MULTIPERIOD MEAN-VARIANCE PORTFOLIO OPTIMIZATION IN MARKOVIAN MARKETS UNDER IMPERFECT INFORMATION

by

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A Thesis Submitted to the
Graduate School of Engineering
in Partial Fulfillment of the Requirements for
the Degree of

Master of Science

in

Industrial Engineering

Koç University

September, 2004

Koç University

Graduate School of Sciences and Engineering

This is to certify that I have examined this copy of a master's thesis by

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and have found that it is complete and satisfactory in all respects,
and that any and all revisions required by the final
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Date: 30.9.2004

To my Family

ABSTRACT

In this thesis, we analyze multiperiod portfolio optimization problems in stochastic markets where periodic returns are serially correlated and information flow is imperfect. Serial correlation and imperfect information flow are constructed by using two processes, one of which is observable and the other is hidden. Both processes are assumed to be Markov chains. The market consists of a riskless asset and several risky assets whose returns directly depend on state of the unobserved market process. The state of the unobserved stochastic market in a certain period depends on the prevailing economic, social and other relevant factors. We consider two different models that describe the imperfect information flow between the unobserved stochastic market and the observed process. Considering such a stochastic market modulated by a hidden Markov chain, the multiperiod mean-variance formulation is solved by using dynamic programming. The explicit optimal solution is obtained, and some illustrative cases which demonstrate the application of the solution procedure are given.

Keywords: Multiperiod Portfolio Optimization, Mean-Variance Models, Dynamic Programming, Hidden Markov Chain, Imperfect Information

ACKNOWLEDGMENTS

First, I would like to express my deepest gratitude to Prof. Süleyman Özekici for his supervision, support and time spent in all steps of the development of this work. I am proud of being one of his students who have been able to learn from his deep knowledge and experience.

I would like to thank Assist. Prof. Alper Demir for his comments on the final phase of this study.

I am grateful to Assist. Prof. Lerzan Örmeci and Assist. Prof. Deniz Aksen for taking part in my thesis committee, for critical reading of this thesis and for their valuable suggestions and comments.

I also would like to thank my friends Bahadır, Davut, Doruk, Erhan, Sezen, Tuğrul, Uğur and Negar for the beautiful atmosphere they have created around me.

Finally, I would like to thank my family for giving me continuous moral support throughout my whole life.

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NOMENCLATURE

Y	Observed market process
Z	Stochastic (unobserved) market process
$Q(a,b)$	Transition matrix of Z
$\hat{Q}(i,j)$	Transition matrix of Y
r_f	Return of the riskless asset
$R_k^e(a)$	Excess return of the kth asset in state a
$r_k^e(a)$	Expected value of excess return of the kth asset in state a
$\hat{r}_k(i)$	Expected value of return of the kth asset given that Y is in state i
σ_{ij}	Covariance between ith and jth asset returns
$\hat{\sigma}_{kl}(i)$	Covariance between kth and lth asset returns given that Y is in state i
$O(i,a)$	Observation matrix
$E(a,i)$	Emission matrix
$E_i[\bullet]$	Expectation given that the initial market state is i
P	Probability
u	Vector representing the amounts invested in risky assets
$Var_i(\bullet)$	Variance given that the initial state is i
X_n	Money available for investment at the nth period

Chapter 1

INTRODUCTION

Portfolio management can be defined as the process of allocating wealth among different assets such that an allocation which suits investor's risk and return preferences is determined. Moreover, an investment policy describes an investor's decisions about which portion of his wealth to invest in each asset during the investment horizon. Typically, when an investor constructs his investment policy, what is known is the initial amount of capital, not the returns of assets. If all assets had deterministic returns, then the optimal investment policy would be simply to invest all the wealth in the asset with the highest return. However, in reality the returns are stochastic, so investors are faced with the problem of portfolio selection. Moreover, optimal investment policy determination depends on many factors such as investment time horizon, characteristics of the market and the objective of the decision maker. Portfolio theory has been extensively used to provide solutions for the portfolio selection problem. In this thesis, we consider the multiperiod portfolio selection problem in a stochastic market where the information flow is imperfect and the returns of the assets are modulated by a hidden Markov chain. The main objective is to come up with an optimal analytical solution to the multiperiod mean-variance formulation for this problem.

Origin of modern portfolio theory is credited to Harry Markowitz for his pioneering paper [1] that appeared in 1952. Markowitz's model, the so-called classical mean-variance model, is a parametric optimization model for the single period portfolio selection problem which provides analytical solutions for an investor trying to maximize his expected wealth without exceeding a predetermined risk level, or alternatively, for an investor trying to minimize his risk ensuring a predetermined wealth.

After the introduction of the classical mean-variance model, one of the most immediate future research areas involved the multiperiod portfolio selection problem. Especially by considering long term investors who invest continuously rather than for a single period, many researchers tried to adapt the classical mean-variance model or similar models for the multiperiod case. In most of the multiperiod models, it is assumed that the return of a specified asset in a specified period is independent of the return of the same asset and all other assets in previous periods. However, in a more realistic approach some sort of dependence among the returns should be considered. Moreover, most of the studies including Markowitz's mean-variance model assume a free market in which all investors have access to all information perfectly. But, in reality investors act according to imperfect information because of either inability to access all information or inability to grasp all cause and effect relationships. In this study, this dependence among returns and imperfect information is modeled by assuming that there exist hidden economic factors that determine behaviors of the market. Therefore, the market is taken as a hidden Markov chain.

Imperfection in information flow is set up through a probabilistic relationship between the observed and unobserved market processes. The probabilistic relationship is defined in two different types of models by the observation matrix O and the emission matrix E. In the observation matrix, $O(i, a)$ denotes the probability that the unobserved market process is in state a in a period given that observed market process is in state i in that period. In the emission matrix, $E(a, i)$ denotes the probability that the observed market process is in state i in a period given that unobserved market process is in state a in that period. We suppose that there exists a single observation matrix O or an emission matrix E which govern the relationship between the observed and unobserved market processes. In this study, we do not deal with the problem of estimating the probabilities of these matrices, but assume that they are given.

The thesis is organized as follows: A literature survey on multiperiod portfolio optimization and hidden Markov models in portfolio optimization is given in Chapter 2. Chapter 3 describes the stochastic structure of the market in which serial correlation among returns and imperfect information is assumed. The mean-variance problem formulation in a perfectly observable market for generating efficient multiperiod portfolio policies is given in Chapter 4. The solution of the problem that is found by dynamic programming is given in Chapter 5. The mean-variance problem formulation in an imperfectly observable market and the corresponding dynamic programming solution are given in Chapter 6. Some illustrative cases demonstrating the application of the analytical solutions are given in Chapter 7. Chapter 8 presents the concluding remarks and possible further research topics. Finally, MATLAB codes used for solving the problems and explicit forms of some equations can be found in Appendix A and B.

Chapter 2

LITERATURE SURVEY

Portfolio management is one of the major study areas in financial engineering. Portfolio management deals with the portfolio selection problem faced by an investor who wants to allocate his wealth among investment opportunities according to his risk and return preferences. Modern portfolio theory has its origins in the work of Harry Markowitz [1] which was published in 1952 and won him a Nobel prize in economics in 1990. In his paper, Markowitz introduced the first systematic treatment of investors' conflicting objectives of high return versus low risk. Markowitz's model called single-period mean-variance model aims to maximize an investor's expected wealth without exceeding a predetermined risk level or, alternatively, aims to minimize his risk ensuring a predetermined wealth. The model has explicit solutions and provides the set of efficient portfolios. After Markowitz's classical mean-variance model, huge amount of related research on portfolio management has been developed. In this survey, we will mention about some major researches on portfolio management with emphasis on mean-variance models and multiperiod formulations.

In a survey paper, Steinbach [2] reviews the mean-variance models in financial portfolio analysis. This survey refers to 208 papers which shows the diversity of different models and approaches used to analyze this problem for both single period and multiperiod cases.

Merton [3] studies the derivation of the mean-variance efficient portfolio frontier analytically. In his paper, the efficient portfolio frontiers are derived explicitly, and the characteristics of the frontiers are verified. The mutual fund theorem is proved by showing that any efficient portfolio can be attained by a linear combination of two other efficient portfolios ("mutual funds"). Later, he studies the efficient portfolio set when one of the assets is riskless. Two mutual funds can be chosen in such a way that one fund holds only the riskless security and the other fund contains only risky assets. So, by using the mutual fund theorem, Merton explains the traditional way of finding the efficient frontier when one of the assets is riskless as graphing the efficient frontier for risky assets only, and then drawing a line from the intercept tangent to the efficient frontier

In portfolio management, the safety-first approach was developed by Roy [4] in 1952 as an alternative to the classical mean-variance approach. The objective of the safety-first approach is to minimize the probability that the terminal wealth of an investor is below a preselected amount. Roy names the event that causes an investor's wealth to fall below a disaster level as a dread event. Moreover, the principle of safety-first is to reduce the chance of such a dread event as much as possible. In the model, the objective function is defined as the upper bound of the probability of a dread event by using the Chebyshev's inequality given that only the first and second moments of return distributions are known. Later, the problem of holding n assets and the special case of the problem with two assets are analyzed in more detail. Levy and Sarnat [5] show that a special case of the safety-first approach provides the same optimal portfolios as the classical mean-variance approach does.

After Markowitz's single-period mean-variance model, a lot of research has been done on the multiperiod portfolio selection problem. One of the first multiperiod models is the portfolio revision approach developed by Smith [6]. He extends the existing Markowitz/Sharpe model which forms a basis for selecting and revising portfolios. The Markowitz/Sharpe model finds how to select a portfolio only at a single point in time, so by using this model an investor should constantly change his investment holdings such that his portfolio is efficient, and this will result in excessive portfolio turnover. Therefore, brokerage fees and taxes will substantially reduce the portfolio yield of this model. Smith extended the Markowitz/Sharpe model to a transition model which is an adaptive type of mechanism performed at finite intervals. According to the suggested technique, a transition should be made only if its expected dollar return exceeds the dollar cost of the transition, which consists of brokerage fees and the associated taxes that must be paid by the investor. The technique is applied to 150 common stocks between 1957 and 1964. Portfolio yields, which result from the revision procedure, are compared with similar performance measures from unrevised portfolios. The result is that higher portfolio yields can be achieved by revising portfolios using the Smith's transition model.

One of the common techniques used in solving the multiperiod portfolio selection problem is dynamic programming. Mossin [7] analyzed the multiperiod problems using a dynamic programming approach. Mossin states that formulation of the portfolio selection problem in terms of portfolio rate of return obscures the absolute size of the portfolio to be taken into consideration. Therefore, in his multiperiod theory he focuses on the development of total wealth through an investment horizon. He first analyzes the single-period problem in which an investor makes his portfolio decision at the beginning of a period and waits without making any changes in his decision until the end of the period when returns are realized. The investor then makes the next period decision according to the wealth level at the previous period. Here, the investor makes his decisions at the beginning of each period such that the expected utility of his final wealth at the end of the investment horizon is maximized. Moreover, Mossin explains that sequential portfolio decisions are contingent upon the outcomes of previous periods and take into account the information regarding future probability distributions. After the last period decision is made, by using a backward recursion procedure, an optimal first-period decision is determined assuming statistical independence among yields in different periods and without taking transaction costs into account. He also considers an "myopic" investor who makes decisions considering only the wealth and probability distributions of returns at the beginning of a period and aims to maximize expected utility of the wealth at the end of that period disregarding the following periods completely. In other words, the investor makes a series of single-period decisions rather than a sequence of decisions. Mossin emphasizes that a myopic investment strategy can be optimal for utilities which are logarithmic and power functions. Mossin also studies whether there can exist an optimal stationary portfolio policy such that proportions of wealth invested in each asset in each period are the same. He states that an optimal stationary portfolio policy cannot exist if yield distributions are not stationary.

After the portfolio revision approach introduced by Smith [6], Chen et al. [8] have developed it to what they call a portfolio revision process. In this process, an investor revises his initial portfolio periodically to adapt to changing conditions. They claim that investment decisions are usually made starting with a portfolio rather than cash, so some assets must be liquidated to permit investment in others. In order to include the expected transfer costs incurred in transition, a single-period portfolio revision model is formulated. It is assumed that a portfolio is revised when new information becomes available and the marginal utility of revision equals the marginal cost of revision. Analytical results of the model are compared to Smith's [6] target portfolio. The comparison shows that Smith's model does not consider the multiperiod aspect, and thus suggests a controlled transition approach which is inferior to the true optimal solution obtained in the portfolio revision process. The single-period portfolio revision model is then extended to the multiperiod case in a dynamic programming framework, and compared to Mossin's [7] dynamic portfolio selection model through two numerical examples. One of the examples assumes an investor starting with cash (portfolio selection problem), and the other assumes an investor starting with a portfolio (portfolio revision problem). Both examples are two-asset and two-period problems. It is stated that Mossin's model has to be modified since investors starting with a portfolio of assets are locked in, so transfer costs have to be taken into account. Finally, the multiperiod, multiasset case is also discussed.

Samuelson [9] formulates and solves a generalized multiperiod portfolio selection model, corresponding to lifetime planning of consumption and investment decisions. He asserts that the present lifetime model shows that at early, or any, stages of life investing for many periods does not introduce extra tolerance for being risky. In his paper, a stochastic programming problem that needs to be solved simultaneously for optimal saving-consumption and portfolio selection decisions over time is derived. Then, the optimal decisions as a function of initial wealth are obtained. The model is applied first to a problem with one riskless asset and then to problems involving risky assets. Cases where the utility functions are isoelastic are also analyzed. He finds out that the optimal portfolio decision is independent of wealth at each stage and independent of all consumption-saving decisions for isoelastic marginal utility functions. Moreover, for isoelastic marginal utilities, the model shows that investors have the same risk tolerance at all stages of life.

Dumas and Luciano [10] focuses on an imperfection in financial markets. They study the dynamic portfolio choice problem under transaction costs. The model considers an investor who accumulates wealth without consuming until some terminal point in time and has the objective of maximizing the expected utility from his terminal consumption. They model a continuous time portfolio selection, and provide necessary conditions which must be satisfied when it is optimal to refrain from trading, and which must prevail when trading takes place. Even though the model assumes that transaction costs are proportional to the value of the trade, the authors emphasize that the model has analytical solution whether the transaction costs are fixed or of mixed character. The theory of optimal regulated Brownian motion is used to calculate the portfolio policy in the form of two control barriers, between which portfolio proportions are allowed to fluctuate. An exact analytical solution is obtained. Finite horizon and infinite horizon solutions in the absence and presence of transaction costs are compared. Moreover, deviations from a base case are examined in the dimensions of increasing transaction costs, increasing risk aversion and increasing risk. They find out that increased transaction costs do not bias the optimal portfolio one way or the other, and that there is very little interaction between transaction costs and risk aversion.

Roy [11] studies dynamic portfolio choice for survival under uncertainty and develops a discrete dynamic optimization model where the objective is to maximize the long-run probability of survival through risk portfolio choice over time. There is a given minimum withdrawal requirement (subsistence consumption). The investor survives only if his wealth is large enough to meet this requirement every period over an infinite horizon. If the wealth level is less than the subsistence consumption, the investor is said to be ruined, otherwise the investor is said to survive. In each period, the investor withdraws part of the current wealth and allocates the rest between a risky and a riskless asset. If the returns of the risky assets are assumed to be independent and identically distributed with continuous density, the existence of a stationary optimal policy is proved and the dynamic programming equation is given which yields the maximum survival probability and the stationary optimal policies. The stationary optimal policies point out variable risk preference ranging from extreme 'risk-loving' behavior for low levels of wealth to 'risk-averse' behavior for high levels of wealth. Moreover, in the model there exists lower and upper critical levels of initial wealth. Regardless of what actions the investor takes, he cannot survive, if the initial wealth is below the lower critical level, and he survives with probability one by choosing to concentrate all investment on the riskless asset in every period if the initial wealth is above the upper critical level. Roy shows that between these two critical levels, the maximum survival probability is continuous and strictly increasing in current wealth.

Ehrlich and Hamlen [12] solve the stochastic portfolio consumption control problem under the assumption that individuals follow precommitment strategies over finite intervals of time. The assumption seems to be valid if we consider that in reality it is too costly for an investor to collect current information and immediately make appropriate changes to his investment strategies. Therefore, the precommitment approach is an alternative to Merton's continuous time stochastic dynamic control problem, which assumes instantaneous feedback and costless revisions of choices all along the time axis. The investor can invest a proportion of his wealth in a riskless asset and the rest in a portfolio of risky assets equivalent to the market portfolio. It is shown that under precommitment the intertemporal consumption growth path would be a relatively smooth function of the risk-free rate of return, time preference, and the coefficient of relative risk aversion, and independent of the portfolio's risk parameters. Moreover, it is shown that investors tend to hold portfolios that are a function of their expected risk and return parameters, but are independent of their wealth levels and risk preferences.

Bodily and White [13] study the optimal consumption and portfolio mixture for a discrete time, discrete state preference model. The investor's current wealth and past consumption experience through a summary descriptor of past consumption determines his preferences for future consumption. Relations between the optimal consumption and investment decisions, which depend on an investor's preferences and future expectations on returns, are found. The preferences are subjective and represented by a von Neumann-Morgenstern utility function. After stating the investor's problem and developing the model, the modeling flexibility of this approach is illustrated by an example. Based on certain assumptions the following results about the properties of admissible strategies and policy implications are found: The optimal expected utility over a finite planning horizon is nondecreasing in initial wealth and non-increasing in summary descriptor. The optimal consumption level does not decrease as wealth and summary descriptor increase. The optimal fraction of investment in the risky opportunity does not decrease as wealth increases.

Elton and Gruber [14] compare selecting portfolios on the basis of the geometric mean of future multiperiod returns against selecting portfolios on the basis of the expected utility of multiperiod returns. They show that when the ability of the investor to revise his portfolio is considered, each of these rules is only appropriate under a very restrictive set of conditions. The objective is assumed to maximize the expected utility of the investor's wealth at some terminal time. The analysis is performed both when return distributions are unchanging over time and when they change in a regular pattern. Maximization of the geometric mean leads to the maximization of the expected utility of terminal wealth when the investor's utility function is logarithmic and the distribution of returns in all future periods are constant over time or the distribution of returns in any period is expected to be the first-period returns multiplied by a constant, or raised to a power or both. The selection of portfolios that maximize the expected value of a utility function in terms of return and risk is appropriate whenever either utility functions are quadratic or returns are normally distributed.

Li and Ng [15] consider the mean-variance formulation in multiperiod portfolio selection and derive an analytical optimal portfolio policy and an analytical expression of the mean-variance efficient frontier. They aim to apply Markowitz's single-period mean-variance formulation into a multiperiod framework. They extend of the existing literature dealing with risk management in dynamic portfolio selection. Their model assumes independence of returns over time. Dynamic programming is used for this multiperiod portfolio selection problem. A separable auxiliary problem generating the same efficient frontier with the classical models is used to solve the mean-variance formulation. As a special case, a model with a riskless asset is discussed.

Leippold et al. [16] are concerned with a geometric approach to discrete time multi-period mean-variance portfolio optimization of assets and liabilities that largely simplifies the mathematical analysis and the economic interpretation of such model settings. For portfolios consisting of both assets and liabilities, closed form solutions are obtained by using the geometric approach to dynamic mean-variance optimization. The objective is defined as a function of the surplus of final total assets and liabilities. The asset only model mean-variance problem used in Li and Ng [15] can be represented in terms of simple products of some single period orthogonal returns. The usefulness of the geometric representation of multiperiod optimal policies and mean-variance frontiers are discussed by the authors.

The works cited so far do not consider statistical dependence between returns over the periods. Due to complexities in calculations, serially correlated returns which exist in real life has not withdrawn much attention.

Hakansson and Liu [17] study the capital growth model in which investment returns are statistically dependent on returns in previous periods. Their model is different than the classical mean-variance model since it has a logarithmic utility function, which means risk aversion so that risk factors are not taken into account directly. It is assumed that the investor makes decisions at discrete points which may be unequally spaced in time. He can invest in a riskless and risky assets in each period. The returns in a given period depend on the change in the general condition of the economic environment. The transition probabilities are constants which implies that the economy obeys a non-stationary Markov process. The Markov chain formed by the transition probabilities is assumed to be irreducible and ergodic. All investments are realizable in cash at the end of each period, and taxes and conversion costs are proportional to the amount invested. The returns from risky assets in a given period can depend on the state of the economy both at the beginning and at the end of the period. The amount of capital available at each decision point depends on the amount at the previous decision point. An optimal investment strategy is obtained on the basis of a slightly generalized and weakened version of the rational criterion that more is preferred to less in the very long run. The optimal policy obtained is myopic and maximizes the long run growth rate.

Hakansson [18] considers Mossin's [7] work whose model isolates the class of utility functions of terminal wealth. Hakansson's model is similar to the model built in his past work [17] and involves two versions: one with serial correlation of returns and the other without the serial correlation. Solutions of these models show that Mossin's [7] conclusions are true only in a limited sense even when returns are serially independent. When investment returns in the various periods are statistically dependent, only the logarithmic function provides utility functions of short-run wealth, which are myopic. He assumes stochastically constant returns to scale, perfect liquidity, divisibility of assets at each decision point, absence of transaction costs, withdrawals, capital additions, taxes and short sales.

Hakansson [19] extends the standard portfolio selection model to the multiperiod case. He also extends the results of multiperiod mean-variance approach based on average compound return in the stationary two-asset case. According to the results, the set of efficient portfolios in any one period decrease as the horizon increases and converge to a single efficient sequence. If an investor wants to maximize the expected average compound return over N periods, $N \geq 2$, then there exists a unique, single-period von Neumann-Morgenstern utility function defined on wealth which is consistent with this objective. This utility function implies risk aversion and does not in general produce a mean-variance efficient portfolio in the single-period case. When N is large, the set of average compound returns, which are mean-variance efficient, can be exactly or approximately obtained only with a subset of the terminal functions, which induce myopic single-period utility functions. The growth-optimal portfolio is demonstrated to be efficient in the limit. It is then found that only the riskless portfolio sequence will generally be efficient with respect to both single-period and total return as well as the long-run average compound return. This paper indicates rapid convergence of the long-run efficient portfolios to N-period efficient portfolios and later presents implications of graphic analysis by use of indifference curves. Mean-variance formulations of average compound return over two or more periods imply risk aversion without reference to the variance. They are consistent with von Neumann-Morgenstern utility theory, and they imply decreasing absolute risk aversion, automatically insure solvency. Therefore, the investor's survival imply that myopic investment behavior is optimal.

Hernández-Hernández and Marcus [20] investigate the existence of optimal stationary policies, which maximize the long run average reward, for infinite horizon risk sensitive Markov control processes with denumerable state space, unbounded cost function, and long run average cost. Using the vanishing discount approach, they prove there exist optimal stationary policies, and then derive an optimal stationary policy for a given utility function.

Bieleceki et al. [21] extend standard dynamic programming results for the risk sensitive optimal control of discrete time Markov chains to a new class of models. The state space of the Markov control model is finite and consists of a set of possible factor values. The transition matrix of the underlying Markov chain is assumed to be irreducible and strictly positive, so that same kinds of dynamic programming results found in the existing discrete time risk sensitive control theory literature still remain valid. A portfolio's return is a combination of asset returns which depend on the factor's state both at the beginning and at the end of the period. The optimal trading strategy is characterized in terms of a dynamic programming equation. The results are applied to the financial problem of managing a portfolio of assets affected by Markovian microeconomic and macroeconomic factors, where the investor seeks to maximize the portfolio's risk adjusted growth rate. Finally, optimal stationary policies are given, and some illustrative cases are presented. The model used in this paper resembles the model constructed in this thesis since in both of the models macroeconomic factors on the market play the crucial role in obtaining an optimal trading strategy.

In multiperiod portfolio optimization, stochastic markets represented by Markov chains are used in Çakmak and Özekici [22] and in Çelikyurt and Özekici [23]. Çakmak and Özekici [22] present a multiperiod mean-variance model where the model parameters change with a stochastic market. According to the state of the market during any period, the mean vector and covariance matrix of the random returns of risky assets change. The stochastic market follows a Markov chain. To obtain explicit formulations of the efficient frontier, dynamic programming is used. Moreover, numerical examples are presented to demonstrate the application of the procedure. Çelikyurt and Özekici [23] analyze the multiperiod mean-variance model given in Çakmak and Özekici [22] by considering safety-first approach, coefficient of variation and quadratic utility functions. Using dynamic programming, efficient frontiers and optimal portfolio management policies are obtained. Finally, several examples are given to demonstrate the procedure with an interpretation of the optimal policies.

Even though hidden Markov models (HMMs) are one of the important tools in speech recognition, bioinformatics, gene prediction etc., they have been used in portfolio optimization only very recently. In 2002, Elliott et al. [24] use a HMM to describe stock price movements in order to find optimal portfolio trading strategy that maximizes the expected terminal wealth. The model considers discrete time description of the stock prices since the authors aim to provide trading strategies at significant times where a change of the stock price requires a rebalancing of the portfolio. By using Expected Maximization (EM) algorithm, historical data are trained and the hidden Markov model is estimated. A numerical example involving a single risky asset (German stock index, Xetra-DAX) and a riskless asset is given. Even though the model suffers from not considering transaction costs, and non-divisibility of asset units, the terminal wealth increases in accordance with the proposed optimal trading strategy.

Imperfect information is also a new concept in finance like HMMs. Sources of imperfection in financial markets are usually transaction costs, taxes, indivisibility of assets, etc. Imperfect information concept appears to be in game theory and used for sequential games where a player does not know exactly what actions other players take. Well-known economist Stiglitz [25], who was World Bank Senior Vice President and Chief Economist between February 1997 and February 2000, focuses on imperfect information. He emphasizes that obtaining information is imperfect, costly and there are major information asymmetries. Moreover, he believes that understanding imperfect information is one of the most important breaks from the past, and provides explanations to some of the basic characteristics of a market economy. The study on credit rationing by Stiglitz and Weiss [26] presents the first theoretical justification of true credit rationing by considering the effect of imperfect information in markets. In addition, Stiglitz [27] explains the observed phenomena of price dispersions and advertising effects at the equilibrium of product markets, which cannot be explained by traditional models of competition with perfect information.

This literature survey presents information about major research papers on portfolio optimization, especially on multiperiod formulations and on objectives other than the mean-variance trade-off. This thesis involves the application of multiperiod portfolio optimization under imperfect information. It extends the model of multiperiod mean-variance portfolio optimization in Markovian markets given in Çakmak and Özekici [22]. The model and its parameters depend on a stochastic market. The market involves one riskless and m risky assets. Depending on underlying economic factors, the mean vector and the covariance matrix of the asset returns change because the state of a stochastic market changes. The solution of the mean-variance formulation is found by solving an auxiliary problem with a dynamic programming technique. The explicit optimal solution is obtained for both the auxiliary problem and the main problem, and some illustrative cases which demonstrate the application of the solution procedure are given.

Chapter 3

STOCHASTIC MARKET MODULATED BY A HIDDEN MARKOV CHAIN

Financial markets are stochastic in the sense that value of an asset changes with time. Once stochastic behavior of assets is formalized, appropriate portfolios are constructed to take advantage of their stochastic nature. Formalization of stochastic behavior of assets usually depends on the assumption that the return of a financial instrument in a certain period is independent of the return of that instrument in previous periods so that the multiperiod portfolio model and its solution becomes simple. However, in a realistic setting returns of financial instruments are often serially correlated.

Moreover, performance of the financial market depends on some economic factors. For instance, one can consider gross domestic product growth rate, interest rate, exchange rate, inflation rate, etc. as factors affecting performance of an economy. In our model, the financial market involves several risky assets and a riskless asset. Exact distributions of these assets are unknown to investors in the market, but they know some of the parameters of these distributions; like means, variances and covariances. In our model, distributions of returns are not directly affected by economic factors, but their parameters are. These factors determine in which state the market functions, so there is some kind of a relationship between the market and its states. Moreover, state of the market in a certain period depends on the states in previous periods, because economic factors changing over time are correlated. As the market process changes over time, the changing factors alter the state of the market; hence, distributions of returns change due to the change in its parameters. Therefore, distributions of returns directly depend on states of the market rather than the factors affecting the market. Thus, property of serially correlated returns is carried through a market process rather than time series models such as Box-Jenkins autoregressive-moving average (ARMA) model. However, the market process is hidden to us, and this is the main concern in this thesis.

In this study, portfolio management in markets where rates of returns are serially correlated and information flow is imperfect is analyzed. Serial correlation and imperfect information flow are constructed by using observed and unobserved processes. Both processes are assumed to be Markov chains, so the state of the market in a period depends only on the state of the last period, which is the well-known property of the Markov chain. We let Z_n denote the state of the unobserved market process in period n, so that $Z = \{Z_n;\ n = 0, 1, 2, \dots\}$ is a Markov chain with some transition matrix Q and some state space $F = \{a, b, c, ...\}$ consisting of n_z states. Then, the transition probabilities are given as

$$P\{Z_{n+1} = b \mid Z_n = a\} = Q(a, b). \tag{3.1}$$

Moreover, we let Y_n denote the state of the observed market process in period n, so that $Y = \{Y_n;\ n = 0, 1, 2, \dots\}$ is a Markov chain with some transition matrix $\hat{Q}$ and some state space $E = \{i,\ j, ...\}$ consisting of n_y states. Then, the transition probabilities are given as

$$P\{Y_{n+1} = j \mid Y_n = i\} = \hat{Q}(i, j). \tag{3.2}$$

It is assumed that the state Y_n of any period is known at the beginning of that period. The market functions according to the unobserved process whose states depend on various economic factors; however, investors in the market can only see the observed process.

The relationship between the stochastic market and the distribution of the returns is such that the distribution of the return of risky assets in a period depends only on the unobserved state of the market in that period. In other words, if we let R denote the random variable representing the return of an asset, then $R(a)$ denotes the return of this asset in any period where a is the state of the unobserved market in that period. Therefore, the expected value and the variance of the return of an asset depend only on the states of the unobserved market process. Thus, when the state of the market in two different periods is the same, the expected return, variance and covariance matrix of assets in these periods will be the same. In this way, one needs to generate the means, the variances and the covariances of the returns for all assets only for the states of the unobserved market process. Otherwise, all of these parameters should be generated for all assets and for all periods, and this would sharply increase the computational complexity of the model when investment horizon is long. Therefore, one can easily generate parameters for assets for all states of the unobserved market process where number of states of the process is much less than the number of periods in a long investment horizon.

The market consists of one riskless asset with a known return r_f and m risky assets with random returns $R(a) = (R_1(a), R_2(a), ..., R_m(a))$ in state a of the unobserved market process. We let $r_k(a) = E[R_k(a)]$ denote the mean return of the kth asset in state a and $\sigma_{kl}(a) = \text{Cov}(R_k(a), R_l(a))$ denote the covariance between kth and lth asset returns in state a. The riskless asset is typically a cash bond and our setting allows for two possible cases for its return. In a truly riskless scenario, the return of the cash bond depends on the observed state of the market since it is known to the investor with certainty. This allows us to assume that riskless lending or borrowing is possible with return $r_f(i)$ if the observed market state is i. In such a case, the excess return of the kth asset is

$$R_k^e = R_k(Z_n) - r_f(Y_n) \tag{3.3}$$

in period n. Another scenario is obtained by assuming that the return of the cash bond depends on the true state of the stochastic market such that lending or borrowing is possible with return $r_f(a)$ if the unobserved market is in state a. Note that this implies a random return $r_f(Z_n)$ in period n since Z is a hidden process. Now, the excess return of the kth asset is

$$R_k^e = R_k(Z_n) - r_f(Z_n) \tag{3.4}$$

in period n. The first scenario corresponds to a market with a fixed-return cash bond (which is the typical case) while the second case corresponds to a market with a variable-return cash bond. Our analysis will generally focus on the first scenario, but we will also point out how our results should be adjusted for the second case. From the expressions given above, it can be concluded that $r_f(i)$ or $r_f(a)$ is a scalar and $r(a) = (r_1(a), r_2(a), ..., r_m(a))$ is a row vector for all a. For any column vector z, z' denotes the row vector representing its transpose.

The main theme of this study is that investors in the market do not know how the market behaves exactly, simply because they do not have perfect information about the market. In other words, we argue against the efficient market theory which claims that investors in the market receive and act on all of the information as soon as it becomes available. Therefore, investors in the market do observe the market but do not perfectly know when, where, why and how the market responds; so, the market is inefficient. To capture the inefficient behavior of the market, the market in this thesis is composed into two processes one of which is observable and the other is hidden. Certainly, these two processes have some kind of relationship. To identify the relationship between the observed and unobserved market processes, two models are considered:

Model I: In this model, we consider an investor in the market who cannot see the stochastic market process. The market process Z is a hidden Markov chain, and the investor does not know exactly in which state the market is by looking at only the observed process Y. While explaining the relationship between the market process Z and the observation process Y, we assume that the relationship is described probabilistically by an observation matrix O. The observation matrix O involves conditional probabilities of $O(i, a)$ which denotes the probability that the unobserved market process is in state a in a period given that observed market process is in state i in that period. The conditional probabilities are assumed to satisfy

$$P\{Z_n = a \mid Y_n, Y_{n-1}, ..., Y_0\} = P\{Z_n = a \mid Y_n\} \tag{3.5}$$

with the observation matrix

$$O(i, a) = P\{Z_n = a \mid Y_n = i\}. \tag{3.6}$$

One of the important property of the observation matrix is that the actual state of the stochastic market depends only on the last state of the observed process. Other definitions of the observation matrix may be possible, for instance, by letting O depend on more than the last state of the observation process Y. However, the definition given in (3.6) lets the observation matrix to be small to provide ease in estimating the observation matrix and complexity of calculations. Moreover, once the state spaces of the market processes and time intervals are clearly defined, it is acceptable to assume that by looking at the current state of the observation process one can identify probabilistically in which state the unobserved market process can be. Another important property of the observation matrix O is that row sums of the matrix must add up to 1. In other words, if an investor in the market observes a state i, then that state must have a corresponding unobserved state of a in the state space F of the unobserved process.

Model II: In the second model, an investor in the market first identifies which state of the market process Z has effect on which state of the observation process Y, and estimates corresponding probabilities for each combination of these states. In this model, we assume this information is known and given by the emission matrix E. The emission matrix E involves conditional probabilities on the observation process in any period given the unobserved market process in that period. This is given by

$$E(a,i) = P\{Y_n = i \mid Z_n = a\}. \tag{3.7}$$

One can infer that row sums of the emission matrix E must also add up to 1. In other words, if the market is in some state a, then there must be some state i that is observed.

We assume that the investor determines the observation matrix O from the emission matrix E by taking weighted averages so that

$$O(i,a) = \frac{E(a,i)}{\sum\limits_{a \in F} E(a,i)}. \tag{3.8}$$

After selecting one of the models described, we identify the appropriate observation matrix from (3.6) or (3.8). We now define conditional expected returns $\hat{r}_k(i)$ and covariances $\hat{\sigma}_{kl}(i)$ given that state i is observed. We let $\hat{r}_k(i) = E\left[R_k(Z_n) \mid Y_n = i\right]$ denote the mean return of the kth asset given that the observed process is in state i and $\hat{\sigma}_{kl}(i) = \mathrm{Cov}(R_k(Z_n), R_l(Z_n) \mid Y_n = i)$ denote the covariance between kth and lth asset returns given that the observed process is in state i. The amount of investor's wealth at period n is denoted by X_n and X_T denotes the final wealth. The vector $u = (u_1, u_2, ..., u_m)$ gives the amounts invested in risky assets $(1, 2, ..., m)$ at period n; that is, it denotes the investment strategy. To determine the wealth available for investment at the beginning of each period, we use a wealth dynamics equation. The amounts invested in each risky asset are multiplied by the corresponding asset returns and the remaining amount is invested in the risk-free asset so that it is multiplied by the prevailing risk-free return. The wealth dynamics equation constitutes a constraint in a multiperiod model and it is written as

$$\begin{aligned} X_{n+1}(u) &= R(Z_n)'u + \left(X_n - 1'u\right) r_f(Y_n) \\ &= r_f(Y_n) X_n + (R(Z_n)' - 1' r_f(Y_n))u \end{aligned} \tag{3.9}$$

where $1 = (1, 1, ..., 1)$ is the column vector consisting of 1's. Note that we should replace $r_f(Y_n)$ by $r_f(Z_n)$ in these equations if the return of the cash bond depends on Z.

In this thesis, our model is based on the following assumptions:

- Unlimited borrowing and lending at the prevailing return of the riskless asset in any period are possible,
- Short selling is allowed for all assets in all periods,
- No capital additions or withdrawals are allowed during the investment horizon,
- Transaction costs are negligible,
- For all states of the unobserved market process, expected returns of each asset including the risk free asset and covariance matrices are known,
- Number of states in the observed and unobserved processes are known,
- Transition matrix Q of the unobserved market process is known.

Chapter 4

MEAN-VARIANCE MODEL FORMULATIONS IN A PERFECTLY OBSERVABLE MARKET

In the next two chapters, we assume that the market is perfectly observable so that $Z_n = Y_n$ in all periods n. Therefore, the excess return of asset k in period n can be written as

$$R_k^e = R_k(Z_n) - r_f(Y_n) = R_k(Y_n) - r_f(Y_n) = R_k^e(Y_n).$$

The results obtained for perfectly observable market will then be extended to our case with imperfect information.

The classical mean-variance model in a single period, introduced by Harry Markowitz, constructs the framework of the trade-off between the expected return and the variance of the return of a portfolio. The model involves a market with m assets having known expected returns $r = (r_1, r_2, ..., r_m)$ and covariances σ_{ij} for $i, j = 1, 2, ..., m$. An investment portfolio consisting of m assets is defined by a set of m weights u_i, $i = 1, 2, ..., m$, that sum up to 1. The mean-variance model solves the portfolio selection problem through a quadratic programming formulation:

$$\begin{aligned} MV(\mu) \quad : \quad & \min \sum_{i,j=1}^{m} u_i u_j \sigma_{ij} \\ & s.t. \sum_{i=1}^{m} u_i r_i = \mu \\ & \sum_{i=1}^{m} u_i = 1 \end{aligned} \tag{4.1}$$

The formulation, given in (4.1), finds the best allocation of wealth among m assets with the objective of minimizing the portfolio risk while yielding a desired level of expected portfolio return μ. This formulation allows short selling since there are no non-negativity constraints. Moreover, the formulation assumes perfect information flow in the market since returns $\{r_i\}$ and covariances $\{\sigma_{ij}\}$ are known and taken directly into in the model.

The formulation can be solved analytically using Lagrange multipliers so that efficient frontiers can be obtained which show how much risk corresponds to a specified return level μ. Markowitz also adapted this formulation to an alternative portfolio selection problem which considers an investor who wants to maximize his return while keeping his risk below a predetermined risk level σ.

In the multiperiod setting, both formulations of Markowitz requires the wealth dynamics equation given in (3.9) as a constraint to control the wealth level for investment at the beginning of each period. Equivalent mean-variance formulations corresponding to Markowitz's portfolio selection problems, given that the initial market state is i, are $P1(\sigma)$ and $P2(\mu)$ shown in (4.2) and (4.3). We let $E_i[Z] = E[Z \mid Y_0 = i]$ and $\text{Var}_i(Z) = E_i[Z^2] - E_i[Z]^2$ denote the conditional expectation and variance of any random variable Z given that the initial market state is i.

$$\begin{aligned} P1(\sigma) \quad : \quad & \max \quad E_i[X_T] \\ & \text{s.t.} \quad \text{Var}_i(X_T) \leq \sigma \\ & \qquad X_{n+1}(u) = r_f(Y_n) X_n + R^e(Y_n)' u \end{aligned} \tag{4.2}$$

$$\begin{aligned} P2(\mu) \quad : \quad & \min \quad \text{Var}_i(X_T) \\ & \text{s.t.} \quad E_i[X_T] \geq \mu \\ & \qquad X_{n+1}(u) = r_f(Y_n) X_n + R^e(Y_n)' u \end{aligned} \tag{4.3}$$

The multiperiod mean-variance formulations given in (4.2) and (4.3) do not have straightforward solutions as in the single period case. In order to obtain the analytical solutions, dynamic programming can be used. However, $P1(\sigma)$ and $P2(\mu)$ are not separable so cannot be solved using dynamic programming. An equivalent formulation to both $P1(\sigma)$ and $P2(\mu)$ is

$$\begin{aligned} P3(\omega) \quad : \quad & \max \quad E_i[X_T] - \omega \text{Var}_i(X_T) \\ & \text{s.t.} \quad X_{n+1}(u) = r_f(Y_n) X_n + R^e(Y_n)' u \end{aligned} \tag{4.4}$$

where $\omega > 0$. Since $P3(\omega)$ is equivalent to both $P1(\sigma)$ and $P2(\mu)$, there are one-to-one relationships between the three parameters σ, μ and ω. Therefore, once $P3(\omega)$ is solved parametrically for ω, it is sufficient to set $\text{Var}_i(X_T) = \sigma^2$ and $E_i[X_T] = \mu$ to identify which

ω gives the optimal solution of $P1(\sigma)$and $P2(\mu)$ respectively. The optimal solution of $P1(\sigma)$ and $P2(\mu)$ is called the minimum-variance point on the efficient frontier which is a the mean versus standard deviation graph, i.e. $E_i[X_T]$ versus $\sqrt{\text{Var}_i(X_T)}$ graph. Efficient portfolios on the efficient frontier can be obtained by changing the value of ω in the objective function of (4.4).

However, like $P1(\sigma)$ and $P2(\mu)$ we cannot solve $P3(\omega)$ by using dynamic programming since it is not separable. Therefore, instead of solving $P3(\omega)$, a tractable auxiliary problem $P4(\lambda, \omega)$ whose optimal solution is the same as $P3(\omega)$ when $\lambda = 1 + 2wE_i[X_T]$ is used. We obtain the $E_i[X_T]$ using the optimal solution of $P4(\lambda, \omega)$ which is

$$\begin{aligned} P4(\lambda,\omega) \quad : \quad \max \quad & E_i\left[-\omega X_T^2 + \lambda X_T\right] \\ \text{s.t.} \quad & X_{n+1}(u) = r_f(Y_n)X_n + R^e(Y_n)'u \end{aligned} \tag{4.5}$$

where ω is a positive parameter so that it can be taken out of the objective function to get the modified formulation

$$\begin{aligned} P4(\lambda,\omega) \quad : \quad \max \quad & \omega E_i\left[-X_T^2 + \frac{\lambda}{\omega}X_T\right] \\ \text{s.t.} \quad & X_{n+1}(u) = r_f(Y_n)X_n + R^e(Y_n)'u \end{aligned} \tag{4.6}$$

for all λ and $\omega > 0$. Analyzing (4.6) shows that the optimal policy will be a function of $\gamma = \lambda/\omega$. Since $P4(\lambda, \omega)$ is separable in the sense of dynamic programming, its formulation given in (4.5) will be used to solve the dynamic multiperiod portfolio selection problem.

The important relationship between these four formulations is that the optimal solution sets of former problems are included in the optimal solution sets of later formulations so that the solutions of former problems can be obtained from $P4(\lambda, \omega)$. In other words, solving $P4(\lambda, \omega)$ means solving $P3(\omega)$ which in turn means solving both $P1(\sigma)$ and $P2(\mu)$ for σ and μ associated with ω.

Chapter 5

DYNAMIC PROGRAMMING FORMULATION IN A PERFECTLY OBSERVABLE MARKET

Dynamic programming is used in the derivation of the optimal solution of the multiperiod mean-variance problem, the details of which are given in Çakmak and Özekici [22]. The auxiliary problem $P4(\lambda, \omega)$ involves the maximization of a simple expected utility function using only the first two moments of the terminal wealth at the end of the investment horizon. In order to solve $P4(\lambda, \omega)$, we define $v_n(i, x)$ as the optimal expected utility using the optimal policy given that the observed market state is i and the amount of money available for investment is x at period n. Then, the dynamic programming equation becomes

$$v_n(i, x) = \max_u E\left[v_{n+1}(Y_{n+1}, X_{n+1}(u)) \,|Y_n = i\right] \tag{5.1}$$

which can be rewritten as

$$v_n(i, x) = \max_u \sum_{j \in E} Q(i, j) \, E\left[v_{n+1}\left(j, r_f(i)\, x + R^e(i)' u\right)\right] \tag{5.2}$$

for $n = 0, 1, 2, \cdots, T-1$ with the boundary condition $v_T(i, x) = -\omega x^2 + \lambda x$ for all $i \in E$. The solution for this problem is found by solving the dynamic programming equation recursively.

Before writing down the optimal solution, we need to introduce some terminology and notation used in the derivation of the solution. We define the matrix

$$V(i) = E\left[R^e(i)\, R^e(i)'\right] \tag{5.3}$$

for any state $i \in E$. The covariance matrix $\sigma(i)$ is assumed to be positive definite for all $i \in E$ which is a justified assumption since

$$z'\sigma(i)z = E[(z_1 R_1(i) + z_2 R_2(i) + \cdots + z_m R_m(i))^2] \geq 0 \tag{5.4}$$

for any vector $z = (z_1, z_2, \cdots, z_m)$. This property of $\sigma(i)$ is inherited by $V(i)$ such that for any $i \in E$, $V(i) = \sigma(i) + r^e(i)\, r^e(i)'$ is a positive definite matrix.

We now define $f(i)$, $g(i)$ and $h(i)$, which are functions of asset returns, for a given state i as

$$f(i) = r_f(i)^2 [1 - h(i)] \tag{5.5}$$

$$g(i) = r_f(i) [1 - h(i)] \tag{5.6}$$

where

$$h(i) = r^e(i)' V^{-1}(i) r^e(i) . \tag{5.7}$$

It turns out that for any $i \in E$, $f(i), g(i) > 0$ and $0 < h(i) < 1$.

For any matrix M and vector f, we define the matrix M_f such that

$$M_f(i,j) = M(i,j) f(j) \tag{5.8}$$

for $i, j \in E$ and the vector $\overline{M}$ such that

$$\overline{M}(i) = \sum_{j \in E} M(i,j) . \tag{5.9}$$

With this notation, M_f^n is the nth power of M_f, and $\overline{M_f^n}$ is simply the vector obtained by adding the columns of the matrix M_f^n for $n \geq 0$. It follows that $\overline{M_f^0} = 1$ when $n = 0$ and $\overline{M}_f = Mf$ when $n = 1$.

If a, b and c are three vectors, then $(a/b) \bullet c$ denotes the vector where$((a/b) \bullet c)(i) = (a(i)/b(i))c(i)$. Using these notations, we define

$$h_n(i) = \frac{\overline{Q_g^n}(i)}{\overline{Q_f^n}(i)} h(i) \tag{5.10}$$

$$\overline{h}_n(i) = \left(\frac{\overline{Q_g^n}(i)}{\overline{Q_f^n}(i)} \right)^2 h(i) . \tag{5.11}$$

After defining the notations and terminology, we will present the main results of Çakmak and Özekici [22] without presenting their proofs which can be found in the original paper. We let x_0 denote the initial wealth which is assumed to be known.

The optimal solution of $P4(\lambda, \omega)$ is

$$v_n(i,x) = -\omega_n(i) x^2 + \lambda_n(i) x + \alpha_n(i) \tag{5.12}$$

and the corresponding optimal policy maximizing the objective function is

$$u_n(i,x) = \left[\frac{1}{2} \left(\frac{\lambda}{\omega} \right) \frac{\overline{Q_g^{T-n-1}}(i)}{\overline{Q_f^{T-n-1}}(i)} - r_f(i) x \right] V^{-1}(i) r^e(i) \tag{5.13}$$

where

$$\omega_n(i) = \omega \overline{Q}_f^{T-n-1}(i) f(i) \tag{5.14}$$

$$\lambda_n(i) = \lambda \overline{Q}_g^{T-n-1}(i) g(i) \tag{5.15}$$

$$\alpha_n(i) = \sum_{k=n+2}^{T} Q^{k-n-2} \overline{Q}_{\bar{\alpha}_k}(i) + \bar{\alpha}_{n+1}(i) \tag{5.16}$$

and

$$\bar{\alpha}_n(i) = \frac{\left(\lambda \overline{Q}_g^{T-n}(i)\right)^2}{4\omega \overline{Q}_f^{T-n}(i)} h(i) \tag{5.17}$$

for $n = 0, 1, \cdots, T-1$. In (5.16), the summation on the right hand side vanishes if $n = T-1$.

The optimal investment policy $u_n(i, x)$ in (5.13) gives the amount of money that should be invested in each asset at period n given the market state i and the current wealth x. This formula shows that the amounts invested in risky assets are determined based on investor's attitude toward risk, reflected in the first term inside the parenthesis, and investor's current wealth, reflected in the second term inside the parenthesis. The first term can be calculated before the investment process starts whereas the second term is calculated at every time period when the current wealth is observed. By substituting (5.13) into the wealth dynamic equation given in (3.9) and then taking expectations of X_n and X_n^2, we obtain

$$E_i[X_n] = \overline{Q}_g^{n-1}(i) g(i) x_0 + \frac{\lambda}{2\omega} \sum_{k=1}^{n} Q^{k-1} \left(\overline{Q}_g^{n-k} \bullet h_{T-k}\right)(i) \tag{5.18}$$

$$E_i\left[X_n^2\right] = \overline{Q}_f^{n-1}(i) f(i) x_0^2 + \left(\frac{\lambda}{2\omega}\right)^2 \sum_{k=1}^{n} Q^{k-1} \left(\overline{Q}_f^{n-k} \bullet \overline{h}_{T-k}\right)(i) \tag{5.19}$$

for $n = 1, \cdots, T$.

If we define

$$a_1(i) = \overline{Q}_g^{T-1}(i) g(i) \tag{5.20}$$

$$a_2(i) = \overline{Q}_f^{T-1}(i) f(i) \tag{5.21}$$

$$b(i) = \frac{1}{2} \sum_{k=1}^{T} Q^{k-1} \left(\frac{\left(\overline{Q}_g^{T-k}\right)^2}{\overline{Q}_f^{T-k}} \bullet h \right)(i) \tag{5.22}$$

then the optimal solution satisfies the simplified expressions

$$E_i[X_T] = a_1(i) x_0 + b(i)\gamma \tag{5.23}$$

$$E_i\left[X_T^2\right] = a_2(i) x_0^2 + \frac{1}{2} b(i) \gamma^2 \tag{5.24}$$

where $\gamma = \lambda/\omega$. Hence, the variance of the terminal wealth is

$$\mathrm{Var}_i\left(X_T\right) = \left(a_2(i) - a_1(i)^2\right)x_0^2 - 2a_1(i)b(i)x_0\gamma + \left(\frac{1}{2} - b(i)\right)b(i)\gamma^2. \tag{5.25}$$

With respect to our multiperiod portfolio optimization problem, $E_i\left[X_T\right]$ is the expected wealth (or the expected return when $x_0 = 1$) at the end of the investment horizon and $\mathrm{Var}_i\left(X_T\right)$ measures the risk of the final wealth. From now on, in order to simplify the notation a_1, a_2 and b are going to be used in stead of $a_1\left(i\right), a_2\left(i\right)$ and $b\left(i\right)$. Moreover, a_1, a_2 and b still depends only on the initial observed market state i.

After finding the optimal solution for $P4(\lambda,\omega)$, the next step is to obtain the optimal solution of $P3(\omega)$ which is found to be

$$u_n\left(i,x\right) = \left[\left(\frac{1+2\omega a_1 x_0}{2\omega\left(1-2b\right)}\right)\frac{\overline{Q}_g^{T-n-1}\left(i\right)}{\overline{Q}_f^{T-n-1}\left(i\right)} - r_f\left(i\right)x\right]V^{-1}\left(i\right)r^e\left(i\right) \tag{5.26}$$

for $n = 0, 1, \cdots, T-1$ such that

$$E_i\left[X_T\right] = \frac{a_1 x_0}{1-2b} + \frac{b}{\omega\left(1-2b\right)} \tag{5.27}$$

$$\mathrm{Var}_i\left(X_T\right) = \left(a_2 - \frac{{a_1}^2}{1-2b}\right)x_0^2 + \frac{b}{2\omega^2\left(1-2b\right)}. \tag{5.28}$$

The optimal solutions of $P1(\sigma)$ and $P2(\mu)$ are obtained from $P3(\omega)$ by taking

$$\omega = \sqrt{\frac{b}{2\left[\left(1-2b\right)\sigma - \left[\left(1-2b\right)a_2 - {a_1}^2\right]x_0^2\right]}} \tag{5.29}$$

for $P1(\sigma)$, and

$$\omega = \frac{b}{\left(1-2b\right)\mu - a_1 x_0} \tag{5.30}$$

for $P2(\mu)$.

Finally, the mean-variance efficient frontier is found to be

$$\mathrm{Var}_i(X_T) = \left(a_2 - \frac{a_1^2}{1-2b}\right)x_0^2 + \frac{\left[\left(1-2b\right)E_i\left[X_T\right] - a_1 x_0\right]^2}{2b\left(1-2b\right)} \tag{5.31}$$

defined for $E_i\left[X_T\right] \geq a_1 x_0/\left(1-2b\right)$. The minimum variance point of the efficient frontier is found by minimizing the expression for variance in (5.25) with respect to γ. This point has a gamma value of $(2a_1x_0)/(1-2b)$ which implies that the γ value has to be greater than $(2a_1x_0)/(1-2b)$ so as to get portfolios on the efficient frontier. For the minimum variance portfolio

$$E_i[X_T] = \frac{a_1 x_0}{1-2b} \tag{5.32}$$

$$\text{Var}_i(X_T) = \left(a_2 - \frac{{a_1}^2}{1-2b}\right) x_0^2. \tag{5.33}$$

In order to identify the efficient frontier, one should first calculate $f(i)$, $g(i)$ and $h(i)$ for all states using (5.5)-(5.7). Then, a_1, a_2 and b for the given initial state i of the market should be computed using (5.20)-(5.22). The associated ω can be calculated in terms of σ or μ using (5.29) or (5.30). Finally, substituting the associated ω into (5.26) yields the optimal multiperiod portfolio policy for $P1(\sigma)$ or $P2(\mu)$ which leads to the expectation and the variance of the final wealth given in (5.27) and (5.28) respectively.

Chapter 6

MEAN-VARIANCE MODEL FORMULATIONS UNDER IMPERFECT INFORMATION

Mean-variance model formulations under imperfect information are technically very similar to the formulations under perfect information given in Chapters 4 and 5 except for the fact that formulations under imperfect information consider two market processes, one of which is hidden to investors in the market. Since the investors cannot see the real process, they act according to what they observe, and most of the time they do not get what they have expected because they calculate their expectations using the information that is assumed to be perfectly known. To explain this kind of information discrepancy, it is necessary to introduce some form of imperfection.

6.1 Information Structure

Imperfection in information flow is set up through a probabilistic relationship between the observed and unobserved market processes. The probabilistic relationship is defined by the observation matrix O defined in (3.6) and the emission matrix E defined in (3.7). In the observation matrix, $O(i, a)$ denotes the probability that the unobserved market process is in state a in a period given that observed market process is in state i in that period. In the emission matrix, $E(a, i)$ denotes the probability that the observed market process is in state i in a period given that unobserved market process is in state a in that period. We suppose that there exists a single observation matrix O or an emission matrix E which govern the relationship between the observed and unobserved market processes. However, we should note that it is difficult to find the true matrices since they are hidden to investors. They can be estimated by training an output using Baum-Welch algorithm; however, Baum-Welch algorithm suffers from local optima, sensitivity to initial parameter settings and large amount of training data. Better estimation technique needs to be developed by using Bayesian statistics, but we leave this to statisticians. Therefore, instead of estimating them

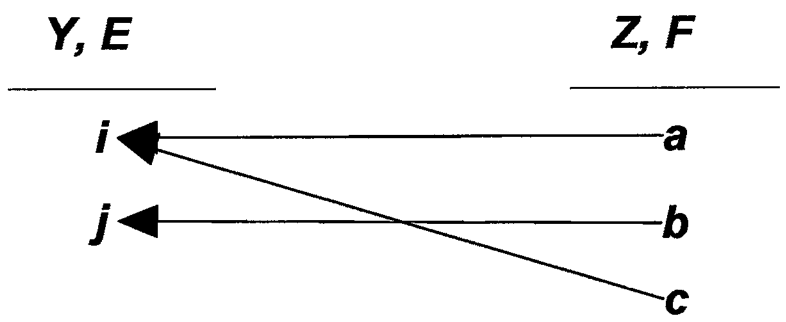

Figure 6.1: Model I Illustration

directly we developed two models assuming that one of them is known. In the first model, O is assumed to be given; moreover, in the second model E is assumed to be given and O is obtained from E.

6.1.1 *Model I*

In the first model, an investor cannot see the stochastic market process, but the transition matrix Q of the market process is known. The stochastic market is a hidden Markov chain, and the investor does not know exactly in which state the actual market is by looking at only the observed process. However, if the investor could see the stochastic market process, he would know exactly in which state the observed market is. Moreover, investors in the market behave depending on the observed market process, so we need to figure out a way to model the relationship between the two processes. We assume that there is a known set of unobserved market states that show a given observed state to the investor exactly. In other words, we let $s(i)$ denote the set of market states in F which shows the observed state i in E. For example, the relationship can be described as in Fig. 6.1. In this figure, the stochastic market process Z has three market states $F = \{a, b, c\}$, and the observed market process Y has two market states $E = \{i, j\}$. Investors in the market can only see the observed market states $\{i, j\}$. In order to observe state i, the stochastic market must be in either state a or state c. Moreover, state j is observed only when the stochastic market is in the state b. In other words, $s(i)$ consists of states $\{a, c\}$ and $s(j)$ consists of state $\{b\}$.

Information about this set of market states is captured by a binary function I which is defined as

$$I(a,i) = \begin{cases} 1, & a \in s(i) \\ 0, & \text{otherwise.} \end{cases} \tag{6.1}$$

Here, we note that the intersection of $s(i)$'s is empty so that $s(i) \cap s(j) = \phi$ whenever $i \neq j$. This is because if the stochastic process Z is in state a, then the observed process Y is in state i with probability one; so, each state a in F can correspond to only a single state i in E. In this model, we also assume that the observation matrix O is known. With these assumptions, the investors in the market observe a state that is probabilistically reflected by the unobserved market and choose a portfolio based on their observation.

We now focus on the observation process Y and determine its transition matrix $\hat{Q}$ defined in (3.2) as

$$\begin{aligned}
\hat{Q}(i,j) &= P\{Y_{n+1} = j \mid Y_n = i\} &(6.2)\\
&= \sum_{a,b \in F} P\{Y_{n+1} = j, Z_{n+1} = b, Z_n = a \mid Y_n = i\} &(6.3)\\
&= \sum_{a,b \in F} P\{Z_n = a \mid Y_n = i\} P\{Z_{n+1} = b \mid Z_n = a\} &(6.4)\\
&\qquad \bullet\, P\{Y_{n+1} = j \mid Z_{n+1} = b\} &\\
&= \sum_{a,b \in F} O(i,a) Q(a,b) I(b,j) &(6.5)\\
&= \sum_{a \in F, b \in s(j)} O(i,a) Q(a,b). &(6.6)
\end{aligned}$$

Given any observation matrix O and relationship $\{s(i); i \in E\}$, one can easily determine the transition matrix $\hat{Q}$ of Y.

6.1.2 Model II

In the second model, an investor does not see the market process. He first tries to identify which state in the market process Z has effect on which state in the observation process Y and determines the corresponding probabilities for each one of these states. In this model, we assume this information is known and given by the emission matrix E defined in (3.7). For example, the relationship can be described in Fig. 6.2. In this figure, stochastic market process Z and the observed market process Y has two market states $F = \{a, b\}$ and $E = \{i, j\}$ respectively. State i is observed with probability $E(a,i)$ when the stochastic process is in state a or with probability $E(b,i)$ when the process is in state b. Moreover,

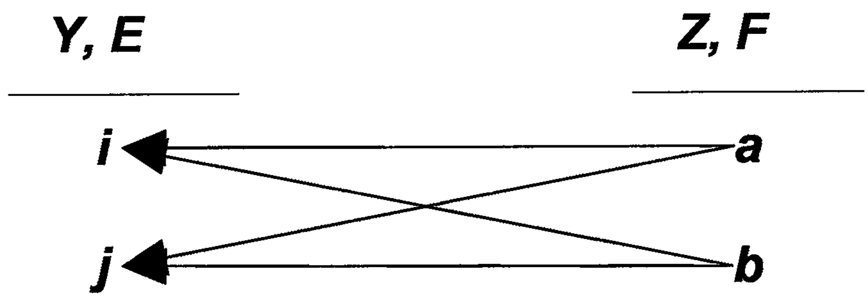

Figure 6.2: Model II Illustration

state j is observed with probability $E(a,j)$ when the stochastic process is in state a or with probability $E(b,j)$ when the process is in state b. In other words, $s(i)$ and $s(j)$ now consist of the states $\{a,b\}$.

Here, each market state a in F may correspond to more than one state i in E, because we assume through (3.7) that by observing the market process in state a, we can only say that the unobserved market process is in state i with probability $E(a,i)$. Moreover, unlike the first model we do not assume that the observation matrix O is known. The observation matrix O is determined from the emission matrix E using

$$O(i,a) = \frac{E(a,i)}{\sum_{a\in F} E(a,i)}. \tag{6.7}$$

We can now determine the transition matrix $\hat{Q}$ of the observation process Y as

$$\begin{aligned}
\hat{Q}(i,j) &= P\{Y_{n+1}=j \mid Y_n=i\} && (6.8)\\
&= \sum_{a,b\in F} P\{Y_{n+1}=j, Z_{n+1}=b, Z_n=a \mid Y_n=i\} && (6.9)\\
&= \sum_{a,b\in F} P\{Z_n=a \mid Y_n=i\}P\{Z_{n+1}=b \mid Z_n=a\} && (6.10)\\
&\qquad\qquad \bullet\, P\{Y_{n+1}=j \mid Z_{n+1}=b\} &&\\
&= \sum_{a,b\in F} O(i,a)Q(a,b)E(b,j). && (6.11)
\end{aligned}$$

6.2 Dynamic Programming Formulation under Imperfect Information

After selecting the appropriate model, we focus on the covariance matrix and mean vector of the assets for the observed process Y. Covariance of assets' returns can be calculated by using

$$\begin{aligned} \widehat{\text{Cov}}_{kl}(i) &= \text{Cov}(R_k(Z_n), R_l(Z_n) \mid Y_n = i) \qquad (6.12) \\ &= E\left[R_k(Z_n)R_l(Z_n) \mid Y_n = i\right] \qquad (6.13) \\ &\quad - E\left[R_k(Z_n) \mid Y_n = i\right] E\left[R_l(Z_n) \mid Y_n = i\right]. \end{aligned}$$

By computing the first element on the right hand side of (6.13) as

$$\begin{aligned} E\left[R_k(Z_n)R_l(Z_n) \mid Y_n = i\right] &= \sum_{a \in F} P\{Z_n = a \mid Y_n = i\} E\left[R_k(a)R_l(a)\right] \qquad (6.14) \\ &= \sum_{a \in F} O(i,a)(\text{Cov}(R_k(a), R_l(a)) \qquad (6.15) \\ &\quad + E\left[R_k(a)\right] E\left[R_l(a)\right]) \\ &= \sum_{a \in F} O(i,a)\text{Cov}(R_k(a), R_l(a)) \qquad (6.16) \\ &\quad + \sum_{a \in F} O(i,a) E\left[R_k(a)\right] E\left[R_l(a)\right] \\ &= \sum_{a \in F} O(i,a)\sigma_{kl}(a) + \sum_{a \in F} O(i,a) r_k(a) r_l(a) \qquad (6.17) \end{aligned}$$

and the second element on the right hand side of (6.13) as

$$\begin{aligned} E\left[R_k(Z_n) \mid Y_n = i\right] &= \sum_{a \in F} P\{Z_n = a \mid Y_n = i\} E\left[R_k(a)\right] \qquad (6.18) \\ &= \sum_{a \in F} O(i,a) r_k(a) \qquad (6.19) \end{aligned}$$

(6.13) becomes

$$\widehat{\text{Cov}}_{kl}(i) = \hat{\sigma}_{kl}(i) + \hat{r}_{kl}(i) - \hat{r}_k(i)\hat{r}_l(i) \qquad (6.20)$$

where

$$\begin{aligned} \hat{\sigma}_{kl}(i) &= \sum_{a \in F} O(i,a)\sigma_{kl}(a) \qquad (6.21) \\ \hat{r}_{kl}(i) &= \sum_{a \in F} O(i,a) r_k(a) r_l(a) \qquad (6.22) \\ \hat{r}_k(i) &= \sum_{a \in F} O(i,a) r_k(a). \qquad (6.23) \end{aligned}$$

The matrix $\hat{V}(i)$ defined for the optimal solution in (5.3) is

$$\begin{aligned}
\hat{V}_{kl}(i) &= E\left[R_k^e R_l^e \mid Y_n = i\right] && (6.24)\\
&= E\left[(R_k(Z_n) - r_f(Y_n))(R_l(Z_n) - r_f(Y_n)) \mid Y_n = i\right] && (6.25)\\
&= E[R_k(Z_n)R_l(Z_n) \mid Y_n = i] - r_f(i)E[R_k(Z_n) + R_l(Z_n) \mid Y_n = i] && (6.26)\\
&\quad + r_f^2(i)\\
&= \hat{\sigma}_{kl}(i) + \hat{r}_{kl}(i) - \sum_{a\in F} O(i,a)r_f(i)(r_k(a) + r_l(a)) + \sum_{a\in F} O(i,a)r_f^2(i) && (6.27)\\
&= \hat{\sigma}_{kl}(i) + \sum_{a\in F} O(i,a)r_k(a)r_l(a) - \sum_{a\in F} O(i,a)r_f(i)(r_k(a) + r_l(a)) && (6.28)\\
&\quad + \sum_{a\in F} O(i,a)r_f^2(i)\\
&= \hat{\sigma}_{kl}(i) + \sum_{a\in F} O(i,a)[r_k(a)r_l(a) - r_f(i)(r_k(a) + r_l(a)) + r_f^2(i)] && (6.29)\\
&= \hat{\sigma}_{kl}(i) + \sum_{a\in F} O(i,a)[(r_k(a) - r_f(i))(r_l(a) - r_f(i))] && (6.30)\\
&= \hat{\sigma}_{kl}(i) + \hat{r}_{kl}^e(i) && (6.31)
\end{aligned}$$

where

$$\hat{r}_{kl}^e(i) = \sum_{a\in F} O(i,a)[(r_k(a) - r_f(i))(r_l(a) - r_f(i))]. \qquad (6.32)$$

Using a similar analysis one can show that $\hat{r}_{kl}^e$ should be defined as

$$\hat{r}_{kl}^e(i) = \sum_{a\in F} O(i,a)[(r_k(a) - r_f(a))(r_l(a) - r_f(a))] \qquad (6.33)$$

for all i in (6.31) if the return of the cash bond depends on Z.

Moreover, $f(i)$ in (5.5), $g(i)$ in (5.6) and $h(i)$ in (5.7) become

$$\hat{f}(i) = \hat{r}_f(i)^2\left[1 - \hat{h}(i)\right] \qquad (6.34)$$

$$\hat{g}(i) = \hat{r}_f(i)\left[1 - \hat{h}(i)\right] \qquad (6.35)$$

where

$$\begin{aligned}
\hat{r}_f(i) &= E[r_f(Y_n) \mid Y_n = i] = r_f(i) && (6.36)\\
\hat{h}(i) &= \hat{r}^e(i)'\,\hat{V}^{-1}(i)\,\hat{r}^e(i) && (6.37)
\end{aligned}$$

and

$$\begin{aligned}
\hat{r}_k^e(i) &= E\left[R_k^e \mid Y_n = i\right] && (6.38)\\
&= E\left[R_k(Z_n) - r_f(Y_n) \mid Y_n = i\right] && (6.39)\\
&= \sum_{a\in F} O(i,a)(r_k(a) - r_f(i)) && (6.40)\\
&= \hat{r}_k(i) - r_f(i). && (6.41)
\end{aligned}$$

Note that $\hat{r}_f$ and $\hat{r}_k^e$ should be defined as

$$\begin{aligned}
\hat{r}_f(i) &= E[r_f(Z_n) \mid Y_n = i] = \sum_{a\in F} O(i,a) r_f(a) && (6.42)\\
\hat{r}_k^e(i) &= \sum_{a\in F} O(i,a)(r_k(a) - r_f(a)) && (6.43)
\end{aligned}$$

for all i if the return of the cash bond depends on Z.

Also, $a_1(i)$ in (5.20), $a_2(i)$ in (5.21) and $b(i)$ in (5.22) are rewritten as

$$\begin{aligned}
\hat{a}_1(i) &= \overline{\hat{Q}}_{\hat{g}}^{T-1}(i)\,\hat{g}(i) && (6.44)\\
\hat{a}_2(i) &= \overline{\hat{Q}}_{\hat{f}}^{T-1}(i)\,\hat{f}(i) && (6.45)\\
\hat{b}(i) &= \frac{1}{2}\sum_{k=1}^{T} \hat{Q}^{k-1}\left(\frac{\left(\overline{\hat{Q}}_{\hat{g}}^{T-k}\right)^2}{\overline{\hat{Q}}_{\hat{f}}^{T-k}} \bullet \hat{h}\right)(i). && (6.46)
\end{aligned}$$

After calculating $\hat{f}(i)$, $\hat{g}(i)$ and $\hat{h}(i)$ for all states by using (5.5)-(5.7), and $\hat{a}_1(i)$, $\hat{a}_2(i)$ and $\hat{b}(i)$ for all states, the mean-variance efficient frontier can be obtained from

$$\mathrm{Var}_i(X_T) = \left(\hat{a}_2 - \frac{\hat{a}_1^2}{1-2\hat{b}}\right)x_0^2 + \frac{\left[\left(1-2\hat{b}\right)E_i\left[X_T\right] - \hat{a}_1 x_0\right]^2}{2\hat{b}\left(1-2\hat{b}\right)} \qquad (6.47)$$

defined for $E_i\left[X_T\right] \geq \hat{a}_1 x_0 / \left(1-2\hat{b}\right)$. The minimum variance point of the efficient frontier is found by using (5.32) and (5.33) as

$$\begin{aligned}
E_i\left[X_T\right] &= \frac{\hat{a}_1 x_0}{1-2\hat{b}} && (6.48)\\
\mathrm{Var}_i\left(X_T\right) &= \left(\hat{a}_2 - \frac{\hat{a}_1{}^2}{1-2\hat{b}}\right)x_0^2. && (6.49)
\end{aligned}$$

To find the optimal portfolio policy, by using (5.29) and (5.30), $\hat{\omega}$ is chosen as

$$\hat{\omega} = \sqrt{\frac{\hat{b}}{2\left[\left(1-2\hat{b}\right)\sigma - \left[\left(1-2\hat{b}\right)\hat{a}_2 - \hat{a}_1{}^2\right] x_0^2\right]}} \tag{6.50}$$

for $P1(\sigma)$ for a given σ, and

$$\hat{\omega} = \frac{\hat{b}}{\left(1-2\hat{b}\right)\mu - \hat{a}_1 x_0} \tag{6.51}$$

for $P2(\mu)$ for a given μ. By using (5.26) and the appropriate $\hat{\omega}$, the optimal policy is found as

$$\hat{u}_n(i,x) = \left[\left(\frac{1+2\hat{\omega}\hat{a}_1 x_0}{2\hat{\omega}\left(1-2\hat{b}\right)}\right)\frac{\overline{\hat{Q}}_{\hat{g}}^{T-n-1}(i)}{\overline{\hat{Q}}_{\hat{f}}^{T-n-1}(i)} - \hat{r}_f(i)\,x\right]\hat{V}^{-1}(i)\,\hat{r}^e(i)\,. \tag{6.52}$$

Chapter 7

NUMERICAL ILLUSTRATIONS

In this chapter, exemplary cases are presented to illustrate the application of the analytical solutions developed in this thesis. All of the numerical illustrations are done by MATLAB 6.5 in IBM G40 laptop computer with 2.4 GHz Pentium CPU and 256 MB of RAM. In order to capture the best view of the plots of each run in the fastest way, outputs are saved to Microsoft Excel 2002 from MATLAB because overlapping of many different graphs causes scale problems in MATLAB and to handle this problem the same run has to be performed many times with different graphical parameter settings. MATLAB codes are given in Appendix A.

In this chapter, Section 7.1 and Section 7.2 present illustrations on the first model and the second model respectively. All of our numerical illustrations focus on the effects of the level of information on the efficient frontier. That is why the observation and emission matrices will be parametrized by a single variable p and sensitivity analysis will be performed. This p value represents, in a probabilistic sense, the level of information that is available to the investors.

7.1 Numerical Illustrations on Model I

Case I.1: In this case, for the sake of simplicity, we consider a market with a single risky asset and a riskless asset where the market is modulated by a hidden Markov chain. The stochastic market process Z has three states $F = \{1, 2, 3\}$ where the states are represented generically by the letter a, and the observed market process Y has two states $E = \{1, 2\}$ where the states are represented generically by the letter i. The relationship between the states of the stochastic market process Z and the observed market process Y is assumed to be known as in Fig. 7.1. Therefore, investors can observe state $i = 1$ when the stochastic market is in either state $a = 1$ or state $a = 2$. Moreover, state $i = 2$ is observed only when the stochastic market is in state $a = 3$. In other words, $s(i = 1)$ consists of states

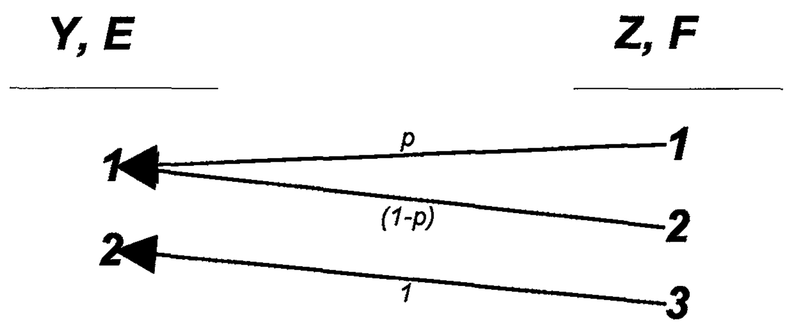

Figure 7.1: Case I.1 Illustration

$\{a = 1, a = 2\}$ so that $s(1) = \{1, 2\}$, and $s(i = 2)$ consists of state $\{a = 3\}$ so that $s(2) = \{3\}$. Regarding the relationship between the stochastic process and the observed process, the observation matrix O is assumed to be given as

$$O(i,a) = \begin{bmatrix} p & (1-p) & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{7.1}$$

where p denotes the probability that the unobserved market process is in state $a = 1$ in a period given that observed market process is in state $i = 1$ in that period. When $p = 0$, probability that the unobserved market process is in state $a = 1$ in a period given that observed market process is in state $i = 1$ in that period becomes zero. In other words, state $i = 1$ can be observed only when the unobserved process is in state $a = 2$. This situation makes state $a = 1$ unnecessary; moreover, in a multiperiod setting, once the stochastic process enters state $a = 1$ the state that can be observed is not clear. Similar argument can be done for $p = 1$. Therefore, in order for our setting described in Fig. 7.1 to work well in a multiperiod problem, p is allowed to take values strictly between 0 and 1. The observation matrix O given in (7.1) can be defined in various ways, but for the sake of simplicity we define it with a single parameter p. The reason is that we would like to perform a parametric analysis to see the effect of the observation matrix O on the efficient frontiers clearly. The observation matrix clearly represent the level of information existing in the market place. With this O matrix, we can say that when we observe state $i = 1$, the stochastic market is in state $a = 1$ with probability p and in state $a = 2$ with probability $(1 - p)$. Since the observation matrix is determined completely by the value of p, it represents the level of information that is available. As p increases, we are more informed that the state of the unobserved process is $a = 1$ given that the state $i = 1$ is observed. In all of our calculations, once the general formulations of efficient frontiers are obtained as a function of p, they are plotted and compared for different values of p.

We assume that the investor has one unit wealth available for investment at the beginning of the planning horizon, and the number of investment periods is known. We consider the problem of an investor who wants to allocate his wealth among a risky asset and a riskless asset such that at the end of the investment horizon expected wealth of the investor is maximized.

The expected value r and the variance σ^2 of the return of the risky asset for each state are given in Table 7.1. Moreover, rate of return per unit risk $\frac{(r(a)-1)}{\sigma(a)}$ of the risky asset for each state is given in Table 7.2. The return r_f of the riskless asset is given in Table 7.3. By comparing the rate of return per unit risk $\frac{(r(a)-1)}{\sigma(a)}$, we decide which state falls into

Table 7.1: Means and variances of returns (Case I.1)

State a	$r(a)$	$\sigma^2(a)$
1	1.08	0.0009
2	1.04	0.0006
3	1.02	0.0003

Table 7.2: Rate of return per unit risk of the risky asset (Case I.1)

State a	$\frac{(r(a)-1)}{\sigma(a)}$
1	2.67
2	1.63
3	1.15

good, ordinary and bad categorizations. Rate of return per unit risk is the greatest in state $a = 1$ and the lowest in state $a = 3$; hence, states $a = 1$, $a = 2$ and $a = 3$ represent good, ordinary and bad scenarios for the market respectively. What is meant by a good scenario

Table 7.3: Return of the riskless asset (Case I.1)

State i	$r_f(i)$
1	1.030
2	1.005

for the market is the time, for instance, when the economy booms, and by a bad scenario for the market is the time, for instance, when the economy is in stagnation.

The transition probability matrix Q of the hidden Markov chain that the stochastic market process follows is given as

$$Q(a,b) = \begin{bmatrix} 0.90 & 0.09 & 0.01 \\ 0.80 & 0.15 & 0.05 \\ 0.70 & 0.20 & 0.10 \end{bmatrix}. \tag{7.2}$$

The transition matrix Q is selected such that the stochastic process visits the first state more than the others, because we would like to have some opinion about the behavior of the stochastic process while comparing the efficient frontiers with different observation matrices. In our illustrations, we used the same transition matrix because we would like to immunize the effect of transition matrix on the efficient frontiers. Our aim is to study the effect of level of information on the frontiers.

After entering O, Q, r_f, r, σ and setting the relationship between states of the stochastic and the observed process, MATLAB code runs for 53 seconds. At the end of the run, we obtained calculations presented below and data points required to plot the efficient frontiers.

From (6.6), the transition probability matrix $\hat{Q}$ of the observed Markov chain is calculated as

$$\hat{Q}(i,j) = \begin{bmatrix} 0.04p + 0.95 & -0.04p + 0.05 \\ 0.9 & 0.1 \end{bmatrix}. \tag{7.3}$$

Then, by using (6.24), $\hat{V}(i)$ for each state i is computed to be

$$\hat{V}(1) = [0.0027p + 0.0007], \hat{V}(2) = [0.000525].$$

By using the expressions given in (6.34)-(6.37), the vectors $\hat{f}(i)$, $\hat{g}(i)$ and $\hat{h}(i)$ are calculated as follows

$$\hat{f}(i) = \begin{bmatrix} 1.0609 - \frac{1.0609(0.04p+0.01)^2}{(0.0027p+0.0007)} \\ 0.5772 \end{bmatrix} \quad (7.4)$$

$$\hat{g}(i) = \begin{bmatrix} 1.03 - \frac{1.03(0.04p+0.01)^2}{(0.0027p+0.0007)} \\ 0.5743 \end{bmatrix} \quad (7.5)$$

$$\hat{h}(i) = \begin{bmatrix} \frac{(0.04p+0.01)^2}{(0.0027p+0.0007)} \\ 0.4286 \end{bmatrix} \quad (7.6)$$

for $i = 1, 2$. In order to find the optimal analytical solutions for the multiperiod portfolio problems we only need to calculate $\hat{a}_1(i)$, $\hat{a}_2(i)$ and $\hat{b}(i)$ defined in (6.44)-(6.46). The variance of the terminal wealth defined in (6.47) is used to plot the efficient frontiers and it depends only on values of $\hat{a}_1(i)$, $\hat{a}_2(i)$ and $\hat{b}(i)$. Moreover, values of $\hat{a}_1(i)$, $\hat{a}_2(i)$ and $\hat{b}(i)$ depend on p for both $i = 1$ and $i = 2$. Since the open form of these values are very long, rather than giving the explicit formulations we present the plots of $\hat{a}_1(i)$, $\hat{a}_2(i)$ and $\hat{b}(i)$ for $i = 1$ while p is increased from 0.005 to 0.995 by using increments of 0.005 in Fig 7.2- Fig 7.4. As an example we include the explicit form of $\hat{a}_1(i)$ in Appendix B.1 for $i = 1$.

Note that $\hat{a}_1(i = 1)$ is obtained by multiplying $\overline{\hat{Q}}^3_{\hat{g}}(i = 1)$ with $\hat{g}(i = 1)$ as shown in (6.44). As it is seen in Fig 7.2, $\hat{a}_1$ is 0.568 for $p = 0.005$ and $\hat{a}_1$ is 0.006 for $p = 0.995$. Similarly, $\hat{a}_2(i = 1)$ is obtained by multiplying $\overline{\hat{Q}}^3_{\hat{f}}(i = 1)$ with $\hat{f}(i = 1)$ as shown in (6.45).

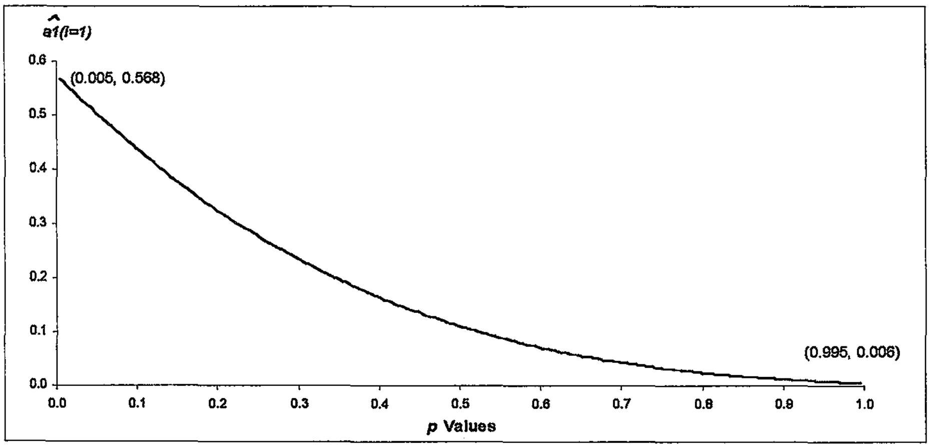

Figure 7.2: $\hat{a}_1(i = 1)$ (Case I.1, $T = 4$)

As it is seen in Fig 7.3, $\hat{a}_2$ is 0.638 for $p = 0.005$ and $\hat{a}_2$ is 0.007 for $p = 0.995$. Likewise,

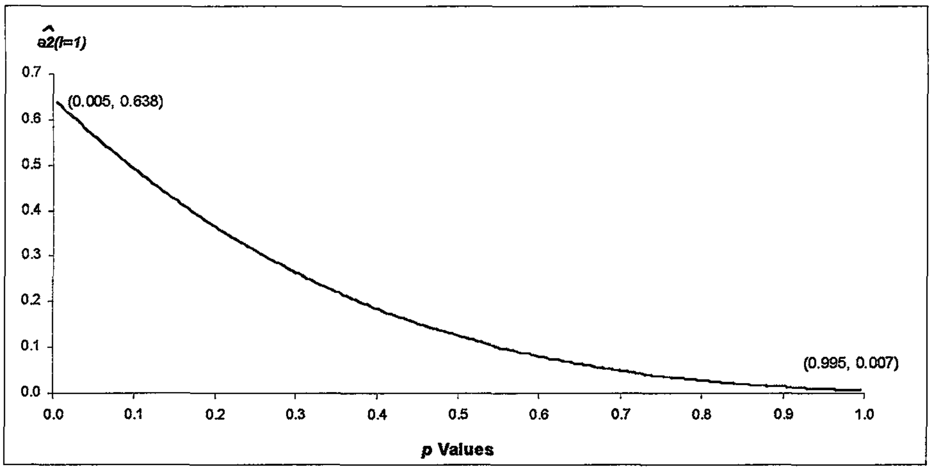

Figure 7.3: $\hat{a}_2(i = 1)$ (Case I.1, $T = 4$)

$\hat{b}(i = 1)$ is obtained from (6.46) and it is equal to 0.247 for $p = 0.005$ and 0.497 for $p = 0.995$ as it is seen in Fig 7.4. In order to maximize a concave utility function, $\hat{b}$ must be less than 1/2 as shown in Çakmak and Özekici [22], and Fig 7.4 justifies that this is true for this case.

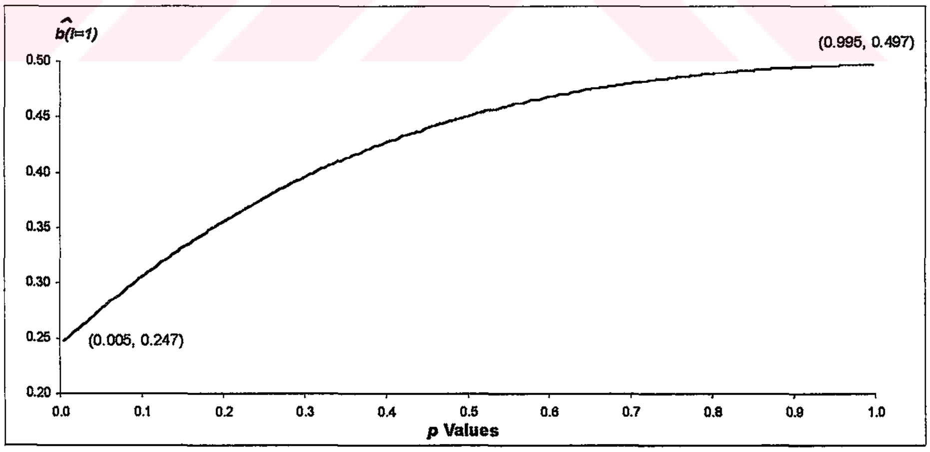

Figure 7.4: $\hat{b}(i = 1)$ (Case I.1, $T = 4$)

As it is for $i = 1$, $\hat{a}_1(i)$, $\hat{a}_2(i)$ and $\hat{b}(i)$ formulas also depend on p for $i = 2$, even though $\hat{f}(i)$, $\hat{g}(i)$ and $\hat{h}(i)$ calculations do not depend on p for $i = 2$ as shown in (7.4)-(7.6). The

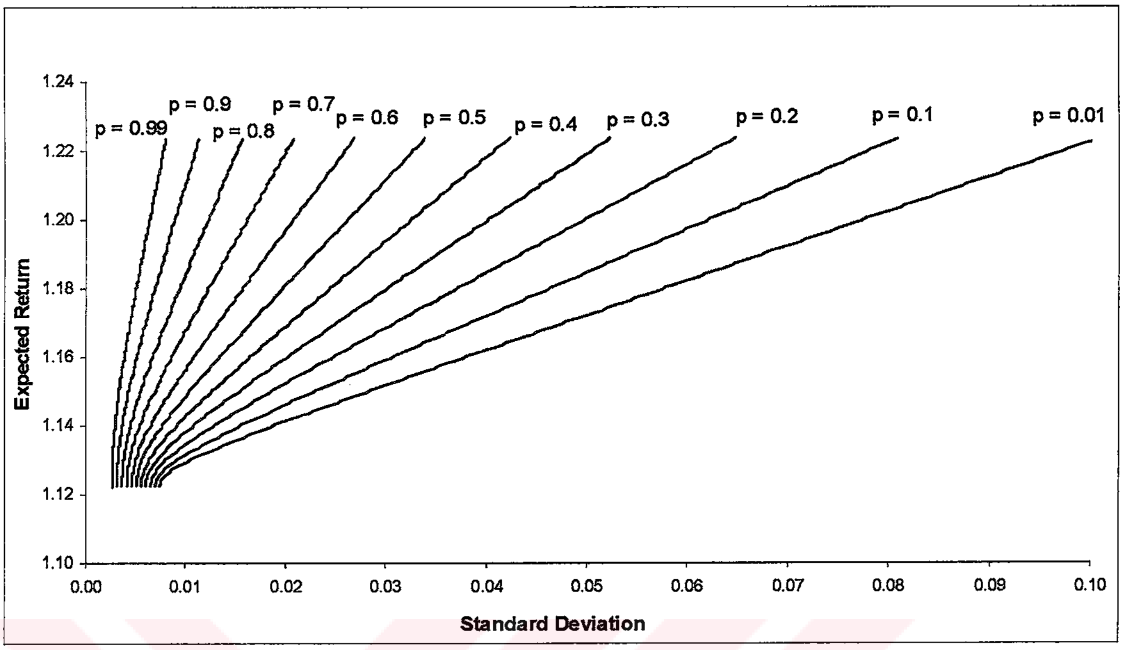

Figure 7.5: Efficient frontiers (Case I.1, $T = 4$, $i = 1$)

reason is any nth power of $\hat{Q}^n_{\hat{g}}$ is calculated through a multiplication of $\hat{Q}(i, j)$ with $\hat{g}(j)$ as defined in (5.8), and this causes each row of the matrix $\hat{Q}^n_{\hat{g}}$ to have at least a single entry which depends on p. Later, the vector $\overline{\hat{Q}}^n_{\hat{g}}$ is calculated by summing entries in all columns of the matrix $\hat{Q}^n_{\hat{g}}$ for each row as defined in (5.9). So, all entries of the vector of $\overline{\hat{Q}}^n_{\hat{g}}$ at any power n depend on p. This argument is also true for $\overline{\hat{Q}}^n_{\hat{f}}$ since the same procedures are used. Hence, $\hat{a}_1(i)$, $\hat{a}_2(i)$ and $\hat{b}(i)$ formulas depend on p for each i because they result from array multiplications of $\overline{\hat{Q}}^n_{\hat{g}}$ and/or $\overline{\hat{Q}}^n_{\hat{f}}$ at some power n.

After calculating $\hat{a}_1(i)$, $\hat{a}_2(i)$ and $\hat{b}(i)$, the efficient frontiers that an investor observes at time zero for $T = 4$ given that the initial state of the observed process is in $i = 1$ is shown in Fig. 7.5 by using (6.47).

In Fig. 7.5, what draws attention first is that minimum-variance portfolios for all p values have nonzero risk over four periods. To look at the minimum-variance portfolios shown in this figure more closely, Fig. 7.6 is presented. The smallest standard deviation among the minimum-variance portfolios is obtained when $p = 0.99$, and it has an expected final wealth of 1.1223 with a standard deviation of $0.0027 > 0$. They are found using (6.48)

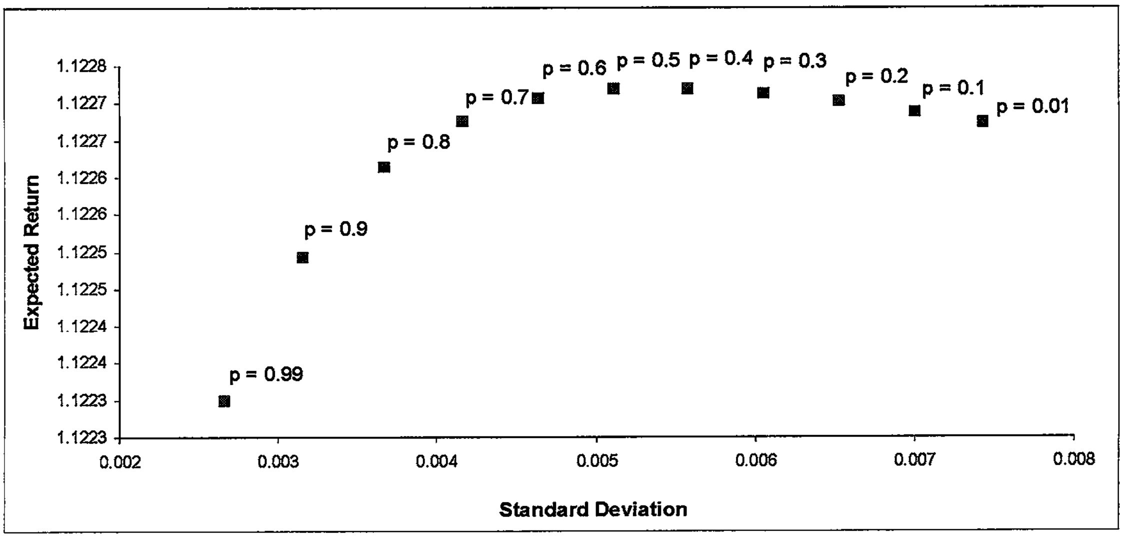

Figure 7.6: Minimum-variance portfolios (Case I.1, $T = 4$, $i = 1$)

and (6.49). In fact, having nonzero risk over four periods should not be surprising because even when an investor puts all his wealth in the risk-free asset over two periods or more, the return of his investment is random since the return of the risk-free asset depends on the state of the market which changes stochastically over time. Even when $p = 1$, that is an investor who knows exactly that if he observes state $i = 1$ then the stochastic market is exactly in state $a = 1$, the investor does not know exactly which state he will observe after the period he is currently in. In other words, uncertainty that exists naturally in transitions of the states of the stochastic market makes the return of multiperiod investing even in risk-free asset risky.

Another important observation from Fig. 7.5 is that as p increases from 0 to 1 for a given standard deviation a greater return is obtained. As p increases from 0 to 1, probability that the unobserved market process is in state $a = 1$ in a period given that observed market process is in state $i = 1$ in that period increases. Moreover from (7.3), for $p = 0.99$, the transition probability matrix $\hat{Q}$ of the observed process is calculated as

$$\hat{Q}(i,j) = \begin{bmatrix} 0.9896 & 0.0104 \\ 0.9000 & 0.1000 \end{bmatrix}.$$

In other words, the observed process visits the state $i = 1$ 98.96% of the time if it was in state $i = 1$, or 90% of the time if it was in state $i = 2$ in the previous period. From the matrix $\hat{Q}$ given in (7.3), we know that as p increases from 0 to 1 investors observe state $i = 1$ more often. In addition, from (7.1) as p increases from 0 to 1 the probability p that the unobserved market process is in state $a = 1$ when they observe the market process is in state $i = 1$ increases correspondingly. Therefore, investors face $a = 1$ state, which is the good scenario, more often, and this provides higher returns as p increases from 0 to 1.

It is also observed from Fig. 7.5 that as p increases from 0 to 1, a lower standard deviation is obtained for a given expected return. It is even more interesting if we consider a special case where we compare the minimum-variance portfolios shown in Fig. 7.6. As shown in Table 7.4, as p increases from 0 to 1, the standard deviation σ_p of the portfolios decreases. In order to understand how the standard deviation σ_p of the minimum-variance

Table 7.4: Standard deviations of the minimum-variance portfolio returns (Case I.1, $T = 4$, $i = 1$)

p	σ_p
0.99	0.00266
0.50	0.00511
0.01	0.00743

portfolios can be decreased, we should look at how these minimum-variance portfolios are constructed. As shown in Table 7.5, we see that in order to achieve an expected return of 2 at the end of investment horizon, the investor has to invest in the risky asset 11.88 units of wealth if $p = 0.99$, and 22.99 units of wealth if $p = 0.01$ at $T = 0$. Therefore, as p increases from 0 to 1, investor carries less risk because the same expected return level μ can be achieved by investing less in the risky asset.

Table 7.5: Amount of wealth invested in the risky asset (Case I.1, $T = 4$, $i = 1$)

p	$u_0(i = 1, x_0 = 1)$
0.99	11.88
0.01	22.99

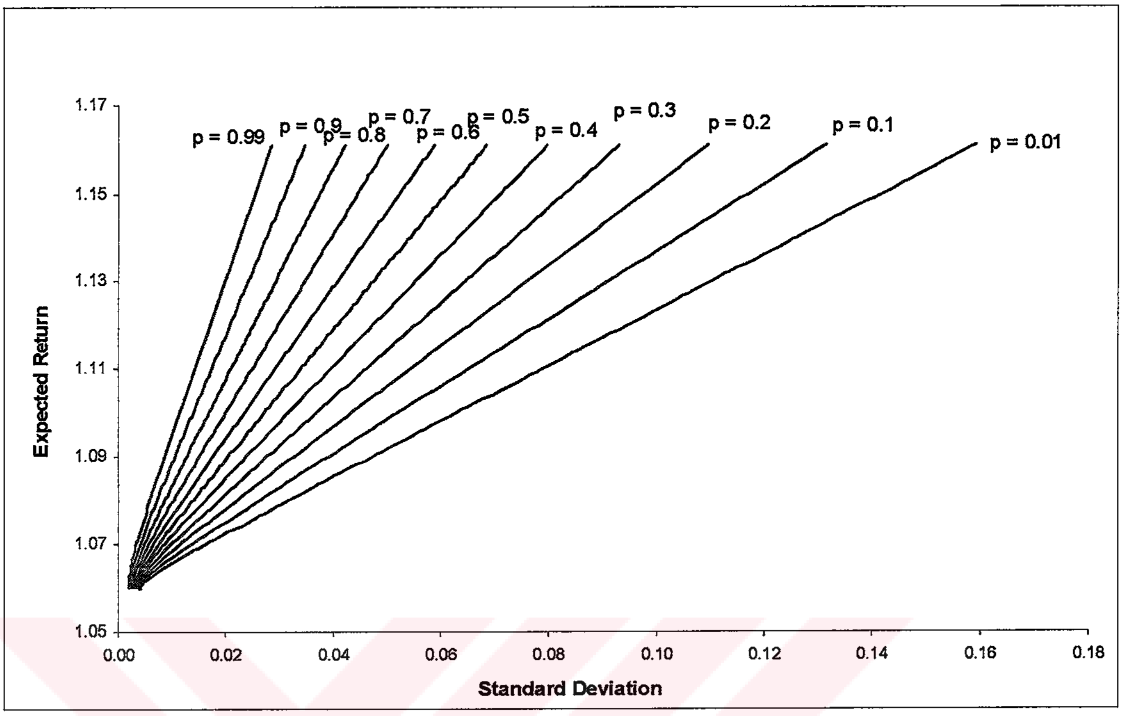

Figure 7.7: Efficient frontiers (Case I.2, $T = 2$, $i = 1$)

Finally, another observation that can be made from Fig.7.5 is that the efficient frontiers look like they do not intersect with each other. To investigate this phenomenon, we obtained the efficient frontier equation in terms of p. However, this equation is quite complicated and does not have a nice well-known form as seen in Appendix B.2. For $T = 4$, their intersections cannot be solved by MATLAB. However, for $T = 2$, we are able to solve the efficient frontier equation in terms of p for a given standard deviation. We analyze this situation next as another case.

Case I.2: In this case, everything except the investment horizon T is kept the same as in the Case I.1. Our aim is to solve the efficient frontier equation in terms of p for a given standard deviation for $T = 2$. The efficient frontiers for this case are given in Fig. 7.7 for different values of p. Before going into deeper analysis of efficient portfolios for a given standard deviation, one may want to look at how efficient frontiers are plotted while expected return is increased from the returns of the minimum-variance portfolios by using increments of 0.001. In Fig. 7.7, we see that efficient frontiers plotted starting for

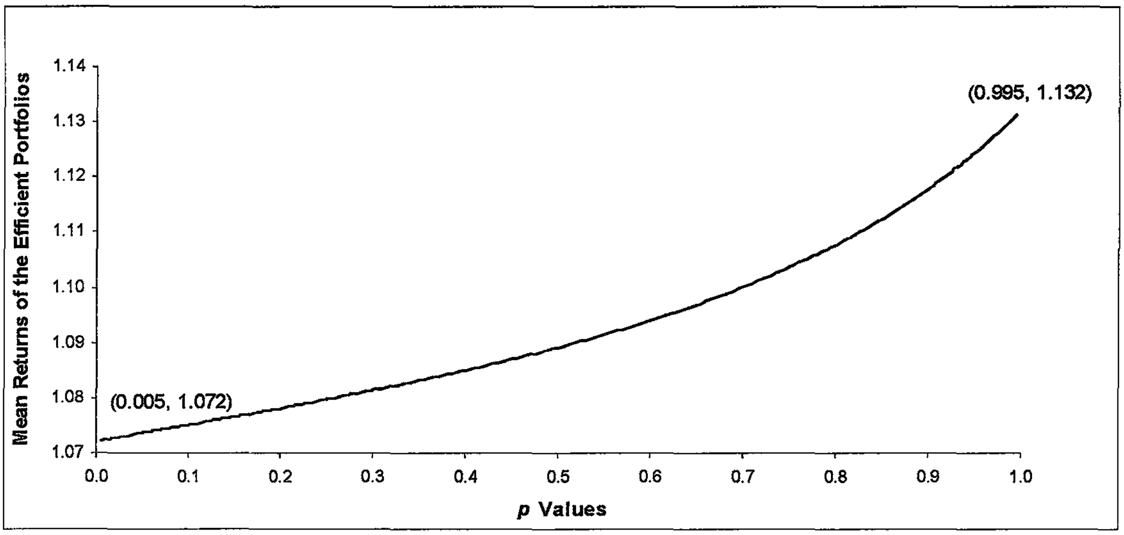

Figure 7.8: Mean returns in terms of p (Case I.2, $T = 2$, $i = 1, \sigma_p = 0.02$)

$p = 0.01$ and ending for $p = 0.99$ do not intersect each other. To look at this more closely, the efficient frontier equation in terms of p is solved for a given standard deviation of 0.02, and expected returns that corresponds to the standard deviation of 0.02 are plotted while p is increased from 0.005 to 0.995 by using increments of 0.005 as shown in Fig. 7.8. In this figure, we see that for the same amount of risk as p increases from 0.005 to 0.995 the investor's expected return increases by $1.132 - 1.072 = 0.06$. Even tough Fig. 7.8 seems to be an increasing function, in order to make sure that efficient frontiers do not intersect for a given standard deviation of 0.02 at every p values between 0.005 and 0.995, we have to look at the first derivative of the efficient frontier equation with respect to p.

In Fig. 7.9, we see that for every p value between 0.005 and 0.995 in increments of 0.005 the first derivative of mean returns is positive. Therefore, since the first derivative of the curve in Fig. 7.8 is positive at all points, it has a positive slope everywhere. This means that efficient frontiers shown in Fig. 7.7 do not intersect for a given standard deviation of 0.02, because at $\sigma = 0.02$ mean returns constantly increases while p increases from 0.005 to 0.995. So, there are no two points on the efficient frontiers which have the same mean return and standard deviation of 0.02. Moreover, slope of the efficient frontiers increases at an increasing rate while p increases from 0.005 to 0.995, so this means for a given standard

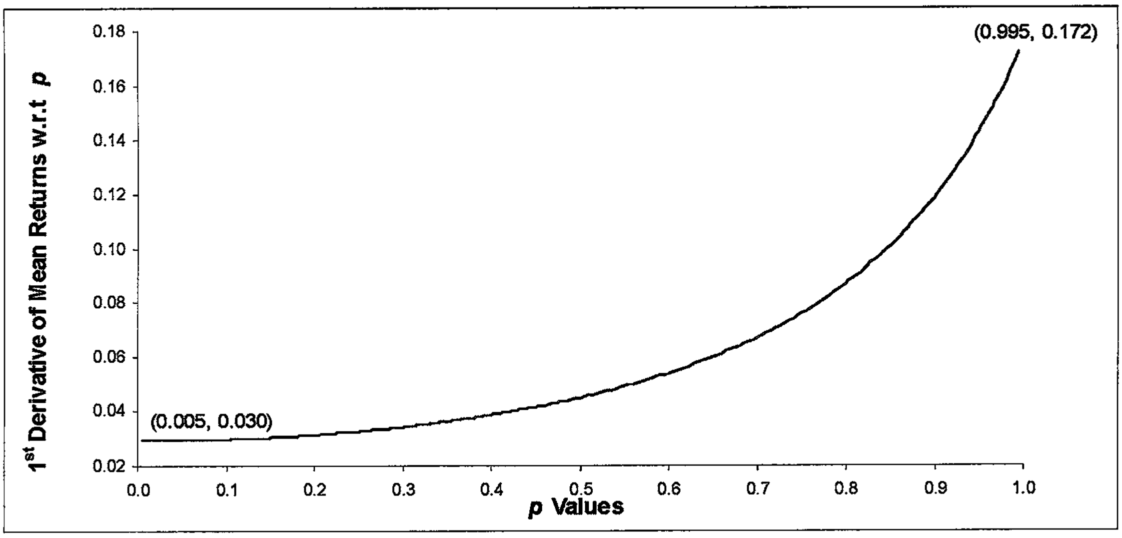

Figure 7.9: The first derivative of mean returns in terms of p (Case I.2, $T = 2$, $i = 1$, $\sigma_p = 0.02$)

deviation of 0.02 an investor earns returns at greater rates as p increases. In other words, the more the investor knows that the unobserved market process is in state $a = 1$ in a period given that observed market process is in state $i = 1$ in that period, the more return he earns at a given risk level. This is simply because we set not only the unobserved process such that it visits state $a = 1$ more often than the other states by the determining entries of the Q matrix in (7.1) but also the return r_f of the riskless asset, the expected value r and the standard deviation σ of the return of the risky asset for state $a = 1$ make it more attractive in terms of returns earned per one unit of risk taken. Therefore, the more investors observe state $i = 1$ in a period and the unobserved market is in state $a = 1$ in that period, they will obtain better investment returns at the end of that period.

7.1.1 Comparison of the Model Assuming a Perfectly Observable Market Using Model I

In this section, numerical illustrations of Model I presented in Section 7.1 are compared with the results obtained by solving the same problem using the formulations of the model assuming a perfectly observable market presented in Chapters 4 and 5.

In fact, the model assuming a perfectly observable market is a special case of Model I. In order to obtain results of the model assuming a perfectly observable market from Model

I, the following conditions must be satisfied:

- The observation matrix O must be an identity matrix,
- Each state in the unobserved process corresponds to one and only one state associated with it in the observed process.

The cases analyzed in Section 7.1 consider a market with three unobserved and two observed states, so two unobserved states $\{a = 1, 2\}$ share the same observed state $\{i = 1\}$. Therefore, the second condition is not satisfied. So, we expect the model assuming a perfectly observable market to perform poorer than Model I in providing efficient portfolios.

The efficient frontiers obtained under perfect information shown in Fig. 7.10 - 7.12 represent the solution of the model assuming a perfectly observable market. In this case, there is only one process which is perfectly observable so that $Z_n = Y_n$ in all periods n. Therefore, the results in Chapter 5 are used to obtain the efficient frontier corresponding to this case.

In order to compare the results of model I with the model assuming a perfectly observable market,both models should have the same number of states, transition matrix Q, means r and variances σ^2 of returns of the risky asset. Note that when $p = 1$, state $a = 2$ of the unobserved process is not possible and the state space F must also have only two states. Therefore, the model does not really converge to the model under perfect information as p increases to 1.

For $T = 4$, that is Case I.1, the problem is solved in 5.5 seconds using the model assuming a perfectly observable market, and the efficient frontiers are plotted as shown in Fig. 7.10. Here, we first note that the frontier obtained assuming perfect information intersect with the frontiers obtained from assuming imperfect information. Another important observation is that minimum-variance portfolios obtained for some p value has greater return than the minimum-variance portfolio obtained from the model assuming perfectly observable market. Moreover, in the figure it seems that for a given level of standard deviation returns earned by assuming perfect observation of the market is always less than the returns earned when p is close to 1. We searched this phenomenon for $T = 2$ case more deeply.

For $T = 2$, the problem is solved in 5.4 seconds using the model assuming a perfectly observable market, and the efficient frontiers are plotted as shown in Fig. 7.11. The pattern

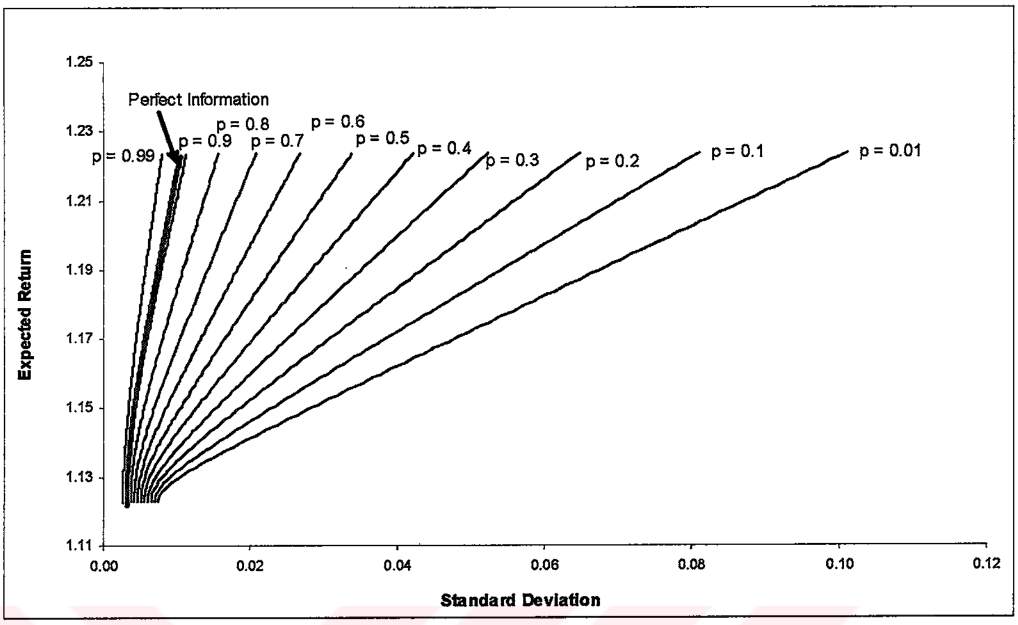

Figure 7.10: Efficient frontiers with perfect and imperfect information (Case I.1, $T = 4$, $i = 1$)

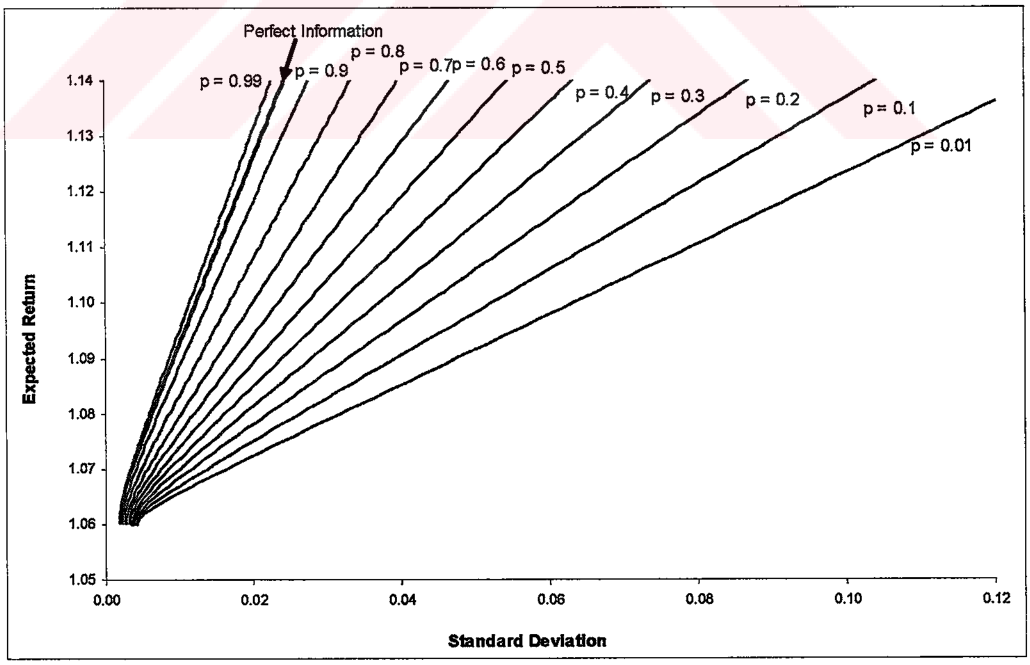

Figure 7.11: Efficient frontiers with perfect and imperfect information (Case I.2, $T = 2$, $i = 1$)

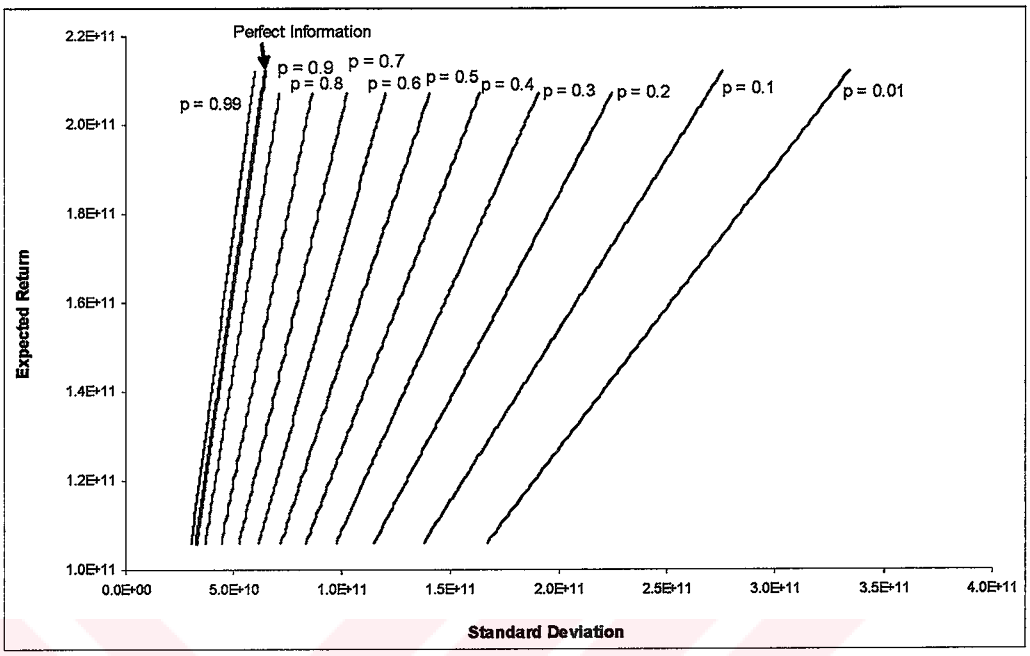

Figure 7.12: Efficient frontiers at greater σ_p's (Case I.2, $T = 2$, $i = 1$)

of frontiers for $T = 2$ is very similar to that for $T = 4$, except for the fact that for the same level of standard deviation and for the same level of information, that is for the same p, investing for $T = 2$ provides less return than investing for $T = 4$. In other words, to earn the same level of return an investor faces less risk when he invests for longer time periods. For instance, when the investor aims to earn an expected return of 1.20 at the end of investment horizon and has p value of 0.99, he faces approximately a standard deviation of 0.006 if he invests for $T = 4$ periods and approximately a standard deviation of 0.04 if he invests for $T = 2$ periods.

In addition, for $T = 2$, that is Case I.2, we investigate the behavior of the frontier obtained assuming perfect observation of the market as standard deviation goes to infinity. So, Fig. 7.12 shows the efficient frontiers which are plotted in the order of 10^{12} standard deviations. In this figure, for some $p = (1 - \varepsilon)$, where ε is a very small number, one can find an efficient portfolio such that for a given level of standard deviation it provides greater return than the efficient portfolio obtained from assuming perfect observation of the market.

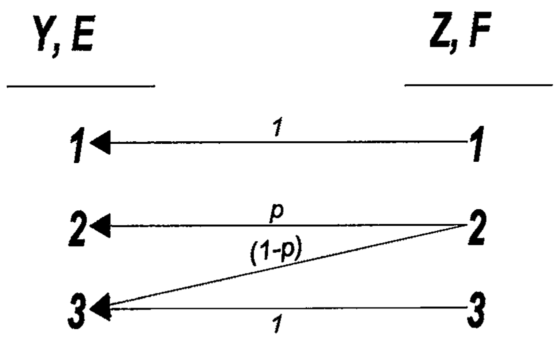

Figure 7.13: Case II.1 Illustration

7.2 Numerical Illustrations on Model II

Case II.1: In this case, as in Section 7.1, we consider a market with a single risky asset and a riskless asset where the market is modulated by a hidden Markov chain. Both the stochastic market process Z and the observed market process Y have three states $F = \{1, 2, 3\}$ and $E = \{1, 2, 3\}$ where the states are represented generically by the letters a and i for E and F respectively. The relationship between the states of the stochastic market process Z and the observed market process Y is assumed to be known as in Fig. 7.13.

In this model, we assume that investors know the probability of observing state i of the observed market process Y in a period given state a of the stochastic market process Z in that period. Therefore, as shown in Fig. 7.13, when the stochastic market is in state $a = 1$ investors can observe state $i = 1$ with probability $E(a = 1, i = 1) = 1$, and when the market is in state $a = 3$, state $i = 3$ is observed with probability $E(a = 3, i = 3) = 1$ because when the stochastic market is in either state $a = 1$ or $a = 3$, it corresponds to a single state i in the process Y. Moreover, when the stochastic market is in state $a = 2$ investors can observe state $i = 2$ with probability $E(a = 2, i = 2) = p$, and state $i = 3$ with probability $E(a = 2, i = 3) = (1 - p)$ because when the stochastic market is in state $a = 2$, only state $i = 2$ or $i = 3$ in the process Y can be observed and sum of probabilities of observing these states must be 1. Therefore, the relationship between the stochastic process and the observed process is modeled by the emission matrix E as

$$E(a,i) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & p & 1-p \\ 0 & 0 & 1 \end{bmatrix}. \tag{7.7}$$

From (7.7), an investor can determine the observation matrix O by taking weighted averages defined in (3.8) as

$$O(i,a) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{p}{p} & 0 \\ 0 & \frac{1-p}{2-p} & \frac{1}{2-p} \end{bmatrix}. \tag{7.8}$$

For $p = 0$, as shown in (3.8), $0/0$ indefiniteness occurs in $O(i=2, a=2)$, so p is defined for $0 < p \leq 1$ in all of our calculations. Once the general formulations of efficient frontiers are obtained for different values of p, they are plotted and compared with each other.

Like the cases in Section 7.1, the investor starts with one unit of wealth at $T = 0$. Moreover, the expected value r and the standard deviation σ of the return of the risky asset for each state are the same with the previous cases analyzed and shown in Table 7.1. Hence, categorization of the states has not changed; states $a = 1$, $a = 2$ and $a = 3$ still represent good, ordinary, and bad scenarios for the market respectively. Return of the riskless asset is now given in Table 7.6 for $i = 1, 2$, and 3. Finally, the transition matrix Q of the hidden

Table 7.6: Return of the riskless asset (Case II.1)

State i	$r_f(i)$
1	1.030
2	1.020
3	1.005

Markov chain that the stochastic market process follows is kept unchanged and is shown in Table 7.2. With the inputs given up to this point, the MATLAB code is run for 391 seconds and all necessary calculations to plot the efficient frontiers are obtained.

From (6.6), the transition probability matrix $\hat{Q}$ of the observed Markov chain is calculated as

$$\hat{Q}(i,j) = \begin{bmatrix} 0.9 & 0.09p & 0.1 - 0.09p \\ 0.8 & 0.15p & 0.2 - 0.15p \\ \frac{0.8(1-p)+0.7}{(2-p)} & \frac{0.15p(1-p)+0.2p}{(2-p)} & \frac{(1-p)(0.2-0.15p)+(0.3-0.2p)}{(2-p)} \end{bmatrix}.$$

Then, by using (6.24), $\hat{V}(i)$ is computed to be

$$\hat{V}(1) = [0.0034],\ \hat{V}(2) = [0.001],\ \hat{V}(3) = \left[\frac{0.0018(1-p) + 0.000525}{(2-p)})\right].$$

for each state $i = 1, 2$, and 3. By using the definitions given in (6.34)-(6.37), the vectors $\hat{f}(i)$, $\hat{g}(i)$ and $\hat{h}(i)$ are calculated as follows

$$\hat{f}(i) = \begin{bmatrix} 0.2808 \\ 0.6242 \\ 1.01 - \frac{1.01(\frac{1.04(1-p)+1.02}{(2-p)} - 1.005)^2}{(\frac{0.0018(1-p)+0.000525}{(2-p)})} \end{bmatrix} \tag{7.9}$$

$$\hat{g}(i) = \begin{bmatrix} 0.2726 \\ 0.6120 \\ 1.005 - \frac{1.005(\frac{1.04(1-p)+1.02}{(2-p)} - 1.005)^2}{(\frac{0.0018(1-p)+0.000525}{(2-p)})} \end{bmatrix} \tag{7.10}$$

$$\hat{h}(i) = \begin{bmatrix} 0.7353 \\ 0.4 \\ \frac{(\frac{1.04(1-p)+1.02}{(2-p)} - 1.005)^2}{(\frac{0.0018(1-p)+0.000525}{(2-p)})} \end{bmatrix}. \tag{7.11}$$

for $i = 1, 2$, and 3.

To obtain the efficient frontier equation we only need to calculate $\hat{a}_1(i)$, $\hat{a}_2(i)$ and $\hat{b}(i)$ defined in (6.44)-(6.46). Because of their definitions, $\hat{a}_1(i)$, $\hat{a}_2(i)$ and $\hat{b}(i)$ formulas depend on p for all i. In addition, since the open form of these values are very long, rather then giving the explicit formulations, we present the plots of $\hat{a}_1(i)$, $\hat{a}_2(i)$ and $\hat{b}(i)$ for $i = 1$ while p is increased from 0.005 to 1 by using increments of 0.005 in Fig 7.14 - Fig 7.16.

Firstly, $\hat{a}_1(i = 1)$ is calculated from (6.44), and $\hat{a}_1$ is 0.0071 for $p = 0.005$ and $\hat{a}_1$ is 0.0082 for $p = 1$ as it is seen in Fig 7.14. Then, $\hat{a}_2(i = 1)$ is calculated from (6.45), and $\hat{a}_2$ is 0.0078 for $p = 0.005$ and $\hat{a}_2$ is 0.0092 for $p = 1$ as it is seen in Fig 7.15. Likewise, $\hat{b}(i = 1)$ is obtained from (6.46) and it takes the value 0.4968 for $p = 0.005$ and 0.4963 for $p = 1$ as it is seen in Fig 7.16. Moreover, $\hat{b}$ satisfies the condition of being less than 1/2 for all p values.

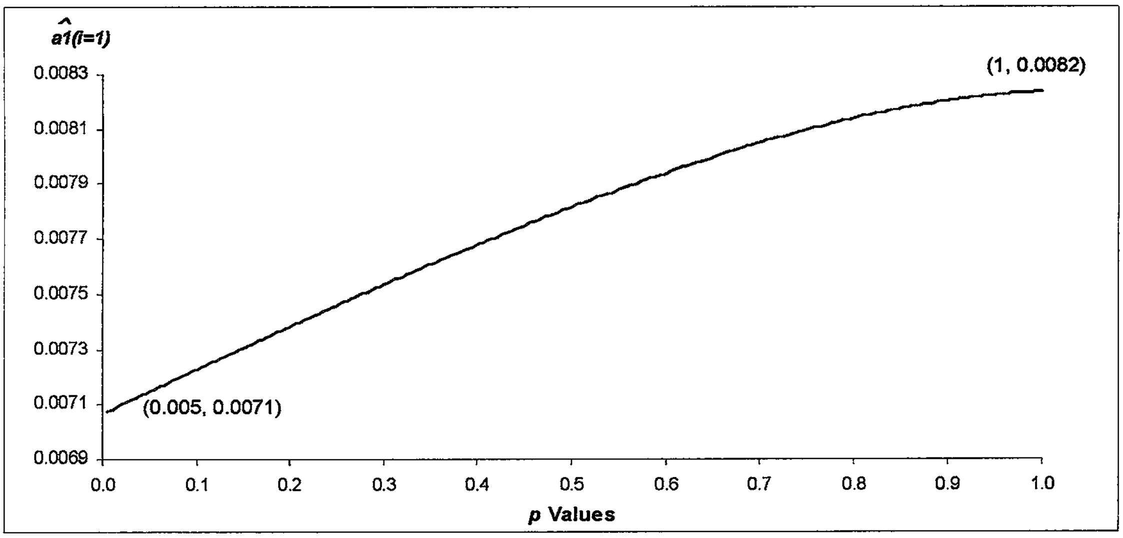

Figure 7.14: $\hat{a}_1(i = 1)$ (Case II.1, $T = 4$)

After calculating $\hat{a}_1(i)$, $\hat{a}_2(i)$ and $\hat{b}(i)$, the efficient frontiers that an investor observes at time zero for $T = 4$ given that the initial state of the observed process is in $i = 1$ is shown in Fig. 7.17 by using (6.47). In this figure, what draws attention first is that all efficient portfolios on the frontiers have nonzero risk over four periods as it is in the first model shown in Fig. 7.5 .

To look at the minimum-variance portfolios shown in Fig. 7.17 more closely, Fig. 7.18 is presented. The standard deviation and return of the minimum-variance portfolios are found by using (6.48) and (6.49). The smallest standard deviation among the minimum-variance portfolios is obtained when $p = 1$, and it has the expected final wealth of 1.11391 with the standard deviation of $0.00414 > 0$. As it is explained in Section 7.1, having nonzero risk over four periods is due to the uncertainty that exists naturally in transitions of the states of the stochastic market.

Moreover, in both Fig. 7.5 and Fig. 7.17, as p increases from 0 to 1 for a given standard deviation a greater return is obtained. As p increases from 0 to 1, probability that the observed market process is in state $i = 2$ in a period given that the unobserved market process is in state $a = 2$ in that period increases. Moreover, the transition matrix $\hat{Q}$ of the observed process becomes equal to the transition matrix Q of the stochastic process given

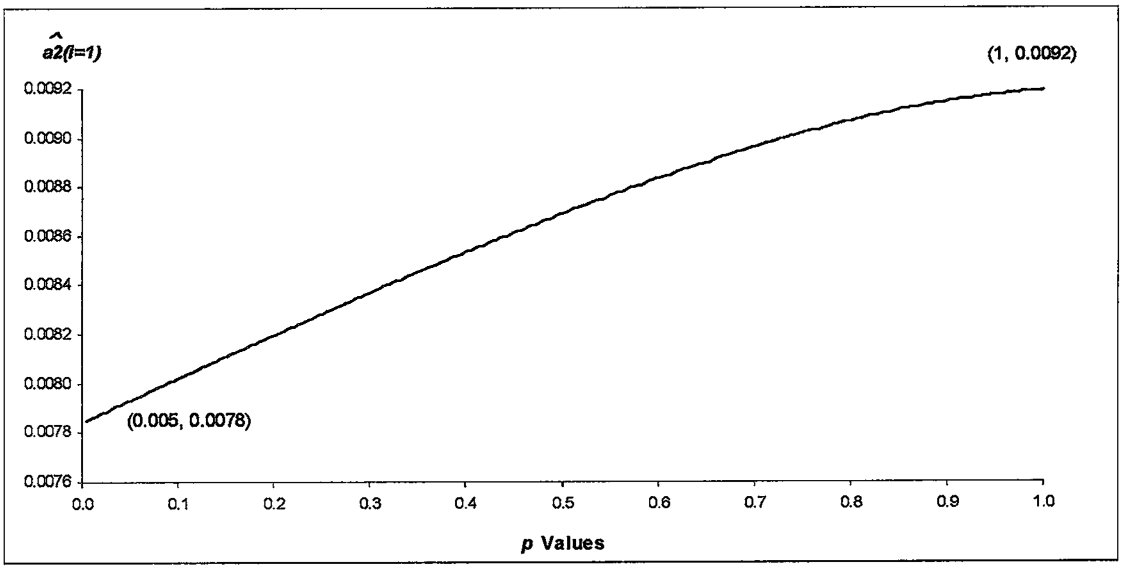

Figure 7.15: $\hat{a}_2(i = 1)$ (Case II.1, $T = 4$)

in (7.2)

$$\hat{Q}(i,j) = \begin{bmatrix} 0.90 & 0.09 & 0.01 \\ 0.80 & 0.15 & 0.05 \\ 0.70 & 0.20 & 0.10 \end{bmatrix} \tag{7.12}$$

for $p = 1$, and $\hat{Q}$ becomes

$$\hat{Q}(i,j) = \begin{bmatrix} 0.9000 & 0.0009 & 0.0991 \\ 0.8000 & 0.0015 & 0.1985 \\ 0.7497 & 0.0018 & 0.2485 \end{bmatrix} \tag{7.13}$$

for $p = 0.01$. Here, we notice that as p increases probability of visiting state $i = 2$ from states $i = 1, 2, 3$ becomes greater than the probability of visiting state $i = 3$ from these states. For instance, when the observed process is in state $i = 1$, it will visit state $i = 2$ 9% of the time for $p = 1$ and 0.09% of the time for $p = 0.01$ in the next period.

We can look at effect of p on the minimum-variance portfolios shown in Fig. 7.18. As it is in Section 7.1, we see that as p increases from 0 to 1 the returns r_p of the minimum-variance portfolios increase, and standard deviation σ_p of the portfolios decreases as shown in Table 7.7. Because of the portfolio effect explained in Section 7.1, the standard deviation σ_p of the minimum-variance portfolio for $p = 1$ is lower than σ_p of the portfolio for $p = 0.01$,

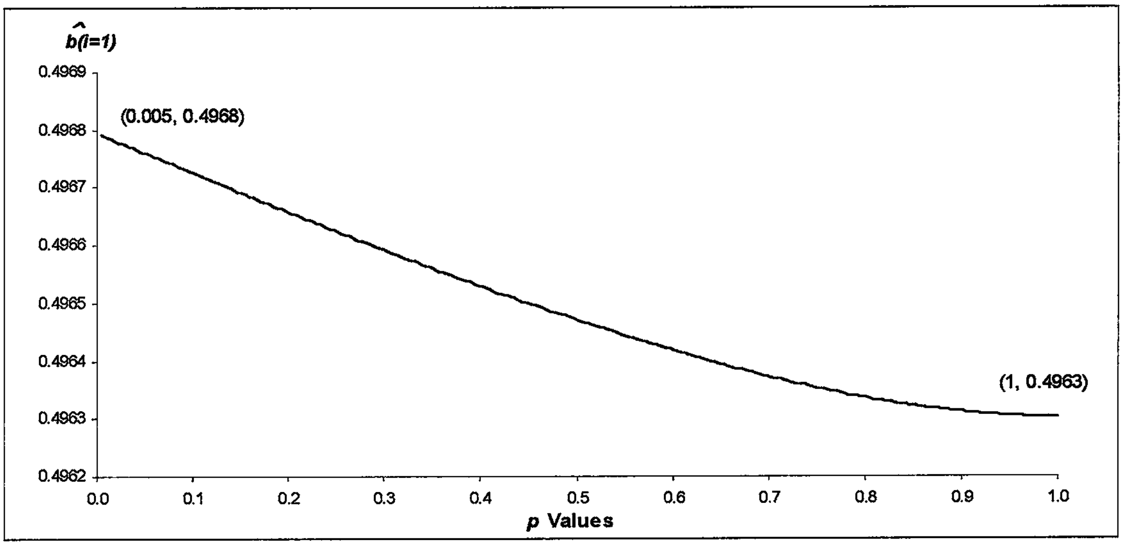

Figure 7.16: $\hat{b}(i = 1)$ (Case II.1, $T = 4$)

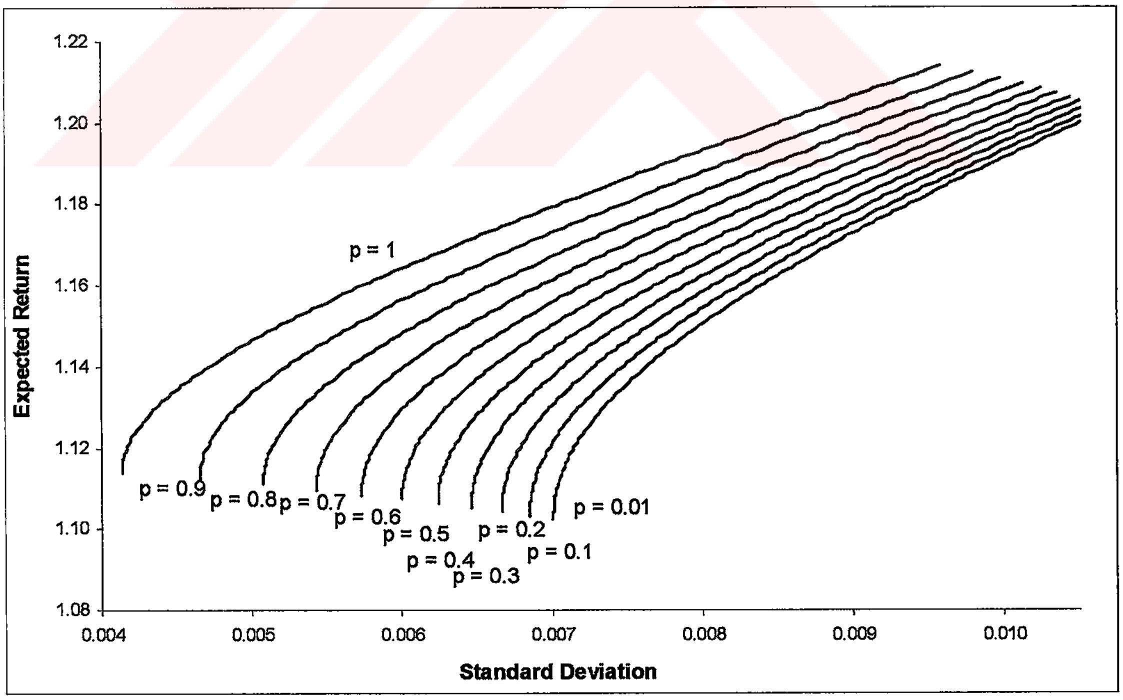

Figure 7.17: Efficient frontiers (Case II.1, $T = 4$, $i = 1$)

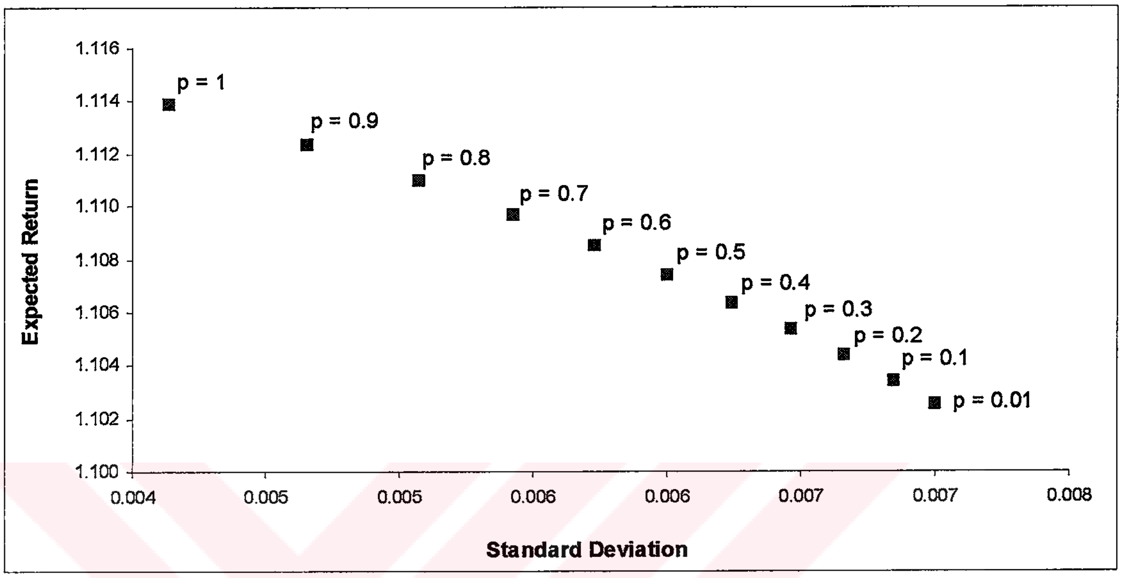

Figure 7.18: Minimum-variance portfolios (Case II.1, $T = 4$, $i = 1$)

Table 7.7: Means and standard deviations of the minimum-variance portfolio returns (Case II.1, $T = 4$, $i = 1$)

p	r_p	σ_p
1.00	1.11390	0.00414
0.50	1.10743	0.00600
0.01	1.10254	0.00700

while the return r_p for $p = 1$ is greater than r_p for $p = 0.01$. For the problem $P2(\mu = 2)$, optimal investment policy for $T = 0$ is calculated by using (6.52), and in Table 7.8 one can see how much is invested into the risky asset given that the initial wealth is one unit. In order to achieve an expected return of 2 at the end of the investment horizon, the investor has to invest in the risky asset 12.09 units of wealth if $p = 1$, and 12.36 units of wealth if $p = 0.01$ at $T = 0$. Investing less in the risky asset and still earning the same expected return level μ makes investing when $p = 1$ more attractive than $p = 0.01$.

Table 7.8: Amount of wealth invested in the risky asset (Case II.1, $T = 4$, $i = 1$)

p	$u_0(i = 1, x_0 = 1)$
1.00	12.09
0.01	12.24

Finally, we investigate whether the efficient frontiers shown in Fig. 7.17 intersect with each other. In order to do this, we need to analyze the efficient frontier equation in terms of p. However, this equation is quite long (22 pages) and does not have a nice well-known form. Since we cannot solve the efficient frontier equation in terms of p for a given standard deviation for $T = 4$, we solved it for $T = 2$. This is illustrated as our next case.

Case II.2: In this case, everything except the investment horizon T is kept the same with the Case II.1. The efficient frontier for this case is given in Fig. 7.19 for different values of p. Our aim is to solve the efficient frontier equation in terms of p for a given standard deviation for $T = 2$. Before going into deeper analysis of efficient frontiers for a given standard deviation of 0.02, one may want to look at how efficient frontiers are plotted while expected return is increased from the returns of the minimum-variance portfolios by using increments of 0.001. The main difference in the characteristics of the efficient frontiers between Fig. 7.19 and Fig. 7.7 in Section 7.1, is that the frontiers in the former case seems to converge to a single line whereas in the latter case they do not converge as we have shown in Fig. 7.12. To see whether the efficient frontiers in Fig. 7.19 converge as we suspect, we analyze their behaviors as the standard deviation of the portfolios σ_p goes to infinity. As shown in Fig. 7.20, which is plotted in the order of 10^{12} standard deviations, the efficient frontiers do converge to a single curve.

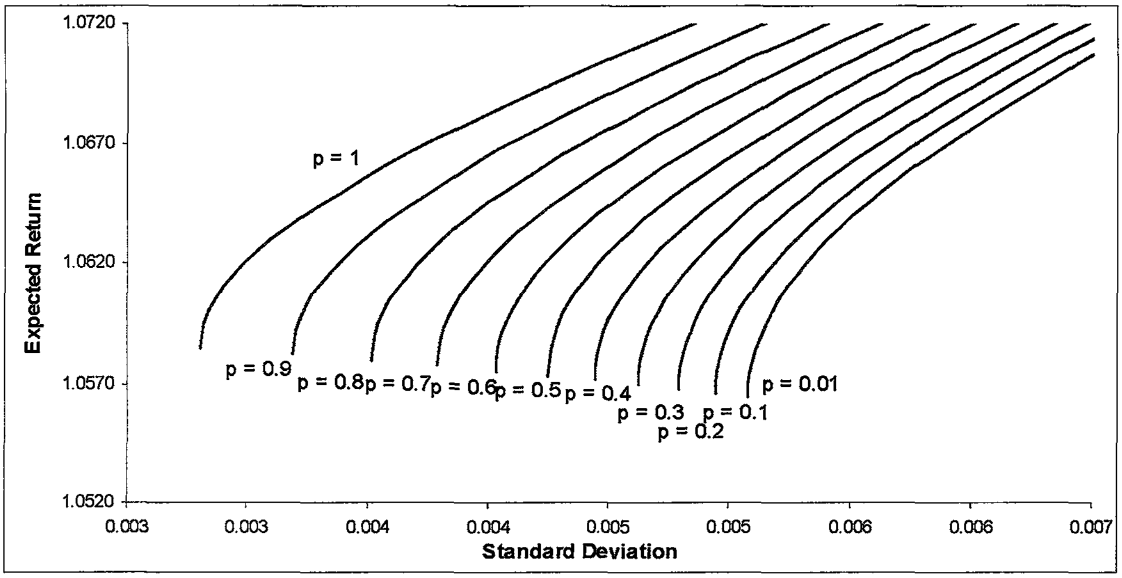

Figure 7.19: Efficient frontiers (Case II.2, $T = 2$, $i = 1$)

Moreover, since the frontiers are converging, we suspect that they intersect with each other. Therefore, the efficient frontier equation in terms of p is solved for a given standard deviation of 0.02, and expected returns that corresponds to the standard deviation of 0.02 are plotted while p is increased from 0.005 to 1 by using increments of 0.005 as shown in Fig. 7.21. In this figure, we see that for the same amount of risk as p increases from 0.005 to 1 the investor's expected return increases by $1.1261 - 1.1241 = 0.002$. If we take the first derivative of the efficient frontier equation with respect to p, the slope of the function changes in the order of 10^{-3} as shown in Fig. 7.22. Moreover, we see in Fig. 7.22 that the first derivative of mean returns in terms of p is positive for every p values between 0.005 and 0.995 in increments of 0.005 as it was in Fig. 7.9. This means that efficient frontiers shown in Fig. 7.19 do not intersect for a given standard deviation of 0.02, because mean returns constantly increase while p increases from 0.005 to 1. So, there are no two points on the efficient frontiers which have the same mean return and standard deviation of 0.02.

7.2.1 Comparison of the Model Assuming a Perfectly Observable Market Using Model II

In this section, numerical illustrations of Model II presented in Section 7.2 are compared with the results obtained by solving the same problem using the formulations of the model

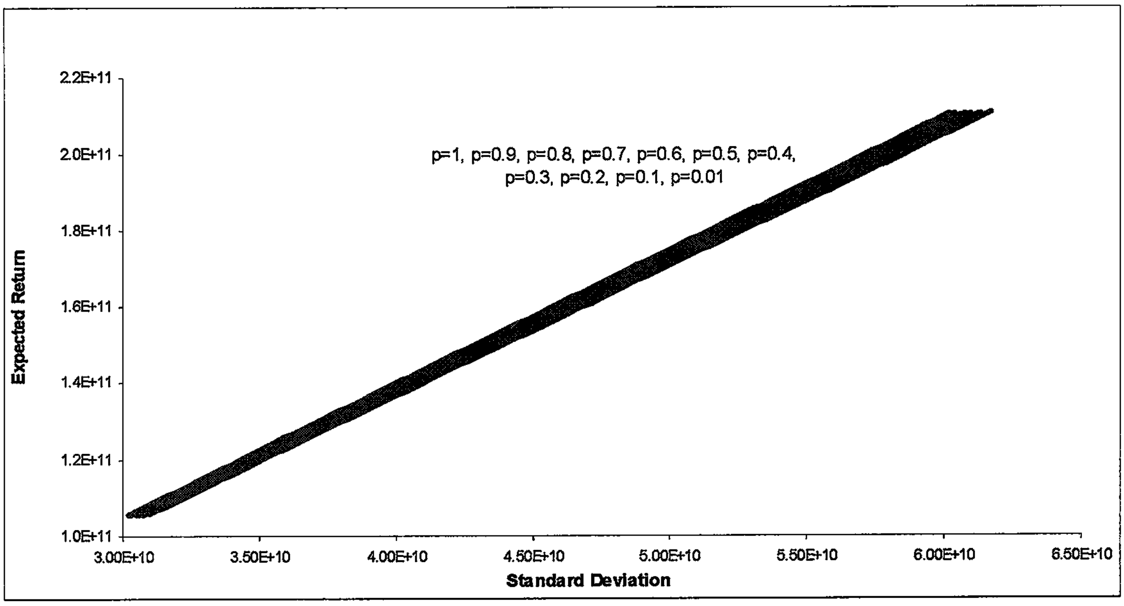

Figure 7.20: Efficient frontiers at greater σ_p's (Case II.2, $T = 2$, $i = 1$)

assuming perfectly observable market presented in Chapters 4 and 5.

In fact, the model assuming a perfectly observable market is a special case of Model II. In order to obtain results of the model assuming a perfectly observable market from Model II, the following conditions must be satisfied:

- The emission matrix E must be an identity matrix,
- Each state in the unobserved process corresponds to one and only one state associated with it in the observed process.

The cases analyzed in Section 7.2 consider a market with three unobserved and three observed states; moreover, the emission matrix becomes identity matrix and one to one mapping of observed and unobserved states is satisfied, when $p = 1$. Therefore, results of model II when $p = 1$ is expected to be the same with the results of the model assuming a perfectly observable market.

For $T = 4$, the model assuming a perfectly observable market is solved in 5.6 seconds, and the efficient frontiers are plotted as shown in Fig. 7.23. Here, we first note that the

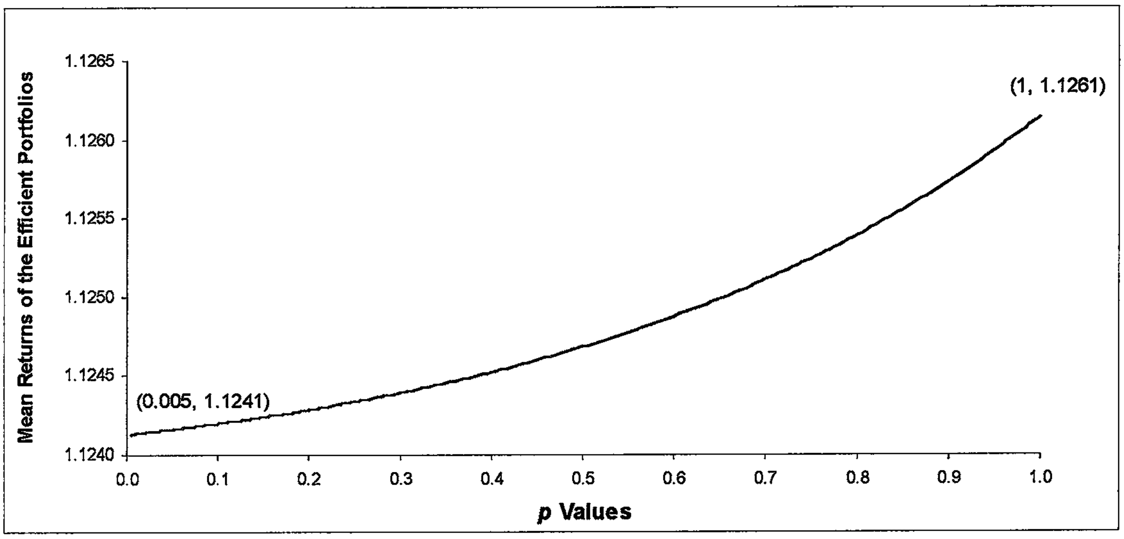

Figure 7.21: Mean returns in terms of p (Case II.2, $T = 2$, $i = 1, \sigma_p = 0.02$)

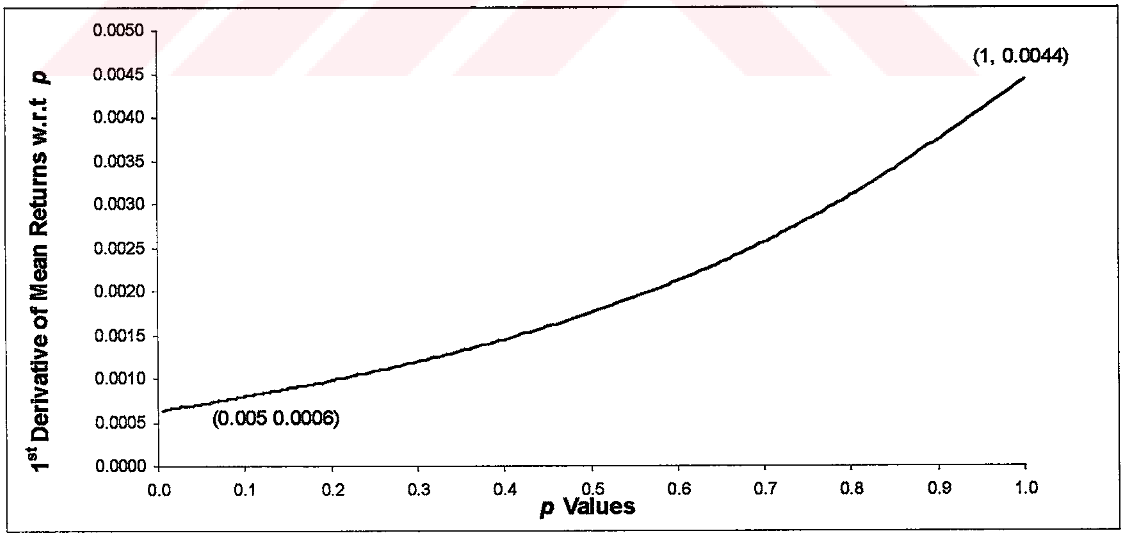

Figure 7.22: The first derivative of mean returns in terms of p (Case II.2, $T = 2$, $i = 1$, $\sigma_p = 0.02$)

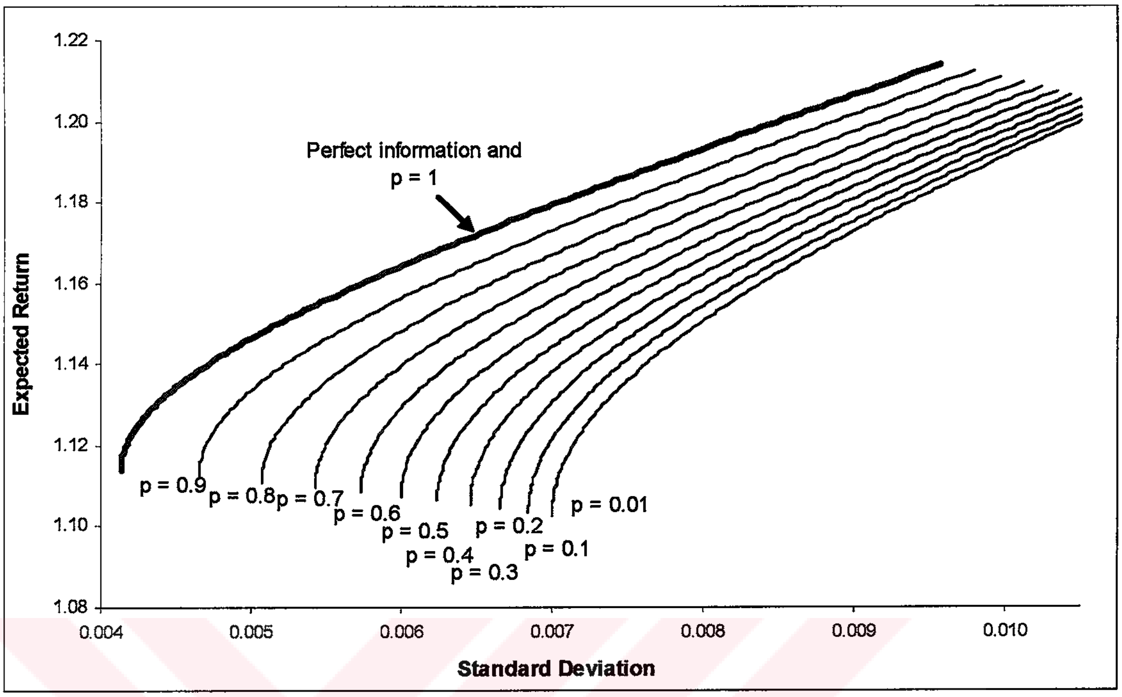

Figure 7.23: Efficient frontiers with perfect and imperfect information (Case II.1, $T = 4$, $i = 1$)

frontier obtained assuming perfect information coincide exactly with the frontier obtained when $p = 1$. Moreover, an efficient frontier with greater p dominate a frontier with lower p; in other words, as p increases from 0 to 1, an investor takes less risk for the same level of expected return. We searched the behaviors of the frontiers also for $T = 2$ case.

For $T = 2$, the solution is obtained in 5.5 seconds, and the efficient frontiers are plotted as shown in Fig. 7.24. The pattern of frontiers for $T = 2$ is very similar to that for $T = 4$, except for the fact that for the same level of standard deviation and for the same level of information, that is for the same p, investing for $T = 2$ provides less return than investing for $T = 4$. In other words, to earn the same level of return an investors face less risk when he invests for longer time periods. For instance, when the investor aims to earn an expected return of 1.16 at the end of investment horizon and has p value of 0.01, he faces approximately a standard deviation of 0.008 if he invests for $T = 4$ periods and approximately a standard deviation of 0.03 if he invests for $T = 2$ periods.

In addition, for $T = 2$ case we investigate the behavior of the frontiers obtained assuming

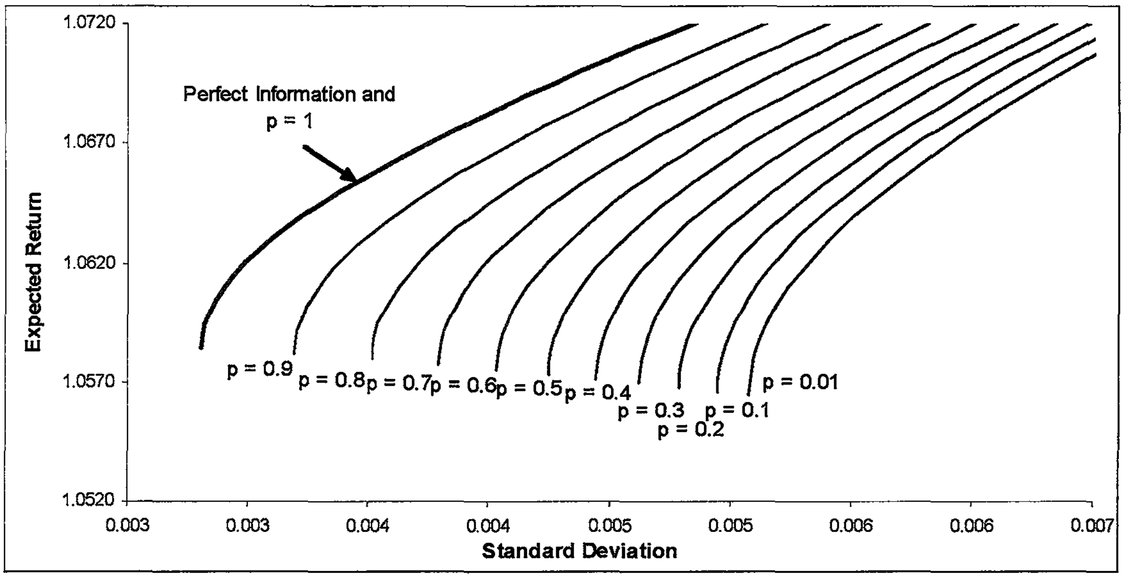

Figure 7.24: Efficient frontiers with perfect and imperfect information (Case II.2, $T = 2$, $i = 1$)

perfect observation of the market as standard deviation goes to infinity. So, Fig. 7.25 shows the efficient frontiers which are plotted in the order of 10^{12} standard deviations. In this figure, we see that all efficient frontiers converge to a single frontier.

Case II.3: In this case, we consider a market with three risky assets and one riskless asset where the market is modulated by a hidden Markov chain. Both the stochastic market process Z and the observed market process Y have four states $F = \{1, 2, 3, 4\}$ and $E = \{1, 2, 3, 4\}$ where the states are represented generically by the letters a and i for E and F respectively. The relationship between the states of the stochastic market process Z and the observed market process Y is assumed to be known as in Fig. 7.26. Therefore, the relationship between the stochastic process and the observed process is modeled by the emission matrix E as

$$E(a,i) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.2 & 0.8 & 0 \\ 0 & 0.3 & 0.7 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{7.14}$$

From (7.7), an investor can determine the observation matrix O by taking weighted averages

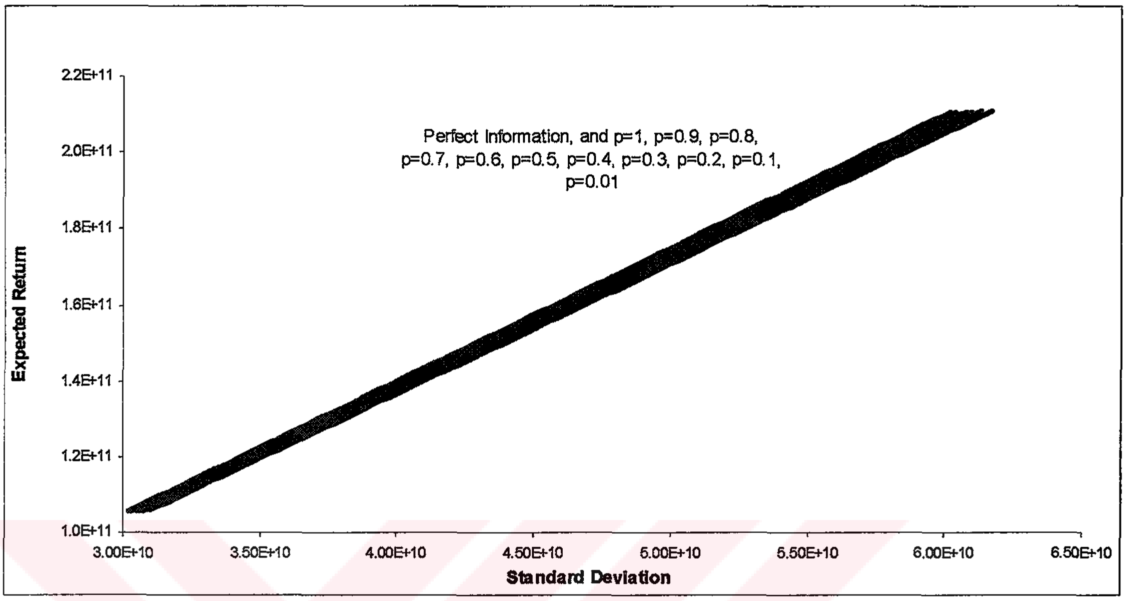

Figure 7.25: Efficient frontiers at greater σ_p's (Case II.2, $T = 2$, $i = 1$)

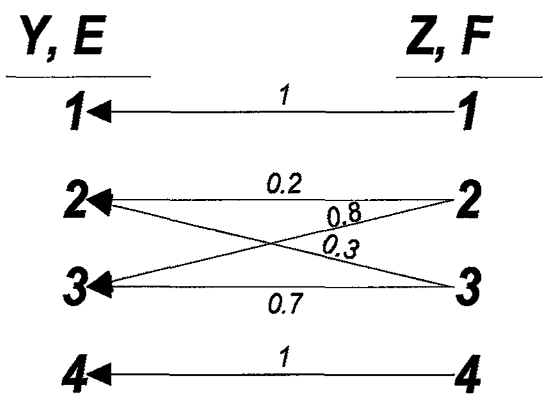

Figure 7.26: Case II.3 Illustration

defined in (3.8) as

$$O(i,a) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2/5 & 3/5 & 0 \\ 0 & 8/15 & 7/15 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad (7.15)$$

The investor starts with one unit of wealth at $T = 0$. The expected value r of the return of the risky assets for each state are given in Table 7.9. The return r_f of the riskless asset is given in Table 7.10.

Table 7.9: Mean returns (Case II.3)

State a	$r_1(a)$	$r_2(a)$	$r_3(a)$
1	0.9162	0.8558	0.8751
2	0.9690	0.9970	0.9691
3	1.0318	1.0668	1.0802
4	1.1160	1.1704	1.1297

Table 7.10: Return of the riskless asset (Case II.3)

State i	$r_f(i)$
1	1.0008
2	1.0018
3	1.0038
4	1.0048

Moreover, covariance matrices $\sigma(a)$ of the returns of the risky assets in the order of 10^{-3}

are given as

$$\sigma(1) = \begin{bmatrix} 2.927 & -0.513 & -0.361 \\ -0.513 & 8.979 & 1.304 \\ -0.361 & 1.304 & 4.365 \end{bmatrix}, \sigma(2) = \begin{bmatrix} 9.762 & -2.506 & -1.553 \\ -2.506 & 9.461 & -2.309 \\ -1.553 & -2.309 & 6.649 \end{bmatrix}$$

$$\sigma(3) = \begin{bmatrix} 12.641 & -3.664 & -3.492 \\ -3.664 & 14.714 & 8.258 \\ -3.492 & 8.258 & 15.136 \end{bmatrix}, \sigma(4) = \begin{bmatrix} 8.202 & 3.119 & 2.282 \\ 3.119 & 18.438 & 5.821 \\ 2.282 & 5.821 & 10.355 \end{bmatrix}$$

for each state $a = 1, 2, 3$ and 4.

Finally, the transition probability matrix Q of the hidden Markov chain that the stochastic market process follows is given as

$$Q(a,b) = \begin{bmatrix} 0.23 & 0.18 & 0.12 & 0.47 \\ 0.23 & 0.23 & 0.08 & 0.46 \\ 0.30 & 0 & 0.30 & 0.40 \\ 0.37 & 0.37 & 0.21 & 0.05 \end{bmatrix}. \quad (7.16)$$

With the inputs given up to this point, the MATLAB code is run for 9.3 seconds and all necessary calculations to plot the efficient frontiers are obtained.

From (6.6), the transition probability matrix $\hat{Q}$ of the observed Markov chain is calculated as

$$\hat{Q}(i,j) = \begin{bmatrix} 0.2353 & 0.0706 & 0.2235 & 0.4706 \\ 0.2723 & 0.0817 & 0.2214 & 0.4246 \\ 0.2631 & 0.0789 & 0.2252 & 0.4328 \\ 0.3684 & 0.1368 & 0.4421 & 0.0526 \end{bmatrix}.$$

Then, by using (6.24), $\hat{V}(i)$ is computed to be

$$\hat{V}(1) = \begin{bmatrix} 0.0101 & 0.0118 & 0.0103 \\ 0.0118 & 0.0300 & 0.0195 \\ 0.0103 & 0.0195 & 0.0202 \end{bmatrix}, \hat{V}(2) = \begin{bmatrix} 0.0125 & -0.0020 & -0.0009 \\ -0.0020 & 0.0152 & 0.0072 \\ -0.0009 & 0.0072 & 0.0159 \end{bmatrix}$$

$$\hat{V}(3) = \begin{bmatrix} 0.0121 & -0.0021 & -0.0008 \\ -0.0021 & 0.0138 & 0.0050 \\ -0.0008 & 0.0050 & 0.0140 \end{bmatrix}, \hat{V}(4) = \begin{bmatrix} 0.0206 & 0.0215 & 0.0162 \\ 0.0215 & 0.0459 & 0.0265 \\ 0.0162 & 0.0265 & 0.0260 \end{bmatrix}$$

for each state $i = 1, 2, 3$ and 4.

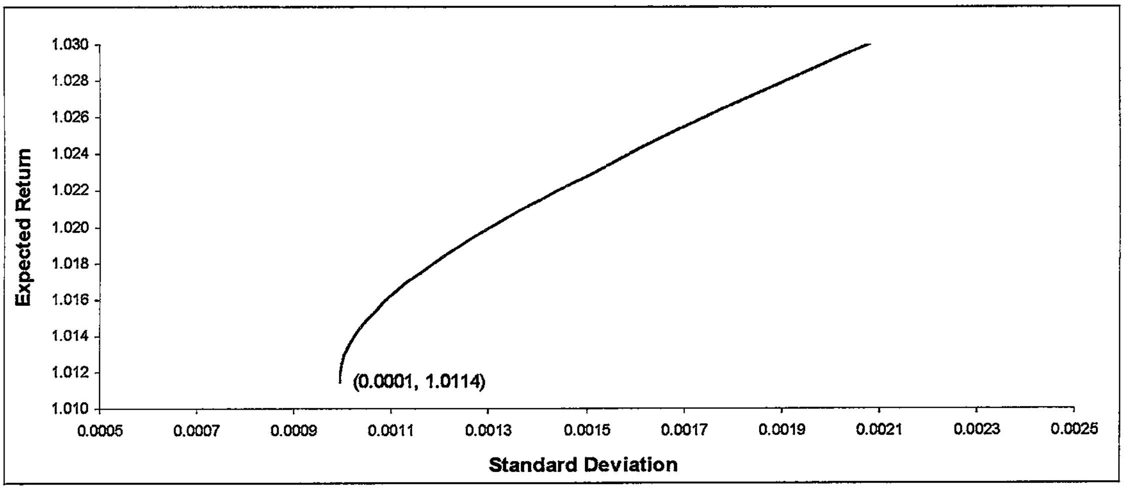

Figure 7.27: Efficient frontier (Case II.3, $T = 4$, $i = 1$)

By using the definitions given in (6.34)-(6.37), the vectors $\hat{f}(i)$, $\hat{g}(i)$ and $\hat{h}(i)$ are calculated as follows

$$\hat{f}(i) = \begin{bmatrix} 0.1061 \\ 0.8831 \\ 0.9538 \\ 0.2656 \end{bmatrix}, \ \hat{g}(i) = \begin{bmatrix} 0.1060 \\ 0.8816 \\ 0.9502 \\ 0.2643 \end{bmatrix}, \ \hat{h}(i) = \begin{bmatrix} 0.8941 \\ 0.1200 \\ 0.0534 \\ 0.7370 \end{bmatrix}$$

for $i = 1, 2, 3$ and 4.

Then, by using the vectors $\hat{f}(i)$, $\hat{g}(i)$ and $\hat{h}(i)$ and the transition probability matrix $\hat{Q}$ of the observed Markov chain, $\hat{a}_1(i)$, $\hat{a}_2(i)$ and $\hat{b}(i)$ defined in (6.44)-(6.46) are calculated as

$$\hat{a}_1(i) = \begin{bmatrix} 0.0097 \\ 0.0801 \\ 0.0869 \\ 0.0316 \end{bmatrix}, \ \hat{a}_2(i) = \begin{bmatrix} 0.0098 \\ 0.0811 \\ 0.0882 \\ 0.0321 \end{bmatrix}, \ \hat{b}(i) = \begin{bmatrix} 0.4952 \\ 0.4604 \\ 0.4572 \\ 0.4844 \end{bmatrix}$$

for $i = 1, 2, 3$ and 4.

After calculating $\hat{a}_1(i)$, $\hat{a}_2(i)$ and $\hat{b}(i)$, the efficient frontier that an investor observes at time zero for $T = 4$ given that the initial state of the observed process is in state $i = 1$ is shown in Fig. 7.27 by using (6.47).

The optimal initial portfolio for $\hat{\mu} = 1.04^4 \cong 1.1699$ is calculated by using (6.52) as

$$\hat{u}_0(1,1) = \begin{bmatrix} -0.5823 \\ -0.2368 \\ -0.4598 \end{bmatrix}$$

for initial state of $i = 1$ and wealth of $x_0 = 1$. Therefore, at time zero the investor should shortsell three risky assets and invest in the riskless asset by $0.5823 + 0.2368 + 0.4598 + 1 = 2.278$ in the initial period. State $i = 1$ can be observed only when the unobserved process is in state $a = 1$ and the mean returns of the risky asset is less than the mean return of the riskless asset in state $a = 1$ as it is seen in the first row of the Table 7.9. Therefore, the investor prefers to shortsell the risky assets and invest in the riskless asset to earn $\hat{\mu} \cong 1.1699$. For the same $\hat{\mu}$, the optimal initial portfolio is calculated as

$$\hat{u}_0(3,1) = \begin{bmatrix} -0.0219 \\ 0.2721 \\ 0.1074 \end{bmatrix}$$

for initial state of $i = 3$ and wealth of $x_0 = 1$. Therefore, at time zero the investor should shortsell the first risky asset by 0.0219 and invest in the other risky assets by 0.2721 and 0.1074 and the riskless asset by $0.0219 - 0.2721 - 0.1074 + 1 = 0.6424$ in the initial period. State $i = 3$ can be observed when the unobserved process is in state $a = 2$ with probability 0.8 and in state $a = 3$ with probability 0.7. In order to explain why the first asset is shortselled, we should look at the mean returns of the assets given that the observed process is in state $i = 3$ rather than looking at the returns in state $a = 2$ or in state $a = 3$. One can calculate mean excess returns of the assets given that the observed process is in state $i = 3$ as

$$\hat{r}^e(i) = \begin{bmatrix} -0.0055 \\ 0.0258 \\ 0.0171 \end{bmatrix}.$$

Therefore, the investor should shortsell the first risky asset since it has a negative excess return of -0.0055. He invests in the other risky assets which offer positive excess return.

Chapter 8

CONCLUSION

In this thesis, a multiperiod portfolio optimization problem in a stochastic market with imperfect information flow is studied. The market consists of risky assets and a riskless asset whose returns depend on the economic conditions that define the states of a Markov chain; therefore, they are serially correlated with each other via prevailing market conditions which are hidden to investors. The main objective is to come up with the optimal solution to the multiperiod mean-variance formulation.

First, the stochastic market process modulated by a hidden Markov chain is described, and two models regarding the relationship between the observed market process and the actual stochastic market process are constructed. The multiperiod mean-variance model formulation in a perfectly observable market and the corresponding dynamic programming formulation are given. These are then extended to the multiperiod mean-variance model formulation in an imperfectly observable market. The efficient frontier equations are obtained. Finally, some numerical illustrations are provided to demonstrate the solution procedure and efficient frontiers for different levels of information.

In our multiperiod model, one can find the optimal investment policy for the entire planning horizon; however, policies other than the current period are not implemented. At the beginning of each period, the problem is resolved with the new input data if necessary, and only the optimal investment policy for the current period is implemented. Moreover, the optimal investment policy depends on the transition matrix of the hidden Markov chain, observation or emission matrices, mean return of risky assets, return of the risk free asset and covariance matrices of returns of risky assets. In this thesis, we mainly focus on the effect of observation and emission matrices on the optimal investment policy, so the other factors are kept unchanged in the numerical illustrations.

In this study, we assumed that mean return of each asset including the risk free asset, covariance matrices, and the transition matrix of the unobserved market process are known.

Moreover, number of states in the observed and unobserved processes are assumed to be given. To relax these assumptions one can use Bayesian analysis in future work.

In addition, our study does not consider transaction costs which could have a significant effect on the optimal solution of the problem in reality, so transaction costs can be taken into account. However, the inclusion of transaction costs will change the wealth dynamics equation. This may make it very difficult to find an explicit solution to the optimization problem $P4(\lambda, \omega)$ using dynamic programming, and one may have to use other computational methods involving simulation or heuristic procedures.

Finally, our study considers discrete and finite time horizons, so continuous time and infinite time horizon models can also be studied. In the continuous time setting, note that another important issue is the determination of the time at which the portfolio should be changed. This will further complicate the formulation and solution of the problem.

Appendix A

MATLAB CODES

A.1 Model I

```
%Enter the covariance matrix of returns of the risky assets
   for each state "a".
   %The most riskiest state of the stochastic market is a=1.
   %The least riskiest state of the stochastic market is a=3.
   covariance{1} = [0.0009];
   covariance{2} = [0.0006];
   covariance{3} = [0.0003];
   %Mean returns of assets for each state "a".
   %Please be aware that they are not rate of returns but returns (1+rate of
return).
   returnn{1} = [1.08 ];%small_r_sub_k_(a)
   returnn{2} = [1.04 ];%small_r_sub_k_(a)
   returnn{3} = [1.02];%small_r_sub_k_(a)
   %Return of the riskless asset for each state "a".
   returnn_free{1}=[1.03];%small_r_sub_f_(a)
   returnn_free{2}=[1.02];%small_r_sub_f_(a)
   returnn_free{3}=[1.005];%small_r_sub_f_(a)
   %Enter the return of the riskless asset observed:  r_sub_f{i}
   %Please note that you should enter returnn_free{i} where i=1,2, ..ny.
   returnn_free{1}=[1.03];%small_r_sub_f_(i)
   returnn_free{2}=[1.005];
   %Enter the investor's expected wealth at the end of investment horizon.
   %"mu" is used for determining optimal portfolio policy at time zero for
   %the next period.
```

```
mu=2
% m :number of risky assets
m=length(covariance{1});
%Transition matrix of the unobserved market process Z
%which is assumed to be a Markov chain :  Q(a,b)
Q=[.9 .09 .01;.8 .15 .05; .7 .2 .1];
%Conditional probability:  O(i,a) = P{Z=a|Y=i}
% O :Observation Matrix which involves conditional probabilites defined
p=0.01;
O=[p (1-p) 0; 0 0 1];
%Number of observed "ny" and unobserved "nz" states
ny=size(O,1);
nz=size(O,2);
%Number of time periods till the investment horizon ends
T=4;
%Set up the information structure
s{1}=[1 , 2];
s{2}=[3];
I=zeros(nz,ny);
for j=1:ny
for a=1:nz
for b=s{j}
I(a,j)=I(a,j) + Q(a,b);
end
end
end
%Calculate the probability transition matrix of the observed market
%process "Q_hat"
Q_hat=zeros(ny,ny);
for i=1:ny
for j=1:ny
```

```
for a=1:nz
Q_hat(i,j)=Q_hat(i,j)+O(i,a)*I(a,j);
end
end
end
%Store "covariance, returnn,..  etc." matrices into single matrices
%"covariances, returns, ..etc."
covariances=cell(nz,1);
returns=cell(nz,1);
for a=1:nz
covariances(a)=covariance(a);
returns(a)=returnn(a);
end
returns_free=cell(1,ny);
for i=1:ny
returns_free(1,i)=returnn_free(i);
end
%==================
%Now we will find the covariance matrix when we start at any state in
%Y denoted by "i",
%and when we arrive a state in Z denoted by "a".
mycov=zeros(m,m,ny); %cov_hat_sub_k_l
cov=covmultip(O,covariances);%calculates sigma_hat_k_l_(i)
ret1=ret1multip(O,returns);%calculates returns_hat_sub_k_l_(i)
%calculates returns_hat_k_(i)*returns_hat_l_(i)
ret2=ret2multip(O,r_hat_single(O,returns))
for i=1:ny
for k=1:m
for l=1:m
mycov(k,l,i)=(cov{i}(k,l)+ret1{i}(k,l)-ret2{i}(k,l)) ;
end
```

```
end
end
%==================
V_hat=zeros(m,m,ny);
%calculates returns_excess_hat_sub_k_l_(i)
ret3=ret3multip(0,returns,returns_free)
ret4=returns_free;%calculates returns_free_k(i)
ret5=r_hat_single(0, returns);%calculates returns_hat_k(i)
for i=1:ny
for k=1:m
for l=1:m
V_hat(k,l,i)= cov{i}(k,l) + ret3{i}(k,l) ;%V_hat
end
end
end
for i=1:ny
h_hat{i}= (ret5{i}-ret4{i})*inv(V_hat(:,:,i))*(ret5{i}-ret4{i})';
g_hat{i}= ret4{i}*(1-h_hat{i});
f_hat{i}= (ret4{i}^2)*(1-h_hat{i});
all_g_hat(i,1)=g_hat(i);
all_f_hat(i,1)=f_hat(i);
all_h_hat(i,1)=h_hat{i};
end
%Calculate Q_hat_sub_fhat_bar and Q_hat_sub_ghat_bar for any power.
%We will use them to calculate a1_hat, a2_hat and b_hat.
Q_sub_ghat_power_zero_bar=ones(ny,1);
for i=1:ny
Q_sub_ghat_power_bar(i,1)=0;
for j=1:ny
Q_sub_ghat_power(i,j,1)=Q_sub_ghat_power_zero_bar(j,1)*Q_hat(i,j)*g_hat{j};
Q_sub_ghat_power_bar(i,1)=Q_sub_ghat_power_bar(i,1)
```

```
                                    +Q_sub_ghat_power(i,j,1);
end
end
for t=2:T
for i=1:ny
Q_sub_ghat_power_bar(i,t)=0;
for j=1:ny
Q_sub_ghat_power(i,j,t)=Q_sub_ghat_power_bar(j,t-1)*Q_hat(i,j)*g_hat{j};
Q_sub_ghat_power_bar(i,t)=Q_sub_ghat_power_bar(i,t)
                                    +Q_sub_ghat_power(i,j,t);
end
end
end
%------------
Q_sub_fhat_power_zero_bar=ones(ny,1);
for i=1:ny
Q_sub_fhat_power_bar(i,1)=0;
for j=1:ny
Q_sub_fhat_power(i,j,1)=Q_sub_fhat_power_zero_bar(j,1)*Q_hat(i,j)*f_hat{j};
Q_sub_fhat_power_bar(i,1)=Q_sub_fhat_power_bar(i,1)
                                    +Q_sub_fhat_power(i,j,1);
end
end
for t=2:T
for i=1:ny
Q_sub_fhat_power_bar(i,t)=0;
for j=1:ny
Q_sub_fhat_power(i,j,t)=Q_sub_fhat_power_bar(j,t-1)*Q_hat(i,j)*f_hat{j};
Q_sub_fhat_power_bar(i,t)=Q_sub_fhat_power_bar(i,t)
                                    +Q_sub_fhat_power(i,j,t);
end
```

```
end
end
%Let's calculate a1_hat and a2_hat for each state i.
a1_hat=dot_product(Q_hat,Q_sub_ghat_power_bar(:,T-1),all_g_hat);
a2_hat=dot_product(Q_hat,Q_sub_fhat_power_bar(:,T-1),all_f_hat);
%Let's calculate b_hat for each state i.
for i=1:ny
for k=1:T-1
upper_term{k}=Q_sub_ghat_power_bar(:,T-k);
lower_term{k}=Q_sub_fhat_power_bar(:,T-k);
inner_term_of_dot_product(i,1,k)=(upper_term{k}(i,1)^2/lower_term{k}(i,1));
end
end
for k=T
upper_term{k}=ones(ny,1);
lower_term{k}=ones(ny,1);
for i=1:ny
inner_term_of_dot_product(i,1,k)=(upper_term{k}(i,1)^2/lower_term{k}(i,1));
end
end
for k=1:T
term_of_dot_product(:,1,k)=inner_term_of_dot_product(:,1,k).*(all_h_hat);
end
for k=2:T
Q_times_term_of_dot_product(:,:,k)=Q_hat^(k-1)*term_of_dot_product(:,1,k);
end
Q_power_zero=eye(ny);
for k=1
Q_times_term_of_dot_product(:,:,k)=Q_power_zero(:,:)
                                        *term_of_dot_product(:,1,k);
end
```

```
b_hat=0;
for k=1:T
b_hat=b_hat+0.5*(Q_times_term_of_dot_product(:,:,k));
k=k+1;
end
%==========
%Let's find efficient portfolios.
x=1;%Initial wealth
syms expected_return
port_variance=cell(nz,1);
for i=1:ny
port_variance{i}=((a2_hat(i,1)-(a1_hat(i,1)^2/(1-2*b_hat(i,1))))*x^2
+ (((1-2*b_hat(i,1))*expected_return)-a1_hat(i,1)*x)^2
/(2*(b_hat(i,1)-2*b_hat(i,1)^2)));
end
for i=1:ny
;%standard deviation of the minimum variance portfolio
x2{i}=sqrt((a2_hat(i)-(a1_hat(i)^2/(1-2*b_hat(i))))*(x^2))
%expected return of the minimum variance portfolio
y2{i}=(a1_hat(i)*x)/(1-2*b_hat(i));
end
for i=1:ny
d=1;
for any_expected_return=(y2{i}):0.001:(1.2*(y2{i}))
any_std{i}(1,d)=subs(sqrt(port_variance{i}),
                    expected_return,any_expected_return);
d=d+1;
end
end
for i=1:ny
any_expected_return_vectorr{i}=(y2{i}):0.001:(1.2*(y2{i}));
```

```
end
%SAVE any_std{1} output as an excel file
saved_output_y_axes=any_expected_return_vectorr{1};
saved_output_x_axes=any_std{1};
save mymulti13_y.xls saved_output_y_axes -ascii -tabs
save mymulti13_x.xls saved_output_x_axes -ascii -tabs
%How should we invest 1 Dollar (x=1) at t=0 for having "mu"
%amount of wealth at the end of T period ?
%For solving this, we use solve P2 problem:  Minimizing variance.
%This is why we entered mu at the beginning of the code.
%for a given mu find w used in optimal port.  policy.  x=1
for i=1:ny
ww(i,1)=b_hat(i,1)/((1-2*b_hat(i,1))*mu-a1_hat(i,1)*x);
lambda_star(i,1)=(1+2*ww(i,1)*a1_hat(i,1)*x)/(ww(i,1)*(1-2*b_hat(i,1)));
upper1=Q_bar((Q_sub(Q_hat,all_g_hat))^(T-1));
lower1=Q_bar((Q_sub(Q_hat,all_f_hat))^(T-1));
u{i}= ((lambda_star(i,1)/2)*(upper1(i,1)/lower1(i,1)) - ret4{1,i}.*x)*
(inv(V_hat(:,:,i))*(ret5{i}-ret4{i})'); %n=0.
end
%Plot efficient frontiers for each state i.
for i=1:ny
figure
%plot((any_std{i}),(any_expected_return_vectorr{i}),(x2{i}),(y2{i}),'+')
plot((any_std{i}),(any_expected_return_vectorr{i}))
title(['Plot of The Efficient Frontier i=',num2str(i),' for T=',
                              num2str(T)],'FontSize',10)
text(((x2{i})),((y2{i})+0.005),['Optimal Portfolio Strategy at t=0:
u(i:',num2str(i),'x:',num2str(x),') = (',num2str((u{i})'),')'],
                                                      'FontSize',9)
text(((x2{i})),((y2{i})),['\leftarrow min var point:(', num2str((x2{i})),
' , ',num2str((y2{i})),')'],'FontSize',9)
```

```
xlabel('Std.  Dev.')
ylabel('Expected Return')
end
```

A.2 Model II

```
%Enter the covariance matrix of returns of the risky assets
   for each state "a".
   %The most riskiest state of the stochastic market is a=1.
   %The least riskiest state of the stochastic market is a=3.
   covariance{1} = [0.0009];
   covariance{2} = [0.0006];
   covariance{3} = [0.0003];
   %Mean returns of assets for each state "a".
   %Please be aware that they are not rate of returns but returns (1+rate of
return).
   returnn{1} = [1.08 ];%small_r_sub_k_(a)
   returnn{2} = [1.04 ];%small_r_sub_k_(a)
   returnn{3} = [1.02];%small_r_sub_k_(a)
   %Enter the return of the riskless asset observed:  r_sub_f{i}
   %Please note that you should enter returnn_free{i} where i=1,2, ..ny.
   returnn_free{1}=[1.03];%small_r_sub_f_(a)
   returnn_free{2}=[1.02];%small_r_sub_f_(a)
   returnn_free{3}=[1.005];%small_r_sub_f_(a)
   %Enter the investor's expected wealth at the end of investment horizon.
   %"mu" is used for determining optimal portfolio policy at time zero for
   %the next period.
   mu=2
   % m :number of risky assets
   m=length(covariance{1});
   %Transition matrix of the unobserved market process Z
   %which is assumed to be a Markov chain :  Q(a,b)
```

```
Q=[.9 .09 .01;.8 .15 .05; .7 .2 .1];
%E :Emission Matrix which involves conditional probabilites of P{Y=i|Z=a}
p=0.01;
E=[1 0 0; 0 p (1-p); 0 0 1];
%Number of states
ny=size(E,2);
nz=size(E,1);
%Conditional probability:  P{Z=a|Y=i}
% O :Observation Matrix which involves conditional probabilites defined
% above.  It is calculated according to a defined rule (weighted sum).
E_column_sum=zeros(1,ny);
for i=1:ny
for a=1:nz
E_column_sum(1,i) = E_column_sum(1,i)+E(a,i) ;
end
end
for i=1:ny
for a=1:nz
O(i,a)=E(a,i)/E_column_sum(1,i) ;
end
end
%Number of time periods till the investment horizon ends
T=4;
%Calculate the probability transition matrix of the observed market
%process "Q_hat"
Q_hat=zeros(ny,ny);
Q_hat_inner=zeros(nz,ny);
for a=1:nz
for j=1:ny
for b=1:nz
Q_hat_inner(a,j)=Q_hat_inner(a,j)+Q(a,b)*E(b,j);
```

```
end
end
end
for i=1:ny
for j=1:ny
for a=1:nz
Q_hat(i,j) = Q_hat(i,j) + O(i,a)*Q_hat_inner(a,j);
end
end
end
%Store "covariance, returnn,..  etc." matrices into single matrices
%"covariances, returns, ..etc."
covariances=cell(nz,1);
returns=cell(nz,1);
for a=1:nz
covariances(a)=covariance(a);
returns(a)=returnn(a);
end
returns_free=cell(1,ny);
for i=1:ny
returns_free(1,i)=returnn_free(i);
end
%===================
%Now we will find the covariance matrix when we start at any state in Y
%denoted by "i",
%and when we arrive a state in Z denoted by "a".
mycov=zeros(m,m,ny); %cov_hat_sub_k_l
cov=covmultip(O,covariances);%calculates sigma_hat_k_l_(i)
ret1=ret1multip(O,returns);%calculates returns_hat_sub_k_l_(i)
%calculates returns_hat_k_(i)*returns_hat_l_(i)
ret2=ret2multip(O,r_hat_single(O,returns));
```

```
for i=1:ny
for k=1:m
for l=1:m
mycov(k,l,i)=(cov{i}(k,l)+ret1{i}(k,l)-ret2{i}(k,l)) ;
end
end
end
%==================
V_hat=zeros(m,m,ny);
%ret3 calculates returns_excess_hat_sub_k_l_(i)
ret3=ret3multip(0,returns,returns_free);
ret4=returns_free;%calculates returns_free_k(i)
ret5=r_hat_single(0, returns);%calculates returns_hat_k(i)
for i=1:ny
for k=1:m
for l=1:m
V_hat(k,l,i)= cov{i}(k,l) + ret3{i}(k,l) ;%V_hat
end
end
end
for i=1:ny
h_hat{i}= (ret5{i}-ret4{i})*inv(V_hat(:,:,i))*(ret5{i}-ret4{i})';
g_hat{i}= ret4{i}*(1-h_hat{i});
f_hat{i}= (ret4{i}^2)*(1-h_hat{i});
all_g_hat(i,1)=g_hat(i);
all_f_hat(i,1)=f_hat(i);
all_h_hat(i,1)=h_hat{i};
end
%Calculate Q_hat_sub_fhat_bar and Q_hat_sub_ghat_bar for any power.
%We will use them to calculate a1_hat, a2_hat and b_hat.
Q_sub_ghat_power_zero_bar=ones(ny,1);
```

```
for i=1:ny
Q_sub_ghat_power_bar(i,1)=0;
for j=1:ny
Q_sub_ghat_power(i,j,1)=Q_sub_ghat_power_zero_bar(j,1)*Q_hat(i,j)*g_hat{j};
Q_sub_ghat_power_bar(i,1)=Q_sub_ghat_power_bar(i,1)
                                   +Q_sub_ghat_power(i,j,1);
end
end
for t=2:T
for i=1:ny
Q_sub_ghat_power_bar(i,t)=0;
for j=1:ny
Q_sub_ghat_power(i,j,t)=Q_sub_ghat_power_bar(j,t-1)*Q_hat(i,j)*g_hat{j};
Q_sub_ghat_power_bar(i,t)=Q_sub_ghat_power_bar(i,t)
                                   +Q_sub_ghat_power(i,j,t);
end
end
end
%------------
Q_sub_fhat_power_zero_bar=ones(ny,1);
for i=1:ny
Q_sub_fhat_power_bar(i,1)=0;
for j=1:ny
Q_sub_fhat_power(i,j,1)=Q_sub_fhat_power_zero_bar(j,1)*Q_hat(i,j)*f_hat{j};
Q_sub_fhat_power_bar(i,1)=Q_sub_fhat_power_bar(i,1)
                                   +Q_sub_fhat_power(i,j,1);
end
end
for t=2:T
for i=1:ny
Q_sub_fhat_power_bar(i,t)=0;
```

```
for j=1:ny
Q_sub_fhat_power(i,j,t)=Q_sub_fhat_power_bar(j,t-1)*Q_hat(i,j)*f_hat{j};
Q_sub_fhat_power_bar(i,t)=Q_sub_fhat_power_bar(i,t)
                                    +Q_sub_fhat_power(i,j,t);
end
end
end
%Let's calculate a1_hat and a2_hat for each state i.
a1_hat=dot_product(Q_hat,Q_sub_ghat_power_bar(:,T-1),all_g_hat);
a2_hat=dot_product(Q_hat,Q_sub_fhat_power_bar(:,T-1),all_f_hat);
%Let's calculate b_hat for each state i.
for i=1:ny
for k=1:T-1
upper_term{k}=Q_sub_ghat_power_bar(:,T-k);
lower_term{k}=Q_sub_fhat_power_bar(:,T-k);
inner_term_of_dot_product(i,1,k)=(upper_term{k}(i,1)^2/lower_term{k}(i,1));
end
end
for k=T
upper_term{k}=ones(ny,1);
lower_term{k}=ones(ny,1);
for i=1:ny
inner_term_of_dot_product(i,1,k)=(upper_term{k}(i,1)^2/lower_term{k}(i,1));
end
end
for k=1:T
term_of_dot_product(:,1,k)=inner_term_of_dot_product(:,1,k).*(all_h_hat);
end
for k=2:T
Q_times_term_of_dot_product(:,:,k)=Q_hat^(k-1)*term_of_dot_product(:,1,k);
end
```

```
Q_power_zero=eye(ny);
for k=1
Q_times_term_of_dot_product(:,:,k)=Q_power_zero(:,:)
                                    *term_of_dot_product(:,1,k);
end
b_hat=0;
for k=1:T
b_hat=b_hat+0.5*(Q_times_term_of_dot_product(:,:,k));
k=k+1;
end
%==========
%Let's find efficient portfolios.
x=1;%Initial wealth
syms expected_return
port_variance=cell(nz,1);
for i=1:ny
port_variance{i}=((a2_hat(i,1)-(a1_hat(i,1)^2/(1-2*b_hat(i,1))))*x^2
+ (((1-2*b_hat(i,1))*expected_return)-a1_hat(i,1)*x)^2
/(2*(b_hat(i,1)-2*b_hat(i,1)^2)));
end
for i=1:ny
%standard deviation of the minimum variance portfolio
x2{i}=sqrt((a2_hat(i)-(a1_hat(i)^2/(1-2*b_hat(i))))*(x^2));
%expected return of the minimum variance portfolio
y2{i}=(a1_hat(i)*x)/(1-2*b_hat(i));
end
for i=1:ny
d=1;
for any_expected_return=(y2{i}):0.001:(1.2*(y2{i}))
any_std{i}(1,d)=subs(sqrt(port_variance{i}),
        expected_return,any_expected_return);
```

```
d=d+1;
end
end
for i=1:ny
any_expected_return_vectorr{i}=(y2{i}):0.001:(1.2*(y2{i}));
end
%SAVE any_std{1} output as an excel file
saved_output_y_axes=any_expected_return_vectorr{1};
saved_output_x_axes=any_std{1};
save mymulti13_y.xls saved_output_y_axes -ascii -tabs
save mymulti13_x.xls saved_output_x_axes -ascii -tabs
%How should we invest 1 Dollar (x=1) at t=0 for having "mu"
%amount of wealth at the end of T period ?
%For solving this, we use solve P2 problem:  Minimizing variance.
%This is why we entered mu at the beginning of the code.
%for a given mu find w used in optimal port.  policy.  x=1
for i=1:ny
ww(i,1)=b_hat(i,1)/((1-2*b_hat(i,1))*mu-a1_hat(i,1)*x);
lambda_star(i,1)=(1+2*ww(i,1)*a1_hat(i,1)*x)/(ww(i,1)*(1-2*b_hat(i,1)));
upper1=Q_bar((Q_sub(Q_hat,all_g_hat))^(T-1));
lower1=Q_bar((Q_sub(Q_hat,all_f_hat))^(T-1));
u{i}= ((lambda_star(i,1)/2)*(upper1(i,1)/lower1(i,1)) - ret4{1,i}.*x)
*(inv(V_hat(:,:,i))*(ret5{i}-ret4{i})'); %n=0.
end
%Plot efficient frontiers for each state i.
for i=1:ny
figure
%plot((any_std{i}),(any_expected_return_vectorr{i}),(x2{i}),(y2{i}),'+')
plot((any_std{i}),(any_expected_return_vectorr{i}))
title(['Plot of The Efficient Frontier i=',num2str(i),' for T=',
                              num2str(T)],'FontSize',10)
```

```
text(((x2{i})),((y2{i})+0.005),['Optimal Portfolio Strategy at t=0:
u(i:',num2str(i),'x:',num2str(x),') = (',num2str((u{i})'),')'],
                                'FontSize',9)
text(((x2{i})),((y2{i})),['\leftarrow min var point:(', num2str((x2{i})),'
, ',
num2str((y2{i})),')'],'FontSize',9)
xlabel('Std.  Dev.')
ylabel('Expected Return')
end
```

A.3 Subroutines Used

A.3.1 covmultip

```
function covmultip=f(O,covariances);
%function used in calculating sigma_hat_kl_(i)
m=length(covariances{1});
ny=size(O,1);
nz=size(O,2);
for i=1:ny
for k=1:m
for l=1:m
covmultip{i}(k,l)=0;
for a=1:nz
covmultip{i}(k,l)= covmultip{i}(k,l)+O(i,a).*deal(covariances{a}(k,l));
a=a+1;
end
end
end
end
```

A.3.2 dot_product

```
function dot_product=f(Q,Q_sub_bar,h);
   %function used in calculating array multiplication.  That is
   %i'th elements of two vectors are multiplied and result
   %entered into ith element of a new vector.
   ny=size(Q,1);
   nz=size(Q,2);
   dot_product=zeros(ny,1);
   for i=1:ny
   dot_product(i)=Q_sub_bar(i)*h{i};
   end
```

A.3.3 Q_bar

```
function Q_bar=f(Q);
   %function used in calculating Q_bar
   ny=size(Q,1);
   nz=size(Q,2);
   for i=1:ny
   Q_bar(i,1)=0;
   for j=1:nz
   Q_bar(i,1)=Q_bar(i,1)+Q(i,j);
   j=j+1;
   end
   end
```

A.3.4 Q_sub

```
function Q_sub=f(Q,g);
   %function used in calculating Q_bar
   ny=size(Q,1);
   nz=size(Q,2);
   for i=1:ny
```

```
for j=1:nz
Q_sub(i,j)=Q(i,j)*g{j};
end
end
```

A.3.5 r_hat_single

```
function r_hat_single=f(O,returns);
%function used in calculating r_hat_k_(i)
m=size(returns{1},2);
ny=size(O,1);
nz=size(O,2);
for i=1:ny
for k=1:m
r_hat_single{i}(1,k)=0;
for a=1:nz
r_hat_single{i}(1,k)=r_hat_single {i}(1,k)+O(i,a).*deal(returns{a}(1,k));
a=a+1;
end
end
end
```

A.3.6 ret1multip

```
function ret1multip=f(O,returns);
%function used in calculating sigma_kl_(i)
m=size(returns{1},2);
ny=size(O,1);
nz=size(O,2);
for i=1:ny
for k=1:m
for l=1:m
ret1multip{i}(k,l)=0;
```

```
for a=1:nz
ret1multip{i}(k,l)= ret1multip{i}(k,l)+O(i,a).*deal(returns{a}(1,k))
                                        .*deal(returns{a}(1,l));
a=a+1;
end
end
end
end
```

A.3.7 ret2multip

```
function ret2multip=f(O,data_calculated_by_r_hat_single);
  %function used in calculating r_hat_k_(i)*r_hat_l_(i)
  m=length(data_calculated_by_r_hat_single{1});
  ny=size(O,1);
  nz=size(O,2);
  for i=1:ny;
  for k=1:m;
  for l=1:m;
  ret2multip{i}(k,l)=0;
  for a=1:nz;
  ret2multip{i}(k,l)=ret2multip{i}(k,l)+O(i,a).*deal(data_calculated
  _by_r_hat_single{i}(1,k)).*deal(data_calculated_by_r_hat_single{i}(1,l));
  a=a+1;
  end
  end
  end
  end
```

A.3.8 ret3multip

```
function ret3multip=f(O,returns,returns_free);
  %function used in calculating return_excess_hat_sub_kl_(i)
```

```
m=size(returns{1},2);
ny=size(O,1);
nz=size(O,2);
for i=1:ny
for k=1:m
for l=1:m
ret3multip{i}(k,l)=0;
for a=1:nz
ret3multip{i}(k,l)=ret3multip{i}(k,l)+O(i,a).*deal(
returns{a}(1,k)-returns_free{i}).*deal(returns{a}(1,l)-returns_free{i});
a=a+1;
end
end
end
end
```

Appendix B

EXPLICIT FORM OF SOME EQUATIONS

B.1 $\hat{a}_1(i=1)$ *in Case I.1*

```
((((1/25*p + 19/20)*(103/100 - 103/100*(1/25*p + 1/100)^2/(27/10000*p + 7/10000)) -
191098327984407958487381399578233/8318957063997814557067224023285000*p +
191098327984407958487381399578233/6655165651198251645653779218628000)*
(1/25*p + 19/20)*(103/100 - 103/100*(1/25*p + 1/100)^2/(27/10000*p + 7/10000)) +
191098327984407958487381399578233/332758282559912582282688960931400*
(3275767607314797596247908067412311/3327582825599125822826889609314000 -
927/1000*(1/25*p + 1/100)^2/(27/10000*p + 7/10000))*(- 1/25*p + 1/20))*(1/25*p +
19/20)*
(103/100 - 103/100*(1/25*p + 1/100)^2/(27/10000*p + 7/10000)) +
191098327984407958487381399578233/332758282559912582282688960931400*
((9/10*(1/25*p + 19/20)*(103/100 - 103/100*(1/25*p + 1/100)^2/(27/10000*p + 7/10000)) -
1719884951859671626386432596204097/83189570639978145570672240232850000*p +
1719884951859671626386432596204097/66551656511982516456537792186280000)*
(103/100 - 103/100*(1/25*p + 1/100)^2/(27/10000*p + 7/10000)) +
625993712623342485782450674984133110458376358760953070547111826463/11072807461222262221
464453375826450382550764638207459555550596000000 -
177148150041546177517802557409021991/3327582825599125822826889609314000000*
(1/25*p + 1/100)^2/(27/10000*p + 7/10000))*( - 1/25*p + 1/20))*
(103/100 - 103/100*(1/25*p + 1/100)^2/(27/10000*p + 7/10000))
```

B.2 Mean Returns in Terms of p in Case I.1

```
(1/2) / (-
385644819532628970328963363188431711552942433659688249155331951491996740433740
641201199174354922338036812208331010469914762111721858909747600578317849276447
027398969786140990790652199242459944496327732641501322512694744708471210457743
6125482524377629101269029224448000000000000000000000000*p -
176250633822132656547317600265168730001618142740741053511332221926591031668847
519131948952026164909529544996877194628407827865726123586913436242201309372130
436759830188693609614248682111122680368975556829322383807558202181390883621477
3984564808109172600468453248204800000000000000000000000*p^2 +
27811305703399463793612839873565476782359583509904875575917139004442728249325
123342311834791168839679919024464110062791531878149618205178559176503272969845
814957291814493441925451664381896721427461567347510140680213321697059756561168
633127908942459074113557485322224000000000000000000*p^7 +
439376255102965452884524498236499583097104040685201656902551199201415654395381
910989179427592044580460814897928008022218851338348473134577567016801830104083
329254977667382496023664974152952776082792435916811874901843842123696585990337
46911191787954774901528270471168000000000000000000*p^8 +
204165681055353867625395741124711890343662774961774659116445907867726276915164
665273044260997155867539888174243165115009005911097306731094046774422911863377
307990102399207582080678481158803655375114176336969484571206257669817800411889
8993538750283532895568333583155200000000000000000*p^9 -
271455688633280388546902835255915434717462374918079894104783066629663899222181
441658243163117850939010931810696713758268302490952226083871806745053427314814
127983746337344237524794756550287709012370385435936756037967739629154059255645
620530368622287194963407599042560000000000000000000*p^6 -
465599443410017648830389238125041323320734894509767684840656853887266580951582
402839985552182340243379789014848642553947566632991710318259713126612317431521
755036080181850457577764214019424172880272138646299545368934245696617883982422
7259506695746348423409619802521600000000000000000000000*p^3 ...
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