BLACK-SCHOLES MODEL AND ITS USE IN DIFFERENT SCENARIOS

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by

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ABSTRACT

In this thesis we study applications of stochastic calculus to models in financial mathematics. In particular, we consider the famous Black-Scholes model for option pricing, and also models for currency exchange and dividend payments.

ÖZET

Bu tezde stokastik kalkülüs uygulamalarını finansal matematik modellerine uygulayacağız. Özellikle, opsiyon fiyatlama için olan ünlü Black-Scholes modelini ve aynı zamanda kambiyo kuru ve temettü ödemeleri modellerini gözden geçireceğiz.

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1 Introduction

At the end of 19th century and the beginning of 20th century, the measure theory and axiomatic probability theory has been established by the leading mathematicians of the time among which were Henri Lebesgue (1875-1941), Emile Borel (1871-1956) and Andre Kolmogorov (1903-1987). Based on their axiomatic foundations, the prominent Japanese mathematician Kiyoti Ito (1915- 2008) developed what is today called Ito calculus. With these developments, the second half of 20th century has witnessed an unpredictable boom in the applications of mathematical theories in the financial settings.

In 1952, Harry Markovitz laid the groundwork for the theory of portfolio selection in his Ph.D thesis based on mean return of the stocks. After him, in 1969, Robert Merton introduced stochastic calculus into the study of stochastic calculus into the study of finance. At the same time as Merton's work and with Merton's assistance, Fischer Black and Myron Scholes has developed their celebrated option pricing formulae [3]. Based on the significance of this work, in 1997, the importance of their model was honoured world wide when Myron Scholes and Robert Merton received the Nobel Prize for Economics. Unfortunately, Fisher Black died in 1995, or he would have also received the award [4]. This model provided for the first time a solid solution to an important practical problem of finding a fair price for a European call option, i.e. the right to buy one share of a given stock at a specified price and time.

In this thesis, we will derive the Black-Scholes model in two ways. First we will derive the Black-Scholes formula first by so called martingale representation theorem. Then we will derive the formula by the assumption of self-financing portfolio and during the way we get a deterministic parabolic partial differential equation whose solution is the price of the option. Later, we will apply this framework to interest rates and continuous and periodic dividend payments. The rest of the thesis is as follows. In section 2 we will give the preliminary definitions. In section 3 we will derive the Black-Scholes option pricing formula first by martingale approach and then by PDE approach. In section 4 we will apply this framework to determine fair interest rates and fair continuous and periodic dividend payments. Then we conclude the thesis at section 5 by shortcomings of this model and new perspectives on pricing instruments.

2 Preliminaries

Definition 1. *A σ-algebra Σ is a collection of subsets of S which has the following properties:*

- $S \in \Sigma$
- *If $A \in \Sigma$ then $A^c \in \Sigma$*
- *If $(A_n,\, n \in \mathbb{N})$ n is a sequence of sets where each $A_n \in \Sigma$ then $\bigcup_{n \in N} A_n \in \Sigma$*

Definition 2. *A measure on (S, Σ) is a mapping m from Σ to $[0, \infty]$ which satisfies*

- $m(\emptyset) = 0$
- *(σ-additivity) If $(A_n,\, n \in \mathbb{N})$ n is a sequence of sets where each $A_n \in \Sigma$ and if these sets are mutually disjoint, i.e. $A_n \bigcap A_m = \emptyset$ if $m \neq n$, then*

$$m(\textstyle\bigcup_{n \in \mathbb{N}} A_n) = \sum_{n=1}^{\infty} m(A_n)$$

In general if m is a measure on (S, Σ), we call (S, Σ, m) *a measure space.*

Definition 3. *A probability measure on (S, Σ) is a measure which has total mass* 1.

In the case of probability measures we usually use a different notation and write Ω instead of S, $\mathcal{F}$ instead of Σ and P instead of m. The triple $(\Omega, \mathcal{F}, P)$ is called a *probability space.* Ω is called the *sample space* and elements of Ω are called *outcomes.* Sets in the σ-algebra are called *events.* If $A \in \mathcal{F}$, then $P(A)$ is called the *probability* of the event A. We always have $0 \leq P(A) \leq 1$.

Definition 4. *We define the* expectation *of X (when it exists) by*

$$\mathbb{E}(X) = \int_{\Omega} X(w) P(dw)$$

This exists if and only if X is integrable*, i.e.*

$$\mathbb{E}(|X|) < \infty$$

Definition 5. *Suppose that Σ_1 and Σ_2 are given σ-algebras of S_1 and S_2, respectively. A function f from S_1 to S_2 is said to be (Σ_1-Σ_2) measurable if for all $B \in \Sigma_2$ $f^{-1}(B) \in \Sigma_1$.*

If we have a probability space $(\Omega, \mathcal{F}, P)$ a *random variable* is defined to be a *measurable* function X from $(\Omega, \mathcal{F})$ to $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

Definition 6. *Let* $(\Omega, \mathcal{F}, P)$ *be a probability space. A stochastic process* $(X_t)_{t\in\mathcal{T}}$ *is a family of random variables defined on this space.* $\mathcal{T}$ *is called the index set. We talk of discrete time stochastic processes when* $\mathcal{T}$ *is a discrete set. Similarly, we talk of continuous time stochastic processes when* $\mathcal{I}$ *is an interval. Usually,* $\mathcal{T}$ *stands for time.*

We need to be able to model the flow of information in time. The standard way doing this is to use a filtration of sub-σ-algebras.

Definition 7. *A discrete filtration is a sequence of* $(\mathcal{F}_n, n \in \mathbb{Z}_+)$ *of sub-*σ*-algebras of* $\mathcal{F}$ *such that each*

$$\mathcal{F}_n \subseteq \mathcal{F}_{n+1}$$

Continuous filtration is defined analogously.

Definition 8. *Let* $(\mathcal{F}_n, n \in \mathbb{Z}_+)$ *be a filtration. A stochastic process* $Y = (Y_n, n \in \mathbb{Z}_+)$ *is said to be adapted to its filtration if each* Y_n *is* $\mathcal{F}_n$ *measurable. Continuous case is defined analogously.*

Theorem 9. (Radon-Nikodym) *If* P *and* Q *are finite measures on the measurable space* $(\Omega,\mathcal{F})$ *and* $Q(A) = 0$ *whenever* $A \in \mathcal{F}$ *and* $P(A) = 0$*, then there exists a positive measurable function* Y *on* Ω *such that*

$$Q(A) = \int_A Y dP, \tag{2.1}$$

for each $A \in \Sigma$*. Moreover, any* Σ *measurable function* Z *on* Ω *satisfying* (2.1) *for all* $A \in \mathcal{F}$ *is equal to* Y *almost everywhere.*
Here Y *is called the density of* Q *relative to* P *and denoted by* $\frac{dQ}{dP}$*, which is Radon-Nikodym derivative of* Q *relative to* P *[7].*

Definition 10. (Conditional Expectation) *If* X *is an integrable random variable on* $(\Omega, \mathcal{F}, P)$ *and* $\mathcal{G}$ *is a* σ*-algebra on* Ω *such that* $\mathcal{G} \subset \mathcal{G}$*, then there exists a* $\mathcal{G}$ *measurable integrable random variable on* $(\Omega, \mathcal{F}, P)$*,* $\mathbb{E}[X|\mathcal{G}]$*, such that*

$$\int_A \mathbb{E}[X|\mathcal{G}]dP = \int_A X dP \tag{2.2}$$

for all $A \in \mathcal{G}$*, then* $Y = \mathbb{E}[X|\mathcal{G}]$ *almost surely in* $(\Omega, \mathcal{F}, P)$*. When* Y *is a random variable, we let* $\mathbb{E}[X|Y] = \mathbb{E}[X|\mathcal{F}_Y]$*, where* $\mathcal{F}_Y$ *stands for the* σ*-algebra generated by the random variable* Y*. We call* $\mathbb{E}[X|\mathcal{G}]$ *and* $\mathbb{E}[X|Y]$ *the conditional expectations of* X *given* $\mathcal{G}$ *and* Y *respectively.*

Definition 11. (Cumulative Distribution Function) *The cumulative distribution function F_X of X is a Borel function from $\mathbb{R}$ to $[0,1]$. It is defined by*

$$F_X(x) = p_X((-\infty, x]) = P(X \leq x)$$

Definition 12. (Convergence Modes) *Let $(X_n, n \in \mathbb{N})$ be a sequence of random variables. There are four different ways in which it can converge to a random variable X, i.e. four different ways of giving meaning to the idea of "$lim_{n\to\infty}X_n = X$"*

- almost sure convergence*: In this case the numbers $X_n(w) \to X(w)$ as $n \to \infty$ for all $w \in \Omega$ except for a possible set of measure zero where convergence fails.*
- convergence in mean square*: In this case we require that*

$$lim_{n\to\infty}\mathbb{E}(|X_n - X|^2) = 0$$

- convergence in probability*: Here we require that for all $c > 0$*

$$lim_{n\to\infty}P(|X_n - X| > c) = 0$$

- convergence in distribution*: If F_n is the cdf of each X_n and F is the cdf of X, we require in this case that $F_n(x) \to F(x)$ as $lim_{n\to\infty}$ for all $x \in \mathbb{R}$ where $F(x)$ is continuous.*

Definition 13. (Brownian Motion) *Let $(\Omega, \mathcal{F}, P)$ be a probability space. A real-valued stochastic process $B = (B(t), t \geq 0)$ is called a Brownian motion if it satisfies the following - for all $0 \leq s < t < \infty$:*

- *$B(0) = 0(a.s.)$,*
- *$B(t) - B(s) \sim N(0, t - s)$,*
- *$B(t) - B(s)$ is independent of $\sigma\{B(u), 0 \leq u \leq s\}$,*
- *For (almost all) $w \in \Omega$, the mapping from $\mathbb{R}^+$ to $\mathbb{R}$ given by $t \to B(t)(w)$ is continuous.*

Definition 14. (Martingales) *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(\mathcal{F}_n, n \in \mathbb{Z}_+)$ be a filtration on this space. A stochastic process $M = (M_n, n \in \mathbb{R}^+)$ is said to be a (discrete time) martingale if*

- *it is adapted*

- *it is integrable (i.e. each* $\mathbb{E}(|X_n|) < \infty$ *)*
- $\mathbb{E}(X_n|\mathcal{F}_{n-1}) = X_{n-1}$ *for all* $n \in \mathbb{N}$

Theorem 15. (Girsanov Theorem) *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space and let* $(B_t)_{0\leq t\leq T}$ *be an* $\mathcal{F}$*-Brownian motion.*
Let $F \in \mathcal{P}_2(T)$ *then its* stochastic exponential *is the process* $\varepsilon_F = (\varepsilon_F(t), 0 \leq t \leq T)$ *where*

$$\varepsilon_F(t) = \exp\Big\{\int_0^t F(s)dB(s) - \frac{1}{2}\int_0^t F(s)^2 ds\Big\} \tag{2.3}$$

and this process satisfies the SDE

$$d\varepsilon_F(t) = F(t)\varepsilon_F(t)dB(t)$$

Furthermore we may define a new probability measure Q *on* $(\Omega, \mathcal{F})$ *which is equivalent to* P *and which has Radon-Nikodym derivative:*

$$\frac{dQ}{dP} = \varepsilon_F(T)$$

Girsanov's theorem *then tells us that* $(W(t), 0 \leq t \leq T)$ *is a Brownian motion on* $(\Omega, \mathcal{F}, Q)$ *where for each* $0 \leq t \leq T$*,*

$$W(t) = B(t) - \int_0^t F(s)ds \tag{2.4}$$

[5].

Definition 16. *We define* $\mathcal{H}_2(T)$ *to be the set of all processes* F *which satisfy the following:*

- $(F(t), 0 \leq t \leq T)$ *is adapted, i.e. each* $F(t)$ *is* $\mathcal{F}_t$*-measurable.*
- $||F||_T^2 = (||F||_T)^2 = \int_0^T \mathbb{E}(F(s)^2)ds < \infty$

Definition 17. *Similarly, we define* $\mathcal{P}_2(T)$ *to be the set of all processes* F *which satisfy the following:*

- $(F(t), 0 \leq t \leq T)$ *is adapted, i.e. each* $F(t)$ *is* $\mathcal{F}_t$*-measurable.*
- $\int_0^T \mathbb{E}(F(s)^2)ds < \infty$ *with probability* 1*.*

Definition 18. *We define the class of Ito processes* $(M(t), 0 \leq t \leq T)$ *where each*

$$M(t) = M(0) + \int_0^t F(s)dB(s) + \int_0^t G(s)ds \tag{2.5}$$

Here $M(0)$ *is an* $\mathcal{F}_0$*-measurable random variable so* $(M(0) = M_0)$, $F \in \mathcal{P}_2(T)$, *so the second term is a stochastic integral. The third term is a random Lebesgue integral, so we assume that* $G = (G(t), 0 \leq t \leq T)$ *is an adapted process such that the Lebesgue integral* $\int_0^t G(s)(w)ds$ *exists for all* $t \in [0, T]$ *and for (almost all)* $w \in \Omega$, *then each* $\int_0^t G(s)ds$ *is a* $\mathcal{F}_t$*-measurable random variable defined by*

$$\Big(\int_0^t G(s)ds\Big)(w) = \Big(\int_0^t G(s)ds(w)\Big)$$

Theorem 19. (Ito's Formula) *If* $f \in C_{1,2}$ *and* $M = (M(t), t \geq 0)$ *is an Ito process with stochastic differential* (2.5) *then* $(f(t, M(t)), 0 \leq t \leq T)$ *is an Ito process with stochastic differential*

$$df(t, M(t)) = \frac{\partial f}{\partial t}(t, M(t))dt + \frac{\partial f}{\partial x}(t, M(t))dM(t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, M(t))F(t)^2 dt$$

[5].

Theorem 20. (Martingale Representation Theorem) *If* $(M(t), 0 \leq t \leq T)$ *is a square-integrable martingale then there exists* $F \in \mathcal{H}_2(T)$ *such that for each* $0 \leq t \leq T$,

$$M(t) = \mathbb{E}(M(0)) + \int_0^t F(s)dB(s) \tag{2.6}$$

[5].

3 The Black-Scholes Model

Our setting will assume that the interval $[0, T]$, where T is the terminal date for all relevant academic activitiy. For simplicity we will work with a model which only involves two financial assets. These will be

- A risk-free security (a bond or bank account) whose value at time t is $A(t)$. We assume that the principal $A(0)$ is invested at a fixed interest rate $r > 0$ and so the formula for continuous compound interest yields

  $$A(t) = A(0)e^{rt}.$$

  Hence, $\frac{dA(t)}{dt} = rA(0)e^{rt} = rA(t)$ and we will often write this as $dA(t) = rA(t)dt$. For simplicitiy, we will assume that $A(0) = 1$

- A stock whose value at time t is $S(t)$.

We investigate the return of $S(t)$ and over a small time period $\delta(t)$ this is $\frac{\delta S(t)}{S(t)}$ where $\delta S(t) = S(t + \delta t) - S(t)$. If S behaves like A, we would write $\frac{\delta S(t)}{S(t)} = \mu \delta t$, where $\mu \in \mathbb{R}$. Since we need to include a factor which describes the random behaviour of stock prices and so we introduce an adapted process $(Y(t), 0 \leq t \leq T)$ and write

$$\frac{\delta S(t)}{S(t)} = \mu \delta t + \delta Y(t),$$

so that

$$\delta S(t) = \mu S(t) \delta t + S(t) \delta Y(t). \tag{3.1}$$

There is still debate as to which is the "best choice" for the process Y. In the Black-Scholes model we choose $Y(t) = \sigma B(t)$. Here $\sigma > 0$ and $B = (B(t), 0 \leq t \leq T)$ is a Brownian motion which is adapted to the filtration $(\mathcal{F}_t, 0 \leq t \leq T)$. In this model we take the formal limit of (3.1) as $\delta t \to 0$ and interpret the result as a stochastic differential equation (SDE) in the Ito sense. We then obtain

$$dS(t) = \mu S(t)dt + \sigma S(t)dB(t) \tag{3.2}$$

with initial condition $S(0) = S_0$ which is the stock price when the investor enters the market. $S = (S(t), 0 \leq t \leq T)$ is then an adapted process. Using Ito's formula we get that the unique solution to (3.2) is

$$S(t) = S_0 \exp\{\sigma B(t) + (\mu - \frac{1}{2}\sigma^2)t\}, \tag{3.3}$$

for all $0 \leq t \leq T$.

The parameter μ is called the *stock drift*. It measures the logarithmic rate of return of the stock in the absence of noise. The parameter σ is called the *volatility*. It measures the strength of the random fluctuations in the stock price. For further investigation, we need the following definitions:

Definition 21. *A contingent claim X is a $\mathcal{F}_T$-measurable random variable. If $X = f(S(T))$ for some measurable function f then X is called a European contingent claim or ECC e.g. a European call option $X = (S(T) - E)^+$ where k is the exercise price.*

Definition 22. *A portfolio is a pair (ϕ, ψ) of real-valued adapted process $\phi = (\phi(t), 0 \leq t \leq T)$ and $\psi = (\psi(t), 0 \leq t \leq T)$. We interpret $\phi(t)$ as the number of stocks held at time t and $\psi(t)$ as the number of bonds or bank accounts held at time t where $V = (V(t), 0 \leq t \leq T)$ is the corresponding adapted wealth process defined by*

$$V(t) = \psi(t)A(t) + \phi(t)S(t),$$

Definition 23. *Given a contingent claim X, the portfolio (ϕ, ψ) is said to be replicating if $V(T) = X$.*

Definition 24. *A portfolio (ϕ, ψ) is said to be self-financing if*

$$dV(T) = \psi(t)dA(t) + \phi(t)dS(t)$$

Definition 25. *We define the discounted wealth process and discounted stock process as $\tilde{V}$ and $\tilde{S}$ respectively, where $\tilde{V}(t) = A(t)^{-1}V(t) = e^{-rt}V(t)$ and $\tilde{S}(t) = e^{-rt}S(t)$*

Definition 26. *An ECC is attainable if there exists a self-financing portfolio ϕ such that*

$$X = V_\phi(T)$$

In this case ϕ is called a replicating strategy.

Definition 27. *A market is said to be complete if every ECC is attainable.*

Definition 28. *A self-financing strategy ϕ is said to be an arbitrage opportunity if*

$$V_\phi(0) = 0, V_\phi(T) \geq 0 \text{ and } P(V_\phi(T) > 0) > 0.$$

A market is said to be arbitrage-free if there are no arbitrage opportunities. Namely, if arbitrage opportunities exist, there is a non-zero probability that your portfolio can create wealth with no investment.

Theorem 29. (First Fundamental Theorem of Asset Pricing) *The market is arbitrage-free if and only if there exists at least one martingale measure [6].*

Theorem 30. (Second Fundamental Theorem of Asset Pricing) *An arbitrage-free market is complete if and only if there exists a unique martingale measure [6].*

3.1 Deriving the Black-Scholes Formula by Martingale Approach

In this section we will derive the equivalent probability measure Q and the corresponding claim martingale to reach our ultimate goal of reaching the formula. First, we need the corresponding measure.

Theorem 31. *There exists an equivalent probability measure Q under which $(\tilde{S}(t), 0 \leq t \leq T)$ is a martingale. It is obtained by taking $F(t) = \frac{r-\mu}{\sigma}$ in (2.6) for all $(0 \leq t \leq T)$. We then have*

$$d\tilde{S}(t) = \sigma \tilde{S}(t) dW(t) \tag{3.4}$$

Proof. We note that if $F(t) = \frac{r-\mu}{\sigma}$ then $dW(t) = dB(t) - (\frac{r-\mu}{\sigma})dt$.
We are given that $dS(t) = \mu S(t)dt + \sigma S(t)dB(t)$ and $\tilde{S}(t) = e^{-rt}S(t)$. By Ito's formula

$$\begin{aligned} d\tilde{S}(t) &= -re^{-rt}S(t)dt + e^{-rt}dS(t) \\ &= (\mu - r)\tilde{S}(t)dt + \sigma\tilde{S}(t)dB(t) \\ &= (\mu - r)\tilde{S}(t)dt + \sigma\tilde{S}(t)dW(t) + (r-\mu)\tilde{S}(t)dt \\ &= \sigma\tilde{S}(t)dW(t) \end{aligned}$$

So we have derived (3.4). Since W is a Q-Brownian motion, it follows that $\tilde{S}$ is a Q-martingale, indeed $\tilde{S}$ is a Q-geometric Brownian motion with

$$\tilde{S}(t) = S(0)\exp\{\sigma W(t) - \frac{1}{2}\sigma^2 t\}$$

□

The next step in the Black-Scholes theory by the martingale approach is, as the name implies, to obtain a martingale from the contingent claim X. We assume that X is square-integrable with respect to the measure Q, i.e. $\mathbb{E}_Q(X^2) \leq \infty$. We will turn the random variable X into a Q-martingale in a two stage process. First discount X so that X becomes $e^{-rT}X$ and then condition to define

$$\tilde{X}(t) = e^{-rT}\mathbb{E}_Q(X|\mathcal{F}_t)$$

Lemma 32. $\tilde{X} = (\tilde{X}(t), 0 \leq t \leq T)$ *is a square-integrable Q-martingale.*

Proof. The process $(\tilde{X}(t), 0 \leq t \leq T)$ is clearly adapted. It satisfies the martingale property as follows, if $0 \leq s \leq t \leq T$

$$\tilde{X}(s) = \mathbb{E}_Q(e^{-rT}X|\mathcal{F}_s) = \mathbb{E}_Q(\mathbb{E}_Q(e^{-rT}X|\mathcal{F}_s)|\mathcal{F}_t) = \mathbb{E}_Q(\tilde{X}|\mathcal{F}_s)$$

Square-integrability follows from conditional Jensen inequality as follows:

$$\mathbb{E}_Q(\tilde{X}(t)^2) = e^{-2rT}\mathbb{E}_Q(\mathbb{E}_Q(X|\mathcal{F}_t)^2) \leq e^{-2rT}\mathbb{E}_Q(\mathbb{E}_Q(X^2|\mathcal{F})) = e^{-2rT}\mathbb{E}_Q(X^2) < \infty$$

As the process $\tilde{X}$ is square-integrable, it is automatically integrable. □

We call the process $\tilde{X}$ the *claim martingale*. As it is square-integrable we can apply the (2.3) in the probability space $(\Omega, \mathcal{F}, Q)$ to deduce that there exists a process $\delta = (\delta(t), 0 \leq t \leq T)$ in $\mathcal{H}_2(T)$ such that for each $0 \leq t \leq T$,

$$\tilde{X}(t) = \tilde{X}(0) + \int_0^t \delta(u)dW(u) = \tilde{X}(0) + \int_0^t \gamma(u)d\tilde{S}(u), \tag{3.5}$$

where $\gamma(t) = \frac{\delta(t)}{\sigma\tilde{S}(t)}$ and we have used (3.4). We can rewrite (3.5) in stochastic differential form to obtain

$$d\tilde{X}(t) = \gamma(t)d\tilde{S}(t) \tag{3.6}$$

3.1.1 The Black-Scholes Portfolio

In this section, we will derive the corresponding portfolio based on the ideas above to show how to hedge and price an arbitrary contingent claim X. We define the *Black-Scholes portfolio* by

$$\phi(t) = \gamma(t); \quad \psi(t) = \tilde{X}(t) - \gamma(t)\tilde{S}(t), \tag{3.7}$$

for all $0 \leq s \leq t$. Its value at time t is

$$\begin{aligned} V(t) &= \gamma(t)S(t) + (\tilde{X}(t) - \gamma(t)\tilde{S}(t)A(t)) \\ &= \gamma(t)A(t)\tilde{S}(t) + (\tilde{X}(t) - \gamma(t)\tilde{S}(t)A(t) \\ &= \tilde{X}(t)A(t) \end{aligned} \tag{3.8}$$

Theorem 33. *The Black-Scholes portfolio is self-financing and replicating.*

Proof. The portfolio is replicating since by (3.8)

$$\begin{aligned} V(T) &= A(T)\tilde{X}(T) \\ &= e^{rT}e^{-rT}\mathbb{E}_Q(X|\mathcal{F}_T) \\ &= X, \end{aligned} \tag{3.9}$$

where we have used the fact that X is $\mathcal{F}_T$-measurable.

To see that the portfolio is self-financing, we find the stochastic differential of (3.8) using Ito's formula and apply (3.6) and (3.7) to obtain

$$\begin{aligned} dV(t) &= \tilde{X}(t)dA(t) + A(t)d\tilde{X}(t) \\ &= (\psi(t) + \gamma(t)\tilde{S}(t))dA(t) + \gamma(t)A(t)d\tilde{S}(t) \\ &= \psi(t)dA(t) + \gamma(t)[\tilde{S}(t)dA(t) + A(t)d\tilde{S}(t)] \\ &= \psi(t)dA(t) + \gamma(t)d[A(t)\tilde{S}(t)] \\ &= \psi(t)dA(t) + \phi(t)dS(t). \end{aligned} \tag{3.10}$$

Hence, the portfolio is self-financing, as well. □

From this we get that the *arbitrage price* of the option at time t is

$$\begin{aligned} V(t) &= A(t)\tilde{X}(t) \\ &= e^{rt}\mathbb{E}_Q(e^{-rT}X|\mathcal{F}_t) \\ &= e^{-r(T-t)}\mathbb{E}_Q(X|\mathcal{F}_t) \end{aligned} \tag{3.11}$$

i.e. to attempt to sell the option at this time at a greater or lower price creates arbitrage opportunities. This can be understood from the following scenarios as follows: Suppose the price of an ECC X is π'. If $\pi' > \pi_X(0)$, the owner should sell the option and invest $\pi_X(0)$ into a replicating portfolio ϕ. Then at time T, the value of the portfolio is $V_\phi(T) = X$ which is what we'd have if he'd kept the option. So he's made $\pi' - \pi_X(0)$ risk-free profit. If $\pi' < \pi_X(0)$, the option should be bought and kept until time T when it matures. Its value is now X which is what the buyer would have if the buyer had invested $\pi_X(0)$ into a replicating portfolio at time zero. So again a risk-free profit has been made.
In particular, the price of the option at time zero is

$$V(0) = e^{-rT}\mathbb{E}_Q(X) \tag{3.12}$$

3.1.2 Pricing a European Call Option via Martingale Approach

In this section we will price the European call option via the tools that we have developed above. We begin by considering the case of a general ECC, so $X = f(S(T))$ where f is a Borel measurable function from$[0, \infty)$ to $\mathbb{R}$. Then (3.12) becomes

$$V(0) = e^{-rT}\mathbb{E}_Q(f(S(T)))$$

Then by using (3.4) we have

$$\begin{aligned} S(T) &= A(T)\tilde{S}(T) \\ &= e^{rT}S(0)\exp\{\sigma W(T) - \frac{1}{2}\sigma^2 T\} \\ &= S(0)e^{U+rT}, \end{aligned}$$

where $U = \sigma W(T) - \frac{1}{2}\sigma^2 T$. Since under the measure Q, $W(T) \sim N(0,T)$, it follows that $U \sim N(-\frac{1}{2}\sigma^2 T, \sigma^2 T)$. By letting $S(0) = S$, we thus obtain

$$V(0) = \frac{e^{-rT}}{\sigma\sqrt{2\pi T}} \int_{-\infty}^{\infty} f(Se^{x+rT}) \exp\{-\frac{(x + \frac{1}{2}\sigma^2 T)^2}{2\sigma^2 T}\} \, dx \tag{3.13}$$

This is the *Black-Scholes pricing formula* for a general European contingent claim.

Now we will apply this to a European call option. In this case $f(x) = (x-E)^+$, where E is the exercise price. In particular, $f(Se^{x+rT}) = \max\{Ee^{x+rT} - E, 0\} \neq 0$ if and only if $x > \log\frac{E}{S} - rT$. Pulling the factor of e^{-rT} through the integral in (3.13), we then obtain

$$V(0) = \frac{1}{\sigma\sqrt{2\pi T}} \int_{\log(\frac{E}{S})-rT}^{\infty} (Se^x - Ee^{-rT}) \exp\{-\frac{(x+\frac{1}{2}\sigma^2 T)^2}{2\sigma^2 T}\} \, dx \tag{3.14}$$

Let Φ be the cdf of a standard normal, i.e. $\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{1}{2}u^2} \, du$, then we proceed as follows:
First split (3.14) into two integrals. In the first integral we have $e^x \exp\{-\frac{(x+\frac{1}{2}\sigma^2 T)^2}{2\sigma^2 T}\} = \exp\{-\frac{(x-\frac{1}{2}\sigma^2 T)^2}{2\sigma^2 T}\}$ Now substitute $y_1 = \frac{x-\frac{\sigma^2 T}{2}}{\sigma\sqrt{T}}$ into the first integral and $y_2 = \frac{x+\frac{\sigma^2 T}{2}}{\sigma\sqrt{T}}$ into the second one. After making these substitutions we have

$$\begin{aligned} V(0) &= S\frac{1}{\sqrt{2\pi}} \int_{-d_1}^{\infty} e^{-\frac{y_1^2}{2}} \, dy_1 - Ee^{-rT}\frac{1}{\sqrt{2\pi}} \int_{-d_2}^{\infty} e^{-\frac{y_2^2}{2}} \, dy_2 \\ &= E\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d1} e^{-\frac{u_1^2}{2}} \, du_1 - Ee^{-rT}\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_2} e^{-\frac{u_2^2}{2}} \, du_2 \end{aligned}$$

Hence we get the following:

$$V(0) = S\Phi\left(\frac{\log(\frac{S}{E}) + (r+\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right) - Ee^{-rT}\Phi\left(\frac{\log(\frac{S}{E}) + (r-\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}\right) \tag{3.15}$$

Equation (3.15) is the celebrated *Black-Scholes pricing formulae for a European call option*. It is often written as

$$V(0) = S\Phi(d1) - Ee^{-rT}\Phi(d2), \tag{3.16}$$

where $d_1 = \frac{\log(\frac{S}{E})+(r+\frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$ and $d_2 = d_1 - \sigma\sqrt{T}$.

3.2 Deriving the Black-Scholes Formula by PDE Approach

This section is mostly based on the derivation described in [2]. The model we have in (3.2) is an Ito process. Therefore, we let the function $F(S,t)$ be twice differentiable in S and in t. Applying the Ito lemma we obtain:

$$dF(S,t) = \frac{\partial F}{\partial S}dS + \frac{\partial F}{\partial t}dt + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 F}{\partial S^2}dt \tag{3.17}$$

Plugging into equation (3.17) for dS we have

$$dF(S,t) = \sigma S dB \frac{\partial F}{\partial S} + (\mu S \frac{\partial F}{\partial S} + \frac{\partial F}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 F}{\partial S^2})dt \tag{3.18}$$

Now set up a portfolio long one option , F, and short an amount $\frac{\partial F}{\partial S}$ stock. The value of the portfolio, π, is

$$\pi = F - \frac{\partial F}{\partial S}S \tag{3.19}$$

The change, $d\pi$ in the value of this portfolio over a small time interval dt is given by

$$d\pi = dF - \frac{\partial F}{\partial S}dS \tag{3.20}$$

Hence we get

$$d\pi = \sigma S dB \frac{\partial F}{\partial S}dB + (\mu S \frac{\partial F}{\partial S} + \frac{\partial F}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 F}{\partial S^2})dt - \frac{\partial F}{\partial S}(\mu S dt + \sigma S dB) \tag{3.21}$$

This simplifies to

$$d\pi = (\frac{\partial F}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 F}{\partial S^2})dt \tag{3.22}$$

At this point we note that this portfolio is completely riskless because it does not contain the random Brownian motion term. Since this portfolio contains no risk it must earn the same as other short-term risk-free securities. If it earned more than this, arbitrageurs could make a profit by selling the risk-free securities and using the proceeds to buy this portfolio. If the portfolio earned less arbitrageurs could make a riskless profit by selling the portfolio and buying the risk-free securities. It follows for a riskless portfolio that

$$d\pi = r\pi dt \tag{3.23}$$

where r is the risk free interest rate. Substituting for $d\pi$ and π we get

$$(\frac{\partial F}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 F}{\partial S^2})dt = r(F - S\frac{\partial F}{\partial S})dt \tag{3.24}$$

Further simplification yields the Black-Scholes differential equation

$$\frac{\partial F}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 F}{\partial S^2} + rS\frac{\partial F}{\partial S} - rF = 0 \tag{3.25}$$

In order to solve the Black-Scholes equation we will change the equation into an equation that we can work with. For this we change the variables as follows:

$$\begin{aligned}
S &= Ee^x, \text{ where E stands for exercise price} && (3.26)\\
t &= T - \frac{\tau}{\frac{1}{2}\sigma^2} && (3.27)\\
F &= Ef(x,\tau) && (3.28)
\end{aligned}$$

Using the chain rule from Calculus for transforming parital derivatives for functions of two variables we have

$$\begin{aligned}
\frac{\partial F}{\partial S} &= \frac{\partial F}{\partial x}\frac{\partial x}{\partial S} + \frac{\partial F}{\partial \tau}\frac{\partial \tau}{\partial S} && (3.29)\\
\frac{\partial F}{\partial t} &= \frac{\partial F}{\partial x}\frac{\partial x}{\partial \tau} + \frac{\partial F}{\partial \tau}\frac{\partial \tau}{\partial t} && (3.30)
\end{aligned}$$

By the corresponding transformations we have the following

$$\frac{\partial \tau}{\partial t} = -\frac{1}{2}\sigma^2 \qquad \frac{\partial x}{\partial t} = 0 \qquad \frac{\partial x}{\partial S} = \frac{1}{S} \qquad \frac{\partial \tau}{\partial S} = 0 \qquad (3.31)$$

Plugging these into the partial derivative terms we get

$$\begin{aligned}
\frac{\partial F}{\partial S} &= \frac{1}{S}\frac{\partial F}{\partial x} && (3.32)\\
\frac{\partial F}{\partial t} &= -\frac{1}{2}\sigma^2\frac{\partial F}{\partial \tau} && (3.33)\\
\frac{\partial^2 F}{\partial S^2} &= \frac{1}{S^2}\frac{1}{S^2}\frac{\partial^2 F}{\partial x^2} - \frac{1}{S^2}\frac{\partial F}{\partial x} && (3.34)
\end{aligned}$$

Substituting these into the *Black-Scholes partial differential equation* gives the differential equation

$$\frac{\partial F}{\partial \tau} = \frac{\partial^2 F}{\partial x^2} + (k-1)\frac{\partial F}{\partial x} - kF \qquad (3.35)$$

where

$$k = \frac{r}{\frac{1}{2}\sigma^2}$$

The initial condition $C(S,T) = \max(S-E,0)$ is transformed into

$$v(x,0) = \max(e^x - 1, 0)$$

Now we apply another change of variable and let

$$v(x,\tau) = e^{\alpha x + \beta \tau}u(x,\tau)$$

Then by simple differentiation we have

$$\begin{aligned}
\frac{\partial v}{\partial \tau} &= \beta e^{\alpha x+\beta\tau}u + e^{\alpha x+\beta\tau}\frac{\partial u}{\partial \tau}\\
\frac{\partial v}{\partial x} &= \alpha e^{\alpha x+\beta\tau}u + e^{\alpha x+\beta\tau}\frac{\partial u}{\partial x}\\
\frac{\partial^2 v}{\partial x^2} &= \alpha(\alpha e^{\alpha x+\beta\tau}u + e^{\alpha x+\beta\tau}\frac{\partial u}{\partial x}) + \alpha e^{\alpha x+\beta\tau}\frac{\partial u}{\partial x} + e^{\alpha x+\beta\tau}\frac{\partial^2 u}{\partial x^2}
\end{aligned}$$

Substituting these partials into equation (3.35) yields

$$\beta u + \frac{\partial u}{\partial \tau} = \alpha^2 u + 2\alpha \frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} + (k-1)(\alpha u + \frac{\partial u}{\partial x}) - ku$$

We can get rid of the u terms and the $\frac{\partial u}{\partial x}$ terms by carefully choosing values of α and β such that

$$\beta = \alpha^2 + (k-1)\alpha - k$$

and

$$2\alpha + k - 1 = 0$$

We can rearrange these equations so they can be written

$$\begin{aligned} \alpha &= -\frac{1}{2}(k-1) \\ \beta &= -\frac{1}{4}(k+1)^2 \end{aligned}$$

We now have the transformation from v to u as

$$v = e^{-\frac{1}{2}(k-1)x - \frac{1}{4}(k+1)^2\tau} u(x, \tau)$$

resulting in the diffusion equation

$$\frac{du}{d\tau} = \frac{d^2 u}{dx^2} \text{ for } -\infty < x < \infty, \tau > 0 \tag{3.36}$$

Our initial condition has now been changed as well to

$$u_0 = u(x, 0) = \max(e^{\frac{1}{2}(k+1)x} - e^{\frac{1}{2}(k-1)x}, 0) \tag{3.37}$$

We let $u(x,\tau) = w(\frac{x}{\sqrt{\tau}})$ i.e. we find a solution of the heat equation of the form

$$\begin{aligned}
u(x,\tau) &= w(\frac{x}{\sqrt{\tau}}) = w(p) \\
u_t &= w'(\frac{x}{\sqrt{\tau}}) \\
u_x &= \frac{1}{\sqrt{\tau}} w'(\frac{x}{\sqrt{\tau}}) \\
u_{xx} &= \frac{1}{\sqrt{\tau}} \frac{1}{\sqrt{\tau}} \cdot w''(\frac{x}{\sqrt{\tau}}) \\
-\frac{x}{2\tau^{\frac{3}{2}}} w'(\frac{x}{\sqrt{\tau}}) - \frac{1}{t} w''(\frac{x}{\sqrt{\tau}}) &= 0 \\
\frac{x}{2\sqrt{\tau}} w'(\frac{x}{\sqrt{\tau}}) + w''(\frac{x}{\sqrt{\tau}}) &= 0 \\
p &= \frac{x}{\sqrt{\tau}} \\
\frac{1}{2} \cdot p \cdot w'(p) + w''(p) &= 0 \\
\frac{w''(p)}{w'(p)} &= -\frac{1}{2} p \\
\ln w'(p) &= -\frac{1}{4} p^2 + C \\
w'(p) &= \exp -\frac{p^2}{4} \cdot C \\
w(s) &= c_1 \int_{-\infty}^{s} \exp -\frac{p^2}{4} dp + C_2
\end{aligned}$$

Hence we get the equation

$$Q(x,\tau) = \int_{-\infty}^{\frac{x}{\sqrt{\tau}}} \exp^{-\frac{p^2}{4}} dp$$

By linearity/superposition we get that

$$u(x,\tau) = \int_{-\infty}^{\infty} Q(x-s,\tau) C(s) ds$$

We also want that the initial condition is satisfied. Namely, we want

$$u(x,0) = \int_{-\infty}^{\infty} C(s) Q(x-s,0) ds = f(x)$$

Since

$$lim_{t \to 0^+} Q(x-s,\tau) = \begin{cases} 0 & \text{if } x < s \\ \sqrt{4\pi} & \text{if} x > s \end{cases}$$

We get that

$$u(x,t) = \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty} f'(s) Q(x-s,\tau) ds$$

Using integration by parts and noting the asymptotic behaviour of these functions we get that

$$u(x,\tau) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} f(s) e^{\frac{(x-s)^2}{4\tau}} ds, \text{ where } t > 0$$

Hence we get the well-known solution to the (3.36) as

$$u(x,\tau) = \frac{1}{\sqrt{4\pi\tau}} \int_{-\infty}^{\infty} u_0(s) e^{-\frac{(x-s)^2}{4\tau}} ds \tag{3.38}$$

where $u_0(x,0)$ is given by equation (3.37). In order to solve this integral it is very convenient to make a change of variable

$$y = \frac{s-x}{\sqrt{2\tau}}$$

so that

$$u(x,\tau) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u_0(y\sqrt{2\tau}+x) e^{-\frac{y^2}{2}} dy$$

Substituting our initial condition into this equation results in

$$u(x,\tau) = \frac{1}{\sqrt{2\pi}} \int_{-\frac{x}{\sqrt{2\tau}}}^{\infty} e^{\frac{1}{2}(k+1)(y\sqrt{2\tau}+x)} e^{-\frac{y^2}{2}} dy - \frac{1}{\sqrt{2\pi}} \int_{-\frac{x}{\sqrt{2\tau}}}^{\infty} e^{\frac{1}{2}(k-1)(y\sqrt{2\tau}+x)} e^{-\frac{y^2}{2}} dy$$

In order to solve this we will solve each integral separately. The first integral can be solved by completing the square in the exponent. The exponent of the first integral is

$$-\frac{1}{2}y^2 + \frac{1}{2}(k+1)(x+y\sqrt{2\pi})$$

Factoring out the $-\frac{1}{2}$ gives us

$$-\frac{1}{2}(y^2 - [k+1]y\sqrt{2\tau} - [k+1]x)$$

Separating out the term that is not a function of y, and adding and subtracting terms to set up a perfect square yields

$$\frac{1}{2}(k+1)x - \frac{1}{2}\left[y^2 - [k+1]y\sqrt{2\tau} + \left(\frac{[k+1]\sqrt{2\tau}}{2}\right)^2 - \left(\frac{[k+1]\sqrt{2\tau}}{2}\right)^2\right]$$

which can be written

$$\frac{1}{2}(k+1)x - \frac{1}{2}(y - \frac{[k+1]\sqrt{2\tau}}{2})^2 + \frac{1}{2}(\frac{[k+1]\sqrt{2\tau}}{2})^2$$

and simplified to

$$\frac{1}{2}(k+1)x - \frac{1}{2}(y - \frac{[k-1]\sqrt{2\tau}}{2})^2 + \frac{(k+1)^2\tau}{4}$$

Thus the first integral reduces to

$$I_1 = \frac{e^{\frac{1}{2}(k+1)x}}{\sqrt{2\pi}} \int_{-\frac{x}{\sqrt{2\tau}}}^{\infty} e^{\frac{1}{4}(k+1)^2\tau} e^{-\frac{1}{2}(y-\frac{1}{2}[k+1]\sqrt{2\tau})^2} dy$$

Now substituting

$$z = y - \frac{1}{2}[k+1]\sqrt{2\tau}$$

results in

$$\begin{aligned} I_1 &= \frac{e^{\frac{1}{2}(k+1)x} + \frac{1}{4}(k+1)^2\tau}{\sqrt{2\pi}} \int_{-\frac{x}{\sqrt{2\tau}}-\frac{1}{2}(k+1)\sqrt{2\tau}}^{\infty} e^{-\frac{1}{2}z^2} dz \\ &= \frac{e^{\frac{1}{2}(k+1)x} + \frac{1}{4}(k+1)^2\tau}{\sqrt{2\pi}} \Phi(d_1) \end{aligned}$$

where

$$d_1 = -\frac{x}{\sqrt{2\tau}} - \frac{1}{2}(k+1)\sqrt{2\tau}$$

and

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}y^2}$$

is the cumulative distribution function for the normal distribution.
The calculation of the second integral I_2 is identical to that of I_1 except that $(k-1)$ replaces $(k+1)$ throughout. Finally, we work our way backwards with

$$v(x,\tau) = e^{-\frac{1}{2}(k-1)x-\frac{1}{4}(k+1)^2\tau} u(x,\tau)$$

and then substituting the inverse transformations

$$\begin{aligned} x &= \log\left(\frac{S}{E}\right) \\ \tau &= \frac{1}{2}\sigma^2(T-t) \\ C &= Ev(x,\tau) \end{aligned}$$

we finally obtain the desired result

$$C(S,t) = S \cdot \Phi(d_1) - Ee^{-r(T-t)} \cdot \Phi(d_2)$$

where

$$d_1 = \frac{\log(\frac{S}{E}) + (r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

and

$$d_2 = \frac{\log(\frac{S}{E}) + (r-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

Hence the martingale approach and the PDE approach coincide as to be expected if you take $t = 0$ as your initial time.

3.2.1 Application of the Black-Scholes PDE to the Hedging Process

Our first application of the pde (3.25) is to hedging. Recall that we have constructed a portfolio $V(t) = \psi(t)A(t) + \phi(t)S(t)$ where the process $\phi(t)$ has been the hedging process. It turns out that if we differentiate the retrieved function $F(t,x)$ with respect to x we get the hedging process $\phi(t)$ itself. We will state this as our next theorem. For that we will need the following lemmas.

Lemma 34. *A necessary and sufficient condition for the discounted stock process $(\tilde{S}(t), 0 \leq t \leq T)$ to be a martingale is that $\mu = r$ where μ and r are stock drift and the interest rate as above.*

Proof. We are given that $dS(t) = \mu S(t)dt + \sigma S(t)dB(t)$ and $\tilde{S}(t) = e^{-rt}S(t)$. By Ito's formula we have

$$\begin{aligned} d\tilde{S}(t) &= -re^{-rt}S(t)dt + e^{-rt}dS(t) \\ &= (\mu - r)\tilde{S}(t)dt + \sigma\tilde{S}(t)dB(t) \end{aligned}$$

Hence $\tilde{S}$ will be a martingale if and only if $r = \mu$. This completes the proof. □

Lemma 35. *A portfolio with associated wealth process $V = (V(t), 0 \leq t \leq T)$ is self-financing if and only if the discounted wealth process satisfies*

$$d\tilde{V}(t) = \phi(t)d\tilde{S}(t) \tag{3.39}$$

Proof. First we note that for any portfolio (ψ, ϕ) the wealth at time t is $V(t) = \psi(t)e^{rt} + \phi(t)S(t)$ and so

$$\tilde{V}(t) = e^{-rt}V(t) = \psi(t) + \phi(t)\tilde{S}(t)$$

Now assume that the portfolio is self-financing so that $dV(t) = r\psi(t)e^{rt}dt + \phi(t)dS(t)$. Then by Ito's product formula

$$\begin{aligned} d\tilde{V}(t) &= -re^{-rt}V(t)dt + e^{-rt}dV(t) \\ &= -r\tilde{V}(t)dt + r\psi(t)dt + \phi(t)[\mu\tilde{S}(t)dt + \sigma\tilde{S}(t)dB(t)] \\ &= \phi(t)(\mu - r)\tilde{S}(t)dt + \phi(t)\sigma\tilde{S}(t)dB(t) \\ &= \phi(t)d\tilde{S}(t) \end{aligned}$$

Conversely suppose that $d\tilde{V}(t) = \phi(t)d\tilde{S}(t)$ Hence by Ito's product formula we have

$$d\tilde{V}(t) = \phi(t)(\mu - r)\tilde{S}(t)dt + \phi(t)\sigma\tilde{S}(t)dB(t)$$

Since $V(t) = e^{rt}\tilde{V}(t)$, Ito's product formula yields

$$
\begin{aligned}
dV(t) &= e^{rt}d\tilde{V}(t) + re^{rt}\tilde{V}(t)dt \\
&= \phi(t)(\mu - r)S(t)dt + \phi(t)\sigma S(t)dB(t) + rV(t)dt \\
&= \phi(t)(\mu - r)S(t)dt + \phi(t)\sigma S(t)dB(t) + r\psi(t)e^{rt}dt + r\phi(t)S(t)dt \\
&= r\psi(t)e^{rt}dt + \phi(t)[\mu S(t)dt + \sigma S(t)dB(t)] \\
&= \psi(t)dA(t) + \phi(t)dS(t)
\end{aligned}
$$

and so the portfolio is self-financing as was required. □

Theorem 36. *For each* $0 \leq t \leq T$,

$$\phi(t) = \frac{\partial F}{\partial x}(t, S(t))$$

Proof. Define $\tilde{F}(t,x) = e^{-rt}F(t, xe^{rt})$ Using the (3.25) we differentiate the function $\tilde{F}(t,x)$ as follows:

$$
\begin{aligned}
\frac{\partial \tilde{F}}{\partial t} &= -re^{-rt}F(t, xe^{rt}) + e^{-rt}\frac{\partial F}{\partial t}(t, xe^{rt}) + rx\frac{\partial F}{\partial x}(t, xe^{rt}) \\
&= e^{-rt}\big[\frac{\partial F}{\partial t}(t, xe^{rt}) - rF(t, xe^{rt}) + rxe^{rt}\frac{\partial F}{\partial x}(t, xe^{rt})\big] \\
&= -\frac{1}{2}e^{-rt}\sigma^2 x^2 e^{2rt}\frac{\partial^2 F}{\partial x^2}(t, xe^{rt}) \text{ by (3.25)} \\
&= -\frac{1}{2}\sigma^2 x^2 \frac{\partial^2 \tilde{F}}{\partial x^2}(t, x)
\end{aligned}
$$

Applying Ito's formula and using (3.4) we obtain

$$
\begin{aligned}
\tilde{F}(T, \tilde{S}(T)) - \tilde{F}(0, \tilde{S}(0)) &= \int_0^T \frac{\partial \tilde{F}}{\partial u}(u, \tilde{S}(u))du \\
&+ \int_0^T \sigma\tilde{S}(u)\frac{\partial F}{\partial x}(u, \tilde{S}(u))dW(u) + \int_0^T \frac{1}{2}\sigma^2\tilde{S}(u)^2\frac{\partial^2 \tilde{F}}{\partial x^2}(u, S(u))du \\
&= \int_0^T \frac{\partial \tilde{F}}{\partial x}(u, \tilde{S}(u))\sigma\tilde{S}(u)dW(u) \\
&= \int_0^T \frac{\partial \tilde{F}}{\partial x}(u, \tilde{S}(u))d\tilde{S}(u)
\end{aligned}
$$

Since $\tilde{V}(t) = \tilde{F}(t, S(t))$, we have shown that

$$\tilde{V}(T) = V(0) + \int_0^T \frac{\partial \tilde{F}}{\partial x}(u, \tilde{S}(u))d\tilde{S}(u)$$

Now since Black-Scholes portfolio is self-financing we have also by (3.39) $\tilde{V}(T) = V(0) + \int_0^T \phi(u) d\tilde{S}(u)$ and so we deduce that

$$\phi(t) = \frac{\partial F}{\partial x}(t, \tilde{S}(t)) = \frac{\partial F}{\partial x}(t, S(t))$$

$\square$

3.2.2 Hedging of the Black-Scholes Portfolio in a European Call Option Scenario

The exact mathematical details in a European call option scenario of the *hedging* concept appears below, but the basic idea is that we can in effect *duplicate* our option by a portfolio consisting of continually changing holdings of a risk-free bond and of the stock on which the call is written.
To be more specific the derivatives of the function $F(t, x)$ of (3.25) with respect to various variables are called the *Greeks*. By direct differentiation we get the following two of them namely *delta* and *theta*, which are defined respectively as follows:

$$\begin{aligned} F_x(x,t) &= \Phi(d_1(T-t,x)) \\ F_t(x,t) &= -rEe^{-r(T-t)}\Phi(d_2(T-t,x)) - \frac{\sigma x}{2\sqrt{T-t}}\Phi'(d_1(T-t,x)) \end{aligned}$$

Because both Φ and Φ' are always positive, delta is always positive and theta is always negative. Another of the Greeks is *gamma*, which is

$$F_{xx}(x,t) = \Phi'(d_1(T-t,x))\frac{\partial}{\partial x}d_1(T-t,x) = \frac{1}{\sigma x\sqrt{T-t}}\Phi'(d_1(T-t,x))$$

If at time t the stock price is x, then the hedging process for the European call option calls for holding $F_x(x,t)$ shares of stock, a position whose value is $x \cdot F_x = x \cdot \Phi(d_1)$. The hedging portfolio value is $F(x,t) = x \cdot (d_1) - Ee^{-r(T-t)}\Phi(d_2)$, and since $x \cdot F_x(x,t)$ of this value is invested in stock, the amount invested in the money market must be

$$F(x,t) - x \cdot F_x(x,t) = -Ee^{-r(T-t)}\Phi(d_-),$$

a negative number. To hedge a short position in a call option, one must borrow money. To hedge a long position in a call option, one does the opposite. Namely, to hedge a long position one should hold $-F_x$ shares of stock (i.e., have a short position in stock) and invest $Ke^{-r(T-t)}\Phi(d_2)$ in the money market account.
Because delta and gamma are positive, for fixed t, the function $F(x,t)$ is increasing and convex in the variable x. Suppose at time t the stock price is x_1 and we wish to take a long position in the position and hedge it. We do this by purchasing the option for $F(x_1,t)$, shorting $F_x(x_1,t)$ shares of stock, which generates income $x_1 \cdot F_x(x_1,t)$ and investing the difference,

$$M = x_1 \cdot F_x(x_1,t) - F(x_1,t)$$

in the money market account. We wish to consider the sensitivity to stock price changes of portfolio that has these three components: long option, short stock, and long money market account. The initial portfolio value

$$F(x_1, t) - x_1 \cdot F_x(x_1, t) + M$$

is zero at the moment t when we set up these positions. If the stock price were to instantaneously fall to x_0 and we do not change our positions in the stock or money market account, then the value of the option we hold would fall to $F(x_0, t)$ and the liablity due to our short position in stock would decrease to $x_0 \cdot F_x(x_1, t)$. Our total portfolio value, including M in the money market account, would be

$$F(x_0, t) - x_0 \cdot F_x(x_1, x) + M = F(x_0, t) - F_x(x_1, t) \cdot (x_0 - x_1) - F(x_1, t)$$

This is the difference at x_0 between the curve $y = F(x, t)$ and the straight line $y = F_x(x_1, t)(x - x_1) + F(x_1, t)$. Because this difference is positive, our portfolio benefits from an instantaneous drop in the stock price. On the other hand, if the stock price were to instantaneously rise to x_2 and we do not change our positions in the stock or money market account, then the value of the option would rise to $F(x_2, t)$ and the liability due to our short position in stock would increase to $x_2 \cdot F_x(x_1, t)$. Our total portfolio value, including M in the money market account, would be

$$F(x_2, t) - x_2 \cdot F_x(t, x_1) + M = F(x_2, t) - F_x(x_1, t) \cdot (x_2 - x_1) - F(x_1, t)$$

This is the difference at x_2 between the curve $y = F(t, x)$ and the straight line $y = F_x(x_1, t)(x - x_1) + F(x_1, t)$. This difference is positive so our portfolio benefits from an instantaneous rise in the stock price.

The portfolio we have set up is said to be *delta-neutral* and *long gamma*. The portfolio is long gamma because it benefits from the convexity of $F(t, x)$ as described above. If there is an instantaneous rise or fall in the stock price, the value of the portfolio increases. A long gamma portfolio is profitable in high stock volatility.

Delta-neutral refers to the fact that the line $y = F_x(x_1, t)(x - x_1) + F(x_1, t)$ is tangent to the curve $y = F(t, x)$. Therefore, the straight line is a good approximation to the option price *for small stock price moves*. If the straight line were steeper than the option price curve at the starting point x_1, then we would be *shortdelta*; an upward move in the stock price would hurt the portfolio because the liability from the short position in stock would rise faster than the value of the option. On the other hand, a downward move would increase the portfolio value because the option price would fall more slowly than the rate of decrease in the liability from the short stock position. Unless, a trader has a view on the market, he tries to set up portfolios that are delta-neutral. If he expects high-volatiliy, he would at the same time try to choose the portfolio to be long gamma.

The essence of the hedging argument is that if the stock price is a really a geometric Brownian motion and we have determined the right value of the volatility σ, then so long as we continuously rebalance our portfolio, all these effects exactly cancel! Furthermore, based on the model, the more volatile stocks offer more opportunity for profit from the portfolio that hedges a long call position with a short stock position, and hence the call is more expensive. The derivative of the option price with respect to the volatility σ is called *vega*, and it is positive. As volatility increases, so do option prices in the Black-Scholes model [8].

4 Variations on Black-Scholes Model

4.1 Foreign Exchange

We would like to be able to price and hedge claims in different currencies. This introduces a new complication - the exchange rate between currencies. We will consider US and UK investors seeking to hold assets in either currency. For simplicity we will just deal with risk-free assets in either currency. So
$(A(t), 0 \leq \mathrm{t} \leq \mathrm{T})$ is a dollar cash bond.
$(D(t), 0 \leq \mathrm{t} \leq \mathrm{T})$ is a Sterling cash bond.
$(E(t), 0 \leq \mathrm{t} \leq \mathrm{T})$ is the exchange rate,
i.e. $E(t)$ is the value of one pound in dollars at time t.

We assume the following dynamical behaviour of these quantities

$$\begin{aligned} A(t) &= e^{rt} \\ D(t) &= e^{ut} \\ E(t) &= E_0 \exp\{\mu t + \sigma B(t)\}, \end{aligned}$$

so the risk-free assets A and D are continuously compounded at rates r and u (respectively), while E evolves as a geometric Brownian motion where E_0 is a constant. We take the point of view of a dollar investor who wants to hold dollar-valued options that speculate on the future value of Sterling. Frim his point of view neither the exchange rate E nor the the Sterling bond D are tradeable assets, but their product is so we define $S(t) = E(t)D(t)$
We can now apply the Black-Scholes methodology within the dollar market to the two assets A and S. The first step is to find a measure Q_d under which the discounted product $\tilde{S}(t) = A(t)^{-1}S(t)$ becomes a martingale. We have by Ito's formula

$$d\tilde{S}(t) = \sigma \tilde{S}(t)(dB(t) - \theta dt),$$

where $\theta = \frac{r-\mu-u-\frac{1}{2}\sigma^2}{\sigma}$. We apply Girsanov's theorem and find that $\frac{dQ_d}{dP} = \exp\{\theta B(T) - \frac{1}{2}\theta^2 T\}$ and $(W(t), 0 \leq t \leq T)$ is a Q_d-Brownian motion where $W(t) = B(t) - \theta \cdot t$ so that $d\tilde{S}(t) = \tilde{S}(t)\sigma dW(t)$.
It then follows that the arbitrage price at time t of a contingent claim X (priced in dollars) is

$$V(t) = e^{-r(T-t)}\mathbb{E}_Q(X|\mathcal{F}_t)$$

We will study this further in the case where X is a forward contract to buy one unit of Sterling at time T for the price k. So the value of the claim at the terminal date is $X = E(T) - k$ and so

$$V(t) = e^{-r(T-t)}\mathbb{E}_Q(E(T) - k|\mathcal{F}_t)$$

A forward contract costs nothing at time zero and so we must have

$$0 = V(0) = e^{-rT}\mathbb{E}_Q(E(T) - k) = e^{-rT}\mathbb{E}_Q(E(T)) - e^{-rT}k$$

and hence the (arbitrage-free) strike price must be $k = \mathbb{E}_Q(E(T))$. We can simplify this even further. Writing $E(T)$ in terms of the Q_d-Brownian motion W we find that

$$E(T) = E_0 \exp\{\sigma W(T) + (r - u - \frac{1}{2}\sigma^2)T\},$$

so the calculation of $\mathbb{E}_Q(E(T))$ boils down to that of the moment generating function for a normal distribution. Hence we get that

$$E = e^{(r-u)T} E_0$$

and so

$$V(t) = e^{-r(T-t)} \mathbb{E}_Q(E(T) - e^{(r-u)T} E_0 | \mathcal{F}_t) \tag{4.1}$$

Our next step in the dollar market is to find the precise hedging portfolio (ϕ, ψ) whose value at time t is $V(t) = \phi(t)S(t) + \psi(t)A(t)$. To do this we use the fact that for each $0 \leq t \leq T$, $E(t) = D(t)^{-1}S(t) = D(t)^{-1}A(t)\tilde{S}(t)$. Now $D(T)^{-1} = e^{-u(T-t)}D(t)^{-1}$ and $A(T) = e^{r(T-t)}A(t)$. Since $\tilde{S}$ is a Q_d-martingale we have

$$\begin{aligned}
\mathbb{E}_Q(E(T)|\mathcal{F}_t) &= e^{(r-u)(T-t)} D(t)^{-1} A(t) \mathbb{E}_Q(\tilde{S}(T)|\mathcal{F}_t) \\
&= e^{(r-u)(T-t)} D(t)^{-1} A(t) \tilde{S}(t) \\
&= e^{(r-u)(T-t)} E(t)
\end{aligned}$$

Substituting into (4.1) we obtain

$$\begin{aligned}
V(t) &= e^{-u(T-t)} E(t) - e^{rt-uT} E_0 \\
&= e^{-uT}(e^{ut} E(t) - e^{rt} E_0)
\end{aligned}$$

If we discount in dollars we have $\tilde{V}(t) = e^{-rt}V(t)$ and so

$$\begin{aligned}
\tilde{V}(t) &= e^{-uT}(e^{-(r-u)t} E(t) - E_0) \\
&= e^{-uT} \tilde{S}(t) - e^{-uT} E_0
\end{aligned}$$

From this we see that the portfolio is constant, i.e. for all $0 \leq t \leq T$, $\phi(t) = e^{-uT}$ and $\psi(t) = -e^{-uT} E_0$.

So far we have investigated this transaction from the point of view of the dollar investigator. We now take the perspective of the Sterling investor. From this point of view, there are two tradeable assets, these being the Sterling bond $D = (D(t), 0 \leq t \leq T)$ and the Sterling value of the dollar bond which is $Z = (Z(t), 0 \leq t \leq T)$ where each $Z(t) = E(t)^{-1}A(t)$. The discounted Sterling value at time t is $\tilde{Z}(t) = D(t)^{-1}Z(t)$. Now we get the corresponding equivalent measure as follows

$$\begin{aligned}
\tilde{Z}(t) &= D(t)^{-1} E(t)^{-1} A(t) \\
&= E_0^{-1} \exp\{-\sigma B(t) - (\mu + u - r)t\}
\end{aligned}$$

Hence

$$d\tilde{Z}(t) = \tilde{Z}(t)\big(- \sigma dB(t) - \big(\mu + u - r - \frac{1}{2}\sigma^2\big) dt\big)$$

and by Girsanov's theorem the measure Q_p under which $\tilde{Z}$ is a Q_p-martingale which is given by

$$\frac{dQ_p}{dP} = \exp\left\{\lambda B(t) - \frac{1}{2}\lambda^2 T\right\}$$

where $\lambda = \frac{r+\frac{1}{2}\sigma^2-\mu-u}{\sigma}$ and so W' is a Q_p-Brownian motion where for each $0 \leq t \leq T$,

$$W'(t) = B(t) - \lambda t$$

So the arbitrage-price at time t in Stirling of the claim X (which we recall is priced in dollars) is

$$\begin{aligned} U(t) &= D(t)\mathbb{E}_Q(D(T)^{-1}E(T)^{-1}X|\mathcal{F}_t) && (4.2) \\ &= e^{-u(T-t)}\mathbb{E}_Q(E(T)^{-1}X|\mathcal{F}_t) && (4.3) \end{aligned}$$

4.2 Dividends

4.2.1 Continuous Dividend Payments

Our final variation on the standard Black-Scholes theory will deal with dividends. We will work with continuous time and real-valued asset prices. We will begin by making the assumption that a dividend is paid *continuously* at a fixed rate c where $0 \leq c \leq 1$. So in the time period $[t, t+\delta t]$, the owner of the stock receives a dividend payment of $cS(t)\delta(t)$.
The difficulty now is that $S(t)$ no longer represents the true worth of the asset at time t as it does not take into account of the accumulated dividends up to that time. In other words S is not a *tradeable asset*. The solution to this difficulty is to translate the problem into one which involves tradeable assets. We make the further assumption that whenever a dividend is paid then it is immediately reinvested in the stock. So the dividend $cS(t)\delta(t)$ is used to buy $c\delta(t)$ units of stock. We thus construct a new tradeable asset $Z = (Z(t), 0 \leq t \leq T)$ whose return at time t is

$$\begin{aligned}\frac{\delta Z(t)}{Z(t)} &= c\delta t + \frac{\delta S(t)}{S(t)} \\ &= c\delta t + \mu\delta t + \sigma\delta B(t),\end{aligned}$$

so that

$$\delta Z(t) = (\mu + c)Z(t)\delta t + \sigma Z(t)\delta B(t)$$

hence by letting $\delta t \to 0$ we get the following SDE in the Ito sense

$$dZ(t) = (\mu + c)Z(t)dt + \sigma Z(t)dB(t)$$

We thus see that at time t rather than holding one unit of stock we have e^{ct} units with total value

$$\begin{aligned}Z(t) &= e^{ct}S_t \\ &= e^{ct}S_0 \exp\left\{\sigma B(t) + \left(\mu - \frac{1}{2}\sigma^2\right)t\right\}\end{aligned}$$

Hence by Ito's formula we get that

$$dZ(t) = e^{ct}dS(t) + cZ(t)dt \tag{4.4}$$

Our strategy will be to regard Z as a single asset which pays no dividends. We now apply Black-Scholes theory but with the asset Z taking the place of S. Any portfolio which (at time t) contains $\phi(t)e^{ct}$ units of original stock $S(t)$ and $\psi(t)$ units of risk-free asset $A(t)$ can be thought of as a portfolio which contains $\phi(t)$ units of the new asset $Z(t)$ and $\psi(t)$ units of $A(t)$. Note that in this context, for (ϕ, ψ) to be self-financing, we require

$$\begin{aligned}dV(t) &= \phi(t)dZ(t) + \psi(t)dA(t) \\ &= \phi(t)e^{ct}dS(t) + c\phi(t)Z(t)dt + \psi(t)dA(t)\end{aligned}$$

by (4.4). We will now price a European call option with strike price k and maturity date T which is written on the dividend paying stock. First define $\tilde{Z}(t) = A(t)^{-1}Z(t) = e^{-rt}Z(t)$. We then have by Ito's formula

$$d\tilde{Z}(t) = \sigma\tilde{Z}(t)dB(t) + (\mu + c - r)\tilde{Z}(t)dt$$

We now apply Girsanov's theorem and take $F(t) = \frac{r-\mu-c}{\sigma}$ in (2.3) to obtain the equivalent probability measure Q then $W = (W(t), 0 \leq t \leq T)$ is a Q-Brownian motion where $W(t) = B(t) + \frac{\mu+c-r}{\sigma}t$ and

$$d\tilde{Z}(t) = \sigma\tilde{Z}(t)dW(t)$$

The value of the replicating portfolio at time t (and hence the arbitrage price of the option) is

$$V(t) = e^{-r(T-t)}\mathbb{E}_Q((S(T) - k)^+|\mathcal{F}_t)$$

Taking into consideration that $\tilde{Z}$ (and not $\tilde{S}$) which is a martingale under the probability measure Q and so we can write

$$\begin{aligned} V(t) &= e^{-r(T-t)}\mathbb{E}_Q((e^{-cT}Z(T) - k)^+|\mathcal{F}_t) \\ V(t) &= e^{-r(T-t)}e^{-cT}\mathbb{E}_Q((Z(T) - ke^{cT})^+|\mathcal{F}_t) \end{aligned}$$

Hence by denoting $G(t) = e^{(r-c)(T-t)}S(t)$ and $\theta = T - t$ we get that

$$V(t) = e^{-r\theta}\left\{G(t)\Phi\left(\frac{\log\left(\frac{G(t)}{k}\right) + \frac{\sigma^2\theta}{2}}{\sigma\sqrt{\theta}}\right) - k\Phi\left(\frac{\log\left(\frac{G(t)}{k}\right) - \frac{\sigma^2\theta}{2}}{\sigma\sqrt{\theta}}\right)\right\}$$

where Φ denotes the standard normal distribution as usual.

4.2.2 Periodic Dividend Payments

In the real world, dividend payments are paid at regular intervals, say at times $T_1, T_2, ...$ At each time T_i the holder of the stock receives a payout of $cS(T_i)$ where $0 < c < 1$. Between payouts, the stock price will evolve according to the usual geometric Brownian motion model. We model the arbitrage-free model of the stock price in this context as

$$S(t) = S_0(1 - c)^{N(t)}\exp\left\{\sigma B(t) + \left(\mu - \frac{1}{2}\sigma^2\right)t\right\},$$

where $N(t) = \max i, T_i \leq t$. Again we have the problem that S is not a tradeable asset and this can again be remedied by reinvesting the dividend payment into stock. So at each time T_i our total holding of stock will increase by a factor of $(1 - c)^{-1}$ and we work with the process $Z = (Z(t), 0 \leq t \leq T)$ where

$$Z(t) = (1 - c)^{-N(t)}S(t) = S_0\exp\left\{\sigma B(t) + \left(\mu - \frac{1}{2}\sigma^2\right)t\right\}$$

We will thus apply the Black-Scholes model wih the assets Z and A instead of S and A. The analysis is fairly similar to that in the standard Black-Scholes case. So the equivalent martingale measure Q is again obtained by taking $F(t) = \frac{r-\mu}{\sigma}$ in (2.3). To illustrate the use of the theory in this case, we will apply it to find the fair price K of a forward contract to buy one unit of periodic dividend paying stock at time T. So the claim is $X = S(T) - K$ and as usual the arbitrage price at time t is

$$\begin{aligned} V(t) &= e^{-r(T-t)}\mathbb{E}_Q((S(T) - K)|\mathcal{F}_t) \\ &= e^{-r(T-t)}\mathbb{E}_Q(((1-c)^{N(T)}Z(T) - K)|\mathcal{F}_t) \\ &= (1-c)^{N(T)}Z(t) - Ke^{-r(T-t)}, \end{aligned}$$

where we have used the fact that $\tilde{Z}$ is a Q-martingale where each $\tilde{Z}(t) = e^{-rt}Z(t)$. Now since $V(0) = 0$, we immediately deduce that

$$K = e^{rT}(1-c)^{N(T)}Z(0) = e^{rT}(1-c)^{N(T)}S_0$$

and get the arbitrage free fair price K of a forward contract.

5 Conclusion and Further Remarks

This thesis has dealt with developing the Black-Scholes approach to pricing and hedging of contingent claims and we have seen how to extend this to take account of both exchange rate complications and dividend payments. There are more extensions in this framework like *exotic options* such as digital and multistage options. These are still of basic ECC type. There are also important examples of options where the pay-offs are not ECC claims but which may depend on the whole history of the stock price $(S(t), 0 \leq t \leq T)$. These include American options and Asian options where $X = \left(\frac{1}{T}\int_0^T S(t)dt - K\right)^+$.

The Black-Scholes paradigm is extremely powerful and has launched a revolution in option pricing. However there are some problems with it. One of these is the main assumption - that the stock price is a geometric Brownian motion - is a very crude assumption. In recent years there has been a lot of work on replacing Brownian motion with more general stochastic processes which more accurately model the behaviour of the market. One particular direction that has been investigated by a number of workers is to use a general Levy process - i.e. a process that has independent and stationary increments - instead of Brownian motion. The paths of such a process are no longer continuous and "the discontinuous jumps in the corresponding stock price are a reflection of supply and demand shocks to the economy" [1] A major difficulty with moving away from Brownian motion is that the market is is *incomplete*, i.e. martingale measures are no longer unique and so the second fundamental theorem of asset pricing is no longer valid. Further studies on this research is to be found in [1].

References

[1] Applebaum, D. *Levy processes and stochastic calculus.* Cambridge, England. Cambridge University Press, 2004.

[2] Baxter, M.; Rennie A. *Financial calculus: an introduction to derivative pricing.* Cambridge, England: Cambridge University Press, 1996.

[3] Black F.; Scholes M. *The pricing of options and corporate liabilities.* The Journal of Political Economy. Vol. 81, No. 3. (May - Jun., 1973), pp. 637-654.

[4] Hull, J. *Options, futures and other derivatives.* Upper Saddle, River, New Jersey. Prentice Hall, 2000.

[5] Karatzas, I.;Shreve S.E. *Brownian motion and stochastic calculus.* 2nd ed. Graduate Texts in Mathematics, 113. Springer, New York, 1991.

[6] Karatzas, I.;Shreve S.E. *Methods of mathematical finance.* Graduate Texts in Mathematics. Springer, New York, 2001.

[7] Rudin, W. *Real and complex analysis.* Mc-Graw Hill, New York.

[8] Shreve S.E. *Stochastic calculus for finance II continuous time models.* 2nd ed. Finance, Springer, New York, 2008.