SEPARATELY ANALYTIC FUNCTIONS AND GENERALIZATIONS OF HARTOGS' THEOREM

by

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Abstract

Let $\mathcal{D}$ and $\mathcal{G}$ be arbitrary Stein manifolds, $E \subset \mathcal{D}$ and $F \subset \mathcal{G}$ compact sets, and $X = (E \times \mathcal{G}) \cup (\mathcal{D} \times F)$. Under certain general hypothesis it is proved that a function f on X which is separately analytic, i.e. for which $f(z, w)$ is analytic in z in $\mathcal{D}$ for any fixed $w \in F$ and analytic in w in $\mathcal{G}$ for any fixed $z \in E$, extends to an analytic function in some open neighbourhood $\tilde{X}$ of X.

HER DEĞİŞKENE GÖRE ANALİTİK FONKSİYONLAR VE HARTOGS TEOREMİNİN GENELLEŞTİRMELERİ

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Özet

$\mathcal{D}$ ve $\mathcal{G}$ Stein manifoldları, $E \subset \mathcal{D}$ ve $F \subset \mathcal{G}$ kompakt kümeler, ve $X = (E \times \mathcal{G}) \cup (\mathcal{D} \times F)$ olsun. Belirlenmiş bir hipotez altında, X kümesi üzerinde tanımlı, değişkene göre analitik ($w \in F$ sabit tutulduğunda, z değişkenine göre $\mathcal{D}$'de analitik ve $z \in E$ sabit tutulduğunda, w değişkenine göre $\mathcal{G}$'de analitik) bir f fonksiyonunun, X kümesinin açık bir komşuluğunda analitik bir fonksiyona genişletilebileceği ispatlanmıştır.

to my family

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Contents

CHAPTER 1

INTRODUCTION

Classical theorem of Hartogs' [2] (cf. also [3]) asserts that a function defined in a domain $\mathcal{D} \subset \mathbb{C}^n$ and having the property that it is analytic with respect to each variable when the remaining variables are held constant is then analytic in $\mathcal{D}$. Hukuhara [4] raised the question of whether the following strengthening of Hartogs' theorem holds.

Problem 1: Let $\mathcal{D}$ and $\mathcal{G}$ be Stein manifolds. Characterize those compact sets $E \subset \mathcal{D}$ having the property that every function $f(z, w)$ which is defined on $\mathcal{D} \times \mathcal{G}$ and analytic in z in $\mathcal{D}$ for every fixed $w \in \mathcal{G}$ and analytic in w in $\mathcal{G}$ for every fixed $z \in E$ is then analytic in $\mathcal{D} \times \mathcal{G}$.

This problem was solved in [16], [20], and [19]. Zakharyuta obtained the solution as a consequence of a general theorem in [23]. And we will give the solution to this problem as a corollary in Chapter 5.

Siciak in [17] posed and partially solved the following general problem.

Problem 2: Let $\mathcal{D}$ and $\mathcal{G}$ be Stein manifolds and $E \subset \mathcal{D}$ and $F \subset \mathcal{G}$. Give conditions on E and F so that each function which is separately analytic on $X = (E \times \mathcal{G}) \cup (\mathcal{D} \times F)$ extends to a function analytic in some open neighbourhood of X.

In Chapter 5 as a main Theorem 5.1.1 we will present a complete solution to Problem 2 which was proved by Zakharyuta in [23].

The basis of our considerations is comprised of the methods of extremal plurisubharmonic functions (cf. [17], [24], [21]). Sufficient information about relative extremal functions can be found in Chapter 4.

We will introduce a method for construction of bases that is based on Hilbert methods which have been applied to spaces of analytic functions in Chapter 2.

Moreover, information about spaces of analytic functions will be represented in Chapter 3.

CHAPTER 2

SOME INFORMATION ON FUNCTIONAL ANALYSIS

In this chapter, we will mention some preliminary concepts about functional analysis. In section 2.1, locally convex space will be introduced ([9]) and definition of a nuclear space will be given.

In section 2.2, Hilbert scales will be introduced ([8], [10]) and some theorems that construct a common basis for pairs of Hilbert spaces will be discussed [22].

2.1 Locally Convex Spaces

Let X be a non-empty set. We will define a *topology* on X as a system τ of subsets of X which has the properties:

1. X and $\emptyset$ are open.

2. The union of every collection of open sets is open.

3. The intersection of any two open sets is open.

The elements of τ are called open sets. A topological space (X, τ) is a set X with a topology τ.

A topological space X is called a *Hausdorff space* if for each distinct pair x, y in X there exist disjoint open sets U_x and U_y with $x \in U_x$ and $y \in U_y$. Later on, we will assume the topological spaces to be Hausdorff.

By a topological vector space, we will mean a $\mathbb{F}$-vector space E with a topology τ for which addition $+ : E \times E \to E$ and scalar multiplication $\cdot : \mathbb{F} \times E \to E$ are continuous in τ.

A collection $\tau' \subset \tau$ is a *base* for τ if every member of τ that is, every open set is a union of members of τ'. A collection γ of neighbourhoods of a point $x \in X$ is a *local base at* x if every neighbourhood of x contains a member of γ. In the vector space context, the term *local base* will always mean a local base at 0.

Definition 2.1.1. *A topological vector space E, is called a locally convex space if there is a local base $\mathcal{B}$ whose members are convex.*

A *locally convex topology*, on a $\mathbb{F}$-vector space E, is a topology τ on E for which (E, τ) is a locally convex space.

We now introduce a special class of locally convex spaces which will be important to us.

Definition 2.1.2. *Let E and F be Banach spaces and $A : E \to F$ be a linear map. If there exist sequences $(\lambda_j)_{j\in\mathbb{N}}$ in E' and $(\beta_j)_{j\in\mathbb{N}}$ in F such that $\sum_{j\in\mathbb{N}} \|\lambda_j\|\|\beta_j\| < \infty$, so that*

$$Ax = \sum_{j\in\mathbb{N}} \lambda_j(x)\beta_j \quad for\, all\ x \in E, \tag{2.1}$$

then A is called a nuclear operator. And (2.1) is said to be a nuclear representation of A.

Let E be a locally convex space and p be a semi-norm on E. A norm is defined on the quotient space E/N_p by $\|x+N_p\|_p := p(x)$, where $N_p = \{x \in E : p(x) = 0\}$. The space $E_p := \left(\widehat{E/N_p}, \|\|_p\right)$ is called the *local Banach space* for the semi-norm p. We have $\|\iota^p(x)\|_p = p(x)$, for all $x \in E$, where ι^p is the canonical map, $\iota^p : E \to E_p$, $\iota^p(x) := x + N_p$. Note that if p and q are semi-norms on E with $q \geq p$, then the identity map on E induces a continuous linear map $\iota^p_q : E_q \to E_p$ between the local Banach spaces determined by the relation $\iota^p_q \circ \iota^p = \iota^q$. Now, if for each continuous semi-norm p on E there exists a continuous semi-norm q satisfying $q \geq p$, so that $\iota^p_q : E_q \to E_p$ is nuclear, then E is called a *nuclear space*. For example, $A(\Omega)$, $C^\infty(\Omega)$.

We shall say that a topological vector space X is *imbedded in* a topological vector space Y, if there exists an injective linear continuous mapping $i : X \longrightarrow Y$.

We denote this imbedding by $X \hookrightarrow Y$. In addition, a bounded linear operator $T : V \to W$ between normed spaces V and W is said to be a *compact* or *completely continuous* operator, if it maps every set bounded in the norm of V to a set relatively compact in W.

2.2 Hilbert Pairs and Scales

Lemma 2.2.1. *(see e.g. [21]) Let H_0, H_1 be a pair of Hilbert spaces with a linear dense compact embedding $H_1 \hookrightarrow H_0$. Then there exists a system $\{e_k\} \subset H_1$ which is a common orthogonal basis in H_1 and H_0 such that*

$$\|e_k\|_{H_0} = 1, \quad \mu_k = \mu_k(H_0, H_1) := \|e_k\|_{H_1} \uparrow \infty. \tag{2.2}$$

Proof. Let H_0, H_1 be a pair of Hilbert spaces. Define the operator $J : H_1 \hookrightarrow H_0$ as $Jx \equiv x$ for any $x \in H_1$. Then J is a linear dense compact imbedding.
For any $x \in H_1$, $y \in H_0$ the adjoint operator $J^* : H_0 \to H_1$ is defined as

$$\langle Jx, y\rangle_{H_0} = \langle x, J^*y\rangle_{H_1}$$

Now, define $A := J^*J$. Then, A is self-adjoint since, $A^* = (J^*J)^* = J^*J = A$. If both x and y are elements of H_1, then since $x = Jx$,

$$\langle x, y\rangle_{H_0} = \langle Jx, y\rangle_{H_0} = \langle Jx, Jy\rangle_{H_0} = \langle x, J^*Jy\rangle_{H_1} = \langle x, Ay\rangle_{H_1} = \langle Ax, y\rangle_{H_1},$$

where the last equality follows since A is self-adjoint.

A is compact since it is the superposition of a continuous and compact operator. Also, since $\langle x, y\rangle_{H_0} = \langle Ax, y\rangle_{H_1}$, for any $x \in H_1$, $\langle Ax, x\rangle_{H_1} \geq 0$ as $\langle Ax, x\rangle_{H_1} = 0$ if and only if $x = 0$.

Therefore, A is a compact, self-adjoint, strictly positively defined operator. Hence there exists a complete orthonormalized sequence of eigenvectors $\{g_k\}$:

$$Ag_k = \lambda_k g_k,\ k \in \mathbb{N},\ \lambda_k > 0,\ \lambda_k \to 0.$$

Take the sequence of eigenvalues $\lambda_k \downarrow 0$. Then,

$$\langle g_k, g_j\rangle_{H_0} = \langle Ag_k, g_j\rangle_{H_1} = \langle \lambda_k g_k, g_j\rangle = \lambda_k \delta_{kj}.$$

Thus, $\|g_k\|_{H_0} = \sqrt{\lambda_k}$, $\|g_k\|_{H_1} = 1$ and $\{g_k\}$ is a common orthogonal basis in H_1 and H_0. To renormalize this system, let $e_k := \frac{1}{\sqrt{\lambda_k}} g_k$. Clearly, $\{e_k\}$ is also a common orthogonal basis in H_1 and H_0 such that

$$\|e_k\|_{H_0} = 1, \quad \|e_k\|_{H_1} = \mu_k = \mu_k(H_0, H_1) \uparrow \infty \quad as \quad k \to \infty \tag{2.3}$$

where $\mu_k = \frac{1}{\sqrt{\lambda_k}}$. □

Definition 2.2.1. *A family of Banach spaces X_α, $\alpha_0 \le \alpha \le \beta_0$ is called a scale of Banach spaces if for arbitrary $\alpha_0 \le \alpha < \beta \le \alpha_1$, the following conditions are met:*

(i) $X_\beta \hookrightarrow X_\alpha$,

(ii) $||x||_{X_\gamma} \le C(\alpha, \beta, \gamma)(||x||_{X_\alpha})^{1-\tau(\gamma)}(||x||_{X_\beta})^{\tau(\gamma)}$ *with* $\tau(\gamma) = \frac{\gamma-\alpha}{\beta-\alpha}$, $\alpha < \gamma < \beta$

for any $x \in X_\beta$.

Let $H_\alpha = H_0^{1-\alpha} H_1^\alpha$, $\alpha \in (-\infty, \infty)$, be a *Hilbert scale* generated by Hilbert spaces with dense imbedding $H_1 \hookrightarrow H_0$ ([8], [10]). If this imbedding is compact, then the scale $\{H_\alpha\}$ can be described transparently, since in this case there is a *common orthogonal basis* $\{e_k\}$ for H_0 and H_1, normalized in H_0 and enumerated by non-decreasing of norms in H_1 as in (2.3). Using this basis the scale is determined by the norms

$$||x||_{H_\alpha} := \Big(\sum_{k\in\mathbb{N}} |\zeta_k|^2 \mu_k^{2\alpha}\Big)^{1/2}, \ x = \sum_{k\in\mathbb{N}} \zeta_k e_k. \tag{2.4}$$

In the case $\alpha \ge 0$ the space H_α consists of $x \in H_0$ with finite norm (2.4) while in the case $\alpha < 0$ the space H_α is the completion of H_0 by the norm (2.4).

Lemma 2.2.2. *Let H be a Hilbert space densely and completely continuously embedded in a Banach space X. Then there are Hilbert spaces H_0 and $\tilde{H}_0$ with continuous imbeddings*

$$\tilde{H}_0 \hookrightarrow X \hookrightarrow H_0$$

such that there exists a common orthogonal basis $\{e_n\}$ in the spaces H, H_0 and $\tilde{H}_0$ which satisfies the conditions

$$\|e_n\|_{H_0} = 1, \quad \|e_n\|_H = \mu_n, \quad \|e_n\|_{\tilde{H}_0} \le Cn^2$$

where $\{\mu_n\}$ is any nondecreasing sequence of positive numbers and C is a positive constant.

Proof. With the help of the natural embedding into H, put the functional $x^* \in X^*$ (where X^* is the dual space of X) into correspondence with the element $x' \in H$ such that $x^*(x) = (x, x')_H$ for $x \in H$. Then we obtain the triple of spaces

$$X' \hookrightarrow H \hookrightarrow X$$

By a result of Raimi (cf. [14]), there exist an orthogonal basis $\{g_n\}$ in H and a numerical sequence $h_n \downarrow 0$ such that

$$\|g_n\|_{X'} \leq \frac{1}{h_n}, \quad |(g_n, x)| \leq h_n \|x\|_{X'}, \quad x \in X'.$$

Now set $e_n = \mu_n \, g_n$, where $\mu_n = \frac{n}{h_n}$. Then the Hilbert spaces H_0 and $\tilde{H}_0$ are obtained as the completion of H with respect to the Hilbert norms

$$\left(\sum |(e_n, x)|^2\right)^{1/2} = \|x\|_{H_0}, \quad \left(\sum |(e_n, x)|^2 n^2\right)^{1/2} = \|x\|_{\tilde{H}_0}.$$

Clearly, these are the spaces sought. $\square$

CHAPTER 3

SPACES OF ANALYTIC FUNCTIONS

In this chapter we will define Stein manifolds following [3], then introduce spaces of analytic functions.

3.1 Stein Manifolds

A Hausdorff topological space Ω is called a *manifold of dimension* n if any point in Ω has a neighborhood which is homeomorphic to an open set in $\mathbb{R}^n$.

Definition 3.1.1. *A manifold Ω (of dimension $2n$) is called a complex analytic manifold (of complex dimension n) if there is a given family $\mathcal{F}$ of homeomorphisms κ, called complex analytic coordinate systems, of open sets $\Omega_\kappa \subset \Omega$ on open sets $\tilde{\Omega}_\kappa \subset \mathbb{C}^n$ such that*

(i) *If κ and $\kappa' \in \mathcal{F}$, then the mapping*

$$\kappa'\kappa^{-1} : \kappa(\Omega_\kappa \cap \Omega_{\kappa'}) \longrightarrow \kappa'(\Omega_\kappa \cap \Omega_{\kappa'})$$

defines an analytic mapping,

(ii) $\cup_{\kappa \in \mathcal{F}} \Omega_\kappa = \Omega$,

(iii) *If κ_0 is a homeomorphism of an open set $\Omega_{\kappa_0} \subset \Omega$ onto an open set in $\mathbb{C}^n$ and the mapping $\kappa\kappa_0^{-1} : \kappa_0(\Omega_\kappa \cap \Omega_{\kappa_0}) \longrightarrow \kappa(\Omega_\kappa \cap \Omega_{\kappa_0})$ and its inverse are analytic for every $\kappa \in \mathcal{F}$, then $\kappa_0 \in \mathcal{F}$.*

In fact, the condition (iii) is in a way superfluous. For if $\mathcal{F}$ satisfies (i) and (ii), we can extend $\mathcal{F}$ in one and only one way to a family $\mathcal{F}'$ satisfying (i), (ii), (iii).

Indeed, the only such family $\mathcal{F}'$ is the set of all mappings satisfying the condition (iii) relative to $\mathcal{F}$. A complex analytic structure can thus be defined by an arbitrary family $\mathcal{F}$ satisfying (i) and (ii), but if the condition (iii) is dropped, there are many families defining the same structure. Such a family is called a *complete set of complex analytic coordinate systems*, and two such sets are equivalent if they define the same structure.

We say that n complex valued functions $(z_1, ..., z_n)$ defined in a neighborhood of a point $w \in \Omega$ are a *local coordinate system* at w if they define a mapping of a neighbourhood of w into $\mathbb{C}^n$ which is a coordinate system in the sense defined above.

Definition 3.1.2. *A closed subset V of a complex analytic manifold Ω of dimension n is called an analytic submanifold of dimension m if for each $v \in V$, there exist a neighborhood U of v and local coordinates $f_1, ..., f_n$ such that $U \cap V = \{z \in U : f_{m+1}(z) = ... = f_n(z) = 0\}$.*

Note that, the notion of a submanifold is local.

Ω is called *countable at infinity* if there exists a countable family of compact subsets $\{K_i : i \in \mathbb{N}\}$ such that each compact subset of Ω is contained in some K_i.

Definition 3.1.3. *A complex analytic manifold Ω of dimension n which is countable at infinity is called Stein manifold if*

(i) *Ω is holomorphically convex, that is the $A(\Omega)$-hull*

$$\widehat{K}_\Omega := \Big\{ z \in \Omega : |f(z)| \leq \sup_K \{|f| \text{ for all } f \in A(\Omega)\} \Big\}$$

is a compact subset of Ω for every compact subset K of Ω.

(ii) *For given two points z_1, z_2 with $z_1 \neq z_2$, there exists a function $f \in A(\Omega)$ such that $f(z_1) \neq f(z_2)$.*

(iii) *For every $z \in \Omega$, there exist n analytic functions on Ω $f_1, ..., f_n$ which form a coordinate system at z.*

Due to the following theorem, a Stein manifold can be represented as a submanifold of $\mathbb{C}^N$ where N is sufficiently large.

Theorem 3.1.1. *([3]) Any Stein manifold of dimension n is isomorphic to an analytic submanifold of $\mathbb{C}^{2n+1}$.*

In the following definition and theorems (till the end of this section), we require all manifolds to be *connected* and *countable at infinity* without making this assumption explicitly in every statement. We shall say that a manifold $\tilde{\Omega}$ is a *holomorphic extension* of another manifold Ω if

(i) Ω is an open subset of $\tilde{\Omega}$.

(ii) The analytic structure of Ω is induced by that in $\tilde{\Omega}$.

(iii) For every $f \in A(\Omega)$ one can find $\tilde{f} \in A(\tilde{\Omega})$ such that $f = \tilde{f}$ in Ω.

Theorem 3.1.2. *If Ω is a Stein manifold, and $\tilde{\Omega}$ is a holomorphic extension of Ω, then $\Omega = \tilde{\Omega}$.*

Stein manifolds are maximal not only in the sense that they have no holomorphic extensions, but also in the sense that, if one can find a holomorphic extension which is a Stein manifold, it contains all natural holomorphic extensions:

Theorem 3.1.3. *Let Ω_1 and Ω_2 be holomorphic extensions of Ω, and assume that Ω_1 is a Stein manifold and that functions in $A(\Omega_2)$ give local coordinates everywhere in Ω_2 and separate points in Ω_2. Then there is an analytic isomorphism φ of Ω_2 into Ω_1 which is the identity on Ω; and moreover, if Ω_2 is a Stein manifold it is an isomorphism onto Ω_1.*

Hence, there is, apart from isomorphisms, at most one holomorphic extension of Ω which is a Stein manifold. When such an extension exists, it is called *envelope of holomorphy* of Ω.

We shall now give a sufficient condition for the existence of an envelope of holomorphy.

Definition 3.1.4. *A complex manifold* Ω *of dimension* n *is called a Riemann domain if analytic functions separate points in* Ω *and there is an analytic map*

$$\varphi : \Omega \to \mathbb{C}^n$$

which is everwhere regular, that is, locally an isomorphism.

Theorem 3.1.4. *Let* $\mathcal{D}$ *be a domain on a Stein manifold* Ω*. Then there exists an envelope of holomorphy* $\tilde{\mathcal{D}}$ *of the domain* $\mathcal{D}$*, therewith* $\tilde{\mathcal{D}}$ *is a Riemann domain over* Ω*.*

3.2 Spaces of Analytic Functions

Let Ω be a complex manifold. $A(\Omega)$ is the space of all analytic functions on Ω with the topology of uniform convergence on compact subsets of Ω, i.e. with the locally convex topology generated by semi-norms

$$|x|_K := \max\{|x(z)| : z \in K\} \tag{3.1}$$

where K is any compact subset of Ω. If Ω is countable at infinity, then $A(\Omega)$ is a Fréchet space whose topology is given by the sequence of seminorms $\{|x|_{K_s}\}_{s=1}^{\infty}$ where $K_s \subset intK_{s+1}$ and $\cup_s K_s = \Omega$.

Let E be an arbitrary subset of Ω. By $\mathcal{G}(E) = \mathcal{G}_\Omega(E)$, we denote the collection of all open neighborhoods of E in Ω. For $D_f, D_g \in \mathcal{G}(E)$, the functions $f \in A(D_f)$ and $g \in A(D_g)$ are said to be equivalent $(f \sim g)$ if there exists a $D \in \mathcal{G}(E)$ such that $D \subset D_f \cap D_g$ and $f \equiv g$ on D. A germ of analytic functions, briefly (analytic) germ, is an eqivalence class obtained by the relation $\sim$. If x is a germ on E and $f \in x$ then we say that f represents the germ x. We denote by $A(E)$ the locally convex space of all germs on E endowed with the inductive limit topology

$$A(E) = \lim \operatorname{ind}_{D \in \mathcal{G}(E)} A(D)$$

that is, the finest topology on $A(E)$ for which all natural mappings

$$J_{D,E} : A(D) \longrightarrow A(E) \ , D \in \mathcal{G}(E)$$

are continuous.

Let K be a compact set in Ω and $J : A(K) \longrightarrow C(K)$ be the natural homomorphism of restriction. We denote by $AC(K)$, the Banach space obtained by the completion of $J(A(K))$ in $C(K)$ according to the norm defined in (3.1).

3.2.1 Duality

Let Ω be a Stein manifold. Elements of conjugate space $A'(\Omega) = A(\Omega)^*$, that is linear continuous functionals on $A(\Omega)$, are called *analytic functionals* (*on* Ω). Analytic functionals have a significant part in the investigation of structure of analytic functions, especially in the basis problem.

If E is an arbitrary subset of Stein manifold Ω then the natural map

$$J^* = J^*(E, \Omega) \, : \, A(E)^* \to A'(\Omega), \tag{3.2}$$

that transforms a functional $x^* \in A(E)^*$ to its restriction on $A(\Omega)$, is a linear continuous map. In the case when E is a Runge set in Ω, that is $A(\Omega)$ is dense in $A(E)$, the map in (3.2) is an imbedding.

3.2.2 GKS- Duality

The following result, due to Grothendieck, Köthe, and Silva (see [1, 5, 7, 18]) allows us to realize the space $A(E)^*$, for any set $E \subset \overline{\mathbb{C}}$, as the space of analytic functions $A(E^*)$ where $E^* := \overline{\mathbb{C}} \setminus E$ with the assumption that all germs of $A(E)$ are equal to zero at the point ∞ if $\infty \in E$.

Theorem 3.2.1. *For any set $E \subset \overline{\mathbb{C}}$ there exists an isomorphism $\gamma : A(E)^* \to A(E^*)$ such that the following formula holds*

$$x^*(x) = \int_\Gamma x'(\zeta) x(\zeta) d\zeta, \; x \in A(E),$$

where $x' = \gamma(x^*)$, $\Gamma = \Gamma(x, x')$ is a rectifiable contour separating the singularities of the analytic germs x and x^*.

In several complex variables, there is no similar universal representation of $A(E)^*$ as a space of analytic functions.

3.2.3 The Dual Form of Cartan Theorem

Let M be a closed analytic submanifold of Stein manifold Ω. Then according to Cartan theorem the restriction operator

$$R : A(\Omega) \to A(M) : Rx = x|M, \ x \in A(\Omega),$$

is a surjection. The adjoint operator $R^* : A(M)^* \to A(\Omega)^*$ maps any functional $\varphi \in A(M)^*$ to $\psi = \varphi \circ R \in A(\Omega)^*$. Using the theorem about dual relation between endomorphisms and monomorphisms we get the following dual version of Cartan theorem:

Proposition 3.2.1. *The adjoint operator $R^* : A(M)^* \to A(\Omega)^*$ of the restriction operator $R : A(\Omega) \to A(M)$ is an isomorphic embedding.*

CHAPTER 4

SOME INFORMATION ON PLURIPOTENTIAL THEORY

In this chapter, we first present some fundamental properties of plurisubharmonic functions. Then we define the relative extremal function and give some elementary properties of the function. For further study of plurisubharmonic functions the reader can consult [6], [12], [11] and [25].

4.1 Plurisubharmonic Functions

Let $z = (z_1, \dots, z_n) \in \mathbb{C}^n$. The two norms on $\mathbb{C}^n$ that we shall be using are the *Euclidean norm*

$$\|z\| = \left(\sum_{k=1}^{n} |z_k|^2 \right)^{1/2}$$

and the *maximum norm*

$$|z| = \max\{|z_1|, \dots, |z_n|\}.$$

Note that these norms are equivalent and $|z| \leq \|z\| \leq \sqrt{n}|z|$.
Let $a \in \mathbb{C}^n$ and $r > 0$. The *open polydisc*, with center at a, and radius r, is the set $U(a, r) = \{z \in \mathbb{C}^n : |z - a| < r\}$.
Let Ω be an open subset of $\mathbb{C}^n$, and let $u : \Omega \longrightarrow [-\infty, \infty)$ be an upper semicontinuous function which is not identically $-\infty$ on any connected component of Ω. The function u is said to be *plurisubharmonic* if for each $a \in \Omega$ and $b \in \mathbb{C}^n$, the function $\lambda \longmapsto u(a + \lambda b)$ is subharmonic or identically $-\infty$ on every component of the set $\{\lambda \in \mathbb{C} : a + \lambda b \in \Omega\}$. We denote by $\mathcal{PSH}(\Omega)$ the set of all plurisubharmonic functions in Ω.

The following theorem can be taken as an equivalent definition of plurisubharmonic functions.

Theorem 4.1.1. *Let $u : \Omega \longrightarrow [-\infty, \infty)$ be upper semicontinuous and not identically $-\infty$ on any connected component of $\Omega \subset \mathbb{C}^n$. Then $u \in \mathcal{PSH}(\Omega)$ if and only if for each $a \in \Omega$ and $b \in \mathbb{C}^n$ such that*

$$\{a + \lambda b : \lambda \in \mathbb{C}, |\lambda| \leq 1\} \subset \Omega,$$

we have

$$u(a) \leq \frac{1}{2\pi} \int_0^{2\pi} u(a + e^{it}b)dt. \tag{4.1}$$

It should be noted that plurisubharmonicity is a local property and we can define a *plurisubharmonic function on a complex submanifold* of $\mathbb{C}^n$. Indeed, let M be a complex submanifold of $\mathbb{C}^n$. A function $u : M \to [-\infty, \infty)$ is said to be *plurisubharmonic* on M if for each $a \in M$ there exists a neighbourhood U of a in $\mathbb{C}^n$ and a function $\tilde{u} \in \mathcal{PSH}(U)$ such that $\tilde{u}|U \cap M = u$. It is obvious that most basic properties of plurisubharmonic functions defined on open subsets of $\mathbb{C}^n$ can be easily transferred to the case of plurisubharmonic functions on complex submanifolds.

Let $\Omega \subset \mathbb{C}^n$ be open. If $\Omega \neq \mathbb{C}^n$, define

$$\Omega_\epsilon := \{z \in \Omega : \quad dist(z, \partial\Omega) > \epsilon\}$$

for $\epsilon > 0$. If $\Omega = \mathbb{C}^n$, we set $\Omega_\epsilon = \mathbb{C}^n$. The following theorem is known as *main approximation theorem* for plurisubharmonic functions.

Theorem 4.1.2. *Let Ω be an open subset of $\mathbb{C}^n$, and let $u \in \mathcal{PSH}(\Omega)$. Then there exists $u_\epsilon \in \mathcal{C}^\infty \cap \mathcal{PSH}(\Omega_\epsilon)$ such that u_ϵ decreases with decreasing ϵ, and $\lim_{\epsilon \to 0} u_\epsilon(z) = u(z)$ for each $z \in \Omega$.*

Theorem 4.1.3. *Let Ω be an open subset of $\mathbb{C}^n$*

(i) *If $u, v \in \mathcal{PSH}(\Omega)$ then $\max(u, v) \in \mathcal{PSH}(\Omega)$.*

(ii) *The family $\mathcal{PSH}(\Omega)$ is a convex cone, i.e. if α, β are non-negative numbers and $u, v \in \mathcal{PSH}(\Omega)$, then $\alpha u + \beta v \in \mathcal{PSH}(\Omega)$.*

(iii) *If Ω is connected and $\{u_j\}_{j\in\mathbb{N}} \subset \mathcal{PSH}(\Omega)$ is a decreasing sequence then $u = \lim_{j\to\infty} u_j \in \mathcal{PSH}(\Omega)$ or $u \equiv -\infty$.*

(iv) *Let $\{u_\alpha\}_{\alpha\in A} \subset \mathcal{PSH}(\Omega)$ be such that its upper envelope $u = \sup_{\alpha\in A} u_\alpha$ is locally bounded above. Then the upper semicontinuous regularization $u^*(y) = \limsup_{\substack{z\to y \\ z\in\Omega}} u(z)$, $y \in \Omega$ is plurisubharmonic in Ω.*

Proposition 4.1.1. *Let Ω be a domain in $\mathbb{C}^n$. Let $V \subset \Omega$ be an open subset. If $u \in \mathcal{PSH}(\Omega)$, $v \in \mathcal{PSH}(V)$, and*

$$\limsup_{z\to y} v(z) \leq u(y), \ y \in \partial V \cap \Omega, \tag{4.2}$$

then

$$w = \begin{cases} \max\{u, v\} & \text{in } V \\ u & \text{in } \Omega \setminus V \end{cases}$$

is plurisubharmonic in Ω.

Proof. The boundary condition (4.2) on v ensures that w is upper semicontinuous on Ω. By Theorem 4.1.3 (i) w satisfies the local submean inequality (4.1) at each $z \in V$, and it does also when $z \in \Omega \setminus V$ since $w \geq u$ on Ω. □

4.2 The Relative Extremal Functions

Let $\mathcal{D}$ be a Stein manifold of dimension n and E is a compact subset of $\mathcal{D}$. The *relative extremal function* for E in $\mathcal{D}$ is defined as

$$\omega^0(z) = \omega^0(\mathcal{D}, E, z) = \sup\{u(z) : u \in \mathcal{PSH}(D), u|_E \leq 0, u \leq 1\} \tag{4.3}$$

Note that the function $(\omega^0(z))^*$ is plurisubharmonic in $\mathcal{D}$. From now on we will denote the upper semicontinuous regularization of $\omega^0(z)$ by $\omega(z) = \omega(\mathcal{D}, E, z)$ and the class of all functions $u \in \mathcal{PSH}(D)$ satisfying the inequalities $u|_E \leq 0$, $u \leq 1$

by $P(E, \mathcal{D})$.

We also define the functions

$$\tilde{\omega}(z) = \tilde{\omega}(\mathcal{D}, E, z) = \lim_{s\to\infty} \omega(\mathcal{D}_s, E, z), z \in \mathcal{D}, \tag{4.4}$$

$$\tilde{\omega^0}(z) = \tilde{\omega^0}(\mathcal{D}, E, z) = \lim_{s\to\infty} \omega^0(\mathcal{D}_s, E, z), z \in \mathcal{D}. \tag{4.5}$$

where $\mathcal{D}_s$ is a sequence of open sets in $\mathcal{D}$ with $\mathcal{D}_s \Subset \mathcal{D}_{s+1}$ and $\mathcal{D} = \cup_1^\infty \mathcal{D}_s$.

One of the main notions in our work is that of pluriregularity of compact sets in Stein manifolds.

Definition 4.2.1. *A compact set E in a Stein manifold Ω is called pluriregular with respect to its open neighbourhood $D \subset \Omega$ if $\omega(G, E, z) \equiv 0$ on E for every open neighbourhood $E \subset G \Subset D$. A compact set E is called pluriregular if for any its open neighborhood D the set E is pluriregular with respect to the holomorphic hull $\tilde{D}$ of D, that is,*

$$\omega(\tilde{D}, E, z) \equiv 0, \quad z \in E$$

If a compact set E is holomorphically convex in $\mathcal{D}$, that is, $\hat{E}_{\mathcal{D}} = E$, then it is sufficient to verify the condition 4.2.1 in only one pseudoconvex neighbourhood $\mathcal{G} \Subset \mathcal{D}$ of E.

Definition 4.2.2. *An open set D in a Stein manifold Ω is called pluriregular (or hyperconvex) if there exists a function $u \in \mathcal{PSH}(D)$ such that $u(z) < 0$, $z \in D$ with $\lim u(z) = 0$ as $z \to \partial D$.*

Definition 4.2.3. *An open set D in a Stein manifold Ω is called strongly pluriregular if there exists a pseudoconvex open neighbourhood $\mathcal{G} \Supset D$ and a continuous plurisubharmonic function $u(z)$ in $\mathcal{G}$ such that $D = \{z \in \mathcal{G} : u(z) < 0\}$. Such a function u is called an exhaustion function for $\mathcal{G}$.*

The following monotonicity property of the relative extremal function is a direct consequence of the definition.

Proposition 4.2.1. *If $E_1 \subset E_2 \subset \mathcal{D}_2 \subset \mathcal{D}_1$, then the inequality*

$$\omega^0(\mathcal{D}_1, E_1, z) \leq \omega^0(\mathcal{D}_2, E_2, z)$$

holds where both functions are defined.

Proposition 4.2.2. *Let Ω be an open set in $\mathbb{C}^n$, and let $E \subset \Omega$. Suppose that M and m are two real numbers such that $M > m$. If $u \in \mathcal{PSH}(\Omega)$ satisfies the conditions*

$$u \leq M \quad on \quad \Omega \quad and \quad u \leq m \quad on\ E,$$

then

$$u \leq M(1+\omega) - m\omega.$$

Proof. Since the function

$$\frac{u-M}{M-m} \in \mathcal{PSH}(\Omega)$$

as an immediate consequence of the definition of the relative extremal function, we have

$$\frac{u-M}{M-m} \leq \omega^0(z) \leq \omega(z)$$

and this is equivalent to the desired estimate. □

There follows an analog of the Hadamard theorem on three surfaces, which is called also *Two Constants Theorem for analytic functions.*([17])

Proposition 4.2.3. *Let $\mathcal{D}$ be a Stein manifold, E be compact set in $\mathcal{D}$, and $f \in A(\mathcal{D})$ with $|f|_E \leq M_0$ and $|f|_{\mathcal{D}} \leq M_1$. Then*

$$|f|_{\mathcal{D}_\alpha} \leq (M_0)^{1-\alpha}(M_1)^{\alpha}, \quad \alpha \in (0,1),$$

where $\mathcal{D}_\alpha = \{z \in \mathcal{D} : \omega(\mathcal{D}, E, z) < \alpha\}$.

Proposition 4.2.4. *Let Ω be a strongly pluriregular subset of $\mathbb{C}^n$, and let E be a compact subset of Ω. Suppose that $\{\Omega_j\}$ is an increasing sequence of open subsets of Ω such that $\Omega = \cup_{j=1}^{\infty}\Omega_j$ and $K \subset \Omega_1$. Then*

$$\lim_{j\to\infty} \omega^0(\Omega_j, E, z) = \omega^0(\Omega, E, z) \quad z \in \Omega.$$

Proof. Take a point $z_0 \in \Omega$. We may suppose, without loss of generality, that $E \cup \{z_0\} \subset \Omega_1$. Let $\varrho < 0$ be an exhaustion function (see Definition 4.2.3) for Ω such that $\varrho \leq -1$ on E.

Take $\varepsilon \in (0,1)$ such that $\varrho(z_0) < -\varepsilon$. There exists $j_0 \in \mathbf{N}$ for which the open set $A = \varrho^{-1}((-\infty, -\varepsilon))$ is relatively compact in Ω_{j_0}. Take $u \in \mathcal{PSH}(\Omega_{j_0})$ such that $u \leq 1$ on Ω_{j_0} and $u \leq 0$ on E. Then

$$v(z) = \begin{cases} \max\{u(z) - \varepsilon, \varrho(z)\} & z \in A \\ \varrho(z) & z \in \Omega \backslash A \end{cases}$$

defines a plurisubharmonic function; moreover, $v|E \leq 0$ and $v \leq 1$. Thus $v(z_0) \leq \omega^0(z_0)$. Since u was an arbitrary member of the family P(E,Ω_{j_0}), we have

$$\omega^0(\Omega_{j_0}, E, z_0) - \varepsilon \leq \omega^0(\Omega, E, z_0).$$

By proposition 4.2.1,

$$\omega^0(\Omega_j, E, z_0) - \varepsilon \leq \omega^0(\Omega, E, z_0) \leq \omega^0(\Omega_j, E, z_0)$$

for all $j \geq j_0$. As ε can be made arbitrarily small, the result follows. □

Lemma 4.2.1. *Let $\mathcal{D}$ be a Stein manifold and E be a compact set in $\mathcal{D}$. Then for any sequence of compact sets $\{E_s\}$ in $\mathcal{D}$ with $E_{s+1} \Subset E_s$*

$$\omega(\mathcal{D}, E_s, z) \uparrow \omega^0(\mathcal{D}, E, z), z \in \mathcal{D}$$

where $E = \cap_1^\infty E_s$.

Proof. Clearly, $\omega(\mathcal{D}, E_1, z) \leq \omega(\mathcal{D}, E_2, z) \leq \ldots$, hence the limit exists. Set $\psi(z) = \lim_{s\to\infty} \omega(\mathcal{D}, E_s, z)$. Since ω is a plurisubharmonic function in $\mathcal{D}$ with $\omega \leq 1$ and $\omega|_{E_s} \leq 0$, hence $\omega|E \leq 0$; $\omega(\mathcal{D}, E_s, z)$ belongs to $P(E, \mathcal{D})$. Then for some s, by the definition of the relative extremal function we have $\omega(\mathcal{D}, E_s, z) \leq \omega^0(\mathcal{D}, E, z)$, thus $\psi(z) \leq \omega^0(\mathcal{D}, E, z)$.

Now we need to prove the other side of the inequality. Let u be an arbitrary function in $P(E, \mathcal{D})$. By the upper semicontinuity of u, given any $\varepsilon > 0$ the set $U = \{z \in \mathcal{D} : u(z) < \varepsilon\}$ is an open neighbourhood of E and then there exists

$s_0 = s_0(\varepsilon)$ such that for each $s \geq s_0$, $E_s \subset U$. Hence, if $s \geq s_0$ and $z \in \mathcal{D}$ the function $v(z) = \frac{u(z)-\varepsilon}{1-\varepsilon}$ becomes plurisubharmonic with $v(z)|E_s \leq 0$ and $v(z) \leq 1$, therefore, we have

$$\begin{aligned} u(z) &\leq (1-\varepsilon)\omega^0(\mathcal{D}, E_s, z) + \varepsilon \\ &\leq (1-\varepsilon)\omega(\mathcal{D}, E_s, z) + \varepsilon \end{aligned}$$

Taking limits of both sides as $s \to \infty$ and then as $\varepsilon \to 0$, we obtain that $u(z) \leq \psi(z)$ for any $u \in P(E, \mathcal{D})$, whence we have $\omega^0(\mathcal{D}, E, z) \leq \psi(z)$ which proves the desired equality. □

4.3 Pluripolar Sets

In this section we will give the definition of pluripolarity of a set and consider some properties of pluripolar sets.

Definition 4.3.1. *A set A is called pluripolar in an open set $\mathcal{D}$ in a Stein manifold Ω if there exists a function $\psi \in \mathcal{PSH}(\mathcal{D})$ satisfying $\psi(z) \equiv -\infty$ on A but $\psi \not\equiv -\infty$ on any connected component of $\mathcal{D}$.*

Lemma 4.3.1. *Let E_j be pluripolar sets in $\mathcal{D} \Subset \Omega$. Then the set $E = \cup_1^\infty E_j$ is also pluripolar in $\mathcal{D}$.*

Proof. Let $\psi_j \in \mathcal{PSH}(\mathcal{D})$ with $\psi_j \not\equiv -\infty$ on any connected component $\mathcal{D}'_E$ of $\mathcal{D}$ but with $\psi_j \equiv -\infty$ on E_j. Take the increasing sequence of open sets $\{\mathcal{D}_s\}$ with $\mathcal{D}_s \Subset \mathcal{D}_{s+1}$ and $\mathcal{D} = \cup_1^\infty \mathcal{D}_s$. Now set

$$c_{j,s} = \sup\{\psi_j(z) \,:\, z \in \mathcal{D}_s\}$$

In every connected component $\mathcal{D}'_E$ of $\mathcal{D}$ there exists at least one point a_k at which $\psi_j(a_k) \neq -\infty$ for all $j = 1, 2, \ldots$. Since in any coordinate neighborhood $U \subset \mathcal{D}'_E$ the functions ψ_j are subharmonic in the corresponding real local coordinates, the set $L = \{z \in U \,:\, \psi_j(z) = -\infty,\ j = 1, 2, \ldots\}$ has Lebesque measure zero in the local coordinates. Therefore $U \backslash L = \emptyset$. Now choose a sequence of positive numbers

$\{\beta_j\}$ so that both of the series

$$\sum_{j=1}^{\infty} \beta_j |c_{j,s}|, \qquad \sum_{j,k} \beta_j |\psi_j(a_k)|$$

converges. We will show that the series

$$\sum_{j=1}^{\infty} \beta_j \, \psi_j(z) \to \psi(z) \qquad \text{in } \mathcal{D}$$

where $\psi \in \mathcal{PSH}(\mathcal{D})$. For $z \in \mathcal{D}_s$ we can rewrite the above series as the sum of two series

$$\sum \beta_j \, c_{j,s} \; - \sum_{j=1}^{\infty} \beta_j \, (c_{j,s} - \psi_j(z))$$

where the first one converges by the assumption and the second one is a series of nonnegative terms which converges in $\mathcal{D}_s$. Also, ψ is plurisubharmonic in $\mathcal{D}_s$ since it is the limit of the decreasing sequence of its partial sums, which are plurisubharmonic. But s was arbitrary, so the series converges in $\mathcal{D}$ to a function $\psi \in \mathcal{PSH}(D)$. Moreover, $\psi(z) = -\infty$ on E by construction and $\psi(z) \not\equiv -\infty$ on any component $\mathcal{D}'_E$ of $\mathcal{D}$ since $\psi(a_k) = \sum \beta_j \psi_j(a_k) \neq -\infty$ by the assumption.
Hence E is also a pluripolar set in $\mathcal{D}$. □

Lemma 4.3.2. *Let $\mathcal{D}$ be a Stein manifold and E be a compact subset of $\mathcal{D}$. Then the following conditions are equivalent:*

(i) $\omega(\mathcal{G}, E, z) \equiv 1$ *in $\mathcal{G}$ for any open set $\mathcal{G}$ with $E \subset \mathcal{G} \Subset \mathcal{D}$;*

(ii) *E is pluripolar in $\mathcal{D}$.*

Proof. (i)⇒(ii)Let $\mathcal{G}$ be an open pseudoconvex set with $E \subset \mathcal{G} \Subset \mathcal{D}$. Take a sequence of pluriregular compact sets $\{E_s\}$, $E_{s+1} \Subset E_s$, $E = \cap_1^{\infty} E_s$. Then by Lemma 4.2.1, $\omega(\mathcal{D}, E_s, z) \uparrow \omega^0(\mathcal{D}, E, z)$; and by hypothesis

$$\limsup_{\zeta \to z} \omega^0(\mathcal{G}, E, \zeta) = \omega(\mathcal{G}, E, z) \equiv 1.$$

Therefore, by the theorem of Brelot-Cartan ([15],Chapter 3,Theorem 3.4.2) the set $\{z \in \mathcal{G} \, : \, \omega^0(\mathcal{G}, E, z) < 1\}$ has zero capacity (if $\mathcal{G}$ is considered to be a $2n$-dimensional real manifold). Consequently there exists a point a_j in each connected component $\mathcal{G}_j$ of $\mathcal{G}$ such that $\omega^0(\mathcal{G}, E, a_j) = 1$. Now choose a sequence of positive numbers $\{\gamma_s\}$ so that $\sum_{s=1}^{\infty} \gamma_s = \infty$ and $\sum_{s=1}^{\infty} \gamma_s(1 - \omega(\mathcal{G}, E_s, z)) < \infty$. Then the function $\Psi(z) = \sum_{s=1}^{\infty} \gamma_s(\omega(\mathcal{G}, E_s, z) - 1)$ is plurisubharmonic in $\mathcal{G}$ as the sum of a series with non-positive plurisubharmonic terms, and the condition on the second series guarantees that $\Psi(z) \equiv -\infty$ on E while $\Psi(a_j) \neq -\infty$. Thus E is pluripolar in $\mathcal{G}$.

(ii)$\Rightarrow$(i)Let $\mathcal{G}$ be an open set with $E \subset \mathcal{G} \Subset \mathcal{D}$. Take an open set $\mathcal{G}'$ with $\mathcal{G} \Subset \mathcal{G}' \Subset \mathcal{D}$. Then by the hypothesis there exists a function $\Psi \in \mathcal{PSH}(\mathcal{D})$ such that $\Psi(z) \equiv -\infty$ on E and $\Psi \not\equiv -\infty$ on any connected component of $\mathcal{D}$. Now we consider a function

$$u_s(z) = \frac{\Psi(z) + s}{a + s} \quad z \in \mathcal{G}, \tag{4.6}$$

where $a = \sup\{\Psi(z) : z \in \mathcal{G}\}$. Clearly $u_s \in \mathcal{PSH}(\mathcal{G})$ and $u_s \leq \omega(\mathcal{G}, E, z)$ for $z \in \mathcal{G}$ and for all $s > |a|$. Thus, taking a point $\zeta \in \mathcal{G}$ such that $\Psi(\zeta) \neq -\infty$, we obtain by (4.6) that $\omega(\mathcal{D}, E, \zeta) = 1$. Since by the hypothesis we can find such a point in every connected component of $\mathcal{G}$ we have $\omega(\mathcal{D}, E, z) \equiv 1$ in $\mathcal{G}$ by the maximum principle. □

Proposition 4.3.1. *Let $\mathcal{D}$ be a Stein manifold and E a compact subset of $\mathcal{D}$ such that $E \cap \mathcal{D}'$ is not pluripolar in $\mathcal{D}'$ for any connected component $\mathcal{D}'$ of $\mathcal{D}$. Then for any $f \in A(\mathcal{D})$, $f|E \equiv 0$ implies that $f \equiv 0$ i.e. E is a set of uniqueness for the class $A(\mathcal{D})$.*

Proof. Assume that there is a function $f \in A(\mathcal{D})$ with $f|E \equiv 0$ but $f \not\equiv 0$ in some connected component $\mathcal{D}'$. Then the function $u(z) = \ln|f(z)|$ is plurisubharmonic in $\mathcal{D}'$ and $u \equiv -\infty$ on $E \cap \mathcal{D}'$ but $u(z) \not\equiv -\infty$ in $\mathcal{D}'$, i.e. E is pluripolar in $\mathcal{D}'$. □

Proposition 4.3.2. *Let $\mathcal{D}$ be a strongly pluriregular open set on a Stein manifold Ω and $E \subset \mathcal{D}$ be a pluriregular compact set. Then the function $\omega(\mathcal{D}, E, z)$ extends to a continuous plurisubharmonic function in an open set $\mathcal{D}' \Supset \mathcal{D}$.*

Proof. Let $\mathcal{D}'$ be a pseudoconvex open set with $\mathcal{D} \Subset \mathcal{D}' \Subset \Omega$ and $u \in C(\mathcal{D}') \cap \mathcal{PSH}(\mathcal{D}')$ such that $\mathcal{D} = \{z \in \mathcal{D}' : u(z) < 0\}$. Take $\gamma = -\sup\{u(z) : z \in E\}$.

Then the function

$$v(z) = \begin{cases} \omega(\mathcal{D}, E, z) & z \in \mathcal{D} \\ 1 + \frac{1}{\gamma} u(z) & z \in \mathcal{D}' \backslash \mathcal{D} \end{cases}$$

is the desired extension. $\square$

CHAPTER 5

SEPARATELY ANALYTIC FUNCTIONS

5.1 Separate Analyticity

Definition 5.1.1. *Let $\mathcal{D}$ and $\mathcal{G}$ be Stein manifolds and $E \subset \mathcal{D}$, $F \subset \mathcal{G}$. We shall say that a function $f(z,w)$ defined on the set $X = (E \times \mathcal{G}) \cup (\mathcal{D} \times F)$ is separately analytic on X if it is analytic in z in $\mathcal{D}$ for each $w \in F$ and analytic in w in $\mathcal{G}$ for each $z \in E$.*

The next theorem gives conditions on E and F providing an answer to Problem 2.

Theorem 5.1.1. *Let $\mathcal{D}$ and $\mathcal{G}$ be Stein manifolds and $E \subset \mathcal{D}$, $F \subset \mathcal{G}$ compact sets with the property that $\hat{E}_{\mathcal{D}}$ and $\hat{F}_{\mathcal{G}}$ are pluriregular in $\mathcal{D}$ and $\mathcal{G}$, respectively. Let $f(z,w)$ be a function separately analytic on the set $X = (E \times \mathcal{G}) \cup (\mathcal{D} \times F)$. Then the function f extends uniquely to a function $\tilde{f}$ which is analytic in the open set*

$$\tilde{X} = \{(z,w) \in (\mathcal{D} \times \mathcal{G}) \,:\, \omega(\mathcal{D}, E, z) + \omega(\mathcal{G}, F, w) < 1\}$$

containing X.

Theorem 5.1.1 will be obtained as a consequence of the following more general result.

Theorem 5.1.2. *Let $\mathcal{D}$, $\mathcal{G}$, E, F and X be as in Theorem 5.1.1, with E is not pluripolar in each connected component of $\mathcal{D}$. Let $f(z,w)$ be a separately analytic function on $X = (E \times \mathcal{G}) \cup (\mathcal{D} \times F)$ which is locally bounded on $E \times \mathcal{G}$. Then there* *is a function $\tilde{f}$ which is analytic in the open set*

$$\tilde{X} = \{(z, w) \in (\mathcal{D} \times \mathcal{G}) \,:\, \tilde{\omega}(\mathcal{D}, E, z) + \omega(\mathcal{G}, F, w) < 1\}$$

and agrees with f on $X \cap \tilde{X}$. The set $\tilde{X}$ is a neighbourhood of $\mathcal{D} \times F$ and if, moreover E satisfies the condition in Theorem 5.1.1 then $\tilde{X}$ is also a neighbourhood of X.

Proof. By Lemma 4.3.2 the hypothesis on E guarantees the existence of an open set B with $\hat{E}_{\mathcal{D}} \subset B \Subset \mathcal{D}$ such that $\omega(B, E, z) \not\equiv 1$ in every connected component of B.

Set $K = \hat{E}_{\mathcal{D}}$, $L = \hat{F}_{\mathcal{G}}$ and choose two sequences of strongly pluriregular open sets with smooth boundaries $\{\mathcal{D}^{(s)}\}$ and $\{\mathcal{G}^{(s)}\}$ such that

$$K \subset B \Subset \mathcal{D}^{(1)} \Subset \ldots \Subset \mathcal{D}^{(s)} \Subset \ldots \Subset \mathcal{D}, \quad \mathcal{D} = \cup_{s=1}^{\infty} \mathcal{D}^{(s)}, \tag{5.1}$$

$$L \subset \mathcal{G}^{(1)} \Subset \ldots \Subset \mathcal{G}^{(s)} \Subset \ldots \Subset \mathcal{G}, \quad \mathcal{G} = \cup_{s=1}^{\infty} \mathcal{G}^{(s)}, \tag{5.2}$$

and also such that $\mathcal{D}^{(s)}$ and $\mathcal{G}^{(s)}$ have no connected components not intersecting E and F, respectively. Then by construction, for any s we have

$$\omega(\mathcal{D}^{(s)}, E, z) \not\equiv 1 \tag{5.3}$$

in each connected component of $\mathcal{D}^{(s)}$.

Now fix $s \geq 2$ and consider the Hilbert space G_1 of all functions $x(w)$ analytic in $\mathcal{G}^{(s)}$ with finite norm

$$\|x\|_{G_1} = \left(\int_{\mathcal{G}^{(s)}} |x(w)|^2 \, d\sigma_w \right)^{1/2}$$

where $d\sigma_w$ is the volume element in $\mathcal{G}^{(s)}$ with a fixed Hermitian metric on Ω ([3]). And similarly construct the Hilbert space H_1 for the domain $\mathcal{D}^{(s)}$. These spaces are related by continuous imbeddings

$$A(\overline{\mathcal{D}^{(s)}}) \hookrightarrow H_1 \hookrightarrow A(\mathcal{D}^{(s)}), \quad A(\overline{\mathcal{G}^{(s)}}) \hookrightarrow G_1 \hookrightarrow A(\mathcal{G}^{(s)}). \tag{5.4}$$

By a result of Pietsch [13], since $A(L)$ is nuclear, there is a Hilbert space G_0 with continuous and dense imbeddings

$$A(L) \hookrightarrow G_0 \hookrightarrow AC(L). \tag{5.5}$$

Then by Lemma 2.2.1 there is a common orthogonal basis $\{g_l\}$ in the spaces G_1 and G_0 satisfying

$$\|g_l\|_{G_0} = 1, \quad \|g_l\|_{G_1} = \nu_l \uparrow \infty \quad l \in \mathbb{N} \tag{5.6}$$

On the other hand, using Lemma 2.2.2 we choose two Hilbert spaces H_0 and $\tilde{H}_0$ with continuous and dense imbeddings

$$H_1 \hookrightarrow A(\mathcal{D}^{(s)}) \hookrightarrow \tilde{H}_0 \hookrightarrow AC(K) \hookrightarrow H_0 \tag{5.7}$$

and such that there exists a common orthogonal basis $\{h_k\}$ for the spaces H_1, H_0 and $\tilde{H}_0$ with

$$\|h_k\|_{H_0} = 1, \quad \|h_k\|_{H_1} = \mu_k \uparrow \infty, \quad \|h_k\|_{\tilde{H}_0} \leq Ck^2 \tag{5.8}$$

where $C < \infty$.

Now, let $\mathcal{G}_\alpha^{(s)} = \{w \in \mathcal{G}^{(s)} : \omega(\mathcal{G}^{(s)}, F, w) < \alpha\}$ for $0 < \alpha < 1$, be the sublevel open sets. The continuity of the imbedding $G_1 \hookrightarrow A(\mathcal{G}^{(s)})$ implies that for any q, $0 < q < 1$, there exists a constant $C_q > 0$ such that for any $x \in G_1 \hookrightarrow A(\mathcal{G}^{(s)})$,

$$|x|_{\mathcal{G}_q^{(s)}} \leq C_q \, \|x\|_{G_1}.$$

In particular,

$$|g_l|_{\mathcal{G}_q^{(s)}} \leq C_q \, \|g_l\|_{G_1} = C_q \, \nu_l \tag{5.9}$$

since $\|g_l\|_{G_1} = \nu_l$. The continuity of the imbedding $G_0 \hookrightarrow AC(L)$ implies that, for any $x \in G_0 \hookrightarrow AC(L)$,

$$|x|_L \leq C \, \|x\|_{G_0}.$$

In particular,

$$|g_l|_L \leq \|g_l\|_{G_0} = C \tag{5.10}$$

since $\|g_l\|_{G_0} = 1$. For any $\alpha < 1$ and $\varepsilon > 0$, we choose q with $\beta := \alpha/q < \alpha + \varepsilon$. Then using Proposition 4.2.3 for g_l and the relations (5.9), (5.10) we get,

$$|g_l|_{\mathcal{G}_{q\beta}^{(s)}} \leq (|g_l|_L)^{1-\beta} \, (|g_l|_{\mathcal{G}_q^{(s)}})^\beta \leq C^{1-\beta} \, (C_q \, \nu_l)^\beta = C^{1-\beta} \, C_q^\beta \, \nu_l^\beta$$

Therefore

$$|g_l|_{\mathcal{G}_\alpha^{(s)}} \le N\nu_l^{\alpha+\varepsilon}, \quad \textit{for all} \quad l \in \mathbb{N} \tag{5.11}$$

holds with a constant $N = N(\alpha, \varepsilon)$. In a similar way we can obtain the following estimate for the common orthogonal basis $\{h_k\}$,

$$|h_k|_{\mathcal{D}_\alpha^{(s-1)}} \le C'\, k^2\, (\mu_k)^\alpha, \quad \textit{for all} \quad k \in \mathbb{N}, \tag{5.12}$$

where $\mathcal{D}_\alpha^{(s)} = \{z \in \mathcal{D}^{(s)} \,:\, \omega(\mathcal{D}^{(s)}, E, z) < \alpha\}$ for $0 < \alpha < 1$.

It is known (see, e.g. [21]) that the asymptotic estimate

$$\ln \mu_k \asymp k^{1/n}, \quad \ln \nu_k = k^{1/m}, \quad k \to \infty, \tag{5.13}$$

where n and m are the dimensions of the spaces $\mathcal{D}$ and $\mathcal{G}$.

Now fix $z \in E$. By the hypothesis of separate analyticity, the function $\varphi_z(w) := f(z,w)$ belongs to the space $A(\mathcal{G}) \subset A(\mathcal{G}^{(s)}) \subset G_1$. Therefore it has an expansion in terms of the basis $\{g_l\}$:

$$f(z,w) = \sum_{l=1}^{\infty} b_l(z) g_l(w), \tag{5.14}$$

converging in the norm in G_1. By the assumption of local boundedness of f on $E \times \mathcal{G}$ together with the orthogonality of the system, we derive the estimate for the coefficients $b_l(z)$:

$$|b_l(z)| = \frac{1}{(\nu_l)^2} \left| \int_{\mathcal{G}^{(s)}} f(z,w)\, \overline{g_l(w)}\, d\sigma_w \right| \le \frac{C_f}{\nu_l}, \quad z \in E,\, l \in \mathbb{N} \tag{5.15}$$

The estimates (5.11), (5.15) and (5.13) imply that the series (5.14) converges uniformly on the set $E \times \mathcal{G}_\alpha^{(s-1)}$ for $0 < \alpha < 1$.

For reasons of symmetry there is an expansion

$$f(z,w) = \sum_{k=1}^{\infty} a_k(w) h_k(z), \quad w \in F \tag{5.16}$$

but since we are not assuming the local boundedness of f on $\mathcal{D} \times F$ the bound on the coefficients may not be uniform in w. Still, we have the estimate, by Hölder's Inequality:

$$|a_k(w)| = \frac{1}{(\mu_k)^2} \left| \int_{\mathcal{D}^{(s)}} f(z,w)\, \overline{h_k(z)}\, d\sigma_z \right| \le \frac{M(w)}{\mu_k}, \quad w \in F,\, k \in \mathbb{N}. \tag{5.17}$$

where $M(w) < \infty$.

Below we show that this bound can be replaced by a uniform one.

Consider now a set of functionals $\{h_k^*\} \subset H_0^*$, which constitutes the biorthogonal system to the system h_k, by the continuity of the imbedding $H_0^* \hookrightarrow AC(K)^*$ there exists a constant B such that

$$\|h_k^*\|_{AC(K)^*} \leq B\,\|h_k^*\|_{H_0^*} = B < \infty \tag{5.18}$$

where $\|h_k^*\|_{H_0^*} = 1$ by (5.8). By Hahn-Banach Theorem, each of the functionals h_k^* extends with the same norm, from $AC(K)$ to a linear functional $\tilde{h}_k^* \in B(E)^*$. Now, applying these functionals to the series (5.14) for any fixed $w \in \mathcal{G}^{(s-1)}$, we obtain an expansion

$$a_k(w) = h_k^*(f) = \sum_{l=1}^{\infty} h_k^*(b_l(z))g_l(w) = \sum_{l=1}^{\infty} a_{kl}g_l(w), \quad w \in \mathcal{G}^{(s-1)},\ k \in \mathbf{N}. \tag{5.19}$$

By (5.15) and (5.18) we see that

$$|a_{kl}| = |h_k^*(b_l)| \leq \|\tilde{h}_k^*\|_{B(E)^*}\|b_l\|_{B(E)} \leq \frac{C'_f}{\nu_l} \tag{5.20}$$

It follows from (5.20) and (5.11) that the series (5.19) converges uniformly inside the open set $\mathcal{G}^{(s)}$. Thus each function $a_k(w)$ is also analytic in $\mathcal{G}^{(s)}$, and since $\mathcal{G}^{(s-1)} \Subset \mathcal{G}_\alpha^{(s)}$ for some α with $0 < \alpha < 1$, we also have that

$$|a_k(w)| \leq C''(f) < \infty, \quad w \in \mathcal{G}^{(s-1)}. \tag{5.21}$$

Consider the sequence of functions

$$\psi_k = \frac{\ln|a_k(w)|}{\ln(\mu_k)}.$$

Since for each k the functions $a_k(w)$ are analytic in $\mathcal{G}^{(s-1)}$, ψ_k are plurisubharmonic functions in $\mathcal{G}^{(s-1)}$. Moreover, by (5.13) and (5.21)this sequence converges uniformly in $\mathcal{G}^{(s-1)}$, and on the other hand,

$$\limsup_{k\to\infty} \psi_k(w) \leq -1, \quad w \in F.$$

Then using Hartogs Lemma on sequences of plurisubharmonic functions, for any $\varepsilon > 0$ there exists a number $k_0(\varepsilon)$ such that

$$\psi_k(w) < -1 + \varepsilon, \quad k \geq k_0(\varepsilon), \quad w \in F$$

This shows the existence of a constant $C = C(\varepsilon)$ such that

$$|a_k(w)| \leq \frac{C(\varepsilon)}{\mu_k^{1-\varepsilon}}, \quad w \in F, \quad k \in \mathbb{N}. \tag{5.22}$$

Applying Proposition 4.2.3 with (5.21) and (5.22) we get,

$$|a_k(w)|_{\mathcal{G}_\alpha^{(s-1)}} \leq \frac{C(f,\varepsilon)}{\mu_k^{\alpha-\varepsilon}}, \; 0 < \alpha < 1, \; k \in \mathbb{N}. \tag{5.23}$$

Consequently, we obtain a uniform estimate on the coefficients of the series (5.16). Combining this result together with (5.12) and (5.13) we conclude that the series (5.16) converges uniformly inside the domain $\tilde{X}_s = \cup_{\alpha \in (0,1)} (\mathcal{D}_\alpha^{(s)} \times \mathcal{G}_{1-\alpha}^{(s-1)})$. Thus its sum defines a function $\varphi_s(z,w)$ analytic in $\tilde{X}_s$.

Furthermore, $\varphi_s(z,w) = f(z,w)$ for all $(z,w) \in (\mathcal{D}^{(s)} \times F) \cup (E \times \mathcal{G}^{(s-1)})$. Indeed, if $(z,w) \in (\mathcal{D}^{(s)} \times F)$ then from the representation of $f(z,w)$ as the series (5.16) it follows. On the other hand, if $(z,w) \in (E \times \mathcal{G}^{(s-1)})$ by the absolute and uniform convergence of the series we obtain,

$$\varphi_s(z,w) = \sum_{k=1}^{\infty} a_k(w)\, h_k(z) = \sum_k \big(\sum_l a_{kl}\, g_l(w) \big)\, h_k(z)$$

$$= \sum_l \big(\sum_k h_k^*(b_l(z))\, h_k(z) \big) g_l(w) = \sum_{l=1}^{\infty} b_l(z)\, g_l(w) = f(z,w), \quad (z,w) \in E \times \mathcal{G}^{s-1}$$

which gives the desired equality.

For any fixed $w \in \mathcal{G}_\alpha^{(s-1)}$ since E is a set of uniqueness for the class $A(\mathcal{D})$ by Proposition 4.3.1, the equality

$$f(z,w) = \varphi_s(z,w) = \varphi_{s+1}(z,w), \quad z \in E,$$

implies $\varphi_s(z,w) = \varphi_{s+1}(z,w)$ in $\mathcal{D}_{1-\alpha}^{(s)}$. Therefore,

$$\varphi_s(z,w) = \varphi_{s+1}(z,w) \quad (z,w) \in \tilde{X}_s, \quad s \in \mathbb{N}$$

Since, $\tilde{X} = \cup_{s=1}^{\infty} \tilde{X}_s$ by the definition of $\tilde{\omega}$, there is an analytic function $\varphi(z,w)$ in $\tilde{X}$ agreeing with $f(z,w)$ on $X \cap \tilde{X}$.

By the pluriregularity of the set $\hat{F}_{\mathcal{G}}$ we have the inclusions $F \subset \mathcal{G}_\alpha^{(s)}$ for $0 < \alpha < 1$, $s \in \mathbb{N}$. Hence,

$$\mathcal{D}_\alpha^{(s)} \times F \subset \mathcal{D}_\alpha^{(s)} \times \mathcal{G}_{1-\alpha}^{(s-1)}, \quad 0 < \alpha < 1, \quad s \in \mathbb{N}. \tag{5.24}$$

Furthermore, by the observation (5.3), $\mathcal{D}^{(s)} = \cup_{\alpha \in (0,1)} \mathcal{D}_\alpha^{(s)}$, which together with (5.24) implies that,

$$\mathcal{D} \times F \subset \cup_{s=1}^{\infty} \cup_{\alpha \in (0,1)} \mathcal{D}_\alpha^{(s)} \times F \subset \cup_{s=1}^{\infty} \cup_{\alpha \in (0,1)} \mathcal{D}_\alpha^{(s)} \times \mathcal{G}_{1-\alpha}^{(s-1)} = \tilde{X}.$$

If in addition, the compact set E satisfies the hypothesis of the theorem, i.e. if $\hat{E}_{\mathcal{D}}$ is pluriregular, then E and F play symmetric roles, thus $E \times \mathcal{G} \subset \tilde{X}$, and hence $X \subset \tilde{X}$. $\square$

Now, with the lemma below we show that the assumption about boundedness of f can be removed.

Lemma 5.1.1. *Let $\mathcal{D}$ and $\mathcal{G}$ be Stein manifolds and $E \subset \mathcal{D}$, $F \subset \mathcal{G}$ compact sets with the property that E is not pluripolar in each connected component of $\mathcal{D}$ and $\hat{F}_{\mathcal{G}}$ is pluriregular in $\mathcal{G}$. Let $f(z,w)$ be a function separately analytic on the set $X = (E \times \mathcal{G}) \cup (\mathcal{D} \times F)$. Then there exists a function $\varphi(z,w)$ analytic in an open neighbourhood of the set $\mathcal{D} \times F$ and agreeing with $f(z,w)$ on $\mathcal{D} \times F$. In particular, $f(z,w)$ is locally bounded in $\mathcal{D} \times F$.*

Proof. Let $\mathcal{G}_1$ be any open pseudoconvex neighbourhood of the compact set F with $\mathcal{G}_1 \Subset \mathcal{G}$ and such that every connected component of G_1 intersects F. For each $m \in \mathbb{N}$ we introduce the set

$$E_m = \left\{ z \in E \, : \, \sup_{w \in \mathcal{G}_1} |f(z,w)| \leq m \right\}$$

Since by hypothesis the function f is analytic in $\mathcal{G}$ for any fixed $z \in E$, we have $\sup \left\{ |f(z,w)| : w \in \mathcal{G}_1 \right\} \leq C < \infty$ and so $E = \cup_1^\infty E_m$, $E_m \subset E_{m+1}$ for $m = 1, 2, \ldots$

By Lemma 4.3.1 there exists an $m \in \mathbb{N}$ such that the compact set E_m is not pluripolar in $\mathcal{D}$. By the construction of E_m the function f is locally bounded on $E_m \times \mathcal{G}_1$. Thus all of the hypothesis of Theorem 5.1.2 are satisfied if we put E_m and $\mathcal{G}_1$ in place of E and $\mathcal{G}$, respectively. Therefore there exists a function $\varphi(z,w)$ on the set

$$Y = \{(z,w) \in \mathcal{D} \times \mathcal{G}_1 \,:\, \tilde{\omega}(\mathcal{D}, E_m, z) + \omega(\mathcal{G}_1, F, w) < 1\},$$

which is an open neighbourhood of $\mathcal{D} \times F$ with $\varphi(z,w) \equiv f(z,w)$ on $\mathcal{D} \times F$. In particular, the local boundedness of $f(z,w)$ on $\mathcal{D} \times F$ has been proved. □

Proof. (*of Theorem* 5.1.1) Observe that by the pluriregularity of the compact set $\hat{E}_{\mathcal{D}}$, the set E is not pluripolar in every connected component of $\mathcal{D}$. Interchanging the roles of $\mathcal{D}$ and E with $\mathcal{G}$ and F, Lemma 5.1.1 implies that f is locally bounded on $E \times \mathcal{G}$. Then by applying Theorem 5.1.2 we complete the proof since $\tilde{\omega}(\mathcal{D}, E, z) = \omega(\mathcal{D}, E, z)$ by the hypothesis and $\tilde{X}$ is an open neighbourhood of X. □

Below, as a corollary we give the solution to Problem 1.

Corollary 5.1.1. *Let $\mathcal{D}$ and $\mathcal{G}$ be Stein manifolds. Let $E \subset \mathcal{D}$ be a compact set which is not pluripolar in every connected component of $\mathcal{D}$. Let the function $f(z,w)$ be defined on $\mathcal{D} \times \mathcal{G}$, analytic in z in $\mathcal{D}$ for every fixed $w \in \mathcal{G}$, analytic in w in $\mathcal{G}$ for every fixed $z \in E$. Then $f(z,w)$ is analytic in $\mathcal{D} \times \mathcal{G}$.*

Proof. Without loss of generality we may assume that $\mathcal{G}$ is connected. Choose a sequence of domains $\{\mathcal{G}_s\}$ with $\mathcal{G}_s \Subset \mathcal{G}_{s+1}$ and $\mathcal{G} = \cup_1^\infty \mathcal{G}_s$ such that $F_s = \bar{\mathcal{G}}_s$ are pluriregular compact sets. Applying Lemma 5.1.1, we see that $f(z,w)$ extends analytically to a neighbourhood of the set $\mathcal{D} \times F_s$ and is analytic in $\mathcal{D} \times \mathcal{G}_s$ in the variables (z,w). Since s is arbitrary it follows that $f(z,w)$ is analytic in $\mathcal{D} \times \mathcal{G}$. □

It has been proved in [20] and [19] that the hypothesis on E is essential. If the hypothesis is not satisfied, there exists a function $f(z,w)$ satisfying the hypothesis of the corollary which is not analytic in $\mathcal{D} \times \mathcal{G}$.

Definition 5.1.2. *A Stein manifold $\mathcal{D}$ is called pluricopolar if $\omega(\mathcal{D}, K, z) \equiv 0$ in $\mathcal{D}$ for any compact set $K \subset \mathcal{D}$.*

Corollary 5.1.2. *If in the hypothesis of Theorem 5.1.1 at least one of the manifolds $\mathcal{D}$ or $\mathcal{G}$ is pluricopolar, then $f(z,w)$ extends to an analytic function $\varphi(z,w)$ in $\mathcal{D} \times \mathcal{G}$.*

Proof. Without loss of generality assume that $\mathcal{D}$ is copolar, i.e. $\omega(\mathcal{D}, E, z) \equiv 0$ for all $z \in \mathcal{D}$. Then,

$$\tilde{X} = \big\{(z,w) \in \mathcal{D} \times \mathcal{G} \,:\, \omega(\mathcal{G}, F, w) < 1\big\} \supset \mathcal{D} \times \mathcal{G}.$$

Therefore, $\tilde{X} = \mathcal{D} \times \mathcal{G}$. □

Theorem 5.1.3. *Let $\mathcal{D}$ and $\mathcal{G}$ be Stein manifolds at least one of which is pluricopolar. Let E be a compact set in $\mathcal{D}$ which is not pluripolar in any connected component of $\mathcal{D}$ and F be a similar set in $\mathcal{G}$. Then any separately analytic function f on $X = (\mathcal{D} \times F) \cup (E \times \mathcal{G})$ extends uniquely to a function analytic in $\mathcal{D} \times \mathcal{G}$.*

For the proof of the theorem we give a stronger version of Theorem 5.1.2 which has an analogous proof.

Lemma 5.1.2. *Let $\mathcal{D}$ and $\mathcal{G}$ be Stein manifolds, and assume that $E \subset \mathcal{D}$ and $F \subset \mathcal{G}$ are compact sets satisfying the hypothesis of Theorem 5.1.3. Let $f(z,w)$ be a separately analytic function on $X = (\mathcal{D} \times F) \cup (E \times \mathcal{G})$ which is locally bounded in $E \times \mathcal{G}$. Let*

$$\tilde{X} = \big\{(z,w) \in \mathcal{D} \times \mathcal{G} \,:\, \tilde{\omega}(\mathcal{D}, E, z) + \tilde{\omega}(\mathcal{G}, F, w) < 1\big\}.$$

Then there exists a unique function $\varphi \in A(\tilde{X})$ which agrees with f on $X \cap \tilde{X}$.

Proof. (*of Theorem* 5.1.3) Without loss of generality assume that $\mathcal{D}$ is pluricopolar. Let $\mathcal{G}_s$ be a sequence of open sets as in (5.2). For fixed s, by a similar construction as in the proof of Lemma 5.1.1 there exists a compact set $E' \subset E$ which is not pluripolar in every connected component of the set $\mathcal{G}_s$ and such that f is locally bounded on $E' \times \mathcal{G}_s$. Using Lemma 5.1.2 we obtain a function $\varphi_s \in A(\tilde{X}_s)$ which agrees with f on $X \cap \tilde{X}_s$ where

$$\tilde{X}_s = \left\{ (z,w) \in \mathcal{D} \times \mathcal{G}_s \,:\, \tilde{\omega}(\mathcal{D}, E', z) + \tilde{\omega}(\mathcal{G}, F, w) < 1 \right\}.$$

Since $\mathcal{D}$ is pluricopolar, we have $\tilde{\omega}(\mathcal{D}, E', z) \equiv 0$; on the other hand, $\tilde{\omega}(\mathcal{G}_s, F, w) \not\equiv 1$ on any connected component of $\mathcal{G}_s$. Therefore, $\tilde{X}_s = \mathcal{D} \times \mathcal{G}_s$. Finally, the function $\varphi \in A(\mathcal{D} \times \mathcal{G})$ defined by $\varphi(z,w) = \varphi_s(z,w)$ for $(z,w) \in \tilde{X}_s$ is the desired function which completes the proof of the theorem. $\square$

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