqrt{\Delta_s}\tau}{2})}$$

$$B(\tau) = \frac{-(u^2+iu)}{(\kappa - iu\rho_{x\nu}\theta) + \sqrt{\Delta}\coth(\frac{\sqrt{\Delta}\tau}{2})}$$

$$C_s(\tau) = \frac{a(-u^2+iu)}{\alpha - (1+ui)\rho_{x\zeta}\beta + \sqrt{\Delta_{cs}}\coth(\frac{\sqrt{\Delta_{cs}}\tau}{2})}$$

$$C(\tau) = \frac{-a(u^2+iu)}{(\alpha - ui\rho_{x\zeta}\beta) + \sqrt{\Delta_c}\coth(\frac{\sqrt{\Delta_c}\tau}{2})}$$

The components regarding subscript s are defined under stock price risk adjusted measure $\mathbb{P}^{\mathbb{S}}$ as indicated in (4.16). The characteristic function for measure $\mathbb{P}^{S}$ is found by:

(i) Apply Theorem 4.1 to have a new characteristic function under $\mathbb{P}^{S}$:

$$\mathbb{E}(\frac{e^{iu\ln S(t)}e^{\ln S(t)}}{\varphi(-i)}) = \mathbb{E}(\frac{e^{(iu+1)\ln S(t)}}{\varphi(-i)})$$
$$= \mathbb{E}(\frac{e^{i(u-i)\ln S(t)}}{\varphi(-i)})$$

(ii) Substituting $u - i = v$ we have new charateristic function $\mathbb{E}(e^{iv\ln S(t)})$.

(iii) Applying this directly to the option pay-off we obtain new components of the characteristic function under $\mathbb{P}^{\mathbb{S}}$.

4.6.9 Extending Jump Heston Model (Bates Model) with Skewed-Normal Jumps

A fundamental part of finance theory is based on the assumption, either explicitly or implicitly, that the asset returns have multivariate or univariate normal probability distribution [1]. Nevertheless, many published studies indicate that this assumption is not valid empirically and is not observed in the financial data analysis. Moreover, models based on this assumption fail to satisfactorily fit real-world data. Thus, its use for the pricing of many financial instruments is questionable.

The skewed-normal distribution introduced by [4] could be a good candidate as it is flexible to adjust for either negative or positive skewness of the data. As found in [34] that currencies whose changes are more sensitive to negative market jumps provide significantly higher expected returns. Because of that, it is worth to study such a case for pricing of many derivatives, particularly options as well, as the underlier for these instruments are directly exchange rate or stock price, which have a well-observed skewness. We can change this assumption by switching to skewed-normal distribution, where we had a very brief introduction at Section 4.6.9.1, introduced by [4].

The characteristic function of this distribution derived by [43, 4]:

$$\varphi_Z(u) = e^{\frac{-u^2}{2}} \left(1 + iG(\frac{u\delta}{\sqrt{1+\delta^2}})\right) \tag{4.97}$$

$$G(x) = \int_0^x \sqrt{\frac{2}{\pi}} e^{\frac{v^2}{2}} dv$$

By using the characteristic function of skewed normal distribution (4.97), the characteristic function of $\log S_t$ (4.92), where we study log-normal jumps, and its characteristic function after $\mathbb{P}^S$ measure change will change accordingly as follows:

$$f_{SN} = e^{A(\tau)+B(\tau)v_t+C(\tau)\zeta+iu\log S_t+\lambda\tau\left[\exp\frac{-u^2}{2}\left(1+iG(\frac{u\delta}{\sqrt{1+\delta^2}})\right)-1\right]} \tag{4.98}$$

$$f^s_{SN} = e^{A_s(\tau)+B_s(\tau)v_t+C_s(\tau)\zeta+iu\log S_t+\lambda\tau\left[\exp\frac{-(u-i)^2}{2}\left(1+iG(\frac{(u-i)\delta}{\sqrt{1+\delta^2}})\right)-1\right]} \tag{4.99}$$

4.6.9.1 Skewed-Normal Distribution

A random variable Z has a skewed-normal distribution with parameter α, denoted by $Z \sim SN(\alpha)$. Its density is given by $f(z,\alpha) = 2\Phi(\alpha z)\phi(z)$ where Φ and ϕ are cumulative distribution function and density function of standard normal random variate, $z, \alpha \in \mathbb{R}$. The general properties of this distribution are [4]:

(i) $\mathcal{SN}(0) = \mathcal{N}(0,1)$

(ii) If $Z \sim \mathcal{SN}(\alpha)$ then $-Z \sim \mathcal{SN}(-\alpha)$

(iii) As $\lambda \to \mp\infty$, $\mathcal{SN}(\alpha)$ tends to the half-normal distribution, $\mp|X|$ when $X \sim \mathcal{N}(0,1)$

(iiii) If $Z \sim \mathcal{SN}(\alpha)$ then $Z^2 \sim \chi_1^2$

The characteristic function of this distribution is [43] :

$$\Phi_Z(t) = e^{\frac{-t^2}{2}}(1 + iG(\delta t)) \tag{4.100}$$

where $\delta = \frac{\alpha}{1+\alpha^2}$, and for $x \geq 0$, we have

$$\begin{aligned} G(x) &= \int_0^x \sqrt{\frac{2}{\pi}} e^{\frac{u^2}{2}du} \\ G(-x) &= -G(x) \end{aligned} \tag{4.101}$$

4.6.10 Extension of Heston Model with a Microstructure Adjustment on underlying Asset Price Process

It's stated in many studies [34] that microstructure contaminates prices due to trading noise, which is significant especially in small company stock prices. In financial markets where distress exists could also create significantly microstructure affected prices which makes it crucial to consider this effect in pricing of any instrument having the stock price as the underlier. For that reason, the modification of equity options which takes the stock price as the underlier is worthwile to study. Therefore, we correct the stock price possibly contaminated by trading noise and rederive the option price implied by this price process. This extension could also be reflected in credit risk via trading noise cleaned credit spreads and PD.

In order to model the microstructure effect, we assumed the liquidity or trading noise effect behaving like ICIR process as assumed in [30] for bond prices.

$$\Lambda(t) = \int_0^t \lambda(s)ds \tag{4.102}$$

$$d\lambda_t = \kappa(\beta - \lambda_t)dt + \zeta\sqrt{\lambda_t}dW_t$$

Then our microstructure corrected stock price could be defined:

$$\tilde{S}_t = e^{-\Lambda(t)}S_t = L_t S_t$$

Here the trade noise correction serves as an illiquidity adjustment and is supposed to discount the stock price since illiquidity is a factor to be compensated by a premium in order to avoid loss or reduction in profits of investors caused by lack of trade [30]. Taking into account the price impact of microstructure noise modelled by a ICIR process, we rederive the option pricing model of stochastic volatility with jumps. We write the option payoff for corrected price as:

$$V_t = \mathbb{E}(e^{-r(T-t)}(\tilde{S_T} - K)|\mathcal{F}_t)^+ \tag{4.103}$$

In terms of stochastic volatility models terms we can rederive the pricing formula using the model we have used in (4.55) after some analytical effort.

The derivation starts by rewriting the respective probabilities for S and risk neutral measure. These probabilities are familiar but these are defined for noise/illiquidity adjusted stock price $\tilde{S}_t$

$$\mathbb{P}^s(\tilde{S}_T > K) = F^s, \quad \mathbb{P}(S_T > K) = F.$$

Using illiquidity adjustment L_t term and setting $x = \log(L)$ we have:

$$F_s = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \int_0^\infty Re\left(\frac{e^{-iu(\log(K)-x)}\Phi^s(u)f(x)}{iu}dudx\right) \tag{4.104}$$

$$F = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \int_0^\infty Re\left(\frac{e^{-iu(\log(K)-x)}\Phi(u)f(x)}{iu}dudx\right). \tag{4.105}$$

Since the L_t is a ICIR process, the density could be obtained by Fourier inverting charateristic function. Using Fubini theorem we can further write

$$F = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty e^{iux}f(x)dx \int_0^\infty Re\left(\frac{e^{-iu\log(K)}}{iu}\Phi(u)\right)du$$

First element in the above double integral is actually characteristic function of $x = \log(L_t) = \Lambda(t)$ which is defined to be $\Phi^L(u)$. Further we can write the formula for probabilities as the product of two characteristic functions and Fourier inversion component as

$$F^L = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re\left(\frac{e^{-iu\log(K)}}{iu}\Phi(u)\Phi^L(u)du\right)$$

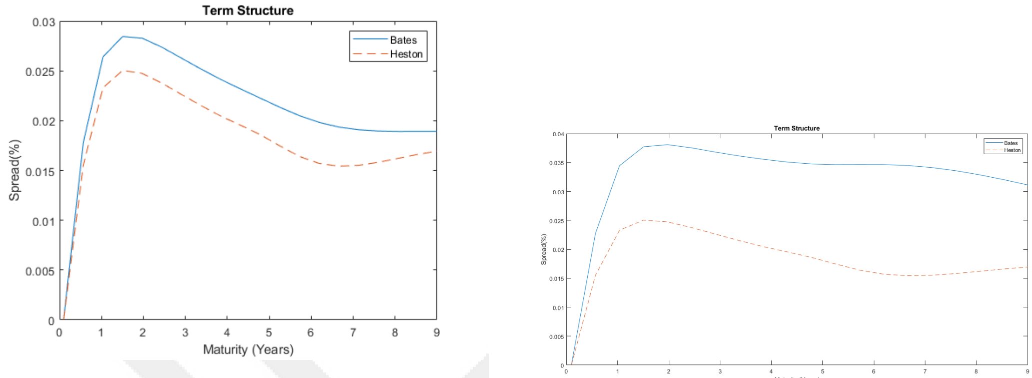
and

$$F^L_s = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re\left(\frac{e^{-iu\log(K)}}{iu}\Phi(u)^s\Phi^L(u)du\right).$$

The final option price is

$$V_t = S_t F_s^L - K e^{-r(T-t)} F^L$$

This result is expected to help us in investigating the impact trading noise or illiquidity on credit spreads, PD and the price of the option itself. We can see the effect from figure 4.2 below.



(a) Liquidty Unadjusted CDS Spread Bates Model

(b) Liquidty Adjusted CDS Spread Bates Model

Figure 4.2: Bates Model Liquidity Adjusted Spread

4.7 Credit Risk Estimation under Structural Models

4.7.1 Merton Type Models

Credit risk estimation using structural models has been popular for decades. In the context of macrofinancial risk, the Merton model has become a tool for even for regulators. The model relies on the fact that market actors know and process information efficiently under information flow being instant [21]. For that reason, any financial or political news or information regarding any company, sector could be equally reflected in the price of the equity. This requires, the default probability to be a function of market value of the company.

The model uses normally distributed asset return process and market value of assets have log-normal distribution, which is at the heart of Black-Scholes and Merton Model (BSM). Thus, our asset value process will be the basic log-normal value process of BSM model:

$$\frac{dA_t}{A_t} = \mu dt + \sigma dW_t$$

The same process at risk neutral measure can be written as:

$$\frac{dA_t}{A_t} = rdt + \sigma dW_t$$

We may then write

$$A_T = A_t \exp\big((r - 0.5\sigma^2)(T - t) + \sigma(W_T - W_t)\big)$$

Based on the asset price process we write the market value of the company as

$$E_t = A_t N(d_1) - Ke^{-r(T-t)}N(d_2), \tag{4.106}$$

where the terms d_1 and d_2 are

$$d_1 = \frac{\log(\frac{A_t}{L}) + (r + 0.5\sigma^2)(T - t)}{\sigma(T - t)}$$
$$d_2 = \frac{\log(\frac{A_t}{L}) + (r - 0.5\sigma^2)(T - t)}{\sigma(T - t)}$$

respectively.

In this model, a company's market value is assumed to have the properties of an option contract. This has strong financial ground such that; a company's equity can be seen as an option contract with shareholders of the same company. Because, in the case of a default the shareholders take the remaining assets of the firm after bondholders or creditors are satisfied financially. We can thus formulate the pay-off of this transaction as:

$$E_t = e^{-r(T-t)} \max(\{A_T - L, 0\} \tag{4.107}$$

Since this is similar to the payoff of an option, it is reasonable to determine the market value of a company by an option pricing formula where BSM and its extensions are convenient. Given the market value of a company's equity, we can calculate the default probability through the option valuation formula and its components.

In the BSM model, probability of default under risk-neutral measure is defined by

$$\mathbb{P}(A_T < L) = \mathbb{P}(A_t e^{(r-0.5\sigma^2)(T-t)+\sigma(W_T-W_t)} < L), \tag{4.108}$$

Real world probability measure is defined by

$$\mathbb{P}(A_T < L) = \mathbb{P}(A_t e^{(\mu-0.5\sigma^2)(T-t)+\sigma(W_T-W_t)} < L). \tag{4.109}$$

Then the difference between default barrier L and A_T under risk-neutral measure and actual measure is going to be respectively where we it is called Distance-to-Default (DD)

$$DD_T^{RN} = \frac{\log(\frac{A_t}{L}) + (r - 0.5\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

$$DD_T = \frac{\log(\frac{A_t}{L}) + (\mu - 0.5\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

Then we calculate risk-neutral default probability

$$\mathbb{P}^{RN}(A_T \leq L) = \mathbb{P}\left(\frac{\log(\frac{A_t}{L}) + (r - 0.5\sigma^2)(T-t)}{\sigma_A\sqrt{T-t}}\right)$$

Then we calculate real world default probability

$$\mathbb{P}^{RW}(A_T \leq L) = \mathbb{P}\left(\frac{\log(\frac{A_t}{L}) + (\mu - 0.5\sigma^2)(T-t)}{\sigma_A\sqrt{T-t}}\right)$$

4.7.2 Estimating Unobserved Parameters in Structural Credit Risk Models

The major issue regarding the structural models is the estimation of parameters which is a similar challenge to BSM; however in structural models, depending on the underlying process, there are at least one unknown parameter, unobserved asset value A_t is common then σ_A or more is possible. Since the equity model of the company is an option, it is possible to assume the company's market value as the option value. Therefore, in order to find the A_t and σ_A we have to use either a statistical estimation methodology such as maximum likelihood estimation (MLE) or filtering using historical market values of stock prices. This is still an indirect method since market values are unobserved or a possible two equation system based methodology as we have two unknowns.

For these purposes we can start with the equation system based estimation method [31]. This method exploits the Ito's Lemma in order to reach an equation system. To be specific, it is assumed that equity price itself has a log-normal process with different diffusion parameter.

As the equity process is an option we can write the its PDE using Feynman-Kac theorem applying Ito's lemma similar to BSM equation [49]

$$dE_t = \Big(- rE_t + \frac{\partial E_t}{\partial t}dt + r\frac{\partial E_t}{\partial A_t}dt + \frac{1}{2}\frac{\partial^2 E_t}{\partial A_t^2}\sigma_A^2\Big)dt + \sigma_A\frac{\partial E_t}{\partial A_t}dW_t \tag{4.110}$$

Assuming E_t has a log-normal process as suggested above we can write:

$$dE_t = E_t\sigma_{E_t}dW_t$$

This means equtity process E_t is a martingale. From BSM model discounted asset price process is a martingale as well. Thus, the elements under dt terms is going to be zero which is also a direct result of Feynman-Kac [49]. Hence, we are left with second part of (4.110) and (4.111) where it's convenient to equate these two, since they have to be equal by assumption. Then we obtain the condition

$$E_t\sigma_{E_t}dW_t = \sigma_A\frac{\partial E_t}{\partial A_t}dW_t \tag{4.111}$$

Besides, we have (4.106) which is the result of option payoff

$$E_t = A_tN(d_1) - Ke^{-r(T-t)}N(d_2) \tag{4.112}$$

Finally, we have these two equation system to be solved numerically to estimate A_t and σ_A. This methodology has been used and generated for BSM model since it is more convenient to generate a system of equations with only a GBM modelled equity value process. The usage of the structural model is similar for the other Levy processes with some subtleties.

A generic algorithm for estimating A_t for structural models will be:

(i) Calculate call option price at root A_{t-1} and Δ_C of the same option at A_{t-1}

(ii) Apply Newton-Raphson formula at each iteration,

$$A_t = A_{t-1} - \frac{C(A_t)}{\Delta_C(A_t)}$$

- In case BSM is used this step will be complemented by

$$\sigma_{A_t} = \frac{\sigma_{E_t}\Delta_C A_t}{E_t}$$

due to (4.111) and Newton-Raphson step will be,

$$A_t = A_{t-1} - \frac{C(A_t, \sigma_{A_t})}{\Delta_C(A_t, \sigma_{A_t})}$$

(iii) Then recalculate the Call price using A_t and Δ_C using P_1 if underlying process is Variance Gamma (4.21) or F^s either from (4.93) or (4.55) if underlying process is stochastic volatility,

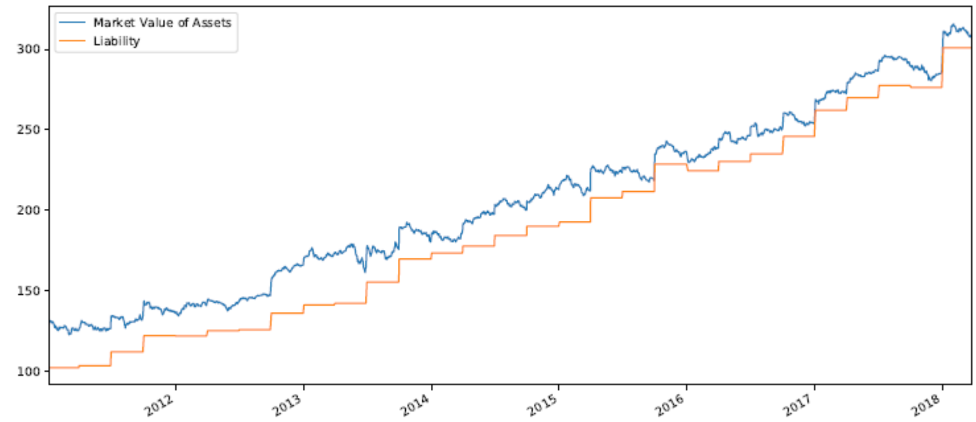
- Again in BSM model this step will be recalculate the Call price using A_t, σ_{A_t} and Δ_C using $N(d_1(A_t, \sigma_{A_t}))$ from (4.112).

(iv) Then if $\epsilon = A_t - A_{t-1}$ the preset convergence level ϵ is achieved stop iteration,

(v) Calculate $PD(t,T) = \mathbb{P}(A_T < L_T)$ using P_2 from (4.21) if Variance Gamma process is used or F from (4.56) or (4.94) if stochastic volatility is used.

- Again in BSM model this will be Calculate $PD(t,T) = \mathbb{P}(A_T < L_T)$ using the $N(d_2)$ from (4.112).

The Δ_C will always be P_1 from (4.21) due to the relationship we derived in equation (E.3). This algorithm presents a way to estimate the unobserved parameters of the model. However, it's also possible to use a MLE method to estimate the same parameters [31]. After applying this algorithm to our extended Bates affine factor stochastic volatility model, we obtain following results:



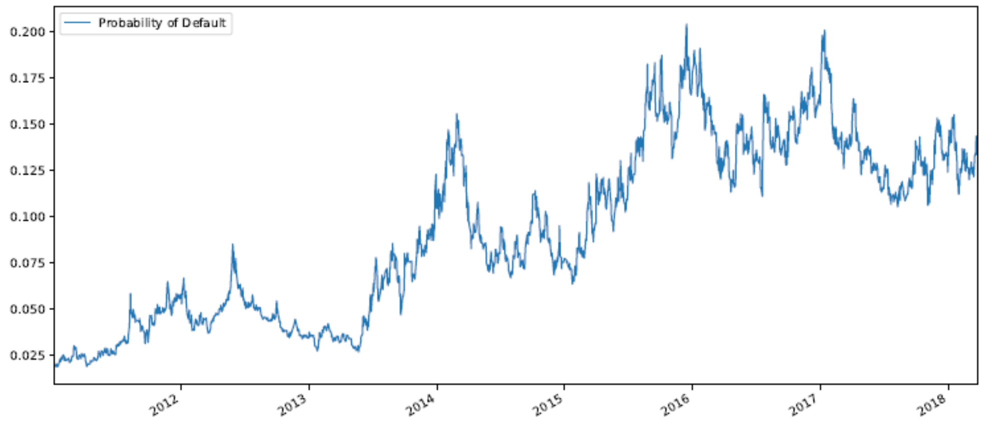


Figure 4.3: Bates Affine Model market Value of Assets $A(t)$ & PD-Akbank

We see that model consistently works and estimates $A(t)$ however, the PD seems to overestimate compared to Merton model. We will see later that Variance Gamma model also estimates higher PD than Merton model. This situation is possibly due to model parameter calibration and numerical sensitivity of fourier inversion during PD calculation in Bates affine model. Nevertheless, the model calculates higher PD than Brownian Motion Merton model and is more prudent to gauge risk accumulation.

4.8 Credit Risk Estimation under Variance Gamma Environment

4.8.1 Merton Type Credit Risk Estimation

As explained in the previous section Merton model application in Brownian Motion environment is more straightforward than other Levy processes. In Variance Gamma environment due to its pure jump nature it is not trivial to apply the same reasoning. The system of equations (4.112) and (4.111) alleviates the estimation of unobserved asset value $A(t)$ and volatility σ_A considerably. This equation system obtained by introducing (4.110). The same reasoning could be applied to Variance Gamma case, however we have totally 4 parameters $A(t), \theta_A, \sigma_A$ and ν_A. In additon we ought to find θ_E, σ_E and ν_E as it is required that equity process follows its own Variance Gamma to apply Merton's analogy. In order to apply structural credit risk estimation algorithm in previous section, we write $E(t)$ to be a exponential martingale below:

$$E(t) = E(0)e^{-t\phi(-i)_E + X_E(t)} \tag{4.113}$$

The $E(t)$ equity of the company, which is the option price in Merton context, is a function of $A(t)$ and the Ito's lemma for this Variance Gamma functional is:

$$\begin{aligned} dE(t) = \Big(&- rE(t) + \frac{\partial E(t)}{\partial t} + r\frac{\partial E(t)}{\partial A(t)}A(t) \qquad (4.114) \\ &+ \int_{-\infty}^{\infty} \Big[E(A_{t_-}e^{X_{A(t)}}, t) - E(A_{t_-}, t) \\ &- \frac{\partial E(A(t), t)}{\partial A(t)}(e^{X_A} - 1) \Big] \nu(dx) \Big) dt + \frac{\partial E(t)}{\partial A(t)} dX_{A(t)} A(t) \end{aligned}$$

$$dE(t) = E(t)dX_E(t) \tag{4.115}$$

Similar to Brownian Motion case the Variance Gamma implied call option price could be found by the setting dt part, the variance gamma PIDE equal to zero. Then we are left with most right of the (4.114) and (4.115). In order to reconcile both processes since they are supposed to be the same, we have to equate these terms.

$$\frac{\partial E(t)}{\partial A(t)} dX_{A(t)} A(t) = E(t)dX_E(t) \tag{4.116}$$

Using (E.3) in Appendix E we can write that $\frac{\partial E(t)}{\partial A(t)} = F_{VG}(d_1)$ under Variance Gamma process. After that we obtain system equations

$$E(t)\theta_E = F_{VG}(d_1)\theta_A A(t)$$
$$E(t)\sigma_E = F_{VG}(d_1)\sigma_A A(t)$$
$$\nu_A = \nu_E$$

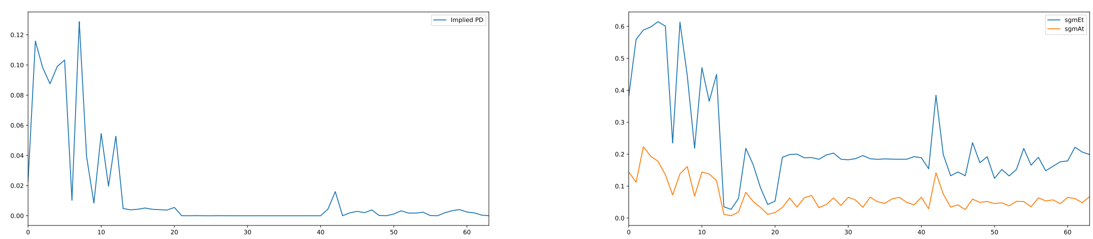
We set the gamma scale parameters to be the same since we assume $\sigma-$ algebra $\mathcal{G}$ generated by the subordinator in both variance gamma processes are the same as they belong to same company.

We already have from (4.21) and adapting it yields

$$E(A(t), \theta_{A(t)}, \sigma_{A(t)}, D, r) = A(t)F_{VG}(d_1) - De^{-rt}F_{VG}(d_2) \tag{4.117}$$

As a result we complete the set of four equations for four unknown parameters as we desired. The problem is that we have to calibrate the equity process $E(t)$ to log-returns of stock prices to find $\theta_E, \sigma_E, \nu_E$ or we could resort to equity options of the company which is better to find risk-neutral parameters lastly it's possible to use parameters calibrated to CDS spreads, if exists for the company which are risk neutral as well.

We construct a hypothetical application in this regard and we found that the well-known Merton model algorithm for contingent claims analysis works fairly well in our Variance Gamma case too. As we can see from figures below 4.4b and 4.4a Variance Gamma credit risk model presented acceptable results such that, hypothetical equity and unobserved asset prices have an expected trend as well as PD.



(a) DB VG Merton Model PD Plot (b) DB VG Merton Model Sigma Et & At

Figure 4.4: VG Structural Model Estimated Market Value of Assets $A(t)$ & PD

In order to implement these numerical calculations we need model risk-neutral parameters where we obtained from fitting Variance Gamma barrier model to CDS prices of Deutsche Bank for 2012. The equity values and distress barrier; liabilities are specified 50 whereas market capitalization values are adjusted to range between 30 and 10 since they are supposed come from equity market.

Another empirical study is on the data [1] for the period 2011-2017. We compare Variance Gamma Merton model and Brownian Motion Merton model where the latter is conventional model used for structural PD estimation. We see from figure 4.5 below that the Variance Gamma model is clearly superior to the Brownian Motion model due to possibly fat tail property. Because in Brownian Motion model, PD almost vanishes for low risk area however, Variance Gamma model shows a plausible PD level for the liability level.

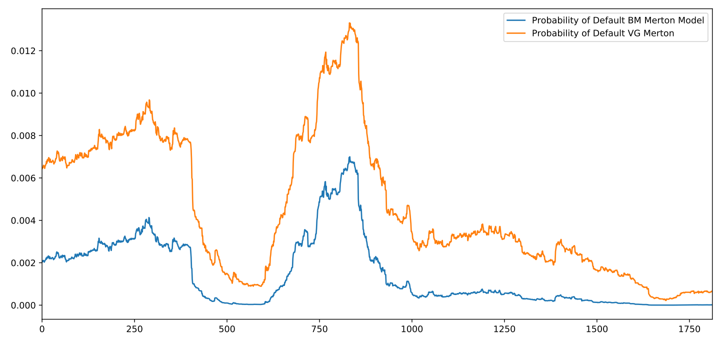


Figure 4.5: VG Merton Model and BM Merton Model PD-Akbank

4.8.2 Barrier Type Credit Risk Estimation

These models are important in the sense that they could provide more realistic improvement over Merton's structural credit risk model while preserving the similar default probability measurement process or metrics. This model has first been introduced in [8], where default is defined as the knock-out option. More formally, default could be defined as a stopping time as expressed in (4.118) where the stopping time is connected to the minimum of GBM to be less than liability threshold L_t within time to maturity. However, as our random variable is a time changed Brownian Motion process, we will have to re-adjust the stopping time with respect to Variance Gamma. This can be defined

$$\tau = \inf\left(t : e^{X(t)} \leq L_t\right), 0 < t < T \tag{4.118}$$

[1] Akbank's (fifth largest bank in Turkey in terms of asset magnitude) balance sheet liabilities and market value of equity for 2011-2017.

where

$$X(t) = \theta\gamma + \sigma W(\gamma)$$

as first defined in (4.8)

$$PD(t,T) = E(\mathbb{1}_{\{t<\tau<T\}}) = \mathbb{P}(\tau < T)$$

where asset price of the company is below the liability L_t. Here we will use time changed BMs defined in the previous section as the underlying process for the assets of the company. The liability L_t could be either constant, a deterministic function of time or a stochastic process.

The credit risk measurement here has the same underlying stochastic process, hence the same probabilistic structure with down-and-out barrier options except the underlying structure is a two factor variance gamma process. Thus for any pricing or measurement purpose the joint distribution of BM and its maximum/minimum plus the distribution function of time change is required.

4.8.2.1 Flat Barrier

We have outlined probabilistic properties regarding joint distribution of Brownian Motion and its maximum/minimum. This model is introduced by [8] to find the bond price based on the default probability implied by asset price underlied by minimum Brownian Motion . We further extend this model by changing Brownian Motion to variance gamma process whose probabilistic properties are outlined in (4.5.8). Thus, given the debt/distress threshold L or L_t we concisely write default probability implied by Variance Gamma (VG) model:

$$PD(t,T) = \int_0^\infty P\Big(M_\gamma - (r - \phi(-i))\tau \leq \log\left(\frac{A(t)}{L}\right) |\gamma = g\Big) f(g)dg \quad (4.119)$$

Using suggestion in [26] and formula in [29] where we give derivation at the appendix

F, we have final formula for $PD(t,T)$ as follows:

$$PD(t,T) = \int_0^\infty \left(N\left(\frac{\ln\left(\frac{L}{A(t)}\right) - \mu - \theta}{\sigma\sqrt{\gamma}} \right) + e^{\frac{2\tilde{\theta}}{\sigma^2}\ln\left(\frac{L}{A(t)}\right)} N\left(\frac{\ln\left(\frac{L}{A(t)}\right) - \mu + \theta}{\sigma\sqrt{\gamma}} \right) \right) \times \frac{g^{\tau/\nu-1}e^{-g/\nu}}{\nu^{\tau/\nu}\Gamma(\tau/\nu)} dg \tag{4.120}$$

where $\mu = (r - q - \phi(-i))\tau$.

We can further simplify the formula below to a semi-closed form formula using equation (4.24) as follows:

$$PD(\tau) = F_{VG}\left(\ln\left(\frac{L}{A(t)}\right) - \mu, -\theta, \tau \right) + e^{-\frac{2\theta}{\sigma_A^2}\ln\left(\frac{L}{A(t)}\right)} F_{VG}\left(\ln\left(\frac{L}{A(t)}\right) - \mu, \theta, \tau \right), \tag{4.121}$$

where $\tau = T - t$.

4.8.2.2 Stochastic Barrier

In order to model financial distress position of a company, the assumption of constant liability could be abandoned. However, it is important to model liability realistically which enforces to find a market value of debt structure. As we use Variance Gamma in many modelling purposes in this study, we can base dynamic stochastic liability structure on Variance Gamma process as well. A similar idea was used in [25] considering Brownian Motion for stochastic liability. Therefore, the proposed methodology and analytical results of previous section enables us to find Default Probabilities for a stochastic debt environment.

There will be dependence between market value of assets and liabilities of the same company. Therefore, dependence or correlation structure could also be implemented either by using the same stochastic time change or imposing correlation between brownian motions of assets and liabilities.

We start with general structure and derive the PD in the case of stochastic barrier then we continue with affine factor model. To be consistent we assume Variance Gamma value process for both market value of assets and liabilities. Thus, we can propose a model such as:

$$D_t = D_0 \exp\Big((r - \phi_L(-i))t + X_D(t)\Big),$$

where $X(t) = \theta_D \gamma + \sigma_D W_D(\gamma)$

$$A(t) = A_0 \exp\Big((r - q - \phi_A(-i))t + X_A(t)\Big).$$

where $X_A(t) = \theta_A \gamma + \sigma_A W_A(\gamma)$ and $dW_A dW_D = \rho dt$.

In this model we use same gamma subordinator for the Brownian Motion as market evaluates the financial information for a company simultaneously thus more or less the assets and liabilities have a connection due to risk evaluation. Moreover, the Brownian Motion part is also correlated as it is the main driver of asset and liability value processes.

We may also assume same stochastic time change for assets and liabilities since the market absorbes the infomation regarding the company's financial standing. Therefore trading time could be correlated. This will make the derivation more tractable. As a result the same γ subordinator and correlated brownian motion assumptions lead to correlation in the Variance Gamma environment

$$\rho_{i,j} = \frac{\theta_i \theta_j + \sigma_i \sigma_j \rho}{\sqrt{(\theta_i^2 + \sigma_i^2)(\theta_j^2 + \sigma_j^2)}} \tag{4.122}$$

The formal definition of the default is the event that the assets are below a liability barrier at any time until debt maturity as in (4.108). As defined in (4.118)

$$\tau = \inf\{t : e^{X(t)} \le D(t)\}, \quad t < \tau < T$$

$$PD(t,T) = \mathbb{E}(\mathbb{1}_{\{t<\tau<T\}})$$

$$= \mathbb{P}(\tau < T)$$

However, in this case the liability is stochastic as well which reflects the market valuation of debts especially when the firm has an issued debt in the market. We can formally write probability of default

$$PD(t,T) = \mathbb{P}(\underline{M}_{A(t)} \le D(t)).$$

where $\underline{M}_{A(t)}$ is the minimum of X_A.

However, it could be more convenient to use a solvency level barrier for both mathematical tractability and solvency is a key financial ratio followed by the market. For these purposes, we can target a level L such that values below that barrier will send the company to default more formally we can define:

$$PD(t,T) = \mathbb{P}\left(\frac{A(t)}{D(t)} \leq L\right). \tag{4.123}$$

Using the solvency barrier we write the process $L(t)$ where the derivation is given in G,

$$L(t) = \frac{A_0}{D_0} e^{\left(\phi_D(-i)-\phi_A(-i)-\phi_{AD}(-i)\right)t+(\theta_A-\theta_D)\gamma+\sigma_A W_A(\gamma)-\sigma_D W_D(\gamma)}.$$

Remark 4.2. Here we write the variance gamma process in terms of drift and brownian motion components since it is more intuitive to write in this form rather than single variance gamma process for later derivations.

Given the solvency barrier L the PD can be written as:

$$PD(0,t) = \mathbb{P}\left(\frac{A_0}{D_0} e^{\left(\phi_D(-i)-\phi_A(-i)-\phi_{AD}(-i)\right)t+(\theta_A-\theta_D)\gamma+\min(\sigma_A W_A(\gamma)-\sigma_D W_D(\gamma))} \leq L\right)$$

We can redefine composite Variance Gamma processes in the exponent above by first defining W_D of the debt as the correlated Brownian Motion conditioned on $\gamma = g$

$$W_D(g) = \rho W_A(g) + \sqrt{1-\rho^2} W(g)$$

$$W_{AD}(g) = \sigma_A W_A(g) - \sigma_D W_D(g) \tag{4.124}$$

We write composite diffusion parameter of the subordinated Brownian Motion

$$\sigma_{AD} = \sqrt{\sigma_A^2 + \sigma_D^2 - 2\sigma_A\sigma_D\rho} \tag{4.125}$$

$$\theta_{AD} = \theta_A - \theta_D \tag{4.126}$$

Finally, we can write

$$PD(0,t) = \mathbb{P}\left(\frac{A_0}{D_0} e^{\left(\phi_D(-i)-\phi_A(-i)-\phi_{AD}(-i)\right)t+\theta_{AD}\gamma+\sigma_{AD}\underline{M}_{AD}(\gamma)} \leq L\right)$$

After this setup we can use formulas (4.121) to calculate PD for stochastic liability barrier:

$$PD(t,T) = \int_0^\infty \left(N\left(\frac{\log(\frac{L}{L_0}) + \tilde{\theta}_{AD} g}{\sigma_{AD}\sqrt{g}}\right) + e^{\frac{-2\underline{m}\tilde{\theta}_{AD}}{\sigma^2_{AD}}\log(\frac{L}{L_0})} N\left(\frac{\log\left(\frac{L}{L_0}\right) - \tilde{\theta}_{AD} g}{\sigma_{AD}\sqrt{g}}\right)\right) \times \frac{g^{t/\nu-1}e^{-g/\nu}}{\nu^{t/\nu}\Gamma(t/\nu)} dg \tag{4.127}$$

where we define components,

$$\mu = (\phi_D(-i) - \phi_A(-i) - \phi_{AD}(-i) - q)\tau$$
$$L(0) = \frac{A(0)}{D(0)}$$
$$\underline{m} = \log\frac{L}{L_0}.$$

4.8.3 Affine Factor Variance Gamma Barrier Model

In order to construct a dependence between asset price processes, it is convenient to construct the link with the state of the economy by means of an affine factor decomposition. Similar to what we have done in (4.5.6) we will modify the single Variance Gamma process as an affine model of two variance gamma processes. Consistent with our general affine factor variance gamma model framework we have our decomposition:

$$X_t = Y_t + cZ(t), \tag{4.128}$$
$$Y(t;\sigma,\gamma,\theta) = \theta_1\gamma_1(t;1,\nu) + \sigma_1 W(\gamma_1(t;1;\nu))$$
$$Z(t;\sigma,\gamma,\theta) = \theta_2\gamma_2(t;1,\nu) + \sigma_2 W(\gamma_2(t;1;\nu))$$

This model could be used for cases which requires dependence between counterparties and the underlying asset of the derivative. Thanks to affine structure, we would be modelling possible dependencies without resorting to methods such as copulas which are difficult to calibrate numerically. However, the challenge of this model will be its complexity due to many components. We can write default probability process for this model:

$$P(\tau < T) = \mathbb{P}\left(A_t e^{(r-\phi(-i))\tau + \underline{M}_{X(\tau)}} < L\right)$$

In order to find a solution we can follow two ways: first is to write variance gamma as the difference between two gamma processes and apply this to the affine factor sum, second is to exploit the results of infimum of brownian motion . The steps for these methods are:

(i) We can force the linear combination of the variance gamma processes $Y(t) + cZ(t)$ to be another variance gamma process. This means the making convolution to be closed under variance gamma . As we indicate at the 4.10.3 there are some conditions defined rigorously for variance gamma process closed under convolution. If we satisfy these constraints, we would have another variance gamma with parameters coming from variance gamma components of linear combination. As a result we have a final variance gamma process using the aforementioned constraints and (4.128)

$$Y(t) + cZ(t) = VG\left(\theta_1 + c\theta_2, \sqrt{\sigma_1^2 + c^2\sigma_2^2}, \frac{\nu_1\nu_2}{\nu_1 + \nu_2}\right) \tag{4.129}$$

This leads to the following correlation coefficient for any two assets in the variance gamma environment:

$$\rho_{i,j}^{VG} = \frac{c_i c_j \left(\theta_Z^2 \nu_Z + \sigma_Z^2\right) t}{\sqrt{(\theta_i^2 + \sigma_i^2)(\theta_j^2 + \sigma_j^2)}} \tag{4.130}$$

Using the sum $X(t) = Y(t) + cZ(t)$ and working through its minimum we can use formula (4.121) again to calculate PD under this linear combination.

Then after convolution, (4.121) will become:

$$\int_0^\infty \left(N\left(\frac{\log(\frac{L}{A_t}) + \tilde{\theta}_{yz} g}{\sigma_{yz}\sqrt{g}}\right) + e^{\frac{-2\tilde{\theta}_{yz}\log(\frac{L}{A_t})}{\sigma_{yz}^2}} N\left(\frac{\log(\frac{L}{A_t}) - \tilde{\theta} g}{\sigma_{yz}\sqrt{g}}\right)\right) \times \frac{g^{\tau/\nu_{yz}-1} e^{-g/\nu_{yz}}}{\nu_{yz}^{\tau/\nu_{yz}} g(\tau/\nu_{yz})} dg \tag{4.131}$$

where

$$\begin{aligned}
\mu_{yz} &= r - q - (\phi_y(-i) + \phi_z(-i)) \\
\nu_{yz} &= \frac{\nu_y \nu_z}{\nu_y + \nu_z} \\
\tilde{\theta}_{yz} &= \theta_y + c\theta_z - \mu_{yz} \\
\sigma_{yz} &= \sqrt{\sigma_y^2 + c^2\sigma_z^2}.
\end{aligned}$$

Again to simplify and improve this result, we again use (4.24) using necessary parameters form the formula above as follows:

$$PD(\tau) = F_{VG}\Bigg(\ln\Big(\frac{L}{A(t)}\Big) - \mu_{yz}, -\theta_{yz}, \tau \Bigg) + e^{\frac{-2\theta_{yz}\ln(\frac{L}{A(t)})-\mu_{yz}}{\sigma_{yz}^2}} F_{VG}\Bigg(\ln\Big(\frac{L}{A(t)}\Big) - \mu_{yz}, \theta_{yz}, \tau \Bigg) \quad (4.132)$$

where $t < \tau < T$.

(ii) The other method proposed is to regroup the negative and positive parts of the linear combination and to write a new variance gamma process which can be written as the difference of two Gamma processes [17]. We can write this proposal:

$$Y(t) + cZ(t) = \gamma_{1p}(\nu_p, \mu_{1p}) + c\gamma_{2p}(\nu_p, \mu_{2p}) - (\gamma_{1n}(\nu_n, \mu_1 n) + c\gamma_{2n}(\nu_n, \mu_{2n})) \quad (4.133)$$

However, there is another complication here is that Gamma distributed random variables are also subject to some constraints to be closed under convolution. This constraint is [2] for two independent Gamma variates $X_1 \sim Ga(\mu_1, \nu)$ and $X_2 \sim Ga(\mu_2, \nu)$ we have the sum $X_1 + X_2 \sim Ga(\mu_1 + \mu_2, \nu)$.

Therefore, Gamma distribution is closed under convolution as long as scale coefficient is the same for both Gamma processes. Hence, we will have to find a way to obtain an exact if not an approximate Gamma variate without forcing the Gamma components of the negative and positive variates of (4.133) to keep scale parameters ν the same. This could give more flexibility.

The approximate way to obtain two Gamma processes from (4.133) could be to use *Welch-Satterthwaite* equation which gives an approximate distribution for sum of gamma random variables with different parameters, for which a convolution is not closed unless same scale parameters is assumed. Using the approximation formula derived in [54, 47], we obtain for our case:

$$p_c = \mu_1\nu_1 + \sum_{j=2}^{N} c_j\mu_j\nu_j$$

$$\nu_c = \frac{\mu_1\nu_1^2 + \sum_{j=2}^{N} c_j\mu_j\nu_j^2}{p_c}$$

$$\mu_c = \frac{p_c^2}{\mu_1\nu_1^2 + \sum_{j=2}^{N} c_j\mu_j\nu_j^2}$$

After this approximation we end-up with variance gamma process since we have difference of two variance gamma processes finally. This result enables us to use (4.121) via redefinition of parameters accordingly.

4.8.4 Affine Factor Variance Gamma Stochastic Barrier Model

As given in (4.128) we have two components in the linear setup

$$X_t = Y_t + cZ(t)$$

where,

$$Y(t;\sigma,\gamma,\theta) = \theta_1\gamma_1(t;1,\nu) + \sigma_1 W(\gamma_1(t;1;\nu))$$
$$Z(t;\sigma,\gamma,\theta) = \theta_2\gamma_2(t;1,\nu) + \sigma_2 W(\gamma_2(t;1;\nu))$$

Using this setup and equations (4.124)– (4.125)–(4.126) and (4.131) we obtain

$$\int_0^\infty \left(N\left(\frac{\log(\frac{L}{A_t}) + \tilde{\theta}_{AD}\gamma}{\sigma_{AD}\gamma}\right) + e^{\frac{-2\tilde{\theta}_{AD}\log(\frac{L}{A_t})}{\sigma_{AD}^2}} N\left(\frac{\log(\frac{L}{A_t}) - \tilde{\theta}_{AD}\gamma}{\sigma_{AD}\gamma}\right)\right) \times \frac{g^{\tau/\nu_{yz}} e^{-g/\nu_{yz}}}{\nu_{yz}^{\tau/\nu_{yz}}\Gamma(\tau/\nu_{yz})} dg \tag{4.134}$$

where we define,

$$\mu_{AD} = (\phi_A(-i) - \phi_D(-i) - \phi_{AD}(-i) - q)$$
$$\phi_A = \phi_y(-i) + \phi_z(-i)$$
$$\nu_A = \frac{\nu_y\nu_z}{\nu_y + \nu_z}$$
$$\tilde{\theta}_{AD} = \theta_A - \theta_D - \mu_{AD}$$
$$\theta_A = \theta_y + c\theta_z$$
$$\sigma_A = \sqrt{\sigma_y^2 + c^2\sigma_z^2}$$
$$\sigma_{AD} = \sqrt{\sigma_A^2 + \sigma_D^2 - 2\sigma_A\sigma_D\rho}$$

The derivation is a combination of factor variance gamma and stochastic barrier model. Here we use the first method proposed for affine factor barrier model which uses convolution of variance gamma process under constraint.

4.9 Monte Carlo Framework For Variance Gamma Credit Risk Model

In order to justify the validity of (4.121) it is useful to resort to Monte Carlo (MC) Simulation. However, the path dependency of the PD process here makes it slightly more difficult to implement a safe MC simulation. As given in [51] the price of an option or a derivative with barrier feature, it is more trivial to solve the PIDE for the Levy process or use a MC simulation. The use of plain MC involves approximation of the First Passage Time $\tau \cong \inf[t \geq 0 : A_t \leq L]$ with discretized interval; $\Delta t = t_k - t_{k-1}$ and $\tau^{\Delta t} \cong \min[t_k \geq 0 : A_{t_k} \leq L]$. However, under this approach the process might cross the barrier and return back, this corresponds to a sudden default and survival in credit risk terms, which might be missed during a plain simulation MC simulation due to discrete time interval. In order to capture this event, we could either calculate $\mathbb{P}(\inf A_{t,t+\Delta} < L | A_t > L, A_{t+\Delta t} > L)$ or we can calculate $\mathbb{P}(\inf A(s)_{t<s<T})$ to evaluate the minimum of $A(t)$ under entire maturity. We can calculate this probability by the help of the conditional probability:

$$
\begin{aligned}
\mathbb{P}(e^{M(t)} \leq m | W(t) = w) &= 1 - \int_m^w \frac{e^{2\mu(w-\mu)} 2(w - 2\mu)}{t} d\mu \\
&= e^{\frac{2m(w-m)}{t}}
\end{aligned}
$$

where $M(t) = \inf W(t)$ This corresponds to the distribution of Brownian Bridge with the interval $[0, t]$ and $W(t) = w, W(0) = 0, M(t) \leq \log(L)$. We can extend this approach to the time changed/subordinated brownian motion case by conditioning on the subordinator.

We can write this probability under the variance gamma case:

$$
\mathbb{P}(e^{M(\gamma)} \leq m | W(\gamma) = w) = \int_0^\infty e^{\frac{2m(w-m)}{g}} g^{\alpha-1} e^{\frac{-g}{\alpha}} \frac{\beta^\alpha}{\Gamma(\alpha)} dg \tag{4.135}
$$

This expression could be calculated under numerical quadrature. However, we can obtain a semi-closed form formula where we start by presenting the formula in [13]:

$$
\Gamma_\alpha(\beta) = 2^{\frac{\alpha}{2}} K_\alpha(2\sqrt{b}) \tag{4.136}
$$

This function is called generalized incomplete gamma function and $K_\nu(y)$ is the modified Bessel function of the second kind.

This formula is the closed form of general expression

$$\Gamma_\alpha(b) = \int_0^\infty e^{-t-\frac{b}{t}} t^{\alpha-1} dt$$

We can observe that after some substiution we can directly use this expression for the integral (4.135). The substiution follows:

$$b = \frac{2m(m-w)}{\nu}, \alpha = \frac{\tau}{\nu}, g = \nu t$$

This follows:

$$\mathbb{P}(e^{M(\gamma)} \leq m | W(\gamma) = w) = \int_0^\infty e^{\frac{2m(w-m)}{g}} g^{\alpha-1} e^{\frac{-g}{\alpha}} \frac{\beta^{-\alpha}}{\Gamma(\alpha)} dg \tag{4.137}$$

$$= \int_0^\infty e^{\frac{-b}{t}-t} t^{\alpha-1} \frac{\nu^{\alpha-1-\frac{\tau}{\nu}}}{\Gamma(\frac{\tau}{\nu})} \nu dt$$

Given that $\alpha = \frac{\tau}{\nu}$ and νdt term we obtain

$$\mathbb{P}(e^{M(\gamma)} \leq m | W(\gamma) = w) = \int_0^\infty e^{\frac{-b}{t}-t} t^{\frac{\tau}{\nu}} \frac{1}{\Gamma(\frac{\tau}{\nu})} dt$$

using (4.136) the expression above yields

$$\mathbb{P}(M_X(\gamma) \leq \underline{m}) = \Gamma_{\frac{2m(m-w)}{\nu}} \left(\frac{\tau}{\nu}\right) \tag{4.138}$$

$$= 2\Big(\frac{|2m(m-w)|}{\nu}\Big)^{\frac{\tau}{2\nu}} K_{\frac{\tau}{\nu}} \left(2\sqrt{\frac{|2m(m-w)|}{\nu}}\right)$$

Now we can use this equation to generate $\inf X(\gamma)$. Since we have $P(M_X(\gamma) \leq \underline{m})$, it is possible to use inverse transform method to generate $M_X(\gamma)$. To proceed we first generate $U \sim U(0,1)$ then solve (4.138)to find the root b.

$$\Gamma_{\frac{2m(m-w)}{\nu}} \left(\frac{\tau}{\nu}\right) = 2\Big(\frac{|2m(m-w)|}{\nu}\Big)^{\frac{\tau}{2\nu}} K_{\frac{\tau}{\nu}} \left(2\sqrt{\frac{|2m(m-w)|}{\nu}}\right) = U$$

After solving for b, we simulate $X(\gamma)$ and plug it in below as w then exploit the substitution to find the $M_X(\gamma)$ which is m below. After solving the quadratic equation below we obtain m.

$$\frac{2m(m-w)}{\nu} = b$$

$$2m^2 - 2mw - b\nu = 0$$

$$m = \frac{2w \pm \sqrt{4w^2 + 8b\nu}}{4}$$

$$m = \frac{w - \sqrt{w^2 + \frac{b\nu}{4}}}{2}$$

This secures that $M_X(\gamma)$ is less than $X(\gamma)$ which is a necessary condition. Otherwise, solving (4.138) directly for m might cause numerical instabilities. The main idea behind this approach comes from simulation of $\inf W(t)$ where we have given a detailed derivation at the appendix A. After finding $M_X(\gamma)$ we are ready to calculate default probability of assets having an exponential variance gamma process. This could be calculated by

$$\mathbb{P}\Big(A(\tau) \leq L|\mathcal{F}_t\Big) = \mathbb{P}\left(\frac{A(t)}{L}e^{(r-\phi(-i))\tau+M_X(\gamma)} \leq 1\right) \tag{4.139}$$

where $M_X(\gamma) = \inf(W(\gamma) + \theta(\gamma))$

Then we write the MC algorithm in summary as follows

Algorithm 1 Monte Carlo algorithm for Simulating Minimum of Variance Gamma and PD calculation.

1: **procedure** VGPDMCSIM(ν, τ, U, Z)

2: Generate r.v. $U \leftarrow U(0,1)$,

3: Use,

$$\Gamma_{\frac{\tau}{\nu}}(b) = 2b^{\frac{\tau}{2\nu}} K_{\frac{\tau}{\nu}}\Big(2\sqrt{b}\Big) = U$$

4: Using inverse transform find b,

$$\Gamma^{-1}_{\frac{\tau}{\nu}}(U) = b,$$

5: Simulate Variance Gamma r.v by Brownian Motion subordination,

$$W(\gamma) = Z\gamma + \theta\gamma,$$

$Z \sim N(0,1)$ and $\gamma = Ga(\frac{\tau}{\nu}, \nu)$

6: Find $M_X(\gamma) = m$ by using following equation

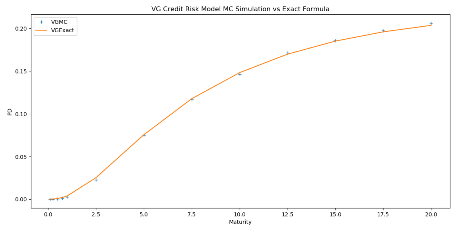
$$m = \frac{w - \sqrt{w^2 + \frac{b\nu}{4}}}{2},$$

7: After obtaining many simulated m calculate $PD(\tau)$ given liability L as in (4.139)

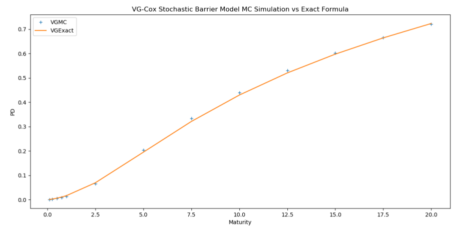
$$\mathbb{P}\Big(A(\tau) \leq L|\mathcal{F}_t\Big) = \mathbb{P}\left(\frac{A(t)}{L}e^{(r-\phi(-i))\tau+M_X(\gamma)} \leq 1\right),$$

where $\tau = T - t$.

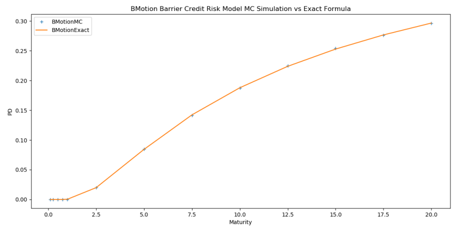
A hypothetic model with exact and MC simulated PD calculations have presented in figures 4.6a and 4.6b below. We can see a very close almost the same results for exact formula and MC simulation procedure outlined above.



(a) VG Model Cox Barrier PD: At=100.0, L=50.0, r=0.04,q=0.0, ν=0.5,σ=0.2,θ =-0.08



(b) VG Model Cox Barrier PD: At=100.0, L=50.0, r=0.04,q=0.0, ν=0.5,σ=0.2,θ =-0.08



(c) BM Black-Cox Barrier PD: At=100.0, L=50.0 r=0.04,q=0.0, σ=0.2

Figure 4.6: VG Black-Cox and BM Black-Cox Monte Carlo vs Analytical Formula

4.10 Stress Testing in Counterparty Credit Risk

4.10.1 Introduction

The stress testing in Counterparty Credit Risk setting is not a straightforward problem. Linking default probability to state of economy is not trivial. Although, we construct an affine factor Variance Gamma model containing systematic factor in our Counterparty Credit Risk setting, this setup still is does not contain a decent structure to account for some macro variables that drive the state of the economy i.e. GDP, inflation, exchange rate etc.

Moreover, still we can not insert key market variables such as CDS spreads, VIX index. The reason we have to insert the these variables into Counterparty Credit Risk framework is that this type of a credit risk model will enable macrofinancial stress testing and scenario analysis.

4.10.2 Variance Gamma Macro Credit Risk Model

Given in the introduction, our purpose is to construct a model which enables stress testing. For that reason, we decide to use [52] type model which is mainly used in [28] for macroeconomic stress testing. This model has a structural Merton model type construction and is studied by Vasicek (1997) to model porfolio loss distribution. The model assumes that the asset return of a company has the structure:

$$R_{it} = \sqrt{1-\rho^2}Z_{it} + \rho U_t \tag{4.140}$$

where Z_{it} is idisyncratic factor for i^{th} company's asset return and U_t is the common factor representing the state of the economy. Both random variables are $N(0,1)$ and independent. Our contribution to this structure is to use variance gamma process for both random variables Z and U in order to make the asset return distribution more skewed and fatter tails than the Normal. This has similarities what we have studied in affine factor variance gamma model. Therefore, the results tools and formulas related to this model could also be used in this construction when needed.

However, the construction is more akin to [40] which is a one factor variance gamma copula. We can write model:

$$R_{it} = \sqrt{1-\rho^2} Z_{it}^{VG} + \rho U_t^{VG} \tag{4.141}$$

where

$$\begin{aligned} U_t &\sim VG(\rho\theta, \sigma, \frac{\nu}{1-\rho}) \\ Z_{it} &\sim VG(\sqrt{1-\rho^2}\theta, \sigma, \frac{\nu}{\rho}) \\ R_{it} &\sim VG(\theta, \sigma, \nu) \end{aligned} \tag{4.142}$$

The parameters are chosen so that final distribution is (4.142) this can be shown using the result in [40] about the sum of two Variance Gamma processes for which we have given a derivation at Appendix (D):

$$X_1 \sim VG(\theta_1, \sigma_1, \nu_1)$$

$$X_2 \sim VG(\theta_2, \sigma_2, \nu_2)$$

Then we have,

$$X_1 + X_2 \sim VG\left(\theta_1 + \theta_2, \sqrt{\sigma_1^2 + \sigma_2^2}, \frac{\nu_1 \nu_2}{\nu_1 + \nu_2}\right) \tag{4.143}$$

As a result we have a final variance gamma process using the aforementioned constraints and (4.128)

$$I(t) + cZ(t) = VG\left(\theta_I + c\theta_Z, \sqrt{\sigma_Z^2 + c^2\sigma_Z^2}, \frac{\nu_I \nu_Z}{\nu_I + \nu_Z}\right) \tag{4.144}$$

As mentioned using result (4.143) we obtain (4.142). Now we can detail how we obtain (4.142) using (4.143). The (4.142) is the final Variance Gamma process we want to achieve given its decomposition (4.141). First, our conjectured gamma scale parameter is ν and we have our parameters:

$$\nu_1, \nu_2 = (\frac{\nu}{1-\rho}, \frac{\nu}{\rho})$$

Using (4.143)

$$\nu = \frac{\nu_1 \nu_2}{\nu_1 + \nu_2} = \frac{\frac{\nu^2}{\rho(1-\rho)}}{\nu(\frac{1}{1-\rho} + \frac{1}{\rho})}$$

Second, for the drift θ we have

$$\theta_{R_{it}} = \sqrt{1-\rho^2}\theta_{Z_{it}}^{VG} + \rho\theta_{U_t}^{VG}$$

From properties given at (4.142) we obtain

$$\theta_{R_{it}} = (1-\rho^2)\theta + \rho^2\theta = \theta$$

Third, we set σ parameter same for both U_t and Z_{it}.

This completes the derivation of (4.142) where we desired to have a convenient parametrization for the systematic and idiosyncratic components of R_{it} return process . However, we are free to choose different parameters for U_t and Z_{it}.

Moreover, considering our specific model the ρ dependence coefficient of R_i, R_j returns of any two company's assets become:

$$\rho_{R_{it},R_{jt}} = \frac{\rho_i\rho_j(\theta_i^2\frac{\nu_i^2}{(1-\rho_i)^2} + \sigma_i^2)}{\sigma_i\sigma_j} \tag{4.145}$$

$$\rho_{R_{it},R_{jt}} = \frac{\rho_i\rho_j(\theta_j^2\frac{\nu_j^2}{(1-\rho_j)^2} + \sigma_j^2)}{\sigma_i\sigma_j} \tag{4.146}$$

This is a direct result of our constrained setting about two uncorrelated variance gamma processes. The reason for this construction is to have more degrees of freedom and obtain a return distribution that has the same distribution as the state of the economy and idiosyncratic components.

This modelling enables to integrate macro-economic and key market variables as control variables for the common factor U_t. As we will see we can calibrate and use the model for credit risk measurement similar to [40].

After this point, general asset return model based on Variance Gamma , we can start a Credit Risk Model with state of economy components. Our dependent key variable has some form of default probability variable such as PD extracted from CDS, non-performing loan (NPL) ratio or PDs direct from a portfolio. This variable is actually probability of a binary event which is linked to the asset return of a company staying below a threshold T, namely default threshold:

$$\mathbb{P}(Y_t = 1) = \mathbb{P}(R_{it} < T)$$

In order to integrate this default event with the state of the economy, macroeconomic variables and key financial variables, some form of model is needed. For that purpose we can start with a linear model of key economic variables such as:

$$I_t = \beta_0 + \beta_1 X_{1t} + \beta_2 X_{2t} + \cdots + \beta_k X_{kt} \tag{4.147}$$

Equivalently,

$$I_t = X_t \beta$$

where X_t is a $(N \times K)$ matrix and β is a $(k \times 1)$ vector of coefficients. We link this linear combination with the default threshold T by equating them below [28]:

$$\mathbb{P}(R_{it} < T) = \mathbb{P}\left(\rho U_t + \sqrt{1-\rho^2} Z_{it} < X_t \beta\right) = \Psi_{VG}(X_t \beta)$$

where Ψ_{VG} is the cumulative distribution function of VG random variable. We can further write the same default probability by conditioning on the state of the economy U_t as:

$$\begin{aligned} \mathbb{P}(R_{it} < T) &= \mathbb{P}\left(\rho U_t + \sqrt{1-\rho^2} Z_{it} < X_t \beta\right) \\ &= \int_{-\infty}^{\infty} \mathbb{P}\left(Z_{it} < \frac{X_t\beta - \rho u}{\sqrt{1-\rho^2}} \middle| U_t = u\right) f(u) du \\ &= \int_{-\infty}^{\infty} \Psi_{VG_{Z_{it}}}\Big(\frac{X_t\beta - \rho u}{\sqrt{1-\rho^2}}\Big) \psi_{VG_u}(u) du \end{aligned} \tag{4.148}$$

The next task is, using this formula and the data, to estimate the parameters of whole process. Since we have a PD function in hand, it's possible to derive a function for estimation.

4.10.3 Variance Gamma Vasicek Factor Model PD Estimation

As we mentioned in previous section a calibration must be implemented to make further predictions, stress tests and simulations regarding PD. However, the calibration is not straightforward and a statistical estimation method must be used. Therefore, we first propose a non-linear least squares (NLS) method, however we have (k) parameters form linear model and 4 parameters from variance gamma model.

In our model, construction we adjust the parameters of variance gamma process so that an estimation is possible with minimum effort and higher degrees of freedom. Because we have a combination of two variance gamma random variables meaning

7 different parameters at the beginning. Now in our adjusted model we have 4 parameters to estimate. As we have a model for Probability of Default we can select variable proxy for PD to calibrate the β coefficients of the linear model. We now write estimation function of the model:

$$PD(\theta, \nu, \sigma, \beta_1, \beta_2 \ldots \beta_j) = \sqrt{\frac{\sum_{i=1}^{N}(PD^i_{Company} - CDS^i_{Model})^2}{N}} \tag{4.149}$$

The last equation is more convenient to work with as it is more tractable mathematically and more lenient in terms of numerical optimization. Therefore, after optimizing (4.149) we can obtain parameters of (4.147). The important issue is the estimation of variance gamma process parameters $\rho, \theta, \sigma, \nu$ which will be solved in next section. We implement a hypothetical stres test experiment for Variance Gamma model below:

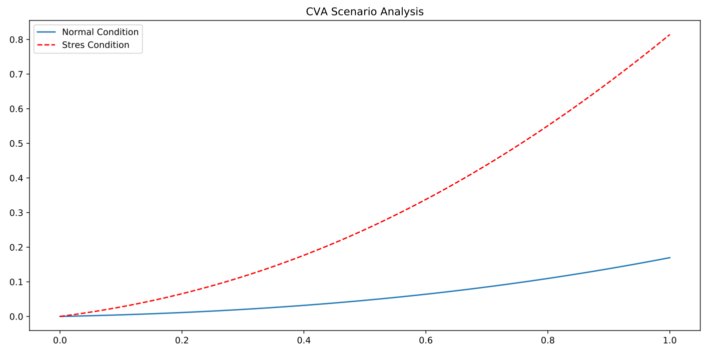


Figure 4.7: Variance Gamma Model Stress Test Exercise-Double Volatility Scenario

In this exercise, we double systematic volatility to see the effect on Counterparty Credit Risk . We observe an almost ten fold increase in CVA. It is trivial to see that common factor shows its role on both counterparty PD and the value of underlier to account wrong way risk.

4.11 Calibration of Parameters for Variance Gamma Process in General

The calibration is a delicate issue and is important for estimation, simulation and prediction tasks. Calibration of models has many aspects; such as calibration to market prices of the instruments, direct estimation from time series using Method of Moments (MME), Maximum Likelihood Estimation (MLE) and linear, non-linear filtering techniques. We start with a modified MME method which was developed in [16] for variance gamma and normal inverse gaussian processes. Because these will be inputs for MLE estimation of variance gamma and normal inverse gaussian one-factor credit risk models. We can start with writing moments of variance gamma process in terms of $\theta^2\nu = \kappa$ since this is a very common terms among first four moments:

$$\begin{aligned}
\mu_1 &= \theta t \\
\mu_2 &= \sigma^2(\kappa+1) \\
\mu_3 &= \sigma^2\theta\nu(2\kappa+3)t \\
\mu_4 &= 3\sigma^4 t\Bigg(\nu(2\Big(\kappa+1)^2-1\Big)+t(\kappa+1)^2\Bigg)
\end{aligned}$$

We can further set $(\kappa+1)^2 = \mathcal{E}$ and define new idenntities in terms of moments such that:

$$\frac{\mu_4}{3\mu_2^2}-1 = \frac{3\sigma^4 t\Bigg(\nu(2\mathcal{E}-1)+t\mathcal{E}\Bigg)}{3\sigma^4 t^2\mathcal{E}}-1 = \frac{2\nu}{t}-\frac{\nu}{\mathcal{E}t} \tag{4.150}$$

$$\frac{\mu_3}{\mu_2\mu_1} = \frac{2(2\sqrt{\mathcal{E}}+1)}{\sqrt{\mathcal{E}}} \tag{4.151}$$

Using (4.150) and setting $\frac{\mu_4}{3\mu_2^2}-1 = A$ and (4.151) and setting $\frac{\mu_3}{\mu_2\mu_1} = C$ we can derive ν using these relationships. We obtain the relationships below by elaborating over equations:

$$\mathcal{E} = \frac{\nu}{2\nu - At}$$

$$\nu = \frac{C\sqrt{\mathcal{E}}}{2\sqrt{\mathcal{E}}+1} = \frac{C}{2+\frac{1}{\sqrt{\mathcal{E}}}}$$

Using (4.150)

$$\frac{1}{\sqrt{\mathcal{E}}} = \frac{2-At}{\nu} \Rightarrow \nu\Big(\sqrt{2-\frac{At}{\nu}}\Big)+2\nu = C \Rightarrow \sqrt{2\nu^2-At\nu} = C-2\nu$$

Then we can see that

$$2\nu^2+\nu(At-4C)+C^2 = 0 \Rightarrow \nu = \frac{4C-At\mp\sqrt{(At-4C)^2-8C^2}}{4}$$

The A and C terms are for asymptotic moments, and are substituted by sample moments at which the methodology of MME is based on. Further we continue to derive moments as we have formula for ν and it's linked to $\mathcal{E}$ and this also specifies or is the component of other moments. Therefore finally we conclude:

$$\nu = \frac{\frac{4\hat{\mu}_3}{\hat{\mu}_2\hat{\mu}_1} - \left(\frac{\hat{\mu}_4}{3\hat{\mu}_2}^2 - 1\right)t \mp \sqrt{\left(\left(\frac{\hat{\mu}_4}{3\hat{\mu}_2^2} - 1\right)t - 4\frac{\hat{\mu}_3}{\hat{\mu}_2\hat{\mu}_1}\right)^2 - 8\left(\frac{\hat{\mu}_3}{\hat{\mu}_2\hat{\mu}_1}\right)^2}}{4}$$

$$\theta = \frac{\hat{\mu}_1}{t}$$

$$\sigma = \frac{\hat{\mu}_2}{t\sqrt{\frac{\nu}{2\nu - \left(\frac{\hat{\mu}_4}{3\hat{\mu}_2^2} - 1\right)t}}}$$

Now as we have derived the variance gamma process distribution parameters θ, ν, σ, we will then optimize NLS function (4.149), conditioned on variance gamma process parameters calculated here, derived in 4.10.3 to estimate the macro-financial variable coefficients.

If our purpose is to estimate the parameters θ, ν, σ through MLE we can write down likelihood function as follows:

$$\ell(\rho, \nu, \theta, \sigma) = \prod_{t=1}^{T} \psi_{VG}(\theta, \nu, \sigma \mid x_t, \tau) \tag{4.152}$$

Then log-likelihood function is

$$\ell(\rho, \nu, \theta, \sigma) = \sum_{t=1}^{T} \psi_{VG}(\theta, \nu, \sigma \mid x_t, \tau) \tag{4.153}$$

where for example regarding daily data, we can set τ=1. For more detailed analysis regarding high frequency data the study [16] could be used as reference.

Here ψ shows the density of Variance Gamma process and we use the equation (4.25) as a semi-closed form for this density. After setting up the log-likelihood function we can either numerically optimize it given the data or optimize through analytical derivatives.

4.12 Calibration of Parameters under Affine Factor Structure: A Case Study

The calibration procedure in this model is different since we use Credit Default Swap (CDS) spreads or option prices as the variable to model and as our modelling architecture is different from plain Variance Gamma equity return process. In terms of implementation methodology; we will use non-linear least squares to coincide the market

CDS spreads or option prices with that of proposed model. In our empirical application, we used the dataset of [32] which use CDS spreads of the companies; Deutsche Bank (DB), ENIspa [2] and Brent future call prices as of 24, June 2014. The reason is, we followed their affine factor decomposition methodology although with major diferences such as considering Variance Gamma process different from their normal inverse gaussian one and additionally using Variance Gamma Black-Cox structural model in the affine model context. Thus, for comparison and benchmarking purposes at the same time we use their dataset. Moreover, we give a parameter surface graph which shows the model's ability to fit a full dataset. For that purpose we use again CDS spreads for Deutsche Bank for December of 2012 due to data availability. We present the parameter surface in Figure 4.9.

The non-linear least squares method is a common method to find parameters of risk-neutral measure in that sense. As a benchmark market variable, we will use CDS spreads. This objective function is:

$$f(\theta, \nu, \sigma, ...) = \sqrt{\frac{\sum_{i=1}^{N}(CDS^i_{Market} - CDS^i_{Model})^2}{N}} \tag{4.154}$$

where N shows number of maturities in the term structure and CDS_{Model} is the CDS spread generated by the model, whereas CDS_{Market} is the market quotes for that.

To calibrate Brent futures market call prices we used Variance Gamma option price formula of equation (4.21). The calibration procedure is similar to above such that the objective function is:

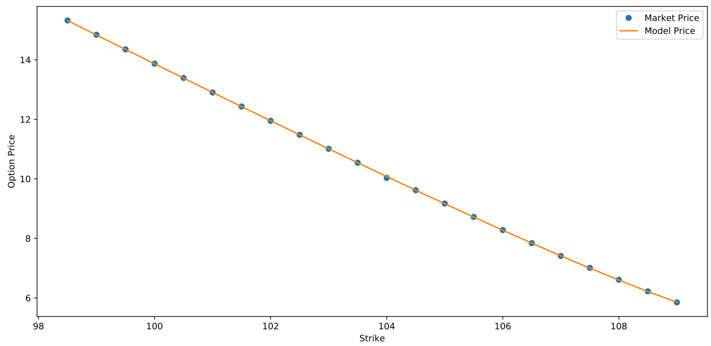
$$f(\theta, \nu, \sigma, ...) = \sqrt{\frac{\sum_{j=1}^{M}(Call^j_{Market} - Call^j_{Model})^2}{M}} \tag{4.155}$$

Here M shows number available calls for various strikes. We can see from Figure (4.8) and Table 4.1 above that the calibration produced almost perfect fit for call prices. The results turn out to be slightly better than our followed study [32].

[2] ENIspa is an Italian oil and gas company whereas Deutsche Bank is the largest bank in Germany operating globally.

Table 4.1: VG Parameter Estimates for Brent Futures Options

Option	Market Price	Model Price	Strike
1	15.32	15.317	98.5
2	14.84	14.832	99.0
3	14.35	14.348	99.5
4	13.87	13.865	100.0
5	13.39	13.384	100.5
6	12.90	12.904	101.0
7	12.43	12.427	101.5
8	11.95	11.952	102.0
9	11.48	11.479	102.5
10	11.01	11.009	103.0
11	10.54	10.542	103.5
12	10.03	10.079	104.0
13	9.62	9.619	104.5
14	9.17	9.165	105.0
15	8.72	8.717	105.5
16	8.28	8.274	106.0
17	7.84	7.834	106.5
18	7.41	7.414	107.0
19	7.01	7.00	107.5
20	6.61	6.599	108.0
21	6.22	6.215	108.5
22	5.85	5.858	109.0



Parameters: ν=2.99, σ=0.081, θ =0.04, r =0.0045, q =0.0032
Figure 4.8: Brent Futures Call Fit for VG Call Option Model

In order to implement non-linear least squares optimization we have switched between *Levenberg-Marquardt* and *Nelder-Mead* algorithm for a better fit results. As indicated and used in [35] *Nelder-Mead* is a derivative free optimization method and generally a preferred tool for calibration purposes whereas *Levenberg-Marquardt* is a widely used optimization algorithm for curve fitting. As given below, the parameter surface graph shows a fairly good fit to the CDS spread term structure.

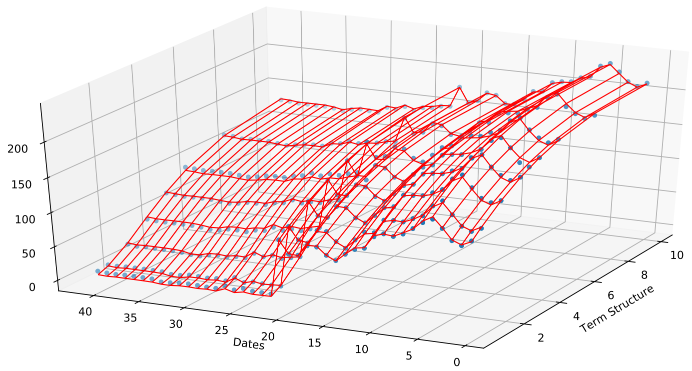
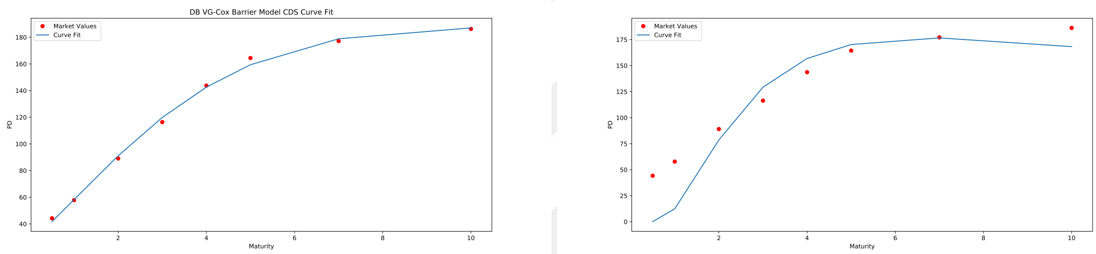


Figure 4.9: VG Black-Cox Model CDS calibration Surface

This fairly large parameter and prediction space is obtained by fitting the CDS, as mentioned before, to the data of Deutsche Bank (DB) as of December, 2012. The term structure we used encompasses 6m, 1y, 2y, 3y, 4y, 5y, 7y and 10y as data allows. This shows the flexibility of our variance gamma Black-Cox structural PD model and the success of *Levenberg-Marquardt* algorithm for curve fitting.

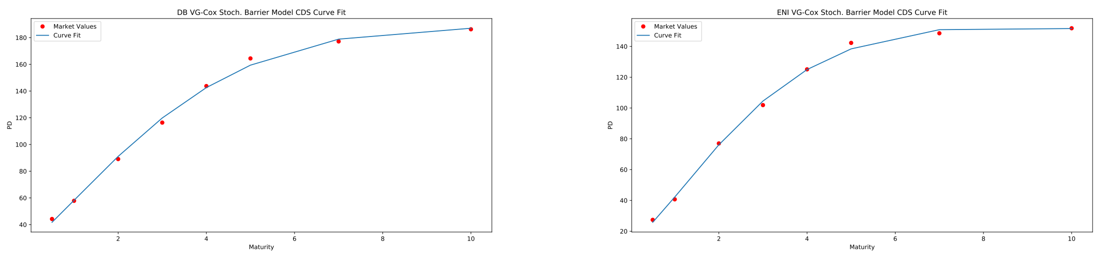
In order to make a comparison with our benchmark paper [32], we implemented another calibration using the same CDS term structure for at the date 26 June 2014 for Deutsche Bank and ENI. The term structure covers 6m, 1y, 2y, 3y, 4y, 5y, 7y and 10y. The parameters from this calibration will be used Counterparty Credit Risk calculation in CVA estimation section.

However, we also checked the fitting flexbility of the model using some of the data in [35]. We can see successful fits in both flat barrier model 4.10a and stochastic barrier model 4.11b below. The Table 4.2 confirms fairly good fit our barrier model numerically. In the last row, the table also shows unobserved implied asset value of the company. We see that, allowing the model to specify asset value internally produces a better RMSE as we have one more parameter to fit the curve. However, we observe that the difference is marginal as well.



(a) Parameters: ν=1.4,σ=0.22,θ =0.0045,r =0.032,q =0.021 (b) Parameters: θ=0.0167, σ=0.234

Figure 4.10: VG Black-Cox and BM Black-Cox Credit Risk Model CDS Fit



(a) Deutsche Bank (DB): ν=1.33,θ_A=-0.6529,θ_D=-0.6504 σ_A= 0.245,σ_D=0.416,ρ_{AD}=0.932 (b) ENI: ν=0.82,θ_A=-1.4836,θ_D=-1.4874σ_A= 0.457,σ_D= 0.644,ρ_{AD}=0.982

Figure 4.11: VG Stochastic Barrier Credit Risk Model Fit

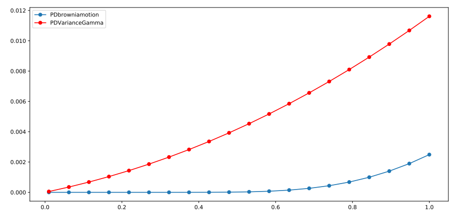
We check whether the model is also working better compared to its brownian motion counterpart, namely Black-Cox model, and we calibrated this model to the same data. We can see from 4.10b above that our variance gamma barrier model is superior than brownian motion barrier model as well due to larger parametric structure of Variance Gamma and its ability to generate plausible spreads even for very short-term.

Table 4.2: VG Barrier Credit Risk Model CDS Market Fit

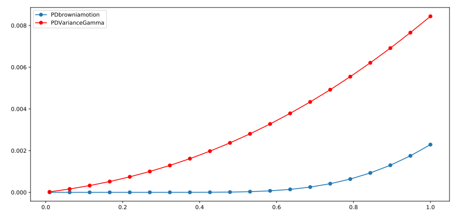
Term Str.	Market CDS Spread (DB)	Model CDS Spread	Term Str.	Market CDS Spread (ENI)	Model CDS Spread
6m	44.23	41.73	6m	27.39	25.78
1y	57.83	58.22	1y	40.71	42.20
2y	89.07	91.08	2y	77.00	76.12
3y	116.37	119.84	3y	101.89	104.52
4y	143.73	142.59	4y	125.11	124.98
5y	164.43	159.39	5y	142.28	138.40
7y	177.15	178.87	7y	148.53	150.85
10y	186.25	187.02	10y	151.79	151.60

Company	RMSEmerton %	Market Val. of Assets	RMSEnomerton %	Market Val. of Assets	Liability Barrier
DB	2.567537	83.55	2.584384	100.0	50.0
ENI	2.031100	64.92	2.133691	100.0	50.0

In terms of PD estimation, we use our pre-calibrated VG Black-Cox barrier type credit risk estimation to account for a sudden default event which is more realistic by definition [32, 35]. We can observe this from figures 4.12a and 4.12b below that there is a clear PD vanishing at very near short-terms for brownian motion Black-Cox model whereas variance gamma model displays byfar larger PD estimates which is usable in practice.



(a) PD for Deutsche Bank



(b) PD for ENISpa

Figure 4.12: VG Black-Cox & BM Black-Cox Short Term PD

We expect same results in spreads as well since we derive spreads from PD term structure. In order to calculate CVA, we used Brent Crude oil options prices as the derivative contract for the same date. The model architecture we have used is the affine factor convolution constrained variance gamma which accounts for default dependence between parties plus dependence of counterparty and underlier. This dependence is reflected in the ENI SpA which is an Italian multinational oil and gas company, the counterparty in the study and the Brent future option, the underlier.

4.12.1 Calibration of Variance Gamma Factor Components

Next issues are the calibration of affine factor model parameters $I(t)$ and $Z(t)$ and reconciling the equality in distribution of $X(t) = I(t) + cZ(t)$. In order to find the parameters of $I(t)$ and $Z(t)$ we have first matched the correlation (4.130) implied by variance gamma process to the empirical correlation matrix of related companies'. For our estimation purposes it's sufficient to use 3 companies' data. However it's possible to extend this procedure to many companies.

This helps us find the parameters $c_i, c_j, \theta_Z, \nu_Z, \sigma_Z$ as a first step of procedure. The last three of this set is the parameters of systematic component $Z(t)$. The empirical correlation calibration procedure will lead to set of equations below:

$$\rho_{12}\sqrt{(\theta_1^2+\sigma_1^2)(\theta_2^2+\sigma_2^2)} = c_1 c_2 \sqrt{\theta_Z^2 \nu_Z + \sigma_Z^2} \tag{4.156}$$

$$\rho_{13}\sqrt{(\theta_1^2+\sigma_1^2)(\theta_3^2+\sigma_3^2)} = c_1 c_3 \sqrt{\theta_Z^2 \nu_Z + \sigma_Z^2}$$

$$\rho_{23}\sqrt{(\theta_2^2+\sigma_2^2)(\theta_3^2+\sigma_3^2)} = c_2 c_3 \sqrt{\theta_Z^2 \nu_Z + \sigma_Z^2}$$

After solving this system of equations, we obtain 5 parameters mentioned above to be used for finding idiosyncratic component parameters. The solution is implemented using standard least squares and Broyden-Fletcher-Goldfarb-Shanno (BFGS) algorithm and Conjugate Gradient (CG) method interchangeably.

Here the parameters $\theta_i, \nu_i, \sigma_i$ correspond to ones calibrated to CDS spreads and option prices. The second step is, using (4.128) convolution constraints, to derive the parameters of idiosynratic process $I(t)$. The convolution constrains lead to the solutions for parameters of $I(t)$ for j^{th} company in the multivariate variance gamma setting :

$$\theta^j_{I(t)} = \sigma_{X(t)} - c_j \theta_{Z(t)} \tag{4.157}$$

$$\sigma^j_{I(t)} = \sqrt{\sigma^2_{X(t)} - c_j^2 \theta^2_{Z(t)}}$$

$$\nu^j_{I(t)} = \frac{\nu_{Z(t)} \nu_j}{\nu_{Z(t)} - \nu_j}$$

We see from table below that the empirical correlation matrix fit is very successful with RMSE=1e-6 which means (4.156) is solved. The rest of the parameters is derived according to (4.157).

Table 4.3: VG Idiosyncratic and Systematic Component Parameter Estimates

Company	$\hat{\rho}$	ρ	c_j	ν	ν_S	ν_Z,θ_Z,σ_Z	σ_S	θ_S
1	0.534878	0.534890	0.197950	1.295654	1.698915	5.458517	0.186767	-0.015145
2	0.218622	0.218627	0.320822	0.700210	0.803249	-0.019799	0.159392	-0.046399
3	0.360111	0.360111	0.071632	2.996702	6.644505	0.325164	0.077847	0.041472

4.12.2 Verification of Distributional Equality

We solve the problem of checking distributional equality that secures the sanity of factor decomposition; $X(t) = I(t) + cZ(t)$ via [15]'s COS method and convolution formula from probability theory. The details of the theory and implementation the COS method could be found both [15] and [24].

As well known, convolution of independent random variables has the result [7]

$$\mathbb{P}\Big((X = Y + Z) \leq x\Big) = \int_{\Omega_Z} \mathbb{P}\left(Y \leq x - Z \Big| Z = z\right) dP_Z(\omega)$$

Then the cdf of X can be written if $f(z)$, the density of Z, is defined as in [7]

$$F(x) = \int_{-\infty}^{x} F_y\left((x - z)\Big| Z(t) = z\right) f(z)dz$$

Then the density of x, $f(x)$, will be:

$$f(x) = \int_{-\infty}^{x} f_y\left(x - z\right) f(z)dz \tag{4.158}$$

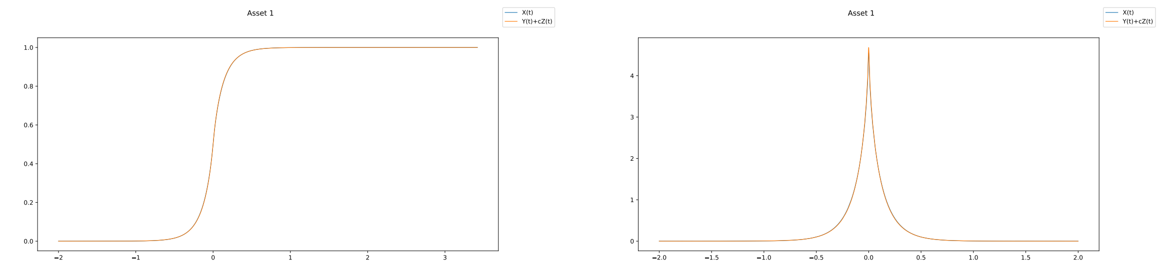
In our case where we define convolution $X = I + cZ$ we obtain

$$f(x) = \int_{-\infty}^{x} f_I\left(x - c_j z\right) f(z)dz \tag{4.159}$$

Therefore if we get equality of densities for all quantiles with a negligible numerical error we could be safe to use the affine factor decomposition. Particularly in terms of calculations involving probabilities. After calibrating models to data and using COS method and convolution we obtain quite satisfactory results for density equality under variance gamma convolution constraints for e.g. first asset of calibration data set.

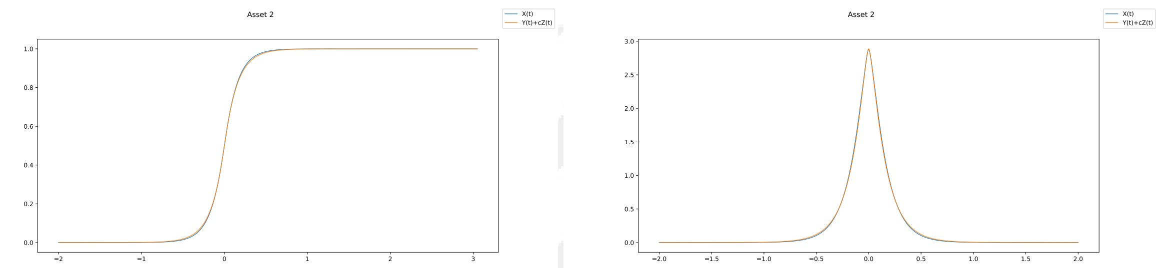
We used Wilcoxon signed-rank test which show whether two samples come from same distribution and Kolmogorov-Smirnov (K-S) test for whether two distributions are same by comparing empirical CDFs. The test results show robust P-value=0.844 for Wilcoxon and P-value=0.96 for K-S test. P-value=0.7 regarding asset 1 in our data set the same test results are Pvalue=0.7 for Wilcoxon and P-value=0.8 for K-S test.

In figures (4.15a, (4.15b)) below, we see almost perfect quantile-quantile match for asset 1 and asset 2 in the dataset we use. Therefore, we soundly see the equality in distribution is satisfied for asset 2 and asset 1 with their affine factor decompositions. For calculations involving convolution method we used closed form densities in equation (4.24). For COS method we used variance gamma characteristic function in equation (4.13).



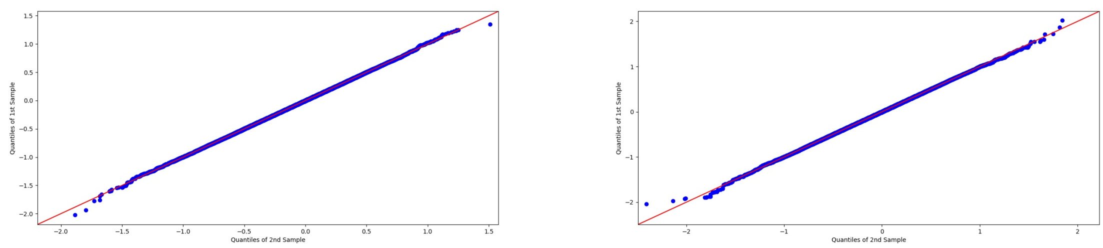
(a) VG Cos Method CDF Asset 1 (b) VG Cos Method PDF Asset 1

Figure 4.13: VG Affine Factor Model Convolution Asset 1



(a) VG Convolution Method CDF Asset 1 (b) VG Convolution Method PDF Asset 2

Figure 4.14: VG Affine Factor Model Convolution Asset 2



(a) VG QQ-Plot Asset2: I(t)+cS(t) vs X(t); Wilcoxon, K-S Tests P-val=0.844,0.96 (b) VG QQ-Plot Asset1: I(t)+cS(t) vs X(t); Wilcoxon, K-S Tests P-val=0.7,0.8

Figure 4.15: VG Convoluted Processes QQ-Plots

CHAPTER 5

COUNTERPARTY CREDIT RISK ESTIMATION USING LEVY PROCESSES

5.1 Introduction

Thus far we modify and study a variety of models in terms of PD calculation both in stochastic volatility subordinated models and subordinated infinite activity pure jump models. We define and work out Counterparty Credit Risk in detail at first chapter. As mentioned, it requires significant computation effort and credit risk estimation of both counterparties plus wrong way risk to be taken into consideration. This chapter is constructed to study and apply proposed methods in previous chapters to estimate key Counterparty Credit Risk measure CVA, wrong way risk. The latter is important to consider due to possible dependence between the underlying process of the derivative contract and the credit risk of the counterparty. As we explain in previous chapters on the structrural credit risk models, our extended affine factor methodology of [32, 33] enables us to model the link, asset value dependence between the counterparties of the derivative transaction and the derivative itself.

Therefore, a detailed estimation methodology concerning the Counterparty Credit Risk and calibration of the proposed models with the market price and default data, such as CDS spreads, is key to estimate the risk measure. In this chapter we plan to set-out possible methods to implement these targets.

5.2 CVA Estimation and Application in Affine Factor Levy Models

The general definition and formula for CVA is given in (2.6). Given this methology we can first write a general formula for the Levy models we study so far. Therefore we can start with variance gamma process. As first defined in (2.6) we can recall it as follows:

$$CVA(t) = \int_t^T \Big((1-R)e^{-\int_s^T r(u)du}V_+(T)\Big)\, dPD_s \tag{5.1}$$

We compactly write

$$CVA(t) = (1-R)\int_t^T V(s)dPD_s \tag{5.2}$$

Here $V(s)$ is the value of the derivative at time of the valuation and this extends until the maturity due to the definition of CVA.

$$CVA(t)_1 = (1-R)\sum_{j=1}^{N}(1-PD^1_{t_j})1 - PD^2_{t_j}V_{t_j} \tag{5.3}$$

$$CVA(t)_2 = (1-R)\sum_{j=1}^{N}PD^1_{t_j}(1-PD^2_{t_j})V_{t_j} \tag{5.4}$$

where the subscripts show the CVA of both parties in the transaction.

Hence, for each discrete time point we will have to re-evaluate the derivative price and $PD_j,\ PD_k$, for the $j^{th},\ k^{th}$ company. More elaborately and considering the Levy processes we have studied do far containing the affine factor framework proposed by [32] we write:

$$CVA(t)_j = (1-R)\mathbb{E}(\mathbb{1}_{\{A_{jt}e^{(r-\phi(-i)_X)\tau+Y_j(t)+c_jZ(t)>L_j}\}}\mathbb{1}_{\{A_{kt}e^{(r-\phi(-i)_X)\tau+Y_k(t)+c_kZ(t)<L_k}\}}\times$$
$$D(\tau)V(S(\tau)^+)) \tag{5.5}$$

$$CVA(t)_k = (1-R)\mathbb{E}(\mathbb{1}_{\{A_{jt}e^{(r-\phi(-i)_X)\tau+Y_j(t)+c_jZ(t)<L_j}\}}\mathbb{1}_{\{A_{kt}e^{(r-\phi(-i)_X)\tau+Y_k(t)+c_kZ(t)>L_k}\}}\times$$
$$D(\tau)V(S(\tau)^+)) \tag{5.6}$$

The important distinction here is the connection between counterparties and the derivative underlier through common term $Z(t)$ which corresponds to systematic risk factor. It could be observed that this is a general framework and it's possible to write spefi-cic verisons of it for the Levy processes we have used so far. Starting with variance gamma environment we can materially write the CVA:

$$CVA(t)_j = (1-R)\sum_{i=1}^{T} F_{VG}(Z; A_{jt}, L_j, \sigma_j, \theta_j, \tau_i/\nu_j, \nu_j, c_j)\times \quad (5.7)$$
$$\Big(1 - F_{VG}(Z; A_{kt}, L_k, \sigma_k, \theta_k, \tau_i/\nu_k, \nu_k, c_k\Big) \times V(S_t, K, r, \tau_i, \sigma_u, c_u, \theta_u)$$

$$CVA(t)_k = (1-R)\sum_{i=1}^{T} F_{VG}(Z; A_{jt}, L_j, \sigma_j, \theta_j, \tau_i/\nu_j, \nu_j, c_j)\times \quad (5.8)$$
$$\Big(1 - F_{VG}(Z; A_{kt}, L_k, \sigma_k, \theta_k, \tau_i/\nu_k, \nu_k, c_k\Big) \times V(S_t, K, r, \tau_i, \sigma_u, c_u, \theta_u)$$

Using our factor based variance gamma option pricing formula (4.33) we can rewrite CVA in terms of it as follows:

$$CVA(t)_j = (1-R)\int_{-\infty}^{\infty} F_{VGj}(d_{2j}(z))\Big(1 - F_{VGk}(d_{2k}(z))\Big) f_Z(z) V(\tau, z) dz \quad (5.9)$$

$$CVA(t)_k = (1-R)\int_{-\infty}^{\infty} \Big(1 - F_{VGj}(d_{2j}(z))\Big) F_{VGk}(d_{2k}(z) f_Z(z) V(\tau, z) dz \quad (5.10)$$

Here the last term $V(\tau, z)$ is the derivative transaction for which the two counterparties are in deal. These integrals could be calculated using numerical quadrature. However due to the complex nature of variance gamma density, where in practice the PD term F_{VG} also has another integral inside as given in (4.19). Thus, it's more practical and safer to use inversion of characteristic function for recovering density. For this purpose it's possible to use in particular the COS method of [15] to calculate the densities and integrals defined at (5.9) and (5.10).

For a plain vanilla option or a forward option contract we can derive a semi-closed form formula using characteristic functions as follows:

$$CVA(t)_j = (1-R)\left(1 - \int_{-\infty}^{\infty} Re\left(\frac{e^{-iu\log(\frac{L}{A_{jt}})}\phi_{VG_j}}{iu} du\right)\right)\times \quad (5.11)$$
$$\left(1 - \int_{-\infty}^{\infty} Re\left(\frac{e^{-iu\log(\frac{L}{A_{kt}})}\phi_{VG_k}}{iu} du\right)\right) \times C(S_t, K, r, q, \sigma, \theta, \nu, \tau)$$

This will be compactly written as a product of two PDs and the call option price:

$$CVA(t)_j = (1-R) \times (1 - PD_j) \times PD_k \times C\Big(S_t, K, r, q, \sigma, \theta, \nu, c, \tau\Big)$$
$$CVA(t)_k = (1-R) \times (1 - PD_k) \times PD_j \times C\Big(S_t, K, r, q, \sigma, \theta, \nu, c, \tau\Big)$$

The derivation of these final formulas involves dealing with characteristic functions and some algebric arrangements, thus we give the detailed derivation in Appendix C. In Heston/Bates environment the CVA will be similarly:

$$CVA(t)_j = (1-R)\sum_{i=1}^{T}\left[\frac{1}{2}+\frac{1}{\pi}\int_0^\infty Re\left(\frac{e^{-iu\ln(K)}f(\log A_{jt}, v_t, \zeta_t, \tau_l, c_j, u)}{iu}du\right)\right] \times\left(1-\left[\frac{1}{2}+\frac{1}{\pi}\int_0^\infty Re\left(\frac{e^{-iu\ln(K)}f(\log A_{kt}, v_t, \zeta_t, \tau_l, c_k, u)}{iu}du\right)\right]\right)\times V(S_t, K, r, \tau_l, \sigma_u, c_u, \theta_u) \tag{5.12}$$

$$CVA(t)_k = (1-R)\sum_{l=1}^{T}\left[\frac{1}{2}+\frac{1}{\pi}\int_0^\infty Re\left(\frac{e^{-iu\ln(K)}f(\log A_{jt}, v_t, \zeta_t, \tau_l, c_j, u)}{iu}du\right)\right] \times\left(1-\left[\frac{1}{2}+\frac{1}{\pi}\int_0^\infty Re\left(\frac{e^{-iu\ln(K)}f(\log A_{kt}, v_t, \zeta_t, \tau_l, c_k, u)}{iu}du\right)\right]\right)\times V(S_t, K, r, \tau_l, \sigma_u, c_u, \theta_u) \tag{5.13}$$

where f is the density stated in (4.96). We can see an example of an IRS swap CVA with our variance gamma barrier model:

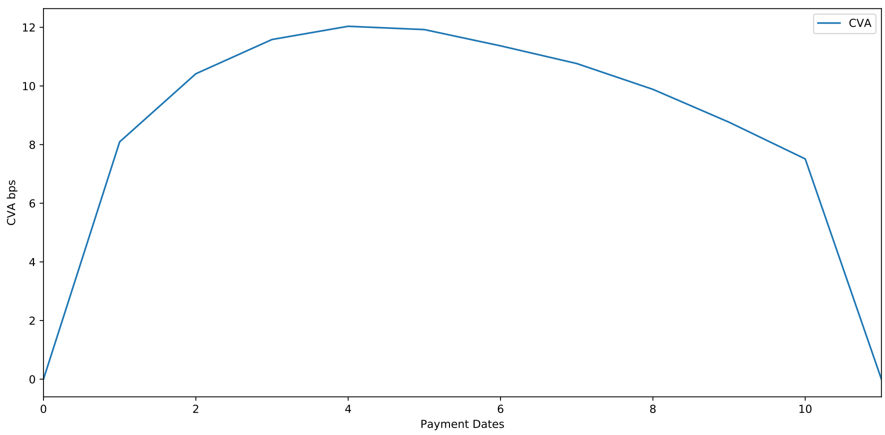


Figure 5.1: An IRS Swap CVA Example

The PD is calculated using Variance Gamma barrier as we mentioned and the simulation is implemented using the generic Counterparty Credit Risk algorithm we set-out in first chapter. For that purpose we calibrated CIR model to 1 year FED treasury bond rate series for necessary interest rate simulations. The calibration is done with our analytic derivation formula we present at Appendix B.

Additionally we have figure below an example of CVA calculations regarding variance gamma model and affine factor augmented Bates model we have derived in this study.

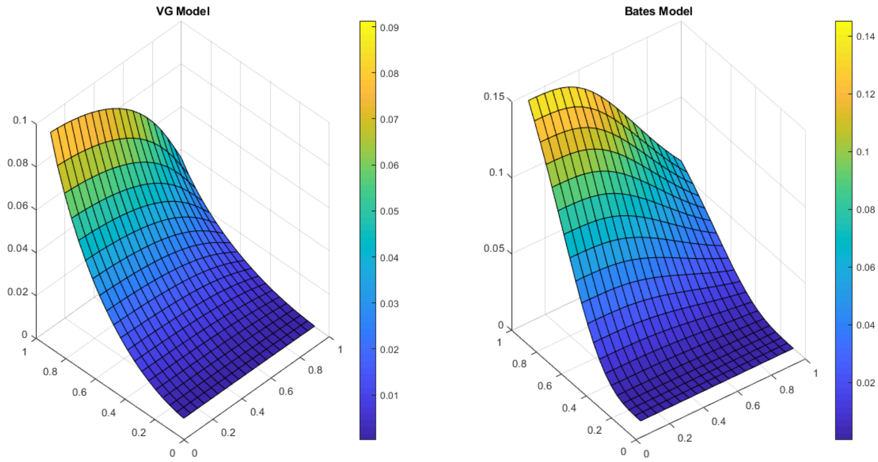


Figure 5.2: CVA for Variance Gamma and Bates Stochastic Volatility Models

As we can see our augmented Bates model displays a more steep CVA rise as liability level is increased. However, variance gamma model shows a less vanishing of CVA compared to augmented Bates model in lower risk liability levels. This is possibly higher PD calculated by using Bates model as the default model. It is worth to note that in Figure 4.3 the similar results is observed as well that the Bates model overestimates the default probability for even low risk area.

5.3 CVA VaR and Expected Shortfall (ES)

As it's well-known from Basel-II and Basel-III regulators require capital to be allocated for unexpected lossed from market risk via the Value-at-Risk (VaR) measure in terms of model based risk measurement. We recall that VaR is used to measure maximum loss at a given confidence level over a given time horizon. In mathematical terms:

$$VaR(X) = \sup \left\{ X \leq F(1-\alpha)^{-1} \right\}$$

The term VaR of CVA refers to the potential unexpected losses that a bank could face is case of a deterioration of the credit quality of its counterparty, at a given confidence level $(1-\alpha)$ and over given horizon. In Basel-III the VaR of CVA is constrained to be linked to changes in counterparty credit spreads not to any possible market variables such as exchange rate, interest rates, commodities or any other reference market variable[33]. However, the correct specification of VaR of any risk measure requires multivariate distribution of risk factors and VaR. Therefore, counterparty credit spread and the key risk factors in the pricing of the derivative contract should be taken into consideration during the calculation of CVA VaR. The regulatory VaR calculation in terms of CVA could be calculated with the same methodology of market risk VaR, which requires the emprirical or theoretical joint distribution of CVA, market risk factors and credit spread of the counterparties.

After full distribution of these risk factors are obtained we can calculate VaR or ES. These factors to be simulated are based on real world measure as the scenarios are subject to real world distribution. After these scenarios are calculated many times, M times, we will be ready to apply the scenarios for derivative price calculation. This requires the simulation based calculation of CVA to be repeated M times to get a CVA distribution of which VaR is to be calculated.

We see an example in figure below for an empirical distribution of CVA. In this distribution CVA-VaR and CVA-ES for an Interest Rate (IRS) swap are calculated. Since this exercise is based on interest rate scenarios, we use our CIR process analytical calibration formula derived in Appendix B to simulate interest rate scenarios.

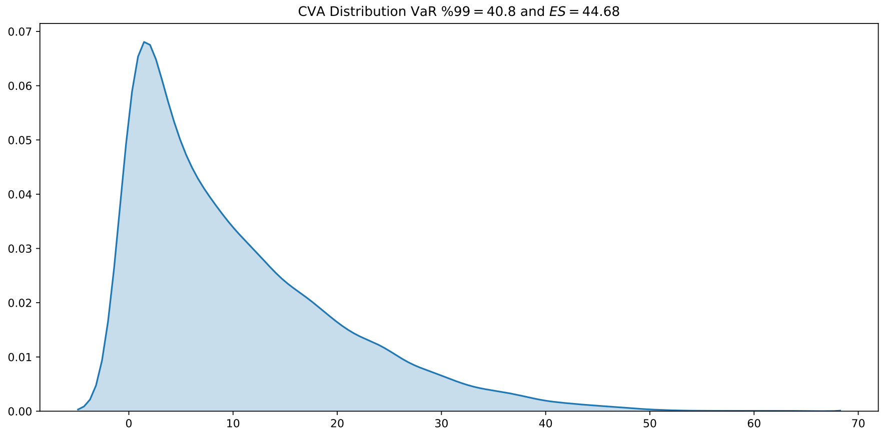


Figure 5.3: Empirical CVA Distribution of an IRS Swap

The figure displays a loss distribution as the CVA is a expected loss. The benefit of this empirical distrinbution is that it enables us to calculate risk measures such as VaR and ES. In this hypothetical example we calculate the VaR($\%99$)=40.8 whereas the more strict measure which is suggested by Basel-III, ES($\%99$) is calculated to be 44.68.

CHAPTER 6

CONCLUSION

In this study we aimed to find out some formulas enabling efficient simulation of Counterparty Credit Risk in the context of CVA and EPE. The results we reached shows some absolute efficiency. However, this efficiency could be improved by complementing it with other efficiency methods found in the literature such as quantization and brownian motion local time methods. Naturally, these efficiency methodologies are in the form of analytical methodologies and algorithmic type not a hardware aid is used, the efficiency of this type could be used for further optimization. We plan to study these additional methodologies in further studies.

Our second purpose is to find a Credit Risk model of structural type flexible in terms of PD calculation and calibration to CDS spreads. Since this is an implicit option model, it's also necessary for the model to satisfy the market features such as skew, smile and fat tail property of log returns, which is a criticized drawback of Black-Scholes model.

For these reasons we propose Levy processes with jumps, particularly subordinated processes which are said to be promising in many studies such as [3, 36] and many others. Among those we choose variance gamma process for its mathematical tractability and flexibility. We observe that variance gamma process is quite sucessful to satisfy volatility smile, skew and fat tail in log equity distributions. In Variance Gamma based credit risk environment we re-derive Variance Gamma option price of [36] particular to our purpose of deriving a default dependent structural framework. We see that this model fits market option price quotes perfectly. Thus, we confirm the accurateness of our reformulation in addition to replicating results in literature. We then extend this alternative derivation method to linear factor decomposition allowing default dependence between counterparties.

Moreover in credit risk context, we develop another model of Variance Gamma process with barrier feature. Although estimations of credit risk in a barrier context have been done for some time, they have been mostly implemented via numerical methods such as solutions of PIDE, Wiener-Hopf factorization or Monte Carlo simulations of barrier type. For this purpose we make a slight change on Black-Cox barrier model by introducing time change through Variance Gamma process. After model construction and testing for various parametric cases, we calibrated it to the real CDS data. We observe that it successfully fits to the CDS spread even slighly better than its normal inverse gaussian counterpart compared to study of [32] and remarkably better than Brownian Motion based Black-Cox barrier model. Since our main purpose was to use a flexible model for Counterparty Credit Risk measures, we wanted to transform the Variance Gamma barrier model into a type that can integrate default dependence of counterparties or dependence between PD and underlying asset.

For achieving this purpose we follow [32, 6] affine linear factor decomposition which could be used for the purposes above. We saw that this calibrates to empirical correlation structure very well and the distriubutional equality between original process and its convolution by the factors match almost equally in distribution. Then we presented an example calculation of CVA calculation carried out by this model in hand. In terms of variance gamma based credit risk models, we additionally developed a modified Merton model for this process having features similar to original Merton model. We see that it works quite accurate and fast.

In addition to Variance Gamma based models, we continue to construct models of stochastic volatility time change case. We see that these models catch volatiliy skew and smile as well. We integrated default dependence and dependence between processes through affine factor decomposition of stochastic volatiliy time change. The integration was done through a modification of Bates model and is brand new in that sense.

This model was also seen to integrate dependence between counterparties successfully and could also be used in Counterparty Credit Risk calculation. However calibration is a bit challenge. As a variant of this model, we add a liquidity adjustment factor to see whether a liquidity drought could impose a change in PD and a rise in implied spread by the model. We saw that this absolutely shows expected changes in spreads and PD when there is liquidity drought or liquidity abundance.

Our last purpose is to integrate the state of economy to the CVA calculation. The economic crisis, recession or growth states could implicitly effect the PD of counterparties or risk factors e.g. interest rate, exchange rate scenarios. Therefore, these factors should have been integrated to the Counterparty Credit Risk calculations.

For that reason we develop model through modifying [28] study which was an application of Vasicek credit risk model [52] into the stress testing. Our modification is to introduce variance gamma process this model. Since this model contained normally distributed random variables, we expected that our new model could be better in fitting with real nature of market data and economic state. Our stress testing model of variance gamma Vasicek type is also a good candidate for integrating stress testing to CVA calculations. Moreover, we also conclude that our affine factor augmented stochastic volatiliy model could be used for these purposes and for pricing purpose.

Finally, we conclude that our variance gamma Black-Cox barrier model could be safely used in any purpose of structural credit risk calculations and fitting to credit spread term structure.

For further research, we plan to extend brownian motion local time approach of [9] and possibly the quantization approach of [39] to our efficient CVA calculation framework. Moreover, we aim to extend our efficient Counterparty Credit Risk calculation setup for the cases bias introduced by SDE simulation methods of Euler-Maruyama and Milstein.

Lastly, we aim to improve our Levy stochastic volatility models in the context of affine factor decomposition and find approximations for factorizations related to First Passage Time of Levy subordinated processes. Extending these processes to the stress testing framework as we do for variance gamma process will be our another further project.

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APPENDIX A

DERIVATION OF MAXIMUM OF BROWNIAN MOTION SIMULATION

The simulation of maximum of Brownian Motion is not a trivial process and is dependent on simulation of a Brownian Motion path. The derivation here straihgtforward and useful, we used this procedure for deriving minimum of variance gamma as well in the study.

After using conditional distribution maximum of Brownian Motion and using Brownian Motion path as given, we can simulate the maximum. We can start by writing conditional distribution of Brownian Motion and its maximum: As given in [49] we can write

$$
\begin{aligned}
f_{M(t)|W(t)}(m|w) &= \frac{f_{M(t),W(t)}(m,w)}{f_{W(t)}(w)} \\
&= \frac{2(2m-w)}{t\sqrt{2\pi t}}\sqrt{2\pi t}e^{-\frac{(2m-w)^2}{2t}+\frac{w^2}{2t}} \\
&= \frac{2(2m-w)}{t}e^{\frac{-2m(m-w)}{t}}
\end{aligned}
$$

Using this conditional density we can derive the maximum process:

$$
P(M_t \leq m|W(t)=w)\int_w^m = \frac{2(2u-w)}{t}e^{\frac{-2u(u-w)}{t}}du = 1 - e^{\frac{-2m(m-w)}{t}}
$$

Then given that any CDF is uniformly distributed we can start by simulating uniform distribution and continue by inverse transformation method:

$$
\begin{aligned}
1 - e^{\frac{-2m(m-w)}{t}} &= U \\
\frac{-2m^2 + 2mw}{t} &= 1 - \ln(U) \\
2m^2 - 2mw + t(1 - \ln(U)) &= 2m^2 - 2mw + C = 0 \\
C &= t(1 - \ln(U))
\end{aligned}
$$

This is standart quadratic function then it is trivial to find the roots

$$
\begin{aligned}
\Delta &= 4w^2 - 8C \\
m_{1,2} &= \frac{2w \pm \sqrt{\Delta}}{4}
\end{aligned}
\tag{A.1}
$$

Then using the positive root from this final formula we can easily obtain maximum of Brownian Motion. This exact solution for the roots of maximum process will help us simulate maximum of Brownian Motion without using any numerical root finding method. The summary of the method is

(i) Simulate a Brownian Motion Path

(ii) Simulate uniform random variable $U(0, 1)$

(iii) Then use A.1 to find the roots m_1 & m_2

(iv) Select the postive root as the simulated maximum value

APPENDIX B

DERIVATION FOR ANALYTIC CALIBRATION OF COX-INGERSOLL-ROSS (CIR) SDE

Although CIR process has transition density of non-central χ^2 distribution and we can write log-likelihood based on that, we can not analytically derive the optimal set of parameters that maximize the log-likelihood. Instead we can use an approximation methodology where we can set-up an Euler discretization $t_n - t_{n-1} = \Delta t$ and given this small grid; the discretized CIR process will be:

$$dY_t = a(b - Y_{t-1})\Delta t + \sigma\sqrt{Y_{t-1}\Delta t}\epsilon$$

The transition density of CIR process becomes:

$$p(S_t|S_{t-1}) = \frac{1}{\sqrt{2\pi\sigma^2\Delta t Y_{t-1}}} e^{\frac{-(Y_t-(Y_{t-1}(1-a)+ab))^2}{2\sigma^2\Delta t Y_{t-1}}}$$

$N(Y_{t-1}(1 - a\Delta t) + ab\Delta t, \sigma^2\Delta t Y_{t-1})$. Then given that Gaussian transition density, we can find parameters which optimize the log-likelihood. The log-likelihood can be written as:

$$L = \log L(a, b, \sigma, Y_{t-1}) = -(T-1)\ln\sigma - \frac{1}{2}\sum_{t=2}^{T} Y_{t-1} - \frac{(T-1)}{2}\ln 2\pi - \frac{1}{2}\sum_{t=1}^{T}\frac{(Y_t - \mu)^2}{\sigma^2 Y_{t-1}}$$

$$\mu = Y_{t-1}(1 - a) + ab$$

Therefore, first step is to obtain first order condition(FOC) for parameter b:

$$\begin{aligned}\frac{\partial L}{\partial b} &= \sum_{t=2}^{T}\frac{Y_t - \mu}{\sigma^2 Y_{t-1}} a \\ &= a\sum_{t=2}^{T}\frac{Y_t - Y_{t-1}}{\sigma^2 Y_{t-1}\Delta t} + \frac{a^2\Delta t(T-1)}{\sigma^2\Delta t} - a^2 b = 0\end{aligned}$$

Using FOC we will have:

$$a = \frac{\sum_{t=2}^{T} \frac{Y_t - Y_{t-1}}{Y_{t-1}\Delta t}}{1 - T + \sum_{t=2}^{T} \frac{b}{Y_{t-1}}}$$

We can parametrize this for further use:

$$a = \frac{A}{\Delta t(B + bC)}$$

Then FOC for parameter a will be:

$$\frac{\partial L}{\partial a} = \sum_{t=2}^{T} \frac{(Y_t - \mu)(Y_{t-1} - b)}{\sigma^2 \Delta t Y_{t-1}} = 0$$

This will be:

$$\sum_{t=2}^{T}\left(\frac{Y_t - Y_{t-1} + a\Delta t Y_{t-1} - ab\Delta t}{\sigma^2 \Delta t} + \frac{b(1 - a\Delta t)(T-1)}{\sigma^2 \Delta t}\right) = b\sum_{t=2}^{T}(\frac{Y_t}{\sigma^2 \Delta t Y_{t-1}})$$

$$= (Y_T - Y_1) + a\sum_{t=2}^{T}(Y_{t-1}) + 2abB - bB + ab^2C = bF$$

$$= D + aE + 2abB - bB + ab^2C = bF$$

$$= D + \frac{A}{\Delta t(B + bC)}(E + 2bB + b^2C) - bB = bF$$

$$F = \sum_{t=2}^{T}(\frac{Y_t}{\sigma^2 \Delta t Y_{t-1}})$$

Then we can further simplify:

$$b(CD\Delta t + 2AB\Delta t - B^2\Delta t - BF\Delta t) + b^2(A\Delta t - BC\Delta t - FC\Delta t) + G = b^2\tilde{A} + b\tilde{B} + G = 0$$

Here we can write $\tilde{A}$

$$\tilde{A} = \Delta t C\left(\sum_{t=2}^{T}(\frac{Y_t}{Y_{t-1}} - 1) - (1 - T + \sum_{t=2}^{T}\frac{Y_t}{Y_{t-1}})\right)$$

Using the fact that $\sum_{t=2}^{T} 1 = T - 1$

$$\tilde{A} = \Delta t C\left(\sum_{t=2}^{T}(\frac{Y_t}{Y_{t-1}} - 1) - (\sum_{t=2}^{T}\frac{Y_t}{Y_{t-1}} - 1)\right) = 0$$

Since $\tilde{A}$ is literally 0 we have a linear equation:

$$b\tilde{B} + G = 0$$

whose solution

$$b = -\frac{G}{\tilde{B}} \tag{B.1}$$

Using this result for b we will have:

$$a = \frac{A\tilde{B}}{\Delta t(B\tilde{B} - GC)} \tag{B.2}$$

where

$$\tilde{B} = (CD\Delta t + 2AB\Delta t - B^2\Delta t - BF\Delta t)$$

Then FOC for parameter σ will be:

$$\frac{\partial L}{\partial \sigma} = -\frac{(T-1)}{\sigma} + \sum_{t=2}^{T} \frac{(Y_t - \mu)^2 2\sigma Y_{t-1}}{\sigma^4 Y_{t-1}^2} = 0$$

This will finally give us:

$$\sigma^2 = \frac{1}{T-1} \sum_{y=2}^{T} \frac{(Y_t - \mu)^2}{\Delta t Y_{t-1}}$$

where μ can be calculated using (B.1)–(B.2) and (B.1).

APPENDIX C

PROOF OF LEMMA 3.2

The moment generating function (MGF) of CIR process can be written as given in [49]:

$$\mathbb{E}(e^{uX(t)}) = \left(\frac{1}{1-2v(t)u}\right)^{\frac{2a}{\sigma^2}} e^{\left(\frac{e^{-bt}uX(0)}{1-2v(t)u}\right)} \tag{C.1}$$

The $v(t)$ term is given as:

$$v(t) = \frac{\sigma^2}{4b}\left(1-e^{-bt}\right) \tag{C.2}$$

moreover, $a_k, b, c \in R$ and $\sigma \in R^+$. As given in Section 3 the $Y(t)$ and $Z(t)$ are ICIR processes an were given by:

$$\begin{aligned} Z(t) &= \int_0^\infty z(s)ds \\ Y(t) &= \int_0^\infty y(s)ds \\ dy_t &= (a_1 - by_t)dt + \sigma_v\sqrt{y(t)}dW_t^y \\ dz_t &= (a_2 - bz(t))dt + \sigma_v\sqrt{z(t)}dW_t^z \end{aligned}$$

Here the coefficients are different from 4.47 and their equivalents are $a_{1,2} = \kappa\beta_{1,2}$ and $b = \kappa$, $\sigma_v = \sigma_v$ here.

The linear combination $X(t) = Y(t) + cZ(t)$ can be written as:

$$X(t) = \int_0^\infty y(s) + cz(s)ds$$

To prove that $X(t)$ is a ICIR process, we have to prove that $y(t) + cz(t)$ is also a CIR process. For that purpose we will use MGF of CIR process since the equality in distributions will require same MGF. Thus, we write the MGF of linear combination below, starting by $Y(t)$. Since $Y(t)$ and $Z(t)$ are independent by assumption, we can separately write their MGFs

$$\mathbb{E}(e^{uY(t)}) = \left(\frac{1}{1-2v(t)u}\right)^{\frac{2a_1}{\sigma_v^2}} e^{\left(\frac{e^{-bt}u(Y(0))}{1-2v(t)u}\right)}.$$

Here $Z(t)$ process is linearly transformed to $cZ(t)$ and we can proceed by writing the MGF for $cZ(t)$ as

$$\mathbb{E}(e^{ucZ(t)}) = \left(\frac{1}{1-2v(t)u}\right)^{\frac{2ca_2}{\sigma_v^2}} e^{\left(\frac{e^{-bt}u(cZ(0))}{1-2v(t)u}\right)}.$$

This yields the joint MGF as

$$\mathbb{E}(e^{u(Y(t)+cZ(t))}) = \left(\frac{1}{1-2v(t)u}\right)^{\frac{2(a_1+ca_2)}{\sigma_v^2}} e^{\left(\frac{e^{-bt}u(Y(0)+cZ(0))}{1-2v(t)u}\right)}. \qquad \text{(C.3)}$$

As we can see the MGF of affine model matches general CIR MGF form as long as the volatility and adjustment coefficient of CIR processes $y(t)$ and $z(t)$ are the same. Thus, satisfying this constraint and using the structure of MGF we can produce a convoluted ICIR process and it will have the SDE

$$dx_t = d(y(t)+cz(t)) = (a_1 + ca_2 - b(y(t)+cz(t)))dt + \sigma_v\sqrt{y(t)+cz(t)}dW_t^z$$

More compactly we can write:

$$dx_t = (a_x - bx(t))dt + \sigma_v\sqrt{x(t)}dW_t^z$$

We can use then charateristic function for ICIR suited for the affine transformation. The characteristic function will be

$$\begin{aligned}\varphi(u,\tau) &= A(u,\tau)e^{B(u,\tau)X(0)} \qquad \text{(C.4)}\\ &= A(u,\tau)e^{B(u,\tau)[Y(0)+cZ(0)]}.\end{aligned}$$

APPENDIX D

VARIANCE GAMMA AND NORMAL INVERSE GAUSSIAN CONVOLUTION DERIVATION

As stated in [40] and [45] if certain conditions are satisfied, it is possible to have a variance gamma process after summing i.i.d variance gamma processes, which is variance gamma process is closed under convolution under some required conditions [45].

We can start with presenting Laplace transform [45] of variance gamma process as it is very useful to show independence, convolution etc.

$$L_{VG}(z) = e^{\mu z}\left(\frac{\alpha^2-\beta^2}{\alpha^2-(\beta+z)^2}\right)^{\lambda} \tag{D.1}$$

Then keeping the parameters α and β fixed it's possible to arrive another variance gamma process with Laplace transform [45]:

$$L_{VG}(z) = L_{VG_1}(z)L_{VG_2}(z) = e^{\mu_1 z}\left(\frac{\alpha^2-\beta^2}{\alpha^2-(\beta+z)^2}\right)^{\lambda_1} e^{\mu_2 z}\left(\frac{\alpha^2-\beta^2}{\alpha^2-(\beta+z)^2}\right)^{\lambda_2}$$
$$L_{VG}(z) = e^{(\mu_1+\mu_2)z}\left(\frac{\alpha^2-\beta^2}{\alpha^2-(\beta+z)^2}\right)^{(\lambda_1+\lambda_2)}$$

Therefore we have another variance gamma process with new parameters still the same α and β

$$VG(\mu_1+\mu_2, \alpha, \beta, \lambda_1+\lambda_2)$$

If we re-parametrize the variance gamma as in [51] so that it is harmonius with our regular notation we have:

$$\alpha = \frac{\sqrt{\theta^2 + \frac{2\sigma^2}{\nu}}}{\sigma^2}$$
$$\beta = \frac{\theta}{\sigma^2}$$
$$\lambda = \frac{1}{\nu}$$

Assume that our two iid variance gamma processes are:

$$VG_1 = \theta_1\gamma + \sigma_1 W(\gamma)$$
$$VG_2 = \theta_2\gamma + \sigma_2 W(\gamma)$$

For two iid variance gamma process the parameters we have used for Laplace transform will be:

$$\alpha_1 = \frac{\sqrt{\theta_1^2 + \frac{2\sigma_1^2}{\nu_1}}}{\sigma_1^2}, \quad \beta_1 = \frac{\theta_1}{\sigma_1^2}$$
$$\alpha_2 = \frac{\sqrt{\theta_2^2 + \frac{2\sigma_2^2}{\nu_2}}}{\sigma_2^2}, \quad \beta_2 = \frac{\theta_2}{\sigma_2^2}$$

Then the sum could be in the form:

$$VG_1 + VG_2 = VG = (\theta_1 + \theta_2)\gamma + \sqrt{(\sigma_1^2 + \sigma_2^2)}W(\gamma) \tag{D.2}$$

Then the α and β will be:

$$\alpha = \frac{\sqrt{(\theta_1 + \theta_2)^2 + \frac{2\sigma_1^2 + \sigma_2^2}{\nu}}}{\sigma_1^2 + \sigma_2^2} \tag{D.3}$$
$$\beta = \frac{\theta_1 + \theta_2}{\sigma_1^2 + \sigma_2^2}$$
$$\lambda = \frac{1}{\nu_1} + \frac{1}{\nu_2} = \frac{\nu_1 + \nu_2}{\nu_1\nu_2}$$

In order to have this same α and β we must have

$$\frac{\theta_1 + \theta_2}{\sigma_1^2 + \sigma_2^2} = \frac{\theta_2}{\sigma_2^2} = \frac{\theta_1}{\sigma_1^2} = C_1 \tag{D.4}$$

Then

$$\theta_2 = C_1\sigma_2^2$$
$$\theta_1 = C_1\sigma_1^2$$

As a result

$$\frac{C(\sigma_1^2+\sigma_2^2)}{\sigma_1^2+\sigma_2^2} = C_1 \tag{D.5}$$

This satisfies the β equivalence, for α equivalence using α_1 and α_2 we can write:

$$\begin{aligned} \sigma_1^2\nu_1 &= C_2 \\ \sigma_2^2\nu_2 &= C_2 \end{aligned} \tag{D.6}$$

For the sum; using (D.3) we can write:

$$\alpha = \frac{\sqrt{C_1 + \frac{2(\sigma_1^2+\sigma_2^2)(\nu_1+\nu_2)}{\nu_1\nu_2}}}{\sigma_1^2+\sigma_2^2} \tag{D.7}$$

This could be simplified:

$$\alpha = \sqrt{C_1 + \frac{2(\nu_1+\nu_2)}{(\sigma_1^2+\sigma_2^2)\nu_1\nu_2}} \tag{D.8}$$

Then using (D.6) we will have:

$$\alpha = \sqrt{C_1 + \frac{2(\nu_1+\nu_2)}{C_2(\nu_2+\nu_1)}}$$

Finally

$$\alpha = \sqrt{C_1 + \frac{2}{C_2}}$$

which is the same as α_1 and α_2 as required.

Therefore new variance gamma which is the sum two iid variance gamma process will be:

$$VG(\theta_1+\theta_2, \sqrt{\sigma_1^2+\sigma_2^2}, \frac{\nu_1\nu_2}{\nu_1+\nu_2}) \tag{D.9}$$

The second part of the proof involves normal inverse gaussian process which is the other infinite activity pure jump Levy process. Start with Laplace transform [45]

$$L_{NIG}(z) = e^{\mu z + \left(\sqrt{\alpha^2-\beta^2} - \sqrt{\alpha^2-(\beta+z)^2}\right)\delta} \tag{D.10}$$

Our new two iid normal inverse gaussian process will be in the form:

$$\begin{aligned} NIG_1 &= \theta_1\iota + \sigma_1 W(\iota) \\ NIG_2 &= \theta_2\iota + \sigma_2 W(\iota) \end{aligned}$$

Similar to Variance Gamma process the sum of two normal inverse gaussian process will have Laplace transform:

$$L_{NIG_1}(z)L_{NIG_1}(z) = e^{(\mu_1+\mu_2)z+\left(\sqrt{\alpha^2-\beta^2}-\sqrt{\alpha^2-(\beta+z)^2}\right)(\delta_1+\delta_2)} \tag{D.11}$$

Therefore, as long as we keep α and β parameters same, the normal inverse gaussian process will be closed under convolution, as a result the sum of two normal inverse gaussian processes will be normal inverse gaussian with parameters:

$$NG\left(\theta_1+\theta_2, \sqrt{\sigma_1^2+\sigma_1^2}, \frac{\nu_1\nu_2}{\nu_1+\nu_2}\right) \tag{D.12}$$

$$\alpha_1 = \frac{\sqrt{\theta_1^2+\frac{2\sigma_1^2}{\nu_1}}}{\sigma_1^2}, \quad \beta_1 = \frac{\theta_1}{\sigma_1^2}$$

$$\alpha_2 = \frac{\sqrt{\theta_2^2+\frac{2\sigma_2^2}{\nu_2}}}{\sigma_2^2}, \quad \beta_2 = \frac{\theta_2}{\sigma_2^2}$$

As we can see this result is the same as variance gamma process. The reason for that is that the Laplace transform requires α and β parameters to be the same as the two iid normal inverse gaussian processes. Moreover, the α and β formulas in normal inverse gaussian process is exactly the same as variance gamma . These requirements are as in the variance gamma :

$$\theta_2 = C_1\sigma_2^2, \tag{D.13}$$

$$\theta_1 = C_1\sigma_1^2,$$

$$\sigma_1^2\nu_1 = C_2,$$

$$\sigma_2^2\nu_2 = C_2. \tag{D.14}$$

If we elaborate further, we see that the requirements imply the same structure of variance gamma process convolution. A useful application of this could be in variance gamma option pricing. Write two variance gamma processes driving returns of stocks in the equity market such that

$$S_1 = X_{S_1} + C_{S_1}Z,$$

$$S_2 = X_{S_2} + C_{S_2}Z.$$

Here, X_{S_1} and X_{S_2} terms are independent idiosynratic components respectively for each stock and Z term is the systematic component. This framework is the same as we do in 4.5.6. However, this time we restrict the system to be still variance gamma process and derive the parameters that sustain it. As a result we have the respective return process for two stocks having the same systematic component.

$$S_1 = VG\Big(\theta_{S_1} + C_{S_1}\theta_Z = \theta_1, \sigma_{S_1}^2 + C_{S_1}^2\sigma_Z^2 = \sigma_1^2, \frac{\nu_{S_1}\nu_Z}{\nu_{S_1}+\nu_Z} = \nu_1\Big)$$

$$S_2 = VG\Big(\theta_{S_2} + C_{S_2}\theta_Z = \theta_2, \sigma_{S_2}^2 + C_{S_2}^2\sigma_Z^2 = \sigma_2^2, \frac{\nu_{S_2}\nu_Z}{\nu_{S_2}+\nu_Z} = \nu_2\Big). \qquad \text{(D.15)}$$

Using the conditions in (D.13) we will obtain a set of formulas for parameters that form the system. From (D.15) we see that the variance of the stocks can both be written as:

$$\sigma_1^2 = \sigma_{S_1}^2(1 + \frac{\nu_{S_1}}{\nu_Z})$$

$$\sigma_2^2 = \sigma_{S_2}^2(1 + \frac{\nu_{S_2}}{\nu_Z})$$

Further using the (D.13) and after tedious algebra we obtain:

$$\frac{\nu_{S_1}}{\nu_{S_2}} = \frac{\theta_2\sigma_1^4}{\theta_1\sigma_2^4} = R_c, R = \frac{\theta_1\sigma_2^2}{\theta_2\sigma_1^2}$$

We also know that:

$$\frac{\sigma_{S_2}^2}{\sigma_{S_2}^2} = \frac{R^2}{R_c} = R_{S_{12}}$$

Using all these we see that

$$\sigma_{S_1}^2 = \frac{\sigma_1^2 - R^2\sigma_2^2}{1 - R^2R_{S_{12}}}$$

and

$$\sigma_{S_2}^2 = \frac{\sigma_1^2 - R^2\sigma_2^2}{1 - R^2R_{S_{12}}}R_{S_{12}}$$

Further using the (D.13) again we obtain:

$$\theta_{S_1} = \frac{\theta_1 - R\theta_2}{1 - RR_N}, \quad R_N = \frac{\sigma_{S_2}^2}{\sigma_{S_1}^2}R$$

Collecting all above and after some effort we have:

$$\nu_{S_1} = \frac{RR_\nu - 1}{R_\nu D_\nu}, \quad D_\nu = \frac{\nu_1\nu_2}{\nu_1 - \nu_2}$$

$$\nu_{S_2} = R_\nu\nu_{S_1}, \quad c\nu_Z = \frac{\nu_1\nu_{S_1}}{\nu_1 - \nu_{S_1}}$$

$$\theta_Z = \frac{\theta_1 - \theta_{S_1}}{C_{S_1}}, \quad \sigma_Z = \sqrt{\frac{\sigma_1^2 - \sigma_{S_1}^2}{C_{S_1}^2}}$$

APPENDIX E

DERIVATION HESTON/BATES MODEL DELTA

First, the option value is written as usual:

$$\mathbb{E}((S_T - K)^+) = \mathbb{E}((S_T - K)\mathbb{1}_{\{S_T > K\}}) \tag{E.1}$$

Then the derivative of this expectation is:

$$\begin{aligned}\frac{\partial \mathbb{E}((S_T - K)\mathbb{1}_{\{S_T > K\}})}{\partial S_t} &= \mathbb{E}^S(\mathbb{1}_{\{S_T > K\}}|\mathcal{F}_t) + \mathbb{E}(\delta((S_T - K)e^{-r\tau})S_T e^{-r\tau}|\mathcal{F}_t) \\ &\quad - \mathbb{E}(Ke^{-r\tau}\delta((S_T - K)e^{-r\tau})|\mathcal{F}_t)\end{aligned} \tag{E.2}$$

Here δ is Dirac function and is the derivative of Heaviside function $\mathbb{1}_{\{S_T > K\}}$ This leads to the result:

$$\frac{\partial \mathbb{E}((S_T - K)\mathbb{1}_{\{S_T > K\}})}{\partial S} = \frac{\partial C}{\partial S} = \mathbb{E}^S(\mathbb{1}_{\{S_T > K\}}|F_t) = P^s(S_T > K) = F_s \tag{E.3}$$

We have this result since the Dirac terms are equal to 1 when $S_T = K$. This leads to the formula using (E.2):

$$S_t p^s(S_t) = Ke^{-r\tau}p(Ke^{-r\tau})$$

where p^s and p correspond to densities under risk neutral measure $\mathbb{P}$ and stock risk-adjusted $\mathbb{P}^S$ measure respectively.

Then given this general conclusion, we can see that numerically Heston Call will also have $\Delta = F_s$

As we have derived previously, see (4.92) the affine Heston Call with Jump/Bates

Model has final formula and parameters:

$$V(S_t, v_t, \zeta_t, K, r, \tau, \rho_{x\zeta}, \rho_{xv}, \alpha, \beta, \theta, \kappa, a, \sigma_v, \eta) = S_t F_s - e^{-r\tau} K F$$
$$F_s = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty Re\left(\frac{e^{-iu\log(K)} f_s(\log S_t, v_t, \zeta_t, \tau, u)}{iu} du\right)$$
$$F = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty Re\left(\frac{e^{-iu\log(K)} f(\log S_t, v_t, \zeta_t, \tau, u)}{iu} du\right)$$
$$f^s = \exp\left\{A(\tau) + B(\tau)v_t + C(\tau)\zeta + iu\log S_t + \lambda\tau\left[(1+\mu)^{ui+1}\exp\left(\frac{\sigma_{jp}^2(ui-u^2)}{2}\right) - 1\right]\right\}$$
$$f = \exp\left\{A(\tau) + B(\tau)v_t + C(\tau)\zeta + iu\log S_t + \lambda\tau\left[(1+\mu)^{ui}\exp\left(\frac{-\sigma_{jp}^2(u^2+ui)}{2}\right) - 1\right]\right\}$$

The greek of the call can be calculated:

$$\frac{\partial C}{\partial S} = F_s + S_t iu F_s - Ke^{-r\tau}\frac{iuF}{S_t}$$

The result (E.2) over Heston/Bates Model implies

$$S_t \frac{1}{2\pi}\int_{-\infty}^{\infty} Re\left(e^{-iu\log(K)} f_s(\log S_t, v_t, \zeta_t, \tau, u) du\right) -$$
$$Ke^{-r\tau}\frac{1}{2\pi}\int_{-\infty}^{\infty} Re\left(e^{-iu\log(K)} f(\log S_t, v_t, \zeta_t, \tau, u) du\right) = 0$$

Therefore,

$$\frac{\partial C}{\partial S} = F_s$$

APPENDIX F

DERIVATION OF DISTRIBUTION FUNCTION OF BROWNIAN MOTION MINIMUM

Corollary:

$$\widehat{W}(T) = \alpha T + W(T)$$

As given in [49]:

$$\begin{aligned} \tilde{P}\{\widehat{M}(T) \leq m\} &= \tilde{P}\{M(T) + \alpha T \leq m\} \\ &= \Phi(\frac{m - \alpha T}{\sqrt{T}}) - e^{2\alpha m}\Phi(\frac{-\alpha T - m}{\sqrt{T}}) \end{aligned} \tag{F.1}$$

and

$$f(m, w) = -\frac{2(2m-w)}{T\sqrt{2\pi T}} e^{\frac{-1}{2T}}(2m-w)^2 = \frac{2(w-2m)}{T\sqrt{2\pi T}} e^{\frac{-1}{2T}(2m-w)^2}$$

where $w \geq m, m \leq 0$. and $\widehat{M}(T) = \sup \widehat{W}(t), 0 < t < T$

Proof:

$$\begin{aligned} \tilde{P}\{\inf W(t) + \alpha T \leq m\} &= \tilde{P}\{\sup -W(t) - \alpha T \geq -m\} \\ &= \tilde{P}\{\sup -W(t) - \alpha T \geq -m\} \\ = \tilde{P}\{\sup W(t) - \alpha T \geq -m\} &= \tilde{P}\{M(T) - \alpha T \geq -m\} \\ &= 1 - \tilde{P}\{M(T) - \alpha T \leq -m\} \end{aligned}$$

We could write above equalities via reflection property of brownian motion and using the fact that $-W(t) \sim W(t)$ [29]. Then using (F.1) (i.e. $\alpha = -\alpha$ and m = -m) we can write:

$$1 - \tilde{P}\{M(T) - \alpha T \leq -m\} = 1 - \Phi\left(\frac{-m + \alpha T}{\sqrt{T}}\right) + e^{-2\alpha m}\Phi\left(\frac{\alpha T + m}{\sqrt{T}}\right)$$

Then finally our first equality becomes:

$$\begin{aligned}\tilde{P}\{\inf W(t) + \alpha T \leq m\} &= 1 - \tilde{P}\{M(T) - \alpha T \leq -m\} \\ &= \Phi\left(\frac{m - \alpha T}{\sqrt{T}}\right) + e^{2\alpha m}\Phi\left(\frac{m + \alpha T}{\sqrt{T}}\right)\end{aligned}$$

APPENDIX G

SOLVENCY PROCESS DERIVATION

Proof. We have pure jump process with infinite activity defined below:

$$X(t) = X(0) + \int_0^t b(s)ds + \int_0^t \int_R y dJ(y,t)$$

Define exponential Levy process

$$S(t) = S(0)e^{X(t)}$$

This has an alternative reprsesentation via Ito's lemma:

$$S(t) = S(0) + \int_0^t S(u)dX(u) \tag{G.1}$$

where $dX(t) = b(t)dt + ydJ(t)$

$R(t)$ is an exponential Levy process of the form:

$$R(t) = \frac{A(0)}{D(0)} e^{\Big((\phi_D(-i)-\phi_A(-i))t + X_A(t) - X_D(t)\Big)}$$

Let's assume $X_A(t) - X_D(t) = X(t)$ then Applying Ito's lemma for Levy processes as given in [51] we obtain:

$$\begin{aligned} R(X(t),t) - R(X(0),0) = \int_0^t \int_{-\infty}^{\infty} \Big[& R(X(s),s) - R(X(s_-),s) \\ & -R(s)(e^y - 1)\nu(dy)\Big] ds \\ & + \int_0^t \Big(\phi_D(-i) - \phi_A(-i)\Big) R(s) ds \\ + \int_0^t \int_{-\infty}^{\infty} & \Big[f(X(s),s) - f(X(s_-),s)\Big] d\tilde{J}(y,s) \end{aligned} \tag{G.2}$$

Using the fact that below formula defines expectation with respect to jump measure we finally obtain:

$$\mathbb{E}(X(t)) = \int_{-\infty}^{\infty} (e^y - 1)\nu(dy) = \phi_X(-i) = \phi_{X_A - X_D}(-i)$$

Then we get:

$$R(X(t),t) = R(X(0),0) + \int_0^t \Big(\phi_D(-i) - \phi_A(-i) - \phi_{X_A-X_D}(-i)\Big) R(X(s_-))ds +$$
$$\int_0^t \int_{-\infty}^{\infty} R(X(s_-)) \Big[e^{X(s)-X(s_-)} - 1\Big] d\tilde{J}(y,s)$$

Then given (G.1), we can see that it has the same structure with the equation above. Using that the result follows:

$$R(t) = R(0)\exp(\Big(\phi_D(-i) - \phi_A(-i) - \phi_{X_A-X_D}(-i)\Big)t + X(t))$$

where we have defined $X(t) = X_A(t) - X_D(t)$. □

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