FUZZY LINEAR PROGRAMMING: REVIEW AND IMPLEMENTATION

by

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to my family

&

Serhat

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FUZZY LINEAR PROGRAMMING: REVIEW AND IMPLEMENTATION

Esra Dervişoğlu

Abstract

Most of the time we encounter problems where quantities cannot be expressed by crisp numbers; instead, the use of vague qualifications is preferred. The Fuzzy Set Theory (FST), developed by Lotfi A. Zadeh, is an effective framework which can be used in the solution of such problems. In FST, the objects belong to a set to a degree between [0, 1]. The degree of belongingness is referred to as the membership degree. FST states that utilizing this membership degree is more profitable than just partitioning the objects dichotomously as in classical bivalent (two-valued) set theory. In most of the problems, where linear programming can be applied, the decision maker chooses to state the inequalities and the coefficients used in objective function and constraints by vague expressions. Fuzzy Linear Programming, based on the FST, is developed to model and solve such problems. In this thesis, the proposed approach, in the most general case, first compares the fuzzy left-hand side with the fuzzy right-hand side and the fuzzy objective function with a fuzzy goal by means of a membership function based on the fuzzy relation using "min" function. After determination of these membership functions associated with the constraints and the objective function, a new auxiliary problem is formed. The obtained auxiliary problem is a non-linear fractional programming problem with its nominator and denominator are defined by linear functions. The optimal solution of such a problem can be found by solving a sequence of linear programs.

In this study, the solution approaches present in the literature for fuzzy linear programming are categorized, some points which are unclear is identified and tried to be improved, and finally the proposed solution methodology is applied to so-called fuzzy analytical hierarchy process.

BULANIK DOĞRUSAL PROGRAMLAMA: İNCELEME VE UYGULAMA

Esra Dervişoğlu

Özet

Çoğu zaman, niceliklerin kesin sayılarla ifade edilmeyip, muğlak nitelemelerin tercih edildiği problemlerle karşılaşmaktayız. Lotfi A. Zadeh tarafından geliştirilmiş olan Bulanık Kümeler Kuramı (BKK) bu tip problemlerin çözümünde kullanılabilecek etkin bir çerçeve sağlamaktadır. BKK'de kavramlar, bir kümeye [0,1] aralığında bir derece ile bağlıdırlar. Bu bağlılık derecesi, aitlik derecesi olarak adlandırılır. BKK kavramların, iki-değerli klasik kümeler kuramında olduğu gibi ikiye ayrılmasındansa aitlik derecesi ile tanımlamasının daha faydalı olduğunu ortaya koymaktadır. Doğrusal programlama yönteminin uygulanabileceği bir çok problemde, karar verici, amaç fonksiyonu ile kısıtlarda kullanılan katsayıları ve eşitsizlikleri, muğlak nitelemeler ile ifade etmeyi tercih etmektedir. Bulanık doğrusal programlama yöntemi, BKK'yi temel alarak, bu tip problemlerin modellenmesi ve çözümlenmesi için geliştirilmiştir. Bu tezde ele alınan problem yapısında çözüm, en genel haliyle, ilk olarak kısıtların sol taraflarının değerini kısıtların sağ tarafının değeri ile, amaç fonksiyonunun değerini dışarıdan verilen bir değerle minimum fonksiyonunu temel alarak karşılaştırmaya dayanmaktadır. Bu aitlik fonksiyonlarının elde edilmesinden sonra, yeni bir yardımcı problem tanımlanmaktadır. aitlik derecesi bulunmaya çalışılmaktadır. Elde edilen yardımcı problem, payı ve paydası doğrusal olan kesirli doğrusal olmayan programlama problemine dönüşmektedir. Bu yeni problemin en iyi çözümü, bir seri doğrusal programlama problemi çözülerek elde edilir.

Bu çalışmada, literatürde önerilen kimi yöntemlerin sınıflandırılması, açık olmayan kimi noktaların aydınlatılması iyileştirmelerin yapılması, son olarak da söz konusu yöntemin analitik hiyerarşi prosesine uygulanması yapılmıştır.

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Chapter 1

Introduction

1.1 Elementary Concepts in Fuzzy Set Theory

Fuzzy set theory derives from the logic underlying the modes of reasoning, which are approximate rather than exact. The importance of fuzzy logic derives from the fact that the modes of human reasoning, and especially, common sense reasoning is approximate in nature; both do not well-define boundaries of referred objects, such as *young man*, *high temperature*, *big size*, and so on. Futhermore, everything is a matter of degree; such as *not quite young*, *high to some extend*, and so on. In 1965, Zadeh states this phenomenon "A fuzzy set is a collection of objects that might belong to the set to a degree, varying from 1 for full belongingness to 0 for full non-belongingness, through all intermediate values " and come up with the fuzzy set theory [41].

A paradigm is a set of rules and regulations which defines boundaries and tells us what to do to be successful in solving problems within these boundaries. Thus fuzzy set theory is a superset of a paradigm shift from the conventional bivalent (two-valued) set theory. Bivalent set theory can be somewhat limiting if a humanistic problem is described mathematically. For example, Figure 1.1 illustrates the bivalent sets to characterise the temperature of a room.

The most obvious limiting feature of bivalent sets can be seen clearly from Figure 1.1; in that they are mutually exclusive. For example, "cold" and "cool" is divided by 1 degrees celcius and at that boundary they have *the same membership*; i.e, they are true at *the same degree*. The conventional (bivalent) set theory represents the

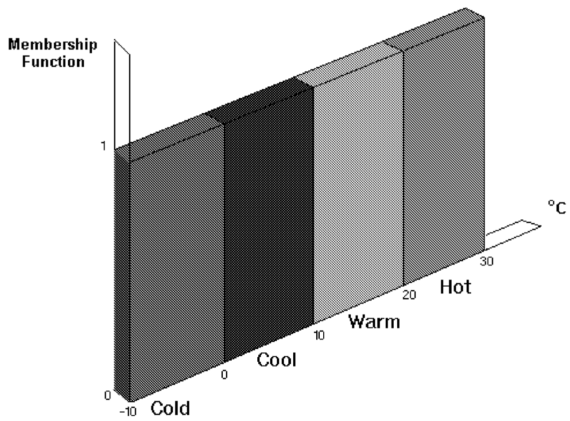

Figure 1.1: Bivalent sets to chacterize the temperature of a room.

temperature in this way; however, the natural system differs. The natural system can be described more accurately by the fuzzy set theory. Figure 1.2 shows how fuzzy sets can describe the natural system [46].

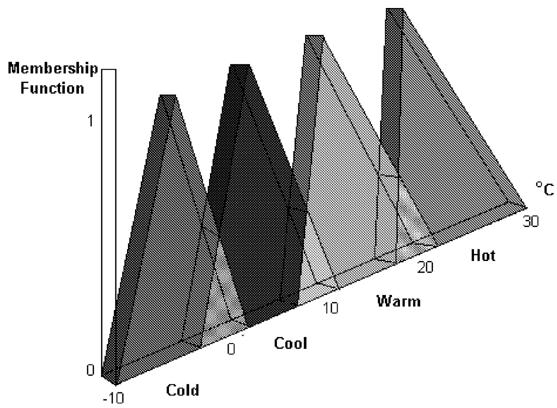

Figure 1.2: Fuzzy sets to chacterize the temperature of a room.

The characteristic function of a fuzzy set, often called the membership function, is a function, whose range is an ordered membership set containing more than two (often a continuum of) values (typically, the unit interval). Therefore, a fuzzy set is often understood as a function. This conception has been a source of criticism from mathematicians [3] as functions are already well-known, and a theory of functions already exists. However, the novelty of the fuzzy set theory, as first proposed by Zadeh, is to treat functions as if they were subsets of their domains. Such functions are intersection, union, complement, inclusion, and so on. By Zadeh they are extended so as to combine functions ranging on an ordered membership set. For example, in elementary fuzzy set theory, the union of functions is performed by taking their pointwise maximum, intersection by pointwise minimum, complementation by means of an order-reversing automorphism of the membership scale, and set-inclusion by the pointwise inequality between functions. Fuzzy set theory indeed closely connects to many-valued logics that appeared in 1930s, if degrees of membership are understood as degrees of truth, intersection as conjuction, union as disjunction, complementation as negation and set-inclusion as implication [9].

Additionally, fuzzy set theory has a number of branches such as fuzzy mathematical programming, fuzzy decision analysis, fuzzy pattern recognition, fuzzy topology, and so on. Here we will deal with fuzzy linear programming (FLP) which is a branch of fuzzy mathematical programming. Before going on to FLP, it is useful to give basic definitions, properties and operations about fuzzy sets. For a more extensive discussion of these elemantary concepts, we refer the reader to [9].

Definition 1.1.1 If X is a collection of objects denoted generically by x, then a fuzzy set $\tilde{A}$ in X is a set of ordered pairs:

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x))|x \in X\}, \tag{1.1}$$

where $\mu_{\tilde{A}}(x)$ is called the membership function or grade of membership (also degree of compatibility or degree of truth) of x in $\tilde{A}$ that maps X to the membership space M. When M contains only the two points 0 and 1, $\tilde{A}$ is a nonfuzzy set and $\mu_{\tilde{A}}(x)$ is identical to the characteristic function in conventional set theory. The range of the membership function is a subset of the nonnegative real numbers whose supremum is finite. Elements with a zero degree of membership function are normally not listed [44].

Example 1.1.1 $\tilde{A} =$ "real numbers around 10"

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)|x \in X\},$$

where

$$\mu_{\tilde{A}}(x) = \begin{cases} 0, & x < 4, x > 16 \\ \frac{x-4}{6}, & 4 \leq x \leq 10 \\ \frac{10-x}{6} & 10 < x \leq 16. \end{cases}$$

The membership function of fuzzy set $\tilde{A} =$ "real numbers around 10" is shown in Figure 1.3:

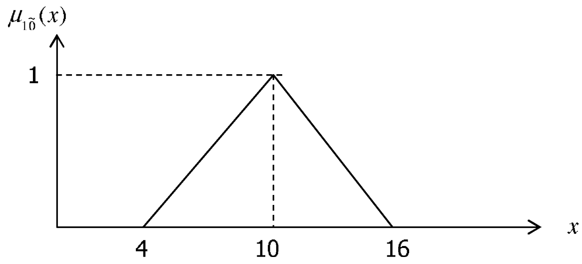

Figure 1.3: Real numbers around 10.

Definition 1.1.2 The fuzzy set $\tilde{A}$ is called normal if $\sup \mu_{\tilde{A}}(x) = 1$. Otherwise, if it is not empty, it can easily be normalized by dividing $\mu_{\tilde{A}}(x)$ by $\sup \mu_{\tilde{A}}(x) = 1$.

From now on if not stated, it is assumed that fuzzy sets are normalized and represented as in equation (1.1.)

Definition 1.1.3 The support of a fuzzy set $\tilde{A}$, $S(\tilde{A})$, is the crisp set of all $x \in X$ such that $\mu_{\tilde{A}}(x) > 0$.

Definition 1.1.4 The height of a fuzzy set $\tilde{A}$, $Hgt(\tilde{A})$, is given by

$$Hgt(\tilde{A}) = \sup\{\mu_{\tilde{A}}(x) | x \in X\}.$$

Definition 1.1.5 The crisp set of elements that belong to the fuzzy set $\tilde{A}$ at least to the degree α is called the α-level set (α-cut):

$$\tilde{A}_\alpha = \{x \in X | \mu_{\tilde{A}}(x) \geq \alpha\}.$$

Figure 1.4 shows the α-cut of $\tilde{A}$:

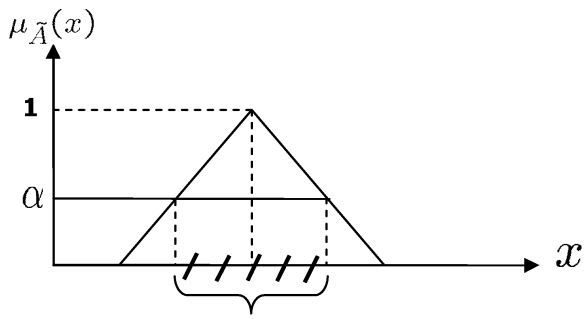

Figure 1.4: α-cut of $\tilde{A}$.

The operators for the intersection and union are named as t-norms and t-conorms respectively. For the intersection of fuzzy sets, the *min*-operator and the algebraic product have been suggested. A general class of operators for the intersection of fuzzy sets is called triangular norms or t-norms, which are operators with two arguments from $[0,1] \times [0,1]$ that satisfy the following conditions:

- $t(0,0) = 0;\ \ t(\mu_{\tilde{A}}(x), 1) = t(1, \mu_{\tilde{A}}(x)) = \mu_{\tilde{A}}, \quad x \in X$ (boundary)
- $t(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) \leq t(\mu_{\tilde{C}}(x), \mu_{\tilde{D}}(x))$ if $\mu_{\tilde{A}}(x) \leq \mu_{\tilde{C}}(x)$ and $\mu_{\tilde{B}}(x) \leq \mu_{\tilde{D}}(x)$ (monotonicity)
- $t(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) = t(\mu_{\tilde{B}}(x), \mu_{\tilde{A}}(x))$ (commutativity)
- $t(\mu_{\tilde{A}}(x), t(\mu_{\tilde{B}}(x), \mu_{\tilde{C}}(x))) = t(t(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x), \mu_{\tilde{C}}(x))$ (associativity)

The most used t-norm is the *min* operator.

For the union of fuzzy sets, the max-operator and the algebraic sum have also been suggested. A general class of aggregation operators for the union of fuzzy sets called triangular conorms or t-conorms (s-norms), which are operators with two arguments that map from $[0,1] \times [0,1]$ into $(0,1]$ that satisfy the following conditions:

- $s(1,1) = 1;\ \ s(\mu_{\tilde{A}}(x), 0) = s(0, \mu_{\tilde{A}}(x)) = \mu_{\tilde{A}}, \quad x \in X$ (boundary)
- $s(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) \leq s(\mu_{\tilde{C}}(x), \mu_{\tilde{D}}(x))$ if $\mu_{\tilde{A}}(x) \leq \mu_{\tilde{C}}(x)$ and $\mu_{\tilde{B}}(x) \leq \mu_{\tilde{D}}(x)$ (monotonicity)
- $s(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) = s(\mu_{\tilde{B}}(x), \mu_{\tilde{A}}(x))$ (commutativity)
- $s(\mu_{\tilde{A}}(x), s(\mu_{\tilde{B}}(x), \mu_{\tilde{C}}(x))) = s(s(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x), \mu_{\tilde{C}}(x))$ (associativity)

The most used t-conorm is the max operator.

One of the most fundamental concepts in fuzzy set theory, which can be used to generalize crisp mathematical concepts to fuzzy sets, is the extension principle. The extension principle can be defined as follows:

Definition 1.1.6 Let X be a cartesian product of universes $X = X_1 \times ... \times X_r$, and $\tilde{A}_1, ..., \tilde{A}_r$ be r fuzzy sets in $X_1, ..., X_r$, respectively. Suppose, f is a mapping from X to a universe Y, $y = f(x_1, ..., x_r)$. Then the extension principle allows us to define a fuzzy set $\tilde{B}$ in Y by

$$\tilde{B} = \{(y, \mu_{\tilde{B}}(y)) | y = f(x_1, ..., x_r), (x_1, ..., x_r) \in X\}, \quad (1.2)$$

where

$$\mu_{\tilde{B}}(y) = \begin{cases} \sup_{(x_1,...,x_r)\in f^{-1}(y)} & \min\{\mu_{\tilde{A}_1}(x_1), ..., \mu_{\tilde{A}_r}(x_r)\}, \quad \text{if} \quad f^{-1}(y) \neq 0, \\ 0, & \text{otherwise,} \end{cases}$$

where f^{-1} is the inverse of f.

For $r = 1$, the extension principle reduces to

$$\mu_{\tilde{B}}(y) = \begin{cases} \sup_{x\in f^{-1}(y)} \quad \mu_{\tilde{A}}(x) & \text{if} \quad f^{-1}(y) \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

The membership functions for fuzzy sets can be in any form (as far as convexity is preserved) but for the most applications, linear membership fuctions are prefered since the convexity is preserved throughtout the operations. The most widely used fuzzy number forms are triangular and trapezoidal forms. Triangular fuzzy numbers are represented as below:

$$\tilde{X} = (x, \underline{x}, \bar{x}),$$

where x is the center value, $\underline{x}$ is the left spread and $\bar{x}$ is the right spread. A triangular fuzzy number $\tilde{X}$ is illustrated in Figure 1.5.

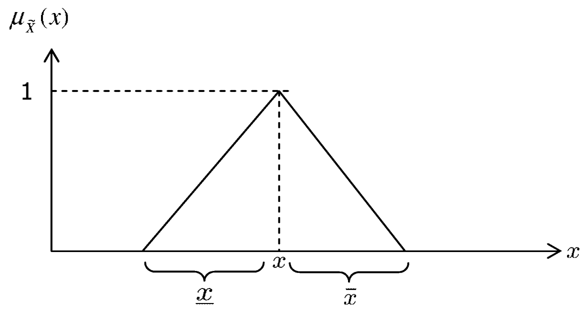

Figure 1.5: A triangular fuzzy number $\tilde{X}$.

The membership function of a triangular fuzzy number, $\tilde{X}$ can be formed as follows:

$$\mu_{\tilde{X}}(t) = \begin{cases} \frac{t-(x-\underline{x})}{\underline{x}}, & \text{if} \quad x - \underline{x} < t < x, \\ 1, & \text{if} \quad t = x, \\ \frac{(x+\overline{x})-t}{\overline{x}}, & \text{if} \quad x + \overline{x} > t > x \\ 0, & \text{otherwise.} \end{cases}$$

Trapezoidal fuzzy numbers are represented as below:

$$\tilde{X} = (x_l, x_r, \underline{x}, \overline{x}),$$

where x_l is the left most value where membership grade equals to 1, x_r is the right most value where membership grade equals to 1, and $\underline{x}$ is the left spread and $\overline{x}$ is the right spread. A trapezoidal fuzzy number $\tilde{X}$ is illustrated in Figure 1.6.

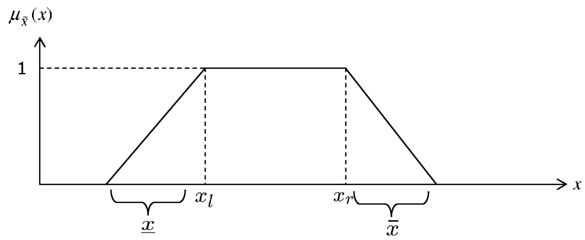

Figure 1.6: A trapezoidal fuzzy number $\tilde{X}$.

The membership function of a trapezoidal fuzzy number $\tilde{X}$ can be formed as follows:

$$\mu_{\tilde{X}}(t) = \begin{cases} \frac{t-(x_l-\underline{x})}{\underline{x}}, & \text{if} \quad x_l - \underline{x} < t < x_l, \\ 1, & \text{if} \quad x_l < t < x_r, \\ \frac{(x_r+\overline{x})-t}{\overline{x}}, & \text{if} \quad x_r + \overline{x} > t > x_r \\ 0, & \text{otherwise.} \end{cases}$$

Fuzzy sets have their own arithmetic to carry on opearations for calculations. Some basic fuzzy operations are discussed below:

For triangular fuzzy numbers, fuzzy addition is carried out as follows:

$$\tilde{X} = (x, \underline{x}, \bar{x}),$$

$$\tilde{Y} = (y, \underline{y}, \bar{y}),$$

$$\tilde{X} + \tilde{Y} = (x + y, \underline{x} + \underline{y}, \bar{x} + \bar{y}).$$

For trapezoidal fuzzy numbers, fuzzy addition is carried out as follows:

$$\tilde{X} = (x_l, x_r, \underline{x}, \bar{x})$$

$$\tilde{Y} = (y_l, y_r, \underline{y}, \bar{y})$$

$$\tilde{X} + \tilde{Y} = (x_l + y_l, x_r + y_r, \underline{x} + \underline{y}, \bar{x} + \bar{y})$$

For triangular fuzzy numbers, subtraction is carried out as follows:

$$\tilde{X} = (x, \underline{x}, \bar{x})$$

$$\tilde{Y} = (y, \underline{y}, \bar{y})$$

$$\tilde{X} - \tilde{Y} = (x - y, \underline{x} + \bar{y}, \bar{x} + \underline{y})$$

For trapezoidal fuzzy numbers, subtraction is carried out as follows:

$$\tilde{X} = (x_l, x_r, \underline{x}, \bar{x})$$

$$\tilde{Y} = (y_l, y_r, \underline{y}, \bar{y})$$

$$\tilde{X} - \tilde{Y} = (x_l - y_r, x_l - y_r, \underline{x} + \bar{y}, \bar{x} + \underline{y})$$

Given a fuzzy number, $\tilde{X}$ and a scalar $k \in R$, the multiplication of a fuzzy number and a scalar is carried out as follows:

(i) $k = 0, \quad k \cdot \tilde{X} \triangleq 0$

(ii) $k \neq 0, \quad \tilde{Z} \triangleq k \cdot \tilde{X} \quad$ iff $\quad \mu_{\tilde{Z}}(z) = \mu_{\tilde{x}}(\frac{z}{k}) \quad \forall z \in Z$

For triangular fuzzy numbers:

$$\tilde{X} = (x, \underline{x}, \bar{x})$$

$$k \cdot \tilde{X} = \begin{cases} (kx, k\underline{x}, k\bar{x}), & k > 0 \\ 0, & k = 0 \\ (kx, -k\bar{x}, -k\underline{x}), & k < 0 \end{cases} \tag{1.3}$$

For trapezoidal fuzzy numbers:

$$\tilde{X} = (x_l, x_r, \underline{x}, \bar{x})$$

$$k \cdot \tilde{X} = \begin{cases} (kx_l, kx_r, k\underline{x}, k\bar{x}) & , \ k > 0 \\ 0 & , \ k = 0 \\ (kx_l, kx_r, -k\bar{x}, -k\underline{x}) & , \ k < 0 \end{cases}$$

1.2 Preliminaries of Fuzzy Linear Programming

A Linear Programming (LP) problem is a special case of a Mathematical Programming problem. From an application perspective, mathematical (and therefore, linear) programming is an optimization tool, which allows the rationalization of many managerial and/or technological decisions required by contemporary techno-socio-economic applications. From an analytical perspective, a mathematical program attempts to identify an extreme (minimum or maximum) point of a function, which furthermore satisfies a set of constraints. Linear programming is the specialization of the mathematical programming to the case where both the objective function and the problem constraints are linear.

General formulation of a linear program is as follows:

$$\begin{array}{lll} \text{Objective Function} & \max / \min & cx \\ \text{Technological Constraints} & \text{s.t.} & Ax \begin{pmatrix} \leq \\ \geq \\ = \end{pmatrix} b, \\ \text{Sign Restrictions} & & \begin{pmatrix} x \geq 0 \\ x \leq 0 \\ x \ \ \text{urs} \end{pmatrix}, \end{array}$$

where c is a vector (objective function coefficient vector) consisting of c_j's, $j = 1, ..., n$, A is a matrix (left-hand side coefficients / technological coefficients matrix) consisting of a_{ij}'s, $i = 1, ..., m$, $j = 1, ..., n$; b is a vector (right-hand side value vector) consisting of b_i's, $i = 1, ..., m$, x is a vector consisting of x_j's, $j = 1, ..., n$, and *urs* means unrestricted in sign.

An important factor for the applicability of the mathematical programming methodology in various application contexts, is the computational tractability of the resulting analytical models. This tractability requirement translates to the existence of effective and efficient algorithmic procedures to provide a systematic and fast solution to these models. For linear programming problems, there are powerful computational methods such as the simplex algorithm and the interior point algorithms.

As stated above, linear programs allow us to rationalize decisions which can truely be rationalized if the problem is stated as real as it behaves in the actual world. Most of the time, vague, subjective and imprecise parameters are converted to crisp numbers by several assumptions and then used in LP problems. In such cases, fuzzy set theory is a very useful tool to state the problem more realistically. The parameters of a linear programming problem can be stated as fuzzy numbers. As well as parameters, inequalities can be given as fuzzy statements. Such a problem is called a Fuzzy Linear Programming (FLP) problem. Of course, it is not necessary to state every component as fuzzy. Depending on the problem, the fuzziness can be only at the objective function, at the left-hand side, at the right-hand side, and at the inequality relation or fuzziness can be at any combination of those. If there is only fuzziness at the inequality relation or at the right-hand side, the model is identified as flexible programming problem.

The general formulation of a fuzzy linear program is as follows:

$$\begin{array}{lll} \text{Objective Function} & \max / \min & \tilde{c}x \\ \text{Technological Constraints} & \text{s.t.} & \tilde{A}x \begin{pmatrix} \tilde{\leq} \\ \tilde{\geq} \\ \tilde{=} \end{pmatrix} \tilde{b}, \\ \text{Sign Restrictions} & & \begin{pmatrix} x \geq 0 \\ x \leq 0 \\ x \ \text{ urs} \end{pmatrix}, \end{array}$$

where $\tilde{c}$ is a vector consisting of $\tilde{c}_j$'s with membership functions $\mu_{\tilde{c}_j}$, $j = 1, ..., n$, $\tilde{A}$ is a matrix consisting of $\tilde{a}_{ij}$'s with membership functions $\mu_{\tilde{a}_{ij}}$, $i = 1, ..., m$, $j = 1, ..., n$, $\tilde{b}$ is a vector consisting of $\tilde{b}_i$'s with membership functions $\mu_{\tilde{b}_i}$, $i = 1, ..., m$, x is vector consisting of x_j's, $j = 1, ..., n$; and *urs* means unrestricted in sign. Due to the complex structure of fuzzy numbers, the fuzzy linear programming (FLP) problem cannot be solved by means of conventional algorithms. Special approaches are needed. Optimization in a fuzzy environment has been first defined by Bellman and Zadeh [4], and their work constitutes the main idea behind the most of the approaches. According to the Bellman and Zadeh [4], the fuzzy decision set can be defined as follows: Let X be the set of alternatives that contain the solution of a given optimization problem; that is, the problem is feasible. Let C_i be the fuzzy domain defined by the i^{th} constraint, $i = 1, ..., m$. Let G_j be the fuzzy domain of the j^{th} goal, $j = 1, ..., J$.

A fuzzy decision is the fuzzy set D on X defined by the following:

$$D = (\cap_{i=1}^{m} C_i) \cap (\cap_{j=1}^{J} G_j).$$

Let μ_D denote a membership function, for the fuzzy decision set membership function is defined as

$$\mu_D(x) = \{\mu_{C_i}(x), \mu_{G_j}(x), i = 1, ..., m, j = 1, ..., J\} \quad \forall x \in X.$$

The final decision, x^f, is chosen from the maximal decision set:

$$M_f = \{x^f \quad |\mu_D(x^f) \geq \mu_D(x)\}.$$

The fuzzy decision set can be very complex due to the structure of the FLP problem. First, the number of fuzzy components - -objective, left-hand side, right-hand side and inequality- - used at time same time effects complexity. Then, the types of membership functions - -linear, exponential, logarithmic- - and the types of t-norms - -min, product- - effect the determination of fuzzy decision set. Yet, most of the time it will not be possible to determine the fuzzy decision set. Because of this difficulty, the studies in the literature focus on the specific cases of the FLP problem. The applications of fuzzy linear program to many decision problems are still untouched.

1.3 Outline

In this thesis, first an extensive literature survey of FLP is presented in Chapter 2. Then in Chapter 3, an improved solution methodology is proposed for a specific FLP problem. Following that to show the applicability of the proposed solution methodology Chapter 4 represented that fuzzy set theory can be used in analytical hierachy process and that problem can be solved by the proposed solution method. Finally, the conclusions and future research is represented in Chapter 5.

Chapter 2

Literature Review

Following the pioneering work of Zadeh [41] on fuzzy sets, many articles have appeared in the literature. The first approach dealing with fuzzy mathematical programming is proposed by Bellman and Zadeh [4]. Consequently, the fuzzy linear programming became an attractive research area. If a simple categorization is made, it is seen that the majority of the research is focused on the use of fuzzy sets in the objective function and in the right-hand side values of constraints. As stated in Section 1.2, this type of FLP models are called flexible programming models. Below, there is a review of solution methods for FLP problems presented in the literature. The solution methods are categorized according to three model types: models with fuzzy objective, models with fuzzy right-hand side (or fuzzy inequality), and models with various combination of fuzzy components. In the following sections we will investigate each model in further detail. Finally, a section is dedicated to the review and discussion of methodologies in the literature.

2.1 Models with Fuzzy Objectives

A FLP problem with a fuzzy objective is modelled as:

$$\begin{aligned} \max \quad & \tilde{c}x \\ \text{s.t.} \quad & Ax \leq b, \\ & x \geq 0, \end{aligned} \tag{2.1}$$

where A is an $m \times n$ matrix of real numbers, b is a vector of real numbers and $\tilde{c}$ is a vector of fuzzy numbers, $\tilde{c}_j$ with the membership function $\mu_{\tilde{c}_j}$, $j = 1, ..., n$.

To solve (2.1), several approaches have been proposed in the literature. They are discussed in the following subsections.

2.1.1 Multiobjective Approach

The multiobjective approach is orginally proposed by Verdegay et al. [33]. Since we were not able to access the original paper we referred to the papers [31,36,37] which have cited it. In [31, 36, 37] FLP problems in (2.1) with triangular fuzzy objective function coefficients, $\tilde{c}_j = (c_j, \underline{c_j}, \overline{c_j})$, are investigated. The membership function of $\tilde{c}_j$ is defined as follows:

$$\mu_{\tilde{c}_j}(t) = \begin{cases} 0, & \text{if } c_j + \bar{c_j} \leq t \text{ or } t \leq c_j - \underline{c_j}, \\ h_j(t), & \text{if } c_j - \underline{c_j} \leq t \leq c_j, \\ g_j(t), & \text{if } c_j \leq t \leq c_j + \bar{c_j} \end{cases} \qquad j = 1, ..., n,$$

where h_j and g_j are assumed to be strictly increasing and decreasing continuous functions, respectively, with $h_j(c_j) = g_j(c_j) = 1, j = 1, ..., n$.

For the solution of the considered problem, multiobjective approach first considers the $(1-\alpha)$-cut of $\tilde{c}_j$, $\alpha \in [0, 1]$,

$$\forall t \in R, \quad \mu_{c_j}(t) \geq 1 - \alpha \quad \Leftrightarrow h_j^{-1}(1-\alpha) \leq t \leq g_i^{-1}(1-\alpha).$$

Later Verdegay et al. [33] proposes that a solution to the considered FLP problem is the parametric solution of the following multiobjective parametric LP problem:

$$\begin{aligned} \max \quad & [c^1x, c^2x, ..., c^{2^n}x] \\ \text{s.t.} \quad & Ax \leq b, \qquad && x \geq 0 \\ & c^k \in E(1-\alpha), && k = 1, ..., 2^n \\ & \alpha \in [0, 1] \end{aligned} \tag{2.2}$$

where $E(1-\alpha)$, for each $\alpha \in [0, 1]$, is the set of vectors in R^n such that each set of its components is either in the upper bound, $h_j^{-1}(1-\alpha)$, or in the lower bound, $g_j^{-1}(1-\alpha)$, of the respective $(1-\alpha)$-cut, that is, $\forall k = 1, 2, ..., 2^n$. To find a parametric solution to the given LP, any classical multiobjective LP approach (vector-maximum method, interactive techniques) can be used.

Hence, in multiobjective approach the decision variable, x is determined based on an auxiliary mathematical programming problem in which all possible combinations of the boundaries of the objective function coefficients (note that there are 2^n such combinations) are treated as objective functions. The auxiliary multiobjective LP is solved for each α-cut parametrically. Note that obtained solution is parametric not unique.

2.1.2 Interval Arithmetic Approach

The interval arithmetic approach is originally proposed by Tanaka et al. [30] and it is also cited in papers [31, 36, 37]. Tanaka et al. [30] investigate FLP problems as in (2.1) with triangular fuzzy objective function coefficients, $\tilde{c}_j = (c_j, \underline{c_j}, \bar{c_j})$.

For the solution, Tanaka et al. [30] consider the α-cuts of the fuzzy numbers and propose that the solution of the following biobjective parametric linear programming problem is the solution of the considered FLP problem:

$$\begin{aligned} \max \quad & z'(\alpha) = (z^l(x,\alpha), z^c(x,\alpha)) \\ \text{s.t} \quad & Ax \leq b, \\ & x \geq 0, \\ & \alpha \in [0,1], \end{aligned} \tag{2.3}$$

where $z^l(x,\alpha)$ and $z^c(x,\alpha)$ are defined by

$$z^l(x,\alpha) = \sum_{j=1}^{n} (c_j - \alpha \underline{c_j}) x_j$$

and

$$z^c(x,\alpha) = \frac{1}{2} \sum_{j=1}^{n} (2c_j + \alpha(\overline{c_j} - \underline{c_j})) x_j.$$

Note that, $z^c(x,\alpha)$ corresponds to the center whenever the objective function coefficients are symmetric fuzzy numbers. Also note that, the interval arithmetic approach is also a multiobjective approach where the objective functions are determined by two special cases of objective function: the lower bound of coefficients and the center value of the coefficients for the given α-cut.

2.1.3 Possibilistic Approach

The possibilistic approach is originally proposed by Tanaka et al. [27, 29] and also cited in [31, 35]. Tanaka et al. [27, 29] investigate FLP problems as in (2.1) with triangular fuzzy objective function coefficients, $\tilde{c}_j = (c_j, \underline{c_j}, \bar{c_j})$.

For the solution the following two properties are used:

(i) The objective function value

$$\tilde{z} = \tilde{c_1}x_1 + \tilde{c_2}x_2 + ... + \tilde{c_n}x_n$$

is also a triangular fuzzy number with the following membership function

$$\mu(y) = \begin{cases} 1 - (|2y - (2c - \underline{c} + \bar{c})|x)/(\underline{c} + \bar{c})x, & x > 0, \\ 1, & x = 0, y = 0, \\ 0, & x = 0, y \neq 0, \end{cases}$$

where $c = (c_1, ..., c_n)$, $\underline{c} = (\underline{c_1}, ..., \underline{c_n})$ and $\bar{c} = (\overline{c_1}, ..., \overline{c_n})$.

(ii) The maximization of a fuzzy set such as $\tilde{z}$ can be also be done by means of

$$\max \;\; w_1 y^U x + w_2 y^L x$$

where $w_1 + w_2 = 1$, $w_1, w_2 \in [0, 1]$, $y^U = c + \bar{c}$ and $y^L = c - \underline{c}$.

Due to the properties stated above, it is proposed that the solution of the considered FLP problem is the solution of the following auxiliary conventional LP problem:

$$\begin{aligned} \max \quad & w_1 c^U x + w_2 c^L x \\ \text{s.t.} \quad & Ax \leq b, \\ & x \geq 0, \end{aligned}$$

where (w_1, w_2) is a pair of values that the decision maker must choose according to his wishes with $w_1, w_2 \in [0, 1]$ and $w_1 + w_2 = 1$.

Solution idea of possibilistic approach is very similar to the multiobjective approach and interval arithmetic approach. Possibilistic approach considers the weighted mean of the lower and upper bounds off the objective function coefficients.

2.1.4 Stratified Piecewise Reduction Approach

Stratified piecewise reduction approach is proposed by [17] and cited in [31, 35]. In [17,31,35] FLP problems as in (2.1) are considered with triangular fuzzy objective function coefficients, $\tilde{c}_j = (c_j, \underline{c_j}, \bar{c}_j)$. Triangular fuzzy numbers are modelled by means of nested intervals, defined by means of a set

$$\tilde{c}_j = \{[c_j^L, c_j^U]^i/\alpha_i; i = 1, ..., m\}, \quad \forall j = 1, ..., n,$$

where each having a membership degree $\alpha_i \in [0,1], i = 1, ..., m$, c_j^L is the lower value and c_j^U is the upper value of α_i-cut, verifying

$$\forall \alpha_1, \alpha_2 \in [0,1], \quad \alpha_1 \geq \alpha_2 \Rightarrow [c_j^L, c_j^U]^1 \subseteq [c_j^L, c_j^U]^2.$$

For each $\alpha_i, i = 1, ..., m$, the following auxiliary problem is proposed as a solution to the considered FLP problem:

$$\begin{aligned} \text{Max} \quad & \lambda \\ \text{s.t.} \quad & f_1[c^{L\alpha}x] \geq \lambda \\ & f_2[c^{U\alpha}x] \geq \lambda, \\ & Ax \leq b, \\ & x \geq 0, \\ & \lambda \geq 0, \end{aligned} \tag{2.4}$$

where $f_1[c^{L\alpha}x]$ and $f_2[c^{U\alpha}x]$ are memberships formed as

$$\begin{aligned} f_1[c^{L\alpha}x] &= (c^{L\alpha}x - z_{\text{min}}^{U\alpha})/(z_{\text{min}}^{*\alpha} - z_{\text{min}}^{U\alpha}) \quad \text{if} \quad z_{\text{min}}^{U\alpha} \leq c^{L\alpha}x \leq z_{\text{min}}^{*\alpha}, \\ f_2[c^{U\alpha}x] &= (c^{U\alpha}x - z_{\text{max}}^{U\alpha})/(z_{\text{max}}^{*\alpha} - z_{\text{max}}^{U\alpha}) \quad \text{if} \quad z_{\text{max}}^{U\alpha} \leq c^{U\alpha}x \leq z_{\text{max}}^{*\alpha}. \end{aligned}$$

On the other hand, by definition

$$z_{\text{min}}^{*\alpha} = c^{L\alpha}(x_{\text{min}}^*) = \max\{c^{L\alpha}x | x \in X\}; z_{\text{min}}^{U\alpha} = c^{L\alpha}(x_{\text{max}}^*),$$

$$z_{\text{max}}^{*\alpha} = c^{U\alpha}(x_{\text{max}}^*) = \max\{c^{U\alpha}x | x \in X\}; z_{\text{max}}^{U\alpha} = c^{L\alpha}(x_{\text{min}}^*),$$

with

$$X = \{x \in R^n | Ax \leq b, x \geq 0\}.$$

The ultimate solution can be found by the intersection of the solutions obtained by solving the auxiliary models (2.4) for each $\alpha_i, i = 1, ..., m$, and that can be done by the following model:

$$\begin{aligned}
\text{Max} \quad & \lambda \\
\text{s.t.} \quad & (z^{*\alpha}_{\min} - z^{U\alpha}_{\min})\lambda - c^{L\alpha}x \leq -z^{U\alpha}_{\min}, \\
& (z^{*\alpha}_{\max} - z^{U\alpha}_{\max})\lambda - c^{U\alpha}x \leq -z^{U\alpha}_{\max}, \\
& Ax \leq b, \\
& x \geq 0, \\
& \lambda \geq 0.
\end{aligned} \tag{2.5}$$

It is evident that

$$f_1[c^{L\alpha}x^*] = f_2[c^{U\alpha}x^*] = \lambda^*$$

with (λ^*, x^*) being the optimal solution of the auxiliary model (2.5).

Note that, in stratified piecewise reduction approach two objective functions that are based on the lower and upper bounds of the coefficients that are obtained for a specific α-cut is considered. The realization degrees of the objective functions are maximized simultaneously.

2.1.5 Progressive Reduction Approach

In [17] FLP problems as in (2.1) are considered. For the solution, a very similar algorithm to the stratified piecewise reduction approach, called the progressive reduction approach, is proposed. In this approach α-cuts are not used, but the reduction idea is the same. The objective function $\tilde{c}x$ is denoted by $z(x)$. Objective function coefficients are given as interval fuzzy numbers, $\tilde{c}_j = (c^L_j, c^U_j)$, where c^L_j is the lower value and c^U_j is the upper value of fuzzy number $\tilde{c}_j$.

Author [17] proposes to reduce the many infinite objective functions by extreme positioning to the two extreme objective functions $z_{\min}(x)$ and $z_{\max}(x)$, where the first one is the minimum value that an objective function can take and the latter, maximum. Then objective function becomes:

$$\max_{x\in X}\begin{pmatrix} z_{\min}(x) \\ z_{\max}(x) \end{pmatrix} = \max_{x\in X}\begin{pmatrix} c^{L'} \cdot x \\ c^{U'} \cdot x \end{pmatrix}. \tag{2.6}$$

The complete solution set of problem (2.6) is a subset of complete solution of the considered FLP problem. To determine a compromise solution of the problem (2.6), a procedure proposed by Zimmermann [43] is used. According to the procedure, at first the maximum objective values $z^*_{\min}$ and $z^*_{\max}$ are determined as solution of the conventional linear programming problems:

$$z^*_{\min} = z_{\min}(x^*_{\min}) = \max_{x \in X} z_{\min}(x),$$
$$z^*_{\max} = z_{\max}(x^*_{\max}) = \max_{x \in X} z_{\max}(x).$$

If the optimal solutions $x^*_{\min}$ or $x^*_{\max}$ are not determined unequivocally, the solution vectors which are most diverging should be chosen. Denoting $\bar{z}_{\min} = z_{\min}(x^*_{\max})$ and $\bar{z}_{\max} = z_{\max}(x^*_{\min})$, it can be assumed that a decision maker is only willing to accept a solution x which has the properties $z_{\min}(x) \geq \bar{z}_{\min}$ and $z_{\max}(x) \geq \bar{z}_{\max}$. From here the following linear membership function can be formed:

$$f_{zk}(x) = \begin{cases} \frac{z_k(x) - \bar{z}_k}{z^*_k - \bar{z}_k}, & \text{with } \bar{z}_k \leq z_k(x) \leq z^*_k \quad k = \{\min, \max\} \\ 0, & \text{otherwise} \end{cases}$$

Then a new problem is formed as follows:

$$\begin{aligned} \max \quad & \begin{pmatrix} f_{z_{\min}}(x) \\ f_{z_{\max}}(x) \end{pmatrix} \\ \text{s.t.} \quad & Ax \leq b, \\ & x \geq 0 \end{aligned} \tag{2.7}$$

Actually the problem defined by the equation (2.7) has the same complete solution as vector optimization problem defined by equation (2.6). To calculate a compromise solution of the problem defined by the equation (2.7), the *min* operator is used as t-norm:

$$\lambda(x) = \min(f_{z_{\min}}(x), f_{z_{\max}}(x))$$

The value $\lambda(x)$ can be interpreted as an expression of total satisfaction on the part of the decision maker, who intends to improve both objectives as well. Then a compromise solution of (2.7) can be calculated by solving the LP problem

$$\begin{aligned} \max \quad & \lambda \\ \text{s.t.} \quad & (z^*_{\min} - \bar{z}_{\min})\lambda - z(x) \leq -\bar{z}_{\min}, \\ & (z^*_{\max} - \bar{z}_{\max})\lambda - z(x) \leq -\bar{z}_{\max}, \\ & A.x \leq b, \\ & \lambda \geq 0, \\ & x \geq 0. \end{aligned}$$

2.1.6 Compromise Objective Function Approach

In [17], a solution method to an FLP problem as in (2.1) with objective function coefficients given as interval fuzzy number, $\tilde{c_j} = [c_j^L, c_j^U]$, $j = 1, ..., n$, is proposed. The main idea of the approach is to transfer the infinitely many objective functions into a single compromise objective function. The simplest way of doing this is to choose a single representative, $\hat{c_j}$ for each interval $[c_j^L, c_j^U]$. Then compromise solution of the FLP problem (2.1) can be obtained by solving the LP problem:

$$\begin{array}{ll} \max & \sum_{j=1}^{n} \hat{c}_j x \\ \text{s.t.} & Ax \leq b, \\ & x \geq 0, \end{array}$$

where $\hat{c}_j$ is assumed to be a value with the highest 'chance of realization' in interval $[c_j^L, c_j^U]$. To determine a representative of a interval, different approaches have been proposed; such as, interval median, rule of Hurwicz, state of a nature as representatives(propobability)-expected value, mode, and so on [17].

2.1.7 Comparison Approach

In [31] an FLP problem as in (2.1) is considered. For the solution, a comparison approach based on the fuzzy ranking methods [5–8,37,39,40] is used. After applying the comparison approach, the following conventional LP problem is obtained whose solution is proposed as the solution of (2.1):

$$\begin{array}{lll} \max & \sum_{j=1}^{n} g(\tilde{c}_j) xj & \\ \text{s.t.} & \sum_{j=1}^{n} a_{ij} x_j \leq b_i & i = 1, \ldots, m \\ & x_j \geq 0, & j = 1, \ldots, n \end{array}$$

where g is a ranking function. For example, if g is defined by Average's index and objective function coefficients are given as trapezoidal fuzzy numbers, $\tilde{c}_j = (c_{l_j}, c_{r_j}, \underline{c_j}, \overline{c_j})$, the proposed LP becomes the following auxiliary problem:

$$\begin{array}{lll} \max & \sum_{j=1}^{n} (c_{l_j} - \frac{\underline{c_j}}{t+1} + \lambda(c_{r_j} - c_{l_j}) + \frac{\overline{c_j} + \underline{c_j}}{t+1})) x_j & \\ \text{s.t.} & \sum_{j=1}^{n} a_{ij} x_j \leq b_i & i = 1, \ldots, m \\ & x_j \geq 0, & j = 1, \ldots, n \end{array}$$

where λ and t fixed by the decision maker.

2.1.8 Fuzzy Aspiration Level Approach

In [18] Rommelfanger considers an FLP problem as in (2.1). For the solution, an approach totally different from the ones decribed so far is proposed. The approach will not be stated here in detail since additional knowledge about how to deal with fuzzy left-hand side, fuzzy right-hand side and fuzzy inequality is necessary. However, the basic idea behind the approach is to treat objective function as a constraint. In order to do so, first a fuzzy aspiration level for the objective function is defined, $\tilde{N}$, and then fuzzy objective is converted to a fuzzy constraint as $\tilde{N} \tilde{\leq} \tilde{Z}(x)$. After that, some assumptions are made to deal with constraints. Finally for the compromise solution of (2.1) an LP problem is proposed.

2.2 Models with Fuzzy Right-Hand Side or Fuzzy Inequality

This section investigates the proposed solution methodologies for two FLP problems. The first problem involves only fuzzy right-hand sides and is modeled as below:

$$\begin{aligned} \max \quad & cx \\ \text{s.t.} \quad & Ax \leq \tilde{b}, \\ & x \geq 0, \end{aligned} \tag{2.8}$$

where c is a vector of real numbers , A is a $m \times n$ matrix of real numbers, $\tilde{b}$ is a vector of fuzzy numbers consisting of $\tilde{b_i}$'s with membership functions $\mu_{\tilde{b_i}}$. The second problem involves only fuzzy inequalities and is modeled as below:

$$\begin{aligned} \max \quad & cx \\ \text{s.t.} \quad & Ax \tilde{\leq} b, \\ & x \geq 0, \end{aligned} \tag{2.9}$$

where c is a vector of real numbers, A is a $m \times n$ matrix of real numbers, b is a vector of real numbers and the inequality relation is given as fuzzy.

FLP problems with either fuzzy right-hand sides or fuzzy inequalities, but not both, are nearly the same. To our knowledge, there is no special property that helps to distinguish them.Therefore, the appropiate model is selected by the decision maker's preference. The case when the decision maker chooses to use both fuzzy right-hand sides and fuzzy ineuqalities is investigated in the next section. For the FLP problems (2.9) and (2.8), different solution approaches have been proposed. Some of them are listed below:

1. In [31], FLP problems (2.9) and (2.8) are considered which are originally studied by Zimmermann [42]. The solution of the FLP problems are found by solving a proper LP problem. To obtain such an LP model, a fuzzy goal z_0, a tolerance p_0 for fuzzy goal and tolerances p_i for each constraint are taken from the decision maker. Constraints are evaluated between the given tolerance limits to a degree, which shows the membership grade. The objective function is compared with the given fuzzy goal and then, treated as a constraint. Finally to combine the constraints and derive a solution that satisfies all of the constraints, *min* operator is used as t-norm. In other words, a minimum membership functions is maximized. This approach leads to the following LP:

$$\begin{aligned} \max \quad & \lambda \\ \text{s.t.} \quad & cx \geq z_0 - (1-\lambda)p_0, \\ & Ax \leq b + (1-\lambda)p, \\ & x \geq 0, \\ & \lambda \in [0,1], \end{aligned} \tag{2.10}$$

   where p is a vector consisting of p_i's, $i = 1, ..., m$. The optimal solution of the LP problem(λ^*, x_j^*) is also the optimal solution of considered FLP problem.

2. In [38] by Werners, the same problem and the solution approach in [42] (the previous approach) is considered. To make the problem more realistic, determination of fuzzy goals and tolerances has been changed. The following LP problem is proposed for the solution of (2.9) and (2.8):

$$\begin{aligned} \max \quad & \lambda \\ \text{s.t.} \quad & cx \geq Z^1 - (1-\lambda)(Z^1 - Z^0), \\ & Ax \leq b + (1-\lambda)p, \\ & x \geq 0, \\ & \lambda \in [0,1], \end{aligned} \tag{2.11}$$

   where

$$Z^0 = \inf(\max_{x \in X} cx) = Z^*(\alpha = 1)$$

and

$$Z^1 = \sup(\max_{x \in X} cx) = Z^*(\alpha = 0),$$

with $X = \{x | Ax \leq b + (1-\alpha)p, x \in [0,1]\}, A \in R^{m \times n}, b, p \in R^m, x \in R^n$.

Here, the α corresponds to the α-cut of fuzzy set associated with the relation $A x \tilde{\leq} b$.

3. In [32] Verdegay considers the problem (2.9). Tolerances for each constraint, denoted by the vector p, are taken from the decision maker. Then for the solution of (2.9) the following parametric LP problem is proposed:

$$\begin{aligned} \max \quad & cx \\ \text{s.t.} \quad & Ax \leq b + (1-\alpha)p, \\ & x \geq 0, \\ & \alpha \in [0,1]. \end{aligned} \tag{2.12}$$

Note that the solution is a parametric solution and there is no unique solution.

4. In [36] fuzzy integer linear programming problem is studied. The considered FILP problem is as follows:

$$\begin{aligned} \text{Max} \quad & cx \\ \text{s.t.} \quad & Ax \tilde{\leq} b, \\ & x \geq 0, \\ & x \in \mathcal{N}, \end{aligned} \tag{2.13}$$

where c is a vector of real numbers, A is a $m \times n$ matrix of real numbers, b is a vector of real numbers and the inequality relation is given as fuzzy.

The solution of problem (2.13) can be found by implementing the same methodology as in previous approach and for the solution following ILP problem is proposed:

$$\begin{aligned} \max \quad & cx \\ \text{s.t.} \quad & Ax \leq b + (1-\alpha)p, \\ & x \geq 0, \\ & \alpha \in [0,1], \\ & x \in \mathcal{N}. \end{aligned}$$

2.3 Models with Various Combinations of Fuzzy Components

In [10] a penalty method is proposed to solve a FLP problem. All the coefficients used in the solution methodology is assumed to be fuzzy numbers.

First the following crisp linear programming problem is considered:

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & Ax \leq b, \\ & x \geq 0, \end{aligned}$$

where c and b are vectors, and A is a $m \times n$ matrix.

Secondly, constraints are removed and put into objective function by subtracting the following penalty term from the objective function

$$d_i \max(0, A_i x - b_i),$$

where each $d_i > 0$ is the cost per unit of violation of the right-hand side values. Then the objective function becomes

$$f(x) = c^T x - d^T \max(0, Ax - b).$$

Then every coefficient is replaced with a predetermined fuzzy number. Then every fuzzy number is characterized by its α-cuts. The author of [10] makes the assumption, "The expected midpoint of the fuzzy number as the basis of comparing two fuzzy numbers makes sense for a decision maker whose utility for an interval of possible values is the midpoint of the interval". Then he states a new optimization problem

$$\max \quad EA(\tilde{f}(x)) = EA(\tilde{c}^T x - \tilde{d}^T \max(0, \tilde{A}x - \tilde{b})),$$

where "EA" means "expected average". When the α-cuts of fuzzy numbers are taken and the expected average is calculated, $f(x)$ is written as:

$$\tilde{f}_\alpha^+(x) = (\tilde{c}_\alpha^+)^T x - (\tilde{d}_\alpha^-)^T \max(0, \tilde{A}_\alpha^- x - (\tilde{b}_\alpha^+))$$

$$\tilde{f}_\alpha^-(x) = (\tilde{c}_\alpha^-)^T x - (\tilde{d}_\alpha^+)^T \max(0, \tilde{A}_\alpha^+ x - (\tilde{b}_\alpha^-))$$

Finally, the following optimization problem is formed:

$$\max \quad EA(\tilde{f}(x)) = \frac{1}{2} \int_0^1 (\tilde{f}_\alpha^-(x) + \tilde{f}_\alpha^+(x)) d\alpha.$$

The obtained problem is concave and solution to the problem can be found by the gradient ascent algorithm.

Another combination is presented in [34]. In [34] the solution methodologies for two types of problems (problem with fuzzy inequality, problem with fuzzy left-hand side and fuzzy right-hand side) are investigated, and later by combining them a general methodology is given. Given solution methodology for the FLP problem with fuzzy left-hand side and fuzzy right-hand side is based on the study of Tanaka [29]. The model is given as:

$$\begin{aligned} \max \quad & cx \\ \text{s.t.} \quad & \tilde{A}x \leq \tilde{b}, \\ & x \geq 0, \end{aligned} \tag{2.14}$$

where $\tilde{A}$ is a $m \times n$ matrix and $\tilde{b}$ is a vector consisting of fuzzy numbers $\tilde{a}_{ij}$ and $\tilde{b}_i$, respectively.

For triangular fuzzy numbers defined as $\tilde{a}_{ij} = (a_{ij}, \underline{a_{ij}}, \bar{a_{ij}})$, the following auxiliary LP is proposed to solve the problem (2.14):

$$\begin{aligned} \max \quad & cx \\ \text{s.t.} \quad & \textstyle\sum_{j=1}^{n}[(1-1/2h)(a_{ij}+\bar{a}_{ij})+1/2h(a_{ij}-\underline{a}_{ij})]x \\ & \qquad \leq (1-1/2h)(b_i+\bar{b}_i)+1/2h(b_i-\underline{b}_i), \\ & [1/2h(a_{ij}+\bar{a}_{ij})+(1-1/2h)(a_{ij}-\underline{a}_{ij})]x \\ & \qquad \leq 1/2h(b_i+\bar{b}_i)+(1-1/2h)(b_i-\underline{b}_i), \\ & x \geq 0, \end{aligned}$$

where $h \in (0,1]$ is a level or degree of optimism being specified by the decision maker a priori.

2.4 Summary and Discussion

An FLP problem with all components (except the decision variables) being fuzzy is given below:

$$\begin{aligned} \text{Max} \quad & \tilde{c}x \\ \text{s.t.} \quad & \tilde{A}x \ \tilde{\leq} \ \tilde{b} \\ & x \geq 0 \end{aligned} \tag{2.15}$$

where $\tilde{c}$ is a vector, $\tilde{A}$ is a $m \times n$ matrix and $\tilde{b}$ is a vector consisting of fuzzy numbers with the membership functions $\mu_{\tilde{c}}$, $\mu_{\tilde{A}}$, and $\mu_{\tilde{b}}$ respectively, and $\tilde{\leq}$ is a fuzzy inequality relation.

In the literature, a single solution methodology is not given for FLP problem (2.15). Many solution approaches are provided showing how to deal with fuzziness in different components. By combining two or more, a solution can be found to the problem (2.15). This section summarizes the widely used methodologies that are proposed to deal with fuzziness in different components. Moreover, by stating the superiorities or deficiencies of the approaches given in the previous sections, a general methodology for a compromise solution is discussed.

Flexible programming constraints are of type either

$$Ax \;\tilde{\leq}\; b \tag{2.16}$$

or

$$Ax \;\leq\; \tilde{b} \tag{2.17}$$

where in equation (2.16) only the inequality relation is fuzzy and in equation (2.17) only the right-hand side is fuzzy. These two formulations have nearly the same meaning from a decision maker's point of view. Hence, the solution methodologies proposed are nearly the same. Commonly there are two methods: the first one tries to get unique solution by converting problem into a conventional linear programming problem; the second, finds a solution by converting the problem into a parametric linear programming problem. In the first method, the objective function is also considered, and a tolerance level (fuzzy goal) is determined for it. By determining membership grades for constraints and objective function, the models (2.10 and 2.11) are obtained for the unique solution. In the second method, only the soft constraints are considered; the constraints are written according to their α-cuts and parametric linear programming problem is obtained for the solution as presented in equation (2.12). The first one's advantage is that, it has a unique solution and there is no need for a decision maker; however, the second one gives a set of solutions and for the final solution decision maker is needed.

In an FLP model, fuzzy left-hand side, $\tilde{A}$ is a $m \times n$ matrix consisting of fuzzy numbers, $\tilde{a}_{ij}$, $i = 1, ..., m, j = 1, ..., n$. Dealing with fuzzy left-hand also varies in literature, and does not have a unique method. There are three common methods to deal with them: the first is α-cuts, the second is indexes (such as Yager's, Adamo's [9]) (note that they are also names as ranking functions), and the third aggregating them. Taking the α-cut produces one or two problems to solve. Using indexes helps to reduce the fuzzy numbers to crisp ones. Aggregating (as rows) allows to treat them as a single fuzzy number. In order to aggregate them, in each constraint every fuzzy number is multiplied by the decision variable and then added. In the case of triangular numbers such as $\tilde{a}_{ij} = (a_{ij}, \underline{a}_{ij}, \bar{a}_{ij})$, the aggregated fuzzy number for each constraint can be represented as follows:

$$\widetilde{a_i x} = (\sum_{j=1}^{n} a_{ij}x_j, \sum_{j=1}^{n} \underline{a}_{ij}x_j, \sum_{j=1}^{n} \bar{a}_{ij}x_j).$$

Fuzzy relations is a well studied subject in literature to compare fuzzy sets. The inequality relation, $\tilde{\leq}$, in the constraints, $\tilde{A}x\tilde{\leq}\tilde{b}$, has been studied in the papers [6,14,18–21,28]. In most of these approaches fuzzy constraints $\tilde{A}_i(x)\tilde{\leq}\tilde{b}_i$ are replaced by one or two crisp linear constraints. However, this procedure has the disadvantage that the fuzzy constraints turn to crisp constraints with no regard to the objectives [18]. In [18, 20, 26], a more flexible interpretation, which considers the objective function, is proposed. In FLP problems as in (2.15), fuzzy inequality relations can be treated by one of the above approaches.

Fuzzy objectives is the most studied branch of fuzzy mathematical programming. In Section 2.1 the approaches that are proposed to deal with fuzzy objectives are listed: multiobjective approach, interval approach, possibilistic approach, stratified piecewise reduction approach, progressive reduction approach, comparison approach, compromise objective function approach, and fuzzy aspiration level. In FLP problems as in (2.15), one of these approaches can be used. It cannot be said that one of the approaches is superior to the other since each focuses on different points and makes different assumptions. However, even intiutively, it can be understood that some have shortfalls. For example, the compromise objective function approach proposes to use a crisp number representing the fuzzy number. The solution method has nothing to do with FLP; solution method is related to deriving a single value from an infinitely many alternatives.

So far this thesis discussed methods to deal with fuzziness when it is in one of the components of the model. A compromise solution to the FLP problem (2.15) can be found by applying the appropriate alternatives together. For example, if the membership function is used in one component, the other components should also be represented by membership functions. Similarly, if indexes are used for one component, in all components, indexes should be used. In the case when membership functions are used, based on [4], the following model for the compromise solution of the FLP problem (2.15) can be used:

$$\begin{aligned} \max \quad & \lambda \\ \text{s.t.} \quad & \mu_O(x) \geq \lambda, \\ & \mu_{C_i}(x) \geq \lambda, \quad i = 1, ..., m, \\ & x \geq 0, \\ & \lambda \in [0, 1] \end{aligned}$$

where $\mu_O(x)$ is the membership function of the objective function and $\mu_{C_i}(x), i = 1, ..., m$ are the membership functions of the constraints.

Chapter 3

Improved Solution Methodology

Although 40 years have passed since the launch of fuzzy set theory, there are many problems that have not been answered yet in the application of fuzzy sets to problems. One of them is the solution of an FLP problem with all components are fuzzy. There exists some solution methodologies in the literature but they neither exactly state the solution, nor interpret the solution obtained. This chapter aims at investigating the solution methodologies that have been proposed so far and come up with a thorough solution methodology. The following sections present the proposed improved solution methodology.

3.1 Problem Statement

Mathematical programming (MP) models are used to determine the best alternative among others such that an objective function is optimized. Linear programming models form a special class of MP models. In LP problems all the constraints and the objective function are linear. The parameters in classical LP problems are given as crisp numbers. However, it is neither easy nor realistic to define them exactly, since most of the time real world parameters are vague, subjective, and imprecise. Fuzzy set theory is an excellent tool in handling vagueness, subjectivity and imprecision in the parameters and in the inequalities. Therefore, LP problems can be modelled by fuzzy numbers and fuzzy inequalities. Such LP problems including fuzzy components are called Fuzzy Linear Programming(FLP) problems. There are various FLP problem types according to the combination of fuzzy components used. Among these problems the ones which only include fuzzy-right hand side or fuzzy inequality or fuzzy objective is the most common ones [9,18,21,28,31,35–37,43,44], because they can be solved quite easily when compared to other combinations. While FLP problems are superior to LP problems in representing the real world, an FLP problem cannot be solved as easily as LP problems.

In this thesis we will deal with the most general form of the FLP problem which has fuzzy objective, fuzzy left-hand side , fuzzy right-hand side and fuzzy inequalities. The considered FLP problem can be modelled as follows:

$$\begin{aligned} \text{Max} \quad & \tilde{c}x \\ \text{s.t.} \quad & \tilde{A}x \tilde{\leq} \tilde{b}, \\ & x \geq 0, \end{aligned} \tag{3.1}$$

where c is a vector consisting of $\tilde{c}_j$'s with membership function $\mu_{\tilde{c}_j}$, $j = 1, ..., n$; A is a matrix consisting of $\tilde{a}_{ij}$'s with membership function $\mu_{\tilde{a}_{ij}}$, $i = 1, ..., m$, $j = 1, ..., n$; b is a vector consisting of $\tilde{b}_i$'s with membership function $\mu_{\tilde{b}_i}$, $i = 1, ..., m$; x is a vector consisting of x_j's, $j = 1, ..., n$; and $\tilde{\leq}$ is a fuzzy less than or equal to relation.

The proposed methodology will try to find out the x vector which maximizes the satisfaction degree of the constraints and at the same time yields the best objective function value. Note that, in this thesis, we will assume the following:

- The fuzzy numbers are given as normal triangular fuzzy numbers such as $\tilde{u} = (u, \underline{u}, \bar{u})$ where $u, \underline{u}, \bar{u}$ and $(u - \underline{u})$ are positive real numbers.
- Eventhough it is highly restrictive, we will deal with only "addition" as the linear operator in the objective function and the constraints. Note thatfor the general case some manipulations are required which are actually not difficult to determine. However, because of the time constraint we only dealt with the restricted case. As an example of such manipulations, the readers may refer to the Section 4.1, in which subtraction in a constraint is treated with special care.
- For the intersection of fuzzy sets and inequality relations *min* operator is used as t-norm.

Now, the proposed solution will be presented in the following sections.

3.2 Aggregating Objective Function Coefficients and Left-Hand Side Coefficients

In (3.1) each fuzzy objective function coefficient, $\tilde{c}_j$, and each left-hand side coefficient, $\tilde{a}_{ij}$ is given as fuzzy numbers, which are defined by membership functions. As a first step, the membership functions for the objective function and for the left-hand side are calculated by using fuzzy arithmetics. In other words, for each inner product, a triangular fuzzy number is determined by using fuzzy multiplication and addition respectively [18].

The decision variable, x is a vector consisting of positive crisp numbers; x_j's, $j = 1, ..., n$; i.e., they can be treated as positive scalars for fuzzy multiplication. With this property, each fuzzy number in the objective function and in the left-hand side can be multiplied by its decision variable, x_j and a new fuzzy number can be obtained. Then the objective function and each row of the left-hand side can be converted to a single fuzzy number by using fuzzy addition. After applying fuzzy arithmetics, the fuzzy objective and the fuzzy left hand side can be represented as a triangular fuzzy number as shown below, respectively:

$$\tilde{cx} = \Big(\; c_1x_1 + \cdots + c_nx_n \quad , \;\; \underline{c}_1x_1 + \cdots + \underline{c}_nx_n \quad , \;\; \bar{c}_1x_1 + \cdots + \bar{c}_nx_n \; \Big),$$

$$\tilde{A}x = (\widetilde{a_ix}) = \Big(\; a_{i1}x_1 + \cdots + a_{in}x_n, \;\; \underline{a}_{i1}x_1 + \cdots + \underline{a}_{in}x_n, \;\; \bar{a}_{i1}x_1 + \cdots + \bar{a}_{in}x_n \; \Big).$$

From now on, for simplicity, fuzzy objective and fuzzy left-hand side numbers will be shown as follows, respectively:

$$\widetilde{cx} = (cx, \underline{cx}, \overline{cx}), \tag{3.2}$$

$$\widetilde{a_ix} = (a_ix, \underline{a_ix}, \overline{a_ix}). \tag{3.3}$$

Their membership functions will also be formulated as follows, respectively:

$$\mu_{\tilde{cx}}(t) = \begin{cases} 0, & \text{if} \;\; t < (cx - \underline{cx}) \;\; \text{and} \;\; t > (cx + \overline{cx}), \\ \frac{t-(cx-\underline{cx})}{\underline{cx}}, & \text{if} \;\; (cx - \underline{cx}) \leq t < cx, \\ 1, & \text{if} \;\; t = cx, \\ \frac{(cx+\overline{cx})-t}{\overline{cx}}, & \text{if} \;\; cx < t \leq (cx + \overline{cx}), \end{cases}$$

$$\mu_{a_i \tilde{x}}(t) = \begin{cases} 0, & \text{if } \ t < (a_i x - \underline{a_i x}) \ \text{ and } \ t > (a_i x + \overline{a_i x}), \\ \frac{t - (a_i x - \underline{a_i x})}{\underline{a_i x}}, & \text{if } \ (a_i x - \underline{a_i x}) \leq t < a_i x, \\ 1, & \text{if } \ t = a_i x, \\ \frac{(a_i x + \overline{a_i x}) - t}{\overline{a_i x}}, & \text{if } \ a_i x < t \leq (a_i x + \overline{a_i x}). \end{cases}$$

3.3 Comparison of Fuzzy Left-Hand Side with Fuzzy Right-Hand Side

After aggregating the objective function coefficients and the left-hand side coefficients, (3.1) becomes

$$\begin{aligned} \max \quad & \widetilde{cx} \\ \text{s.t.} \quad & \widetilde{a_i x} \tilde{\leq} \tilde{b}, \quad i = 1, ..., m, \\ & x \geq 0, \end{aligned} \tag{3.4}$$

where $\widetilde{cx}$ and $\widetilde{a_i x}$ are triangular fuzzy numbers as defined in (3.2) and (3.3) respectively, and $\tilde{b}$ is a vector of triangular fuzzy numbers and each member can be represented as follows:

$$\tilde{b_i} = (b_i, \underline{b_i}, \bar{b_i}), \quad i = 1, ..., m.$$

As seen in the model (3.4), both sides of the constraints are fuzzy numbers and the inequality is also fuzzy. So two sides of the constraint can be compared via the fuzzy inequality relation, and the membership function can be defined for it. For comparing two fuzzy numbers, several relations have been defined in literature [5, 14, 21, 22, 29]. Here for the comparison of fuzzy numbers via fuzzy less than or equal to relation, a t-norm *min* function is used. Applying the *min* function for each constraint of (3.4) the following membership function is obtained:

$$\begin{aligned} \mu_{\tilde{\leq}}(\widetilde{a_i x}, \tilde{b_i}) &= \sup\{\min(\mu_{\widetilde{a_i x}}(u), \mu_{\tilde{b_i}}(v) | u \leq v)\} \\ &= \begin{cases} 1 & \text{if } \ 0 < x, a_i x \leq b_i, \\ \frac{b_i + \bar{b_i} - (a_i x - \underline{a_i x})}{\underline{a_i x} + \bar{b_i}} & \text{if } \ b_i < a_i x, (a_i x - \underline{a_i x}) \leq b_i + \bar{b_i}, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

The membership function for constraints can be illustrated as in Figure 3.1

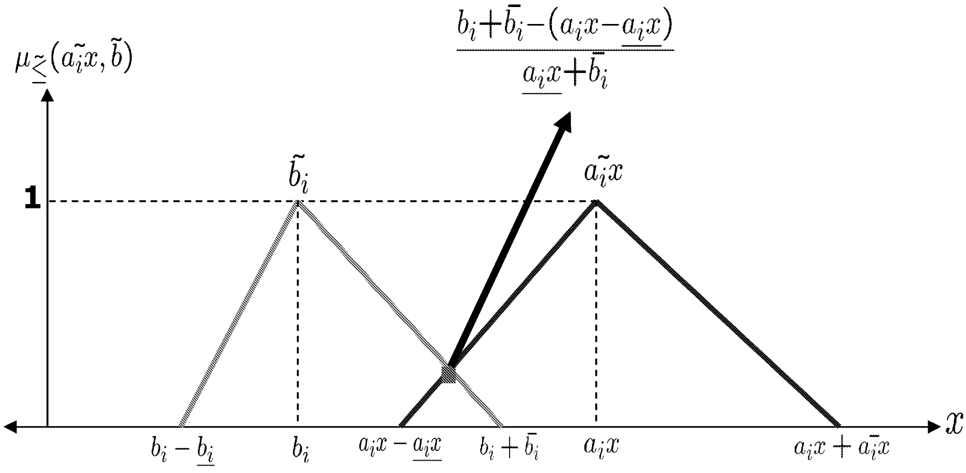

Figure 3.1: Membership function for constraints.

3.4 Feasible Solution

The feasible solution in a linear programming problem is formed by the intersection of the constraints. For FLP problems, feasible solution can be defined by the same manner.

Definition 3.4.1 Let $\mu_{\widetilde{a_ix}}$ and $\mu_{\tilde{b_i}}$, $i = 1, ..., m$, be the membership functions defined for the fuzzy quantities $\widetilde{a_ix}$ and $\tilde{b_i}$, respectively. Let $\mu_{\lesssim}(\widetilde{a_ix}, \tilde{b_i})$ be the membership function for the constraints. Let *min* function be used for intersection.
A fuzzy set $\tilde{X}$, defined for x vector by a membership function $\mu_{\tilde{X}}$,

$$\mu_{\tilde{X}}(x) = \begin{cases} \min\left(\mu_{\lesssim}(\widetilde{a_1x}), \cdots, \mu_{\lesssim}(\widetilde{a_mx})\right) & \text{if } x \geq 0, \\ 0 & \text{otherwise,} \end{cases}$$

is called the feasible solution of the FLP problem (3.1) [15].

3.5 Comparison of Fuzzy Objective with a Given Fuzzy Goal

Up to now it is shown that FLP problem (3.1) has a feasible solution which is actually a fuzzy set with membership function $\mu_{\tilde{X}}$, and it is also shown that fuzzy objective function can be represented as fuzzy number, $\widetilde{cx}$. To reach an optimal solution, fuzzy set of the feasible region and fuzzy set of objective function have to be aggregated. In order to do that, the objective function should be treated like a fuzzy constraint. Since the right-hand side for the constraint formed by the objective function is not known unlike the constraints (as discussed in Section 3.3), a special treatment is required. Here, a fuzzy goal is assumed to be given by a modeller or the decision maker. Since FLP problem (3.1) is a maximization problem, the higher objective function values are more desirable. So if a fuzzy goal such as $\tilde{d}$ is assumed to be given, the following comparison can be formed:

$$\tilde{cx} \tilde{\geq} \tilde{d}. \tag{3.5}$$

The membership function for a fuzzy goal $\tilde{d} = (d, \underline{d}, \infty)$ is defined by a membership function $\mu_{\tilde{d}}$:

$$\mu_{\tilde{d}}(t) = \begin{cases} 0, & \text{if } \ t < (d - \underline{d}), \\ \frac{t-(d-\underline{d})}{\underline{d}}, & \text{if } \ (d - \underline{d}) \leq t \leq d, \\ 1, & \text{if } \ t > d. \end{cases} \tag{3.6}$$

Determination of a fuzzy goal is a very important issue which will be investigated thoroughly in Subsection 3.7.1. The membership function for the constraint (3.5) can be obtained by applying the same procedure as in Section 3.3:

$$\begin{aligned} \mu_{\tilde{\geq}}(\tilde{cx}, \tilde{d}) &= \sup\{\min(\mu_{\tilde{cx}}(u), \mu_{\tilde{d}}(v) | u \geq v)\} \\ &= \begin{cases} 1, & \text{if } \ 0 < x, d \leq cx, \\ \frac{(cx+\bar{c}x)-(d-\underline{d})}{\bar{c}x+\underline{d}}, & \text{if } \ cx < d, \ d - \underline{d} \leq cx + \bar{c}x, \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \tag{3.7}$$

The membership function (3.7) for the relation (3.5) can be illustrated as in Figure 3.2:

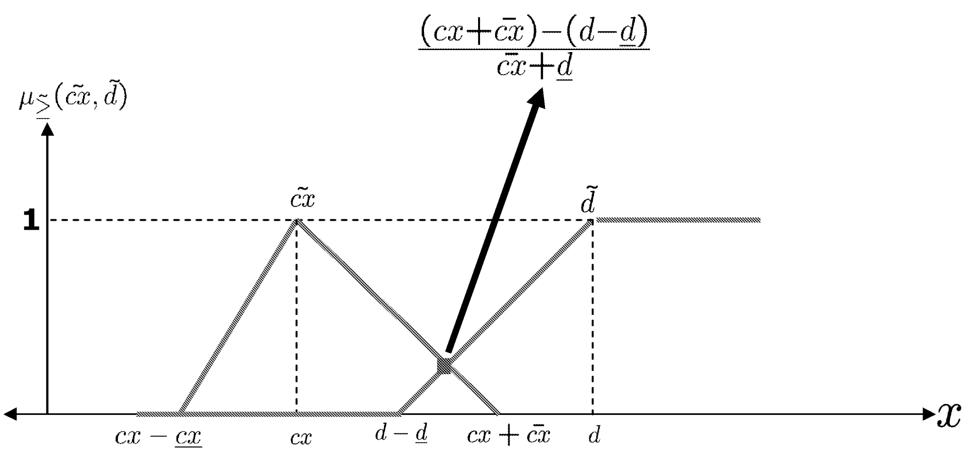

Figure 3.2: Membership function (3.7) for relation (3.5).

3.6 Optimal Solution

The previous sections have indicated that an optimal solution can be found by the intersection of fuzzy set of the feasible solution and fuzzy set of objective function. It was also shown, how to obtain those fuzzy sets and define their membership functions. Given these information, optimal solution can be defined as follows:

Definition 3.6.1 Let $\mu_{\gtrsim}(\tilde{cx}, \tilde{d})$ and $\mu_{\tilde{X}}(x)$ be the membership functions for the fuzzy set of the objective and the fuzzy set of the feasible solution, respectively.
A fuzzy set $\tilde{X}^*$ with the membership function $\mu_{\tilde{X}^*}$, defined for all $x \in R^n$ by

$$\mu_{\tilde{X}^*}(x) = \min\left(\mu_{\gtrsim}(\tilde{cx}, \tilde{d}), \mu_{\tilde{X}}(x)\right),$$

is called the optimal solution of the FLP problem (3.1).

A vector $x^* \in R^n$ with the property

$$\mu_{\tilde{X}^*}(x^*) = Hgt(\tilde{X}^*)$$

is the max-optimal solution [15]. This max-optimal solution is the satisficing solution of the FLP problem (3.1) with the highest degree of membership. A vector x^* can be found by solving the problem :

$$\max\min\left\{\mu_{\gtrsim}(\tilde{cx}, \tilde{d}), \mu_{\lesssim}(\widetilde{a_i x}, \tilde{b}_i) | x \geq 0, i = 1, ..., m\right\}. \tag{3.8}$$

Problem (3.8) can be converted to an auxiliary optimization problem:

$$\begin{aligned} \max \quad & \lambda \\ \text{s.t.} \quad & \mu_{\gtrsim}(\tilde{cx}, \tilde{d}) \geq \lambda, \\ & \mu_{\tilde{X}}(x) \geq \lambda, \\ & x \geq 0, \\ & \lambda \in [0, 1], \end{aligned} \tag{3.9}$$

where λ is an auxiliary variable defined to find the maximum membership degree that satisfies all the constraints, i.e. membership functions of objective and constraints. The optimal solution of (3.9) is (λ^*, x^*).

3.7 Further Discussions

[15, 16] discuss how to reach max-optimal solution; however, some points, which are crucial, are not studied. These are: the determination of the fuzzy goal and the structure of the proposed auxiliary optimization problem (3.9) and the solution method for the problem. Determination of a fuzzy goal is important for the proposed solution methodology since many max-optimal solutions can be found according to the fuzzy goal chosen. The structure of the proposed auxiliary optimization is also important since it effects the solvability of the system; i.e., determination of the max-optimal solution.

3.7.1 Determination of The Fuzzy Goal

To our knowledge, a detailed discussion about the determination of the fuzzy goal is not given in the literature [15, 16]. This section investigates the determination of a suitable fuzzy goal.

In problem (3.9), the fuzzy goal $\tilde{d}$ is the factor that restricts the model to a single solution λ^*. λ^* cannot be increased more since it is the maximum value where membership function of objective, $\mu_{\gtrsim}(\tilde{cx}, \tilde{d})$, coincides with one or more of the membership function of the constraints, $\mu_{\lesssim}(\widetilde{a_i x}, \tilde{b}_i)$, and above this value, *min* function defined for the fuzzy relations used in objective and constraints are not satisfied anymore. Without fuzzy goal $\tilde{d}$, it would not be possible to find value such as λ^*, since the constraint $\mu_{\gtrsim}(\tilde{cx}, \tilde{d})$ cannot be formed. So the existence of a suitable fuzzy goal $\tilde{d}$ is crucial. Moreover, it is crucial to select an appropriate value. The problem (3.9) will give different λ values for the different fuzzy goals $\tilde{d}$. So it is useful to study the possible fuzzy goal values.

For the FLP problem (3.1) the following two cases give the minimum objective function value and maximum objective function value:

- The minimum objective function value, $z_{\min}$, is determined via the LP problem:

$$\begin{aligned} \max \quad & cx - \underline{cx} \\ \text{s.t.} \quad & a_i x + \overline{a_i x} \leq b_i - \underline{b_i}, \quad i = 1, ..., m, \\ & x \geq 0, \end{aligned} \tag{3.10}$$

- The maximum objective function value, $z_{\max}$, is determined via the LP problem:

$$\begin{aligned} \max \quad & cx + \overline{cx} \\ \text{s.t.} \quad & a_i x - \underline{a_i x} \leq b_i + \overline{b_i}, \quad i = 1, ..., m, \\ & x \geq 0. \end{aligned} \tag{3.11}$$

If a fuzzy goal $\tilde{d} = (d, \underline{d}, \infty)$, is defined by a membership function such as (3.6), d and $d - \underline{d}$ values chacterize it. d and $d - \underline{d}$ values can be named as upper limit and lower limit, respectively. So fuzzy function values are spread between these limits with different membership degrees. As defined in section 3.5, the fuzzy goal is used to treat the fuzzy objective as a constrained. In the light of these observations it is resonable to determine the upper limit of the fuzzy goal, d, as the maximum objective function value, $z_{\max}$, and the lower limit of the fuzzy goal, $d - \tilde{d}$, as the minimum objective function value, $z_{\min}$.

It should be mentioned that the proposed limits for the fuzzy goal do not yield an ultimate max-optimal solution. It serves as a useful starting point for the decision maker. It is reasonable to use those limits. The following conclusions can be made for those limits:

- If the upper limit of the fuzzy goal is less than the minimum objective function value, then λ is always 1:

$$d < z_{\min} \quad \rightarrow \quad d < \underline{cx} \quad \rightarrow \quad \lambda = 1.$$

- If the lower limit of the fuzzy goal is greater than the maximum objective function value, then λ is always 0:

$$d - \underline{d} > z_{\max} \quad \rightarrow \quad d - \underline{d} > \overline{cx} \quad \rightarrow \quad \lambda = 0.$$

3.7.2 Auxiliary Optimization Model

In the solution methodology, the other important point is the auxiliary optimization problem (3.9). In [15,16] no information is given about the solution methodology for the cases except one-dimensional FLP problem. In this study, a multi-dimensional

FLP problem as (3.1) is considered. For such a problem the auxiliary optimization problem (3.9), in open form can be written as follows:

$$\begin{aligned}
\max \quad & \lambda \\
\text{s.t.} \quad & \frac{(cx+\bar{c}x)-(d-\underline{d})}{\bar{c}x+\underline{d}} \geq \lambda, \\
& \frac{b_i+\bar{b_i}-(a_ix-\underline{a_ix})}{\underline{a_ix}+\bar{b_i}} \geq \lambda, \quad i=1,...,m \\
& x \geq 0, \\
& \lambda \in [0,1].
\end{aligned} \tag{3.12}$$

This is a non-linear fractional programming problem where the numerator and the denominator of each fraction are given by linear functions. Notice that for a fixed λ, (3.12) becomes a linear programming problem. To solve such a problem, λ is gradually increased and for each fixed λ, an LP is solved. This process continues until no feasible solution exists. The result of the last feasible solution (λ^*, x^*), is recorded as the max-optimal solution.

In the following section, the proposed improved solution methodology, determination of a fuzzy goal, and the solution of an auxiliary problem are studied on numerical examples.

3.8 Numerical Examples

In this chapter, three FLP problems are considered and solved by the improved solution methodology described in the previous sections. For the solutions, GAMS solver is used [45]. Two GAMS codes are used. The first one is a general LP code and used to calculate the results of (3.10) and (3.11). The second one consists of two parts: the main code where the auxiliary model (3.12) is present and the data file (.inc file) where the coefficients are written. The main code except from auxiliary model contains the following algorithm to reach λ^*:

Step 1: $\epsilon = 0.01$, $\lambda := \epsilon$, set $\lambda^* := \lambda$

Step 2: Solve model (3.12) for λ. If there is no feasible solution stop, display $\lambda*$; otherwise, set $\lambda^* := \lambda$ and go to Step 3

Step 3: $\lambda := \lambda + \epsilon$, check $\lambda < 1$ if true go to Step 2.

The GAMS solver ran on a notebook with Celeron CPU 2.20 Ghz and 240 MB

RAM. Run time is not recorded for the numerical examples given here since its ϵ is kept constant as 0.01 and for the considered small-sized problems the computation times are negligible.

Example 3.8.1 This is the example 1 of [29]. The considered FLP problem is:

$$\begin{aligned} \max \quad & \tilde{25}x_1 + \tilde{8}x_2 \\ \text{s.t.} \quad & \tilde{15}x_1 + \tilde{34}x_2 \tilde{\leq} \tilde{800}, \\ & \tilde{20}x_1 + \tilde{10}x_2 \tilde{\leq} \tilde{430}, \\ & x_1, x_2 \geq 0, \end{aligned} \tag{3.13}$$

where coefficients are $\tilde{c}_1 = (25, 2, 2)$, $\tilde{c}_2 = (8, 1, 1)$, $\tilde{a}_{11} = (15, 3, 3)$, $\tilde{a}_{12} = (34, 2, 2)$, $\tilde{a}_{21} = (20, 1, 1)$, $\tilde{a}_{22} = (10, 3, 3)$, $\tilde{b}_1 = (800, 50, 50)$ and $\tilde{b}_2 = (430, 50, 50)$.

For the solution, first, problems (3.10) and (3.11) are formed, and $z_{\min}$ and $z_{\max}$ values are calculated, respectively, by using the first GAMS code. Based on those values, the fuzzy goal is derived and the data file of second GAMS code is filled. Then the main code is executed. The results obtained during those steps are given in Table 3.1. The result can be interpreted as: "The highest membership degree where constraints and objective are satisfied at the same time is 0.64 if the fuzzy goal is defined by $(682.1, 265.9, 0)$ ".

$z_{\min}$	$z_{\max}$	$\tilde{d}$	λ^*	x_1^*	x_2^*	$\widetilde{cx}^*(\lambda^*)$
416.2	682.1	(682.1, 265.9, 0)	0.64	22.798	0	586.365

Table 3.1: Results of (3.13).

The results of the example (3.13) cannot be compared with the results in [29] since for the solution different approaches have been used.

Example 3.8.2 As a second example, the following FLP problem is considered:

$$\begin{aligned} \max \quad & \tilde{1}x_1 + \tilde{2}x_2 \\ \text{s.t.} \quad & \tilde{2}x_1 + \tilde{1}x_2 \tilde{\leq} \tilde{6}, \\ & \tilde{1}x_1 + \tilde{3}x_2 \tilde{\leq} \tilde{9}, \\ & x_1, x_2 \geq 0, \end{aligned} \tag{3.14}$$

where coefficients are $c_1 = a_{12} = a_{21} = (1, 0.5, 0.5)$, $c_2 = a_{11} = (2, 1, 1)$, $a_{22} = (3, 1, 1)$, $b_1 = (6, 2, 2)$ and $b_2 = (9, 3, 3)$.

The solution procedure for problem (3.14) is the same as the one given for example (3.13). Based on that solution procedure, the obtained results are given in Table 3.2:

$z_{\min}$	$z_{\max}$	$\tilde{d}$	λ^*	x_1^*	x_2^*	$\widetilde{cx}^*(\lambda^*)$
1.589	22.286	(22.286, 20.697, 0)	0.50	3.039	3.256	11.9368

Table 3.2: Results of (3.14).

For the different values of fuzzy goal, different results can be obtained. Some are shown in Table 3.3:

$\tilde{d}$	λ^*	x_1^*	x_2^*	$\widetilde{cx}^*(\lambda^*)$
(22.286, 14.286, 0)	0.39	3.459	3.470	13.572
(22.286, 11.286, 0)	0.33	3.682	3.674	14.724
(22.286, 8.286, 0)	0.25	4.104	3.792	16.072
(22.286, 5.286, 0)	0.17	4.513	4.068	17.899
(22.286, 2.286, 0)	0.07	5.207	4.277	20.160

Table 3.3: Results of (3.14) for different fuzzy goals.

Table 3.3 indicates that, if the upper limit of fuzzy goal fixed at $z_{\max}$ and lower limit gradually decreased, the λ^* value and objective function value decreases. However, there is no guarantee.

If the lower limit of fuzzy goal is defined as greater than the $z_{\max}$, for example $\tilde{d} = (25, 2, 0)$, the result is $\lambda^* = 0$, which is the expected result as defined in Section 3.7.1.

If the upper limit of fuzzy goal is defined as lower than the $z_{\min}$, for example $\tilde{d} = (1, 0.5, 0)$, the result is $\lambda^* = 1$, which is the expected result as defined in section 3.7.1.

Example 3.8.3 As a third example the following FLP problem is considered:

$$\begin{aligned}
\max \quad & \tilde{4.5}x_1 + \tilde{5.5}x_2 + \tilde{6.5}x_3 + \tilde{7}x_4 \\
\text{s.t.} \quad & \tilde{10}x_1 + \tilde{12}x_2 + \tilde{8}x_3 + \tilde{18}x_4 \tilde{\leq} \tilde{60000}, \\
& \tilde{14}x_1 + \tilde{11}x_2 + \tilde{10}x_3 + \tilde{17}x_4 \tilde{\leq} \tilde{64500}, \\
& \tilde{13}x_1 + \tilde{10}x_2 + \tilde{11}x_3 + \tilde{20}x_4 \tilde{\leq} \tilde{65000}, \\
& \tilde{1}x_1 \tilde{\leq} \tilde{825}, \\
& \tilde{1}x_2 \tilde{\leq} \tilde{1850}, \\
& \tilde{1}x_3 \tilde{\leq} \tilde{1250}, \\
& \tilde{1}x_4 \tilde{\leq} \tilde{1050}, \\
& x_1, x_2, x_3, x_4 \geq 0,
\end{aligned} \tag{3.15}$$

where coefficients are $c_1 = (4.5, 1, 1)$, $c_2 = (5.5, 1, 1)$, $c_3 = (6.5, 1, 1)$, $c_4 = (7, 2, 2)$, $a_{11} = a_{23} = a_{32} = (10, 2, 3)$, $a_{12} = (12, 5, 1)$, $a_{13} = (8, 3, 4)$, $a_{14} = (18, 5, 2)$, $a_{21} = (14, 3, 3)$, $a_{22} = a_{33} = (11, 4, 3)$, $a_{24} = (17, 3, 4)$, $a_{31} = (13, 5, 8)$, $a_{34} = (20, 5, 6)$, $a_{41} = a_{52} = a_{63} = a_{74} = (1, 0, 0)$, $b_1 = (60000, 5000, 5000)$, $b_2 = (64500, 4500, 4500)$, $b_3 = (65000, 3000, 3000)$, $b_4 = (825, 175, 175)$, $b_5 = (1850, 150, 150)$, $b_6 = (1250, 150, 150)$ and $b_1 = (1050, 250, 250)$.

The solution procedure for problem (3.15) is same as the one given for example (3.13). Based on that solution procedure, the obtained results are given in Table 3.4:

$z_{\min}$	$z_{\max}$	$\tilde{d}$	λ^*	x_1^*	x_2^*	x_3^*	x_4^*	$\widetilde{cx^*}(\lambda^*)$
18411.538	40700	(40700, 22888.462, 0)	0.66	884.5	1901	1301	1135	33517.923

Table 3.4: Results of (3.15).

Chapter 4

Fuzzy AHP Problems

Analytical Hierarchy Process (AHP) is an approach developed by Saaty [23] for dealing with complex multi-criteria decision problems. A major component of AHP methodology is the priority structure which can be local or global [1]. This study deals with local priorities - the priority of an element in a certain level with respect to an element in a level immediately above it. To reach local priorities, AHP uses a comparison scale and a pairwise comparison matrix such as A:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix},$$

where a_{ij} , $i = 1, ..., n, j = 1, ..., n$, a single value from comparison scale (usually 1-9 scale) showing the strength of alternative i to alternative j. Since the comparison is also made between alternative j and alternative i, the pairwise comparison matrix can be written as:

$$A = \begin{bmatrix} 1 & a_{12} & \cdots & a_{1n} \\ \frac{1}{a_{12}} & 1 & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{1}{a_{1n}} & \frac{1}{a_{n2}} & \cdots & 1 \end{bmatrix}. \tag{4.1}$$

The relation between the local priority vector $w^T = [w_1 \ \ ... \ \ w_n]$ and comparison matrix is as follows:

$$a_{ij} = \frac{w_i}{w_j}, \tag{4.2}$$

where $\sum_{j=1}^{n} w_j = 1$.

The priority vector is the principle eigenvector of comparison matrix [1]. Also the relation $Aw = nw$ holds, where n is the number of elements being compared, when the comparison matrix is perfectly consistent which means, all the elements of comparison matrix satifies the condition [1]:

$$a_{ij} = a_{ik}a_{kj} \quad \forall i, j, k = 1, ..., n.$$

In literature, more information can be found about the consistency of matrix [23–25].

The given relations above hold for the comparison matrix consisting of a_{ij}'s which are crisp numbers. However, as mentioned in the fuzzy set theory, determining the parameters as a single value is not so easy and also does not reflect the real world. In [1, 2, 11, 12] authors consider the interval pairwise comparison judgements to overcome this problem. In [1, 2] the author proposes Preference Programming; whereas in [11, 12] the author proposes fuzzy linear programming to derive crisp priority vector.

This study considers a comparison matrix, such as $\tilde{A}$, defined via using fuzzy numbers in order to reflect the parameters more realisticly:

$$\tilde{A} = \begin{bmatrix} 1 & \tilde{a}_{12} & \cdots & \tilde{a}_{1n} \\ \frac{1}{\tilde{a}_{12}} & 1 & \cdots & \tilde{a}_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{1}{\tilde{a}_{1n}} & \frac{1}{\tilde{a}_{n2}} & \cdots & 1 \end{bmatrix}, \tag{4.3}$$

where $\tilde{a}_{ij} = (a_{ij}, \underline{a_{ij}}, \overline{a_{ij}})$ are normal symmetric triangular fuzzy numbers with the following properties: $a_{ij} \geq 0$, $\underline{a_{ij}} = \overline{a_{ij}} \geq 0$ and $a_{ij} - \underline{a_{ij}} \geq 0$. This study also tries to find out the crisp priority vector with the highest satisfaction degree. From now on, the defined problem will be referred to as the fuzzy AHP problem.

To solve the fuzzy AHP problem, the following steps are applied. First the fuzzy relations are defined for comparison matrix. Then the relations are stated as fuzzy linear programming problem. Finally, the obtained FLP problem is solved by the solution method discussed in Chapter 3.

4.1 Defining Fuzzy Relations for Comparison Matrix

The relation (4.2) can be defined for the fuzzy AHP problem as follows:

$$\tilde{a}_{ij} \ \tilde{=} \ \frac{w_i}{w_j}. \tag{4.4}$$

The fuzzy equality (4.4) can be written as two fuzzy inequalities:

$$\begin{aligned} \tilde{a}_{ij} w_j - w_i \ &\tilde{\leq} \ \tilde{0}, \\ \tilde{a}_{ij} w_j - w_i &\tilde{\geq} \ \tilde{0}. \end{aligned} \tag{4.5}$$

The right-hand side of the fuzzy inequality is defined by a fuzzy value $\tilde{0} = (0, 0, 0)$.

4.2 Stating The Relations as Fuzzy Linear Programming Problem

In [1] the author discusses that priority vector can be derived if $\frac{1}{2}n(n-1)$ entries of comparison matrix (4.1) is known. Here the upper triangle of comparison matrix is used. For each element of the upper triangle, the fuzzy inequality relation as in (4.5) can be written. So $n(n-1)$ fuzzy constraints are obtained. Besides these constraints there are two additional ones based on the structure of the problem. First one derives from the fact that the sum of the priorities must add up to 1; and second, each priority must be greater than or equal to 0, i.e., decision variables must be greater than or equal to 0. When the constraints are written, the obtained problem is a linear programming problem without the objective function as follows:

$$\begin{aligned} &\tilde{a}_{ij} w_j - w_i \ \tilde{\leq} \ \tilde{0}, \quad i < j, \\ &\tilde{a}_{ij} w_j - w_i \tilde{\geq} \ \tilde{0}, \quad\quad i = 1, ..., n, j = 1, ..., n, \\ &\textstyle\sum_{j=1}^{n} w_j = 1, \\ &w_j \geq 0. \end{aligned} \tag{4.6}$$

The crisp numbers in coeffients, i.e., $(-1, -1, ..., -1)$, can be represented as fuzzy numbers with spreads being equal to 0. Then the obtained problem is a FLP problem with fuzzy left-hand sides, fuzzy right-hand sides and fuzzy inequalities:

$$\begin{aligned} &\textstyle\sum_{j=1}^{n} \tilde{k}_{ij} w_j \ \tilde{\leq} \ \tilde{0}, \quad i = 1, ..., n, \\ &\textstyle\sum_{j=1}^{n} w_j = 1, \\ &w_j \geq 0, \end{aligned} \tag{4.7}$$

where k_{ij} corresponds to the left-hand side coefficients in (4.6).

4.3 Solving The Obtained Fuzzy Linear Programming Problem

The obtained problem is a FLP linear programming problem without an objective function. It can be solved based on the solution methodology defined in Section 3. The solution for problem (4.7) can be obtained by applying the following steps:

- The fuzzy left-hand side of the the constraints are aggregated to a single fuzzy number by using fuzzy arithmetics. The crisp constraint remains untouched.
- For the comparison of the fuzzy left-hand side and the fuzzy right-hand side by fuzzy less than or equal to relation, min function is used and membership function is formed:

$$\mu_{\tilde{\leq}}(\widetilde{k_i w} \tilde{\leq} \tilde{0}).$$

- A new problem is formed:

$$\begin{aligned}&\mu_{\tilde{\leq}}(\widetilde{k_i w} \tilde{\leq} \tilde{0}),\\&\textstyle\sum_{j=1}^{n} w_j = 1,\\&w_j \geq 0.\end{aligned}$$

- The obtained problem is rewritten by defining an artificial variable, λ:

$$\begin{aligned}\max\quad & \lambda\\ \text{s.t.}\quad & \mu_{\tilde{\leq}}(\widetilde{k_i w} \tilde{\leq} \tilde{0}) \geq \lambda, \quad i = 1, ..., n,\\ & \textstyle\sum_{j=1}^{n} w_j = 1,\\ & w_j \geq 0,\\ & \lambda \in [0, 1].\end{aligned} \tag{4.8}$$

- The resulting problem (4.8) is a non-linear fractional programming problem and it can be solved by the same approach discussed in Section 3.7.2.
- The solution of (4.8) is (λ^*, w^*), where λ^* is the maximum degree of membership that satisfies all the constraint and w^* is the resulting priority vector.

The solution, (λ^*, w^*), of the fuzzy AHP problem (4.3) can be interpreted as follows: the given fuzzy comparison matrix is consistent to a degree λ^* with w^* being its priority vector. In other words, two important results are obtained: one, the priority vector, w^* is determined, second, it is determined that the given fuzzy comparison matrix is consistent to a degree λ^*.

Although the solution methodologies are similar, FLP problems and fuzzy AHP problems differ. The main difference comes from the absence of the objective function in fuzzy AHP; second, the structure of fuzzy relations. In fuzzy AHP problem, fuzzy relations are self-restricting since the same constraint is defined both by a greater than or equal to and less than or equal to relations. The third difference comes from the crisp constraint which also restricts the solution.

In the next section a numerical example is represented to show the application of the proposed solution methodology.

4.4 Numerical Example

To show the result of proposed solution methodology, the two examples in [1] are choosen by making the assumption that given intervals in [1] are symmetric triangular fuzzy numbers.

The solutions are carried out by the same solver and conditions defined in Section 3.8. Also the code for the auxiliary problem very similar to the one used in Section 3.8.

Example 4.4.1 The fuzzy comparison matrix (upper triangle) is given as:

$$\tilde{A} = \begin{bmatrix} 1 & \tilde{1.5} & \tilde{4} \\ 0 & 1 & \tilde{2.5} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & (1.5, 0.5, 0.5) & (4, 2, 2) \\ 0 & 1 & (2.5, 0.5, 0.5) \\ 0 & 0 & 1 \end{bmatrix}. \tag{4.9}$$

When problem (4.9) is solved by the proposed algorithm, the following results are obtained.

The results in Table (4.1) indicate that given problem (4.9) is consistent to a degree 0.93, which shows a considerably high satisfaction level.

Example 4.4.2 The fuzzy comparison matrix (upper triangle) is given as:

λ	0.93
w_1	0.542
w_2	0.341
w_3	0.315

Table 4.1: Results of (4.9).

$$\tilde{A} = \begin{bmatrix} 1 & \tilde{1.5} & \tilde{8.5} \\ 0 & 1 & \tilde{2.5} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & (1.5, 0.5, 0.5) & (8, 0.5, 0.5) \\ 0 & 1 & (2.5, 0.5, 0.5) \\ 0 & 0 & 1 \end{bmatrix}. \quad (4.10)$$

When problem (4.10) is solved by the proposed algorithm, the λ^* value becomes 0. This result indicates that given problem (4.9) is inconsistent and the given fuzzy pairwise comparison matrix should be revised.

Chapter 5

Conclusion and Future Research

Fuzzy set theory is developed to express vague, imprecise, subjective parameters more acurately and realistically. In other words, fuzzy set theory can cope with the natural expressions better than the bivalent (conventional) set theory. The superiority of fuzzy set theory lies in the fact that it does not only see objects as black and white but also greys between. This speciality thus attracts reasearchers to use fuzzy sets in their problems. Linear programming problems are such problems. The linear programming problems whose coefficients, inequality relations are defined by fuzzy sets are called as Fuzzy Linear Programming problems. Since fuzzy linear programming is a new concept when compared to linear programming, no unified solution methodology exists.

Deriving from that knowledge, this study, first aims to review the literature and combine the solution methodologies proposed so far. Secondly, it detect the deficiencies in proposed solution methodologies and over come those deficiencies, make propositions that improves the solution. Thirdly, as an application, analytical hierarchy process is chosen since the nature of this process is very appropriate to fuzzy set theory and has not been solved by means of a proposed algorithm before.

As a future research the proposed solution methodology will tried to be applied to fuzzy numbers whose membership functions are not defined linearly. Also as a future research the restrictive case, just use of "addition" operator, will tried to be relaxed.

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