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TECHNICAL EFFICIENCY ESTIMATION IN SELECTED TURKISH PRIVATE MANUFACTURING INDUSTRIES: STOCHASTIC FRONTIER APPROACH WITH A CROSS-SECTIONAL MODEL

Önder YERLİKAYA
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Tez Danışmanı: Yrd. Doç. Dr. M. Kutluğhan Savaş ÖKTE

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İNGİLİZCE İKTİSAT Bilim Dalında **2504010009** numaralı ÖNDER YERLİKAYA'NIN hazırladığı **"TECHNICAL EFFICIENCY ESTIMATON IN SELECTED TURKISH PRIVATE MANUFACTURING INDUSTRIES: STOCHASTIC FRONTIER APPROACH WITH A CROSS-SECTIONAL MODEL"** konulu **YÜKSEK LİSANS / ~~DOKTORA~~ TEZİ** ile ilgili **TEZ SAVUNMA SINAVI,** Lisansüstü Öğretim Yönetmeliği'nin 10.Maddesi uyarınca **09.07.2004 CUMA** günü saat **14.00'de** yapılmış, sorulan sorulara alınan cevaplar sonunda adayın tezinin ....Kabulü......'ne* **OYBİRLİĞİ / ~~OYÇOKLUĞUYLA~~** karar verilmiştir.

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Besim Ömer Paşa Caddesi Beyazıt-34452-İstanbul
Tel:0212 440 00 00Fax: 0212 520 85 04

ÖZ

Üretim birimlerinin verimlilik performansı hem iktisatçılar hem de politika yapıcılar için önemlidir. Bir üretim biriminin verimliliğinden anlaşılan tüm üretim faktörlerinin verimliliğini kapsayan toplam faktör verimliliğidir. Toplam faktör verimliliğinin kaynaklarınının araştırılması ve bu kaynakların toplam faktör verimliliğine olan etkilerinin saptanması verimlilik arttırmada politika yapıcılara önemli ipuçları sağlar. Solow'un *artık* yaklaşımına göre toplam faktör verimliliğinin tek kaynağı teknik ilerlemedir. Son yıllardaki çalışmalar, teknik ilerlemenin yanında teknik etkinlik düzeyindeki değişimlerin de -üretim biriminin veri teknoloji ile gerçekleştirebileceği üretim düzeyi ile gerçekte ürettiği düzey arasındaki fark- toplam faktör verimliliğine katkıda bulunabileceğini göstermiştir. Bu çalışmada, Türkiye özel imalat sanayii üçlü ana iktisadi faaliyet kollarında toplam faktör verimliliğinin kaynağı olarak teknik etkinlik, stokastik üretim sınırı yaklaşımı ile yatay-kesit veri kullanılarak tahmin edilmektedir. 1985 ve 1990 yıllarında, reel katma değer büyüme haddindeki değişimlerin üretim faktörlerinin büyüme haddi ile açıklanamayan kısmını, diğer bir deyişle artıktaki değişimlerin çok büyük bir bölümünü teknik etkinsizlik açıklamaktadır. 1995 yılında ise bu değişimler, rassal şoklar tarafından açıklanmaktadır.

ABSTRACT

The productivity performance of production units is important for both economists and policy-makers. When we talk about productivity, we refer to total factor productivity which is a productivity measure including all production factors. The investigation of sources of total factor productivity growth and the effects of the sources of variation on total factor productivity growth allow policy-makers to understand the facts in improving the productivity. According to Solow *residual* approach, the unique source of total factor productivity is technical progress. Recent studies reveal that along with technical progress, variations in technical efficiency -the gap between the production unit's potential output with existing technology and its actual output- can also contribute to total factor productivity growth. In this study, technical efficiency as a source of total factor productivity growth, is estimated by means of stochastic production frontier approach with cross-sectional data in Turkish private manufacturing industries at three-digit level. In 1985 and 1990, the vast majority of variations in the growth rate of real value added not explained by variations in the growth rate of production factors, namely variations in residual, are clarified by technical inefficiency. In 1995, these variations are clarified by random shocks.

PREFACE

The prominent purpose of this study is to demonstrate that technical inefficiency and random shocks along with technical progress can be source of variatons in real value added. In this context, technical inefficiency and random effects are tried to be distinguished in Turkish private manufacturing industries.

Productivity and total factor productivity concepts are discussed first and total factor productivity measurement is included in the study. Sources of productivity growth are discussed in main two titles, technical change and economic efficiency change. Technical efficiency as a component of economic efficiency and its estimation are discussed in detail. Theoretical structure of the two main approaches, the non-parametric Data Envelopment Analysis and econometric stochastic frontier approach, to efficiency estimation are tried to be given.

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Önder YERLİKAYA

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LIST OF ABBREVIATIONS

BCC : Banker, Charnes and Cooper
CCR : Charnes, Cooper and Rhodes
CR : Concentration Ratio
COLS : Corrected Ordinary Least Squares
CRS : Constant Returns to Scale
CSS : Cornwell, Schmidt and Sickles
DEA : Data Envelopment Analysis
DMU : Decision Making Unit
FDH : Free Disposal Hull
GLS : Generalized Least Squares
ISIC : International Standart Industrial Classification
L^{SDI} : Strong Disposability of Input Set
L^{WDI} : Weak Disposability of Input Set
LSDV : Least Squares with Dummy Variables
MOLS : Modified Ordinary Least Squares
NIRS : Non-Increasing Returns to Scale
OLS : Ordinary Least Squares
P^{SDO} : Strong Disposability of Output Set
P^{WDO} : Weak Disposability of Output Set
S.I.S : State Institute of Statistics
TE : Technical Efficiency
TFP : Total Factor Productivity
TFPG : Total Factor Productivity Growth
VRS : Variable Returns to Scale

INTRODUCTION

The performance of production units are essential for both economists and policy makers. When we discuss the performance of production units, we usually describe them as being more or less productive. By the productivity of a production unit, we refer to the total factor productivity which is a productivity measure including all factors of production, and we mean the ratio of its output to its input. This ratio is easy to obtain in the case of a single-output and single-input. In the more likely event, the production unit uses several inputs to produce several outputs, inputs and outputs must be aggregated into two unique scalars. Thus, productivity measures the relation between various outputs and various inputs.

The measurement of total factor productivity growth is based on economic theory of production. The theory consists of a production function with constant returns to scale together with necessary conditions for producer equilibrium. Output and input quantities entering into the production function are identifed with real product and real factor input. Marginal rates of substitution are identified with corresponding price ratios. Movements along the production function may be separated from shifts in the production function. Shifts in the production function are identifed with changes in total factor productivity.[1]

While there are many ways to measure TFP, the most often used input and output indexes are Laspayres, Paasche, Fisher, Törnqvist and Divisia indexes. Output (input) prices are assigned as weigths to the respective outputs (inputs) in constructing quantity indexes relative to selected base years. Output (input) quantities are assigned as weights to the respective outputs (inputs) in constructing price indexes relative to selected base years. Laspayres indexes overestimate the changes in TFP aggregates whereas Paasche indexes underestimate the changes in TFP aggregates. In order to correct these deficiencies, Fisher (1922) defined an index

[1] D.W. Jorgenson, Z. Griliches, "Explanation of Productivity Change," **The Review of Economic Studies**, Vol. 34, 1967, p. 249.

number formula which is the geometric mean of Laspayres and Paasche indexes. In addition to this, Divisia index is often used in productivity studies. Divisia output (input) index depends on the revenue (cost) share weighted by the percentage growth in outputs (inputs). On the other hand, Törnqvist index which offers a discrete approximation to Divisia index is based on a translog technology. Construction of the index using a continuous translog cost function at two discrete points in time is done by using the quadratic approximation lemma.

The first emprical attempt to measure TFP was made by Tinbergen in 1942, in a remarkable but neglected article in which estimates are presented for four countries, including the United States, for a fourty-four year period. Then, the concept of TFP was further elaborated by Kendrick in 1951. He measures TFP by means of a distribution equation.[2]

It was Solow (1957) who used an explicit production function to measure TFP. Solow measured TFP by using a production function consisting of two inputs, capital and labor, with a constant returns to scale technology and an autonomous and neutral technical change. Solow obtained the residual, the difference between growth rate of real output and weighted growth rates of capital and labor, as conventionally measured.

Jorgenson and Griliches (1967) based their measurement of TFP on social accounting system. Within this system, the prices are identified with implicit deflators, measurement of both outputs and inputs is based on market transactions, prices reflect private benefits and private costs. Costless part of any change in the pattern of productive activity is attributed to change in TFP.

The Malmquist productivity index was introduced by Caves, Christensen and Diewert (1982). They extended an idea of Malmquist (1953), who in a consumer

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[2] Erhan Yıldırım "Total Factor Productivity Growth in Turkish Manufacturing Industries Between 1963-1983: An Analysis," **METU Studies in Development**, Vol. 16, No. 3-4, 1989, pp. 68-69.

context used ratios of input distance functions to construct an input quantity index.[3] An input distance function describes the production technology by looking at a minimal proportional contraction of the input vector given an output vector whereas an output distance function describes it by looking at a maximal proportional expansion of the output vector given an input vector.[4]

In Solow residual approach, technical progress is considered to be the unique source of TFP growth. Recent developments acknowledge that along with technical progress, variations in technical efficiency can also contribute to productivity growth.[5] Measurement of technical efficiency is essenatial for at least three reasons as put forward by Lovell (1993). First of all, inefficiency measures are success performance indicators which allow us to make comparisons across similar units. Second, by measuring the variations in efficiency levels, we can explore the sources of efficiency and productivity differentials. Third, efficiency analyses provide policy implications that guide improvement of efficiency.[6]

The beginning point for any discussion of frontiers and efficiency measurement is the work of Farrell (1957) who provided definitions and computational framework for technical efficiency. Following him, researchers applying frontier estimation techniques represent technology by a bounding function rather than fitting an average function through observed data. The methodology elaborated by Farrell (1957) represents the best practice technology production of a product, defined in terms of the maximum real output producible given available real inputs. Technical inefficiency of a production unit is measured by its deviation from

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[3] Rikard Althin, "Measurement of Productivity Changes: Two Malmquist Index Approaches," **Journal of Productivity Analysis**, Vol. 16, 2001, p. 107.

[4] Tim Coelli, et. al., **An Introduction to Efficiency and Productivity Analysis**, Kluwer Academic Publishers, Dordrecht, 1998, p. 222.

[5] Sangho Kim, Gwangho Han, "A Decomposition of Total Factor Productivity Growth in Korean Manufacturing Industries: A Stochastic Frontier Approach," **Journal of Productivity Analysis**, Vol. 16, 2001, p. 269.

[6] C.A. Knox Lovell, "Production Frontiers and Productive Efficiency," in **The Measurement of Productive Efficiency: Techniques and Applications**, Ed. by. Harold O. Fried, C.A. Knox Lovell, and Shelton S. Schmidt, Oxford University Press, New York, 1993, p. 5.

the function, measured on a ray from the origin through the actual production point to the frontier.[7]

Estimation of frontiers begins with the work of Farrell (1957). Afterwards, the idea was developed by Farrell and Fieldhouse (1962), Afriat (1972) and tested by Aigner and Chu (1968), Seitz (1971), Richmond (1974) and Førsund and Jansen (1977). Their methods are based on deterministic frontiers. Then, Aigner, Lovell and Schmidt (1977) and Meeusen and van den Broeck (1977) introduced independently the stochastic production frontier model, based on econometric methods. They established a production frontier model consisting of a production function of the usual regression type with an error term composed of two components, $\varepsilon_i = v_i - u_i$. The first error component, v_i captures the effects of statistical noise and the second error component, u_i captures the effects of technical inefficiency. Aigner, Lovell and Schmidt assigned half-normal and exponential distributions for technical inefficiency term, Meeusen and van den Broeck assigned an exponential distribution. The half-normal and exponential distributions are one-sided and single-parameter distributions. Then, researchers developed more flexible two-parameter distributions for technical inefficiency component. Greene (1980a, 1980b) proposed a gamma distribution and Stevenson (1980) proposed both gamma and truncated-normal distributions.

Førsund, Lovell and Schmidt (1980) stated in their paper that the main weakness of the stochastic frontier model is due to the fact that it is not possible to decompose individual residuals into their two components. The best one can do is to obtain an estimate of mean efficiency over the sample. Two years later, Jondrow, Lovell, Materov and Schmidt (1982) provided a solution, either the mean or the mode of conditional distribution $[u_i \backslash v_i - u_i]$, to the problem of decomposing the

7 Finn R. Førsund, et. al., "A Survey of Frontier Production Functions and of Their Relationship to Efficiency Measurement," **Journal of Econometrics**, Vol. 13, No. 1, 1980, p. 8.

residuals into inefficiency and statistical noise components. The model can be estimated by either cross-sectional or panel data.

Cross-sectional data provides a snapshot of production units in a specific point in time whereas panel data provides an evidence on their performance through a sequence of time periods. The use of panel data in modelling the production function dates back to Mundlak (1961). Pitt and Lee (1981) were the first to use panel data to estimate firm-specific efficiency levels by means of econometric methods and extended cross-sectional maximum likelihood estimation techniques to panel data.

Non-parametric approach dating back to Farrell (1957) employes mathematical programming techniques in constructing production frontiers and measures the efficiency of a production unit relative to all other production units with restriction that all production units lie on or below the efficient frontier. Non-parametric linear programming methods were introduced by Afriat (1972) but did not gain popularity until Charnes, Cooper and Rhodes (1978) proposed a formal model. This model is called as Data Envelopment Analysis, DEA, and based on the construction of a piecewise linear frontier function that envelops data set as tightly as possible.

This study involves four sections. Section 1 gives the definitions of productivity and total factor productivity and discusses the most common total factor productivity growth index numbers Laspayres, Paasche, Fisher and Törnqvist, and some other indexes, Divisia and Malmquist, together with some approaches to the measurement of total factor productivity are introduced. Section 2 discusses the nature of technical change and economic efficiency as sources of productivity growth. Determinants of efficiency along with Farrell's efficiency definitions and measurement are also discussed in this section. Section 3 provides theoretical structure of production frontiers and efficiency estimation. In this context, two main approaches, non-parametric Data Envelopment Analysis and econometric stochastic frontier approach, are discussed in detail. We compare the two methods. Section 4 contains an empirical analysis with cross-sectional data in relation to selected Turkish private manufacturing industries. The cross-sectional units are defined at three-digit level International Standart Industrial Classification ISIC (Rev. 2) codes.

1. THEORETICAL CONCEPTS AND THE MEASUREMENT OF TOTAL FACTOR PRODUCTIVITY

> "The story of productivity, the ratio of output to input is at heart the record of man's efforts to raise himself from poverty."[1]

Productivity concept is based on the economic theory of production which in turn based on the production function. In this context, productivity is defined as the ratio of output to input, measuring the relationship between various outputs and various inputs.

$$\text{Productivity} = \frac{\text{Output}}{\text{Input}} \tag{1.1}$$

High productivity corresponds to produce more with given inputs or to obtain a given level of output with less inputs. It is obvious that there are two problems in this context: maximization and minimization. The definition given above in (1.1) measures the productivity in physical terms. In addition to this, productivity can be measured in economic terms. That is,

$$\text{Productivity} = \frac{\text{Value of Output}}{\text{Value of Input}} \tag{1.2}$$

The economic definition of productivity given in (1.2) is expected to be greater than unity. On the other hand, the physical definition of productivity is less than or equal to unity since there are losses in transformation of inputs to outputs. If a single output and a single input are involved in production, the productivity ratio is in physical terms. However, when there are multiple outputs and multiple inputs they must be aggregated in common units by means of value or cost weights. The result is

[1] John W. Kendrick, **Productivity Trends in the United States**, Princeton University Press, Princeton, 1961, p. 3.

an economic rather than a physical ratio. A productivity ratio may be changed when the price or unit cost of an output or input is changed, even though the physical ratio is unchanged.[2]

When we talk about productivity, it is usually referred to total factor productivity, TFP. TFP is a productivity measure including all inputs. Other measures of productivity such as a labour productivity, barrels per day in a petroleum refinery, vehicles per day in an auto assembly line, tons per day in a steel mill are what is known as partial measures of productivitiy. These partial productivity measures can provide misleading indication of overall productivity when considered in isolation.[3]

Productivity is very important, especially for developing countries with scarce sources. For instance, labor productivity is usually considered to be a better measure of welfare because it bears a strong correspondence to per-capita income. Productivity raises profitability which in turn raises the labourers' per-capita income. Productivity is the key for welfare and development.

1.1 Total Factor Productivity Index Numbers

An index number is defined as a real number that measures changes in a set of related variables from a base period to a current period, and provides a quantitative description of change over time. An index number is similar in purpose to other summary statistics in that it provides a useful summary of the data under consideration. In order to make comparisons across firms, industries, regions and countries as well as over time and place, index numbers are widely used. They are the most commonly used instruments to measure changes in levels of various

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[2] J.R. Norsworthy, S.L. Jang, "Emprical Measurement and Analysis of Productivity and Technological Change: Applications in High-Technology and Service Industries," in **Contributions to Economic Analysis** Ed. by. D.W. Jorgenson, J.-J. Laffont, North-Holland, Amsterdam, 1992, p. 8.

[3] Tim Coelli, et. al., **An Introduction to Efficiency and Productivity Analysis**, Kluwer Academic Publishers, Dordrecht, 1998, p. 3.

economic variables. Economic indicators such as consumer price indexes, price deflators for national income aggregates, financial indexes, indexes of import and export prices are examples calculated by means of index numbers.[4]

We can construct an index number, $V_{s,t}$ in order to calculate value change from base to current period by the following formula:

$$V_{s,t} = \frac{\sum_{i=1}^{N} p_i^t q_i^t}{\sum_{i=1}^{N} p_i^s q_i^s} \qquad (1.1.1)$$

Let p_i^j and q_i^j denote the price and quantity of the i-th commodity, $i = 1,...,N$ in the j-th period, $j = (s,t)$. s and t refer to base and current periods, respectively and this notation will be used hereafter. The value index measures the change in the value of the basket of quantities of N commodities from base to current period. Since the value index is the result of changes in both prices and quantities in a single commodity case, decomposition of relative effects of price and quantity on the value index is:[5]

$$V_{s,t} = \frac{p^t q^t}{p^s q^s} = \frac{p^t}{p^s}\frac{q^t}{q^s} \qquad (1.1.2)$$

It is obvious from (1.1.2) that there is no index number problem. When there is more than one commodity an aggregation problem arises and the quantities for different inputs and outputs must be aggregated, leading to use of index number

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[4] Tim Coelli, et. al., **An Introduction to Efficiency and Productivity Analysis**, Kluwer Academic Publishers, Dordrecht, 1998, pp. 69-70.

[5] **Ibid.**, pp. 71-72.

formulas. The problem is to combine N different measures of price or quantity changes into a single real number.[6]

This section will focus on three concepts of total factor productivity growth, on how these three concepts are used to define total factor productivity growth and on the definitions of total factor productivity growth indexes widely used, especially Laspayres, Paasche, Fisher and Törnqvist.

Total factor productivity is the rate of transformation of total input into total output which is measured by the ratio of total output to total input. For simplicity, consider the case of one input and one output. Total factor productivity is then defined as:

$$TFP = q_1^t / x_1^t = g^t \tag{1.1.3}$$

where g^t is the output-input coefficient, q_1^t is the quantity of output 1 and x_1^t is the quantity of input 1 in period t. The first concept of total factor productivity growth is the ratio of output-input coefficient in period t to output-input coefficient in period s:[7]

$$TFPG_1 = \left(q_1^t / x_1^t\right) / \left(q_1^s / x_1^s\right) = g^t / g^s \tag{1.1.4}$$

The second concept of total factor productivity growth is the ratio of the rate of output growth to the rate of input growth.[8]

$$TFPG_2 = \left(q_1^t / q_1^s\right) / \left(x_1^t / x_1^s\right) \tag{1.1.5}$$

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[6] **Ibid.**, pp. 71-72.

[7] W. Erwin Diewert, Alice O. Nakamura, "Index Number Concepts, Measures and Decompositions of Productivity Growth," **Journal of Productivity Analysis**, Vol. 19, 2003, p. 128.

[8] **Ibid.**, p. 129.

It is seen from (1.1.4) and (1.1.5) that the first and the second concepts are expressed with different definitions but they are essentially the same.

The third concept relates total factor productivity growth to total revenue and total cost. Total revenues in periods s and t are:

$$TR^s = p_1^s q_1^s \tag{1.1.6}$$

$$TR^t = p_1^t q_1^t \tag{1.1.7}$$

where p_1 denotes price of output 1[9].

Total costs in periods s and t are:

$$TC^s = w_1^s x_1^s \tag{1.1.8}$$

$$TC^t = w_1^t x_1^t \tag{1.1.9}$$

where w_1 denotes price of input 1[10].

Dividing total revenue ratio by output price ratio for periods s and t produce:[11]

$$\left(TR^t / TR^s\right)/\left(p_1^t / p_1^s\right) = \left(p_1^t q_1^t / p_1^s q_1^s\right)/\left(p_1^t / p_1^s\right) = q_1^t / q_1^s \tag{1.1.10}$$

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[9] **Ibid.**, p. 129.

[10] **Ibid.**, p. 129.

[11] **Ibid.**, p. 129.

Similarly, dividing total cost ratio by input price ratio for periods s and t produce:

$$\left(TC^t / TC^s\right) / \left(w_1^t / w_1^s\right) = \left(w_1^t x_1^t / w_1^s x_1^s\right) / \left(w_1^t / w_1^s\right) = x_1^t / x_1^s \tag{1.1.11}$$

The third concept of total factor productivity measure is then defined as:[12]

$$TFPG_3 = \left[\frac{\left(TR^t / TR^s\right)}{\left(p_1^t / p_1^s\right)}\right] / \left[\frac{\left(TC^t / TC^s\right)}{\left(w_1^t / w_1^s\right)}\right] \tag{1.1.12}$$

After introducing three concepts of total factor productivity, we need to consider the realistic case in which multiple inputs are utilized in production and multiple outputs are obtained at the end of the production process. We begin by defining Laspayres, Paasche, Fisher and Törnqvist output and input quantity index number formulations.

The Laspayres quantity index uses the base period prices as weights. The Laspayres output and input quantity indexes are defined as:[13]

$$Q_{output}^L = \frac{\sum_{i=1}^{N} p_i^s q_i^t}{\sum_{i=1}^{N} p_i^s q_i^s} \tag{1.1.13}$$

$$Q_{input}^L = \frac{\sum_{i=1}^{M} w_i^s x_i^t}{\sum_{i=1}^{M} w_i^s x_i^s} \tag{1.1.14}$$

[12] **Ibid.**, pp. 129.

[13] Coelli, **op.cit.**, p. 75.

The Paasche output quantity index uses the current period prices as weights. Paasche output and input quantity indexes are defined as:[14]

$$Q^{P}_{output} = \frac{\sum_{i=1}^{N} p_i^t q_i^t}{\sum_{i=1}^{N} p_i^t q_i^s} \qquad (1.1.15)$$

$$Q^{P}_{input} = \frac{\sum_{i=1}^{M} w_i^t x_i^t}{\sum_{i=1}^{M} w_i^t x_i^s} \qquad (1.1.16)$$

Since Laspayres and Paasche quantity indexes are two extreme cases Irving Fisher defined a geometric mean of these two indexes as a possible index number formula. Fisher output and input quantity indexes are computed by the following formulae:[15]

$$Q^{F}_{output} = \sqrt{Q^{L}_{output} Q^{P}_{output}} \qquad (1.1.17)$$

$$Q^{F}_{input} = \sqrt{Q^{L}_{input} Q^{P}_{input}} \qquad (1.1.18)$$

Since a quantity index comprises a measure of growth between base and current periods, the first and the second concepts can be used to define total factor productivity growth. In index number literature, total factor productivity growth index is defined as the ratio of output quantity index to input quantity index. So, we can construct Laspayres, Paasche and Fisher total factor productivity growth indexes as follows:[16]

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[14] Coelli, **op.cit.,** p. 75.

[15] Coelli, **op.cit.,** p. 73.

[16] Diewert, **op.cit.,** p. 131.

$$TFPG^{L} = \frac{Q^{L}_{output}}{Q^{L}_{input}} \tag{1.1.19}$$

$$TFPG^{P} = \frac{Q^{P}_{output}}{Q^{P}_{input}} \tag{1.1.20}$$

$$TFPG^{F} = \frac{Q^{F}_{output}}{Q^{F}_{input}} \tag{1.1.21}$$

In order to define total factor productivity growth by using the third concept, price indexes are needed, too. Simply interchanging prices and quantities, Laspayres, Paasche and Fisher price indexes can easily be defined.

The Laspayres price index uses base period quantities as weights. It measures the relative costs of maintaining base period standarts in the base period and in the current period.

The Laspayres output and input price indexes are defined as:

$$P^{L}_{output} = \frac{\sum_{i=1}^{N} p_i^t q_i^s}{\sum_{i=1}^{N} p_i^s q_i^s} = \sum_{i=1}^{N} \frac{p_i^t}{p_i^s} . \omega_i^s \tag{1.1.22}$$

$$P^{L}_{input} = \frac{\sum_{i=1}^{M} w_i^t x_i^s}{\sum_{i=1}^{M} w_i^s x_i^s} = \sum_{i=1}^{M} \frac{w_i^t}{w_i^s} . \upsilon_i^s \tag{1.1.23}$$

where $\omega_i^s = p_i^s q_i^s / \sum_{i=1}^{N} p_i^s q_i^s$ is the value share of i-th commodity, and $\upsilon_i^s = w_i^s x_i^s / \sum_{i=1}^{M} w_i^s x_i^s$ is the cost share of the i-th input in the base period.[17]

On the contrary, the Paasche price index uses current period quantities as weights. It measures the relative costs of maintaining current period standarts in the base period and in the current period.

The Paasche output and input price indexes are defined as:

$$P_{output}^{P} = \frac{\sum_{i=1}^{N} p_i^t q_i^t}{\sum_{i=1}^{N} p_i^s q_i^t} = \frac{1}{\sum_{i=1}^{N} \frac{p_i^s}{p_i^t} . \omega_i^t} \qquad (1.1.24)$$

$$P_{input}^{P} = \frac{\sum_{i=1}^{M} w_i^t x_i^t}{\sum_{i=1}^{M} w_i^s x_i^t} = \frac{1}{\sum_{i=1}^{M} \frac{w_i^s}{w_i^t} . \upsilon_i^t} \qquad (1.1.25)$$

where $\omega_i^t = p_i^t q_i^t / \sum_{i=1}^{N} p_i^t q_i^t$ is the value share of the i-th commodity, and $\upsilon_i^t = w_i^t x_i^t / \sum_{i=1}^{M} w_i^t x_i^t$ is the cost share of the i-th input in the current period. It is obvious from the second part of the equation (1.1.25) that Paasche price index is weighted harmonic mean of price relatives.[18]

[17] Coelli, **op.cit.**, p. 72.

[18] Coelli, **op.cit.**, pp. 73.

Laspayres price indexes overstate the rise in prices since they do not take substitution effects into consideration as commodities in the basket become more expensive relative to other commodities, whereas Paasche price indexes understate the rise in prices. Because of different weights used for the Laspayres and Paasche indexes, the two indexes produce different results for the same periods. Unless the periods being compared are too far apart, Laspayres and Paasche indexes give similar results. The greater the length of the periods being compared, the greater the price and quantity movements and differences between the two indexes.[19]

Similarly, we can define Fisher output and input price indexes as follows:

$$P^{F}_{output} = \sqrt{P^{L}_{output} P^{P}_{output}} \quad (1.1.26)$$

$$P^{F}_{input} = \sqrt{P^{L}_{input} P^{P}_{input}} \quad (1.1.27)$$

If *product rule*[20] is satisfied, a price index is implicit counterpart of a quantity index. Due to the fact that Laspayres and Paasche indexes satisfy the product rule, the Laspayres quantity index is the implicit counterpart of the Paasche price index and the Paasche quantity index is the implicit counterpart of the Laspayres price index. Also, Fisher indexes satisfy the product rule that the Fisher price index is the implicit counterpart of the Fisher quantity index.[21] Symbolically:

$$Q^{P}_{output} P^{L}_{output} = Q^{L}_{output} P^{P}_{output} = Q^{F}_{output} P^{F}_{output} = TR^{t} / TR^{s} \quad (1.1.28)$$

[19] <http://www.stats.govt.nz/domino/external/omni/omni.nsf/00000000000000000000000000000000/a96c1a4be99a752acc2569030012ac29?OpenDocument>

[20] This is also known as weak factor reversal test and implies that the product of the output quantity and output price indexes must equal to the nominal or total revenue ratio, the product of the input quantity and input price indexes must equal to the total cost ratio.

[21] Diewert, **op.cit.**, p. 132.

$$Q^P_{input}P^L_{input} = Q^L_{input}P^P_{input} = Q^F_{input}P^F_{input} = TC^t / TC^s \tag{1.1.29}$$

These equalities can be used to define total factor productivity growth indexes in the context of the third concept. Rearranging (1.1.28) and (1.1.29) Laspayres, Paasche and Fisher total factor productivity growth indexes can be obtained by:[22]

$$TFPG^L = \frac{Q^L_{output}}{Q^L_{input}} = \frac{\left(TR^t / TR^s\right) / P^P_{output}}{\left(TC^t / TC^s\right) / P^P_{input}} \tag{1.1.30}$$

$$TFPG^P = \frac{Q^P_{output}}{Q^P_{input}} = \frac{\left(TR^t / TR^s\right) / P^L_{output}}{\left(TC^t / TC^s\right) / P^L_{input}} \tag{1.1.31}$$

$$TFPG^F = \frac{Q^F_{output}}{Q^F_{input}} = \frac{\left(TR^t / TR^s\right) / P^F_{output}}{\left(TC^t / TC^s\right) / P^F_{input}} \tag{1.1.32}$$

Many other total factor productivity growth indexes can be constructed besides Laspayres, Paasche and Fisher TFPG indexes by any output/input quantity and price indexes that satisfy the product rule so that; $Q^I_{output}P^I_{output} = \left(TR^t / TR^s\right)$ and $Q^I_{input}P^I_{input} = \left(TC^t / TC^s\right)$. Total factor productivity growth index is in general:[23]

$$TFPG^I = \frac{Q^I_{output}}{Q^I_{input}} = \frac{\left(TR^t / TR^s\right) / P^I_{output}}{\left(TC^t / TC^s\right) / P^I_{input}} \tag{1.1.33}$$

One of the most commonly used index number formula in the measurement of total factor productivity is the Törnqvist Index. It is a weighted geometric average

[22] Diewert, **op.cit.**, p. 133.

[23] Diewert, **op.cit.**, p. 134.

of the price or quantity relatives, with weights given by the simple average of the value shares in periods s and t. Then, the Törnqvist output and input quantity indexes are defined as:

$$Q^T_{Output} = \prod_{i=1}^{N} \left[\frac{q_i^t}{q_i^s} \right]^{\frac{\omega_i^t + \omega_i^s}{2}} \tag{1.1.34}$$

$$Q^T_{Input} = \prod_{i=1}^{M} \left[\frac{x_i^t}{x_i^s} \right]^{\frac{\upsilon_i^t + \upsilon_i^s}{2}} \tag{1.1.35}$$

where $\omega_i^t = p_i^t q_i^t / \sum_{i=1}^{N} p_i^t q_i^t$ is the value share of the i-th commodity in current period, $\omega_i^s = p_i^s q_i^s / \sum_{i=1}^{N} p_i^s q_i^s$ is the value share of the i-th commodity in base period, $\upsilon_i^t = w_i^t x_i^t / \sum_{i=1}^{M} w_i^t x_i^t$ is the cost share of the i-th input in current period and $\upsilon_i^s = w_i^s x_i^s / \sum_{i=1}^{M} w_i^s x_i^s$ is the cost share of the i-th input in base period.[24]

Then, the Törnqvist total factor productivity growth index can easily be constructed as follows:

$$TFPG^T = \frac{Q^T_{Output}}{Q^T_{Input}} = \frac{\prod_{i=1}^{N} \left[\frac{q_i^t}{q_i^s} \right]^{\frac{\omega_i^t + \omega_i^s}{2}}}{\prod_{i=1}^{M} \left[\frac{x_i^t}{x_i^s} \right]^{\frac{\upsilon_i^t + \upsilon_i^s}{2}}} \tag{1.1.36}$$

[24] Coelli, **op.cit.**, p. 74.

The Törnqvist indexes are usually presented and applied in log-change forms. The Törnqvist total factor productivity growth index in its logarithmic form is then defined as:[25]

$$\ln TFPG^{T}=\frac{1}{2}\sum_{i=1}^{N}\left(\omega_i^s+\omega_i^t\right)\left(\ln q_i^t-\ln q_i^s\right)-\frac{1}{2}\sum_{i=1}^{M}\left(\upsilon_i^s+\upsilon_i^t\right)\left(\ln x_i^t-\ln x_i^s\right) \qquad (1.1.37)$$

So far, it has been introduced various types of price, quantity and total factor produvtivity growth indexes. Actually, the problem is to choose which functional form should be used. There are two main approaches in choosing among the different functional forms for total factor productivity growth indexes. These are the test or axiomatic approach and the exact index number approach.

Index number theorists have been trying to propose a list of mathematical properties that a price (quantity) index should satisfy. Then, the product test rule is applied to solve for the functional form of the quantity (price) index. These mathematical properties, that are the common sense properties of good index numbers, are the index number theory tests or axioms. The other approach in order to determine a total factor productivity growth index is the exact index number approach which is based on a producer behavioral model.

1.1.1 The Test or Axiomatic Approach

As mentioned before in Section 1.1 product rule indicates that the product of the quantity and price indexes must equal the nominal or total revenue ratio for periods s and t.

$$PQ=TR^{t}/TR^{s} \qquad (1.1.1.1)$$

[25] Coelli, **op.cit.**, p. 88.

If the functional form of P (Q) is given, then applying the product rule the functional form of Q (P) is defined implicitly by using the following expressions:[26]

$$Q = \left(TR^t / TR^s\right) / P \qquad \text{and} \qquad P = \left(TR^t / TR^s\right) / Q \qquad (1.1.1.2)$$

Since index number theorists first concentrated on the determination of the functional form for a price index that satisfy the product rule, it follows the same process in which once the functional form for price index is determined, then functional forms for quantity and total factor productivity growth indexes are determined.

Before introducing some of these axiomatic tests, it is assumed that every component of each price and quantity vector is positive for s and t. The price index is denoted by $P\left(p^s, p^t, q^s, q^t\right)$ so that $p^t = \left(p_1^t, ..., p_N^t\right)$, $p^s = \left(p_1^s, ..., p_N^s\right)$ and it will be used throughout this section.[27]

Positivity:

$P\left(p^s, p^t, q^s, q^t\right) > 0$ Each component of the price vector is positive.

Continuity:

$P\left(p^s, p^t, q^s, q^t\right)$ is a continuous function of its arguments which corresponds to differentiability.

The Identity or Constant Prices Test:

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[26] W. Erwin Diewert, Alice O. Nakamura, "Index Number Concepts, Measures and Decompositions of Productivity Growth," **Journal of Productivity Analysis**, Vol. 19, 2003, p. 138.

[27] W.E. Diewert, "Fisher Ideal Output, Input and Productivity Indexes Revisited," in **Essays in Index Number Theory Vol. I**, Ed. by. W.E. Diewert, Alice O. Nakamura, North-Holland, Amsterdam, 1993, p. 317.

$P(p^s, p^t, q^s, q^t) = 1$ so that $p^s = p^t$. This means that if prices are equal in periods s and t, then the price index should equal unity regardless of the quantity vectors for these periods.

The Constant Basket Test:

$P(p^s, p^t, q^s, q^t) = \sum_{i=1}^{N} p_i^t q_i / \sum_{j=1}^{N} p_j^s q_j$ so that $q^s = q^t$. This test states that if quantities are constant during the two time periods, then the price index should equal the ratio of constant basket of quantities priced at period t to the value of constant basket of quantities priced at period s.

Proportionality in Current Prices Test:

$P(p^s, \lambda p^t, q^s, q^t) = \lambda P(p^s, p^t, q^s, q^t)$ for $\lambda > 0$. For a scalar multiplication of each element of p^t by λ, the new price index is λ times the old price index.

Inverse Proportionality in Base Prices:

$P(\lambda p^s, p^t, q^s, q^t) = \lambda^{-1} P(p^s, p^t, q^s, q^t)$ for $\lambda > 0$. For a scalar multiplication of each element of p^s by λ, the new price index is equal to the old price index divided by λ.

Invariance to Proportional Changes in Current Quantities:

$P(p^s, p^t, q^s, \lambda q^t) = P(p^s, p^t, q^s, q^t)$ for $\lambda > 0$. For a scalar multiplication of each element of q^t by λ the price index does not change.

Invariance to Proportional Changes in Base Quantities:

$P(p^s, p^t, \lambda q^s, q^t) = P(p^s, p^t, q^s, q^t)$ for $\lambda > 0$. For a scalar multiplication of each element of q^s by λ the price index does not change.

Commodity Reversal Test:

$P\left(p^{\sim s}, p^{\sim t}, q^{\sim s}, q^{\sim t}\right)=P\left(p^{s}, p^{t}, q^{s}, q^{t}\right)$. $p^{\sim s}$ and $p^{\sim t}$ indicate the permutation of the components of p^{s} and p^{t}. In the same way $q^{\sim s}$ and $q^{\sim t}$ indicate the permutation of components of q^{s} and q^{t}. This test states that if the ordering of the commodities is changed the value of the price index does not change.

Commensurability Test:

$P\left(\alpha_1 p_1^s,\ldots,\alpha_N p_N^s;\alpha_1 p_1^t,\ldots,\alpha_N p_N^t;\alpha_1^{-1} q_1^s,\ldots,\alpha_N^{-1} q_N^s;\alpha_1^{-1} q_N^t,\ldots,\alpha_N^{-1} q_N^t\right)=$

$P\left(p_1^s,\ldots,p_N^s;p_1^t,\ldots,p_N^t;q_1^s,\ldots,q_N^s;q_1^t,\ldots,q_N^t\right)$ for $\alpha_1,\ldots,\alpha_N>0$. If the units of measurement for each commodity are changed, the value of the price index does not change.

Time Reversal Test:

$P\left(p^{t}, p^{s}, q^{t}, q^{s}\right)=1/P\left(p^{s}, p^{t}, q^{s}, q^{t}\right)$. If the price and the quantity vectors for periods s and t are interchanged, the new price index is the old price index to the power of -1.

Quantity Reversal Test:

$P\left(p^{s}, p^{t}, q^{s}, q^{t}\right)=P\left(p^{s}, p^{t}, q^{t}, q^{s}\right)$. If the quantity vectors for periods s and t are interchanged, the price index does not change.

Price Reversal Test:

$$\frac{p^t q^t}{p^s q^s P\left(p^{s}, p^{t}, q^{s}, q^{t}\right)}=\frac{p^s q^t}{p^t q^s P\left(p^{t}, p^{s}, q^{s}, q^{t}\right)}$$

Mean Value Test for Prices:

$$\min_i\left\{p_i^t/p_i^s : i=1,\ldots,N\right\}\leq P\left(p^{s}, p^{t}, q^{s}, q^{t}\right)\leq \max_i\left\{p_i^t/p_i^s : i=1,\ldots,N\right\}$$

This test states that the price index must lie between the minimum price ratio and the maximum price ratio.

Mean Value Test for Quantities:

$$\min_i \left\{ q_i^t / q_i^s : i = 1,...,N \right\} \le p^t q^t / p^s q^s P\left(p^s, p^t, q^s, q^t\right) \le \max_i \left\{ q_i^t / q_i^s : i = 1,...,N \right\}$$

Using the product rule, the implicit quantity index defined in terms of price index above must lie between the minimum quantity ratio and the maximum quantity ratio.

Paasche and Laspayres Bounding Test:

$$\left(p^t q^s / p^s q^s\right) \le P\left(p^s, p^t, q^s, q^t\right) \le \left(p^t q^t / p^s q^t\right)$$

$$\left(p^t q^t / p^s q^t\right) \le P\left(p^s, p^t, q^s, q^t\right) \le \left(p^t q^s / p^s q^s\right)$$

The price index must satisfy at least one of the above inequalities, namely the price index must lie between Laspayres and Paasche price indexes.

Monotonicity in Current Prices:

$P\left(p^s, p^t, q^s, q^t\right) < P\left(p^s, p, q^s, q^t\right)$ if $p^t < p$.

If period t prices increase, then the new price index must be larger than the old price index. So, $P\left(p^s, p^t, q^s, q^t\right)$ is increasing in the arguments of p^t.

Monotonicity in Base Prices:

$P\left(p^s, p^t, q^s, q^t\right) > P\left(p, p^t, q^s, q^t\right)$ if $p > p^s$.

If period s prices increases, then the new price index must be smaller than the old price index. So, $P\left(p^s, p^t, q^s, q^t\right)$ is decreasing in the arguments of p^s.

Monotonicity in Current Quantities:

$$\frac{p^t q^t}{p^s q^s P\left(p^s, p^t, q^s, q^t\right)} < \frac{p^t q}{p^s q^s P\left(p^s, p^t, q^s, q\right)} \text{ if } q^t < q.$$

If period t quantities increase, then the new implicit quantity index defined in terms of price index must increase. So, $p^t q^t / \left(p^s q^s P\left(p^s, p^t, q^s, q^t\right)\right)$ is increasing in the arguments of q^t.

Monotonicity in Base Quantities:

$$\frac{p^t q^t}{p^s q^s P\left(p^s, p^t, q^s, q^t\right)} > \frac{p^t q^t}{p^s q P\left(p^s, p^t, q, q^t\right)} \text{ if } q > q^s .$$

If period s quantities increase, then the new implicit quantitiy index defined in terms of price index must decrease. So, $p^t q^t / \left(p^s q^s P\left(p^s, p^t, q^s, q^t\right)\right)$ is decreasing in the arguments of q^s. Fisher price index satisfies all these tests. Paasche and Laspayres price indexes fail only time reversal, quantity reversal and price reversal tests. In this context, Paasche and Laspayres price indexes exhibit good performance. The Törnqvist price index fails constant basket test, mean value test for quantities, quantity reversal test, price reversal test, Paasche and Laspayres bounding test, monotonicity in current prices test, monotonicity in base prices test, monotonicity in current quantities test and monotonicity in base prices test.[28]

1.1.2 Exact Index Numbers Approach

Another approach in order to determine the functional form for a measure of total factor productivity growth is to derive the index based on a producer behavioral model. This section includes derivation of an index based on a producer behavioral model according to Diewert (1976).

Let a firm's production function at time t be $q_1 = F^t\left(q_2, ..., q_N, x_1, ..., x_M\right)$ for $t = 0, 1, ..., T$. At an existing level of technology at time t, the firm can produce q_1 units of output 1 if it also produces q_2 units of output 2,..., and q_N units of output N using x_1 units of input 1,..., and x_M units of input M. The production function can also be used to define total cost function.[29] That is:

[28] **Ibid.**, pp. 321-330.

[29] W. Erwin Diewert, Alice O. Nakamura, "Index Number Concepts, Measures and Decompositions of Productivity Growth," **Journal of Productivity Analysis**, Vol. 19, 2003, p. 139.

$$C^t\left(q_1,\ldots,q_N,w_1,\ldots,w_M\right)\equiv\min_x\sum_{i=1}^{M}w_ix_i : q_1=F^t\left(q_2,\ldots,q_N,x_1,\ldots,x_M\right) \tag{1.1.2.1}$$

If the assumption of cost minimization is imposed, the total cost of production, C^t, is equal to the minimum cost for periods $t=0,1,\ldots,T$. Therefore,

$$C^t\equiv\sum_{i=1}^{M}w_i^tx_i^t=C^t\left(q_1^t,\ldots,q_N^t,w_1^t,\ldots,w_M^t\right) \tag{1.1.2.2}$$

In order to relate the cost functions to the periods $t=0,1,\ldots,T$, it is assumed that:

$$C^t\left(q_1^t,\ldots,q_N^t,w_1^t,\ldots,w_M^t\right)=\frac{1}{a^t}C\left(q_1^t,\ldots,q_N^t,w_1^t,\ldots,w_M^t\right) \tag{1.1.2.3}$$

where $a^t>0$ demonstrates period t relative efficiency parameter and C demonstrates an atemporal cost function[30].

Taking the natural logarithm of both sides of (1.1.2.3),

$$\ln C^t\left(q_1^t,\ldots,q_N^t,w_1^t,\ldots,w_M^t\right)=-\ln a^t+\ln C\left(q_1^t,\ldots,q_N^t,w_1^t,\ldots,w_M^t\right) \tag{1.1.2.4}$$

If a translog cost function is imposed for $\ln C$, the right-hand side of (1.1.2.3) becomes:[31]

$$\ln C^t\left(q_1^t,\ldots,q_N^t,w_1^t,\ldots,w_M^t\right)=\alpha_0+\sum_{n=1}^{N}\beta_n\ln q_n^t+\sum_{m=1}^{M}\gamma_m\ln w_m^t$$

$$+\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\theta_{ij}\ln q_i^t\ln q_j^t+\frac{1}{2}\sum_{i=1}^{M}\sum_{j=1}^{M}\delta_{ij}\ln w_i^t\ln w_j^t+\sum_{n=1}^{N}\sum_{m=1}^{M}\zeta_{nm}\ln q_n^t\ln w_m^t \tag{1.1.2.5}$$

[30] **Ibid.**, pp. 139-140.

[31] **Ibid.**, p. 140.

$\theta_{ij} = \theta_{ji}$ for $1 \le i < j \le N$

$\delta_{ij} = \delta_{ji}$ for $1 \le i < j \le M$

If the firm is minimizing costs, the demand for input x_m at period t is:

$$x_m^t = \partial C^t \left(q_1^t, \ldots, q_N^t, w_1^t, \ldots, w_M^t\right) / \partial w_m \qquad m = 1, \ldots, M \qquad t = 1, \ldots, T \tag{1.1.2.6}$$

Using Theil's global version and Kloek's quadratic approximation lemma,[32]

$$f\left(z^1 - z^0\right) = \frac{1}{2}\left[\nabla f\left(z^1\right) + \nabla f\left(z^0\right)\right]^T \left(z^1 - z^0\right) \tag{1.1.2.7}$$

where $\nabla f\left(z^r\right)$ is gradient vector of f evaluated at z^r[33] since $\ln C^t$ is a quadratic function in the variables $\ln q_1, \ldots, \ln q_N$ and $\ln w_1, \ldots, \ln w_M$ $\ln C^t - \ln C^s$ yields:[34]

$$\ln C^t - \ln C^s = \frac{1}{2}\sum_{n=1}^{N}\left[q_n^t \frac{\partial \ln C^t}{\partial q_n}\left(q^t, w^t\right) + q_n^s \frac{\partial \ln C^s}{\partial q_n}\left(q^s, w^s\right)\right]\ln\left(q_n^t / q_n^s\right)$$
$$+\frac{1}{2}\sum_{m=1}^{M}\left[\left(w_m^t x_m^t / C^t\right) + \left(w_m^s x_m^s / C^s\right)\right]\ln\left(w_m^t / w_m^s\right) - \ln\left(a^t / a^s\right) \tag{1.1.2.8}$$

An additional behavioral assumption, profit maximizing behavior, can be imposed. Solving the following maximization problem,

$$\max_{q_1, \ldots, q_N} \left\{\sum_{n=1}^{N} p_n^t q_n - C^t\left(q_1, \ldots, q_N, w_1^t, \ldots, w_M^t\right)\right\} \tag{1.1.2.9}$$

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32 W.E. Diewert, "Exact and Superlative Index Numbers," in **Essays in Index Number Theory: Volume 1,** Ed. by. W.E. Diewert, Alice O. Nakamura, North-Holland, Amsterdam, 1993, p. 225.

33 It is applied if and only if the quadratic function is defined by (1.1.2.5).

34 W. Erwin Diewert, Alice O. Nakamura, "Index Number Concepts, Measures and Decompositions of Productivity Growth," **Journal of Productivity Analysis,** Vol. 19, 2003, pp. 140-141.

where $p_n^t = \partial C^t\left(q_1^t,...,q_N^t,w_1^t,...,w_M^t\right)/\partial q_n$ due to the fact that competitive price taking behavior is assumed. Thus, (1.1.2.8) can be rewritten as follows:

$$\ln C^t - \ln C^s = \frac{1}{2}\sum_{n=1}^{N}\left[\left(p_n^t q_n^t / C^t\right)+\left(p_n^s q_n^s / C^s\right)\right]\ln\left(q_n^t / q_n^s\right)$$

$$+\frac{1}{2}\sum_{m=1}^{M}\left[\left(w_m^t x_m^t / C^t\right)+\left(w_m^s x_m^s / C^s\right)\right]\ln\left(w_m^t / w_m^s\right)-\ln\left(a^t / a^s\right) \qquad (1.1.2.10)$$

Solving equation (1.1.2.10) for $\left(a^t / a^s\right)$ which denotes for productivity change,

$$\frac{a^t}{a^s}=\frac{\prod_{n=1}^{N}\left(q_n^t / q_n^s\right)^{\frac{1}{2}\left[\left(p_n^t q_n^t / C^t\right)+\left(p_n^s q_n^s / C^s\right)\right]}}{\tilde{Q}_{Input}^T} \qquad (1.1.2.11)$$

$\tilde{Q}_{Input}^T$ is the implicit Törnqvist input quantity index.[35] Since costs, output and input prices and quantities can be observed, it is possible to measure productivity change in the above manner.[36]

1.2 The Malmquist Index

Within the DEA framework, Caves, Christensen and Diewert (1982) established a link between Farrell efficiency measures and total factor productivity indices by proposing a productivity index based on Malmquist (1953). The Malmquist index is defined by means of distance functions. In order to define multi-input, multi-output production technology without the need to specify a behavioural assumptions as in the case of exact index numbers approach distance functions are needed. An input

[35] The implicit Törnqvist input quantity index, $\tilde{Q}_{Input}^T$ is equal to $\left(C^t / C^s\right)/P_{Input}^T$ where P_{Input}^T is the Törnqvist input price index.

[36] Diewert, **op.cit.**, pp. 141-142.

distance function describes the production technology by looking at a minimal proportional contraction of the input vector, given an output vector whereas an output distance function describes it by looking at a maximal proportional expansion of the output vector, given an input vector.[37]

The production technology is defined by set S using output sets, $P(x)$ represents the set of all output vectors, q. q is produced using the input vector, x. This is shown as:

$$P(x) = \{q : x \text{ can produce } q\} \quad (1.2.1)$$

For each x, $P(x)$ is assumed to satisfy the following properties:

- $0 \in P(x)$: nothing can be produced out of a given set of inputs.
- Non-zero output levels cannot be produced from zero level of inputs.
- $P(x)$ satisfies strong disposability of outputs: if $q \in P(x)$ and $q^* \leq q$ then $q^* \in P(x)$
- $P(x)$ satisfies strong disposability of inputs: if q can be produced from x, then q can be produced from any $x^* \geq x$
- $P(x)$ is closed.
- $P(x)$ is bounded: we cannot produce unlimited levels of outputs with given inputs.
- $P(x)$ is convex: commodities are continuously divisible.[38]

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[37] Tim Coelli, et al., **An Introduction to Efficiency and Productivity Analysis**, Kluwer Academic Publishers, Dordrecht, 1998, p. 62.

[38] **Ibid.**, p. 62.

Then, the output distance function at time period t is defined as:

$$D_0^t\left(x^t,q^t\right)=\inf\left\{\theta:\left(x^t,q^t/\theta\right)\in P^t\left(x\right)\right\}^{39} \tag{1.2.2}$$

The output distance function has the following properties:

- $D_o^t\left(x^t,q^t\right)$ is non-decreasing in q and increasing in x.
- $D_o^t\left(x^t,q^t\right)$ is linearly homogeneous in q.
- $\left(x^t,q^t\right)\in P^t\left(x\right)$ (if q belongs to the production possibility set of x) if and only if $D_o^t\left(x^t,q^t\right)\leq 1$.
- $D_o^t\left(x^t,q^t\right)=1$ if q belongs to the frontier of the production possibility set.[40]

In order to define a Malmquist output-based productivity index, it is necessary to relate an input-output vector $\left(x^t,q^t\right)$ at time period t to the technology $P^{t+1}\left(x\right)$ in the following period. It is defined as:

$$D_o^{t+1}\left(x^t,q^t\right)=\inf\left\{\theta:\left(x^t,q^t/\theta\right)\in P^{t+1}\left(x\right)\right\} \tag{1.2.3}$$

Similarly, we can define $D_o^t\left(x^{t+1},q^{t+1}\right)$, $D_o^t\left(x^t,q^t\right)$ and $D_o^{t+1}\left(x^{t+1},q^{t+1}\right)$ respectively[41],

$$D_o^t\left(x^{t+1},q^{t+1}\right)=\inf\left\{\theta:\left(x^{t+1},q^{t+1}/\theta\right)\in P^t\left(x\right)\right\} \tag{1.2.4}$$

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[39] inf stands for "infimum." This allows for the possibility that minimum does not exist. $\theta=+\infty$ is possible.

[40] Coelli, **op.cit.**, p. 63.

[41] Rolf Färe, et. al., "Productivity Developments in Swedish Hospitals: A Malmquist Output Index Approach," in **Data Envelopment Analysis: Theory, Methodology and Application**, Ed. by. Abraham Charnes, William W. Cooper, Arie Y. Lewin, Lawrence M. Seiford, Kluwer Academic Publishers, Dordrecht, 1994, p. 256.

$$D_o^t\left(x^t,q^t\right)=\inf\left\{\theta:\left(x^t,q^t/\theta\right)\in P^t\left(x\right)\right\} \tag{1.2.5}$$

$$D_o^{t+1}\left(x^{t+1},q^{t+1}\right)=\inf\left\{\theta:\left(x^{t+1},q^{t+1}/\theta\right)\in P^{t+1}\left(x\right)\right\} \tag{1.2.6}$$

Then, Malmquist output-based index is defined by the following notation:

$$M_o^{t+1}\left(x^{t+1},q^{t+1},x^t,q^t\right)=\left[\frac{D_o^t\left(x^{t+1},q^{t+1}\right)}{D_o^t\left(x^t,q^t\right)}\frac{D_o^{t+1}\left(x^{t+1},q^{t+1}\right)}{D_o^{t+1}\left(x^t,q^t\right)}\right]^{1/2} \tag{1.2.7}$$

Equation (1.2.7) is the geometric mean of the two indexes. The first part of the right hand side of the equation in the brackets is the ratio that compares the performance of data from period t to period $t+1$ relative to the existing production technology at time period t and the second part is the ratio that compares the performance of the same data relative to the existing production technology at time period $t+1$.[42]

An alternative way of stating Malmquist output-based produvtivity index allows for decomposing the effects of technical progress and technical efficiency between the two periods. This is shown in notational form:

$$M_o^{t+1}\left(x^{t+1},q^{t+1},x^t,q^t\right)=\frac{D_o^{t+1}\left(x^{t+1},q^{t+1}\right)}{D_o^t\left(x^t,q^t\right)}\left[\frac{D_o^t\left(x^{t+1},q^{t+1}\right)}{D_o^{t+1}\left(x^{t+1},q^{t+1}\right)}\frac{D_o^t\left(x^t,q^t\right)}{D_o^{t+1}\left(x^t,q^t\right)}\right]^{1/2} \tag{1.2.8}$$

The first term in equation (1.2.8) represents the change in technical efficiency. If this term has a value of unity there is no change in technical efficiency. A value greater (less) than unity indicates that technical efficiency improves (deteriorates). The second term, the geometric mean of the two indexes, represents

[42] C.A. Knox Lovell, "Production Frontiers and Productive Efficiency," in **The Measurement of Productive Efficiency: Techniques and Applications**, Ed. by. Harold O. Fried, C.A. Knox Lovell, Shelton S. Schmidt, Oxford University Press, New York, 1993, p. 51.

the technical progress. If this term has a value of unity technical change does not occur. A value greater (less) than unity indicates that technical change is progressive (regressive).[43]

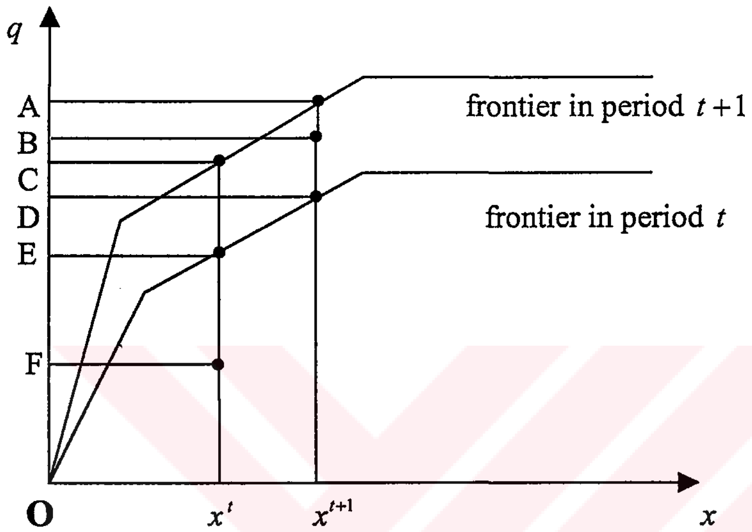

Figure 1. Malmquist output-based productivity index

Source: Rolf Färe, et. al., "Productivity Developments in Swedish Hospitals: A Malmquist Output Index Approach," in **Data Envelopment Analysis: Theory, Methodology and Applications**, Ed. by. Abraham Charnes, William W. Cooper, Arie Y. Lewin, Lawrance M. Seiford, Kluwer Academic Publishers, Dordrecht, 1994, p. 258.

For illustrative purposes, consider Figure 1. Assume that for each period the firm operates below the frontier. In period t the firm produces at point F denoted by distance OF and in period $t+1$ it produces at point B denoted by distance OB. Then, the effects of technical progress and change in technical efficiency can be demonstrated in terms of distances.

$$\text{Technical efficiency change} = \frac{(OB/OA)}{(OF/OE)} \qquad (1.2.9)$$

[43] **Ibid.**, p. 52.

$$\text{Technical change} = \left[\frac{(OB/OE)}{(OB/OA)} \frac{(OF/OE)}{(OF/OC)} \right]^{1/2} \quad (1.2.10)$$

Distance functions are calculated as solutions to linear programming problems. It is assumed that there are K observations, $k=1,...,K$ on $m=1,...,M$ inputs in each period $t=1,...,T$ to produce $n=1,...,N$ outputs. It is also assumed that the number of observations does not change over time, $K^t = K$. Then the technology, that serves as constraints to the linear programming problem in period t is described as follows:

$$P^t(x) = \left\{ (x^t, q^t) : \sum_{k=1}^{K} \lambda^{k,t} x_m^{k,t} \le x_m^t, \quad m=1,...,M \right. \quad (1.2.11)$$

$$\sum_{k=1}^{K} \lambda^{k,t} q_n^{k,t} \ge q_n^t \qquad n=1,...,N$$

$$\sum_{k=1}^{K} \lambda^{k,t} \le 1; \lambda^{k,t} \ge 0 \quad k=1,...,K$$

where λ is an intensity variable. The relative productivity change of observation k' between period t and $t+1$ is calculated by $D_o^t\left(x^{k',t}, q^{k',t}\right)$ and $D_o^t\left(x^{k',t+1}, q^{k',t+1}\right)$ as solutions to the following progamming problems:[44]

$$D_o^t\left(x^{k',t}, q^{k',t}\right) = \min_{\theta,\lambda} \theta \quad (1.2.12)$$

subject to

[44] Färe, op.cit., pp. 258-259.

$$\sum_{k=1}^{K} \lambda^{k,t} x_m^{k,t} \le x_m^{k',t} \qquad m = 1,\ldots,M$$

$$\sum_{k=1}^{K} \lambda^{k,t} q_n^{k,t} \ge q_n^{k',t} / \theta \qquad n = 1,\ldots,N$$

$$\sum_{k=1}^{K} \lambda^{k,t} \le 1; \lambda^{k,t} \ge 0 \qquad k = 1,\ldots,K$$

On the other hand, $D_o^t\left(x^{k',t+1}, q^{k',t+1}\right)$ is calculated as follows:

$$D_o^t\left(x^{k',t+1}, q^{k',t+1}\right) = \min_{\theta,\lambda} \theta \qquad (1.2.13)$$

subject to

$$\sum_{k=1}^{K} \lambda^{k,t} x_m^{k,t} \le x_m^{k',t+1} \qquad m = 1,\ldots,M$$

$$\sum_{k=1}^{K} \lambda^{k,t} q_n^{k,t} \ge q_n^{k',t+1} / \theta \qquad n = 1,\ldots,N$$

$$\sum_{k=1}^{K} \lambda^{k,t} \le 1; \lambda^{k,t} \ge 0 \qquad k = 1,\ldots,K$$

1.3 The Divisia Index

The Divisia or chain-link index is widely used in measuring economic aggregates and especially in studying the sources of economic growth and has received increasing attention in recent years. It is believed to be appropriate for decomposing the sources of economic growth into the components which can be explained by increased supply of inputs and technical change.[45]

The Divisia index is a weighted sum of growth rates, where the weights are the components' share in total value. Divisia's derivation of the price and quantity indexes can be summarized by the following procedure.

Let $p_i = (p_1, \ldots, p_n)$ and $q_i = (q_1, \ldots, q_n)$ be the output price and output quantity vectors. Prices and quantities are functions of time; $p_i(t)$, $q_i(t)$. Total expenditure at time t is:

$$TR(t) = \sum_{i=1}^{N} p_i(t) q_i(t) = p(t) q(t) \tag{1.3.1}$$

The rate of change in total expenditure is the time derivative of (1.3.1).

$$\partial \left[\sum_{i=1}^{N} p_i(t) q_i(t) \right] / \partial t = \sum_{i=1}^{N} p_i(t) \partial q_i(t) / \partial t + \sum_{i=1}^{N} q_i(t) \partial p_i(t) / \partial t \tag{1.3.2}$$

Dividing both sides of the above equation through by $p(t)q(t)$, we obtain the following identity:

$$\frac{d(TR)}{TRdt} = \sum_{i=1}^{N} \frac{p_i(t) q_i(t) \frac{\partial q_i(t)}{q_i(t) \partial t}}{p(t) q(t)} + \sum_{i=1}^{N} \frac{q_i(t) p_i(t) \frac{\partial p_i(t)}{p_i(t) \partial t}}{p(t) q(t)} \tag{1.3.3}$$

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[45] D. Usher, "The Suitability Of The Divisia Index For The Measurement Of Economic Aggregates," **Review of Income and Wealth,** Vol. 20, 1973, p. 273.

Since $\frac{p_i(t)q_i(t)}{p(t)q(t)}$ is the cost share of the total expenditure function, $TR(t)$, if given $s_i(t)$ is the cost share of the i-th commodity then,[46]

$$s_i(t)=\frac{p_i(t)q_i(t)}{p(t)q(t)}, \tag{1.3.4}$$

$$\frac{d(TR)}{TRdt}=\sum_{i=1}^{N}s_i(t)\left[\frac{\partial q_i(t)}{q_i(t)\partial t}+\frac{\partial p_i(t)}{p_i(t)\partial t}\right] \tag{1.3.5}$$

The above derivation of growth of total expenditure is decomposed into a share weighted rate of change of quantities and a share weighted rate of change of prices. Integrating both sides of (1.3.5) in interval $[0,T]$ and putting the result in exponential form, we obtain:

$$TR(T)=TR(0)\underbrace{\exp\left\{\int_0^T\left[\sum_{i=1}^{N}s_i(t)\frac{\partial q_i(t)}{q_i(t)\partial t}\right]dt\right\}}_{\text{Divisia Quantity Index}}\underbrace{\exp\left\{\int_0^T\left[\sum_{i=1}^{N}s_i(t)\frac{\partial p_i(t)}{p_i(t)\partial t}\right]dt\right\}}_{\text{Divisia Price Index}} \tag{1.3.6}$$

Consider the following continuous, twice differentiable production function with one output and N inputs, that is homogenous of degree one and all inputs are paid with respect to their marginal products.

$$Q(t)=F\left(x_1(t),\ldots,x_n(t);t\right) \tag{1.3.7}$$

[46] W.E. Diewert, "The Economic Theory of Index Numbers: A Survey," in **Essays in Index Number Theory Volume 1**, Ed. by. W. Erwin Diewert, Alice O. Nakamura, North Holland, Amsterdam, 1993, p. 217.

where $Q(t)$ is the output, x_i is the i-th input and t is time. Taking the logarithm of $F(.)$ and differentiating with respect to time, we obtain the following equation:

$$\frac{dQ(t)}{Q(t)dt} = \frac{\partial F_1}{F\partial x_1(t)}\frac{dx_1(t)}{dt} + \ldots + \frac{\partial F_N}{F\partial x_N(t)}\frac{dx_N(t)}{dt} + \frac{\partial F}{F\partial t} \tag{1.3.8}$$

Rearranging (1.3.8),

$$\frac{dQ}{Qdt} = \sum_{i=1}^{N}\left[\frac{x_i}{F}\frac{\partial F}{\partial x_i}\frac{dx_i}{x_i dt}\right] + \frac{\partial F}{Fdt} \tag{1.3.9}$$

where $\frac{x_i}{F}\frac{\partial F}{\partial x_i}$ is the share of the i-th input in total product. Replacing it with β_i, we obtain the following partial differential equation:

$$\frac{dQ}{Qdt} = \sum_{i=1}^{N}\left[\beta_i(t)\frac{dx_i}{x_i dt}\right] + \frac{\partial F}{Fdt} \tag{1.3.10}$$

Then, the growth of the Divisia index of total factor productivity, $\frac{\partial A}{Adt}$ is:

$$\frac{\partial F}{Fdt} = \frac{\partial A}{Adt} = \frac{dQ}{Qdt} - \sum_{i=1}^{N}\left[\beta_i(t)\frac{dx_i}{x_i dt}\right] \tag{1.3.11}$$

Integrating both sides of (1.3.11) in interval $[0,T]$, we obtain the index of total factor productivity growth:

$$\frac{A(T)}{A(0)} = \frac{Q(T)}{Q(0)}\exp\left\{-\int_0^T \sum_{i=1}^{N}\beta_i(t)\frac{dx_i}{x_i dt}dt\right\} \tag{1.3.12}$$

The above derivation of the Divisia index assumes that quantities and factor shares are continuous functions of time. However, economic data is not collected on a continuous time basis. Therefore, Divisia indexes must be approximated by using discrete time data.[47]

In order to get a discrete approximation to a continuous Divisia index, we assume that factor shares are constant and do not fluctuate over time. The constant share, $\overline{\beta}_i$ must satisfy the following relation:

$$\int_0^T \overline{\beta}_i \frac{dx_i}{x_i dt} dt = \int_0^T \beta_i(t) \frac{dx_i}{x_i dt} dt \qquad (1.3.13)$$

Integrating left-hand side of the equation (1.3.13),

$$\overline{\beta}_i \ln\left[x_i(T)/x_i(0)\right] = \int_0^T \beta_i(t) \frac{dx_i}{x_i dt} dt \qquad (1.3.14)$$

If we substitute (1.3.14) into (1.3.12), we obtain:[48]

$$\frac{A(T)}{A(0)} = \frac{Q(T)}{Q(0)} / \prod_{i=1}^{N} \overline{\beta}_i \frac{x_i(T)}{x_i(0)} \qquad (1.3.15)$$

Consequently, using data only from period 0 to T we can calculate the Divisia index of total factor productivity growth.

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[47] Spencer Star, Robert E. Hall, "An Approximate Divisia Index of Total Factor Productivity," **Econometrica,** Vol. 44, No. 2, 1976, pp. 257-258.

[48] **Ibid.,** p. 258-259.

1.4 Jorgenson and Griliches's Social Accounting System

According to Jorgenson and Griliches, measurement of total factor productivity is based on the social accounting system. Within this framework, the prices are identified with implicit deflators, measurements of both output and input is based on market transactions, prices reflect private benefits and private costs. Costless part of any change in the pattern of productive activity is attributed to change in total factor productivity. Changes in applied technology, managerial efficieny, industrial organization, etc. can be accounted for changes in total factor producitvity.[49]

To define the social accounting system the following notation is introduced.

Q_i is the quantity of the i-th output,

X_j is the quantity of the j-th input,

p_i is the price of the i-th output,

w_j is the price of the j-th input,

$i = 1 \ldots M$;

$j = 1 \ldots N$;

For each accounting period, the fundamental identity is:

$$p_1 Q_1 + \ldots + p_M Q_M = w_1 X_1 + \ldots + w_N X_N \qquad (1.4.1)$$

which equates the total value of output to the total value of input.[50]

In order to be able to define total factor productivity, differentiate totally the identity with respect to time and divide both sides by the corresponding total value. That is:

[49] D.W. Jorgenson, Z. Griliches, "The Explanation of Productivity Change," **The Review of Economic Studies**, Vol. 34, 1967, pp. 250-251.

[50] **Ibid.**, p. 251.

$$\frac{(p_1Q_1)\frac{\left[\frac{dp_1Q_1}{dt}+\frac{dQ_1p_1}{dt}\right]}{(p_1Q_1)}+(p_2Q_2)\frac{\left[\frac{dp_2Q_2}{dt}+\frac{dQ_2p_2}{dt}\right]}{(p_2Q_2)}+\ldots+(p_MQ_M)\frac{\left[\frac{dp_MQ_M}{dt}+\frac{dQ_Mp_M}{dt}\right]}{(p_MQ_M)}}{\sum_{i=1}^{M}p_iQ_i}$$

$$=\frac{(w_1X_1)\frac{\left[\frac{dw_1X_1}{dt}+\frac{dX_1w_1}{dt}\right]}{(w_1X_1)}+(w_2X_2)\frac{\left[\frac{dw_2X_2}{dt}+\frac{dX_2w_2}{dt}\right]}{(w_2X_2)}+\ldots+(w_NX_N)\frac{\left[\frac{dw_NX_N}{dt}+\frac{dX_Nw_N}{dt}\right]}{(w_NX_N)}}{\sum_{j=1}^{N}w_jX_j}$$

(1.4.2)

Rearranging the identity:

$$\left[\frac{p_1Q_1}{\sum_{i=1}^{M}p_iQ_i}\right]\left[\frac{dp_1}{p_1dt}+\frac{dQ_1}{Q_1dt}\right]+\left[\frac{p_2Q_2}{\sum_{i=1}^{M}p_iQ_i}\right]\left[\frac{dp_2}{p_2dt}+\frac{dQ_2}{Q_2dt}\right]+\ldots+\left[\frac{p_MQ_M}{\sum_{i=1}^{M}p_iQ_i}\right]\left[\frac{dp_M}{p_Mdt}+\frac{dQ_M}{Q_Mdt}\right]$$

$$=\left[\frac{w_1X_1}{\sum_{j=1}^{N}w_jX_j}\right]\left[\frac{dw_1}{w_1dt}+\frac{dX_1}{X_1dt}\right]+\left[\frac{w_2X_2}{\sum_{j=1}^{N}w_jX_j}\right]\left[\frac{dw_2}{w_2dt}+\frac{dX_2}{X_2dt}\right]+\ldots+\left[\frac{w_NX_N}{\sum_{j=1}^{n}w_jX_j}\right]\left[\frac{dw_N}{w_Ndt}+\frac{dX_N}{X_Ndt}\right]$$

(1.4.3)

$$\sum_{i=1}^{M}\omega_i\left[\frac{dp_i}{p_idt}+\frac{dQ_i}{Q_idt}\right]=\sum_{j=1}^{N}v_j\left[\frac{dw_j}{w_jdt}+\frac{dX_j}{X_jdt}\right] \quad (1.4.4)$$

where $\omega_i=\left[\frac{p_iQ_i}{\sum_{i=1}^{M}p_iQ_i}\right]$ and $v_j=\left[\frac{w_jX_j}{\sum_{j=1}^{N}w_jX_j}\right]$

ω_i and v_j[51] denote the relative shares of the value of the i-th output in the value of total output and the value of the j-th input in the value of total input, respectively. The identity reflects the fact that weighted average of the sum of rates of growth of output prices and quantities is equal to weighted average of the sum of rates of growth of input prices and quantities.

Since the rate of growth of total output quantity index is defined as sum of the weighted averages of the rates of growth of individual products, quantity index of total output can be formulated as follows:

$$\frac{dQ}{Qdt} = \sum_{i=1}^{M} \omega_i \frac{dQ_i}{Q_i dt} \tag{1.4.5}$$

Similarly, the rate of growth of total input quantity index is sum of the weighted averages of the rates of growth of individual factor inputs.[52]

$$\frac{dX}{Xdt} = \sum_{j=1}^{N} v_j \frac{dX_j}{X_j dt} \tag{1.4.6}$$

Total factor productivity can be defined in terms of Divisia quantity indexes of total output and total input. Total factor productivity index, represented by A, is the ratio of the quantity of total output to the quantity of total input.

$$A = \frac{Q}{X} \tag{1.4.7}$$

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[51] ω_i, $v_i \geq 0$, and $\sum_{i=1}^{M} \omega_i = \sum_{j=1}^{N} v_j = 1$

[52] Jorgenson, op.cit., p. 252.

where Q and X denote Divisia quantity index of total output and total input, respectively. It is easy to derive the rate of growth of total factor productivity by means of logarithmic differentiation:[53]

$$\frac{dA}{Adt}=\frac{dQ}{Qdt}-\frac{dX}{Xdt}=\sum\omega_i\frac{dQ_i}{Q_i dt}-\sum v_j\frac{dX_j}{X_j dt} \quad (1.4.8)$$

Alternatively, the rate of growth of total factor productivity can also be computed by using price indexes. Total factor productivity index is the ratio of the Divisia input price index, W^D, to Divisia output price index, P^D.

$$A=\frac{W^D}{P^D} \quad (1.4.9)$$

The rate of growth of Divisia price indexes for total output and total input are as follows:

$$\frac{dP^D}{P^D dt}=\sum_{i=1}^{M}\omega_i\frac{dp_i}{p_i dt} \quad (1.4.10)$$

$$\frac{dW^D}{W^D dt}=\sum_{j=1}^{N}v_j\frac{dw_j}{w_j dt} \quad (1.4.11)$$

The rate of total factor productivity growth is then:[54]

$$\frac{dA}{Adt}=\frac{dW^D}{W^D dt}-\frac{dP^D}{P^D dt}=\sum_{j=1}^{N}v_j\frac{dw_j}{w_j dt}-\sum_{i=1}^{M}\omega_i\frac{dp_i}{p_i dt} \quad (1.4.12)$$

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[53] Jorgenson, **op.cit.,** p. 252.

[54] Jorgenson, **op.cit.,** p. 252.

So far, total factor productivity has been defined as the ratio of quantity to price indexes. An economic interpretation of this definition can be developed from the theory of production. Constant returns to scale prevails in the production and all marginal rates of substitution between pairs of inputs and outputs are equal to price ratios. Production function in implicit form is:[55]

$$F\left(Q_1, Q_2, \ldots, Q_M; X_1, X_2, \ldots, X_N\right) = 0 \qquad (1.4.13)$$

Total factor productivity is then the difference between weighted average rates of growth of outputs and inputs:

$$\frac{dA}{Adt} = \sum \left[\frac{F_i Q_i}{\sum_{i=1}^{M} F_i Q_i} \frac{dQ_i}{Q_i dt} \right] - \sum \left[\frac{F_j X_j}{\sum_{j=1}^{N} F_j X_j} \frac{dX_j}{X_j dt} \right] \qquad (1.4.14)$$

where $F_i = \frac{\partial F}{\partial Q_i}$, $F_j = \frac{\partial F}{\partial X_j}$. From the theory of production, all marginal rates of substitution between pairs of inputs and outputs are equal to price ratios, the necessary conditions for producer equilibrium. That is:[56]

$$F_i = \frac{\partial F}{\partial Q_i} = p_i \text{ and } F_j = \frac{\partial F}{\partial X_j} = w_j \qquad (1.4.15)$$

which corresponds to the definition given in (1.4.8).

[55] Jorgenson, **op.cit.**, p. 253.

[56] Jorgenson, **op.cit.**, p. 253.

1.5 Kendrick's Arithmetic Approach

John W. Kendrick's work entitled "Productivity Trends in the United States" provides an arithmetic measure of total productivity over the 1899-1953 period for the US economy. Consequences of this study pertaining to the US economy is not of our interest but Kendrick's procedure merits close scrutiny.

Kendrick's production equation for a fully-integrated industry with a single product is defined as:

$$Q = C\left(w_0 L + i_0 K\right) \tag{1.5.1}$$

where C is the arithmetic index of productivity, w_0 is the real wage rate and i_0 is the real rate of return in base period, L is the labor input, K is the capital input including land in a given year. L and K is in terms of physical units. Since all prices have been divided by the price of output in the base year Q corresponds to output of an industry in both physical and value terms in base year price. One of the basic assumptions is the constant input prices, which reflects the marginal products. Kendrick uses these constant prices as weights in the formulation of his production equation.[57]

Manipulating equation (1.5.1), Kendrick's index is as follows:

$$C = \frac{Q}{w_0 L + i_0 K} \tag{1.5.2}$$

[57] Evsey D. Domar, "On Total Productivity and All That," **Journal of Political Economy**, Vol. 70 No. 6, 1962, p. 598.

It is obvious from equation (1.5.2) that C is the average productivity of an arithmetic combination of labor and capital (K includes land) and is called as total factor productivity.[58]

Kendrick follows an ideal procedure in constructing his index. For the whole economy, output is expressed as net national product at factor cost. He used the stock of capital net of depreciation and excluded both the current depreciation and the cost of materials from both sides of the production equation. Kendrick's price of capital is average rate of return. However, he uses the interest rate on government bonds for governmental capital. Aggregate labor input is estimated by a sum weighted by the wage rate in each industry recognizing differences in labor quality on the inter-industrial level. These differences are attributed to differences in sex, physical prowess, and mental ability.[59]

1.6 Solow Residual Approach

There are many ways in the measurement of total factor productivty. Index numbers approach has been discussed in previous sections. This section focuses on the use of Solow's general index of techinal change.

Assuming that an aggregate production function exists and it is specified accurately and inputs are properly measured, (we do not go into details on errors in measuring the factors of production and errors resulting from omitting relevant factors), aggregate, two-factor, twice-differentiable production function can be written as:

$$Q = F(K, L, t) \tag{1.6.1}$$

[58] **Ibid.,** p. 599.

[59] **Ibid.,** pp. 601-602.

where Q, K and L represent aggregate output, capital and labor input, respectively. t appears in the function to allow for technical change. Solow uses the phrase "technical change" as a shorthand expression for any kind of shift in the production function.[60] Slowdowns, speedups, innovations, improvements in the number of skilled labor appear as techinal change.

Solow's general index of technical change is disembodied[61] and requires three restrictive assumptions: Hicks-neutral technical change, constant returns to scale, and perfect competition in both product and factor markets.[62]

Technical change is neutral if marginal rates of substitution between the pairs of factors are constant and if factor shares in total output are unaffected. In this context, the production function can be defined in the following form.

$$Q = A(t) f(K,L) \tag{1.6.2}$$

where $A(t)$ is a measure of disembodied technical change and $f(K,L)$ is the function homogeneous of degree one. Taking the natural logarithm of the production function,

$$\ln Q = \ln A + \ln f(K,L) \tag{1.6.3}$$

Differentiating equation (1.6.3) totally with respect to time, we obtain:

[60] Robert M. Solow, "Technical Change and the Aggregate Production Function," **Review of Economics and Statistics,** Vol. 39, 1957, p. 312.

[61] Technical change is said to be disembodied if changes in technology affect output independently of the ages of machines.

[62] Badi H. Baltagi, James M. Griffin, "A General Index of Technical Change," **Journal of Political Economy,** Vol. 96, No.1, 1988, p. 23.

$$\frac{dQ}{Qdt}=\frac{dA}{Adt}+\frac{K}{Q}\frac{\partial f}{\partial K}\frac{dK}{Kdt}+\frac{L}{Q}\frac{\partial f}{\partial L}\frac{dL}{Ldt} \tag{1.6.4}$$

where $\frac{K}{Q}\frac{\partial f}{\partial K}$ and $\frac{L}{Q}\frac{\partial f}{\partial L}$ denote capital and labor shares in total output, respectively[63]. Replacing them with α and β:

$$\frac{dQ}{Qdt}=\frac{dA}{Adt}+\alpha\frac{dK}{Kdt}+\beta\frac{dL}{Ldt} \quad \text{or} \tag{1.6.5}$$

$$\frac{dA}{Adt}=\frac{dQ}{Qdt}-\alpha\frac{dK}{Kdt}-\beta\frac{dL}{Ldt} \tag{1.6.6}$$

Rate of growth of output is the sum of the rate of growth of each factor weighted by their respective shares in total output and the rate of growth of output attributable to technical change only[64]. In order to calculate equation (1.6.6), Solow assumed that time derivatives could be approximated by discrete changes. In this respect, equation (1.6.6) is equivalent to a Divisia index of productivity growth.[65]

According to Solow residual approach, technical progress is the unique source of total factor productivity growth. The basic result is that the residual measures the shift in the production function. Solow thought of his method as a way

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[63] Since the production function exhibits constant returs to scale, α and β sum to 1. Under constant returns to scale and perfect competition in both product and factor markets, α and β are exact measures of the elasticity of the production function with respect to capital and labor, respectively.

[64] Ramu Ramanathan, **Lecture Notes in Economics and Mathematical Systems: Intorduction to the Theory of Economic Growth,** Springer-Verlag, New York, 1982, p. 73.

[65] S. Grosskopf, "Efficiency and Productivity" in **The Measurement of Productive Efficiency: Techniques and Applications**, Ed. by. Harold O. Fried, C.A. Knox Lovell, Shelton S. Schmidt, Oxford Universtiy Press, 1993, p. 171.

to measure the trend in productivity. In years of expansion, the residual is large and in years of recession it is low or even negative.[66]

It is also possible to express the production function in intensive form due to homogeneity assumption. Let $q = \frac{Q}{L}$, $k = \frac{K}{L}$, $\beta = 1 - \alpha$; equation (1.6.5) becomes:

$$\frac{dq}{qdt} = \frac{dA}{Adt} + \alpha \frac{dk}{kdt} \qquad (1.6.7)$$

It is seen from equation (1.6.7) that technical change can be estimated from time series of output per unit of labor, capital per unit of labor, and the share of capital in total output.[67]

[66] Robert E. Hall, "Invariance Properties of Solow's Productivity Residual," **NBER Working paper No: 3034**, 1989, p. 1.

[67] Robert M. Solow, "Technical Change and the Aggregate Production Function," **Review of Economics and Statistics**, Vol. 39, 1957, p. 313.

2. SOURCES OF PRODUCTIVITY GROWTH

Economic growth is a complex process. Although growth of physical capital stock and the laborforce tend to be more visible, productivity growth plays a significant role in economic growth. Well, the sources of productivity growth is important as well as productivity growth. There are two main sorts of answer to the question that what are the sources of productivity growth: The first one is technological progress through technological innovation and knowledge and the other one is economic efficiency which can be decomposed into technical, allocative and scale efficiency.

In a seminal paper, Solow (1957) attributed the whole residual to technological change only. Later, researchers reduced this contribution by allowing changes in labor quality and correcting mismeasurements of capital and output. In spite of this fact, residual still accounts for around a third of economic growth.

In the neo-classical growth model of Solow and Swan, diminishing returns to capital accumulation would eventually make any growth in excess of the rate of technological progress self-limiting. Endogenous growth models that include capital as well as innovation[1] come to the same conclusion. In these models, the incentive to innovate determines the rate of technological progress which in turn determines the economy's long-run growth rate independently of the amount of capital.[2]

The view that technological progress is dominant determinant of long-run growth in an economy is also the consensus view of the mainstream textbooks. Blanchard (1997, p.461, 496) states that "growth must ultimately be due to technological progress, and the rate of output growth in the steady state is independent of the saving rate". Also, David Romer (1996, p.95) states that "the

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[1] See P.M. Romer, "Endogenous Technological Change," **Journal of Political Economy**, Vol. 98, 1990, pp. 71-102. and G.M. Grossman, and E. Helpman, **Innovation and Growth in the Global Economy**, Cambridge, MA: MIT Press, 1991, chapter 5.

[2] Peter Howitt, and Philippe Aghion, "Capital Accumulation and Innovation as Complementary Factors in Long-Run Growth," **Journal of Economic Growth**, Vol. 3, 1998, p. 111.

driving force of growth is the accumulation of knowledge...capital accumulation is not central to growth".[3]

Farrell (1957) defines economic efficiency and its two sources: technical efficiency and allocative efficiency. If inefficient use of the combination which has been chosen can be eliminated, it will be possible to have a higher productivity level.[4] Technical inefficiency arises from excessive input usage which in turn costs are not minimized and profits are not maximized. Allocative inefficiency arises from combining inputs in the wrong proportions which in turn costs are not minimized and profits are not maximized.

Both technical and allocative efficiency are necessary but not sufficient for profit maximization. If we leave the assumption that production prevails under constant returns to scale, the firm could be scale inefficient. Scale efficiency enables spreading of fixed costs over a larger amount of output. In larger organizations, laborforce can specialize and run larger masses without having to reset equipment. As time passes, the laborforce can learn how to operate new production techniques with increasing skills and hence can increase their productivity through learning.[5]

The following sub-sections considers technical change and economic efficiency as a source of productivity growth. Technical characteristics of production process and relative factor price movements as two major sets of factors that have influences on total factor productivity growth are analyzed in detail. Then, the main focus of this section, technical change and economic efficiency and its components, are discussed in detail.

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[3] **Ibid.**, p. 112.

[4] David G. Mayes, **Sources of Productivity Growth**, Ed. by. David G. Mayes, Cambridge University Press, Cambridge, 1996, p. 6.

[5] **Ibid.**, p. 7.

2.1 Technical Change and its Nature

Technical characteristics of the production process and the relative factor price movements are two major sets of factors that have influences on total factor productivity growth.

Technical characteristics of the production process can be divided into four components: The efficiency of production, which will be discussed in the next section in more detail, the bias in technical change, the elasticity of substitution, and the scale of operation of the production process.[6]

2.1.1 Technical Bias

Changes in the shape of the isoquants[7] represent bias in technical change, which is reflected by a greater movement of the isoquant to one axis than another towards the origin. However, neutral technical change is measured by the extent of the parallel shifts of the isoquant towards the origin. In Figure 2, the shift from Q_1 to Q_2 demonstrates a neutral technical change whereas the shift from Q_1 to Q_3 demonstrates proportionally greater saving of labor than for all other production techniques. These shifts towards the origin tend to increase the productivity of one factor at the expense of slowing down that of another.

[6] M. Ishaq Nadiri, "Some Approaches to the Theory and Measurement of Total Factor Productivity: A Survey," **Journal of Economic Literature,** Vol. 8, No. 4, 1970, p. 1141.

[7] An isoquant is the locus of all input combinations that yields the same level of output.

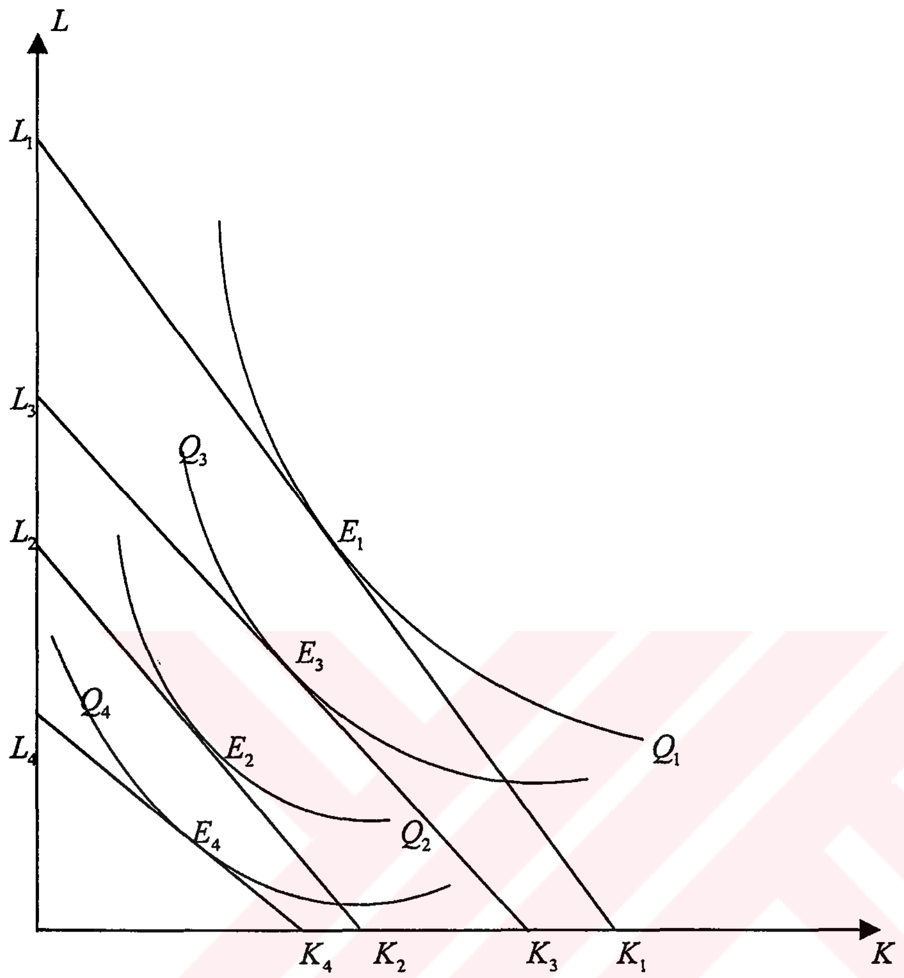

Figure 2. Unit Capital Requirements

Source: M. Ishaq Nadiri, "Some Approaches to the Theory and Measurement of Total Factor Productivity: A Survey", **Journal of Economic Literature,** Vol. 8, No. 4, 1970, p. 1142.

There is no complete definition of bias in technical change. There are several ways of defining technical bias. Hicks, Harrod and Solow proposed different definitions for technical bias. In Hicksian sense, technological change is neutral whenever factor proportions are constant and the marginal substitution between capital and labor is also constant.[8]

Hicks proposed a definition of technical bias along a constant capital-labor ratio.

[8] Ramu Ramanathan, **Lecture Notes in Economics and Mathematical Systems: Intorduction to the Theory of Economic Growth,** Springer-Verlag, New York, 1982, p. 75.

$$\left[\frac{\partial(KF_K)/(LF_L)}{\partial t}\right]_{(K/L)\text{constant}} = 0 \text{ if technical change is neutral,}$$

$$\left[\frac{\partial(KF_K)/(LF_L)}{\partial t}\right]_{(K/L)\text{constant}} > 0 \text{ if technical change is labor-saving,}$$

$$\left[\frac{\partial(KF_K)/(LF_L)}{\partial t}\right]_{(K/L)\text{constant}} < 0 \text{ if technical change is capital-saving}$$ [9]

where $F_K = \partial F / \partial K$ and $F_L = \partial F / \partial L$.

Technical change is Harrod neutral whenever the capital-output ratio is constant, then the marginal product of capital is also constant leaving distribution of income constant.[10]

Harrod proposed a definition of technical bias along a constant capital-output ratio.

$$\left[\frac{\partial(KF_K)/(LF_L)}{\partial t}\right]_{(K/Q)\text{constant}} = 0 \text{ if technical change is neutral,}$$

$$\left[\frac{\partial(KF_K)/(LF_L)}{\partial t}\right]_{(K/Q)\text{constant}} > 0 \text{ if technical change is labor-saving,}$$

$$\left[\frac{\partial(KF_K)/(LF_L)}{\partial t}\right]_{(K/Q)\text{constant}} < 0 \text{ if technical change is capital-saving.}$$ [11]

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[9] Nadiri, **op.cit.,** p. 1142.

[10] Ramanathan, **op.cit.,** p. 78.

[11] Nadiri, **op.cit.,** p. 1143.

Solow proposed a definition of technical bias along a constant labor-output ratio.

$$\left[\frac{\partial\left(KF_K\right)/\left(LF_L\right)}{\partial t}\right]_{(L/Q)\text{constant}} = 0 \text{ if technical change is neutral,}$$

$$\left[\frac{\partial\left(KF_K\right)/\left(LF_L\right)}{\partial t}\right]_{(L/Q)\text{constant}} > 0 \text{ if technical change is labor-saving,}$$

$$\left[\frac{\partial\left(KF_K\right)/\left(LF_L\right)}{\partial t}\right]_{(L/Q)\text{constant}} < 0 \text{ if technical change is capital-saving}$$ [12]

Biases in technical change are measured by the relative change in per capita capital when relative factor prices are kept constant. Proprotional form of this measure is:

$$B = \frac{d\left(K/L\right)}{dt}\left(L/K\right) \tag{2.1.1.1}$$

This kind of measure is the answer to the question that how much per capita capital changes if technical advance occurs alone in the production process. It measures the extent to which points on each curve with the same slope move closer to one axis than another. Technical change is neutral if B is equal to zero. When B is greater (less) than zero, the bias in technical change is labor-saving (capital-saving) reflecting the fact that the proportionate saving in labor (capital) is greater than the proprotionate saving in capital (labor).[13]

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[12] Nadiri, **op.cit.**, p. 1143.

[13] W.E Salter, with an addendum by W.B. Reddaway, **Productivity and Technical Change**, Cambridge Universtiy Press, Cambridge, 1966, pp. 31-32.

2.1.2 Elasticity of Substitution

Another characteristic of technical change is the elasticity of substitution. Elasticity of substitution is a pure number that measures the ease of exchanging factors and the rate at which substitution takes place.[14] It is defined as the ratio of the percentage change in factor proportions to the percentage change in relative factor prices. Mathematically:

$$\sigma = \frac{d(K/L)/(K/L)}{d(w/r)/(w/r)} \qquad (2.1.2.1)$$

The importance of the elasticity of substitution arises from determining how far changes in relative factor prices are effective in adding to the rates of productivity increase which are established by technical advance only. Any technical progress that makes capital and labor more easily substitutable increases the elasticity of substitution whereas technical progresses limiting the substitution possibilities reduce the elasticity of substitution, giving rise to change in the curvature of the isoquants. This is illustrated by the movement of the isoquant from Q_1 to Q_4 in Figure 2. For instance, developments in robotics allow greater freedom in the substitution of capital equipment for labor and increase the elasticity of substitution. Advances which are applied over a small range of the production function reduce the elasticity of substitution.[15]

2.1.3 Scale Effects

The scale of operation of the production process corresponds to the saving of both inputs due to an increase in the scale of operations of the economy or to the reduction

[14] James M. Henderson, and Richard E. Quandt, **Microeconomic Theory: A Mathematical Approach**, 3rd. ed., McGraw-Hill International, 1980, p. 73.

[15] Salter, **op.cit.**, p. 34.

in the average cost of production as the firms' output expand. The scale effect can be viewed by the movement of the isoquant from Q_1 to Q_2 providing that $Q_2 > Q_1$.

Through learning, labor can be adapted to operation of new production techniques with increasing skills and therefore leading to increase in their productivity level. Specialization and multi-skilling make laborforce more capable of doing a number of jobs and abolish idle time. Most of the firms produce a range of products and production of one product benefits from that of another. This can be due to joint use of equipment or joint use of labor. Lessons taken from one production process can be applied in another process. A firm having all these interlinkages will tend to be more productive.[16]

2.1.4 Relative Factor Prices

Besides the technical characteristics of the production process, changing in relative factor prices affect factor productivity via their effect on (K/L). Its effect is due to increase the rate of productivity growth of one factor at the expense of slowing down that of another. Effectiveness of changing relative factor prices depends on the elasticity of substitution. When the elasticity of substitution is zero, changes in relative factor prices have no effect. When the elasticity of substitution is infinite, changes in relative factor prices have a large effect on productivity. If labor is becoming more expensive relative to capital and the elasticity of substitution is large, the rate of growth of labor productivity will increase; at the same time the rate of growth of capital productivity will decrease. Reverse is true if capital is becoming more expensive relative to labor.[17]

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[16] David G. Mayes, **Sources of Productivity Growth,** Ed. by. David G. Mayes, Cambridge University Press, 1996, p. 7.

[17] Salter, **op.cit.,** p. 40.

Consequently, observed productivity movements are the net result of these five influences represented by the efficiency of production, the bias in technical change, the elasticity of substitution, the scale effects of the production process and the relative factor prices.

2.2 Economic Efficieny: Definitions and Farrell's Measurement

Economic efficiency is the ability of a firm's production of maximum potential output from a given input mix. It reflects the comparison between observed and maximum potential values of outputs and inputs. The comparison can be in the form of the ratio of observed to maximum potential output from a given input mix or the ratio of minimum potential to observed inputs to produce given output.[18]

Remember the definition of production function. It gives the maximum potential output that can be produced from a given input mix. Similarly, a cost function gives the minimum level of cost to produce some level of output at given input prices. A profit function gives the maximum profit that is obtained by given output and input prices. Since these functions set a limit to the range of possible observations, the concept of "frontier" may be more appropriate.[19] Points that lie on the frontiers denote efficient points. The points below the production and profit frontiers and the points above the cost frontier denote inefficient points.

In a competitive economy, firms will operate at the most efficient point which satisfies the objective of profit maximization. The firms are assumed to be price-takers and no firm has influence on prices set by the market. With full information in both product and factor markets, each firm will choose the optimal levels of inputs and outputs realizing their profit-maximizing objective. Thus, all firms will produce

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[18] C.A. Knox Lovell, "Production Frontiers and Productive Efficiency," in **The Measurement of Productive Efficiency** Ed. by. Harold O. Fried, C.A. Knox Lovell, Shelton S. Schmidt, Oxford University Press, New York, 1993, p. 4.

[19] Finn R. Førsund, et. al., "A Survey of Frontier Production Functions and of Their Relationship to Efficiency Measurement," **Journal of Econometrics**, Vol. 13, 1980, p. 5.

the same output with identical technologies and the same allocation of resources representing full efficiency for the economy. But in reality economies differ from that hypothetical environment. Even if firms are fully efficient and so are the economies, they are exposed to external shocks. The maximum potential is therefore never be realized in practice. Thus, it may be appropriate to discuss in the framework of inefficieny.[20]

The measurement of (in)efficiency as well as technical progress, in an industry is important for economic theorists and policy-makers due to the fact that it is a good performance indicator. If the separate effects of technical progress and economic efficiency on productivity growth are decomposed, sources of variation in productivity growth can be explored and thus various measures can be implemented to achieve high rates of productivity growth.[21]

The discussion of (in)efficiency measurement begins with Farrell's (1957) work on definitions and computational framework for both technical and allocative efficiency. The measurement of economic efficiency has been based on the assumption that the efficient production function is known. It is an effort to compare the observed performance of a firm with some postulated standart of full efficiency. The efficient production function is represented by an isoquant, so the problem is to estimate an efficient isoquant from a scatter diagram based on the observations of inputs and outputs of a number of firms.[22]

Assuming a firm utilizing two factors of production, x_1 and x_2 to produce output q under constant returns to scale. The production frontier is:

$$q = f(x_1, x_2) \qquad (2.2.1)$$

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[20] David Mayes, et. al., **Inefficiency in Industry**, Harvester Wheatsheaf, London, 1994, p. 13.

[21] Førsund, **op.cit.**, p. 5.

[22] M.J. Farrell, "The Measurement of Productive Efficiency," **Journal of the Royal Statistical Society Series A (General)**, Part 3, 1957, p. 255.

We can rewrite (2.2.1) due to constant returns to scale as:

$$f\left(x_1/q, x_2/q\right) = 1 \qquad (2.2.2)$$

So the unit isoquant, II`, can be sketched as follows:

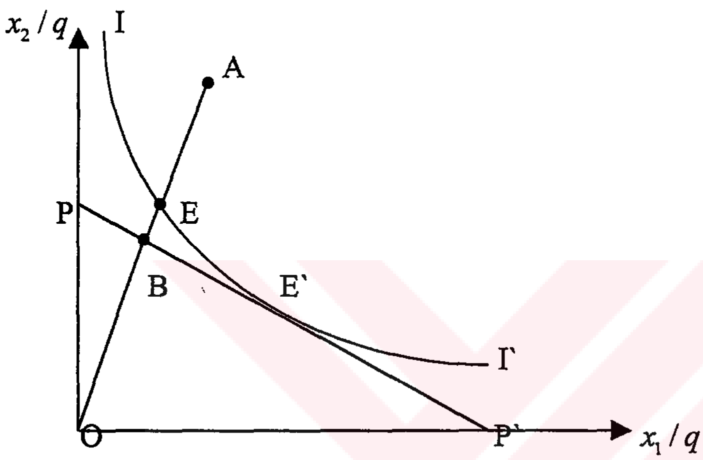

Figure 3. Technical and Allocative Efficiencies

Source: Tim Coelli, et. al., **An Introduction to Efficiency and Productivity Analysis**, Kluwer Academic Publishers, Dordrecht, 1998, p. 135.

Farrell's input-oriented efficiency measures can be illustrated by the above figure. PP` line shows the ratio of input prices. If the firm uses (x_1, x_2) units to produce q, point A represents $(x_1/q, x_2/q)$. The point E that lies on the unit isoquant represent an efficient combination of two factors of production in the same ratio as A. It is obvious that the point E produces the same output as A using only a fraction (OE/OA) as much of each factor or produces (OA/OE) times as much output from the same inputs. Thus, the ratio (OE/OA) is called as technical efficiency of the firm. Input-oriented technical efficiency measure reveals the answer in reply to by how much input quantities can be proportionally reduced without changing the output quantities produed.

The slope of the isoquant at E' is equal to the ratio of prices of the two factors of production; so optimality condition is realized. Thus, the costs of production at E' is only a fraction (OB/OE) of those at E. This ratio, (OB/OE), is called as allocative efficiency of E. Every point on the isoquant II' shows that the firm is technically full efficient. But it may have allocative inefficiency due to utilizing inputs in the wrong proportions. Therefore, it is obvious from Figure 3 that while keeping technical efficiency constant it is possible to increase allocative efficiency or decrease allocative inefficiency by moving from E to E'. The product of the technical and the allocative efficiency is called as overall economic efficiency, which is the ratio (OB/OA).[23]

Farrell's output-oriented efficiency measures can be illustrated by Figure 4. below.

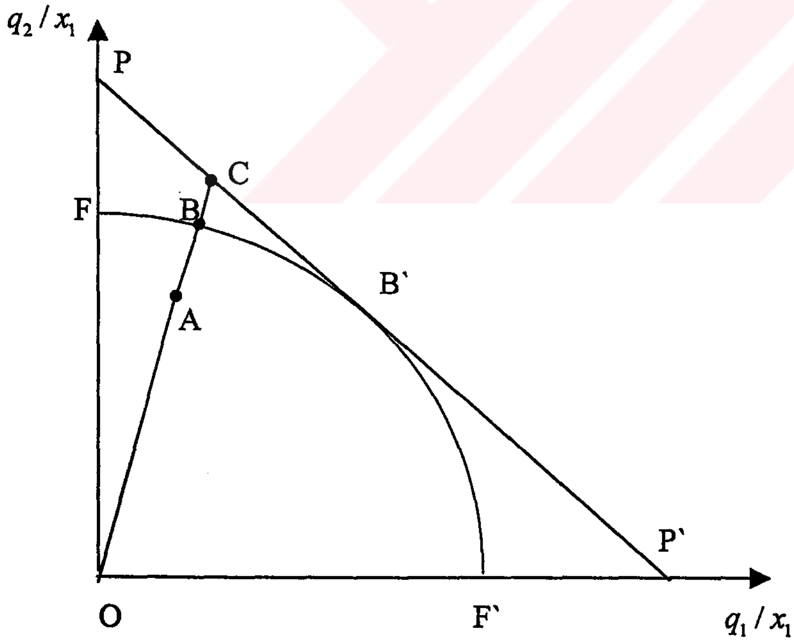

Figure 4. Technical and Allocative Efficiencies from an Output Orientation

Source: Tim Coelli, et. al., **An Introduction to Efficiency and Productivity Analysis**, Kluwer Academic Publishers, Dordrecht, 1998, p. 138.

[23] **Ibid.**, pp .254-255.

From Figure 4, the production consists of two outputs and a single input. If we hold the quantity of input fixed at a particular level, the technology can be demonstrated by a production possibility curve, FF'. PP' shows isorevenue line. The point A represents an inefficient firm's production. Output-oriented technical efficiency is the ratio (OA/OB). Output-oriented technical efficiency measure reveals the answer in reply to by how much output quantities can be proportionally expanded without changing the input quantities used. Allocative efficiency is defined by the ratio (OB/OC). Overall economic efficiency is then defined by the product of technical efficiency and allocative efficiency, that is (OA/OC).[24]

Let x be an input vector $x \equiv (x_1, x_2, ..., x_n)$ and w be an input price vector $w \equiv (w_1, w_2, ..., w_n)$. Suppose that only one product, q (shows actual production level) is produced and that is sold at fixed price $p > 0$. Production technology and maximum potential output is represented by $f(x)$ and assumed to be continuously differentiable.

An efficient production technology is also represented by a cost function. That is:

$$C(q, w) = \min_x \{wx \setminus f(x) \geq q, x \geq 0\} \tag{2.2.3}$$

that shows the minimum cost required to produce q at input prices w. Another representation of efficient production technology is provided by profit function. That is:

$$\pi(p, w) = \max_{x,y} \{pq - wx \setminus f(x) \geq q, x \geq 0, q \geq 0\} \tag{2.2.4}$$

[24] Tim Coelli, et. al., **An Introduction to Efficiency and Productivity Analysis**, Kluwer Academic Publishers, Dordrecht, 1998, pp. 137-139.

that shows the maximum profit available at input prices w and output price p. If $q = f(x)$, then the firm is said to be technically efficient, and if $q < f(x)$ it is said to be technically inefficient. Technical inefficiency results from excessive input usage so costs are not minimized that is, $wx > C(q,w)$ and profits are not maximized that is, $pq - wx < \pi(p,w)$.[25]

Koopmans gives a formal definition of technical efficiency:

> "A producer is technically efficient if an increase in any output requires a reduction in at least one other output or an increase in at least one input, and if a reduction in any input requires an increase in at least one other input or a reduction in at least one output. Thus, a technically inefficient producer could produce the same outputs with less of at least one input, or could use the same inputs to produce more of at least one output."

The firm is said to be allocatively efficient if $\frac{f_i(x)}{f_j(x)} = \frac{w_i}{w_j}$ and allocatively inefficient if $\frac{f_i(x)}{f_j(x)} \neq \frac{w_i}{w_j}$. As noted above, since allocative inefficiency is due to utilizing inputs in the wrong proportions, costs are not minimized and profits are not maximized.[26]

Both technical and allocative efficiency are necessary but not sufficient for profit maximization. If we leave the assumption that production prevails under constant returns to scale, the firm could be scale inefficient. The firm is said to be scale efficient if $p = C(q,w)$ and scale inefficient if $p \neq C(q,w)$. Profits are maximized and costs are minimized if and only if the firm is technically, allocatively and scale efficient. If the scale is too large or too small to minimize costs, the firm

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[25] Førsund, **op.cit.**, p. 6.

[26] Førsund, **op.cit.**, pp. 6-7.

will have scale inefficiency. In addition to that, highly complex hierarchical and bureaucratic structure can be also counted as a source of scale inefficiency.[27]

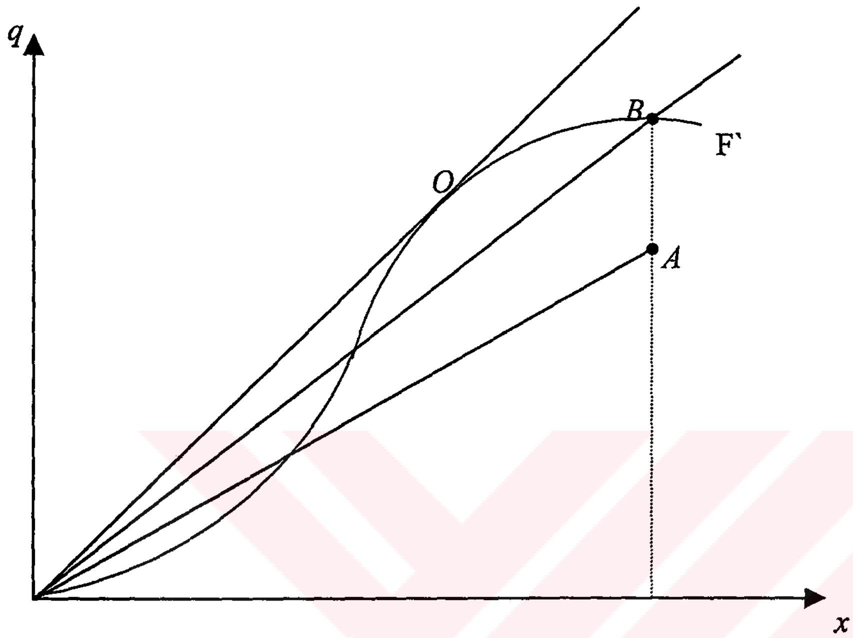

Figure 5. Productivity and Scale Economies

Source: Tim Coelli, et. al., **An Introduction to Efficiency and Productivity Analysis,** Kluwer Academic Publishers, Dordrecht, 1998, p. 5.

In Figure 5, a ray through the origin is used to measure the productivity at a particular point. The slope of the ray is (q/x) which corresponds to a measure of productivity. The production unit can reach a higher productivity level by moving from point A to point B by noting the fact that the slope of the ray connecting the origin to point B is greater than the slope of the ray connecting the origin to point A. However, by moving to point O, the production unit can reach maximum possible productivity. This latter movement is an example of exploiting scale economies. The point O is the point of technically optimal scale. Operation at any other point on the production frontier results in lower productivity.

[27] Mayes, **op.cit.,** p. 21.

2.3 Determinants of Inefficiency

Under competitive conditions inefficieny is removed. In reality, markets have imperfections and it is reasonable to measure and determine the extent to which these imperfections occur. In this context, imperfections can be categorized in three sub-sections: market structure, spatial disparities, and managerial and organisational influences.

2.3.1 Market Structure

It is stated that if there is perfect competition in the economy there won't be any problem of inefficiency. In reality, economies differ from that hypothetical environment. Products are not identical, firms sell their products for different prices, firms differ in productivity, allocation of resources is not optimal, and so on. In this context, it is sensible to identify the sources and the degree of imperfections.

First way of deciding if an industry exhibit free competition is to look at the market structure. Whether the market exhibits monopolistic or oligopolistic structure or not is important at this stage and firm concentration, market share and barriers to entry, which constitute elements of market structure, are good indicators in analyzing that matter.[28]

Large firms can dominate the market and the production can be concentrated in the hands of a few large companies. For example, cartels can restrict smaller firms to compete through pricing policies without any explicit collusion. So, it can be hypothesized that in an industry where firm concentration is high these firms will tend to be more inefficient. This is not to strictly say that there is a consistent positive linear relationship between firm concentration and inefficiency. Emprical studies indicate a curvilinear relationship between firm concentration and inefficiency.

[28] David Mayes, Christopher Harris, Melanie Lansbury, **Inefficiency in Industry**, Harvester Wheatsheaf, London, 1994, pp. 122-123.

Unless the market is segmented, spread of prices will be low and the market exhibits fairly low imperfections. Thus, in a concentrated industry firms could be efficient due to high standardized products whereas in a non-concentrated industry firms could be inefficient because the market is segmented.[29]

Measurement of concentration in an industry can be stated with two different viewpoints. One is to measure the proportion of output attributable to top X firms in the industry. The other one is to determine the number of firms that comprise X per cent of the industry.[30]

The concentration ratio is expressed in terms of CR_X which indicates the percentage of the market share controlled by the biggest X firms. For instance, $CR_2 = 60\%$ denotes the fact that top 2 firms control 60 percent of the market. Most widely used concentration ratio for judging the extent to which market exhibits imperfections and oligopolistic structure is CR_4. A CR_4 of over 50% is generally considered to be a tight oligopoly and a CR_4 between 25% and 50% is considered to be a loose oligopoly. In addition to that, the market is considered to be super-tight oligopoly if a CR_3 is of over 90% or CR_2 of over 80%.[31]

Since CR_X does not indicate the relative sizes of the firms an alternative measuring tool, the Herfindahl index, is introduced to determine the maket structure. The Herfindahl index is given by:

$$H = \sum_i s_i^2 \qquad (2.3.1.1)$$

[29] **Ibid.**, p. 124.

[30] **Ibid.**, p. 124.

[31] <http://www.oligopolywatch.com/2003/08/15.html>

where s_i indicates the percentage market share of the i-th firm in the industry. The higher the index, the more the concentration and less open market comptetition. Maximum score, that is 10.000, is observed in the case of monoploy because a monopoly has 100% of the market and square of 100 is 10.000. In duoploy case where each firm has a market shares of 50% , H index will be $50_1^2 + 50_2^2 = 5000$. On the contrary, an industry having 100 firms that each has market shares of 1% will have a score of $1_1^2 + 1_2^2 + ... + 1_{99}^2 + 1_{100}^2 = 100$. A 1000-2000 value of the index generally indicates intense competition. Thus, the Herfindahl index gives a better indication of the relative market control of the largest firms.[32]

Another way of deciding if an industry exhibits free competition is to look at firms' market share. It is the most widely used indicator of the firms' degree of monopoly power. The higher the market share, the higher the monopoly power the firm has. Low market share usually defines that the firm is under strong competitive pressure.[33]

The absolute degree of market power depends on the firm's demand elasticity conditions and varies with demand inelasticity. The simple raising of the single price and price discrimination have effects on prices. Varying demand conditions allow for firm to divide the market into different consumer groups and to charge different prices for each group.

The extent of competition could also be measured by the openness of the market to imports and this can be done through the ratio of imports to sales. Theoretically, inefficiency decreases as the market becomes more open to foreign competition. Inefficient firms will be unable to compete and go out of the market. When relative comparison is made between domestic and foreign firms, a high import ratio could indicate that the industry is inefficient. Because there may be

[32] < http://www.oligopolywatch.com/2003/08/15.html>

[33] William G. Shepherd, **The Economics of Industrial Organization**, 3rd. ed., Prentice-Hall International, New Jersey, 1990, p. 62.

excess market share which the foreign firms are able to fill. Thus, it is reasonable to take into account the industry's ability to export, the ratio of exports to sales. The greater the ability of an industry to export, the greater its international competitiveness and hence the lower the inefficiency. [34]

Ease of entry and exit is another element of market structure and plays a critical role in determining market structure. Entry is the addition of new firms into the market. On the other hand, exit is the subtraction of incumbent firms from the market.

Since free entry and exit is a property of perfectly competitive market by defining the ease or difficulty of entry and exit, a market can be classified as competitive or not. Imperfections in the market can be due to existence of both barriers to entry and exit.

Barriers to entry are some conditions that prevent new firms from entering a market. The sources of barriers to entry come in two main categories. These are exogeneous and endogeneous sources. Exogeneous sources are the ones that arise from the market conditions and outside the control of the new entrants. Endogeneous sources are strategic actions which dominant firm takes against potential new entrants.[35]

High capital requirement, especially in capital-intensive industries, where minimum efficient scale is large prevents entry into the industry. Scale economies may encourage new firms to penetrate the industry leading to an increase in total capacity of the industry. This causes prices to fall which in turn reduces the gain to the new entrants. If economies of scale is large, the incumbent firms tend to have large market shares and entrants may be exposed to sharp penalties brought into force by incumbent firms. Product differentiation is another source of barrier to entry

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[34] Mayes, **op.cit.,** p. 129.

[35] Shepherd, **op.cit.,** p. 273.

and arises from advertising and other marketing strategies. The incumbent firms may have brand loyalty so may have advantages over the new entrants. Thus, new entrants may not try to establish their brand names. Absolute cost advantage can also be counted as an exogeneous source and arises from differential wage rates, superior talent, random luck, and historical accident.[36]

By taking retaliation and preemptive actions, creating and expanding excess capacity, increasing selling expenses in order to strengthen brand loyalty, strategic patenting, controlling many critical resources incumbent firms may prevent new firms from competition. All these actions taken by the incumbent firms constitute endogeneous sources of barrier to entry.[37]

Exit an industry is another consideration in understanding a firm's tendency to enter an industry. If the exit an industry is costly due to high sunk costs that cannot be recouped, there is less incentive to enter the industry. So, costs of exit serve as a barrier to entry just as do costs of entering an industry. Therefore, industries that show high levels of entry and exit present lower levels of inefficiency because new firms will push out inefficient firms.[38]

2.3.2 Spatial Disparities

Spatial disparities can be mentioned as another factor that have influence on industrial inefficiency. There are two main ways in which spatial disparities have influence on industrial efficiency. First one arises from the fact that inputs can disperse differently across the country and quality of products can vary considerably. For instance, this effect can be easily seen in agricultural sector. Variation in wheather conditions will affect the quality of agricultural products which in turn has

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[36] Shepherd, **op.cit.**, pp. 274-275.

[37] Shepherd, **op.cit.**, pp. 275-276.

[38] Dennis W. Carlton, Jeffrey M. Perloff, **Modern Industrial Organization**, 2nd. ed., Harper Collins, 1994, p. 111.

influences on food manufacturing industries. Not only agricultural sector but also services sector may be affected by spatial disparities. Labor skills in different labor markets can vary across the country so business services may be higher in the main business center. Thus, differences in distribution of production across the country cause efficiency differentials because of differences in productive conditions.[39]

Second one arises from demand conditions. If demand is very localized or market size is limited, there is a possibility for a considerable variation in competitive conditions. Monopolistic competition may exist in rural zones whereas relatively higher competition may exist in urban areas.[40]

2.3.3 Managerial and Organizational Influences

Consider two identical firms with the same scale of operation, using the same technology and same combination of inputs to produce a homogeneous product. The differences in output produced by two identical firms could only be attributed to differences in how the two firms are organized and managed. This source of inefficiency is called as X-inefficiency explained by non-optimal behaviour of firms i.e., firms producing at a scale where costs are not minimized and profits are not maximized.[41]

Motivational factors, proper incentives, reorganization of production process that costitute X-efficiency play a critical role in improving the efficiency and so does the productivity. Motivational factors are important because the relation between inputs and outputs does not show a predetermination. There are four reasons why given inputs cannot be transformed into predetermined outputs: 1) contracts for labor are incomplete, 2) not all production factors are marketed, 3) the production function

[39] David Mayes, Christopher Harris, Melanie Lansbury, **Inefficiency in Industry**, Harvester Wheatsheaf, London, 1994, p. 140.

[40] **Ibid.**, p. 140.

[41] **Ibid.**, p. 14.

is not completely specified, 4) interdependence and uncertainty force firms to collaborate implicitly with each other and to imitate each other with respect to technique. When the motivation is weak, the firm managers will slack in their operations and will not search for cost-reducing methods. On the other hand, competitive pressures lead to cost reductions, and the absence of such pressures causes costs to increase. In addition to motivational factors, empirical evidence suggests that increased efficiency and productivity may arise from psychological factors. The empirical findings point out that small working groups are more productive than larger groups, working groups made up of friends are more productive than those made up of non-friends, groups that are generally supervised are more efficient than those that are closely supervised and groups that are given more information about the importance of their task are more efficient than those given less information.[42]

The nature of the management, the environment in which the firm operates and the incentives employed are also important. A well-designed and proper operated incentives can lead working tempo to vary both between different workers and different departments without any need to change purchasable inputs per unit. For instance, improvement in relations between labor and management can create a positive motivation among workers.[43]

[42] Harvey Leibenstein, "Allocative Efficiency vs. X-Efficiency," **American Economic Review**, Vol. 56, June 1966, pp. 397-409.

[43] **Ibid.**, p. 401.

3. PRODUCTION FRONTIERS AND EFFICIENCY ESTIMATION

Production frontiers have been estimated by means of many different methods. There are two main methods that have been used: Data Envelopment Analysis (DEA) and Stochastic Frontiers Approach. DEA includes mathematical programming techniques, on the other hand stochastic frontiers involve econometric methods. Classification can also be made according to the way that whether the function is specified as a parametric function of inputs or non-parametric, a statistical model of the relationship between observed output and the frontier, and a deterministic or a random. Eigth permutations of these possibilities can be taken into consideration.

Attempts to estimate the production frontiers began with the work of Farrell (1957). Afterwards, the idea was developed by Farrell and Fieldhouse (1962), Afriat (1972) and tested by Aigner and Chu (1968), Seitz (1971), Richmond (1974) and Førsund and Jansen (1977). Their methods are based on deterministic frontiers. The major drawback of their approach is that their models do not allow for random events in the production process which are beyond the control of the production units. Aigner, Lovell and Schmidt (1977) and Meeusen and van den Broeck (1977) overcame this problem by independently proposing the stochastic production frontier model. They estabilished a production frontier model consisting of a production function of the usual regression type with an error term composed of two components. The first component involves random events (such as weather, luck, machine performance, input supply breakdowns, and other events beyond the control of the production units) and is assumed to be normally distributed; the second component involves non-positive technical inefficiency and is assumed to have a one-sided distribution. Non-parametric approach dating back to Farrell (1957) employs mathematical programming techniques in constructing production frontiers and measures the efficiency of a production unit relative to all other production units with restriction that all production units lie on or below the efficient frontier. In contrast to parametric approach, non-parametric approach does not require any assumptions about the functional form for the frontier technology.

Since Cobb and Douglas started running regressions the empirical analysis of production frontiers has been based on a least squares statistical methodolgy. Efficient scores have been neglected in favor of less efficient scores but more likely values. All attention has been directed from extreme scores to average scores.

In this section, econometric methods, mathematical programming techniques and deterministic frontier models are introduced and discussed. These two approaches use different techniques to calculate the efficiency scores. The econometric approach is stochastic and attributes all variation in output not explained by variation in inputs to some combination of random events and technical ineffciency. Mathematical programming approach lumps random events and technical inefficiency together and calls the combination inefficiency. Deterministic production frontier models do not specify the separated effects of random events and technical inefficiency on the variation in output and attributes all variation in output not explained by variation in inputs to technical inefficiency only.

The econometric approach and deterministic frontier models are parametric. They are open to the effects of misspecification of functional form with inefficiency and technology. The mathematical programming approach is non-parametric and less inclined to this type of specification error in this sense.

3.1 Mathematical Programming Approach: Simple Data Envelopment Analysis Model

The mathematical programming approach to the costruction of production frontiers and the measurement of efficiency is done through Data Envelopment Analysis, DEA. It envelops a data set as tightly as possible subject to some assumptions about the structure of the production technology and makes no accommodation for statistical noise. DEA consists of techniques, based on the observed input/output data, for measuring the relative efficiency of Decision Making Units (DMU*s*). DEA provides some techniques that serve as a new way of obtaining empirical estimates of production functions and efficient production possibility surfaces. These techniques are non-parametric.[1]

DMU is an organization under consideration in DEA and regarded as the entity responsible for converting inputs into outputs and whose performances to be evaluated. DMU*s* may involve banks, schools, hospitals, and various public enterprises.

DEA begins with the Edwardo Rhode's Ph.D. dissertation research (1978), which was about an educational program for disadvantaged students, at Carnegie Mellon University's School of Urban and Public Affairs under the supervision of W.W. Cooper. Rhode's research then resulted in the formulation of A. Charnes, W.W. Cooper and E. Rhode, known as the basic CCR model. CCR model uses the mathematical programming optimization method to generalize the Farrell's single output/input technical efficiency to the multiple output/input technical efficiency score by constructing a single virtual output to a single virtual input relative efficiency measure for each DMU. For a particular DMU, the ratio of single virtual output to single virtual input yields a measure of efficiency, that can be compared with other DMU*s* in the system. The relative technical efficiency score of any DMU is calculated by means of forming a ratio of a weighted sum of outputs to a weighted sum of inputs. Multipliers for both outputs and inputs are selected so that Pareto efficiency measure of each DMU is obtained subject to constraint that none of DMU can have a relative efficiency score greater than one.[2]

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[1] Lawrance M. Seiford, Robert M. Thrall, "Recent Developments in DEA: The Mathematical Programming Approach to Frontier Analysis," **Journal of Econometrics**, Vol. 46, 1990, p. 8.

[2] **Data Envelopment Analysis: Theory, Methodology and Application**, Ed. by. Abraham Charnes, William W. Cooper, Arie Y. Lewin, Lawrence M. Seiford, Kluwer Academic Publishers, Dordrecht, 1994, p. 3-6.

$$\text{Virtual Output} = u_1 q_{1o} + u_2 q_{2o} + \ldots + u_m q_{mo}$$

$$\text{Virtual Input} = \nu_1 x_{1o} + \nu_2 x_{2o} + \ldots + \nu_n x_{no}$$

where o ranges over $1, 2, \ldots, K$.

The multipliers to construct a single virtual output and a single virtual input from multiple outputs and inputs are interpreted in three different ways. First interpretation of the multipliers are the shadow prices of outputs and inputs. These shadow prices are the optimal values of appropriate Lagrange multipliers associated with the linear programming formulations of the appropriate DEA models in CCR model. Multipliers are non-negative weights as in the theory of index numbers in the second sense. In the third sense, multipliers are the parameters of a suitable production frontier implicit in the data. This sense was proposed first by Farrell (1957) for single output and single input and formed the non-parametric models of efficiency measurement.[3]

Simple DEA model corresponds to a model which assumes constant returns to scale in the production, strong disposibility of inputs and outputs, convexity of the set of feasible input-output combinations and it has input-orientation. All these assumptions can be relaxed by adding some constraints to the simple DEA mathematical programming problem.[4]

3.1.1 Input-Orientation

Assuming that there are n inputs and m outputs for each of K producers, DMU*s* use input vector $x \in R_+^n$ to produce output vector $q \in R_+^m$. The $n \times K$ input matrix, X, and $m \times K$ output matrix, Q, represent the data for all DMU*s*.

[3] Jati K. Sengupta, **Dynamics of Data Envelopment Analysis: Theory of Systems Efficiency**, Kluwer Academic Publishers, Dordrecht, 1995, pp. 1-2.

[4] C.A. Knox Lovell, "Production Frontiers and Productive Efficiency," in **The Measurement of Productive Efficiency: Techniques and Applications**, Ed. by Harold O. Fried, C.A. Knox Lovell, Shelton S. Schmidt, Oxford University Press, New York, 1993, p. 28.

$$X = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1K} \\ x_{21} & x_{22} & \cdots & x_{2K} \\ \vdots & \vdots & \vdots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nK} \end{pmatrix} \qquad Q = \begin{pmatrix} q_{11} & q_{12} & \cdots & q_{1K} \\ q_{21} & q_{22} & \cdots & q_{2K} \\ \vdots & \vdots & \vdots & \vdots \\ q_{m1} & q_{m2} & \cdots & q_{mK} \end{pmatrix}$$

The object is to measure the performance of each DMU relative to the most efficient DMU in the sample of K producers.

Three different forms of the DEA linear programming problem is possible to be demonstrated. Via the ratio $u'q_i / v'x_i$, a measure of the ratio of all outputs to all inputs can be obtained, where u is $m \times 1$ vector of output weights and v is $n \times 1$ vector of input weights. The optimal weights are obtained by solving the following mathematical programming problem:

$$\max_{u,v}(u'q_i / v'x_i), \tag{3.1.1.1}$$

subject to

$$u'q_j / v'x_j \leq 1, \qquad j = 1,2,...,K$$
$$u, v \geq 0$$

where q_i and x_i are column vectors for the *i*-th DMU.

The above mathematical programming problem replies to optimal values for u and v, such that the efficiency measure for i-*th* DMU is maximized subject to constrained that all efficiency scores must be less than or equal to unity. Infinite number of solutions problem arises with this "ratio form" of DEA model. To eliminate this problem, $v'x_i = 1$ is imposed.

$$\max_{\mu,\nu}(\mu' q_i), \tag{3.1.1.2}$$

subject to

$$\nu' x_i = 1,$$
$$\mu' q_j - \nu' x_j \leq 0, \qquad j = 1, 2, \ldots, K$$
$$\mu, \nu \geq 0$$

The above form is known as the "multiplier form" of the DEA mathematical programming problem.[5]

The "envelopment form" of the DEA mathematical programming problem can be derived as follows:

$$\min_{\theta,\lambda} \theta, \tag{3.1.1.3}$$

subject to

$$-q_i + Q\lambda \geq 0,$$
$$\theta x_i - X\lambda \geq 0,$$
$$\lambda \geq 0$$

where θ is scalar and λ is Kx1 intensity vector. Problem (3.1.1.3) is solved K times to generate K optimal values of (θ, λ). The efficiency score is measured by θ. It satisfies $\theta \leq 1$ and with a value of 1 indicating a technically efficient firm according to the Farrell definition.[6]

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[5] Tim Coelli, et. al., **An Introduction to Efficiency and Productivity Analysis,** Kluwer Academic Publishers, Dordrecht, 1998, p. 141.

[6] **Ibid.,** p. 141.

3.1.2 Output-Orientation

The above mathematical programming problems are input-oriented. Output-oriented versions can be obtained by replacing equation (3.1.1.1) with a minimization problem, equation (3.1.1.2) with a minimization multiplier problem, and equation (3.1.1.3) with a maximization envelopment problem.[7]

The output-oriented ratio form of DEA mathematical programming problem is:[8]

$$\min_{u,v}(v'x_i / u'q_i) \qquad (3.1.2.1)$$

subject to

$$v'x_j / u'q_j \geq 1, \qquad j = 1,2,...,K$$
$$u, v \geq 0$$

Imposing $u'q_i = 1$, the output-oriented multiplier form of DEA mathematical programming problem is:[9]

$$\min_{\upsilon,u}(\upsilon' x_i), \qquad (3.1.2.2)$$

subject to

$$u'q_i = 1,$$

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[7] C.A. Knox Lovell, "Production Frontiers and Productive Efficiency," in **The Measurement of Productive Efficiency: Techniques and Applications**, Ed. by. Harold O. Fried, C.A. Knox Lovell, Shelton S. Schmidt, Oxford University Press, New York, 1993, p. 28.

[8] **Ibid.**, p. 27.

[9] **Ibid.**, p. 27.

$$\upsilon' x_j - u' q_j \geq 0, \quad j = 1,2,...,K$$

$$\upsilon, u \geq 0$$

The output-oriented envelopment form of DEA mathematical programming problem is:[10]

$$\max_{\theta,\lambda} \theta, \qquad (3.1.2.3)$$

subject to

$$X\lambda \leq x_i,$$

$$\theta q_i \leq Q\lambda,$$

$$\lambda \geq 0$$

$\theta = 1$ is necessary but not sufficient for technical efficiency since $(\theta q_i, x_i)$ may contain slacks. Koopmans gives a formal definition of technical efficiency stating the fact that a producer is only technically efficient if it operates on the production frontier and slacks are zero. The condition that $\theta = 1$ is referred to as radial efficiency and also called as technical efficiency since a value of $\theta < 1$ corresponds to the fact that all inputs can be simultaneously reduced without changing the input proportions. Because production possibility set allows for maximum $1 - \theta$ proportional reduction. On the other hand, any further reductions associated with non-zero slacks will necessarily change the input proportions. So, the inefficiencies associated wih non-zero slacks are referred to as mix inefficiencies.[11]

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[10] **Ibid.**, p. 28.

[11] William W. Cooper, Lawrance M. Seiford, Kaoru Tone, **Data Envelopment Analysis: A Comprehensive Text with Models, Applications, References and DEA-Solver Software**, Kluwer Academic Publishers, Dordrecht, 2000, p. 45.

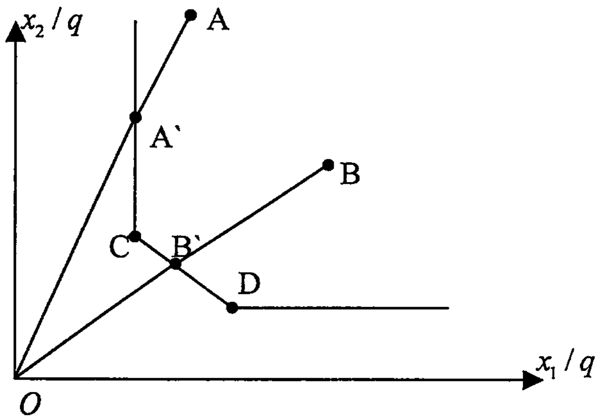

Figure 6. Input Slacks

Source: Tim Coelli, et. al., **An Introduction to Efficiency and Productivity Analysis**, Kluwer Academic Publishers, Dordrecht, 1998, p. 143.

In the above figure, the Farrell measures of technical efficiency are given by $(OA`/OA)$ for producer $A$. The producer $A$ could still produce the same amount of output by reducing the amount of input $x_2$ (that is, the amount of $CA`$). This is known as input slack.

The assumption that constant returns to scale prevails in the production is most commonly relaxed. Constant returns to scale assumption is only appropriate when all DMU*s* are operating at an optimal scale. However, market imperfections may cause a DMU to be not operating at optimal scale. When all DMU*s* are not operating at optimal scale, the use of constant returns to scale assumption deteriorates technical efficiency scores. If the technology is variable returns to scale, a scale efficiency score can be obtained by conducting both a constant returns to scale and a variable returs to scale DEA models. If there is a difference between the constant returns to scale and the variable returns to scale technical efficiency scores, this states that the firm has scale inefficiency. So, the use of variable returns to scale assumption in DEA model will give more accurate technical efficiency measures.[12]

[12] Coelli, **op.cit.**, p. 150-151.

The above CCR model in equation (3.1.1.3) can be modified by adding the constraint $K'\lambda = 1$ where K is Kx1 vector of ones. So, the model in equation (3.1.1.3) becomes as follows; known as BCC (Banker, Charnes and Cooper) model.

$$\min_{\theta,\lambda} \theta, \tag{3.1.2.4}$$

subject to

$$-q_i + Q\lambda \geq 0,$$
$$\theta x_i - X\lambda \geq 0,$$
$$K'\lambda = 1,$$
$$\lambda \geq 0$$

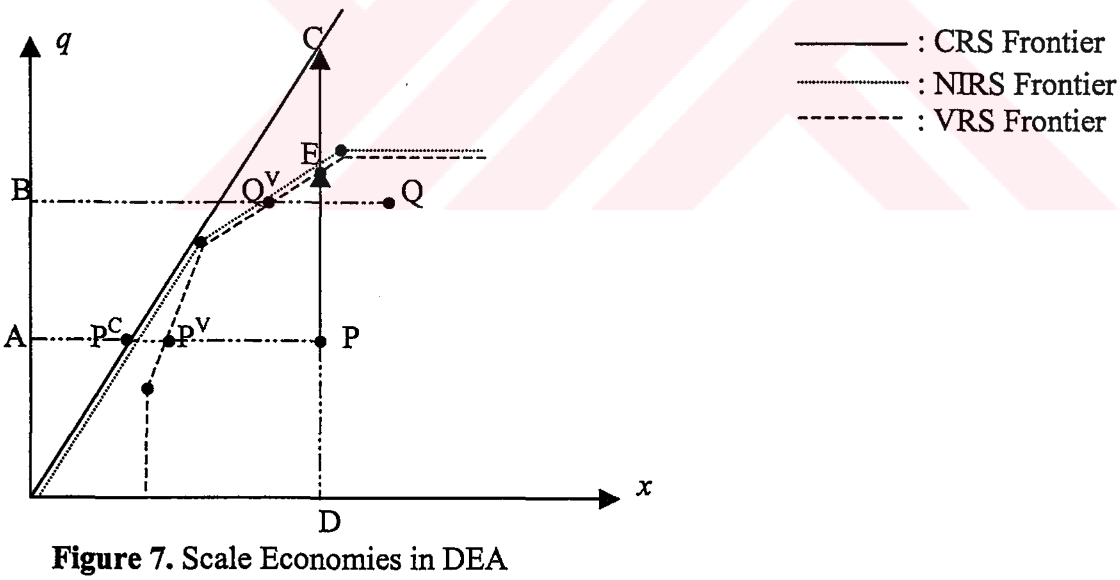

Figure 7. Scale Economies in DEA

Source: Tim Coelli, et. al., **An Introduction to Efficiency and Productivity Analysis**, Kluwer Academic Publishers, Dordrecht, 1998, p. 152.

Constant returns to scale, variable returns to scale and non-incerasing returns to scale frontiers are illustrated in Figure 7. *CRS*, *VRS* and *NIRS* correspond to constant returns to scale, variable returns to scale and non-increasing returns to scale, respectively. In Figure 7, under *CRS* the input-oriented technical inefficiency of the point P is the distance PP^C whereas under VRS the technical inefficiency is only PP^V. The difference between these two distances, $P^C P^V$ indicates scale inefficiency. If we are to demonstrate technical and scale efficiencies in ratio terms, that is:

$$TE_{CRS} = AP^C / AP \tag{3.1.2.5}$$

$$TE_{VRS} = AP^V / AP \tag{3.1.2.6}$$

$$SE = AP^C / AP^V \Rightarrow TE_{CRS} = TE_{VRS} SE \tag{3.1.2.7}$$

where TE_{CRS} , TE_{VRS} and SE correspond to CRS technical efficiency, VRS technical efficiency and scale efficiency, respectively. From equation (3.1.2.7), CRS technical efficiency measure has two components so that it can be decomposed into pure technical efficiency and scale efficiency.[13]

In Figure 7, the input-oriented technical efficiency measures for the producer which is on the point P are the same for constant returns to scale and non-inreasing returns to scale tehnologies, that is $\left(AP^C / AP\right)$. The input-oriented technical efficiency measures for the producer which is on the point Q are the same for non-increasing returns to scale and variable returns to scale technologies, that is $\left(BQ^V / BQ\right)$. The output-oriented technical effiicency measures for that producer which is on the point P are the same for non-increasing returns to scale and varibale returns to scale technologies, but lower for constant returns to scale technology since $(DP / DC) < (DP / DE)$.

In spite of the fact that CRS technical efficiency measure is decomposed into pure technical efficiency and scale efficiency, scale efficiency score does not indicate whether the DMU is operating in an area of increasing or decreasing returns to scale. This fact can be overcome by introducing another constraint into equation (3.1.2.4).

[13] Coelli, op.cit., p.151.

Non-increasing returns to scale is modeled by adding $K'\lambda \leq 1$ constraint into the equation (3.1.2.4). So, the model in equation (3.1.2.4) becomes;

$$\min_{\theta,\lambda} \theta, \tag{3.1.2.8}$$

subject to

$$-q_i + Q\lambda \geq 0,$$
$$\theta x_i - X\lambda \geq 0,$$
$$K'\lambda \leq 1,$$
$$\lambda \geq 0$$

If *NIRS* technical efficiency score is equal to *VRS* technical efficiency score, then decreasing returns to scale prevails in the production for that DMU. The point Q corresponds to that case. If they are not equal, then increasing returns to scale exists for that DMU as is the case for the point P in Figure 7.[14]

The assumption of strong disposibility of inputs and outputs can also be relaxed. Weak disposibility of inputs and outputs is modelled by replacing the constraints $-q_i + Q\lambda \geq 0$ and $\theta x_i - X\lambda \geq 0$ in equation (3.1.1.3) with $q_i = \alpha Q\lambda$ and $\beta\theta x_i = X\lambda$, $\alpha, \beta \in (0,1]$:[15]

$$\min_{\theta,\lambda} \theta, \tag{3.1.2.9}$$

subject to,

$$q_i = \alpha Q\lambda,$$

[14] Coelli, **op.cit.**, pp. 151-152.

[15] Lovell, **op.cit.**, p. 30.

$\beta\theta x_i = X\lambda,$

$\lambda \geq 0 \qquad \lambda_i \in \{0,1\} \qquad i = 1,\dots,K$

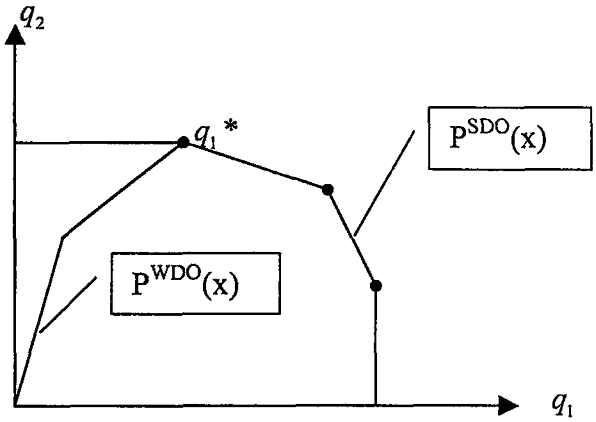

Figure 8. Strong and weak disposibility of outputs

Source: C.A. Knox Lovell, "Production Frontiers and Productive Efficiency," in **The Measurement of Productive Efficiency: Techniques and Applications**, Ed. by. Harold O. Fried, C.A. Knox Lovell, Shelton S. Schmidt, Oxford University Press, New York, 1993, p. 31.

In Figure 8, $P^{WDO}(x)$ and $P^{SDO}(x)$ denote weak disposable and strong disposable output sets, respectively. Output q_1 is not freely disposable throughout the positive part relative to the output set $P^{WDO}(x)$. Any reduction in q_1 beneath q_1* requires either a reduction in q_2 or an increase in input usage. Both of them are costly, therefore disposal is not free.[16]

[16] Lovell, **op.cit.**, p. 30.

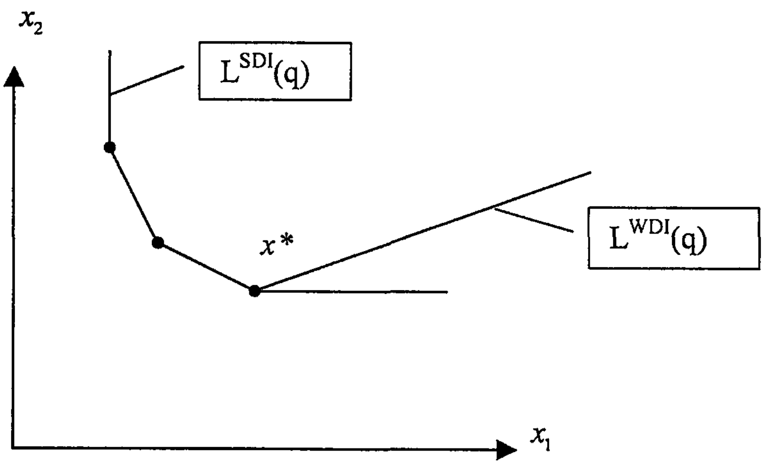

Figure 9. Strong and weak disposability of inputs

Source: Lovell, C.A. Knox., "Production Frontiers and Productive Efficiency," in **The Measurement of Productive Efficiency: Techniques and Applications**, Ed. by. Harold O. Fried, C.A. Knox Lovell, Shelton S. Schmidt, Oxford University Press, New York, 1993, p. 31.

In Figure 9, $L^{WDI}(q)$ and $L^{SDI}(q)$ denote weak disposable and strong disposable input sets, respectively. Input x_1 is not freely disposable through the positive part relative to input set $L^{WDI}(q)$ owing to the fact that an increase in x_1 reduces output or requires an increase in x_2 in order to maintain same level of output. By imposing weak and strong disposability constraints, a comparison about the nature of disposability of outputs and inputs can be made.[17]

The assumption of convexity of output sets and input sets can be relaxed by adding $K'\lambda = 1$ constraint into the equation (3.1.1.3). A free disposal hull, FDH, relaxes the convexity of $P(x)$ and of $L(q)$ but maintains the assumptions of strong disposability of outputs and inputs and variable returns to scale. Adding $K'\lambda = 1$ constraint into the equation (3.1.1.3) the programming model becomes:[18]

[17] Lovell, **op.cit.,** pp. 30-31.

[18] Lovell, **op.cit.,** pp. 31-32.

$$\min_{\theta,\lambda} \theta, \qquad (3.1.2.10)$$

subject to

$$-q_i + Q\lambda \geq 0,$$

$$\theta x_i - X\lambda \geq 0,$$

$$K'\lambda = 1, \qquad \lambda_i \in \{0,1\} \qquad i = 1,...,K$$

$$\lambda \geq 0$$

DEA and FDH frontiers are compared in Figure 10. The FDH frontier envelops the data more tightly and has a more restirictive notion of domination than the DEA frontier. A DMU is FDH-dominated by a single observed efficient DMU, since $K'\lambda = 1$ and $\lambda_i \in \{0,1\}$ whereas a DMU is DEA-dominated by a fictitious observation defined as a linear combination of a set of efficient DMU*s*. In Figure 10, $L^{FDH}(q) \subseteq L^{DEA}(q)$ and $\text{Eff}\, L^{FDH}(q)$ consists of the input vectors x^B, x^C and x^D. Inefficient input vector x^A is dominated by x^B, and has Debreu-Farrell technical efficiency score λ^A with slack in input x_2.[19]

[19] Lovell, **op.cit.**, pp. 32-33.

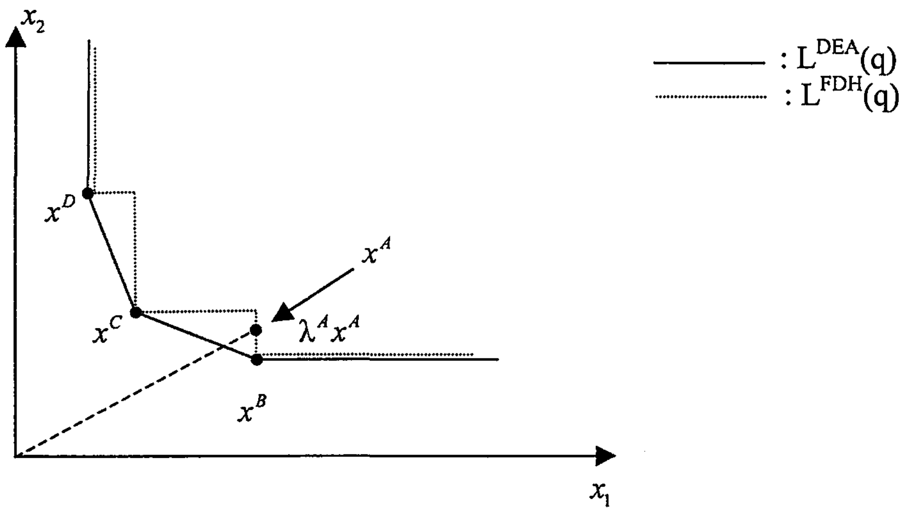

Figure. 10 Technical efficiency measurement relative to DEA and FDH frontiers

Source: C.A. Knox Lovell, "Production Frontiers and Productive Efficiency," in **The Measurement of Productive Efficiency: Techniques and Applications**, Ed. by. Harold O. Fried, C.A. Knox Lovell, Shelton S. Schmidt, Oxford University Press, New York, 1993, p. 32.

3.2 Deterministic Production Frontiers

Deterministic production frontiers attribute all variation in output not explained by variation in inputs to technical inefficiency only and do not allow for statistical noise. Deterministic production frontier model is written as:

$$Q_i = f(x_i;\beta)\exp\{-u_i\} \tag{3.2.1}$$

where β, the parameter vector, and $\exp\{-u_i\}$ represent the structure of the production frontier and technical efficiency, respectively. Since technical efficiency score is bounded by one, that is technical efficiency ≤ 1, it is guaranteed that $u_i \geq 0$. And $u_i \geq 0$ guarantees that $Q_i \leq f(x_i;\beta)$. Assuming that $f(x_i;\beta)$ takes the log-linear Cobb-Douglas form, the deterministic production frontier model becomes as follows:[20]

[20] Subal C. Kumbhakar, C.A. Knox Lovell, **Stochastic Frontier Analysis**, Cambridge University Press, New York, 2003, p. 66.

$$\ln Q_i = \beta_0 + \sum_n \beta_n \ln x_{ni} - u_i \qquad (3.2.2)$$

The object is to obtain the estimates of the parameters vector and of $u_i s$ which are then used to obtain the estimates of technical efficiencies for each production units. In this context, three methods are introduced: Goal programming, corrected ordinary least squares and modified ordinary least squares.

3.2.1 Goal Programming

Aigner and Chu (1968) showed that the above deterministic production frontier model can be rewritten as mathematical programming models. The first model is the linear programming problem, in which the goal is to select a non-negative parameter vector that minimizes the sum of deviations beneath the parametric frontier. The second model is the quadratic programming problem, in which the goal is to select a non-negative parameter vector that minimizes the sum of squared deviations beneath the parameter frontier. The resulting deviations are then converted into technical efficiency measures.[21]

In the original paper, Aigner and Chu attribute all differences in technical efficiency to disturbance term. Errors of mesurement in all variables are neglected. For simplicity, one output-two input Cobb-Douglas model is assumed. The model is:

$$Q = A x_1^{\alpha} x_2^{\beta} u \qquad (3.2.1.1)$$

where x_1 and x_2 are inputs, Q is output, A is the technology parameter and u is the disturbance term. The problem is to obtain estimates of the parameters A, α and β.[22]

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[21] **Ibid.**, p. 66.

[22] D.J. Aigner, S.F. Chu, "On the Estimating the Industry Production Function," **The American Economic Review**, Vol. 58, No. 4, p. 8.

The above model can be extended from one-output and two-inputs to multiple-outputs and multiple-inputs. The production frontier model in Cobb-Douglas form can be written as follows:

$$Q_i = \beta_0 x_i^{\beta_n} \exp\{-u_i\} \tag{3.2.1.2}$$

Taking the natural logarithm of (3.2.1.2),

$$\ln Q_i = \beta_0 + \sum_{n=1}^{N} \beta_n \ln x_{ni} - u_i \tag{3.2.1.3}$$

where $i = 1, 2, ..., K$ and $n = 1, 2, ..., N$.

The linear programming problem that minimizes the sum of deviations beneath the parametric frontier is:[23]

$$\min \sum_{i=1}^{K} u_i \tag{3.2.1.4}$$

subject to

$$\beta_0 + \sum_{n=1}^{N} \beta_n \ln x_{ni} \geq \ln Q_i$$

The second model is a quadratic programming problem that minimizes the sum of squared proportionate deviations of the observed output of each producer beneath the parameter frontier. The quadratic programming problem is:[24]

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[23] Kumbhakar, **op.cit.**, p. 67.

[24] Kumbhakar, **op.cit.**, p. 67.

$$\min \sum_{i=1}^{K} u_i^2 \qquad (3.2.1.5)$$

subject to

$$\beta_0 + \sum_{n=1}^{N} \beta_n \ln x_{ni} \geq \ln Q_i$$

In parametric approach, the ability to describe the frontier function in a simple mathematical form and the ability to accommodate non-constant returns to scale are some advantages over non-parametric approach. In spite of the fact that mathematical form may be too simple, the parametric function imposes structure on the frontier that may be unwarranted. Also, homogenous Cobb-Douglas case can be relaxed.[25]

Limitation on the number of observations that may be technically efficient is one of the disadvantages of the goal programming approach to frontier estimation. When the linear programming model is used to calculate the parameter vector in homogenous Cobb-Douglas case, there will only be as technically efficient observations as there are parameters to be estimated.[26]

Another disadvantage of the goal programming approach is the fact that it is extremely sensitive to outliers. One solution to this problem is to dismiss some observations as suggested by Aigner and Chu. If the rate of change of the estimates with respect to successful dismissals decreases, the suggestion will be useful.[27]

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[25] Finn R. Førsund, C.A. Knox Lovell, Peter Schmidt, "A Survey of Frontier Production Functions and of Their Relationship to Efficiency Measurement," **Journal of Econometrics**, Vol. 13, 1980, p. 10.

[26] **Ibid.**, p. 10.

[27] **Ibid.**

The major disadvantage of the goal programming apporach is that no statistical assumptions and interpretations are made about the parameters since the parameters are calculated rather than estimated. Mathematical programming processes produce parameters without statistical inferences.[28]

Schmidt (1976) made assumptions to those made in usual regression context, that is, $u_i s$ are independently and identically distributed with finite mean μ and finite variance σ^2. ($\mu > 0$ since $u \geq 0$) If it is further assumed away problems of correlation among $u_i s$ and $x_i s$ and disturbances in the first-order conditions are independent of $u_i s$ then the model in (3.2.1.3) may be properly estimated by least squares.[29]

If ordinary least squares is applied to the model in (3.2.1.3) by virtue of the assumptions mentioned above, the estimates of $\beta_i s$ will be best linear unbiased and computed standart error for this estimate is appropriate whereas the estimator of β_0 will be biased. The estimator of β_0 will be best linear unbiased estimator of $(\beta_0 - \mu)$. Then the model in (3.2.1.3) becomes

$$\ln Q_i = (\beta_0 - \mu) + \sum_{n=1}^{N} \beta_n \ln x_{ni} - u_i^* \qquad (3.2.1.6)$$

where $u_i^* = u_i - \mu$ are independently and identically distributed with mean of zero and variance of σ^2. Although ordinary least squares estimators will not be normally distributed in finite samples, they will be asymptotically distributed. This

[28] Kumbhakar, **op.cit.**, p. 67.

[29] Peter Schmidt, "On the Statistical Estimation of Parameter Frontier Production Functions," **The Review of Economics and Statistics**, Vol. 58, No. 2, 1976, p. 238.

corresponds to the fact that the usual tests of hypotheses including $\beta_i s$ are asymptotically valid.[30]

Schmidt (1976) pointed out that a statistical interpretation can be given to goal programming approach if a particular distribution is assumed for the disturbance term. Then the model can be estimated by maximum likelihood procedure. This increases the "asymptotic efficiency"[31] of estimators relative to ordinary least squares depending on the distribution of $u_i s$.[32]

Maximum likelihood estimation requires a particular assumption about the distribution of the disturbance term. Assuming that $u_i s$ follows an exponential distribution:

$$f(u) = \sigma_u^{-1} \exp\{-u / \sigma_u\} \tag{3.2.1.7}$$

where σ_u is the standard deviation of this distribution. Then, the log likelihood function is:

$$\ln L = K \ln \sigma - \sigma^{-1} \sum_{i=1}^{K} | u_i | \tag{3.2.1.8}$$

The log-likelihood function in (3.2.1.8) is maximized by minimizing the sum of absolute residuals. Schmidt (1976) stated that parameters calculated by Aigner and Chu's linear programming technique are maximum likelihood estimates of the

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[30] **Ibid.**, pp. 238-239.

[31] See Jack Jonston and John Dinardo, "**Econometric Methods**", Mc Graw-Hill International, 1997, p. 64.

[32] Schmidt, **op.cit.**, p. 238.

parameters of the deterministic production frontier if an exponential distribution is imposed on the disturbance term.[33]

Assuming that $u_i s$ follow a half-normal distribution,

$$f(u) = \frac{2}{\sigma_u \sqrt{2\pi}} \exp\{-\frac{u^2}{2\sigma_u^2}\} \tag{3.2.1.9}$$

where σ_u is the standard deviation and σ_u^2 is the variance of this distribution. Then, the log- likelihood function is:

$$\ln L = \text{constant} - \frac{1}{2}\ln\sigma_u^2 - \frac{1}{2\sigma_u^2}\sum_{i=1}^{K} u_i^2 \tag{3.2.1.10}$$

The log-likelihood function in (3.2.1.10) is maximized by minimizing the sum of squared residuals. Schmidt (1976) stated that the parameters calculated bu Aigner and Chu's quadratic programming technique are maximum likelihood estimates of the parameters of deterministic production frontier if a half-normal distribution is imposed on the disturbance term.[34]

A problem with maximum likelihood estimation arises since the range of dependent variable depends on the parameters to be estimated which violates a regularity condition for maximum likelihood estimation. The range of $\ln Q_i$ is $(-\infty,\ \beta_0 + \sum_{n=1}^{N} \beta_n \ln x_{ni})$ which depends on β_0 and $\beta_n s$. Therefore, the usual theorems cannot be invoked to determine the asymptotic distributions of parameter estimates.[35]

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[33] Kumbhakar, **op.cit.,** p. 67.

[34] Kumbhakar, **op.cit.,** p. 68.

[35] Schmidt, **op.cit.,** p. 239.

Then, Greene (1980) showed that if a gamma distribution is assumed for the disturbance term, deterministic frontier model satisfies all regularity conditions for maximum likelihood estimation. Aigner and Chu's linear and quadratic programming problems have maximum likelihood estimation counterparts and have uncertain statistical properties. On the other hand, Greene's maximum likelihood problem that have desirable statistical properties has no known goal programming counterpart.[36]

3.2.2 Corrected Ordinary Least Squares: COLS

In the discussion of Farrell's original paper, Winsten (1957) suggested that the deterministic production frontier model in (3.2.2) could be estimated by corrected ordinary least squares, COLS, estimation procedure.[37]

COLS makes no assumption about the non-positive efficiency component. It estimates the consistent and unbiased technology parameters in (3.2.2) by means of ordinary least squares and then corrects the biased OLS intercept by shifting it up to ensure that estimated frontier bounds the data from above that means that all corrected residuals are non-positive and at least one is zero.[38]

The COLS intercept parameter is estimated consistently by:

$$\hat{\beta}_0^* = \hat{\beta}_0 + \max_i \{\hat{u}_i\} \qquad (3.2.2.1)$$

[36] C.A. Knox Lovell, "Production Frontiers and Productive Efficiency," in **The Measurement of Productive Efficiency: Techniques and Applications**, Ed. by. Harold O. Fried, C.A. Knox Lovell, Shelton S. Schmidt, Oxford University Press, New York, 1993, p. 37.

[37] Subal C. Kumbhakar, C.A. Knox Lovell, **Stochastic Frontier Analysis**, Cambridge University Press, New York, 2003, p. 70.

[38] C.A. Knox Lovell, "Production Frontiers and Productive Efficiency," in **The Measurement of Productive Efficiency: Techniques and Applications**, Ed. by. Harold O. Fried, C.A. Knox Lovell, Shelton S. Schmidt, Oxford University Press, New York, 1993, p. 21.

where $\hat{u}_i s$ are OLS residuals and $\hat{\beta}_0$ is intercept parameter estimated by OLS. The OLS residuals are corrected by:

$$-\hat{u}_i^* = \hat{u}_i - \max_i \{\hat{u}_i\} \qquad (3.2.2.2)$$

where $\hat{u}_i^* s$ are non-positive COLS residuals which are used to obtain consistent estimates of technical efficiency.[39]

Although COLS estimation procedure is easy to apply, it has a disadvantage that estimated production frontier is parallel to the OLS regression due to the fact that only OLS intercept is corrected, leaving the technology parameters unchanged. As a result of this deficiency, the structure of best practice production technology is the same as the structure of technology of less efficient producers. This deficiency causes to the fact that COLS generates the same efficiency ranking as OLS does. So, efficiency magnitudes must be taken into account as well as a ranking.[40]

3.2.3 Modified Ordinary Least Squares: MOLS

MOLS was first introduced by Richmond (1974). Unlike COLS, it makes an assumption about the non-positive efficiency component. The most popular and widely used assumptions are half-normal and exponential distributions. The reason for the assumptions that the disturbance term follows an exponential or a half-normal distribution is that technical efficiency is expected to follow one of these distributions with increasing degrees of technical inefficiency being increasingly less likely.[41]

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[39] Kumbhakar, **op.cit.**, p. 70

[40] Lovell, **op.cit.**, p. 22.

[41] Subal C. Kumbhakar, C.A. Knox Lovell, **Stochastic Frontier Analysis**, Cambridge University Press, New York, 2003, p. 71.

Like COLS, technology parameters are first estimated by OLS. Then, the estimated OLS intercept parameter is modified by shifting it up by plus the estimated mean of $u_i s$, that are extracted from the moments of OLS residuals. The MOLS intercept parameter is estimated by:

$$\hat{\beta}_0^{**} = \hat{\beta}_0 + E\{\hat{u}_i\} \quad (3.2.3.1)$$

where $E\{\hat{u}_i\}$ is the mean of the estimated OLS residuals. The OLS residuals are modified in the opposite direction.

$$-\hat{u}_i^{**} = \hat{u}_i - E\{\hat{u}_i\} \quad (3.2.3.2)$$

Then, modified OLS residuals are used to obtain consistent estimates of technical efficiency scores.[42]

Although MOLS is easy to apply like COLS, it has a disadvantage that if a producer has a sufficiently large positive OLS residual, it is possible that $[\hat{u}_i - E\{\hat{u}_i\}] > 1$, meaning that technical efficiency score is larger than 1. It is also possible that modification of OLS shifts the intercept parameter so that no producer is technically efficient. Since estimated production frontier is parallel to OLS regression, the structure of best practice production technology is the same as the structure of technology of less efficient producers like COLS. This fact causes to the fact that MOLS also generates the same efficiency ranking as OLS does.[43]

[42] **Ibid.**, p. 71.

[43] C.A. Knox Lovell, "Production Frontiers and Productive Efficiency," in **The Measurement of Productive Efficiency: Techniques and Applications**, Ed. by. Harold O. Fried, C.A. Knox Lovell, Shelton S. Schmidt, Oxford University Press, New York, 1993, p. 22.

3.3 Stochastic Production Frontiers

Aigner, Lovell and Schmidt (1977) and Meesuen and van den Broeck (1977) independently proposed a stochastic production frontier model in which an additional random error is added to non-negative random variable given in equation (3.2.2). The motivation behind this idea was that deviations from the production frontier may be due to random events, such as bad weather, luck, poor machine performance, strikes, input supply breakdowns, labor-market conflicts, etc., and not be under the control of the production units. Differently saying that every production unit owns its production frontier and that frontier is exposed to random evets.[44]

Unlike deterministic models, stochastic production frontier model allows for random events, measurement errors and variables omitted from the specified production technology. The stochastic production frontier model in log-linear Cobb-Douglas form is written as follows:

$$\ln Q_i = \beta_0 + \sum_n \beta_n \ln x_{ni} + v_i - u_i \qquad v_i - u_i = \varepsilon_i \tag{3.3.1}$$

where v_i accounts for random events, measurement erros and some other variables not included into the model, u_i represents technical efficiency; $u_i \geq 0$ whereas v_i is unrestricted. Since the error term in equation (4.3.1) has two components, the stochastic production frontier model is also called as composed error model.

The random error component, v_i is assumed to be independently and identically distributed as $N \sim (0, \sigma_v^2)$, the technical efficiency component, u_i is

[44] William H. Greene, "The Econometric Approach to Efficiency Analysis," in **The Measurement of Productive Efficiency: Techniques and Applications**, Ed. by. Harold O. Fried, C.A. Knox Lovell, Shelton S. Schmidt, Oxford University Press, New York, 1993, p. 76.

assumed to be distributed independently of v_i and to follow one-sided distribution.[45] The most common one-sided distributions are exponential, half-normal, truncated-normal and gamma distributions.

The random error component follows a symmteric distribution whereas technical efficiency component follows an asymmetric distribution. Therefore, the error term ε_i is asymmetric. Assuming that v_i and u_i are distributed independently of regressors, OLS estimation of (3.3.1) provides consistent estimates of the $\beta_n s$ but not of β_0.[46] When $\sigma_v^2 = 0$ the model in (3.3.1) becomes a deterministic frontier model. On the other hand, when $\sigma_u^2 = 0$ the model becomes Zellner, Kmenta and Dréze (1966) stochastic production function model.[47]

The presence of technical inefficiency can be tested by the null hypothesis $H_0 : \sigma_u^2 = 0$. (If $\sigma_u^2 = 0$, $u_i \equiv 0$ for every i) If $u_i = 0$ then $\varepsilon_i = v_i$ and the data do not indicate technical inefficiency. If $u_i > 0$, then ε_i is negatively skewed and the data indicate technical inefficiency. Thus, a test for the presence of inefficiency in the data can be based on the sample skewness of the OLS residuals. This is defined as:

$$\sqrt{b_1} = \frac{m_3}{\sqrt{m_2^3}} \tag{3.3.2}$$

where m_2 and m_3 are the second and third moments of OLS residuals, respectively. $m_3 < 0$ implies that OLS residuals are negatively-skewed and this corresponds to the fact that there is evidence for technical inefficiency in tha data. On the other hand,

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[45] Dennis Aigner, C.A. Knox Lovell, Peter Schmidt, "Formulation and Estimation of Stochastic Frontier Production Function Models," **Journal of Econometrics**, Vol. 6, 1977, p. 24.

[46] Subal C. Kumbhakar, C.A. Knox Lovell, **Stochastic Frontier Analysis**, Cambridge University Press, New York, 2003, p. 73.

[47] Aigner, **op.cit.**, p. 24.

$m_3 > 0$ implies that OLS residuals are positively-skewed and this corresponds to the fact that the model is misspecified.[48]

Estimating the technology parameters, $\beta_n s$, and the estimates of technical efficiency of each production unit are the two objects. Assuming that $u_i s$ are distributed independently of the regressors, OLS provides consistent estimates of technology parameters but not intercept parameter. In order to estimate the technical efficiency of each production unit, separate estimates of v_i and u_i are required which in turn requires distributional assumptions on the random error and technical efficiency components. In order to obtain the estimates of $\beta_n s$ and random error and technical efficiency components, two-step procedure is followed. In the first step, OLS is used to obtain the estimates of $\beta_n s$ and in the second step maximum likelihood estimation procedure is followed to obtain the estimates of intercept parameter and variances of two error components.[49]

3.3.1 Cross-Sectional Models

In stochastic production frontier estimation, efficiency measurement can be analyzed by means of either cross-sectional or panel data. Cross-sectional data requires strong distributional assumptions on both random error component and technical inefficiency component.

In accordance with the stochastic production frontier model given in equation (3.3.1), the following distributional assumptions are made:

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[48] Kumbhakar, **op.cit.**, p. 73

[49] Subal C. Kumbhakar, C.A.Knox Lovell, **Stochastic Frontier Analysis**, Cambridge University Press, 2000, p. 74.

1) The random error component v_i is assumed to be distributed as $N(0,\sigma_v^2)$
2) The technical inefficiency component is assumed to be distributed as exponentinal.
3) v_i and u_i are assumed to be distributed identically and independently of each other and of the regressors.[50]

The density functions for u_i and v_i are:

$$f(u)=\sigma_u^{-1}\exp\{-u\sigma_u^{-1}\} \tag{3.3.1.1}$$

$$f(v)=\frac{1}{\sigma_v\sqrt{2\pi}}\exp\left\{\frac{-v^2}{2\sigma_v^2}\right\} \tag{3.3.1.2}$$

Due to the third assumption, the joint density function of u_i and v_i is just the product of their individual density functions. That is:

$$f(u,v)=\frac{1}{\sqrt{2\pi}\sigma_v\sigma_u}\exp\left\{-u\sigma_u^{-1}-\frac{v^2}{2\sigma_v^2}\right\} \tag{3.3.1.3}$$

Replacing v with $(u+\varepsilon)$, the joint density function for u_i and ε is:

$$f(u,\varepsilon)=\frac{1}{\sqrt{2\pi}\sigma_v\sigma_u}\exp\left\{-u\sigma_u^{-1}-\frac{(u+\varepsilon)^2}{2\sigma_v^2}\right\} \tag{3.3.1.4}$$

The marginal density function of ε is obtained by integrating (3.3.1.4). That is:

[50] **Ibid.,** p. 80.

$$f(\varepsilon)=\int_0^{\infty} f(u,\varepsilon)\,du = \sigma_u^{-1}\Phi\left\{-\varepsilon\sigma_v^{-1}-\sigma_v\sigma_u^{-1}\right\}\exp\left\{\varepsilon\sigma_u^{-1}+\frac{\sigma_v^2}{2\sigma_u^2}\right\} \quad (3.3.1.5)$$

where Φ is the standart normal cumulative distribution function. $f(\varepsilon)$ is distributed as asymmetrically with mean and variance:[51]

$$E(\varepsilon)=-E(u)=-\sigma_u \quad (3.3.1.6)$$

$$V(\varepsilon)=\sigma_u^2+\sigma_v^2 \quad (3.3.1.7)$$

The log-likelihood function for a sample of K production units is writen as:

$$\ln L = \text{constant-}K\ln\sigma_u + K\left\{\frac{\sigma_v^2}{2\sigma_u^2}\right\}+\sum_{i=1}^{K}\ln\Phi(-A)+\sum_{i=1}^{K}\frac{\varepsilon_i}{\sigma_u} \quad (3.3.1.8)$$

where $A=-\tilde{\mu}\sigma_v^{-1}$ and $\tilde{\mu}=-\varepsilon-\left(\sigma_v^2\sigma_u^{-1}\right)$. The maximum likelihood estimates of all parameters are obtained by maximizing (3.3.1.8) with respect to σ_u and σ_v.[52]

Point estimates of technical efficiency are obtained by either the mean or the mode of the conditional distribution of u given ε. The conditional distribution of u given ε is:

$$f(u\setminus\varepsilon)=\frac{f(u,\varepsilon)}{f(\varepsilon)}=\frac{1}{\sqrt{2\pi}\sigma_v\Phi\left(-\tilde{\mu}\sigma_v^{-1}\right)}\exp\left\{-\frac{(u-\tilde{\mu})^2}{2\sigma^2}\right\} \quad (3.3.1.9)$$

which is distributed as $N^{+}\left(\tilde{\mu},\sigma_v^2\right)$. The mean and the mode of (3.3.1.9) are:

[51] **Ibid.**, p. 80.

[52] **Ibid.**, p. 80.

$$E\left[u_i \setminus \varepsilon_i\right] = \tilde{\mu}_i + \sigma_v \left\{ \frac{\varphi\left(-\tilde{\mu}_i \sigma_v^{-1}\right)}{\Phi\left(-\tilde{\mu}_i \sigma_v^{-1}\right)} \right\} \tag{3.3.1.10}$$

$$M\left[u_i \setminus \varepsilon_i\right] = \begin{cases} \tilde{\mu}_i & \text{if } \tilde{\mu}_i \geq 0, \\ 0 & \text{otherwise} \end{cases} \tag{3.3.1.11}$$

where φ is the standart normal density function. Then, producer-specific estimates of technical efficiency can be obtained by substituting either (3.3.1.10) or (3.3.1.11) into the equation $TE = \exp\{-\hat{u}_i\}$ where $\hat{u}_i$ is either $E\left[u_i \setminus \varepsilon_i\right]$ or $M\left[u_i \setminus \varepsilon_i\right]$.[53]

As in the normal- exponential model, the normal- half normal model assumes that the random error component, v_i is distributed as $N\left(0, \sigma_v^2\right)$ and v_i and u_i are distributed identically and independently of each other and of the regressors. The normal- half normal model differs from normal-exponential model in the fact that the technical efficiency component, u_i is assumed to be distributed as half normally, $N^+\left(0, \sigma_u^2\right)$.

The density function for v is given in equation (3.3.1.2). The density function for u is:

$$f(u) = \frac{2}{\sqrt{2\pi}\sigma_u} \exp\left\{\frac{-u^2}{2\sigma_u^2}\right\} \tag{3.3.1.12}$$

Due to the fact that v_i and u_i are distributed identically and independently of each other, the joint density function for u and v is the product of their individual density functions. That is:

[53] **Ibid.**, p. 82.

$$f(u,v)=\frac{1}{\pi\sigma_u\sigma_v}\exp\left\{\frac{-u^2}{2\sigma_u^2}-\frac{v^2}{2\sigma_v^2}\right\} \quad (3.3.1.13)$$

Replacing v with $(u+\varepsilon)$, the joint density function for u and ε is,

$$f(u,\varepsilon)=\frac{1}{\pi\sigma_u\sigma_v}\exp\left\{\frac{-u^2}{2\sigma_u^2}-\frac{(u+\varepsilon)^2}{2\sigma_v^2}\right\} \quad (3.3.1.14)$$

The marginal density function of ε is obtained by integrating (3.3.1.14). That is:

$$f(\varepsilon)=\int_0^\infty f(u,\varepsilon)\,du=2\sigma^{-1}\varphi\left(\varepsilon\sigma^{-1}\right)\Phi\left(-\varepsilon\lambda\sigma^{-1}\right) \quad (3.3.1.15)$$

where $\sigma=\sqrt{\sigma_u^2+\sigma_v^2}$, $\lambda=\sigma_u\sigma_v^{-1}$ and φ and Φ are standart normal density and cumulative distribution functions. λ can be interpreted as an indicator for the relative contributions of u and v to ε. $f(\varepsilon)$ is distributed as asymmetrically with mean and variance:[54]

$$E(\varepsilon)=-E(u)=-\sigma_u\sqrt{\frac{2}{\pi}} \quad (3.3.1.16)$$

$$V(\varepsilon)=\sigma_u^2\left(\frac{\pi-2}{\pi}\right)+\sigma_v^2 \quad (3.3.1.17)$$

As σ_u^2 goes to zero or σ_v^2 goes to infinity, λ goes to zero and symmetric error component dominates the technical inefficiency component in the determination of ε. In that case, the model turns into OLS production function model with no technical inefficiency. As σ_v^2 goes to zero or σ_u^2 goes to infinity, λ goes to infinity

[54] **Ibid.**, p. 75.

and technical inefficiency component dominates the random error component. In that case, the model turns into deterministic production frontier model with no statistical noise.[55] Battese and Corra (1977) suggest the parameter $\gamma = \sigma_u^2 \sigma^{-2}$ has a value between zero and one whereas the λ-parameter could be any non-negative value.[56]

The log-likelihood function for a sample of K production units is:

$$\ln L = \text{constant} - K \ln \sigma + \sum_{i=1}^{K} \ln \Phi\left(-\lambda \sigma^{-1} \varepsilon_i\right) - \frac{\sum_{i=1}^{K} \varepsilon_i^2}{2\sigma^2} \qquad (3.3.1.18)$$

The maximum likelihood estimates of all parameters are obtained by maximizing (3.3.1.18) with respect to the parameters.[57]

Point estimates of technical efficiency are obtained by either the mean or the mode of the conditional distribution of u given ε. The conditional distribution of u given ε is:[58]

$$f(u \backslash \varepsilon) = \frac{f(u, \varepsilon)}{f(\varepsilon)} = \frac{1}{\sqrt{2\pi}\sigma_*} \exp\left\{\frac{-(u - \mu_*)^2}{2\sigma_*^2}\right\}\left[1 - \Phi\left(-\mu_* \sigma_*^{-1}\right)\right]^{-1} \qquad (3.3.1.19)$$

where $\mu_* = \frac{-\varepsilon \sigma_u^2}{\sigma^2}$ and $\sigma_*^2 = \frac{\sigma_u^2 \sigma_v^2}{\sigma^2}$

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[55] Dennis Aigner, C.A. Knox Lovell, Peter Schmidt, "Formulation and Estimation of Stochastic Frontier Production Function Models," **Journal of Econometrics**, Vol. 6, 1977, p. 26.

[56] A value of γ of zero indicates that all deviations from the frontier are due to statistical noise, while a value of one indicates that all deviations are due to technical efficiency. It should be noticed that γ is not equal to the ratio of the variance of technical inefficiency effects to the total residual variance. Notice that total residual variance is $V(\varepsilon) = \sigma_u^2\left(\frac{\pi - 2}{\pi}\right) + \sigma_v^2$

[57] Kumbahakar, **op.cit.**, p. 77.

[58] Kumbahakar, **op.cit.**, pp. 77-78.

The mean and the mode of (4.3.1.19) are:[59]

$$E[u_i \backslash \varepsilon_i] = \mu_* + \sigma_* \left[\frac{\varphi(-\mu_* \sigma_*^{-1})}{1 - \Phi(-\mu_* \sigma_*^{-1})} \right] = \sigma_* \left[\frac{\varphi(\varepsilon\lambda\sigma^{-1})}{1 - \Phi(\varepsilon\lambda\sigma^{-1})} - (\varepsilon\lambda\sigma^{-1}) \right] \qquad (3.3.1.20)$$

$$M[u_i \backslash \varepsilon_i] = \begin{cases} -\varepsilon_i \dfrac{\sigma_u^2}{\sigma^2} & \text{if } \varepsilon_i \le 0, \\ 0 & \varepsilon_i > 0 \end{cases} \qquad (3.3.1.21)$$

where $-\mu_* \sigma_*^{-1} = \varepsilon\lambda\sigma^{-1}$ and $\lambda = \sigma_u \sigma_v^{-1}$.

Producer-specific estimates of technical efficiency can be obtained by substituting either (3.3.1.20) or (3.3.1.21) into the equation $TE = \exp\{-\hat{u}_i\}$ where $\hat{u}_i$ is either $E[u_i \backslash \varepsilon_i]$ or $M[u_i \backslash \varepsilon_i]$.

Another one-sided distribution is truncated normal distribution. The truncated normal distribution generalizes the half-normal distribution and contains an additional parameter, μ to be estimated. μ is the mode of the normal distribution which is truncated below at zero. In relation with the normal- truncated normal distribution, the following assumptions are made:

1) v_i is distributed as normally,

2) u_i is distributed as $N^+(\mu, \sigma_u^2)$, and

3) v_i and u_i are distributed independently and identically of each other and of regressors.[60]

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[59] J. Jondrow, et. al., "On the Estimation of Technical Inefficiency in the Stochastic Frontier Production Function Model", **Journal of Econometrics**, Vol. 19, No. 2/3, 1982, p. 235.

[60] Kumbhakar, **op.cit.**, p. 83.

The density function for v_i is given in equation (3.3.1.2). The density function for u is:

$$f(u)=\frac{1}{\sqrt{2\pi}\sigma_u\Phi\left(\mu\sigma_u^{-1}\right)}\exp\left\{\frac{-(u-\mu)^2}{2\sigma_u^2}\right\} \tag{3.3.1.22}$$

where μ is the mode of the normal distribution, truncated below zero. $f(u)$ is the density of a normally distributed variable with non-zero mean, truncated below zero. If we set $\mu=0$, the density function in (3.3.1.22) turns into the half-normal density function.[61]

The joint density function u_i and v_i is the product of their individual density functions.

$$f(u,v)=\frac{1}{2\pi\sigma_u\sigma_v\Phi\left(\mu\sigma_u^{-1}\right)}\exp\left\{\frac{-(u-\mu)^2}{2\sigma_u^2}-\frac{v^2}{2\sigma_v^2}\right\} \tag{3.3.1.23}$$

Replacing v with $(u+\varepsilon)$, the joint density function for u_i and ε is:

$$f(u,\varepsilon)=\frac{1}{2\pi\sigma_u\sigma_v\Phi\left(\mu\sigma_u^{-1}\right)}\exp\left\{\frac{-(u-\mu)^2}{2\sigma_u^2}-\frac{(u+\varepsilon)^2}{2\sigma_v^2}\right\} \tag{3.3.1.24}$$

The marginal density function for ε is obtained by integrating (3.3.1.24).

$$f(\varepsilon)=\int_0^\infty f(u,\varepsilon)\,du=\sigma^{-1}\varphi\left(\frac{\mu+\varepsilon}{\sigma}\right)\Phi\left(\frac{\mu}{\sigma\lambda}-\varepsilon\lambda\sigma^{-1}\right)\left[\Phi\left(\mu\sigma_u^{-1}\right)\right]^{-1} \tag{3.3.1.25}$$

[61] Kumbhakar, **op.cit.**, p. 83.

$f(\varepsilon)$ is distributed as asymmetrically with mean and variance:[62]

$$E(\varepsilon) = -E(u) = -\frac{\mu a}{2} - \frac{\sigma_u a}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(\mu\sigma_u^{-1}\right)^2\right\} \tag{3.3.1.26}$$

$$V(\varepsilon) = \frac{a\mu^2}{2}\left(\frac{2-a}{2}\right) + \frac{a\sigma_u^2}{2}\left(\frac{\pi - a}{\pi}\right) + \sigma_v^2 \tag{3.3.1.27}$$

where $a = \left[\Phi\left(\mu\sigma_u^{-1}\right)\right]^{-1}$.

The log-likelihood function for a sample of K production units is:

$$\ln L = \text{constant} - K\ln\sigma - K\ln\Phi\left(\mu\sigma_u^{-1}\right) + \sum_{i=1}^{K}\ln\Phi\left(\frac{\mu}{\sigma\lambda} - \lambda\sigma^{-1}\varepsilon_i\right) - \frac{1}{2}\sum_{i=1}^{K}\left(\frac{\varepsilon_i + \mu}{\sigma}\right)^2$$

(3.3.1.28)

where $\sigma_u = \dfrac{\sigma\lambda}{\sqrt{1+\lambda^2}}$. The maximum likelihood estimates of all parameters are obtained by maximizing (3.3.1.28) with respect to the parameters.[63]

Point estimates of technical efficiency are obtained by either the mean or the mode of the conditional distribution of u given ε. The conditional distribution of u given ε is:

$$f(u \backslash \varepsilon) = \frac{f(u,\varepsilon)}{f(\varepsilon)} = \frac{1}{\sqrt{2\pi}\sigma_*\left[1 - \Phi\left(-\tilde{\mu}\sigma_*^{-1}\right)\right]} \exp\left\{\frac{-\left(u - \tilde{\mu}\right)^2}{2\sigma_*^2}\right\} \tag{3.3.1.29}$$

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[62] Rodney E. Stevenson, "Likelihood Functions for Generalized Stochastic Frontier Estimation," **Journal of Econometrics**, Vol. 13, 1980, p. 60.

[63] Kumbhakar, **op.cit.**, p. 85.

where $\tilde{\mu}_i = \frac{\left(\mu\sigma_v^2 - \varepsilon_i\sigma_u^2\right)}{\sigma^2}$, $\sigma_*^2 = \frac{\sigma_u^2\sigma_v^2}{\sigma^2}$ and $f\left(u \setminus \varepsilon\right)$ is distributed as $N^+\left(\tilde{\mu}_i, \sigma_*^2\right)$[64]

The mean and the mode of (3.3.1.29) are:[65]

$$E\left[u_i \setminus \varepsilon_i\right] = \sigma_*\left[\tilde{\mu}_i\sigma_*^{-1} + \frac{\varphi\left(\tilde{\mu}_i\sigma_*^{-1}\right)}{1-\Phi\left(-\tilde{\mu}_i\sigma_*^{-1}\right)}\right] \qquad (3.3.1.30)$$

$$M\left[u_i \setminus \varepsilon_i\right] = \begin{cases} \tilde{\mu}_i & \text{if } \tilde{\mu}_i \geq 0, \\ 0 & \text{otherwise} \end{cases} \qquad (3.3.1.31)$$

Producer-specific estimates of technical efficiency can be obtained by substituting either (3.3.1.30) or (3.3.1.31) into the equation $TE = \exp\left\{-\hat{u}_i\right\}$ where $\hat{u}_i$ is either $E\left[u_i \setminus \varepsilon_i\right]$ or $M\left[u_i \setminus \varepsilon_i\right]$.

Just as in the previous generalizations, the normal-gamma model can be formulated by assuming that $u_i s$ follow a gamma distribution. The normal-gamma generalization was first proposed by Greene (1980a, b) and Stevenson (1980) and extended by Greene (1990). In relation with the normal-gamma model, the following assumptions are made: 1) v_i is assumed to be distributed as independently and identically, $v_i \sim N(0, \sigma_v^2)$. 2) u_i is distributed as gamma. 3) u_i and v_i are distributed independently of each other and of the regressors.[66]

The density function for v is given in (3.3.1.2). The gamma density function for u is:

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[64] Kumbhakar, **op.cit.,** p. 85.

[65] Kumbhakar, **op.cit.,** p. 86.

[66] Kumbhakar, **op.cit.,** p. 86.

$$f(u)=\frac{u^m}{\Gamma(m+1)\sigma_u^{m+1}}\exp\{-u\sigma_u^{-1}\} \qquad m>-1 \tag{3.3.1.32}$$

The joint density function for u and v is the product of their individual density functions.

$$f(u,v)=\frac{u^m}{\sqrt{2\pi}\sigma_v\Gamma(m+1)\sigma_u^{m+1}}\exp\left\{-u\sigma_u^{-1}-\frac{v^2}{2\sigma_v^2}\right\} \tag{3.3.1.33}$$

$$f(u,\varepsilon)=\frac{u^m}{\sqrt{2\pi}\sigma_v\Gamma(m+1)\sigma_u^{m+1}}\exp\left\{-u\sigma_u^{-1}-\frac{(u+\varepsilon)^2}{2\sigma_v^2}\right\} \tag{3.3.1.34}$$

The marginal density function of ε is:

$$f(\varepsilon)=\int_0^\infty f(u,\varepsilon)\,du=\frac{\sigma_v^m}{\sigma_u^{m+1}\sqrt{2\pi}\Gamma(m+1)}\exp\left\{\varepsilon\sigma_u^{-1}+\frac{\sigma_v^2}{2\sigma_u^2}\right\}\int_w^\infty (t-w)^m\exp\left\{\frac{-t^2}{2}\right\}dt \tag{3.3.1.35}$$

where $w=(\varepsilon\sigma_v^{-1})+(\sigma_v\sigma_u^{-1})$. $f(\varepsilon)$ is distributed as asymmetrically with mean and variance:[67]

$$E(\varepsilon)=-E(u)=-(m+1)\sigma_u \tag{3.3.1.36}$$

$$V(\varepsilon)=\sigma_v^2+(m+1)\sigma_u^2 \tag{3.3.1.37}$$

In equation (3.3.1.35), the marginal density function contains an integral term leading to some problems in estimation. Beckers and Hammond (1987) developed a

[67] Kumbhakar, **op.cit.**, p. 88.

closed-form expression for (3.3.1.35) in which case no need m to be integer values.[68] Thus, equation (3.3.1.35) can be rewritten as:[69]

$$f(\varepsilon)=\int_0^\infty f(u,\varepsilon)\,du=\frac{\exp\left\{\frac{-\varepsilon^2}{2\sigma_v^2}\right\}}{\sigma_u^{m+1}\sigma_v\sqrt{2\pi}\Gamma(m+1)}\int_0^\infty u^m\exp\left\{-u\sigma_u^{-1}-\frac{\varepsilon u}{\sigma_v^2}-\frac{u^2}{2\sigma_v^2}\right\}du \tag{3.3.1.38}$$

The log-likelihood function for a sample of K production units is,

$$\ln L=\text{constant}-K\ln\Gamma(m+1)-(m+1)K\ln\sigma_u+K\left(\frac{\sigma_v^2}{2\sigma_u^2}\right)$$
$$+\sum_{i=1}^{K}\varepsilon_i\sigma_u^{-1}+\sum_{i=1}^{K}\ln\Phi\left(\frac{-\left(\varepsilon_i+\sigma_v^2\sigma_u^{-1}\right)}{\sigma_v}\right)+\sum_{i=1}^{K}\ln h(m,\varepsilon_i) \tag{3.3.1.39}$$

where $h(m,\varepsilon_i)=E\left[z^m\setminus z>0,\varepsilon_i\right]$ and $z\approx N\left[-\left(\varepsilon_i+\sigma_v^2\sigma_u^{-1}\right),\sigma_v^2\right]$. The maximum likelihood estimates of all parameters are obtained by maximizing (3.3.1.39) with respect to the parameters.[70]

The conditional distribution of u given ε is needed to obtain the estimates of the technical efficiency of each producer.

$$f(u\setminus\varepsilon)=\frac{f(u,\varepsilon)}{f(\varepsilon)}=\frac{u^m\exp\left\{-u\sigma_u^{-1}-\frac{\varepsilon u}{\sigma_v^2}-\frac{u^2}{2\sigma_v^2}\right\}}{\int_0^\infty u^m\exp\left\{-u\sigma_u^{-1}-\frac{\varepsilon u}{\sigma_v^2}-\frac{u^2}{2\sigma_v^2}\right\}du} \tag{3.3.1.40}$$

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[68] Kumbhakar, **op.cit.**, p. 88.

[69] D.E. Beckers, C.J. Hammond, "A Tractable Likelihood Function for the Normal-Gamma Stochastic Frontier Model," **Economics Letters**, Vol. 24, 1987, p. 35.

[70] Kumbhakar, **op.cit.**, p. 89.

The mean of (3.3.1.40) is:

$$E\left[u_i \setminus \varepsilon_i\right] = \frac{h\left(m+1, \varepsilon_i\right)}{h\left(m, \varepsilon_i\right)} \tag{3.3.1.41}$$

which is used to obtain producer-specific estimates of technical efficiency.[71]

3.3.2 Panel Data Models

In the discussion of cross-sectional models, there are K production units that are available for the estimation of the stochastic production frontier. If a number of production units are available over a number of time periods, the data obtained are called as panel data. Panel data have some advantages over cross-sectional data in the estimation process. In this context, panel data allows for the investigation of both technical change and technical efficiency over time.[72]

Having panel data set makes the one capable of relaxing the strong distributional assumptions used with cross-sectional models and of obtaining the estimates of the technical efficiency scores with more desirable properties.[73]

Cross-sectional models are affected by three serious difficulties. First of all, the estimation of the model and the separation of technical inefficiency from statistical noise requires strong distributional assumptions on technical inefficiency component. It is not clear the robustness of the results for each distributional assumptions. It is also useful to note that not everyone agrees that skewness should be regarded as evidence of technical inefficiency. Second, maximum likelihood

[71] Subal C. Kumbhakar, C.A. Knox Lovell, **Stochastic Frontier Analysis**, Cambridge University Press, New York, 2003, p. 89.

[72] Tim Coelli, et. al., **An Introduction to Efficiency and Productivity Analysis**, Kluwer Academic Publishers, Dordrecht, 1998, p. 202.

[73] Kumbhakar, **op.cit.**, p. 95.

estimation also requires the assumption that technical efficiency component is independent of the regressors. It may be incorrect to assume that technical efficiency component is independent of the regressors if the production unit knows its level of efficiency. Third, the variance of the conditional mean or the conditional mode of $(u_i \setminus \varepsilon_i)$ for each production unit does not vanish as the size of the cross-section increases.[74] All of these restrictions can be avoided if we have a panel data set. More observations on each production unit can serve as a substitute for strong distributional assumptions and can also serve as a substitute for the independence assumption. Adding more observations on each production unit generates consistent estimates of the technical efficiency as the number of time periods goes to infinity.[75]

Panel data models in the stochastic frontier literature can be divided into two main groups. The first group assumes that technical efficiency is time-invariant and the second group assumes that technical efficiency varies over time, that is time-varying. Each of these two groups can be divided into many sub-groups depending on whether any distributional assumptions or functional forms are imposed on the error and technical efficiency components or not.[76]

3.3.2.1 Time-Invariant Models

Time-invariant technical efficiency model is estimated by Schmidt and Sickles' (1984) method. In their model, only the intercept varies over production units, differences in the intercept are interpreted as differing efficiency levels. They do not make strong distributional assumptions about random error and efficiency components; and also any assumption of independency between technical efficiency

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[74] Peter Schmidt, Robin C. Sickles, "Production Frontiers and Panel Data," **Journal of Business and Economic Statistics**, Vol. 2, No. 4, 1984, p. 367.

[75] Kumbhakar, **op.cit.**, p. 96.

[76] Subal C. Kumbhakar, et. al., "Temporal Patterns of Technical Efficiency: Results from Competing Models," **International Journal of Industrial Organization**, Vol. 15, 1997, p. 598.

and regressors, however the assumption that technical efficiency is time-invariant seems unrealistic.[77]

Let Q_{it} and X_{it} be the vector of outputs and inputs of the production unit, $i=(1,...,K)$ at time $t=(1,...,T)$ respectively. Given the production function $f(.)$ in Cobb-Douglas form, the general stochastic production frontier model with time-invariant technical efficiency can be written as follows:

$$\ln Q_{it} = \beta_0 + \sum_n \beta_n \ln X_{nit} + v_{it} - u_i \qquad (3.3.2.1.1)$$

where v_{it} represents random error component and $u_i \geq 0$ represents technical efficiency component. The structure of the production function is assumed so that no technical change is allowed. The model given in equation (3.3.2.1.1) is similar to a traditional panel data model with producer effects but with no time effects. The parameters of the model and the technical efficiency can be estimated in a number of different ways.[78]

If $u_i s$ are treated as firm-specific constants, the model can be estimated by ordinary least squares as a fixed-effects model. In relation to the fixed-effects model given in equation (3.3.2.1.1), the following assumptions are made:

1) v_{it} is uncorrelated with regressors and assumed to be identically and independently distributed as $\left(0, \sigma_v^2\right)$.

2) u_i is assumed to be correlated with regressors or with v_{it}, and no distributional assumption is made for u_i.[79]

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[77] Christopher Cornwell, et. al., "Production Frontiers with Cross-Sectional and Time Series Variation in Efficiency Levels," **Journal of Econometrics**, Vol. 46, 1990, p. 186.

[78] Kumbhakar, **op.cit.**, p. 97.

[79] Kumbhakar, **op.cit.**, p. 98.

Let $E(u_i) = \mu > 0$ and define,

$\beta_0^* = \beta_0 - \mu$ and $u_i^* = u_i - \mu$

so that $u_i^* s$ are identically and independently distributed with mean 0. Then, the model in (3.3.2.1.1) becomes:

$$\ln Q_{it} = \beta_0^* + \sum_n \beta_n \ln X_{nit} + v_{it} - u_i^* \tag{3.3.2.1.2}$$

where the technical efficiency and random error components have zero means. Later, defining the following equations,

$\beta_{0i} = \beta_0 - u_i = \beta_0^* - u_i^*$, the model in (3.3.2.1.2) becomes:

$$\ln Q_{it} = \beta_{0i} + \sum_n \beta_n \ln X_{nit} + v_{it} \tag{3.3.2.1.3}$$

where β_{0i} are producer-specific intercepts.[80] Estimation is employed by supressing β_0 and estimating K producer-specific intercepts, by retaining β_0 and estimating $(K-1)$ producer-specific intercepts or by applying the within transformation in which all data are expressed in terms of deviations from producer means and the K intercepts are recovered as means of producer residuals. Each of these variants is referred as least squares with dummy variables, LSDV.[81]

After the estimation, a normalization process is employed.

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[80] Schmidt, **op.cit.**, p. 368.

[81] Kumbhakar, **op.cit.**, p. 98.

$$\hat{\beta}_0 = \max_i \left\{ \hat{\beta}_{0i} \right\} \tag{3.3.2.1.4}$$

and u_i is estimated from:

$$\hat{u}_i = \hat{\beta}_0 - \hat{\beta}_{0i} \tag{3.3.2.1.5}$$

which guarantees that all $\hat{u}_i \geq 0$. Then, producer-specific estimates of technical efficiency are given by:[82]

$$TE = \exp\left\{ -\hat{u}_i \right\} \tag{3.3.2.1.6}$$

In the fixed effects model, at least one production unit is assumed to be fully efficient and the other production units' technical efficiencies are measured relative to fully efficient production unit(s).[83]

The main advantage of the LSDV is that consistency of the estimates of the $\beta_n s$ is not depend on the uncorrelatedness of the regressors and the random error and the technical effiency components. It also does not depend on the distributional assumptions on random error and technical efficiency components. The LSDV estimates of the $\beta_n s$ are consistent as either the size of the cross-sections or the number of time periods goes to infinity. However, consistency of the individual estimated $\beta_{0i} s$ requires that the number of time periods goes to infinity and the estimates are consistent for u_i as both the size of the cross-sections and the number of time periods go to infinity.[84]

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[82] Kumbhakar, **op.cit.**, p. 98.

[83] Kumbhakar, **op.cit.**, p. 98.

[84] Schmidt, **op.cit.**, p. 368.

One drawback of this approach is the presence of time-invariant attributes of the production units, such as capital stock, regulatory environment, location or some other characteristics. That is, it captures the effects that vary across production units but that are invariant over time.[85]

In the fixed-effects approach, it is assumed that $u_i s$ are correlated with the regressors. If $u_i s$ are assumed to be uncorrelated with regressors, then a random-effects approach might be preferable.[86] The following assumptions are made:

1) u_i is assumed to be randomly distributed with constant mean and variance and be uncorrelated with v_{it}.

2) Not any distributional assumption is made for u_i.

3) It is assumed that v_{it} has zero mean and constant variance.[87]

The model given in (3.3.2.1.1.1) can be rewritten as follows:

$$\ln Q_{it} = [\beta_0 - \mu] + \sum_n \beta_n \ln X_{nit} + v_{it} - [u_i - \mu] \qquad (3.3.2.1.7)$$

or

$$\ln Q_{it} = \beta_0^* + \sum_n \beta_n \ln X_{nit} + v_{it} - u_i^* \qquad (3.3.2.1.8)$$

where β_0^* does not depend on i since μ is a positive constant. The model in (3.3.2.1.2) can be estimated by the standart two-step generalized least squares, GLS

[85] William. H. Greene, "The Econometric Approach to Efficiency Analysis," in **The Measurement of Productive Efficiency: Techniques and Applications**, Ed. by. Harold O. Fried, C.A. Knox Lovell, Shelton S. Schmidt, Oxford Universtiy Press, 1993, p. 84.

[86] **Ibid.**, p. 84.

[87] Schmidt, **op.cit.**, p. 369.

method. In the first step, OLS is used to estimate all parameters. The variances of $u_i s$ and v_{it} are estimated by several methods documented in the literature. In the second step, β_0^* and the $\beta_n s$ are reestimated by means of feasible GLS.

After β_0^* and $\beta_n s$ being estimated by means of GLS, the u_i^* can be estimated from the residuals. That is:

$$\hat{u}_i^* = \frac{1}{T}\sum_{t=1}^{T}\left(\ln Q_{it} - \hat{\beta}_0^* - \sum_n \hat{\beta}_n \ln X_{nit}\right) \qquad (3.3.2.1.9)$$

Estimates of the u_i are obtained through the normalization.

$$\hat{u}_i = \max_i \left\{\hat{u}_i^*\right\} - \hat{u}_i^* \qquad (3.3.2.1.10)$$

which can be substituted into (3.3.2.1.6) to obtain producer-specific estimates of technical efficiency.[88]

Like the fixed-effects approach, at least one production unit is assumed to be fully efficient and the other production units' technical efficiencies are measured relative to fully efficient production unit(s).[89]

An alternative estimator of u_i^* is the best linear unbiased predictor, BLUP. That is given by,

$$\tilde{u}_i^* = \frac{-\hat{\sigma}_u^2}{\hat{\sigma}_v^2 + T\hat{\sigma}_u^2}\sum_{t=1}^{T}\left(\ln Q_{it} - \hat{\beta}_0^* - \sum_n \hat{\beta}_n \ln X_{nit}\right) \qquad (3.3.2.1.11)$$

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[88] **Ibid.,** p. 101.

[89] **Ibid.,** p. 101.

Estimates of u_i are obtained by normalization.[90]

$$\tilde{u}_i = \max_i \{\tilde{u}_i^*\} - \tilde{u}_i^* \qquad (3.3.2.1.12)$$

For large T, (3.3.2.1.3) and (3.3.2.1.5) are equivalent. (3.3.2.1.5) is consistent as well as (3.3.2.1.3) as T and K go to infinity. Consistent estimation of σ_u^2 requires $K \to \infty$. Thus GLS is preferable when K is large. When T is large and K is small, GLS is useless. When T and K are both large, GLS is feasible but not more efficient that within transformation.[91]

The other method which is applied to random effects model is maximum likelihood estimation. The maximum likelihood estimation of stochastic production frontier panel data model with time-invariant approach is similar to the procedure applied to cross-sectional data, and makes distributional assumptions about v_{it} and u_i.

3.3.2.2 Time-Varying Model

The assumption of time-invariant technical efficiency is very restrictive. Imposing this restriction without testing its appropriateness may cause to inconsistent estimators for the parameters.[92]

Cornwell, Schmidt and Sickles (1990) and Kumbhakar (1990) are the first to propose a stochastic production frontier panel data models with time-varying technical efficiency. The model developed by Cornwell, Schmidt and Sickles (CSS)

[90] Kumbhakar, **op.cit.**, p. 101.

[91] Schmidt, **op.cit.**, p. 369.

[92] Subal C. Kumbhakar, "Production Frontiers, Panel Data, and Time-Varying Technical Inefficiency," **Journal of Econometrics**, Vol. 46, 1990, p. 201.

allow technical efficiency to vary over time by specifying technical efficiency term as a quadratic function of time.[93]

The model proposed in (3.3.2.1.1) becomes:

$$\ln Q_{it} = \beta_{0t} + \sum_{n} \beta_n \ln X_{nit} + v_{it} - u_{it} \quad (3.3.2.2.1)$$

$$= \ln Q_{it} = \beta_{it} + \sum_{n} \beta_n \ln X_{nit} + v_{it} \quad (3.3.2.2.2)$$

where $\beta_{it} = \beta_{0t} - u_{it}$ is the intercept for production unit i in period t, β_{0t} is the production frontier intercept common to all production units. Since technical efficiency term varies over time, any arbitrary pattern of temporal change for it can be specified.[94]

CSS allow technical efficiency to vary over time by specifying technical efficiency term as a quadratic function of time. They model the technical efficiency (including the intercept) as follows:

$$\beta_{it} = \theta_{i1} + \theta_{i2} t + \theta_{i3} t^2 \quad (3.3.2.2.3)$$

where θ_{i1}, θ_{i2} and θ_{i3} are unknown parameters to be estimated. The efficiency, β_{it} is a quadratic function of time and it varies across firms.[95]

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93 Subal C. Kumbhakar, "Production Frontiers, Panel Data, and Time-Varying Technical Inefficiency," **Journal of Econometrics**, 1990, 46, p. 202.

94 Subal C. Kumbhakar, C.A. Knox Lovell, **Stochastic Frontier Analysis**, Cambridge University Press, 2000, p. 108.

95 Subal C. Kumbhakar, et. al., "Temporal Patterns of Technical Efficiency: Results from Competing Models," **International Journal of Industrial Organization**, Vol. 15, 1997, pp. 600-601.

Following Schmidt and Sickles (1984), the frontier intercept at time t, β_t and the producer-specific level of efficiency for production unit i in period t, u_{it} are estimated respectively as:

$$\hat{\beta}_t = \max_i \left\{\hat{\beta}_{it}\right\} \qquad \text{for } t = 1, 2, \ldots, T \tag{3.3.2.2.4}$$

and

$$\hat{u}_{it} = \hat{\beta}_t - \hat{\beta}_{it} \tag{3.3.2.2.5}$$

then, technical efficiency for the production unit i at time t is calculated by:

$$TE_{it} = \exp\left\{-\hat{u}_{it}\right\} \tag{3.3.2.2.6}$$

βs can be estimated by means of within estimators, generalised least squares, Hausman and Taylor instrumental variable estimator and maximum likelihood methods.[96]

If $\theta_{i2} = \theta_{i3} = 0$ for every i, the model collapses to the time-invariant technical efficiency model. If $\theta_{i2} = \theta_2$ and $\theta_{i3} = \theta_3$ for every i, the model collapses to fixed-effects model with producer-specific intercepts θ_{i1} and a quadratic function of time common to all production units given by $\theta_2 t + \theta_3 t^2$. This corresponds to the fact that technical efficiency is producer-specific and varies over time in the same manner for all production units. Quadratic time term captures the effects of technical change.[97]

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[96] **Ibid.**, p. 601.

[97] Subal C. Kumbhakar, C.A. Knox Lovell, **Stochastic Frontier Analysis**, Cambridge University Press, 2000, p. 108.

An alternative specification of u_{it} is introduced by Lee and Schmidt (1993). That is:

$$\beta_{it} = \theta_t \delta_i \quad (3.3.2.2.7)$$

where $\beta_{it} = \beta_{0t} - u_{it}$ and $\theta_t s$ are unknown parameters to be estimated. It is more flexible than the CSS model since no functional form is assumed but it is less flexible that CSS model in the sense that temporal pattern of technical efficiency is assumed to be the same for all production units. This model is useful and reasonable for short panels. For identification purposes θ is normalized by allowing $\theta_1 = 1$. The model is non-linear and allows for the inclusion of time or firm-invariant variables in the specification.[98]

v_{it} are assumed to be identically and independently distributed with zero mean and variance σ_v^2. The parameters β, θ_t, σ_v^2 and δ_i are estimated by means of either fixed-effects or random-effects approaches[99] where in the latter case, σ_δ^2 is estimated instead of δ_i. The frontier intercept at time t, β_t and the producer-specific level of inefficiency for production unit i in period t, u_{it} and the technical efficiency for the production unit i in period t, TE_{it} are obtained through (3.3.2.2.4)-(3.3.2.2.6).[100]

Kumbhakar (1990) suggested a stochastic frontier model for panel data according to following time-varying specification:

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[98] Subal C. Kumbhakar, et. al., "Temporal Patterns of Technical Efficiency: Results from Competing Models," **International Journal of Industrial Organization**, Vol. 15, 1997, p. 601.

[99] See Young Hoon Lee, Peter Schmidt, "A Production Frontier Model with Flexible Temporal Variation in Technical Efficiency," in **The Measurement of Productive Efficiency: Techniques and Applications**, Ed. by. Harold O. Fried, C.A. Knox Lovell, and Shelton S. Schmidt, Oxford University Press, New York, 1993, pp. 240-246 for fixed-effects and random-effects estimation.

[100] Subal C. Kumbhakar, et. al., "Temporal Patterns of Technical Efficiency: Results from Competing Models," **International Journal of Industrial Organization**, Vol. 15, 1997, p. 601.

$$u_{it} = \left[1 + \exp\left\{\gamma t + \delta t^2\right\}\right]^{-1} u_i \qquad (3.3.2.2.8)$$

where γ and δ are unknown parameters to be estinated and u_i is assumed to have half-normal distribution.[101]

The function u_{it} satisfies the properties 1) $0 \le u_{it} \le 1$ and 2) u_{it} can be monotonically increasing or decreasing, and concave or convex depending on the signs and the magnitudes of the parameters γ and δ. Time-invariant technical efficiency can be tested by conducting a hypothesis $H_0 : \gamma = \delta = 0$ [102]

Another alternative specification of u_{it} is introduced by Battese and Coelli (1992). The technical inefficiency effects are assumed to be defined by:

$$u_{it} = \exp\left\{-\eta\left(t - T\right)\right\} u_i \qquad (3.3.2.2.9)$$

where η is a single unknown parameter to be estimated and u_i is assumed to be identically and independently distributed as truncated normal, $N \sim \left(\mu, \sigma_u^2\right)$. Technical efficiency is allowed to vary over time but temporal pattern of inefficiency is assumed to be the same for all production units.[103] If the estimated value of η is positive, technical efficiency has increased at a decreasing rate. If it is negative, technical efficiency has decreased at an inreasing rate. When $\eta = 0$, the model collapses to the common time-invariant technical inefficiency model.[104] The

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[101] Tim Coelli, et. al., **An Introduction to Efficiency and Productivity Analysis,** Kluwer Academic Publishers, Dordrecht, 1998, p. 203.

[102] Subal C. Kumbhakar, C.A. Knox Lovell, **Stochastic Frontier Analysis,** Cambridge University Press, 2000, p. 112.

[103] Subal C. Kumbhakar, et. al., "Temporal Patterns of Technical Efficiency: Results from Competing Models," **International Journal of Industrial Organization,** Vol. 15, 1997, pp. 601-602.

[104] Pelin Kale, "Three Essays on Technical Efficiency in Turkish Manufacturing Industries," **Unpublished Ph.D Dissertation,** Ankara, 2001, p. 85.

assumptions that technical efficiency is time-invariant and allows half-normal distribution can be tested by conducting hypotheses through the use of generalized likelihood ratio tests[105], $H_0 : \eta = 0$ for the former case and $H_0 : \mu = 0$ for the latter case.[106]

In (3.2.2.9) if the i-th production unit is observed in the last period of the panel, then $u_{it} = u_i$. For earlier time periods, technical efficiency effect for i-th production unit at time t is the product of the technical inefficiency effect at the last period of the panel and the value of exponential function defined in (3.3.2.2.9). The value of the exponential function depends on the parameter η and the number of time periods before the last period of the panel.[107]

[105] The generalized likelihood-ratio test statistic, λ is calculated by: $\lambda = -2\left[\ln\{L(H_0)\} - \ln\{L(H_1)\}\right]$ where $L(H_0)$ and $L(H_1)$ are the values of the likelihood function under the null and alternative hypotheses, respectively.

[106] Coelli, **op.cit.**, p. 205.

[107] Coelli, **op.cit.**, p. 204.

4. AN EMPIRICAL STUDY ON SELECTED TURKISH PRIVATE MANUFACTURING INDUSTRIES

In previous sections, the theoretical structure of different approaches to efficiency estimation is discussed. As an empirical application, we select 22 Turkish private manufacturing industries for three cross-sections, 1985, 1990 and 1995, at three-digit level according to International Standart Industrial Classification of All Economic Activities, Second Revision, (ISIC Rev. 2) in order to estimate technical efficiency levels for each industry. The industries we select is listed by Table 1. in the appendix.

The approach to estimate the technical efficiency levels for each industry we select is stochastic frontier approach. We assume Cobb-Douglas model in the log-linear form. That is:

$$\ln Q_i = \beta_0 + \beta_1 \ln L_i + \beta_2 \ln K_i + \beta_3 \ln E_i + v_i - u_i \qquad v_i - u_i = \varepsilon_i \tag{4.1}$$

where Q_i is real value added[1], L_i is the labor input (number of hours worked), K_i is the capital input measured by the total capacity of power equipment, E_i is the electricity consumption measured in kWh. All data set used in this study are obtained from State Institute of Statistics, S.I.S. β_0, β_1, β_2 and β_3 are the parameters to be estimated, v_i accounts for random events, measurement errors and some other variables not included into the model, u_i represents technical efficiency; $u_i \geq 0$ whereas v_i is unrestricted. We assume half-normal distribution for technical inefficiency term. v_i is assumed to follow a normal distribution.

[1] Nominal value added is deflated by private manufacturing sector wholesale price index, measured at 1987 prices.

The maximum likelihood estimation is performed by computer program FRONTIER Version 4.1 written by Tim Coelli.[2] FRONTIER Version 4.1 assumes a linear functional form. Thus if anyone wishes to estimate a Cobb-Douglas production function, all input and output data must be loged before creating the data file for the program to use.[3]

The final maximum likelihood estimates are listed below:

Table 5.1 The final maximum likelihood estimates

	1985			1990			1995		
	Coefficient	**Standart-Error**	**t-ratio**	**Coefficient**	**Standart-Error**	**t-ratio**	**Coefficient**	**Standart-Error**	**t-ratio**
beta 0	5,7762	2,0009	2,8867	5,3679	0,8704	6,167	11,4262	2,1894	5,2187
beta 1	-0,00012	-0,000023	-5,3306	-0,0000026	0,0000035	-0,7416	-0,00009	0,000046	-1,9479
beta 2	0,9532	0,0132	71,8519	0,9782	0,0358	27,2739	0,673	0,1213	5,5474
beta 3	0,000055	0,0000101	5,4549	0,0000082	0,000065	0,127	0,000047	0,000047	1,0166
sigma-square	0,5786	0,2038	2,839	0,7134	0,1864	3,8261	0,9052	0,3694	2,45
gamma	0,9999	-0,00018	5551,6228	0,9999	0,000005	196842,23	0,9155	0,1075	8,5161

Table 5.2 Log-likelihood functions and likelihood ratio tests

	1985	1990	1995
Log-likelihood function	-12,097	-11,9566	-19,4897
Likelihood-ratio test of the one-sided error (λ)	8,6879	9,1527	1,1972

The critical value for a sample size of 22 at 5% significance level is 2, 074. In this context, the parameters β_0, β_1, β_2, β_3, σ^2 and γ exceed this critical t-value and thus, are all significant in 1985. The coefficients of labor and electricity consumption is so small that they are negligible. The coefficient of capital input is very strong. That means that capital input has a very strong effect on real value added whereas labor and electricity consumption have almost no effect.

[2] The computer program is available at <http://www.uq.edu.au/economics/cepa/frontier.htm>

[3] T.J. Coelli, **A Guide to FRONTIER Version 4.1: A Computer Program for Frontier Production Function Estimation**, CEPA Working Paper 96/07, Department of Econometrics, University of New England, Armidale, 1996a, p. 9.

Likelihood ratio test requires the estimation of the model under both null and alternative hypotheses. Under the null hypothesis, $H_0 : \gamma = 0$, the model is equivalent to average response function without the technical inefficiency effect. This test statistic is calculated as follows:

$$\lambda = -2\left[\ln\{L(H_0)\} - \ln\{L(H_1)\}\right] \tag{4.2}$$

where $L(H_0)$ and $L(H_1)$ are the values of the likelihood function under the null and alternative hypotheses, respectively.[4] If H_0 is true, this test statistics is usually assumed to be asymptotically distributed as a mixed- chi-square random variable with degrees of freedom equal to the number of restrictions involved. The calculation of the critical value for one-sided generalised likelihood-ratio test of $H_0 : \gamma = 0$ vs. $H_1 : \gamma > 0$ is simple. The critical value for a test of size α is equal to the value $\chi_1^2(2\alpha)$. Reject $H_0 : \gamma = 0$ in favor of $H_1 : \gamma > 0$ if λ exceeds $\chi_1^2(2\alpha)$.[5] We restrict μ to be zero for allowing technical inefficiency term to be distributed as half-normally. Thus, at 5% significant level with degrees of freedom equal to one, $\chi_1^2(2\alpha) = 2{,}71$. In 1985, λ exceeds 2,71 and thus we reject $H_0 : \gamma = 0$. It is obvious from Table 5.1 that the estimate of γ is almost 1. This result indicates that the vast majority of residual variation is due to inefficiency effect, and the random error is approximately zero. Hence, the traditional average response function is not an adequate respresentation of the data and stochastic frontier model is not significantly different from the deterministic frontier model with no random error involved. Technical efficiency estimates are given in Table 2. in the appendix. The industries according to ISIC (Rev.2) 311, 321, 322, 323, 324, 331, 332, 361, 381, 382 and 384 are below the average of the selected industries' technical efficiency level, 59,3% in 1985. The least efficient or the most inefficient industry is 322 having 19,1%

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[4] Tim Coelli, et. al., **An Introduction to Efficiency and Productivity Analysis**, Kluwer Academic Publishers, Dordrecht, 1998, p. 191.

[5] **Ibid.**, p. 192.

efficiency level. The most efficient industries are 314 and 352 having 99,9% efficiency level relative to rest of the industries.

When it comes to 1990 estimates, the parameters β_0, β_2, σ^2 and γ are significant at %95 confidence level whereas the parameters β_1 and β_3 are insignificant. Thus, labor and electricity consumption have no effect on real value added. The coefficient of capital is very strong meaning that it has a very strong effect on real value added. Likelihood-ratio test of the one-sided error, λ exceeds the critical value of this test statistics, and thus we reject $H_0 : \gamma = 0$. This result states that the vast majority of residual variation is due to inefficiency effect, and the random error is approximately zero. Hence, the traditional average response function is not an adequate respresentation of the data and stochastic frontier model is not significantly different from the deterministic frontier model as in 1985.

The industries according to ISIC (Rev.2) 312, 321, 322, 323, 324, 331, 332, 361, 369, 381, 382 and 383 are below the average of the selected industries' technical efficiency level, 62,4% in 1990. The least efficient or the most inefficient industry is 323 having 13,2% efficiency level. The most efficient industries are 352 and 384 having 99,7% efficiency level relative to the rest of the industries. 321, 322, 323, 324, 331, 332, 361, 381 and 382 are still below the average technical efficiency level in 1990. 311 and 384 which are below the average level in 1985 have a striking increase in efficiency with more than 40%. On the other hand, 383 which is approximately 30% above the average level in 1985 has a striking decrease in efficiency with more than 50%.

In 1995, the parameters β_0, β_2, σ^2 and γ are significant at 95% confidence level whereas the parameters β_1 and β_3 are insignificant as in the 1990. Hence, labor and electricity consumption have no effect on real value added. The capital input has a positive effect on real value added but less relative to years 1985 and 1990. Likelihood-ratio test of the one-sided error, λ fails to exceed the critical value of this test statistics, and thus we accept $H_0 : \gamma = 0$. This result states that the vast majority of residual variation is due to the random effect which dominates the technical efficiency component in the determination of total residual. Therefore, we are back to an OLS production function model with no technical inefficiency. The OLS estimates are listed below:

Table 5.3 The OLS estimates

	1995		
	Coefficient	**Standart-Error**	**t-ratio**
beta 0	5,0044	2,3241	2,1532
beta 1	-0,00012	0,000036	-3,4088
beta 2	0,9485	0,1295	7,3228
beta 3	0,000094	0,000044	2,133
sigma-square	0,3189		

The parameters β_0, β_1, β_2, β_3 are all significant at 95% confidence level. The coefficients of labor input and electricity consumption is so small that they are negligible. The coefficient of capital input is very strong. That means that capital input has a very strong effect on real value added as in the years 1985 and 1990.

CONCLUSION

According to Solow *residual* approach, technical progress is the unique source of total factor productivity growth. Recent developments acknowledge that along with technical progres, changes in technical efficiency can also contribute to total factor productivity growth. The investigation of sources of total factor productivity growth and the effects of the sources of variation on total factor productivity growth allow policy-makers to understand the facts in improving the productivity. In this study, technical efficiency is estimated by means of stochastic production frontier approach with three cross-sections, 1985, 1990, and 1995 in selected Turkish private manufacturing industries at three-digit level.

Technical efficiency as a source of total factor productivity growth can be estimated by two main methods: Data Envelopment Analysis which includes mathematical programming techniques, and Stochastic Frontier Approach which employes econometric methods.

The results of the model we established point out that capital has a very strong effect on the determination of real value added in selected Turkish private manufacturing industries during all the years taken into consideration. For all years, the capital coefficients are above 90%.

The contribution of labor to real value added is so small that it is negligible in 1985. It is also noticeable that its effect on real value added is negative. This brings the result into mind that labor productivity is negligible but negative in 1985. In 1990, it can be stated that there is no contribution of labor input to real value added according to statistical results. In 1995, its effect on real value added is negligible but negative which in turn corresponds to the conclusion that its productivity is negligible but negative as in 1985. All these results from the fact that laborforce is not adequately qualified and hidden-unemployment is high for selected years in Turkish private manufacturing industries.

The contribution of electiricty consumption to real value added is so small that it is negligible but positive in 1985. This states that its productivity is positive but negligible. In 1990, it can be stated that there is no contribution of electricity consumption to real value added according to statistical results. In 1995, its effet on real value added is positive but negligible which in turn corresponds to the conclusion that its productivity is positive but negligible as in 1985.

In 1985 and 1990, the vast majority of residual variation is due to inefficiency effect, and the random shocks are approximately zero. Technical inefficiency effects dominates the random effects. That is to say that all variation in output not associated with variation in inputs is attributed to technical inefficiency. In 1995, next to the crisis year, the vast majority of residual variation is due to random shocks, all variation in output not associated with variation in inputs is attributed to random shocks. The crisis in 1994 have strong effects on Turkish private manufacturing industries so that it changes the relative contribution of sources of residual variation in the determination of output variation.

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APPENDIX

Table 1.

INDUSTRY	ISIC Codes (Rev. 2)
Food Manufacturing	311
Manufacture of food products not elswhere classified	312
Beverages Industries	313
Tobacco Manufactures	314
Manufacture of Textiles	321
Manufacture of wearing apparel, except footwear	322
Manufacture of leather and products of leather, leather substitutes and fur, except footwear and wearing apparel	323
Manufacture of footwear, except vulcanize or moulded rubber of plastic footwear	324
Manufacture of wood and wood cork products, except furniture	331
Manufacture of furniture and fixtures, except primarily of metals	332
Manufacture of paper and paper products	341
Printing, publishing and allied industries	342
Manufacture of industrial chemicals	351
Manufacture of other chemicals	352
Manufacture of rubber products	355
Manufacture of pottery, china and earthenware	361
Manufacture of other non-metalic mineral products	369
Iron and steel basic industries	371
Manufacture of fabricated metal products, except machinery and equipment	381
Manufacture of machinery except electrical	382
Manufacture of electrical machinery, apparatus, appliances and supplies	383
Manufacture of transport equipment	384

Table 2. Technical Efficiency Estimates

ISIC Rev.2	% Technical Efficiency in 1985	% Technical Efficiency in 1990	% Average of 1985 and 1990
311	50,7	96,7	70,0
312	63,9	37,1	48,7
313	88,8	97,7	93,1
314	99,9	89,9	94,8
321	38,3	39,9	39,1
322	19,1	35,6	26,1
323	33,2	13,2	20,9
324	24,5	33,3	28,6
331	24,3	34,5	29,0
332	19,3	35,9	26,3
341	71,4	93,6	81,7
342	65	93,6	78,0
351	94,3	99,4	96,8
352	99,9	99,7	99,8
355	73,8	92,5	82,6
361	51,1	37,4	43,7
369	69,8	37,9	51,4
371	91,5	95,4	93,4
381	41,1	37,4	39,2
382	39,9	38	38,9
383	90,4	38	58,6
384	53,5	99,7	73,0
%Average Techical Efficiency	**59,3**	**62,4**	**60,8**