Bundle Pricing in Two-Stage Supply Chains

by

Burak Gökgür

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This is to certify that I have examined this copy of a doctoral dissertation by

Burak Gökgür

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and that any and all revisions required by the final

examining committee have been made.

Committee Members:

Prof. Dr. Selçuk Karabatı, (Advisor)

Prof. Dr. Gürhan Kök

Assoc. Prof. Dr. Murat Usman

Prof. Dr. Refik Güllü

Assoc. Prof. Dr. Okan Örsan Özener

Date:

To Mom and Dad...

ABSTRACT

What is the right promotion strategy a firm should pursue to stimulate demand and increase revenue? Many retailers and researchers propose a broad range of promotion strategies to answer this question. In this dissertation, we investigate the bundle pricing practice, as an alternative and less-studied form of promotion strategy.

In the first part of the dissertation, we study the one-time bundle pricing problem in a single-seller/single-buyer supply chain where the seller sells two different products. One of the products has excess inventory and the seller aims to clear the inventory by offering the one-time bundle offer. The main research objective of this chapter is to examine when it is beneficial to extend a one-time bundle offer to the buyer in lieu of an individual price reduction for the product with the excess inventory. To address this question, we model a game in a Stackelberg setting. Through a range of numerical experiments, our results suggest that when the targeted inventory liquidation quantity is not too high, the seller can be better off with the bundle offer.

In the second part, we study a multi-segment market in which a retailer aims to clear the inventory of a product by bundling it with a second product which is independently valued by the consumers, i.e., neither substitute nor complement. We investigate the question of how dynamic and segment-specific bundle pricing impacts retailers revenue. We develop a revenue model that integrates the dynamic and segment-specific aspects of the pricing decisions, and present a computational study to analyze their revenue impact relative to a price promotion for the individual item only. The results indicate that the bundle offers are most effective when the initial inventory of the item under consideration is high. The results also demonstrate that dynamic pricing is beneficial when the initial inventory of the item is low. An additional revenue improvement is observed when the price of the bundle is dynamically optimized. Segment-specific

pricing has no direct impact on revenue when prices are static; segment-specific and dynamic pricing, however, can bring about substantial revenue improvements that are an increasing function of the initial inventory level of the item. We also consider the correlation in consumers valuations of the bundled products and show that dynamically priced and segment-specific bundle offers yield a robust revenue performance, reducing the negative impact of positive correlation in consumers valuations of the products on the retailer's revenue.

In the third part of the dissertation, we explore the impact of strategic customers and the degree of substitutability among the products on the long-run per period revenue that the retailer can achieve through temporary bundle offers. We develop a strategic customer model to characterize the optimal purchasing and consumption policies in a multi-product setting. We address the question of how a strategic customers purchasing behavior, and characteristics of products in the bundle affect the retailer's revenue. We derive a closed-form expression of the optimal bundle price for a given promotion frequency in a setting where customers do not purchase the two products separately. Our results demonstrate that retailers should adjust their promotion policy (i.e., depth and frequency) with respect to 1) the characteristics of products that form the bundle, and 2) the market structure. Our findings suggest that when the products show a degree of substitutability, the retailer should present bundle offers with higher discount levels, and less frequently. When the bundle offer includes complementary products, the retailer should extend bundle offers with smaller discounts and more frequently.

ÖZETÇE

Firmalar, satışlarını hızlandırmak ve gelirlerini arttırmak için nasıl bir promosyon stratejisi takip etmeliler? Bu soru, birçok perakende şirketi ve araştırmacı tarafından cevaplandırılmaya çalışılmakta ve farklı promosyon politikaları geliştirilmektedir. Bu tezde, daha az çalışılmış ve alternatif bir promosyon yöntemi olan paket fiyatlandırma konusu ele alınmıştır. Bu tez, perakendecilerin paket fiyatlandırma süreçlerindeki farklı uygulamaları ve olguları ele alan üç ayrı çalışmadan oluşmaktadır.

İlk çalışmada, paket indirim modeli işletmeler arası (B2B) tedarik zincirinde incelenmiştir. Satıcının bir ürününde stok fazlası bulunmaktadır. Satıcı fazla olan stoğu bir defalık sunacağı paket indirimi ile eritmeyi amaçlamaktadır. Bu almadaki temel araştırma sorusu: Hangi durumlarda paket indirimi, tekli-fiyat indirim modeline göre satıcının elde ettiği gelir göz önüne alındığında daha iyi bir performans sergiler? Bu soruya yanıt bulabilmek için oyun teorisi modelleme yaklaşımını takip edilmiş, satıcının ve alıcının modelleri Stackelberg teorisi ile geliştirilmiştir. Yürütülen sayısal deneyler sonucunda, stok seviyesinin çok yüksek olmadığı durumlarda, satıcının paket indirim modeli ile daha fazla gelir elde edebildiği gözlemlenmiştir.

İkinci çalışmada, perakendeci bir ürününü çoklu segment yapısında bulunan müşteri havuzuna ikinci bir ürün ile beraber paket indirimi kampanyası ile satarak, envanterini istenilen seviyeye düşürmeyi amaçlamaktadır. Ayrca, paket indirimini oluşturacak ikinci ürünün, birinci ürün ile tamamlayıcı veya yerine geçici ilişkisi olmadığı varsayılmıştır. Bu çalışmada dinamik ve segment-bazlı paket fiyatlandırma politikasının perakendecinin gelirine olan etkileri incelenmiştir. Bu amaçla dinamik ve segment-bazlı fiyat kararlarını da dikkate alan bir gelir modeli geliştirilmiş, ve sayısal çalışma aracılığı ile gelire olan etkisi tekli-fiyat indirimi ile karşılaştırmalı olarak analiz edilmiştir. Paket indirim modelinin, ilgili ürünün stok seviyesi yüksek olduğu

zaman etkinliği gözlemlenmiştir. Dinamik fiyatlandırma politikasının ise, ilgili stok seviyesinin az olduğu durumlarda etkili olduğu görülmüştür. Paket indirim modeli dinamik bir şekilde fiyatlandırıldığı zaman perakendecinin gelirinin belirgin bir ekilde arttığı gözlemlenmiştir. Segment-bazlı fiyatlandırmanın tek başına gelir artırımı için yeterli olmadığı, dinamik fiyatlandırma ile beraber kullanımında ise etkin olduğu tespit edilmiştir. Paket indirim modeli hem dinamik bir şekilde hem de segment-bazlı fiyatlandırıldığı zaman müşterilerin çekince fiyatlarındaki korelasyon gözetilmeksizin gürbüz bir performans sergilediği gözlemlenmiştir.

Üçüncü çalışmada ise, stratejik müşterilerin ve paket indirimini oluşturan ürünlerin ilişkisinin, perakendecinin belirli aralıklarla müşterisine sunduğu paket indirim modelinden uzun dönem gelirine olan etkisi incelenmiştir. Bu amaçla, stratejik müşteri modeli geliştirilmiş ve müşterinin satın alma ve tüketim politikalarının analitik yapısı gösterilmiştir. Müşterilerin satın alma ve tüketim davranışları göz önünde bulundurularak, perakendecinin gelir modeli geliştirilmiştir. Müşterilerin sadece paket indiriminden yararlandığı durum için, perakendecinin gelirini en yüksekleyecek paket fiyatının yapısı elde edilmiştir. Yürütülen saysal deneyler sonucunda, birbirinin yerine geçebilecek ürünler paket indirimini oluşturduğu durumda, perakendecinin indirim miktarını arttırması ve paket indirimini seyrek aralıklarla müteriye sunması gerektiği gözlemlenmiştir. Paket indirimini oluturan ürünlerin birbirlerini tamamlayıcı ürünler olduğu durumda ise, perakendecinin az indirim oranı ve sık indirim politikası ile gelirini arttırdığı incelenmiştir.

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Chapter 1

INTRODUCTION

The pricing decision is one of the most challenging decisions that retailers face. Finding the best pricing strategy has always been an intriguing area for retailers and researchers. Retailers employ a variety of pricing strategies. There are several reasons for which retailers offer discounted prices to customers such as to price discriminate among heterogeneous customers with respect to willingness-to-pay values, and to manage products' inventory by stimulating sales. In many retail settings, retailers carry different products and need to consider the interplay between the products in designing pricing (or, promotion) strategies. In this spirit, this dissertation investigates three valuable problems for researchers and retailers.

In Chapter 2, we study a one-time bundle offer design problem in a single-seller/single-buyer supply chain setting where the seller sells two different products to the buyer. We present a game theoretical model in which the seller's aim is to employ a one-time forward-buy incentive to shift a certain amount of a product's inventory. The seller determines the bundle price to entice the buyer to purchase the targeted forward-buy amount. The buyer, in turn, determines how much to purchase from the bundle offer. We first characterize the buyer's optimal response to the seller's bundle offer and solve the retailer's profit maximization problem in a Stackelberg game framework. In this study, we answer the following research question: when is it beneficial to extend a one-time bundle offer to the buyer in lieu of a price reduction for an individual product? For benchmarking purposes, we compare the bundle discount with the individual discount in terms of the profit that the retailer can achieve. We conclude that when the targeted forward-buy quantity is not too high, for certain demand and cost structures, the seller can be better off with the bundle offer relative to a temporary reduction in the price of the product that results in the same forward-buy quantity. Our results suggest that the bundle offer can be an effective alternative in creating forward-but incentives for product with a low demand and high price. When imbalances arise between the products' demand, the seller faces a large one-time delivery to create a one-time forward-buy incentive through the bundle discount.

The main contributions of the first chapter are to develop a detailed analytical model in a single-seller/single-buyer supply chain setting and to demonstrate, through a range of numerical experiments that the demand and cost structures of the products in the bundle can have a significant impact on the profit that the retailer can achieve. We believe that the insights of this study will be of value to sellers in understanding relationship between products in creating forward-buy incentives.

In Chapter 3, we consider bundling in the context of targeted promotions and analyze the interplay between the promotion strategy (discount for a single product vs. the bundle) and segment-specific and dynamic pricing decisions. we examine the case of a retailer that aims to reduce the excess inventory of a certain product (referred to as the primary product) by making it part of a bundle offer formed with another product (referred to as the secondary product). Assuming that the retailer has the necessary information to cluster its customers into different segments in terms of their valuations of the primary and secondary products, we study the revenue impact of dynamic and targeted bundle pricing decisions. This study points to important revenue-enhancing pricing strategies for a retailer that wants to clear the inventory of the item under consideration (i.e., the primary product). The results demonstrate that, particularly when the initial inventory of the primary product is high, the potential of the bundle offers to improve the revenue is significant and indicate an additional revenue opportunity when the price of the bundle is dynamically optimized. This study illustrates that segment-specific dynamic pricing brings about substantial revenue improvements that are an increasing function of the initial inventory level of the item while static and segment-specific pricing have no direct impact on revenue.This study also shows that dynamically priced and segment-specific bundle offers yield robust revenue performance, mitigating a potentially revenue-diminishing impact of positive correlations in consumers valuations of the primary and secondary products.

The main contributions of the second chapter are to develop a detailed revenue model that considers different pricing strategies and to examine a number of factors and their relationship that have not been addressed in the literature. This study explores the interplay between bundling, dynamic pricing, and personalized promotions. The insights driven in the chapter could be of value to the retailers employing a variety of strategies to stimulate demand and increase revenue.

In Chapter 4, we study a consumer stockpiling and a retailer's bundle pricing problems for storable products (e.g., processed foods, salted snacks). The market includes strategic and myopic customers for the bundle offer. While myopic customers satisfy their immediate demand, strategic customers can stockpile the products for future consumption but they incur inventory holding costs. We model the customer's problem as a discrete dynamic programming over an infinite time horizon. In a discount period, strategic customers stockpile both products based on the current bundle price and expectations for the next bundle period. In turn, the retailer incorporates consumers' purchasing and consumption policies into the pricing problem and determines the depth and frequency of the bundle promotion to maximize the average per period revenue over the length of the stockpiling cycle. We first analytically prove the optimal purchasing structure of a consumer, that is a state-dependent threshold policy. Our results suggest that when the degree of substitutability among the products increases, the retailer should employ less frequent promotion with relatively higher discount level. If the degree of complementarity among the products increases, our results indicate that the retailer should employ the bundle promotion more often with relatively lower discount level.

The main contributions of the third essay are to establish the structure of optimal purchasing and consumption policies of a strategic consumer in response to the retailer's temporary bundle discount, and to derive a closed-form expressions for the consumer's optimal stockpiling strategy when the retailer follows bimodal pricing policy. From the retailer's perspective, our insights are valuable for retailers in designing temporary promotions (i.e., bundle discount and frequency) in a multi-product setting to price discriminate among heterogeneous customers.

The rest of this dissertation is organized as follows: Chapter 2, 3, and 4 address the first, second, and third essays, respectively. An overall conclusion of the dissertation and future work are provided in Chapter 5. Additionally, all the necessary proofs and supplementary materials for each chapter are given in the appendix devoted to the corresponding chapter.

Chapter 2

TRADE PROMOTIONS WITH ONE-TIME BUNDLE DISCOUNTS

2.1 Introduction

Supply chain trade promotions are temporary incentives offered by sellers to defend their products against competition by stimulating sales. They increase the profitability of product categories, and reduce inventory levels by shifting some of their inventory to the buyers (Chopra and Meindl 2014).

Sellers, influenced by the intensity of competition, particularly in the consumer packaged goods (CPG) industry, allocate increasingly more resources to trade promotions. In the United States, more than one-fifth of products are sold in promotion periods (The Nielsen Company 2015). A similar escalation of promotional activities is observed in Europe where one in four of all products is sold during promotion periods (IRi 2015). Nijs et al. (2009) report that trade promotion spending exceeds $75 billion annually and comprises 60% of the marketing budgets for CPG companies. However, trade promotions clearly increase supply chain inventory costs (Chopra and Meindl 2014), and only 33% of trade promotions actually help companies increase their sales revenue (The Nielsen Company 2015).

As a short-term trade promotion strategy, the individual price-reduction scheme is heavily employed by sellers (The Nielsen Company 2015). In the individual price reduction scheme, promotional products are sold to the retailer at lower prices than their list prices. There are two potential consequences of price reductions in a supply chain (Blattberg and Briesch. 2012): a higher *pass-through rate*, i.e., the proportion of the sellers discount that is passed to the end consumers by the buyer, and an increased *forward-buying* behavior that emerges when buyers purchase the products in larger quantities compared to their regular order quantities in the absence of discounted prices. Primarily, sellers may offer a temporary price incentive to stimulate sales, particularly if buyers pass some portion of the price discount to the end consumers. However, when faced with overstocking, they may also rely on price discounts to shift their inventory to the buyers.

As a form of trade promotion, bundling is the practice of offering two or more products as a single unit at a price that is lower than the sum of the individual price of the products (Stremersch and Tellis 2002). In their review of price discount practices, Munson and Rosenblatt (1998) report that 63% of the studied firms use multiple-item aggregation (i.e., bundling). Current business-to-consumer (B2C) promotional offers partly rely on bundling across various product categories such as dishwashing products (e.g., dishwasher detergents and rinse agents), beverages and snacks (e.g., soft drinks and savory snacks), and personal care products (e.g., razor blades and shaving foams). We note that in the business-to-business (B2B) setting that we study in this chapter, the bundle offer can be a contractual requirement on the relative purchase quantities of the bundled products, and the products may not necessarily be delivered in physically bundled packages, as observed in the B2C setting. Although the primary factor for the attractiveness of a bundle offer is the price discount, a bundle offer may also lead to a streamlined procurement process through the coordination of orders (see Nalebuff 2003 and The Boston Consulting Group 2008, for a detailed discussion of additional factors influencing buyers' response to bundle offers).

Although bundle offers have the potential to bring about significant benefits for the sellers, decisions that lead to effective bundle offers are multi-layered, and are not trivial. The sellers need to determine the products to be bundled along with their relative quantities in the bundle. Equally important are the bundling strategy decisions: sellers can choose to sell the bundle along with the individual products (i.e., mixed-bundling) or to sell only the bundle (i.e., pure-bundling). The sellers also need to determine the bundle price, which plays a key role in shaping buyers' reaction to the bundle offer (see Stremersch and Tellis 2002, and Nalebuff 2003 for a comprehensive overview of the bundling terminology). The relative quantity of products in a bundle, which we will refer to as the *bundle composition*, has an impact on the effectiveness of the bundle offer and is closely tied to the bundle pricing decisions. Therefore, more effective bundle offers can be designed when pricing and bundle composition decisions are made jointly.

In this study, we consider the one-time bundle offer design problem in a single-seller/single-buyer supply chain setting where the seller sells two different products to the buyer. We specifically address the case where the seller's objective is to shift a certain amount of a product's inventory to fend off imminent competitive threats by boosting the sales of the product. In the subsequent discussion, the product for which an inventory shift is planned will be referred to as the *target product*, and the second product in the bundle will be referred to as the *secondary product*. In other words, we consider the case where the seller aims to create a forward-buy incentive for the target product by designing a one-time bundle offer for the target and secondary products. We assume that the buyer faces price-sensitive demands for both products. We also assume that information about the demand functions and cost parameters is publicly available. This assumption is actually not very restrictive: as part of the information required for supply chain transactions, the seller already has access to order quantity and the annual demand data of the buyer from previous exchanges with different price levels. If the holding cost rate is not specific to the company, i.e., if the buyer's holding cost rate is equal to the average rate of the industry in which she operates, the seller can readily infer the buyer's demand functions and ordering cost parameters.

We attempt to answer an important question that the seller faces in creating forward-buy incentives: when is it beneficial to extend a one-time bundle offer to the buyer in lieu of price reduction for an individual product? For certain demand and cost structures, we show that when the targeted forward-buy quantity is not too high, the seller can be better off with the bundle offer relative to a temporary reduction in the price of the target product that results in the same forward-buy quantity.

The structure of this chapter is as follows. First, we review the relevant literature in Section 2.2. Next, in Section 2.3.1, we present a detailed description of the problem setting. We discuss buyer and seller problems in the base case with no discount in Section 2.3.2, and one-time price reductions without and with bundling in Sections 2.3.3 and 2.3.4, respectively. A numerical example is presented in Section 2.4. In Section 2.5, based on an extensive numerical study, we report the profit performances of one-time price reductions for individual products and product bundles, and discuss the managerial implications of the observed differences. We conclude in Section 2.6 with a summary of our findings.

2.2 Literature Review

In this section, we review research streams that address bundling issues from the perspective of operations management and related fields, and present an overview of the previous research on one-time price reduction models in two-stage supply chains.

The operations management literature primarily addresses inventory management and the pricing dimensions of bundling in the B2C setting. Hanson and Martin (1990) develop a mixed-integer linear programming model to find the optimal bundle prices in a setting where the profit-maximizing firm sets prices and customers make purchases to maximize their surplus. Ernst and Kouvelis (1999) consider an inventory control problem in a mixed-bundling setting, and determine the optimal inventory levels of individual products and bundles. Bitran and Ferrer (2007) propose a non-linear integer programming model to determine the optimal composition and price of a bundle of technological goods in a competitive environment. They define the bundle composition problem as determining the products to be included in a bundle so as to maximize the expected profits. Gürler et al. (2009) and Bulut et al. (2009) analyze a dynamic bundle pricing problem, and find that the performance of bundling strategies depends on the demand structure and the initial inventory levels. McCardle et al. (2007) consider the bundling problem from the retail side and, employing uniformly distributed reservation prices, show that the profitability of bundling depends on the cost structures and demand correlation of the products. Bhargava (2012) studies a bundling problem in a vertical channel setting where a retailer purchases and bundles products from different manufacturers, and demonstrates that offering individual products always dominates pure bundling. Cao et al. (2015) study a 2-stage channel where the retailer has the option of selling the manufacturers product alone, or bundling it with the one of its own private-label products. Cao et al. (2015) state that the channel context has a direct impact on retailer performance, and demonstrate how the retailer can induce a lower wholesale price by using the bundling option as a strategic leverage.

One-time price reductions have been extensively studied in the B2B context as a supply chain coordination tool (see Munson and Rosenblatt 1998, Li and Wang 2007, and Ramasesh 2010 for comprehensive reviews of the related literature). Ardalan (1988) and Sarker and Al Kindi (2006) analyze the buyers optimal ordering policy when the supplier offers a one-time price reduction under the assumption that annual demand rate is exogenously given. Ardalan (1988) models a general case where the on-hand inventory level is allowed to be non-negative when the reduced price is in effect. Sarker and Al Kindi (2006) extend the study of Ardalan (1988) by considering the situation where the buyer can benefit from the one-time price reduction in multiple replenishment cycles. In the price-sensitive demand setting, Abad (1988) and Arcelus and Srinivasan (1998) model the buyer's profit-maximization problem when a one-time price reduction is offered by the seller. In both studies, the wholesale price of the seller is assumed to be constant, and demand is considered to be a decreasing function of the price the buyer (e.g., a retailer) charges her customers. Arcelus et al. (2001) provide a comparison of the price discount and the delay of payment sales promotion strategies and analyze the retailers profit-maximizing strategy in response to these promotions. Su and Geunes (2012) demonstrate that even though the trade promotions can result in higher operations costs, these costs may be more than offset by increased revenues yielding supply chain profits even exceeding that under an everyday low price strategy.

In this study, we assume that the seller is not constrained by a promotion budget (see Berger and Bechwati 2001 for a discussion of the promotional expenditure allocation decisions when the seller has to operate under such a constraint). We also assume that the two products that can be bundled are given, and refer to the bundle composition problem as the determination of the units of each product in the bundle. We consider the bundling as a mechanism for a seller to design trade promotions in a single-seller/single-buyer supply chain with price-dependent demands, and investigate the impact of bundle composition decisions on the profitability of the one-time bundle offer from both the sellers and the buyers perspectives.

2.3 Model

2.3.1 Problem Setting and Notation

We consider a two-stage supply chain with a single seller that sells two different products to a buyer. Let i be the product index, where $i = 1, 2$, refers to the target and secondary products, respectively (see Section **??** for definitions of the target and secondary products). Both the seller and the buyer incur purchasing, inventory holding and ordering costs, and the supply chain revenue is generated through price-sensitive product demands.

We consider the case where product demands are independent. As discussed in Adams and Yellen (1976), McAfee et al. (1989), and Venkatesh and Kamakura (2003), when the valuations of the products to be bundled are negatively (positively) correlated, the revenue performance of the bundle offers improve (deteriorate). Because our main analysis involves a comparison with an alternative scheme (i.e., one-time only price reduction for the target product) that involves a single price-sensitive demand function, the independent product setting provides a more objective basis for performance comparison by eliminating the positive or negative impact of the demand correlation of the bundled products.

In addition to the demand model we introduce in this section, a wide range of supply chain models with price-sensitive demand functions are discussed in the literature. For example, Lim (2013) considers the problem of jointly determining the price and order quantity for the retailer under statistical uncertainties of the parameters defining the demand and purchase cost functions, and transforms the problem into a convex optimization program. In this study, mainly due to the tractability issues, we assume that the parameters of the demand model do not involve any statistical uncertainty.

The seller's goal is to design a one-time forward-buy incentive that involves a temporary price reduction for a bundle of the target and secondary products. The seller can express the targeted forward-buy quantity as a multiple of the buyer's regular order quantity for the target product, denoted by $Q_{R,1}$. In other words, the targeted forward-buy quantity can be set equal to $tQ_{R,1}$, where $t >= 1$ is the number of regular order quantity cycles that the seller wants the buyer to forward-buy. We note that t can assume fractional values. For example, if the buyer's regular order quantity for the target product is 10, i.e., $Q_{R,1} = 10$, and the targeted forward-buy quantity is equal to 15, then the number of periods that the seller wants the buyer to forward-buy is equal to 1.5, or $t = 1.5$. We also note that the value of t is set exogenously, driven by the factors that necessitate the seller's consideration of a trade promotion.

The seller announces the wholesale bundle price, w_B, by selecting the bundle discount, β, and the bundle composition, (b_1, b_2). The quantity of product type i in the bundle is equal to b_i, $i = 1, 2$, and the wholesale bundle price is computed as $w_B = (1 - \beta)(\sum_{i=1}^{2} b_i w_i)$ where w_i is the regular wholesale price of product $i, i = 1, 2$, when no discount is offered by the seller. The seller determines w_B to entice the buyer to purchase the targeted forward-buy quantity while maximizing his profit with the bundle offer. The buyer, in turn, chooses an optimal bundle quantity to be purchased, Q_B, to maximize her profit with the one-time bundle offer.

We first model the decisions of the supply chain participants when no price reduction is offered by the seller. In this base model, the seller sets the wholesale prices of the two products independently (which will be referred as the "regular" wholesale prices) to maximize his profit per unit time (e.g., profit per year) by taking into consideration the buyer's response to the wholesale prices. We then characterize the buyer's optimal purchase quantity decisions under a one-time price reduction for the target product only, and the bundle of the target and secondary products, and incorporate them into the seller's profit-maximization problem in a Stackelberg game framework.

The parameters and decision variables of all three problems can be listed as follows:

Parameters:

$D_i(\cdot)$: demand per unit time of product i is a function of the price p_i the buyer charges her customers: $D_i(p_i) = A_i - \gamma_i p_i$, where A_i is the demand function constant and γ_i is the elasticity parameter of the demand function for product i, $i = 1, 2$,

S_i: buyer's fixed cost of placing an order for product i, $i = 1, 2$,

s_i: seller's fixed cost of placing an order for product i, $i = 1, 2$,

F: holding cost rate per unit time,

c_i: seller's unit procurement (or manufacturing) cost for product i, $i = 1, 2$,

t: targeted number of forward-buy periods for product 1 (i.e., the target product) exogenously set by the seller with $t \geq 1$.

The seller's decision variables:

w_i: regular wholesale price for product i, where $c_i \leq w_i < \frac{A_i}{\gamma_i}$, $i = 1, 2$,

$w_{I,1}$: wholesale price of product 1 (i.e., the target product) when a one-time price reduction is offered to the buyer to create a forward-buy incentive where $0 \leq w_{I,1} < w_1$,

b_i: quantity of product i in the bundle, where $b_i \in \mathbb{Z}^+$, $i = 1, 2$,

β: wholesale discount rate with the one-time bundle offer, where $0 \leq \beta < 1$, leading to a bundle wholesale price w_B of $(1 - \beta) \sum_{i=1}^{2} b_i w_i$.

n_i: seller's lot-size multiplier for product i, where $n_i \in \mathbb{Z}^+$, $i = 1, 2$.

The buyer's decision variables:

$Q_{R,i}$: regular order quantity of product $i, i = 1, 2$, when no discount is offered by the seller,

$Q_{I,1}$: order quantity of the target product 1 (i.e., the target product), when a one-time individual discount is offered by the seller,

Q_B: bundle order quantity when a one-time bundle discount is offered by the seller.

$p_{R,i}$: buyer's regular selling price for product i, where $w_i < p_{R,i} < \frac{A_i}{\gamma_i}, i = 1, 2,$, when no discount is offered by the seller,

$p_{I,1}$: buyer's selling price for units of product 1, purchased with the one-time individual discount where $w_1 < p_{I,1} < \frac{A_1}{\gamma_1}$,

$p_{B,i}$: buyer's selling price for units of product $i, i = 1, 2$, purchased with the one-time bundle discount where $w_i < p_{B,i} < \frac{A_i}{\gamma_i}, i = 1, 2$.

In the remainder of the study, we assume that the setup costs do not change when the bundle offer option is selected. When the buyer simultaneously purchases the target and secondary products via the bundle offer, the associated setup cost will be equal to the sum of the individual setup costs of the target and secondary products. With this assumption, we leave the coordination benefits of bundling out of the analysis, and eliminate the risk of putting the individual price discount scheme in a particularly disadvantageous position.

We will also carry out the derivation of the results under the assumption that the boundary conditions imposed on the price decision variables are in effect.

2.3.2 The Base Case

In this section, we consider the case where the seller does not offer a temporary price reduction. The results of this section will be used to set the benchmark profit levels for the supply chain participants.

The Buyer's Pricing and Ordering Decisions

Abad (1988) and Arcelus and Srinivasan (1998) derive the optimal pricing and order quantity decisions of the buyer for the case where the demand is price-sensitive. We use the same approach to model the buyer's response to wholesale prices, $w_i, i = 1, 2$, set by the seller. The buyer's profit per unit time function for product i, $H_{R,i}(p_{R,i}, Q_{R,i}, w_i), i = 1, 2$, can be expressed as follows:

$$H_{R,i}(p_{R,i}, Q_{R,i}, w_i) = \quad (p_{R,i} - w_i) D_i(p_{R,i}) - w_i F \frac{Q_{R,i}}{2} - S_i \frac{D_i(p_{R,i})}{Q_{R,i}}, i = 1, 2. \quad (2.1)$$

The first component of the buyer's profit function is the revenue minus the cost of sales when the buyer sells the product i at price $p_{R,i}, i = 1, 2$, and the seller's wholesale price for product i is $w_i, i = 1, 2$. In turn, the second and third components of the buyer's profit function are the holding and ordering costs per unit time, respectively. For fixed values of $Q_{R,i}$ and w_i for product $i, i = 1, 2$, $H_{R,i}(p_{R,i}, Q_{R,i}, w_i)$ is concave in $p_{R,i}$, and the optimal selling price, $p^*_{R,i}(Q_{R,i}, w_i)$, that maximizes $H_{R,i}(p_{R,i}, Q_{R,i}, w_i)$, $i = 1, 2$, can be expressed as follows:

$$p^*_{R,i}(Q_{R,i}, w_i) = \quad \frac{1}{2}\left(\frac{A_i}{\gamma_i} + \frac{S_i}{Q_{R,i}} + w_i\right), i = 1, 2. \quad (2.2)$$

For fixed values of $p_{R,i}$ and w_i, $i = 1, 2$, and when $w_i \leq p_{R,i} < \frac{A_i}{\gamma_i}$, $H_{R,i}(p_{R,i}, Q_{R,i}, w_i)$ can be readily shown to be concave in $Q_{R,i}$. With the concavity property of the objective function, the optimal order quantity, $Q^*_{R,i}(p_{R,i}, w_i)$, that maximizes $H_{R,i}(p_{R,i}, Q_{R,i}, w_i)$ can now be written as:

$$Q^*_{R,i}(p_{R,i}, w_i) = \quad \sqrt{\frac{2S_i D_i(p_{R,i})}{w_i F}}, i = 1, 2. \quad (2.3)$$

Using the properties of the buyer's profit function presented in A.1, and also as proven in Arcelus and Srinivasan (1998), it can be shown that buyer's optimal order quantity $Q^*_{R,i} \equiv Q^*_{R,i}\left(p^*_{R,i}, w_i\right), i = 1, 2$, and selling price $p^*_{R,i} \equiv p^*_{R,i}\left(Q^*_{R,i}, w_i\right), i = 1, 2$, should jointly satisfy Equations (2.2) and (2.3). Although the optimal selling prices or order quantities can be expressed solely in terms of problem parameters by substituting Equation (2.2) into Equation (2.3), or vice versa, the resulting expressions do not directly lead to closed-form solutions. The optimal values, on the other hand, can be readily obtained by employing numerical methods.

The Seller's Wholesale Pricing Decision

In this section, we discuss the seller's profit-maximization problem for product $i, i = 1, 2$. To render the analysis tractable, we consider the case for a given value of the lot-size multiplier, $n_i \geq 1, i, i = 1, 2$. Once a procedure has been developed to solve the profit-maximization problem with a fixed lot size multiplier, the optimal lot size multiplier can be determined with a simple search procedure.

When the wholesale price is a decision variable, the seller's optimization problem for product $i, i = 1, 2$, can be stated as

$$\max \quad H_{S,i}\left(p_{R,i}, Q_{R,i}, w_i\right) = \left(w_i - c_i\right) D_i\left(p_{R,i}\right) - c_i F \tfrac{(n_i-1)Q_{R,i}}{2} - s_i \tfrac{D_i\left(p_{R,i}\right)}{n_i Q_{R,i}} \tag{2.4}$$

subject to

$$Q_{R,i} = \sqrt{\tfrac{2S_i D_i\left(p_{R,i}\right)}{w_i F}}, i = 1, 2, \tag{2.5}$$

$$p_{R,i} = \tfrac{1}{2}\left(\tfrac{A_i}{\gamma_i} + \tfrac{S_i}{Q_{R,i}} + w_i\right), i = 1, 2, \tag{2.6}$$

$$c_i \leq w_i < \tfrac{A_i}{\gamma_i}, i = 1, 2, \tag{2.7}$$

where $H_{S,i}\left(p_{R,i}, Q_{R,i}, w_i\right)$ is the seller's profit per unit time when he sets the wholesale price as w_i. As captured in constraints (2.5) and (2.6), the buyer sets the selling price as $p_{R,i}$, and chooses $Q_{R,i}$ as her regular order quantity (see A.1). By incorporating Equation (2.6) into the objective function and Equation (2.5), the seller's problem can be simplified as

$$\begin{aligned} \max \quad H_{S,i}\left(Q_{R,i}, w_i\right) = \tfrac{1}{2}\Big(& \left(A_i - \tfrac{S_i \gamma_i}{Q_{R,i}} - w_i \gamma_i\right)\left(w_i - c_i - \tfrac{s_i}{n_i Q_{R,i}}\right) \\ & - c_i F\left(n_i - 1\right) Q_{R,i}\Big) \end{aligned} \tag{2.8}$$

subject to

$$Q_{R,i} = \sqrt{\tfrac{S_i}{w_i F}\left(A_i - \gamma_i\left(w_i + \tfrac{S_i}{Q_{R,i}}\right)\right)}, i = 1, 2, \tag{2.9}$$

$$c_i \leq w_i \leq \tfrac{A_i}{\gamma_i}, i = 1, 2. \tag{2.10}$$

$H(Q_{R,i}, w_i)$, $i = 1, 2$, is concave in w_i for a fixed value of $Q_{R,i}, i = 1, 2$, and for a fixed value of $Q_{R,i}$, $i = 1, 2$, using the concavity property of the objective function, the optimal wholesale price of the seller, $w_i^*(Q_{R,i})$, can be written as:

$$w_i^*(Q_{R,i}) = \frac{1}{2}\left(\frac{A_i}{\gamma_i} + \frac{s_i - n_i S_i}{n_i Q_{R,i}} + c_i\right), i = 1, 2. \tag{2.11}$$

Although the uniqueness of the $w_i^*(Q_{R,i}), i = 1, 2$, value for a fixed $Q_{R,i}, i = 1, 2$, can be readily shown, it is not possible to express the overall optimal solution in closed form (see Jungkyu et al. 2011 for a discussion of a similar problem). We therefore solve the above problem using a numerical method (Sahinidis 2014) by using the joint solution of Equations (2.9) and (2.11) as the starting point.

In the remainder of the study, to simplify the exposition, the optimal profits of the no-discount case will be presented as $H_{R,i}^*$ in lieu of $H_{R,i}^*\left(p_{R,i}^*, Q_{R_i}^*, w_i^*\right), i = 1, 2$, for the buyer, and as $H_{S,i}^*$ in lieu of $H_{S,i}^*\left(p_{R,i}^*, Q_{R_i}^*, w_i^*\right), i = 1, 2$, for the seller.

2.3.3 The One-Time Individual Discount Case

In this section, we consider the case where the seller's objective is to create a forward-buy incentive for exactly $tQ_{R,1}$ units of the target product with a one-time individual discount. We note that because the seller is not interested in creating a forward-buy opportunity for the secondary product, the analysis we present in this section will focus solely on the one-time discounted price of the target product. As also assumed in Arcelus and Srinivasan (1998), the one-time discounted price will be available at the regular replenishment time of the buyer. We also assume that the buyer will sell all purchased products through the authorized channels, i.e., she will not consider the option of selling some part of her inventory to gray markets.

The Buyer's Pricing and Ordering Decisions

The buyer's objective is again profit-maximization, however, her profit per unit time now has two components: 1) profit generated with the sales of the units purchased with the discount, and 2) profit achieved in the remainder of the unit time.

Arcelus and Srinivasan (1998) present a model to find the buyer's optimal order quantity and selling price when the seller extends a one-time discounted price for the product under consideration, i.e., the target product. Following a similar approach, when a one-time individual discount is available for the target product, the buyer's total profit per unit time, $H_{R,I}\left(Q_{I,1}, p_{I,1}, w_{I,1}\right)$, can be expressed as follows:

$$H_{R,I}\left(Q_{I,1}, p_{I,1}, w_{I,1}\right) = \quad \left(p_{I,1} - w_{I,1}\right) Q_{I,1} - w_{I,1} F \frac{\left(Q_{I,1}\right)^2}{2D_1\left(p_{I,1}\right)} - S_1 + \left(1 - \frac{Q_{I,1}}{D_1\left(p_{I,1}\right)}\right) H^*_{R,1} \tag{2.12}$$

The first term in (2.12) represents the profit obtained by selling $Q_{I,1}$ units of the target product purchased at the discounted price. The second term in (2.12) is the average holding cost of the target product in the period in which the items purchased with the discounted price are completely sold. The third term in (2.12) is the buyer's setup cost for the items she purchases with the discounted price. We note that the setup cost can be more accurately expressed with an indicator function as $S_1 I(Q_{I,1} > 0)$, however, to simplify the exposition, we will simply write it as S_1. The fourth term in (2.12) represents the profit obtained in the remainder of unit time where the order quantity and the selling price are $Q^*_{R,1}$ and $p^*_{R,1}$, respectively. In the fourth term of (2.12), we implicitly assume that $Q_{I,1}$ is less than the annual demand of the target product, however, when this is not the case, the unit time in the definition of $D_1\left(p_{I,1}\right)$ can be adjusted to have $\frac{Q_{I,1}}{D_1\left(p_{I,1}\right)} \leq 1$. On the other hand, when the one-time discount is not attractive, the buyer can choose $Q^*_{I,1} = 0$, and (2.12) reverts to the buyer's profit per unit time in the no-discount case.

For fixed values of $Q_{I,1}$ and $w_{I,1}$, $H_{R,I}\left(Q_{I,1}, p_{I,1}, w_{I,1}\right)$ is concave in $p_{I,1}$. Therefore, for fixed values of $Q_{I,1}$ and $w_{I,1}$, the optimal selling price for the target product, $p^*_{I,1}\left(Q_{I,1}, w_{I,1}\right)$, that maximizes $H_{R,I}\left(Q_{I,1}, p_{I,1}, w_{I,1}\right)$ can be expressed as:

$$p^*_{I,1}\left(Q_{I,1}, w_{I,1}\right) \quad = \frac{A_1}{\gamma_1} - \frac{\sqrt{\gamma_1^3\left(F Q_{I,1} w_{I,1} + 2H^*_{R,1}\right)}}{\sqrt{2}\gamma_1^2}. \tag{2.13}$$

Similarly, for fixed values of $p_{I,1}$ and $w_{I,1}$, $H_{R,I}\left(Q_{I,1}, p_{I,1}, w_{I,1}\right)$ is concave in $Q_{I,1}$, and the optimal order quantity with the discounted price of the target product,

$Q^*_{I,1}(p_{I,1}, w_{I,1})$, can be written as:

$$Q^*_{I,1}(p_{I,1}, w_{I,1}) = \frac{(p_{I,1}-w_{I,1})(A_1-\gamma_1 p_{I,1})-H^*_{R,1}}{Fw_{I,1}}. \tag{2.14}$$

For a given one-time discounted wholesale price of product 1, $w_{I,1}$, the buyer's optimal order quantity, $Q^*_{I,1}(p_{I,1}, w_{I,1})$, and the optimal selling price, $p^*_{I,1}(Q_{I,1}, w_{I,1})$, can be obtained by jointly solving Equations (2.13) and (2.14) (as shown in Arcelus and Srinivasan 1998).

The Seller's Problem

With the one-time individual discount, the seller aims to create a forward-buy incentive for exactly $tQ^*_{R,1}$ units of the target product. Therefore, the seller can determine the wholesale price of the target product, $w_{I,1}$ by setting the RHS of Equation (2.14) equal to $tQ_{R,1}$, and solving it for $w_{I,1}$:

$$w_{I,1}(t) = \frac{p^*_{I,1}\left(A_1-\gamma_1 p^*_{I,1}\right)-H^*_{R,1}}{A_1+FtQ^*_{R,1}-\gamma_1 p^*_{I,1}}. \tag{2.15}$$

Assuming that $tQ^*_{R,1}$ units of the target product are delivered in a single lot, the seller's profit per unit time function can be expressed as:

$$\begin{aligned} H_{S,I}\left(tQ^*_{R,1}, p^*_{I,1}, w_{I,1}(t)\right) &= (w_{I,1}(t)-c_1)\,tQ^*_{R,1}-s_1 \\ &+ \left(1-\frac{tQ^*_{R,1}}{D_1\left(p^*_{I,1}\right)}\right)H^*_{S,1} \end{aligned} \tag{2.16}$$

The first term in (2.16) represents the profit obtained by selling $tQ^*_{R,1}$ units of the target product with the discounted wholesale price. The second term in (2.16) is the fixed ordering cost of the seller for the target product in the promotion period. The third term in (2.16) represents the seller's profit obtained from the target product in the remainder of the unit time after the units purchased with the discounted price are all sold. As shown in (2.15), $w_{I,1}(t)$ is a function of $p^*_{I,1}$. Therefore, the optimal wholesale price for product 1, $w^*_{I,1}(t)$, should satisfy the following conditions:

$$w^*_{I,1}(t) = \frac{p^*_{I,1}\left(A_1 - \gamma_1 p^*_{I,1}\right) - H^*_{R,1}}{A_1 + FtQ^*_{R,1} - \gamma_1 p^*_{I,1}}, \tag{2.17}$$

$$p^*_{I,1} = \frac{A_1}{\gamma_1} - \frac{\sqrt{\gamma_1^3\left(FQ_{I,1}w^*_{I,1}(t) + 2H^*_{R,1}\right)}}{\sqrt{2}\gamma_1^2}. \tag{2.18}$$

In A.2, we provide a proof of the existence of unique values of $w^*_{I,1}$ and $p^*_{I,1}$ when Equalities (2.17) and (2.18) are solved jointly.

2.3.4 The One-Time Bundle Discount Case

In this section, we consider the case where the seller offers a one-time bundle discount to create a forward-buy incentive. The seller designs the bundle discount such that the forward-quantity of the target product is exactly $tQ_{R,1}$ units. As we also assumed in the previous section, the one-time bundle discount will be available at the regular replenishment time of the buyer.

The Buyer's Problem

The buyer's objective is to maximize the total profit he will obtain with the sales of the target and secondary products he purchases with the bundle discount plus the profit he will achieve in the remainder of the unit time. When a bundle offer with composition (b_1, b_2) and discount β is extended to the buyer, her profit per unit time can be expressed as follows:

$$\begin{aligned} H_{R,B}\left(Q_B, b_1, b_2, p_{B,1}, p_{B,2}, \beta\right) = \; & Q_B\left(\left(\sum_{i=1}^{2} b_i p_{B,i}\right) - (1-\beta)\left(\sum_{i=1}^{2} b_i w^*_i\right)\right) \\ & - \sum_{i=1}^{2}\left((1-\beta)\, w^*_i F \frac{(Q_B b_i)^2}{2D_i\left(p_{B,i}\right)} + S_i\right) \\ & + \sum_{i=1}^{2}\left(\left(1 - \frac{Q_B b_i}{D_i\left(p_{B,i}\right)}\right) H^*_{R,i}\right) \end{aligned} \tag{2.19}$$

The first term in (2.19) represents the buyer's profit when she sells $Q_B b_1$ units of the target product and $Q_B b_2$ units of the secondary product at prices $p_{B,1}$ and $p_{B,2}$, respectively. The second term in (2.19) is the buyer's holding and fixed ordering costs when she decides to purchase $Q_B b_1$ units of the target product and $Q_B b_2$ units of the secondary product. Finally, the third term in (2.19) is the buyer's profit in the remainder of the unit time after the units purchased with the bundle discount have been sold to the customers. We note that, for the remainder of the unit time, the buyer's profit will be equal to the profit she achieves in the no-discount case as discussed in Section 2.3.2.

For given Q_B, b_1, b_2 and β values, $H_{R,B}\left(Q_B, b_1, b_2, p_{B,1}, p_{B,2}, \beta\right)$ can be readily shown to be concave in $p_{B,i}$, and the optimal selling prices, $p^*_{B,i}\left(Q_B, b_i, \beta\right)$, $i, i = 1, 2$ that maximize $H_{R,B}\left(Q_B, b_1, b_2, p_{B,1}, p_{B,2}, \beta\right)$ can then be expressed as

$$p^*_{B,i}\left(Q_B, b_i, \beta\right) = \frac{2A_i\gamma_i - \sqrt{2\gamma_i^3\left(2H^*_{R,i} - b_i F Q_B w^*_i\left(\beta - 1\right)\right)}}{2\gamma_i^2}, i = 1, 2. \quad (2.20)$$

Similarly, for given $p_{B,1}$, $p_{B,2}$, b_1, b_2 and β values $H_{R,B}\left(Q_B, b_1, b_2, p_{B,1}, p_{B,2}, \beta\right)$ is concave in Q_B. The buyer's optimal bundle quantity when the seller extends a one-time bundle discount can be expressed as

$$\begin{aligned} Q^*_B\left(b_1, b_2, p_{B,1}, p_{B,2}, \beta\right) = & \frac{b_2 H^*_{R,2} D_1\left(p_{B,1}\right)}{F\left(\beta - 1\right)\left(b_2^2 w^*_2 D_1\left(p_{B,1}\right) + b_1^2 w^*_1 D_2\left(p_{B,2}\right)\right)} \\ & -\Big(\left(\left(b_1\left(p_{B,1} + w^*_1\left(\beta - 1\right)\right) + b_2\left(p_{B,2} + w^*_2\left(\beta - 1\right)\right)\right) D_1\left(p_{B,1}\right)\right) \\ & \Big)\frac{D_2\left(p_{B,2}\right)}{F\left(\beta - 1\right)\left(b_2^2 w^*_2 D_1\left(p_{B,1}\right) + b_1^2 w^*_1 D_2\left(p_{B,2}\right)\right)} \\ & + \frac{\left(b_1 H^*_{R,1} D_2\left(p_{B,2}\right)\right)}{F\left(\beta - 1\right)\left(b_2^2 w^*_2 D_1\left(p_{B,1}\right) + b_1^2 w^*_1 D_2\left(p_{B,2}\right)\right)}. \end{aligned} \quad (2.21)$$

For a given bundle composition (b_1, b_2), and bundle discount β, the buyer's optimal selling prices, $p^*_{B,1}\left(Q_B, b_1, \beta\right)$ and $p^*_{B,2}\left(Q_B, b_2, \beta\right)$, and optimal bundle quantity, $Q^*_B\left(b_1, b_2, p_{B,1}, p_{B,2}, \beta\right)$, can be derived by solving Equations (2.20) and (2.21) jointly. In other words, the buyer's optimal selling prices and bundle order quantity satisfy the following conditions:

$$\begin{aligned}
p^*_{B,1} &= \frac{2A_1\gamma_1 - \sqrt{2\gamma_1^3\left(2H^*_{R,1} - b_1 F Q^*_B w^*_1 (\beta - 1)\right)}}{2\gamma_1^2} \\
p^*_{B,2} &= \frac{2A_2\gamma_2 - \sqrt{2\gamma_2^3\left(2H^*_{R,2} - b_2 F Q^*_B w^*_2 (\beta - 1)\right)}}{2\gamma_2^2} \\
Q^*_B &= \frac{b_2 H^*_{R,2} D_1\left(p^*_{B,1}\right)}{F(\beta-1)\left(b_2^2 w_2^* D_1\left(p^*_{B,1}\right) + b_1^2 w_1^* D_2\left(p^*_{B,2}\right)\right)} \\
&\quad - \frac{\left(\left(b_1\left(p^*_{B,1} + w^*_1(\beta-1)\right) + b_2\left(p^*_{B,2} + w^*_2(\beta-1)\right)\right) D_1\left(p^*_{B,1}\right)\right) D_2\left(p^*_{B,2}\right)}{F(\beta-1)\left(b_2^2 w_2^* D_1\left(p^*_{B,1}\right) + b_1^2 w_1^* D_2\left(p^*_{B,2}\right)\right)} \\
&\quad + \frac{\left(b_1 H^*_{R,1} D_2\left(p^*_{B,2}\right)\right)}{F(\beta-1)\left(b_2^2 w_2^* D_1\left(p^*_{B,1}\right) + b_1^2 w_1^* D_2\left(p^*_{B,2}\right)\right)} \qquad (2.22)
\end{aligned}$$

Unlike the no-discount and one-time discount cases, it is not possible to prove the existence of a unique set of $p^*_{B,1}, p^*_{B,2}$ and Q^*_B values that satisfy these conditions. However, $p^*_{B,1}, p^*_{B,2}$ and Q^*_B can be readily obtained by employing numerical methods.

The Seller's Problem

The seller's objective is to create a forward-buy incentive for a purchase quantity of the target product that is equal to exactly $tQ^*_{R,1}$ units. To determine the bundle discount for a given bundle composition (b_1, b_2) that will make the buyer purchase exactly $tQ^*_{R,1}$ units of the target product, the seller incorporates the buyer's response for the bundle discount into his problem so that $Q^*_B b_1 = tQ^*_{R,1}$. The bundle discount rate $\beta(t, b_1, b_2)$ which guarantees that $Q^*_B b_1 = tQ^*_{R,1}$ can be determined by setting $Q^*_B(b_1, b_2, p_{B,1}, p_{B,2}, \beta)$ of Equation (2.21) equal to $tQ^*_{R,1}$ and then solving it for β:

$$\beta(t, b_1, b_2) = D_2\left(p^*_{B,2}\right)\left(\frac{\left(b_2\left(b_2 F Q^*_{R,1} t w^*_2 + b_1 H^*_{R,2}\right) D_1\left(p^*_{B,1}\right) + b_1\left(b_1 F Q^*_{R,1} t w^*_1 + b_1 H^*_1 + \left(-b_1 p^*_{B,1} - b_2 p^*_{B,2} + b_1 w^*_1 + b_2 w^*_2\right) D_1\left(p^*_{B,1}\right)\right)\right)}{\left(b_2^2 F Q^*_{R,1} t w^*_2 D_1\left(p^*_{B,1}\right) + b_1\left(b_1 F Q^*_{R,1} t w^*_1 + \left(b_1 w^*_1 + b_2 w^*_2\right) D_1\left(p^*_{B,1}\right)\right) D_2\left(p^*_{B,2}\right)\right)}\right) \qquad (2.23)$$

The seller's profit function can now be expressed as:

$$\begin{aligned} H_{S,B}\left(Q_B, p^*_{B,1}, p^*_{B,2}, b_1, b_2, \beta\left(t, b_1, b_2\right)\right) \quad = \quad & Q_B\left(\left(1-\beta\left(t, b_1, b_2\right)\right)\left(b_1 w^*_1 + b_2 w^*_2\right)\right) \\ & +Q_B\left(-b_1 c_1 - b_2 c_2\right) \\ & -s_1 - s_2 \\ & +\sum_{i=1}^{2}\left(1-\frac{Q_B b_i}{D_i\left(p^*_{B,i}\right)}\right) H^*_{S,i}. \end{aligned} \tag{2.24}$$

The first term in (2.24) represents the seller's profit when $Q^*_B b_1 = tQ^*_{R,1}$ units of the target product and $Q^*_B b_2$ or $tQ^*_{R,1}\frac{b_2}{b_1}$ units of the secondary product are sold to the buyer with a wholesale discount rate of $\beta\left(t, b_1, b_2\right)$. The second and third terms in (2.24) are the seller's fixed ordering costs for the target and secondary products sold with the bundle discount, respectively. The third term in (2.24) represents the regular profit the seller will obtain in the remainder of the unit time with the sales of the target and secondary products.

As shown in (2.23), $\beta\left(t, b_1, b_2\right)$ is a function of $p^*_{B,1}$ and $p^*_{B,2}$. Therefore, the bundle discount rate, $\beta\left(t, b_1, b_2\right)$, should satisfy the following conditions:

$$\begin{aligned} \beta\left(t, b_1, b_2\right) \quad = \quad & \\ & \frac{\left(b_2\left(b_2 F Q^*_{R,1} t w^*_2 + b_1 H^*_{R,2}\right) D_1\left(p^*_{B,1}\right)\right)}{\left(b_2^2 F Q^*_{R,1} t w^*_2 D_1\left(p^*_{B,1}\right) + b_1\left(b_1 F Q^*_{R,1} t w^*_1 + \left(b_1 w^*_1 + b_2 w^*_2\right) D_1\left(p^*_{B,1}\right)\right) D_2\left(p^*_{B,2}\right)\right)} \\ & + \frac{\left(b_1\left(b_1 F Q^*_{R,1} t w^*_1 + b_1 H^*_1 + \left(-b_1 p^*_{B,1} - b_2 p^*_{B,2} + b_1 w^*_1 + b_2 w^*_2\right) D_1\left(p^*_{B,1}\right)\right) D_2\left(p^*_{B,2}\right)\right)}{\left(b_2^2 F Q^*_{R,1} t w^*_2 D_1\left(p^*_{B,1}\right) + b_1\left(b_1 F Q^*_{R,1} t w^*_1 + \left(b_1 w^*_1 + b_2 w^*_2\right) D_1\left(p^*_{B,1}\right)\right) D_2\left(p^*_{B,2}\right)\right)}, \end{aligned} \tag{2.25}$$

$$\begin{aligned} p^*_{B,1} \quad &= \quad \frac{2A_1\gamma_1 - \sqrt{2\gamma_1^3\left(2H^*_{R,1} - b_1 F Q^*_B w^*_1\left(\beta\left(t, b_1, b_2\right) - 1\right)\right)}}{2\gamma_1^2}, \\ p^*_{B,2} \quad &= \quad \frac{2A_2\gamma_2 - \sqrt{2\gamma_2^3\left(2H^*_{R,2} - b_2 F Q^*_B w^*_2\left(\beta\left(t, b_1, b_2\right) - 1\right)\right)}}{2\gamma_2^2}. \end{aligned} \tag{2.26}$$

Similar to the buyer's problem, although it is not trivial to prove the existence of a unique set of $\beta\left(t, b_1, b_2\right)$, $p^*_{B,1}$ and $p^*_{B,2}$ values that satisfy the above conditions, they can be readily obtained by employing numerical methods.

2.4 A Numerical Example

In this section, we present a numerical illustration of the no-discount and the one-time individual discount cases to set the background for the computational analysis we will present in the next section.

The underlying model for the no-discount and the one-time individual discount problems can be viewed as an extension of the model presented in Arcelus and Srinivasan (1998), and we will base our discussion on the numerical example presented in Section 4 of Arcelus and Srinivasan (1998). We consider a target product, i.e., product 1, with a demand function given by $49000 - 3000\{p_{R,1}, p_{I,1}\}$. The remaining parameters of the problem are as follows: $S_1 = 10$, $w_1 = 10$, and $F = 0.25$. Arcelus and Srinivasan (1998) do not consider the supplier's pricing problem, and for $w_1 = 10$, they determine the buyer's optimal price as $p^*_{R,1} = 13.185$, and order quantity as $Q_{R,1} = 275$, (or, 274.89 to be more precise) yielding an annual profit of $29,395.10$ for the buyer. The conditions we present in Section 2.3.2, which are identical to conditions (4) in Arcelus and Srinivasan (1998), yield the same results. Since we also consider the supplier's profit-maximization problem, we extend the example presented in Arcelus and Srinivasan (1998) by first assuming that $s_1 = S_1 = 10$ and then computing the seller's unit procurement cost for the target product, c_1, which would make his optimal wholesale price in the no-discount case, w_1, equal to 10. From Equation (2.10), we obtain the equality $10 = \frac{1}{2}\left(\frac{49000}{3000} + \frac{10 - n_1 10}{n_1 275} + c_1\right)$, where n_1 is the supplier's lot size multiplier. With a simple search over the possible values of n_1, and we pick $n_1 = 2$ and $c_1 = 3.68$, yielding the maximum profit for the seller under the constraint that $w_1^* = 10$. In other words, when $c_1 = 3.68$, to maximize his profit, the seller sets the wholesale price as $w_1^* = 10$, and chooses his lot size multiplier as $n_1 = 2$. The annual profit of the seller can be computed as follows:

$$\begin{aligned} H_{S,1}\left(p_{R,1}, Q_{R,1}, w_1\right) &= \left(w_1 - c_1\right) D_1\left(p_{R,1}\right) - c_1 F \frac{(n_1-1)Q_{R,1}}{2} - s_1 \frac{D_1\left(p_{R,1}\right)}{n_1 Q_{R,1}} \\ &= (10-3.68)(49,000 - 3,000 \times 13.185) \\ &\quad -3.68 \times 0.25 \times \frac{(2-1)275}{2} - 10\frac{49,000 - 3,000 \times 13.185}{2 \times 275} \\ &= 59,394.17 \end{aligned} \tag{2.27}$$

Let us now consider the case with $t = 3.865$, i.e., the problem instance where the seller's objective is to create a forward-buy incentive such that an order for $275 \times 3.865 = 1063$ units is placed by the buyer. In the numerical example of Arcelus and Srinivasan (1998), this instance corresponds to the case where the seller offers a wholesale price of 9.8 where the annual profits of the seller and the buyer turn out to be 59,303.10 and 29,529.40, respectively. (We note that the reported profits in Table 1 of Arcelus and Srinivasan (1998) cover a period of two years. To determine the buyer's annual profit when $w_1 = 9.8$, we adjust the reported profit by $29,395.10$ [i.e., the buyer's annual profit when $w_1 = 10$]: $29,529.40 = 58,924.50 - 29,395.10$.)

We now turn to the model we have presented in Section 2.3.3. We assume that seller's unit cost of the target product is $c_1 = 3.68$, as we computed earlier. To create a forward-buy incentive with $t = 3.865$, the seller determines $w^*_{I,1}(3.865)$ by solving the following set of equalities (as given in Equations (2.17) and (2.18)):

$$\begin{aligned} w^*_{I,1}(3.865) &= \frac{p^*_{I,1}\left(49,000 - 3,000 p^*_{I,1}\right) - 29,395.10}{49,000 + 0.25 \times 3.865 \times 275 p^*_{I,1}}, \\ p^*_{I,1} &= \frac{49,000}{3,000} - \frac{\sqrt{3,000^3\left(0.25 \times 295 w^*_{I,1}(3.865) + 2 \times 29,395.10\right)}}{\sqrt{2} \times 3,000^2}. \end{aligned}$$

The solution of the above equalities results in $w^*_{I,1}(3.865) = 9.8$ and $p^*_{I,1} = 13.135$. We note that when $w^*_{I,1} = 9.8$, Arcelus and Srinivasan (1998) compute the buyer's order quantity as 1063 and, when the target forward buy quantity is chosen as 1063, our model computes the wholesale price that would make the buyer order exactly 1063 units as $w^*_{I,1} = 9.8$. The annual profit of the seller can now be computed using

Equation (2.16):

$$\begin{aligned} H_{S,I}\left(tQ^*_{R,1}, p^*_{I,1}, w_{I,1}(t)\right) &= \left(w_{I,1}(t) - c_1\right) tQ^*_{R,1} - s_1 + \left(1 - \frac{tQ^*_{R,1}}{D_1\left(p^*_{I,1}\right)}\right) H^*_{S,1} \\ &= (9.8 - 3.68)(3.865 \times 275) - 10 \\ &\quad + \left(\frac{3.865 \times 275}{49,000 - 3,000 \times 13.135}\right) 59,394.17 \\ &= 59,309.60 \end{aligned}$$

Because the buyer lowers the price of the product from 13.185 to 13.135 when the wholesale price drops to 9.8, the demand for the target product increases. However, the increase in the demand does not compensate for the impact of lower wholesale price (9.8 instead of 10), and the seller achieves a lower annual profit (59,309.60 instead of 59,394.17). We note that, to create a forward-buy incentive, the seller has to move away from his optimal wholesale price, and therefore the profit decrease is an anticipated outcome.

Let us now consider a secondary product with the following demand function and parameters: $D_2(p_2) = 30,000 - 1,500p_2$, $S_2 = 2$, $s_2 = 2$, and $c_2 = 2$. In the no-discount case (see Section 2.3.3), the buyer sets the retail price as $p^*_{R,2} = 17.520$, and places orders for $Q_{R,2} = 63$ units. The seller sets the wholesale price as $w^*_2 = 15.00$, and chooses a lot size multiplier of $n_2 = 1$. With the secondary product, and in the no-discount case, the seller and the buyer achieve annual profits of 18,482.00 and 9,138.20, respectively.

We now consider the case where the seller attempts to create a forward-buy incentive for the target product with a one-time bundle discount. Let $(b_1 = 1, b_2 = 1)$ be the bundle composition. To create an incentive for a forward-buy quantity of exactly 1063 units of the target product, or for $t = 3.865$, the seller determines the bundle discount $\beta\,(3.865, 1, 1)$ by solving the set of equalities given in (2.26). The solution of the equalities results in $\beta\,(3.865, 1, 1) = 0.0435$, and retail prices of $p^*_{B,1} = 13.140$, and $p^*_{B,2} = 17.280$. With the one-time bundle offer, the seller, for the target and secondary products, achieves annual profits of 59,227.30 and 18,736.60, respectively, corresponding to a total annual profit of 77,963.90. In comparison to the one-time individual discount case, where the seller achieves an annual profit of 59,309.60 + 18,482.00 = 77,791.60, the one-time bundle offer results in a higher annual profit.

With the one-time bundle discount, the buyer, on the other hand, achieves a total annual profit of 29,572.50 + 9,225.20 = 38,797.70, which is similarly higher than the profit she achieves with the one-time individual discount: 29,529.40 + 9,138.20 = 38,667.60, and participates in the trade promotion.

On the other hand, when the factors that necessitate the forward-buy incentive are not in effect, the seller would not convert the one-time bundle discount into a permanent bundle discount (by not selling the two products independently) because it would lower her annual profit to 77,832.00 (calculations not shown here for the sake of brevity) which is lower than her annual profit in the no-discount case (i.e., 59,394.17 + 18,482.00 or 77,876.17).

In the Computational Analysis section, we will explore the product characteristics that make the seller, who needs to create a forward-buy incentive, prefer the one-time bundle offer over a one-time individual price discount offer.

2.5 Computational Analysis

In this section, we present the results of a set of numerical experiments. We first describe the scheme we have employed to create the problem instances. We then present the results and conclude with a detailed analysis of the results.

2.5.1 Problem Instances

We consider three distinct sets of problem instances where all parameters, except for the buyer's fixed costs of placing an order with the seller, are randomly drawn from the three collections of uniform distributions presented in Table 2.1.

In reporting the results of the numerical experiments, we use two critical ratios to characterize the problem instances: 1) the ratio of the price of the target product to the price of the secondary product in the no-discount case (i.e., $\frac{p_{R,1}}{p_{R,2}}$), and 2) the ratio of the demand of the target product to the demand of the secondary product in the

Table 2.1: Parameter distributions in the numerical experiments.

	A_1	A_2	γ_1	γ_2	c_1	c_2	s_1	s_2	S_1	S_2
Dist_1	[27500, 32500]	[8500, 10000]	[400, 800]	[30, 90]	[10, 30]	[35, 55]	[100, 150]	[400, 500]	125	450
Dist_2	[27500, 32500]	[20000, 25000]	[300, 750]	[100, 350]	[20, 40]	[50, 90]	[300, 400]	[400, 500]	350	450
Dist_3	[27500, 35000]	[15000, 25000]	[100, 200]	[100, 300]	[65, 85]	[40, 80]	[300, 500]	[300, 400]	400	350

no-discount case (i.e., $\frac{D_1(p_{R,1})}{D_2(p_{R,2})}$). These two ratios help us to cluster problem instances based on the demand and price levels of the target and secondary products, paving the way for a meaningful interpretation of the results for the practitioners.

In Figure 2.1, we present a summary of the problem instances over the ranges of the two ratios: the horizontal (vertical) axis represents the range for the $\ln\left(\frac{p_{R,1}}{p_{R,2}}\right)$ ($\ln\left(\frac{D_1(p_{R,1})}{D_2(p_{R,2})}\right)$) values in the problem instances we consider. We report the ranges in the logarithmic scale to divide the problem instance space into sub-ranges with equal sizes. (We also report the linear scale values of the ratios next to their logarithmic values along the vertical axis.) In Figure 2.1, the problem instance space is divided into 16 clusters. We refer to these clusters using the combinations of vertical and horizontal labels $(i, j), i, j \in \{2-, 1-, 1+, 2+\}$. For example, cluster $(1+, 2+)$ refers to the problem instances where the $\ln\left(\frac{D_1(p_{R,1})}{D_2(p_{R,2})}\right)$ ratio is the $(0, 0.752)$ interval, and the $\ln(\frac{p_{R,1}}{p_{R,2}})$ ratio is the $(0.752, 1.504)$ interval (or, using the linear scale, the $\frac{D_1(p_{R,1})}{D_2(p_{R,2})}$ ratio is the $(1, 2.12)$ interval, and the $\frac{p_{R,1}}{p_{R,2}}$ ratio is the $(2.12, 4.5)$ interval).

With each of Dist_1, Dist_2, and Dist_3 of Table 2.1, we randomly generated 1,000 problem instances whose $\ln\left(\frac{p_{R,1}}{p_{R,2}}\right)$ and $\ln\left(\frac{D_1(p_{R,1})}{D_2(p_{R,2})}\right)$ ratios were in the $(-1.504, 1.504)$ range. For every problem instance that belonged to a specific $(i, j), i, j \in \{2-, 1-, 1+, 2+\}$ cluster, where i represents the range for the demand ratio and j represents the range for the price ratio, another problem was created by switching the parameters of the target and secondary products. With this problem generation scheme, we created a set of $3 \times 1,000 \times 2 = 6,000$ problem instances. For each cluster, we report the total number of problem instances in the ranges that define the cluster (the number in the upper row) along with the contributions of the three distributions of Table 2.1 to the total number of problem instances (the numbers in the lower row). For example, in cluster $(1+, 2+)$, we have a total of 194 problem instances, and Dist_1, Dist_2, and Dist_3 contribute 5, 138, and 51 problem instances, respectively.

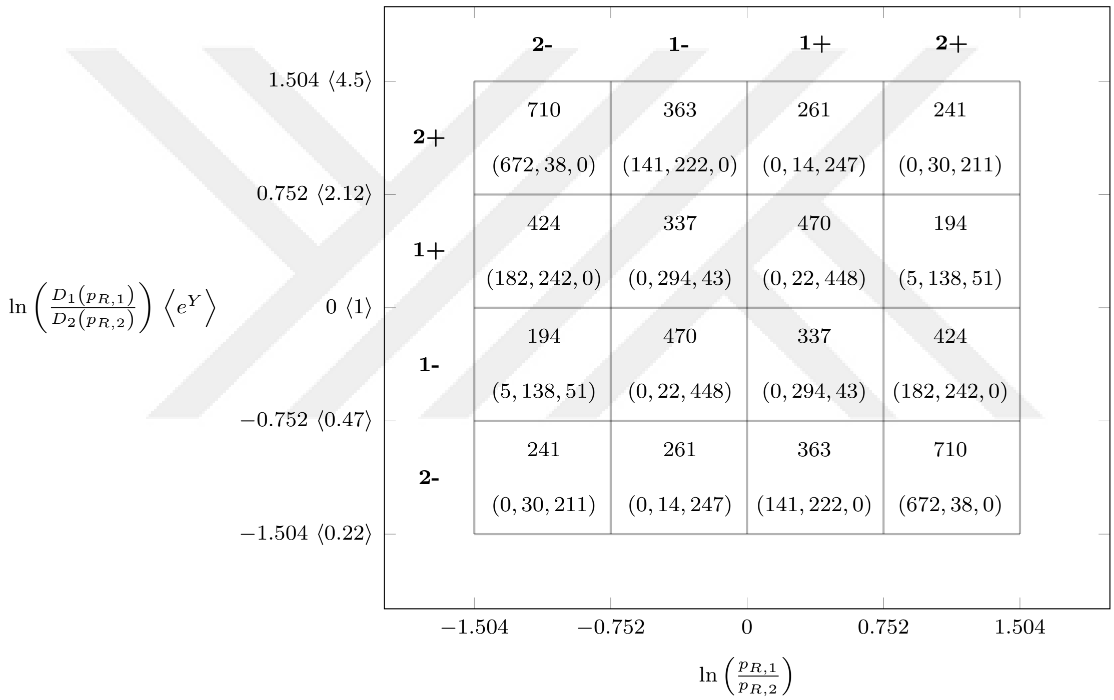

Figure 2.1: The distribution of 6,000 problem instances over the problem clusters.

In the numerical experiments, we consider forward-buy incentives for $t = 2$ and $t = 3$ and with bundle compositions $(b_1 = 1, b_2 = 1)$ and $(b_1 = 2, b_2 = 1)$. As we discuss in the subsequent section, when the seller attempts to create a forward-buy incentive for the target product with a bundle composition where $b_2 > b_1$, e.g., $(b_1 = 1, b_2 = 2)$, he usually ends up with a large forward-buy quantity for the secondary product as well, rendering the designed forward-buy incentive impractical due to the large delivery requirement for the secondary product.

In Figure 2.2, we report the percentage of problem instances where the bundle offer is preferred by the buyer over an individual discount for $t = 2$ and $t = 3$ and bundle compositions ($b_1 = 1, b_2 = 1$) and ($b_1 = 2, b_2 = 1$). The values reported in Figure 2.2 illustrate that in the majority of the problem clusters, the bundle discount is attractive from the buyer's perspective. The lowest preference percentage (66.7%) is observed in problem cluster $(2-, 1+)$ with $t = 2$ and ($b_1 = 1, b_2 = 2$). In other clusters, particularly when the bundle composition is given by ($b_1 = 1, b_2 = 1$), the buyer prefers the bundle discount in almost all of the problem instances. In the remainder of the study, we report the average values of the performance indicators over the problem instances in which the buyer prefers the bundle discount over a one-time individual product discount. We also note that the values of the performance indicators are tightly clustered around their means, and the limits of the 95% confidence intervals are within 1% of the average values.

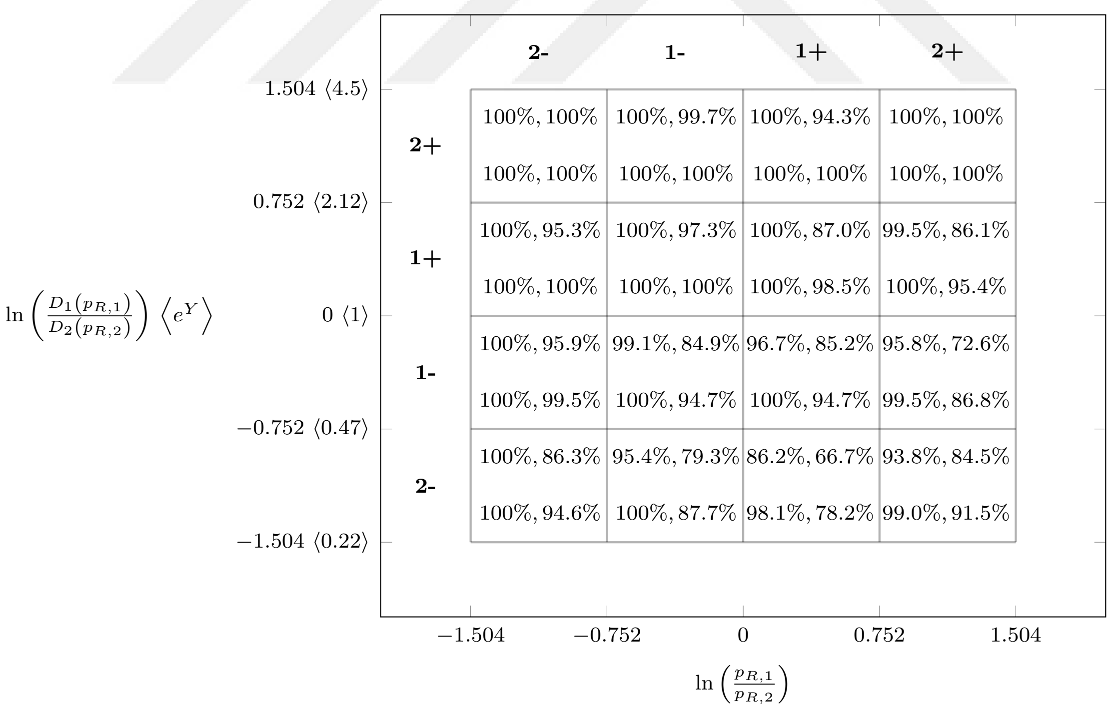

Figure 2.2: Percentage of problem instances where the buyer prefers the bundle discount over the individual product discount (first row for $t = 2$ and ($b_1 = 1, b_2 = 1$), and $t = 2$ and ($b_1 = 2, b_2 = 1$), respectively; second row for $t = 3$ and ($b_1 = 1, b_2 = 1$), and $t = 3$ and ($b_1 = 2, b_2 = 1$)), respectively.

2.5.2 Results of the Numerical Experiments

We report the results of the numerical experiments in Figures 2.3, 2.4, 2.5, and 2.6. For each cluster of problem instances, we report the results using the following format:

In the top row, we report the percentage point changes in the profits of the seller and the buyer, respectively, when a bundle discount is offered, in lieu of an individual discount, to create a forward-buy incentive for the target product and for the selected value of t: $\left(\left(\frac{H^*_{S,B}}{H^*_{S,I}}-1\right)\times 100\%, \left(\frac{H^*_{R,B}}{H^*_{R,I}}-1\right)\times 100\%\right)$.

In the second row, we report the percentage point changes in the demands of the target and secondary products, respectively, when a bundle discount is offered to create a forward-buy incentive for the target product and for the selected value of t: $\left(\left(\frac{D_1\left(p^*_{B,1}\right)}{D_1\left(p^*_{R,1}\right)}-1\right)\times 100\%, \left(\frac{D_2\left(p^*_{B,2}\right)}{D_2\left(p^*_{R,2}\right)}-1\right)\times 100\%\right)$. We note that the new demand levels will be observed until the seller sells all the units she has procured with the bundle offer; upon depletion of these units, the prices, and therefore the demand levels, of the target and secondary products will revert to their regular levels.

In the third row, we report the amounts of the target and secondary products sold during the promotion period as a percentage of their respective annual demands when a bundle discount is offered:

$$\left(\frac{tQ^*_{R,1}}{D_1\left(p^*_{R,1}\right)+\frac{tQ^*_{R,1}}{D_1\left(p^*_{B,1}\right)}\left(D_1\left(p^*_{B,1}\right)-D_1\left(p^*_{R,1}\right)\right)}, \frac{\frac{tQ^*_{R,1}}{b^*_1}b^*_2}{D_2\left(p^*_{R,2}\right)+\frac{\frac{tQ^*_{R,1}}{b^*_1}b^*_2}{D_2\left(p^*_{B,2}\right)}\left(D_2\left(p^*_{B,2}\right)-D_2\left(p^*_{R,2}\right)\right)}\right).$$

We note that the annual demand for each product is now equal to the regular annual demand plus the incremental demand the supply chain experiences during the promotion period.

And finally in the last row, we report the number of forward-buy periods for the secondary product when the seller designs the bundle offer to create a forward-buy incentive for the target product and for the selected value of t: $\frac{\frac{tQ^*_{R,1}}{b^*_1}b^*_2}{Q^*_{R,2}}$.

2.5.3 Analysis of the Results

The question of whether more effective trade promotions can be designed with a one-time bundle discount in lieu of a one-time individual product discount can be answered affirmatively only when both the seller and the buyer experience a profit increase relative to the one-time individual product discount case, and the resulting solution can be practically implemented. We will discuss the results presented in Figures 2.3, 2.4, 2.5, and 2.6 in light of this observation.

We first consider the problems with $t = 2$ in Figures 2.3 and 2.4. The profit increases for both the seller and the buyer relative to the one-time individual product discount case, reported in the top row, are particularly substantial when the target product has a higher demand and a lower price relative to those of the secondary product, i.e., in problem clusters $(1+, 2-)$, $(2+, 2-)$, $(1+, 1-)$ and $(2+, 1-)$. These problem clusters can be characterized by the combination of a target product which is relatively fast-moving and has a lower price, and a secondary product which is relatively slow-moving and has a higher price. The profit increase both parties experience is mostly due to the changes in the secondary product's demand. For example, when the bundle composition is $(b_1 = 1, b_2 = 1)$ (Figure 2.3), in cluster $(2+, 1-)$, the demand of the secondary product temporarily increases by 23.4% as a result of the new price the buyer sets after she receives the bundle discount. The interplay between the demand increase and the bundle composition is quite strong: when the bundle composition is changed to $(b_1 = 2, b_2 = 1)$ (Figure 2.4), again in cluster $(2+, 1-)$, the temporary increase in the demand of the secondary product drops to 8.5%.

Although the objective of the bundle discount is to create a forward-buy incentive for the target product, the bundle offer may require the buyer to forward-buy units of the secondary product as well. When the bundle composition is selected as $(b_1 = 1, b_2 = 1)$ (Figure 2.3), the forward-buy quantity for the secondary product (reported as a percentage of its annual demand in the third row) and the number of forward-buy periods (reported in the fourth row) are observed to be substantially high in problem clusters $(1+, 2-)$, $(2+, 2-)$, $(1+, 1-)$ and $(2+, 1-)$. Changing the bundle composition to ($b_1 = 2, b_2 = 1$) (Figure 2.4) helps to reduce the forward-buy quantity of the secondary product at the expense of decreasing the relative profit increases over the individual product discount alternative. For example, in cluster $(1+, 1-)$, changing the bundle composition to ($b_1 = 2, b_2 = 1$) reduces the increase in the seller's profit from 3.1% to 1.5%, making, on the other hand, the bundle offer easier to implement by reducing the number of forward-buy periods of the secondary product from 2.9 to 1.4.

We now turn to problems with $t = 3$ in Figures 2.5 and 2.6. In line with the results for $t = 2$, the profit increase is again higher in problem clusters $(1+, 2-)$, $(1+, 1-)$ and $(2+, 1-)$ where the target product has a higher demand and a lower price than the secondary product. With $t = 3$, the seller achieves higher profits in problem clusters $(1-, 1+)$, $(2-, 1+)$ and $(1-, 2+)$, too. Particularly when the bundle composition ($b_1 = 2, b_2 = 1$) is used (Figure 2.6), the resulting purchase quantities for the secondary product are identical to the regular purchase quantities, rendering the bundle discount highly implementable.

As illustrated in Figures 2.3, 2.4, 2.5, and 2.6, one-time bundle discount can be an efficient alternative to creating forward-buy incentives. However, the ease of implementation of the bundle discount scheme depends on the characteristics of the bundled products along with the bundle composition decisions and the length of the target forward-buy period, t.

When the target product has a lower demand and a higher price (i.e., problem clusters $(1-, 1+)$, $(2-, 1+)$, $(1-, 2+)$ and $(2-, 2+)$), the bundle discount scheme can be readily implemented when the target forward-buy period, i.e., t, is not too high. For instance, in cluster $(2-, 1+)$ (Figure 2.3), the profits of the seller and the buyer increase by 1.1% and 0.4%, respectively, and the resulting number of forward-buy periods of the secondary product is 1.2, i.e., with the bundle discount scheme, the required delivery quantity of the secondary product is not too different from its regular delivery quantity. Again in the same problem cluster $(2-, 1+)$, when the value of t increases to 3 and the bundle composition is ($b_1 = 2$, $b_2 = 1$) (Figure 2.6), the seller substantially increases his profit, and the number of forward-buy periods of the secondary product is equal to 0.9, i.e., the amount of the secondary product delivered with the bundle offer is actually less than the buyer's regular order quantity for the secondary product.

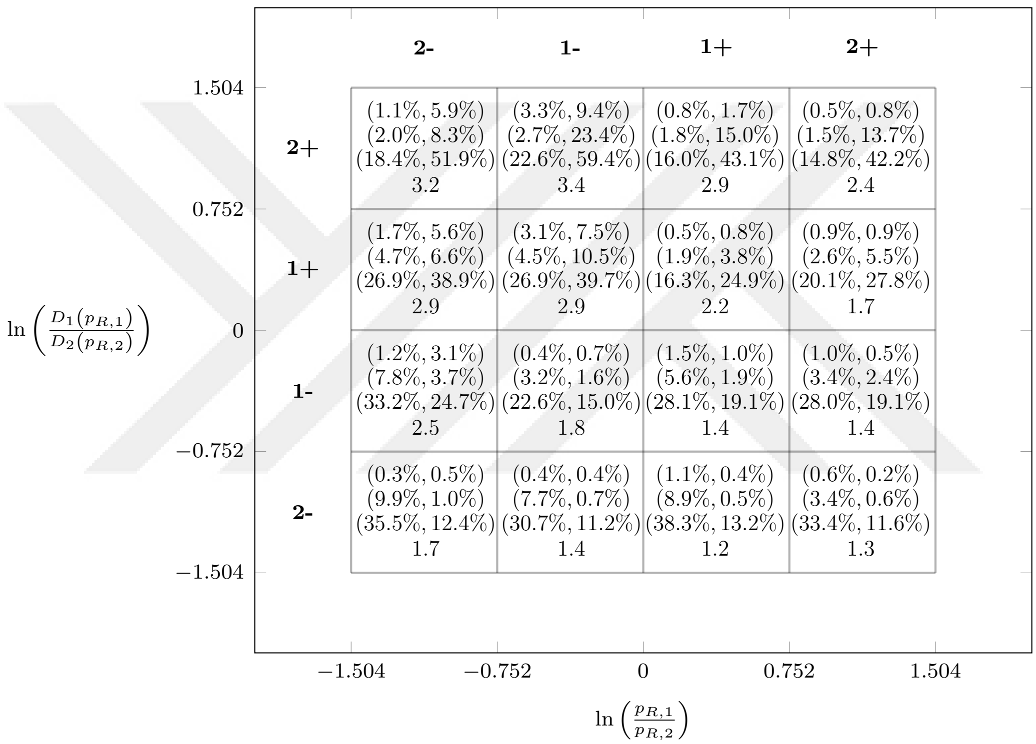

Figure 2.3: Summary of numerical results for $t = 2$ and $(b_1 = 1, b_2 = 1)$ (see Section 2.5.2 for the description of the Figure's format).

2.6 Conclusion

Although the negative impact of forward-buying behavior on supply chain inventory costs is well understood, firms continue to offer trade promotions to stimulate sales or to defend their products against competition (Chopra and Meindl 2014). In this study, we address the question of whether one-time bundle discounts, as an alternative to one-time individual product discounts, result in more effective trade promotions.

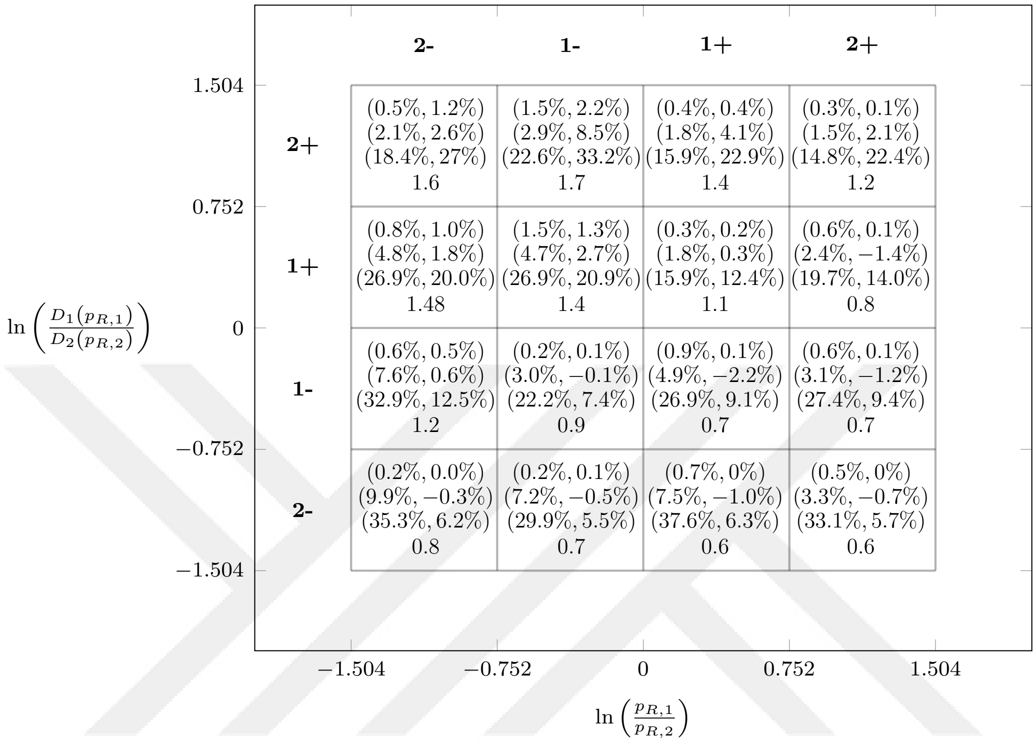

Figure 2.4: Summary of numerical results for $t = 2$ and $(b_1 = 2, b_2 = 1)$ (see Section 2.5.2 for the description of the Figure's format).

Our findings suggest that, for a seller who wants to create a forward-buy incentive, the bundle discount scheme can actually be a more effective alternative. However, the degree of effectiveness and ease of implementation of the bundle discount scheme depend on the characteristics of the bundled products along with the bundle composition decisions.

When a seller aims to create a forward-buy incentive for a product with a relatively low demand and high price, our results indicate that, particularly when the bundle composition is carefully selected, the bundle discount scheme may not require a large inventory shift of the secondary product. Our findings suggest that for products with a low demand and high price, the bundle discount scheme can be an effective alternative to creating forward-buy incentives. On the other hand, when a seller attempts to create a one-time bundle discount-based forward-buy incentive for a product whose

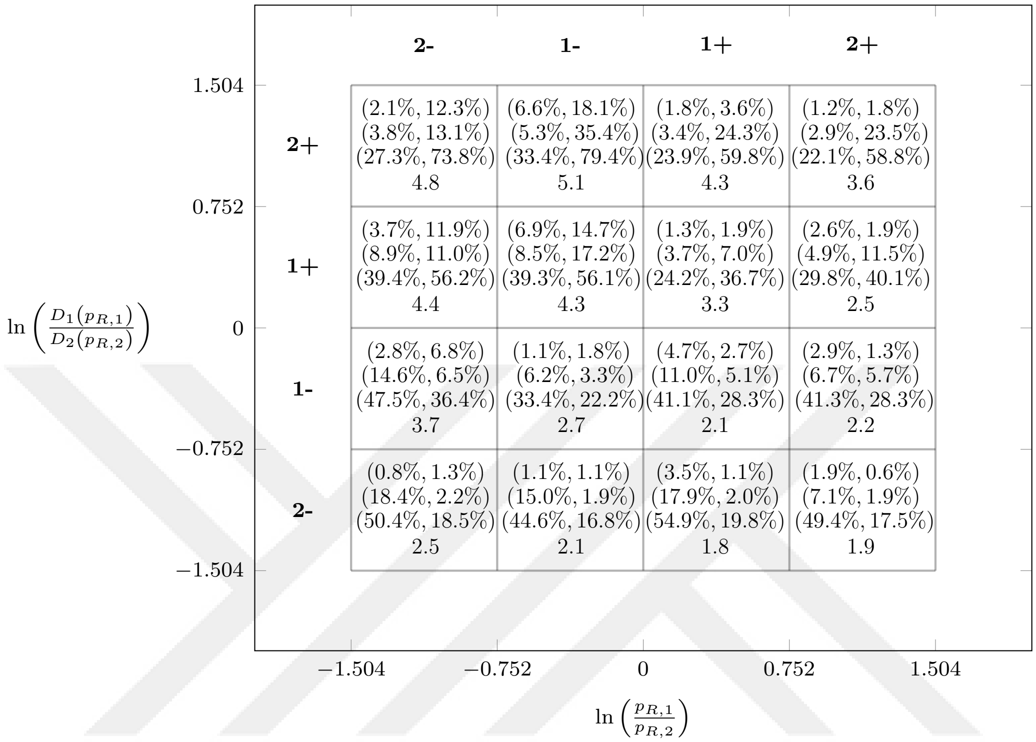

Figure 2.5: Summary of numerical results for $t = 3$ and $(b_1 = 1, b_2 = 1)$ (see Section 2.5.2 for the description of the Figure's format).

demand is larger than the demand of the other product in the bundle, our results show that the seller may face a large one-time delivery requirement for the secondary product as well. If this requirement is not offset by the bundle composition decisions, the large one-time delivery of the secondary product may burden the supply chain with additional procurement/manufacturing and inventory costs, rendering the bundle discount scheme difficult to implement.

Our assumption, that the setup costs do not change when the bundle offer option is selected, leaves the coordination benefits of bundling out of the analysis and puts the one-time bundle discount scheme in a disadvantageous position. The observation that a one-time bundle discount scheme can be more effective even under this rather restrictive assumption points out that one-time bundle discounts can be even more effective in cases where bundling results in additional savings through the coordination

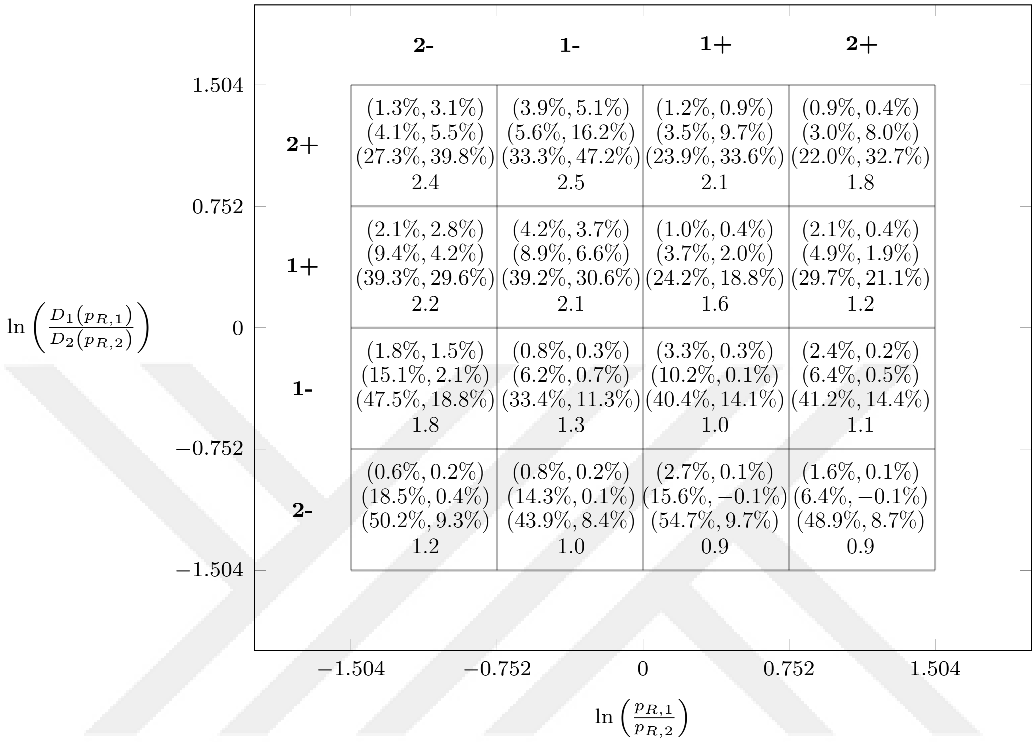

Figure 2.6: Summary of numerical results for $t = 3$ and $(b_1 = 2, b_2 = 1)$ (see Section 2.5.2 for the description of the Figure's format).

of setups.

In this chapter, we study the bundle discount scheme in a business-to-business setting and do not include consumers' response to bundle offers in its scope. The business-to-customer setting also presents a rich set of research problems with regard to the design of the bundle discounts. The central questions are again the selection of the products to be bundled, bundle composition, and pricing. Analysis of these questions, however, requires models that are different to those presented in this study, particularly when the customers are heterogeneous in terms of their reservation prices.

Chapter 3

DYNAMIC AND TARGETED BUNDLE PRICING OF TWO INDEPENDENTLY VALUED PRODUCTS

3.1 Introduction

Advances in information technology and data analytics have enabled retailers to understand their customers' buying behavior at a very granular level and paved the way for the implementation of promotions targeted at the individual customer level. Mobile advertising that relies on these advances plays a key role in personalized promotions, in the form of emailed offers, SMS, app or banner messages, and is expected to reach a volume above $60 B by 2019 (eMarketer, 2015). Personalized promotions seem to be highly valued by the retail customers: in a recent survey of more than 1,500 consumers in the United States and the United Kingdom, 65% of the respondents said that they are more likely to shop at a retailer in-store or online that sends them relevant and personalized promotions (Accenture Interactive, 2016).

The UK retailer Tesco has been a leader in the design of data-driven loyalty programs that are based on customers' purchasing patterns. At Tesco, customer segments are formed through market basket analyses, and targeted promotion activities are extended to individual segments (Humby et al., 2004; Davenport et al., 2011). A similar personalized promotion approach is implemented by the French grocery retailer Carrefour (IBM Retailer Solutions, 2009). Major retailers such as Safeway and Kroger offer their customers individualized prices by tracking customers' locations in the store or by observing customers' purchases in real time when they use the barcode reader on their phone to skip the checkout process (Clifford, 2012; Farnham, 2013). In Safeway's "just for U" program, digital coupons for personalized deals are extended to individual consumers either online or through the “just for U” app. The CEO of Safeway, Steve Burd, told analysts in 2013 (Ross, 2013): “There’s going to come a point where our shelf pricing is pretty irrelevant because we can be so personalized in what we offer people.”

Retailers engage in such promotional activities to stimulate demand and increase revenue. Although some of the promotional activities, such as loyalty programs, are designed with a long-term perspective, a typical promotional activity’s goal is to stimulate the short-term demand. Short-term demand stimulation can help a retailer improve its bottom line and, in certain cases, addresses issues that may arise due to operational inefficiencies. The management of excess inventories is a typical example. According to a study sponsored by IBM (Webber et al., 2011), retailers, such as the German retailer Metro Group and the Dutch retailer Albert Heijn, practice dynamic pricing and promotional activities to stimulate demand for products with shorter remaining shelf-lives.

Pricing strategies such as dynamic pricing (Elmaghraby and Keskinocak, 2003) or promotional activities like cross-selling (Aydin and Ziya, 2008) can be instrumental when a retailer aims to reduce excess inventories while maximizing the revenue. Targeted (or, personalized) pricing, i.e., the practice of offering an individualized discount to a customer based on her purchase history (Arora et al., 2008), can be an additional strategy for stimulating demand. For example, as quoted in (Farnham, 2013), Safeway tries to sell wipes to customers that are observed buying baby food but not wipes via personalized deals and coupons they can download. The discounts load directly to the loyalty card, so the discount is already taken by the time customers get to the register.

In this study, as an alternative and relatively less-studied form of promotion strategy, we focus on bundling, i.e., the practice of selling two or more different products in a package for a price that is lower than the sum of the individual prices of the products (Stremersch and Tellis, 2002). Bundling, as an effective price discrimination tool, reduces heterogeneity in customers’ product valuations and enables retailers to extract a larger share of the available surplus (Salinger, 1995). We consider bundling in the context of targeted promotions, and analyze the interplay between the promotion strategy (discount for a single product vs. the bundle) and segment-specific and dynamic pricing decisions. We examine the case of a retailer that aims to reduce the excess inventory of a product with a shorter remaining shelf-life (referred to as the primary product) by making it part of a bundle offer formed with another product which has a relatively longer remaining shelf-life (referred to as the secondary product). We consider the case where 1) the primary and secondary products belong to related categories such as baby food and wipes (as seen in the Safeway example), or yogurt and oatmeal products, and 2) are independently valued, i.e., are neither complements or substitutes. Assuming that the retailer has the ability to cluster its customers into different segments in terms of their valuations of the primary and secondary products, we study the revenue impact of dynamic and targeted bundle pricing decisions.

Our study points to important revenue-enhancing pricing strategies for a retailer that wants to clear the inventory of the item under consideration (i.e., the primary product). The results demonstrate that, particularly when the initial inventory of the primary product is high, the capacity of the bundle offers to improve the revenue is significant and indicate an additional revenue potential when the price of the bundle is dynamically optimized. Our study illustrates that segment-specific dynamic pricing brings about substantial revenue improvements that are an increasing function of the initial level inventory of the item while static and segment-specific pricing have no direct impact on revenue. Our study also shows that dynamically priced and segment-specific bundle offers yield a robust revenue performance, mitigating a potentially revenue-diminishing impact of the positive correlation in consumers' valuations of the primary and secondary products.

This chapter is organized as follows. We review the relevant research streams and summarize our contributions in the next section. In Section 3.3, a detailed description of the problem setting, along with the characterization of the customer arrival process and the segmentation scheme, is presented. Section 3.4 discusses revenue maximization models that capture the retailer's various options for bundle pricing over a promotion period (interchangeably, planning horizon) under specific product valuation distributions of customers. An extensive computational study (Section 3.5) discusses the revenue impact of the retailer's pricing strategies that are examined in this study. Finally, Section 3.6 summarizes key findings that are posited, and discusses the main limitations of the model and possible extensions of the studied research questions.

3.2 Literature Review

In this section, to position our contributions, we review research streams that are relevant for the key aspects of our model, i.e., bundling, segment-specific or targeted pricing, and dynamic pricing.

Stremersch and Tellis (2002) identify two forms of bundling: price bundling (the sale of two or more separate products in a package at a discount), and product bundling (the integration and sale of two or more separate products or services at any price). Price bundling, which is the focal strategy in our study, reduces heterogeneity in customers' product valuations and enables the retailers to extract a larger share of the available surplus (Adams and Yellen, 1976; Salinger, 1995). Banciu and Odegaard (2016) study a single-period bundle pricing problem when the underlying valuations of the bundle components are dependent. They model the joint density of reservation prices by using copula theory and demonstrate that under the pure bundling strategy, i.e., when products that are part of the bundle offer are not individually available for sale, and when the products have relatively small marginal costs, the seller is better off by bundling products that have a negative association between their valuations. Cataldo and Ferrer (2017) consider a firm's multiple bundle composition and pricing problem, and find that a bundle's optimal price depends on the compositions of the bundles offered by the firm, and not on their prices, and on the composition and price of all the competitors' bundles.

Bulut et al. (2009) address a dynamic bundle pricing problem with limited inventories. To the best of our knowledge, this is the closest study to our own. Bulut et al. (2009) show that bundling is effective when the product valuations are negatively correlated and the starting inventory levels are high. When the starting inventory levels of the bundled products are equal and in excess of average demand, Bulut et al. (2009) demonstrate that most of the benefits of bundling can be achieved through pure bundling. Our study differs from Bulut et al. (2009) along two dimensions. First, we consider the bundle pricing problem in the context of segment-specific or targeted pricing. Second, the market structure we introduce enables us to express the correlation in product valuations in terms of segment sizes. In addition, we present a number of structural properties of the revenue maximizing bundle discounts in relation to the inventory levels and the number of remaining periods in the planning horizon.

Bundling differs from cross-selling (also referred to as up-selling, as described in Aydin and Ziya, 2008) practices in that the individual discounted prices of the products that form the bundle cannot be identified by the customers with a bundle offer, whereas, with cross-selling, the discounted price of the promoted product is directly observable. Additionally, the bundle offer is extended before the customer completes a purchase, the cross-selling offer is made available after the customer commits to the purchase of a product (Netessine et al., 2006).

Access to exponentially increasing personalized consumer information and advances in the retailers' data-analytics capabilities have paved the way for market segmentation at the individual consumer level. Personalized pricing is a typical strategy retailers follow to exploit the available data. Acquisti and Varian (2005) show that when the consumers are myopic or if anonymizing technologies are too costly for them, it might be attractive to use personalized prices. In today's retail setting, anonymizing technologies may not be too costly: a consumer who is part of a loyalty program (such as the "just for U" program we referred to in Section **??**) may choose not to run the app that tracks her purchases in the store. Acquisti and Varian (2005) also demonstrate that personalized prices can be effective when high-value consumers can be offered a price-service package that is perceived to be more valuable than the one offered to low-value consumers. In a simulation study, Shiller (2014) shows that using demographics alone to tailor prices improves profits by 0.8%. Golrezaei et al. (2014) study a personalized, choice-based assortment optimization problem and, with actual sales data from an online retailer, show that location-based personalization can lead to over 10% revenue improvement. Bimpikis et al. (2016) study targeted advertising and show that the optimal fraction of a firm's marketing budget that is targeted to a specific consumer is an increasing function of her network centrality. Esteves and Resende (2016) consider a two-firm, two-product setting where each firm's set of potential buyers is composed of two distinct segments of equal size, and show that more advertising to the weak market can only arise in equilibrium if the firms can simultaneously target price and advertising content. Sahni et al. (2016) analyze randomized experiments on emailed targeted discount offers and argue that that, in addition to being tools for price discrimination, emailed offers result in cross-category spillover to the firm's non-promoted products.

In terms of its legal implications, personalized pricing or price discrimination can be associated with (1) exploitative abuse (i.e., when the firm offers different prices to different customers), (2) exclusionary effects (i.e., when the firm makes available a set of related offerings with fixed prices associated with each), and (3) distortionary effects (i.e., when the firm charges different prices in different segments) (Murthi and Sarkar, 2003). In a recent note presented to the OECD, the Federal Trade Commission of the United States (FTC) stated that "price discrimination enhances market competition," and "it is often viewed as efficient" (The Federal Trade Commission, 2016). The FTC's position implies that price discrimination is acceptable as long as it does not have anti-competitive implications.

Dynamic pricing of limited inventories has been a very active research area starting with the seminal papers of Gallego and van Ryzin (1994), Gallego and van Ryzin (1997), and Bitran and Mondschein (1997). Reviews of the early literature are presented in Elmaghraby and Keskinocak (2003), and Bitran and Caldentey (2003). Shen and Su (2007) present a review of the literature on dynamic pricing problems with strategic customers, and Aviv et al. (2009) present a review of the research stream that addresses the adverse impact of strategic customer behavior. The scope of the dynamic pricing literature has been expanded with models that consider additional features such as dynamic pricing and learning in a changing environment (Farias and Van Roy, 2010; Den Boer, 2015), dynamic pricing with time-varying demand function (Chen and Farias, 2013), dynamic pricing with uncertain production cost (Sibdari and Pyke, 2014), dynamic pricing model for multi-class problem in the airline industry (Otero and Akhavan-Tabatabaei, 2015), dynamic pricing of primary product and ancillary services (Odegaard and Wilson, 2016), joint dynamic pricing and inventory management with strategic customers (Chen and Shi, 2017), and dynamic pricing and ordering of perishable products (Herbon and Khmelnitsky, 2017).

In their extensive review of the recent literature on dynamic pricing with multiple products, competition, and limited demand information, Chen and Chen (2015) point out that the existing models on personalized pricing are static in nature and do not address the inventory issues, and state that dynamic personalized pricing with inventory consideration is an exciting future research topic. In this study, we present a model that contributes to the literature by considering the optimization of the segment-specific and dynamic pricing decisions of a retailer that aims to clear the inventory of an item with a bundle offer. For benchmarking purposes, we also consider a segment-specific and dynamic pricing problem for the primary product only. For the specific retail setting we consider in the study, and to the best of our knowledge, our study also contributes to the literature by investigating the interactions among bundling, dynamic pricing and targeted promotions, and by illustrating certain structural properties of the optimal bundle discounts in a multi-period setting.

3.3 Problem Setting

We consider a retailer that operates in a market with multiple and identifiable customer segments. The segments are heterogeneous in terms of their valuation distributions for the primary and secondary products that form the bundle. The retailer plans to liquidate the excess inventory of the primary product through a dynamically priced and targeted bundle offer over the planning horizon while maximizing the expected total revenue. The planning horizon is finite and is divided into T decision epochs. At the start of each decision epoch (interchangeably, period), in consideration of the available units in the inventory and the remaining time in the planning horizon, the retailer chooses the discount level for the bundle offer. Without any loss of generality, we assume that the bundle offer does not require any physical integration of the two products, and therefore it is made available until the inventory of the primary product is depleted. We also assume that the initial inventory of the secondary product is large enough to meet the demand during the planning horizon or that there exists a fast replenishment option.

3.3.1 Arrival process and the sequence of events in each period

We consider a Poisson arrival process and assume that, at most, one customer arrives in each period (see Netessine et al., 2006, for similar assumptions). At the start of a period and in consideration of the current market structure – i.e., the size of each segment, the valuation distributions of each segment for the primary and secondary products and the prices of the products – the retailer determines 1) the segments that will be informed about the bundle offer and 2) the price of the bundle offer which is allowed to be unique for each informed segment.

If a customer arrival takes place in a period and the arriving customer belongs to a segment that has been informed about the bundle offer, she considers the following purchase options: 1) bundle of the primary and secondary products, 2) primary product only, 3) secondary product only, and 4) no purchase. A customer whose segment has not been informed about the bundle offer has the identical options, however, since the customer is not eligible for the bundle offer, the price of the bundle is simply equal to the sum of the individual prices of the primary and secondary products in that particular period.

We assume that each customer chooses the purchase option that maximizes her net utility. The net utility for individual products is expressed as the difference between the customer's valuation of the product and the price of the product. For the bundle offer, the net utility is considered to be additive (as in Adams and Yellen, 1976) and equal to the sum of the valuations minus the price of the bundle.

3.3.2 Customer segmentation and identification of arrival rates

In parallel with models where choice probabilities are expressed in terms of segment-specific distributions (Kamakura and Russell, 1989), we assume that the customers of a product can be clustered into a finite number of groups, and product valuations (i.e., willingness-to-pay values) of customers in a specific group can be represented by a random variable. To keep the exposition simple, we consider two groups of customers for each of the primary and secondary products: customers with low (L) valuations and customers with high (H) valuations. Let $\overline{R}_{L,P}$ and $\overline{R}_{H,P}$ ($\overline{R}_{L,S}$ and $\overline{R}_{H,S}$) be the random variables for the primary (secondary) product that capture the customers' valuations with levels low and high, respectively. We present a detailed discussion of the stochastic ordering of these random variables in Section 3.4.1 where we also impose the requirement that $\mathbf{E}\left[\overline{R}_{H,P}\right] \geq \mathbf{E}\left[\overline{R}_{L,P}\right]$ and $\mathbf{E}\left[\overline{R}_{H,S}\right] \geq \mathbf{E}\left[\overline{R}_{L,S}\right]$.

When customers' valuations for the primary and secondary products are jointly considered, the market can be divided into four segments where each segment $n \in N = \{HH, HL, LH, LL\}$ is associated with a pair of random variables that reflect the valuations of customers for the primary and secondary products. For example, the segment HL represents the group of customers whose valuations for the primary and secondary products can be captured by random variables $\overline{R}_{H,P}$ and $\overline{R}_{L,S}$, respectively. The segment structure we use and the models we develop based on this structure can be readily extended to cases where the customers can be assigned to more than two groups with respect to their valuations of a product. For example, in a setting with product valuations at levels low, medium and high, the cardinality of the segment set N will be $3^2 = 9$.

The revenue models we develop in this study assume that the segment sizes are known by the retailer. A typical approach for the identification of segment sizes is to analyze the price levels at which the sale transactions with individual customers are realized (Brin et al., 1997; Silverstein et al., 1998; Boztug and Reutterer, 2008). To identify the customers segments in our setting, a simple procedure that relies on association rules can be carried out by studying four specific transaction scenarios: Let C_{HH} denote the count of the sale transactions in the market basket data where a customer has purchased the primary and the secondary products jointly at their regular prices, C_{LL} denote the count of the sale transactions in which the primary and secondary products are purchased jointly at their discounted prices or are not present in the transaction, and C_{HL} (C_{LH}) denote number of transactions in which the primary (secondary) product is purchased at its regular price and the secondary (primary) product is purchased at a discounted price or is not present in the transaction. Let δ_n denote the fraction of customers that belong to segment n, $n \in N$. Once the number of transactions for each scenario are tallied, the relative segment sizes can be readily computed:

$$\delta_n = \frac{C_n}{\sum_{k \in N} C_k}, \quad n \in N = \{HH, HL, LH, LL\} \tag{3.1}$$

3.4 The model

In this section, we first describe two revenue maximization models that capture the retailer's two options for the liquidation of the excess inventory of the primary product over the planning horizon: 1) the targeted dynamic pricing of the primary product (TDP), and 2) the targeted dynamic pricing of a bundle offer that is formed with the primary and secondary products (TDB). Subsequently, to determine the impact of targeted pricing, we consider two special cases of these models: 1) non-targeted dynamic pricing of the primary product (DP), and 2) non-targeted dynamic pricing of the bundle offer (DB).

3.4.1 Customers' Product Valuations

As defined in Section 3.3.2, let $\overline{R}_{l,j}, l \in \{L, H\}$ and $j \in \{P, S\}$, be the random variable denoting the valuation of the group of customers that assign a value at level l to product j. With a slight abuse of notation, we let $R_{n,P}, R_{n,S}$ and $R_{n,B}, n \in N$ be the random variable that denotes the valuation of a customer segment n for the primary product, secondary product, and the bundle of the two products, respectively. Also let $F_{n,j}(\cdot)$, $\overline{F}_{n,j}(\cdot)$ and $f_{n,j}(\cdot)$, $n \in N$ and $j \in \{P, S\}$, be the cumulative distribution, the complementary cumulative distribution, and the probability density functions, respectively, of random variable $R_{n,j}$. For example, when $n = HL$ and $j = S$, $F_{n,S}(\cdot)$, $\overline{F}_{n,S}(\cdot)$ and $f_{n,S}(\cdot)$ will be associated with random variable $R_{HL,S}$.

In parallel with the literature on customers' valuation of bundle offers, and under the assumption that primary and secondary products are not complements or substitutes, we adopt the additivity assumption for the valuation of the bundle (Adams and Yellen, 1976) the random variable that represents the valuation of segment $n \in N$ for the bundle, $R_{n,B}$, is equal to the sum of the valuations of individual products in the bundle: $R_{HH,B} = \overline{R}_{H,P} + \overline{R}_{H,S}$, $R_{HL,B} = \overline{R}_{H,P} + \overline{R}_{L,S}$, $R_{LH,B} = \overline{R}_{L,P} + \overline{R}_{H,S}$, and $R_{LL,B} = \overline{R}_{L,P} + \overline{R}_{L,S}$.

We first define the generalized failure rate function and failure rate ordering, and then list three assumptions about the properties of the valuation distributions $F_{n,j}, n \in N$ and $j \in \{P, S\}$.

Definition 1. *The generalized failure rate of a continuous random variable is given by* $\frac{xf(x)}{1-F(x)}$ *where* $f(\cdot)$ *and* $F(\cdot)$ *denote the probability density and the cumulative distribution functions of the random variable, respectively.*

Definition 2. *For two cumulative distribution functions* $F_1(\cdot)$ *and* $F_2(\cdot)$ *(with probability density functions* $f_1(\cdot)$ *and* $f_2(\cdot)$*, respectively), if* $\frac{f_1(x)}{1-F_1(x)} < \frac{f_2(x)}{1-F_2(x)}$, $\forall x$*, then* $F_1(\cdot)$ *strictly dominates* $F_2(\cdot)$ *in failure rate ordering (denoted as* $F_1(\cdot) >_{fr} F_2(\cdot)$*).*

Assumption 1. $F_{n,j}, n \in N$ *and* $j \in \{P, S\}$*, are twice continuously differentiable, strictly increasing functions, and they all have the same non-negative support.*

Assumption 2. *$F_{n,j}, n \in N$ and $j \in \{P,S\}$, have strictly increasing generalized failure rates.*

Assumption 3. *For $n_1 \in \{HH, HL\}$ and $n_2 \in \{LL, LH\}$, $F_{n_1,P}(\cdot) >_{fr} F_{n_2,P}(\cdot)$, and for $n_1 \in \{HH, LH\}$ and $n_2 \in \{LL, HL\}$, $F_{n_1,S}(\cdot) >_{fr} F_{n_2,S}(\cdot)$.*

Assumption 2 is required to provide some regularity on the objective function, such as the unimodality of the revenue function in the bundle price. This assumption is satisfied by a variety of probability distributions such as Weibull and Gamma distributions. Similarly, Assumption 3 represents the case where the customers are less (more) price-sensitive when they have a high (low) product valuation. For the primary (secondary) product in the bundle, Assumption 3 implies that the valuation of a customer that belongs to segment HH or HL (HH or LH) stochastically dominates that of a customer that belongs to segment LL or LH (HL or LL) (for probability distributions which have increasing generalized failure rate property, and for a comparison of various assumptions about product valuations in revenue management, see Ziya et al. (2004), Lariviere (2006) and Banciu and Mirchandani (2013)).

3.4.2 Purchase Probabilities

Let p_P and p_S be regular prices of the primary and secondary products, respectively. We assume that p_P and p_S are the revenue-maximizing prices when the prices are kept constant during the planning horizon. Also, let $d^P_{n,t}$ ($d^B_{n,t}$) be the discount the retailer offers for the primary product (the bundle of the two products) for segment $n \in N$ in period $t, t = 1, 2, \ldots, T$, where T is the length of the planning horizon, and t denotes the number of periods to go, i.e., the number of periods the current period is away from the end of the planning horizon. We assume that $0 \leq d^P_{n,t} < p_P$ and $0 \leq d^B_{n,t} < \min\{p_P, p_S\}$. The last assumption ensures that in the mixed-bundling setting where the bundle offer and the individual products are simultaneously available to the customers, the probability that a customer will purchase only one of the primary or the secondary products is positive.

We also assume that the retailer offers a bundle discount if and only if she does not offer an individual discount for the primary product, i.e., $d^B_{n,t} > 0 \Leftrightarrow d^P_{n,t} = 0, n \in N, t = 1, 2, \ldots, T$. This assumption allows the analysis we present in the subsequent sections to set apart the revenue impact of the individual product discount and the bundle discount. Although it is technically possible to offer discounts for an individual product along with the bundle discount, because the bundle discount makes sense only when $d^B_{n,t} \geq d^P_{n,t}, n \in N, t = 1, 2, \ldots, T$, the individual product discount has a marginal revenue impact in the presence of a meaningful bundle offer.

We now turn to the calculation of the purchase probabilities of an arriving customer from segment $n \in N$. To simplify the exposition in the remainder of this section, we will drop the time index $t, t = 1, 2, \ldots, T$, whenever doing so will not cause confusion. Let α^P_n (α^S_n) be the probability that the customer will purchase the primary (secondary) product only, α^{PS}_n be the probability that the customer will purchase the primary and secondary products simultaneously when $d^P_n > 0$ and $d^B_n = 0$ (i.e., in the case of individual discount for the primary product), and α^B_n be the probability that the customer will purchase the bundle of the primary and secondary products when $d^P_n = 0$ and $d^B_n > 0$. Also let $\alpha^{\emptyset}_n, n \in N$ and $t = 1, 2, \ldots, T$, be the no-purchase probability.

We first consider the case of $d^B_n > 0$ and $d^P_n = 0, n \in N$. As also assumed in Bulut et al. (2009) and Gürler et al. (2009), we assume that each arriving customer selects the purchase option that maximizes her net surplus. The arriving customer purchases only the primary product when 1) the primary product provides a non-negative net utility, *and* 2) the secondary product presents a negative net utility, *and* 3) the bundle offer delivers a lower net utility than the primary product itself. Therefore, α^P_n can be calculated as follows:

$$\begin{aligned}\alpha_n^P &= Pr\{p_P - d_n^P \leq R_{n,P}, p_S \geq R_{n,S},\\ &\quad R_{n,P} - (p_P - d_n^P) \geq R_{n,B} - (p_P + p_S - d_n^B)\},\\ &= Pr\{p_P - d_n^P \leq R_{n,P}, p_S \geq R_{n,S},\\ &\quad R_{n,P} - (p_P - d_n^P) \geq R_{n,P} + R_{n,S} - (p_P + p_S - d_n^B)\},\\ &= Pr\{p_P - d_n^P \leq R_{n,P}, p_S \geq R_{n,S}, d_n^P + (p_S - d_n^B) \geq R_{n,S}\},\\ &= Pr\{p_P - d_n^P \leq R_{n,P}, p_S - (d_n^B - d_n^P) \geq R_{n,S}\}.\end{aligned}$$

By definition, $d_n^B > 0$ and $d_n^P = 0, n \in N$, and, when a bundle offer is extended, the probability of a customer purchasing only the primary product is given as $\alpha_n^P = Pr\{p_P \leq R_{n,P}, p_S - d_n^B \geq R_{n,S}\}$. Similarly, if there is no bundle offer, the probability of the purchase of only the primary product is equal to $\alpha_n^P = Pr\{p_P - d_n^P \leq R_{n,P}, p_S \geq R_{n,S}\}$.

The arriving customer purchases only the secondary product when 1) the secondary product provides a non-negative net utility, *and* 2) the primary product presents a negative net utility, *and* 3) the bundle offer delivers a lower net utility than the secondary product itself. Using arguments similar to those presented for α_n^P, α_n^S can be calculated as follows:

$$\alpha_n^S = Pr\{p_S \leq R_{n,S}, p_p - d_n^p \geq R_{n,P}, R_{n,S} - p_S \geq R_{n,B} - (p_P + p_S - d_n^B)\}.$$

When the retailer extends a bundle offer, a customer's purchase probability of the secondary product only is given as $\alpha_n^S = Pr\{p_S \leq R_{n,S}, p_P - d_n^B \geq R_{n,P}\}$. If there is no bundle offer, the purchase probability of the secondary product only is equal to $\alpha_n^S = Pr\{p_S \leq R_{n,S}, p_P - d_n^P \geq R_{n,P}\}$. The arriving customer does not purchase anything when all three options provide negative net utilities:

$$\alpha_n^\emptyset = Pr\{p_P - d_n^P \geq R_{n,P}, p_S \geq R_{n,S}, p_P + p_S - d_n^B \geq R_{n,B}\}.$$

When there is no bundle offer, a customer's no-purchase probability is given as $\alpha_n^\emptyset = Pr\{p_P - d_n^P \geq R_{n,P}, p_S \geq R_{n,S}\}$, and, with a bundle offer, the no-purchase probability is calculated as $\alpha_n^\emptyset = Pr\{p_P \geq R_{n,P}, p_S \geq R_{n,S}, p_P + p_S - d_n^B \geq R_{n,B}\}$.

Table 3.1: Purchase probabilities for segment $n \in N$.

Discount Strategy	Purchased Products	Probability
	Primary only	$\alpha_n^P = \overline{F}_{n,P}(p_P) F_{n,S}(p_S - d_n^B)$
Bundle Offer:	Secondary only	$\alpha_n^S = F_{n,P}(p_P - d_n^B) \overline{F}_{n,S}(p_S)$
$d_n^B > 0$ and $d_n^P = 0$	No-Purchase	$\alpha_n^\emptyset = \int_0^{p_P} F_{n,S}(\min\{p_S, p_P + p_S - d_n^B - x\}) f_{n,P}(x) dx$
	Bundle	$\alpha_n^B = 1 - \alpha_n^P - \alpha_n^S - \alpha_n^\emptyset$
	Primary only	$\alpha_n^P = \overline{F}_{n,P}(p_P - d_n^P) F_{n,S}(p_S)$
Individual Discount:	Secondary only	$\alpha_n^S = F_{n,P}(p_P - d_{n,}^P) \overline{F}_{n,S}(p_S)$
$d_n^B = 0$ and $d_n^P > 0$	No-Purchase	$\alpha_n^\emptyset = F_{n,P}(p_P - d_n^P) F_{n,S}(p_S)$
	Primary and Secondary	$\alpha_n^{PS} = 1 - \alpha_n^P - \alpha_n^S - \alpha_n^\emptyset$

In the first four rows of Table 3.1, we illustrate how the purchase probabilities are expressed under the bundle offer and in terms of the cumulative and complementary cumulative distributions, and the probability density functions of random variables $R_{n,j}$, $n \in N$ and $j \in \{P, S\}$. In the last four rows of the same table, we present the purchase probabilities for the case of $d_n^B = 0$ and $d_n^P > 0, n \in N$, which can be computed in a similar manner.

3.4.3 The Firm's Revenue Maximization Problem

In this section we first present a general formulation of the firm's revenue maximization problem. We then summarize how the general formulation can be mapped into the problems of interest that are discussed in the introductory paragraph of Section 3.4.

Let $V_t(y)$ denote the retailer's expected revenue when starting in period t with an inventory of y units of the primary product. As noted earlier, the initial inventory of the secondary product is assumed to be large enough to meet the demand during the planning horizon, and therefore the inventory level of the secondary product is not considered to be part of the problem's state space. For $y > 0$ and $t \geq 1$, the optimality equations can be expressed as

$$V_t(y) = \max_{d^B_{n,t}, n\in N} \sum_{n\in N} \delta_n \left(\begin{array}{c} \alpha^P_{n,t}\left(p_P + V_{t-1}(y-1)\right) \\ +\alpha^S_{n,t}\Big(p_S + V_{t-1}(y)\Big) \\ +\alpha^B_{n,t}\left(p_P + p_S - d^B_{n,t} + V_{t-1}(y-1)\right) \\ +\alpha^{\emptyset}_{n,t} V_{t-1}(y) \end{array} \right)$$

for the bundle offer case, and as

$$V_t(y) = \max_{d^P_{n,t}, n\in N} \sum_{n\in N} \delta_n \left(\begin{array}{c} \alpha^P_{n,t}\left(p_P - d^P_{n,t} + V_{t-1}(y-1)\right) \\ +\alpha^S_{n,t}\Big(p_S + V_{t-1}(y)\Big) \\ +\alpha^{PS}_{n,t}\left(p_P - d^P_{n,t} + p_S + V_{t-1}(y-1)\right) \\ +\alpha^{\emptyset}_{n,t} V_{t-1}(y) \end{array} \right)$$

for the case of individual product discount. As boundary conditions, we have $V_t(0) = 0, t = 1, 2, ..., T$, and when $t = 0$, denoting the end of the planning horizon, $V_0(\cdot) = 0$, i.e., without any loss of generality, we assume that the salvage value of the primary product is equal to zero.

The first term of $V_t(y)$ is the revenue-to-go in the event that the arriving customer purchases the primary product, the second term is the revenue-to-go if she purchases the secondary product, the third term represents the event that the arriving customer purchases both products together, and the fourth term is the revenue-to-go in the event that she leaves without a purchase.

We first consider model DP, i.e., the non-targeted dynamic pricing of the primary product. In DP, we solve the optimization problem with the inclusion of the following constraints: $d^P_{n,t} = d^P_t$ and $d^B_{n,t} = 0, n \in N, t = 0, 1, \ldots, T$, where $d^P_t, t = 0, 1, \ldots, T$, are the new decision variables.

In model DB, where the bundle discount is non-targeted and dynamically adjusted, the optimization problem is solved with the inclusion of the following constraints: $d^P_{n,t} = 0$ and $d^B_{n,t} = d^B_t, n \in N, t = 0, 1, \ldots, T$, where $d^B_t, t = 0, 1, \ldots, T$, are the new decision variables.

In model TDP, where the pricing of the primary product is targeted and dynamically adjusted, the optimization problem is solved with the inclusion of the following constraints: $d^P_{n,t} \geq 0$ and $d^B_{n,t} = 0, n \in N, t = 0, 1, \ldots, T$.

Finally, in model TDB, the optimization problem is solved with the inclusion of constraints $d^P_{n,t} = 0$ and $d^B_{n,t} \geq 0, n \in N, t = 0, 1, \ldots, T$, to allow for targeted and dynamically adjusted bundle pricing.

Let V^*_m be the optimal revenue with model $m, m \in \{DP, DB, TDP, TDB\}$. Because the solution of a problem with non-targeted pricing is also feasible for the targeted pricing case, we have that $V^*_{DP} \leq V^*_{TDP}$ and $V^*_{DB} \leq V^*_{TDB}$. The direction of the relationship between V^*_{DP} and V^*_{DB} or V^*_{TDP} and V^*_{TDB}, on the other hand, depends on the distributions of customers' product valuations, and is the focal point of the computational analysis we will present in Section 3.5.

3.4.4 Properties of the Value Function and the Optimal Bundle Discount

In this section, we discuss how the value function $V_t(y)$ and the expected value of carrying one unit of the primary product into the next period depend on $t, t = 1, 2, \ldots, T$, and $y, y = 1, 2, \ldots, Y$, where y denotes the number of units of the primary product that are available at the start of a period, and Y denotes the initial inventory of the primary product, i.e., the number of units of the primary product that are available at $t = T$. Propositions 1, 2, 3, and 4 demonstrate the monotonicity properties of the value function, and Propositions 5 and 6 characterize the behavior of the optimal bundle discount with respect to the primary product's inventory level and the remaining time to the end of the planning horizon. The proofs of Propositions 1 to 6 are presented in Appendix B.2 of the online supplement.

Proposition 1. *$V_t(y)$ is a non-decreasing function of t: $V_{t-1}(y) \leq V_t(y), t = 1, 2, \ldots, T$, and $y = 1, 2, \ldots, Y$, i.e., the higher the number of periods-to-go, the larger the value of having y units in the inventory.*

Proposition 2. *The expected marginal value of the primary product decreases as the number of periods-to-go decreases: $V_t(y) - V_t(y-1) \geq V_{t-1}(y) - V_{t-1}(y-1), t = 1, 2, \ldots, T$, and $y = 1, 2, \ldots, Y$.*

Proposition 3. *The expected marginal value of the primary product increases as the available number of units of the primary product decreases:* $V_t(y) - V_t(y-1) \geq V_t(y+1) - V_t(y)$, $t = 1, 2, \ldots, T$, *and* $y = 1, 2, \ldots, Y$.

Proposition 4. $V_t(y)$ *is a convex and increasing function of* t*:* $V_{t+1}(y) - V_t(y) \leq V_{t+2}(y) - V_{t+1}(y)$, $t = 1, 2, \ldots, T$, *and* $y = 1, 2, \ldots, Y$.

Proposition 2 states that the marginal expected revenue of the primary product is higher when there are more periods to sell it. In Propositions 3 and 4, we show that, for a given number of periods-to-go, the expected revenue is a concave function of the available number of units of the primary product and, for a given level of the primary product's inventory, the expected revenue is a convex function of the number of periods-to-go, respectively.

Proposition 5. *The optimal bundle discount is a non-decreasing function of the primary product's inventory level:* $d^B_{n,t}(y) \leq d^B_{n,t}(y+1)$, $t = 1, 2, \ldots, T$, *and* $y = 1, 2, \ldots, Y$.

Proposition 6. *The optimal bundle discount is a non-increasing function of* t*:* $d^{B*}_{n,t}(y) \leq d^{B*}_{n,t-1}(y)$, $t = 1, 2, \ldots, T$, *and* $y = 1, 2, \ldots, Y$, *i.e., when there are* y *units of the primary product in the inventory, the higher the number of periods-to-go, the smaller the optimal value of the bundle discount.*

Propositions 6 and 7 show that for a given number of periods-to-go, the optimal bundle discount increases as the primary product's inventory level increases and, for a given level of the primary product's inventory, the optimal bundle discount is a decreasing function of the number of periods-to-go, respectively. The above listed properties of the value function and the optimal values of the bundle discounts play a key role in the efficient solution of the problem instances considered in the computational study by reducing the range over which the discount values need to be optimized.

3.5 Computational Study

The revenue models of Section 3.4.3 consider various combinations of pricing decisions (static vs. dynamic, and targeted vs. non-targeted) and discount type selection decisions (individual discount for the primary product vs. the bundle discount for the two products). To study the revenue implications of the resulting eight pricing strategies, we consider a total of 6,075 problem instances. The problem generation scheme presented in Aydin and Ziya (2008) will be used as a basis for creating the problem instances.

In a pricing problem with two products, the covariance of the customers' product valuations is instrumental in understanding the degree of similarity (or dissimilarity) of the products. Before we proceed with a description of our problem generation scheme, we briefly discuss the approach we will employ in capturing the relationship between product valuations: Let P and S be random variables that denote the valuations for the primary and secondary products, respectively, of a randomly chosen customer. Proposition 7 states the relationship between the segment probabilities δ_n, $n \in N$, and valuations P and S (a proof of the proposition is presented in Appendix B.1 of the online supplement).

Proposition 7. *Let $q_{H,P}$ ($q_{L,P}$) denote the fraction of customers who value the primary product high (low): $q_{H,P} = \delta_{HL} + \delta_{HH}$ ($q_{L,P} = \delta_{LH} + \delta_{LL}$). Similarly, let $q_{H,S}$ ($q_{L,S}$) denote the fraction of customers who value the secondary product high (low): $q_{H,S} = \delta_{LH} + \delta_{HH}$ ($q_{L,S} = \delta_{HL} + \delta_{LL}$). The valuations P and S have a negative (positive) covariance, $Cov(P, S)$, if and only if $\frac{\delta_{HH}}{\delta_{LH}+\delta_{HH}} + \frac{\delta_{LL}}{\delta_{HL}+\delta_{LL}} < 1$ ($\frac{\delta_{HH}}{\delta_{LH}+\delta_{HH}} + \frac{\delta_{LL}}{\delta_{HL}+\delta_{LL}} > 1$)* where

$$\begin{aligned} Cov(P,S) &= \left(E[\overline{R}_{H,P}] - E[\overline{R}_{L,P}]\right) \times \left(E[\overline{R}_{H,S}] - E[\overline{R}_{L,S}]\right) \times \\ &\quad q_{H,S}q_{L,S}\left(\frac{\delta_{HH}}{\delta_{LH}+\delta_{HH}} + \frac{\delta_{LL}}{\delta_{HL}+\delta_{LL}} - 1\right). \end{aligned} \tag{3.2}$$

In the analysis of the computational results, we rely on the $\frac{\delta_{HH}}{\delta_{LH}+\delta_{HH}} + \frac{\delta_{LL}}{\delta_{HL}+\delta_{LL}}$ values to cluster the randomly created problems, and present an analysis of the revenue implications of the direction and level of covariance (or correlation) of the customers' product valuations.

We now turn to the problem generation scheme and list its parameters of the along with their value sets in Table 3.2. For all the Weibull random variables listed in the table, the shape parameter is set equal to three (two) for the primary (secondary) product. Once the values for the parameters listed in Table 3.2 are set, the remaining problem parameters can be directly computed by observing the conditions the problem structure imposes. For example, when $\delta_{LH} + \delta_{HH}$, $\frac{\delta_{HH}}{\delta_{LH}+\delta_{HH}}$ and $\frac{\delta_{LL}}{\delta_{LL}+\delta_{HL}}$ are set equal to 0.30, 0.00 and 0.50, respectively, δ_{LL}, δ_{HL}, δ_{LH}, and δ_{HH} can be computed as follows: since $\delta_{HH} = 0.00 \times (\delta_{LH} + \delta_{HH})$ and $\delta_{LL} = 0.50 \times (\delta_{LL} + \delta_{HL})$, we have that $\delta_{HH} = 0.00$ and $\delta_{LH} = 0.30$. Then, from $\delta_{LL} + \delta_{HL} + \delta_{LH} + \delta_{HH} = 1.00$, we have that $\delta_{HL} + \delta_{LL} = 0.70$. Finally, from $\frac{\delta_{LL}}{\delta_{LL}+\delta_{HL}} = 0.50$ it follows that $\delta_{HL} = 0.35$ and $\delta_{LL} = 0.35$.

In Tables 3.3, 3.4, and 3.5, we represent the pricing strategies with a three-field notation: **S**tatic or **D**ynamic pricing/**I**ndividual discount for the primary product or **B**undle discount/**T**argeted or **N**on-**T**argeted pricing. We consider the **S/I/NT** pricing strategy, which has the lowest revenue among the eight possible pricing strategies, to be the base case and report the revenue impact of incremental strategy changes that eventually lead to the **D/B/T** strategy which delivers the largest revenue improvement over the base case (percentage revenue improvement figures that are typeset in boldface in Columns (4) of Tables 3.3 to 3.5). We note that in the computational analysis we first compute the revenue-maximizing prices for the primary and secondary products under the **S/I/NT** pricing strategy, and then use these base prices (i.e., p_P and p_S) in the analysis of the remaining pricing strategies. This approach help us report the true marginal revenue impact of the pricing strategies.

Each of the next three sections (Sections 3.5.1 to 3.5.3) takes a different path in moving from the base case strategy to the top performing strategy to shed light on the marginal impact of the studied pricing strategies under different inventory load factors, i.e., L values. These paths can be identified by superimposing the row headings with

Table 3.2: Problem Instances: Parameters and Value Sets.

Parameter	Definition	Value Set
T	Length of the planning horizon	$\{20, 30, 40\}$
L	Initial inventory load factor of the primary product, i.e., the percentage of demand that can be met with the initial inventory when every arriving customers decides to purchase the primary product. For example, when $T = 30$, and $L = 50\%$, the initial inventory level of the primary product is set equal to $30 \times 0.5 = 15$.	$\{20\%, 50\%, 80\%\}$
$\delta_{LH} + \delta_{HH}$	Fraction of customers that value the secondary product high	$\{0.3, 0.5, 0.7\}$
$\frac{\delta_{HH}}{\delta_{LH}+\delta_{HH}}$	Probability that a customer belongs to segment HH of the primary product given that she assigns a high value to the secondary product	$\{0.0, 0.3, 0.5, 0.7, 1.0\}$
$\frac{\delta_{LL}}{\delta_{HL}+\delta_{LL}}$	Probability that a customer belongs to segment LL of the primary product given that she assigns a low value to the secondary product	$\{0.0, 0.3, 0.5, 0.7, 1.0\}$
$(W_{H,P}, W_{L,P})$	Weibull scale parameters for the $\overline{R}_{H,P}$ and $\overline{R}_{L,P}$ random variables, respectively	$\{(100, 50), (90, 50), (75, 65)\}$
$(W_{H,S}, W_{L,S})$	Weibull scale parameters for the $\overline{R}_{H,S}$, and $\overline{R}_{L,S}$ random variables, respectively	$\{(100, 50), (80, 70), (50, 25)\}$

the column headings. In Table 3.3, for example, the entries in rows **·/B/·** and column **S/·/NT** report the revenue improvement the static and non-targeted bundle discount (**S/B/NT**) brings about over the base case revenue under different values of the inventory load factor (i.e., $L = 20\%$, 50%, and 80%).

We conclude the computational analysis section with a discussion of the interaction of the pricing strategies and the correlation between the customers' valuations of the primary and secondary products in Section 3.5.4.

3.5.1 Revenue Enhancements with Bundle Discounts

In Table 3.3, we study the case where the individual product discount (**·/I/·**) is replaced with the bundle discount (**·/B/·**). In the case of **S/I/NT** (i.e., in Column (1) where the discount is static, for the primary product only and non-targeted), replacing an individual discount for the primary product with a bundle discount (**S/I/NT** → **S/B/NT**) enhances the revenue by 0.58%[1], 1.50%[1], and 3.56%[1] for $L = 20\%$, 50%, and 80%, respectively (to facilitate the transitions, when we refer in the text to a figure from a table we suffix a superscript that is also suffixed to the corresponding figure in the table).

In Column (2) of Table 3.3, **S/·/NT** is replaced by **S/·/T**. **S/I/T** improves the base case revenue by only 0.02%[2], 0.07%[2], and 0.17%[2] for $L = 20\%$, 50%, and 80%, respectively. The rather minimal improvements clearly show that a targeted discount strategy for the individual product is not effective when the prices remain fixed, i.e., static, in the planning horizon. **S/B/T**, on the other hand, improves the revenue, on average over the L range, by 1.92% (average of 0.59%[3], 1.53%[3], and 3.63%[3]). We consider the dynamic discounts in Columns (3) and (4) of Table 3.3. With **D/I/NT**, the base case revenue is enhanced by 4.45%[4], 1.53%[4], and 0.06%[4] for $L = 20\%$, 50%, and 80%, respectively. When dynamic discounts for the individual product are also segment-specific (**D/I/T**), the enhancement levels rise to 5.01%[5], 2.62%[5], and 1.67%[5] for $L = 20\%$, 50%, and 80%, respectively. The bundle discount, when dynamically adjusted in the planning horizon (i.e., **D/B/·**), brings about similar marginal improvements regardless of whether the discounts are segment-specific or not: 1.43%[6] vs. 1.41%[6], 2.31%[6] vs. 2.23%[6], and 4.16%[6] vs. 4.24%[6]

Table 3.3: Revenue impact of discounts: The individual product vs. the bundle.

L		(1)	(2)	(3)	(4)
		$\mathbf{S}/\cdot/\mathbf{NT}$	$\mathbf{S}/\cdot/\mathbf{T}$	$\mathbf{D}/\cdot/\mathbf{NT}$	$\mathbf{D}/\cdot/\mathbf{T}$
20%	$\cdot/\mathbf{I}/\cdot$	Base Case	0.02%[2]	4.45%[4]	5.01%[5]
	$\cdot/\mathbf{B}/\cdot$	0.58%	0.59%[3]	5.88%	**6.42%**
	Δ	*0.58%*[1]	*0.57%*	*1.43%*[6]	*1.41%*[6]
50%	$\cdot/\mathbf{I}/\cdot$	Base Case	0.07%[2]	1.53%[4]	2.62%[5]
	$\cdot/\mathbf{B}/\cdot$	1.50%	1.53%[3]	3.84%	**4.85%**
	Δ	*1.50%*[1]	*1.46%*	*2.31%*[6]	*2.23%*[6]
80%	$\cdot/\mathbf{I}/\cdot$	Base Case	0.17%[2]	0.06%[4]	1.67%[5]
	$\cdot/\mathbf{B}/\cdot$	3.56%	3.63%[3]	4.22%	**5.91%**
	Δ	*3.56%*[1]	*3.46%*	*4.16%*[6]	*4.24%*[6]

for $L = 20\%$, 50%, and 80%, respectively. These figures indicate that the interaction between the bundle discounts and targeted discounts is rather inconsequential. In case of $\mathbf{D}/\mathbf{I}/\cdot$, the marginal contribution of targeted discounts is equal to 0.56% (5.01%[5]−4.45%[4]), 1.10% (2.62%[5]−1.53%[4]), and 1.61% (1.67%[5]−0.06%[4]) for $L = 20\%$, 50%, and 80%, respectively, and remains significantly below the marginal revenue enhancements observed with $\mathbf{D}/\mathbf{B}/\mathbf{NT}$: 1.43%[6], 2.31%[6], and 4.16%[6] for $L = 20\%$, 50%, and 80%, respectively. This observation highlights the fact that, even when a retailer does not have access to technological means that might facilitate segment-specific discount announcements, bundle discounts can still lead to a substantial revenue increase. The results of Table 3.3 also indicate that, independent of other strategies, the revenue impact of bundle offers increases as the initial inventory of the primary product gets higher.

3.5.2 Revenue Enhancements with Dynamic Discounts

In Table 3.4 we examine the marginal impact of dynamic discounts. The figures presented in Column (1) of Table 3.4 demonstrate that pure dynamic discounts ($\mathbf{S}/\mathbf{I}/\mathbf{NT} \rightarrow \mathbf{D}/\mathbf{I}/\mathbf{NT}$) bring about a revenue improvement of 4.45%[1], 1.53%[1], and 0.06%[1] for L = 20%, 50%, and 80%, respectively, over the base case. The figures also illustrate that, irrespective of other strategies, their revenue impact decreases as the initial inventory level of the primary product increases. **D/I/T** or **D/B/T** generates an additional improvement of around 0.50% (0.54% = 4.99%[2] − 4.45%[1] or 0.53% = 5.83%[4] − 5.30%[3]), 1.00% (1.03% = 2.56%[2] − 1.53%[1] or 0.98% = 3.32%[4] − 2.34%[3]), and 1.50% (1.44% = 1.50%[2] − 0.06%[1] or 1.61% = 2.27%[4] − 0.66%[3]) for L = 20%, 50%, and 80%, respectively, regardless of whether the discount is for the individual product only or for the bundle of the two products. In other words, as also observed in Table 3.3, the interaction of dynamic and targeted discounts generates an additional revenue improvement of approximately 0.50%, 1.00% and 1.50% for L = 20%, 50%, and 80%, respectively, over the base case revenue. We study the marginal impact of bundle discounts in Columns (3) and (4) of Table 3.4. A comparison of the marginal revenue improvements in Columns (1) vs. (3) or Columns (2) vs. (4) reveals that the interaction between dynamic and bundle discounts yields an additional revenue improvement of approximately 0.80%: 0.85% = 5.30%[3] − 4.45%[1] or 0.84% = 5.83%[4] − 4.99%[2] for L = 20%; 0.81% = 2.34%[3] − 1.53%[1] or 0.76% = 3.32%[4] − 2.56%[2] for L = 50%; and 0.60% = 0.66%[3] − 0.06%[1] or 0.77% = 2.27%[4] − 1.50%[2] for L = 80%.

With **S/B/NT**, the base case revenue is enhanced, on average, by 0.58%[5], 1.50%[5], and 3.56%[5] for L = 20%, 50%, and 80%, respectively; with **S/B/T**, the enhancement level is equal to 0.59%[6], 1.53%[6], and 3.63%[6] for L = 20%, 50%, and 80%, respectively, indicating a minimal level of interaction between bundle discounts and targeted discounts.

3.5.3 Revenue Enhancements with Targeted Discounts

The results reported in Table 3.5 summarize the observations we have made in light of the figures reported in Sections 3.5.1 and 3.5.2: Targeted static discounts have a negligible revenue impact on their own (Column (1) of Table 3.5). In the four-segment setting we study, targeted static discounts for either the individual product or the bundle of the two products attempt to increase (decrease) the discounts for the segments that are less (more) likely to buy the primary product: while segments LH and LL enjoy larger discounts, segments HL and HH receive lower discounts. With static discounts that are optimized at the start of the planning horizon, however, the

Table 3.4: Revenue impact of discounts: Static vs. dynamic.

L		(1)	(2)	(3)	(4)
		$\cdot/\mathbf{I}/\mathbf{NT}$	$\cdot/\mathbf{I}/\mathbf{T}$	$\cdot/\mathbf{B}/\mathbf{NT}$	$\cdot/\mathbf{B}/\mathbf{T}$
20%	$\mathbf{S}/\cdot/\cdot$	Base Case	0.02%	0.58%[5]	0.59%[6]
	$\mathbf{D}/\cdot/\cdot$	4.45%	5.01%	5.88%	**6.42%**
	Δ	*4.45%*[1]	*4.99%*[2]	*5.30%*[3]	*5.83%*[4]
50%	$\mathbf{S}/\cdot/\cdot$	Base Case	0.07%	1.50%[5]	1.53%[6]
	$\mathbf{D}/\cdot/\cdot$	1.53%	2.62%	3.84%	**4.85%**
	Δ	*1.53%*[1]	*2.56%*[2]	*2.34%*[3]	*3.32%*[4]
80%	$\mathbf{S}/\cdot/\cdot$	Base Case	0.17%	3.56%[5]	3.63%[6]
	$\mathbf{D}/\cdot/\cdot$	0.06%	1.67%	4.22%	**5.91%**
	Δ	*0.06%*[1]	*1.50%*[2]	*0.66%*[3]	*2.27%*[4]

targeted discounts cannot fully benefit from the consideration of the trade-off between lower price and higher demand in segments LH and LL and higher price and lower demand in segments HL and HH. The revenue impact of the targeted discounts becomes significant when they are dynamically optimized by taking the inventory level into consideration at the start of every period (Columns (3) and (4)). As expected, the higher the initial inventory, the larger the revenue impact of dynamic targeted discounts.

Bundle discounts improve the revenue by 0.58%[1], 1.50%[1], and 3.56%[1] for $L = 20\%$, 50%, and 80%, respectively, and they have no interaction with targeted discounts (from the comparison of $\mathbf{S}/\mathbf{B}/\mathbf{NT}$ and $\mathbf{S}/\mathbf{B}/\mathbf{T}$ in Columns (3) and (4) of Table 3.5).

When dynamic discounts are introduced over the base case ($\mathbf{D}/\mathbf{I}/\mathbf{NT}$), the revenue increases by 4.45%[2], 1.53%[2], and 0.06%[2] for $L = 20\%$, 50%, and 80%, respectively, and the interaction of dynamic and targeted discounts generates an additional improvement of approximately 0.50%, 1.00%, and 1.50% (from the comparison of the figures in the Δ rows and Columns (3) and (4) of Table 3.5). And finally, when the available alternatives all fully exploited, i.e., when $\mathbf{D}\big/\mathbf{B}\big/\mathbf{T}$ is used, the revenue improvements rise to

- for $L = 20\%$, i.e., when the level of the initial inventory is low: $6.42\%^3 \approx 0.58\%$ (from $\cdot\big/\mathbf{B}\big/\cdot$) + 4.45% (from $\mathbf{D}\big/\cdot\big/\cdot$) + 0.50% (from $\mathbf{D}\big/\cdot\big/\mathbf{T}$ interaction) + 0.80% (from $\mathbf{D}\big/\mathbf{B}\big/\cdot$ interaction) = 6.33%,

- for $L = 50\%$, i.e., when the level of the initial inventory is medium: $4.85\%^3 \approx 1.50\%$ (from $\cdot\big/\mathbf{B}\big/\cdot$) + 1.53% (from $\mathbf{D}\big/\cdot\big/\cdot$) + 1.00% (from $\mathbf{D}\big/\cdot\big/\mathbf{T}$ interaction) + 0.80% (from $\mathbf{D}\big/\mathbf{B}\big/\cdot$ interaction)= 4.83%, and

- for $L = 80\%$, i.e., when the level of the initial inventory is high: $5.91\%^3 \approx 3.56\%$ (from $\cdot\big/\mathbf{B}\big/\cdot$) + 0.06% (from $\mathbf{D}\big/\cdot\big/\cdot$) + 1.50% (from $\mathbf{D}\big/\cdot\big/\mathbf{T}$ interaction) + 0.80% (from $\mathbf{D}\big/\mathbf{B}\big/\cdot$ interaction) = 5.92%,

as reported in Column (4) of Table 3.5.

The revenue models help us to identify the sources of revenue improvements: When the initial inventory level is low, 70% of the revenue improvement is from dynamic discounts ($4.45\%^2$ vs. $6.42\%^3$); bundle discounts, on their own, bring an additional revenue which is equivalent to one eighth of dynamic discounts' contribution ($4.45\%^2$ vs. $0.58\%^1$). When the initial inventory level is high, dynamic discounts and bundle discounts reverse their roles ($0.06\%^2$ vs. $3.56\%^1$): the contribution of dynamic discounts on their own are negligible, and bundle discounts, alone, generate 60% of the total revenue improvement ($3.56\%^1$ vs. $5.91\%^3$). The observation that dynamic discounts are not effective when the initial inventory level is high is consistent with the earlier findings: Bitran and Mondschein (1997) study a pricing problem with, at most, K price changes in the planning horizon and, when the buyers' reservation-price distribution is time-invariant and the inventory goes to infinity, show that the constant-pricing policy is optimal. Specifically, dynamic discounts are rendered redundant when the inventory goes to infinity, i.e., a single price set at the beginning of the planning horizon suffices to maximize the expected profit.

The revenue improvement from the interaction of dynamic discounts and bundle

Table 3.5: Revenue impact of discounts: Targeted vs. non-targeted.

L		(1)	(2)	(3)	(4)
		S/I/·	**S/B/·**	**D/I/·**	**D/B/·**
20%	**·/·/NT**	Base Case	0.58%[1]	4.45%[2]	5.88%
	·/·/T	0.02%	0.59%	5.01%	**6.42%**[3]
	Δ	*0.02%*	*0.01%*	*0.56%*	*0.54%*
50%	**·/·/NT**	Base Case	1.50%[1]	1.53%[2]	3.84%
	·/·/T	0.07%	1.53%	2.62%	**4.85%**[3]
	Δ	*0.07%*	*0.04%*	*1.10%*	*1.01%*
80%	**·/·/NT**	Base Case	3.56%[1]	0.06%[2]	4.22%
	·/·/T	0.17%	3.63%	1.67%	**5.91%**[3]
	Δ	*0.17%*	*0.07%*	*1.61%*	*1.68%*

discounts is independent of the initial inventory level, and can be considered to be significant (around 0.80%).

Targeted static discounts, for all levels of the initial inventory, do not seem to be effective on their own, and their interaction with bundle discounts does not result in revenue improvement, either. On the other hand, the revenue improvement from the interaction of dynamic discounts and targeted discounts is a linearly increasing function of the initial inventory level, i.e., the larger the initial inventory level, the higher the revenue improvement the interaction brings about.

3.5.4 Interaction of Pricing Strategies and Correlation Between Product Valuations

In Figure **??** we finally consider the influence of the correlation between the customers' product valuations on the observed revenue improvements. As noted in Proposition 7, the sum $\frac{\delta_{HH}}{\delta_{LH}+\delta_{HH}} + \frac{\delta_{LL}}{\delta_{HL}+\delta_{LL}}$ assumes values in the $[0, 2]$ range, and, while a value of zero indicates a negative correlation of one, a value of two indicates a positive correlation of one. We cluster the 6,045 problems of the computational study by dividing the $[0, 2]$

range into five equal subranges of size 0.4. For example, the $[0, 0.4)$ range corresponds to problem instances with very high negative correlations; the $[1.2, 1.6)$ range groups problems that have a medium level of positive correlation. The distribution of the 6,045 problem instances into the subranges are as follows: $[0, 0.4)$ range: 12%, $[0.4, 0.8)$ range: 28%, $[0.8, 1.2)$ range: 28%, $[1.2, 1.6)$ range: 20%, $[1.6, 2.0]$ range: 12%.

The revenue impact of bundle discounts is strongly influenced by the correlation between the product valuations; specifically, the more negative the correlation, the higher the revenue contribution of bundle discounts (Figure 3.1). This observation is in line with the results presented in earlier studies that consider the effect of correlation (Adams and Yellen, 1976; Bulut et al., 2009). The revenue improvement with dynamic and targeted discounts, too, is dependent on the correlation between the product valuations: the larger the value of positive correlation, the higher the revenue contribution of dynamic and targeted discounts (Figure 3.2). The other two drivers of revenue improvement discussed earlier, namely dynamic discounts and the dynamic bundle discounts, do not seem to be affected by the changes in the direction and intensity of the correlation between the valuations of the primary and secondary products (Figures 3.3 and 3.4).

The results presented in Figures 3.1 and 3.2 indicate that the two revenue drivers (bundle discounts, and dynamic and targeted discounts) are influenced by the correlation between the product valuations in opposite directions, and this observation can be exploited to attenuate the potentially negative revenue impact of the correlation: when **D/B/T** is the selected strategy, the overall revenue impact (Figure 3.5) becomes independent of the value of the correlation and remains almost constant in the $[0, 2]$ range of possible correlation values and for all levels of initial inventory. From the practice perspective, when the retailer implements **D/B/T**, the information on the correlation between product valuations becomes less important.

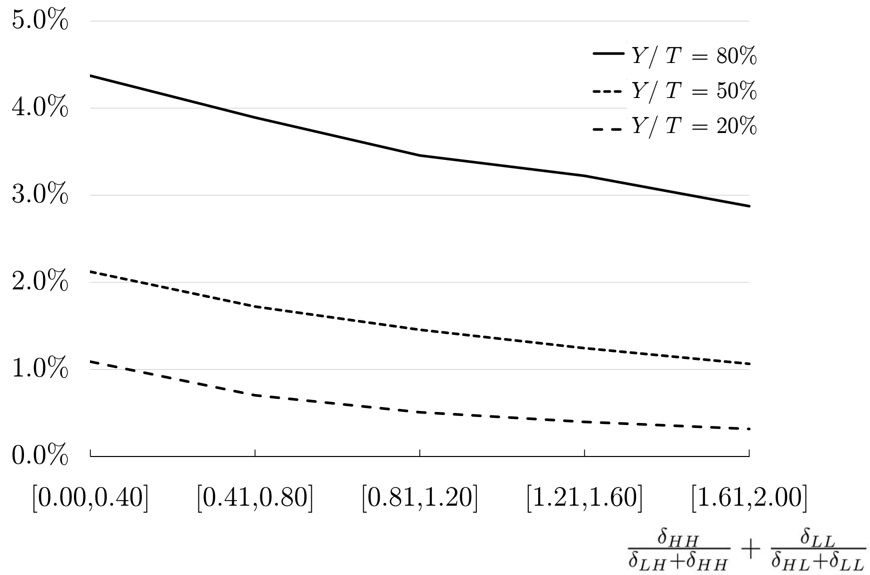

Figure 3.1: Impact of the correlation on average revenue improvement achieved with dynamic discounts (difference between **D/I/NT** and **S/I/NT**).

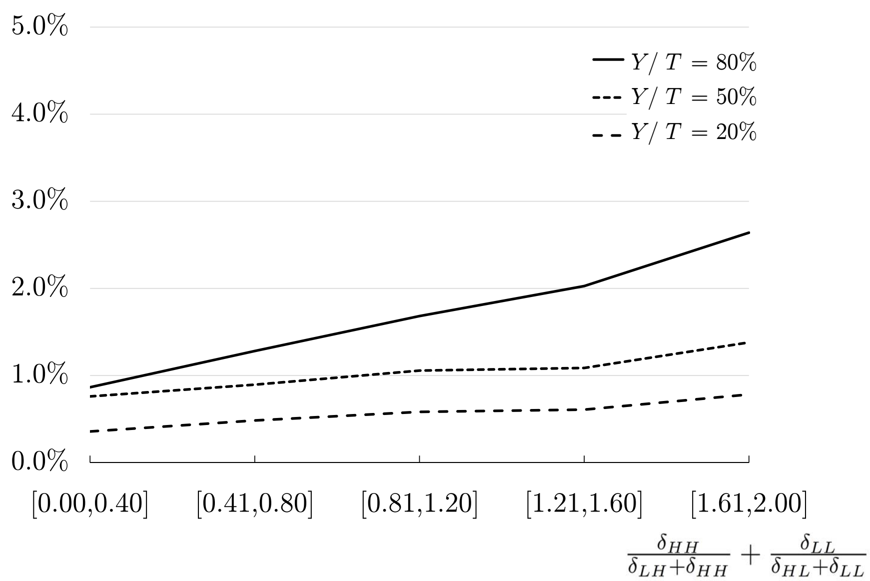

Figure 3.2: Impact of the correlation on average revenue improvement achieved with the interaction of the dynamic and targeted discounts (difference between **D/B/T** and **D/B/NT**).

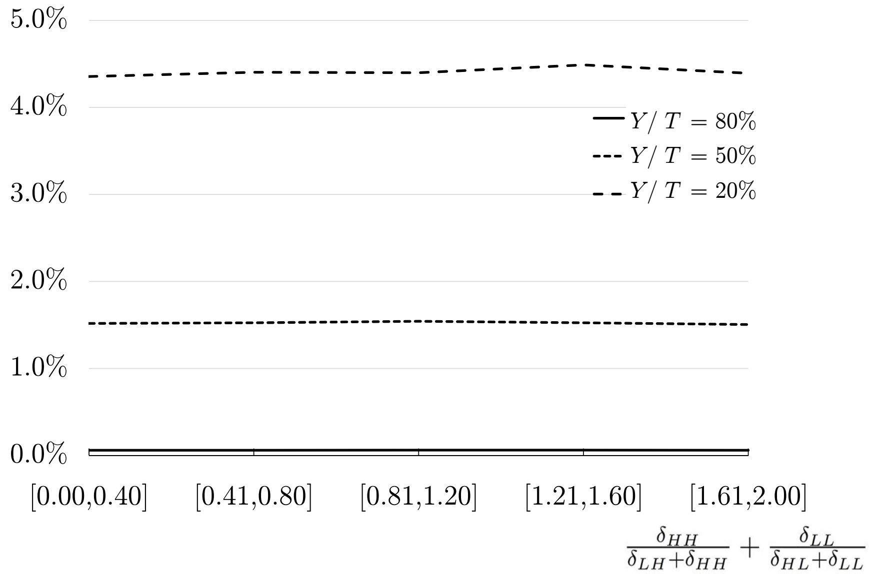

Figure 3.3: Impact of the correlation on average revenue improvement achieved with dynamic discounts (difference between **D/I/NT** and **S/I/NT**).

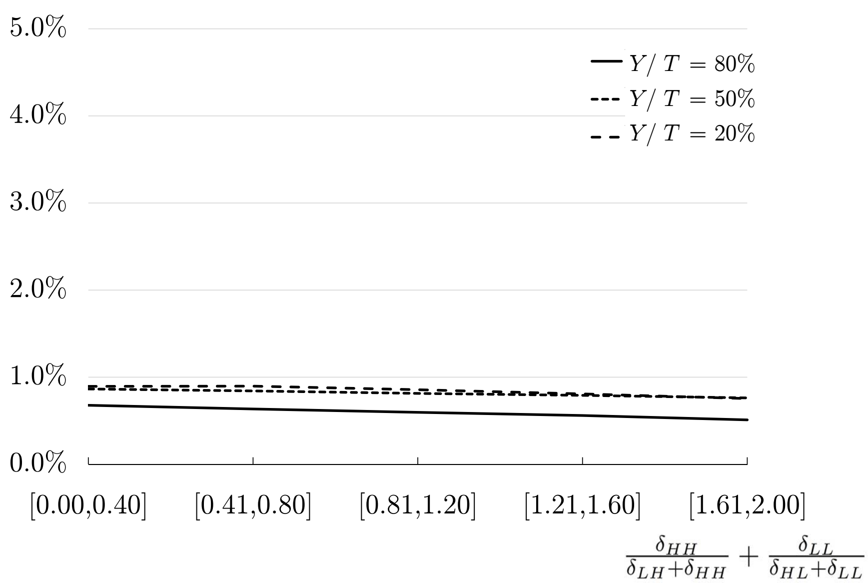

Figure 3.4: Impact of the correlation on average revenue improvement achieved with the interaction of the bundle and targeted discounts (difference between **D/B/T** and **D/I/NT**).

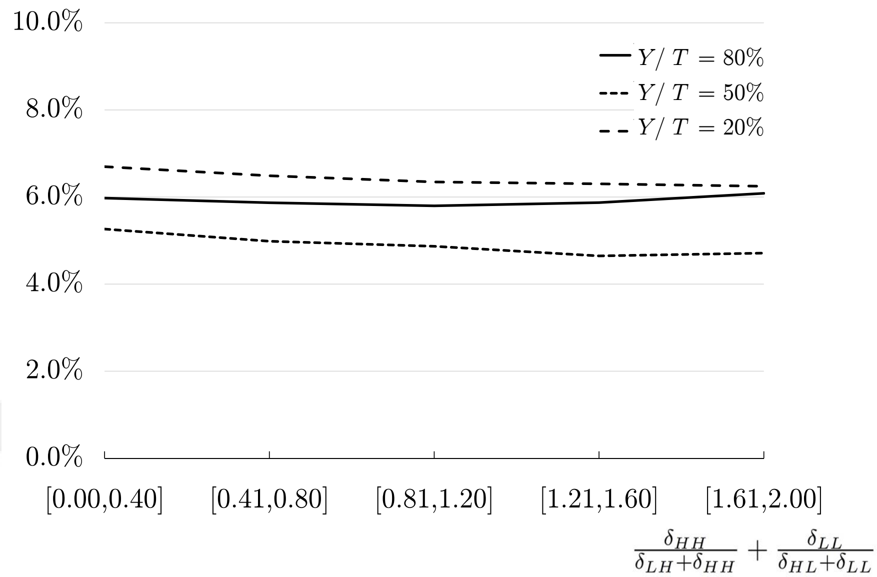

Figure 3.5: Impact of the correlation on average revenue improvement achieved with dynamic and targeted bundle discounts.

3.6 Concluding Remarks

In this study, we consider a setting where a retailer aims to clear the inventory of an item with a bundle offer. We specifically address the question of how the flexibility to announce segment-specific discounts and/or the ability to dynamically change discounts interact with the revenue improvements a retailer can achieve with bundle discounts. We develop a revenue model with dynamic and targeted bundle discounts, and present a computational study that distills the important aspects of the revenue improvement potentials of the studied discount types.

We first illustrate that bundle discounts are most effective when the initial inventory of the promotional item is high and the customers' valuations for the products that form the bundle are negatively correlated: the higher the level of negative correlation, the easier it is for bundle discounts to create additional surplus by attracting the customers who would not be otherwise (i.e., when the bundle offer is not available) purchasing the two products. Dynamic discounts, on the other hand, are most effective when the initial inventory is low. The direction and the level of the correlation between product valuations do not seem to have an impact on dynamic discounts' revenue improvement performance.

Dynamically optimized bundle discounts achieve significant revenue improvements, regardless of the initial inventory level of the promotional product, by virtue of the reciprocal revenue improvement responses of bundle discounts and the dynamic discounts to changing levels of the initial inventory. Their revenue potential is slightly reduced when the products' valuations are positively correlated: because the key impact of dynamic discounts on revenue improvement stems from setting the right price in consideration of the inventory levels, it cannot help bundle discounts create additional surplus by attracting new customers when the products' valuations are positively correlated and the customers who might eventually purchase the two products are mostly in the HH segment.

Targeted static discounts have no direct impact on revenue. We note that the individual products' base prices are optimized for the case of static and non-targeted pricing and, therefore, we report the additional revenue improvement the targeted discounts can bring about. When a bundle offer is extended for two products that have a negative correlation between product valuations, the majority of customers value the primary product high and secondary product low, or the primary product low and secondary product high, i.e., the customers are mostly in the HL and LH segments. The right bundle price is attractive for both customer segments and creates additional surplus for the retailer; targeted discounts that are optimized at the start of the planning horizon cannot bring about an additional revenue, as the LL and HH segments are smaller when the correlation is negative. When product valuations are positively correlated, on the other hand, the customers are mostly in the HH and LL segments. Since most of the customers who would eventually purchase the products are in the HH segment, static and targeted discounts' capacity to increase revenue is again limited, because the optimal bundle discounts are already designed to capture the largest possible surplus from the HH segment.

Dynamic and targeted discounts, on the other hand, can be effective when the initial inventory is high and the customers are mostly in the HH segment, i.e., products' valuations are positively correlated. If the inventory continues to remain high, dynamically adjusted targeted and segment-specific discounts can attract new customers from the LH and HL segments without reducing the revenue that could be generated from the HH segment.

When the retailer employs the available options concurrently, our computational analysis indicates that the resulting revenue improvement can be significant and not dependent on the initial inventory levels or the direction and level of the correlation between product valuations.

The data-analytics capabilities and the information system infrastructure required to offer targeted discounts may necessitate considerable software and hardware investments. Our computational analysis points out that the benefit of targeted discounts is accentuated only when they are dynamically optimized. Consequently, in the context of designing a discount mechanism for a promotional product, a retailer's strategy development road map should guarantee that the targeted discounts, when integrated into the pricing mechanism, confluences with an existing dynamic optimization capability. In line with earlier studies, our numerical results also indicate that bundling can be a very effective instrument for revenue improvement, particularly when the bundled products' valuations are negatively correlated. A quintessential discount scheme for promotional products that encompasses the dynamic and targeted discounts for bundle offers delivers robust revenue improvement, relieving the retailer from the burden of the consideration of the levels of the initial inventory and the correlation between product valuations in setting discount levels.

Our study focuses on the pricing of a bundle formed by a primary (or promotional) product and a secondary product ignoring the inventory considerations for the secondary product. A natural extension of our research would be to integrate the secondary product's inventory level as a state variable into the retailer's revenue model. A further research topic would be to consider the impact of bundle composition decisions, e.g., the units of each product to be included in the bundle offer.

Chapter 4

INTERTEMPORAL BUNDLE PRICING AND CONSUMER STOCKPILING

4.1 Introduction

Price promotion is one of the key drivers to increase sales and stores' traffic in many retail settings such as grocery stores and supermarkets. In the United States, retailers sell more than 20% of products during promotion periods (The Nielsen Company (2015)). A similar trend arises in Europe where retailers achieve between 12% and 25% of sales on promotions (Gedenk et al. (2010)).

Consumers strategically respond to price promotions by changing the time and/or quantity of their purchase (Bell et al. (1999)). For instance, a forward-looking (or, strategic) consumer, who observes a high discount level for a product in the current period may increase her purchases not only to accelerate the consumption level (the consumption effect) but also to stockpile the product for future consumption (the stockpiling effect) because she anticipates a higher price in future periods. For example, in the context of storable packaged goods such as canned tuna or soup, strategic consumers respond to a price promotion by stockpiling the product, because they expect higher future prices (Erdem et al. (2003) and Haviv (2015)). Similarly, Bell et al. (2002) and Ailawadi et al. (2007) support the consumption effect of a price promotion in product categories such as soft drinks and salted snacks.

Empirical studies, too, show that households stockpile products on promotion periods. Pesendorfer (2002) investigates supermarket price reductions for ketchup products in Springfield, Missouri. His main finding is that current demand depends on past prices, which demonsrates the existence of a stockpiling effect. Similarly, Hendel and Nevo (2006) examine the price and demand dynamics of soft drinks, laundry detergents, and yogurt from nine supermarkets in the U.S., and the analysis shows that demand increases during price reductions are, in part, due to the stockpiling effect. Consequently, inter-temporal shifts occur in consumers' purchases due to price promotions, which is consistent with the post-promotional dip in sales reported in the study of Hendel and Nevo (2003). Similar observations about the households' stockpiling behavior are available in panel data analysis of purchase incidence and quantity (Bucklin and Gupta (1992) and Mace and Neslin (2004)).

Retailers recognize the fact that customers behave strategically in their purchasing behavior, and employ different pricing strategies. Prior studies (Pesendorfer (2002), Su (2010), Hendel and Nevo (2013)) demonstrate that firms practice periodic promotions to price discriminate between patient and impatient consumers. Major retailers, such as Kroger and Safeway, implement dynamic (or, combination of Hi-Low and Everyday-Low-Price) pricing strategies in almost 50% of their stores (Ellickson and Misra (2008)).

In today's retailing, retailers employ a variety of pricing strategies to stimulate demand and increase revenue (such as dynamic pricing and cross-selling). When deciding strategies, retailers can exploit the relationship among different categories (i.e., whether they are complements or substitutes). One such strategy that includes multiple products simultaneously is bundling, i.e., the practice of temporarily selling a bundle of different products in a package for a price that is lower than the sum of the regular prices of the products (Stremersch and Tellis (2002)). Retailers commonly use bundling strategies in the context of non-durable consumer goods. Examples include fabric softener and laundry detergent, dishwashing detergent and liquid, candy and ice-cream, and potato chips and vinegar (Desai et al. (2014)).

In this study, we consider the bundling strategy of a retailer in the context of periodic promotions, and investigate the interplay of the relationship among products in the bundle offer, the retailer's promotion strategy (i.e., depth and frequency of the bundle offer) and heterogeneous customers in terms of valuations of the products that form the bundle offer. We examine the case of a monopolist retailer that aims to increase per period revenue by selling two products (herein after products 1 and 2) through temporary bundle offers over an infinite time horizon. The retailer determines a bundle price along with a deal frequency. In turn, based on expectations for the retailer's bundle strategy (i.e., depth and frequency of the bundle offer), consumers choose how many units to purchase and to consume from each product. Consumers can stockpile the products for their future use. We consider the case of non-durable products that may exhibit consumption effects. Examples include processed foods, soft drinks, and salted snacks. Consumers form an expectation for the timing of the next bundle price conditional on the timing of the latest bundle offer. Therefore, the current pricing practice of the retailer holds information about the pricing strategy that the retailer pursue in the next period. We assume that the population is heterogeneous in terms of reservation prices of each product in the bundle. The retailer uses bundling as a mechanism to stimulate demand and increase revenue over an infinite horizon. The retailer does not change the products' regular prices over an infinite horizon. Customers aim to maximize the expected total surplus obtained from purchasing and consuming both products. For benchmarking purposes, we consider the case where the retailer offers constant prices for the two products that form the bundle offer.

Based on our model, we address the following research questions:

1. What is the optimal stockpiling strategy of a customer when the retailer offers a bundle offer?

2. What is the impact of the relationship among products on the consumer's stockpiling strategy?

3. What is the impact of relationship among products on the performance of the retailer's bundle promotion strategy?

4. What is the impact of heterogeneous customers' purchasing behavior on the performance of retailer's bundle promotion strategy?

In order to investigate these questions, we model a consumer's stockpiling problem as a dynamic program which aims to maximize the sum of discounted expected future surplus over an infinite horizon. We analyze the retailer's problem to determine the bundle price and corresponding promotion frequency by maximizing the long-run per period revenue.

From the consumer's perspective, we first analytically prove that the purchasing policy of a consumer is a state-dependent threshold policy. For each inventory level of the products, there exist optimal purchasing levels that maximize the consumer's expected total surplus. We find that per period consumption level of a product is driven by the current price, on-hand inventory level of the product and the expectation for the timing of the next promotion period. We also show that products' purchasing and consumption levels decrease as the degree of substitutability among the two products increases.

From the retailer's perspective, our results provide several managerial insights on bundle (or, multi-product) pricing strategies for storable product categories. Our work sheds some light on optimal promotion designs in a multi-product setting. Through a range of numerical experiments, we show that periodic bundle promotions are useful when the market includes heterogeneous customers with respect to their product valuations. Specifically, our results suggest that when the degree of substitutability among the products that form the bundle increases, the retailer should employ bundle offers with higher discount levels, and less frequently. If the degree of complementarity among the products in the bundle increases, our results indicate that the retailer should employ bundle offers with smaller discount levels and more frequently.

This chapter is organized as follows. We review the relevant research streams and summarize our contributions in the next section. In Section 4.3, a detailed description of the problem setting, along with consumer and retailer problems, is presented. Section 4.4 discusses the revenue impact of the retailer's pricing strategies that are examined in this study. Finally, Section 4.5 summarizes key findings that are posited, and discusses the main limitations of the model and possible extensions of the studied research questions.

4.2 Related Literature

Three main streams of literature are relevant to this study. This study is related to the literature on 1) consumers' response to promotions because we develop a strategic consumer model for periodic bundle discounts and characterize the stockpiling strategy of a strategic consumer. Our work is also related to the literature on 2) retailers' pricing strategies and 3) bundling, as we consider the bundle pricing as a promotion mechanism to stimulate demand and increase revenue.

The first stream of literature focuses on consumers' response to promotions. In the literature, studies have investigated various effects of price promotions on consumer responses, such as stockpiling and purchase acceleration. Bell et al. (1999) and Van Heerde et al. (2000) demonstrate that the demand expansion effect (stockpiling and purchase acceleration) of a promotion is significant than the brand switching. This result is consistent with the finding in studies of Van Heerde et al. (2004) and Chan et al. (2008).

There are papers that analyze the impact of a promotion on consumption acceleration. Ailawadi and Neslin (1998) and Ailawadi et al. (2007) show that customers may accelerate consumption rate in packaged-good product. They find that customers increase consumption rate in some product categories such as bacon, soft drinks, yogurt and salted snacks, but not in staple products such as bathroom tissue and paper towels. They conclude that benefits of promotions, such as consumption acceleration and demand stimulation, can offset the negative aspect of consumers' stockpiling. Sun (2005) finds that promotion increases the consumption rate for ketchup and yogurt. The papers, Gönül and Srinivasan (1996), Erdem et al. (2003) and Hendel and Nevo (2006), develop a structural dynamic forward-looking estimation models, and conclude that consumer expectations have a great impact on stockpiling strategies.

Several theoretical papers examine consumers' stockpiling strategies under price uncertainty for a single product by assuming that prices are given. Meyer and

Assuncao (1990) study a consumer's purchasing problem in response to i.i.d random prices. Assuncao and Meyer (1993) further extend the study of Meyer and Assuncao (1990) by incorporating consumption decisions, and they consider prices that follow a Markovian process. They conclude that the on-hand inventory level may increase the consumption rate of a product. Krishna (1994) analyzes stockpiling policy of a consumer for an arbitrary deal distribution. Ho et al. (1998) find that the consumption rate increases in price fluctuation. Bell et al. (2002) demonstrate that accelerated consumption intensifies price competition.

The second stream of relevant literature concentrates on retailers' pricing policy. Blattberg et al. (1981) and Jeuland and Narasimhan (1985) demonstrate that retailers can use periodic discounts to reduce inventories and to price discriminate when strategic customers stockpile products during promotions. Su (2010) investigates customer stockpiling and retailer pricing problems for a single product. In the model, consumers differ in their consumption rates, holding and fixed shopping costs. Su (2010) shows that, in a rational expectations equilibrium, the seller uses periodic promotions when frequent customers pay relatively more than others. There are other studies that include a consumer's stockpiling strategy into a retailer's pricing policy under commitment (see Hendel and Nevo (2013) and Besbes and Lobel (2015)).

Finally, the third stream of relevant literature is on bundling. Studies have focused on bundling to stimulate sales and increase retailer's revenue. Price discrimination is the most important phenomenon analyzed in the bundling context (see Stigler (1963); Adams and Yellen (1976); Schmalensee (1984); Hanson and Martin (1990); Ernst and Kouvelis (1999); Bulut et al. (2009)).

Another set of papers investigate the impact of the product characteristics (i.e., whether they are complement or substitute) on the retailer's revenue. Mulhern and Leone (1991), Venkatesh and Kamakura (2003) and Bulut et al. (2009) address how relationship among products affects the retailer's pricing strategy and find that bundle promotion can increase sales of both products when they are complements. Leeflang and Parreno-Selva (2012) and Leeflang et al. (2008) also show the empirical evidence of cross-category demand effects of price promotions.

Our study differs from the above literature in several aspects. From the consumer's perspective, the papers above consider stockpiling strategies under the price uncertainty for a single product. In this study, we focus on consumers' stockpiling strategies in a multi-product setting (i.e., bundling). We analyze the stockpiling strategy of a customer in response to temporary bundle discounts. From the retailer's perspective, the papers above consider pricing problem for a single product. In this study, we consider retailer's pricing problem for a bundle of two products. In the bundle literature, studies that consider the relationship among products are the single-period problem. They do not consider impacts of a retailer's promotion frequency and the relationship among the products in the bundle on the retailer's long-run revenue.

To the best of our knowledge, this is the first study to analyze the consumer's stockpiling and retailer pricing policies in a multi-product setting. We first characterize the consumer's purchasing and consumption policies in the context of the bundling. We study the retailer's optimal bundle pricing policy. We analyze how the promotion policy (i.e., depth and frequency of a bundle offer) of a retailer depends on the relationship among products (i.e., whether they are complements or substitutes) and customers' valuations of products in a bundle offer.

4.3 Problem Setting

We consider a retailer that operates in a market that includes different customer segments in terms of products' valuations. The retailer and a mass of customers interact on a periodic (e.g., daily or weekly) basis. Customers periodically visit the retailer's store. For example, grocery stores can consider the period as a week. In each period, the retailer chooses a strategy; either he offers a bundle offer at a discounted price and the products at their regular prices simultaneously or he only presents the products at their regular prices, and each consumer determines both purchasing and consumption quantities of the two products with respect to the current bundle price, on-hand inventory levels of the products and her expectation for the timing of the next bundle discount. We capture consumers' expectations by a temporally dependent distribution. Customers have a common probability distribution function defined over future bundle prices conditional on the current price. Therefore, given customers' purchasing strategies, the retailer determines his promotion strategy (i.e., depth and frequency of bundle offer) to maximize the long-run per period revenue. Similarly, given the belief over the retailer's bundle strategy, customers aim to maximize the expected total surplus.

If a customer arrives to the store and a bundle discount is available then she considers the following purchase options: 1) bundle of the products 1 and 2, 2) product 1 only, 3) product 2 only, 4) bundle of the products and product 1 only, 5) bundle of the products and product 2 together and 6) no purchase. If a bundle discount is not available when the customer arrives to the store, then she considers the following options: 1) products 1 and 2, 2) product 1 only, 3) product 2 only, and 4) no purchase. In this problem setting, consumers make purchasing and consumption decisions by maximizing the expected total surplus, and the retailer determines a bundle discount along with its frequency to maximize the long-run per period revenue through periodic bundle offers. The retailer uses the bundle offer as a mechanism to stimulate demand and increase revenue, and does not change the individual prices of the products. Without any loss of generality, we assume that the bundle offer does not require any physical integration of the two products, and the quantity of each product in the bundle is set to be one.

4.3.1 Market Structure

The market includes heterogeneous customers in terms of reservation prices (or, product valuations) for products that form the bundle offer (i.e., products 1 and 2). We consider two segments for each product: customers with low (L) valuations and with high (H) valuations. Customers with high valuations are the primary consumers of the corresponding product. Their reservation prices (or, willingness-to-pay (WtP)) values are relatively higher, so they are less sensitive to price changes for the product.

Conversely, customers with low valuations are the secondary consumers of the product, because their WtP values are relatively lower and they are more sensitive to price changes for the two products. We consider that customers with low valuations make a purchase only when the retailer offers a discount for the corresponding product.

When customers' valuations for the products 1 and 2 jointly considered, the market can be divided into four segments where each segment $n \in N = \{LL, LH, HL, HH\}$. The segment-$LL$ represents the group of customers who have low valuations for the products that form the bundle offer, whereas the segment-HL represents the fraction of customer who has a relatively high valuation for the product 1 but a low valuation for the product 2.

A customer who belongs to segment-LL purchases the products only when they are on sale. A segment-HL (LH) customer purchases the product 1 (product 2) and does not buy the product 2 (product 1) at their regular prices. She purchases the product 2 (product 1) only when the retailer offers the product at a discounted price. Customers who belong to the segment-HL (LH) can be considered as discount seeker customers for the product 2 (1), and they receive a positive surplus only from a deal for the corresponding product. Customers, who belong to segment-HH, have high valuations for both products and make a purchase at their regular prices. In parallel with the market structure analyzed in the studies of Liu and van Ryzin (2008) and Cachon and Swinney (2011), we consider customers who have low valuations for a product as bargain hunting or discount seeker and they receive a positive surplus only when a deal is presented for the corresponding product.

In our setting, the segment structure we use and the models we develop based on this structure can be readily extended to cases where the customers can be assigned to more than two groups with respect to their valuations of a product. We assume that the retailer observes the proportion of each segment size and that the market structure is stationary over time. In parallel with Brin et al. (1997); Silverstein et al. (1998), we assume that the retailer can identify segment sizes by analyzing the purchasing prices of individual customers. Let δ_n denote the proportion of customers in segment

n, $n \in N$. Without loss of generality, we assume that the market size is normalized to one, so the segment fractions to be consistent with each other and need to satisfy the following:

$$\sum_{n \in N} \delta_n = 1 \tag{4.1}$$

4.3.2 Consumer's Surplus Maximization Problem

Consider a customer from segment n, $n \in N$. The consumer's purchasing and consumption decisions on any shopping trip are associated with the following factors:

1. *The current bundle price, and the regular prices of the products.* Let $Q_{n,j,t}$ and $Q_{n,B,t}$, $n \in N$ and $j \in \{1, 2\}$ denote the number of products purchased separately at price p_j and purchased in bundle form at price p_B, where $p_B \leq \sum_{j=1}^{2} p_j$, by the segment-$n$ consumer in period t.

2. *The positive consumption utility derived from the products.* Let $C_{n,j,t}$, $n \in N$ and $j \in \{1, 2\}$ denote the consumption levels with respect to the p_1, p_2, and p_B in period t for the products 1 and 2, respectively. Positive consumption utility is equal to $U(C_{n,1,t}, C_{n,2,t}, \theta)$, where θ represents the degree of substitutability among the two products in the bundle. We assume that the consumption utility function of a consumer from segment n is given by

$$\begin{aligned} U(C_{n,1,t}, C_{n,2,t}, \theta) \;\; = \;\; & \left(a_{n,1} \times C_{n,1,t} - \gamma_{n,1} \times C_{n,1,t}^2\right) \\ & + \left(a_{n,2} \times C_{n,2,t} - \gamma_{n,2} \times C_{n,2,t}^2\right) \\ & + \left(\theta \times C_{n,1,t} \times C_{n,2,t}\right) \end{aligned} \tag{4.2}$$

where $C_{n,1,t}$ and $C_{n,2,t}$ denote the quantities consumed of products 1 and 2, respectively. The terms $a_{n,1}$ and $a_{n,2}$ and $\gamma_{n,1}$ and $\gamma_{n,2}$ represent the WtP values and saturation coefficients of a consumer who belongs to segment-n for the products 1 and 2, respectively. We assume that the consumption utility function to be concave in consumption quantities. We capture diminishing marginal returns from consumption through parameters $\gamma_{n,1}$ and $\gamma_{n,2}$; i.e., marginal utility from consuming an extra unit of a product decreases, as consumption level of the product increases. Quadratic consumption utility function is commonly used in economics and marketing literature (e.g., Assuncao and Meyer (1993) and Goic et al. (2011)). We assume that $a_{HH,j} \geq a_{n,j}$ $\forall j \in \{1,2\}, n \in N$, $a_{HL,j} \geq a_{LL,j}$ and $a_{LH,j} \geq a_{LL,j}$ $\forall j \in \{1,2\}$ to represent cases that price-sensitive customers have lower valuations relative to customers with higher valuations for the products that form the bundle offer.

3. *Inventory holding cost.* Let $I_{n,j,t}$, $n \in N$ and $j \in \{1,2\}$ denote the number of units in inventory for product j. The initial inventory level for product j in period $t+1$ is simply equal to $I_{n,j,t+1} = I_{n,j,t} + Q_{n,j,t} + Q_{n,B,t} - C_{n,j,t}$. At the end of each period, the customer incurs the total holding cost which is equal to $\sum_{j=1}^{2} h_j \times I_{n,j,t}$, where h_j is the unit holding cost for product j.

4. *Expectations for the bundle promotions.* The customer has a conditional probability distribution function, $F(p_B^{+1} \mid p_{B,t})$, defined over future bundle discounts given the current bundle price. We assume that the bundle price in the next period, p_B^{+1}, depends on the current price, p_B. The customer's belief (or, expectation) for the discount in the next period will be different when the discount is available as opposed to not. Also, we capture the discounted future surplus through the parameter of β.

If a bundle discount is available then the consumer's problem in period t is to select the purchase quantities, $Q_{n,j,t}$ and $Q_{n,B,t}$, and the consumption levels, $C_{n,j,t}$, $n \in N$ and $j \in \{1,2\}$, to maximize the sum of discounted her expected future surpluses. If a bundle discount is not presented to the customer then she purchases products at their regular prices. To simplify the exposition in the remainder of this section, we will drop the time index, t, and focus on consumers' stationary purchasing and consumption policies.

We develop an infinite-horizon dynamic programming model to capture the consumer's problem, and characterize the consumer's long-run purchase and consumption policies with the bundle offer.

Let $V(I_{n,1}, I_{n,2}, p_B)$ denote the maximum expected discounted surplus when a customer in segment n, who has $I_{n,1}$ and $I_{n,2}$ units of starting inventory from the products, observes the bundle price, p_B. The optimality equation of the segment-n

consumer is as follows:

$$V(I_{n,1}, I_{n,2}, p_B) = \max_{\substack{Q_{n,j}\geq 0\\ Q_{n,B}\geq 0\\ C_{n,j}\geq 0}} \left(\begin{array}{c} U\left(C_{n,1}, C_{n,2}, \theta\right) \\ -\left(p_1 Q_{n,1} + p_2 Q_{n,2} + p_B Q_{n,B}\right) \\ -\sum_{j=1}^{2} h_j \left(I_{n,j} + Q_{n,j} + Q_{n,B} - C_{n,j}\right) \\ +\beta\left(V(I_{n,1}^{+1}, I_{n,2}^{+1}, p_B^{+1} \mid p_B)\right) \end{array} \right), \tag{4.3}$$

where $I_{n,j}^{+1} = I_{n,j} + Q_{n,B} + Q_{n,j} - C_{n,j}$ $\forall j \in J$ and $\forall n \in N$.

In equation (4.3), consumer decision variables are the purchase and consumption levels for the two products. Inventory levels of the products and the observed bundle price are the state variables for the model.

The first two terms of $V(I_{n,1}, I_{n,2}, p_B)$ is the customer's net utility derived from purchasing $Q_{n,j}$ and $Q_{n,B}$ units of products, and consuming $C_{n,1}$ units of product 1 and $C_{n,2}$ units of product 2. The third term denotes the cost of carrying the products to next period. The fourth term, in which p_B^{+1} corresponds to beliefs about the bundle price in the next period, represents the customer's future expected net surplus, discounted by β, based on the beliefs about the retailer's pricing strategy. The initial inventory level of a product in the next period is equal to the initial inventory level in the current period plus the quantity purchased and minus the units consumed.

The dynamic programming formulation presented in equation (4.3) yields the optimal individual purchasing, $Q_{n,j}^*\left(I_{n,j}, p_j, p_B\right)$, bundle purchasing, $Q_{n,B}^*\left(I_{n,1}, I_{n,2}, p_B\right)$, and consumption policies, $C_{n,j}^*\left(I_{n,j}, p_j, p_B\right)$, for a customer in segment n.

4.3.3 The Optimal Purchase and Consumption Policies

In this section, we first present the characterization of optimal purchasing and consumption policies of a customer when a bundle offer is presented. Then, we provide some results to demonstrate the impact of problem parameters on the consumer's optimal policies.

Consider a customer who belongs to segment n with $I_{n,1}$ and $I_{n,2}$ units of on-hand inventories of the products. Let $I_{n,j}^*\left(p_j, p_B\right)$ denote the optimal inventory level (before consumption, but after purchase) of product j that maximizes the customer's surplus when the regular prices, p_1 and p_2, and the bundle price, p_B, are presented.

Theorem 1. *When the regular prices, p_1 and p_2, and bundle price p_B are presented to the consumer who has $I_{n,1}$ and $I_{n,2}$ units of inventory of the products, the optimal purchasing policy is as follows:*

$$\begin{aligned} Q^*_{n,B}\left(I_{n,1}, I_{n,2}, p_B\right) &= max\left(min\left(I^*_{n,1}\left(p_1, p_B\right) - I_{n,1}, I^*_{n,2}\left(p_2, p_B\right) - I_{n,2}\right), 0\right), \\ Q^*_{n,j}\left(I_{n,j}, p_{n,j}, p_B\right) &= max\left(I^*_{n,j}\left(p_j, p_B\right) - \left(I_{n,j} + Q^*_{n,B}\left(I_{n,1}, I_{n,2}, p_B\right)\right), 0\right). \end{aligned}$$

Theorem 1 expresses that there exist optimal inventory levels for the two products that the customer wishes to hold before consumption for each corresponding level of the bundle price p_B. The first part of the theorem states that if the inventory level of both products are less than optimal levels then a customer purchases $min\left(\left(I^*_{n,1}\left(p_1, p_B\right) - I_{n,1}\right), \left(I^*_{n,2}\left(p_2, p_B\right) - I_{n,2}\right)\right)$ units through the bundle offer at a price p_B where $p_B \leq p_1 + p_2$. The customer may also purchase one of the products separately along with the bundle purchase if the purchase quantity from the bundle offer is not adequate to raise the optimal inventory level of one of the products. This quantity is equal to $Q^*_{n,j}\left(I_{n,j}, p_{n,j}, p_B\right)$ for product j.

The consumption policy of a consumer, who belongs to segment-n, for product j is in the functional form of $C^*_{n,j}\left(I_{n,j}, p_j, p_B, \theta\right)$, where $C^*_{n,j}\left(I_{n,j}, p_j, p_B, \theta\right)$ denote the optimal consumption level for product j, $j \in \{1, 2\}$, that the consumer should consume with respect to the observed individual and bundle prices when the on-hand inventory level of product j is equal to the units of $I_{n,j}$ and the degree of complementarity among the products is θ. The proof of Theorem 1 is presented in Appendix C.1.1 and C.1.2.

The on-hand inventory level of products plays a crucial role in the consumer's consumption policy. When the on-hand inventory of any of the products is zero, the optimal consumption policy only depends on the observed individual and bundle prices, and the degree of complementarity among products. If the products in the bundle are independent to each other ($\theta = 0$), then their optimal consumption levels are not dependent to each other. In the case of complement ($\theta > 0$) (substitute ($\theta < 0$))

products, consumption levels are higher (lower) compared to the consumption levels of independent products. When the inventory levels of the products are non-negative, the customer considers the current prices and the size of current inventory in determining her consumption policy. In the consumption policy, the value of on-hand inventory corresponds to the customer's expectations about the bundle prices. For instance, if the customer expects to observe high prices in future periods, then she may decrease her consumption rate and keep higher level of inventory of products.

We, now, turn to investigate the impact of problem parameters, such as inventory levels, holding costs and discounting factor, on the optimal purchase and consumption policies presented in Theorem 1.

Proposition 8. *The optimal consumption level for product* j*,* $C^*_{n,j}\left(I_{n,j}, p_j, p_B, \theta\right)$*,* $j \in \{1, 2\}$ *and* $n \in N$*, is a non-decreasing function of the on-hand inventory,* $I_{n,j}$*, for given values of* p_j *and* p_B*.*

Proposition 8 states that the consumption level of a product increases as its inventory increases. The intuition behind this proposition is as follows: consider two consumers who have the same utility structure, but different on-hand inventory levels for product j. Also, assume that consumers anticipate the retailer's pricing policy with certainty in the next two periods. The retailer follows a regular pricing policy in the first period and offers a bundle discount in the second period. Then, the consumer with low on-hand inventory for the product reduces the consumption level to not to purchase the product at a higher price in the next period and waits for the next bundle discount. The consumer with relatively higher on-hand inventory for the product does not reduce the consumption level and continues to consume her regular quantity. This result is applicable for product categories in which consumption rate is flexible. For instance, on-hand inventory increases the consumption rate for product categories such as salted snacks and processed foods. However, consumption rate does not change for staple products such as toilet paper. The utility function presented in (4.2) corresponds to the product categories in which consumption rate is flexible.

Proposition 9. *The optimal inventory level for product* j*,* $I^*_{n,j}(p_j, p_B)$*,* $j \in \{1, 2\}$ *and* $n \in N$*, is a non-decreasing function in the discounting factor,* β*, and non-increasing function in the unit holding cost,* h_j*.*

Proposition 9 expresses that as the carrying cost of a product increases, customers increase consumption rate of a product and, in turn, keep less inventory. Also, we show that the optimal inventory level increases as customers value the expected future utility high in Proposition 9. We can interpret the second result in the following way: strategic customers ($\beta > 0$) carry higher units of inventories compared to myopic customers ($\beta = 0$).

Proposition 10. *The optimal consumption level for product* j*,* $C^*_{n,j}(I_{n,j}, p_j, p_B, \theta)$*,* $j \in \{1, 2\}$ *and* $n \in N$*, is a non-increasing function in the discounting factor,* β*, and is a non-decreasing function in the unit holding cost,* h_j*.*

Proposition 10 states that as the carrying cost of a product increases, customers increase consumption rate of a product and, in turn, keep less inventory. We can consider product categories with high unit holding costs as perishable products. We also demonstrate that the optimal consumption level decreases as customers value the expected future utility high in Proposition 9. We can interpret the first result in the following way: strategic customers ($\beta > 0$) consume less units of a product compared to myopic customers ($\beta = 0$).

The proofs of Propositions 8-10 are presented in Appendix C.1.3-C.1.5. All these findings are consistent with results in a single product setting derived by Assuncao and Meyer (1993).

4.3.4 A Closed-Form Representation of the Customer's Stockpiling Policy

Although Propositions 8-10 are helpful in explaining how problem parameters affect the consumer's optimal purchasing and consumption policies, they do not present any insights about the impact of consumers' bundle offer expectations on the optimal policies.

In this section, we derive a closed-form solution of the consumer's purchasing problem for a given bundling strategy of the retailer. We assume that the retailer follows bimodal pricing strategy, i.e., either the bundle and the individual products are simultaneously available to customers at prices p_1, p_2 and p_B or only the individual products are available to customers at prices p_1 and p_2. The retailer does not change the value of p_B over time. When the bundle discount is presented to customers, they always observe bundle price, p_B.

For benchmarking purposes, we first obtain closed-form expressions of optimal regular consumption levels for products in the bundle. The problem of a segment-n customer is to maximize discounted total surplus received from purchasing and consuming the two products. Let V_n^R be the discounted total surplus of a customer who belongs to segment-n when the retailer employs regular pricing policy. Given the retailer regular pricing policy, V_n^R can be expressed as follows:

$$V_n^R = \sum_{i=0}^{\infty} \beta^i \left(U\left(C_{n,1}^R, C_{n,2}^R, \theta\right) - p_1 Q_{n,1}^R - p_2 Q_{n,2}^R \right) \tag{4.4}$$

In equation (4.4), $C_{n,j}^R$ and $Q_{n,j}^R$ denote the consumption level and purchasing quantity of a segment-n customer for product j. The segment-n customer determines the purchasing and consumption policy for the two products to maximize discounted total surplus. Because the retailer follows the regular pricing policy, the customer does not hold inventory of products, and purchasing quantities of products are equal to the corresponding consumption levels of products, i.e, $Q_{n,1}^R = C_{n,1}^R$ and $Q_{n,2}^R = C_{n,2}^R$, $\forall n \in N$. For fixed values of p_1 and p_2, V_n^R is concave in $C_{n,1}^R$ and $C_{n,2}^R$, and the optimal regular consumption levels of products, $C_{n,1}^{R*}$ and $C_{n,2}^{R*}$, are as follows:

$$\begin{aligned} C_{n,1}^{R*} &= \frac{-2a_{n,1}\gamma_{n,2} + 2p_1\gamma_{n,2} - \theta\left(a_{n,2} - p_2\right)}{\theta^2 - 4\gamma_{n,1}\gamma_{n,2}}, \\ C_{n,2}^{R*} &= \frac{-2a_{n,1}\gamma_{n,1} + 2p_2\gamma_{n,1} - \theta\left(a_{n,1} - p_1\right)}{\theta^2 - 4\gamma_{n,1}\gamma_{n,2}}, \end{aligned} \tag{4.5}$$

where $\theta^2 < 4\gamma_{n,1}\gamma_{n,2}$, $\forall n \in N$.

Equation (4.5) demonstrates the closed-form expressions of the regular consumption rate of the segment-n consumer for the two products. With the retailer's regular pricing policy, a segment-n consumer purchases and consumes $C_{n,1}^{R*}$ units of product 1 and $C_{n,2}^{R*}$ units of product 2 in each period. Thus, $Q_{n,1}^{R*} = C_{n,1}^{R*}$ and $Q_{n,2}^{R*} = C_{n,2}^{R*}$. In parallel with the marketing literature, the equation (4.5) indicates that $C_{n,1}^{R*}$ and $C_{n,2}^{R*}$ increase as the degree of substitutability among the products decreases, i.e., $\frac{\partial C_{n,j}^{R*}}{\partial \theta} > 0, \quad \forall j \in \{1, 2\}$.

In the market structure that we present in Section 4.3.1, when the retailer follows a regular pricing policy, customers' purchasing policies are as follows: Customers who belong to segment-HH receive a positive surplus from purchasing both products at regular prices, p_1 and p_2. Customers in segment-LL receive a positive surplus only from purchasing products at a bundle price, p_B where $p_B \leq p_1 + p_2$. Customers who belong to segment-HL (segment-LH) purchase the product 1 (product 2) at a price p_1 (p_2). Therefore, $Q_{HH,1}^{R*} = C_{HH,1}^{R*} \geq 0$ and $Q_{HH,2}^{R*} = C_{HH,2}^{R*} \geq 0$ for a customer who belongs to segment-HH. For customers in segment-HL (segment-LH), $Q_{HL,1}^{R*} = C_{HL,1}^{R*} \geq 0$ $\left(Q_{LH,1}^{R*} = C_{LH,1}^{R*} = 0\right)$ and $Q_{HL,2}^{R*} = C_{HL,2}^{R*} = 0$ $\left(Q_{LH,2}^{R*} = C_{LH,2}^{R*} \geq 0\right)$. For a customer who belongs to segment-LL, $Q_{LL,1}^{R*} = C_{LL,1}^{R*} = 0$ and $Q_{LL,2}^{R*} = C_{LL,2}^{R*} = 0$.

We now turn to the consumer's consumption and purchasing problem when the retailer offers temporary bundle discounts. With the retailer's on bundle/off bundle pricing strategy, consumers' expectations for a bundle discount can be represented by a first-order Markov process with two price levels, a discounted bundle price, p_B, and regular prices, (p_1, p_2). Let B_t and R_t denote the bundle and the regular pricing actions that the retailer pursue in period t, respectively.

Consumers believe that if the bundle offer is available in the current period, the retailer offers the next bundle discount with probability α_{BB} in the following period. However, if the bundle offer is not presented in the current period, consumers believe that the retailer offers the next bundle discount with probability $1 - \alpha_{RR}$ in the following period. We assume that consumer expectations of bundle are consistent with the retailer's bundling strategy. The corresponding probability function, $f\left(B_{t+1} \mid B_t\right)$, is as follows:

$$\begin{array}{c} \\ B_t \\ R_t \end{array} \begin{array}{c} \begin{array}{cc} B_{t+1} & R_{t+1} \end{array} \\ \left[\begin{array}{cc} \alpha_{BB} & 1-\alpha_{BB} \\ 1-\alpha_{RR} & \alpha_{RR} \end{array} \right] \end{array} \tag{4.6}$$

If the consumer believes that it is high likely that the retailer follows consecutive regular pricing policy then α_{RR} will be higher than 0.5. For instance, when the consumer forms a belief that the retailer does not offer the bundle discount in consecutive periods then α_{BB} is equal to zero. In this section, we consider the case in which the retailer does not offer bundle discount in consecutive periods. It is a reasonable assumption because as we have stated in Section 4.2, retailers follow intertemporal promotions to stimulate demand. Thus, we set α_{BB} equal to zero.

At the start of a bundle period, the retailer sets the price of the bundle offer, p_B. A customer who belongs to segment-n, $n \in N$, makes purchase and consumption decisions based on her beliefs about the timing of the next bundle discount. The customer determines the number of units, $C_{n,j}$, to consume for product j, $j \in \{1, 2\}$ at discounted price p_B in the bundle period. Then, based on the holding cost of the products, h_j, and the expectations about the timing of the next bundle discount, $f\left(B_{t+1} \mid B_t\right)$, the customer determines the number of periods that the products are stockpiled. We assume that customers stockpile the products for equal number of periods (i.e., $T_{1,n} = T_{2,n}\ \forall n \in N$), but the number of stockpiling periods can differ across customer segments. This assumption is not restrictive because customers stockpiles the products through bundle offers and we consider that the bundle offer includes one unit of each product. Therefore, customers adjusts their purchasing policies accordingly in the long-run.

A customer who belongs to segment-n stockpiles the products through bundle offers to last for T_n periods. In a bundle period, the consumer purchases $Q_{n,j} + Q_{n,B}$ units of product j through the bundle and the individual offers, and consumes $\frac{Q_{n,j}+Q_{n,B}}{T_n}$ units of product j in each period until the next bundle discount.

Our aim is to derive the optimal purchasing and consumption policies of a segment-

n consumer given that her expectations of future prices is given by the above Markov process. Following the dynamic programming formulation provided in Equation (4.3), let $V_t^n(0,0,p_B)$ denote the expected total surplus of purchasing, consuming and holding the products, from the beginning of period t in which a bundle discount is presented, until an infinite horizon, given starting inventory level of products is zero for a customer who belongs to segment-n.

We now present a segment-n customer's problem and derive the closed-form expression of the stockpiling and consumption policies. Given the retailer's bimodal pricing policy, the customer's goal is to maximize her expected total surplus until the next bundle period. Therefore, $V_t^n(0,0,p_B)$ is measured at the period at which a bundle discount is presented. $V_t^n(0,0,p_B)$ can be written as follows:

$$\begin{aligned} V_t^n(0,0,p_B) \;=\; & U\left(C_{n,1}, C_{n,2}, \theta\right) - \left(p_1 Q_{n,1} + p_2 Q_{n,2} + p_B Q_{n,B}\right) \\ & - \sum_{j=1}^{2} \left(T_n - 1\right)\left(h_j\left(C_{n,j}\right)\right) \\ & + \sum_{i=t+1}^{T_n-1} \beta^i \alpha_{RR}^{i-1} \left(U\left(C_{n,1}, C_{n,2}, \theta\right)\right) \\ & - \sum_{i=t+1}^{T_n-1} \beta^i \alpha_{RR}^{i-1} \left(\left(T_n - i - 1\right) \sum_{j=1}^{2} h_j C_{n,j}\right) \\ & + \sum_{i=t+2}^{T_n-1} \beta^i \alpha_{RR}^{i-2} \left(1 - \alpha_{RR}\right) V_{t+i}^n((T_n - i)C_{n,1}, (T_n - i)C_{n,2}, p_B) \\ & + \sum_{i=t+T_n}^{\infty} \beta^i \alpha_{RR}^{i-2} \left(U\left(C_{n,1}^{R*}, C_{n,2}^{R*}, \theta\right) - p_1 C_{n,1}^{R*} - p_2 C_{n,2}^{R*}\right). \end{aligned} \tag{4.7}$$

where $C_{n,j} = \frac{Q_{n,j}+Q_{n,B}}{T_n}$ $\forall j \in J$.

In Equation (4.7), the segment-n customer determines purchase quantities and corresponding consumption rates of the products to maximize her expected future surplus when a bundle discount is presented to her. When a bundle discount is available to the customer, she acts strategically and stockpiles the products to last for T_n periods through the bundle offer. $V_t^n(0,0,p_B)$ includes the current surplus and expected surplus of future periods. The current surplus is equal to the utility received from the consumption of products minus the total cost of purchasing and holding products.

The first two terms of $V_t^n(0, 0, p_B)$ denote the utility received from consuming $C_{n,1}$ and $C_{n,2}$ units of products 1 and 2, respectively, and the cost of purchasing $Q_{n,1}$, $Q_{n,2}$ and $Q_{n,B}$ units at prices p_1, p_2 and p_B in the current bundle period t. The third term is the cost of carrying $(T_n - 1) \times C_{n,1}$ and $(T_n - 1) \times C_{n,2}$ units of products to the next period, $t + 1$.

The fourth and fifth terms represent the discounted surplus the customer receives from consuming $C_{n,1}$ and $C_{n,2}$ units of the products minus the carrying costs of stockpiled products during periods in which the retailer does not offer a bundle discount and the customer has sufficient units of inventory of the products purchased from the latest bundle period. The sixth term denotes the discounted surplus of the consumer during the periods in which the customer has sufficient units of products (i.e., between periods $i + 2$ and $T_n - 1$) and anticipates that the retailer offers the bundle discount with probability $(1 - \alpha_{RR})\, \alpha_{RR}^{i-2}$. For instance, If the bundle discount is presented to the customer in period $t + 2$ (i.e., two periods after the latest bundle discount), the customer's expected future surplus is equal to $V_{t+2}^n((T_n - 2)C_{n,1}, (T_n - 2)C_{n,2}, p_B)$. If the bundle discount is not available in period $t + 2$, the customer's surplus will be equal to the utility received from consuming the products minus holding costs of $(T_n - 3) \sum_{j=1}^{2} h_j\,(C_{n,j})$ until the next period. The last term represents the case in which the retailer does not offer a bundle discount in the last T_n periods and the customer depletes inventory of the products and starts to make a purchase at their regular prices and to consume $C_{n,1}^{R*}$ and $C_{n,2}^{R*}$ units of products 1 and 2, respectively. The expected discounted surplus for the period $t + T_n$ is $\beta^{T_n} \times \left(\alpha_{RR}^{T_n-1} \times \left(U\left(C_{n,1}^{R*}, C_{n,2}^{R*}, \theta\right) - \sum_{j=1}^{2} p_j C_{n,j}^{R*}\right) + \alpha_{RR}^{T_n-2} \times (1 - \alpha_{RR})\, V_{t+T_n}^n(0, 0, p_B)\right)$.

Total surplus of a segment-n customer with zero units of inventory, $V_t^n\,(0, 0, p_B)$, is equal to the total surplus when the customer has $I_{n,1}$ and $I_{n,2}$ units of inventory minus the total purchasing cost of $I_{n,1}$ and $I_{n,2}$ units on bundle period. Therefore, $V_t^n\,(I_{n,1}, I_{n,2}, p_B) = V_t^n\,(0, 0, p_B) + (p_1 + p_B)\, I_{n,1} + (p_2 + p_B)\, I_{n,2}$. After some algebraic manipulations we can rewrite the segment-n customer's problem as follows:

$$
\begin{aligned}
V_t^n(0,0,p_B) \quad = \quad & \frac{1}{\left(1-(1-\alpha_{RR})\sum_{i=t+2}^{\infty}\beta^i\alpha_{RR}^{i-2}\right)}\Bigg(\\
& U\left(C_{n,1},C_{n,2},\theta\right)-\left(p_BQ_{n,B}+p_1Q_{n,1}+p_2Q_{n,2}\right) \\
& -\sum_{j=1}^{2}\left(T_n-1\right)\left(h_jC_{n,j}\right) \\
& +\sum_{i=t+1}^{T_n-1}\beta^i\alpha_{RR}^{i-1}\left(U\left(C_{n,1},C_{n,2},\theta\right)\right) \\
& -\sum_{i=t+1}^{T_n-1}\beta^i\alpha_{RR}^{i-1}\left((T_n-i-1)\sum_{j=1}^{2}h_jC_{n,j}\right) \\
& +\sum_{i=t+2}^{T_n-1}\beta^i\alpha_{RR}^{i-1}\left(1-\alpha_{RR}\right)\left((T_n-i)C_{n,1}\left(p_1+p_B\right)\right) \\
& +\sum_{i=t+2}^{T_n-1}\beta^i\alpha_{RR}^{i-1}\left(1-\alpha_{RR}\right)\left((T_n-i)C_{n,2}\left(p_2+p_B\right)\right) \\
& +\sum_{i=t+T_n}^{\infty}\beta^i\alpha_{RR}^{i-1}\left(U\left(C_{n,1}^{R*},C_{n,2}^{R*},\theta\right)-p_1Q_{n,1}^{R*}-p_2Q_{n,2}^{R*}\right)\Bigg). \qquad (4.8)
\end{aligned}
$$

where $C_{n,j} = \frac{Q_{n,j}+Q_{n,B}}{T_n}$ $\forall j \in J$. The customer's problem is intractable when we consider the number of stockpiling periods, T_n, and the purchase quantities $Q_{n,B}$, and $Q_{n,j}$ $\quad j \in \{1,2\}$ jointly. Therefore, we solve the customer's problem for a given value of T_n.

For fixed values of p_B, α_{RR} and T_n, $V_t^n(0,0,p_B)$ is concave in $Q_{n,1}$, $Q_{n,2}$ and $Q_{n,B}$ (The proof is presented in Appendix C.1.6). Optimal purchasing quantities, $Q_{n,1}^*$, $Q_{n,2}^*$

and $Q^*_{n,B}$, that maximize $V^n_t(0,0,p_B)$ can be expressed as follows:

$$
\begin{aligned}
Q^*_{n,B} &= \\
&\frac{a_{n,1}+a_{n,2}+\frac{h_1+h_2+p_B\beta(\alpha_{RR}-1)}{1-\alpha_{RR}\beta}}{2\left(\gamma_{n,1}+\gamma_{n,2}-\theta\right)} \\
&- \frac{\frac{\alpha_{RR}(h_1+h_2+p_B-\beta p_B)(\beta(T_n-1)(\alpha_{RR}-1)-T_n)}{\alpha_{RR}^{T_n}\beta^{T_n}+\alpha_{RR}(-1+\beta(-1+\alpha_{RR}))}}{2\left(\gamma_{n,1}+\gamma_{n,2}-\theta\right)} \\
&- \frac{2\left(Q^*_{n,1}\gamma_{n,1}+Q^*_{n,2}\gamma_{n,2}-\theta\left(Q^*_{n,1}+Q^*_{n,2}\right)\right)}{2\left(\gamma_{n,1}+\gamma_{n,2}-\theta\right)}, \\
Q^*_{n,1} &= \\
&\frac{a_{n,1}+\frac{h_1+p_1\beta(\alpha_{RR}-1)}{1-\alpha_{RR}\beta}}{2\gamma_{n,1}} \\
&- \frac{\frac{\alpha_{RR}(h_1+p_1-\beta p_1)(\beta(T_n-1)(\alpha_{RR}-1)-T_n)}{\alpha_{RR}^{T_n}\beta^{T_n}+\alpha_{RR}(-1+\beta(-1+\alpha_{RR}))}}{2\gamma_{n,1}} \\
&- \frac{2\left(Q^*_{n,B}\gamma_{n,1}-\theta\left(Q^*_{n,B}+Q^*_{n,2}\right)\right)}{2\gamma_{n,1}}, \\
Q^*_{n,2} &= \\
&\frac{a_{n,2}+\frac{h_2+p_2\beta(\alpha_{RR}-1)}{1-\alpha_{RR}\beta}}{2\gamma_{n,2}} \\
&- \frac{\frac{\alpha_{RR}(h_2+p_2-\beta p_2)(\beta(T_n-1)(\alpha_{RR}-1)-T_n)}{\alpha_{RR}^{T_n}\beta^{T_n}+\alpha_{RR}(-1+\beta(-1+\alpha_{RR}))}}{2\gamma_{n,1}} \\
&- \frac{2\left(Q^*_{n,B}\gamma_{n,2}-\theta\left(Q^*_{n,B}+Q^*_{n,1}\right)\right)}{2\gamma_{n,2}}.
\end{aligned}
\tag{4.9}
$$

The equation (4.9) demonstrates the closed-form expressions of the optimal purchasing quantities of products bought at regular prices p_1 and p_2, and at a bundle price, p_B where $p_B \leq p_1 + p_2$. Theorem 1 suggests that either $Q^*_{n,1}$ or $Q^*_{n,2}$ will take the value of zero in the optimal solution. The segment-n customer purchases both products through the bundle offer with units of $Q^*_{n,B}$ at a price p_B, and can make additional purchase from either $Q^*_{n,1}$ units of product 1 or $Q^*_{n,2}$ units of product 2, if purchasing only from the bundle offer does not maximize the expected future surplus of the customer.

Finally, we derive a closed-form expression for the purchasing and consumption policies of a customer who belongs to segment-n. The customer's problem is intractable to derive the structure of the surplus function for stockpiling period T_n, $Q_{n,B}$ and $Q_{n,j}$, $\forall j \in \{1, 2\}$. Therefore, we derive the closed-form expressions for a given value of T_n. We obtain the best value of T_n that yields the maximum surplus for the customer with a simple search procedure. In the next section, we will incorporate the closed-form policies into the retailer's bundle pricing problem.

4.3.5 Retailer's Revenue Maximization Problem

In this section, we present the bundle pricing problem of the retailer. The retailer follows bimodal pricing policy and offers temporary bundle discounts to stimulate demand and increase revenue. Let p_1^* and p_2^* denote regular prices of the products 1 and 2, respectively. We assume that p_1^* and p_2^* are the revenue-maximizing prices when the retailer follows regular pricing policy.

The retailer determines the bundle price, p_B, where $p_B \leq p_1^* + p_2^*$, along with the ρ_B to maximize long-run per period revenue denoted by $\Pi_B(p_B, \alpha_{RR})$. ρ_B is the long-run probability that the retailer offers bundle discount. ρ_B can be computed from the transition matrix presented in (4.6); that is, $\rho_B = 1 - \frac{1}{2-\alpha_{RR}}$. The long-run per period revenue of the retailer, $\Pi_B(p_B, \rho_B)$, can be expressed as follows:

$$\begin{aligned} \Pi_B(p_B, \alpha_{RR}) &= \rho_B \left(\sum_{n \in N} \delta_n \left(p_B Q_{n,B}^* + p_1^* Q_{n,1}^* + p_2^* Q_{n,2}^* \right) \right) \\ &+ (1 - \rho_B) \left(\sum_{n \in N} \delta_n \left(p_1^* Q_{n,1}^{R*} + p_2^* Q_{n,2}^{R*} \right) \right) \end{aligned} \tag{4.10}$$

where $\rho_B = 1 - \frac{1}{2-\alpha_{RR}}$.

$\Pi_B(p_B, \alpha_{RR})$ includes revenue received from the bundle discount and regular pricing policy. In a bundle promotion period, current revenue is equal to the revenue received from selling products through not only the bundle offer but also individual offers.

In Equation (4.10), the first term denotes the revenue received from selling $Q_{n,B}^*$, $Q_{n,1}^*$ and $Q_{n,2}^*$ units of products at prices p_B, p_1 and p_2, respectively. The second term represents the regular revenue received from selling products at their regular prices p_1^* and p_2^*.

We derive the structure of the optimal bundle price, p_B^*, and a closed-form expression for the specific setting where $\gamma_1 = \gamma_2$ and a segment-n customer does not purchase individual products along with the bundle offer, i.e., $Q_{n,B}^* > 0$ and $Q_{n,j}^* = 0 \quad \forall j \in \{1, 2\}$ and $\forall n \in N$.

The retailer's revenue function, $\Pi_B(p_B, \rho_B)$, is a concave function in p_B for a given value of ρ_B (The proof is presented in the Appendix C.1.7). To keep the exposition simple, the optimal bundle price p_B^* that maximizes $\Pi_B(p_B, \alpha_{RR})$ for a given ρ_B is given in the Appendix C.1.7.

The retailer's problem is intractable to derive the structure of the revenue function for the bundle price p_B and ρ_B jointly. Therefore, we derive the closed-form expression of p_B^* for a given value of ρ_B. We obtain the best value of ρ_B that yields the maximum revenue for the retailer by employing a simple numerical search procedure.

We explore managerial implications of the retailer's bundle pricing policy (i.e., depth (p_B^*) and frequency (ρ_B)) through a range of numerical experiments which we present in Section 4.4.

4.4 Computational Analysis

The retailer's problem presented in section 4.3.5 includes the bundle pricing and promotion frequency decisions. To explore the revenue impact of the retailer's bundling strategy with respect to the degree of substitutability among the products, consumers' product valuations and the market structure, we have generated 6,300 different problem instances. For each instance, we have found the optimal bundle price along with the promotion frequency to maximize the retailer's per period revenue. We first describe the scheme we have employed to create the problem instances. We then discuss the results and explore the managerial implications. We list the problem generation parameters along with the corresponding value sets in Table 4.1. We generate problen instances by considering all possible combinations within the sets and the parameter

Table 4.1: Parameter and Value Sets

Parameter	Value set
θ	$\{-1.0, -0.9, -0.8, -0.7, -0.6, -0.5, -0.4, -0.3, -0.2$ $-0.1, 0.0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0\}$
$\frac{\delta_{HH}}{\delta_{HH}+\delta_{LH}}$	$\{0.00, 0.30, 0.50, 0.70, 1.00\}$
$\frac{\delta_{LL}}{\delta_{LL}+\delta_{HL}}$	$\{0.00, 0.30, 0.50, 0.70, 1.00\}$
$(r_{H,1}, r_{L,1})$	$\{(8,2), (4,2)\}$
$(r_{H,2}, r_{L,2})$	$\{(8,2), (4,2)\}$

values.

In the computational analysis, we consider various degrees of substitutability among the products, denoted by θ, in the range of -1.0 and 1.0 with 0.1 increments. When the products in the bundle are perfect complements (substitutes), θ takes the value of 1.0 (-1.0). We cluster the values of θ into equally sized five subranges of size 0.3. The $\{-1.0, -0.9, -0.8, -0.7\}$ ($\{1.0, 0.9, 0.8, 0.7\}$) range corresponds to the case where the products show a high degree of substitutability (complementarity), and we denote this group by $H-$ ($H+$). We represent the products that show a medium level of substitutability (complementarity), $\{-0.6, -0.5, -0.4, -0.3\}$ ($\{0.6, 0.5, 0.4, 0.3\}$), by $M-$ ($M+$). Similarly, the $\{-0.2, -0.1, 0.0, 0.1, 0.2\}$ range corresponds to the in which products are low substitutes, independent and low complements, and we denote this group by O.

We use the market structure scheme presented in Aydin and Ziya (2008) as a basis in generating problem instances. In Table 4.1, $\frac{\delta_{HH}}{\delta_{HH}+\delta_{LH}}$ denote probability that a customer who values the product 1 high given that she assigns a high value to the product 2. $\frac{\delta_{LL}}{\delta_{LL}+\delta_{HL}}$ denote probability that a customer who assigns a low value the product 1 given that she assigns a low value to the product 2. Once these values are given, we can compute the segment fractions.

To keep the representation of market structure simple, we use the product similarity metric proposed in the study of Aydin and Ziya (2008) and we cluster the segment fractions with respect to the sum of $\frac{\delta_{HH}}{\delta_{HH}+\delta_{LH}}$ and $\frac{\delta_{LL}}{\delta_{LL}+\delta_{HL}}$. We cluster the segment fractions into four cases and employ the following notation to represent a cluster: X_Y, where X represents the direction of relationship among the products in terms of product valuations (i.e, products are either dissimilar or similar) and Y denotes the degree of the corresponding relationship among the products.

If sum of the conditional probabilities is less than 0.5 (i.e., $\frac{\delta_{HH}}{\delta_{HH}+\delta_{LH}} + \frac{\delta_{LL}}{\delta_{LL}+\delta_{HL}} < 0.5$) then the market includes a high fraction of customers who purchase either of the products at a regular price of the corresponding product. In this case, customers have a higher valuation only for one of the products. These products are highly dissimilar in terms of customers' valuations. We denote this group by D_{High}. When sum of the conditional probabilities is between 0.5 and 1.0 (i.e., $0.5 \leq \frac{\delta_{HH}}{\delta_{HH}+\delta_{LH}} + \frac{\delta_{LL}}{\delta_{LL}+\delta_{HL}} \leq 1$), these products correspond to the case where the market includes a low fraction of customers who purchase only one of the products at its regular price, and these products show a low degree of dissimilarity in terms of customers' valuations. We denote this group by D_{Low}. If sum of the conditional probabilities is between 1 and 1.5 (i.e., $1.0 \leq \frac{\delta_{HH}}{\delta_{HH}+\delta_{LH}} + \frac{\delta_{LL}}{\delta_{LL}+\delta_{HL}} \leq 1.5$), the fraction of customers who value both products high is relatively lower and we denote this cluster by S_{Low}. Finally, when the summation of the conditional probabilities is greater than 1.5, this corresponds to the case where the fraction of customers who purchase both products is relatively higher. Products demonstrate high similarity in terms of customers' valuations and we denote this group by S_{High}. The distribution of the 6,300 problem instances into the subgroups are as follows: D_{High}: 20%, D_{Low}: 40%, S_{Low}: 20%, S_{High}: 20%.

In Section 4.4.1, we analyze the impacts of the degree of substitutability among the products and the market structure on the bundle revenue performance of the retailer. We report the performance of bundle discount relative to regular case in which the retailer follows no-discount policy.

We set the values of reservation prices, $r_{n,j}$ where $r_{H,1} = a_{HH,1} = a_{HL,1}$, $r_{L,1} = a_{LL,1} = a_{LH,1}$, $r_{H,2} = a_{HH,2} = a_{LH,2}$ and $r_{L,2} = a_{LL,2} = a_{HL,2}$, and the saturation effect, $\gamma_{n,j}$, for each segment and product, $\forall n \in N$, and $\forall j \in \{1, 2\}$, such that customers who value a product high consumes at most four units of the corresponding product and customers who value a product low consumes at most one unit of the corresponding product in a period when the two products are independent ($\theta = 0$). We set $\gamma_{n,j}$ to 1 $\forall n \in N$ and $\forall j \in \{1, 2\}$.

We also consider the case where customers have a strong incentive to stockpile. For this purpose, we set the interest rate, F, to 10%, and compute the holding cost of product j as $h_j = Fp_j$. We assume that the utility of future consumption is almost undiscounted and set to the value of β to 0.999. We assume that consumers can stockpile the products to last for at most T_n periods. It is never optimal stockpile the products to last for infinite number of periods because customers incur holding costs. We can interpret upper bound of T_n as the perishability level of the products. Niraj et al. (2008) demonstrate that 95% of consumers make a purchase to last for at most two periods. In parallel with the empirical findings in the study of Niraj et al. (2008), we set the upper bound value of T_n to 2 for a segment-n customer, where $n \in N$.

4.4.1 Revenue Impact of Temporary Mixed-Bundling Strategy

In this section, we discuss how the degree of substitutability, θ, and the market structure, $\frac{\delta_{HH}}{\delta_{HH}+\delta_{LH}} + \frac{\delta_{LL}}{\delta_{LL}+\delta_{HL}}$, affect the bundling strategy of the retailer.

In Table 4.2 we examine the marginal revenue impact of temporary bundling strategy with respect to different values of θ and $\frac{\delta_{HH}}{\delta_{HH}+\delta_{LH}} + \frac{\delta_{LL}}{\delta_{LL}+\delta_{HL}}$. In Table 4.2, we report the marginal revenue improvement the bundle discount generates over the regular case revenue.

The column (1) of Table 4.2 shows that when the fraction of customers who purchases either of the products is high (i.e, a group of customers who belong to D_{High}), the bundling strategy enhances the revenue, on average, by 1.14% over the regular case. The revenue performance of the bundling strategy decreases as the products gets substitute. When the fraction of customers who purchase either of the products is high (i.e., customers who belong to D_{High}), the retailer does not generate additional revenue through the bundle offer when the products show a higher degree of substitutability.

The column (2) of Table 4.2 indicates the revenue performance of the bundling strategy when the fraction of customers who purchase either of the products relatively decreases (i.e., products demonstrate a low degree of dissimilarity in terms of valuations, (D_{Low})). Bundle discount generates an additional revenue over the case in which products show a higher degree of dissimilarity (i.e., (D_{High})), on average, by 1.55%. The bundle discount enhances the revenue by 1.0% (average of 1.04% and 0.86%) as the degree of substitutability among the products increases. When the fraction of customers who purchase only either of the products decreases (i.e., D_{Low}), the retailer's revenue increases by 2.2% for the products that show a high degree of complementarity (i.e., $H+$) over the case in which the fraction of customers who purchase either of the products is high (i.e., D_{High}).

Table 4.2: Revenue Impact of Bundle Discount: Bundle Discount vs. Regular Pricing

	(1)	(2)	(3)	(4)	
	Market Structure				
θ	D_{High}	D_{Low}	S_{Low}	S_{High}	Average
$H+$	4.69%	6.89%	7.51%	9.37%	7.07%
$M+$	1.30%	3.81%	5.36%	9.60%	4.77%
O	0.00%	1.24%	3.22%	8.32%	2.81%
$M-$	0.00%	1.04%	2.82%	6.90%	2.36%
$H-$	0.00%	0.86%	1.99%	5.85%	1.91%
Average	1.14%	2.69%	4.13%	8.02%	

Table 4.2 presents that the revenue impact of the bundling strategy mainly depends on the heterogeneity in customers' valuations of products and the degree of substi-

tutability among the products, θ. As the fraction of customers who belong to segments δ_{HH} and δ_{LL} (i.e., S_{High}) increases, the revenue impact of bundling enhances. This is because a higher value of $\frac{\delta_{HH}}{\delta_{HH}+\delta_{LH}} + \frac{\delta_{LL}}{\delta_{LL}+\delta_{HL}}$, most customers in the market have either high or low valuations for both products. The retailer entices bargain-hunting customers through the bundle discount. The degree of substitutability among the products plays an important role on the revenue impact of the bundling strategy. As products get substitute (i.e., $\theta < 0$), the per period revenue of bundling and the revenue performance over the regular case decrease. This observation is in parallel with the results presented in earlier studies (Adams and Yellen (1976); Schmalensee (1984); Venkatesh and Kamakura (2003); Bulut et al. (2009)).

In Table 4.2, bundling strategy generates an additional revenue improvement when the products are not only complement but also substitute. This result matches our intuition. In the regular environment, customers simultaneously purchases and consumes complement products. They receive an additional surplus from consuming complement products together and the retailer receives higher revenue from selling complement products. When the products are substitute, the retailer entices customers to purchase the products through the bundle offer, and this results in marginal revenue improvement of bundling for these products over the regular case. Although we observe the revenue improvement over the base case for the substitute products, the revenue performance of the bundle decreases in the degree of substitutability among the products.

We, now, turn to the analysis of bundle discount level, denoted by d_B where $d_B = p_1^* + p_2^* - p_B^*$, and the bundle frequency, ρ_B, that generate retailer's bundle revenue presented in Table 4.2. Table 4.3 reports the corresponding bundle discount and promotion frequency results with respect to different values of θ and $\frac{\delta_{HH}}{\delta_{HH}+\delta_{LH}} + \frac{\delta_{LL}}{\delta_{LL}+\delta_{HL}}$. In the results reported in Table 4.3, we use the following format: the first number in parentheses indicates the corresponding bundle discount, d_B, and the second number shows the long-run probability that the retailer follows a bundling strategy, denoted by ρ_B.

The column (1) of Table 4.3 shows that when the majority of the customers purchase only one product or does not make a purchase in the market (i.e., customers who belong to D_{High}), the retailer offers a bundle offer with probability, on average, 11.4% and a higher level of bundle discount, on average 30.6%, when the products show medium and high degrees of complementarity (i.e., $M+$ and $H+$). When the products show medium and high degrees of substitutability (i.e., $M-$ and $H-$), the retailer does not offer a bundle discount in the long-run. When most of customers purchase only one product or does not make a purchase in the market, the retailer offers bundle discount only for products that show a high degree of complementarity.

Table 4.3: Bundle Discount Level (d_B) and Promotion Frequency (ρ_B)

	(1)	(2)	(3)	(4)	
	Market Structure				
θ	D_{High}	D_{Low}	S_{Low}	S_{High}	Average
$H+$	$(24\%, 0.188)$	$(21.6\%, 0.217)$	$(21.3\%, 0.232)$	$(17.6\%, 0.280)$	$(21.2\%, 0.227)$
$M+$	$(37.2\%, 0.040)$	$(32.2\%, 0.094)$	$(27.1\%, 0.159)$	$(18.0\%, 0.261)$	$(29.3\%, 0.130)$
O	$(-, 0.000)$	$(36.4\%, 0.043)$	$(27.9\%, 0.130)$	$(19.7\%, 0.205)$	$(32.2\%, 0.084)$
$M-$	$(-, 0.000)$	$(37.1\%, 0.032)$	$(30.5\%, 0.088)$	$(25.3\%, 0.119)$	$(34.1\%, 0.054)$
$H-$	$(-, 0.000)$	$(37.1\%, 0.024)$	$(33.6\%, 0.048)$	$(26.5\%, 0.086)$	$(35.0\%, 0.036)$
Average	$(36.85\%, 0.043)$	$(33.03\%, 0.080)$	$(28.04\%, 0.131)$	$(21.35\%, 0.191)$	

In column (2) of Table 4.3, as a result of increase in the fraction of customers who purchase both products, the retailer generates an additional revenue over the regular case. The retailer employs the bundle strategy with the discount level of 33.03% and frequency of 0.08, on average. In addition to the impact of increase in products' valuations, the retailer slightly adjusts the bundle frequency as the degree of substitutability among the products changes. For instance, the retailer decreases the bundle frequency from 0.155 (average of 0.217 and 0.094) to 0.028 (average of 0.032 and 0.024), when we change the products from complements (i.e., $H+$ and $M+$) to substitutes (i.e., $H-$ and $M-$).

In columns (3) and (4) of Table 4.3, as heterogeneity in customers' valuations decreases (i.e., S_{Low} and S_{High}), the retailer decreases the bundle discount level and increases its frequency. The retailer decreases the bundle discount, on average, by 31.3%, and increases the bundle frequency by 45.8%, on average, respectively.

In Table 4.3, we report the average results of bundle discount and frequency with respect to the degree of substitutability among the products and the similarity in products' valuations. On average, our results suggest that the retailer increases (decreases) the bundle discount level and decreases (increases) its frequency for the substitute (complement) products. As the customers' valuations gets higher, the retailer decreases the bundle discount and increases the frequency.

We highlight the three key observations in Section 4.4.1.

1. Our results indicate that bundling strategy brings revenue improvement of around 3.74%. The revenue improvement depends on the heterogeneity in customers' valuations and the degree of substitutability among the products in the bundle offer. The retailer generates an additional revenue when the heterogeneity in customers' valuations is at the lowest level (i.e., S_{High}).

2. Our results suggest that the retailer can increase the revenue when the products that form the bundle offer show a degree of not only complementarity but also substitutability. When the products are substitutes, the marginal revenue improvement with the bundling strategy is 2.14% (average of 2.36% and 1.91%). The impact of bundle strategy increases to 4.88% (average of 7.07%, 4.77% and 2.81%) when the products are not substitute.

3. Our results suggest that the retailer should offer a lower (higher) level discount with relatively higher (lower) frequency when the products show a degree of complementarity (substitutability). On average, the retailer employs the bundling strategy with discount level of 34.57% (average of 34.1% and 35.00%) and frequency of 0.045 (average of 0.054 and 0.036) for substitute products. When

the products show a degree of complementarity (i.e., $\theta \in \{O, M+, H+\}$), the retailer offers the bundling strategy with the discount level of 27.57% (average of 21.2%, 29.3% and 32.2%) and the frequency of 0.147 (average of 0.227, 0.130 and 0.084). The retailer increases the discount level, d_B, by around of 25.3% and decreases the bundle frequency from 0.147 to 0.045 when the products show a degree of substitutability.

4.5 Conclusion

In this chapter, we consider a setting where a retailer employs periodic bundle promotion to stimulate demand and increase revenue. We address questions of how the relationship among products and heterogeneous customer base affect the revenue improvement that the retailer can achieve with bundle offers. We first develop a strategic customer model for the products that form the bundle offer, and characterize the optimal purchasing and consumption policies. We analyze how the problem parameters affect the consumer's stockpiling policies.

We derive the strategic customer's optimal purchasing and consumption policy in response to a bundle offer in a setting where the retailer follows bimodal pricing policy (i.e., on-discount/off-discount strategy). Then, we develop a retailer's revenue model with bimodal pricing policy and present a computational study that demonstrates how bundle promotion design (i.e., bundle discount and frequency) changes with respect to characteristics of products in the bundle and the market structure.

We first illustrate that bundle discounts are most effective when the heterogeneity in customers' valuations for the products decreases. When the heterogeneity decreases, the fraction of customers who purchase only either of the products decreases and the retailer offers a lower level of discount to entice customers. As a result, the retailer generates an additonal revenue as the heterogeneity in customers' valuations decreases. The degree of substitutability among the products plays a moderator role in this relationship, and the retailer achieves a higher revenue with a bundle offer that includes complement products. Our results also suggest that the retailer can be better-off with a bundle offer that consists of substitute products but the corresponding revenue improvement is relatively lower. When the heterogeneity is high in customers' valuations for both products, the retailer generates an additional revenue through the bundle offer only for complement products.

We also analyze how bundle promotion design changes with respect to the characteristics of products in the bundle and heterogeneous customer base. Our results suggest that the retailer should employ temporary bundle offer with a lower (higher) level of discount and higher (lower) frequency when the bundle includes subsitute (complement) products.

Our study focuses on the pricing and timing of a bundle formed by two products. A natural extension of our research would be to incorporate the retailer's replenishment strategies into the revenue model. A further research topic would be to analyze the impact of different promotion policies with respect to the product's inventory level.

Chapter 5

CONCLUSION

This dissertation includes three different problems. In the first problem, we examine the seller's one-time bundle offer design problem to liquidate the excess inventory of one of the products. In the second problem, we investigate how dynamic and segment-specific bundle pricing affects the retailer's revenue, and how these policies interact with each other. In the third problem, we study how the degree of substitutability among the products that form the bundle offer and heterogeneous customer base with strategic customers affect the retailer's bundle pricing and promotion frequency decisions.

In the first part of dissertation, we study a one-time bundle offer problem in a setting where the supply chain includes a single buyer and a single seller. The seller has an excess inventory of a product and aims to reduce that inventory by employing the one-time bundle offer. We evaluate the performance of the bundle offer relative to one-time individual discount. We model the seller's and the buyer's problems in a Stackelberg framework. We first derive the buyer's optimal decisions in response to the seller's one-time bundle offer. Then, we derive the seller's optimal bundle decisions by incorporating the buyer's decisions.

In the first part, we primarily address the question of whether one-time bundle discounts result in more effective trade promotions than one-time individual discounts. Through a range of numerical experiments, we have found that effectiveness of the bundle discount scheme depends on the characteristics of the products that form the bundle offer along with the units of products in the bundle. Major contributions of the first part can be summarized as follows:

1. When the seller creates a forward-buy incentive for a product with relatively low

demand and high price, the bundle composition needs to be carefully selected, for the bundle discount scheme to be an effective alternative.

2. If the seller plans to create a forward-buy incentive for a product whose demand is relatively larger, our findings indicate that the seller needs to face a large one-time delivery requirement for the secondary product to balance the quantity requirements. This additional procurement cost makes the bundle discount difficult to implement.

The problem setting developed in the first problem can be extended in several ways. First, the model can be extended to a setting where the seller faces with stochastic demand for both products. Second, we can consider the inventory level of the second product in the bundle while designing a forward-buy incentive for a product. Then, the seller's problem would be to design a bundle offer with a limited supply of the second product. Third, consumers' response to bundle offer can be incorporated into the one-time bundle offer problem. These are left for future research.

The second part of the dissertation studies dynamic and targeted bundle pricing problem of a retailer that operates in a market with multiple and identifiable customer segments. The segments are heterogeneous in terms of products' valuations. The retailer aims to liquidate the excess inventory of a product through a dynamic and targeted bundle pricing over the planning horizon while maximizing the expected total revenue. We propose a revenue model that integrates the dynamic and segment-specific dimensions of the pricing decisiosn. The way we model customer segments enables us to derive correlation relationship among the products through segment fractions. We derive the structural properties of the retailer's revenue function and the dynamic and targeted bundle prices. Then, we present a numerical study to analyze the revenue impact of dynamic and targeted bundle pricing relative to an individual discount. Major contributions of the second part can be summarized as follows:

1. This research is the first to investigate the interaction between bundling, dynamic pricing, and segment-specific pricing policies. Also, this study is the first to

derive structural properties for the dynamic bundle pricing problem.

2. The degree of effectiveness for the bundle pricing and dynamic pricing depends on the inventory level that the retailer aims to sell. Bundle pricing is most effective when the inventory level is high and dynamic pricing is beneficial when the inventory level is low.

3. Segment-specific and static pricing has almost no direct impact on revenue. However, segment-specific and dynamic pricing can generate additional revenue as a function of the initial inventory level.

4. When bundle offers are segment-specific and dynamically priced, they eliminate the negative impact on the retailer's revenue of positive correlation in customers' valuations.

An interesting topic for future research would be a study that integrates the inventory level of the second product into the retailer's revenue model. In this study, we assume that customers are myopic. Therefore, another topic would be to consider dynamic and targeted bundle pricing with strategic customers. The model can also be extended to a setting where the replenishment of products are possible. Then, the retailer coordinates pricing decisions with inventory procurement, or production decisions. Research questions for this problem can be as follows: How does the optimal frequency and the depth of the bundle promotion vary with the remaining inventory level of products and remaining time to replenishment of products? What are the impacts of different replenishment policies on the frequency and the depth of bundle promotion? Finally, we can consider the retailer's bundle composition problem along with pricing decisions. In the problem setting, there is an online retailer that sells different products to its customers over a finite selling season. Customers arrive sequentially. The retailer determines a subset of products that form a bundle offer with the corresponding price and offers the corresponding bundle offer to each arriving customer. The customer then decides whether to purchase any product or the bundle offer from the offered assortment. The retailer's objective is to maximize expected cumulative revenue over the selling season. We assume that the retailer has limited prior information on customers product preferences and willingness-to-pay values. Personalized (or targeted) bundle offers require estimation of product preferences and willingness-to-pay values by observing customers transactions. Therefore, the retailer faces a trade-off between earning revenue immediately and learning customers attributes for future revenues. This study focuses on the efficient estimation of customer attributes by using transaction history in online retailing and explores the revenue impact of dynamic bundle composition and pricing. These issues are also left for future research.

The third part of the dissertation focuses on intertemporal bundle pricing with strategic customers. We extend the retailer's bundle pricing problem to include strategic customers (i.e., customers change quantity and timing of purchases with respect to beliefs about the retailer's promotion strategy). We examine how strategic customers and the characteristics of products in the bundle affect the revenue that the retailer can achieve through bundle offers. In the problem setting, strategic customers are heterogeneous in terms of products' valuations and the retailer determines the bundle price and promotion frequency to price discriminate among the heterogeneous customers while maximizing long-run per period revenue. We derive closed-form expressions of the customer's optimal purchase quantity and the retailer's optimal bundle price for a setting where customers make purchases through only bundle offers. We also present a numerical study to investigate the impact of strategic customers' valuations and the degree of substitutability among the products that form the bundle offer on the revenue that the retailer achieves. A brief summary of the contributions of this part is as follows:

1. We analytically prove that the customer's optimal purchasing and consumption policies is a state-dependent threshold policy.

2. Although at a preliminary stage, this study is the first to consider the interac-

tion between strategic customers, characteristics of products, and the bundle promotion design in a multi-product setting.

3. Our results suggest that the retailer can be better off with a bundle offer that includes substitute products. However, the degree of revenue improvement depends on the retailer's bundle promotion decisions and customers' valuations.

4. Our findings suggest that, as the degree of substitutability among the products increases, the retailer should employ bundle offers with a higher discount and lower frequency.

5. Our findings indicate that, as the degree of complementarity among the products increases, the retailer should employ bundle offers with a lower discount and higher frequency.

The model developed in the third essay can be extended in several ways. First, in this study, we assume that the retailer does not change discount level of the bundle offer and announces the same price to all customers. A natural extension of our research would be to incorporate the dynamic and segment-specific aspects of pricing decisions. Second, we can incorporate the retailer's replenishment strategies into the revenue model. It would be interesting to examine the impact of different promotion policies with respect to inventory level of the products. Third, it would be interesting to analyze which promotion strategy (i.e., individual discount, pure bundling, or mixed-bundling) dominates other strategies and to what extent in terms of the revenue that the retailer achieves. These issues are left for future studies.

This dissertation extends the state of art in the multiproduct pricing and in the interface of operations management and marketing by analyzing pricing and inventory dimensions that have not been addressed in the literature. We believe that the results driven in this dissertation will be of value to the retailers engaging in promotional activities to stimulate demand and increase revenue. The insights of this dissertation will be valuable in designing alternative efficient promotion schemes in a multi-product setting and help to pave the way for a comprehensive body of literature for academia and practitioners.

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Appendix A

SUPPLEMENTARY MATERIAL FOR: TRADE PROMOTIONS WITH ONE-TIME BUNDLE DISCOUNTS

A.1 The Buyer's Pricing and Ordering Decisions in the No-Discount Case

The buyer's profit function, $H_{R,i}\left(p_{R,i}, Q_{R,i}, w_i\right), i = 1,2$, can be simplified as $H_{R,i}\left(Q_{R,i}, w_i\right)$ by expressing the optimal selling prices in terms of $Q_{R,i}$ and w_i :

$$H_{R,i}\left(p_{R,i}^*\left(Q_{R,i}, w_i\right), Q_{R,i}, w_i\right) \equiv H_{R,i}\left(Q_{R,i}, w_i\right) = \frac{\gamma_i}{4}\left(\frac{A_i}{\gamma_i} - \frac{S_i}{Q_{R,i}} - w_i\right)^2 - w_i F \frac{Q_{R,i}}{2}, i = 1,2. \tag{A.1}$$

As shown in Abad (1988), for a given w_i value, $H_{R,i}\left(Q_{R,i}, w_i\right)$ is convex-concave in $Q_{R,i}, i = 1,2$. Let $\overline{Q}_{R,i}\left(w_i\right) = \frac{3S_i\gamma_i}{2(A_i - \gamma_i w_i)}, i = 1,2$, be the order quantity for product $i, i = 1,2$, at which the second derivative of $H_{R,i}\left(Q_{R,i}, w_i\right)$ with respect to $Q_{R,i}$ is equal to zero. $\overline{Q}_{R,i}\left(w_i\right)$ actually defines the regions in which $H_{R,i}\left(Q_{R,i}, w_i\right), i = 1,2$, is convex or concave:

$$Q_{R,i} \leq \frac{3S_i\gamma_i}{2(A_i - \gamma_i w_i)} \Rightarrow \frac{\partial^2 H_{R,i}\left(Q_{R,i}, w_i\right)}{\partial^2 Q_{R,i}} \geq 0, i = 1,2,$$
$$Q_{R,i} \geq \frac{3S_i\gamma_i}{2(A_i - \gamma_i w_i)} \Rightarrow \frac{\partial^2 H_{R,i}\left(Q_{R,i}, w_i\right)}{\partial^2 Q_{R,i}} \leq 0, i = 1,2.$$

We note that because the selling price of product i should be strictly less than $\frac{A_i}{\gamma_i}, i = 1,2$, the buyer's order quantity $Q_{R,i}, i = 1,2$, is bounded below by $\frac{S_i\gamma_i}{A_i - \gamma_i w_i}, i = 1,2$.

Let $Q_i^+\left(w_i\right)$ be equal to $\arg\max_{Q_{R,i} \geq \frac{3S_i\gamma_i}{2(A_i - \gamma_i w_i)}} H_{R,i}\left(Q_{R,i}, w_i\right)$. Also let $Q_i^-\left(w_i\right)$ be equal to $\arg\max_{\frac{S_i\gamma_i}{A_i - \gamma_i w_i} < Q_{R,i} \leq \frac{3S_i\gamma_i}{2(A_i - \gamma_i w_i)}} H_{R,i}\left(Q_{R,i}, w_i\right)$. Due to the convexity of $H_{R,i}\left(Q_{R,i}, w_i\right)$ in the $\left(\frac{S_i\gamma_i}{A_i - \gamma_i w_i}, \frac{3S_i\gamma_i}{2(A_i - \gamma_i w_i)}\right)$ interval, the overall optimal order quantity $Q_{R,i}^*\left(w_i\right)$ can only be equal to $Q_i^+(w_i)$ or $\frac{S_i\gamma_i}{A_i-\gamma_i w_i}, i=1,2$. Through Equation A.1, it can be readily shown that $H_{R,i}\left(\frac{S_i\gamma_i}{A_i-\gamma_i w_i}, w_i\right) \leq 0 \leq H_{R,i}\left(Q_i^+(w_i), w_i\right), i=1,2$, therefore, $Q_i^+(w_i), i=1,2$, is the unique optimal solution. Because the problem $\max_{Q_{R,i}} H_{R,i}(Q_{R,i}, w_i)$ has a unique solution, we can conclude that the $p^*_{R,i}, i=1,2$, and $Q^*_{R,i}, i=1,2$, values can be obtained by jointly solving the following equalities: $Q^*_{R,i} = \sqrt{\frac{2S_iD_i\left(p^*_{R,i}\right)}{w_iF}}$, and $p^*_{R,i} = \frac{1}{2}\left(\frac{A_i}{\gamma_i} + \frac{S_i}{Q^*_{R,i}} + w_i\right), i=1,2$.

A.2 Existence of Unique $w^*_{I,1}$ and $p^*_{I,1}$ Values

For a given $Q_{I,1}$, the buyer's best response function for the selling price of product 1, $p^*_{I,1}$, is a decreasing and convex function in $w_{I,1}$.

$$\begin{aligned}\frac{\partial p^*_{I,1}}{\partial w_{I,1}} &= -\frac{FQ_{I,1}\gamma_1}{2\sqrt{2}\sqrt{\gamma_1^3\left(FQ_{I,1}w_{I,1}+2H^*_{R,1}\right)}} < 0,\\ \frac{\partial^2 p^*_{I,1}}{\partial^2 w_{I,1}} &= \frac{F^2Q_{I,1}^2\gamma_1^4}{4\sqrt{2}\left(\gamma_1^3\left(FQ_{I,1}w_{I,1}+2H^*_{R,1}\right)\right)^{3/2}} > 0. \end{aligned} \tag{A.2}$$

Similarly, for a fixed value of $Q_{I,1}$, the seller's best response function for the one-time discounted wholesale price for product 1, $w^*_{I,1}$, is a concave function in $p_{I,1}$:

$$\frac{\partial^2 w^*_{I,1}}{\partial^2 p_{I,1}} = -\frac{2\gamma_1\left(FQ_{I,1}\left(A_1+FQ_{I,1}\right)+H^*_{R,1}\gamma_1\right)}{\left(FQ_{I,1}+A_1-\gamma_1 p_{I,1}\right)^3} < 0. \tag{A.3}$$

Figure A.1 illustrates the convex structure of $p^*_{I,1}$ with respect to $w_{I,1}$, in its domain $\left(0, \frac{2\left(A_1^2-\gamma_1 H^*_{R,1}\right)}{FQ_{I,1}\gamma_1}\right)$, and the concave structure of $w^*_{I,1}$ with respect to $p_{I,1}$, in the $\left(\frac{A_1-\sqrt{A_1^2-4\gamma_1 H^*_{R,1}}}{2\gamma_1}, \frac{A_1+\sqrt{A_1^2-4\gamma_1 H^*_{R,1}}}{2\gamma_1}\right)$ range. To prove that there exist unique values of $w^*_{I,1}$ and $p^*_{I,1}$ that satisfy Equations (2.17) and (2.18), it is sufficient to show that $\frac{A_1\gamma_1-\sqrt{H^*_{R,1}\gamma_1^3}}{\gamma_1^2} \geq \frac{A_1-\sqrt{A_1^2-4\gamma_1 H^*_{R,1}}}{2\gamma_1}$ and $\frac{A_1\gamma_1-\sqrt{H^*_{R,1}\gamma_1^3}}{\gamma_1^2} \leq \frac{A_1+\sqrt{A_1^2-4\gamma_1 H^*_{R,1}}}{2\gamma_1}$. After some algebraic manipulations, these two inequalities can be shown to be equivalent to $2\sqrt{H^*_{R,1}\gamma_1} \leq A_1$. Let $R_1(p_1)$ be the total revenue of the buyer when the price of product 1 is p_1, where $R_1(p_1) = p_1 D_1(p_1) = p_1(A_1-\gamma_1 p_1)$. $R_1(p_1)$ is a concave function in p_1, and $p_1^* = \frac{A_1}{2\gamma_1}$. The maximum revenue can now be expressed as

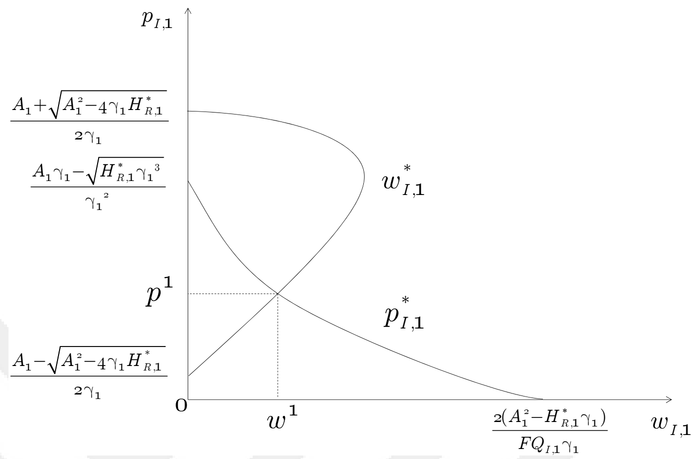

Figure A.1: Plots of $w^*_{I,1}$ and $p^*_{I,1}$ values.

$\frac{A_1}{2\gamma_1}(A_1 - \gamma_1\frac{A_1}{2\gamma_1}) = \frac{A_1^2}{4\gamma_1}$. $R_1\,(p_1^*)$ is always greater than $H^*_{R,1}$, implying that $\frac{A_1^2}{4\gamma_1} \geq H^*_{R,1}$, or $2\sqrt{H^*_{R,1}\gamma_1} \leq A_1$.

Appendix B

SUPPLEMENTARY MATERIAL FOR: DYNAMIC AND TARGETED BUNDLE PRICING OF TWO INDEPENDENTLY VALUED PRODUCTS

B.1 Proof of Proposition 7

Let $Pr\{n \mid l\}$,$n \in \{HL, LH, HH, LL\}$ and $l \in \{Low, High\}$ be the probability that a customer belongs to segment n given that she values the secondary item at level l. $Pr\{n \mid l\}$ values can be expressed as follows: $Pr\{HL \mid Low\} = \frac{\delta_{HL}}{\delta_{HL}+\delta_{LL}}$, $Pr\{LL \mid Low\} = \frac{\delta_{LL}}{\delta_{HL}+\delta_{LL}}$, $Pr\{LH \mid High\} = \frac{\delta_{LH}}{\delta_{LH}+\delta_{HH}}$, and$Pr\{HH \mid High\} = \frac{\delta_{HH}}{\delta_{LH}+\delta_{HH}}$. Also, by definition, $Pr\{n \mid l\}$ values satisfy the following equation:$Pr\{HH \mid High\} + Pr\{LH \mid High\} = Pr\{HL \mid Low\} + Pr\{LL \mid Low\} = 1$. Also let$\mu_{H,P}$ $(\mu_{L,P})$ denote the expected value of the product valuation of customers who value the primary product high (low). Similarly, let $\mu_{H,S}$ $(\mu_{L,S})$ be the expected value of the product valuation of customers who value the secondary product high (low). Let P (S) be the random variable that represents the product valuation of an arriving customer for the primary (secondary) product. Given the $q_{.,.}$ parameters of Proposition 7 and the above conditional probabilities, $E(P)$ $(E(S))$ can be expressed as $q_{H,P}\mu_{H,P} + q_{L,P}\mu_{L,P}$ $(q_{H,S}\mu_{H,S} + q_{L,S}\mu_{L,S})$. Then, the expected value of the multiplication of the two random variables, i.e., $E\left(PS\right)$, can be written as follows

$$\begin{aligned} E\left(PS\right) &= q_{H,S}\Big(\int_0^\infty \int_0^\infty xyf_{HH,S}\left(x\right)\left(Pr\{HH \mid High\}f_{HH,P}\left(y\right)\right)dxdy + \\ &\int_0^\infty \int_0^\infty xyf_{LH,S}\left(x\right)\left(Pr\{LH \mid High\}f_{LH,P}\left(y\right)\right)dxdy\Big) + \\ &q_{L,S}\Big(\int_0^\infty \int_0^\infty xyf_{HL,S}\left(x\right)\left(Pr\{HL \mid Low\}f_{HL,P}\left(y\right)\right)dxdy + \\ &\int_0^\infty \int_0^\infty xyf_{LL,S}\left(x\right)\left(Pr\{LL \mid Low\}f_{LL,P}\left(y\right)\right)dxdy\Big) \qquad \text{(B.1)} \\ E\left(PS\right) &= q_{H,S}\Big(\int_0^\infty y\left(Pr\{HH \mid High\}f_{HH,P}\left(y\right)\right)\int_0^\infty xf_{HH,S}\left(x\right)dxdy + \\ &\int_0^\infty y\left(Pr\{LH \mid High\}f_{LH,P}\left(y\right)\right)\int_0^\infty xf_{LH,S}\left(x\right)dxdy\Big) + \\ &q_{L,S}\Big(\int_0^\infty y\left(Pr\{HL \mid Low\}f_{HL,P}\left(y\right)\right)\int_0^\infty xf_{HL,S}\left(x\right)dxdy + \\ &\int_0^\infty y\left(Pr\{LL \mid Low\}f_{LL,P}\left(y\right)\right)\int_0^\infty xf_{LL,S}\left(x\right)dxdy\Big) \qquad \text{(B.2)} \\ E\left(PS\right) &= q_{H,S}\Big(\int_0^\infty y\left(Pr\{HH \mid High\}f_{HH,P}\left(y\right)\right)\mu_{H,S}dy + \\ &\int_0^\infty y\left(Pr\{LH \mid High\}f_{LH,P}\left(y\right)\right)\mu_{H,S}dy\Big) + \\ &q_{L,S}\Big(\int_0^\infty y\left(Pr\{HL \mid Low\}f_{HL,P}\left(y\right)\right)\mu_{L,S}dy\Big) + \\ &\int_0^\infty y\left(Pr\{LL \mid Low\}f_{LL,P}\left(y\right)\right)\mu_{L,S}dy\Big) \qquad \text{(B.3)} \end{aligned}$$

$$\begin{aligned} E\left(PS\right) &= q_{H,S}\mu_{H,S}\Big(\int_0^\infty y\left(Pr\{HH \mid High\}f_{HH,P}\left(y\right)\right)dy\Big) + \\ &\int_0^\infty y\left(Pr\{LH \mid High\}f_{LH,P}\left(y\right)\right)dy\Big) + \\ &q_{L,S}\mu_{L,S}\Big(\int_0^\infty y\left(Pr\{HL \mid Low\}f_{HL,P}\left(y\right)\right)dy\Big) + \\ &\int_0^\infty y\left(Pr\{LL \mid Low\}f_{LL,P}\left(y\right)\right)dy\Big) \qquad \text{(B.4)} \\ E\left(PS\right) &= q_{H,S}\mu_{H,S}\Big(Pr\{HH \mid High\}\mu_{H,P} + Pr\{LH \mid High\}\mu_{L,P}\Big) + \\ &q_{L,S}\mu_{L,S}\Big(Pr\{HL \mid Low\}\mu_{H,P} + Pr\{LL \mid Low\}\mu_{L,P}\Big). \qquad \text{(B.5)} \end{aligned}$$

The covariance of random variables P and S, $Cov(P,S)$, can now be written as

$$\begin{aligned} Cov(P,S) &= E(PS) - E(P)E(S) \\ &= \Bigg(q_{H,S}\mu_{H,S}\Big(Pr\{HH \mid High\}\mu_{H,P} + Pr\{LH \mid High\}\mu_{L,P} \Big) \\ &\quad q_{L,S}\mu_{L,S}\Big(Pr\{HL \mid Low\}\mu_{H,P} + Pr\{LL \mid Low\}\mu_{L,P} \Big) \Bigg) - \\ &\quad \Big(q_{H,P}\mu_{H,P} + q_{L,P}\mu_{L,P} \Big)\Big(q_{H,S}\mu_{H,S} + q_{L,S}\mu_{L,S} \Big). \end{aligned} \tag{B.6}$$

With algebraic manipulations, $Cov(P,S)$ can be simplified as

$$\begin{aligned} Cov(P,S) &= q_{H,S}\mu_{H,S}\Big(\mu_{H,P}\left(Pr\{HH \mid High\} - q_{H,P} \right) + \\ &\quad \mu_{L,P}\left(Pr\{LH \mid High\} - q_{L,P} \right) \Big) + \\ &\quad q_{L,S}\mu_{L,S}\Big(\mu_{H,P}\left(Pr\{HL \mid Low\} - q_{H,P} \right) + \\ &\quad \mu_{L,P}\left(Pr\{LL \mid Low\} - q_{L,P} \right) \Big). \end{aligned} \tag{B.7}$$

Since $Pr\{HH \mid High\} + Pr\{LH \mid High\} = q_{H,P} + q_{L,P} = Pr\{HL \mid Low\} + Pr\{LL \mid Low\}$, we have $Pr\{HH \mid High\} - q_{H,P} = -\left(Pr\{LH \mid High\} - q_{L,P} \right)$ and $Pr\{HL \mid Low\} - q_{H,P} = -\left(Pr\{LL \mid Low\} - q_{L,P} \right)$. Therefore,

$$\begin{aligned} Cov(P,S) &= \left(\mu_{H,P} - \mu_{L,P} \right)\Big(q_{H,S}\mu_{H,S}\left(Pr\{HH \mid High\} - q_{H,P} \right) + \\ &\quad q_{L,S}\mu_{L,S}\left(Pr\{HL \mid Low\} - q_{H,P} \right) \Big). \end{aligned} \tag{B.8}$$

By using the equality $q_{H,P} = q_{H,S}Pr\{HH \mid High\} + q_{L,S}Pr\{HL \mid Low\}$, we can write that

$$\begin{aligned} Cov(P,S) &= \left(\mu_{H,P} - \mu_{L,P} \right)\left(\mu_{H,S} - \mu_{L,S} \right) q_{H,S}q_{L,S} \times \\ &\quad \left(Pr\{HH \mid High\} + Pr\{LL \mid Low\} - 1 \right). \end{aligned}$$

Since $\mu_{H,P} - \mu_{L,P} > 0$ and due to Assumption 3, we can conclude that if $Pr\{HH \mid High\} + Pr\{LL \mid Low\} > 1$, then P and S are positively correlated, and if $Pr\{HH \mid High\} + Pr\{LL \mid Low\} < 1$, then P and S are negatively correlated.

B.2 Proof of Propositions 1-6

Following Bitran and Mondschein (1997) and Aydin and Ziya (2008), we use an inductive argument on $k = y + t$. Inequalities hold for $k = 0$. As the induction step, we assume that all inequalities hold for all $t + y < k$, and prove that they hold for $t + y = k$. We also let $\Delta(y,t) = V_{t-1}(y) - V_{t-1}(y-1)$ be the expected benefit from carrying one unit of the primary product into $t-1$ when there are y units of the primary product in the inventory.

B.2.1 Proof of Proposition 1

For $t = 1$ the inequality holds. As the induction step, we assume that it holds for t, and show that it holds for $t+1$ as well. By definition, $V_{t+1}(y)$ can be expressed as follows:

$$\begin{aligned} V_{t+1}(y) &= \sum_{n\in N} \delta_n \Big(\alpha^P_{n,t}\left(p_P + V_t(y-1)\right) + \alpha^S_{n,t}\left(p_S + V_t(y)\right) + \\ & \alpha^B_{n,t}\left(p_P + p_S - d^B_{n,t} + V_t(y-1)\right) + \alpha^{\emptyset}_{n,t} V_t(y) \Big). \end{aligned}$$

By using the induction assumption for t, we obtain

$$\begin{aligned} V_{t+1}(y) &\geq \sum_{n\in N} \delta_n \Big(\alpha^P_{n,t}\left(p_P + V_{t-1}(y-1)\right) + \alpha^S_{n,t}\left(p_S + V_{t-1}(y)\right) + \\ & \alpha^B_{n,t}\left(p_P + p_S - d^B_{n,t} + V_{t-1}(y-1)\right) + \alpha^{\emptyset}_{n,t} V_{t-1}(y) \Big), \end{aligned}$$

or

$$V_{t+1}(y) \geq V_t(y).$$

B.2.2 Proof of Proposition 2

Suppose $y > 0$. For a feasible $\overline{d}^B_{n,t}$ value we have

$$\begin{aligned} V_{t+1}(y) &= \sum_{n\in N} \delta_n \Big(V_t(y) + \alpha^B_{n,t}\left(p_P + p_S - \overline{d}^B_{n,t} - \Delta(y,t+1)\right) + \\ & \alpha^S_{n,t}(p_S) + \alpha^P_{n,t}\left(p_P - \Delta(y,t+1)\right) \Big), \end{aligned}$$

which can be rewritten as

$$V_{t+1}(y) - V_t(y) = \sum_{n\in N}\delta_n\Big(\alpha^B_{n,t}\Big(p_P + p_S - \overline{d}^B_{n,t} - \Delta(y,t+1)\Big) + \alpha^S_{n,t}(p_S) + \alpha^P_{n,t}(p_P - \Delta(y,t+1))\Big). \quad (B.9)$$

Because $\overline{d}^B_{n,t}$ is feasible for $V_{t+1}(y+1)$, we have

$$V_{t+1}(y+1) - V_t(y+1) \geq \sum_{n\in N}\delta_n\Big(\alpha^B_{n,t}\Big(p_P + p_S - \overline{d}^B_{n,t} - \Delta(y+1,t+1)\Big) + \alpha^S_{n,t}(p_S) + \alpha^P_{n,t}(p_P - \Delta(y+1,t+1))\Big). \quad (B.10)$$

In light of Proposition 3, Equation B.10 can be rewritten as

$$V_{t+1}(y+1) - V_t(y+1) \geq \sum_{n\in N}\delta_n\Big(\alpha^B_{n,t}\Big(p_P + p_S - \overline{d}^B_{n,t} - \Delta(y,t+1)\Big) + \alpha^S_{n,t}(p_S) + \alpha^P_{n,t}(p_P - \Delta(y,t+1))\Big). \quad (B.11)$$

From Equations B.9 and B.11, we conclude that

$$V_{t+1}(y+1) - V_{t+1}(y) \geq V_t(y+1) - V_t(y).$$

B.2.3 Proof of Proposition 3

Suppose $y > 0$. For a feasible $\overline{d}^B_{n,t}$ value we have

$$V_t(y+2) = \sum_{n\in N}\delta_n\Big(\alpha^B_{n,t}\Big(p_P + p_S - \overline{d}^B_{n,t} - \Delta(y+2,t)\Big) + \alpha^P_{n,t}(p_P - \Delta(y+2,t)) + \alpha^S_{n,t}(p_S)\} + V_{t-1}(y+2)\Big).$$

By subtracting $V_{t-1}(y+1)$ from both sides, we obtain

$$V_t(y+2) - V_{t-1}(y+1) = \sum_{n\in N}\delta_n\Big(\alpha^B_{n,t}\Big(p_P + p_S - \overline{d}^B_{n,t} - \Delta(y+2,t)\Big) + \alpha^P_{n,t}(p_P - \Delta(y+2,t)) + \alpha^S_{n,t}(p_S) + V_{t-1}(y+2) - V_{t-1}(y+1)\Big),$$

or

$$V_t(y+2) - V_{t-1}(y+1) = \sum_{n\in N}\delta_n\Big(\Delta(y+2,t)\left(1-\alpha^P_{n,t}-\alpha^B_{n,t}\right)+\alpha^P_{n,t}(p_P)+ \alpha^S_{n,t}(p_S)+\alpha^B_{n,t}\left(p_P+p_S-\overline{d}^B_{n,t}\right)\Big). \quad (B.12)$$

Because $\overline{d}^B_{n,t}$ is feasible for $V_{t+1}(y+1)$, we have

$$V_{t+1}(y+1) \geq \sum_{n\in N}\delta_n\Big(\alpha^B_{n,t}\left(p_P+p_S-\overline{d}^B_{n,t}-\Delta(y+1,t+1)\right)+ \alpha^P_{n,t}(p_P-\Delta(y+1,t+1))+\alpha^S_{n,t}(p_S)+V_t(y+1)\Big).$$

By subtracting $V_t(y)$ from both sides, we obtain

$$V_{t+1}(y+1) - V_t(y) \geq \sum_{n\in N}\delta_n\Big(\Delta(y+1,t+1)\left(1-\alpha^P_{n,t}-\alpha^B_{n,t}\right)+\alpha^P_{n,t}(p_P)+ \alpha^S_{n,t}(p_S)+\alpha^B_{n,t}\left(p_P+p_S-\overline{d}^B_{n,t}\right)\Big). \quad (B.13)$$

By using the induction assumptions of Propositions 2 and 3, we have $V_t(y+1) - V_t(y) \geq V_{t-1}(y+1) - V_{t-1}(y) \geq V_{t-1}(y+2) - V_{t-1}(y+1)$. Using this inequality along with Equation B.13, we obtain

$$V_{t+1}(y+1) - V_t(y) \geq \sum_{n\in N}\delta_n\Big(\Delta(y+2,t)\left(1-\alpha^P_{n,t}-\alpha^B_{n,t}\right)+\alpha^P_{n,t}(p_P)+ \alpha^S_{n,t}(p_S)+\alpha^B_{n,t}\left(p_P+p_S-\overline{d}^B_{n,t}\right)\Big). \quad (B.14)$$

From Equation B.12 and Inequality B.14, we have

$$V_{t+1}(y+1) - V_t(y) \geq V_t(y+2) - V_{t-1}(y+1). \quad (B.15)$$

By using the induction assumption of Proposition 4, we now have

$$2V_t(y+1) \geq V_{t+1}(y+1) + V_{t-1}(y+1). \quad (B.16)$$

By adding up and rearranging Inequalities B.15 and B.16, we conclude that

$$V_t(y+1) - V_t(y) \geq V_t(y+2) - V_t(y+1).$$

B.2.4 Proof of Proposition 4

Suppose $y > 0$. For a feasible $\overline{d}^B_{n,t}$ value we have

$$\begin{aligned} V_{t+2}(y) &= \sum_{n\in N} \delta_n \Big(V_{t+1}(y) + \alpha^B_{n,t} \left(p_P + p_S - \overline{d}^B_{n,t} - \Delta(y, t+2) \right) + \\ & \alpha^P_{n,t} \left(p_P - \Delta(y, t+2) \right) + \alpha^S_{n,t} \left(p_S \right) \Big), \end{aligned}$$

which can be rewritten as

$$\begin{aligned} V_{t+2}(y) - V_{t+1}(y) &= \sum_{n\in N} \delta_n \Big(\alpha^B_{n,t} \left(p_P + p_S - \overline{d}^B_{n,t} - \Delta(y, t+2) \right) + \\ & \alpha^P_{n,t} \left(p_P - \Delta(y, t+2) \right) + \alpha^S_{n,t} \left(p_S \right) \Big). \end{aligned} \tag{B.17}$$

Because $\overline{d}^B_{n,t}$ is feasible for $V_{t+1}(y)$, we have

$$\begin{aligned} V_{t+1}(y) - V_t(y) &\geq \sum_{n\in N} \delta_n \Big(\alpha^B_{n,t} \left(p_P + p_S - \overline{d}^B_{n,t} - \Delta(y, t+1) \right) + \\ & \alpha^P_{n,t} \left(p_P - \Delta(y, t+1) + \right) \alpha^S_{n,t} \left(p_S \right) \Big). \end{aligned} \tag{B.18}$$

By using the induction assumption of Proposition 2, Inequality B.18 can be written as

$$\begin{aligned} V_{t+1}(y) - V_t(y) &\geq \sum_{n\in N} \delta_n \Big(\alpha^B_{n,t} \left(p_P + p_S - \overline{d}^B_{n,t} - \Delta(y, t+2) \right) + \\ & \alpha^P_{n,t} \left(p_P - \Delta(y, t+2) \right) + \alpha^S_{n,t} \left(p_S \right) \Big). \end{aligned} \tag{B.19}$$

From Equation B.17 and Inequality B.19, we conclude that

$$V_{t+1}(y) - V_t(y) \geq V_{t+2}(y) - V_{t+1}(y).$$

B.2.5 Proof of Propositions 5 and 6

For each segment n, $n \in N$, we first define the function $\Pi_s\left(d^B_{n,t}, \Delta(y,t)\right)$ that denotes the expected marginal revenue obtained from one unit of the primary product when the bundle discount is $d^B_{n,t}$, and the expected benefit from carrying one unit of the primary product into $t-1$ when there are y units of the primary product in the inventory is $\Delta(y,t)$:

$$\Pi_s\left(d^B_{n,t}, \Delta(y,t)\right) = \alpha^B_n\left(p_P + p_S - d^B_{n,t} - \Delta(y,t)\right) + \alpha^P_n\left(p_P - \Delta(y,t)\right) + \alpha^S_n\left(p_S\right).$$

We also let $d^B_{n,t}\left(\Delta(y,t)\right)$, $n \in N$, be the bundle discount that maximizes the expected marginal revenue obtained from one unit of the primary product:

$$d^B_{n,t}\left(\Delta(y,t)\right) = inf\left\{d^B_{n,t} : \Pi_s\left(d^B_{n,t}, \Delta(y,t)\right) \geq \Pi_s\left(d^B_{n,t}, \Delta(y,t)\right), \forall d^B_{n,t}\right\}.$$

Let C_s denote, for each segment $n \in N$, the set of feasible bundle discount and marginal value tuples:

$$C_s := \left\{\left(d^B_{n,t}, \Delta(y,t)\right) : \Delta(y,t) > 0, p_P + p_S - \Delta(y,t) > d^B_{n,t} \geq 0\right\}.$$

We will prove that $d^B_{n,t}\left(\Delta(y,t)\right)$ is decreasing in $\Delta(y,t)$. By Theorem 8.1 on page 124 of Porteus (2002), it is sufficient to show that $\Pi_s\left(d^B_{n,t}, \Delta(y,t)\right)$ is submodular on C_s. Therefore, we need to prove that the following inequality is true for any $x = (x_1, x_2) \in C_s$ and $z = (z_1, z_2) \in C_s$:

$$\Pi_s(x \wedge z) + \Pi_s(x \vee z) \leq \Pi_s(x) + \Pi_s(z), \qquad \text{(B.20)}$$

where $x \wedge z = (min(x_1, z_1), min(x_2, z_2))$ and $x \vee z = (max(x_1, z_1), max(x_2, z_2))$. To show the desired result, we need to consider two cases:

1. $x_1 \geq z_1$ and $x_2 \geq z_2$

   Since $\Pi_s(x \wedge z) = \Pi_s(z)$ and $\Pi_s(x \vee z) = \Pi_s(x)$, Inequality (B.20) holds.

$x_1 \geq z_1$ and $x_2 < z_2$

By using the definition of $\Pi_s\left(d^B_{n,t}, \Delta\right)$ and after re-arranging its terms, Inequality (B.20) can be simplified as

$$(z_2 - x_2)\left(\alpha^B_n(z_1) + \alpha^P_n(z_1)\right) + (x_2 - z_2)\left(\alpha^B_n(x_1) + \alpha^P_n(x_1)\right) \leq 0. \qquad \text{(B.21)}$$

We first note that

$$\begin{aligned}
\alpha_n^B\left(d_{n,t}^B\right) &= Pr\{R_{n,S}+R_{n,P} > p_P+p_S-d_{n,t}^B, \\
&\quad R_{n,S}+R_{n,P}-(p_P+p_S-d_{n,t}^B) > R_{n,P}-p_P, \\
&\quad R_{n,S}+R_{n,P}-(p_P+p_S-d_{n,t}^B) > R_{n,S}-p_S\}, \\
&= Pr\{R_{n,S} > p_S-d_{n,t}^B, R_{n,P} > max\left(p_p-d_{n,t}^B, p_p-d_{n,t}^B+p_S-R_{n,S}\right)\}, \\
&= Pr\{R_{n,P} > max\left(p_p-d_{n,t}^B, p_p-d_{n,t}^B+p_S-R_{n,S}\right) \mid R_{n,S} > p_S-d_{n,t}^B\} \times \\
&\quad Pr\left\{R_{n,S} > p_S-d_{n,t}^B\right\},
\end{aligned}$$

and

$$\begin{aligned}
\alpha_n^P\left(d_{n,t}^B\right) &= Pr\{R_{n,P} \geq p_P, R_{n,S} \leq p_S-d_{n,t}^B\}, \\
&= Pr\left\{R_{n,P} \geq p_P\right\} \times \left(1-Pr\left\{R_{n,S} > p_S-d_{n,t}^B\right\}\right), \\
&= Pr\left\{R_{n,P} \geq p_P\right\} - Pr\left\{R_{n,P} \geq p_P\right\} \times Pr\left\{R_{n,S} > p_S-d_{n,t}^B\right\}.
\end{aligned}$$

$\alpha_n^B\left(d_{n,t}^B\right)$ is increasing, and $\alpha_n^P\left(d_{n,t}^B\right)$ is decreasing in $d_{n,t}^B$. However, $\alpha_n^B\left(d_{n,t}^B\right)+\alpha_n^P\left(d_{n,t}^B\right)$ is increasing in $d_{n,t}^B$. Because $x_1 \geq z_1$, $x_2 < z_2$, it follows that $\alpha_n^B(x_1)+\alpha_n^P(x_1) \geq \alpha_n^B(z_1)+\alpha_n^P(z_1)$, and, therefore, Inequality (B.21) holds.

We have shown that $\Pi_s\left(d_{n,t}^B, \Delta(y,t)\right)$ is submodular on C_s and $d_{n,t}^B(\Delta(y,t))$ is decreasing in $\Delta(y,t)$. Note that $\Delta(y,t)$ is increasing in t and decreasing in y (by Propositions 2 and 3). Therefore, we conclude that $d_{n,t}^{B*}(\Delta(y,t))$ is decreasing in t and increasing in y.

Appendix C

SUPPLEMENTARY MATERIAL FOR: INTERTEMPORAL BUNDLE PRICING AND CONSUMER STOCKPILING

C.1 Proof of Theorem 1

To prove Theorem 1, we first need to characterize the structure of the consumer's value function, $V(I_{n,1}, I_{n,2}, p_B)$, with respect to $I_{n,1}$ and $I_{n,2}$. Let $v(I_{n,1}, I_{n,2}, p_B) = \int_0^\infty V(I_{n,1}, I_{n,2}, x)\,\partial F(x \mid p_B)$ be the expected value of $V\left(I_{n,1}, I_{n,2}, p_B^{+1}\right)$ given the current price p_B. Also, let $w(I_{n,1}, I_{n,2}, p_B) = -h_1(I_{n,1}) - h_2(I_{n,2}) + \beta v(I_{n,1}, I_{n,2}, p_B)$. The consumer problem can be written as follows:

$$V(I_{n,1}, I_{n,2}, p_B) = \max_{\substack{Q_{n,j}\geq 0\\ Q_{n,B}\geq 0\\ C_{n,j}\geq 0}} \begin{pmatrix} U(C_{n,1}, C_{n,2}, \theta) - (p_1 Q_{n,1} + p_2 Q_{n,2} + p_B Q_{n,B}) \\ +w(I_{n,1}^{+1}, I_{n,2}^{+1}, p_B) \end{pmatrix}, \tag{C.1}$$

where $I_{n,j}^{+1} = I_{n,j} + Q_{n,B} + Q_{n,j} - C_{n,j}\ \forall j \in J$ and $\forall n \in N$.

The following lemma characterizes the structure of consumer's value function, $V(I_{n,1}, I_{n,2}, p_B)$, with respect to $I_{n,1}$ and $I_{n,2}$.

Lemma 1. *If $U(C_{n,1}, C_{n,2}, \theta)$ is a jointly concave function in $C_{n,1}$ and $C_{n,2}$, and h_j is a convex function in $I_{n,j}$, then $V(I_{n,1}, I_{n,2}, p_B)$, $v(I_{n,1}, I_{n,2}, p_B)$ and $w(I_{n,1}, I_{n,2}, p_B)$ are concave in $I_{n,1}$ and $I_{n,2}$ for every p_B and n.*

Proof. We use inductive argument on $V\left(I_{n,1}, I_{n,2}, p_B^{+1}\right)$. Suppose that $V\left(I_{n,1}, I_{n,2}, p_B^{+1}\right)$ is a jointly concave function in $I_{n,1}$ and $I_{n,2}$ for every p_B and n.

$$J(I_{n,1}, I_{n,2}, p_B) = \begin{pmatrix} U(C_{n,1}, C_{n,2}, \theta) - (p_1 Q_{n,1} + p_2 Q_{n,2} + p_B Q_{n,B}) \\ -\sum_{j=1}^{2} h_j(I_{n,j} + Q_{n,j} + Q_{n,B} - C_{n,j}) \\ +\beta v(I_{n,1} + Q_{n,1} + Q_{n,B} - C_{n,1}, I_{n,2} + Q_{n,2} + Q_{n,B} - C_{n,2}, p_B) \end{pmatrix},$$

where $I_{n,j} + Q_{n,j} + Q_{n,B} - C_{n,j} \geq 0 \ \forall j \in J$ and $\forall n \in N$.

Then, the function is jointly concave for every values of p_B and n. This is because the sum of concave functions is concave and the Hessian matrix is as follows:

$$\mathbf{H}\left(J(I_{n,1}, I_{n,2}, p_B)\right) = \begin{bmatrix} \frac{\partial^2 J(I_{n,1}, I_{n,2}, p_B)}{\partial^2 I_{n,1}} \leq 0 & \frac{\partial^2 J(I_{n,1}, I_{n,2}, p_B)}{\partial I_{n,1}\partial I_{n,2}} = 0 \\ \frac{\partial^2 J(I_{n,1}, I_{n,2}, p_B)}{\partial I_{n,2}\partial I_{n,1}} = 0 & \frac{\partial^2 J(I_{n,1}, I_{n,2}, p_B)}{\partial^2 I_{n,2}} \leq 0 \end{bmatrix}$$

Following the principal minor method, we can conclude that the Hessian matrix is negative semi-definite. To complete the induction argument, assume that the number of periods is finite, and let $V_1(I_{n,1}, I_{n,2}, p_B) = U\left(C_{n,1}, C_{n,2}, \theta\right) - \left(p_1 Q_{n,1} + p_2 Q_{n,2} + p_B Q_{n,B}\right)$ the one-period-to-go function and $C_{n,j} = I_{n,j} + Q_{n,j} + Q_{n,B}$, $\forall j \in J$. Then,

$$\begin{aligned} V_1(I_{n,1}, I_{n,2}, p_B) &= U\left(I_{n,1} + Q_{n,1} + Q_{n,B}, I_{n,2} + Q_{n,2} + Q_{n,B}, \theta\right) \\ &- \left(p_1 Q_{n,1} + p_2 Q_{n,2} + p_B Q_{n,B}\right). \end{aligned}$$

$U\left(C_{n,1}, C_{n,2}, \theta\right)$ is a jointly concave function in $C_{n,1}$ and $C_{n,2}$. Therefore, $V_1(I_{n,1}, I_{n,2}, p_B)$ is a jointly concave function in $I_{n,1}$ and $I_{n,2}$. Using the induction argument as $t \to \infty$ completes the concavity proof of $V(I_{n,1}, I_{n,2}, p_B)$ in $I_{n,1}$ and $I_{n,2}$. Concavity of $v\left(I_{n,1}, I_{n,2}, p_B\right)$ and $w\left(I_{n,1}, I_{n,2}, p_B\right)$ follows because sum of concave function is itself concave. $\square$

Lemma 1 indicates that there exists an optimal inventory level for each product j, denoted as $I^*_{n,j}\left(p_j, p_B\right) \forall j \in J$, that maximizes the segment-n customer's total surplus for each level of p_1, p_2 and p_B. Therefore, when the retailer announces the bundle discount at a price p_B, the segment-n customer makes purchase decision based on the on-hand inventory levels of the products. We first derive the structure of optimal purchasing policy of the segment-n customer. Then, we present the analysis for the optimal consumption policy of the segment-n customer.

C.1.1 Proof of Optimal Purchasing Policy

Let $I_{n,1}$ and $I_{n,2}$ denote the current inventory levels (after consumption, but before purchase) for products 1 and 2, respectively. We investigate a segment-n customer's purchasing policy in four different cases based on the inventory levels of the products. Four different cases are: i) $I_{n,1} \geq I^*_{n,1}(p_1, p_B)$ and $I_{n,2} \geq I^*_{n,2}(p_2, p_B)$, ii) $I_{n,1} \geq I^*_{n,1}(p_1, p_B)$ and $I_{n,2} < I^*_{n,2}(p_2, p_B)$, iii) $I_{n,1} < I^*_{n,1}(p_1, p_B)$ and $I_{n,2} \geq I^*_{n,2}(p_2, p_B)$ and iv) $I_{n,1} < I^*_{n,1}(p_1, p_B)$ and $I_{n,2} < I^*_{n,2}(p_2, p_B)$.

Case i) is trivial. If the on-hand quantities of the products exceed the optimal inventory levels, driven by the current p_B along with the p_1 and p_2, then the customer does not make any purchase. In case ii), the customer's on-hand quantities are larger than the optimal level for the product 1 but less than the product 2. Therefore, the customer makes a purchase only for the product 2. Similarly, in case iii), the customer purchases only the product 1. In case iv), customer's on-hand inventory levels are less than the optimal quantities for both products, and the customer needs to make a purchase for both products. When the retailer announces a bundle discount, she first tries to fulfill the need of both products through the bundle offer at a discounted price, p_B. If either of the product's inventory level still remains below the optimal quantity then the customer makes an individual purchase. Let $Q^*_{n,B}(I_{n,1}, I_{n,2}, p_B)$ denote the optimal bundle quantity that the segment-n customer purchases at a price p_B. Similarly, let $Q^*_{n,j}(I_{n,j}, p_{n,j}, p_B)$ be the optimal purchase quantity of the product j that the segment-n customer buys at a price p_j. Thus, the structure of optimal purchasing policy for product j, $j \in \{1, 2\}$, and for the bundle offer is as follows:

$$\begin{aligned} Q^*_{n,B}(I_{n,1}, I_{n,2}, p_B) &= max\left(min\left(I^*_{n,1}(p_1, p_B) - I_{n,1}, I^*_{n,2}(p_2, p_B) - I_{n,2}\right), 0\right), \quad \text{(C.2)} \\ Q^*_{n,j}(I_{n,j}, p_{n,j}, p_B) &= max\left(I^*_{n,j}(p_j, p_B) - \left(I_{n,j} + Q^*_{n,B}(I_{n,1}, I_{n,2}, p_B)\right), 0\right) \forall j \in \{1, 2\} \end{aligned}$$

C.1.2 Proof of Optimal Consumption Policy

Lagrangian formulation of the segment-n problem presented in Equation C.1 is as follows:

$$L(I_{n,1}, I_{n,2}, p_B, \lambda, \mu, \eta) = \begin{pmatrix} V(I_{n,1}, I_{n,2}, p_B) \\ +\sum_{j=1}^{2} \lambda_j (I_{n,j} + Q_{n,j} + Q_{n,B} - C_{n,j}) \\ +\sum_{j=1}^{2} \mu_j Q_{n,j} + \mu_B Q_{n,B} + \sum_{j=1}^{2} \eta_j C_{n,j} \end{pmatrix}, \quad \text{(C.3)}$$

where λ_j, μ_j, and μ_B are the Lagrange multipliers associated with the constraints $I_{n,j} + Q_{n,j} + Q_{n,B} - C_{n,j} \geq 0$, $Q_{n,j} \geq 0$ and $Q_{n,B} \geq 0$, respectively.

The three first-order conditions of the $L(I_{n,1}, I_{n,2}, p_B, \lambda, \mu, \eta)$ are:

$$\begin{aligned} \frac{\partial L(I_{n,1}, I_{n,2}, p_B, \lambda, \mu, \eta)}{\partial C_{n,j}} &= \frac{\partial U\left(C_{n,1}, C_{n,2}, \theta\right)}{\partial C_{n,j}} \\ &- \frac{\partial w\left(I_{n,1} + Q_{n,1} + Q_{n,B} - C_{n,1}, I_{n,2} + Q_{n,2} + Q_{n,B} - C_{n,2}\right)}{\partial C_{n,j}} \\ &- \lambda_j = 0, \forall j \in J. \end{aligned} \tag{C.4}$$

$$\begin{aligned} \frac{\partial L(I_{n,1}, I_{n,2}, p_B, \lambda, \mu, \eta)}{\partial Q_{n,B}} &= -p_B \\ &+ \frac{\partial w\left(I_{n,1} + Q_{n,1} + Q_{n,B} - C_{n,1}, I_{n,2} + Q_{n,2} + Q_{n,B} - C_{n,2}\right)}{\partial Q_{n,B}} \\ &+ \lambda_1 + \lambda_2 + \mu_B = 0, \forall j \in J. \end{aligned} \tag{C.5}$$

$$\begin{aligned} \frac{\partial L(I_{n,1}, I_{n,2}, p_B, \lambda, \mu, \eta)}{\partial Q_{n,j}} &= -p_j \\ &+ \frac{\partial w\left(I_{n,1} + Q_{n,1} + Q_{n,B} - C_{n,1}, I_{n,2} + Q_{n,2} + Q_{n,B} - C_{n,2}\right)}{\partial Q_{n,j}} \\ &+ \lambda_j + \mu_j = 0, \forall j \in J. \end{aligned} \tag{C.6}$$

In the analysis, we do not consider the Lagrange multiplier $\eta_j > 0$, $\forall j \in \{1, 2\}$ because consumers receives utility from consuming the products, and when $C_{n,j} > 0$, η_j is equal to 0 for product j.

As we have presented in Section C.1.1, the consumer purchases either only bundle of products or one of the individual products along with the bundle of products. Therefore, for purchasing policies, we concentrate on cases where $(\mu_B = 0, \mu_1 > 0, \mu_2 > 0)$, $(\mu_B = 0, \mu_1 = 0, \mu_2 > 0)$ and $(\mu_B = 0, \mu_1 > 0, \mu_2 = 0)$. For consumption related Lagrange multipliers λ_j, $\forall j \in J$. Either λ_j or $I_{n,j} + Q_{n,j} + Q_{n,B} - C_{n,j}$ equal to zero for product j, $\forall j \in J$. We need to investigate different cases of λ_j for each purchasing scenario. To keep the exposition simple, we present two cases in terms of purchasing policy: $(\mu_B = 0, \mu_1 > 0, \mu_2 > 0)$ and $(\mu_B = 0, \mu_1 = 0, \mu_2 > 0)$. For each purchasing strategy, four different consumption cases can occur: i) $\mu_B = 0$, $\mu_1 > 0$, $\mu_2 > 0$, $\lambda_1 = 0$

and $\lambda_2 = 0$, ii) $\mu_B = 0$, $\mu_1 > 0$, $\mu_2 > 0$, $\lambda_1 > 0$ and $\lambda_2 = 0$, iii) $\mu_B = 0$, $\mu_1 > 0$, $\mu_2 > 0$, $\lambda_1 = 0$ and $\lambda_2 > 0$, iv) $\mu_B = 0$, $\mu_1 > 0$, $\mu_2 > 0$, $\lambda_1 > 0$ and $\lambda_2 > 0$.

Case i). $(C_{n,j} < I_{n,j} + Q_{n,B}, \forall j \in J)$. Summation of first-order conditions (C.4) and (C.5) yields $\frac{\partial U(C_{n,1},C_{n,2},\theta)}{\partial C_{n,1}} + \frac{\partial U(C_{n,1},C_{n,2},\theta)}{\partial C_{n,2}} = p_B$, and $c_j\left(I_{n,j}, p_j, p_B, \theta\right)$ is the solution of the summation of (C.4) and (C.5).

Case ii). $(C_{n,1} = I_{n,1} + Q_{n,B})$ and $(C_{n,2} < I_{n,2} + Q_{n,B})$. Summation of first-order conditions (C.4) and (C.5) yields $\frac{\partial U(C_{n,1},C_{n,2},\theta)}{\partial C_{n,1}} + \frac{\partial U(C_{n,1},C_{n,2},\theta)}{\partial C_{n,2}} = p_B$, and $c_j\left(I_{n,j}, p_j, p_B, \theta\right)$ is the solution of the summation of (C.4) and (C.5) where $c_1\left(I_{n,1}, p_j, p_B, \theta\right) = I_{n,1} + Q_{n,B}$.

Case iii). $(C_{n,1} < I_{n,1} + Q_{n,B})$ and $(C_{n,2} = I_{n,2} + Q_{n,B})$. Summation of first-order conditions (C.4) and (C.5) yields $\frac{\partial U(C_{n,1},C_{n,2},\theta)}{\partial C_{n,1}} + \frac{\partial U(C_{n,1},C_{n,2},\theta)}{\partial C_{n,2}} = p_B$, and $c_j\left(I_{n,j}, p_j, p_B, \theta\right)$ is the solution of the summation of (C.4) and (C.5) where $c_2\left(I_{n,2}, p_j, p_B, \theta\right) = I_{n,2} + Q_{n,B}$.

Case iv). $(C_{n,1} = I_{n,1} + Q_{n,B})$ and $(C_{n,2} = I_{n,2} + Q_{n,B})$. Summation of first-order conditions (C.4) and (C.5) yields $\frac{\partial U(C_{n,1},C_{n,2},\theta)}{\partial C_{n,1}} + \frac{\partial U(C_{n,1},C_{n,2},\theta)}{\partial C_{n,2}} = p_B$, and $c_j\left(I_{n,j}, p_j, p_B, \theta\right)$ is the solution of the summation of (C.4) and (C.5) where $c_j\left(I_{n,j}, p_j, p_B, \theta\right) = I_{n,2} + Q_{n,B}$ and $Q_{n,B} = I_{n,1} - C_{n,1} = I_{n,2} - C_{n,2}$. This case corresponds to a final inventory of zero for both products.

When the customer's purchasing policy is to buy one of the individual products along with the bundle of products, $(\mu_B = 0, \mu_1 = 0, \mu_2 > 0)$ or $(\mu_B = 0, \mu_1 > 0, \mu_2 = 0)$, we obtain the same results for all four cases. In summary, when a customer purchases products either at a bundle price or at a regular price, her consumption policy is driven by on-hand inventory levels of products, current prices of products and the bundle offer, and the degree of substitutability among products in the bundle offer. The impact of the degree of substitutability among products on consumption levels is explicit; that is, $\frac{\partial U(C_{n,1},C_{n,2},\theta)}{\partial \theta} = C_{n,1} \times C_{n,2}$. Therefore, as products get complement, total utility received from consuming the products together increases.

C.1.3 Proof of Proposition 8

When the existing inventory levels of the products are greater than the optimal quantities, the segment-n customer does not make a purchase. This situation corresponds to case where $I_{n,1} \geq I^*_{n,1}(p_1, p_B)$ and $I_{n,2} \geq I^*_{n,2}(p_2, p_B)$. In this case, $C^*_{n,j}(I_{n,j}, p_j, p_B, \theta)$ is the solution of the first-order condition presented in (C.4). To derive the behavior of $C^*_{n,j}(I_{n,j}, p_j, p_B, \theta)$ with respect to $I_{n,j}$, we first need to find partial derivatives of $\frac{\partial^2 L(I_{n,1}, I_{n,2}, p_B, \lambda, \mu, \eta)}{\partial^2 C_{n,j}}$ and $\frac{\partial^2 L(I_{n,1}, I_{n,2}, p_B, \lambda, \mu, \eta)}{\partial C_{n,j} \partial I_{n,j}}$. Then, we use the Implicit Function Theorem to find the slope of partial derivatives. Partial derivatives are as follows:

$$\begin{aligned} \frac{\partial^2 L(I_{n,1}, I_{n,2}, p_B, \lambda, \mu, \eta)}{\partial^2 C_{n,j}} &= \frac{\partial^2 U(C_{n,1}, C_{n,2}, \theta)}{\partial^2 C_{n,j}} \\ &+ \frac{\partial^2 w(I_{n,1} + Q_{n,1} + Q_{n,B} - C_{n,1}, I_{n,2} + Q_{n,2} + Q_{n,B} - C_{n,2})}{\partial^2 C_{n,j}} \\ \frac{\partial^2 L(I_{n,1}, I_{n,2}, p_B, \lambda, \mu, \eta)}{\partial C_{n,j} \partial I_{n,j}} &= \\ &- \frac{\partial^2 w(I_{n,1} + Q_{n,1} + Q_{n,B} - C_{n,1}, I_{n,2} + Q_{n,2} + Q_{n,B} - C_{n,2})}{\partial C_{n,j} \partial I_{n,j}} \end{aligned} \tag{C.7}$$

By using the Implicit Function Theorem, the slope of partial derivatives can be written as:

$$\begin{aligned} -\frac{\frac{\partial^2 L(I_{n,1}, I_{n,2}, p_B, \lambda, \mu, \eta)}{\partial C_{n,j} \partial I_{n,j}}}{\frac{\partial^2 L(I_{n,1}, I_{n,2}, p_B, \lambda, \mu, \eta)}{\partial^2 C_{n,j}}} &= -\frac{\partial C_{n,j}}{\partial I_{n,j}} \\ &= \frac{\frac{\partial^2 w\left(I_{n,1}+Q_{n,1}+Q_{n,B}-C_{n,1}, I_{n,2}+Q_{n,2}+Q_{n,B}-C_{n,2}\right)}{\partial C_{n,j} \partial I_{n,j}}}{\frac{\partial^2 U(C_{n,1}, C_{n,2}, \theta)}{\partial^2 C_{n,j}} + \frac{\partial^2 w\left(I_{n,1}+Q_{n,1}+Q_{n,B}-C_{n,1}, I_{n,2}+Q_{n,2}+Q_{n,B}-C_{n,2}\right)}{\partial^2 C_{n,j}}} \geq 0 \end{aligned} \tag{C.8}$$

Because $U(C_{n,1}, C_{n,2}, \theta)$ and $w(I_{n,1} + Q_{n,1} + Q_{n,B} - C_{n,1}, I_{n,2} + Q_{n,2} + Q_{n,B} - C_{n,2})$ are concave, we conclude that $C^*_{n,j}(I_{n,j}, p_j, p_B, \theta)$ is non-decreasing in $I_{n,j}$ for product j.

C.1.4 Proof of Proposition 9

To analyze the behavior of $I^*_{n,j}(p_j, p_B)$ with respect to h_j and β, It is sufficient to investigate the behavior of $Q_{n,B}$ with respect tot h_j and β, because $Q_{n,B}$ determines the level of $I^*_{n,j}(p_j, p_B)$. Therefore, we first calculate the partial derivatives of $\frac{\partial^2 L(I_{n,1},I_{n,2},p_B,\lambda,\mu,\eta)}{\partial^2 Q_{n,B}}$, $\frac{\partial^2 L(I_{n,1},I_{n,2},p_B,\lambda,\mu,\eta)}{\partial Q_{n,B}\partial\beta}$ and $\frac{\partial^2 L(I_{n,1},I_{n,2},p_B,\lambda,\mu,\eta)}{\partial Q_{n,B}\partial h_j}$. Partial derivatives are as follows:

$$\begin{aligned}
\frac{\partial^2 L(I_{n,1},I_{n,2},p_B,\lambda,\mu,\eta)}{\partial^2 Q_{n,B}} &= 2\frac{\partial^2 w\left(I_{n,1}+Q_{n,1}+Q_{n,B}-C_{n,1}, I_{n,2}+Q_{n,2}+Q_{n,B}-C_{n,2}\right)}{\partial^2 Q_{n,B}} \\
\frac{\partial^2 L(I_{n,1},I_{n,2},p_B,\lambda,\mu,\eta)}{\partial Q_{n,B}\partial\beta} &= 2\frac{\partial v\left(I_{n,1}+Q_{n,1}+Q_{n,B}-C_{n,1}, I_{n,2}+Q_{n,2}+Q_{n,B}-C_{n,2}, p_B\right)}{\partial Q_{n,B}\partial\beta} \\
\frac{\partial^2 L(I_{n,1},I_{n,2},p_B,\lambda,\mu,\eta)}{\partial Q_{n,B}\partial h_j} &= -1 \quad \forall j \in J
\end{aligned} \tag{C.9}$$

By following the Implicit Function Theorem, the slope of partial derivatives are as follows:

$$\begin{aligned}
-\frac{\frac{\partial^2 L(I_{n,1},I_{n,2},p_B,\lambda,\mu,\eta)}{\partial Q_{n,B}\partial\beta}}{\frac{\partial^2 L(I_{n,1},I_{n,2},p_B,\lambda,\mu,\eta)}{\partial^2 Q_{n,B}}} &= -\frac{\partial Q_{n,B}}{\partial\beta} \\
&= -\frac{\frac{\partial v\left(I_{n,1}+Q_{n,1}+Q_{n,B}-C_{n,1},I_{n,2}+Q_{n,2}+Q_{n,B}-C_{n,2},p_B\right)}{\partial Q_{n,B}\partial\beta}}{\frac{\partial^2 w\left(I_{n,1}+Q_{n,1}+Q_{n,B}-C_{n,1},I_{n,2}+Q_{n,2}+Q_{n,B}-C_{n,2}\right)}{\partial^2 Q_{n,B}}} \geq 0
\end{aligned} \tag{C.10}$$

$\frac{\partial v}{\partial Q_{n,B}\partial\beta}$ is positive because $\frac{\partial w}{\partial Q_{n,B}} = p_B$ when $Q_{n,B} > 0$ from the first-order condition presented in (C.5) and $\frac{\partial v}{\partial Q_{n,B}\partial\beta} \geq \frac{\partial w}{\partial Q_{n,B}} = p_B \geq 0$. Also, $\frac{\partial^2 w}{\partial^2 Q_{n,B}} < 0$. Therefore, the optimal inventory levels, $I^*_{n,1}$ and $I^*_{n,2}$, are non-decreasing in β.

$$\begin{aligned}
-\frac{\frac{\partial^2 L(I_{n,1},I_{n,2},p_B,\lambda,\mu,\eta)}{\partial Q_{n,B}\partial h_j}}{\frac{\partial^2 L(I_{n,1},I_{n,2},p_B,\lambda,\mu,\eta)}{\partial^2 Q_{n,B}}} &= -\frac{\partial Q_{n,B}}{\partial h_j} \\
&= -\frac{(-1)}{\frac{\partial^2 w\left(I_{n,1}+Q_{n,1}+Q_{n,B}-C_{n,1},I_{n,2}+Q_{n,2}+Q_{n,B}-C_{n,2}\right)}{\partial^2 Q_{n,B}}} \leq 0
\end{aligned} \tag{C.11}$$

$\frac{\partial^2 w}{\partial^2 Q_{n,B}} < 0$. Therefore, the optimal inventory levels, $I^*_{n,1}$ and $I^*_{n,2}$, are non-increasing in h_1 and h_2, respectively.

C.1.5 Proof of Proposition 10

To analyze the behavior of $C^*_{n,j}(I_{n,j}, p_j, p_B)$ with respect to h_j and β, we first calculate the partial derivatives of $\frac{\partial^2 L(I_{n,1},I_{n,2},p_B,\lambda,\mu,\eta)}{\partial^2 C_{n,j}}$, $\frac{\partial^2 L(I_{n,1},I_{n,2},p_B,\lambda,\mu,\eta)}{\partial C_{n,j}\partial\beta}$ and $\frac{\partial^2 L(I_{n,1},I_{n,2},p_B,\lambda,\mu,\eta)}{\partial C_{n,j}\partial h_j}$.

Partial derivatives are as follows:

$$\frac{\partial^2 L(I_{n,1}, I_{n,2}, p_B, \lambda, \mu, \eta)}{\partial^2 C_{n,j}} = \frac{\partial^2 U\left(C_{n,1}, C_{n,2}, \theta\right)}{\partial^2 C_{n,j}} + \frac{\partial^2 w\left(I_{n,1}+Q_{n,1}+Q_{n,B}-C_{n,1}, I_{n,2}+Q_{n,2}+Q_{n,B}-C_{n,2}\right)}{\partial^2 C_{n,j}},$$

$$\frac{\partial^2 L(I_{n,1}, I_{n,2}, p_B, \lambda, \mu, \eta)}{\partial C_{n,j}\partial\beta} = - \frac{\partial v\left(I_{n,1}+Q_{n,1}+Q_{n,B}-C_{n,1}, I_{n,2}+Q_{n,2}+Q_{n,B}-C_{n,2}, p_B\right)}{\partial C_{n,j}\partial\beta}$$

$$\frac{\partial^2 L(I_{n,1}, I_{n,2}, p_B, \lambda, \mu, \eta)}{\partial C_{n,j}\partial h_j} = 1 \quad \forall j \in J \tag{C.12}$$

By using the Implicit Function Theorem, the slope of partial derivatives can be written as:

$$-\frac{\frac{\partial^2 L(I_{n,1}, I_{n,2}, p_B, \lambda, \mu, \eta)}{\partial C_{n,j}\partial\beta}}{\frac{\partial^2 L(I_{n,1}, I_{n,2}, p_B, \lambda, \mu, \eta)}{\partial^2 C_{n,j}}} = -\frac{\partial C_{n,j}}{\partial\beta} \tag{C.13}$$

$$= -\frac{-\frac{\partial v\left(I_{n,1}+Q_{n,1}+Q_{n,B}-C_{n,1}, I_{n,2}+Q_{n,2}+Q_{n,B}-C_{n,2}, p_B\right)}{\partial C_{n,j}\partial\beta}}{\frac{\partial^2 U(C_{n,1}, C_{n,2}, \theta)+\partial^2 w\left(I_{n,1}+Q_{n,1}+Q_{n,B}-C_{n,1}, I_{n,2}+Q_{n,2}+Q_{n,B}-C_{n,2}\right)}{\partial^2 C_{n,j}}} \leq 0$$

$\frac{\partial v}{\partial C_{n,j}\partial\beta} > 0$ and $w\left(I_{n,1}+Q_{n,1}+Q_{n,B}-C_{n,1}, I_{n,2}+Q_{n,2}+Q_{n,B}-C_{n,2}\right)$ is concave in inventory levels of $I_{n,1}^{+1}$ and $I_{n,2}^{+1}$ as we discuss in Lemma 1, where $I_{n,j}^{+1} = I_{n,j} + Q_{n,j} + Q_{n,B} - C_{n,j}$ for product j. Therefore, the optimal consumption levels, $C_{n,1}^*$ and $C_{n,2}^*$, are non-increasing in β.

$$-\frac{\frac{\partial^2 L(I_{n,1}, I_{n,2}, p_B, \lambda, \mu, \eta)}{\partial C_{n,j}\partial h_j}}{\frac{\partial^2 L(I_{n,1}, I_{n,2}, p_B, \lambda, \mu, \eta)}{\partial^2 C_{n,j}}} = -\frac{\partial C_{n,j}}{\partial h_j} \tag{C.14}$$

$$= -\frac{1}{\frac{\partial^2 U(C_{n,1}, C_{n,2}, \theta)+\partial^2 w\left(I_{n,1}+Q_{n,1}+Q_{n,B}-C_{n,1}, I_{n,2}+Q_{n,2}+Q_{n,B}-C_{n,2}\right)}{\partial^2 C_{n,j}}} \geq 0$$

Because $w\left(I_{n,1}+Q_{n,1}+Q_{n,B}-C_{n,1}, I_{n,2}+Q_{n,2}+Q_{n,B}-C_{n,2}\right)$ is concave in inventory levels of $I_{n,1}^{+1}$ and $I_{n,2}^{+1}$ as we discuss in Lemma 1, where $I_{n,j}^{+1} = I_{n,j} + Q_{n,j} + Q_{n,B} - C_{n,j}$ for product j, the optimal consumption levels, $C_{n,1}^*$ and $C_{n,2}^*$, are non-decreasing in h_1 and h_2, respectively.

C.1.6 Segment-n Consumers' Optimal Purchase Quantities

$$\frac{\partial^2 V_t^n(0,0,p_B)}{\partial^2 Q_{n,B}} = \frac{2\left(\alpha_{RR}^{T_n}\beta^{T_n} + \alpha_{RR}\left(-1+\beta\left(-1+\alpha_{RR}\right)\right)\right)\left(\gamma_{n,1}+\gamma_{n,2}-\theta\right)}{\alpha_{RR}\left(-1+\beta\right)\left(-1+\beta\left(-1+\alpha_{RR}\right)\right)} \tag{C.15}$$

Equation (C.15) shows the second derivative of the segment-n customer's value function with respect to bundle purchase quantity, $Q_{n,B}$. The denominator in equation (C.15) is positive because the multiplication of $(-1+\beta)$ and $(-1+\beta(-1+\alpha_{RR}))$ is positive. For $Q_{n,B} \geq 0$, the term $(\gamma_{n,1}+\gamma_{n,2}-\theta)$ must be greater than or equal to 0. Therefore, we need to show that the term $\left(\alpha_{RR}^{T_n}\beta^{T_n}+\alpha_{RR}(-1+\beta(-1+\alpha_{RR}))\right) \leq 0$ to conclude that $V_t^n(0,0,p_B)$ is concave in $Q_{n,B}$.

$$\begin{aligned}\left(\alpha_{RR}^{T_n}\beta^{T_n}+\alpha_{RR}(-1+\beta(-1+\alpha_{RR}))\right) &= \alpha_{RR}^{T_n}\beta^{T_n}-\alpha_{RR}+\beta\alpha_{RR}(\alpha_{RR}-1)\\ &= \beta\alpha_{RR}\left((\beta\alpha_{RR})^{T_n-1}+\alpha_{RR}-1\right)-\alpha_{RR}\\ &= \alpha_{RR}\left(\beta\left((\beta\alpha_{RR})^{T_n-1}+\alpha_{RR}-1\right)-1\right)\end{aligned}$$

where $\beta \approx 1$, and so

$$\begin{aligned}\left(\alpha_{RR}^{T_n}\beta^{T_n}+\alpha_{RR}(-1+\beta(-1+\alpha_{RR}))\right) &= \alpha_{RR}\left((\alpha_{RR})^{T_n-1}+\alpha_{RR}-2\right)\\ &= \alpha_{RR}\left(\alpha_{RR}\left(\alpha_{RR}^{T_n-2}+1\right)-2\right)<0\end{aligned}$$

Therefore, we can conclude that $\frac{\partial^2 V_t^n(0,0,p_B)}{\partial^2 Q_{n,B}} < 0$.

$$\frac{\partial^2 V_t^n(0,0,p_B)}{\partial^2 Q_{n,j}} = \frac{2\left(1+\frac{\alpha_{RR}^{T-1}\beta^T}{-1+\beta(-1+\alpha_{RR})}\right)\gamma_{n,j}}{-1+\beta} \tag{C.16}$$

Because β takes value between 0 and 1, the denominator of equation (C.16) is negative. The equation (C.16) is undefined when β takes the value of 1. Therefore, we analyze the sign of numerator of the equation given in (C.16), and it is sufficient to prove that $\left(\frac{\alpha_{RR}^{T-1}\beta^T}{-1+\beta(-1+\alpha_{RR})}\right)$ is between -1 and 0 to demonstrate the concavity of $C_{n,j}$ in $V_t^n(0,0,p_B)$.

The denominator, $-1+\beta(-1+\alpha_{RR})$, is negative because β and α_{RR} take values between 0 and 1. Then, $-2 < -1+\beta(-1+\alpha_{RR}) < -1$ and $0 < \alpha_{RR}^{T-1}\beta^T < 1$. We do not consider the $\leq$ case for both terms because neither β nor α_{RR} takes values of 0 and 1, so $-1 < \left(\frac{\alpha_{RR}^{T-1}\beta^T}{-1+\beta(-1+\alpha_{RR})}\right) < 0$.

Therefore, the numerator of $\frac{\partial^2 V_t^n(0,0,p_B)}{\partial^2 C_{n,j}}$, $2\left(1+\frac{\alpha_{RR}^{T-1}\beta^T}{-1+\beta(-1+\alpha_{RR})}\right)\gamma_{n,j}$, is positive and the denominator is negative, we can conclude that $\frac{\partial^2 V_t^n(0,0,p_B)}{\partial^2 C_{n,j}} < 0$.

C.1.7 The Concavity Structure and Closed-Form Expression of Bundle Price, p_B^*

$$
\begin{aligned}
\frac{\partial^2 \Pi_B\left(p_B, \rho_B\right)}{\partial^2 p_B} &= \frac{1}{\left(-2+\alpha_{RR}\right)\left(\gamma_1+\gamma_2-\theta\right)}\Bigg(\\
&\delta_{HH}\left(\frac{(\beta-1)\alpha_{RR}(\beta(\alpha_{RR}-1)(T_{HH}-1)-T_{HH})}{\alpha_{RR}^{T_{HH}}\beta^{T_{HH}}+\alpha_{RR}(\beta(\alpha_{RR}-1)-1)}+\frac{\beta(\alpha_{RR}-1)}{1-\beta\alpha_{RR}}\right) \\
&+\ \delta_{HL}\left(\frac{(\beta-1)\alpha_{RR}(\beta(\alpha_{RR}-1)(T_{HL}-1)-T_{HL})}{\alpha_{RR}^{T_{HL}}\beta^{T_{HL}}+\alpha_{RR}(\beta(\alpha_{RR}-1)-1)}+\frac{\beta(\alpha_{RR}-1)}{1-\beta\alpha_{RR}}\right) \\
&+\ \delta_{LH}\left(\frac{(\beta-1)\alpha_{RR}(\beta(\alpha_{RR}-1)(T_{LH}-1)-T_{LH})}{\alpha_{RR}^{T_{LH}}\beta^{T_{LH}}+\alpha_{RR}(\beta(\alpha_{RR}-1)-1)}+\frac{\beta(\alpha_{RR}-1)}{1-\beta\alpha_{RR}}\right) \\
&+\ \delta_{LL}\left(\frac{(\beta-1)\alpha_{RR}(\beta(\alpha_{RR}-1)(T_{LL}-1)-T_{LL})}{\alpha_{RR}^{T_{LL}}\beta^{T_{HH}}+\alpha_{RR}(\beta(\alpha_{RR}-1)-1)}+\frac{\beta(\alpha_{RR}-1)}{1-\beta\alpha_{RR}}\right)\Bigg)
\end{aligned}
$$

All terms presented in (C.17) are in the same structure so it is sufficient to show the sign of one term to prove the concavity of $\Pi_B\left(p_B, \rho_B\right)$ in p_B for a given value of ρ_B.

Let consider terms belongs the segment HH. The second component is negative (i.e., $\frac{\beta(\alpha_{RR}-1)}{1-\beta\alpha_{RR}}$), because $0<\alpha_{RR}<1$. The numerator of the first component, $(\beta-1)\alpha_{RR}(\beta(\alpha_{RR}-1)(T_{HH}-1)-T_{HH})$, is also negative because $(\beta-1)<0$, $(\alpha_{RR}-1)<0$ and $(T_{HH}-1)-T_{HH}<0$. In the problem setting, we consider strategic customers, so we can assume that $\beta\rightarrow 1$. Then, the sign of the denominator of the first component is $\alpha_{RR}^{T_{HH}}+\alpha_{RR}\left(\alpha_{RR}-2\right)<0$ because $0<\alpha_{RR}<1$. All components belong to each segment have the same structure with the negative sign. Therefore, we can conclude that the revenue function of the retailer is a concave function in the bundle price as follows:

$$
\frac{\partial^2 \Pi_B\left(p_B, \rho_B\right)}{\partial^2 p_B} \quad < \quad 0 \quad where\left(\gamma_1+\gamma_2<\theta\right).
$$

For the sake of clarity in representation, we present the optimal bundle price in a compact form where you can find the explicit definition of each corresponding term. The optimal bundle price, p_B^*, is as follows:

$$
p_B^* \quad = \quad \frac{N_1+N_2+N_3+N_4}{2\left(D_1+D_2+D_3+D_4\right)} \qquad \text{(C.17)}
$$

where

$$
\begin{aligned}
N_1 &= -\delta_{HH}\left(a_{HH,1}+a_{HH,2}\right)-\delta_{HL}\left(a_{HL,1}+a_{HL,2}\right)-\delta_{LH}\left(a_{LH,1}+a_{LH,2}\right)\\
&\quad -\delta_{LL}\left(a_{LL,1}+a_{LL,2}\right)\\
N_2 &= \frac{h_1}{\beta\alpha_{RR}-1}\left(\delta_{HH}+\delta_{HL}+\delta_{LH}+\delta_{LL}\right)+\frac{h_2}{\beta\alpha_{RR}-1}\left(\delta_{HH}+\delta_{HL}+\delta_{LH}+\delta_{LL}\right)\\
N_3 &= \frac{h_1\alpha_{RR}(\beta(\alpha_{RR}-1)(T_{LL}-1)-T_{LL})}{\alpha_{RR}^{T_{LL}}\beta^{T_{LL}}+\alpha_{RR}(\beta(\alpha_{RR}-1)-1)}\left(\delta_{LL}+\delta_{LH}+\delta_{HL}+\delta_{HH}\right)\\
N_4 &= \frac{h_2\alpha_{RR}(\beta(\alpha_{RR}-1)(T_{LL}-1)-T_{LL})}{\alpha_{RR}^{T_{LL}}\beta^{T_{LL}}+\alpha_{RR}(\beta(\alpha_{RR}-1)-1)}\left(\delta_{LL}+\delta_{LH}+\delta_{HL}+\delta_{HH}\right)\\
D_1 &= (\beta-1)\beta\alpha_{RR}^2\left(\frac{\delta_{LL}(T_{LL}-1)}{\alpha_{RR}^{T_{LL}}\beta^{T_{LL}}+\alpha_{RR}(\beta(\alpha_{RR}-1)-1)}\right)\\
&\quad +(\beta-1)\beta\alpha_{RR}^2\left(\frac{\delta_{LH}(T_{LH}-1)}{\alpha_{RR}^{T_{LH}}\beta^{T_{LH}}+\alpha_{RR}(\beta(\alpha_{RR}-1)-1)}\right)\\
&\quad +(\beta-1)\beta\alpha_{RR}^2\left(\frac{\delta_{HL}(T_{HL}-1)}{\alpha_{RR}^{T_{HL}}\beta^{T_{HL}}+\alpha_{RR}(\beta(\alpha_{RR}-1)-1)}\right)\\
D_2 &= \alpha_{RR}\left(-\frac{(\beta-1)\delta_{LL}(\beta(T_{LL}-1)+T_{LL})}{\alpha_{RR}^{T_{LL}}\beta^{T_{LL}}+\alpha_{RR}(\beta(\alpha_{RR}-1)-1)}-\frac{(\beta-1)\delta_{LH}(\beta(T_{LH}-1)+T_{LH})}{\alpha_{RR}^{T_{LH}}\beta^{T_{LH}}+\alpha_{RR}(\beta(\alpha_{RR}-1)-1)}\right)\\
D_3 &= \alpha_{RR}\left(-\frac{(\beta-1)\delta_{HL}(\beta(T_{HL}-1)+T_{HL})}{\alpha_{RR}^{T_{HL}}\beta^{T_{HL}}+\alpha_{RR}(\beta(\alpha_{RR}-1)-1)}+\frac{\beta\delta_{HL}}{1-\beta\alpha_{RR}}+\frac{\beta\delta_{LH}}{1-\beta\alpha_{RR}}+\frac{\beta\delta_{LL}}{1-\beta\alpha_{RR}}\right)\\
D_4 &= \frac{(1-\alpha_{RR})\delta_{HH}\alpha_{RR}^{T_{HH}}\beta^{T_{HH}+1}}{(\beta\alpha_{RR}-1)\left(\alpha_{RR}^{T_{HH}}\beta^{T_{HH}}+\alpha_{RR}(\beta(\alpha_{RR}-1)-1)\right)}\\
&\quad +\frac{\alpha_{RR}\delta_{HH}\left(\beta^2(1-\alpha_{RR})(\beta\alpha_{RR}-2)+(\beta-1)T_{HH}(\beta(\alpha_{RR}-1)-1)(\beta\alpha_{RR}-1)\right)}{(\beta\alpha_{RR}-1)\left(\alpha_{RR}^{T_{HH}}\beta^{T_{HH}}+\alpha_{RR}(\beta(\alpha_{RR}-1)-1)\right)}
\end{aligned}
$$