ON INCREASING (NON) STATIONARY LEVY PROCESSES: AN EASY APPROACH USING RENEWAL PROCESSES.

by

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to my family

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ON INCREASING (NON) STATIONARY LEVY PROCESSES: AN EASY APPROACH USING RENEWAL PROCESSES.

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Abstract

Increasing (non)stationary Levy processes are widely used in Operations Research and Engineering. The main areas of applications of these stochastic processes are insurance mathematics, inventory control and maintenance. Special and well known instances of these processes are (non) stationary Poisson and compound Poisson processes. Since in textbooks (increasing) Levy processes are mostly regarded as special instances of continuous time martingales the main properties of Levy processes are derived by applying general results available for martingales. However, understanding the theory of martingales requires a deep insight into the theory of stochastic processes and so it might be difficult to understand the proofs of the main properties of increasing Levy processes. Therefore the main purpose of this study is to relate increasing Levy processes to simpler stochastic processes and give simpler proofs of the main properties. Fortunately there is a natural way linking increasing Levy processes to random processes occurring within renewal theory. Using this (sample path) approach and applying properties of random processes occurring within renewal theory we are able to analyze the undershoot and overshoot random process of an increasing Levy process. By a similar approach the (asymptotic) properties of the hitting time at level r can also be derived. Next to well known results we also derive new results in this thesis. In particular we extend Lorden's inequality for the renewal function and the residual life process to both the expected hitting time and the expected overshoot of an increasing Levy process at level r.

ARTAN VE DURAĞAN LEVY SÜREÇLERİ: YENİLEME TEORİSİ İLE YAKLAŞIM

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Özet

Artan durağan (olmayan) Levy süreçleri, Yöneylem Araştırması ve Mühendisliklerde sıkça kullanılır. Bu stokastik süreçlerinin temel uygulama alanlarından bazıları sigorta matematiği, envanter kontrolü ve bakımdır. Durağan (olmayan) Poisson ve bileşik Poisson süreçleri özel ve oldukça tanınan örnekleridir. Kitaplarda, (artan) Levy süreçleri, sürekli zamanlı Martingale'lerin özel bir kolu olarak görüldüğü için Levy süreçlerinin temel özellikleri, genel Martingale bulguları uygulanarak elde edilmiştir. Ancak Martingale teorisini anlamak için Stokastik süreçler teorisi üzerine derin bir bilgiye sahip olmak gerekmektedir ve dolayısıyla artan Levy süreçlerinin temel özelliklerinin kanıtlarını anlamak oldukça zor olabilir. Bu nedenle bu çalışmanın temel amacı artan Levy süreçlerini daha basit Stokastik süreçlere benzetmek ve temel özellikleri için daha basit kanıtlar sağlamaktır. Neyseki Yenileme süreçleri sayesinde, Artan Levy süreçlerini Stokastik süreçlere bağlayan doğal bir yol vardır. Bu yolu kullanarak ve Yenileme süreçlerinin özelliklerini uygulayarak, artan Levy süreçlerinin eksik kalan (undershoot) ve aşma (overshoot) rastgele sürecini analiz etmeyi başardık. Benzer bir yöntemle vurma zamanının r seviyesindeki asimtotik özelliklerini de çıkarabildik. Bu tezde bilinen sonuçlara ek olarak yeni sonuçlar da bulduk. Yenileme süreçleri ve Artık Yaşam süreçleri için olan Lorden eşitliğini hem artan Levy süreçlerinin r seviyesindeki vurma zamanı hemde aşma süreci için genişlettik.

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CHAPTER 1

INTRODUCTION AND MOTIVATION

The theory of stochastic processes with stationary and independent increments (nowadays called Levy processes) are a branch of modern probability theory and cover a large class of well known stochastic processes such as Poisson processes, compound Poisson processes and Brownian motion. Nowadays Levy processes serve as a modeling tool in areas such as mathematical finance, risk estimations, optimal stopping problems, inventory theory and maintenance. These processes are named after the French mathematician Paul Lévy who played a crucial role in developing the theory of these processes. In this master thesis we only derive by means of a new technique the main properties of increasing Levy processes which are processes with stationary and nonnegative stationary increments. Although a subclass of Levy processes these processes are also important since they serve as building blocks in inventory control and maintenance. Most of the contributions of the theory of Levy processes was made between 1930 to 1940s by Paul Lévy (France), Alexander Khintchine (Russia), Kiyosi Ito (Japan) and Bruno de Finett (Italy). Despite its importance nowadays there are only a few recently published books on this topic. The main references are Bertoin (cf. [1]), Kyprianou (cf. [11]) and Sato (cf. [18]). Since these books cover the general theory of Levy processes with real increments and this theory is strongly related to the theory of continuous time martingale understanding these books require a deep knowledge of martingale theory and are therefore difficult to read.

1.1 Contribution of the thesis.

The primary purpose of this study is to derive the most important properties of increasing Levy processes by an easier technique. Although most of the results presented in this study are well known properties of increasing Levy processes, it is in general difficult to understand these proofs. As already mentioned most of these properties were proved by using martingale theory. In our study, we use a different approach to verify these properties by approximating a continuous time increasing Levy process by a sequence of renewal processes and making use of well known results for renewal processes. This approach is also mentioned in [19] and [2] trying to justify results for increasing Levy processes by comparing them to the results for renewal processes without using a firm mathematical foundation. Also by using this approach we derive some new asymptotic results.

1.2 Outline of the thesis

The thesis is structured as follows. In Chapter 2 the relevant theory of increasing Levy processes is discussed. We first start in Section 2.1 with some basic definitions and results. Also in this section we introduce the non-stationary version of an increasing Levy process. In Section 2.2 we discuss the overshoot and undershoot random process and the hitting time of level r of a non-stationary and stationary increasing Levy process. This section is divided into four parts: In Subsection 2.2.1 we show the sample path relation of the above random variables for a non-stationary increasing Levy process with the same random variables in its stationary version. In Subsection 2.2.2 the asymptotic behavior of the hitting time of level r is discussed relating it to a renewal function and in Subsection 2.2.3 the cdf and asymptotic behavior of the overshoot and undershoot random variables at level r are derived by means of the same renewal approximation approach. Finally in Subsection 2.2.4 we consider the fractional part of the hitting time at level r and derive some asymptotic results for this fractional random variable by means of renewal theory. We end this thesis with a conclusion listed in Chapter 3.

CHAPTER 2

INCREASING (NON)STATIONARY LEVY PROCESSES

In this study we derive the most important properties of increasing Levy processes. These properties are proved approximating an (increasing) Levy process by a sequence of renewal processes and using well known results for renewal processes.

2.1 Basic definitions and results for increasing Levy processes.

In this section we first introduce some basic definitions.

Definition 1 *A sequence of random variables* $\mathbf{X}_n, n \in \mathbb{N}$, *on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *is converging almost surely to a random variable* $\mathbf{X}$ *on* $(\Omega, \mathcal{F}, \mathbb{P})$ *(notation* $\mathbf{X}_n \stackrel{a.s}{\rightarrow} \mathbf{X}$ *) if*

$$\mathbb{P}(\omega \in \Omega : \lim_{n\uparrow\infty} \mathbf{X}_n(\omega) = \mathbf{X}(\omega)) = 1. \tag{2.1}$$

If the sequence $\mathbf{X}_n$ *is a decreasing sequence of random variables on* $(\Omega, \mathcal{F}, \mathbb{P})$ *(i.e* $\mathbf{X}_n \geq \mathbf{X}_{n+1}$*), then we denote this by* $\mathbf{X}_n \downarrow^{a.s} \mathbf{X}$. *A sequence of random variables* $\mathbf{X}_n, n \in \mathbb{N}$, *on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *is converging in probability to a random variable* $\mathbf{X}$ *on* $(\Omega, \mathcal{F}, \mathbb{P})$ *(notation* $\mathbf{X}_n \stackrel{\mathcal{P}}{\rightarrow} \mathbf{X}$*) if*

$$\lim_{n\uparrow\infty} \mathbb{P}(\omega \in \Omega : \mid \mathbf{X}_n(\omega) - \mathbf{X}(\omega) \mid > \epsilon) = 0 \tag{2.2}$$

for every $\epsilon > 0$. *A stochastic process* $\mathbf{X} = \{\mathbf{X}(t) : t \geq 0\}$ *is called continuous in probability if for every* $s \geq 0$ *and every sequence* $t_n, n \in \mathbb{N}$ *converging to* s *it follows that*

$$\mathbf{X}(t_n) \stackrel{\mathcal{P}}{\rightarrow} \mathbf{X}(s). \tag{2.3}$$

We are now able to introduce the definition of an increasing Levy process.

Definition 2 *A stochastic process* $\mathbf{X} = \{\mathbf{X}(t) : t \geq 0\}$ *is called an increasing Levy process if*

1. *The process* $\mathbf{X}$ *is continuous in probability.*

2. $\mathbf{X}(0) = 0$ *and* $\mathbf{X}(t) \geq 0$ *for every* $t > 0$.

3. *The process* $\mathbf{X}$ *has independent and stationary increments.*

It is shown in Theorem 14.20 of [4] (see also Theorem 30 of [13]) that there exists a unique modification of the above process $\mathbf{X}$ having right continuous sample paths with left-hand limits. A stochastic process with sample paths having these properties is called a càdlàg process. In the remainder of this thesis we will use this modification and additionally assume that

$$\mu_1 := \mathbb{E}(\mathbf{X}(1)) \tag{2.4}$$

is finite and positive. To start with our analysis we first derive an elementary result satisfied by an increasing Levy process. Observe $\sigma^2(\mathbf{Z})$ denotes the variance of a random variable $\mathbf{Z}$ and

$$\mathbf{Z}_1 \stackrel{d}{=} \mathbf{Z}_2$$

means that the random variables $\mathbf{Z}_1$ and $\mathbf{Z}_2$ having the same cdf. Also $\mathbb{R}_+ := [0, \infty)$ and a function is called continuous on $\mathbb{R}_+$ if it is continuous on $(0, \infty)$ and right continuous in 0.

Lemma 1 *If the stochastic process* $\mathbf{X}$ *is an increasing Levy process satisfying* $\mu_1 := \mathbb{E}(\mathbf{X}(1))$ *is finite and positive, then*

$$\mathbb{E}(\mathbf{X}(t)) = \mu_1 t \tag{2.5}$$

for every $t \geq 0$. *If additionally the second moment* $\mu_2 := \mathbb{E}(\mathbf{X}(1)^2)$ *is finite, then*

$$\sigma^2(\mathbf{X}(t)) = t\sigma^2(\mathbf{X}(1)). \tag{2.6}$$

Proof. By the independent and stationary increments of an increasing Levy process it follows that

$$\mathbf{X}(t+s) \stackrel{d}{=} \mathbf{X}(t) + \mathbf{X}(s) \tag{2.7}$$

for every $s, t > 0$ and so

$$\mathbb{E}(\mathbf{X}(t+s)) = \mathbb{E}(\mathbf{X}(s)) + \mathbb{E}(\mathbf{X}(t)). \tag{2.8}$$

Since the sample paths of the process $\mathbf{X}$ are increasing and $\mathbf{X}(0) = 0$ we obtain $0 \leq \mathbf{X}(t) \leq \mathbf{X}(t+h)$ for every $t, h \geq 0$. This implies using relation (2.4) and (2.8) and μ_1 finite that the function $t \mapsto \mathbb{E}(\mathbf{X}(t))$ is finite for every $t \geq 0$. We will now show that this function is also continuous on $\mathbb{R}_+$. By the definition of an increasing Levy process we know that

$$\mathbf{X}(t) \xrightarrow{\mathcal{P}} \mathbf{X}(s)(t \to s)$$

for $s > 0$ and

$$\mathbf{X}(t) \xrightarrow{\mathcal{P}} \mathbf{X}(0)(t \downarrow 0).$$

Since $0 \leq \mathbf{X}(t) \leq \mathbf{X}(t+h)$ for every $t, h \geq 0$ and $\mathbb{E}(\mathbf{X}(t)$ is finite for every $t > 0$ the conditions of the dominated convergence in probability theorem hold (see Theorem 1.3.6 of [15]) or the Appendix) and we may conclude

$$\lim_{t \to s} \mathbb{E}(\mathbf{X}(t)) = \mathbb{E}(\mathbf{X}(s))$$

for $s > 0$ and

$$\lim_{t \downarrow 0} \mathbb{E}(\mathbf{X}(t)) = \mathbb{E}(\mathbf{X}(0)) = 0.$$

This shows the continuity of the function $t \mapsto \mathbb{E}(\mathbf{X}(t))$ on $\mathbb{R}_+$. The continuity of this function in combination with a standard approximation argument applied to the so-called Cauchy functional equation in relation (2.8) (see Theorem 1.4 of [6]) finally yields

$$\mathbb{E}(\mathbf{X}(t)) = \mu_1 t \tag{2.9}$$

for every $t \geq 0$ and so relation (2.5) is verified. To verify relation (2.6) we observe by the stationary and independent increments of an increasing Levy process that

$$\sigma^2(\mathbf{X}(t+s)) = \sigma^2(\mathbf{X}(t) + \mathbf{X}(s)) = \sigma^2(\mathbf{X}(t)) + \sigma^2(\mathbf{X}(s)). \tag{2.10}$$

Since $\mathbf{X}$ is continuous in probability it follows for every continuous function f on $\mathbb{R}_+$ (cf. [3]) that

$$f(\mathbf{X}(t)) \xrightarrow{\mathcal{P}} f(\mathbf{X}(s))(t \to s)$$

for $s > 0$ and

$$f(\mathbf{X}(t)) \xrightarrow{\mathcal{P}} f(\mathbf{X}(0))(t \downarrow 0).$$

This shows for $s > 0$

$$\mathbf{X}(t)^2 \xrightarrow{\mathcal{P}} \mathbf{X}(s)^2(t \to s) \tag{2.11}$$

and

$$\mathbf{X}(t)^2 \xrightarrow{\mathcal{P}} \mathbf{X}(0)^2 = 0 \ (t \downarrow 0).$$

By a similar argument as used before using μ_1 and μ_2 finite and hence $\sigma^2(\mathbf{X}(1))$ is finite we know that the function $t \mapsto \sigma^2(\mathbf{X}(t))$ is finite. Since

$$\mathbb{E}(\mathbf{X}(t)^2) = \sigma^2(\mathbf{X}(t)) + \mu_1^2 t^2 \tag{2.12}$$

it also follows that the function $t \mapsto \mathbb{E}(\mathbf{X}(t)^2)$ is finite. Now again by the monotonicity of the sample paths $t \mapsto \mathbf{X}(t)^2$ and relation (2.11) the conditions of the dominated convergence in probability theorem hold. Hence we may conclude that the function $t \mapsto \mathbb{E}(\mathbf{X}(t)^2)$ is continuous on $\mathbb{R}_+$ and so by relation (2.12) the function $t \mapsto \sigma^2(\mathbf{X}(t))$ is continuous on $\mathbb{R}_+$. Again applying Theorem 1.4 of [6] yields relation (2.6). □

Since for any increasing Levy process $\mathbf{X}$ with μ_1 positive and finite the process $\mathbf{Y} = \{\mathbf{Y}(t) : t \geq 0\}$ given by

$$\mathbf{Y}(t) = \mu_1^{-1}\mathbf{X}(t)$$

is again an increasing Levy process we may assume without loss of generality that $\mu_1 = 1$. In this thesis we always assume that we are dealing with a (càdlàg) increasing Levy process with $\mu_1 = 1$. To discuss some well known properties of the cdf F_t of the random variable $\mathbf{X}(t)$ we first introduce for completeness the following definition. (cf. [19])

Definition 3 *A random variable $\mathbf{Z}$ is said to be infinitely divisible if for every $n \in \mathbb{N}$ one can find some sequence of independent and identically distributed random variables $\mathbf{Z}_k, 1 \leq k \leq n$ such that*

$$\mathbf{Z} \stackrel{d}{=} \sum\nolimits_{k=1}^{n} \mathbf{Z}_k$$

Clearly the property of infinite divisibility is a property of the cdf of a random variable and so it is also common to call the cdf infinitely divisible. By Definitions 2 and 3 and writing

$$\mathbf{X}(t) = \sum_{k=1}^{n} \mathbf{X}\left(\frac{kt}{n}\right) - \mathbf{X}\left(\frac{(k-1)t}{n}\right)$$

it is clear that for every $t > 0$ the cdf F_t of the random variable $\mathbf{X}(t)$ is infinitely divisible. Introducing the probability Laplace-Stieltjes transform (pLSt) $\pi_t : \mathbb{R}_+ \to \mathbb{R}_+$ of the nonnegative random variable $\mathbf{X}(t)$ given by

$$\pi_t(\alpha) := \mathbb{E}(\exp(-\alpha \mathbf{X}(t))) \tag{2.13}$$

one can show the following well-known result (cf. [19]).

Lemma 2 *It follows for every $\alpha \geq 0$ and $t > 0$ that*

$$\pi_t(\alpha) = \pi_1(\alpha)^t. \tag{2.14}$$

Proof. We will first check that the function

$$t \mapsto \ln \pi_t(\alpha) = \ln(\mathbb{E}(\exp(-\alpha \mathbf{X}(t)))$$

is finite and continuous on $\mathbb{R}_+$. Since the stochastic process $\mathbf{X}$ is continuous in probability and the function $t \mapsto \exp(-\alpha t)$ is continuous it follows that

$$\exp(-\alpha \mathbf{X}(t)) \xrightarrow{\mathcal{P}} \exp(-\alpha \mathbf{X}(s)) \ (t \to s)$$

and

$$\exp(-\alpha \mathbf{X}(t)) \xrightarrow{\mathcal{P}} \exp(-\alpha \mathbf{X}(0)) = 1 \ (t \downarrow 0)$$

Also for every $\alpha \geq 0$

$$0 \leq \exp(-\alpha \mathbf{X}(t)) \leq 1$$

and so the function $t \mapsto \mathbb{E}(\exp(-\alpha \mathbf{X}(t))$ is finite. By the monotonicity of the sample paths of the stochastic process $\mathbf{X}$ we may conclude that

$$0 \leq \exp(-\alpha \mathbf{X}(t+h)) \leq \exp(-\alpha \mathbf{X}(t))$$

for every $t, h \geq 0$ and combining these observations the conditions of the dominated convergence in probability theorem are satisfied. Hence the function $t \mapsto \pi_t(\alpha)$ is continuous on $\mathbb{R}_+$ and using $\mathbf{X}(t)$ is finite with probability one implying $\pi_t(\alpha) > 0$ for every $\alpha \geq 0$ and $t > 0$ also the function $t \mapsto \ln(\pi_t(\alpha))$ is continuous on $\mathbb{R}_+$. Using now the independent and stationary increments of an increasing Levy process $\mathbf{X}$ we obtain for $t, s > 0$ and $\alpha \geq 0$ that

$$
\begin{aligned}
\ln(\pi_{t+s}(\alpha)) &= \ln(\mathbb{E}(\exp(-\alpha\mathbf{X}(t+s)))) \\
&= \ln(\mathbb{E}(\exp(-\alpha(\mathbf{X}(t)+\mathbf{X}(s)))) \\
&= \ln(\mathbb{E}(\exp(-\alpha\mathbf{X}(t)))\mathbb{E}(\exp(-\alpha\mathbf{X}(s)))) \qquad (2.15) \\
&= \ln(\mathbb{E}(\exp(-\alpha\mathbf{X}(t))))+\ln(\mathbb{E}(\exp(-\alpha\mathbf{X}(s)))) \\
&= \ln(\pi_t(\alpha)) + \ln(\pi_s(\alpha))
\end{aligned}
$$

This shows that the function $t \mapsto \ln \pi_t(\alpha)$ is a continuous solution of the Cauchy functional equation in relation (2.15) and by Theorem 1.4 of [6] the desired result follows. □

By investigating in more detail the Laplace-Stieltjes transform $\alpha \mapsto \pi_1(\alpha)$ one can show the following informative representation of the pLST π_t. To prove this result we need the following definition (cf. [19], [5]).

Definition 4 *A function $f : (0, \infty) \to \mathbb{R}$ is called completely monotone if for every $x > 0$ the nth derivative $f^{(n)}(x), n \in \mathbb{Z}_+$ of the function f exists and $(-1)^n f^{(n)}(x) \geq 0$ for every $x > 0$ and $n \in \mathbb{Z}_+$.*

In Definition 4 it is assumed that $(-1)^0 f^{(0)}(x) := f(x)$. In the next result we give a complete characterization of all completely monotone functions. This characterization is called Bernsteins theorem. A proof of this important result can be found in [5] or [20].

Lemma 3 *A real valued function* $f:(0,\infty)\to\mathbb{R}$ *is completely monotone if and only if there exists some right continous function* H *on* $\mathbb{R}$ *satisfying* $H(x)=0$ *for* $x<0$ *satisfying*

$$f(x)=\int_{0-}^{\infty}\exp(-xy)dH(y)$$

for every $x>0$.

It is now possible to prove the following detailed representation for the pLST π_t.

Lemma 4 *For every* $t>0$ *and* $\alpha\geq 0$

$$\pi_t(\alpha)=\exp\left(-t\int_{0-}^{\infty}\frac{1-\exp(-\alpha x)}{x}dK_1(x)\right) \tag{2.16}$$

with $K_1:\mathbb{R}\mapsto\mathbb{R}$ *a right continuous cdf satisfying* $K_1(x)=0$ *for every* $x<0$.

Proof. By Lemma 2 we only need to verify for every $\alpha\geq 0$ that

$$\pi_1(\alpha)=\exp\left(-\int_{0-}^{\infty}\frac{1-\exp(-\alpha x)}{x}dK_1(x)\right).$$

Since π_t is the pLST of the random variable $\mathbf{X}(t)$ it follows that the function $\alpha\mapsto\pi_t(\alpha)$ is completely monotone on $\mathbb{R}_+$. This shows for every $t>0$ that the function

$$\alpha\mapsto-\frac{1}{t}\pi_t'(\alpha)$$

is also completely monotone on $\mathbb{R}_+$ and by Lemma 2

$$\lim_{t\downarrow 0}-\frac{1}{t}\pi_t'(\alpha)=\lim_{t\downarrow 0}-\pi_1^{t-1}(\alpha)\pi_1'(\alpha)=-\frac{\pi_1'(\alpha)}{\pi_1(\alpha)}. \tag{2.17}$$

By the continuity theorem for pLST transforms and Lemma 3 it can be shown that the limit of completely monotone functions is again completely monotone and this implies by relation (2.17) and $-\frac{1}{t}\pi_t'(\alpha)$ is completely monotone that the function $\rho:\mathbb{R}_+\mapsto\mathbb{R}_+$ given by

$$\rho(\alpha)=-\frac{\pi_1'(\alpha)}{\pi_1(\alpha)} \tag{2.18}$$

is completely monotone. To write π_1 as a function of the completely monotone function ρ we observe by relation (2.18) and $\pi_1(0)=1$ that by standard calculus

$$\ln\pi_1(\alpha)=-\int_0^{\alpha}\rho(s)ds.$$

Hence it follows for every $\alpha \geq 0$ and $t > 0$ that

$$\pi_1(\alpha) = \exp\left(-\int_0^\alpha \rho(s)ds\right). \tag{2.19}$$

Applying Lemma 3 to the completely monotone function ρ we may conclude that there exists some right continuous increasing function $K_1 : \mathbb{R} \mapsto \mathbb{R}$ with $K_1(x) = 0$ for every $x < 0$ satisfying

$$\rho(\alpha) = \int_{0-}^\infty \exp(-\alpha x) dK_1(x) < \infty$$

for every $\alpha > 0$. Since $\mu_1 = 1$ and by relation (2.18)

$$\rho(0) = -\frac{\pi_1'(0)}{\pi_1(0)} = \frac{\mu_1}{1} = 1$$

it follows that K_1 is a cdf. Combining this with relation (2.19) yields

$$\pi_1(\alpha) = \exp\left(-\int_0^\alpha \int_{0-}^\infty \exp(-sx) dK_1(x) ds\right). \tag{2.20}$$

To complete our proof observe by Fubini's theorem that

$$\begin{aligned} \textstyle\int_0^\alpha \int_{0-}^\infty \exp(-sx)dK_1(x)ds &= \textstyle\int_{0-}^\infty \int_0^\alpha \exp(-sx)dsdK_1(x) \\ &= \textstyle\int_{0-}^\infty \frac{1-\exp(-\alpha x)}{x} dK_1(x) \end{aligned}$$

and substituting this in relation (2.20) we obtain relation (2.16). □

The function K_1 in the above representation is called the canonical function associated with the pLST π_1. In the remainder it is always assumed without loss of generality that the left extremity $l_{\mathbf{X}(1)}$of the random variable $\mathbf{X}(1)$ given by

$$l_{\mathbf{X}(1)} := \inf\{x \geq 0 : x \text{ is a point of increase of the cdf } F_1\} = 0.$$

It can be shown (cf. [19]) that

$$l_{\mathbf{x}(1)} = \lim_{s \uparrow \infty} \rho(s) = K_1(0).$$

Hence for an increasing Levy process satisfying $\mu_1 = 1$ and $l_{\mathbf{x}(1)} = 0$ we obtain that K_1 is a right continuous cdf satisfying $K_1(0) = 0$. The next result plays a crucial role in showing the correctness of our approximation technique using renewal processes.

Lemma 5 *If the stochastic process* $\mathbf{X}$ *is an increasing Levy process with* $\mu_1 = 1$*, then for every* $x_1, x_2 > 0$

$$\lim_{t \downarrow 0} \frac{F_t(x_1 + x_2) - F_t(x_1)}{t} = \int_{x_1}^{x_1+x_2} u^{-1} dK_1(u) \tag{2.21}$$

with K_1 *a cdf on* $\mathbb{R}_+$*. Moreover,*

$$\lim_{t \downarrow 0} \frac{1 - F_t(x_1)}{t} = \int_{x_1}^{\infty} u^{-1} dK_1(u). \tag{2.22}$$

Proof. By relation (2.19) and Lemma 2 we obtain

$$\begin{aligned} -\int_{0-}^{\infty} x \exp(-\alpha x) dF_t(x) &= \pi_t'(\alpha) \\ &= -t\rho(\alpha) \exp(-t \textstyle\int_0^\alpha \rho(s) ds) \\ &= -t\rho(\alpha)\pi_t(\alpha) \\ &= -t \textstyle\int_{0-}^{\infty} \exp(-\alpha x) dK_1(x) \pi_t(\alpha). \end{aligned} \tag{2.23}$$

with K_1 a cdf on $\mathbb{R}_+$. Introducing for every $t > 0$ the function $L_t : [0, \infty) \to [0, \infty)$ given by

$$L_t(x) := t^{-1} \int_0^x u dF_t(u) \tag{2.24}$$

it follows by relation (2.23) and Laplace inversion that

$$L_t(x) = \int_0^x F_t(x - u) dK_1(u). \tag{2.25}$$

for every $t > 0$. By relation (2.25) the value $L_t(x)$ can be seen as

$$L_t(x) = \mathbb{P}(\mathbf{X}(t) + \mathbf{Z} \leq x) \tag{2.26}$$

with the random variable $\mathbf{X}(t)$ independent of the nonnegative random variable $\mathbf{Z}$ and

$$\mathbb{P}(\mathbf{Z} \leq x) = K_1(x).$$

This shows that the function $x \mapsto L_t(x)$ is a cdf on $\mathbb{R}_+$ for every $t > 0$. Since $\mathbf{X}$ has increasing sample paths it also follows using relation (2.26) that the function $t \mapsto L_t(x)$ is decreasing for every fixed x. Finally, by the continuity in probability and $\mathbf{X}(0) = 0$ we obtain applying again relation (2.26) that

$$\lim_{t \downarrow 0} L_t(x) = \mathbb{P}(\mathbf{Z} \leq x) = K_1(x). \tag{2.27}$$

Hence this shows (as we already know) that the function $x \mapsto K_1(x)$ is a cdf and so

$$\int_x^\infty u^{-1} dK_1(u) < \infty$$

for every $x > 0$. Since for every $x_1, x_2 > 0$ it follows by relation (2.24) that

$$\frac{F_t(x_1 + x_2) - F_t(x_1)}{t} = \int_{x_1}^{x_1+x_2} u^{-1} dL_t(u) \tag{2.28}$$

we obtain by relation (2.27) and the Helly-Bray lemma (cf. [12]) that

$$\lim_{t \downarrow 0} \frac{F_t(x_1 + x_2) - F_t(x_1)}{t} = \lim_{t \downarrow 0} \int_{x_1}^{x_1+x_2} u^{-1} dL_t(u) = \int_{x_1}^{x_1+x_2} u^{-1} dK_1(u).$$

By the above observations and applying the extended Helly-Bray Lemma (cf. [12]) we also obtain that

$$\lim_{t \downarrow 0} \frac{1 - F_t(x_1)}{t} = \int_{x_1}^\infty u^{-1} dK_1(u)$$

and this shows the result. □

To describe non-stationary behavior of an increasing Levy process we introduce a so-called time transformation function.

Definition 5 *A function $\nu : [0, \infty) \mapsto \mathbb{R}_+$ is called a time transformation function if ν is strictly increasing and continuous on $\mathbb{R}_+$ and it satisfies $\nu(0) = 0$ and $\nu(\infty) = \infty$.*

For any time transformation function ν the increasing inverse function $\nu^{\leftarrow} : [0, \infty) \mapsto \mathbb{R}$ is given by (cf. [14])

$$\nu^{\leftarrow}(s) := \inf\{t \geq 0 : \nu(t) \geq s\}. \tag{2.29}$$

Since ν is strictly increasing and continous it follows that $\nu^{\leftarrow}$ is also strictly increasing and continous and it satisfies

$$\nu(\nu^{\leftarrow}(s)) = s \tag{2.30}$$

for every $s \geq 0$. To capture non-stationarity in increasing Levy processes we introduce the next definition.

Definition 6 *A stochastic process $\mathbf{X}_\nu = \{\mathbf{X}_\nu(t) : t \geq 0\}$ is an increasing Levy process with time transformation ν if there exists some increasing Levy process $\mathbf{X}$ with $\mu_1 = 1$ satisfying*

$$\mathbf{X}_\nu(t) = \mathbf{X}(\nu(t))$$

for every $t \geq 0$.

By the definition of an increasing Levy process $\mathbf{X}_\nu$ with continuous time transformation function ν it is easy to see that the stochastic process $\mathbf{X}_\nu$ is càdlàg, has independent, non-stationary and nonnegative increments and is continuous in probability. Also it satisfies $\mathbf{X}_\nu(0) = 0$ and by Lemma 1 we obtain

$$\mathbb{E}(\mathbf{X}_\nu(t)) = \nu(t). \tag{2.31}$$

and

$$\sigma^2(\mathbf{X}_\nu(t)) = \nu(t)\sigma^2(\mathbf{X}(1)) = \nu(t)(\mu_2 - 1). \tag{2.32}$$

In a lot of applications like maintenance and inventory control one is interested in the so-called hitting time $\mathbf{T}_\nu(r)$ at level r of an increasing Levy process with time transformation function ν given by

$$\mathbf{T}_\nu(r) := \inf\{t \geq 0 : \mathbf{X}_\nu(t) > r\}. \tag{2.33}$$

In case we consider the special case of an increasing Levy process this hitting time is denoted by $\mathbf{T}(r)$. Since the sample paths of the (càdlàg) Levy process are increasing and hence the event $\{\mathbf{T}_\nu(r) > t\}$ is the same as $\{\mathbf{X}_\nu(t) \leq r\}$ we immediately obtain that the cdf of $\mathbf{T}_\nu(r)$ is given by

$$\mathbb{P}(\mathbf{T}_\nu(r) > t) = \mathbb{P}(\mathbf{X}(\nu(t)) \leq r). \tag{2.34}$$

Since the process $\mathbf{X}$ is continuous in probability and ν is a continuous function this implies by relation (2.34) that the function

$$t \mapsto \mathbb{P}(\mathbf{T}_\nu(r) > t)$$

is also continuous on $\mathbb{R}_+$ for every $r > 0$. For any increasing Levy process it is also known that it is a jump process (cf. [13]) and so the same holds for an increasing Levy process with time transformation function ν. Hence for a Levy process one can also introduce the so-called overshoot random variable $\mathbf{W}_\nu(r)$ and undershoot random variable $\mathbf{V}_\nu(r)$ at level r. The overshoot stochastic process $\mathbf{W}_\nu = \{\mathbf{W}_\nu(r) : r \geq 0\}$ is given by

$$\mathbf{W}_\nu(r) := \mathbf{X}_\nu(\mathbf{T}_\nu(r)) - r, \tag{2.35}$$

while the undershoot stochastic process $\mathbf{V}_\nu = \{\mathbf{V}_\nu(r) : r \geq 0\}$ is given by

$$\mathbf{V}_\nu(r) := r - \mathbf{X}_\nu(\mathbf{T}_\nu(r)^-). \tag{2.36}$$

The notation $\mathbf{X}_\nu(\mathbf{T}_\nu(r)^-)$ means by definition

$$\mathbf{X}_\nu(\mathbf{T}_\nu(r)^-) := \lim_{h\downarrow 0} \mathbf{X}_\nu(\mathbf{T}_\nu(r) - h). \tag{2.37}$$

For an increasing Levy process we denote the overshoot at level r by $\mathbf{W}(r)$ and the undershoot by $\mathbf{V}(r)$. In the next section we will study in detail the undershoot, overshoot and hitting time variables in an increasing Levy process.

2.2 On properties of the overshoot, undershoot and hitting time

To start this section we first relate in the next subsection by means of an elementary sample path analysis the overshoot, undershoot and hitting time of an increasing Levy process with time transformation function ν to the corresponding random variables describing these processes in an increasing Levy process. At the same time we identify an embedded partial sum process within an increasing Levy process which enables us to use well-known properties of renewal processes in analyzing the overshoot, undershoot and hitting time of an increasing Levy process.

2.2.1 Sample path properties of the overshoot, undershoot and hitting time

To start our discussion of the properties of the undershoot, overshoot and hitting time random variable at level r of an increasing Levy process with time transformation function ν we show in the next lemma that without loss of generality we may restrict ourselves in our analysis to the same random variables in an increasing Levy process.

Lemma 6 *For any Levy process $\mathbf{X}_\nu$ with time transformation function ν and $r > 0$ we have*

$$\mathbf{T}_\nu(r) = \nu^{\leftarrow}(\mathbf{T}(r)) \tag{2.38}$$

and

$$\mathbf{W}_\nu(r) = \mathbf{W}(r), \mathbf{V}_\nu(r) = \mathbf{V}(r). \tag{2.39}$$

Proof. To verify relation (2.38) it follows applying relation (2.30) and $\nu^{\leftarrow}$ is strictly increasing that

$$\begin{aligned}
\mathbf{T}_\nu(r) &= \inf\{t \geq 0 : \mathbf{X}(\nu(t)) > r\} \\
&= \nu^{\leftarrow}(\inf\{\nu(t) \geq 0 : \mathbf{X}(\nu(t)) > r\}) \\
&= \nu^{\leftarrow}(\inf\{t \geq 0 : \mathbf{X}(t) > r\}).
\end{aligned}$$

To show the second equality we observe by the definition of $\mathbf{W}_\nu(r)$ in relation (2.35) and applying relations (2.30) and (2.38) that

$$\begin{aligned}
\mathbf{W}_\nu(r) &= \mathbf{X}_\nu(\nu^{\leftarrow}(\mathbf{T}(r))) - r \\
&= \mathbf{X}(\nu(\nu^{\leftarrow}(\mathbf{T}(r)))) - r \\
&= \mathbf{X}(\mathbf{T}(r)) - r.
\end{aligned}$$

This shows the first part of relation (2.39). To prove the result for the undershoot we observe by the definition of $\mathbf{V}_\nu(r)$ in relation (2.36), again relation (2.30) and (2.38) the strict monotonicity and continuity of ν that

$$\begin{aligned}
\mathbf{V}_\nu(r) &= r - \lim_{h\downarrow 0} \mathbf{X}_\nu(\mathbf{T}_\nu(r) - h) \\
&= r - \lim_{h\downarrow 0} \mathbf{X}_\nu(\nu^{\leftarrow}(\mathbf{T}(r)) - h) \\
&= r - \lim_{h\downarrow 0} \mathbf{X}(\nu(\nu^{\leftarrow}(\mathbf{T}(r)) - h)) \\
&= r - \mathbf{X}(\mathbf{T}(r)^{-})
\end{aligned}$$

and this shows the second part of relation (2.39). □

To analyze the behavior of the random variables $\mathbf{T}(r)$ belonging to an increasing Levy process $\mathbf{X}$ we consider for any $h > 0$ the Levy process $\mathbf{X}$ sampled at the times nh, $n \in \mathbb{Z}_+$. Introduce for this sampled version the hitting time of level $r > 0$ given by

$$\mathbf{T}_h(r) := \inf\{nh : \mathbf{X}(nh) > r\} = h \inf\{n \in \mathbb{Z}_+ : \mathbf{X}(nh) > r\}. \tag{2.40}$$

To relate $\mathbf{T}_h(r)$ to $\mathbf{T}(r)$ it is easy to see using the monotonicity of the sample path of the process $\mathbf{X}$ that

$$\mathbf{T}_h(r) = h\lfloor \mathbf{T}(r)h^{-1}\rfloor + h \tag{2.41}$$

with $\lfloor . \rfloor$ denoting the lower entier function. This shows

$$\mathbf{T}_h(r) - h \leq \mathbf{T}(r) \leq \mathbf{T}_h(r) \tag{2.42}$$

for every $h \geq 0$.

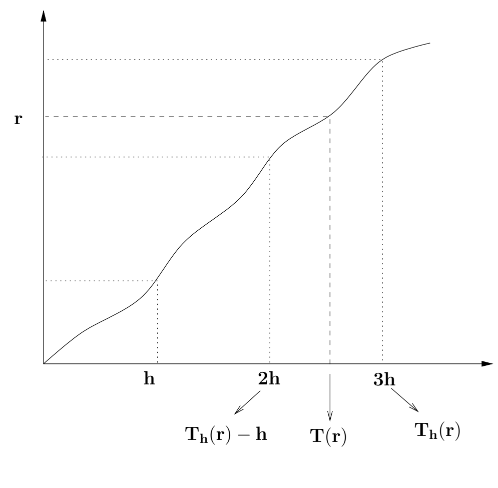

Figure 2.1: Hitting time at level r of an increasing Levy process

Also for ν a time transformation function and hence $\nu^{\leftarrow}$ is increasing it follows by relations (2.38) and (2.42) that

$$\nu^{\leftarrow}(\mathbf{T}_h(r) - h) \leq \mathbf{T}_\nu(r) \leq \nu^{\leftarrow}(\mathbf{T}_h(r)). \tag{2.43}$$

To relate the random variable $\mathbf{T}_h(r)$ to a renewal process we observe the following. Clearly it follows for every $n \in \mathbb{Z}_+$ that

$$\mathbf{X}(nh) = \sum\nolimits_{k=0}^{n} \mathbf{Y}_k(h) \tag{2.44}$$

with

$$\mathbf{Y}_k(h) := \mathbf{X}(kh) - \mathbf{X}((k-1)h) \tag{2.45}$$

for every $k \in \mathbb{N}$ and $\mathbf{Y}_0(h) = 0$. Introducing

$$F_t(x) := \mathbb{P}(\mathbf{X}(t) \leq x) \tag{2.46}$$

and using the definition of an increasing Levy process it follows that the random variables $\mathbf{Y}_k(h), k \in \mathbb{N}$ are independent and identically distributed with cdf F_h. Considering now the renewal process $\mathbf{N}_h := \{\mathbf{N}_h(t) : t \geq 0\}$ generated by the independent and identically distributed random variables $\mathbf{Y}_k(h)$ given by

$$\begin{aligned} \mathbf{N}_h(t) &:= \sup\{n \in \mathbb{Z}_+ : \mathbf{X}(nh) \leq t\} \\ &= \sup\{n \in \mathbb{Z}_+ : \textstyle\sum_{k=0}^{n} \mathbf{Y}_k(h) \leq t\} \end{aligned} \tag{2.47}$$

we obtain by relation (2.40) that

$$\mathbf{T}_h(r) = h(\mathbf{N}_h(r) + 1). \tag{2.48}$$

In the next subsection we will derive some properties of the hitting time $\mathbf{T}_\nu(r)$.

2.2.2 On the behavior of the hitting time $\mathbf{T}_\nu(r)$ as $r \uparrow \infty$.

In this subsection we will derive asymptotic results for the hitting time $\mathbf{T}(r)$ by relating these random variables to the random variables $\mathbf{T}_h(r), h > 0$. By relation (2.40) it follows for every $r > 0$ that the random variables $\mathbf{T}_{2^{-m}}(r), m \in \mathbb{N}$ are decreasing in m

and this implies for every $m \in \mathbb{Z}_+$ that

$$0 \leq \mathbf{T}_{2^{-m}}(r) \leq \mathbf{T}_1(r).$$

Also by relation (2.42) it is clear that

$$\mathbf{T}_{2^{-m}}(r) \overset{a.s}{\downarrow} \mathbf{T}(r)(m \uparrow \infty) \tag{2.49}$$

and hence by Theorem 25.2 of [3]

$$\mathbf{T}_{2^{-m}}(r) \overset{\mathcal{P}}{\to} \mathbf{T}(r)(m \uparrow \infty) \tag{2.50}$$

Finally by relation (2.48) and using $\mathbb{E}(\mathbf{N}_1(r))$ is finite for every r we know that $\mathbb{E}(\mathbf{T}_1(r))$ is finite for every r. Hence the conditions of the dominated convergence in probability theorem hold and we may conclude that

$$\lim_{m \uparrow \infty} \mathbb{E}(\mathbf{T}_{2^{-m}}(r)) \downarrow \mathbb{E}(\mathbf{T}(r)). \tag{2.51}$$

Using the monotonicity of the random variables $\mathbf{T}_{2^{-m}}(r), m \in \mathbb{N}$ we could have also applied directly the dominated convergence theorem of Lebesgue to justify relation (2.51). By a similar argument using ν is strictly increasing and continuous, relation (2.43) and Lemma 6 we obtain that

$$\lim_{m \uparrow \infty} \mathbb{E}(\nu^{\leftarrow}(\mathbf{T}_{2^{-m}}(r)) \downarrow \mathbb{E}(\mathbf{T}_\nu(r)). \tag{2.52}$$

In the next result we first analyze the first order asymptotic behavior of the hitting time $\mathbf{T}(r)$ of an increasing Levy process.

Lemma 7 *For any increasing Levy process* $\mathbf{X}$ *satisfying* $\mu_1 = 1$ *it follows*

$$\lim_{r \uparrow \infty} \frac{\mathbf{T}(r)}{r} \overset{a.s}{=} 1 \tag{2.53}$$

and

$$\lim_{r \uparrow \infty} \frac{\mathbb{E}(\mathbf{T}(r))}{r} = 1. \tag{2.54}$$

Proof. By relation (2.48) and inequality (2.42) applied to $h = 1$ we obtain

$$\mathbf{N}_1(r) \leq \mathbf{T}(r) \leq \mathbf{N}_1(r) + 1. \tag{2.55}$$

Using Kolmogorovs strong law of large numbers it can be easily shown that

$$\lim_{r \uparrow \infty} \frac{\mathbf{N}_1(r)}{r} \overset{a.s}{=} \mathbb{E}(\mathbf{X}(1))^{-1} = 1. \tag{2.56}$$

and this applied to relation (2.55) yields relation (2.53). To prove the second asymptotic expectation result we observe by relation (2.55) that

$$\mathbb{E}(\mathbf{N}_1(r)) \leq \mathbb{E}(\mathbf{T}(r)) \leq \mathbb{E}(\mathbf{N}_1(r) + 1) \tag{2.57}$$

Applying in relation (2.57) the weak renewal theorem (cf. [9]) to the renewal process $\mathbf{N}_1$ and using $\mathbb{E}(\mathbf{X}(1)) = \mu_1 = 1$ the result in relation (2.54) follows. □

The above first order asymptotic results can be easily extended to $\mathbb{E}(\mathbf{T}(r)^p)$ for any $p \in \mathbb{N}$. Observe we need to use this extension to prove first order asymptotic results for increasing Levy processes with a so-called regularly varying time transformation function ν. It can be shown by standard techniques from renewal theory that $\mathbb{E}(\mathbf{N}_1(r)^p)$ is finite for every $r > 0$ and

$$\lim_{r \uparrow \infty} \frac{\mathbb{E}(\mathbf{N}_1(r)^p)}{r^p} = 1. \tag{2.58}$$

(see Exercise 16 and 17 of Chapter 5 of [9]). Since by relation (2.42) we know

$$(\mathbf{T}_1(r) - 1)^p \leq \mathbf{T}(r)^p \leq \mathbf{T}_1(r)^p$$

this shows

$$\mathbb{E}((\mathbf{T}_1(r) - 1)^p) \leq \mathbb{E}(\mathbf{T}(r)^p) \leq \mathbb{E}(\mathbf{T}_1(r)^p).$$

Applying now relations (2.58) and (2.48) yields

$$\lim_{r \uparrow \infty} \frac{\mathbb{E}(\mathbf{T}(r)^p)}{r^p} = 1 \tag{2.59}$$

Up to now we did not give an explicit integral representation for the expectation $\mathbb{E}(\mathbf{T}_\nu(r))$. Using

$$\mathbb{E}(\mathbf{X}) = \int_0^\infty \mathbb{P}(\mathbf{X} > t)dt$$

for any nonnegative random variable $\mathbf{X}$ and relation (2.34) we obtain

$$\mathbb{E}(\mathbf{T}_\nu(r)) = \int_0^\infty \mathbb{P}(\mathbf{T}_\nu(r) > t)dt = \int_0^\infty \mathbb{P}(\mathbf{X}(\nu(t)) \leq r)dt. \tag{2.60}$$

Using this representation and $t \mapsto \mathbb{P}(\mathbf{X}(\nu(t)) \leq r)$ is a decreasing function one can also give an alternative calculus derivation of relations (2.51), (2.52) and (2.54). Observe the function

$$r \mapsto \int_0^\infty \exp(-qt)\mathbb{P}(\mathbf{X}(t) \leq r)dt$$

is called within the theory of Levy processes the q-potential measure (cf. [11]) and so for $q = 0$ using relation (2.60) we obtain $\mathbb{E}(\mathbf{T}(r))$.

In the above lemma we gave a first order asymptotic result for the hitting time $\mathbf{T}(r)$ of an increasing Levy process. To extend this result to the hitting time $\mathbf{T}_\nu(r)$ of a increasing Levy process with time transformation function ν, it follows by relation (2.38) that we need to analyze the behavior of the random variable $\nu^{\leftarrow}(\mathbf{T}(r))$. To analyze in the next result the expected asymptotic behavior of the hitting time $\mathbf{T}_\nu(r)$ for a subclass of the increasing Levy processes with time transformation function ν, we introduce the following class of functions well known within extreme value theory. (cf. [6], [8])

Definition 7 *A function $f : \mathbb{R}_+ \to \mathbb{R}$ is called regularly varying at infinity if for every $x > 0$*

$$\lim_{t\uparrow\infty} \frac{f(tx)}{f(t)} = x^\alpha \tag{2.61}$$

for some $\alpha \in \mathbb{R}$. The number α is called the index of regular variation and the class of regularly varying functions with index α is denoted by RV_α^∞.

For regularly varying functions the following two fundamental results exists. The first result is called the uniform convergence theorem for regularly varying functions, while the second is called the representation theorem for regularly varying functions. For its proof we refer to [6] or [8].

Lemma 8 *If the function $f : \mathbb{R}_+ \to \mathbb{R}$ is a regularly varying function with index $\alpha \in \mathbb{R}$, then the convergence in relation (2.61) to its limit holds uniformly for any $x \in [a, b]$ satisfying $0 < a < b < \infty$.*

The representation theorem is given by the following result.

Lemma 9 *If the function $f : \mathbb{R}_+ \to \mathbb{R}$ is a regularly varying function at infinity with index $\alpha \in \mathbb{R}$, then there exist measurable functions $a : \mathbb{R}_+ \to \mathbb{R}$ and $c : \mathbb{R}_+ \to \mathbb{R}$ with*

$$\lim_{t\uparrow\infty} c(t) = c_0, 0 < c_0 < \infty \text{ and } \lim_{t\uparrow\infty} a(t) = \alpha \tag{2.62}$$

and $t_0 > 0$ such that for every $t > t_0$

$$f(t) = c(t) \exp\left(\int_{t_0}^{t} \frac{a(s)}{s} ds\right). \tag{2.63}$$

Conversely, if relation (2.63) holds with a and c satisfying (2.62), then the function $f : \mathbb{R}_+ \to \mathbb{R}$ is a regularly varying function at infinity with index α.

It is now possible to show the following first order asymptotic results for the hitting time $\mathbf{T}_\nu(r)$ of an increasing Levy process with a regularly varying time transformation function ν.

Lemma 10 *If the stochastic process $\mathbf{X}_\nu$ is an increasing Levy process $\mathbf{X}_\nu$ with time transformation function ν regularly varying at infinity with index $0 < \alpha < \infty$ then*

$$\lim_{r\uparrow\infty} \frac{\mathbf{T}_\nu(r)}{\nu^{\leftarrow}(r)} \overset{a.s}{=} 1 \tag{2.64}$$

and

$$\lim_{r\uparrow\infty} \frac{\mathbb{E}(\mathbf{T}_\nu(r))}{\nu^{\leftarrow}(r)} = 1. \tag{2.65}$$

Proof. It can be shown (cf. [8]) for any strictly increasing and continuous function ν regularly varying at infinity with index $0 < \alpha < \infty$ that $\nu^{\leftarrow}$ is a regularly varying function at infinity with index α^{-1}. By relations (2.38) and (2.53) and the uniform convergence theorem for regularly varying functions we therefore obtain

$$\lim_{r\uparrow\infty} \frac{\mathbf{T}_\nu(r)}{\nu^{\leftarrow}(r)} = \lim_{r\uparrow\infty} \frac{\nu^{\leftarrow}(\mathbf{T}(r))}{\nu^{\leftarrow}(r)} = \lim_{r\uparrow\infty} \frac{\nu^{\leftarrow}\left(\frac{\mathbf{T}(r)}{r} r\right)}{\nu^{\leftarrow}(r)} \overset{a.s}{=} 1.$$

This proves relation (2.64). To verify relation (2.65) it follows by relation (2.64) and lemma of Fatou (cf. [21]) that

$$\liminf_{r\uparrow\infty} \frac{\mathbb{E}(\mathbf{T}_\nu(r))}{\nu^{\leftarrow}(r)} \geq 1. \tag{2.66}$$

To show the desired result it is now sufficient to prove that

$$\limsup_{r\uparrow\infty} \frac{\mathbb{E}(\mathbf{T}_\nu(r))}{\nu^{\leftarrow}(r)} \leq 1. \tag{2.67}$$

To verify this we observe the following . Since $\nu^{\leftarrow}$ is regularly varying with index $0 < \alpha^{-1} < \infty$ it follows by relation (2.62) and (2.63) that for any $x > 1$ and $k > \alpha^{-1}$ with $k \in \mathbb{N}$ there exists some r_0 satisfying

$$\frac{\nu^{\leftarrow}(xr)}{\nu^{\leftarrow}(r)} \leq 2x^k$$

for every $r \geq r_0$. This implies using $\nu^{\leftarrow}$ is increasing that

$$\frac{\nu^{\leftarrow}(\mathbf{T}(r))}{\nu^{\leftarrow}(r)} \leq \frac{\nu^{\leftarrow}(r+\mathbf{T}(r))}{\nu^{\leftarrow}(r)} \leq 2\left(1+\mathbf{T}(r)r^{-1}\right)^k$$

Considering now for $r \geq r_0$ the nonnegative random variables

$$2\left(1+\mathbf{T}(r)r^{-1}\right)^k - \nu^{\leftarrow}(\mathbf{T}(r))\nu^{\leftarrow}(r)^{-1}$$

we obtain by Fatou's lemma and relation (2.53)

$$2^{k+1} - 1 \leq \liminf_{r\uparrow\infty}(2\mathbb{E}((1+\mathbf{T}(r)r^{-1})^k) - \frac{\mathbb{E}(\nu^{\leftarrow}(\mathbf{T}(r)))}{\nu^{\leftarrow}(r)} \tag{2.68}$$

By relation (2.59) it follows that

$$\lim_{r\uparrow\infty} \mathbb{E}((1+\mathbf{T}(r)r^{-1})^k) = 2^k$$

and so

$$\begin{aligned} &\liminf_{r\uparrow\infty}(2\mathbb{E}((1+\mathbf{T}(r)r^{-1})^k) - \mathbb{E}\left(\nu^{\leftarrow}(\mathbf{T}(r))\nu^{\leftarrow}(r)^{-1}\right) \\ = \quad & 2^{k+1} - \limsup_{r\uparrow\infty} \mathbb{E}\left(\nu^{\leftarrow}(\mathbf{T}(r))\nu^{\leftarrow}(r)^{-1}\right). \end{aligned}$$

This implies using relation (2.68) that

$$\limsup_{r\uparrow\infty} \mathbb{E}\left(\nu^{\leftarrow}(\mathbf{T}(r))\nu^{\leftarrow}(r)^{-1}\right) \leq 1$$

and by relation (2.38) we obtain

$$\limsup_{r\uparrow\infty} \frac{\mathbb{E}(\mathbf{T}_\nu(r))}{\nu^{\leftarrow}(r)} = \limsup_{r\uparrow\infty} \frac{\mathbb{E}(\nu^{\leftarrow}(\mathbf{T}(r))}{\nu^{\leftarrow}(r)} \leq 1.$$

This verifies relation (2.67) and we have shown the result. □

To prove stronger asymptotic results we first need the following definition (cf. [4]).

Definition 8 *The nonnegative random variable* $\mathbf{X}$ *is said to be distributed on some lattice* $L_d = \{nd : n \in \mathbb{Z}_+\}$ *with* $d > 0$ *if* $\sum_{n=0}^{\infty} \mathbb{P}\{\ \mathbf{X} = nd\} = 1$ *and there is no smaller lattice having this property. The corresponding cdf* F *of* $\mathbf{X}$ *is then called a lattice distribution. If the nonnegative random variable* $\mathbf{X}$ *is not distributed on a lattice* L_d *for any* $d > 0$ *its associated cdf is called non-lattice.*

Without proof we now mention the following characterization result for a lattice distribution in terms of its characteristic function. A proof of this result can be found in Lemma 3. Chapter 15.1 of [20] or in Corollary 3.6.3 of [10]).

Lemma 11 *If* $\mathbf{X}$ *is a nonnegative random variable and* $\varphi : \mathbb{R} \to \mathbb{C}$ *its characteristic function given by*

$$\varphi(u) = \mathbb{E}(\exp(iu\mathbf{X}))$$

then the cdf of a nonnegative random variable $\mathbf{X}$ *is a lattice distribution if and only if there exists some* $u_0 > 0$ *satisfying* $\varphi(u_0) = 1$.

Using Lemma 11 it is now easy to derive the following result.

Lemma 12 *If* $\mathbf{X}$ *is an increasing Levy process then the cdf* F_1 *of the random variable* $\mathbf{X}(1)$ *is non-lattice if and only if the cdf* F_t *of the random variable* $\mathbf{X}(t)$ *is non-lattice for every* $t > 0$.

Proof. By a similar proof as for the probability Laplace-Stieltjes transform (see Lemma 2) it can be shown (cf. [19]) for an increasing Levy process $\mathbf{X}$ with $\varphi_t(u) := \mathbb{E}(\exp(iu\mathbf{X}(t))$ that

$$\varphi_t(u) = \varphi_1(u)^t \tag{2.69}$$

Using relation (2.69) and Lemma 11 the desired result follows. □

Using Lemma 12 together with relation (2.42) one can now show the following refinement of Lemma 7.

Lemma 13 *If* $\mathbf{X}$ *is an increasing Levy process with non-lattice cdf* F_1 *then for every* $s > 0$

$$\lim_{r\uparrow\infty} \mathbb{E}(\mathbf{T}(r+s) - \mathbf{T}(r)) = s$$

and for μ_2 *finite*

$$\lim_{r\uparrow\infty} \mathbb{E}\mathbf{T}(r) - r = \frac{\sigma^2(\mathbf{X}(1))}{2}.$$

Proof. Since F_1 is non-lattice it follows by Lemma 12 that F_h is non-lattice for every $h > 0$. This shows by the strong renewal theorem for non-lattice distributions (cf. [4], [20]) that

$$\lim_{r\uparrow\infty} \mathbb{E}(\mathbf{N}_h(r+s)) - \mathbb{E}(\mathbf{N}_h(r)) = \frac{s}{h}.$$

Applying relation (2.48) we obtain for every $h > 0$ that

$$\lim_{r\uparrow\infty} d_h(r,s) = s \tag{2.70}$$

with $d_h(r,s) := \mathbb{E}(\mathbf{T}_h(r+s)) - \mathbb{E}(\mathbf{T}_h(r))$. Since by relation (2.42) it follows for every $h > 0$ that

$$d_h(r,s) - h \leq \mathbb{E}(\mathbf{T}(r+s)) - \mathbb{E}(\mathbf{T}(r)) \leq d_h(r,s) + h$$

we obtain by relation (2.70) that the function $r \to \mathbb{E}(\mathbf{T}(r+s)) - \mathbb{E}(\mathbf{T}(r))$ has a limit κ and this limit satisfies $s - h \leq \kappa \leq s + h$ for every $h > 0$. Hence $\kappa = s$ and the result is proved. To show the other result we observe by relation (2.42) and (2.48) that for every $h > 0$

$$h(\mathbb{E}(\mathbf{N}_h(r)) - rh^{-1}) \leq \mathbb{E}(\mathbf{T}(r)) - r \leq h(\mathbb{E}(\mathbf{N}_h(r)) - rh^{-1}) + h. \tag{2.71}$$

Since it is well known (cf. [17], [20]) for F_h non-lattice and having a finite second moment that for every $h > 0$

$$\lim_{r\uparrow\infty} \mathbb{E}(\mathbf{N}_h(r)) - rh^{-1} = \frac{\mathbb{E}(\mathbf{X}(h)^2)}{2\mathbb{E}(\mathbf{X}(h))^2} = \frac{\sigma^2(\mathbf{X}(h))}{2h^2} + \frac{1}{2}$$

we obtain by Lemma 1 that

$$\lim_{r\uparrow\infty} \mathbb{E}(\mathbf{N}_h(r)) - rh^{-1}) = \frac{\sigma^2(\mathbf{X}(1))}{2h} + \frac{1}{2}$$

This implies by relation (2.71) that the function $r \mapsto \mathbb{E}(\mathbf{T}(r)) - r$ has a limit ϑ and this limit satisfies

$$\frac{\sigma^2(\mathbf{X}(1))}{2} + \frac{1}{2}h \leq \vartheta \leq \frac{\sigma^2(\mathbf{X}(1))}{2} + 1\frac{1}{2}h$$

Hence by letting $h \downarrow 0$ we obtain the desired result. □

Looking at the above result one might also be interested in the strong asymptotic behavior of the hitting time of an increasing Levy process with time transformation function ν and so we need to analyze under which conditions one can analyze the asymptotic behavior of the difference

$$\mathbb{E}(\mathbf{T}_\nu(r+s)) - \mathbb{E}(\mathbf{T}_\nu(r))$$

for $r \uparrow \infty$. By relation (2.38) this boils down to analyzing the asymptotic behavior of the difference

$$\mathbb{E}(\nu^{\leftarrow}(\mathbf{T}(r+s)) - \mathbb{E}(\nu^{\leftarrow}(\mathbf{T}(r))$$

This behavior is not investigated in this thesis and will be a topic for future research. It is conjectured for $a(t) = \nu^{\leftarrow'}(t)$ the derivative of a regularly varying function ν with index $0 < \alpha < 1$ and F_1 non-lattice that the following result holds.

$$\lim_{t\uparrow\infty} \frac{\mathbb{E}(\mathbf{T}_\nu(r+s)) - \mathbb{E}(\mathbf{T}_\nu(r))}{a(r)} = h.$$

Next to the above asymptotic results one can also prove an extension of Lorden's inequality for renewal functions to increasing Levy processes with time transformation function ν. These inequalities give an upperbound and lower bound on the expectation of the hitting times for all values of r and are useful in applications.

Lemma 14 *If the stochastic process* $\mathbf{X}$ *is an increasing Levy process and the second moment* μ_2 *of the random variable* $\mathbf{X}(1)$ *is finite, then for every* $r \geq 0$

$$r \leq \mathbb{E}(\mathbf{T}(r)) \leq r + \sigma^2(\mathbf{X}(1)).$$

Moreover, if $\mathbf{X}_\nu$ *is an increasing Levy process with a convex time transformation function* ν *and the second moment of the random variable* $\mathbf{X}_\nu(1)$ *is finite, then*

$$\mathbb{E}(\mathbf{T}_\nu(r)) \leq \nu^{\leftarrow}(r + \sigma^2(\mathbf{X}(1)))$$

while for ν *concave*

$$\mathbb{E}(\mathbf{T}_\nu(r)) \geq \nu^{\leftarrow}(r).$$

Proof. By Lorden's inequality for the renewal function (for different easy proofs see [7] or [17]) we know that

$$\frac{r}{h} \leq \mathbb{E}(\mathbf{N}_h(r)) + 1 \leq \frac{r}{h} + \frac{\mathbb{E}(\mathbf{X}(h)^2)}{h^2} = \frac{r}{h} + \frac{\sigma^2(\mathbf{X}(h))}{h^2} + 1.$$

Hence by relation (2.48) and Lemma 1 we obtain for every $h > 0$

$$r \leq \mathbb{E}(\mathbf{T}_h(r)) \leq r + \sigma^2(\mathbf{X}(1)) + h.$$

Applying now relation (2.42) yields for every $h > 0$ that

$$r - h \leq \mathbb{E}(\mathbf{T}_h(r)) - h \leq \mathbb{E}(\mathbf{T}(r)) \leq \mathbb{E}(\mathbf{T}_h(r)) \leq r + \sigma^2(\mathbf{X}(1)) + h$$

Letting $h \downarrow 0$ shows the desired result. To show the second result we first observe that ν convex implies $\nu^{\leftarrow}$ is concave. Applying now Jensen's inequality and relation (2.38)

$$\mathbb{E}(\mathbf{T}_\nu(r)) = \mathbb{E}(\nu^{\leftarrow}(\mathbf{T}(r)) \leq \nu^{\leftarrow}(\mathbb{E}(\mathbf{T}(r)))$$

Applying now the first part yields the desired result. A similar proof using Jensen's inequality and the first part applies to the last formula. □

Finally we list in this section a so-called central limit theorem for the hitting time $\mathbf{T}(r)$ of an increasing Levy process. To do so we need the following definition (cf. [4]).

Definition 9 *The sequence of random variables $\mathbf{Z}_n, n \in \mathbb{N}$ with cdf F_n converges in distribution to the random variable $\mathbf{Z}$ (notation $\mathbf{Z}_n \stackrel{d}{\to} \mathbf{Z}$) if*

$$\lim_{n\uparrow\infty} F_n(x) = F(x)$$

at all continuity points $x \in \mathbb{R}$ of the cdf F of the random variable $\mathbf{Z}$.

Since any distribution function F on $\mathbb{R}$ is nonnegative and bounded above by 1 and increasing it can be shown that the set of discontinuity points of the cdf F is at most countable (cf. [20], [4])

Lemma 15 *For any increasing Levy process $\mathbf{X}$ satisfying $\mu_1 = 1$ and $\mu_2 = \mathbb{E}(\mathbf{X}(1)^2)$ is finite it follows that*

$$\frac{\mathbf{T}(r) - r}{\sqrt{r\,(\mu_2 - 1)}} \stackrel{d}{\to} \mathbf{Z} \tag{2.72}$$

with $\mathbf{Z}$ a random variable having a standard normal distribution.

Proof. Since $\mu_1 = 1$ it follows by Lemma 1 that

$$\sigma^2(\mathbf{X}(h)) = h(\mu_2 - 1) > 0$$

Substituting this in Theorem 7.1 of [9]) we obtain

$$\frac{h\mathbf{N}_h(r) - r}{\sqrt{r\,(\mu_2 - 1)}} \stackrel{d}{\to} \mathbf{Z} \tag{2.73}$$

By relation (2.42) we also know that

$$\frac{h\mathbf{N}_h\,(r) - r}{\sqrt{r\,(\mu_2 - 1)}} \leq \frac{\mathbf{T}(r) - r}{\sqrt{r\,(\mu_2 - 1)}} \leq \frac{h\mathbf{N}_h(r) + h - r}{\sqrt{r\,(\mu_2 - 1)}} \tag{2.74}$$

and since from relation (2.73) it is also easy to show that (a special case of Slutsky theorem (cf. [15])) that

$$\frac{h\mathbf{N}_h(r) + h - r}{\sqrt{r\,(\mu_2 - 1)}} \stackrel{d}{\to} \mathbf{Z}$$

the desired result follows from combining the previous inequalities. □

Looking at the above result one might also be interested in the distributional convergence of the normalized random variable

$$\frac{\mathbf{T}_\nu(r) - \mathbb{E}(\mathbf{T}_\nu(r))}{\sigma(\mathbf{T}_\nu(r))}$$

To show a result for this random variable we probably need to apply a Lindeberg-Feller type central limit theorem (cf. [20]), [4]). One might also use a central limit theorem for martingales. This question will be a topic of future research. In the next subsection we will investigate the properties of the overshoot and undershoot of any Levy process with or without time transformation function ν.

2.2.3 On the overshoot and undershoot stochastic process in an increasing Levy process

In this subsection we will first prove an important sample path result for the overshoot at level r of an increasing Levy process (with $\mu_1 = 1$) relating it to the overshoot generated by a renewal process. This result turns out to be crucial in justifying our approximation technique. Remember in relation (2.35) the overshoot of an increasing Levy process $\mathbf{X}_\nu$ with time transformation function ν is given by

$$\mathbf{W}_\nu(r) = \mathbf{X}_\nu(\mathbf{T}_\nu(r)) - r \tag{2.75}$$

and since by lemma 6 we know that

$$\mathbf{W}_\nu(r) = \mathbf{W}(r) \tag{2.76}$$

we know that we can restrict ourselves to the overshoot at level r in an increasing Levy process. To approximate the overshoot random variables introduce for any r the overshoot $\mathbf{W}_h(r)$ at level r of an increasing Levy process sampled at the time points $nh, n \in \mathbb{Z}_+$ given by

$$\mathbf{W}_h(r) := \mathbf{X}(\mathbf{T}_h(r)) - r \tag{2.77}$$

It is now easy to show the following result.

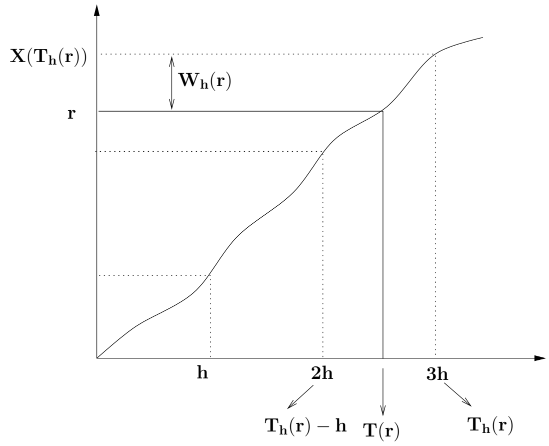

Figure 2.2: Overshoot at level r of an increasing Levy process

Lemma 16 *For every increasing Levy process* $\mathbf{X}$ *and* $r, h > 0$ *it follows that*

$$\mathbf{W}(r) \leq \mathbf{W}_h(r) \leq \mathbf{W}(r) + \mathbf{X}(\mathbf{T}(r) + h) - \mathbf{X}(\mathbf{T}(r)) \tag{2.78}$$

with the random variable $\mathbf{X}(\mathbf{T}(r) + h) - \mathbf{X}(\mathbf{T}(r))$ *independent of* $\mathbf{W}(r)$ *and*

$$\mathbf{X}(\mathbf{T}(r) + h) - \mathbf{X}(\mathbf{T}(r)) \stackrel{d}{=} \mathbf{X}(h). \tag{2.79}$$

Proof. Since the process $\mathbf{X}$ has increasing sample paths and by relation (2.42) we know that

$$\mathbf{T}_h(r) - h \leq \mathbf{T}(r) \leq \mathbf{T}_h(r)$$

for every $h > 0$ it follows that

$$\mathbf{W}(r) \leq \mathbf{X}(\mathbf{T}_h(r)) - r = \mathbf{W}_h(r). \tag{2.80}$$

To show the other inequality we observe using again the monotonicity of the sample paths and the above inequality for the hitting times that

$$\mathbf{W}_h(r) = \mathbf{X}(\mathbf{T}_h(r)) - r \leq \mathbf{X}(\mathbf{T}(r) + h) - r. \tag{2.81}$$

Since $\mathbf{T}(r)$ is a $\mathcal{F}_t$-stopping time of the Levy process $\mathbf{X}$ and each Levy process renews itself at a finite stopping time (see theorem 32 of [13]) it follows that

$$\begin{aligned} \mathbf{X}(\mathbf{T}(r) + h) - r &= \mathbf{W}(r) + \mathbf{X}(\mathbf{T}(r) + h) - \mathbf{X}(\mathbf{T}(r)) \\ &\stackrel{d}{=} \mathbf{W}(r) + \mathbf{Y}(h) \end{aligned} \tag{2.82}$$

with $\mathbf{Y}(h) \stackrel{d}{=} \mathbf{X}(h)$ independent of $\mathbf{W}(r)$. Applying relations (2.81) and (2.82) yields the result. $\square$

In the next result we prove a weaker sample path result for the undershoot at level r. Also this result is crucial in justifying the correctness of our approximation technique. Remember in relation (2.35) the undershoot of an increasing Levy process with time transformation function ν, $\mathbf{X}_\nu$ is given by

$$\mathbf{V}_\nu(r) = r - \mathbf{X}_\nu(\mathbf{T}_\nu(r)^-) \tag{2.83}$$

and since by lemma 6

$$\mathbf{V}_\nu(r) = \mathbf{V}(r) \tag{2.84}$$

we know that we can restrict ourselves to the undershoot at level r in an increasing Levy process. To approximate these random variables introduce for every r the undershoot $\mathbf{V}_h(r)$ at level r of an increasing Levy process sampled at the time points $nh, n \in \mathbb{Z}_+$ given by

$$\mathbf{V}_h(r) := r - \mathbf{X}(h\mathbf{N}_h(r)) \tag{2.85}$$

with $\mathbf{N}_h$ the renewal process associated with the arrival times $\mathbf{X}(nh), n \in \mathbb{Z}_+$. By relation (2.48) we know that

$$\mathbf{T}_h(r) - h = h\mathbf{N}_h(r)$$

and so it follows that

$$\mathbf{V}_h(r) = r - \mathbf{X}(\mathbf{T}_h(r) - h). \tag{2.86}$$

One can now show the following sample path result.

Lemma 17 *For every increasing Levy process* $\mathbf{X}$ *and* $r, h > 0$ *it follows that*

$$\mathbf{V}(r) \leq \mathbf{V}_h(r) \leq r - \mathbf{X}(\mathbf{T}(r) - h). \tag{2.87}$$

Proof. Since by relation (2.40) it is obvious for every $h > 0$ that

$$\mathbf{X}(\mathbf{T}_h(r) - h) < r$$

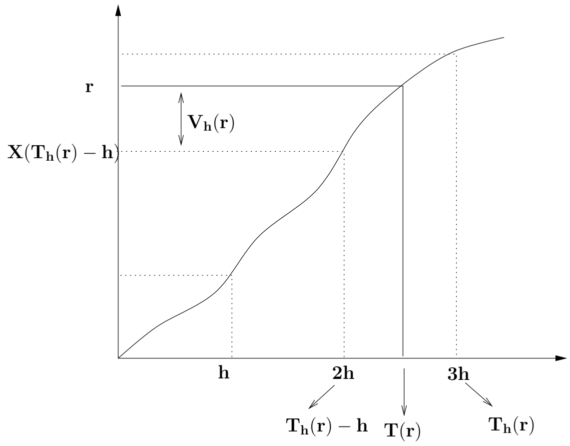

Figure 2.3: Undershoot at level r of an increasing Levy process

we obtain by the monotonicity of the sample paths of an increasing Levy process $\mathbf{X}$, the definition of $\mathbf{T}(r)^-$ and relation (2.42) that

$$\mathbf{T}(r)^- > \mathbf{T}_h(r) - h \geq \mathbf{T}(r) - h. \tag{2.88}$$

Applying again the monotonicity of the sample paths of the stochastic process $\mathbf{X}$ and the definition of the undershoot it follows using relation (2.88) that

$$\mathbf{V}(r) = r - \mathbf{X}(\mathbf{T}(r)^-) \leq r - \mathbf{X}(\mathbf{T}_h(r) - h) \leq r - \mathbf{X}(\mathbf{T}(r) - h).$$

This shows the result. $\square$

Since for fixed $r > 0$ the random variables $\mathbf{T}_{2^{-m}}(r)$, $m \in \mathbb{N}$ are decreasing in m it follows by the increasing sample paths of the increasing Levy process $\mathbf{X}$ that the overshoot random variables

$$\mathbf{W}_{2^{-m}}(r) = \mathbf{X}(\mathbf{T}_{2^{-m}}(r)) - r$$

are decreasing in m and so

$$0 \leq \mathbf{W}_{2^{-m}}(r) \leq \mathbf{W}_1(r)$$

for every $m \in \mathbb{Z}_+$. Moreover, by relation (2.78) and $\mathbf{X}(\mathbf{T}(r) + h) - \mathbf{X}(\mathbf{T}(r) \stackrel{a.s}{\to} 0$ $(h \downarrow 0)$ we obtain that the sequence $\mathbf{W}_{2^{-m}}(r)$ converges a.s to $\mathbf{W}(r)$ and hence we have verified that

$$\mathbf{W}_{2^{-m}}(r) \stackrel{a.s}{\downarrow} \mathbf{W}(r)(m \uparrow \infty) \tag{2.89}$$

This shows by Theorem 25.2 of [3] that

$$\mathbf{W}_{2^{-m}}(r) \stackrel{\mathcal{P}}{\to} \mathbf{W}(r) \ (m \uparrow \infty) \text{ and } \mathbf{W}_{2^{-m}}(r) \stackrel{d}{\to} \mathbf{W}(r) \ (m \uparrow \infty).$$

Since $\mathbf{W}_{2^{-m}}(r)$, $m \in \mathbb{N}$ is a sequence of decreasing random variables we know additionally

$$\mathbb{P}(\mathbf{W}_{2^{-m}}(r) \leq x) \uparrow \mathbb{P}(\mathbf{W}(r) \leq x) \ (m \uparrow \infty) \tag{2.90}$$

for any continuity point x of the cdf of the random variable $\mathbf{W}(r)$. This monotonicity property might be interesting from a computational point of view. Moreover, by a standard application of Walds identity within renewal theory or by the unicity of the solution of a renewal type equation (cf. [9]) it follows that $\mathbb{E}(\mathbf{W}_1(r))$ is finite and so the conditions of the dominated convergence in probability theorem hold. This shows

$$\lim_{m\uparrow\infty} \mathbb{E}(\mathbf{W}_{2^{-m}}(r)) \downarrow \mathbb{E}(\mathbf{W}(r)) \tag{2.91}$$

Again we could have applied directly the dominated convergence result of Lebesgue to justify relation (2.91). To show a similar result for the undershoot process we observe for every $m \in \mathbb{N}$ that by relation (2.47)

$$2^{-m}\mathbf{N}_{2^{-m}}(r) = 2^{-m} \sup\{n \in \mathbb{Z}_+ : \mathbf{X}(2^{-m}n) \leq r\}$$

and this shows by the increasing sample paths of the Levy process $\mathbf{X}$ that the random variables $2^{-m}\mathbf{N}_{2^{-m}}(r)$ are increasing in m. Hence it follows that the undershoot random variables

$$\mathbf{V}_{2^{-m}}(r) = r - \mathbf{X}(2^{-m}\mathbf{N}_{2^{-m}}(r))$$

are decreasing in m and by Lemma 17 we obtain

$$\mathbf{V}_{2^{-m}}(r) \stackrel{a.s}{\downarrow} \mathbf{V}(r) \ (m \uparrow \infty) \tag{2.92}$$

Hence by Theorem 25.2 of [3] it follows

$$\mathbf{V}_{2^{-m}}(r) \stackrel{\mathcal{P}}{\to} \mathbf{V}(r) \ (m \uparrow \infty) \text{ and } \mathbf{V}_{2^{-m}}(r) \stackrel{d}{\to} \mathbf{V}(r) \ (m \uparrow \infty)$$

and by the monotonicity property of the random variables $\mathbf{V}_{2^{-m}}(r)$, $m \in \mathbb{N}$ we additionally obtain

$$\mathbb{P}(\mathbf{V}_{2^{-m}}(r) \leq x) \uparrow \mathbb{P}(\mathbf{V}(r) \leq x) \tag{2.93}$$

for every continuity point x of the cdf of the random variable $\mathbf{V}(r)$. The conditions of the dominated convergence in probability theorem again hold and so for the undershoot random variable we may also conclude that

$$\lim_{m\uparrow\infty} \mathbb{E}(\mathbf{V}_{2^{-m}}(r)) \downarrow \mathbb{E}(\mathbf{V}(r)). \tag{2.94}$$

To determine the joint cdf of the random vector $(\mathbf{V}(r), \mathbf{W}(r))$ for r fixed we observe using $\mathbf{V}_{2^{-m}}(r) \downarrow^{a.s} \mathbf{V}(r)$ and $\mathbf{W}_{2^{-m}}(r) \downarrow^{a.s} \mathbf{W}(r)$ that

$$\{\mathbf{V}(r) \geq z, \mathbf{W}(r) \geq x\} = \cap_{m\in\mathbb{N}}\{\mathbf{V}_{2^{-m}}(r) \geq z, \mathbf{W}_{2^{-m}}(r) \geq x\} \tag{2.95}$$

for every $r > z$ and $x \geq 0$. It can now be easily shown by means of a sample path approach (cf. [9] or Figure 2.4) that

$$\{\mathbf{V}_{2^{-m}}(r) \geq z, \mathbf{W}_{2^{-m}}(r) \geq x\} = \{\mathbf{W}_{2^{-m}}(r - z) \geq x + z\}.$$

Also by a similar argument as used in relation (2.95) we know that

$$\cap_{m\in\mathbb{N}}\{\mathbf{W}_{2^{-m}}(r - z) \geq x + z\} = \{\mathbf{W}(r - z) \geq x + z\}. \tag{2.96}$$

Hence by relations (2.95) and (2.96) we obtain the important set relation

$$\{\mathbf{V}(r) \geq z, \mathbf{W}(r) \geq x\} = \{\mathbf{W}(r - z) \geq x + z\}. \tag{2.97}$$

To determine the cdf of the overshoot $\mathbf{W}(r)$ and undershoot $\mathbf{V}(r)$ of an increasing Levy process at a given level r it is by relations (2.90) and (2.93) natural to determine first the easier cdf of the overshoot $\mathbf{W}_{2^{-m}}(r)$ and undershoot $\mathbf{V}_{2^{-m}}(r)$ of the embedded renewal process at the time points $n2^{-m}$ and then compute its limit for $m \uparrow \infty$. This approach is mathematically more simpler than applying the compensation formula for predictable processes applied to the predictable Levy jump process $\Delta\mathbf{X}$ as done in [1]. Also the results mentioned in Lemma 16 and 17 are sufficient to determine that the stochastic under and overshoot process are converging in distribution to a limiting ran-

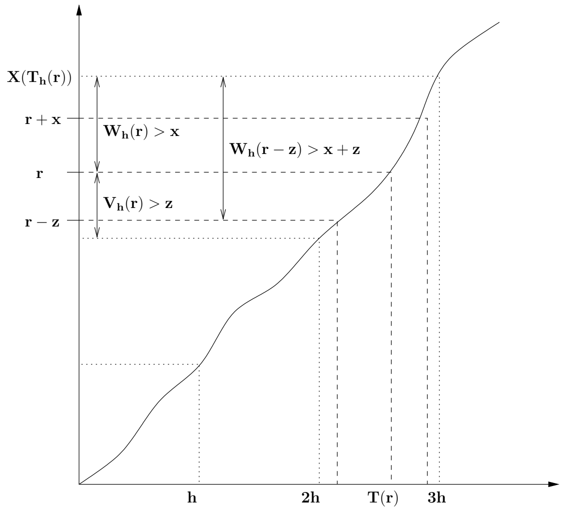

Figure 2.4: The relation between the overshoot and the joint distribution of overshoot and undershoot

dom variable and at the same time determine the form of these limiting distributions. The way to do this is to use asymptotic results for renewal processes, the extended continuity theorem for Laplace-Stieltjes transforms and Lemmas 16 and 17. Observe such a approximation procedure is also mentioned in [2] to justify the form of the limit distributions (reversing limit operations $r \uparrow \infty$ and $h \downarrow 0$) without giving a correctness proof of the proposed procedure. Actually the authors suggest in [2] that justifying this reversal is not straight forward. In [2] it is then later proved that the suggested forms are correct by using the mathematical much more difficult compensation formulas for predictable processes and asymptotic expansions. This observation was one of the incentives to start the present study on increasing Levy processes. A similar approach is also followed in Section 5 of Chapter 7 of [19] without a proper mathematical proof. Observe also that understanding increasing Levy processes is important due to the range of applications of these processes to applied oriented Operations research and Engineering applications like maintenance (see for example [16], insurance mathematics and inventory control [17]. In the next result we will first identify the cdf of the undershoot $\mathbf{V}_h(r)$ and overshoot $\mathbf{W}_h(r)$ at level r.

Lemma 18 *Let* $\mathbf{X}$ *be an increasing Levy process and introduce for* $h > 0$ *the renewal function* $U_{\infty,h} : \mathbb{R}_+ \to \mathbb{R}$ *given by*

$$U_{\infty,h}(y) = \mathbb{E}(\mathbf{N}_h(y)) = \sum\nolimits_{n=0}^{\infty} \mathbb{P}(\mathbf{X}(nh) \leq x)$$

1. *If for a given* $x > 0$ *the function* $g_0 : \mathbb{R}_+ \mapsto \mathbb{R}_+$ *is defined by*

$$g_0(z) = 1 - F_h(z + x),$$

then it follows that

$$\mathbb{P}\left(\mathbf{W}_h\left(r\right) > x\right) = \int_{0-}^{r} g_0(r-y)dU_{\infty,h}(y). \tag{2.98}$$

2. *If for a given* $x > 0$ *the function* $g_1 : \mathbb{R}_+ \mapsto \mathbb{R}_+$ *is defined by*

$$g_1(z) = \begin{cases} 0 & \text{if } z \leq x \\ 1 - F_h(z) & \text{if } z > x \end{cases}$$

then it follows for $r > x$ *that*

$$\mathbb{P}\left(\mathbf{V}_h\left(r\right) > x\right) = \int_{0}^{r} g_1(r-y)dU_{\infty,h}(y)$$

while $\mathbb{P}\left(\mathbf{V}_h\left(r\right) > x\right) = 0$ *for* $r < x$.

Proof. To derive the first equation we first observe by the total law of probability and the renewal argument applied to the regenerative overshoot process $\mathbf{W}_h = \{\mathbf{W}_h(r) : r \geq 0\}$ with regeneration points $\mathbf{X}(nh), n \in \mathbb{N}$ that

$$\begin{aligned}
\mathbb{P}\left(\mathbf{W}_h\left(r\right) > x\right) &= \mathbb{P}\left(\mathbf{W}_h\left(r\right) > x, \mathbf{X}(h) \leq r\right) + \mathbb{P}\left(\mathbf{W}_h\left(r\right) > x, \mathbf{X}(h) > r\right) \\
&= \textstyle\int_0^r \mathbb{P}\left(\mathbf{W}_h\left(r\right) > x | \mathbf{X}(h) = y\right) dF_h\left(y\right) + \mathbb{P}\left(\mathbf{W}_h\left(r\right) > x, \mathbf{X}(h) > r\right) \\
&= \textstyle\int_0^r \mathbb{P}(\mathbf{W}_h\left(r-y\right) > x) dF_h\left(y\right) + \mathbb{P}\left(\mathbf{W}_h\left(r\right) > x, \mathbf{X}(h) > r\right)
\end{aligned} \tag{2.99}$$

Observe if the event $\mathbf{X}(h) > r$ happens then clearly $\mathbf{T}_h(r) = h$ and this shows

$$\mathbf{W}_h(r) = \mathbf{X}(\mathbf{T}_h(r)) - r = \mathbf{X}(h) - r$$

Hence it follows that

$$\{\mathbf{W}_h\,(r) > x, \mathbf{X}(h) > r\} = \{\mathbf{X}(h) - r > x, \mathbf{X}(h) > r\} = \{\mathbf{X}(h) > r + x\}$$

and so

$$g_0(r) := \mathbb{P}(\mathbf{W}_h\,(r) > x, \mathbf{X}(h) > r) = 1 - F_h(r + x)$$

Hence by relation (2.99) the function $\nu(r) = \mathbb{P}\,(\mathbf{W}_h\,(r) > x)$ satisfies the renewal type equation

$$\nu(r) = \int_0^r \nu(r - y)dF_h(y) + g_0(r)$$

By Theorem 4.1 of [9] this renewal type equation has a unique solution and it given by

$$\nu(r) = \int_{0-}^r g_0(r - y)dU_{\infty,h}(y) = \int_{0-}^r (1 - F_h(r + x - y))dU_{\infty,h}(y)$$

To determine the undershoot we observe again by the renewal argument that

$$\begin{aligned}\mathbb{P}(\mathbf{V}_h(r) > x) &= \mathbb{P}(\mathbf{V}_h(r) > x, \mathbf{X}(h) \le r) + \mathbb{P}(\mathbf{V}_h(r) > x, \mathbf{X}(h) > r) \\ &= \textstyle\int_0^r \mathbb{P}(\mathbf{V}_h\,(r - y) > x)dF_h\,(y) + \mathbb{P}\,(\mathbf{V}_h\,(r) > x, \mathbf{X}(h) > r)\end{aligned}$$

It follows in case $\mathbf{X}(h) > r$ that $\mathbf{V}_h\,(r) = r$ and this shows that

$$g_1(r) := \mathbb{P}\,(\mathbf{V}_h\,(r) > x, \mathbf{X}(h) > r) = \begin{cases} 0 & \text{if } r \le x \\ 1 - F_h(r) & \text{if } r > x \end{cases}$$

Hence the function $\sigma(r) = \mathbb{P}(\mathbf{V}_h(r) \le x)$ satisfies the renewal type equation

$$\sigma(r) = \int_0^r \sigma(r - y)dF_h\,(y) + g_1(r)$$

Again we obtain by the unicity of solution of the renewal-type equation that

$$\sigma(r) = \int_{0-}^r g_1(r - y)dU_{\infty,h}(y)$$

and this proves the result. $\square$

From Lemma 18 and applying the strong renewal theorem for non-lattice distributions the following well known result for the over and undershoot stochastic process in renewal theory is easy to derive.

Lemma 19 *If* $\mathbf{X}$ *is an increasing Levy process with non-lattice cdf* F_1 *and* $\mu_1 = 1$*, then for every* $h > 0$ *there exists some random variables* $\mathbf{W}_h(\infty)$ *and* $\mathbf{V}_h(\infty)$ *satisfying*

$$\mathbf{W}_h(r) \xrightarrow{d} \mathbf{W}_h(\infty)(r \to \infty), \mathbf{V}_h(r) \xrightarrow{d} \mathbf{V}_h(\infty)(r \to \infty)$$

and

$$\mathbb{P}(\mathbf{V}_h(\infty) > x) = \mathbb{P}(\mathbf{W}_h(\infty) > x) = \frac{1}{h} \int_x^\infty 1 - F_h(y)dy. \tag{2.100}$$

Proof. By Lemma 12 the cdf F_h associated with the renewal function $x \mapsto U_{\infty,h}(x)$ is non-lattice for every $h > 0$. Since it is easy to see that the function g_0 in Lemma 18 is directly Riemann integrable (cf. [9]) we obtain by the strong renewal theorem (cf. [9]) using $\mathbb{E}(\mathbf{X}(h)) = h$ that

$$\begin{aligned} \lim_{r\uparrow\infty} \int_{0-}^r g_0(r-y)dU_{\infty,h}(y) &= \tfrac{1}{\mathbb{E}(\mathbf{X}(h))} \int_0^\infty g_0(z)dz \\ &= \tfrac{1}{h} \int_x^\infty 1 - F_h(z)dz. \end{aligned}$$

By a similar argument it follows that

$$\begin{aligned} \lim_{r\uparrow\infty} \int_0^r g_1(r-y)dU_{\infty,h}(y) &= \tfrac{1}{\mathbb{E}(\mathbf{X}(h))} \int_0^\infty g_1(z)dz \\ &= \tfrac{1}{h} \int_x^\infty 1 - F_h(z)dz \end{aligned}$$

Applying now Lemma 18 yields the desired result. □

We will now use Lemma 16 and the previous lemma in combination with the continuity theorem for Laplace-Stieltjes transforms (see Appendix) to show that under certain conditions the limit distribution of the overshoot process $\mathbf{W}$ of an increasing Levy process exists and at the same time identify its form.

Lemma 20 *If* $\mathbf{X}$ *is an increasing Levy process with non-lattice cdf* F_1 *and* $\mathbb{E}(\mathbf{X}(1)) = 1$*, then there exits some random variable* $\mathbf{W}(\infty)$ *satisfying*

$$\mathbf{W}(r) \xrightarrow{d} \mathbf{W}(\infty)$$

and

$$\mathbb{P}(\mathbf{W}(\infty) \leq x) = \int_0^x \int_z^\infty u^{-1} dK_1(u)dz.$$

Proof. By Lemma 16 it follows for every $\alpha \geq 0, h > 0$ and $r > 0$ that

$$\mathbb{E}(\exp(-\alpha\mathbf{W}(r))\mathbb{E}(\exp(-\alpha\mathbf{X}(h))) \leq \mathbb{E}(\exp(-\alpha\mathbf{W}_h(r)) \leq \mathbb{E}(\exp(-\alpha\mathbf{W}(r)).$$

and so

$$\mathbb{E}(\exp(-\alpha\mathbf{W}_h(r)) \leq \mathbb{E}(\exp(-\alpha\mathbf{W}(r)) \leq \mathbb{E}(\exp(-\alpha\mathbf{W}_h(r))\mathbb{E}(\exp(-\alpha\mathbf{X}(h)))^{-1} \tag{2.101}$$

Since by relation (2.19)

$$\mathbb{E}(\exp(-\alpha\mathbf{X}(h)) = \exp(-h\int_0^\alpha \rho(s)ds) \tag{2.102}$$

it follows using relation (2.101) that

$$\mathbb{E}(\exp(-\alpha\mathbf{W}_h(r)) \leq \mathbb{E}(\exp(-\alpha\mathbf{W}(r)) \leq \mathbb{E}(\exp(-\alpha\mathbf{W}_h(r))\exp\left(h\int_0^\alpha \rho(s)ds\right). \tag{2.103}$$

In the remainder of the proof we will show that $\lim_{r\uparrow\infty}\mathbb{E}(\exp(-\alpha\mathbf{W}(r)$ exists and identify its limit. Applying Lemma 19 and the continuity theorem for pLST (cf. [19] or Appendix) we obtain

$$\lim_{r\uparrow\infty}\mathbb{E}(\exp(-\alpha\mathbf{W}_h(r))) = \mathbb{E}(\exp(-\alpha\mathbf{W}_h(\infty))) = \frac{1-\mathbb{E}(\exp(-\alpha\mathbf{X}(h))}{\alpha h}. \tag{2.104}$$

Introduce now for every $\alpha \geq 0$ the functions

$$\overline{L}(\alpha) := \limsup_{r\uparrow\infty}\mathbb{E}(\exp(-\alpha\mathbf{W}(r))$$

and

$$\underline{L}(\alpha) := \liminf_{r\uparrow\infty}\mathbb{E}(\exp(-\alpha\mathbf{W}(r))$$

By relations (2.103) and (2.104) we obtain for every $h > 0$ that

$$\mathbb{E}(\exp(-\alpha\mathbf{W}_h(\infty))) \leq \underline{L}(\alpha) \leq \overline{L}(\alpha) \leq \mathbb{E}(\exp(-\alpha\mathbf{W}_h(\infty)))\exp(h\int_0^\alpha \rho(s)ds).$$

This shows taking $h \downarrow 0$ in relation (2.104) that

$$\alpha^{-1}\int_0^\alpha \rho(s)ds \leq \underline{L}(\alpha) \leq \overline{L}(\alpha) \leq \alpha^{-1}\int_0^\alpha \rho(s)ds.$$

Hence $\lim_{r\uparrow\infty} \mathbb{E}(\exp(-\alpha \mathbf{W}_h(r))$ exists for every $\alpha \geq 0$ and

$$\lim_{r\uparrow\infty} \mathbb{E}(\exp(-\alpha \mathbf{W}_h(r)) = \alpha^{-1} \int_0^\alpha \rho(s)ds. \tag{2.105}$$

Since by relation (2.19)

$$\rho(s) = \int_{0-}^\infty \exp(-sx)dK_1(x) \tag{2.106}$$

and K_1 is a cdf on $\mathbb{R}_+$ (see lemma 5) we obtain that the function ρ is right continuous in 0 satisfying $\rho(0) = 1$ and continuous on $(0, \infty)$. This implies

$$\lim_{\alpha\downarrow 0} \alpha^{-1} \int_0^\alpha \rho(s)ds = \rho(0) = 1$$

and we have verified that

$$\alpha \mapsto \alpha^{-1} \int_0^\alpha \rho(s)ds$$

is right continuous in 0. To identify the cdf with pLST $\alpha \mapsto \alpha^{-1} \int_0^\alpha \rho(s)ds$ we observe by Fubinis theorem that

$$\begin{aligned}
\alpha^{-1} \int_0^\alpha \rho(s)ds &= \alpha^{-1} \int_0^\alpha \int_0^\infty \exp(-sx)dK_1(x)ds \\
&= \alpha^{-1} \int_0^\infty \frac{1-\exp(-\alpha x)}{x} dK_1(x) \\
&= \int_0^\infty \int_0^x \exp(-\alpha v)dv x^{-1}dK_1(x) \\
&= \int_0^\infty \exp(-\alpha v) \int_v^\infty x^{-1}dK_1(x)dv
\end{aligned}$$

Hence the cdf F with density $f(\nu) = \int_\nu^\infty x^{-1}dK_1(x)$ has pLST

$$\alpha \mapsto \alpha^{-1} \int_0^\alpha \rho(s)ds$$

and the conditions of the continuity theorem are satisfied. Using now relation (2.105) the desired result follows. $\square$

We now show a similar result for the undershoot process.

Lemma 21 *If* $\mathbf{X}$ *is an increasing Levy process with non-lattice cdf* F_1 *and* $\mathbb{E}(\mathbf{X}(1)) = 1$, *then there exits some random variable* $\mathbf{V}(\infty)$ *satisfying*

$$\mathbf{V}(r) \stackrel{d}{\rightarrow} \mathbf{V}(\infty)$$

and

$$\mathbb{P}(\mathbf{V}(\infty) \leq x) = \int_0^x \int_z^\infty u^{-1} dK_1(u) dz.$$

Proof. It follows by Lemma 17 that

$$\mathbb{P}(\mathbf{V}(r) > x) \leq \mathbb{P}(\mathbf{V}_h(r) > x)$$

for every $h > 0$. This shows by Lemma 19 that

$$\limsup_{r\uparrow\infty} \mathbb{P}(\mathbf{V}(r) > x) \leq \frac{1}{h} \int_x^\infty 1 - F_h(y) dy$$

for every $h > 0$. Taking $h \downarrow 0$ and using Lemma 5 we obtain

$$\limsup_{r\uparrow\infty} \mathbb{P}(\mathbf{V}(r) > x) \leq \int_x^\infty \int_y^\infty u^{-1} dK_1(u) dy = \mathbb{P}(\mathbf{W}(\infty) > x)$$

To show that

$$\liminf_{r\uparrow\infty} \mathbb{P}(\mathbf{V}(r) > x) \geq \mathbb{P}(\mathbf{W}(\infty) > x)$$

and hence prove the result we observe using relation (2.97) that for every $z, h > 0$ and $r > x + h$

$$\begin{aligned} \mathbb{P}(\mathbf{V}(r) > x) &\geq \mathbb{P}(\mathbf{V}(r) \geq x + h) \\ &\geq \mathbb{P}(\mathbf{V}(r) \geq x + h, \mathbf{W}(r) \geq z) \qquad (2.107) \\ &= \mathbb{P}(\mathbf{W}(r - x - h) \geq x + h + z) \end{aligned}$$

Since by Lemma 20 the cdf of the random variable $\mathbf{W}(\infty)$ is continuous and $\mathbf{W}(r) \stackrel{d}{\rightarrow} \mathbf{W}(\infty)$ it follows by relation (2.107) for every $z, h > 0$ that

$$\begin{aligned} \liminf_{r\uparrow\infty} \mathbb{P}(\mathbf{V}(r) > x) &\geq \liminf_{r\uparrow\infty} \mathbb{P}(\mathbf{W}(r - x - h) \geq x + h + z) \\ &= \mathbb{P}(\mathbf{W}(\infty) \geq x + h + z). \end{aligned} \qquad (2.108)$$

Again by the continuity of the cdf of $\mathbf{W}(\infty)$ we obtain

$$\lim_{z+h\downarrow 0} \mathbb{P}(\mathbf{W}(\infty) \geq x+h+z) = \mathbb{P}(\mathbf{W}(\infty) > x)$$

and this shows by relation (2.108) that

$$\liminf_{r\uparrow\infty} \mathbb{P}(\mathbf{V}(r) > x) \geq \mathbb{P}(\mathbf{W}(\infty) > x),$$

thus showing the desired result. $\square$

Applying Lemma 5 and 18 one can show the following formula for the overshoot and undershoot of an increasing Levy process with time transformation ν. In this approach we avoid the more common and mathematically more difficult approach using the complicated compensation formula for predictable processes applied to the jump process $\Delta\mathbf{X}$ of the Levy process $\mathbf{X}$ as discussed without proof in Section O5 of [1] (see also Section 5.2 of [11])

Lemma 22 *If $\mathbf{X}_\nu$ is an increasing Levy process with time transformation ν, then*

$$\mathbb{P}(\mathbf{W}_\nu(r) > x) = \int_{0^-}^{r} \int_{r+x-y}^{\infty} u^{-1} dK_1(u) dU_\infty(y)$$

and

$$\mathbb{P}(\mathbf{V}_\nu(r) > x) = \begin{cases} 0 & \text{if } r \leq x \\ \int_{0^-}^{r-x} \int_{r-y}^{\infty} u^{-1} dK_1(u) dU_\infty(y) & \text{if } r > x \end{cases}$$

with

$$U_\infty(y) := \int_0^\infty \mathbb{P}(\mathbf{X}(t) \leq y) dy = E(\mathbf{T}(y))$$

Proof. By Lemma 6 we only need to show the result for an increasing Levy process $\mathbf{X}$. It follows from relations (2.42) and (2.48) and the random variables $2^{-m}\mathbf{N}_{2^{-m}}(y)$ are increasing in m that for every $y > 0$

$$\lim_{m\uparrow\infty} 2^{-m} U_{\infty,2^{-m}}(y) = \lim_{m\uparrow\infty} 2^{-m} \mathbb{E}(\mathbf{N}_{2^{-m}}(y)) \uparrow \mathbb{E}(\mathbf{T}(y)) = U_\infty(y)$$

Also by Lemma 5 we obtain for every $x > 0$ that

$$\lim_{m\uparrow\infty} 2^m (1 - F_{2^{-m}}(x)) = \int_x^\infty u^{-1} dK_1(u).$$

This implies by relation (2.90) and Lemma 18 that for every $x > 0$

$$\begin{aligned}
\mathbb{P}(\mathbf{W}(r) > x) &=\downarrow \lim_{m\uparrow\infty} \mathbb{P}(\mathbf{W}_{2^{-m}}(r) > x) \\
&= \lim_{m\uparrow\infty} \int_{0-}^{r} 2^m(1 - F_{2^{-m}}(r + x - y))2^{-m} dU_{\infty,2^{-m}}(y) \\
&= \int_{0-}^{r} \int_{r+x-y}^{\infty} u^{-1} dK_1(u) dU_\infty(y)
\end{aligned}$$

To prove the result for the undershoot we observe for $r > x$ using relation (2.93) and again Lemma 18 that

$$\begin{aligned}
\mathbb{P}(\mathbf{V}(r) > x) &=\downarrow \lim_{m\uparrow\infty} \mathbb{P}(\mathbf{V}_{2^{-m}}(r) > x) \\
&= \lim_{m\uparrow\infty} \int_{0-}^{r-x} 2^m(1 - F_{2^{-m}}(r - y))2^{-m} dU_{\infty,2^{-m}}(y) \\
&= \int_{0-}^{r-x} \int_{r-y}^{\infty} u^{-1} dK_1(u) dU_\infty(y)
\end{aligned}$$

and this shows the result. □

Using relation (2.97) and by Lemma 22 the cfd of the random variable $\mathbf{W}(r)$ is continuous on $(0, \infty)$ we obtain immediately from Lemma 22 the joint distribution of the overshoot and undershoot at level r.

Lemma 23 *It follows for any Levy process $\mathbf{X}_\nu$ with time transformation function ν that for $r > z$*

$$\mathbb{P}(\mathbf{V}_\nu(r) > z, \mathbf{W}_\nu(r) > x) = \int_{0-}^{r-z} \int_{r+x-y}^{\infty} u^{-1} dK_1(u) dU_\infty(y).$$

We next continue with expectation results. An alternative way to verify the first result is by observing that $\mathbf{T}(r)$ is a stopping time of an increasing Levy process $\mathbf{X}$ and the process $\mathbf{M} = \{\mathbf{M}_t : t \geq 0\}$ given by

$$\mathbf{M}_t = \mathbf{X}(\nu(t)) - \nu(t)$$

is a martingale. Apply now to this martingale Doob's optional sampling theorem for continous time martingales (see Theorem 14.12 of [4].)

Lemma 24 *It follows for the overshoot process $\mathbf{W}_\nu = \{\mathbf{W}_\nu(r) : r \geq 0\}$ that for any $r > 0$*

$$\mathbb{E}(\mathbf{W}_\nu(r)) = \mathbb{E}(\mathbf{T}(r)) - r$$

Proof. By Lemma 6 we may restrict ourselves to the overshoot of an increasing Levy process. Applying lemma 16 we obtain

$$\mathbb{E}(\mathbf{W}(r)) \leq \mathbb{E}(\mathbf{W}_h(r)) \leq \mathbb{E}(\mathbf{W}(r)) + h$$

and so

$$\mathbb{E}(\mathbf{W}_h(r)) - h \leq \mathbb{E}(\mathbf{W}(r)) \leq \mathbb{E}(\mathbf{W}_h(r)) \tag{2.109}$$

for every $h > 0$. By relations (2.45) and (2.48) it follows

$$\mathbb{E}(\mathbf{W}_h(r)) = \mathbb{E}(\mathbf{X}(h(\mathbf{N}_h(r) + 1)) - r = \mathbb{E}\left(\sum\nolimits_{k=1}^{\mathbf{N}_h(r)+1} \mathbf{Y}_k(h)\right) - r$$

Since $\mathbf{N}_h(r) + 1$ is a stopping time with respect to the sequence $\mathbf{Y}_k(h), k \in \mathbb{N}$ we may use Wald's identity (for an alternative proof using the renewal argument see page 167 of [9]) and so

$$\mathbb{E}(\mathbf{W}_h(r)) = h\mathbb{E}(\mathbf{N}_h(r) + 1) - r$$

Finally

$$\mathbb{E}(\mathbf{N}_h(r) + 1) = \mathbb{E}\left(\sum\nolimits_{k=0}^{\infty} \mathbf{1}_{\{\mathbf{X}(kh) \leq r\}}\right) = \sum\nolimits_{k=0}^{\infty} \mathbb{P}(\mathbf{X}(kh) \leq r)$$

and so we obtain by relation (2.109)

$$h\sum\nolimits_{k=1}^{\infty} \mathbb{P}(\mathbf{X}(kh) \leq r) - r \leq \mathbb{E}(\mathbf{W}(r)) \leq h\sum\nolimits_{k=0}^{\infty} \mathbb{P}(\mathbf{X}(kh) \leq r) - r \tag{2.110}$$

for every $h > 0$. Since the process $\mathbf{X}$ is continuous in probability this yields taking $h \downarrow 0$ in relation (2.110) and using relation (2.60) that

$$\mathbb{E}(\mathbf{W}(r)) = \mathbb{E}(\mathbf{T}(r)) - r.$$

This shows the desired result. □

By Lemmas 24 and 13 it follows immediately for F_1 non-lattice and μ_2 finite that

$$\lim_{r\uparrow\infty} \mathbb{E}(\mathbf{W}_\nu(r)) = \frac{\mu_2 - 1}{2} \tag{2.111}$$

Also by Lemmas 24 and 14 we obtain for every $r > 0$ that

$$0 \leq \mathbb{E}(\mathbf{W}_\nu(r)) \leq \mu_2 - 1 \tag{2.112}$$

Finally we like to determine the joint distribution of the overshoot and the hitting time. To compute this we first need the following result.

Lemma 25 *If* $\mathbf{X}$ *is an increasing Levy process with* $\mathbb{E}(\mathbf{X}(1)) = 1$*, then*

$$\mathbb{P}(\mathbf{T}(r) \leq t, \mathbf{W}(r) \leq y) \quad = \quad \int_0^r \int_{r-x}^{r+y-x} u^{-1} dK_1(u) dU_t(x)$$

with $U_t(x) = \int_0^t F_\nu(x) dv$.

Proof. By relations (2.41) and (2.78) we obtain for every $t > 0, y > 0$

$$\lim_{h \downarrow 0} \mathbb{P}(\mathbf{T}_h(r) \leq h(\lfloor th^{-1} \rfloor + 1), \mathbf{W}_h(r) \leq y) \quad = \quad \mathbb{P}(\mathbf{T}(r) \leq t, \mathbf{W}(r) \leq y)$$

To analyze the probability $\mathbb{P}(\mathbf{T}_h(r) \leq h(\lfloor th^{-1} \rfloor + 1), \mathbf{W}_h(r) \leq y)$ we observe by relation (2.40) that

$$\begin{aligned} \mathbb{P}(\mathbf{T}_h(r) \leq h(\lfloor th^{-1} \rfloor + 1), \mathbf{W}_h(r) \leq y) \quad &= \quad \mathbb{P}(\mathbf{N}_h(r) \leq \lfloor th^{-1} \rfloor, \mathbf{W}_h(r) \leq y) \\ &= \quad \sum_{k=0}^{\lfloor th^{-1} \rfloor} \mathbb{P}(\mathbf{N}_h(r) = k, \mathbf{W}_h(r) \leq y) \end{aligned}$$

To analyze the above sum we observe for $k = 0$ that

$$\mathbb{P}(\mathbf{N}_h(r) = 0, \mathbf{W}_h(r) \leq y\} \quad = \quad \mathbb{P}(r < \mathbf{X}(h) \leq r + y)$$

Moreover, for $k \geq 1$ we obtain

$$\begin{aligned} \mathbb{P}(\mathbf{N}_h(r) = k, \mathbf{W}_h(r) \leq y) \quad &= \quad \mathbb{P}(\mathbf{X}(kh) \leq r, r < \mathbf{X}((k+1)h) \leq r + y) \\ &= \quad \int_0^r \mathbb{P}(r < x + \mathbf{X}((k+1)h) - \mathbf{X}(kh) \leq r + y) dF_{kh}(x) \\ &= \quad \int_0^r \mathbb{P}(r - x < \mathbf{X}(h) \leq r + y - x) dF_{kh}(x) \end{aligned}$$

Introducing now for every $h > 0$ the set of measures $U_{t,h}$ given by

$$U_{t,h}(x) \quad = \quad h \sum_{k=0}^{\lfloor th^{-1} \rfloor} F_{kh}(x)$$

we obtain that

$$\mathbb{P}(\mathbf{T}_h(r) \leq h(\lfloor th^{-1} \rfloor + 1), \mathbf{W}_h(r) \leq y) \quad = \quad \int_0^r \frac{\mathbb{P}(r-x<\mathbf{X}(h)\leq r+y-x)}{h} dU_{t,h}(x)$$

By a simple monotonicity argument using the increasing sample paths of the standard Levy process it follows

$$\lim_{h\downarrow 0} U_{t,h}(x) \quad = \quad \int_0^t F_\nu(x) dv \quad := \quad U_t(x)$$

Applying now Lemma 5 yields

$$\begin{aligned} \lim_{h\downarrow 0} \mathbb{P}(\mathbf{T}_h(r) \leq h(\lfloor th^{-1} \rfloor + 1), \mathbf{W}_h(r) \leq y) \quad &= \quad \lim_{h\downarrow 0} \int_0^r \frac{\mathbb{P}(r-x<\mathbf{X}(h)\leq r+y-x)}{h} dU_{t,h}(x) \\ &= \quad \int_0^r \int_{r-x}^{r+y-x} u^{-1} dK_1(u) dU_t(x) \end{aligned}$$

and hence we have shown the following result. □

For any Levy process $\mathbf{X}_\nu$ with time transformation ν we obtain by relations (2.38) and (2.39) that

$$\mathbb{P}(\mathbf{T}_\nu(r) \leq t, \mathbf{W}_{\nu,}(r) \leq y) \quad = \quad \mathbb{P}(\mathbf{T}(r) \leq \nu(t), \mathbf{W}(r) \leq y) \tag{2.113}$$

and applying Lemma 25 yields

$$\mathbb{P}(\mathbf{T}_\nu(r) \leq t, \mathbf{W}_\nu(r) \leq y) \quad = \quad \int_0^r \int_{\lambda r-x}^{\lambda r+y-x} u^{-1} dK_1(u) dU_{\nu(t)}(x) \tag{2.114}$$

By using the cumulative distribution function of the overshoot it is easy to determine the joint distribution of the overshoot and the undershoot. Finally we determine the joint distribution of the overshoot, undershoot and the hitting time

Lemma 26 *If* $\mathbf{X}$ *is an increasing Levy process with* $\mathbb{E}(\mathbf{X}(1)) = 1$*, then it follows that*

$$\mathbb{P}((\mathbf{V}(r) > x, (\mathbf{W}(r) > y, (\mathbf{T}(r) \leq t) \quad = \quad \int_0^r \int_{r+y-a}^\infty u^{-1} dK_1(u) dU_t(x)$$

with $U_t(x) = \int_0^t F_\nu(x) d\nu$.

Proof. By relation (2.97) it is easy to see that

$$\{\mathbf{V}(r) > x, \mathbf{W}(r) > y, \mathbf{T}(r) \leq t\} = \{\mathbf{W}(r-x) > x+y, \mathbf{T}(r) \leq t\}.$$

which implies,

$$\mathbb{P}((\mathbf{V}(r) > x, \mathbf{W}(r) > y, \mathbf{T}(r) \leq t) = \mathbb{P}(\mathbf{W}(r-x) > x+y, \mathbf{T}(r) \leq t)$$

Now by relations (2.41), (2.78) and (2.87) we obtain for every $t > 0, y > 0$

$$\lim_{h\downarrow 0} \mathbb{P}(\mathbf{T}_h(r) \leq h(\lfloor th^{-1} \rfloor + 1), \mathbf{W}_h(r-x) > x+y) = \mathbb{P}(\mathbf{T}(r) \leq t, \mathbf{W}(r-x) > x+y)$$

To analyse the probability $\mathbb{P}(\mathbf{T}_h(r) \leq h(\lfloor th^{-1} \rfloor + 1), \mathbf{W}_h(r) > y)$ we observe by relation (2.40) that

$$\begin{aligned}\mathbb{P}(\mathbf{T}_h(r) \leq h(\lfloor th^{-1} \rfloor + 1), \mathbf{W}_h(r-x) > x+y) &= \mathbb{P}(\mathbf{N}_h(r) \leq \lfloor th^{-1} \rfloor, \mathbf{W}_h(r-x) > x+y) \\ &= \sum_{k=0}^{\lfloor th^{-1} \rfloor,} \mathbb{P}(\mathbf{N}_h(r) = k, \mathbf{W}_h(r-x) > x+y)\end{aligned}$$

To analyse the above sum we observe for $k = 0$ that

$$\begin{aligned}\mathbb{P}(\mathbf{N}_h(r) = 0, \mathbf{W}_h(r-x) > x+y) &= \mathbb{P}(\mathbf{N}_h(r) = 0, \mathbf{W}_h(r) > y) \\ &= \mathbb{P}(\mathbf{X}_1(h) > r+y)\end{aligned}$$

Moreover, for $k \geq 1$ we obtain, if $r - x > S_k$

$$\begin{aligned}\mathbb{P}(\mathbf{N}_h(r) = k, \mathbf{W}_h(r-x) > x+y) &= \mathbb{P}(\mathbf{X}(kh) \leq r-x, \mathbf{X}((k+1)\,h) > r+y) \\ &= \int_0^{r-x} \mathbb{P}(a + \mathbf{X}((k+1)\,h) - \mathbf{X}(kh) > r+y) dF_{kh}(a) \\ &= \int_0^{r-x} \mathbb{P}(\mathbf{X}(h) > r+y-a) dF_{kh}(a)\end{aligned}$$

Introducing now for every $h > 0$ the set of measures $U_{t,h}$ given by

$$U_{t,h}(a) = h \sum_{k=0}^{\lfloor th^{-1} \rfloor} F_{kh}(a)$$

we obtain that

$$\mathbb{P}(\mathbf{T}_h(r) \leq h(\lfloor th^{-1} \rfloor + 1), \mathbf{W}_h(r-x) > x+y) = \int_0^{r-x} \frac{\mathbb{P}(\mathbf{X}(h) > r+y-a)}{h} dU_{t,h}(a)$$

By a simple monotonicity argument using the increasing sample paths of the standard Levy process it follows

$$\lim_{h\downarrow 0} U_{t,h}(x) \quad = \quad \int_0^t F_\nu(x)dv \quad := \quad U_t(x)$$

Applying now Lemma 5 yields

$$\begin{aligned}\lim_{h\downarrow 0} \mathbb{P}(\mathbf{T}_h(r) \leq h(\lfloor th^{-1}\rfloor + 1), \mathbf{W}_h(r-x) > y + x) \quad &= \quad \lim_{h\downarrow 0} \int_0^r \frac{\mathbb{P}(\mathbf{X}(h) > r+y-a)}{h} dU_{t,h}(a) \\ &= \quad \int_0^r \int_{r+y-a}^\infty u^{-1} dK_1(u) dU_t(x)\end{aligned}$$

and hence we have shown the following result. □

For any increasing Levy process $\mathbf{X}_\nu$ with time transformation ν we obtain by relations (2.38) and (2.39) that

$$\mathbb{P}(\mathbf{T}_\nu(r) \leq t, \mathbf{W}_\nu(r) > y, \mathbf{V}_\nu\,(r) > x) \quad = \quad \mathbb{P}(\mathbf{T}(r) \leq \nu(t), \mathbf{W}(r) > y, \mathbf{V}\,(r) > x) \tag{2.115}$$

and applying Lemma 26 yields

$$\mathbb{P}(\mathbf{T}_\nu(r) \leq t, \mathbf{W}_\nu(r) > y\mathbf{V}_\nu\,(r) > x) \quad = \quad \int_0^r \int_{\lambda r+y-a}^\infty u^{-1} dK_1(u) dU_{\nu(t)}(x) \tag{2.116}$$

In the next subsection we are considering the cdf of the fractional h-part of the hitting time $\mathbf{T}(r)$.

2.2.4 On properties of the Fractional Part

Introduce now for every $x \geq 0$ and $h > 0$ the fractional h-part

$$\mathcal{F}_h(x) \quad := \quad x - h\lfloor xh^{-1}\rfloor$$

Clearly $0 \leq \mathcal{F}_h(x) \leq h$ and by its definition

$$\mathbf{T}(r) \quad = \quad \mathcal{F}_h(\mathbf{T}(r)) + h\lfloor \mathbf{T}(r)h^{-1}\rfloor \tag{2.117}$$

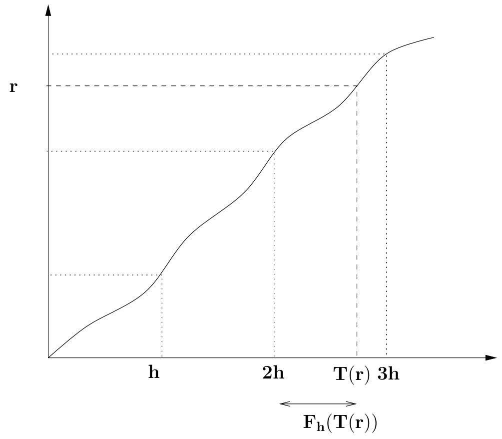

Figure 2.5: Fractional h-part of the hitting time

Also by relations (2.41) and (2.48) it follows that

$$h\lfloor \mathbf{T}(r)h^{-1}\rfloor \quad = \quad \mathbf{T}_h(r) - h \quad = \quad h\mathbf{N}_h(r)$$

For the h-fractional part one can now show the following result.

Lemma 27 *If* $\mathbf{N}_h = \{\mathbf{N}_h(t) : t \geq 0\}, h > 0$ *is the renewal process generated by the partial sum process* $\mathbf{S}_n = \mathbf{X}(nh), n \in \mathbb{N} \cup \{0\}$ *then for* $r > 0$

$$\mathbb{P}(\mathcal{F}_h(\mathbf{T}(r)) \leq t) \quad = \quad \int_{0-}^{r} (1 - F_t(r - x))dU_h(x)$$

for every $0 < t < h$ *with* $U(x) = \mathbb{E}(\mathbf{N}_h(x)) = \sum_{k=0}^{\infty} F_h^{k*}(x)$ *the well-known renewal function.*

Proof. By the definition of $\mathcal{F}_h(\mathbf{T}(r))$ we obtain for every $0 < t < h$ that

$$\mathbb{P}(\mathcal{F}_h(\mathbf{T}(r)) \leq t) \quad = \quad \mathbb{P}(\cup_{k=0}^{\infty}\{kh \leq \mathbf{T}(r) \leq kh + t\}) \quad = \quad \sum_{k=0}^{\infty} \mathbb{P}(kh \leq \mathbf{T}(r) \leq kh + t)$$

Since the cdf of $\mathbf{T}(r)$ is continuous it follows

$$\{kh \leq \mathbf{T}(r) \leq kh + t\} \quad \stackrel{a.s}{=} \quad \{kh < \mathbf{T}(r) \leq kh + t\} \quad = \quad \{\mathbf{X}(kh) \leq r < \mathbf{X}(kh + t\}$$

and so

$$\mathbb{P}(\cup_{k=0}^{\infty}\{kh \leq \mathbf{T}(r) \leq kh + t) \quad = \quad \sum_{k=0}^{\infty} \mathbb{P}(\mathbf{X}(kh) \leq r < \mathbf{X}(kh + t). \tag{2.118}$$

Using now $\mathbf{X}(kh+t) \stackrel{d}{=} \mathbf{X}(kh) + \mathbf{Z}(t)$ with $\mathbf{Z}(t) \stackrel{d}{=} \mathbf{X}(t)$ and $\mathbf{Z}(t)$ independent of $\mathbf{X}$ yields

$$\begin{aligned}
\mathbb{P}(\mathbf{X}(kh) \leq r < \mathbf{X}(kh+t) &= \mathbb{P}(\mathbf{X}(kh) \leq r < \mathbf{X}(kh) + \mathbf{Z}(t)) \\
&= \int_0^r \mathbb{P}(\mathbf{X}(kh \leq r < \mathbf{X}(kh) + \mathbf{Z}(t) | \mathbf{X}(kh) = x) dF_h^{k*}(x) \\
&= \int_0^r \mathbb{P}(r < x + \mathbf{Z}(t)) dF_h^{k*}(x) \\
&= \int_0^r \mathbb{P}(\mathbf{Z}(t) > r - x) dF_h^{k*}(x) \\
&= \int_0^r 1 - F_t(r-x) dF_h^{k*}(x).
\end{aligned}$$

Finally applying relation (2.118) it follows that

$$\begin{aligned}
\mathbb{P}(\mathcal{F}_h(\mathbf{T}(r)) \leq t) &= \sum_{k=0}^{\infty} \int_0^r 1 - F_t(r-x) dF_h^{k*}(x) \\
&= \int_0^r 1 - F_t(r-x) d \sum_{k=0}^{\infty} F_h^{k*}(x) \\
&= \int_0^r 1 - F_t(r-x) dU_h(x).
\end{aligned}$$

and this shows the desired result. $\square$

Using the strong renewal theorem and $\mathbb{E}(\mathbf{X}(h)) = h$ it follows for F_h non-lattice that by Lemma 27

$$\begin{aligned}
\lim_{r \uparrow \infty} \mathbb{P}(\mathcal{F}_h(\mathbf{T}(r)) \leq t) &= \lim_{r \uparrow \infty} \int_{0-}^r (1 - F_t(r-x)) dU_h(x) \\
&= \frac{1}{h} \int_0^{\infty} (1 - F_t(x)) dx \\
&= \frac{\mathbb{E}[\mathbf{X}(t)]}{h} \\
&= \frac{t}{h}
\end{aligned}$$

Hence this shows for $r \uparrow \infty$ that the random variable $\mathcal{F}_h(\mathbf{T}(r))$ converges in distribution to a random variable uniformly distributed on $(0, h)$.

CHAPTER 3

CONCLUSION

In this thesis we derived some important properties of increasing Levy processes useful in applications and proved these properties by making use of well known results for renewal processes. For some of these properties these proofs are easier than the already known proofs using martingale theory, while other results appear to be new.

APPENDIX

In this appendix we collect some important definitions and results used in this thesis. In the next definition we use the convention that the infimum of an empty set is $+\infty$.

Definition 10 *If $f : \mathbb{R}_+ \to \mathbb{R}$ is a nondecreasing function on $\mathbb{R}_+$ the inverse function of f is given by*

$$f^{\leftarrow}(y) = \inf\{s \geq 0 : f(s) \geq y\}. \tag{1}$$

Lemma 28 *If ν is a time transformation function then the function ν is convex if and only if the function $\nu^{\leftarrow}$ is concave.*

Proof. If ν is convex, then clearly for every t_1, t_2 and $0 < \alpha < 1$ we obtain using $\nu(\nu^{\leftarrow}(s)) = s$ for every $s > 0$ that

$$\begin{aligned} \nu(\alpha\nu^{\leftarrow}(t_1) + (1-\alpha)\nu^{\leftarrow}(t_2)) &\leq \alpha\nu(\nu^{\leftarrow}(t_1)) + (1-\alpha)\nu(\nu^{\leftarrow}(t_2)) \\ &= \alpha t_1 + (1-\alpha)t_2 \end{aligned}$$

This implies using $\nu^{\leftarrow}$ is increasing for any increasing function ν and $\nu^{\leftarrow}(\nu(s)) = s$ for every $s > 0$ that

$$\begin{aligned} \alpha\nu^{\leftarrow}(t_1) + (1-\alpha)\nu^{\leftarrow}(t_2) &= \nu^{\leftarrow}(\nu(\alpha\nu^{\leftarrow}(t_1) + (1-\alpha)\nu^{\leftarrow}(t_2))) \\ &\leq \nu^{\leftarrow}(\alpha t_1 + (1-\alpha)t_2) \end{aligned}$$

and we have verified that $\nu^{\leftarrow}$ is concave. A similar type of proof can be given to the reverse implication. $\square$

A very important convergence result called the dominated convergence in probability theorem is listed in the next result.

Lemma 29 *If $|\mathbf{X}_n| \leq |\mathbf{Y}|$ for every $n \in \mathbb{N}$, $\mathbb{E}(|\mathbf{Y}|) < \infty$ and $\mathbf{X}_n \xrightarrow{\mathcal{P}} \mathbf{X}$, then*

$$\lim_{n\uparrow\infty} \mathbb{E}(|\ \mathbf{X}_n - \mathbf{X}\ |) = 0 \tag{2}$$

Proof. We first show that $|\mathbf{X}| \leq |\mathbf{Y}|$ a.s. Since $|\mathbf{Y}| \geq |\mathbf{X}_n|$ for every $n \in \mathbb{N}$ it follows for every $\delta > 0$ that

$$\{| \mathbf{X} |>| \mathbf{Y} | +\delta\} \subseteq \{| \mathbf{X} |>| \mathbf{X}_n | +\delta\} \subseteq \{| \mathbf{X} - \mathbf{X}_n | +\delta\}.$$

By the above inclusions we obtain

$$\mathbb{P}\left(|\mathbf{X}| > |\mathbf{Y}| + \delta\right) \leq \mathbb{P}\left(|\mathbf{X}| > |\mathbf{X}_n| + \delta\right) \leq \mathbb{P}\left(|\mathbf{X} - \mathbf{X}_n| > \delta\right)$$

and using $\mathbf{X}_n \xrightarrow{\mathcal{P}} \mathbf{X}$ and $\delta > 0$ is arbitrary, this shows $P\left(|\mathbf{X}| > |\mathbf{Y}|\right) = 0$. Hence we have shown that

$$|\mathbf{X}| \leq |\mathbf{Y}| \text{ a.s.} \tag{3}$$

To show relation (2) we introduce for notational convenience the random variables $\mathbf{Y}_n := \mathbf{X}_n - \mathbf{X}$. Now it follows for any given $\epsilon > 0$ that

$$\begin{aligned} \mathbb{E}(| \mathbf{Y}_n|) &= \mathbb{E}(| \mathbf{Y}_n| \, \mathbf{1}_{\{|\mathbf{Y}_n|>2m\}}) + \mathbb{E}(| \mathbf{Y}_n| \, \mathbf{1}_{\{\epsilon<|\mathbf{Y}_n|\leq 2m\}}) + \mathbb{E}(| \mathbf{Y}_n| \, \mathbf{1}_{\{|\mathbf{Y}_n|\leq\epsilon\}}) \\ &\leq \mathbb{E}(| \mathbf{Y}_n| \, \mathbf{1}_{\{|\mathbf{Y}_n|>2m\}}) + 2m\mathbb{P}(\mid \mathbf{Y}_n \mid> \epsilon) + \epsilon. \end{aligned} \tag{4}$$

for any and $m \in \mathbb{N}$. Looking at the first term in relation (4) we observe by relation (3) and our assumption that

$$| \mathbf{Y}_n |\leq| \mathbf{X} | + | \mathbf{X}_n |\leq 2 | \mathbf{Y} | \text{ a.s.}$$

Hence it follows that $| \mathbf{Y}_n | \mathbf{1}_{\{|\mathbf{Y}_n|\geq 2m\}} \leq 2 | \mathbf{Y} | \mathbf{1}_{\{|\mathbf{Y}|>m\}}$ and this shows

$$\mathbb{E}(| \mathbf{Y}_n | \, \mathbf{1}_{\{|\mathbf{Y}_n|\geq 2m\}}) \leq 2\mathbb{E}(| \mathbf{Y} | \, \mathbf{1}_{\{|\mathbf{Y}|\geq m\}}) \tag{5}$$

for every $n \in \mathbb{N}$. Since $\mathbb{E}(| \mathbf{Y} |)$ is finite one can now find some $m_0 \in \mathbb{N}$ satisfying $\mathbb{E}(|\mathbf{Y} | \mathbf{1}_{\{|\mathbf{Y}|\geq m_0\}} \leq \frac{\epsilon}{2}$ and this shows by relation (5) that

$$\mathbb{E}(| \mathbf{Y}_n | \, \mathbf{1}_{\{|\mathbf{Y}_n|\geq 2m_0\}}) \leq \epsilon \tag{6}$$

for every $n \in \mathbb{N}$. Looking finally at the second term in relation (4) and using $\mathbf{Y}_n \xrightarrow{\mathcal{P}} 0$ one can find some $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ if holds that

$$2m_0\mathbb{P}(\mid \mathbf{Y}_n \mid> \epsilon) \leq \epsilon \tag{7}$$

for every $n \geq n_0$. Combing the above two terms in relations (6) and (7) and substituting this in relation (4) we have shown for every $\epsilon > 0$ that there exists some n_0 such that that for every $n \geq n_0$ it follows that $\mathbb{E}(|\mathbf{Y}_n|) \leq 3\epsilon$. This shows the desired result. $\square$

We finally mention the continuity theorem without proof. For a proof the reader is referred to [20])

Lemma 30 ***Continuity Theorem*** *For $n \in \mathbb{N}$ let X_n be a random variable with pLSt π_n.*

(i) If $X_n \xrightarrow{d} X$ as $n \to \infty$ then $\lim_{n\to\infty} \pi_n(s) = \pi_X(s)$ for $s \geq 0$

(ii) If $\lim_{n\to\infty} \pi_n(s) = \pi(s)$ for $s \geq 0$ with π a function that is (right-) continuous at zero, then π is the pLSt of a random variable X and $X_n \xrightarrow{d} X$ as $n \to \infty$

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