JOINT INVENTORY AND PRICING DECISIONS IN RETAIL INDUSTRY

by

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JOINT INVENTORY AND PRICING DECISIONS IN RETAIL INDUSTRY

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...canım anneme ve ablama, ne söylesem az, aslında bu tez onlara ait

ve

Engin'e, bana hep destek olduğu için...

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JOINT INVENTORY AND PRICING DECISIONS IN RETAIL INDUSTRY

Abstract

In various industries, managers face the problem of setting prices dynamically over time and determining the replenishment quantities by a fixed deadline so as to maximize the expected profit over a finite and short selling horizon. This problem is especially significant for the retail industries which sell products with short life cycles and price dependent demand. In this thesis, it is assumed that a firm sells a single product over a selling season that is divided into a finite number of discrete time periods. At the beginning of each period, the firm has the option of replenishing the inventory and determining a new price for the product. The replenishment lead time is zero and unmet demand is lost where demand is sensitive to price. There is no fixed charge for ordering, and the total variable ordering cost is proportional to the ordering quantity. Similarly, the inventory holding cost incurred in each period is proportional to the end-of-period inventory. Unsold items at the end of the last period have a salvage value per unit. In this study, this joint inventory-pricing problem is analyzed, and solution methods are presented. In particular, we propose an efficient solution method that is very fast and yields solutions very close to optimality.

Keywords: Pricing, inventory, base-stock policy, fixed point iteration.

PERAKENDE SEKTÖRÜNDE ORTAK STOK VE FİYAT BELİRLEME KARARLARI

Özet

Birçok endüstride, yöneticiler kısa bir satış sezonu içinde karlarını enbüyütmek amacıyla fiyatları dinamik olarak değiştirme ve tedarik miktarlarını belirleme problemiyle karşı karşıyadırlar. Bu problem kısa raf ömrüne sahip ve talebi fiyata duyarlı mallar satan perakende sektörü için daha da önemlidir. Bu tezde, firmanın malı satacak sonlu sayıda ve eşit uzunlukta periyodunun olduğu varsayılmaktadır. Her periyodun başında firmanın stok yenileme seçeneği ve fiyatı değiştirme imkanı bulunmaktadır. Tedarik süresi sıfırdır ve karşılanmayan talep kaybedilmektedir. Talep fiyata duyarlıdır. Tedariğin sabit maliyeti yoktur, değişken maliyeti ise tedarik miktarıyla doğru orantılıdır. Benzer biçimde, stok tutma maliyeti de periyot sonunda elde kalan stok miktarıyla doğru orantılıdır. Sezon sonunda satılamayan ürünlerin satılamayan ürün miktarıyla doğru orantılı bir hurda değeri vardır. Bu çalışmada, bu sayılan varsayımlara sahip ortak stok ve fiyat belirleme problemi analiz edilmekte ve çözüm yöntemleri gösterilmektedir. Ayrıca, çok hızlı ve eniyiye çok yakın sonuçlar veren bir çözüm yöntemi de önerilmektedir.

Anahtar kelimeler: Fiyatlandırma, envanter yönetimi, baz stok politikası, değişmez nokta algoritması.

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Chapter 1

Introduction

In various industries, managers face the problem of setting prices dynamically over time and determining the replenishment quantities by a fixed deadline so as to maximize the expected profit over a finite and short selling horizon. This problem is especially significant for the retail industries which sell products with short life cycles and price dependent demand.

In recent years, the way firms operate changed dramatically. Nowadays, firms can do many of their tasks online, with high speed, low cost and high accuracy. Especially, e-commerce decreased the costs of firms considerably. For instance, changing the prices of the products online has virtually no cost. Similarly, availability of electronic price tags decreased the cost of changing prices at brick-and-mortar companies. Besides, the success of revenue management in the airline and hospital industries shows that dynamic pricing is a very profitable tool in addition to having a low cost. Therefore, dynamic pricing is now a more viable option for firms than it was 20 years ago. According to Chan et al (2004), most of the industry giants, like Amazon and Dell, now utilize dynamic pricing tools and profit from them.

Despite the success of dynamic pricing, integration of inventory management and pricing is still new to many companies. However, this integration is not only useful but also crucial for profitability. When these decisions are not linked but kept separately, the benefits of global optimization is lost. These benefits are especially important for sellers of fashion products and retailers because the season is short in fashion industry. Moreover, the demand is sensitive to price, products become obsolete rapidly, and the cost of the loss of customer goodwill is very significant. In such an environment, incorrect decisions about pricing and replenishment have much deeper impacts. Thus, especially these industries need successful pricing and inventory management policies, leading not only to increased profits but also to higher customer satisfaction.

In this thesis, this problem of joint determination of replenishment quantities and prices problem is considered when the firm has a finite number of periods in the season. It is assumed that a firm sells a single product over a selling season that is divided into a finite number of discrete time periods. At the beginning of each period, the firm has the option of replenishing the inventory and determining a new price for the product. The replenishment lead time is zero and unmet demand is lost where demand is sensitive to price. There is no fixed charge for ordering, and the total variable ordering cost is proportional to the ordering quantity. Similarly, the inventory holding cost incurred in each period is proportional to the end-of-period inventory. Unsold items at the end of the last period have a salvage value per unit. In this study, this joint inventory-pricing problem is analyzed, and solution methods are presented. In particular, we propose an efficient solution method that is very fast and yields solutions very close to optimality.

Chapter 2

Literature Review

Many researchers in marketing science, operations management and economics consider the dynamic pricing problem from different points of views. For a broad overview of the research conducted in marketing science, the reader should refer to the review by Eliashberg and Steinberg (1991). We are particularly interested in the operations management literature noting that we also incorporate inventory decisions. In order to put the work done in this area into a perspective, we must consider three streams of research: dynamic pricing, inventory management and the research that combines both of them. For a detailed discussion on the inventory management, the reader is referred to Porteus (1990). Moreover, the reviews by Yano and Gilbert (2003), Elmagraby and Keskinocak (2003) and Chan et al. (2004) span the work that combines both fields.

There is a huge amount of work done in the inventory management area when there is a single product reviewed periodically and prices are not considered, i.e. when the prices are fixed and given. The most relevant ones are mentioned here. Veinott (1965) is the first to show that myopic order-up-to policies are optimal for the periodic review policies under certain conditions which will be described later in this thesis. A myopic policy is a policy that maximizes only current profit and ignores future profit. The author also demonstrates that myopic order-up-to levels constitute upper bound on the optimal order-up-to levels. Lau and Lau (1998) examine a special case of this model where demand is normally distributed, the only relevant cost is the variable ordering cost and there are only two periods. The authors show how to find the optimal ordering quantities. They also consider the effect of reordering time and draw managerial insights. They find that the second order opportunity is more important if the product has a low profit c/p ratio and/or a great demand uncertainty. Morton and Pentico (1995) review the heuristics and bounds proposed in the literature for the general multi-period case, offer a new heuristic and test this heuristic along with the existing heuristics in the literature computationally.

Gallego and van Ryzin (1994) consider the problem of determining the optimal price path when inventory replenishment is not allowed, and provide the optimal solution to the continuous time formulation in which the price may change at any time. Because implementing this policy is impractical, they demonstrate that a single fixed price heuristic gives good results and is asymptotically optimal as inventory and/or time approaches infinity. Bitran and Mondschein (1997) and Zhao and Zheng (2000) generalize this model by treating the demand as a nonhomogeneous Poisson process. Bitran and Mondschein (1997) assume that the distribution of the maximum price that customers are willing to pay, also called reservation price, for the product is constant over time. First, they solve the model in continuous time, and then they consider the case where the number of price changes is limited. In the second case, when inventory goes to infinity and the reservation price distribution is invariant over time, a constant pricing strategy is optimal. They show that periodic review policies yield results very close to the optimal. Zhao and Zheng (2000) generalize this model by allowing the reservation price distribution to vary over time. They show that the optimal price is decreasing both in the inventory level and in time when a certain sufficient condition holds. They also verify that when the set of allowable prices is discrete, the optimal policy is defined by a set of threshold points of inventory level. Smith and Achabal (1998) incorporate the effect of inventory in the demand function. They focus on finding the optimal initial inventory and the optimal price trajectory of a single product. They assume a deterministic demand function that depends on the inventory level, as well as price and time. They formulate the optimal policy as one of six possible forms.

The papers discussed up to now assume that either prices are fixed or there is no replenishment option during the season. Nevertheless, there are some researchers who combine inventory replenishment with dynamic pricing. Whitin (1955) conducts the first study that incorporates pricing decisions into the inventory replenishment problem. He examines both deterministic and stochastic demand cases in a single period environment. Mills (1959) is the first researcher to use the additive form of demand uncertainty, that is, demand is a sum of two terms: a deterministic function of price and a stochastic error term. He shows that the single period optimal price is bounded from above by a price called riskless price which is equal to $\frac{\mu+\beta c+\alpha}{2\beta}$, where α and β are the parameters of the deterministic part of the demand, μ is the mean of the error term and finally c is the variable cost of ordering. Alternatively, Karlin and Carr (1960) are the first researchers that use the multiplicative form of demand, in which demand is the product of the error term and the deterministic function of price. They show that the riskless price is a lower bound to the optimal price when demand uncertainty is multiplicative. However, in this case, the riskless price is equal to $\frac{\beta c}{\beta-1}$. Thowsen (1975) considers the additive form of demand uncertainty and proves that when holding and stockout costs are convex, the optimal policy is a base stock policy, also called an order-up-to policy. That is, when the starting inventory is below a base stock, it is optimal to order up to the base stock and set a price that depends on the base stock rather than ordering quantity. When the starting inventory is above the base stock, it is optimal not to order and set a price that depends on the inventory level. Since there is no fixed cost of ordering, when the excess demand is lost, the problem reduces to a multi-period newsboy problem which is reviewed by Petruzzi and Dada (1999). Petruzzi and Dada (1999) investigate combining pricing effects with the newsboy problem in single period and multi-period settings in their paper, providing a general framework. They consider both additive and multiplicative forms of demand uncertainty and indicate how to solve the problem to optimality. They indicate that when the demand distribution is general, a search is needed to find the optimal point. However, when the demand distribution satisfies some conditions, it is easier to find the optimal point.

Thomas (1974) and Chen and Simchi-Levi (2004) incorporate a fixed ordering cost in the previous model that allowed backordering of the demand. Thomas (1974) considers a periodic review model with a fixed ordering cost component to maximize the expected profits over a finite selling horizon. He proposes a policy which he calls an (s, S, p) policy: if inventory level is below s_t at the beginning of the period t, order up to S_t; otherwise, do not order. Price depends only on the initial inventory and t. He gives a counter example for which this policy fails to be optimal. Chen and Simchi-Levi (2004) take the same model and prove that the policy proposed by Thomas is indeed optimal when the random component of the demand is additive. For the general form of demand, they suggest another policy that is not as simple as the order-up-to policy and prove its optimality.

In this thesis, the problem of determining replenishment quantities and prices at the beginning of each period in a finite season is considered. There is a single product, the unmet demand is assumed to be lost, there is no fixed charge for ordering, the variable ordering cost is proportional to ordering quantity, and the unsold items at the end of the last period have a salvage value per unit. That is, the problem analyzed in Petruzzi and Dada (1999) is extended to include salvage value explicitly. In addition to the ones proposed in Petruzzi and Dada (1999), an efficient and fast solution method is proposed. The other important assumptions are that the lead time is zero as in all the other papers reviewed here, and that the inventory holding cost is proportional to the amount of inventory left at the end of the period.

This thesis is organized as follows: in Chapter 3, the model with fixed prices is analyzed as it lays the groundwork for the pricing problem and contains key insights. In addition to the exact solution of the single-period model, the form of the optimal solution, heuristics and bounds for multi-period model are presented in this chapter. Also, the solution to a special case of the multi-period model is analyzed. The next chapter, Chapter 4, extends the model in Chapter 3 to include the pricing decision. In this chapter, the problem is analyzed in one-period and multi-period settings as well as for different forms of price-demand relationships. A heuristic is proposed and this heuristic is tested in Chapter 5. Finally, Chapter 6 concludes this work and gives future research directions.

Chapter 3

Replenishment Problem with Fixed Prices

In this chapter, we assume that the decision maker has the option to replenish the inventory at the beginning of each period. However, the prices cannot be changed: they are given and fixed. This problem provides important insights about the replenishment problem with pricing that will be discussed in Chapter 4.

3.1 One-Period Model

When the number of periods is one, this problem reduces to the well-known newsboy problem. The notation used in this section is as follows:

c	:	Ordering cost
s	:	Salvage value
h	:	Inventory holding cost
b	:	Cost of loss of goodwill
μ	:	Mean of the demand
σ	:	Standard deviation of the demand
p	:	Price
y	:	Inventory level after ordering
f	:	Probability density function of the demand
F	:	Cumulative distribution function of the demand
$\pi(y)$	:	Maximum expected profit for an inventory level after ordering y

It is assumed that $p > c$ and $c > s$. For a fixed inventory level after ordering, y, the expected profit is given as (3.1) when the initial inventory is zero.

$$\pi(y) = p\int_0^y xf(x)dx + p\int_y^\infty yf(x)dx - cy + s\int_0^y (y-x)f(x)dx$$
$$-h\int_0^y (y-x)f(x)dx - b\int_y^\infty (y-x)f(x)dx \tag{3.1}$$

When the demand is less than the initial inventory, the amount of sales is equal to the demand and when the demand is greater than initial inventory, the amount of sales is equal to the initial inventory. Thus, the expression $\int_0^y xf(x)dx + \int_y^\infty yf(x)dx$ is the expected sales and $p\int_0^y xf(x)dx + p\int_y^\infty yf(x)dx$ is the expected revenue from the sales. The term cy is the total ordering cost. When an item is not sold at the end of the season, it is sold at a price s, but firm incurs a holding cost per leftover item. The fourth and fifth terms in the equation, $s\int_0^y (y-x)f(x)dx - h\int_0^y (y-x)f(x)dx$ reflect this expected revenue from salvage of the items and the expected holding cost. Finally, the last term $b\int_y^\infty (y-x)f(x)dx$ is the expected cost of the loss of goodwill.

The second derivative of $\pi(y)$ with respect to y is:

$$\frac{\partial^2\pi(y)}{\partial y^2} = -(p-s+h+b)f(y) \tag{3.2}$$

This term is strictly negative since p is greater than c, which is in turn greater than s, and $f(y)$ is positive. As a result, $\pi(y)$ is concave and there is a unique y that maximizes $\pi(y)$. The optimal inventory level y can be found by equating the first derivative to zero and solving the resulting equation for y. It satisfies the following equation:

$$y = F^{-1}\left(\frac{p-c+b}{p-s+h+b}\right) \tag{3.3}$$

The ratio $\frac{p-c+b}{p-s+h+b}$ is also called the critical ratio of the underage cost to the sum of underage and overage costs, where $p-c+b$ is the underage cost, and $c-s+h$ is the overage cost. For detailed information on the newsboy problem, please see Nahmias (2001).

Since the cost function $\pi(y)$ is concave with respect to y, order-up-to policies are optimal for this problem. For a given initial inventory x, the order-up-to policy can be described as: if x is smaller than the order-up-to level y, order $y - x$ to bring the inventory level after ordering to y; otherwise, do not order. Therefore, maximum expected profit when the inventory level before ordering is x, which is denoted by $\pi'(x)$, becomes:

$$\pi'(x) = \begin{cases} \pi(x) & \text{if } x \geq y \\ \pi(y) & \text{if } x < y \end{cases} \tag{3.4}$$

3.1.1 Special Case: Normal Demand

When the demand distribution is assumed to be normal, the integrals in (3.1) are easily calculated. When X is normal with a mean μ and standard deviation σ, Winkler et al. (1972) show that

$$\int_{-\infty}^{y} xf(x)dx = \mu\Theta\left(\frac{y-\mu}{\sigma}\right) - \sigma\phi\left(\frac{y-\mu}{\sigma}\right) \tag{3.5}$$

In equation (3.5), ϕ and Φ denotes the probability density function and cumulative distribution function of standard normal distribution, respectively. When the lower bound on x is zero, equation (3.5) becomes:

$$\begin{aligned} \int_{0}^{y} xf(x)dx &= \int_{-\infty}^{y} xf(x)dx - \int_{-\infty}^{0} xf(x)dx \\ &= \mu\Theta\left(\frac{y-\mu}{\sigma}\right) - \sigma\phi\left(\frac{y-\mu}{\sigma}\right) - \mu\Theta\left(\frac{-\mu}{\sigma}\right) + \sigma\phi\left(\frac{-\mu}{\sigma}\right) \end{aligned} \tag{3.6}$$

As a result, $\pi(y)$ becomes:

$$\begin{aligned} \pi(y) &= (p-s+h+b)\int_{0}^{y} xf(x)dx \\ &\quad +(p-c+b)y - (p-s+h+b)yF(y) - b\mu \\ &= (p-s+h+b)\left(\mu\Theta\left(\frac{y-\mu}{\sigma}\right) - \sigma\phi\left(\frac{y-\mu}{\sigma}\right) - \mu\Theta\left(\frac{-\mu}{\sigma}\right) + \sigma\phi\left(\frac{-\mu}{\sigma}\right)\right) \\ &\quad +(p-c+b)y - (p-s+b+h)yF(y) - b\mu \end{aligned} \tag{3.7}$$

In this form, $\pi(y)$ is very easy to calculate.

3.2 Two-Period Model

In the two-period model, the decision maker has the opportunity to replenish the item at the beginning of the first and second periods. For the simplicity of calculations, we assume that the initial inventory at the beginning of the first period is zero, without loss of generality. In the next section, the effect of initial inventory will be studied and this assumption will be relaxed. The parameters can differ from period to period; thus, the parameters and the variables take a subscript denoting the period number, t. The total number of periods is denoted by T, which is equal to two in this section. The following assumptions are made about the parameters of the problem:

1. $p_t > c_t \qquad \forall t$
2. $c_t + \sum_{k=t+1}^{T} h_k > s \qquad \forall t$
3. $c_{t+1} < c_t + h_t \qquad \forall t$

The first assumption is made for not making loss. If this assumption does not hold, there is no motive for the firm to make business. If the second assumption does not hold, it may be optimal to buy the product in period t, in order not to sell to the customers but to the salvage market only; as a result, the decision maker buys as many of the product as he/she can in period t, and then sells to the salvage market to make profit. If the third assumption does not hold, the firm tends to buy its requirements in advance and keeps them in inventory until they are sold. Thus, for the replenishment opportunity to be a valuable option to the firm, this assumption must hold. Also, it should be noted that the second and the third assumptions avoid the speculative motive for holding inventory.

Based on the analysis of the single period model, it is known that an order-up-to policy, with an order-up-to level of y_2^*, is optimal for the second period. Table 3.2 summarizes possible values of profit in period 2 according to the values of the demand in period 1, y_1 and y_2^*. The demand in period t is denoted by d_t.

Table 3.2: Period 1 Demand vs Period 2 Profit

Relationship between y_1 and y_2^*	Demand in Period 1 (d_1)	Order Quantity in Period 2	Inventory after Ordering in Period 2	Expected Profit in Period 2
$y_1 < y_2^*$	$d_1 < y_1$	$y_2^* - (y_1 - d_1)$	y_2^*	$\pi(y_2^*) + c_2(y_1 - d_1)$
	$d_1 \geq y_1$	y_2^*	y_2^*	$\pi(y_2^*)$
$y_1 \geq y_2^*$	$d_1 < y_1$ and $y_1 - d_1 \geq y_2^*$	0	$y_1 - d_1$	$\pi(y_1 - d_1)$
	$d_1 < y_1$ and $y_1 - d_1 < y_2^*$	$y_2^* - (y_1 - d_1)$	y_2^*	$\pi(y_2^*) + c_2(y_1 - d_1)$
	$d_1 \geq y_1$	y_2^*	y_2^*	$\pi(y_2^*)$

As a result, when demand in period 1 exceeds the order-up-to level in period 1, y_1, it is optimal to start the second period with y_2^*, and since there is no leftover item from the first period, ordering quantity is y_2^*. If y_2^* is greater than y_1, the second period inventory after ordering is y_2^* anyway, so the ordering quantity is the difference between the leftover items from the first period and y_2^*. However, if y_2^* is smaller than y_1, the second period inventory after ordering changes with the amount of leftover inventory. If the inventory from the first period exceeds y_2^*, it is optimal not to buy anything. Thus, expected profit is:

$$\pi_1(y_1) = \pi_1(y_1|s=0) + \int_0^{max\{0, y_1 - y_2^*\}} (\pi(y_1 - x) + c_2(y_1 - x)) f_1(x)dx$$
$$+ \int_{max\{0, y_1 - y_2^*\}}^{y_1} (\pi(y_2^*) + c_2(y_1 - x)) f_1(x)dx + \int_{y_1}^{\infty} \pi(y_2^*) f_1(x)dx \qquad (3.8)$$

The first term is the expected profit of the first period. This term can be calculated using (3.1); however, since leftover items are salvaged only at the end of the season, s must be taken as zero. The second and the third terms represent the expected profit of the second period when there are leftover items from first period. When y_2^* is greater than y_1, the second term vanishes. The last term represents the expected profit when there is no leftover item at the end of period 1.

Lau and Lau (1998) solve a special case of the model above. In their paper, the demand is normally distributed. The price and the ordering cost are stationary, and the costs of inventory holding, loss of goodwill, as well as the salvage value are taken as zero. The authors also study the effect of the reordering time: it is assumed that the time of the second replenishment option can change from immediately after the first period to the end of the season while the length of the season is kept fixed. For each problem, the time of the second replenishment time is found by exhaustive search.

3.3 Multi-Period Model

The general multi-period problem can be formulated as a stochastic dynamic programming model. The term $\pi'_t(x)$ represents the sum of expected profit of periods t to T when the inventory before ordering at the beginning of period t is x and optimal policy is followed in the following periods. The recursive equations are given as:

$$\pi'_t(x_t) = \max_{y_t \geq x_t} \begin{cases} \pi(y_t|s=0) + c_t x_t + E(\pi'_{t+1}(max\{y_t - d_t, 0\})) & \text{if } 1 \leq t < T \\ \pi(y_t) + c_t x_t & \text{if } t = T \end{cases} \tag{3.9}$$

The terms $\pi(y_t|s = 0) + c_t x_t$ and $\pi(y_t) + c_t x_t$ above represent the expected revenue in period t. The term $E(\pi'_{t+1}(max\{y_t - d_t, 0\})$ represents the maximum expected profit of periods $t + 1$ to period T.

Theorem 3.3.1 *The term* $\pi(y_t|s = 0)+c_t x_t+E(\pi'_{t+1}(max\{y_t-d_t, 0\}))$ *is concave with respect to* y_t.

Proof The proof is done by induction on t. For $t = T$, the expression $\pi(y_t)+c_t x_t$ is the one-period profit which is shown to be concave in Section 3.1. Assume that for t=$k + 1$, $\pi'_{k+1}(x_{k+1})$ is concave. As a result, the term $E(\pi'_{k+1}(.))$ is concave. $\pi(y_k|s = 0)$ is also concave and $c_k x_k$ is constant. The sum of the concave functions and scalars is also concave, so $\pi(y_k|s = 0) + c_k x_k + E(\pi'_{k+1}(max\{y_k - d_k, 0\}))$ is concave. Thus, $\max_{y_k \geq x_k}\{\pi(y_k|s = 0) + c_k x_k + E(\pi'_{k+1}(max\{y_k - d_k, 0\}))\}$ is concave. ∎

As a result, order-up-to policies are optimal for this general case as well. The optimal solution to this problem, i.e. the order-up-to levels for each period, can be found by stochastic dynamic programming when the demand and the inventory levels are discrete. However, when the demand has a continuous distribution, e.g. normal distribution, and the inventory levels can have fractional parts, there is currently no way, to the best of the author's knowledge, of finding the optimal solution exactly, since states come from a continuous set. However, there are some heuristics and bounds which give very good results. Some of these bounds and a heuristic are presented in the following sections. For a more detailed discussion of these bounds and heuristics, please see Morton and Pentico (1995).

3.4 Bounds

3.4.1 Upper Bounds

1. Karlin (1960) proposes the concept of myopic policy in his paper. Myopic policy assumes that any leftover items at the end of a period are salvaged at a cost of c_{t+1}. As a result, optimal order-up-to levels become independent of each other, and the problem reduces to T independent single period problems. The myopic order-up-to level $y_t^{(1)}$ for period t is an upper bound to the optimal order-up-to level y_t^*. That is:

$$y_t^* \leq y_t^{(1)} = F_t^{-1}\left(\frac{p_t - c_t + b_t}{p_t - c_{t+1} + h_t + b_t}\right) \tag{3.10}$$

   When $t = T$, c_{t+1} is taken as s. The proof can be found in Morton and Pentico (1995).

   Veinott (1965) specifies the condition in which myopic order-up-to levels are indeed optimal. If the relation $F_t(x) \geq F_{t+1}(x)$ holds for all $x \geq 0$ and $t = 1, \ldots, T-1$, then myopic order-up-to policy is optimal for the problem. For example, these conditions hold when demand is stationary, i.e. mean and standard deviations are the same for all periods.

2. Consider the case where the demands in periods $t+1$ to T are convoluted to period t, i.e. all demands in these periods occur in period t. The order-up-to

level of period t, $y_t^{(2)}$, for this case increases and becomes an upper bound to y_t^*. Suppose $F_{t,N}$ is the cumulative distribution of demands from periods $t+1$ to T. Then the following relation holds:

$$y_t^* \leq y_t^{(2)} = F_{t,N}^{-1}\left(\frac{p_t - c_t + b_t}{p_t - s + \sum_{i=t}^{T} h_t + b_t}\right) \tag{3.11}$$

where $p_t - c_t + b_t$ is the underage cost and $c_t - s + \sum_{i=t}^{T} h_t$ is the overage cost. Proof can be found in Morton and Pentico (1995).

The smallest of these two upper bounds are taken as y_t^u, the tightest upper bound on y_t^*.

3.4.2 Lower Bound

Let us define the probability $P_{t,j}$ as the probability that no order would be placed from period $t+1$ to $j+1$, conditional on $x_t \leq y_t$. $P_{t,j}$ is equal to:

$$P_{t,j} = \left(\int_0^{y_t - y_{j+1}} f_t(d_t)\left(\int_0^{y_t - d_t - y_{j+1}} f_{t+1}(d_{t+1}) \ldots \left(\int_0^{y_t - d_t - d_{t+1} - \cdots - d_{j-1} - y_{j+1}} f_j(x)dx\right)\right)\right) \tag{3.12}$$

To find the lower bound on y_t^*, we also need to find the expectation of time from $t+1$ until the first order R_t^T, which is given as:

$$R_t^T = \sum_{j=1}^{T-1} P_{t,j} \tag{3.13}$$

The lower bound on y_t^*, y_t^l, is equal to $F_t^{-1}\left(\frac{p-c+b-hR_t^T-(c-s)P_t^T}{p-c+b+h}\right)$ when the parameters are stationary. Proof can be found in Morton and Pentico (1995).

The heuristic proposed by Morton and Pentico linearly interpolates between stockout probabilities implied by y_t^l and y_t^u so as to find the stockout probability of the order-up-to level, i.e:

$$y_t = F_t^{-1}(AF_t(y_t^l) + (1-A)F_t(y_t^u)) \tag{3.14}$$

In this equation, A is a scalar between 0 and 1. The best value of A is found by experimental study. For detailed information, please see Morton and Pentico (1995).

3.5 Summary

In this chapter, the replenishment problem with fixed prices is analyzed. The optimal solution to the one-period problem is given. Next, the optimal solution to the two period problem is stated. The dynamic programming formulations of the general multi-period problem are stated. Considering the difficulty of finding the optimal solution in this general case, some heuristic methods are presented that solves this problem efficiently. The main ideas in this chapter will be used in the next two chapters.

Chapter 4

Replenishment Problem with Pricing

In this chapter, we deal with the case in which the decision maker has the option to decide on the price to be charged in a period in addition to the replenishment quantities. This problem is richer than that of the previous section, but more difficult to explore. The problems in this chapter can be separated into two parts according to the way uncertainty is modeled. Each of the models will be analyzed in detail.

4.1 The Additive Demand Model

In the additive model of demand, stochastic and deterministic parts of the demand are added to each other, that is to say, the demand is modeled as $d_t(p)+\epsilon_t$ where, $d_t(p)$ is a deterministic and decreasing function of the price p and ϵ_t is the stochastic error term. Mills (1959) is the first paper that studies this form of demand uncertainty, and it is widely used since then. The function $d_t(p)$ is taken as $\alpha_t - \beta_t p$ in this study, where α_t and β_t are positive constants. ϵ_t is the stochastic part of the demand and independent of the price. It has a mean μ_t and a standard deviation σ_t, with probability density function f_t and cumulative distribution function F_t. For demand to be nonnegative, ϵ_t must be greater than $\beta_t c - \alpha_t$, the lower bound on $\beta_t p - \alpha_t$, which is in turn the lower bound on ϵ_t.

For analytical tractability, inventory level after ordering, y_t, is expressed as

$d_t(p) + z_t$ where z_t is called the stocking factor by Petruzzi and Dada (1999).

4.1.1 One-Period Model

In this section, the special case of additive model is investigated: there is only one period. This problem is the same as the newsboy problem with pricing decision. At the beginning of the period, the decision maker has to decide on how much to stock and the price to be charged in that period. For simplicity, initial inventory is assumed to be zero. This assumption will be relaxed later. The expected profit of this model, $\pi(z, p)$, for a given stocking factor z and price p is given as:

$$\begin{aligned} \pi(z,p) \;=\; & p\left[\alpha - \beta p + \int_{\beta c-\alpha}^{z} \epsilon f(\epsilon)d\epsilon + \int_{z}^{\infty} z f(\epsilon)d\epsilon\right] \\ & -c(\alpha-\beta p+z) + (s-h)\left[\int_{\beta c-\alpha}^{z}(z-\epsilon)f(\epsilon)d\epsilon\right] \\ & -b\left[\int_{z}^{\infty}(\epsilon - z)f(\epsilon)d\epsilon\right] \end{aligned} \tag{4.1}$$

The first part, $p\left[\alpha - \beta p + \int_{\beta c-\alpha}^{z} \epsilon f(\epsilon)d\epsilon + \int_{z}^{\infty} z f(\epsilon)d\epsilon\right]$, is the expected revenue, equal to the product of price and expected sales. The second part, $c(\alpha-\beta p+z)$, is the ordering cost. The term $s\int_{\beta c-\alpha}^{z}(z-\epsilon)f(\epsilon)d\epsilon$ is the expected revenue from the salvage of the items leftover at the end of the period, whereas $h\int_{\beta c-\alpha}^{z}(z-\epsilon)f(\epsilon)d\epsilon$ is the holding cost incurred. Finally, $b\int_{z}^{\infty}(\epsilon-z)f(\epsilon)d\epsilon$ is the expected loss of goodwill cost.

Upon simplification, expected profit equals:

$$\begin{aligned} \pi(z,p) \;=\; & p\left[\alpha - \beta p + \int_{\beta c-\alpha}^{z} \epsilon f(\epsilon)d\epsilon + \int_{z}^{\infty} z f(\epsilon)d\epsilon\right] \\ & -c(\alpha-\beta p+z) + (s-h)\left[\int_{\beta c-\alpha}^{z}(z-\epsilon)f(\epsilon)d\epsilon\right] \\ & -b\left[\int_{z}^{\infty}(\epsilon - z)f(\epsilon)d\epsilon\right] \end{aligned}$$

$$
\begin{aligned}
&= p(\alpha-\beta p)+p\left[\int_{\beta c-\alpha}^{z}\epsilon f(\epsilon)d\epsilon\right]+p\left[z(1-F(z))\right]-c(\alpha-\beta p)-cz \\
&\quad +(s-h)zF(z)-(s-h)\left[\int_{\beta c-\alpha}^{z}\epsilon f(\epsilon)d\epsilon\right] \\
&\quad -b\left[\mu-\int_{\beta c-\alpha}^{z}\epsilon f(\epsilon)d\epsilon\right]+b\left[z(1-F(z))\right] \\
&= (p-s+h+b)\left[\int_{\beta c-\alpha}^{z}\epsilon f(\epsilon)d\epsilon-z(F(z))\right] \\
&\quad +(p-c+b)z+(\alpha-\beta p)(p-c)-b\mu
\end{aligned}
\tag{4.2}
$$

The following two theorems, Theorem 4.1.1 and Theorem 4.1.2, provides the properties of the optimal solution.

Theorem 4.1.1 *The expected profit $\pi(z,p)$ is concave with respect to z when p is fixed. The reverse is also true: $\pi(z,p)$ is concave with respect to p when z is fixed.*

Proof The first derivative of $\pi(z,p)$ with respect to p is given as:

$$
\begin{aligned}
\frac{\partial\pi(z,p)}{\partial p} &= \frac{\partial\left[(p-s+h+b)\left[\int_{\beta c-\alpha}^{z}\epsilon f(\epsilon)d\epsilon-z(F(z))\right]\right]}{\partial p} \\
&\quad +\frac{\partial\left((p-c+b)z\right)}{\partial p}+\frac{\partial\left((\alpha-\beta p)(p-c)-b\mu\right)}{\partial p} \\
&= \int_{\beta c-\alpha}^{z}\epsilon f(\epsilon)d\epsilon-z(F(z))+z-\beta(p-c)+\alpha-\beta p \\
&= -2\beta p+\int_{\beta c-\alpha}^{z}\epsilon f(\epsilon)d\epsilon+z(1-F(z))+\beta c+\alpha
\end{aligned}
\tag{4.3}
$$

The second derivative of $\pi(z,p)$ with respect to p is given as:

$$
\frac{\partial\pi^2(z,p)}{\partial p^2}=\frac{\partial\left[-2\beta p+\int_{\beta c-\alpha}^{z}\epsilon f(\epsilon)d\epsilon+z(1-F(z))+\beta c+\alpha\right]}{\partial p}=-2\beta \tag{4.4}
$$

Similarly, the first derivative of $\pi(z,p)$ with respect to z is given as:

$$
\frac{\partial\pi(z,p)}{\partial z} = \frac{\partial\left[(p-s+h+b)\left[\int_{\beta c-\alpha}^{z}\epsilon f(\epsilon)d\epsilon-z(F(z))\right]\right]}{\partial z}
$$

$$
\begin{aligned}
&+\frac{\partial\left((p-c+b)z\right)}{\partial z}+\frac{\partial\left((\alpha-\beta p)(p-c)-b\mu\right)}{\partial z} \\
&= (p+h-s+b)\left[zf(z)-zf(z)-F(z)\right]+(p-c+b) \\
&= -(p+h-s+b)F(z)+(p-c+b) \qquad (4.5)
\end{aligned}
$$

The second derivative of $\pi(z,p)$ with respect to z is given as:

$$
\begin{aligned}
\frac{\partial\pi^2(z,p)}{\partial z^2} &= \frac{\partial\left[-(p+h-s+b)F(z)+(p-c+b)\right]}{\partial z} \\
&= -(p+h-s+b)f(z) \qquad (4.6)
\end{aligned}
$$

Leibniz' Rule is used to take the derivative of the term $\int\limits_{\beta c-\alpha}^{z}\epsilon f(\epsilon)d\epsilon$ with respect to z. For detailed information on Leibniz' Rule, please see Nahmias (2001).

Both $\frac{\partial\pi^2(z,p)}{\partial p^2}$ and $\frac{\partial\pi^2(z,p)}{\partial z^2}$ are negative since β and $f(y)$ are positive, and p is greater than c, which is in turn greater than s. Consequently, $\pi(z,p)$ is concave with respect to z when p is fixed, and vice versa. ∎

Theorem 4.1.2 *The optimal p and z make the first derivatives zero; thus, they satisfy the following two equations:*

$$z = F^{-1}\left(\frac{p-c+b}{p-s+h+b}\right) \qquad (4.7)$$

$$p = \frac{\mu-\Theta(z)+\beta c+\alpha}{2\beta} \qquad (4.8)$$

In this equation, $\Theta(z)$ is equal to $\int\limits_{z}^{\infty}(\epsilon-z)f(\epsilon)d\epsilon$ and denotes the expected amount of shortages when the stocking factor is z.

Proof This proof is a direct result of the previous theorem. ∎

The expression for p, (4.8), can be substituted for p in the equation for $\pi(z,p)$ and the resulting term can be solved to find optimal z, as stated by Whitin (1955). The reverse is also true: the expression for z, (4.7), can be substituted for z in the equation for $\pi(z,p)$ and the resulting term can be solved for optimal p.

The following theorem is a slight modification of the theorem stated by Petruzzi and Dada (1999). The difference is that salvage value is considered explicitly, and the possibility of the existence of optimal solutions at the boundaries of feasible region is considered.

Theorem 4.1.3 *In the single period problem, when the demand is additive, optimal policy is to select the price p according to (4.8) and to stock $\alpha - \beta p + z$ where z is determined according to the one of the following three cases:*

1. *If the hazard rate of demand distribution, $r(z)$ which is defined as $\frac{f(z)}{1-F(z)}$, satisfies $\left(2r^2(z) + \frac{\partial r(z)}{\partial z}\right) > 0$, then z is either the largest solution of the equation $\frac{\partial \pi(z,p(z))}{\partial z} = 0$ or one of the boundary points.*

2. *If the conditions $\left(2r^2(z) + \frac{\partial r(z)}{\partial z}\right) > 0$ and $\alpha - \beta c + 2\beta b > 0$ hold at the same time, then z is either the unique solution of the equation $\frac{\partial \pi(z,p(z))}{\partial z} = 0$ or one of the boundary points.*

3. *If $\left(2r^2(z) + \frac{\partial r(z)}{\partial z}\right) > 0$ does not hold, a search is needed to find the optimal z over the feasible region.*

Proof When the expression for p, (4.8), is substituted for p in the equation for expected profit, (4.2), expected profit becomes solely a function of z and can be written as $\pi(z)$. $\pi(z)$ is:

$$\begin{aligned} \pi(z) &= \left(\frac{\mu - \Theta(z) + \beta c + \alpha}{2\beta} - s + h + b\right)[-\Theta(z) + \mu - z] \\ &+ \left(\frac{\mu - \Theta(z) + \beta c + \alpha}{2\beta} - c + b\right) z \\ &-b\mu + \left(\alpha - \beta(\frac{\mu - \Theta(z) + \beta c + \alpha}{2\beta})\right)\left(\frac{\mu - \Theta(z) + \beta c + \alpha}{2\beta} - c\right) \end{aligned} \tag{4.9}$$

The point(s) that maximize $\pi(z)$ are needed. These points are indeed the ones that make the first derivative with respect to z zero. Let us take the derivative of $\pi(z)$ and analyze it. The first derivative of $\pi(z)$ with respect to z is:

$$\begin{aligned} \frac{\partial \pi(z)}{\partial z} &= \frac{\partial\left((\frac{\mu-\Theta(z)+\beta c+\alpha}{2\beta} - s + h + b)[-\Theta(z) + \mu - z]\right)}{\partial z} \\ &+ \frac{\partial\left((\frac{\mu-\Theta(z)+\beta c+\alpha}{2\beta} - c + b)z\right)}{\partial z} \\ &+ \frac{\partial\left((\alpha - \beta(\frac{\mu-\Theta(z)+\beta c+\alpha}{2\beta}))(\frac{\mu-\Theta(z)+\beta c+\alpha}{2\beta} - c) - b\mu\right)}{\partial z} \\ &= \frac{\partial\left(\frac{\mu-\Theta(z)+\beta c+\alpha}{2\beta} - s + h + b\right)}{\partial z}[-\Theta(z) + \mu - z] \\ &+ \left(\frac{\mu - \Theta(z) + \beta c + \alpha}{2\beta} - s + h + b\right)\frac{\partial[-\Theta(z) + \mu - z]}{\partial z} \end{aligned}$$

$$
\begin{aligned}
& +\frac{\partial\left(\frac{\mu-\Theta(z)+\beta c+\alpha}{2\beta}-c+b\right)}{\partial z}z+\frac{\mu-\Theta(z)+\beta c+\alpha}{2\beta}-c+b \\
& +\frac{\partial\left(\alpha-\beta(\frac{\mu-\Theta(z)+\beta c+\alpha}{2\beta})\right)}{\partial z}(\frac{\mu-\Theta(z)+\beta c+\alpha}{2\beta}-c) \\
& +\left(\alpha-\beta(\frac{\mu-\Theta(z)+\beta c+\alpha}{2\beta})\right)\frac{\partial\left(\frac{\mu-\Theta(z)+\beta c+\alpha}{2\beta}-c\right)}{\partial z} \\
= & -\frac{1}{2\beta}\frac{\partial\Theta(z)}{\partial z}\left[-\Theta(z)+\mu-z\right] \\
& +\left(\frac{\mu-\Theta(z)+\beta c+\alpha}{2\beta}-s+h+b\right)\frac{\partial\left[-\Theta(z)+\mu-z\right]}{\partial z} \\
& -\frac{1}{2\beta}\frac{\partial\Theta(z)}{\partial z}z+\frac{\mu-\Theta(z)+\beta c+\alpha}{2\beta}-c+b \\
& +\frac{1}{2}\frac{\partial\Theta(z)}{\partial z}\left(\frac{\mu-\Theta(z)+\beta c+\alpha}{2\beta}-c\right) \\
& -\left(\alpha-\beta(\frac{\mu-\Theta(z)+\beta c+\alpha}{2\beta})\right)\frac{1}{2\beta}\frac{\partial\Theta(z)}{\partial z} \\
= & -\frac{1}{2\beta}(F(z)-1)\left[-\Theta(z)+\mu-z\right] \\
& -\left(\frac{\mu-\Theta(z)+\beta c+\alpha}{2\beta}-s+h+b\right)F(z) \\
& -\frac{1}{2\beta}(F(z)-1)z+\frac{\mu-\Theta(z)+\beta c+\alpha}{2\beta}-c+b \\
& +\frac{1}{2}(F(z)-1)\left(\frac{\mu-\Theta(z)+\beta c+\alpha}{2\beta}-c\right) \\
& -\left(\alpha-\beta(\frac{\mu-\Theta(z)+\beta c+\alpha}{2\beta})\right)\frac{1}{2\beta}(F(z)-1) \\
= & (1-F(z))\left(\begin{array}{l}\frac{1}{2\beta}\left(-\Theta(z)+\mu-z\right)+\frac{1}{2\beta}z-\frac{1}{2}\left(\frac{\mu-\Theta(z)+\beta c+\alpha}{2\beta}-c\right) \\ +\left(\alpha-\beta(\frac{\mu-\Theta(z)+\beta c+\alpha}{2\beta})\right)\frac{1}{2\beta}\end{array}\right) \\
& -\left(\frac{\mu-\Theta(z)+\beta c+\alpha}{2\beta}-s+h+b\right)F(z) \\
& +\frac{\mu-\Theta(z)+\beta c+\alpha}{2\beta}-c+b-s+h+s-h \\
= & (1-F(z))\left(\begin{array}{l}\frac{1}{2\beta}\left(-\Theta(z)+\mu-z\right)+\frac{1}{2\beta}z-\frac{1}{2}\left(\frac{\mu-\Theta(z)+\beta c+\alpha}{2\beta}-c\right) \\ +\left(\alpha-\beta(\frac{\mu-\Theta(z)+\beta c+\alpha}{2\beta})\right)\frac{1}{2\beta}+\frac{\mu-\Theta(z)+\beta c+\alpha}{2\beta}-s+h+b\end{array}\right) \\
& -c+s-h \\
= & (1-F(z))\left(\frac{1}{2\beta}\left(\begin{array}{l}-\Theta(z)+\mu-z+z-\frac{\mu-\Theta(z)+\beta c+\alpha-2\beta c}{2} \\ \alpha-\beta(\frac{\mu-\Theta(z)+\beta c+\alpha}{2\beta})+\mu-\Theta(z)+\beta c+\alpha\end{array}\right)-s+h+b\right) \\
& -c+s-h \\
= & (1-F(z))\left(\frac{1}{2\beta}\left(-\Theta(z)+\mu+\beta c+\alpha\right)-s+h+b\right)-c+s-h
\end{aligned}
$$

The second derivative of $\pi(z)$ with respect to z is:

$$\begin{aligned}
\frac{\partial^2\pi(z)}{\partial z^2} &= \frac{\partial\begin{pmatrix} (1-F(z))\left(\frac{1}{2\beta}(-\Theta(z)+\mu+\beta c+\alpha)-s+h+b\right) \\ -c+s-h \end{pmatrix}}{\partial z} \\
&= \frac{\partial(1-F(z))}{\partial z}\left(\frac{1}{2\beta}(-\Theta(z)+\mu+\beta c+\alpha)-s+h+b\right) \\
&\quad +(1-F(z))\frac{\partial\left(\frac{1}{2\beta}(-\Theta(z)+\mu+\beta c+\alpha)-s+h+b\right)}{\partial z} \\
&= -f(z)\left(\frac{1}{2\beta}(-\Theta(z)+\mu+\beta c+\alpha)-s+h+b\right) \\
&\quad +(1-F(z))\frac{1}{2\beta}(1-F(z)) \\
&= -\frac{f(z)}{2\beta}\begin{pmatrix} -\Theta(z)+\mu+\alpha+\beta(c-2s+2h+2b) \\ -\frac{(1-F(z))^2}{f(z)} \end{pmatrix} \\
&= -\frac{f(z)}{2\beta}\begin{pmatrix} -\Theta(z)+\mu+\alpha+\beta(c-2s+2h+2b) \\ -\frac{1-F(z)}{r(z)} \end{pmatrix} \qquad (4.10)
\end{aligned}$$

In the equation above, $r(z)$ is the hazard rate of the distribution function and defined as $\frac{f(z)}{1-F(z)}$.

The third derivative of $\pi(z)$ with respect to z is:

$$\begin{aligned}
\frac{\partial^3\pi(z)}{\partial z^3} &= -\frac{1}{2\beta}\frac{\partial\left(f(z)\left(-\Theta(z)+\mu+\alpha+\beta(c-2s+2h+2b)-\frac{1-F(z)}{r(z)}\right)\right)}{\partial z} \\
&= -\frac{1}{2\beta}\begin{pmatrix} \frac{\partial f(z)}{\partial z}\left(-\Theta(z)+\mu+\alpha+\beta(c-2s+2h+2b)-\frac{1-F(z)}{r(z)}\right) \\ +f(z)\frac{\partial\left(-\Theta(z)+\mu+\alpha+\beta(c-2s+2h+2b)-\frac{1-F(z)}{r(z)}\right)}{\partial z} \end{pmatrix} \\
&= -\frac{1}{2\beta}\begin{pmatrix} \frac{\partial f(z)}{\partial z}\left(-\Theta(z)+\mu+\alpha+\beta(c-2s+2h+2b)-\frac{1-F(z)}{r(z)}\right) \\ +f(z)\left(-\frac{\partial\Theta(z)}{\partial z}-\frac{\partial(1-F(z))}{\partial z}\frac{1}{r(z)}+(1-F(z))\frac{1}{r^2(z)}\frac{\partial r(z)}{\partial z}\right) \end{pmatrix} \qquad (4.11)
\end{aligned}$$

The value of $\frac{\partial^3\pi(z)}{\partial z^3}$ at the point where $\frac{\partial^2\pi(z)}{\partial z^2}$ is equal to 0 is:

$$\begin{aligned}
\left.\frac{\partial^3\pi(z)}{\partial z^3}\right|_{\frac{\partial^2\pi(z)}{\partial z^2}=0} &= -\frac{f(z)(1-F(z))}{2\beta}\left(1+\frac{f(z)}{(1-F(z))r(z)}+\frac{1}{r^2(z)}\frac{\partial r(z)}{\partial z}\right) \\
&= -\frac{f(z)(1-F(z))}{2\beta r^2(z)}\left(2r^2(z)+\frac{\partial r(z)}{\partial z}\right) \qquad (4.12)
\end{aligned}$$

This term is negative if the condition $\left(2r^2(z)+\frac{\partial r(z)}{\partial z}\right) > 0$ holds. In other words, if this point exists, it is a local maximum point for $\frac{\partial\pi(z)}{\partial z}$. It can also be proved by contradiction that there cannot be more than one point that satisfies the condition $\frac{\partial^2 \pi(z)}{\partial z^2} = 0$ if $\left(2r^2(z) + \frac{\partial r(z)}{\partial z}\right) < 0$ holds. Then, there are two cases according to the existence of a point that satisfies the condition $\frac{\partial^2 \pi(z)}{\partial z^2} = 0$:

1. $\frac{\partial \pi(z)}{\partial z}$ is monotone if there is no z that satisfies $\frac{\partial^2 \pi(z)}{\partial z^2} = 0$. Then, if $\frac{\partial \pi(z)}{\partial z}$ crosses zero, it has one root. If not, $\frac{\partial \pi(z)}{\partial z}$ has no root. Additionally, $\left.\frac{\partial \pi(z)}{\partial z}\right|_{z=z_u} = -c+s-h$ is negative since c+h is greater than s by assumption. Furthermore, the value of $\left.\frac{\partial \pi(z)}{\partial z}\right|_{z=z^l}$ is $\frac{1}{2\beta}(-\mu + \mu + \beta c + \alpha) - c + s - h = \frac{\alpha - \beta c + 2\beta b}{2\beta}$. This value is positive when $\alpha - \beta c + 2\beta b$ is positive. If this condition holds, then $\frac{\partial \pi(z)}{\partial z}$ has exactly one root where $\frac{\partial \pi^2(z)}{\partial z^2}$ changes sign from positive to negative. Otherwise, $\frac{\partial \pi(z)}{\partial z}$ has no roots, maximum of $\pi(z)$ occurs at one of the boundaries of the feasible region.

2. $\frac{\partial \pi(z)}{\partial z}$ is unimodal if there is at least one z that satisfies $\frac{\partial^2 \pi(z)}{\partial z^2} = 0$. Since $\left.\frac{\partial^3 \pi(z)}{\partial z^3}\right|_{\frac{\partial^2 \pi(z)}{\partial z^2}=0} < 0$, there is at most one z that satisfies $\frac{\partial^2 \pi(z)}{\partial z^2} = 0$, on the left of which $\frac{\partial \pi(z)}{\partial z}$ is increasing and on the right, $\frac{\partial \pi(z)}{\partial z}$ is decreasing. $\left.\frac{\partial \pi(z)}{\partial z}\right|_{z=z_u}$ is still negative and $\left.\frac{\partial \pi(z)}{\partial z}\right|_{z=z^l}$ is positive if $\alpha - \beta c + 2\beta b$ is positive. If $\left.\frac{\partial \pi(z)}{\partial z}\right|_{\frac{\partial^2 \pi(z)}{\partial z^2}=0} = \frac{(1-F(z))^2}{2\beta r(z)} - c + s - h$ is negative, $\frac{\partial \pi(z)}{\partial z}$ has no zeros, thus maximum of $\pi(z)$ occurs at one of the boundaries of the feasible region. If $\left.\frac{\partial \pi(z)}{\partial z}\right|_{\frac{\partial^2 \pi(z)}{\partial z^2}=0}$ is positive and $\alpha - \beta c + 2\beta b$ is positive, then $\frac{\partial \pi(z)}{\partial z}$ has only one zero at which $\frac{\partial \pi(z)}{\partial z}$ goes from positive to negative, thus this point corresponds to a global maximum. If $\left.\frac{\partial \pi(z)}{\partial z}\right|_{\frac{\partial^2 \pi(z)}{\partial z^2}=0}$ is positive and $\alpha - \beta c + 2\beta b$ is negative, then $\frac{\partial \pi(z)}{\partial z}$ has two zeros. On the smaller one, $\frac{\partial \pi(z)}{\partial z}$ goes from negative to positive, representing the global minimum. The larger one is the global maximum since there, $\frac{\partial \pi(z)}{\partial z}$ goes from positive to negative.

In brief, if $\left(2r^2(z) + \frac{\partial r(z)}{\partial z}\right)$ is positive, then there exist at most two roots, larger of which corresponds to the global maximum. If there is no root, the global maximum is a boundary point. If, in addition to $\left(2r^2(z) + \frac{\partial r(z)}{\partial z}\right)$ being positive, $\alpha - \beta c + 2\beta b > 0$ holds, then the solution to $\frac{\partial \pi(z)}{\partial z} = 0$ is unique and corresponds to the global optimum. If $\left(2r^2(z) + \frac{\partial r(z)}{\partial z}\right)$ is not positive for all values of z, optimal z cannot be found by the equation $\frac{\partial \pi(z)}{\partial z} = 0$, and a search is needed. ∎

The failure rate of normal distribution is increasing, so $\left(2r^2(z) + \frac{\partial r(z)}{\partial z}\right) > 0$ holds for normal distribution. Thus, the optimality occurs at the largest point that make the first derivative zero.

4.1.2 The Case of Initial Inventory

Thowsen (1975) shows that order-up-to policies are optimal for the general multi-period problem, and obviously also for the single period case. For the single period case, if the optimal solution is (y, p), then when initial inventory is below y, it is optimal to order up to y and charge the price p. When initial inventory x is above y, it is optimal not to order and charge a price that depends on x.

4.1.3 Multi-Period Model

Similar to the replenishment case, the multi-period replenishment with pricing model can be modeled as a stochastic dynamic programming model. In this case, recursive equations take the form:

$$\pi_t(x_t) = \max_{p;z \geq x_t - (\alpha_t - \beta_t p)} \begin{cases} \pi(z, p|s = 0) + c_t x_t + E(\pi_{t+1}(max\{z - d_t, 0\})) & \text{if } t \neq T \\ \pi(z, p) + c_t x_t & \text{if } t = T \end{cases} \tag{4.13}$$

However, as states come from a continuous set, it is very hard to find the optimal policy simply by solving these equations. Therefore, properties of the optimal solution must be investigated. Fortunately, Thowsen (1975) shows that order-up-to policies are optimal for the general multi-period problem, as mentioned in the previous section.

4.2 The Multiplicative Demand Model

When the demand, d_t, is assumed to be multiplicative, it is expressed as $d_t(p)\epsilon_t$, where $d_t(p)$ is the deterministic and decreasing function of price and ϵ_t is the stochastic error term. Karlin and Carr (1960) is the first paper that studies multiplicative from of demand uncertainty, and it has been widely used since then in the dynamic pricing literature. The function $d_t(p)$ is taken as $\alpha_t p^{-\beta_t}$ in this thesis where the term p is the price, α_t is a positive constant, and β_t is a real number strictly greater than one. Also, the error term must be strictly positive for demand to be positive. For analytical tractability, order-up-to level is expressed as $d_t(p)z$, where z is called the stocking factor as in the additive demand case. The lower bound on z is the lower bound on error term, which is zero.

4.2.1 One-Period Model

In this section, a special case of the multiplicative model is investigated: there is only one period. As in the additive one-period problem, this problem is the same as the newsboy problem with pricing decision. The initial inventory is again assumed to be zero, but this assumption will be relaxed later. For a given stocking factor z and price p, the expected profit of this model, $\pi(z,p)$, is given as:

$$\begin{aligned}\pi(z,p) &= p\left[\int_0^z \alpha p^{-\beta}\epsilon f(\epsilon)d\epsilon + \int_z^\infty \alpha p^{-\beta} z f(\epsilon)d\epsilon\right] \\ &\quad -c(\alpha p^{-\beta} z) + (s-h)\left[\int_0^z \alpha p^{-\beta}(z-\epsilon)f(\epsilon)d\epsilon\right] \\ &\quad -b\left[\int_z^\infty \alpha p^{-\beta}(\epsilon - z)f(\epsilon)d\epsilon\right] \qquad (4.14)\end{aligned}$$

The first part, $p\left[\int_0^z \alpha p^{-\beta}\epsilon f(\epsilon)d\epsilon + \int_z^\infty \alpha p^{-\beta} z f(\epsilon)d\epsilon\right]$, is the expected revenue, equal to the product of price and expected sales. The second part, $c(\alpha p^{-\beta} z)$, is the ordering cost. The expression $s\int_0^z \alpha p^{-\beta} z(z-\epsilon)f(\epsilon)d\epsilon$ is the expected revenue from the salvage of the items leftover at the end of the period, whereas $h\int_0^z \alpha p^{-\beta} z(z-\epsilon)f(\epsilon)d\epsilon$ is the holding cost incurred. Finally, $b\int_z^\infty \alpha p^{-\beta} z(\epsilon - z)f(\epsilon)d\epsilon$ is the expected cost of the loss of the goodwill cost.

Upon simplification, the expected profit equals:

$$\begin{aligned}\pi(z,p) &= \alpha p^{-\beta}\left(\begin{array}{l} p\left[\int_0^z \epsilon f(\epsilon)d\epsilon\right] + p\left[z\int_z^\infty f(\epsilon)d\epsilon\right] - cz \\ +(s-h)\left[\int_0^z z f(\epsilon)d\epsilon\right] - (s-h)\left[\int_0^z \epsilon(\epsilon)d\epsilon\right] \\ -b\left[\mu - \int_0^z \epsilon f(\epsilon)d\epsilon\right] + b\left[\int_z^\infty z f(\epsilon)d\epsilon\right] \end{array}\right) \\ &= \alpha p^{-\beta}\left(\begin{array}{l} (p-s+h+b)\left[\int_0^z \epsilon f(\epsilon)d\epsilon\right] + p\left[z(1-F(z))\right] - cz \\ +(s-h)zF(z) - b\mu + b\left[z(1-F(z))\right] \end{array}\right) \\ &= \alpha p^{-\beta}\left(\begin{array}{l} (p-s+h+b)\left[\int_0^z \epsilon f(\epsilon)d\epsilon\right] - (p-s+h+b)zF(z) \\ +(p-c+b)z - b\mu \end{array}\right)\end{aligned}$$

$$= \alpha p^{-\beta}\left((p-s+h+b)\left[\int_0^z (\epsilon - z) f(\epsilon) d\epsilon\right] + (p-c+b)z - b\mu\right)$$

$$= \alpha p^{-\beta}\left(-(p-s+h+b)\Lambda(z) + (p-c+b)z - b\mu\right) \quad (4.15)$$

In this formula, $\Lambda(z)$ is the expected amount of leftovers which is equal to $\int_0^z (z-\epsilon)f(\epsilon)d\epsilon$. The relationship between the expected amount of lost sales, $\theta(z)$, and $\Lambda(z)$ is $\Lambda(z) = \Theta(z) - \mu + z$ which is used in the expected profit calculations. Rearranging (4.15), we obtain:

$$\begin{aligned} \pi(z,p) &= \alpha p^{-\beta}\left(-(p-s+h+b)\Lambda(z) + (p-c+b)(\Lambda(z) - \Theta(z) + \mu) - b\mu\right) \\ &= \alpha p^{-\beta}\left(-(c-s+h)\Lambda(z) - (p-c+b)\Theta(z) + (p-c)\mu\right) \quad (4.16) \end{aligned}$$

Theorem 4.2.1 *The optimal p and z satisfy the following two equations:*

$$z = F^{-1}\left(\frac{p-c+b}{p-s+h+b}\right) \quad (4.17)$$

$$p = \frac{\beta}{\beta-1}\frac{(c-s+h)\Lambda(z) + (b-c)\Theta(z) + c\mu}{\mu - \Theta(z)} \quad (4.18)$$

Proof The first derivative of $\pi(z,p)$ with respect to z is:

$$\begin{aligned} \frac{\partial \pi(z,p)}{\partial z} &= \frac{\partial\left(\alpha p^{-\beta}\left(-(p-s+h+b)\Lambda(z) + (p-c+b)z - b\mu\right)\right)}{\partial z} \\ &= \alpha p^{-\beta}\left(-(p-s+h+b)\frac{\partial \Lambda(z)}{\partial z} + \frac{\partial\left((p-c+b)z - b\mu\right)}{\partial z}\right) \\ &= \alpha p^{-\beta}\left(-(p-s+h+b)F(z) + (p-c+b)\right) \quad (4.19) \end{aligned}$$

The second derivative of $\pi(z,p)$ with respect to z is:

$$\begin{aligned} \frac{\partial \pi^2(z,p)}{\partial z^2} &= \frac{\partial\left(\alpha p^{-\beta}\left(-(p-s+h+b)F(z) + (p-c+b)\right)\right)}{\partial z} \\ &= \alpha p^{-\beta}\left(-(p-s+h+b)f(z)\right) \quad (4.20) \end{aligned}$$

The expression $\frac{\partial \pi^2(z,p)}{\partial z^2}$ is negative. As a result, $\pi(z,p)$ is concave with respect to z for a given p. Therefore, at optimality, the first derivative of $\pi(z,p)$ with respect to z must be zero. As $\alpha p^{-\beta}$ cannot be zero, the expression $-(p-s+h+b)F(z) + (p-c+b)$ is zero; and the following relationship exists between the optimal z and p pair:

$$z = F^{-1}\left(\frac{p-c+b}{p-s+h+b}\right) \tag{4.21}$$

Using the relation $\Theta(z) = \Lambda(z) + \mu - z$, the first derivative of $\pi(z,p)$ with respect to p becomes:

$$\begin{aligned}
\frac{\partial \pi(z,p)}{\partial p} &= \frac{\partial\left(\alpha p^{-\beta}\left((p-s+h+b)(-\Lambda(z)) + (p-c+b)z - b\mu\right)\right)}{\partial p} \\
&= -\beta\alpha p^{-\beta-1}\left((p-s+h+b)(-\Lambda(z)) + (p-c+b)z - b\mu\right) \\
&\quad +\alpha p^{-\beta}\left(-\Lambda(z)+z\right) \\
&= \alpha p^{-\beta-1}\begin{pmatrix} p\beta\Lambda(z) - \beta pz + \beta(-s+h+b)\Lambda(z) \\ +\beta(c-b)z + \beta b\mu - p\Lambda(z) + pz + z \end{pmatrix} \\
&= \alpha p^{-\beta-1}(\beta-1)\begin{pmatrix} p(\Lambda(z)-z) \\ -\frac{\beta}{\beta-1}\left[(-s+h+b)\Lambda(z) + (c-b)z + b\mu\right] \end{pmatrix} \\
&= \alpha p^{-\beta-1}(\beta-1)\begin{pmatrix} p(\Theta(z)-\mu) \\ -\frac{\beta}{\beta-1}\left[(-s+h+b)\Lambda(z) + (b-c)\Theta(z) + c\mu\right] \end{pmatrix} \\
&= \alpha p^{-\beta-1}(\beta-1)(\mu-\Theta(z))\begin{pmatrix} -p \\ +\frac{\beta}{\beta-1}\left[\frac{\begin{matrix}(-s+h+b)\Lambda(z) \\ +(b-c)\Theta(z)+c\mu\end{matrix}}{\mu-\Theta(z)}\right] \end{pmatrix}
\end{aligned} \tag{4.22}$$

The first expression $\alpha p^{-\beta-1}(\beta-1)(\mu-\Theta(z)) > 0$ since α is strictly positive, β is strictly greater than 1 and μ is strictly greater than $\Theta(z)$. When p is smaller than $\frac{\beta}{\beta-1}\left[\frac{(-s+h+b)\Lambda(z)+(b-c)\Theta(z)+c\mu}{\mu-\Theta(z)}\right]$, $\frac{\partial\pi(z,p)}{\partial p}$ is positive, so $\pi(z,p)$ is increasing. When p is greater than $\frac{\beta}{\beta-1}\left[\frac{(-s+h+b)\Lambda(z)+(b-c)\Theta(z)+c\mu}{\mu-\Theta(z)}\right]$, $\frac{\partial\pi(z,p)}{\partial p}$ is negative, so $\pi(z,p)$ is decreasing. As a result, $\pi(z,p)$ is maximized when p is equal to $\frac{\beta}{\beta-1}\left[\frac{(-s+h+b)\Lambda(z)+(b-c)\Theta(z)+c\mu}{\mu-\Theta(z)}\right]$. ∎

The following theorem is a slight modification of the theorem stated by Petruzzi and Dada (1999). The difference is, as in the additive case, that salvage value is considered explicitly, and the possibility of the existence of optimal solutions at the boundaries of feasible region is considered.

Theorem 4.2.2 *In the single period problem, when the demand is multiplicative, optimal policy is to select the price p according to* (4.18) *and to stock* $\alpha_t p^{-\beta_t} z$ *where z is determined according to the following two cases:*

1. *If hazard rate of demand distribution, $r(z)$ which is defined as $\frac{f(z)}{1-F(z)}$, satisfies $\left(2r^2(z)+\frac{\partial r(z)}{\partial z}\right)>0$ and additionally $b>2$, then the optimal z is either the unique solution of the equation $\frac{\partial \pi(z,p)}{\partial p}=0$ or one of the boundary points.*

2. *If either $\left(2r^2(z)+\frac{\partial r(z)}{\partial z}\right)>0$ or $b>2$ does not hold, a search is needed to find the optimal z over the feasible region.*

Proof When the expression for p, (4.8), can be substituted for p in the equation for expected profit, expected profit becomes solely a function of z and can be written as $\pi(z)$. The first derivative of $\pi(z)$ with respect to z is:

$$
\begin{aligned}
\frac{\partial \pi(z)}{\partial z} &= \frac{\partial \alpha p(z)^{-\beta}\left(-(c-s+h)\Lambda(z)-(p(z)-c+b)\Theta(z)+(p(z)-c)\mu\right)}{\partial z} \\
&= \frac{\partial\left(\alpha p(z)^{-\beta}\right)}{\partial z}\left(-(c-s+h)\Lambda(z)-(p(z)-c+b)\Theta(z)+(p(z)-c)\mu\right) \\
&\quad +\alpha p(z)^{-\beta}\frac{\partial\left(-(c-s+h)\Lambda(z)-(p(z)-c+b)\Theta(z)+(p(z)-c)\mu\right)}{\partial z} \\
&= -\beta\alpha p(z)^{-\beta-1}\frac{\partial p(z)}{\partial z}\begin{pmatrix} -(c-s+h)(\Theta(z)-\mu+z) \\ -(p(z)-c+b)\Theta(z)+(p(z)-c)\mu \end{pmatrix} \\
&\quad +\alpha p(z)^{-\beta}\begin{pmatrix} -(c-s+h)F(z)-\frac{\partial p(z)}{\partial z}\Theta(z) \\ -(p(z)-c+b)(F(z)-1)+\mu\frac{\partial p(z)}{\partial z} \end{pmatrix} \\
&= \alpha p(z)^{-\beta-1}\begin{pmatrix} \frac{\partial p(z)}{\partial z}\frac{-\beta}{p(z)}\begin{pmatrix} -(c-s+h)(\Theta(z)-\mu) \\ -(c-s+h)z-(p(z)-c+b)\Theta(z) \\ +(p(z)-c)\mu \end{pmatrix} \\ -(c-s+h)F(z)-\frac{\partial p(z)}{\partial z}\Theta(z) \\ -(p(z)-c+b)(F(z)-1)+\frac{\partial p(z)}{\partial z}\mu \end{pmatrix} \\
&= \alpha p(z)^{-\beta-1}\begin{pmatrix} \frac{\partial p(z)}{\partial z}\frac{-\beta}{p(z)}\begin{pmatrix} -(b-s+h)\Theta(z)+\mu(-s+h) \\ -(c-s+h)z-p(z)(\Theta(z)-\mu) \\ +(p(z)-c)\mu \end{pmatrix} \\ -(c-s+h)F(z)-\frac{\partial p(z)}{\partial z}\Theta(z) \\ -(p(z)-c+b)(F(z)-1)+\frac{\partial p(z)}{\partial z}\mu \end{pmatrix} \\
&= \alpha p(z)^{-\beta-1}\begin{pmatrix} \frac{\partial p(z)}{\partial z}\begin{pmatrix} \beta\frac{(\beta-1)(\mu-\Theta(z))}{\beta} \\ +\beta(\Theta(z)-\mu) \end{pmatrix} \\ -(c-s+h)F(z)-\frac{\partial p(z)}{\partial z}\Theta(z) \\ -(p(z)-c+b)(F(z)-1)+\frac{\partial p(z)}{\partial z}\mu \end{pmatrix} \\
&= \alpha p(z)^{-\beta-1}\begin{pmatrix} \frac{\partial p(z)}{\partial z}\left((\beta-1)(\mu-\Theta(z))+\beta(\Theta(z)-\mu)-(\Theta(z)-\mu)\right) \\ -(c-s+h)F(z)-(p(z)-c+b)(F(z)-1) \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
&= \alpha p(z)^{-\beta-1}\left(-(-s+h+p(z)+b)F(z)+p(z)-c+b\right) \\
&= \alpha p(z)^{-\beta-1}(1-F(z))\left(p(z)-s+h+b+\frac{-c+s-h}{1-F(z)}\right) \qquad (4.23)
\end{aligned}
$$

We need to calculate the roots of $\frac{\partial \pi(z)}{\partial z}$. The term $\alpha p(z)^{-\beta-1}(1-F(z))$ is strictly positive inside the feasible region, so $\left(p(z)-s+h+b+\frac{-c+s-h}{1-F(z)}\right)$ should be investigated. Let us denote this function by $R(z)$. The value of $R(z)$ when z is at its lower bound is:

$$
R(z^l) = \frac{\beta}{\beta-1}\frac{b\mu-(b-c)z^l}{z^l} - s + h + b - c = \frac{1}{\beta-1}\left(b\frac{\beta\mu - z^l}{z^l} + c\right) \qquad (4.24)
$$

This term is positive since $\beta\mu - z^l$ is positive and $\beta - 1$ is positive. Likewise, the value of $R(z)$ when z is at its upper bound is:

$$
\lim_{z\to z^u}\left(p(z)-s+h+b+\frac{-c+s-h}{1-F(z)}\right) = p(z^u)-s+h+b+\frac{-c+s-h}{\underbrace{1-F(z^u)}_{0}} \to -\infty \qquad (4.25)
$$

The first derivative of $R(z)$ with respect to z is:

$$
\begin{aligned}
\frac{\partial R(z)}{\partial z} &= \frac{\partial\left(p(z)-s+h+b+\frac{-c+s-h}{1-F(z)}\right)}{\partial z} \\
&= \frac{\partial p(z)}{\partial z} - (c-s+h)\frac{\partial\left(\frac{1}{1-F(z)}\right)}{\partial z} \\
&= \frac{\partial p(z)}{\partial z} - (c-s+h)\frac{1}{(1-F(z))^2}f(z) \\
&= \left(\frac{\beta}{\beta-1}\frac{b\mu(F(z)-1)+(c-s+h)(\mu-\Theta(z))+(F(z)-1)(c-s+h)z}{(\mu-\Theta(z))^2}\right. \\
&\quad \left. -(c-s+h)\frac{r(z)}{1-F(z)}\right) \qquad (4.26)
\end{aligned}
$$

The second derivative of $R(z)$ with respect to z is:

$$
\begin{aligned}
\frac{\partial R^2(z)}{\partial z^2} &= \frac{\partial\left(\frac{\partial p(z)}{\partial z} - (c-s+h)\frac{r(z)}{1-F(z)}\right)}{\partial z} \\
&= \frac{\partial p^2(z)}{\partial z^2} - (c-s+h)\left(\frac{\partial r(z)}{\partial z}\frac{1}{1-F(z)} + r(z)\frac{r(z)}{1-F(z)}\right) \\
&= \frac{\partial\left(\frac{\beta}{\beta-1}\frac{b\mu(F(z)-1)+(c-s+h)(\mu-\Theta(z))+(F(z)-1)(c-s+h)z}{(\mu-\Theta(z))^2}\right)}{\partial z} \\
&\quad -\frac{(c-s+h)}{1-F(z)}\left(\frac{\partial r(z)}{\partial z} + r^2(z)\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\beta}{\beta-1}\left(\begin{array}{l}\left(\begin{array}{l} b\mu(F(z)-1)+(c-s+h)(\mu-\Theta(z)) \\ +(F(z)-1)(c-s+h)z \end{array}\right)\left(-2\frac{1}{(\mu-\Theta(z))^3}(1-F(z))\right) \\ \left(\begin{array}{l} b\mu f(z)+(c-s+h)(1-F(z)) \\ +f(z)(c-s+h)z+(F(z)-1)(c-s+h) \end{array}\right)\frac{1}{(\mu-\Theta(z))^2} \end{array}\right) \\
&\quad -\frac{(c-s+h)}{1-F(z)}\left(\frac{\partial r(z)}{\partial z}+r^2(z)\right) \\
&= -\frac{\partial p(z)}{\partial z}\left(\frac{2}{\mu-\Theta(z)}(1-F(z))\right) \\
&\quad +\frac{\beta}{\beta-1}f(z)\left((b\mu+(c-s+h)z)\frac{1}{(\mu-\Theta(z))^2}\right) \\
&\quad -\frac{(c-s+h)}{1-F(z)}\left(\frac{\partial r(z)}{\partial z}+r^2(z)\right) \\
&= -\frac{\partial p(z)}{\partial z}\left(\frac{2}{\mu-\Theta(z)}(1-F(z))+r(z)\right) \\
&\quad +\frac{\beta}{\beta-1}f(z)\frac{1}{(\mu-\Theta(z))^2}\left(\frac{(c-s+h)(\mu-\Theta(z))}{1-F(z)}\right) \\
&\quad -\frac{(c-s+h)}{1-F(z)}\left(\frac{\partial r(z)}{\partial z}+r^2(z)\right) \\
&= -\left(\frac{\partial R(z)}{\partial z}+(c-s+h)\frac{r(z)}{1-F(z)}\right)\left(\frac{2}{\mu-\Theta(z)}(1-F(z))+r(z)\right) \\
&\quad +\frac{\beta}{\beta-1}f(z)\frac{1}{(\mu-\Theta(z))^2}\left(\frac{(c-s+h)(\mu-\Theta(z))}{1-F(z)}\right) \\
&\quad -\frac{(c-s+h)}{1-F(z)}\left(\frac{\partial r(z)}{\partial z}+r^2(z)\right) \\
&= -\frac{\partial R(z)}{\partial z}\left(\frac{2}{\mu-\Theta(z)}(1-F(z))+r(z)\right) \\
&\quad -(c-s+h)\left(\begin{array}{l} \frac{r(z)}{1-F(z)}\left(\frac{2}{\mu-\Theta(z)}(1-F(z))+r(z)\right) \\ -\frac{\beta}{\beta-1}r(z)\frac{1}{\mu-\Theta(z)} \\ +\frac{1}{1-F(z)}\left(\frac{\partial r(z)}{\partial z}+r^2(z)\right) \end{array}\right) \\
&= -\frac{\partial R(z)}{\partial z}\left(\frac{2}{\mu-\Theta(z)}(1-F(z))+r(z)\right) \\
&\quad -(c-s+h)\left(\begin{array}{l} \left(2-\frac{\beta}{\beta-1}\right)r(z)\frac{1}{\mu-\Theta(z)} \\ +\frac{1}{1-F(z)}\left(\frac{\partial r(z)}{\partial z}+r^2(z)\right) \end{array}\right) \\
&= -\frac{\partial R(z)}{\partial z}\left(\frac{2(1-F(z))}{\mu-\Theta(z)}+r(z)\right)-(c-s+h)\left(\begin{array}{l} \frac{\beta-2}{\beta-1}\frac{r(z)}{\mu-\Theta(z)} \\ +\frac{\left(\frac{\partial r(z)}{\partial z}+r^2(z)\right)}{1-F(z)} \end{array}\right)
\end{aligned}
$$

Finally, the value of the second derivative of $R(z)$ at the point(s) where its first derivative is zero is:

$$
\left.\frac{\partial R^2(z)}{\partial z^2}\right|_{\frac{\partial R(z)}{\partial z}=0} = -(c-s+h)\left(\frac{\beta-2}{\beta-1}\frac{r(z)}{\mu-\Theta(z)}+\frac{\left(\frac{\partial r(z)}{\partial z}+r^2(z)\right)}{1-F(z)}\right) \quad (4.27)
$$

This value is negative if $b > 2$ and $\left(\frac{\partial r(z)}{\partial z}+r^2(z)\right) > 0$. If this holds, $\frac{\partial \pi(z)}{\partial z}$

crosses zero once. To the left of that point, $\pi(z)$ is increasing and to the right of that point $\pi(z)$ is decreasing since the value of $\frac{\partial \pi(z)}{\partial z}$ at the lower and upper bound of the feasible region is positive and negative, respectively. ∎

4.2.2 Multi-Period Model

Like the additive model, the general multi-period problem when the demand is multiplicative can be modeled as a stochastic dynamic programming model. The recursion formula is given as:

$$\pi_t'(x_t) = \max_{p;z \geq x_t-(\alpha_t-\beta_t p)} \begin{cases} \pi(z,p|s=0) + c_t x_t + E(\pi_{t+1}(\alpha_t p^{-\beta_t} max\{z-d_t,0\})) & \text{if } t \neq T \\ \pi(z,p) + c_t x_t & \text{if } t = T \end{cases} \tag{4.28}$$

However, as states come from a continuous set, it is very hard to find the optimal policy simply by solving these equations. This is an extremely difficult problem. So far, there is no work on this problem that shows the form of the optimal policy or attempts to calculate it.

4.3 Fixed Point Iteration Method

Since finding the optimal order quantities by dynamic programming recursions is hard, we need to resort to another technique. In this thesis, we propose fixed-point search as an effective search method.

4.3.1 Fixed Point Iteration Theory

The fixed point iteration method considers finding the roots of the following n equations:

$$\begin{aligned} x_1 &= f_1(x_1, x_2, \ldots, x_n) \\ x_2 &= f_2(x_1, x_2, \ldots, x_n) \\ &\vdots \\ x_n &= f_n(x_1, x_2, \ldots, x_n) \end{aligned}$$

In matrix notation:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \text{ and } \mathbf{f(x)} = \begin{bmatrix} f_1(x_1, x_2, \ldots, x_n) \\ f_2(x_1, x_2, \ldots, x_n) \\ \vdots \\ f_n(x_1, x_2, \ldots, x_n) \end{bmatrix}$$

Let us denote the solution of these equations by x^*. The fixed point iteration method updates the estimate x by substituting $f(x)$ for x. Specifically, in iteration $n+1$, we calculate iteration is $(x)_{n+1}$=$f((x)_n)$ where $(x)_n$ is the n^{th} guess for x^* and $(x)_0$ is the initial guess.

4.3.2 Fixed Point Iteration for Single-Period Problem

For our problem, in both forms of demand uncertainty, there are two equations that optimal p and z satisfy. Thus, the fixed point iteration can be used to find the optimal point. In the additive case, the steps of the search are:

1. Start from $z_0 = F^{-1}\left(\frac{p_0-c+b}{p_0+h-s+b}\right)$ and $p_0 = \frac{\alpha+\beta c+\mu}{2\beta}$ which is shown to be an upper bound on optimal price by Petruzzi and Dada (1999). The variable z_0 is also an upper bound on optimal z since the derivative of $\Theta(z)$ is nonnegative.

2. Using p_n and z_n, solve $p_{n+1} = \frac{\alpha-\Theta(z_n)+\beta c+\mu}{2\beta}$ and $z_{n+1} = F^{-1}\left(\frac{p_n-c+b}{p_n+h-s+b}\right)$ for p_{n+1} and z_{n+1}.

3. Stop the iterations if the difference between values in the consecutive iterations is less than a pre-specified percentage. Check the boundary points in case of the existence of optimality at boundary points. Otherwise go to Step 2.

Similarly, for the multiplicative case, the steps are:

1. Start from $z_0 = F^{-1}\left(\frac{p_0-c+b}{p_0+h-s+b}\right)$ and $p_0 = \frac{\beta c}{\beta-1}$ which is shown to be a lower bound on optimal price.

2. Solve $z_{n+1} = F^{-1}\left(\frac{p_n-c+b}{p_n+h-s+b}\right)$ and $p_{n+1} = \frac{\beta(c-s+h)\Lambda(z_n)+(b-c)\Theta(z_n)+c\mu}{(\beta-1)(\Theta(z_n)-\mu)}$ consecutively for p_{n+1} and z_{n+1}, using p_n and z_n.

3. Stop the iterations if the difference between values in the consecutive iterations is less than a pre-specified percentage. Check the boundary points in case of the existence of optimality at boundary points. Otherwise go to Step 2.

The following theorem gives the conditions on the convergence of the fixed-point iteration method.

Theorem 4.3.1 *Let x^* be a fixed point of $f(x)$, i.e., the solution to $x = f(x)$ and assume that the components of $f(x)$ are continuously differentiable in some neighborhood around x^*. Let $J(x)$ be the Jacobian matrix for the functions f_1 and f_2, that is:*

$$J(x) = \begin{bmatrix} \frac{\partial f_1(x)}{\partial x_1} & \frac{\partial f_1(x)}{\partial x_2} \\ \frac{\partial f_2(x)}{\partial x_1} & \frac{\partial f_2(x)}{\partial x_2} \end{bmatrix}$$

Further assume that:

$$\parallel J(x^*) \parallel_\infty = \max_{1\leq i\leq 2} \sum_{j=1}^{n} \left| \frac{\partial f_i(x)}{\partial x_j} \right| < 1$$

Then, for $(x)_0$ chosen sufficiently close to x^*, the iteration $(x)_{n+1}$=$f((x)_n)$ will converge to x^*.

Proof See Atkinson (1988). ∎

This theorem proves the convergence if the stated condition is satisfied. However, there is no mention of the rate of the convergence.

In our problem in the additive case, $J(x)$ is given as:

$$\begin{aligned} J(x) &= \begin{bmatrix} \frac{\partial\left(\frac{\alpha-\Theta(z)+\beta c+\mu}{2\beta}\right)}{\partial p} & \frac{\partial\left(\frac{\alpha-\Theta(z)+\beta c+\mu}{2\beta}\right)}{\partial z} \\ \frac{\partial F^{-1}\left(\frac{p-c+b}{p+h-s+b}\right)}{\partial p} & \frac{\partial F^{-1}\left(\frac{p-c+b}{p+h-s+b}\right)}{\partial z} \end{bmatrix} \\ &= \begin{bmatrix} 0 & \frac{1-F(z)}{2\beta} \\ \frac{\partial F^{-1}\left(\frac{p-c+b}{p+h-s+b}\right)}{\partial p} & 0 \end{bmatrix} \end{aligned} \tag{4.29}$$

To prove the convergence for our problem for the additive case, we need to show that $\left|\frac{\partial F^{-1}\left(\frac{p-c+b}{p+h-s+b}\right)}{\partial p}\right| < 1$ at the optimality.

In the multiplicative case, $J(x)$ is given as:

$$J(x) = \begin{bmatrix} \frac{\partial\left(\frac{\beta(c-s+h)\Lambda(z)+(b-c)\Theta(z)+c\mu}{(\beta-1)(\Theta(z)-\mu)}\right)}{\partial p} & \frac{\partial\left(\frac{\beta(c-s+h)\Lambda(z)+(b-c)\Theta(z)+c\mu}{(\beta-1)(\Theta(z)-\mu)}\right)}{\partial z} \\ \frac{\partial F^{-1}\left(\frac{p-c+b}{p+h-s+b}\right)}{\partial p} & \frac{\partial F^{-1}\left(\frac{p-c+b}{p+h-s+b}\right)}{\partial z} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \frac{(b\mu(F(z)-1)+(c-s+h)(\Theta(z)-\mu)+(F(z)-1)(c-s+h)z)}{(\beta-1)(\Theta(z)-\mu)^2} \\ \frac{\partial F^{-1}\left(\frac{p-c+b}{p+h-s+b}\right)}{\partial p} & 0 \end{bmatrix} \quad (4.30)$$

To prove the convergence for the multiplicative case, we need to show that $\left|\frac{\partial F^{-1}\left(\frac{p-c+b}{p+h-s+b}\right)}{\partial p}\right| < 1$ and $\frac{(b\mu(F(z)-1)+(c-s+h)(\Theta(z)-\mu)+(F(z)-1)(c-s+h)z)}{(\beta-1)(\Theta(z)-\mu)^2} < 1$ at the optimality.

4.3.3 Fixed Point Iteration for Multi-Period Replenishment Problem with Pricing

To find the order-up-to levels for the multi-period case, the fixed point iteration method can be mixed with the ideas of myopic policies explained in Section 3.4.1. To be precise, for the periods other than the last period, next period's ordering cost is inserted for the salvage value in the single period model. Then, the fixed point iteration method is applied to the periods separately as if they are independent from each other to find the order-up-levels and prices.

4.4 Summary

In this chapter, the problem introduced in the previous chapter is extended to include the prices. However, the problem is harder in this case. To find the optimal solution even for the single period case requires a search. Moreover, in the multi-period case, no method is proposed that finds the optimal point. The fixed point iteration method is proposed to find the optimal point in both one-period and multi-period case. Nevertheless, its performance is not proved theoretically. In the next chapter, this search method is tested to see its performance for some test problems.

Chapter 5

Experimental Tests

The demand distribution is taken as normal distribution for the experimental tests. The reason for this choice is the fact that normal distribution is a well approximation of most processes in the nature. The algorithms are coded in C++ language. For the approximations of cumulative distribution, probability distribution and inverse cumulative functions of standard normal distribution, the ones given by Abramowitz and Stegun (1965) are used.

As emphasized in the previous sections, an efficient search method is not proposed in the literature for the replenishment with pricing problem in the multi-period case. As a result, the only way to obtain a benchmark against which fixed point iteration algorithm (FPIS) will be compared is doing exhaustive search (ES) by discretizing the inventory levels and demand process. In the multi-period case, since the number of computations in exhaustive search explodes with the number of periods and it becomes difficult to make an exhaustive search to benchmark, the number of periods is chosen as two.

All statistical tests are done with Microsoft Office Excel 2003. The maximum number of iterations of FPIS is selected as 25.

5.1 Values of the Parameters

The values of the parameters that are used in the calculation of reorder times when the prices are fixed are presented in Table 5.1.

Table 5.1: Values of the Parameters for Fixed Prices Case

Parameter	Value
μ	10, 60, 110
σ	1, 5
p_1	10
p_2	10
c_1	1, 2, ..., 9
c_2	1, 2, ..., 9
h_1	1, 3
h_2	1, 3
b_1, b_2	2
s	1

The parameter value combinations that violate the assumptions mentioned in Section 3.2 are discarded. For example; the cases in which $c_1 = 1$, $h_1 = 3$ and $c_2 = 5$ is not considered in the analysis since $c_2 < c_1 + h_1$ must hold. After eliminating the combinations which violate assumptions, a total of 1260 combinations are tested, instead of 1944 combinations.

For the pricing model, the values of the parameters for the one-period demand case are presented in Table 5.2.

Table 5.2: Values of the Parameters for One-Period Case

Parameter	Additive Case	Multiplicative Case
μ	0, 50, 100	10, 50, 100
σ	1, 5	1, 5
c	1, 5, 9	1, 5, 9
h	1, 5	1, 5
b	1, 5	1, 5
s	1, 5, 9	1, 5, 9
α	20, 60	20, 60
β	1, 5	1.5, 5

There are 432 possible combinations of the parameters. However, again the combinations that conflict with the assumptions are eliminated from the analysis. For both additive and multiplicative forms of demand uncertainty, this makes 288 feasible combinations of values of the parameters.

For the multi-period pricing model, the values of the parameters are provided in Table 5.3.

Table 5.3: Values of the Parameters for Multi-Period Case

Parameter	Additive Case	Multiplicative Case
μ_1, μ_2	30, 60	30, 60
σ_1, σ_2,	1, 5	1, 5
c_1, c_2	1, 5, 9	1, 5, 9
h_1, h_2	1, 5	1, 5
b_1, b_2	2	2
s	1	1
α_1, α_2	20, 60	20, 60
β_1, β_2	1, 5	1.5, 3

The number of feasible combinations is 4096 out of 9216.

5.2 Efficiency Tests for Replenishment Problem with Fixed Prices

There is no proposed algorithm for this problem, and the reorder time is found by exhaustive search. Thus, efficiency tests are not applicable here.

5.3 Efficiency Tests for Replenishment Problem with Pricing

First of all, the CPU time of the FPIS is compared with the CPU time of the ES procedure so as to find the computational performance of FPIS. The CPU time is measured by the number of clocks ticks of elapsed processor time, using clock function of Visual C++ Library. Approximately 1000 clock ticks is equal to 1 second, so 1 clock tick can be taken as 1 millisecond. Secondly, the solution found by FPIS is compared against the optimal solution found by the ES procedure. The performance measure selected is the percentage error (PE), which is the ratio of absolute difference of the solution of the fixed point iteration and the solution of the exhaustive search to the solution of the exhaustive search. PE is calculated for every period, t, and for the profits (PE_t^{Pr}), the stocking factors (PE_t^z) and the prices (PE_t^p). ES searches by discretizing the inventory and demand levels whereas FPIS searches over all points in the feasible region. Therefore, in some cases, solution of FPIS turns out to be better than that of ES. Hence, the absolute difference is preferred in the analysis.

5.3.1 Additive Demand One-Period Model

The total CPU time of FPIS is 90 clocks for 288 combinations. However, average CPU time of ES is 27 clocks and maximum CPU time of ES is 90 clocks. The percentage errors are presented in Table 5.4.

Table 5.4: Results for One-Period Additive Demand

	FPIS
Max PE^{Pr}	0.000003694
Average PE^{Pr}	0.000000111
Max PE^z	0.000063625
Average PE^z	0.000006492
Max PE^p	0.028837320
Average PE^p	0.000713173

The average and the maximum percentage errors for all decision variables and profits are very small.

5.3.2 Additive Demand Multi-Period Model

The total CPU time of FPIS is 450 clocks for 4096 combinations. The average and maximum CPU time of ES is 1484 and 6179, respectively. The CPU time of FPIS in multi-period case is 5 times the CPU time of FPIS of one-period case. On the other hand, the CPU time of ES is 50 times that of one-period case. Roughly speaking, the rate of CPU time increase of ES is 10 times that of FPIS. This shows that even if the CPU time of ES seems very reasonable in this situation, it will explode as the number of periods increase.

For the last period, it is known that the formulae used by the FPIS is indeed the formulae that the optimal point satisfies. Thus, for the second period, we expect FPIS to yield similar results as the one-period case in terms of all performance measures. In fact, this is the case. The CPU time, PE_2^z and PE_2^p are presented in Table 5.5. PE_2^{Pr} is not recorded for this case since second period profit is irrelevant.

Table 5.5: Results for Multi-Period Additive Demand

FPIS

Max PE_2^z	0.0009775
Average PE_2^z	0.006342177
Max PE_2^p	0.0181
Average PE_2^p	0.004192741

The results are similar to the one-period model. Period 1's results are presented in Table 5.6.

Table 5.6: Results for Multi-Period Additive Demand

FPIS

Max PE_1^{Pr}	0.290765013
Average PE_1^{Pr}	0.001569642
Max PE_1^z	0.022328125
Average PE_1^z	0.005812635
Max PE_1^p	0.098656
Average PE_1^p	0.025596184

Although the average value PE_1^{Pr} is reasonable, the maximum value of PE_1^{Pr} is not. However, only 139 of 4096 combinations is greater than 1%. In order to determine the effect of parameter values on the error profit, independent sample t-tests are conducted. First we checked whether $\beta_1 + \beta_2$ has a negative impact on PE_1^p. In sample 1, $\beta_1 + \beta_2$ is taken as 10 and in sample 2 as 2. The variables tested are the optimal profit, z and p. With 1-3.81389E-07 significance level, the hypotheses that means are equal are rejected in favor of the hypothesis that PE_1^p is higher when $\beta_1 + \beta_2 = 10$. A similar test is conducted so as to determine the effect of $\alpha_1 + \alpha_2$. We can say that as $\alpha_1 + \alpha_2$ increases, PE_1^p decreases with 1-4,01886E-11 significance level. In fact, as α decreases and β increases, deterministic part of the demand decreases. Also, when the price is high, stochastic part of the demand may not offset the negativity of the deterministic part so total demand becomes negative. However, there cannot be negative demand in reality. If this is the case, firms prefer not to do business. Thus, most of the bad results in PE_1^p is a result of these combinations that test procedure automatically creates but do not exist in reality.

It should be noted that the performance measures of FPIS algorithm in the multi-period case is an indicator of the performance of not only the fixed point iteration search algorithm but also the applying the results of fixed pricing problem to pricing problem.

5.3.3 Multiplicative Demand One-Period Model

The total duration of fixed point iteration is 60 clocks for 288 combinations. However, the average CPU time of ES is 43 clocks and the maximum CPU time of ES is 101 clocks.

The percentage errors are presented in Table 5.7.

Table 5.7: Results for One-Period Multiplicative Demand

FPIS

Max PE^{Pr}	0.000005653
Average PE^{Pr}	0.000000167
Max PE^{z}	0.748285692
Average PE^{z}	0.256781447
Max PE^{p}	0.000773488
Average PE^{p}	0.000116242

The average and the maximum percentage errors for the expected optimal profit and prices are fairly good. However, the percentage errors for optimal stocking factors are very large. This shows that for the optimal profits of test problems are very insensitive to the changes in stocking factor. These differences in percentage errors may also be the result of the properties of the model.

5.3.4 Multiplicative Demand Multi-Period Model

The total CPU time of FPIS is 230 clocks for 4096 combinations in the multiplicative case. The average and maximum CPU time of ES is 2286 and 15993, respectively.

The performance measures of the second period, PE_2^z and PE_2^p, are similar to the one-period case because the procedure used is the same as one-period case. The average and maximum values of these quantities of PE_1^{Pr}, PE_1^z and PE_1^p are presented in Table 5.8.

Table 5.8: Results for Multi-Period Mult. Demand

FPIS

Max PE_1^{Pr}	1.510564950
Average PE_1^{Pr}	0.045335466
Max PE_1^{z}	0.019117241
Average PE_1^{z}	0.006923497
Max PE_1^{p}	0.363565000
Average PE_1^{p}	0.046984073

To see the effect of values of variables in PE_1^{Pr}, paired t-tests are conducted. The effect of β_1 on PE_1^{Pr}, all combinations are separated into two samples, in the first sample, β_1 is 5 and the average value of PE_1^{Pr} is 0.090021857. In the second sample, β_1 is 1 and the average value of PE_1^{Pr} is 0.000649075. The result of the t-test is that we can say that the average value of PE_1^{Pr} is higher when β_1 is 5 with 1-2.90098E-42 significance level. As β increases, the deterministic part of the demand gets closer to 0, thus the overall demand. In real cases, β must be smaller than 5.

5.4 Managerial Insights

5.4.1 Replenishment Problem with Fixed Prices

The replenishment problem with fixed prices is a wide-known problem. There is an extensive literature that deals with it. Lau and Lau (1998) solves this problem when the cost parameters except c is taken as zero and all parameters are stationary. They assume that the demand is normally distributed and there are two periods in the season. Besides observing the behavior of the order variables and profit, they study variable reorder time. The variable reorder time means that time of the second replenishment can be anytime between the start and the end of the season. They assume that the mean and the variance of demand is linearly proportional to the length of the periods, that is the demand uncertainty is uniform over the season. They make the following observations:

1. The optimal reorder time decreases as the ratio (c/p) increases.

2. The optimal reorder time decreases as σ increases.

These observations are not repeated here. As our model incorporates nonstationary parameters, salvage value and costs of the loss of goodwill and holding inventory into the model of Lau and Lau, the effect of these added variables is observed.

In this section, k will represent the ratio of the length of the first period to the length of the season. For example, when k is 0.1 and the length of the season is

10 weeks, the length of the first period is 1 week and the second season is 9 weeks. It is assumed that $0.1 \leq k \leq 0.9$. Just for this section, μ and σ represent the mean and standard deviation of the season's total demand rather than those of the individual periods.

As the first case, the effect of relative values of c_1 and c_2 is tested. To see the effect of the ratio c_1/c_2 on the optimal value of k, an independent samples t-test is done. The results are separated into two samples: in the first sample c_1/c_2 ratio is less than 1, whereas in the second sample c_1/c_2 ratio is greater than 1. The mean of the first and the second sample are 0.102430556 and 0.895722222. With 100% significance level, the hypothesis that the optimal reorder levels in two samples are equal in favor of the hypothesis that the optimal reorder level in the second sample is higher. So, as the value of c_1 increases relative to c_2, it is becomes more attractive to reorder earlier in the season. The decision maker prefers to give the order as early as he/she can.

A t-test is conducted so as to see the effect of standard deviation on the reorder time. With 1-0.102113663 significance level, we can say that the optimal reorder time decreases with standard deviation. Also, with 1-3.2585E-07 significance level, we can say that the optimal reorder time increases with μ.

These results are consistent with the results of Lau and Lau.

5.4.2 Replenishment Problem with Pricing

In this section, the FPIS method proposed in the previous chapter is tested for both single and multi-period settings.

Tests for One-Period Model

The following observations are made about the optimal expected profit:

- It is evident that as μ, α and s increases, and as c, h and b decreases, the expected profit increases. Also, as μ increases, z increases.

- To observe the effect of uncertainty of the demand on the optimal profit of the model, paired t-tests are conducted. The population of combinations are separated into two samples. In the first one, σ is 1, whereas in the second sample, σ is 5. We can reject the hypothesis that expected profits in two samples are equal with 1-1,25611E-23 and 1-9,33936E-16 significance level in favor of the hypothesis that the expected profit in the first sample is higher, in additive and multiplicative cases, respectively.

The following observations are made about the optimal p and z:

- Another paired t-test is conducted so as to verify the effect of σ on the optimal z. In the first sample σ is 1 whereas in the second sample, σ is 5. With 1-1.25611E-23 and 1-9.00213E-08 significance level, we reject the hypothesis that expected profits in two samples are equal in favor of the hypothesis that the optimal z in the first sample is higher, in additive and multiplicative cases, respectively. So, as the uncertainty increases, firm reacts with buying less.
- No significance relation is found between α and the optimal z.
- With 1-9.82797E-39 significance level, we conclude that as the optimal p increases as α increases in the additive case. There is no observable effect of α on optimal price in multiplicative case.

Tests for Multi-Period Model

The following observations are made about the optimal expected profit:

- As α_1, α_2, μ_1 and μ_2 increases the optimal expected profit increases in both additive and multiplicative forms of demand uncertainty. Similarly, as the costs increase, the expected profits decrease.
- As σ_1 increases in the additive model, the expected profit decreases with a significance level of 1-0.151140507 which is not so high. However, as σ_2 increases, we are more confident about the decrease in the optimal profit: the significance level is 1-8,35036E-08. The effect of σ on the multiplicative demand: the significance levels for rejection are 0.217137782 and 0.09178996 for σ_1 and σ_2, respectively. Yet, the effect of the uncertainty in the second period is still higher.

- As β increases in multi-period case, with the same price level, firm has a lower level of demand. Thus expected profit decreases as β increases. This fact is also verified by the results with significance levels of rejection as high as 1.

The following observations are made about the optimal z_1 and p_1:

- The most evident result is that as μ increases, z increases as z is a substitute for μ in the additive demand model. Also, we are 1-0.000525195 sure that p decreases as μ increases for additive demand. This is logical since as the inventory level increases, average level of demand must increase, the only way to increase the average demand is to decrease price. This relationship is not observed for the multiplicative demand case.

- With a 1-0.001705295 significance level for the additive model, we can say that the ratio z_1/z_2 increases as the ratio c_1/c_2 decreases. This means that as it becomes more expensive to buy in period 1, optimal order-up-to levels decrease relatively. For the multiplicative case, we are less confident about this relationship (1-0.098793367), however, it still exists.

5.5 Summary

In this chapter, the solution method proposed in the previous chapter (FPIS) is tested for all single and multi-period, additive and multiplicative problems. The optimal solution is found by exhaustive search (ES) and used as a benchmark. The CPU time of the FPIS outweighs the CPU time of ES. When the solutions are compared, we can say that in most of the cases, the solutions found by FPIS and ES are very close. As a general remark, the results of the additive demand model is better than the results of the multiplicative demand model. In the multi-period case, the results of the first period is worse than the results of the last period. So even if the convergence is not proved for FPIS, it works well for the test problems proposed in this chapter.

Chapter 6

Conclusion and Future Research Directions

In this thesis, the problem of joint pricing and replenishment decisions is analyzed. Even if the properties of the optimal solution is explored by various researchers, no one proposes an efficient method to find it for the general case. This thesis proposes a search method to find the optimal solution as a contribution to this problem. Some test problems are created and this search method is tested for these problems. The results are satisfactory; however, there is still some work to be done.

First of all, the test problems generated in this study may need to be improved. The expected profit seems fairly insensitive to the changes in price and stocking factor in these test problems: large changes in price and stocking factor leads to small changes in expected profit. Therefore, the method may be unsuccessful to find the optimal price and stocking factor even though it seems to find a very close solution to the optimal in terms of the expected profit. However, it should be noted that this insensitivity of the expected profit may also be an inherent property of the model and may be unrelated to the test problems.

Secondly, the proposed heuristic should be tested in more than two periods. Only one reorder during the season is not an uncommon case and has more interesting properties than the general multi-period problem. However, as this method can be used in problems with more than two periods, it should be tested for problems with more than two periods. The limitation is the lack of a benchmark for comparison. The CPU time of exhaustive search method explodes with the number of periods, thus, we could not get any benchmark to compare the performances as there is no other known method of obtaining the optimal solution.

The stochastic parts of the demand are assumed to normally distributed in this study. The convergence of the method is not proved for this distribution, thus the method may fail in some environments. Various researchers in the literature assumed other distributions, like the exponential distribution. An inevitable future research direction is analyzing and testing the problem for other demand distributions.

The method to find the order-up-to levels and prices for the periods other than the last period in multi-period problem uses the idea of myopic policies. However, myopic policies give only an upper bound when the prices are fixed. Another idea is to incorporate the lower bound and use the heuristic proposed in section 3.4.2. Both heuristics can be solved and the best one can be selected as the CPU time of both methods will be similar and negligible. Additionally, the performance of the proposed method should be proved theoretically.

The method that is used to compute the results of the proposed method with the optimal point should also be criticized. When the proposed method finds a better solution than the optimal because of the resolution of the exhaustive search method, the error is still taken as the absolute difference rather than zero. If the error was taken as zero, the performance of the method would look better.

Appendix A

Tables of t Test

The tables below are the tables generated by Microsoft Excel 2003. In the tables, df means degrees of freedom. The numbers in parenthesis next to the independent parameters are the values of the independent variables in the first and second sample, respectively. Other terms are self explanatory.

Table A.1: $(\beta_1 + \beta_2)$ (10-2) vs PE_1^p. Multi-Period Additive

t-Test: Paired Two Sample for Means

	Variable 1	Variable 2
Mean	0.004985961	0.000128384
Variance	0.000980503	1.73987E-08
Number of Observations	1024	1024
Pearson Correlation	0.546517962	
Hypothesized Mean Difference	0	
df	1023	
t Stat	4.975577255	
P(T≤t) one-tail	3.81389E-07	
t Critical one-tail	1.646344496	
P(T≤t) two-tail	7.62779E-07	
t Critical two-tail	1.962285575	

Table A.2: $(\alpha_1 + \alpha_2)$ (40-120) vs PE_1^p. Multi-Period Additive

t-Test: Paired Two Sample for Means

	Variable 1	Variable 2
Mean	0.007020486	0.000847007
Variance	0.000937452	1.69563E-06
Number of Observations	1024	1024
Pearson Correlation	0.436495733	
Hypothesized Mean Difference	0	
df	1023	
t Stat	6.569216019	
P(T≤t) one-tail	4.01886E-11	
t Critical one-tail	1.646344496	
P(T≤t) two-tail	8.03773E-11	
t Critical two-tail	1.962285575	

Table A.3: c_1/c_2 (<1 - >1) vs k. Two-Period Fixed Pricing

t-Test: Two-Sample Assuming Unequal Variances

	Variable 1	Variable 2
Mean	0.895722222	0.102430556
Variance	0.000270425	9.55919E-05
Number of Observations	180	864
Hypothesized Mean Difference	0	
df	206	
t Stat	624.6195439	
P(T≤t) one-tail	0	
t Critical one-tail	1.652284145	
P(T≤t) two-tail	0	
t Critical two-tail	1.971546622	

Table A.4: σ (1-5) vs k. Two-Period Fixed Pricing

t-Test: Paired Two Sample for Means

	Variable 1	Variable 2
Mean	0.333079365	0.307857143
Variance	0.123961886	0.115784749
Number of Observations	630	630
Pearson Correlation	0.929979587	
Hypothesized Mean Difference	0	
df	629	
t Stat	4.867359904	
P(T≤t) one-tail	7.15889E-07	
t Critical one-tail	1.647279747	
P(T≤t) two-tail	1.43178E-06	
t Critical two-tail	1.963742534	

Table A.5: μ (110-10) vs k.Two-Period Fixed Pricing

t-Test: Paired Two Sample for Means

	Variable 1	Variable 2
Mean	0.330714286	0.28797619
Variance	0.127626696	0.100842434
Number of Observations	420	420
Pearson Correlation	0.87450683	
Hypothesized Mean Difference	0	
df	419	
t Stat	5.052710873	
P(T≤t) one-tail	3.2585E-07	
t Critical one-tail	1.648498411	
P(T≤t) two-tail	6.517E-07	
t Critical two-tail	1.965641764	

Table A.6: σ (1-5) vs Optimal Profit, One-Period Additive

t-Test: Paired Two Sample for Means

	Variable 1	Variable 2
Mean	1220.336494	1194.782956
Variance	2893739.387	2836472.021
Number of Observations	144	144
Pearson Correlation	0.999935533	
Hypothesized Mean Difference	0	
df	143	
t Stat	11.97638358	
P(T≤t) one-tail	1.25611E-23	
t Critical one-tail	1.655579144	
P(T≤t) two-tail	2.51222E-23	
t Critical two-tail	1.976692167	

Table A.7: σ (1-5) vs Optimal Profit, One-Period Multiplicative

t-Test: Paired Two Sample for Means

	Variable 1	Variable 2
Mean	150.7214447	144.3420212
Variance	63890.45485	60501.33196
Number of Observations	144	144
Pearson Correlation	0.999780795	
Hypothesized Mean Difference	0	
df	143	
t Stat	8.933282872	
P(T≤t) one-tail	9.33936E-16	
t Critical one-tail	1.655579144	
P(T≤t) two-tail	1.86787E-15	
t Critical two-tail	1.976692167	

Table A.8: σ (5-1) vs z, One-Period Additive

t-Test: Paired Two Sample for Means

	Variable 1	Variable 2
Mean	52,51479167	50,50625
Variance	1773,624599	1696,410775
Number of Observations	144	144
Pearson Correlation	0,998803179	
Hypothesized Mean Difference	0	
df	143	
t Stat	10,76696162	
P(T≤t) one-tail	1,80988E-20	
t Critical one-tail	1,655579144	
P(T≤t) two-tail	3,61976E-20	
t Critical two-tail	1,976692167	

Table A.9: σ (5-1) vs z, One-Period Multiplicative

t-Test: Paired Two Sample for Means

	Variable 1	Variable 2
Mean	54.39041667	53.43583333
Variance	1329.80966	1363.659454
Number of Observations	144	144
Pearson Correlation	0.998461243	
Hypothesized Mean Difference	0	
df	143	
t Stat	5.487853475	
P(T≤t) one-tail	9.00213E-08	
t Critical one-tail	1.655579144	
P(T≤t) two-tail	1.80043E-07	
t Critical two-tail	1.976692167	

Table A.10: α (60-20) vs p, One-Period Additive

t-Test: Paired Two Sample for Means

	Variable 1	Variable 2
Mean	36.84470417	24.7648884
Variance	705.4032403	415.1079901
Number of Observations	144	144
Pearson Correlation	0.975030125	
Hypothesized Mean Difference	0	
df	143	
t Stat	17.94108803	
P(T≤t) one-tail	9.82797E-39	
t Critical one-tail	1.655579144	
P(T≤t) two-tail	1.96559E-38	
t Critical two-tail	1.976692167	

Table A.11: σ_1 (1-5) vs Optimal Profit, Multi-Period Additive

t-Test: Paired Two Sample for Means

	Variable 1	Variable 2
Mean	1841.45633	1823.997088
Variance	2161427.731	2152427.357
Number of Observations	2048	2048
Pearson Correlation	0.864072901	
Hypothesized Mean Difference	0	
df	2047	
t Stat	1.031814292	
P(T≤t) one-tail	0.151140507	
t Critical one-tail	1.645598358	
P(T≤t) two-tail	0.302281013	
t Critical two-tail	1.96112351	

Table A.12: σ_2 (1-5) vs Optimal Profit, Multi-Period Additive

t-Test: Paired Two Sample for Means

	Variable 1	Variable 2
Mean	1850.570593	1814.882825
Variance	2168527.095	2144843.36
Number of Observations	2048	2048
Pearson Correlation	0.978082781	
Hypothesized Mean Difference	0	
df	2047	
t Stat	5.250941997	
P(T≤t) one-tail	8.35036E-08	
t Critical one-tail	1.645598358	
P(T≤t) two-tail	1.67007E-07	
t Critical two-tail	1.96112351	

Table A.13: σ_1 (1-5) vs Optimal Profit, Multi-Period Multiplicative

t-Test: Paired Two Sample for Means

	Variable 1	Variable 2
Mean	294.8920415	288.7153362
Variance	91918.17078	87330.78562
Number of Observations	2048	2048
Pearson Correlation	0.287375849	
Hypothesized Mean Difference	0	
df	2047	
t Stat	0.782050127	
P(T≤t) one-tail	0.217137782	
t Critical one-tail	1.645598358	
P(T≤t) two-tail	0.434275564	
t Critical two-tail	1.96112351	

Table A.14: σ_2 (1-5) vs Optimal Profit, Multi-Period Multiplicative

t-Test: Paired Two Sample for Means

	Variable 1	Variable 2
Mean	295.7454455	287.8619321
Variance	91312.8158	87924.13568
Number of Observations	2048	2048
Pearson Correlation	0.598808835	
Hypothesized Mean Difference	0	
df	2047	
t Stat	1.330262685	
P(T≤t) one-tail	0.09178996	
t Critical one-tail	1.645598358	
P(T≤t) two-tail	0.18357992	
t Critical two-tail	1.96112351	

Table A.15: β_1 (2-5) vs Optimal Profit, Multi-Period Additive

t-Test: Paired Two Sample for Means

	Variable 1	Variable 2
Mean	2588.939862	1076.513556
Variance	2071540.553	1098191.626
Number of Observations	2048	2048
Pearson Correlation	0.74260429	
Hypothesized Mean Difference	0	
df	2047	
t Stat	70.98893253	
P(T≤t) one-tail	0	
t Critical one-tail	1.645598358	
P(T≤t) two-tail	0	
t Critical two-tail	1.96112351	

Table A.16: β_2 (2-5) vs Optimal Profit, Multi-Period Additive

t-Test: Paired Two Sample for Means

	Variable 1	Variable 2
Mean	2584.882244	1080.571174
Variance	2074766.584	1107212.344
Number of Observations	2048	2048
Pearson Correlation	0.783951975	
Hypothesized Mean Difference	0	
df	2047	
t Stat	75.84877443	
P(T≤t) one-tail	0	
t Critical one-tail	1.645598358	
P(T≤t) two-tail	0	
t Critical two-tail	1.96112351	

Table A.17: β_1 (2-5) vs Optimal Profit, Multi-Period Multiplicative

t-Test: Paired Two Sample for Means

	Variable 1	Variable 2
Mean	443.0998401	140.5075375
Variance	99569.16507	33895.46074
Number of Observations	2048	2048
Pearson Correlation	0.031756153	
Hypothesized Mean Difference	0	
df	2047	
t Stat	38.01258971	
P(T≤t) one-tail	5.4289E-240	
t Critical one-tail	1.645598358	
P(T≤t) two-tail	1.0858E-239	
t Critical two-tail	1.96112351	

Table A.18: β_2 (2-5) vs Optimal Profit, Multi-Period Multiplicative

t-Test: Paired Two Sample for Means

	Variable 1	Variable 2
Mean	425.6713498	157.9360279
Variance	87929.47386	55479.95732
Number of Observations	2048	2048
Pearson Correlation	0.422345655	
Hypothesized Mean Difference	0	
df	2047	
t Stat	41.70312778	
P(T≤t) one-tail	5.7217E-276	
t Critical one-tail	1.645598358	
P(T≤t) two-tail	1.1443E-275	
t Critical two-tail	1.96112351	

Table A.19: μ_1 (30-60) vs p_1, Multi-Period Additive

t-Test: Paired Two Sample for Means

	Variable 1	Variable 2
Mean	14.59375	14.0625
Variance	83.34440645	76.7210552
Number of Observations	2048	2048
Pearson Correlation	0.665203767	
Hypothesized Mean Difference	0	
df	2047	
t Stat	3.281373077	
P(T≤t) one-tail	0.000525195	
t Critical one-tail	1.645598358	
P(T≤t) two-tail	0.00105039	
t Critical two-tail	1.96112351	

Table A.20: c_1/c_2 (1.8-0.2) vs z_1/z_2, Multi-Period Additive

t-Test: Two-Sample Assuming Unequal Variances

	Variable 1	Variable 2
Mean	1.199571491	1.110881369
Variance	0.319444109	0.296038138
Number of Observations	512	1024
Hypothesized Mean Difference	0	
df	988	
t Stat	2.935192123	
P(T≤t) one-tail	0.001705295	
t Critical one-tail	1.64639736	
P(T≤t) two-tail	0.00341059	
t Critical two-tail	1.962367918	

Table A.21: c_1/c_2 (1.8-0.2) vs z_1/z_2, Multi-Period Multiplicative

t-Test: Two-Sample Assuming Unequal Variances

	Variable 1	Variable 2
Mean	1.184125437	1.145071168
Variance	0.315925632	0.307708982
Number of Observations	512	1024
Hypothesized Mean Difference	0	
df	1010	
t Stat	1.289306511	
P(T≤t) one-tail	0.098793367	
t Critical one-tail	1.646363703	
P(T≤t) two-tail	0.197586733	
t Critical two-tail	1.962315493	

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