ON RAMIFICATION IN EXTENSIONS OF RATIONAL FUNCTION FIELDS

by

NURDAGÜL ANBAR

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APPROVED BY

Prof. Dr. Henning Stichtenoth (Thesis Supervisor)	....
Prof. Dr. Alev Topuzoğlu	....
Assist. Prof. Dr. Cem Güneri	....
Prof. Dr. Aydın Aytuna	....
Dr. Thomas Pedersen	....

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Nurdagül Anbar

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Abstract

Let $K(x)$ be a rational function field, which is a finite separable extension of the rational function field $K(z)$. In the first part of the thesis, we have studied the number of ramified places of $K(x)$ in $K(x)/K(z)$. Then we have given a formula for the ramification index and the different exponent in the extension $F(x)$ over a function field F, where x satisfies an equation $f(x) = z$ for some $z \in F$ and separable polynomial $f(x) \in K[x]$. In fact, this generalizes the well-known formulas for Kummer and Artin-Schreier extensions.

RASYONEL FONKSİYON CİSİM GENİŞLEMELERİNDEKİ DALLANMALAR

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Anahtar Kelimeler: Fonksiyon cisimleri, fonksiyon cisimlerin genişlemeleri, dallanma indexi, fark kuvveti

Özet

$K(x)$ ve $K(z)$ rasyonel fonksiyon cisimleri olsun; öyle ki $K(x)$, $K(z)$ üzerinde ayrışabilir bir cisim genişlemesidir. Öncelikle, $K(x)$'in, $K(x)/K(z)$ genişlemesindeki dallanmış yerlerin sayısına bakılmıştır. Daha sonra, ayrışabilir bir polinom olan $f(x) \in K[x]$ ve bir fonksiyon cismi olan F'in bir elamanı z için $f(x) = z$ denkliği ile tanımlı $F(x)/F$ genişlemesi ele alınmıştır. Bu cisim genişlemelerindeki dallanma indexleri ve fark kuvvetleri için formüller verilmiştir. Aslında; verilen bu formüller Kummer ve Artin-Scheier genişlemeleri için verilen bilindik formüllerin bir genelleştirilmesidir.

to Mithat and Saniye Anbar

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Introduction

Throughout this thesis, K denotes an algebraically closed field.

Let $K(x)$ be a rational function field and $z = \frac{f(x)}{g(x)} \in K(x) \backslash K$, where $f(x)$ and $g(x)$ have no common factors. Then $K(x)$ is an algebraic extension over the rational function field $K(z)$. In the case of $char K = p > 0$, we assume that not both of $f(x)$ and $g(x)$ lie in $K[x^p]$ so that $K(x)/K(z)$ is a finite separable extension.

Let $n \in \mathbb{Z}$, $n > 1$

Question: For which values $i \in \mathbb{Z}$, can we find $z \in K(x)$ such that $K(x)$ has exactly i ramified places in $K(x)/K(z)$ and $[K(x) : K(z)] = n$?In the first part of this thesis, we give some basic definitions and facts to use in the following chapters to answer that question. In chapter 2, we answer the question for $z \in K[x]$ and any characteristic and in chapter 3, we try to give an answer for $z \in K(x)$ and $char K = 0$.

Let F' be an extension of a function field F such that $F' = F(x)$, where x satisfies the equation $z = x^n$ for $n \geq 2$ with $\gcd(n, p) = 1$ in the case of $p = char K > 0$, or $z = x^p - x$, where $p = char K > 0$ for some $z \in F$. These cases are well-known special types of galois extensions, which are called *Kummer* extensions and *Artin-Schreier* extensions, respectively. For these cases, there are explicit formulas to compute the ramification index and the different exponent of a place of F as follows:

Let $P \in \mathbf{P}_F$, $P' \in \mathbf{P}_{F'}$ with $P' \mid P$ and v_P denote the valuation function corresponding to P. For $z = x^n$,

$$e(P' \mid P) = \frac{n}{r_P} \text{ and } d(P' \mid P) = \frac{n}{r_P} - 1,$$

where $r_P = \gcd\{v_P(z), n\}$. For $z = x^p - x$, P is ramified if and only if $m_P > 0$ and in that case

$$e(P' \mid P) = p \text{ and } d(P' \mid P) = (p-1)(m_P + 1),$$

where m_P is defined by

$$m_P := \left\{ \begin{array}{ll} m & \begin{array}{l} \text{,if there exists } y \in F \text{ satisfying} \\ v_P(z - (y^p - y)) = -m < 0 \text{ with } \gcd(m, p) = 1. \end{array} \\ -1 & \text{,if } v_P(z - (y^p - y)) \geq 0 \text{ for some } y \in F\ . \end{array} \right\}.$$

In the last chapter, we derive these formulas by using the results of chapter 2 and chapter 3 with Abhyankar Lemma. Moreover, we generalize these formulas to some other examples.

1

Preliminaries

Let $K(x)$ be a rational function field and $z = \frac{f(x)}{g(x)} \in K(x) \setminus K$. Then $K(x)$ is an algebraic extension of $K(z)$. The question is whether we can find $z \in K(x)$ such that $[K(x) : K(z)] = n$ and $K(x)/K(z)$ has exactly $i \in \mathbb{N}$ ramified places for given $n \in \mathbb{N}$, where $n \geq 2$. We try to answer this question. But before that we give some facts, which we are going to use in the following chapters.

Definition 1.1. Let F'/F be an algebraic extension of function fields and P be a place of F.
(a) An extension P' of P in F' is said to be tamely ramified (resp. wildly ramified) if $e(P' \mid P) > 1$ and the characteristic of K does not divide $e(P' \mid P)$ (resp. characteristic of K divides $e(P' \mid P)$).
(b) P is said to be totally ramified in F'/F if there exists only one place P' of F' which lies over P such that $e(P' \mid P) = [F' : F]$.

Lemma 1.2 (Strict Triangle Inequality)**.** *Let v be a discrete valuation of F/K and let $x, y \in F$ with $v(x) \neq v(y)$. Then*

$$v(x+y) = \min\{v(x), v(y)\}.$$

Theorem 1.3 (Fundamental Equality)**.** *Let F'/K' be a finite extension of F/K. Let P be a place of F/K and $P_1, \ldots, P_m$ be all the places of F'/K' lying over P. Let $e_i := e(P_i \mid P)$ denote the ramification index and $f_i := f(P_i \mid P)$ denote the relative degree of $P_i \mid P$. Then we have*

$$[F' : F] = \sum_{i=1}^{m} e_i f_i.$$

Corollary 1.4. *Let $K(x)$ be a rational function field and $z = \frac{f(x)}{g(x)} \in K(x) \setminus K$ such that $f(x)$ and $g(x)$ have no common factor. Then $K(x)$ is a finite extension field of $K(z)$ of degree*

$$[K(x) : K(z)] = \max\{\deg g(x), \deg f(x)\}.$$

Proof. Let $z = \frac{f(x)}{g(x)} = \frac{\prod p_i^{e_i}(x)}{\prod q_j^{e_j}(x)}$ for some irreducible polynomials $p_i(x)$, $q_j(x) \in K(x)$ and some e_i, $e_j \in \mathbb{Z}^+$. $[K(x) : K(z)] = \left[K(x) : K\left(\frac{1}{z}\right)\right]$, since $K(z) = K\left(\frac{1}{z}\right)$. If $\deg f(x) < \deg g(x)$, then consider $\frac{1}{z}$. So, without loss of generality, assume that $\deg f(x) \geq \deg g(x)$. Let Q_0 denote the zero of z in $K(z)$. Then the places of $K(x)$ lying over Q_0 are the places corresponding to the irreducible factors of $f(x)$ with $e(P_{p_i} \mid Q_0) = e_i$ and $f(P_{p_i} \mid Q_0) = \deg p_i(x)$, where P_{p_i} denotes the place of $K(x)$ corresponding to $p_i(x)$. So, by Fundamental Equality

$$\begin{aligned} [K(x) : K(z)] &= \sum e_i f_i = \sum e_i \deg p_i(x) \\ &= \deg f(x) = \max\{\deg g(x), \deg f(x)\}. \end{aligned}$$

□

Throughout this thesis, we will assume that K is an algebraically closed field and $K(x)/K(z)$ is a finite separable extension; i.e. if $z = \frac{f(x)}{g(x)}$, then not both of the polynomials $f(x)$ and $g(x)$ lie in $K[x^p]$ in the case of $char K = p > 0$. Since K is an algebraically closed field, an irreducible polynomial of $K[x]$ is of the form $x - a$, for some $a \in K$. Also, there is one to one correspondence between the irreducible polynomials of $K[x]$ and the places of $K(x)$ except the pole of x. So, let P_a (resp. Q_a) denote the place of $K(x)$(resp. $K(z)$) corresponding to the polynomial $x - a$ (resp. $z - a$) and P_∞ (resp. Q_∞) denote the pole of x (resp. z).

Definition 1.5. Let $K(x)$ be a rational function field. Then for a given $n \in \mathbb{N}$, we define

$$\boldsymbol{T}_n := \left\{ \begin{array}{c} i \in \mathbb{Z} \mid \text{ there exists } z \in K[x] \text{ such that } [K(x) : K(z)] = n \text{ and} \\ \text{there exist exactly } i \text{ ramified places of } K(x) \text{ in } K(x)/K(z) \end{array} \right\}$$

$$\boldsymbol{S}_n := \left\{ \begin{array}{c} i \in \mathbb{Z} \mid \text{ there exists } z \in K(x) \text{ such that } [K(x) : K(z)] = n \text{ and} \\ \text{there exist exactly } i \text{ ramified places of } K(x) \text{ in } K(x)/K(z) \end{array} \right\}.$$

Our aim is to determine $\boldsymbol{T}_n$ (resp. $\boldsymbol{S}_n$) in chapter 2 (resp. chapter 3). However, we will give some more facts before that.

Theorem 1.6 (Hurwitz Genus Formula)**.** *Let F/K be a function field of genus g and F'/F be a finite separable extension. Let K' denote the constant field of F' and g' denote the genus of F'/K'. Then we have*

$$2g' - 2 = \frac{[F' : F]}{[K' : K]}(2g - 2) + \deg \mathit{Diff}(F'/F),$$

where $\mathit{Diff}(F'/F)$ denotes the different of F'/F.

Corollary 1.7. *Let $K(x)$ be a rational function field and $z = \frac{f(x)}{g(x)} \in K(x)$ such that $K(x)/K(z)$ is separable. Then $\deg \mathit{Diff}(K(x)/K(z)) = 2n - 2$, where $n = [K(x) : K(z)]$.*

Definition 1.8. Let F'/F be an algebraic extension of function fields. F'/F is said to be ramified (resp. unramified) if at least one place P of F is ramified in F'/F (resp. if all places of F are unramified in F'/F).

Theorem 1.9 (Dedekind's Different Theorem)**.** *Let F'/F be a finite separable extension where F/K (resp. F'/K') is a function field with constant field K (resp. K'). Let Q be a place of F and P be a place of F' lying over Q. Then we have*

(a) $d\,(P \mid Q) \geq e\,(P \mid Q) - 1$

(b) $d\,(P \mid Q) = e\,(P \mid Q) - 1 \Leftrightarrow e\,(P \mid Q)$ *is not divisible by* $\mathrm{char} K$*.*

Corollary 1.10. *With the notation as above, then $P \mid Q$ is ramified if and only if $d\,(P \mid Q) \geq 1$; i.e. $P \leq \mathit{Diff}(K\,(x)\,/K\,(z))$.*

Corollary 1.11. *Let $F/K\,(x)$ be a finite separable extension of the rational function field, having K as a full constant field and $[F\colon K\,(x)] = n \geq 2$. Then $F/K\,(x)$ is ramified.*

Proof. Proof: Let g denote the genus of F. Since $K\,(x)$ is a rational function field, genus of $K\,(x)$ is 0 and since $F/K\,(x)$ is a finite separable extension, by Hurwitz Genus Formula

$$\begin{aligned} 2g - 2 &= [F\colon K\,(x)]\,(-2) + \deg \mathrm{Diff}\,(F/K\,(x)) \\ &= n\,(-2) + \deg \mathrm{Diff}\,(F/K\,(x)) \\ &\Rightarrow \deg \mathrm{Diff}\,(F/K\,(x)) = 2g + 2\,(n-1) > 2g \geq 0 \\ &\Rightarrow \deg \mathrm{Diff}\,(F/K\,(x)) > 0. \end{aligned}$$

Hence, there exists $P \in \mathbf{P}_F$ such that $P \leq Diff\,(F/K\,(x))$. So, P is ramified in $F/K\,(x)$, by Dedekind's Different Theorem. □

Theorem 1.12. *Suppose $F' = F\,(x)$ is a finite separable extension of a function field F with $[F'\colon F] = n$. Let Q be a place of F such that the minimal polynomial $\varphi\,(T)$ of x over F has coefficients in O_Q, where O_Q is the valuation ring corresponding to the place Q, and let P be a place of F' lying over Q. Then $d\,(P \mid Q) \leq v_P\,(\varphi'\,(x))$, where φ' denotes the derivative of φ.*

Theorem 1.13. *Let F'/F be a finite separable extension of function fields and $P \in \mathbf{P}_F$, $P' \in \mathbf{P}_{F'}$ with $P' \mid P$. Suppose that $P' \mid P$ is totally ramified; i.e. $e\,(P' \mid P) = [F'\colon F] = n$. Let $x \in F'$ be a P'-prime element and $\varphi\,(T) \in F\,[T]$ be the minimal polynomial of x over F. Then $d\,(P' \mid P) = v_{P'}\,(\varphi'\,(x))$, where $v_{P'}$ denote the discrete valuation function corresponding to P'.*

Proposition 1.14 (Transitivity of the Different)**.** *Let F''/F', F'/F be function field extensions and $P'' \in \mathbf{P}_{F''}$, $P' \in \mathbf{P}_{F'}$, $P \in \mathbf{P}_F$ with $P'' \mid P' \mid P$. Then*

$$d\,(P'' \mid P) = e\,(P'' \mid P)\,d\,(P' \mid P) + d\,(P'' \mid P')\,.$$

Definition 1.15. Suppose that $p(x)$, $q(x) \in K[x]$ such that

$$p(x) = a_m x^m + a_{m-1} x^{m-1} + \cdots + a_1 x + a_0$$

and

$$q(x) = b_n x^n + b_{n-1} x^{n-1} + \cdots + b_1 x + b_0.$$

where a_m,$b_n \neq 0$ and $m,\ n \in \mathbb{Z}$. Then the resultant of $p(x)$ and $q(x)$, denoted by $R(p(x), q(x))$, is defined as the $(m+n) \times (m+n)$ determinant:

$$\begin{bmatrix} a_m & a_{m-1} & \cdots & \cdots & a_1 & a_0 & \cdots & \cdots & 0 & 0 \\ 0 & a_m & \cdots & \cdots & a_2 & a_1 & \cdots & \cdots & 0 & 0 \\ & & \ddots & & & & & & & \\ & & & \ddots & & & & & & \\ 0 & 0 & \cdots & \cdots & a_m & a_{m-1} & \cdots & \cdots & a_1 & a_0 \\ b_n & b_{n-1} & \cdots & \cdots & b_1 & b_0 & \cdots & \cdots & 0 & 0 \\ & & & & & & \ddots & & & \\ & & & & & & & \ddots & & \\ 0 & 0 & \cdots & \cdots & b_n & b_{n-1} & \cdots & \cdots & b_0 & 0 \\ 0 & 0 & \cdots & \cdots & 0 & b_n & \cdots & \cdots & b_1 & b_0 \end{bmatrix}$$

Definition 1.16. Let $p(x) = x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \in K[x]$ with $\deg p(x) \geq 2$. Then the discriminant of $p(x)$, denoted by $D(p(x))$, is defined by

$$D(p(x)) = (-1)^{\frac{1}{2}n(n-1)} R(p(x), p'(x)),$$

where $p'(x)$ denotes the derivative of $p(x)$.

Lemma 1.17. *Let $p(x)$, $q(x) \in K[x]$. Then $R(p(x), q(x)) = 0$ if and only if $p(x)$ and $q(x)$ have a common root.*

Hence $D(p(x)) = 0$ for $p(x) \in K[x]$ with $\deg p(x) \geq 2$ if and only if $p(x)$ has a factor with multiplicity greater than 1.

Theorem 1.18 (Abhyankar Lemma)**.** *Let F'/F be finite separable extension of function fields. Suppose that $F' = F_1 F_2$, where F_1 and F_2 are intermediate fields $F \subseteq F_1$, $F_2 \subseteq F'$. Let $P' \in \mathbf{P}_{F'}$ and $P \in \mathbf{P}_F$ such that $P' \mid P$ and set $P_i := F_i \cap P'$ for $i = 1$, 2. Assume that at least one of the extensions $P_1 \mid P$ or $P_2 \mid P$ is tame. Then*

$$e(P' \mid P) = lcm\left\{e(P_1|P),\ e(P_2 \mid P)\right\}.$$

2

Ramified Places of $K(x)$ in $K(x)/K(z)$ for $z \in K[x]$

In this chapter, we will investigate $\boldsymbol{T}_n$, where $\boldsymbol{T}_n$ is the set consisting of integers i for which we can find $z \in K[x]$ such that $[K(x) : K(z)] = n$ and $K(x)$ has exactly i ramified places in $K(x)/K(z)$. Let $z = f(x)$ be a monic polynomial of $K[x]$ with $\deg f(x) = n$, where $n \geq 2$. Then $K(x)$ is a field extension of $K(z)$ with $[K(x) : K(z)] = n$ and $\varphi(T) = f(T) - z$ is the minimal polynomial of x over $K(z)$. We assume that $\varphi'(T) = f'(T) \neq 0$ in order that $K(x)/K(z)$ is a separable extension. So, we always take a monic polynomial $f(x) \in K[x] \setminus K[x^p]$, where K is an algebraically closed field.

Lemma 2.1. *Let $K(x)$ be a rational function field and $z = f(x) \in K[x]$ with $\deg f(x) = n \geq 2$. Then the ramified places of $K(x)$ in $K(x)/K(z)$ are the pole P_∞ of x and the places corresponding to the zeros of the derivative of $f(x)$.*

Proof. Let $Q_\infty \in P_{K(z)}$ denote the pole of z and let v_{P_∞} and v_{Q_∞} denote the valuation functions at $x = \infty$ and $z = \infty$, respectively. Then

$$v_{P_\infty}(z) = e(P_\infty \mid Q_\infty) v_{Q_\infty}(z) = -e(P_\infty \mid Q_\infty)$$

and

$$v_{P_\infty}(z) = v_{P_\infty}(f(x)) = -\deg f(x) = -n$$

$\Rightarrow e(P_\infty \mid Q_\infty) = n \geq 2$; i.e. P_∞ is totally ramified. Hence, P_∞ is the only place lying over Q_∞.

Let P be a place of $K(x)$ corresponding to $x - a$ and Q be the place of $K(z)$ such that $Q \subseteq P$; i.e. Q is the place corresponding to $z - f(a) = f(x) - f(a)$. Then $\varphi(T) = f(T) - z$ is the minimal polynomial of x over $K(z)$. Since the coefficients of $f(T)$ lies in K, $\varphi(T) \in O_Q[T]$; i.e. x is integral over O_Q, for all $Q \in \mathbf{P}_{K(z)} \setminus \{Q_\infty\}$.By theorem 1.12,

$$d(P \mid Q) \leq v_P(\varphi'(x)) = v_P(f'(x)) = 0, \text{ for all } a \text{ such that } x - a \nmid f'(x)$$

$$\Rightarrow d(P \mid Q) = 0, \text{ for all } a \text{ such that } x - a \nmid f'(x).$$

Therefore, a place corresponding $x - a$, which is not a divisor of $f'(x)$, is unramified.

Now, let $x - a$ be a divisor of $f'(x)$. Then $x - a \mid f(x) - f(a)$ and $x - a \mid (f(x) - f(a))' = f'(x)$; i.e. $f(x) - f(a) = (x - a)^2 g(x)$, for some $g(x) \in K[x]$. Hence,

$$2 \leq v_P(f(x) - f(a)) = e(P \mid Q) v_Q(f(x) - f(a)) = e(P \mid Q).$$

So, a place corresponding to $x - a$, which is a divisor of $f'(x)$, is ramified. $\square$

Corollary 2.2. *Let $n \in \mathbb{Z}$, with $n \geq 2$. Then $\boldsymbol{T}_n \subseteq \{1, 2, \cdots, n\}$. More precisely, if $z = f(x) \in K[x]$ with $\deg f(x) = n$, then $K(x)/K(z)$ has exactly i ramified places if and only if $f'(x)$ has $i-1$ distinct roots.*

Corollary 2.3. *Let $K(x)$ be a rational function field and $z = f(x) \in K[x]$ with $\deg f(x) = n \geq 2$. Suppose $K(x)$ has only one ramified place P in $K(x)/K(z)$. Then P is the pole P_∞ of x and P is wildly ramified.*

Proof. By lemma 2.1, we know that $e(P_\infty \mid Q_\infty) = n \geq 2$. Hence, the only ramified place of $K(x)$ is P_∞. By Hurwitz Genus Formula,

$$d(P_\infty \mid Q_\infty) = \deg \operatorname{Diff}(F/K(x)) = 2n - 2 \geq n, \text{ since } n \geq 2$$

$$\Rightarrow d(P_\infty \mid Q_\infty) \geq e(P_\infty \mid Q_\infty).$$

So, $P_\infty \mid Q_\infty$ is wildly ramified by Dedekind's Different Theorem. □

Corollary 2.4. *Let $K(x)$ be a rational function field. If $1 \in \boldsymbol{T}_n$, then $p \mid n$, where $n = [K(x) : K(z)]$ and $p = charK$.*

Proof. Suppose $1 \in \boldsymbol{T}_n$. Then the ramified place of $K(x)$ is the pole P_∞ of x, which is wildly ramified by corollary 2.3. Hence, $charK \mid e(P_\infty \mid Q_\infty)$, where $e(P_\infty \mid Q_\infty) = n$. □

So, if $p \nmid n$, then $\boldsymbol{T}_n \subseteq \{2, \cdots, n\}$.

Corollary 2.5. *If $p \mid n$, then $1 \in \boldsymbol{T}_n$.*

Proof. $1 \in \boldsymbol{T}_n$ if and only if $K(x)$ has only one ramified place in $K(x)/K(z)$. Let $z = f(x) = x^n + x$. Then $f'(x) = 1$; i.e. $f'(x)$ has no zero. So, the pole of x is the only ramified place of $K(x)$. □

Corollary 2.6. *If $p \nmid n$, then $2 \in \boldsymbol{T}_n$.*

Proof. Let $z = f(x) = x^n$. Then $f'(x) = nx^{n-1}$. Since $p \nmid n$ and $n \geq 2$, 0 is the only zero of $f'(x)$. So, all the ramified places of $K(x)$ in $K(x)/K(z)$ are the pole and the zero of x. □

Lemma 2.7 ($charK = p > 0$)**.** *Let $z = f(x) = g(x) + h(x)$ be a polynomial over K of degree n, where $g(x) = \sum_{p \nmid i} a_i x^i$ and $h(x) = \sum_{p \mid j} b_j x^j$. Let P_∞ denote the pole of x in $K(x)$ and Q_∞ denote the pole of z in $K(z)$. Then $d(P_\infty \mid Q_\infty) = 2n - \{\deg g(x) + 1\}$.*

Proof. Without loss of generality, we can assume that the constant term of $f(x)$ is 0 so that all $i, j \geq 1$. Let $\varphi(T)$ be the minimal polynomial of $\frac{1}{x}$ over $K(z)$. By lemma 2.1, we know that P_∞ is totally ramified. Hence, $d(P_\infty \mid Q_\infty) = v_{P_\infty}\left(\varphi'\left(\frac{1}{x}\right)\right)$, by Theorem 1.13. So, we will first find $\varphi(T)$ to compute $d(P_\infty \mid Q_\infty)$. Since $K(x) = K\left(\frac{1}{x}\right)$,

$$\left[K\left(\frac{1}{x}\right) : K(z)\right] = [K(x) : K(z)] = n.$$

Therefore, $\deg \varphi(T) = \left[K\left(\frac{1}{x}\right) : K(z)\right] = n$.

$$z = g(x) + h(x) = \sum_{p \nmid i} a_i x^i + \sum_{p \mid j} b_j x^j.$$

Multiply both sides of the equality by $\frac{1}{zx^n}$. Then we have

$$\frac{1}{x^n} = \frac{1}{z}\sum_{p \nmid i} a_i \frac{1}{x^{n-i}} + \frac{1}{z}\sum_{p \mid j} b_j \frac{1}{x^{n-j}}$$

$$\Rightarrow \frac{1}{x^n} - \frac{1}{z}\sum_{p \nmid i} a_i \frac{1}{x^{n-i}} - \frac{1}{z}\sum_{p \mid j} b_j \frac{1}{x^{n-j}} = 0.$$

Let $\gamma(T) = T^n - \frac{1}{z}\sum_{p \nmid i} a_i T^{n-i} - \frac{1}{z}\sum_{p \mid j} b_j T^{n-j} \in K(z)[T]$. Then we have seen that $\gamma\left(\frac{1}{x}\right) = 0$. Since $\deg \gamma(T) = n$, $\varphi(T) = \gamma(T)$. Hence,

$$\varphi'(T) = nT^{n-1} - \frac{1}{z}\sum_{p \nmid i} a_i (n-i) T^{n-i-1} - \frac{1}{z}\sum_{p \mid j} b_j (n-j) T^{n-j-1}.$$

Case(i): if $p \mid n$, then $p \mid n-j$ and $p \nmid n-i$. Hence,

$$\varphi'(T) = -\frac{1}{z}\sum_{p \nmid i} a_i (n-i) T^{n-i-1}.$$

Then

$$\begin{aligned}
v_{P_\infty}\left(\varphi'\left(\frac{1}{x}\right)\right) &= v_{P_\infty}\left(-\frac{1}{z}\sum_{p \nmid i} a_i (n-i) \frac{1}{x^{n-i-1}}\right) \\
&= v_{P_\infty}\left(\frac{1}{z}\right) + v_{P_\infty}\left(\sum_{p \nmid i} a_i (n-i) \frac{1}{x^{n-i-1}}\right) \\
&= \deg f(x) + \min_{p \nmid i,\ a_i \neq 0} \{n-i-1\} \text{ (by Strict Triangle Inequality)} \\
&= n + (n - \deg g(x) - 1) \\
&= 2n - \{\deg g(x) + 1\}.
\end{aligned}$$

Case(ii): if $p \nmid n$, then

$$\varphi'(T) = nT^{n-1} - \frac{1}{z}\sum_{p \nmid i} a_i (n-i) T^{n-i-1} - \frac{1}{z}\sum_{p \mid j} b_j (n-j) T^{n-j-1}.$$

Then

$$\begin{aligned}
& v_{P_\infty}\left(\varphi'\left(\frac{1}{x}\right)\right) \\
= \; & v_{P_\infty}\left(n\frac{1}{x^{n-1}} - \frac{1}{z}\sum_{p\nmid i} a_i\left(n-i\right)\frac{1}{x^{n-i-1}} - \frac{1}{z}\sum_{p\mid j} b_j\left(n-j\right)\frac{1}{x^{n-j-1}}\right) \\
= \; & v_{P_\infty}\left(\frac{1}{z}\right) + v_{P_\infty}\left(nz\frac{1}{x^{n-1}} - \sum_{p\nmid i} a_i\left(n-i\right)\frac{1}{x^{n-i-1}} - \sum_{p\mid j} b_j\left(n-j\right)\frac{1}{x^{n-j-1}}\right) \\
= \; & v_{P_\infty}\left(\frac{1}{z}\right) + v_{P_\infty}\left(\frac{1}{x^n}\right) + v_{P_\infty}\left(nzx - \sum_{p\nmid i} a_i\left(n-i\right)x^{i+1} - \sum_{p\mid j} b_j\left(n-j\right)x^{j+1}\right)
\end{aligned}$$

Now, we first compute

$$\begin{aligned}
& nzx - \sum_{p\nmid i} a_i\left(n-i\right)x^{i+1} - \sum_{p\mid j} b_j\left(n-j\right)x^{j+1} \\
= \; & n\left(\sum_{p\nmid i} a_i x^i + \sum_{p\mid j} b_j x^j\right)x - \sum_{p\nmid i} a_i\left(n-i\right)x^{i+1} - \sum_{p\mid j} b_j\left(n-j\right)x^{j+1} \\
= \; & n\sum_{p\nmid i} a_i x^{i+1} + n\sum_{p\mid j} b_j x^{j+1} - \sum_{p\nmid i} a_i\left(n-i\right)x^{i+1} - \sum_{p\mid j} b_j\left(n-j\right)x^{j+1} \\
= \; & \sum_{p\nmid i} a_i\left(n-\left(n-i\right)\right)x^{i+1} - \sum_{p\mid j} b_j\left(n-\left(n-j\right)\right)x^{j+1} \\
= \; & \sum_{p\nmid i} a_i i x^{i+1} - \sum_{p\mid j} b_j j x^{j+1} \\
= \; & \sum_{p\nmid i} a_i i x^{i+1}.
\end{aligned}$$

Hence,

$$\begin{aligned}
& v_{P_\infty}\left(nzx - \sum_{p\nmid i} a_i\left(n-i\right)x^{i+1} - \sum_{p\mid j} b_j\left(n-j\right)x^{j+1}\right) \\
= \; & v_{P_\infty}\left(\sum_{p\nmid i} a_i i x^{i+1}\right) = \min_{p\nmid i,\ a_i\neq 0}\left\{-i-1\right\} \text{ (by Strict Triangle Inequality)} \\
= \; & -\left(\deg g\left(x\right)+1\right).
\end{aligned}$$

So,

$$\begin{aligned}
v_{P_\infty}\left(\varphi'\left(\frac{1}{x}\right)\right) &= v_{P_\infty}\left(\frac{1}{z}\right) + v_{P_\infty}\left(\frac{1}{x^n}\right) - \left(\deg g\left(x\right)+1\right) \\
&= n + n - \left(\deg g\left(x\right)+1\right) \\
&= 2n - \left(\deg g\left(x\right)+1\right).
\end{aligned}$$

□

When $char K = 0$, then $K(x)/K(z)$ is tame.
Therefore, $d(P_\infty \mid Q_\infty) = e(P_\infty \mid Q_\infty) - 1 = n - 1$.

Claim 2.8. Let $K(x)/K(z)$ be defined as before. Then there is no place P of $K(x)$ such that $d(P \mid Q) = p - 1$, where Q is the place of $K(z)$ lying under P.

Proof. If $char K = 0$, then $d(P \mid Q) \neq -1$. Because, $d(P \mid Q)$ is a non-negative integer. So, assume that $char K = p > 0$ and $d(P \mid Q) = p - 1$. If P is tamely ramified, then $d(P \mid Q) = e(P \mid Q) - 1$ by Dedekind's Different Theorem. Hence, $e(P \mid Q) = p$. But p can not divide the ramification index, since P is tamely ramified. So, P must be wildly ramified; i.e. $p \mid e(P \mid Q)$. Then, by Dedekind's Different Theorem, $d(P \mid Q) \geq e(P \mid Q) \Longrightarrow e(P \mid Q) \leq p-1$; i.e. $p \nmid e(P \mid Q)$. Hence, both cases are impossible. ☐

Proposition 2.9. *Let $K(x)$ be a rational function field and $z = f(x) \in K[x]$ with $\deg f(x) = n \geq 2$ and let $f'(x) = \prod_{\text{for some } i} (x - c_i)^{d_i}$, where c_i's are different roots of $f'(x)$ and d_i's are positive integers. Then $d\left(P_{c_i} \mid Q_{f(c_i)}\right) = d_i$, where P_{c_i} is the place of $K(x)$ corresponding to $x - c_i$ and $Q_{f(c_i)}$ is the place of $K(z)$ lying under P_{c_i}; i.e. the place corresponding to $z - f(c_i)$.*

Proof. Since K is an algebraically closed field, for all $P \in \mathbf{P}_{K(x)}$ $\deg P = 1$. So, by Hurwitz Genus Formula

$$\begin{aligned}
\deg \operatorname{Diff}(K(x)/K(z)) &= \deg \sum_{P \mid Q} d(P \mid Q) P \\
&= \sum_{P \mid Q, P \neq P_\infty} d(P \mid Q) + d(P_\infty \mid Q_\infty) = 2n - 2
\end{aligned}$$

$$\begin{aligned}
\Longrightarrow \quad & \sum_{P \mid Q, P \neq P_\infty} d(P \mid Q) = (2n-2) - d(P_\infty \mid Q_\infty) \\
= \quad & (2n-2) - (2n - (\deg g(x) + 1)) \\
= \quad & \deg g(x) - 1 = \deg f'(x).
\end{aligned}$$

The minimal polynomial of x over $K(z)$ is $\varphi(T) = f(T) - z$. Since $f(T)$ has coefficients in K and P_∞ is the only place of $K(x)$ lying over Q_∞, x is integral over O_Q for all $Q \in \mathbf{P}_{K(z)} \setminus Q_\infty$, where O_Q is the valuation ring corresponding to the place Q. By theorem 1.12

$$d\left(P_{c_i} \mid Q_{f(c_i)}\right) \leq v_{P_{c_i}}(\varphi'(x)) = v_{P_{c_i}}(\varphi'(f'(x))) = d_i.$$

So,

$$\sum_i d\left(P_{c_i} \mid Q_{f(c_i)}\right) \leq \sum_i d_i = \deg f'(x) \Longrightarrow d\left(P_{c_i} \mid Q_{f(c_i)}\right) = d_i, \text{ for all } i.$$

☐

Corollary 2.10. *Let $K(x)/K(z)$ be defined as before with $z = f(x)$. Then $f'(x)$ can not contain a factor $x - \alpha$ with multiplicity $p - 1$.*

Proof. Let P_α denote the place of $K(x)$ corresponding to the factor $x-\alpha$ and Q denote the place of $K(z)$ lying under P_α. $d(P_\alpha \mid Q)$ is equal to multiplicity of $x-\alpha$ in $f'(x)$, by proposition 2.9. But $d(P_\alpha \mid Q) \neq p-1$, by claim 2.8. So, $f'(x)$ can not contain a factor with multiplicity $p-1$. □

Now, we investigate $\boldsymbol{T}_n$ for $char K = 2$. Before giving the general condition, we are going to give some simple examples.

Example 2.11 ($char K = 2$)**.** In this example, P_∞ (resp. Q_∞) denotes the pole of x (resp. the pole of z) and P_α (resp. Q_α) denotes the place of $K(x)$ (resp. $K(z)$) corresponding to the factor $x-\alpha$ (resp. $z-\alpha$).

Let $n = 2$,then $\deg f'(x) = 0$.
Since $p \mid n$, $1 \in \boldsymbol{T}_2$, by corollary 2.5. and since $\deg f'(x) = 0$, $\boldsymbol{T}_2 = \{1\}$, by corollary 2.2.

Let $n = 3$, then $\deg f'(x) = 2$.
$1 \notin \boldsymbol{T}_3$ and $2 \in \boldsymbol{T}_3$, since $2 \nmid 3$, by corollary 2.4 and 2.6.
$3 \notin \boldsymbol{T}_3$: $3 \in \boldsymbol{T}_n$ if and only if $f'(x)$ has two distinct zeros. Then $f'(x)$ must have a factor with multiplicity $1 = p-1$. But, this is impossible, by corollary 2.10.
Hence, $\boldsymbol{T}_3 = \{2\}$.

Let $n = 4$, then $\deg f'(x) \leq 2$.
$1 \in \boldsymbol{T}_4$, since $p \mid n$.
$2 \in \boldsymbol{T}_4$: Let $z = f(x) = x^4 + x^3$. Then $f'(x) = x^2$. So, the ramified places of $K(x)$ are P_∞ and P_0, which lie Q_∞ and Q_0 with $e(P_\infty \mid Q_\infty) = 4$, $e(P_0 \mid Q_0) = 3$, $d(P_\infty \mid Q_\infty) = 4$ and $d(P_0 \mid Q_0) = 2$.
$3 \notin \boldsymbol{T}_4$: Since $f'(x)$ can not have a factor with multiplicity 1.
$4 \notin \boldsymbol{T}_4$: $K(x)$ can have at most $\deg f'(x) + 1 \leq 3$ ramified places.
Hence, $\boldsymbol{T}_4 = \{1, 2\}$.

Let $n = 5$, then $\deg f'(x) = 4$.
$1 \notin \boldsymbol{T}_5$ and $2 \in \boldsymbol{T}_5$, since $2 \nmid 5$.
$3 \in \boldsymbol{T}_5$: Let $z = f(x) = x^5 + x^3 = x^3(x+1)^2$, then $f'(x) = x^4 + x^2 = x^2(x+1)^2$. So, the ramified places of $K(x)$ are P_∞, P_0 and P_1 which lie over Q_∞ and Q_0 with $e(P_\infty \mid Q_\infty) = 5$, $e(P_0 \mid Q_0) = 3$, $e(P_1 \mid Q_0) = 2$, $d(P_\infty \mid Q_\infty) = 4$ and $d(P_0 \mid Q_0) = d(P_1 \mid Q_0) = 2$.
$4, 5 \notin \mathbf{T}_5$: Otherwise, $f'(x)$ has a factor with multiplicity 1.
Hence, $\boldsymbol{T}_5 = \{2, 3\}$.

Now, we are ready to give the general case for $char K = 2$:

Lemma 2.12 ($char K = 2$)**.** *Let $K(x)$ be a rational function field and $n \in \mathbb{Z}$, $n \geq 2$. Then*

$$\boldsymbol{T}_n = \{1, 2, \ldots, k\}, \text{ if } n = 2k$$

and

$$\boldsymbol{T}_n = \{2, \ldots, k\}, \text{ if } n = 2k-1.$$

Proof. If $n = 2k$, then $1 \in \boldsymbol{T}_n$, by corollary 2.5.
If $n = 2k - 1$, then $1 \notin \boldsymbol{T}_n$, by corollary 2.4.

$s \in \boldsymbol{T}_n$, if $s \leq k$: $s \in \boldsymbol{T}_n$ if and only if $f'(x)$ has $s - 1$ distinct zeros, i.e. $f'(x)$ is of the form

$$f'(x) = (x + \alpha_1)^{e_1} (x + \alpha_2)^{e_2} \dots \left(x + \alpha_{(s-1)}\right)^{e_{(s-1)}},$$

where α_i's are distinct elements of K and e_i's are positive even integers so that $f'(x)$ has an antiderivative. Then ramified places of $K(x)$ are P_∞ and P_{α_i}'s, where P_{α_i}'s denote the places corresponding to the factor $(x - \alpha_i)$'s, lying above the places Q_∞ and $Q_{f(\alpha_i)}$ with $d(P_\infty \mid Q_\infty) = 2k - \sum\limits_{1 \leq i \leq s-1} e_i$ and $d\left(P_{\alpha_i} \mid Q_{f(\alpha_i)}\right) = e_i$.

$s \notin \boldsymbol{T}_n$, if $s \geq k+1$: If $s \in \boldsymbol{T}_n$, then $f'(x)$ must have $s-1$ distinct zeros; i.e. $f'(x)$ must contain more than $k - 1$ factors. Since $\deg f'(x) \leq 2k - 2$, $f'(x)$ must have a factor with multiplicity 1. But, $f'(x)$ can not contain a factor with multiplicity $p - 1$, by corollary 2.10. □

Now, we are going to investigate $\boldsymbol{T}_n$ for $char K = 3$. Again before giving the general condition, we will give some examples.

Example 2.13 ($char K = 3$)**.** In this example, P_∞ (resp. Q_∞) denotes the pole of x (resp. the pole of z) and P_α (resp. Q_α) denotes the place of $K(x)$ (resp. $K(z)$) corresponding to the factor $x - \alpha$ (resp. $z - \alpha$).

Let $n = 2$, then $\deg f'(x) = 1$.
$1 \notin \boldsymbol{T}_2$ and $2 \in \boldsymbol{T}_2$, because $3 \nmid 2$.
Hence, $\boldsymbol{T}_2 = \{2\}$.

Let $n = 3$, then $\deg f'(x) \leq 1$.
$1 \in \boldsymbol{T}_3$, since $p \mid n$.
$2 \in \boldsymbol{T}_3$: Let $z = f(x) = x^3 + x^2 = x^2(x + 1)$. Then $f'(x) = 2x$. So, ramified places of $K(x)$ are P_∞ and P_0, which lie over Q_∞ and Q_0 with $e(P_\infty \mid Q_\infty) = 3$, $e(P_0 \mid Q_0) = 2$, $d(P_\infty \mid Q_\infty) = 3$ and $d(P_0 \mid Q_0) = 1$.
$3 \notin \boldsymbol{T}_3$:Since $\deg f'(x) \leq 1$, $f'(x)$ can have at most one zero.
Hence, $\boldsymbol{T}_3 = \{1, 2\}$.

Let $n = 4$, then $\deg f'(x) = 3$.
$1 \notin \boldsymbol{T}_4$ and $2 \in \boldsymbol{T}_4$, because $3 \nmid 4$.
$3 \notin \boldsymbol{T}_4$: $3 \in \boldsymbol{T}_4$ if and only if $f'(x)$ has 2 distinct roots. Since $\deg f'(x) = 3$, one of the zeros must have multiplicity 2. But this is a contradiction to corollary 2.10.
$4 \in \boldsymbol{T}_4$: Let $f'(x) = x^3 + x$. Since no exponent of x is congruent to -1 modulo 3, $f'(x)$ has an antiderivative and since $\gcd(f'(x), f''(x)) = 1$, $f'(x)$ has no multiple root; i.e. $f'(x)$ has 3 distinct zeros, say α_1, α_2, and α_3. Then the ramified places of $K(x)$ are P_∞, P_{α_1}, P_{α_2} and P_{α_3} lying above the places Q_∞, $Q_{f(\alpha_1)}$, $Q_{f(\alpha_2)}$ and $Q_{f(\alpha_3)}$, respectively, with $e(P_\infty \mid Q_\infty) = 4$, $e\left(P_{\alpha_i} \mid Q_{f(\alpha_i)}\right) = 2$, $d(P_\infty \mid Q_\infty) = 3$, $d\left(P_{\alpha_i} \mid Q_{f(\alpha_i)}\right) = 1$.
Hence, $\boldsymbol{T}_4 = \{2, 4\}$.

Now, we can state the lemma which gives the set $\boldsymbol{T}_n$ in the case of $charK = 3$.

Lemma 2.14 ($charK = 3$)**.** *Let $K(x)$ be a rational function field and $n \in \mathbb{Z}$, $n \geq 2$. Then*

$$\begin{aligned} \textbf{(i) } \boldsymbol{T}_n &= \{1, 2, \ldots, n-1\}, \text{ if } 3 \mid n \\ \textbf{(ii) } \boldsymbol{T}_n &= \{2, \ldots, n-2, n\}, \text{ if } 3 \nmid n. \end{aligned}$$

Proof. Let P_∞ (resp. Q_∞) denote the pole of x (resp. the pole of z) and P_α (resp. Q_α) denote the place of $K(x)$ (resp. $K(z)$) corresponding to the factor $x-\alpha$ (resp. $z-\alpha$).

(i) Suppose $3 \mid n$, say $n = 3k$ for some $k \in \mathbb{Z}$. Then $\deg f'(x) \leq 3k-2$.
$1 \in \boldsymbol{T}_n$, since $3 \mid n$.
$3l \in \boldsymbol{T}_n$, $1 \leq l \leq k-1$: $3l \in \boldsymbol{T}_n$ if and only if $f'(x)$ has $3l-1$ distinct zeros. Let

$$f'(x) = x^3\left(x^{3l-2}+1\right) = x^{3l+1}+x^3,$$

Since $3l+1 \equiv 1 \neq -1$ $(mod\ 3)$ and $3 \equiv 0 \neq -1$ $(mod\ 3)$, $f'(x)$ has an antiderivative and since $\left(x^{3l-2}+1\right)' = x^{3l-3}$; i.e. $\gcd\left(x^{3l-2}+1, x^{3l-3}\right) = 1$, $x^{3l-2}+1$ has no multiple roots. Therefore, $f'(x)$ has $3l-1$ distinct zeros.
$3l+1 \in \boldsymbol{T}_n$, $1 \leq l \leq k-1$: Let

$$f'(x) = x^{3l}+x+1.$$

Since $3l \equiv 0 \neq -1$ $(mod\ 3)$, $1 \neq -1$ $(mod\ 3)$, $f'(x)$ has an antiderivative. Also, $f''(x) = 1$ implies that $\gcd(f'(x), f''(x)) = 1$. Therefore $f'(x)$ has $3l$ distinct zeros.
$3l+2 \in \boldsymbol{T}_n$, $0 \leq l \leq k-1$: Let

$$f'(x) = x^{3l+1}+1.$$

Since $3l+1 \equiv 1 \neq -1$ $(mod\ 3)$, $f'(x)$ has an antiderivative and since $f''(x) = x^{3l}$, $\gcd(f'(x), f''(x)) = 1$. So, $f'(x)$ have $3l+1$ distinct zeros.
Notice that $n \notin \boldsymbol{T}_n$, since $\deg f'(x) \leq n-2$.
Hence, $\boldsymbol{T}_n = \{1, 2, \ldots, n-1\}$.

(ii) Suppose $3 \nmid n$. Then either $n = 3k+1$ or $n = 3k+2$, for some $k \in \mathbb{Z}$.
Since $3 \nmid n$, $1 \notin T_n$.

If $n = 3k+1$, then $\deg f'(x) = 3k$
$3l \in \boldsymbol{T}_n$, $1 \leq l \leq k-1$: If $l \geq 2$, then let

$$\begin{aligned} f'(x) &= x^{3(k-l)}(x+\alpha)^3\left(x^{3(l-1)}+x+1\right) \\ &= x^{3k}+\alpha^3x^{3k-3}+x^{3k-3l+4}+x^{3k-3l+3}+\alpha^3x^{3k-3l+1}+\alpha^3x^{3k-3l}, \end{aligned}$$

where $0 \neq \alpha \in K$ is not a zero of $x^{3l-3}+x+1$(we can find such α, since K is an algebraically closed field; i.e. K is infinite). Since $3k \equiv 3k-3 \equiv 3k-3l+3 \equiv 3k-3l \equiv 0 \neq -1$ $(mod\ 3)$, and $3k-3l+4 \equiv 3k-3l+1 \equiv 1 \neq -1$ $(mod\ 3)$, $f'(x)$

has an antiderivative and since $\left(x^{3(l-1)}+x+1\right)'=1$, $x^{3(l-1)}+x+1$ has $3l-3$ distinct zeros. So, $f'(x)$ have $3l-1$ distinct zeros.
If $l=1$, then let

$$f'(x)=x^{3(k-1)}(x+1)^3=x^{3k}+x^{3k-3}.$$

Then $f'(x)$ has $2=3l-1$ distinct zeros.
$3l+1\in \boldsymbol{T}_n$, $1\leq l\leq k$: Let

$$f'(x)=x^{3(k-l)+1}\left(x^{3l-1}+1\right)=x^{3k}+x^{3(k-l)+1}.$$

Since $3k\equiv 0\neq -1\ (mod\ 3)$, and $3(k-l)+1\equiv 1\neq -1\ (mod\ 3)$, $f'(x)$ has an antiderivative and since $\gcd\left(x^{3l-1}+1,\ 2x^{3l-2}\right)=1$, where $2x^{3l-2}=\left(\ x^{3l-1}+1\right)'$, $x^{3l-1}+1$ has $3l-1$ distinct zeros. So, $f'(x)$ has $3l$ distinct zeros.
$3l+2\in \boldsymbol{T}_n$, $0\leq l\leq k-1$: Let

$$f'(x)=x^{3(k-l)}\left(x^{3l}+x+1\right)=x^{3k}+x^{3(k-l)+1}+x^{3(k-l)}.$$

Since $3k\equiv 3(k-l)\equiv 0\neq -1\ (mod\ 3)$ and $3(k-l)+1\equiv 1\neq -1\ (mod\ 3)$, $f'(x)$ has an antiderivative. Since $\left(\ x^{3l}+x+1\right)'=1$, $x^{3l}+x+1$ has $3l$ distinct zeros. Then $f'(x)$ have $3l+1$ distinct zeros.

If $n=3k+2$, then $\deg f'(x)=3k+1$.
$3l\in \boldsymbol{T}_n$, $1\leq l\leq k$: Let

$$f'(x)=x^{3(k-l+1)}\left(x^{3l-2}+1\right)=x^{3k+1}+x^{3(k-l+1)}.$$

Since $3k+1\equiv 1\neq -1\ (mod\ 3)$ and $3(k-l+1)\equiv 0\neq -1\ (mod\ 3)$, $f'(x)$ has an antiderivative. $\left(\ x^{3l-2}+1\right)'=x^{3(l-1)}$. Then $\gcd\left(x^{3l-2}+1,\ x^{3(l-1)}\right)=1$, giving that $x^{3l-2}+1$ has $3l-2$ distinct zeros. So, $f'(x)$ has $3l-1$ distinct zeros.
$3l+1\in \boldsymbol{T}_n$, $1\leq l\leq k-1$: Let

$$f'(x)=x^{3(k-l)}(x+\alpha)^3\left(x^{3l-2}+1\right)=x^{3k+1}+\alpha^3x^{3k-2}+x^{3(k-l+1)}+\alpha^3x^{3(k-l)},$$

where $0\neq\alpha\in K$ is not a zero of $x^{3l-2}+1$. Since $3(k-l+1)\equiv 3(k-l)\equiv 0\neq -1\ (mod\ 3)$, and $3k+1\equiv 3k-2\equiv 1\neq -1\ (mod\ 3)$, $f'(x)$ has an antiderivative and since $\left(\ x^{3l-2}+1\right)'=x^{3(l-1)}$, $x^{3l-2}+1$ has $3l-2$ distinct zeros. Hence, $f'(x)$ have $3l$ distinct zeros.
$3l+2\in \boldsymbol{T}_n$, $0\leq l\leq k$: Let

$$f'(x)=x^{3(k-l)+1}\left(x^{3l}+x^2+1\right)=x^{3k+1}+x^{3(k-l+1)}+x^{3(k-l)+1}.$$

$3(k-l+1)\equiv 0\neq -1\ (mod\ 3)$, and $3k+1\equiv 3(k-l)+1\equiv 1\neq -1\ (mod\ 3)$. Also, $x^{3l}+x^2+1$ has $3l$ distinct zeros since $\gcd\left(x^{3l}+x^2+1,\left(\ x^{3l}+x^2+1\right)'\right)=1$, where $\left(\ x^{3l}+x^2+1\right)'=2x$. So, $f'(x)$ has an antiderivative having $3l+1$ distinct zeros.
Notice that $n-1\notin \boldsymbol{T}_n$. If $n-1\in \boldsymbol{T}_n$, then $f'(x)$ would have $n-2$ distinct zeros, where $\deg f'(x)=n-1$. This implies that $f'(x)$ had to have a factor with a multiplicity $2=p-1$. But this is a contradiction to corollary 2.10.
Hence, $\boldsymbol{T}_n=\{2,\ \ldots,\ n-2,\ n\}$. $\square$

From now on, let p denote a prime number, where $p \geq 5$.

Claim 2.15. If $charK = 0$, or $charK = p > n$, then $K(x)/K(z)$ is tame; i.e. there is no place of $K(x)$, which is wildly ramified in $K(x)/K(z)$.

Proof. Suppose there is a place P of $K(x)$ such that P is wildly ramified in $K(x)/K(z)$. Then $charK \mid e(P \mid Q)$, where Q is the place lying under P. But, by Fundamental Equality, $e(P \mid Q) \leq n < p$. □

Claim 2.16. Let $charK = 0$, or $charK = p > n$. Then $\boldsymbol{T}_n = \{2, \ldots, n-1, n\}$.

Proof. Since $K(x)/K(z)$ is tame, $1 \notin \boldsymbol{T}_n$, by corollary 2.3.
$l \in \boldsymbol{T}_n$, $2 \leq l \leq n$: $l \in \boldsymbol{T}_n$ if and only if $f'(x)$ have $l-1$ distinct zeros. Let

$$f'(x) = x^{n-l+1}\left(x^{l-2}+1\right) = x^{n-1} + x^{n-l+1}.$$

Since $charK = 0$, or $charK = p > n$, $f'(x)$ has an antiderivative. Also, $x^{l-2}+1$ has $l-2$ distinct zeros, since $\gcd\left(x^{l-2}+1, \left(x^{l-2}+1\right)'\right) = 1$. Hence, $f'(x)$ have $l-1$ distinct zeros. □

Claim 2.17. Let $n = p = charK$, then $\boldsymbol{T}_p = \{1, \ldots, p-1\}$.

Proof. $1 \in \boldsymbol{T}_p$, since $n = p$.
$l \in \boldsymbol{T}_p$, $2 \leq l \leq p-1$: $l \in \boldsymbol{T}_p$ if and only if $f'(x)$ have $l-1$ distinct zeros. Let

$$f'(x) = x^{p-l}\left(x^{l-2}+1\right) = x^{p-2} + x^{p-l}.$$

Since $p-2,\ p-l \neq -1 \pmod p$, $f'(x)$ has an antiderivative and since $l-2 \neq 0 \pmod p$, $x^{l-2}+1$ has $l-2$ distinct zeros. Hence, $f'(x)$ have $l-1$ distinct zeros.
$p \notin \boldsymbol{T}_p$: Since $\deg f'(x) \leq p-2$, $f'(x)$ can have at most $p-2$ distinct zeros. □

Lemma 2.18. *Let $n = p+1$, where $p = charK$, then $\boldsymbol{T}_{p+1} = \{2, 4, 5, \ldots, p+1\}$.*

Proof. $1 \notin \boldsymbol{T}_{p+1}$ and $2 \in \boldsymbol{T}_{p+1}$, since $p \nmid p+1$.
$3 \notin \boldsymbol{T}_{p+1}$: Suppose $3 \in \boldsymbol{T}_{p+1}$. Then $f'(x)$ must have 2 distinct factors. Without loss of generality, say one of them is x. Then $f'(x)$ is of the form $f'(x) = (x+\alpha)^k x^{p-k}$, where $\alpha \in K^\times$ and $1 \leq k \leq p-1$, i.e.

$$f'(x) = (x+\alpha)^k x^{p-k} = \left(\sum_{l=0}^{k} \binom{k}{l} \alpha^{k-l} x^l\right) x^{p-k} = \sum_{l=0}^{k} \binom{k}{l} \alpha^{k-l} x^{p-(k-l)}.$$

The coefficient of x^{p-1} must be zero so that $f'(x)$ can have an antiderivative. Since $p-(k-l) = p-1 \iff l = k-1$, the coefficient of $x^{p-1} = \binom{k}{k-1}\alpha = k\alpha = 0$. This implies that $\alpha = 0$, since $1 \leq k \leq p-1$, which is a contradiction to $\alpha \in K^\times$.
$l \in \boldsymbol{T}_{p+1}$; $4 \leq l \leq p+1$: Let

$$f'(x) = x^{p-l+2}\left(x^{l-2}+1\right) = x^p + x^{p-l+2}.$$

$p-l+2 \neq -1 \; (mod \; p)$, since $4 \leq l \leq n = p+1$. So, $f'(x)$ has an antiderivative. Also $l-2 \neq 0 \; (mod \; p)$, since $2 \leq l-2 \leq p-1$; i.e. $x^{l-2}+1$ has $l-2$ distinct zeros since $\gcd\left(x^{l-2}+1, \; (l-2)\, x^{l-3}\right) = 1$, where $(l-2)\, x^{l-3} = \left(x^{l-2}+1\right)'$. Therefore, $f'(x)$ have $l-1$ distinct zeros. □

Now, we consider the case $n = p+k$, where $2 \leq k \leq p-1$. But before that, we continue with some examples.

Example 2.19. Let $n = p+2$, then $\deg f'(x) = p+1$. Then
$1 \notin \boldsymbol{T}_{p+2}$ and $2 \in \boldsymbol{T}_{p+2}$, since $p \nmid p+2$.
$3 \in \boldsymbol{T}_{p+2}$: Let

$$f'(x) = x^p\,(x+1) = x^{p+1} + x^p.$$

Since $p+1 \equiv 1 \; (mod \; p)$ and $p \equiv 0 \; (mod \; p)$, $f'(x)$ has an antiderivative, having 2 distinct roots.
$4 \in \boldsymbol{T}_{p+2}$: Let

$$f'(x) = x^{p-2}\,(x-2)^2\,(x+1) = x^{p+1} - 3x^p + 4x^{p-2}.$$

Since $p-2 \equiv -2 \neq -1 \; (mod \; p)$, $p+1 \equiv 1 \; (mod \; p)$ and $p \equiv 0 \; (mod \; p)$, $f'(x)$ has an antiderivative. Notice that $2 \neq -1$, since $p \geq 5$; i.e. $f'(x)$ have 3 distinct zeros.
$l \in \boldsymbol{T}_{p+2}$; $5 \leq l \leq p+1$: Let

$$f'(x) = x^{p-l+3}\left(x^{l-2}+1\right) = x^{p+1} + x^{p-l+3}.$$

$p-l+3 \equiv -1 \; (mod \; p) \iff l \equiv 4 \; (mod \; p)$, but $5 \leq l \leq n = p+1$. So, this is not possible; i.e. $f'(x)$ has an antiderivative. Also, $l-2 \equiv 0 \; (mod \; p) \iff l = 2$, since $l \leq p+1$. But $l \geq 5$. Hence, $x^{l-2}+1$ has $l-2$ distinct zeros. Therefore, $f'(x)$ have $l-1$ distinct zeros.
$p+2 \in \boldsymbol{T}_{p+2}$: Let

$$f'(x) = x^{p+1} + 1.$$

Since $\gcd\left(f'(x), f''(x)\right) = 1$, $f'(x)$ have $p+1$ distinct zeros.
Hence, $\boldsymbol{T}_{p+2} = \{2, 3, \; \ldots, p+2\}$.

Example 2.20. Let $n = p+3$, then $\deg f'(x) = p+2$.
$1 \notin \boldsymbol{T}_{p+3}$ and $2 \in \boldsymbol{T}_{p+3}$, since $p \nmid p+3$.
$3 \in \boldsymbol{T}_{p+3}$: Let

$$f'(x) = x^{p+1}\,(x+1) = x^{p+2} + x^{p+1}.$$

Since $p+2 \equiv 2 \; (mod \; p)$ and $p+1 \equiv 1 \; (mod \; p)$, $f'(x)$ has an antiderivative, having 2 distinct roots.
$4 \in \boldsymbol{T}_{p+3}$: Let

$$f'(x) = x^p\left(x^2+1\right) = x^{p+2} + x^p.$$

Since $p+2 \equiv 2 \; (mod \; p)$ and $p \equiv 0 \; (mod \; p)$, $f'(x)$ has an antiderivative, having 3 distinct roots.
$5 \in \boldsymbol{T}_{p+3}$: $5 \in \boldsymbol{T}_{p+3}$ if and only if $f'(x)$ have 4 distinct zeros. Let

$$f'(x) = \left(x^3+1\right)(x+\alpha)^{p-1},$$

where $\alpha \in K^{\times}$. Notice that $\gcd\left(x^3+1,\ \left(x^3+1\right)'\right) = 1$, i.e. x^3+1 has 3 distinct roots. Now, we will determine $\alpha \in K^{\times}$ so that the coefficient of x^{p-1} becomes zero.

$$\begin{aligned} f'(x) &= \left(x^3+1\right)(x+\alpha)^{p-1} = \left(x^3+1\right)\left(\sum_{i=0}^{p-1}\binom{p-1}{i}\alpha^i x^{(p-1)-i}\right) \\ &= \left(x^3+1\right)\left(x^{p-1}+(p-1)\alpha x^{p-2}+\ldots+\alpha^{p-1}\right) \end{aligned}$$

Then the coefficient of $x^{p-1} = \frac{(p-1)(p-2)(p-3)}{6}\alpha^3+1$. Since K is algebraically closed, we can solve this equation for α. But, we also want α not to be a root of x^3+1; i.e. we do not want $\alpha^3=-1$ so that $f'(x)$ has 4 distinct zeros. Since $\frac{(p-1)(p-2)(p-3)}{6}\alpha^3+1=0$, $\alpha^3=-1 \iff \frac{(p-1)(p-2)(p-3)}{6}=1 \iff p=4$, which is impossible.
$l \in \boldsymbol{T}_{p+3}$; $6 \le l \le p+1$ $(p \ge 7)$: $l \in \mathbf{T}_{p+3}$ if and only if $f'(x)$ have $l-1$ distinct zeros. Let

$$f'(x) = x^{p-l+4}\left(x^{l-2}+1\right) = x^{p+2}+x^{p-l+4}.$$

$p-l+4 \equiv -1\ (mod\ p) \iff l \equiv 5\ (mod\ p)$, but $6 \le l \le p+1$. So, this is not possible; i.e. $p-l+4 \ne -1\ (mod\ p)$ and $p+2 \equiv 2 \ne -1$, since $p \ge 7$. So, $f'(x)$ has an antiderivative. Also, $l-2 \ne 0\ (mod\ p)$, since $4 \le l-2 \le p-1$. Hence, $x^{l-2}+1$ has $l-2$ distinct zeros. Therefore, $f'(x)$ have $l-1$ distinct zeros.
$p+2 \in \boldsymbol{T}_{p+3}$: Let

$$f'(x) = x^2\left(x^p+x+1\right) = x^{p+2}+x^3+x^2.$$

x^p+x+1 has p distinct zeros since $\left(x^p+x+1\right)' = 1$. Therefore, $f'(x)$ has $p+1$ distinct zeros.
$p+3 \in \boldsymbol{T}_{p+3}$: Let

$$f'(x) = x^{p+2}+1.$$

Since $\gcd\left(f'(x),\ f''(x)\right)=1$,$f'(x)$ has $p+2$ distinct zeros.
Hence, $\boldsymbol{T}_{p+3} = \{2,\ 3,\ \ldots,\ p+3\}$.

Now, we can give the general case.

Lemma 2.21. *Let $n=p+k$, where $2 \le k \le p-1$ and $p \ge 5$, then*

$$\boldsymbol{T}_n = \{2,\ 3,\ \ldots,\ n\}.$$

Proof. Since $2 \le k \le p-1$, $p \nmid n$. So, $1 \notin \boldsymbol{T}_n$ and $2 \in \boldsymbol{T}_n$.
$l \in \boldsymbol{T}_n$; $3 \le l \le k+1$ or $k+3 \le l \le p+1$ or $p+3 \le l \le p+k$: Let

$$f'(x) = x^{p+k-l+1}\left(x^{l-2}+1\right) = x^{p+k-1}+x^{p+k-l+1}.$$

Then $p+k-l+1 \equiv -1\ (mod\ p) \iff l=k+2$. Also, $p+k-1 \ne -1\ (mod\ p)$, because $p+2 \le p+k-1 \le 2p-2$. Hence, $f'(x)$ has an antiderivative. $l-2 \equiv 0\ (mod\ p) \iff l=p+2$. So, $l-2 \ne 0\ (mod\ p)$; i.e., $x^{l-2}+1$ has $l-2$ distinct zeros

because $\gcd\left(x^{l-2}+1,\ \left(x^{l-2}+1\right)'\right)=1$, which shows $f'(x)$ have $l-1$ distinct zeros.
$k+2\in \boldsymbol{T}_n$: $f'(x)$ must have $k+1$ distinct zeros. Let

$$f'(x)=\left(x^k+1\right)(x+\alpha)^{p-1},$$

where $\alpha\in K^\times$. Now, we will determine α so that the coefficient of x^{p-1} becomes zero.

$$f'(x)=\left(x^k+1\right)(x+\alpha)^{p-1}=\left(x^k+1\right)\left(\sum_{i=0}^{p-1}\binom{p-1}{i}\alpha^i x^{(p-1)-i}\right).$$

Then the coefficient of $x^{p-1}=\binom{p-1}{k}\alpha^k+1=0 \iff \alpha^k=-\frac{1}{\binom{p-1}{k}}$. Since K is algebraically closed, we can solve this equation for α. Now, we must show that α is not a root of x^k+1. α is a root of

$$x^k+1\iff \alpha^k+1=0\iff \alpha^k=-1\iff \binom{p-1}{k}=1\iff p=k+1.$$

If $p\neq k+1$, then $f'(x)$ has $k+1$ distinct zeros.
If $p=k+1$, then $\deg f'(x)=p+k-1=2p-2$ and we want $f'(x)$ has p distinct zeros. Let

$$f'(x)=\left(x^2+1\right)\left(x^{p-2}+\alpha\right)^2.$$

We can choose $\alpha\in K^\times$ so that x^2+1 and $x^{p-2}+\alpha$ do not have a common zero. Then

$$f'(x)=\left(x^2+1\right)\left(x^{p-2}+\alpha\right)^2=x^{2p-2}+x^{2p-4}+2\alpha x^p+2\alpha x^{p-2}+\alpha^2x^2+\alpha^2.$$

Since $2p-2\equiv p-2\equiv -2\not\equiv -1\ (mod\ p)$, $2\not\equiv -1\ (mod\ p)$, $p\equiv 0\not\equiv -1\ (mod\ p)$ and $2p-4\equiv -4\not\equiv -1$(since $p\geq 5$), $f'(x)$ has an antiderivative. Also, $f'(x)$ has $p=k+1$ distinct zeros, since $\gcd\left(x^{p-2}+\alpha,\ \left(x^{p-2}+\alpha\right)'\right)=\gcd\left(x^2+1,\ \left(x^2+1\right)'\right)=1$.
$p+2\in \boldsymbol{T}_n$: If $k\neq p-1$, then let

$$f'(x)=x^{k-1}\left(x^p+x+1\right)=x^{p+k-1}+x^k+x^{k-1}.$$

Then $k\not\equiv -1\ (mod\ p)$, and $p+k-1\equiv k-1\equiv -1\ (mod\ p)\iff k\equiv 0\ (mod\ p)$, but $k\leq p-1$. So, $f'(x)$ has an antiderivative.
If $k=p-1$, then let

$$f'(x)=x^{k-1}\left(x^p+x^2+1\right)=x^{p+k-1}+x^{k+1}+x^{k-1}.$$

Then $p+k-1\equiv k-1\equiv -2\ (mod\ p)$ and $k+1\equiv 0\ (mod\ p)$. Also, x^p+x+1 and x^p+x^2+1 have p distinct zeros since $\left(x^p+x+1\right)'=1$ and $\left(x^p+x^2+1\right)'=2x$. So, in both cases, $f'(x)$ has an antiderivative having $p+1$ distinct zeros.
Hence $\boldsymbol{T}_n=\{2,\ 3,\ \ldots,\ n\}$. □

Now, we are going to find $\boldsymbol{T}_n$ for $n>2p$ and $p\nmid n$, where $p=char K\geq 5$. But, we first start with an example.

Example 2.22. We will find $\boldsymbol{T}_n$ for $n=kp+1$, where $k\geq 2$ and $p\geq 5$.
$1\notin \boldsymbol{T}_n$ and $2\in \boldsymbol{T}_n$, since $p\nmid n$.
$3\in \boldsymbol{T}_n$: Let

$$f'(x)=x^{(k-1)p}(x+1)^p=x^{kp}+x^{(k-1)p}.$$

Since $k \geq 2$, $(k-1)\,p \geq p$; i.e. $f'(x)$ has zero as a root. Hence, $f'(x)$ has 2 distinct zeros.
$l \in \boldsymbol{T}_n$; $4 \leq l \leq p+1$ or $p+4 \leq l \leq 2p+1$: $f'(x)$ must have $l-1$ distinct zeros. Let

$$f'(x) = x^{kp-l+2}\left(x^{l-2}+1\right) = x^{kp} + x^{kp-l+2}.$$

$kp - l + 2 \equiv -1 \ (mod\ p) \Longleftrightarrow l = 3$ or $l = p+3$. So, $f'(x)$ has an antiderivative. Also, $l-2 \neq 0 \ (mod\ p)$, since $2 \leq l-2 \leq p-1$ or $p+2 \leq l-2 \leq 2p-1$. So, $x^{l-2}+1$ has $l-2$ distinct zeros. Hence, $f'(x)$ has $l-1$ distinct zeros.
$p+2 \in \boldsymbol{T}_n$: $f'(x)$ must have $p+1$ distinct zeros. Let

$$f'(x) = x^{(k-1)p}\left(x^p + x + 1\right) = x^{kp} + x^{(k-1)p+1} + x^{(k-1)p}.$$

Since $kp \equiv (k-1)\,p \equiv 0 \neq -1 \ (mod\ p)$ and $(k-1)\,p+1 \equiv 1 \neq -1 \ (mod\ p)$, $f'(x)$ has an antiderivative. Since $\left(x^p+x+1\right)' = 1$, x^p+x+1 has p distinct zeros, implying that $f'(x)$ have $p+1$ distinct zeros.
$p+3 \in \boldsymbol{T}_n$: $f'(x)$ must have $p+2$ distinct zeros.
If $k \geq 3$, then let

$$\begin{aligned} f'(x) &= x^{(k-2)p}\left(x^p+x+1\right)(x+1)^p \\ &= x^{kp} + x^{(k-1)p+1} + 2x^{(k-1)p} + x^{(k-2)p+1} + x^{(k-2)p}. \end{aligned}$$

Since $kp \equiv (k-1)\,p \equiv (k-2)\,p \equiv 0 \ (mod\ p)$ and $(k-1)\,p+1 \equiv 1 \ (mod\ p)$, $f'(x)$ has an antiderivative. Notice that -1 is not a root of x^p+x+1. Hence, $f'(x)$ have $p+2$ distinct zeros.
If $k=2$, then $n = 2p+1$ and $\deg f'(x) = 2p$. Let

$$\begin{aligned} f'(x) &= x^{p-3}\left(x^i+\alpha\right)\left(x^j+\beta\right)\left(x^2-1\right)^2 \\ &= x^{2p} - 2x^{2p-2} + x^{2p-4} + \alpha x^{p+j+1} - 2\alpha x^{p+j-1} + \alpha x^{p+j-3} + \beta x^{p+i+1} \\ &\quad -2\beta x^{p+i-1} + \beta x^{p+i-3} + \alpha\beta x^{p+1} - 2\alpha\beta x^{p-1} + \alpha\beta x^{p-3}, \end{aligned}$$

where $i+j = p-1$. Let $i=2$, $j = p-3$ and $\alpha = \frac{1}{2}$, then

$$f'(x) = x^{2p} - \frac{3}{2}x^{2p-2} + \frac{1}{2}x^{2p-6} + \beta x^{p+3} - \frac{3}{2}\beta x^{p+1} + \frac{1}{2}\beta x^{p-3}.$$

If $p \neq 5$, then no exponent of x congruent to -1 modulo p; i.e. $f'(x)$ has an antiderivative. Also, $x^2+\frac{1}{2}$ and x^2-1 do not have a common zero and we can choose $0 \neq \beta \in K$ so that $x^2+\frac{1}{2}$, $x^{p-3}+\beta$ and x^2-1 do not have a common factor. Then $f'(x)$ has $p+2$ distinct zeros.
If $p=5$, then let

$$\begin{aligned} f'(x) &= x\left(x^3+\alpha\right)^2\left(x^3+\beta\right) \\ &= x^{10} + (2\alpha+\beta)\,x^7 + \left(\alpha^2+2\alpha\beta\right)x^4 + \alpha^2\beta x. \end{aligned}$$

If we choose $\alpha,\ \beta \in K^{\times}$ such that $\alpha \neq \beta$ and $\alpha^2 + 2\alpha\beta = 0$, then $f'(x)$ has an antiderivative having $7 = p+2$ distinct zeros.
$2p+2 \in \boldsymbol{T}_n$, $k \geq 3$: $f'(x)$ must have $2p+1$ distinct zeros. Let

$$f'(x) = x^{(k-2)p}\left(x^{2p}+x+1\right) = x^{kp} + x^{(k-2)p+1} + x^{(k-2)p}.$$

Since $(x^{2p}+x+1)'=1$, $x^{2p}+x+1$ has $2p$ distinct zeros. Hence, $f'(x)$ have $2p+1$ distinct zeros.
$2p+3\in \boldsymbol{T}_n$: $f'(x)$ must have $2p+2$ distinct zeros. Let

$$\begin{aligned} f'(x) &= x^{(k-2)p-3}\left(x^2+\frac{1}{2}\right)\left(x^{2p-3}+\beta\right)\left(x^2-1\right)^2 \\ &= x^{kp}-\frac{3}{2}x^{kp-2}+\frac{1}{2}x^{kp-6}+\beta x^{(k-2)p+3}-\frac{3}{2}\beta x^{(k-2)p+1}+\frac{1}{2}\beta x^{(k-2)p-3}. \end{aligned}$$

$kp-2$, $(k-2)p+3$, $(k-2)p+1$, $(k-2)p-3\neq -1\ (mod\ p)$, and if $p\neq 5$, then $kp-6\neq -1\ (mod\ p)$. Hence, no exponent of x is congruent to -1 modulo p; i.e. $f'(x)$ has an antiderivative. $x^2+\frac{1}{2}$ and x^2-1 do not have a common zero and we can choose $\beta\in K^{\times}$ so that $x^2+\frac{1}{2}$, $x^{2p-3}+\beta$ and x^2-1 do not have a common zero. Then $f'(x)$ have $2p+2$ distinct zeros.
If $p=5$ and $k\geq 4$, then let

$$\begin{aligned} f'(x) &= x^{(k-3)p}\left(x^{2p}+x+1\right)(x+\alpha)^p \\ &= x^{kp}+\alpha^p x^{(k-1)p}+x^{(k-2)p+1}+x^{(k-2)p}+\alpha^p x^{(k-3)p+1}+\alpha^p x^{(k-3)p}, \end{aligned}$$

where $0\neq\alpha\in K$ is not a root of $x^{2p+1}+1$. Then no exponent of x is congruent to -1 modulo p. Hence, $f'(x)$ has an antiderivative. Also, $x^{2p}+x+1$ has $2p$ distinct zeros and 0 is a zero of $f'(x)$ since $k-3\geq 1$. So, $f'(x)$ has $2p+2$ distinct zeros.
If $p=5$ and $k=3$, then $\deg f'(x)=15$. Let

$$\begin{aligned} f'(x) &= \left(x^6+1\right)\left(x^3+\alpha\right)^2\left(x^3+\beta\right) \\ &= x^{15}+(2\alpha+\beta)x^{12}+\left(\alpha^2+2\beta+1\right)x^9+\left(\alpha^2\beta+2\alpha+\beta\right)x^6 \\ &\quad +\left(\alpha^2+2\alpha\beta\right)x^3+\alpha^2\beta. \end{aligned}$$

So, we can choose α and β with α, $\beta\neq 0$ and $\alpha\neq\beta$ such that the coefficient of $x^9=\alpha^2+2\beta+1$ becomes zero so that no exponent of x congruent to -1 modulo p; i.e. $f'(x)$ has an antiderivative having $12=2p+3$ distinct zeros.
$2p+l\in\boldsymbol{T}_n$; $4\leq l\leq p$: $f'(x)$ must have $2p+l-1$ distinct zeros. Let

$$f'(x)=x^{(k-2)p-l+2}\left(x^{2p+l-2}+1\right)=x^{kp}+x^{(k-2)p-l+2}.$$

$(k-2)p-l+2\equiv -1\ (mod\ p)\iff l=3$, since $l\leq p$. But $l\geq 4$; i.e. $f'(x)$ has an antiderivative. Since $4\leq l\leq p$, $l-2\neq 0\ (mod\ p)$. Hence, $x^{2p+l-2}+1$ has $2p+l-2$ distinct zeros. So, $f'(x)$ has $2p+l-1$ distinct zeros.
In general;
$sp+1\in\boldsymbol{T}_n$; $2\leq s\leq k$: Let

$$f'(x)=x^{(k-s)p+1}\left(x^{sp-1}+1\right)=x^{kp}+x^{(k-s)p+1}.$$

Since $(k-s)p+1\neq -1\ (mod\ p)$, $f'(x)$ has an antiderivative. Also, $x^{sp-1}+1$ has $sp-1$ distinct zeros, since $sp-1\neq 0\ (mod\ p)$. So, $f'(x)$ has sp distinct zeros.
$sp+2\in\boldsymbol{T}_n$; $2\leq s\leq k-1$: $f'(x)$ must have $sp+1$ distinct zeros. Let

$$f'(x)=x^{(k-s)p}\left(x^{sp}+x+1\right)=x^{kp}+x^{(k-s)p+1}+x^{(k-s)p}.$$

Since $(k-s)p$ and $(k-s)p+1\neq -1\ (mod\ p)$, $f'(x)$ has an antiderivative and since $(x^{sp}+x+1)'=1$, $x^{sp}+x+1$ has sp distinct zeros. Hence, $f'(x)$ has $sp+1$ distinct

zeros.
$sp+3 \in \boldsymbol{T}_n$; $2 \le s \le k-1$: $f'(x)$ must have $sp+2$ distinct zeros. Let

$$\begin{aligned} f'(x) &= x^{(k-s)p-3}\left(x^2+\frac{1}{2}\right)\left(x^{sp-3}+\beta\right)\left(x^2-1\right)^2 \\ &= x^{kp}-\frac{3}{2}x^{kp-2}+\frac{1}{2}x^{kp-6}+\beta x^{(k-s)p+3}-\frac{3}{2}\beta x^{(k-s)p+1}+\frac{1}{2}\beta x^{(k-s)p-3}. \end{aligned}$$

$kp-2$, $(k-2)p+3$, $(k-2)p+1$, $(k-2)p-3 \neq -1\ (mod\ p)$, and $kp-6 \neq -1\ (mod\ p)$ if $p \neq 5$. Hence, if $p \neq 5$ no exponent of x is congruent to -1 modulo p; i.e. $f'(x)$ has an antiderivative with $sp+2$ distinct zeros.
If $p=5$ and $k \ge s+2$, then let

$$\begin{aligned} f'(x) &= x^{(k-s-1)p}\left(x^{sp}+x+1\right)(x+\alpha)^p \\ &= x^{kp}+\alpha^p x^{(k-1)p}+x^{(k-s)p+1}+x^{(k-s)p}+\alpha^p x^{(k-s-1)p+1}+\alpha^p x^{(k-s-1)p}, \end{aligned}$$

where $0 \neq \alpha \in K$ is not a root of $x^{sp}+x+1$. Then no power of x congruent to -1 modulo p; i.e. $f'(x)$ has an antiderivative. Also, $x^{sp}+x+1$ has sp distinct zeros implying that $f'(x)$ have $sp+2$ distinct zeros.
If $p=5$ and $k=s+1$, then $\deg f'(x) = 5(s+1)$. Let

$$\begin{aligned} f'(x) &= x\left(x^{(s-1)5}+1\right)\left(x^3+\alpha\right)^2\left(x^3+\beta\right) = \\ &\left(x^{(s-1)5}+1\right)\left(x^{10}+(2\alpha+\beta)x^7+\left(\alpha^2+2\alpha\beta\right)x^4+\alpha^2\beta x\right). \end{aligned}$$

Then the coefficient of x whose exponent is congruent to -1 modulo p is equal to $\alpha^2+2\alpha\beta$. Hence, we can choose α, $\beta \in K^{\times}$ such that $\alpha \neq \beta$, $\alpha^2+2\alpha\beta=0$ and they are not zeros of $x^{(s-1)5}+1$. Then, $f'(x)$ has an antiderivative with $5s+2$ distinct zeros.
$sp+l \in \boldsymbol{T}_n$; $2 \le s \le k-1$ and $4 \le l \le p$: $f'(x)$ must have $sp+l-1$ distinct zeros. Let

$$f'(x) = x^{(k-s)p-l+2}\left(x^{sp+l-2}+1\right) = x^{kp}+x^{(k-s)p-l+2}.$$

$(k-s)p-l+2 \equiv -1\ (mod\ p) \iff l=3$, since $l \le p$. But we have $l \ge 4$; i.e. $(k-s)p-l+2 \neq -1\ (mod\ p)$. So, $f'(x)$ has an antiderivative. Since $4 \le l \le p$, $l-2 \neq 0\ (mod\ p)$. Then $x^{sp+l-2}+1$ has $sp+l-2$ distinct zeros. Hence, $f'(x)$ have $sp+l-1$ distinct zeros.
Therefore, $\boldsymbol{T}_n = \{2, 3, \ldots, n\}$.

Lemma 2.23. *Let* $n = kp+t+1$; $0 \le t \le p-2$ *and* $k \ge 2$, *then*

$$\boldsymbol{T}_n = \{2, 3, \ldots, n\}.$$

Proof. Since in example 2.22 we give the case when $t=0$, we can assume that $t \ge 1$. Since $p \nmid n$, $1 \notin \boldsymbol{T}_n$ and $2 \in \boldsymbol{T}_n$.
$l \in \boldsymbol{T}_n$; $3 \le l \le t+2$ or $t+4 \le l \le p$: $f'(x)$ must have $l-1$ distinct zeros. Let

$$f'(x) = x^{kp+t-l+2}\left(x^{l-2}+1\right) = x^{kp+t}+x^{kp+t-l+2}.$$

$kp+t-l+2 \equiv -1\ (mod\ p) \iff l=t+3$, since $t, l \le p$. So, $f'(x)$ has an antiderivative. Also, $l-2 \neq 0\ (mod\ p)$, since $1 \le l-2 \le p-2$; i.e. $x^{l-2}+1$ has $l-2$ distinct zeros. Hence, $f'(x)$ has $l-1$ distinct zeros.
$t+3 \in \boldsymbol{T}_n$: Let

$$f'(x) = x^{(k-1)p}\left(x^t+1\right)(x+\alpha)^p = x^{kp+t}+x^{kp}+\alpha^p x^{(k-1)p+t}+\alpha^p x^{(k-1)p}.$$

Since $t \leq p-2$, $kp+t$ and $(k-1)p+t$ can not be congruent to -1 modulo p. So, $f'(x)$ has an antiderivative. Since $k \geq 2$, $(k-1)p \geq p$; i.e. 0 is a root of $f'(x)$. Also, x^t+1 has t distinct zeros, because $1 \leq t \leq p-2 \Longrightarrow t \neq 0 \ (mod\ p)$. We can choose $\alpha \in K^{\times}$ so that α is not a zero of x^t+1 . Hence, $f'(x)$ has $t+2$ distinct zeros.
In general;
$sp+1 \in \boldsymbol{T}_n$; for $1 \leq s \leq k$: Let

$$f'(x) = x^{(k-s)p+t+1}\left(x^{sp-1}+1\right) = x^{kp+t} + x^{(k-s)p+t+1}.$$

$(k-s)p+t+1 \equiv -1 \ (mod\ p) \Longleftrightarrow t = p-2$. Hence, $f'(x)$ has an antiderivative for $1 \leq t \leq p-3$. Since $x^{sp-1}+1$ has $sp-1$ distinct zeros, $f'(x)$ have sp distinct zeros.
If $t = p-2$, then $\deg f'(x) = (k+1)p-2$.
If $k \geq s+1$, then let

$$f'(x) = x^{(k-s)p}\left(x^{sp-2}+1\right)(x+\alpha)^p = x^{(k+1)p-2} + \alpha^p x^{kp-2} + x^{(k-s+1)p} + \alpha^p x^{(k-s)p},$$

where $0 \neq \alpha \in K$ is not a zero of $x^{sp-2}+1$. Since no exponent of x is congruent to -1 modulo p, $f'(x)$ has an antiderivative and since $sp-2 \equiv -2 \neq 0 \ (mod\ p)$, $x^{sp-2}+1$ has $sp-2$ distinct zeros. Hence, $f'(x)$ have sp distinct zeros.
If $k = s$, then $\deg f'(x) = (s+1)p-2$. Let

$$\begin{aligned} f'(x) &= \left(x^{(s-1)p+2}+1\right)\left(x^{p-2}+\alpha\right)^2 \\ &= x^{(s+1)p-2} + 2\alpha x^{sp} + \alpha^p x^{(s-1)p+2} + x^{2p-4} + 2\alpha x^{p-2} + \alpha^2. \end{aligned}$$

Since $(s+1)p-2 \equiv p-2 \equiv -2 \neq -1 \ (mod\ p)$, and $(s-1)p+2$, $2p-4 \neq -1 \ (mod\ p)$(since $p \geq 5$), $f'(x)$ has an antiderivative having sp distinct zeros.
$sp+2 \in \boldsymbol{T}_n$, for $1 \leq s \leq k$: Let

$$f'(x) = x^{(k-s)p+t}\left(x^{sp}+x+1\right) = x^{kp+t} + x^{(k-s)p+t+1} + x^{(k-s)p+t}.$$

$(k-s)p+t \neq -1 \ (mod\ p)$ since $1 \leq t \leq p-2$. $(k-s)p+t+1 \equiv -1 \ (mod\ p) \Longleftrightarrow t = p-2$. Hence, $f'(x)$ has an antiderivative for $1 \leq t \leq p-3$. Since $x^{sp-1}+x+1$ has sp distinct zeros, $f'(x)$ has $sp+1$ distinct zeros.
If $t = p-2$, then let

$$f'(x) = x^{(k-s)p+t}\left(x^{sp}+x^2+1\right) = x^{kp+t} + x^{(k-s+1)p} + x^{(k-s)p+t}.$$

Then, $f'(x)$ has an antiderivative, having $sp+1$ distinct zeros.
$sp+l \in \boldsymbol{T}_n$; $3 \leq l \leq t+1$ and $1 \leq s \leq k$: $f'(x)$ must have $sp+l-1$ distinct zeros. Let

$$f'(x) = x^{(k-s)p+t-l+2}\left(x^{sp+l-2}+1\right) = x^{kp+t} + x^{(k-s)p+t-l+2}.$$

$(k-s)p+t-l+2 \equiv -1 \ (mod\ p) \Longleftrightarrow l = t+3$, since $3 \leq l \leq t+1$ and $2 \leq t \leq p-2$. So, $f'(x)$ has an antiderivative. Also, $sp+\ l-2 \neq 0 \ (mod\ p)$, since $1 \leq l-2 \leq p-3$. So, $x^{sp+l-2}+1$ has $sp+l-2$ distinct zeros. Hence, $f'(x)$ has $sp+l-1$ distinct zeros.
$sp+t+2 \in \boldsymbol{T}_n$; $2 \leq s \leq k-1$: $f'(x)$ must have $sp+t+1$ distinct zeros. Let

$$f'(x) = x^{(k-s)p}\left(x^{sp+t}+1\right) = x^{kp+t} + x^{(k-s)p}.$$

Since $1 \leq t \leq p-2$, $sp+t \neq 0$ and $f'(x)$ has $sp+t+1$ distinct zeros.
$sp+t+3 \in \boldsymbol{T}_n$; $1 \leq s \leq k-1$, $t+3 \leq p-1$: $f'(x)$ must have $sp+t+2$ distinct zeros.
If $k \geq s+2$, then let

$$\begin{aligned} f'(x) &= x^{(k-s-1)p}\left(x^{sp+t}+1\right)(x+\alpha)^p \\ &= x^{kp+t} + \alpha^p x^{(k-1)p+t} + x^{(k-s)p} + \alpha^p x^{(k-s-1)p}. \end{aligned}$$

Since $1 \le t \le p-2$, $kp+t$ and $(k-1)\,p+t$ can not be congruent to -1 $(mod\ p)$. So, $f'(x)$ has an antiderivative. Since $k \ge s+2$, $(k-s-1)\,p \ge p$; i.e. 0 is a root of $f'(x)$. Also, $x^{sp+t}+1$ has $sp+t$ distinct zeros, because $1 \le t \le p-2 \Longrightarrow sp+t \ne 0$ $(mod\ p)$. If we choose $\alpha \in K^{\times}$ so that α is not a zero of $x^{sp+t}+1$, then $f'(x)$ has $sp+t+2$ distinct zeros.
If $k=s+1$, then $\deg f'(x) = (s+1)\,p+t$. Let

$$\begin{aligned} f'(x) &= x^{p-(t+3)}\left(x^{t+2}+\frac{1}{2}\right)\left(x^{sp-(t+3)}+\beta\right)\left(x^{t+2}-1\right)^2 \\ &= x^{(s+1)p+t}-\frac{3}{2}x^{(s+1)p-2}+\frac{1}{2}x^{(s+1)p-2t-6}+\beta x^{p+2t+3} \\ &\quad -\frac{3}{2}\beta x^{p+t+1}+\frac{1}{2}\beta x^{p-t-3}, \end{aligned}$$

where $\beta \in K^{\times}$ such that $x^{t+2}+\frac{1}{2}$, $x^{sp-(t+3)}+\beta$ and $x^{t+2}-1$ do not have a common zero. $(s+1)\,p+t \ne -1$ $(mod\ p)$, since $1 \le t \le p-4$. $(s+1)\,p-2 \equiv -2 \ne -1$ $(mod\ p)$. $(s+1)\,p-2t-6 \equiv -1$ $(mod\ p) \Longleftrightarrow 2t \equiv -5$ $(mod\ p) \Longleftrightarrow 2t = p-5$ since $2 \le 2t \le 2p-8$. $p+2t+3 \equiv -1$ $(mod\ p)$ and $p-t-3 \equiv -1$ $(mod\ p) \Longleftrightarrow t \equiv -2$ $(mod\ p)$ but $1 \le t \le p-4$. Hence, if $2t \ne p-5$, then $f'(x)$ has an antiderivative. Also, $t+2$, $sp-(t+3) \ne 0$ $(mod\ p)$, since $t \le p-4$. Hence, $x^{t+2}+\frac{1}{2}$, $x^{t+2}-1$ have $t+2$ and $x^{sp-(t+3)}+\beta$ has $sp-(t+3)$ distinct zeros without having a common zero. Since $t+3 \le p-1$, $p-(t+3) \ge 1$; i.e. 0 is a root of $f'(x)$. So, $f'(x)$ has $sp+t+2$ distinct zeros.
If $2t = p-5$, then let

$$\begin{aligned} f'(x) &= \left(x^{(s-1)p+t+4}+1\right)\left(x^{2t+3}+\beta\right)^2 \\ &= x^{(s+1)p+t}+2\beta x^{sp+t+2}+\beta^2 x^{(s-1)p+t+4}+x^{4t+6}+2\beta x^{2t+3}+\beta^2, \end{aligned}$$

where $\beta \in K^{\times}$ is not a root of $x^{(s-1)p+t+4}+1$. Then no exponent of x is congruent to -1 modulo p. Hence, $f'(x)$ has an antiderivative. Since $2t+3 = p-2$ and $t+4 = \frac{p+3}{2} \ne 0$ $(mod\ p)$, $f'(x)$ has $[(s-1)\,p+t+4]+[2t+3] = sp+t+2$ distinct zeros.
$sp+l \in \boldsymbol{T}_n$; $t+4 \le l \le p$ and $1 \le s \le k-1$: $f'(x)$ must have $sp+l-1$ distinct zeros.
Let

$$f'(x) = x^{(k-s)p+t-l+2}\left(x^{sp+l-2}+1\right) = x^{kp+t}+x^{(k-s)p+t-l+2}.$$

$(k-s)\,p+t-l+2 \ne -1$ $(mod\ p)$, since $t+4 \le l \le p$ and $1 \le t \le p-5 \Longrightarrow 4 \le l-t \le p-6$. So, $f'(x)$ has an antiderivative. Also, $sp+l-2 \ne 0$ $(mod\ p)$, since $t+4 \le l \le p \Longrightarrow 3 \le l-2 \le p-2$. So, $x^{sp+l-2}+1$ has $sp+l-2$ distinct zeros. Hence, $f'(x)$ have $sp+l-1$ distinct zeros.
Hence, $\boldsymbol{T}_n = \{2,\, 3,\, \ldots,\, n\}$. □

Corollary 2.24. *Let $K(x)$ be a rational function field and $n \in \mathbb{Z}$, $n \ge 2$. If $p \mid n$, then $\boldsymbol{T}_n = \{1,\, 2,\, \ldots,\, n-1\}$, where $p \ge 5$.*

Proof. Since $p \mid n$, $1 \in \boldsymbol{T}_n$ and $\deg f'(x) \le n-2$. So, $f'(x)$ can have at most $n-2$ distinct zeros. Hence, $n \notin \boldsymbol{T}_n$. We can write a polynomial $g(x)$ with degree $n-1$ and whose derivative has $l-1$ distinct zeros for $2 \le l \le n-1$. Let $f(x) = x^n + g(x)$. Then $f'(x)$ has $l-1$ distinct zeros. So, $l \in \boldsymbol{T}_n$ for $2 \le l \le n-1$. □

Now, let's state what we have done so far as a theorem.

Theorem 2.25. *Let $K(x)$ be a rational function field and $n \in \mathbb{Z}$, $n \geq 2$. Then we have:*

(*i*) *for* $\mathrm{char} K = 0$, $\boldsymbol{T}_n = \{2, \ldots, n\}$;

(*ii*) *for* $\mathrm{char} K = 2$, $\boldsymbol{T}_n = \{1, 2, \ldots, k\}$, *if* $n = 2k$ *and* $\boldsymbol{T}_n = \{2, \ldots, k\}$, *if* $n = 2k - 1$;

(*iii*) *for* $\mathrm{char} K = 3$, $\boldsymbol{T}_n = \{1, \ldots, n-1\}$ *if* $3 \mid n$ *and* $\boldsymbol{T}_n = \{2, \ldots, n-2, n\}$ *if* $3 \nmid n$;

(*vi*) *for* $\mathrm{char} K = p \geq 5$, $\boldsymbol{T}_n = \{1, \ldots, n-1\}$ *if* $p \mid n$ *and* $\boldsymbol{T}_n = \{2, \ldots, n\}$ *if* $p \nmid n$ *and* $n \neq p + 1$.
If $n = p + 1$, *then* $\boldsymbol{T}_n = \{2, 4, \ldots, n\}$.

3

Ramified Places of $K(x)$ in $K(x)/K(z)$ for $z \in K(x)$

In this chapter, we are going to investigate $\boldsymbol{S}_n$ for $n \geq 2$ and $charK = 0$, where $\boldsymbol{S}_n$ is the set consisting of integers i for which we can find $z \in K(x)$ such that $[K(x) : K(z)] = n$ and $K(x)$ has exactly i ramified places in $K(x)/K(z)$.

$K(x)/K(z)$ is a finite separable extension with

$$[K(x) : K(z)] = \max\{\deg g(x), \deg f(x)\} = n,$$

where $z = \frac{f(x)}{g(x)} \in K(x)$ for some $f(x)$, $g(x) \in K[x]$ with $\gcd(f(x), g(x)) = 1$. Since $charK = 0$, $K(x)/K(z)$ is a tame extension; i.e. there is no place of $K(x)$ which is wildly ramified in $K(x)/K(z)$. Hence, $1 \notin \boldsymbol{S}_n$. Then $\boldsymbol{S}_n \subseteq \{2, \cdots, 2n-2\}$, by Hurwitz Genus Formula. Now, we try to find what $\boldsymbol{S}_n$ can be by looking at the examples we are going to give.

Since $K(x)/K(z)$ is tame, for all place $P \in \mathbf{P}_{K(x)}$ we have $d(P \mid Q) = e(P \mid Q) - 1$, where Q is the place of $K(z)$ lying under P. When $charK = 0$, we know from chapter 2 that $\{2, \ldots, n\} \subseteq \boldsymbol{S}_n$, since $\boldsymbol{T}_n \subseteq \boldsymbol{S}_n$. So, we are going to give examples $K(x)/K(z)$ where $K(x)$ has $i \geq n+1$ ramified places.

Let $z = \frac{f(x)}{g(x)} \in K(x)$ with $\gcd(f(x), g(x)) = 1$ and $\deg f(x) > \deg g(x)$. Then

$$e(P_\infty \mid Q_\infty) = \deg f(x) - \deg g(x) = k > 0$$

and

$$d(P_\infty \mid Q_\infty) = e(P_\infty \mid Q_\infty) - 1 = k - 1,$$

where P_∞ denote the pole of x in $K(x)$ and Q_∞ denote the pole of z in $K(z)$. So, $K(x)$ can have at most $2n - (k+1)$ ramified places in $K(x)/K(z)$ other than P_∞. Suppose that $g(x)$ has no multiple roots so that the only ramified place lying over the pole of z in $K(x)$ can be P_∞. Let Q be the place of $K(z)$ corresponding to the polynomial $z - c$ and P be a place of $K(x)$ lying over Q. Also, let v_Q and v_P denote corresponding valuation functions, respectively. Then

$$v_P(z - c) = e(P \mid Q)\, v_Q(z - c) = e(P \mid Q).$$

Also,

$$v_P(z - c) = v_P\left(\frac{f(x) - cg(x)}{g(x)}\right).$$

Hence, Q is ramified if and only if $f(x) - cg(x)$ has a factor with multiplicity greater than 1 and this holds if and only if $D(f - cg) = 0$, where $D(f - cg)$ denotes the discriminant of the polynomial $f - cg$.

If $n = 2$, then $\deg \mathrm{Diff}\,(K(x)/K(z)) = 2$. So, $\boldsymbol{S}_2 = \boldsymbol{T}_2 = \{2\}$.

If $n = 3$, then $\deg \mathrm{Diff}\,(K(x)/K(z)) = 4$. We know that $\boldsymbol{T}_3 = \{2, 3\} \subseteq \boldsymbol{S}_3$. We try to find $z \in K(x)$ such that $K(x)/K(z)$ has 4 ramified places. Ramification index of each ramified place must be equal to 1, since $\deg \mathrm{Diff}\,(K(x)/K(z)) = 4$. Let $z = \frac{f(x)}{g(x)} = \frac{x^3+1}{x}$. The places lying over the pole Q_∞ of z are the pole P_∞ of x and the zero P_0 of x with $e(P_\infty \mid Q_\infty) = 2$ and $e(P_0 \mid Q_\infty) = 1$; i.e. there is only one ramified place lying over Q_∞. Since $x^3 + 1$ has distinct roots, there is no ramified place of $K(x)$ lying over Q_0; i.e. all ramified places of $K(z)$, except the place lying over Q_∞, corresponds to the polynomial $z - c$, for some $c \in K^\times$. We find which values of c, the place Q_c is ramified.

$$z - c = \frac{x^3+1}{x} - c = \frac{x^3 - cx + 1}{x}.$$

From above discussion, we know that Q_c is ramified if and only if $D(x^3 - cx + 1) = 0$.

$$\begin{aligned}
D\left(x^3 - cx + 1\right) &= (-1)^{\frac{1}{2}3.2} \det R\left(x^3 - cx + 1,\ \left(x^3 - cx + 1\right)'\right) \\
&= -\det R\left(x^3 - cx + 1,\ 3x^2 - c\right) \\
&= -\det \begin{bmatrix} 1 & 0 & -c & 1 & 0 \\ 0 & 1 & 0 & -c & 1 \\ 3 & 0 & -c & 0 & 0 \\ 0 & 3 & 0 & -c & 0 \\ 0 & 0 & 3 & 0 & -c \end{bmatrix} \\
&= 27 - 4c^3.
\end{aligned}$$

$27 - 4c^3$ has 3 distinct roots, say c_i for $i = 1$, 2 and 3. Then the places Q_{c_i} of $K(z)$ are ramified in $K(x)/K(z)$ with ramification index 1; i.e. $K(z)$ has 4 ramified places in $K(x)/K(z)$. Hence, there are 4 ramified places of $K(x)$ in $K(x)/K(z)$.
So, $\boldsymbol{S}_3 = \{2, 3, 4\}$.

Before returning this example, we will give some more examples.

Example 3.1. Let $n = 4$. Then $K(x)/K(z)$ can have at most 6 ramified places.
$5 \in \boldsymbol{S}_4$: Let $z = \frac{x^4+x}{x+2}$.Then

$$e(P_\infty \mid Q_\infty) = 3 \text{ and } d(P_\infty \mid Q_\infty) = 2.$$

Hence, $K(x)$ can have at most 4 other places which are ramified in $K(x)/K(z)$ by Hurwitz Genus Formula. Notice that there is only one ramified place lying over Q_∞. So, other ramified places must lie over the places Q_c of $K(z)$ corresponding to the polynomial $z + c$, for some $c \in K$.

$$z + c = \frac{x^4 + (c+1)x + 2c}{x+2}.$$

Then

$$
\begin{aligned}
& R\left(x^4+(c+1)\,x+2c,\,\left(x^4+(c+1)\,x+2c\right)'\right)\\
= \; & R\left(x^4+(c+1)\,x+2c,\;4x^3+c+1\right)\\
= \; & \det\begin{bmatrix}
1 & 0 & 0 & c+1 & 2c & 0 & 0\\
0 & 1 & 0 & 0 & c+1 & 2c & 0\\
0 & 0 & 1 & 0 & 0 & c+1 & 2c\\
4 & 0 & 0 & c+1 & 0 & 0 & 0\\
0 & 4 & 0 & 0 & c+1 & 0 & 0\\
0 & 0 & 4 & 0 & 0 & c+1 & 0\\
0 & 0 & 0 & 4 & 0 & 0 & c+1
\end{bmatrix}\\
= \; & -27c^4+1940c^3-162c^2-108c-27.
\end{aligned}
$$

$D(x^4+(c+1)\,x+2c)=0$ if and only if $p\,(c)=-27c^4+1940c^3-162c^2-108c-27$ $=0$. Since the roots of $p\,(c)$ are very complicated, to see that all roots are different we look for the $R\,(p\,(c)\,,\,p'\,(c))$, where $p'\,(c)=-108c^3+5820c^2-324c-108$.

$$
\begin{aligned}
R\,(p\,(c)\,,\,p'\,(c)) \;=\; & \det\begin{bmatrix}
-27 & 1940 & -162 & -108 & -27 & 0 & 0\\
0 & -27 & 1940 & -162 & -108 & -27 & 0\\
0 & 0 & -27 & 1940 & -162 & -108 & -27\\
-108 & 5820 & -324 & -108 & 0 & 0 & 0\\
0 & -108 & 5820 & -324 & -108 & 0 & 0\\
0 & 0 & -108 & 5820 & -324 & -108 & 0\\
0 & 0 & 0 & -108 & 5820 & -324 & -108
\end{bmatrix}\\
=\; & 8180\,557\,825\,676\,673\,024.
\end{aligned}
$$

Since $R\,(p\,(c)\,,\,p'\,(c))\neq 0$, $p\,(c)$ has 4 distinct zeros. Hence, $K\,(z)$ has 5 ramified places in $K\,(x)\,/K\,(z)$, which shows that $K\,(x)$ has 5 ramified places in $K\,(x)\,/K\,(z)$.
$6\in \boldsymbol{S}_4$: Let $z=\frac{x^4+x}{x^2+2}$. Then $e\,(P_\infty\mid Q_\infty)=2$ and $d\,(P_\infty\mid Q_\infty)=1$, implying that P_∞

is ramified in $K(x)/K(z)$. Then $z+c=\frac{x^4+cx^2+x+2c}{x^2+2}$ and

$$\begin{aligned} & R\left(x^4+cx^2+x+2c,\, 4x^3+2cx+1\right) \\ = & \det\begin{bmatrix} 1 & 0 & c & 1 & 2c & 0 & 0 \\ 0 & 1 & 0 & c & 1 & 2c & 0 \\ 0 & 0 & 1 & 0 & c & 1 & 2c \\ 4 & 0 & 2c & 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 2c & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 & 2c & 1 & 0 \\ 0 & 0 & 0 & 4 & 0 & 2c & 1 \end{bmatrix} \\ = & 32c^5-512c^4+2044c^3+288c^2-27. \end{aligned}$$

The roots of the polynomial $32c^5-512c^4+2044c^3+288c^2-27$ are $0.199\,82$, $8.\,067\,8\pm 0.988\,52i$ and $-0.167\,74\pm 0.189\,15i$. Hence, $K(x)$ has exactly 6 ramified places in $K(x)/K(z)$.
So, $\boldsymbol{S}_4=\{2,\,3,\,4,\,5,\,6\}$.

Example 3.2. Let $n=5$. Then $K(x)/K(z)$ can have at most 8 ramified places.
$6\in\boldsymbol{S}_5$: Let $z=\frac{x^5+x}{x+2}$. Then $e(P_\infty\mid Q_\infty)=4$ and $d(P_\infty\mid Q_\infty)=3$. Hence, $K(x)$ can have at most 5 other places which are ramified in $K(x)/K(z)$. $z+c=\frac{x^5+(c+1)x+2c}{x+2}$ and

$$\begin{aligned} & R\left(x^5+(c+1)x+2c,\, 5x^4+c+1\right) \\ = & \det\begin{bmatrix} 1 & 0 & 0 & 0 & c+1 & 2c & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & c+1 & 2c & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & c+1 & 2c & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & c+1 & 2c \\ 5 & 0 & 0 & 0 & c+1 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 & c+1 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 & c+1 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & 0 & 0 & c+1 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 & 0 & 0 & c+1 \end{bmatrix} \\ = & 256c^5+51\,280c^4+2560c^3+2560c^2+1280c+256. \end{aligned}$$

Then the roots of the polynomial $256c^5+51\,280c^4+2560c^3+2560c^2+1280c+256$ are $-200.\,26$, $-0.175\,39\pm 0.120\,88i$ and $0.150\,55\pm 0.295\,62i$; i.e. it has 5 distinct roots. Hence, $K(x)$ has 6 ramified places in $K(x)/K(z)$.

$7 \in \boldsymbol{S}_5$: Let $z = \frac{x^5+x}{x^2+2}$. Then $e(P_\infty \mid Q_\infty) = 3$ and $d(P_\infty \mid Q_\infty) = 2$. $z + c = \frac{x^5+cx^2+x+2c}{x^2+2}$ and

$$
\begin{aligned}
& R\left(x^5+cx^2+x+2c,\ 5x^4+2cx+1\right) \\
= \ & \det \begin{bmatrix} 1 & 0 & 0 & c & 1 & 2c & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & c & 1 & 2c & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & c & 1 & 2c & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & c & 1 & 2c \\ 5 & 0 & 0 & 2c & 1 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 2c & 1 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 2c & 1 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & 0 & 2c & 1 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 & 0 & 2c & 1 \end{bmatrix} \\
= \ & 216c^6 + 58\,973c^4 - 3200c^2 + 256.
\end{aligned}
$$

Then the roots of the polynomial $216c^6 + 58\,973c^4 - 3200c^2 + 256$ are $\pm 16.\,525i$, $-0.215\,65 \pm 0.139\,19i$ and $0.215\,65 \pm 0.139\,19i$. Hence, $K(x)$ has 7 ramified places.
$8 \in \boldsymbol{S}_5$: Let $z = \frac{x^5+x}{x^3+2}$. Then $e(P_\infty \mid Q_\infty) = 2$ and $d(P_\infty \mid Q_\infty) = 1$. $z + c = \frac{x^5+cx^3+x+2c}{x^3+2}$ and

$$
\begin{aligned}
& R\left(x^5+cx^3+x+2c,\ 5x^4+3cx^2+1\right) \\
= \ & \det \begin{bmatrix} 1 & 0 & c & 0 & 1 & 2c & 0 & 0 & 0 \\ 0 & 1 & 0 & c & 0 & 1 & 2c & 0 & 0 \\ 0 & 0 & 1 & 0 & c & 0 & 1 & 2c & 0 \\ 0 & 0 & 0 & 1 & 0 & c & 0 & 1 & 2c \\ 5 & 0 & 3c & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 3c & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 3c & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & 3c & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 & 3c & 0 & 1 \end{bmatrix} \\
= \ & 432c^7 - 3600c^5 + 50\,016c^4 + 8000c^3 - 128c^2 + 256.
\end{aligned}
$$

Then the roots of the polynomial are $-5.\,398\,4$, $2.\,778\,2 \pm 3.\,737\,8i$, $-0.235\,57 \pm 0.176\,60i$ and $0.156\,6 \pm 0.184\,02i$. Hence, $K(x)$ has 8 ramified places.
So, $\boldsymbol{S}_5 = \{2, 3, 4, 5, 6, 7, 8\}$.

Example 3.3. Let $n = 6$. Then $K(x)/K(z)$ can have at most 10 ramified places.
$7 \in \boldsymbol{S}_6$: Let $z = \frac{x^6+x}{x+2}$. Then $e(P_\infty \mid Q_\infty) = 5$ and $d(P_\infty \mid Q_\infty) = 4$. Hence, $K(x)$ can have at most 6 other places which are ramified in $K(x)/K(z)$. $z + c = \frac{x^6+(c+1)x+2c}{x+2}$ and

$$
\begin{aligned}
& R\left(x^6 + (c+1)x + 2c,\ 6x^5 + c + 1\right) \\
= & \det \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & c+1 & 2c & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & c+1 & 2c & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & c+1 & 2c & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & c+1 & 2c & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & c+1 & 2c \\
6 & 0 & 0 & 0 & 0 & c+1 & 0 & 0 & 0 & 0 & 0 \\
0 & 6 & 0 & 0 & 0 & 0 & c+1 & 0 & 0 & 0 & 0 \\
0 & 0 & 6 & 0 & 0 & 0 & 0 & c+1 & 0 & 0 & 0 \\
0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & c+1 & 0 & 0 \\
0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & c+1 & 0 \\
0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & c+1
\end{bmatrix} \\
= & -3125c^6 + 1474\,242c^5 - 46\,875c^4 - 62\,500c^3 - 46\,875c^2 - 18\,750c - 3125.
\end{aligned}
$$

Then roots of the polynomial are $0.457\,67$, $471.\,73$, $-0.200\,07 \pm 0.102\,85i$ and $-1.\,282\,3 \times 10^{-2} - 0.302\,27i$. Hence, $K(x)$ has 7 ramified places in $K(x)/K(z)$.
$8 \in \boldsymbol{S}_6$: Let $z = \frac{x^6+x}{x^2+2}$. Then $e(P_\infty \mid Q_\infty) = 4$ and $d(P_\infty \mid Q_\infty) = 3$. Hence, $K(x)$ can have at most 8 ramified places in $K(x)/K(z)$. $z + c = \frac{x^6+cx^2+x+2c}{x^2+2}$ and

$$R\left(x^6+cx^2+x+2c,\ 6x^5+2cx+1\right)$$

$$= \det\begin{bmatrix} 1 & 0 & 0 & 0 & c & 1 & 2c & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & c & 1 & 2c & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & c & 1 & 2c & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & c & 1 & 2c & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & c & 1 & 2c \\ 6 & 0 & 0 & 0 & 2c & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 & 2c & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 & 0 & 0 & 2c & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 & 0 & 0 & 2c & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 2c & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 2c & 1 \end{bmatrix}$$

$$= 2048c^7+110\,592c^6+1492\,736c^5-172\,800c^4+45\,000c^2-3125.$$

Then the roots of the polynomial are $0.237\,19$, $-27.\,057\pm1.\,727\,1i$, $-0.238\,52\pm0.109\,26i$ and $0.177\,26-0.309\,4i$. Since all of them are distinct, $K(x)$ has 8 ramified places in $K(x)/K(z)$.
$9\in \boldsymbol{S}_6$: Let $z=\frac{x^6+x}{x^3+2}$. Then $e\left(P_\infty \mid Q_\infty\right)=3$ and $d\left(P_\infty \mid Q_\infty\right)=2$. Hence, $K(x)$ can have at most 9 ramified places in $K(x)/K(z)$. $z+c=\frac{x^6+cx^3+x+2c}{x^3+2}$ and

$$R\left(x^6+cx^3+x+2c,\ 6x^5+3cx^2+1\right)$$

$$= \det\begin{bmatrix} 1 & 0 & 0 & c & 0 & 1 & 2c & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & c & 0 & 1 & 2c & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & c & 0 & 1 & 2c & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & c & 0 & 1 & 2c & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & c & 0 & 1 & 2c \\ 6 & 0 & 0 & 3c & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 3c & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 & 0 & 3c & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 & 0 & 3c & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 0 & 0 & 3c & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 3c & 0 & 1 \end{bmatrix}$$

$$= -2916c^8+69\,984c^7-559\,872c^6+1492\,884c^5+2700c^4-108\,000c^3-3125.$$

Then the roots of the polynomial are $0.363\,86$, $7.\,141\,1$, $9.\,713\,6 \times 10^{-2} \pm 0.237\,77i$, $-0.266\,28 \pm 0.129\,77i$ and $8.\,416\,7 \pm 0.641\,14i$. Hence, $K(x)$ has 9 ramified places.
$10 \in \boldsymbol{S}_6$: Let $z = \frac{x^6+x}{x^4+2}$. Then $e(P_\infty \mid Q_\infty) = 2$ and $d(P_\infty \mid Q_\infty) = 1$. Then $z + c = \frac{x^6+cx^4+x+2c}{x^4+2}$ and

$$\begin{aligned}
& R\left(x^6 + cx^4 + x + 2c,\, 6x^5 + 4cx^3 + 1\right) \\
= \; & \det \begin{bmatrix}
1 & 0 & c & 0 & 0 & 1 & 2c & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & c & 0 & 0 & 1 & 2c & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & c & 0 & 0 & 1 & 2c & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & c & 0 & 0 & 1 & 2c & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & c & 0 & 0 & 1 & 2c \\
6 & 0 & 4c & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 6 & 0 & 4c & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 6 & 0 & 4c & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 6 & 0 & 4c & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 6 & 0 & 4c & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 6 & 0 & 4c & 0 & 0 & 1
\end{bmatrix} \\
= \; & 8192c^9 + 221\,184c^7 + 768c^6 + 1492\,884c^5 + 259\,200c^4 - 3000c^3 - 3125.
\end{aligned}$$

Since it has 9 distinct roots, namely $0.263\,76$, $5.\,989\,7 \times 10^{-2} \pm 0.269\,93i$, $0.325\,76 \pm 3.\,406\,8i$, $-0.239\,39 \pm 3.\,946\,4i$ and $-0.278\,15 \pm 0.161\,15i$, $K(x)$ there are 10 ramified places in $K(x)/K(z)$.
So, $\boldsymbol{S}_6 = \{2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

Now, we have enough examples to make the following conjecture.

Conjecture 3.4 ($charK = 0$)**.** Let $K(x)/K(z)$ be a function field extension of $[K(x) : K(z)] = n \geq 3$, where $z = \frac{x^n+x}{x^k+2}$ with $1 \leq k \leq n-2$ and let P_∞ and Q_∞ denote the pole of x and z in $K(x)$ and $K(z)$, respectively. Then $K(x)$ has $n+k$ ramified places in $K(x)/K(z)$. If P is a ramified place of $K(x)$ other than P_∞ and Q is the place of $K(z)$ lying under P, then $d(P \mid Q) = 1$ and $d(P_\infty \mid Q_\infty) = n - (k+1)$.

Corollary 3.5 ($charK = 0$)**.** *Let $K(x)$ be a rational function field. If conjecture 3.4 is true, then we can find $z \in K(x)$ such that $K(x)/K(z)$ has exactly i ramified place for $2 \leq i \leq 2n-2$; i.e. $\boldsymbol{S}_n = \{2, \ldots, 2n-2\}$.*

Now, we are going to investigate the rational function field extension $K(x)/K(z)$, where $z = \frac{x^n+1}{x}$ to give a proof for a part of corollary 3.5. In fact, we have seen this for $n = 3$ at the beginning of this chapter.

Example 3.6. Let $z = \frac{x^4+1}{x} \in K(x)$. There is only one ramified place lying over the pole Q_∞ of z, namely the pole P_∞ of x with $e(P_\infty \mid Q_\infty) = 3$ and $d(P_\infty \mid Q_\infty) = 2$. $z + c = \frac{x^4+cx+1}{x}$. Then

$$
\begin{aligned}
R\left(x^4 + cx + 1,\, 4x^3 + c\right) &= \det \begin{bmatrix} 1 & 0 & 0 & c & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & c & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & c & 1 \\ 4 & 0 & 0 & c & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & c & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & c & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 & c \end{bmatrix} \\
&= -27c^4 + 256 = (-1)\, 3^3 c^4 + 4^4.
\end{aligned}
$$

Since $-27c^4 + 256$ has 4 distinct roots, $K(x)$ has 5 ramified places in $K(x)/K(z)$.

Example 3.7. Let $z = \frac{x^5+1}{x} \in K(x)$. Then $z + c = \frac{x^5+cx+1}{x}$ and

$$
\begin{aligned}
R\left(x^5 + cx + 1,\, 5x^4 + c\right) &= \det \begin{bmatrix} 1 & 0 & 0 & 0 & c & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & c & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & c & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & c & 1 \\ 5 & 0 & 0 & 0 & c & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 & c & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 & c & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & 0 & 0 & c & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 & 0 & 0 & c \end{bmatrix} \\
&= 256c^5 + 3125 = 4^4 c^5 + 5^5.
\end{aligned}
$$

Hence, $K(x)$ has 6 ramified places in $K(x)/K(z)$.

Example 3.8. Let $z = \frac{x^6+1}{x} \in K(x)$. Then $z + c = \frac{x^6+cx+1}{x}$ and

$$R\left(x^6+cx+1,\, 6x^5+c\right) = \det \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & c & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & c & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & c & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & c & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & c & 1 \\ 6 & 0 & 0 & 0 & 0 & c & 0 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 & 0 & c & 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 & 0 & 0 & 0 & c & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & c & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & c & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & c \end{bmatrix}$$

$$= -3125c^6 + 46\,656 = (-1)\,5^5c^6 + 6^6.$$

Hence, $K(x)$ has 7 ramified places in $K(x)/K(z)$.

Now, we can give the general case.

Lemma 3.9. *Let $K(x)$ be a rational function field and $z = \frac{x^n+1}{x} \in K(x)$. Then $K(x)/K(z)$ is a function field extension with $[K(x) : K(z)] = n$, which has exactly $n+1$ ramified places; i.e. $n+1 \in \boldsymbol{S}_n$ for all $n \geq 3$.*

Proof. Places of $K(x)$ lying over the pole Q_∞ of z are the pole P_∞ of x and the zero P_0 of x with $e(P_\infty \mid Q_\infty) = n-1 \geq 2$ and $e(P_0 \mid Q_\infty) = 1$; i.e. the only ramified place lying over Q_∞ is P_∞ with $d(P_\infty \mid Q_\infty) = n-2$. Then, by Hurwitz Genus Formula, $K(x)$ can have at most n ramified places other than P_∞, which must lie over the places of $K(z)$ corresponding to some polynomial $z+c$ for some $c \in K$. For $z + c = \frac{x^n+cx+1}{x}$, $R\left(x^n+cx+1,\, nx^{n-1}+c\right)$

$$
= \det \begin{bmatrix}
1 & 0 & \cdots & \cdots & 0 & c & 1 & \cdots & \cdots & 0 & 0 \\
0 & 1 & \cdots & \cdots & 0 & 0 & c & \cdots & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & & \vdots & \vdots & \vdots & & & \vdots & \vdots \\
\vdots & \vdots & & \ddots & \vdots & \vdots & \vdots & & & \vdots & \vdots \\
0 & 0 & \cdots & \cdots & 1 & 0 & 0 & \cdots & \cdots & c & 1 \\
n & 0 & \cdots & \cdots & 0 & c & 0 & \cdots & \cdots & 0 & 0 \\
0 & n & \cdots & \cdots & 0 & 0 & c & \cdots & \cdots & 0 & 0 \\
\vdots & \vdots & & & \vdots & \vdots & \vdots & \ddots & & \vdots & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & \vdots & & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \cdots & n & 0 & 0 & \cdots & \cdots & c & 0 \\
0 & 0 & \cdots & \cdots & 0 & n & 0 & \cdots & \cdots & 0 & c
\end{bmatrix}
$$

$$
= \det \begin{bmatrix}
1 & 0 & \cdots & \cdots & 0 & c & 1 & \cdots & \cdots & 0 & 0 \\
0 & 1 & \cdots & \cdots & 0 & 0 & c & \cdots & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & & \vdots & \vdots & \vdots & & & \vdots & \vdots \\
\vdots & \vdots & & \ddots & \vdots & \vdots & \vdots & & & \vdots & \vdots \\
0 & 0 & \cdots & \cdots & 1 & 0 & 0 & \cdots & \cdots & c & 1 \\
0 & 0 & \cdots & \cdots & 0 & -(n-1)\,c & -n & \cdots & \cdots & 0 & 0 \\
0 & 0 & \cdots & \cdots & 0 & 0 & -(n-1)\,c & \cdots & \cdots & 0 & 0 \\
\vdots & \vdots & & & \vdots & \vdots & \vdots & \ddots & & \vdots & \vdots \\
\vdots & \vdots & & & \vdots & \vdots & \vdots & & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \cdots & 0 & 0 & 0 & \cdots & \cdots & -(n-1)\,c & -n \\
0 & 0 & \cdots & \cdots & 0 & n & 0 & \cdots & \cdots & 0 & c
\end{bmatrix}
$$

$$
= \det \begin{bmatrix}
-(n-1)\,c & -n & \cdots & \cdots & 0 & 0 \\
0 & -(n-1)\,c & \cdots & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & & \vdots & \vdots \\
\vdots & \vdots & & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \cdots & -(n-1)\,c & -n \\
n & 0 & \cdots & \cdots & 0 & c
\end{bmatrix}
$$

$$= (-1)^{n+1}\, n \det \begin{bmatrix} -n & 0 & \cdots & \cdots & 0 & 0 \\ -(n-1)\,c & -n & \cdots & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & & \vdots & \vdots \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & -n & 0 \\ 0 & 0 & \cdots & \cdots & -(n-1)\,c & -n \end{bmatrix}$$

$$+ (-1)^{n+n}\, c \det \begin{bmatrix} -(n-1)\,c & -n & \cdots & \cdots & 0 & 0 \\ 0 & -(n-1)\,c & \cdots & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & & \vdots & \vdots \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & -(n-1)\,c & -n \\ 0 & 0 & \cdots & \cdots & 0 & -(n-1)\,c \end{bmatrix}$$

$$= (-1)^{n+1}\, n\,(-n)^{n-1} + c\,(-(n-1)\,c)^{n-1} = n^n + (-1)^{n-1}\,(n-1)^{n-1}\, c^n.$$

Hence, $R\left(x^n + cx + 1,\, nx^{n-1} + c\right)$ has n distinct roots; i.e. $K(z)$ has $n+1$ ramified places in $K(x)/K(z)$. So, $K(x)$ has $n+1$ ramified places $K(x)/K(z)$. $\square$

4

A Generalization of Kummer and Artin-Schreier Extensions

Let F' be an extension of a function field F such that $F' = F(x)$, where x satisfies the equation $f(x) = z$ for some $z \in F$ and $f(x) \in K[x]$. We can consider F' as a compositum of the fields F and $K(x)$ over the rational function field $K(z)$. Throughout this chapter, we assume that F' is separable over $K(z)$. Let $P \in \mathbf{P}_F$ and $P' \in \mathbf{P}_{F'}$ such that $P' \mid P$ and let $Q := P' \cap K(z)$ and $Q' := P' \cap K(x)$. Suppose that at least one of the extensions $P \mid Q$ or $Q' \mid Q$ is tame. Then, by Abhyankar Lemma,

$$e(P' \mid Q) = lcm\{e(P \mid Q), e(Q' \mid Q)\} = \frac{e(P \mid Q).e(Q' \mid Q)}{\gcd\{e(P \mid Q), e(Q' \mid Q)\}}.$$

Also, by transitivity of the ramification index,

$$e(P' \mid Q) = e(P' \mid P).e(P \mid Q).$$

Hence,

$$e(P' \mid P) = \frac{e(Q' \mid Q)}{\gcd\{e(P \mid Q), e(Q' \mid Q)\}}.$$

Example 4.1. Let $F' = F(x)$, where $z = x^n$ for some $z \in F$ and $n \geq 2$ with $\gcd(n, p) = 1$ in the case of $p = char K > 0$. Then $F' = F.K(x)$. All the ramified places of $K(x)$ in $K(x)/K(z)$ are the pole P_∞ and the zero P_0 of x, which are totally ramified. Since $\gcd(n, p) = 1$, $K(x)/K(z)$ is tame. Hence,

$$e(P_\infty \mid Q_\infty) = e(P_0 \mid Q_0) = n$$

and

$$d(P_\infty \mid Q_\infty) = d(P_0 \mid Q_0) = n - 1,$$

where Q_∞ and Q_0 denote the pole and the zero of z in $K(z)$, respectively.

Let $P \in \mathbf{P}_F$ such that P is not a pole or a zero of z; i.e. $v_P(z) = 0$ and let $Q \in \mathbf{P}_{K(z)}$ such that $P \mid Q$. Since Q is unramified in $K(x)/K(z)$, i.e. $e(Q' \mid Q) = 1$, where $Q' \in P_{K(x)}$ such that $P' \mid Q' \mid Q$. Hence,

$$e(P' \mid P) = \frac{e(Q' \mid Q)}{\gcd\{e(P \mid Q), e(Q' \mid Q)\}} = \frac{1}{\gcd\{e(P \mid Q), 1\}} = 1.$$

So, if $v_P(z) = 0$, then P is unramified in F', which gives $d(P' \mid P) = 0$. Suppose that P is a zero of z and let P' be a place of F' lying over P. Since the zero Q_0 of z is totally ramified in $K(x)/K(z)$; i.e. P_0 is the only place of $K(x)$ lying over Q_0, $P' \mid P_0 \mid Q_0$ and since $v_P(z) = e(P \mid Q_0) v_{Q_0}(z) = e(P \mid Q_0)$,

$$e(P' \mid P) = \frac{e(P_0 \mid Q_0)}{\gcd\{e(P \mid Q_0), e(P_0 \mid Q_0)\}} = \frac{n}{\gcd\{v_P(z), n\}}.$$

Similarly, let P is a pole of z,and let P' be a place of F' lying over P. Since the pole Q_∞ of z is totally ramified in $K(x)/K(z)$, $P' \mid P_\infty \mid Q_\infty$, and since $v_P(z) = e(P \mid Q_\infty) v_Q(z) = -e(P \mid Q_\infty)$,

$$e(P' \mid P) = \frac{e(P_\infty \mid Q_\infty)}{\gcd\{e(P \mid Q_\infty), e(P_\infty \mid Q_\infty)\}} = \frac{n}{\gcd\{v_P(z), n\}}.$$

When $charK = p > 0$, we have $\gcd(n, p) = 1$; i.e. $P' \mid P$ is tame. Hence,

$$d(P' \mid P) = e(P' \mid P) - 1 = \frac{n}{\gcd\{v_P(z), n\}} - 1.$$

Notice that $\gcd\{e(P \mid Q), 1\} = \gcd\{v_P(z), 1\}$, when $v_P(z) = 0$.

Now, let's summarize what we have done in example 4.1.

Corollary 4.2. *Let F' be an extension of function field F such that $F' = F(x)$, where x satisfies the equation $z = x^n$ for some $z \in F$, $n \geq 2$ with $\gcd(n, p) = 1$ in the case of $charK = p > 0$. Let $P \in \mathbf{P}_F$ and $P' \in \mathbf{P}_{F'}$ be an extension of P. Then*

$$e(P' \mid P) = \frac{n}{r_P} \text{ and } d(P' \mid P) = \frac{n}{r_P} - 1,$$

where $r_P = \gcd\{v_P(z), n\}$.

Example 4.3 ($charK = p > 0$)**.** Let $f(T) = T^{p^n} + a_{n-1}T^{p^{n-1}} + \cdots + a_1T^p + a_0T \in K[T]$. Then $f(z+y) = f(z) + f(y)$ for $z, y \in K$. Now, let $F' = F(x)$, where x satisfies the equation $f(x) = z$ for some $z \in F$ and $a_0 \neq 0$ so that $K(x)/K(z)$ is separable. Suppose that for each $P \in \mathbf{P}_F$, there exists $y \in F$ such that either $v_P(z - f(y)) \geq 0$ or $v_P(z - f(y)) = -m$ for some $m \in \mathbb{Z}$ with $\gcd(m, p) = 1$. Since $y \in F$,

$$F' = F(x - y) = F(x'),$$

where $x' = x - y$. Also, let $z' = z - f(y)$. Then

$$z' = z - f(y) = f(x) - f(y) = f(x - y) = f(x').$$

Let $P \in \mathbf{P}_F$ such that there exists $y \in F$ with $v_P(z - f(y)) \geq 0$ and $P' \in \mathbf{P}_{F'}$ lying over P. Then we can consider F' as a compositum of F and $K(x')$ over the field $K(z')$, where $x' = x - y$ and $z' = f(x')$. Let $Q \in \mathbf{P}_{K(z')}$ and $Q' \in \mathbf{P}_{K(x')}$ such that $P' \mid P \mid Q$ and $P' \mid Q' \mid Q$. $K(x')$ has only one ramified place in $K(x')/K(z')$, namely the pole P_∞ of x', which lies over the pole Q_∞ of z' and it is totally ramified; i.e. $e(P_\infty \mid Q_\infty) = p^n$. Since $v_P(z') \geq 0$, P does not lie over Q_∞. Hence, $e(Q' \mid Q) = 1$, giving that $e(P' \mid P) = 1$.

Now, let $P \in \mathbf{P}_F$ such that there exists $y \in F$ with $v_P(z - f(y)) = -m$ for some $m \in \mathbb{Z}^+$ with $\gcd(m, p) = 1$ and $P' \in \mathbf{P}_{F'}$ lying over P. By the same change of variable, we can consider F' as a compositum of F and $K(x')$ over the field $K(z')$. Since $v_P(z') < 0$, P is a pole of z'. So,we have $P' \mid P \mid Q_\infty$ and $P' \mid P_\infty \mid Q_\infty$. Then

$$v_P(z') = e(P \mid Q_\infty)\, v_Q(z') = -e(P \mid Q_\infty).$$

Since $v_P(z') = -m$ and $\gcd(m, p) = 1$,

$$e(P \mid Q_\infty) = m \text{ and } d(P \mid Q_\infty) = m - 1.$$

Also, $P \mid Q_\infty$ is tame, since $\gcd(m, p) = 1$. Hence,

$$e(P' \mid P) = \frac{e(P_\infty \mid Q_\infty)}{\gcd\{e(P \mid Q_\infty),\, e(P_\infty \mid Q_\infty)\}} = \frac{p^n}{\gcd\{m,\, p^n\}} = p^n;$$

i.e. P is totally ramified in F'/F. By Abhyankar Lemma,

$$e(P' \mid Q_\infty) = lcm\{e(P \mid Q_\infty),\, e(P_\infty \mid Q_\infty)\} = lcm\{m,\, p^n\} = mp^n.$$

Also, by transitivity of ramification index,

$$e(P' \mid Q_\infty) = e(P' \mid P_\infty)\, e(P_\infty \mid Q_\infty).$$

Since $e(P_\infty \mid Q_\infty) = p^n$,

$$e(P' \mid P_\infty) = m \text{ and } d(P' \mid P_\infty) = m - 1.$$

Since P_∞ is the only ramified place in $K(x')/K(z')$, by Hurwitz Genus Formula, $d(P_\infty \mid Q_\infty) = 2(p^n - 1)$.So, by transitivity of different,

$$\begin{aligned} d(P' \mid Q_\infty) &= e(P' \mid P_\infty)\, d(P_\infty \mid Q_\infty) + d(P' \mid P_\infty) \\ &= 2mp^n - m - 1 \end{aligned}$$

and

$$d(P' \mid Q_\infty) = e(P' \mid P)\, d(P \mid Q_\infty) + d(P' \mid P)$$
$$\Longrightarrow d(P' \mid P) = (p^n - 1)(m + 1).$$

Corollary 4.4 ($char K = p > 0$)**.** *Let F' be an extension of function field F such that $F' = F(x)$, where x satisfies the equation $z = x^{p^n} + a_{n-1}x^{p^{n-1}} + \cdots + a_1 x^p + a_0 x$ for some $z \in F$, where $a_i \in K$ for all $i = 0, \cdots, n-1$ with $a_0 \neq 0$. Suppose that for each place $P \in \mathbf{P}_F$, there exists $y \in F$ such that either $v_P(z - f(y)) \geq 0$ or $v_P(z - f(y)) = -m$ for some $m \in \mathbb{Z}^+$ with $\gcd(m, p) = 1$ and suppose that there exists at least one place satisfying $v_P(z - f(y)) = -m$. Then*

(i) $[F' : F] = p^n$,

(ii) *the places $P \in \mathbf{P}_F$, for which there exists $y \in F$ with $v_P(z - f(y)) \geq 0$, are unramified in F'/F and*

(iii) *the places $P \in \mathbf{P}_F$, for which there exists $y \in F$ with $v_P(z - f(y)) = -m$, are totally ramified and $d(P' \mid P) = (p^n - 1)(m + 1)$.*

Remark 4.5. If $n = 1$, then for each place $P \in \mathbf{P}_F$ there exists $y \in F$ such that either $v_P(z - f(y)) \geq 0$ or $v_P(z - f(y)) = -m$ for some $m \in \mathbb{Z}^+$ with $\gcd(m, p) = 1$.

Proof. Suppose $v_P(z - f(y_1)) = -lp$, for some $l \in \mathbb{Z}^+$. Since v_P is onto function, there exists $t \in F$ such that $v_P(t) = -l$. Hence,

$$v_P(z - f(y_1)) = v_P(t^p) \Longrightarrow v_P\left(\frac{z - f(y_1)}{t^p}\right) = 0;$$

i.e. $\frac{z-f(y_1)}{t^p} \in O_P \setminus P$, where O_P is the valuation ring corresponding to v_P and P is the maximal ideal of O_P. Then $\frac{z-f(y_1)}{t^p}(P) \neq 0$. Since O_P/P is a perfect field, there exists $y_2 \in O_P$ such that $\frac{z-f(y_1)}{t^p}(P) = (y_2(P))^p$.

$$\frac{z - f(y_1)}{t^p}(P) = (y_2(P))^p$$

$$\begin{aligned}
&\Longrightarrow \left(\frac{z - f(y_1)}{t^p} - y_2^p\right)(P) = 0 \\
&\Longrightarrow v_P\left(\frac{z - f(y_1)}{t^p} - y_2^p\right) > 0 \\
&\Longrightarrow v_P(t^p) + v_P\left(\frac{z - f(y_1)}{t^p} - y_2^p\right) > v_P(t^p) \\
&\Longrightarrow v_P(z - f(y_1) - t^p y_2^p) > v_P(t^p) = -lp,
\end{aligned}$$

where $f(T) = T^p - T$. Since $\frac{z-f(y_1)}{t^p}(P) = (y_2(P))^p$ and $v_P\left(\frac{z-f(y_1)}{t^p}\right) = 0$, $v_P(y_2) = 0$. So,

$$v_P(ty_2) = v_P(t) = -l > -lp.$$

Also, $v_P(z - (y_1^p - y_1) - t^p y_2^p) > -lp$. Hence,

$$v_P(z - (y_1^p - y_1) - (t^p y_2^p - ty_2)) \geq \min\{v_P(z - (y_1^p - y_1) - t^p y_2^p), v_P(ty_2)\} > -lp.$$

Now, let $y = y_1 + ty_2$. Then

$$z - f(y) = z - ((y_1 + ty_2)^p - (y_1 + ty_2)) = z - (y_1^p - y_1) - (t^p y_2^p - ty_2).$$

Hence, $v_P(z - f(y)) > -lp$. □

Corollary 4.6 ($charK = p > 0$). *Let F' be a extension of function field F such that $F' = F(x)$, where x satisfies the equation $z = x^p - x$ for some $z \in F$. Then for each place $P \in \mathbf{P}_F$, there exists $y \in F$ such that either $v_P(z - f(y)) \geq 0$ or $v_P(z - f(y)) = -m$ for some $m \in \mathbb{Z}^+$ with $\gcd(m, p) = 1$. Then*

(i) *the places $P \in \mathbf{P}_F$, for which there exists $y \in F$ with $v_P(z - f(y)) \geq 0$, are unramified in F'/F and*

(ii) *the places $P \in \mathbf{P}_F$, for which there exists $y \in F$ with $v_P(z - f(y)) = -m$, are totally ramified and $d(P' \mid P) = (p - 1)(m + 1)$.*

In fact, corrollary 4.2 and 4.6 are well-known formulas for Kummer and Artin-Schreier extensions, respectively. Now, we are going to generaralize these formulas for another extension.

Example 4.7. Let F' be an extension of function field F such that $F' = F(x)$, where x satisfies the equation $z = \frac{x^n+1}{x}$ for some $z \in F$ and $2 < n \in \mathbb{Z}$. Let $charK = p$. In the case of $p > 0$, suppose that p is an odd prime. Now, consider F' as a compositum of the fields F and $K(x)$ over the rational function field $K(z)$. Let $P \in \mathbf{P}_F$, $P' \in \mathbf{P}_{F'}$ with $P' \mid P$ and v_P, $v_{P'}$ denote the valuation function corresponding to P and P', respectively.

Case(*i*): Suppose that $p \nmid n,\ n-1$. Let Q_∞ denote the pole of z in $K(z)$ and v_{Q_∞} denote the corresponding valuation function. Then Q_∞ has 2 extensions in $K(x)$, namely the pole P_∞ and the zero P_0 of x with $e(P_\infty \mid Q_\infty) = n-1$ and $e(P_0 \mid Q_\infty) = 1$; i.e. P_∞ is the only ramified place lying over Q_∞ with $d(P_\infty \mid Q_\infty) = n-2$, since $p \nmid n-1$.

Suppose that $v_P(z) < 0$; i.e. P is a pole of z. Since P_∞ and P_0 are the only places lying over Q_∞ in $K(x)$, either $P' \cap K(x) = P_\infty$ or $P' \cap K(x) = P_0$. Say $P' \cap K(x) = P_\infty$; i.e. $v_{P'}(x) < 0$.

$$v_P(z) = e(P \mid Q_\infty)\, v_{Q_\infty}(z) \Longrightarrow e(P \mid Q_\infty) = -v_P(z).$$

Hence, by Abhyankar Lemma, we have

$$e(P' \mid P) = \frac{e(P_\infty \mid Q_\infty)}{\gcd\{e(P \mid Q_\infty),\, e(P_\infty \mid Q_\infty)\}} = \frac{n-1}{\gcd\{v_P(z),\, n-1\}}.$$

Since $p \nmid n-1$, $p \nmid e(P' \mid P)$. Hence,

$$d(P' \mid P) = e(P' \mid P) - 1.$$

Similarly, by Abhyankar Lemma, $e(P' \mid P) = 1$ and $d(P' \mid P) = 0$ when $P' \cap K(x) = P_0$.

Let Q_c be the place of $K(z)$ corresponding to the polynomial $z+c$, for some $c \in K$. We have seen in chapter 3 that Q_c is ramified in $K(x)$ if and only if $x^n + cx + 1$ has multiple roots. This holds if and only if $D(x^n + cx + 1) = n^n + (-1)^{n-1}(n-1)^{n-1} c^n = 0$. In other words, Q_c is ramified in $K(x)$ if and only if c is a root of the polynomial $r(x) = x^n + \left(\frac{-1}{n-1}\right)^{n-1} n^n$. Since $p \nmid n$, $r(x)$ has n distinct roots. By Hurwitz Genus Formula, each ramified place has different index 1. Since $p \neq 2$, each ramified place has ramification index 2. Therefore, Q_c has $n-1$ extension in $K(x)$, by Fundamental Equality.

Say $P_c := P' \cap K(x)$. If $v_P(z+c) > 0$ for some c, which is a root of the polynomial $r(x)$, then either $e(P_c \mid Q_c) = 1$ or 2, by above discussion. If $e(P_c \mid Q_c) = 1$, then $e(P' \mid P) = 1$. If $e(P_c \mid Q_c) = 2$, then

$$e(P' \mid P) = \frac{e(P_c \mid Q_c)}{\gcd\{e(P \mid Q_c),\, e(P_c \mid Q_c)\}} = \frac{2}{\gcd\{v_P(z+c),\, 2\}}.$$

Hence, $P' \mid P$ is ramified if and only if $e(P_c \mid Q_c) = 2$ and $v_P(z+c)$ is not divisible by 2.

Now, let $v_P(z+c) < 0$ for all c, which are the roots of $r(x)$; i.e. Q_c is unramified in $K(x)$. So, $e(P_c \mid Q_c) = 1$. Then $e(P' \mid P) = 1$.

Case(*ii*): Suppose $p \mid n$. Let $n = kp^l$, for some $l \in \mathbb{Z}^+$ with $\gcd(k, p) = 1$. Then $z = \frac{x^n+1}{x} = \frac{\left(x^k+1\right)^{p^l}}{x}$ and $D(x^n + cx + 1) = (-1)^{n-1}(n-1)^{n-1}c^n$. So, Q_∞ and Q_0 are all the ramified places of $K(z)$ in $K(x)/K(z)$. If $P' \mid P$ is ramified, then P is a pole or a zero of z.

Let $v_P(z) < 0$ and $v_{P'}(x) < 0$. Since $p \nmid n-1$, $P_\infty \mid Q_\infty$ is tame. Then

$$e(P' \mid P) = \frac{n-1}{\gcd\{v_P(z), n-1\}} \text{ and } d(P' \mid P) = e(P' \mid P) - 1.$$

If $v_{P'}(x) \geq 0$, then $e(P' \mid P) = 1$.

Since $\gcd(k, p) = 1$, $x^k + 1$ has k distinct roots. Let P_i denote the place of $K(x)$ corresponding to zeros of x^k+1 for $i = 1, \cdots, k$. Let $v_P(z) = m > 0$ and $\gcd(m, p) = 1$. Then P is a zero of z. So, $P' \cap K(x) = P_i$ for some $1 \leq i \leq k$. Then $e(P_i \mid Q_0) = p^l$ and $e(P \mid Q_0) = v_P(z) = m$. Since $p \nmid m$, $P \mid Q_0$ is tame. Hence,

$$e(P' \mid P) = \frac{e(P_i \mid Q_0)}{\gcd\{e(P \mid Q_0), e(P_i \mid Q_0)\}} = \frac{p^l}{\gcd\{m, p^l\}} = p^l .$$

Case(*iii*): Suppose $p \mid n-1$. Then $D(x^n + cx + 1) = n^n$. Since $D(x^n + cx + 1)$ has no zeros, the only ramified place of $K(x)$ in $K(x)/K(z)$ is the pole of x with $e(P_\infty \mid Q_\infty) = n-1$ and $d(P_\infty \mid Q_\infty) = 2(n-1)$, by Hurwitz Genus Formula. Hence, if P is not a pole of z, then $P' \mid P$ is unramified.

Let P be a pole of z; i.e. $v_P(z) < 0$. If P' lies over the zero of x; i.e. $v_{P'}(x) \geq 0$, then $P' \mid P$ is unramified. Suppose $v_{P'}(x) < 0$ and $v_P(z) = -m$ for some $m \in \mathbb{Z}^+$ with $\gcd(m, p) = 1$ so that $P \mid Q_\infty$ is tame, since $e(P \mid Q_\infty) = m$. Then

$$e(P' \mid P) = \frac{e(P_\infty \mid Q_\infty)}{\gcd\{e(P \mid Q_\infty), e(P_\infty \mid Q_\infty)\}} = \frac{n-1}{\gcd\{v_P(z), n-1\}} .$$

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