ADVANCED PLANNING AND SCHEDULING IN SUPPLY CHAIN SYSTEMS

by

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This is to certify that I have examined this copy of a master's thesis by

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and have found that it is complete and satisfactory in all respects,
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To my beloved family

ABSTRACT

Supply chain management covers all of the activities from raw material procurement to final delivery of orders to customers. Since it involves different activities with different nature, planning and scheduling plays an integral part in the operation of the supply chain systems. Traditionally, the supply chain systems were optimized by dividing the planning and scheduling problems into smaller subproblems in order to avoid difficulties in modeling and solution. In addition, the market models and inclusion of logistics related activities also treated separately.

The objective in this thesis is to develop advanced planning and scheduling models for the optimal operation of supply chain systems that include market models, production and inventory models and logistics models

Marketing decisions that affect the level of demand for a product have a very strong impact on the operation of supply chain systems. One of the most important marketing decisions is the price that has a strong influence in the demand level and consequently determines the level of revenue and profit. Therefore, in our perspective, price is considered as a decision variable to adjust the level of demand and counteract the rapid changes in order levels. However, considering price as a decision variable introduces bilinear terms that are nonconvex into the objective function whenever the objective function includes revenue terms.

The logistics decisions and transportation infrastructure imposes some constraints on the operation of a supply chain. The planning and scheduling of operations in production facilities has a profound effect impact on the optimization of supply chains. When all of the activities in a supply chain system are considered in an integrated manner, the resulting model can become intractable due to the fact that these activities occur at different time scales. We also present an advanced modeling approach to effectively integrate these activities.

The resulting planning and scheduling model is a bilinear optimization problem with discrete decision variables. A novel global optimization algorithm is developed to solve this class of problems. The developed advanced planning and scheduling models are illustrated with several examples. The efficiency of the proposed global optimization algorithm is verified on benchmark problems and on a supply chain model.

Keywords: Supply Chain, market model, logistics and transportation model, production model, bilinear global optimization algorithm

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NOMENCLATURE

$\alpha_{j2_i}^{j1_i}$	:	Effective number of units of $j2_i$ required to satisfy the demand for one unit of $j1_i$
θ	:	index of operating modes
Θ_f^n	:	set of operating modes for plant f at node n
$ALT_j^{s,d}$	:	Agreed time of transportation it takes from $s \in NUE_{j_i}$ to $d \in N$ for product j
$B_{\theta,j}^{n,f}(k)$	:	amount produced of product j in mode θ at time k at plant f of node n
$BM_{\theta,j}^{n,f}(k)$	:	integer batch size multiplier
$bp_\theta^{n,f}(k)$	:	$\begin{cases} 1, \text{ if batch size of } B_{\theta,j}^{n,f} \text{ is going to be produced} \\ 0, \text{ otherwise} \end{cases}$
$CM_{KS}^{KT}(ks,kt)$	:	The ratio of material flow of period kt of KT which is mapped to the period ks of KS for continuous mapping
$CML_{K1}^{K2}(k1,k2)$	:	The ratio of material flow of period $k2$ of $K2$ which is mapped to the period $k1$ of $K1$ for continuous mapping regarding material transfer limit
$CR_{\theta,r_j}^{n,f}$	:	consumption rate of raw material j to produce product r_j at plant f of node n in mode θ
$CT_\theta^{n,f}$	:	clean-up time of mode θ at plant f of node n
$DM_{KS}^{KT}(ks,kt)$	:	The ratio of material flow of period kt of KT which is mapped to the period ks KS for discrete mapping
$DML_{K1}^{K2}(k1,k2)$	:	The ratio of material flow of period $k2$ of $K2$ which is mapped to the period $k1$ of $K1$ for discrete mapping regarding material transfer limit
E_n	:	set of external markets which can supply node $n \in N$
e	:	index of external markets
f	:	index of production units in a supply chain node
$f_\theta^{n,f}(k)$	:	$\begin{cases} 1, \text{ if } k \text{ is the time of initiation of mode } \theta \text{ at plant } f \text{ at node } n \\ 0, \text{ otherwise} \end{cases}$
f_j^p	:	Fraction of capacity per period consumed of pipe type p to transport per unit of j
$f_j^{v,t}$	:	Fraction of capacity of vehicle type v in transportation class t that product j consumes
i	:	index of substitute product classes
$I_j^n(k)$	:	available inventory of product j at node $n \in N$ at time k

j_i	:	index of substitute products in class i
j	:	$j = \bigcup_i j_i$
k	:	index of periods in the planning horizon, k $\in \{1,..,K\}$ up to chapter 4
k	:	$k \in \{1,2\}$, index of regions in a partition in chapter 4
$K_{KS}^{KT}(ks)$	:	The ordered subset of periods of KT which are mapped to the period ks of KS
$KO(to^n)$	:	Outsourcing or market selling time scale for $to^n \in TO^n$
$KP(tp^n)$	:	Pipe time scale for $tp^n \in TP^n$
KS	:	The ordered set of periods which contain the period to which other periods are going to be mapped
KT	:	The ordered set of periods which are mapped to the periods in KS
KV	:	Generated common vehicle time scale
$KV(tv)$	:	Generated common vehicle time scale for $tv \in TV$
$KV(tv^n)$	:	Vehicle time scale for $tv^n \in TV^n$
$LT_{v,KV}^{n,d}$	:	Lead time specified in terms of time scale KV
$LT_v^{s,d}$	:	Time of transportation between s $\in NUE_{j_i}$ and $d \in N \cup M$ with vehicle class v
$LT^{s,n}$	:	outsourcing lead time between node $s \in N$ and destination $d \in N \cup M$
m	:	index of markets
M	:	set of markets
$ML_{K1}^{K2}(k1)$	:	The subset of periods of $K2$ which are mapped to the period $k1$ of $K1$ regarding material transfer limit
n	:	index of all nodes excluding external markets and markets
$O_j^n(k)$	:	accumulated undelivered order level for product j at $d \in M$ at time k
p	:	index of pipe types
p_{ij}	:	Associated index of P_{ij}
P_{ij}	:	Number of partitions in the bilinear term of x_i and y_j
$p_\theta^{n,f}(k)$	:	$\begin{cases} 1, & \text{if plant } f \text{ at node } n \text{ is in mode } \theta \text{ at time } k \\ 0, & \text{otherwise} \end{cases}$, for modeling style-1
$p_\theta^{n,f}(k)$	:	$\begin{cases} 1, & \text{if } k \text{ is a production day of mode } \theta \text{ at plant } f \text{ of node } n \\ 0, & \text{otherwise} \end{cases}$ for modeling style-2

$pf_{\theta}^{n,f}(k)$	:	$\begin{cases} 1, & \text{if } k \text{ is the first day of a production campaign of mode } \theta \\ & \text{at plant } f \text{ of node } n \\ 0, & \text{otherwise} \end{cases}$
$pl_{\theta}^{n,f}(k)$	:	$\begin{cases} 1, & \text{if } k \text{ is the last day of a production campaign of mode } \theta \\ & \text{at plant } f \text{ of node } n \\ 0, & \text{otherwise} \end{cases}$
$ps_{\theta}^{n,f}(k)$	:	$\begin{cases} 1, & \text{if batch size variable } B_{\theta,j}^{n,f}(k) \text{ is going to be non-zero} \\ 0, & \text{otherwise} \end{cases}$
$R_{\theta,j}^{n,f}$	:	maximum rate of production for product j in mode θ at production unit f of node n
RM_j	:	Set of products r_j that uses product j as raw material
TO^n	:	Set of set of outsourcing or market selling time scales at node n
TP^n	:	Set of set of type of pipes which use the same pipe time scale at node n
TV	:	Set of set of type of vehicles which use the same inventory time scales
TV^n	:	Set of set of type of vehicles which use the same vehicle time scale at node n
S_n	:	set of markets than can be served by node $n \in N$
$SK_{KS}^{KV}(ks)$	:	The simultaneously starting period of KV for the given period ks of KS, where KS is an inventory time scale
$SK_{KS}^{KV(tv)}(ks)$	:	The simultaneously starting period of $KV(tv)$ for the given period ks of KS where KS is an inventory time scale
$ST_{\theta}^{n,f}$	:	setup time of mode θ at plant f of node n
t_{ij}	:	Estimated value of $x_i y_j$
t_v	:	index of operational modes of vehicle type v
v	:	index of transportation vehicle types
$VI_v^n(k)$	:	available number of vehicles of type $v \in V$ at node $n \in N$ at time k
$VS_{v,t}^{n1,n2}(k)$	:	number of vehicles of type $v \in V$ in mode $t \in T_v$ sent from node $n1 \in N$ to node $n2 \in N \cup M$ at time k
$X1_{\theta}^{n,f}(k)$, $X2_{\theta}^{n,f}(k)$	:	variables which determine the upper bound on batch size of mode θ at plant f at node n based on whether setup time has been incurred or not at time k
$x_i D_{ij}(p_{ij})$	:	coordinates of x_i in for the partitioning generated for $x_i y_j$
$x_i t_{ij}(p_{ij}, k)$	:	value of x in region k of partition p_{ij}

$y_{j1_i,j2_i}^{n,d}(k)$	:	amount of product $j1_i$ transferred from node $n \in N$ to destination $d \in N \cup M$ ordered at time k by outsourcing transportation to satisfy the demand of product $j2_i$
$y_{j1_i,j2_i}^{n,d,v,t}(k)$	:	amount of product $j1_i$ transferred from node $n \in N$ to destination $d \in N \cup M$ at time k in vehicle class $v \in V$ utilizing transportation class $t \in T_v$ to satisfy the demand of product $j2_i$
$y_{j2_i,j1_i}^{n,m}(k)$	:	amount of momentarily delivered $j2_i$ to satisfy the demand of $j1_i$ in market m
$Y1_{\theta}^{n,f}(k)$, $Y2_{\theta}^{n,f}(k)$	:	variables which determine the upper bound on batch size of mode θ at plant f at node n based on whether cleanup time requirement has been started or not at time k
$y_j D_{ij}(p_{ij})$	:	coordinates of y_j in for the partitioning generated for $x_i y_j$
$y_j t_{ij}(p_{ij},k)$	:	value of y in region k of partition p_{ij}
$z_{ij}(p_{ij},k)$	:	$\begin{cases} 1, & \textit{if } x_i \textit{ and } y_j \textit{ assumes a value in region } k \textit{ of partition } p_{ij} \\ 0, & \textit{otherwise} \end{cases}$
$zs_{\theta}^{n,f}(k)$	:	$\begin{cases} 1, & \text{if } X1_{\theta}^{n,f}(k) \text{ is strictly positive} \\ 0, & \text{otherwise} \end{cases}$
$zc_{\theta}^{n,f}(k)$	:	$\begin{cases} 1, & \text{if } Y1_{\theta}^{n,f}(k) \text{ is strictly positive} \\ 0, & \text{otherwise} \end{cases}$

CHAPTER 1 – INTRODUCTION

1.1 Motivation

Supply chain management involves all the steps from raw material procurement to production to distribution and even the end customers, which is also described in the emerging extended enterprise paradigm. From an operational point of view, the coordination of the supply chain involves managing and coordinating many different firms with different operational approaches and environments, which requires close control of all information exchange and material flows as well as other operational interactions.

Supply chain management is defined by Global Supply Chain Forum as follows:

> *"Supply chain management is the integration of key business processes from end user through original raw material suppliers that provides products, services, and information that adds value for customers and other stakeholders. One of the most important transformations of modern business management is that individual businesses do not compete as independent entities, but rather as supply chains. Instead of brand versus brand or store versus store, it is supply chain versus supply chain."* [1]

Lambert and Cooper [1] states that cross-functional integration is necessary and marketing is of critical importance for the sound management of the supply chain systems. They state that the controlling the uncertainty in customer demand, manufacturing processes and supplier performance are critical for the success of the supply chain system. They describe demand management process as one of the keys to success in supply chain management and it is also stressed that the demand management process must balance the customers' requirements with the supply capabilities. This point of view is strengthened by their argument of marketing requirements and production plans should be coordinated on an enterprise-wide basis. Lambert and Cooper also mention customer order fulfillment process. And performing order fulfillment process requires the integration of the manufacturing, distribution and transportation plans from their perspective. Therefore, the logical conclusion is that marketing requirements, manufacturing, distribution and transportation plans must be integrated.

The firms are experiencing increased competition with the effect of globalization. Firms must address customer needs and satisfy the needs with the most suitable products, which are required to be more and more customized in order to keep and strengthen their positions. This, in turn, requires that the supply chain system must be as quick and flexible as possible while maintaining low cost and high quality, which implies that the management of the supply chain is of critical importance.

Demand plays an important role in supply chain management problems since it is the input to the whole supply chain system; the stability and the performance of the supply chain system is closely related to its ability to cope with the possible changes in demand, which implies that operational decisions are becoming more and more dependent on the changes in the market. Demand changes rapidly due to several factors including competition, mass customization and fast communication tools. Demand is not totally independent of the decisions made by the firms that form the supply chain. For example, in some markets demand may be highly dependent on the price and a price regulation might prove useful from an operational perspective. In others substitute products may be accepted with little penalty, and those products may be accepted in some other markets as well, which may imply that the supply chain may take advantage of economies of scale. Therefore, operational decisions are dependent on the structure of the markets; and the information and variables at the disposal of the supply chain that can affect the demand must be considered in operational decisions. This point of view is also consistent with extended enterprise paradigm and the definition of supply chain management, which includes the customers. Since operational decisions are dependent on the structure of the market and structure of the market includes demand behavior and demand may be subject to rapid changes, operational decisions must be made as precise as possible to get the maximum performance of the supply chain system.

The globalization and customization trends offer stiffer competition, and in order to cope with this competition supply chain systems must be as flexible and quick as possible while maintaining quality at low cost. This requires close coordination between the firms that form the supply chain. Furthermore, the operational performance of the supply chain is not independent of demand and therefore, the management of demand is an important success factor in supply chain management; and the information and decision variables at disposal which regulates the demand must be made use of whenever appropriate to increase the preciseness of the operational decisions.

In this thesis, the objective is to develop an integrated supply chain system model that includes market models and increases the accuracy of the operational decisions. Introducing market models in the supply chain model may lead to mixed integer non-linear programming

(MINLP) formulations. Furthermore, if price is included as a decision variable and the amount of material transferred to the market is independent of the price, any objective function including revenue becomes a bilinear function, henceforth making the resulting MINLP formulation non-convex. Therefore, another objective in this thesis is to develop an algorithm for the global optimal solution of the bilinear MINLP supply chain optimization problems.

1.2 Literature Survey

The literature survey consists of two parts due to the nature of the thesis. In the first part supply chain literature is presented and in the second part bilinear optimization approaches are discussed.

1.2.1 Supply Chain Management Literature

Due to the stiffening competition in many markets, supply chain optimization problems have received considerable interest in the past decade, although the pioneering studies have started in 1950s.

Kunreuther and Schrage [2] developed an algorithm for determining pricing and ordering decisions of a firm subject to deterministic demand curve which changes from one period to another. The firm produces only one product and tries to maintain the same price throughout the planning horizon. A detailed view on pricing and marketing decision variables is presented in section 3.1.6.

Chen and Chu [3] developed a mathematical model to solve the matching problem between production and demand when demand possesses the property of linearity. The model is directed towards evaluating the production rate which may affect the inventory level and the optimal sales rate at each point on the planning horizon to realize the cumulative maximum profit. The model includes a manufacturer and linear demand faced by the manufacturer. The applied modeling technique makes it possible to solve for analytical optimal solutions. They conclude that for the supply chain system considered, the demand curve can have a significant effect on the performance of the supply chain in terms of profit.

Timpe and Kallrath [4] developed a multi-period multi-site mixed-integer linear programming (MILP) based supply chain system formulation that models production and distribution in a supply chain. One of the most important features of their model is to associate different time scales to production activities and commercial activities. Their model supports several objective functions including maximizing total sales, minimizing costs and maximizing contribution margin. However, they did not consider price as a driver and also did not account for the transportation systems.

Schneeweiss and Zimmer [5] analyzed the coordination mechanism between a producer and supplier, both having private information. Although they assumed that both the producer and the supplier have private information, they also assumed that these do not behave antagonistically. This paper considers a decentralized coordination mechanism and does not include the transportation and marketing aspects of supply chain systems.

Ishii, Takahashi and Muramatsu [6] analyzed the methods of determining the base stock levels and lead times for production in an integrated production, inventory and distribution system without explicitly modeling transportation and marketing aspects. The integrated production, inventory and distribution system has a pull type operation and consists of a manufacturer, a wholesaler and a retailer. They developed a method to minimize the dead stock levels while preventing stock outs.

Perea-Lopez, Ydstie and Grossmann [7] developed a dynamic hybrid simulation model to analyze the characteristics of the classical decentralized control schemes and showed it is possible to reproduce the problems observed in real systems such as demand amplification problem.

Bose and Pekny [8] compared decentralized and centralized coordination schemes on a single product supply chain. Model predictive control is implemented to meet the target customer service level while minimizing average inventory. They concluded that based on their study centralized coordination scheme provided more satisfactory results than that of the decentralized coordination scheme.

Perea-Lopez, Ydstie and Grossmann [9] developed a decision support model to find the optimal values of decision variables for maximizing profit in a supply chain with multiple plants, wholesalers and retailers based on the implementation of model-predictive control strategy applied on an MILP model.

Mestan [10] presented a model predictive control approach to effectively model discrete-continuous dynamic nature of the supply chain system in different control schemes including centralized, semi-decentralized and decentralized configurations. Stochastic demand behavior is assumed and forecasted by using Kalman filter and then these demand values are passed to the optimization model. This approach is applied for a given length of demand update interval.

Seferlis and Giannelos [11] worked on multi-echelon supply chain systems, and proposed an optimization based control approach for those systems. Their control strategy applies multi-variable model predictive control principles to the whole supply chain system. The objective is to meet the customer demand while minimizing the operating costs.

Lee, Kim and Moon [12] approached production-distribution planning in a supply chain by utilizing a hybrid approach which utilizes both simulation and analytical solutions. They developed a problem solving method which combines both analytic and simulation methods. They applied this procedure on a multi-period, multi-product and multi-shop production-distribution environment.

Lee, So and Tang [13] focused on the value of information sharing in a supply chain, more specifically value of demand information sharing in a two-level supply chain. They quantified the benefits of information sharing and showed that when the demands are significantly correlated using analytical models. This paper concluded that the value of information sharing can be very high.

Lakhal et al. [14] treated supply chain systems as a network of activities. They postulate that a superior supply chain is the one which maximizes the value added by the internal activities throughout the supply chain. An activity is a function which consumes resources to generate products. In their model they classified activities into internal and external activities. Hence, the production distribution network is modeled as a network of activities and the objective is to maximize the value added.

Petrovic, Roy and Petrovic [15] introduced the concept of fuzzy sets to deal with the uncertainties present in the supply chain. They envisioned supply chain as a network of facilities, which carry out the basic supply chain functions such as procurement of raw materials. They constructed a supply chain consisting of serially connected nodes and modeled this system to determine the order levels for each inventory while minimizing total cost for decentralized and partially coordinated supply chain systems. An extension of their work which allows uncertainty in lead times is also presented [16].

Arntzen et al. [17], proposed an MILP model for the design of a multi-product and multi-stage supply chain which takes into account of the interdependence of the production, transportation and inventory.

Williams [18] developed heuristic algorithms for scheduling production and distribution operations in a supply chain network where each station has at most one immediate successor but may have any number of predecessors. The objective is to minimize production and/or distribution schedule while satisfying the end customer demand. Williams [19] also developed a dynamic programming algorithm for determining production and distribution batch sizes at each node in the supply chain similar to his previous work. The objective is to minimize the average cost per period on an infinite horizon, where the cost components are production and inventory holding.

Cohen and Lee [20] proposed a modeling framework for integrated planning throughout a supply chain. They developed submodels for modeling the supply chain and developed a heuristic optimization procedure.

Cohen and Lee [21] developed an MINLP model which is based on economic order quantity techniques. The objective function maximizes the total after tax profit for the manufacturing facilities and distribution centers. The constraints of their formulation are resource and production constraints, which they refer to as “managerial constraints”; and feasibility availability and demand limits, which they refer to as “logical consistency constraints”.

An interesting study regarding the equilibrium of the supply chain network is conducted by Dong, Zhang and Nagurney [22]. They model the optimal behavior of the decision makers in the supply chain and derived equilibrium conditions subject to random demand observed at retailers.

Beamon [23], Min and Zhou [24], and Vakharia [25] et al. provided extensive reviews of supply chain literature. These review papers indicate lack of integrated approach to supply chain management problems that include transportation, production, distribution and marketing.

This thesis addresses the major challenges stated by Lambert and Cooper [1], namely, the integration problem of marketing decisions, manufacturing, distribution and transportation plans by devising an integrated optimization model.

1.2.2 Bilinear Optimization Literature

The optimization models often include bilinear terms due to the nature of these terms. There has been a considerable effort to develop reliable and efficient algorithms for optimization problems with bilinear forms.

McCormick [26] devised a procedure for obtaining tight underestimating convex programs for factorable nonlinear programming problems, which also laid the foundations of the algorithm developed in chapter 4 of this thesis.

Al-Khayyal and Falk [27] developed a branch and bound algorithm to minimize the sum of a biconvex function, defined as the sum of a convex function in x, sum of a convex function in y and a bilinear term in x and y over a closed set. The feasible region may be defined by jointly defined constraints in x and y.

Sherali and Alameddine [28] developed a reformulation-linearization technique based optimization algorithm for bilinear programming problems. Their algorithm proceeds by generating underestimating linear programs for a minimization problem and solving these underestimating linear programs in a branch and bound manner. Later Adams and Sherali [29] extended this approach to find global optimal solutions for mixed-integer bilinear optimization problems.

C.S. Adjiman et al. [30] developed a global optimization algorithm called α-Branch and Bound (αBB) which can handle twice differentiable functions in the objective function and in the constraints. The key idea applied in the algorithm is to generate a converging sequence of upper and lower bounds on the global optimum through convex relaxations of the original problem. The performance of this algorithm is also tested [31].

Sherali and Wang [32] developed a global optimization algorithm for factorable nonconvex programming problems. The algorithm proceeds implementing a branch and bound approach where in each node a linear programming relaxation of the problem is solved which is generated by utilizing various approximation schemes such as Mean-Value Theorem and Chebyshev interpolations coordinated with a Reformulation-Linearization Technique.

Lee and Grossman [33] devised a global optimization algorithm for nonconvex generalized disjunctive programming (GDP) problems. The algorithm makes use of convex underestimating functions of bilinear, linear fractional and concave separable functions in continuous variables to construct the convex hull and then relaxed convex GDP is solved in a two-level branch and bound algorithm. In the first level a discrete branch and bound search is performed on the disjunctions to predict lower bounds whereas in the second level a spatial branch and bound is used to solve nonconvex NLP problems for updating the bounds.

The algorithm developed for bilinear and mixed integer bilinear optimization problems in chapter 4 uses the basic concepts given by McCormick [26], and extends those concepts by controlling the error of estimators for bilinear terms. The resulting estimating problem is a mixed-integer linear problem for both bilinear and mixed integer bilinear problems.

1.3 Outline of the Thesis

The objective of this thesis is to develop an integrated supply chain optimization model which includes marketing decisions, manufacturing, distribution and transportation plans; and to solve the resulting optimization model in an acceptable amount of time.

In this thesis the general supply chain problem is described including the models and their shortcomings, and our approach to the problem is presented in chapter 2.

In chapter 3, the basic processes such as inventory management, transportation and production that are conducted by the supply chain nodes are described and their mathematical models are presented. Different types of transportation and production are also discussed. Then detailed mathematical description of production nodes, non-production nodes, markets and external markets are derived. In the last part of this chapter, modeling techniques that can be used to model supply chain nodes in an enhanced accuracy without unnecessarily increasing the number of variables and constraints are introduced. This is achieved by allowing variables and constraints to be defined on different time scales with different resolutions and synchronizations and then mapping these time scales whenever necessary while defining constraints.

In chapter 4, a global optimization algorithm to solve non-convex bilinear and non-convex bilinear mixed integer problems subject to linear constraints that is observed when the objective function of the supply chain model includes revenue terms while price is defined to be a variable, is developed. The effectiveness of the algorithm is illustrated on benchmark problems.

In chapter 5, the performance of the global optimization algorithm is compared with a commercial solver on a supply chain formulation generated by the modeling techniques developed in this thesis.

In chapter 6, the conclusions and possible future research directions are discussed.

1.4 Contributions of This Work

This thesis addresses the development of an integrated approach to optimize supply chain systems that includes marketing decisions, manufacturing, distribution and transportation plans. The presence of price as a decision variable introduces bilinear terms in the objective function whenever revenue terms are included. Therefore, in order to solve the resulting optimization model in an acceptable amount of time a novel global optimization algorithm for bilinear and bilinear mixed-integer problems is developed.

The major contributions of this work are as follows:

i. Introduction of the detailed market models to the supply chain optimization problem and models of different transportation means operating in the supply chain
ii. Development of advanced modeling techniques in section 3.3 to model the supply chain system with better precision without unnecessarily increasing the number of variables and constraints
iii. Development of the global optimization algorithm for bilinear and mixed integer bilinear problems which can be used to solve various important engineering problems including the supply chain optimization problem introduced in this thesis

The first contribution addresses the challenges presented by Lambert and Cooper [1], the second contribution helps the effective integration of models with different time scales which can have different resolutions and synchronizations, and the third contributions addresses the acceptable solution time objective.

CHAPTER 2 – ANALYSIS OF THE PROBLEM

2.1 Description of the Supply Chain Management Problem

The input to a supply chain system is customer demand and the output of the same system is the products supplied in a typical supply chain system as shown in Fig.2.1. This view of supply chain modeling implies that the supply chain has no regulating response on the demand, which is the sole input of the system. On the other hand, it is known that price can be varied in reality, which affects the demand, the degree of which is determined by the price sensitivity of the market. The structure of the demand inevitably affects operational decisions to be taken in the supply chain of a firm as shown in Fig.2.2.

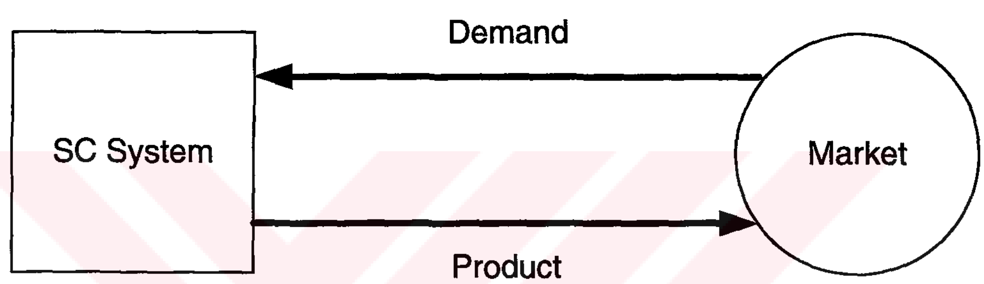

Figure 2.1. Demand is an independent input

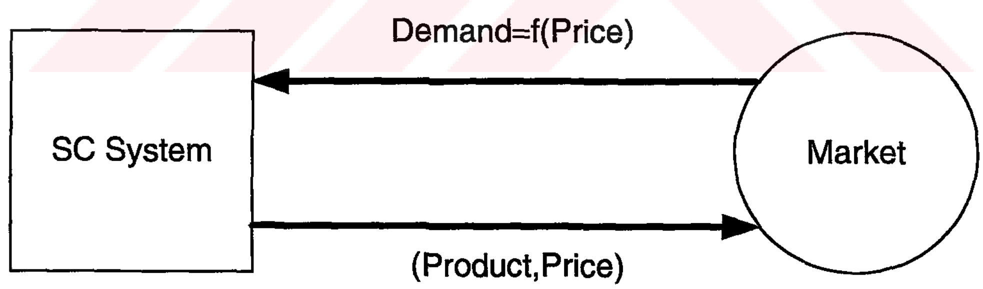

Figure 2.2. Demand is a function of price

The firms usually seek to satisfy the demand of a particular product by substitute products whenever they are short of supply. The acceptability of the substitute products (i.e. whether the customer accepts substitute products or not), and the degree of willingness (i.e. penalties associated to supplying substitute products) is determined by the characteristics of the market that is being served. In addition, two products can be accepted as substitutes in a market, but they may not be accepted as substitutes in another. Therefore, the classes of substitute products are determined by the specific market to be served.

Moreover, the excess inventory of some products in a supply chain at some point during operation need be "dumped out" [6] at a cost or price. Consequently, some nodes in the supply chain may have the ability to dump products.

Usually, the supply side of a supply chain is not a closed system; some nodes in the supply chain system can buy raw materials, semi-finished goods or even finished goods from outside sources. These sources are referred to as external markets or external purchase [4].

Furthermore, in order to move goods from one node of the supply chain to another, there must be some means of transportation. We consider three types of transportation means, pipe, vehicle, and outsourcing. Pipe is a transportation medium which cannot move and is dedicated between two nodes. On the other hand, a vehicle can move and can be used between any two nodes. Both of these may have different operating modes, or configurations. In each configuration, they can carry different bundles of goods in different capacities. Moreover, there may be different types of these transportation media with different capabilities in terms of speed, configuration variability and capacity. Finally, some vehicles may be dedicated to specific routes and they only transport materials on that route. Outsourcing transportation is done by sending an amount of materials or products from one node to another by paying a predetermined fixed price and a variable price charged depending on the amount/quantity transported.

On the production side, a plant may have different operating modes, in each mode several or single type of products can be produced with different production rates. Each mode may require a different setup time and a cleanup time before changing to a different mode, which is different for each node, may be required. Furthermore, campaign production [4] may be allowed. With this building block, and including in-plant inventories models for plants with very complex characteristics can be developed.

In a typical supply chain, there are two types of nodes; the ones which can produce, and the ones which cannot produce. All nodes in the supply chain can serve different markets, and can buy materials from different external markets. When transportation vehicles are used between nodes, the time to transport varies with respect to the vehicle type. Based on these concepts, the conceptual depiction of a sample supply chain from an operational perspective can be seen in Fig.2.3. In this supply chain, there are four plants. Plants 1 and 2 buy raw material from external suppliers and produce semi-finished products to be used at plant 3. On the other hand, plant 4 buys raw materials and directly produced finished goods. The recycling plant receives returned products from retailers and produces, raw materials and semi-finished products and ships them to plants 1 and 4. There are five different markets and each of which is served by some set of supply chain nodes. The external suppliers are external markets that supply raw materials, semi-finished and finished products to the supply chain nodes.

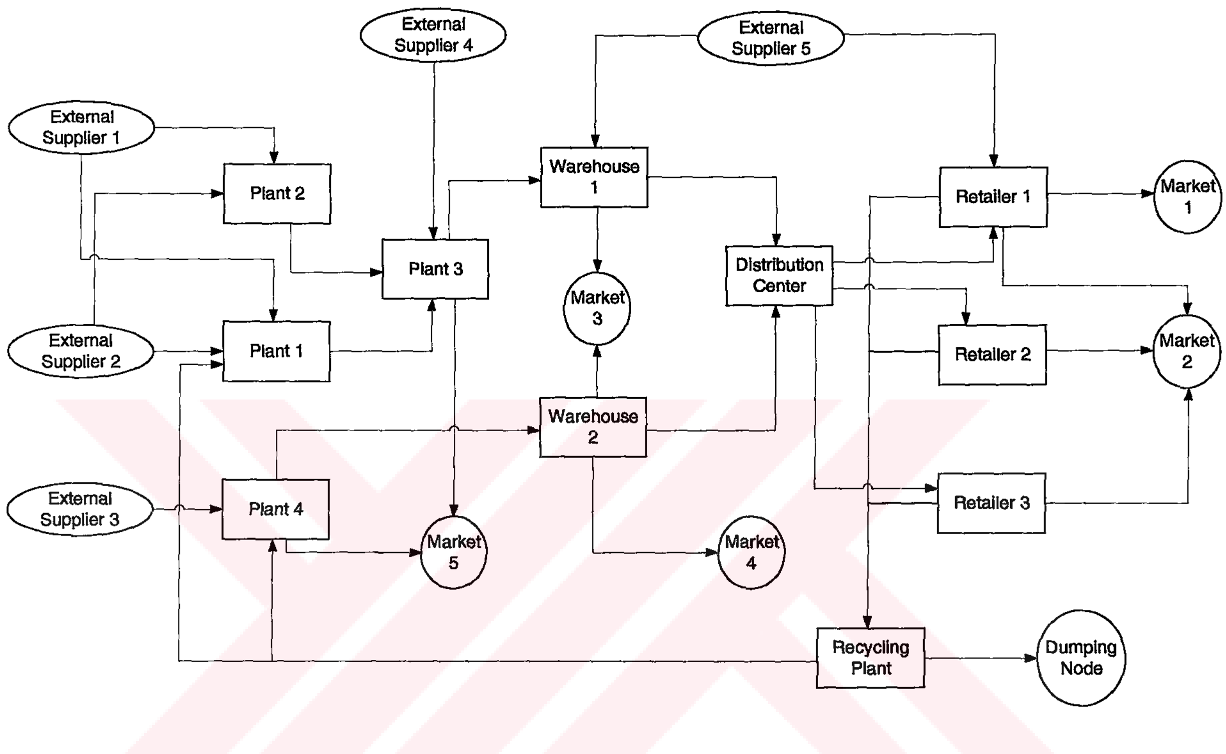

Figure 2.3. Supply Chain from an Operational Perspective

2.2 Models and Shortcomings

The basic shortcomings in the integrated centralized supply chain literature are the inclusion of detailed market models and transportation means, which are becoming more and more important due to the effects of globalization and extended enterprise approaches.

Whenever demand is modeled as a function of price and/or other variables in the control of the supply chain model is significantly simplified especially in terms of modeling the supply chain components which may involve discrete variables and in terms of specifying the constraints that must be imposed. This is due to the fact that whenever price is defined to be a variable and backordering is allowed, the revenue or profit maximization objectives become bilinear, which is non-convex. The optimization of a function involving bilinear terms subject to linear constraints is a difficult problem in terms of computational complexity, hence supply chain literature considered demand as a parameter simplifying the problems while loosing from effective market models. In chapter 4, a global optimization algorithm which provided satisfactory results for bilinear and bilinear mixed-integer linear programming problems are described in order to include realistic market models in supply chain management problems.

Modeling the transportation means in the context of production planning and commercial activities imply that the time resolution on which the variables are declared must be significantly increased since the scheduling of the vehicles usually requires a very high time resolution compared to that of the production planning and commercial activities. A methodology that must be adopted without unnecessarily increasing the number of variables and constraints while including various transportation means is introduced in chapter 3. This methodology can also be used to precisely model and schedule the action critical components of the supply chain without unnecessarily increasing the number of variables and constraints.

2.3 Approach to the Problem

In order to address the requirements described in section 2.1, some abstractions and definitions are made.

It is common that consumers demand specific products to be produced in certain plants. A known example is consumers' preference towards a makes sets made in Japan rather than the ones made elsewhere even for the same product model. Whenever a differentiable characteristic occurs, it represents a different product specification even for the same model. Namely, "Model T100 produced in plant A" and "Model T100 produced in plant B" are different products as well as "Model T100 using component 1 from supplier 5" and "Model T100 using component 1 from supplier 1" are different products whenever consumers can differentiate these. *Therefore, in our context a product can be defined as a good which satisfies a specific demand of consumers and is differentiated by at least one consumer segment from all other products.* It is clear that this definition addresses the demand for some products to be produced in specific plants as mentioned in Kallrath & Timpe [4].

In microeconomic theory [34], it is a known fact that demand varies with price as shown in Fig.2.4. However, the degree of change in demand with respect to a certain amount of price change can vary with different sets of customers. This phenomenon is called price sensitivity. And if price is to be used as a control variable for demand, the function which maps demand value to price must be specified. This function is called demand function.

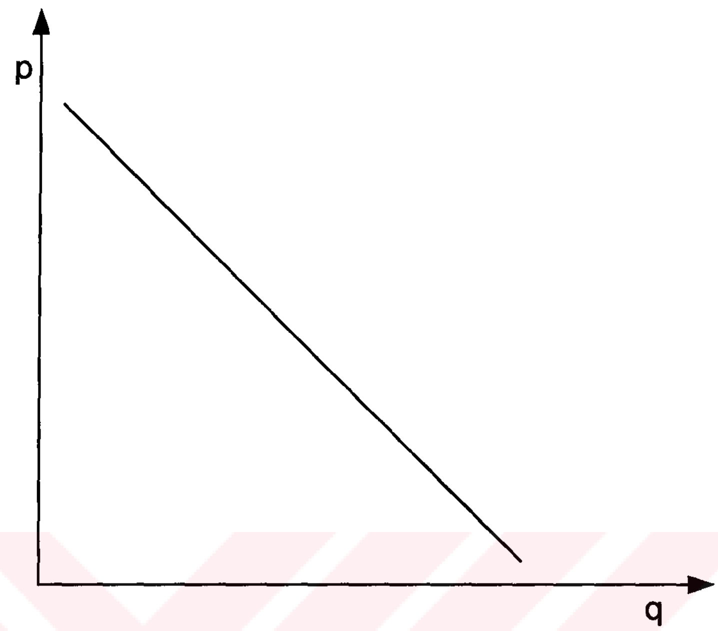

Figure 2.4. Demand as a function of price

In some cases, it is possible that a demand can be satisfied by multiple products. For example, for some customers it may be the case that "Model T100" using component 1 from either supplier can be both acceptable although the one using component 1 from supplier 1 can be more desirable. In this case, the customer demand is assumed to be "Model T100 using component 1 from supplier 1" and "Model T100 using component 1 from supplier 5" is accepted to be a substitute for the first one for this type of customers. On the other hand, for some customers it may be the case that the specific characteristic offered by using component 1 from supplier 1 is crucial and the one which uses component 1 from supplier 5 cannot be acceptable. This implies that for this set of customers, "Model T100 using component 1 from supplier 5" is not a substitute of "Model T100 using component 1 from supplier 1". Therefore, a set of substitute products is determined by the perception of some customers and substitution relationship need not be universal for all customers and need not be transitive. In addition to these, for another set of customers although "Model T100 using component 1 from supplier 5" is a substitute for "Model T100 using component 1 from supplier 1", it may not be as desirable as this substitution is for the first set of customers, i.e. the penalty costs associated with supplying substitute product can be different for different sets of customers.

Based on this framework, in our context, *a market is specified by a demand function for a product which may depend on price, a set of substitute products and the penalty values associated with supplying substitute products.* This basic specification is improved in section 3.2.3.

The dumping nodes are treated as markets, the characteristics of which are determined by what is exactly meant by dumping. For example, if products are sold at a price to another market, then dumping node is simply another market. On the other hand, if waste dumping is considered, then an upper limit on the amount that can be dumped and costs associated with dumping waste must be specified.

The external markets are actually suppliers that are not under supply chain's control, but can provide some raw materials, semi-finished and finished goods to the supply chain nodes. Transportation from an external market to a supply chain node can be done either by the external market or by the vehicles sent by the purchasing supply chain node. In the case where transportation is done by the external market, there is an agreed time of transportation between supply chain nodes and an external market, which specifies the time it takes from the placement of order to delivery. In the case where transportation is done by the vehicles sent by the purchasing supply chain node, the transportation time depends on the vehicles sent. The maximum amount of each material/product that can be supplied by external market to the supply chain nodes is specified. Therefore, whenever an order is to be placed to an external market, the centralized supply chain knows the maximum amount that can be supplied by that external market with certainty.

In a typical supply chain conforming to the description provided in the previous section, there are two types of nodes: the nodes which can convert a product/raw material to another one, and the nodes which cannot. The first type of node is called of type production node (PN), whereas the second type of node of type non-production node (NPN).

In contemporary supply chain systems, it is possible to identify two subsets of type NPN, the nodes which can hold inventory and nodes which cannot hold inventory. The first type of these subsets is a classical inventory holding unit. The second type of these is a rather new concept, and is usually called a cross-docking node. Cross-docking nodes do not hold inventory, but rather act as a transition node, where products carried in one transportation unit is transferred to other transportation units.

There are different type of transportation mediums in a supply chain, each of which with different characteristics. In the previous section, a detailed description of requirements is given. It is worth noting that assigning different speeds to vehicles means that the time to transport materials from one node to another changes with respect to the type of vehicle.

Finally, a plant is a node which can transform raw materials or semi-finished goods into finished goods. There are different operating modes for each plant, and each mode specifies the products that can be produced, and the maximum rate of production for each product. For each mode, there is an associated setup and cleanup time. It must also be noted that campaign production is allowed.

Therefore, the following features are addressed in a supply chain:

1. Product differentiation
2. Markets
    a. Demand function which may depend on price
    b. Substitute products
    c. Penalty costs associated with substitute products
    d. Backordering
3. External markets
    a. Capacity
    b. Agreed lead time
4. Dumping
5. Production nodes
    a. Production modes with specific setup and cleanup times
        i. Products that can be produced
        ii. Rate of production
    b. Campaign production
6. Means of transportation
    a. Vehicle types
        i. Modes of transportation

1. Products that can be carried
2. Associated capacity

ii. Speed

iii. Specific Routes

b. Pipes

i. Associated capacity/rate

c. Outsourcing

7. High precision modeling of the action critical supply chain components

Also note that there is no restriction on material flow, i.e. material flow can be bidirectional.

CHAPTER 3 – INTEGRATED MODEL OF THE SUPPLY CHAIN

A typical supply chain system consists of nodes and associated processes that take place in these nodes. We will develop detailed models for the processes and also the nodes in sections 3.1 and 3.2 respectively. When integrating these processes and nodes the characteristics and the time scales at which the nodes and the processes must be considered. Therefore, we will discuss our modeling approach to integrating different time scales in section 3.3.

3.1 Basic Modeling of Processes

General Framework

Important processes in the supply chain systems include inventory management of raw materials, intermediate and final products, management of orders, material transfers using different transportation mediums, production, and market. The detailed models are discussed in the following subsections.

3.1.1 Inventory Balance

Inventory is the amount of a specific product stored at any node of the supply chain network. Therefore, there is an associated inventory variable for each product at each storage location that can store it. Inventory balance equations are generated for each product and node as shown in Eq.(3.1).

$$
\begin{aligned}
I_{j1_i}^{n}(k+1) = {} & I_{j1_i}^{n}(k) - \sum_{d\in S_n}\sum_{j2_i}\sum_{v}\sum_{t} y_{j1_i,j2_i}^{n,d,v,t}(k) - \sum_{d\in S_n}\sum_{j2_i}\sum_{p} y_{j1_i,j2_i}^{n,d,p}(k) \\
& - \sum_{d\in N}\sum_{v}\sum_{t} y_{j1_i}^{n,d,v,t}(k) - \sum_{d\in N}\sum_{j2_i}\sum_{p} y_{j1_i}^{n,d,p}(k) \\
& + \sum_{s\in N}\sum_{v}\sum_{t} y_{j1_i}^{s,n,v,t}\left(k+1-LT_{v}^{s,n}\right) + \sum_{s\in N}\sum_{p} y_{j1_i}^{s,n,p}(k) \\
& + \sum_{e\in E_n}\sum_{v}\sum_{t} y_{j1_i}^{e,n,v,t}\left(k+1-LT_{v}^{e,n}\right) + \sum_{e\in E_n}\sum_{v}\sum_{t} y_{j1_i}^{e,n}\left(k+1-ALT_{j1_i}^{e,n}\right) \quad \forall k \geq 1
\end{aligned}
\tag{3.1}
$$

Eq.(3.1) indicates that the inventory of product $j1_i$ for the beginning of the next period will be the current inventory of that product minus the amount of that product sent to satisfy the demand of all other substitute products at all destinations in all possible vehicle and transportation mode combinations plus the amount of that product shipped from all sources in all possible vehicle and transportation mode combinations that arrive at the beginning of the next period. The initial inventory levels when k=1 is considered to be known.

Sending a specific product instead of another will only occur while selling products to end customers in a centralized supply chain. Therefore, whenever $n1, n2 \in N$, $y_{j1_i,j2_i}^{n1,n2,v,t}(k)$ can be replaced by $y_{j1_i}^{n1,n2,v,t}(k)$, which, in turn, reduces the number of summation signs. This is observed in third and fourth terms on the right hand side of the first balance equation. Moreover, while purchasing goods from external markets, the supply chain only requests specific types of goods to be sent, and therefore, external markets cannot send substitute products, and this fact eliminates the subscript for substitute goods for the terms involving external markets. The argument of the terms $y_{j1_i}^{s,n,v,t}(k+1-LT_v^{s,n})$, $y_{j1_i}^{e,n,v,t}(k+1-LT_v^{e,n})$ and $y_{j1_i}^{e,n}(k+1-ALT_{j1_i}^{e,n})$ are included in an equation of a specific k whenever $k+1-LT_v^{s,n} \geq 1$, $k+1-LT_v^{e,n} \geq 1$ and $k+1-ALT_{j1_i}^{e,n} \geq 1$, respectively. Finally, the amount of purchased goods from external markets to be transferred by transportation means belonging to the supply chain are referred by $y_{j1_i}^{e,n,v,t}(k+1-LT_v^{e,n})$ variable, whereas the amount of purchased goods from external markets to be transferred by transportation means belonging to the external market are referred by $y_{j1_i}^{e,n}(k+1-ALT_{j1_i}^{e,n})$.

Another consideration about inventory is the physical boundary of the inventory level. Such types of constraints are simple to construct since they only involve inventory variables and parameters for physical limits.

3.1.2 Order Balance

There is a single order balance variable for each market. The order balance variable is only created for the product that is originally demanded by the market. Order balance variable indicates the amount of accumulated unsatisfied demand, hence allowing backordering. Like inventory balance, order balance is governed by two types of equations, one is for initial condition and the other is for modeling the balance relation. For a given market d and a product $j1_i$, order balance equations are generated in equation 3.2.

$$O_{j1_i}^{m}(k+1) = O_{j1_i}^{m}(k) - \sum_{n}\sum_{v}\sum_{t}\sum_{j2_i}\alpha_{j2_i}^{j1_i} * y_{j2_i,j1_i}^{n,m,v,t}\left(k - LT_v^{n,m}\right) - \sum_{n}\sum_{p}\sum_{j2_i}\alpha_{j2_i}^{j1_i} * y_{j2_i,j1_i}^{n,m,p}(k)$$
$$- \sum_{n}\sum_{j2_i}\alpha_{j2_i}^{j1_i} * y_{j2_i,j1_i}^{n,m}(k) + d_{j1_i}^{m}(k) \;\; \forall k \geq 1 \qquad (3.2)$$

Eq.(3.2) models the amount of accumulated unsatisfied demand at the beginning of the next period is the current amount of accumulated unsatisfied demand minus the effective amount of demand satisfied in this period plus the observed demand in this period. The initial orders at k=1 is considered to be known.

The effective number of units or amount of product $j2_i$ required to satisfy the demand for one unit of product $j1_i$ is given by $\alpha_{j2_i}^{j1_i}$ and needs to be explained more. For example, if $j1_i$ and $j2_i$ are television models, then $\alpha_{j2_i}^{j1_i}$ is probably 1. However, if $j1_i$ is a packet of ten kilograms of detergent and $j2_i$ is a packet of five kilograms of detergent then $\alpha_{j2_i}^{j1_i}$ is certainly not 1 but is probably 2. Therefore, $\alpha_{j2_i}^{j1_i}$ is meant to be a scaling factor between substitute products in terms of quantity or amount.

3.1.3 Transportation Means

3.1.3.1 Transportation Vehicle Constraints

In this section, the constraints that govern the movement of vehicles from one node to another node in the supply chain network are going to be introduced. These constraints are referred to as vehicle balance constraints. Constraints related to capacity are going to be elaborated later in section 3.1.4.

Vehicle balance constraints are similar to inventory balance constraints. Basically, they are used to govern the number of vehicles available at each node of the supply chain during the optimization horizon.

It may be the case that transportation of goods is a task that the supply chain must accomplish while buying goods from external markets. Based on these, the vehicle balance equation can be written as follows.

$$
\begin{aligned}
VI_v^n(k+1) = VI_v^n(k) - \sum_{d\in N\cup M\cup E_n} \sum_t VS_{v,t}^{n,d}(k) + \sum_{s\in N} \sum_t VS_{v,t}^{s,n}\left(k+1-LT_v^{s,n}\right) \\
+\sum_e \sum_t VS_{v,t}^{n,e}\left(k+1-LT_v^{e,n}-LT_v^{n,e}\right) \\
+\sum_m \sum_t VS_{v,t}^{n,m}\left(k+1-LT_v^{n,m}-LT_v^{m,n}\right) \forall k \geq 1
\end{aligned}
\tag{3.3}
$$

The terms with lead time subtractions in their time indices are included in a summation whenever their indices evaluates to a value greater than or equal to one. The Eq.(3.3) implies that the number of vehicles of type v at supply chain node n in the next period is equal to the number of vehicles of type v at supply chain node n in this period minus the total number of vehicles of type v sent to other supply chain nodes, plus the number of vehicles sent from other nodes sent to this node $LT_v^{s,n}$ time ago plus the vehicles due to arrive that are sent to external markets and markets. The number of vehicles available at any node in the beginning of period k=1 is known.

Moreover, Eq.(3.3) can be modified to impose routing constraints for vehicle types by modifying VI and VS variables.

3.1.3.2 Pipes

In reality pipes can be modeled as transportation vehicles types with zero lead time. Although it is straightforward to define a variable $VI_p^{n,d}$ and setting it to a constant value defines a pipe, one can only define $VI_p^{n,d}$ to be a parameter for each type of pipe, which is the number of pipes of type p at node n that transfer materials to node d. Different pipe types are necessary to define the capacity differences between different pipes and ability to transport different materials. Pipes are simply modeled as constant numbers and associated material transfer limit constraints are discussed in section 3.1.4.1. There is no need to define balance equations for pipes.

3.1.3.3 Outsourcing

In order to allow the outsourcing of transportation, material transfer variable is modified and new parameters are introduced.

Whenever $n,d \in N$, the variable $y_{j1_i,j2_i}^{n,d}(k)$ can be simplified to $y_{j1_i}^{n,d}(k)$ because the supply chain nodes do not send substitute products to each other in a centralized supply chain.

Furthermore, $y^{n,d}_{j1_i,j2_i}(k)$ and $y^{n,d}_{j1_i}(k)$ can also be specified as a semi-continuous variables whenever the amount of transportation that is outsourced has an upper and lower limit.

The definition of $y^{n,d}_{j1_i,j2_i}(k)$ also implies that changes must be made to inventory and order balance equations, which are given in Eq.(3.4) and Eq.(3.5).

$$\begin{aligned}I^{n}_{j1_i}(k+1)=I^{n}_{j1_i}(k)&-\sum_{d\in S_n}\sum_{j2_i}\sum_{v}\sum_{t}y^{n,d,v,t}_{j1_i,j2_i}(k)-\sum_{d\in S_n}\sum_{j2_i}\sum_{p}y^{n,d,p}_{j1_i,j2_i}(k)-\sum_{d\in S_n}\sum_{j2_i}y^{n,d}_{j1_i,j2_i}(k)\\&-\sum_{d\in N}\sum_{v}\sum_{t}y^{n,d,v,t}_{j1_i}(k)-\sum_{d\in N}\sum_{j2_i}\sum_{p}y^{n,d,p}_{j1_i}(k)-\sum_{d\in N}y^{n,d}_{j1_i}(k)\\&+\sum_{s\in N}\sum_{v}\sum_{t}y^{s,n,v,t}_{j1_i}\left(k+1-LT^{s,n}_{v}\right)+\sum_{s\in N}\sum_{p}y^{s,n,p}_{j1_i}(k)+\sum_{s\in N}y^{s,n}_{j1_i}\left(k+1-LT^{s,n}\right)\\&+\sum_{e\in E_n}\sum_{v}\sum_{t}y^{e,n,v,t}_{j1_i}\left(k+1-LT^{e,n}_{v}\right)+\sum_{e\in E_n}\sum_{v}\sum_{t}y^{e,n}_{j1_i}\left(k+1-ALT^{e,n}_{j1_i}\right)\ \forall k\geq 1\end{aligned}\tag{3.4}$$

$$\begin{aligned}O^{m}_{j1_i}(k+1)=O^{m}_{j1_i}(k)&-\sum_{n}\sum_{v}\sum_{t}\sum_{j2_i}\alpha^{j1_i}_{j2_i}*y^{n,m,v,t}_{j2_i,j1_i}\left(k-LT^{n,m}_{v}\right)-\sum_{n}\sum_{j2_i}\alpha^{j1_i}_{j2_i}*y^{n,m}_{j2_i,j1_i}(k)\\&-\sum_{n}\sum_{j2_i}\alpha^{j1_i}_{j2_i}*y^{n,m}_{j2_i,j1_i}\left(k-LT^{n,d}\right)+d^{m}_{j1_i}(k)\ \forall k\geq 1\end{aligned}\tag{3.5}$$

3.1.4 Product Transfer Limits

3.1.4.1 Material Transfer from a Supply Chain Node

Product transfer from supply chain nodes are governed by two types of constraints. The first one must be written to control the amount transferred from a supply chain node is not more than the inventory level. The second one is imposed in order to control the transfer amount in terms of the available capacity of transportation medium. Since there are two types of transportation vehicles, this logical constraint is imposed in two types of constraints, each of which is for the corresponding transportation medium. Furthermore, a transportation mode may impose two fundamentally different types of capacity restrictions. The first one is that the capacity to carry a product depends on the amount of other products loaded, which happens when products are carried in the same compartment of the vehicle. The second one is that the capacity to carry a product does not depend on amount of other products, which happens when products are carried in different compartments of the vehicle. Therefore, the second constraint can be of four different types. Different combinations of these constraints are possible, and straightforward to model. Eq.(3.6) demonstrate the mathematical model of these concepts.

$$
\begin{aligned}
I_{j1_i}^{n}(k) \geq & \sum_{d\in S_n}\sum_{j2_i}\sum_{v}\sum_{t} y_{j1_i,j2_i}^{n,d,v,t}(k) + \sum_{d\in S_n}\sum_{j2_i}\sum_{p} y_{j1_i,j2_i}^{n,d,p}(k) + \sum_{d\in S_n}\sum_{j2_i} y_{j1_i,j2_i}^{n,d}(k) \\
& + \sum_{d\in N}\sum_{v}\sum_{t} y_{j1_i}^{n,d,v,t}(k) + \sum_{d\in N}\sum_{j2_i}\sum_{p} y_{j1_i}^{n,d,p}(k) + \sum_{d\in N} y_{j1_i}^{n,d}(k) \ \forall k \\
VS_{v,t}^{n,d}(k) \geq & \sum_{j2_i} f_{j1_i}^{v,t} * y_{j1_i,j2_i}^{n,d,v,t}(k) \ \forall k \\
VS_{v,t}^{n,d}(k) \geq & \sum_{j1_i}\sum_{j2_i} f_{j1_i}^{v,t} * y_{j1_i,j2_i}^{n,d,v,t}(k) \ \forall k \\
VI_{p}^{n,d} \geq & \sum_{j2_i} f_{j1_i}^{p} * y_{j1_i,j2_i}^{n,d,p}(k) \ \forall k \\
VI_{p}^{n,d} \geq & \sum_{j1_i}\sum_{j2_i} f_{j1_i}^{p} * y_{j1_i,j2_i}^{n,d,p}(k) \ \forall k
\end{aligned}
\tag{3.6}
$$

The constraint for vehicles is defined as, the total amount transferred of a specific product from a supply chain node to any destination node in a specified transportation configuration of a vehicle type multiplied by the fraction of capacity it consumes from that vehicle in the specified configuration must be less than or equal to the number of vehicles that are sent from the supply chain node to the destination node and the outsourced amount.

3.1.4.2 Material Transfer to a Supply Chain Node from External Markets

As described in transportation vehicle number balance equations, it may be the case that a supply chain node can send vehicles to external markets to transport goods from. The most general type of situation is that the external market offers to transport the goods if external market's transportation resources are to be utilized. The amount to be transported by the external market and the amount to be transported by the transportation means belonging to the supply chain can be specified. These variables were introduced in inventory balance equations and they were $y_{j1_i}^{e,n}\left(k+1-ALT_{j1_i}^{e,n}\right)$ and $y_{j1_i}^{e,n,v,t}\left(k+1-LT_{v}^{e,n}\right)$, respectively. The constraints regarding $y_{j1_i}^{e,n}\left(k+1-ALT_{j1_i}^{e,n}\right)$ are the lower and upper limits on material transfer imposed by the external market if external market is to transport the goods. Therefore, specifying $y_{j_i}^{e,n}\left(k+1-ALT_{j1_i}^{e,n}\right)$ as a continuous or semi-continuous variable is sufficient for modeling purposes. Consequently, these constraints can be written as follows. Here, *LL* and *UL* terms are constants defining lower and upper limits respectively.

$$
0 \vee LL_{j_i}^{e} \leq y_{j_i}^{e,n}(k+1-ALT_{j_i}^{e,n}) \leq UL_{j_i}^{e} \tag{3.7}
$$

On the other hand, the constraint regarding $y_{j1_t}^{e,n,v,t}\left(k+1-LT_v^{e,n}\right)$ must impose that the amount to be transported from the external market *e* to the supply chain node *n* must be less than or equal the total capacity of vehicles in the specified configuration, which is controlled by the Eq.(3.8).

$$\sum_v \sum_t \sum_j f_j^{v,t} * y_j^{e,n,v,t}\left(k\right) \le VS_{v,t}^{n,e}\left(k-LT_v^{n,e}\right) \ \forall k \ge LT_v^{n,e}+1$$
$$y_j^{e,n,v,t}\left(k\right)=0 \ \forall k \le LT_v^{n,e} \tag{3.8}$$

3.1.5 Production Models

The production model is characterized by its modes, the production rates in each mode, and associated setup and cleanup times. Plant must be able to produce in successive periods in a certain mode without incurring successive setup times. Changing from one mode to another requires a cleanup time specific to the current mode and a setup time for the mode in succession. Two different mathematical models are developed for production

3.1.5.1 Modeling Style-1

This modeling style requires that an idle production mode to be defined with zero production capability.

The initiation variable, $f_\theta^{n,f}\left(k\right)$, is introduced for the sole purpose of determining the time setup cost accrues, and is unnecessary if there is no term in involving the setup cost in the objective function. The production model is given in Eq.(3.9).

The logic behind this modeling style is as follows. Whenever a production campaign starts, the setup time is incurred. During the setup time period, the plant must be in the corresponding mode, but the batch sizes are forced to zero by $X1_\theta^{n,f}$ variables. Similarly, during the cleanup time period, the plant must be in the corresponding mode with batch sizes forced to zero by $Y1_\theta^{n,f}$. The satisfaction of setup time requirements are controlled by $X1_\theta^{n,f}$ and $X2_\theta^{n,f}$ variables, whereas the satisfaction of cleanup time requirements are controlled by $Y1_\theta^{n,f}$ and $Y2_\theta^{n,f}$ variables. Whenever, $X1_\theta^{n,f}$ and $Y1_\theta^{n,f}$ are both equal to 1, both setup and cleanup time requirements are satisfied and plant is available to produce batches of allowed products.

$$
\begin{aligned}
&\sum_{\theta} p_{\theta}^{n,f}(k) = 1 \ \forall k \\
&f_{\theta}^{n,f}(1) \geq p_{\theta}^{n,f}(1) \ \forall \theta \\
&f_{\theta}^{n,f}(k) \geq p_{\theta}^{n,f}(k) - p_{\theta}^{n,f}(k-1) \ \forall k \geq 2, \theta \\
&B_{\theta,j}^{n,f}(k) \leq X1_{\theta}^{n,f}(k) * R_{\theta,j}^{n,f} \ \forall k, \theta, j \\
&B_{\theta,j}^{n,f}(k) \leq Y1_{\theta}^{n,f}(k) * R_{\theta,j}^{n,f} \ \forall k, \theta, j \\
&\sum_{t=\max\{1,k-ST_{\theta}^{n,f}\}}^{k} p_{\theta}^{n,f}(t) - ST_{\theta}^{n,f} = X1_{\theta}^{n,f}(k) - X2_{\theta}^{n,f}(k) \ \forall k, \theta \\
&X1_{\theta}^{n,f}(k) \leq zs_{\theta}^{n,f}(k) \ \forall k, \theta \\
&X2_{\theta}^{n,f}(k) \leq \left(1 - zs_{\theta}^{n,f}(k)\right) * ST_{\theta}^{n,f} \ \forall k, \theta \\
&\sum_{t=k}^{\min\{K,k+CT_{\theta}^{n,f}\}} p_{\theta}^{n,f}(t) - CT_{\theta}^{n,f} = Y1_{\theta}^{n,f}(k) - Y2_{\theta}^{n,f}(k) \ \forall k, \theta \\
&Y1_{\theta}^{n,f}(k) \leq zc_{\theta}^{n,f}(k) \ \forall k, \theta \\
&Y2_{\theta}^{n,f}(k) \leq \left(1 - zc_{\theta}^{n,f}(k)\right) * CT_{\theta}^{n,f} \ \forall k, \theta
\end{aligned}
\tag{3.9}
$$

A sample schedule is demonstrated in Table 3.1. The scenario consists of a plant with two operating modes. In the first mode, products 1 and 2 are produced, whereas in the second mode products 1 and 2 are produced. The setup time and cleanup times of both nodes are two periods. The plant is idle initially and then switches to first mode for a horizon of 5 periods and thereafter switches to second mode for 7 periods, including setup and cleanup times. “**_F_**” denotes that the variable is free to be determined by optimization, and the value will only be known after running optimization with other specific parameters, such as costs. However, it must be noted that “**_F_**” values for the initiation variables are 0 if there is an associated cost in the objective function. Furthermore, “**_F_**” values for $zs_{\theta}^{n,f}$ and $zc_{\theta}^{n,f}$ variables are free in this setting.

Furthermore, it is also possible that the production of a product in a mode interferes with the production of another which necessitates to add constraints that state that total production rate must be less than or equal to one. Such a requirement is demonstrated in Eq.(3.9).

$$
\sum_{j} \left(B_{\theta,j}^{n,f}(k) / R_{\theta,j}^{n,f} \right) \leq 1 \ \forall k \tag{3.9}
$$

Table 3.1. Sample Plant Schedule with modeling style-1

	Time													
Variables	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**
$p_0^{n,f}$	1	0	0	0	0	0	0	0	0	0	0	0	0	1
$p_1^{n,f}$	0	1	1	1	1	1	0	0	0	0	0	0	0	0
$p_2^{n,f}$	0	0	0	0	0	0	1	1	1	1	1	1	1	0
$B_{1,1}^{n,f}$	0	0	0	**F**	0	0	0	0	0	0	0	0	0	0
$B_{1,2}^{n,f}$	0	0	0	**F**	0	0	0	0	0	0	0	0	0	0
$B_{2,1}^{n,f}$	0	0	0	0	0	0	0	0	**F**	**F**	**F**	0	0	0
$B_{2,3}^{n,f}$	0	0	0	0	0	0	0	0	**F**	**F**	**F**	0	0	0
$f_1^{n,f}$	**F**	1	**F**	**F**	**F**	**F**	**F**	**F**	**F**	**F**	**F**	**F**	**F**	**F**
$f_2^{n,f}$	**F**	**F**	**F**	**F**	**F**	**F**	1	**F**	**F**	**F**	**F**	**F**	**F**	**F**
$X1_1^{n,f}$	0	0	0	1	1	1	0	0	0	0	0	0	0	0
$X2_1^{n,f}$	2	1	0	0	0	0	0	1	2	2	2	2	2	2
$Y1_1^{n,f}$	0	1	1	1	0	0	0	0	0	0	0	0	0	0
$Y2_1^{n,f}$	0	0	0	0	0	1	2	2	2	2	2	2	2	2
$X1_2^{n,f}$	0	0	0	0	0	0	0	0	1	1	1	0	0	0
$X2_2^{n,f}$	2	2	2	2	2	2	1	0	0	0	0	0	1	2
$Y1_2^{n,f}$	0	0	0	0	0	0	1	1	1	1	1	0	0	0
$Y2_2^{n,f}$	2	2	2	2	1	0	0	0	0	0	0	0	1	2
$zs_1^{n,f}$	0	0	**F**	1	1	1	**F**	0	0	0	0	0	0	0
$zc_1^{n,f}$	**F**	1	1	1	**F**	0	0	0	0	0	0	0	0	0
$zs_2^{n,f}$	0	0	0	0	0	0	0	**F**	1	1	1	**F**	0	0
$zc_2^{n,f}$	0	0	0	0	0	**F**	1	1	1	1	1	**F**	0	0

3.1.5.2 Modeling Style-2

This modeling style requires that an idle production mode must not be defined so that if no production node is active then the all $p_\theta^{n,f}(k)$ variables are zero.

The most significant difference with the first model is the treatment of setup and cleanup requirements. Between the operations of two modes there must be a period of consecutive zeros for all $p_\theta^{n,f}(k)$ variables of length at least the sum of cleanup time of the former mode and the setup time of the latter mode.

The production model is formulated in Eq.(3.10).

$$
\begin{aligned}
& pf_\theta^{n,f}(1) - p_\theta^{n,f}(1) \ge 0 \ \forall f,\theta \\
& pf_\theta^{n,f}(k) - p_\theta^{n,f}(k) + p_\theta^{n,f}(k-1) \ge 0 \ \forall k,f,\theta \\
& pl_\theta^{n,f}(K) - p_\theta^{n,f}(K) \ge 0 \ \forall f,\theta \\
& pl_\theta^{n,f}(k) - p_\theta^{n,f}(k) + p_\theta^{n,f}(k+1) \ge 0 \ \forall k,f,\theta \\
& \sum_\theta p_\theta^{n,f}(k) \le 1 \ \forall k,f,\theta \\
& \sum_{t=1}^{k} \left(pf_\theta^{n,f}(t) - pl_\theta^{n,f}(t) \right) \le 1 \ \forall k,f,\theta \\
& \sum_{t=1}^{k} \left(pf_\theta^{n,f}(t) - pl_\theta^{n,f}(t) \right) \ge 0 \ \forall k,f,\theta \\
& \sum_{\theta' \in \Theta_f^n} \ \sum_{t=\max\left\{1,k-CT_{\theta'}^{n,f}-ST_\theta^{n,f}\right\}}^{k} pl_{\theta'}^{n,f}(t) + pf_\theta^{n,f}(k) \le 1 \ \forall k,f,\theta \\
& B_{\theta,j}^{n,f}(k) \le p_\theta^{n,f}(k) * R_{\theta,j}^{n,f}(k) \ \forall k,f,\theta,j \\
& f_\theta^{n,f}\left(k - ST_\theta^{n,f}\right) - pf_\theta^{n,f}(k) = 0 \ \forall k \ge ST_\theta^{n,f}, f,\theta \\
& f_\theta^{n,f}(k) = 0 \ \forall k < ST_\theta^{n,f}, f,\theta
\end{aligned}
\tag{3.10}
$$

The constraints are not as much intuitive as the former one, and therefore need explanation.

The first two constraints models that if a particular day is a production day, i.e. $p_\theta^{n,f}(k)=1$, then that day must either be the first day of a production campaign, i.e. $pf_\theta^{n,f}(k)=1$, or the day before must also be a production day, i.e. $p_\theta^{n,f}(k-1)=1$. Since there is no other possibility regarding the first day of a production mode the constraints in

Eq.(3.10) must hold. The third and fourth constraints imply that if a particular day is a production day, i.e. $p_{\theta}^{n,f}(k)=1$, then that day must either be the last day of a production campaign, i.e. $pl_{\theta}^{n,f}(k)=1$, or the following must also be a production day, i.e. $p_{\theta}^{n,f}(k+1)=1$. There is no other possibility regarding the last day of production; therefore the third and fourth constraints must hold as well. These four constraints together specify the spread of a production campaign.

The fifth constraint is straightforward, and implies that at most one mode can be active at a time.

The sixth constraint, $\sum_{t=0}^{k}\left(pf_{\theta}^{n,f}(t)-pl_{\theta}^{n,f}(t)\right)\le 1$, is used to express the fact that a production campaign cannot be started before finishing the previous one. Note that, this constraint does not interfere with starting a mode for the first time.

The seventh constraint, $\sum_{t=0}^{k}\left(pf_{\theta}^{n,f}(t)-pl_{\theta}^{n,f}(t)\right)\ge 0$, is used to impose the fact that a campaign cannot finish before it starts.

The eighth constraint, $\sum_{\theta'\in\Theta_f^n}\ \sum_{t=\max\left\{0,k-CT_{\theta'}^{n,f}-ST_{\theta}^{n,f}\right\}}^{k} pl_{\theta'}^{n,f}(t)+pf_{\theta}^{n,f}(k') \le 1$, is used to satisfy the setup time and cleanup time requirements. This constraint models that if a mode starts, then between the last production period of previous mode and the first production period of the current mode, there must be a series of consecutive zeros for variable $pl_{\theta'}^{n,f}(t)$ of length equal to the sum of the cleanup time of the previous mode and the setup time of the current mode.

The ninth constraint, $B_{\theta,j}^{n,f}(k)\le p_{\theta}^{n,f}(k)*R_{\theta,j}^{n,f}(k)$, regulates the sizes of production batches. The tenth and eleventh constraints are used only for marking the beginning of the setup for a particular mode, which is necessary for the determining the time setup cost accrues.

The values of decision variables for the sample plant schedule introduced in the previous section are depicted in Table 3.2.

Table 3.2. Sample Plant Schedule with modeling style-2

	Time													
Variables	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**
$pf_1^{n,f}$	0	0	0	1	0	0	0	0	0	0	0	0	0	0
$p_1^{n,f}$	0	0	0	1	0	0	0	0	0	0	0	0	0	0
$pl_1^{n,f}$	0	0	0	1	0	0	0	0	0	0	0	0	0	0
$pf_2^{n,f}$	0	0	0	0	0	0	0	0	1	0	0	0	0	0
$p_2^{n,f}$	0	0	0	0	0	0	0	0	1	1	1	0	0	0
$pl_2^{n,f}$	0	0	0	0	0	0	0	0	0	0	1	0	0	0
$B_{1,1}^{n,f}$	0	0	0	***F***	0	0	0	0	0	0	0	0	0	0
$B_{1,2}^{n,f}$	0	0	0	***F***	0	0	0	0	0	0	0	0	0	0
$B_{2,1}^{n,f}$	0	0	0	0	0	0	0	0	***F***	***F***	***F***	0	0	0
$B_{2,3}^{n,f}$	0	0	0	0	0	0	0	0	***F***	***F***	***F***	0	0	0
$f_1^{n,f}$	0	1	0	0	0	0	0	0	0	0	0	0	0	0
$f_2^{n,f}$	0	0	0	0	0	0	1	0	0	0	0	0	0	0

Similar to the table of the previous section, "***F***" denotes that the variable is free, and its value is determined after running optimization with other specific parameters, such as costs.

Similar to the first modeling style, it is possible to model the interference of the production of the products in a specific mode with Eq.(3.9).

3.1.5.3 Batch Size Considerations for Production Model

For both of the modeling styles for production described in the previous subsections, it is straightforward to model the production quantity as fixed or a flexible amount between lower and upper limits. The possible cases are as follows.

i. Constant batch size
ii. Flexible batch size

  a. Perfectly flexible between and upper and lower limit or zero
  b. An integral multiple of a fixed amount and batch size is between an upper and lower limit or zero

Constant batch size means that the production quantity is either a fixed positive amount, or is zero.

Perfectly flexible batch size means that the production quantity can take any real value provided that it is in the allowed limits.

3.1.5.3.1 Batch Size Modifications for Modeling Style-1

3.1.5.3.1.1 Constant Batch Size

In order to model production model with constant batch size, parameter $BS_{\theta,j}^{n,f}$ is introduced which defines the constant batch size. Besides, a new variable, $bp_{\theta}^{n,f}(k)$, is introduced either to produce a constant batch size or not to produce when the plant is in the appropriate mode.

Based on these, the modified plant model can be presented in Eq.(3.11).

$$\begin{aligned}
&\sum_{\theta} p_{\theta}^{n,f}(k) = 1 \ \forall k \\
&f_{\theta}^{n,f}(1) \geq p_{\theta}^{n,f}(1) \ \forall \theta \\
&f_{\theta}^{n,f}(k) \geq p_{\theta}^{n,f}(k) - p_{\theta}^{n,f}(k-1) \ \forall k \geq 2, \theta \\
&bp_{\theta}^{n,f}(k) \leq X1_{\theta}^{n,f}(k) * R_{\theta,j}^{n,f} \ \forall k, \theta, j \\
&bp_{\theta}^{n,f}(k) \leq Y1_{\theta}^{n,f}(k) * R_{\theta,j}^{n,f} \ \forall k, \theta, j \\
&\sum_{t=\max\{1,k-ST_{\theta}^{n,f}\}}^{k} p_{\theta}^{n,f}(t) - ST_{\theta}^{n,f} = X1_{\theta}^{n,f}(k) - X2_{\theta}^{n,f}(k) \ \forall k, \theta \\
&X1_{\theta}^{n,f}(k) \leq zs_{\theta}^{n,f}(k) \ \forall k, \theta \\
&X2_{\theta}^{n,f}(k) \leq \left(1 - zs_{\theta}^{n,f}(k)\right) * ST_{\theta}^{n,f} \ \forall k, \theta \\
&\sum_{t=k}^{\min\{K,k+CT_{\theta}^{n,f}\}} p_{\theta}^{n,f}(t) - CT_{\theta}^{n,f} = Y1_{\theta}^{n,f}(k) - Y2_{\theta}^{n,f}(k) \ \forall k, \theta \\
&Y1_{\theta}^{n,f}(k) \leq zc_{\theta}^{n,f}(k) \ \forall k, \theta \\
&Y2_{\theta}^{n,f}(k) \leq \left(1 - zc_{\theta}^{n,f}(k)\right) * CT_{\theta}^{n,f} \ \forall k, \theta \\
&B_{\theta,j}^{n,f}(k) = BS_{\theta,j}^{n,f}(k) * bp_{\theta}^{n,f}(k) \ \forall k, \theta, j
\end{aligned} \tag{3.11}$$

The plant model remains almost the same, the only change is that in fourth and fifth constraints the batch size variable is exchanged by batch production indicator variable and the rates are no more on the right hand side of the constraint.

3.1.5.3.1.2 Flexible Batch Size

Perfectly Flexible Batch Size subject to Upper and Lower Limits

Modeling plant behavior in this case is done in two possible ways. In the first case the upper and lower limits on the batch size are set to $UL_{\theta,j}^{n,f}$ and $LL_{\theta,j}^{n,f}$ respectively, whenever non-zero. In this case, the mathematical description of the batch size choices can be given as follows.

$$B_{\theta,j}^{n,f}(k) = [0] \vee \left[LL_{\theta,j}^{n,f} \leq B_{\theta,j}^{n,f}(k) \leq UL_{\theta,j}^{n,f} \right]$$

Based on these, two possible modeling alternatives can be defined as follows.

i. Defining the batch size variable, $B_{\theta,j}^{n,f}(k)$, as a semi-continuous variable with given upper and lower limit whenever non-zero and setting $R_{\theta,j}^{n,f} = UL_{\theta,j}^{n,f}$ is sufficient. No changes other changes in the model is necessary.

ii. If semi-continuous variables are not going to be utilized, then a binary variable is introduced.

Then, the formulation of plant behavior is given shown in Eq.(3.12).

$$\begin{aligned}
&\sum_{\theta} p_{\theta}^{n,f}(k)=1\ \forall k\\
&f_{\theta}^{n,f}(1)\geq p_{\theta}^{n,f}(1)\ \forall\theta\\
&f_{\theta}^{n,f}(k)\geq p_{\theta}^{n,f}(k)-p_{\theta}^{n,f}(k-1)\ \forall k\geq 2,\theta\\
&ps_{\theta}^{n,f}(k)\leq X1_{\theta}^{n,f}(k)*R_{\theta,j}^{n,f}\ \forall k,\theta,j\\
&ps_{\theta}^{n,f}(k)\leq Y1_{\theta}^{n,f}(k)*R_{\theta,j}^{n,f}\ \forall k,\theta,j\\
&\sum_{t=\max\{1,k-ST_{\theta}^{n,f}\}}^{k} p_{\theta}^{n,f}(t)-ST_{\theta}^{n,f}=X1_{\theta}^{n,f}(k)-X2_{\theta}^{n,f}(k)\ \forall k,\theta\\
&X1_{\theta}^{n,f}(k)\leq zs_{\theta}^{n,f}(k)\ \forall k,\theta\\
&X2_{\theta}^{n,f}(k)\leq\left(1-zs_{\theta}^{n,f}(k)\right)*ST_{\theta}^{n,f}\ \forall k,\theta\\
&\sum_{t=k}^{\min\{K,k+CT_{\theta}^{n,f}\}} p_{\theta}^{n,f}(t)-CT_{\theta}^{n,f}=Y1_{\theta}^{n,f}(k)-Y2_{\theta}^{n,f}(k)\ \forall k,\theta\\
&Y1_{\theta}^{n,f}(k)\leq zc_{\theta}^{n,f}(k)\ \forall k,\theta\\
&Y2_{\theta}^{n,f}(k)\leq\left(1-zc_{\theta}^{n,f}(k)\right)*CT_{\theta}^{n,f}\ \forall k,\theta\\
&LL_{\theta,j}^{n,f}*ps_{\theta}^{n,f}(k)\leq B_{\theta,j}^{n,f}(k)\leq UL_{\theta,j}^{n,f}*ps_{\theta}^{n,f}(k)\ \forall k,\theta,j
\end{aligned}\tag{3.12}$$

The fourth and fifth constraints are changed and the last constraint is added compared to the original model given in Eq.(3.9). Limits on maximum rate parameters are not used in the fourth and fifth constraints and batch size non-zero indicator variable is used on the left hand side. The last constraint models that whenever batch size variable is non-zero then it must conform to the upper and lower bounds.

Integral Multiple Batch Size subject to Upper and Lower Limits

This case is similar to the case with perfectly flexible batch size with the exception that the produced amount must be an integral multiple of a constant, $\omega_{\theta,j}^{n,f}$. In order to model this case, defining an integer batch size multiplier variable $BM_{\theta,j}^{n,f}(k)$ and enforcing Eq.(3.13) is sufficient in addition to the model given in constraints 3.12.

$$B_{\theta,j}^{n,f}(k)=\omega_{\theta,j}^{n,f}*BM_{\theta,j}^{n,f}(k)\ \forall k,\theta,j \tag{3.13}$$

3.1.5.3.2 Batch Size Modifications for Modeling Style-2

3.1.5.3.2.1 Constant Batch Size

Similar to the changes applied in the constant batch size modification of the first modeling style, a constant batch size parameter and a batch production indicator variable is introduced. The model is presented in Eq.(3.14).

$$
\begin{aligned}
& pf_{\theta}^{n,f}(1) - p_{\theta}^{n,f}(1) \geq 0 \ \forall f,\theta \\
& pf_{\theta}^{n,f}(k) - p_{\theta}^{n,f}(k) + p_{\theta}^{n,f}(k-1) \geq 0 \ \forall k,f,\theta \\
& pl_{\theta}^{n,f}(K) - p_{\theta}^{n,f}(K) \geq 0 \ \forall f,\theta \\
& pl_{\theta}^{n,f}(k) - p_{\theta}^{n,f}(k) + p_{\theta}^{n,f}(k+1) \geq 0 \ \forall k,f,\theta \\
& \sum_{\theta} p_{\theta}^{n,f}(k) \leq 1 \ \forall k,f,\theta \\
& \sum_{t=1}^{k} \left(pf_{\theta}^{n,f}(t) - pl_{\theta}^{n,f}(t) \right) \leq 1 \ \forall k,f,\theta \\
& \sum_{t=1}^{k} \left(pf_{\theta}^{n,f}(t) - pl_{\theta}^{n,f}(t) \right) \geq 0 \ \forall k,f,\theta \\
& \sum_{\theta' \in \Theta_f^n} \ \sum_{t=\max\left\{0,k-CT_{\theta'}^{n,f}-ST_{\theta}^{n,f}\right\}}^{k} pl_{\theta'}^{n,f}(t) + pf_{\theta}^{n,f}(k) \leq 1 \ \forall k,f,\theta \\
& f_{\theta}^{n,f}\left(k - ST_{\theta}^{n,f}\right) - pf_{\theta}^{n,f}(k) = 0 \ \forall k \geq ST_{\theta}^{n,f},f,\theta \\
& f_{\theta}^{n,f}(k) = 0 \ \forall k < ST_{\theta}^{n,f},f,\theta \\
& bp_{\theta}^{n,f}(k) \leq p_{\theta}^{n,f}(k) \ \forall k,\theta,j \\
& B_{\theta,j}^{n,f}(k) = BS_{\theta,j}^{n,f}(k) * bp_{\theta}^{n,f}(k) \ \forall k,\theta,j
\end{aligned}
\tag{3.14}
$$

3.1.5.3.2.2 Flexible Batch Size

Perfectly Flexible Batch Size subject to Upper and Lower Limits

Similar to the first modeling style, there are two possible approaches.

i. Defining the batch size variable, $B_{\theta,j}^{n,f}(k)$, as a semi-continuous variable with given upper and lower limit whenever non-zero and setting $R_{\theta,j}^{n,f} = UL_{\theta,j}^{n,f}$ is sufficient. No changes other changes in the model is necessary.
ii. If semi-continuous variables are not going to be utilized, then a binary variable is introduced.

The formulation of plant behavior is given in Eq.(3.15).

$$
\begin{aligned}
& pf_\theta^{n,f}(1) - p_\theta^{n,f}(1) \geq 0 \ \forall f,\theta \\
& pf_\theta^{n,f}(k) - p_\theta^{n,f}(k) + p_\theta^{n,f}(k-1) \geq 0 \ \forall k,f,\theta \\
& pl_\theta^{n,f}(K) - p_\theta^{n,f}(K) \geq 0 \ \forall f,\theta \\
& pl_\theta^{n,f}(k) - p_\theta^{n,f}(k) + p_\theta^{n,f}(k+1) \geq 0 \ \forall k,f,\theta \\
& \sum_\theta p_\theta^{n,f}(k) \leq 1 \ \forall k,f,\theta \\
& \sum_{t=1}^{k}\left(pf_\theta^{n,f}(t) - pl_\theta^{n,f}(t)\right) \leq 1 \ \forall k,f,\theta \\
& \sum_{t=1}^{k}\left(pf_\theta^{n,f}(t) - pl_\theta^{n,f}(t)\right) \geq 0 \ \forall k,f,\theta \\
& \sum_{\theta' \in \Theta_f^n} \ \sum_{t=\max\left\{0,k-CT_{\theta'}^{n,f}-ST_\theta^{n,f}\right\}}^{k} pl_{\theta'}^{n,f}(t) + pf_\theta^{n,f}(k) \ \leq 1 \ \forall k,f,\theta \\
& ps_\theta^{n,f}(k) \leq p_\theta^{n,f}(k) \ \forall k,f,\theta \\
& f_\theta^{n,f}\left(k - ST_\theta^{n,f}\right) - pf_\theta^{n,f}(k) = 0 \ \forall k \geq ST_\theta^{n,f}, f,\theta \\
& f_\theta^{n,f}(k) = 0 \ \forall k < ST_\theta^{n,f}, f,\theta \\
& LL_{\theta,j}^{n,f} * ps_\theta^{n,f}(k) \leq B_{\theta,j}^{n,f}(k) \leq UL_{\theta,j}^{n,f} * ps_\theta^{n,f}(k) \ \forall k,f,\theta,j
\end{aligned}
\tag{3.15}
$$

Integral Multiple Batch Size subject to Upper and Lower Limits

The modifications are exactly the same as those made in the first modeling style. Basically, enforcing Eq.(3.13) and Eq.(3.15) is sufficient.

3.1.5.4 Integrating Production Model with Inventory Constraints

Production and inventory equations are related in two ways: first, whenever a plant produces some products, the inventory of that product increases; second, during production, plant uses up raw materials and, therefore, the inventory of raw materials get lower. In order to model raw material usage, a new set and parameter definition is made.

The inventory of the products that is produced in a production node is the basic inventory equation with one more term, the amount added to inventory by the plants in the production node. The inventory of products that consumed by plant to produce some other products is the basic inventory equation with one more term, the amount that must be subtracted to model material consumption. Therefore, the inventory equation can be written as in Eq.(3.16).

$$\begin{aligned}
I_{j1_i}^{n}(k+1) = I_{j1_i}^{n}(k) &- \sum_{d\in S_n}\sum_{j2_i}\sum_{v}\sum_{t} y_{j1_i,j2_i}^{n,d,v,t}(k) - \sum_{d\in S_n}\sum_{j2_i}\sum_{p} y_{j1_i,j2_i}^{n,d,p}(k) - \sum_{d\in S_n}\sum_{j2_i} y_{j1_i,j2_i}^{n,d}(k) \\
&- \sum_{d\in N}\sum_{v}\sum_{t} y_{j1_i}^{n,d,v,t}(k) - \sum_{d\in N}\sum_{j2_i}\sum_{p} y_{j1_i}^{n,d,p}(k) - \sum_{d\in N} y_{j1_i}^{n,d}(k) \\
&+ \sum_{s\in N}\sum_{v}\sum_{t} y_{j1_i}^{s,n,v,t}\left(k+1-LT_v^{s,n}\right) + \sum_{s\in N}\sum_{p} y_{j1_i}^{s,n,p}(k) + \sum_{s\in N} y_{j1_i}^{s,n}\left(k+1-LT^{s,n}\right) \\
&+ \sum_{e\in E_n}\sum_{v}\sum_{t} y_{j1_i}^{e,n,v,t}\left(k+1-LT_v^{e,n}\right) + \sum_{e\in E_n}\sum_{p} y_{j1_i}^{e,n,p}(k) \\
&+ \sum_{e\in E_n}\sum_{v}\sum_{t} y_{j1_i}^{e,n}\left(k+1-ALT_{j1_i}^{e,n}\right) \\
&+ \sum_{f}\sum_{\theta} B_{\theta,j1_i}^{n,f}(k) - \sum_{f}\sum_{\theta}\sum_{r_{j1_i}} CR_{\theta,r_{j1_i}}^{n,f} * B_{\theta,r_{j1_i}}^{n,f}(k) \quad \forall k \geq 1
\end{aligned} \tag{3.16}$$

Note that, the inventory level equations must be updated as well to prevent material usage more than the available amount. The corresponding product transfer limit constraint is formulated in Eq.(3.17).

$$\begin{aligned}
I_{j1_i}^{n}(k) \geq &\sum_{d\in S_n}\sum_{j2_i}\sum_{v}\sum_{t} y_{j1_i,j2_i}^{n,d,v,t}(k) + \sum_{d\in S_n}\sum_{j2_i}\sum_{p} y_{j1_i,j2_i}^{n,d,p}(k) + \sum_{d\in S_n}\sum_{j2_i} y_{j1_i,j2_i}^{n,d}(k) \\
&+ \sum_{d\in N}\sum_{v}\sum_{t} y_{j1_i}^{n,d,v,t}(k) + \sum_{d\in N}\sum_{j2_i}\sum_{p} y_{j1_i}^{n,d,p}(k) + \sum_{d\in N} y_{j1_i}^{n,d}(k) \\
&+ \sum_{f}\sum_{\theta}\sum_{r_{j1_i}} CR_{\theta,r_{j1_i}}^{n,f} * B_{\theta,r_{j1_i}}^{n,f}(k) \quad \forall k \geq 1
\end{aligned} \tag{3.17}$$

3.1.5.5 Modeling Complex Production Behaviors

With the production model provided in the previous sections and with the usage of inventory and vehicle balance equations, it is possible to construct more complex production models. The approach is to model the complex plant as a system, consisting of production units, transportation units and inventories. A sample plant is given in Fig.3.1.

In the example plant, there are two machines with different capabilities and a press. Machine A converts raw material A to semi-finished A whereas machine B converts raw material B to semi-finished B. The setup and cleanup times are 2 and 1 time units respectively for both machines. The press is capable of processing both semi-finished goods into finished goods. The machines A and B operate only in one production mode and an idle mode whereas press has two production modes and an idle mode. In the first mode, the press processes semi-finished A and in the second mode press processes semi-finished B. The first and second modes require a setup time of 4 and 3 time periods, respectively. The cleanup times of the first modes for the press are zero and one, respectively.

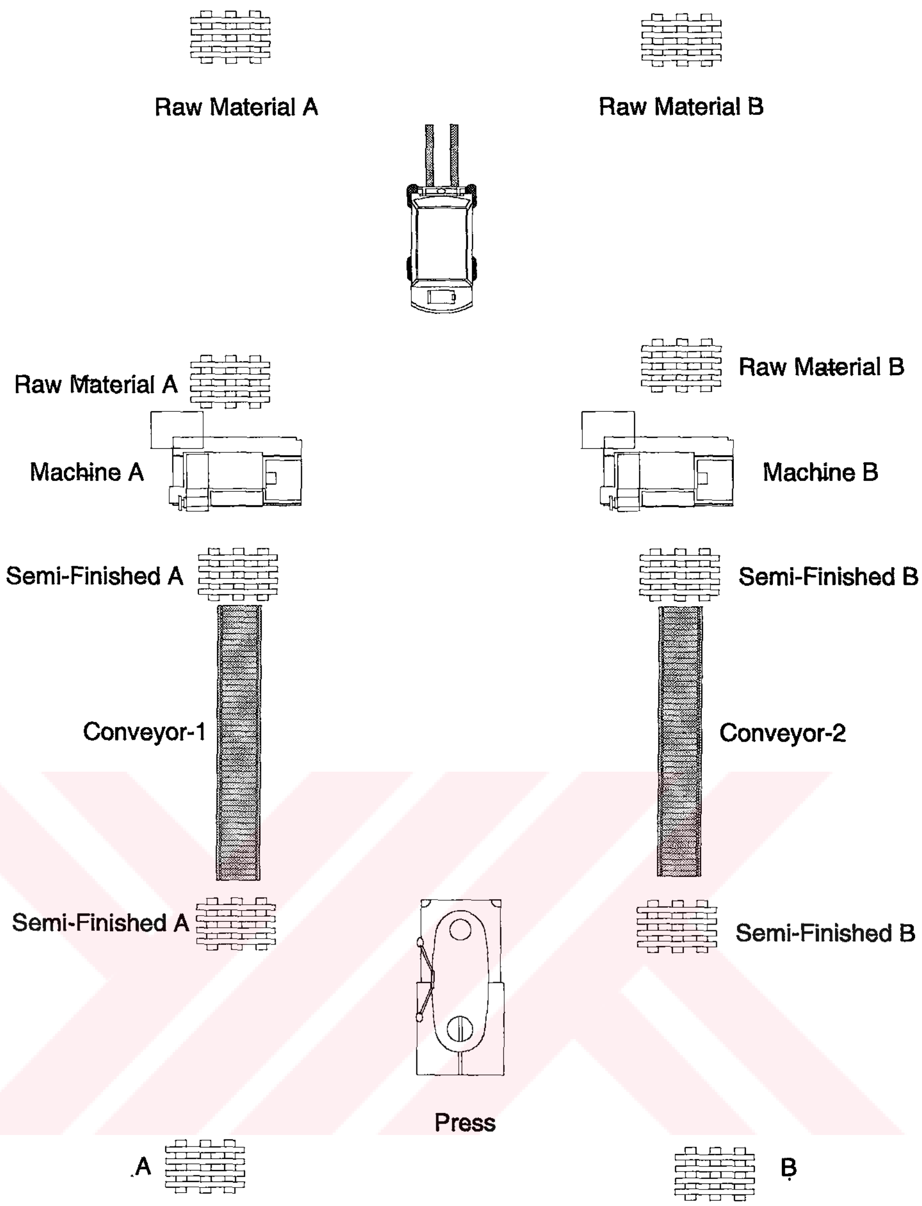

Figure 3.1. Example Plant

In addition to these, work-in-process inventories exist for both machines and the press. Machine A has a maximum production rate of 30 units per time unit and can produce any amount between 0 and 30. On the other hand, machine B can produce any amount between 15 and 45 units per time period. Finally, in the first mode the press produces a constant amount of 50 units per time period, and in the second mode it produces an amount between 30 and 60 provided that the amount is multiple of 15. Conveyors 1 and 2 transport semi-finished good A to press with rate 30 units per time unit and semi-finished good B to press with rate 45 units per time unit, respectively. The raw materials are transported from raw material inventory site to machines by a forklift. A forklift carries raw materials from warehouse to plant. It can carry only one type of raw material at a time. Forklift can carry 20 units of raw material A and 15 units of raw material B per time unit. Time to travel from plant to warehouse is one time unit. The forklift is either at plant or at warehouse. The length of the planning horizon is K time units and all of the machines must be cleaned up at the end of the planning horizon.

Based on this description, the detailed plant model is generated for both modeling styles.

The set of indices are given as follows:

$$k = 1..K$$
$$r = \{rmA, rmB\}$$
$$s = \{smA, smB\}$$
$$p = \{A, B\}$$

3.1.5.5.1 Production Model of Machine A

There are two operating modes for machine A, namely, either idle or in production mode. The idle mode is referred to as mode 0 for modeling style 1 whereas the production mode is referred to as 1. The idle mode is not required in modeling style 2. Therefore, $\theta \in \{0,1\}$ for modeling style1 and $\theta \in \{1\}$ for modeling style 2.

The variables that are used to model machine A in the first modeling style are listed as follows:

$$p_\theta^{n,MA}(k) = \begin{cases} 1, & \text{if machine } A \text{ at node } n \text{ is in mode } \theta \text{ at time } k \\ 0, & \text{otherwise} \end{cases}$$

$B_{1,smA}^{n,MA}(k)$: amount produced of smA in mode 1 at time k at machine A of node n

$$f_\theta^{n,MA}(k) = \begin{cases} 1, & \text{if } k \text{ is the time of initiation of mode } \theta \text{ at machine } A \text{ at node } n \\ 0, & \text{otherwise} \end{cases}$$

$$\left.\begin{array}{l} X1_1^{n,MA}(k), X2_1^{n,MA}(k) \\ Y1_1^{n,MA}(k), Y2_1^{n,MA}(k) \\ zs_1^{n,MA}(k), zc_1^{n,MA}(k) \end{array}\right\} \text{ are as defined in section 3.1.5.1}$$

Based on these variables, the model of A by modeling style 1 is given in Eq.(3.18).

$$\begin{aligned}
&\sum_{\theta} p_{\theta}^{n,MA}(k)=1 \ \forall k \\
&f_{\theta}^{n,MA}(1)\geq p_{\theta}^{n,MA}(1) \ \forall \theta \\
&f_{\theta}^{n,MA}(k)\geq p_{\theta}^{n,MA}(k)-p_{\theta}^{n,MA}(k-1) \ \forall \theta, k\geq 2 \\
&B_{1,smA}^{n,MA}(k)\leq 30 * X1_{1}^{n,MA}(k) \ \forall k \\
&B_{1,smA}^{n,MA}(k)\leq 30 * Y1_{1}^{n,MA}(k) \ \forall k \\
&\sum_{t=\max\{1,k-2\}}^{k} p_{1}^{n,MA}(t)-2 = X1_{1}^{n,MA}(k)-X2_{1}^{n,MA}(k) \ \forall k \\
&X1_{1}^{n,MA}(k)\leq zs_{1}^{n,MA}(k) \ \forall k \\
&X2_{1}^{n,MA}(k)\leq 2*\left(1-zs_{1}^{n,MA}(k)\right) \ \forall k
\end{aligned} \tag{3.18}$$

$$\begin{aligned}
&\sum_{t=k}^{\min\{30,k+1\}} p_{1}^{n,MA}(t)-1 = Y1_{1}^{n,MA}(k)-Y2_{1}^{n,MA}(k) \ \forall k \\
&Y1_{1}^{n,MA}(k)\leq zc_{1}^{n,MA}(k) \ \forall k \\
&Y2_{1}^{n,MA}(k)\leq 1-zc_{1}^{n,MA}(k) \ \forall k
\end{aligned} \tag{3.18}$$

The variables that are used to model machine A in the second modeling style are listed as follows:

$$p_{\theta}^{n,MA}(k)=\begin{cases}1, & \text{if machine } A \text{ at node } n \text{ is in mode } \theta \text{ at time } k\\ 0, & \text{otherwise}\end{cases}$$

$B_{1,smA}^{n,MA}(k)$: amount produced of smA in mode 1 at time k at machine A of node n

$$pl_{\theta}^{n,MA}(k)=\begin{cases}1, & \text{if } k \text{ is the last day of a campaign of mode } \theta \text{ at machine } B \text{ of node } n\\ 0, & \text{otherwise}\end{cases}$$

$$pf_{\theta}^{n,MA}(k)=\begin{cases}1, & \text{if } k \text{ is the first day of a campaign of mode } \theta \text{ at machine } B \text{ of node } n\\ 0, & \text{otherwise}\end{cases}$$

$$f_{\theta}^{n,MA}(k)=\begin{cases}1, & \text{if } k \text{ is the time of initiation of mode } \theta \text{ at machine } B \text{ at node } n\\ 0, & \text{otherwise}\end{cases}$$

Based on these variables, the model of A by modeling style 2 is created in Eq.(3.19).

$$
\begin{aligned}
& pf_\theta^{n,MA}(1) - p_\theta^{n,MA}(1) \geq 0 \ \forall \theta \\
& pf_\theta^{n,MA}(k) - p_\theta^{n,MA}(k) + p_\theta^{n,MA}(k-1) \geq 0 \ \forall \theta, k \geq 1 \\
& pl_\theta^{n,MA}(K) - p_\theta^{n,MA}(K) \geq 0 \ \forall \theta \\
& pl_\theta^{n,MA}(k) - p_\theta^{n,MA}(k) + p_\theta^{n,MA}(k+1) \geq 0 \ \forall \theta, k \leq K-1 \\
& \sum_\theta p_\theta^{n,MA}(k) \leq 1 \ \forall k \\
& \sum_{t=1}^{k} \left(pf_\theta^{n,MA}(t) - pl_\theta^{n,MA}(t) \right) \leq 1 \ \forall \theta, k \\
& \sum_{t=1}^{k} \left(pf_\theta^{n,MA}(t) - pl_\theta^{n,MA}(t) \right) \geq 0 \ \forall \theta, k \\
& \sum_{t=\max\{1,k-3\}}^{k} pl_1^{n,MA}(t) + pf_1^{n,MA}(k) \leq 1 \ \forall k \\
& B_{1,smA}^{n,MA}(k) \leq 30 * p_1^{n,MA}(k) \ \forall k \\
& f_1^{n,MA}(k-2) - pf_1^{n,MA}(k) = 0 \ \forall k \geq 3 \\
& f_1^{n,MA}(k) = 0 \ \forall k \geq 2
\end{aligned} \tag{3.19}
$$

3.1.5.5.2 Production Model of Machine B

Like machine A, there are two operating modes for machine B, namely, either idle or in production mode. The idle mode is referred to as mode 0 for modeling style 1 whereas the production mode is going to be referred to as 1. The idle mode is not required in modeling style 2. Therefore, $\theta \in \{0,1\}$ for modeling style 1 and $\theta \in \{1\}$ for modeling style 2.

The variables that are used to model machine B by utilizing the first modeling style are listed as follows:

$$p_\theta^{n,MB}(k) = \begin{cases} 1, & \text{if machine } B \text{ at node } n \text{ is in mode } \theta \text{ at time } k \\ 0, & \text{otherwise} \end{cases}$$

$$ps_{1,smB}^{n,MB}(k) = \begin{cases} 1, & \text{if machine } B \text{ at node } n \text{ is in mode 1 produces } smB \text{ at time } k \\ 0, & \text{otherwise} \end{cases}$$

$B_{1,smB}^{n,MB}(k)$: amount produced of smB in mode 1 at time k at machine B of node n

$$f_\theta^{n,MB}(k) = \begin{cases} 1, & \text{if } k \text{ is the time of initiation of mode } \theta \text{ at machine } B \text{ at node } n \\ 0, & \text{otherwise} \end{cases}$$

$$\left.\begin{array}{l} X1_1^{n,MB}(k), X2_1^{n,MB}(k) \\ Y1_1^{n,MB}(k), Y2_1^{n,MB}(k) \\ zs_1^{n,MB}(k), zc_1^{n,MB}(k) \end{array}\right\} \text{ are as defined in section 3.1.5.1}$$

Based on these variables, the model of B by modeling style 1 is created in Eq.(3.20).

$$\begin{array}{l} \sum_{\theta} p_{\theta}^{n,MB}(k) = 1 \ \forall k \\ f_{\theta}^{n,MB}(1) \geq p_{\theta}^{n,MB}(1) \ \forall \theta \\ f_{\theta}^{n,MB}(k) \geq p_{\theta}^{n,MB}(k) - p_{\theta}^{n,MB}(k-1) \ \forall \theta, k \geq 2 \\ ps_{1,smB}^{n,MB}(k) \leq X1_1^{n,MB}(k) \ \forall k \\ ps_{1,smB}^{n,MB}(k) \leq Y1_1^{n,MB}(k) \ \forall k \\ \sum_{t=\max\{1,k-2\}}^{k} p_1^{n,MB}(t) - 2 = X1_1^{n,MB}(k) - X2_1^{n,MB}(k) \ \forall k \\ X1_1^{n,MB}(k) \leq zs_1^{n,MB}(k) \ \forall k \\ X2_1^{n,MB}(k) \leq 2 * \left(1 - zs_1^{n,MB}(k)\right) \ \forall k \end{array} \tag{3.20}$$

$$\begin{array}{l} \sum_{t=k}^{\min\{K,k+1\}} p_1^{n,MB}(t) - 1 = Y1_1^{n,MB}(k) - Y2_1^{n,MB}(k) \ \forall k \\ Y1_1^{n,MB}(k) \leq zc_1^{n,MB}(k) \ \forall k \\ Y2_1^{n,MB}(k) \leq 1 - zc_1^{n,MB}(k) \ \forall k \\ 15 * ps_{1,smB}^{n,MB}(k) \leq B_{1,smB}^{n,MB}(k) \leq 45 * ps_{1,smB}^{n,MB}(k) \ \forall k \end{array} \tag{3.20}$$

The variables that are used to model machine B in the second modeling style are listed as follows:

$$p_{\theta}^{n,MB}(k) = \begin{cases} 1, & \text{if machine } B \text{ at node } n \text{ is in mode } \theta \text{ at time } k \\ 0, & \text{otherwise} \end{cases}$$

$$ps_{1,smB}^{n,MB}(k) = \begin{cases} 1, & \text{if machine } B \text{ at node } n \text{ is in mode 1 produces } smB \text{ at time } k \\ 0, & \text{otherwise} \end{cases}$$

$B_{1,smB}^{n,MB}(k)$: amount produced of smB in mode 1 at time k at machine B of node n

$$pl_{\theta}^{n,MB}(k) = \begin{cases} 1, & \text{if } k \text{ is the last day of a production campaign of mode } \theta \text{ at machine } B \text{ of node } n \\ 0, & \text{otherwise} \end{cases}$$

$$pf_{\theta}^{n,MB}(k)=\begin{cases}1, & \text{if } k \text{ is the first day of a production campaign of mode } \theta \text{ at machine } B \text{ of node } n\\ 0, & \text{otherwise}\end{cases}$$

$$f_{\theta}^{n,MB}(k)=\begin{cases}1, & \text{if } k \text{ is the time of initiation of mode } \theta \text{ at machine } B \text{ at node } n\\ 0, & \text{otherwise}\end{cases}$$

Based on these variables, the model of B by modeling style 2 is created in Eq.(3.21).

$$\begin{aligned}
& pf_{\theta}^{n,MB}(1)-p_{\theta}^{n,MB}(1)\geq 0\ \forall\theta\\
& pf_{\theta}^{n,MB}(k)-p_{\theta}^{n,MB}(k)+p_{\theta}^{n,MB}(k-1)\geq 0\ \forall\theta,k\geq 1\\
& pl_{\theta}^{n,MB}(K)-p_{\theta}^{n,MB}(K)\geq 0\ \forall\theta\\
& pl_{\theta}^{n,MB}(k)-p_{\theta}^{n,MB}(k)+p_{\theta}^{n,MB}(k+1)\geq 0\ \forall\theta,k\leq K-1\\
& \sum_{\theta}p_{\theta}^{n,MB}(k)\leq 1\ \forall k\\
& \sum_{t=1}^{k}\left(pf_{\theta}^{n,MB}(t)-pl_{\theta}^{n,MB}(t)\right)\leq 1\ \forall\theta,k\\
& \sum_{t=1}^{k}\left(pf_{\theta}^{n,MB}(t)-pl_{\theta}^{n,MB}(t)\right)\geq 0\ \forall\theta,k\\
& \sum_{t=\max\{1,k-3\}}^{k} pl_{1}^{n,MB}(t)+pf_{1}^{n,MB}(k)\leq 1\ \forall k\\
& f_{1}^{n,MB}(k-2)-pf_{1}^{n,MB}(k)=0\ \forall k\geq 3\\
& ps_{1,smA}^{n,MB}(k)\leq p_{1}^{n,MB}(k)\ \forall k\\
& 15*ps_{1,smB}^{n,MB}(k)\leq B_{1,smB}^{n,MB}(k)\leq 45*ps_{1,smB}^{n,MB}(k)\ \forall k
\end{aligned}\tag{3.21}$$

3.1.5.5.3 Production Model of Press

There are three operating modes for press: the idle mode is referred to as mode 0 whereas the production modes of products A and B are referred to as 1 and 2 respectively. Therefore, $\theta\in\{0,1,2\}$ for modeling style 1. However, since idle mode is not used in modeling style 2, $\theta\in\{1,2\}$ for modeling style 2.

The variables that are used to model press in the first modeling style are listed as follows:

$$p_\theta^{n,PR}(k)=\begin{cases}1, & \text{if press at node } n \text{ is in mode } \theta \text{ at time } k\\ 0, & \text{otherwise}\end{cases}$$

$$bp_{1,A}^{n,PR}(k)=\begin{cases}1, & \text{if press at node } n \text{ is in mode 1 batch produces } A \text{ at time } k\\ 0, & \text{otherwise}\end{cases}$$

$$ps_{2,B}^{n,PR}(k)=\begin{cases}1, & \text{if press at node } n \text{ is in mode 2 produces } B \text{ at time } k\\ 0, & \text{otherwise}\end{cases}$$

$$f_\theta^{n,PR}(k)=\begin{cases}1, & \text{if } k \text{ is the time of initiation of mode } \theta \text{ at press at node } n\\ 0, & \text{otherwise}\end{cases}$$

$B_{1,A}^{n,PR}(k)$: amount produced of product A in mode 1 at time k at press of node n

$B_{2,B}^{n,PR}(k)$: amount produced of product B in mode 2 at time k at press of node n

$BM_{2,B}^{n,PR}(k)$: integer batch size multiplier at time k for product B in mode 2 at press of node n

$$\left.\begin{array}{l} X1_1^{n,MB}(k), X2_1^{n,MB}(k)\\ Y1_1^{n,MB}(k), Y2_1^{n,MB}(k)\\ zs_1^{n,MB}(k), zc_1^{n,MB}(k)\end{array}\right\}\text{ are as defined in section 3.1.5.1}$$

Based on these variables, the model of press by modeling style 1 is created in Eq.(3.22).

$$\begin{aligned}
&\sum_\theta p_\theta^{n,PR}(k)=1\ \forall k\\
&f_\theta^{n,PR}(1)\geq p_\theta^{n,PR}(1)\ \forall\theta\\
&f_\theta^{n,PR}(k)\geq p_\theta^{n,PR}(k)-p_\theta^{n,PR}(k-1)\ \forall\theta,k\geq 2\\
&bp_{1,A}^{n,PR}(k)\leq X1_1^{n,PR}(k)\ \forall k\\
&bp_{1,A}^{n,PR}(k)\leq Y1_1^{n,PR}(k)\ \forall k\\
&ps_{2,B}^{n,PR}(k)\leq X1_2^{n,PR}(k)\ \forall k\\
&ps_{2,B}^{n,PR}(k)\leq Y1_2^{n,PR}(k)\ \forall k\\
&\sum_{t=\max\{1,k-4\}}^{k} p_1^{n,PR}(t)-4=X1_1^{n,PR}(k)-X2_1^{n,PR}(k)\ \forall k\\
&X1_1^{n,PR}(k)\leq zs_1^{n,PR}(k)\ \forall k\\
&X2_1^{n,PR}(k)\leq 4*(1-zs_1^{n,PR}(k))\ \forall k\\
&\sum_{t=\max\{1,k-3\}}^{k} p_2^{n,PR}(t)-3=X1_2^{n,PR}(k)-X2_2^{n,PR}(k)\ \forall k\\
&X1_2^{n,PR}(k)\leq zs_2^{n,PR}(k)\ \forall k\\
&X2_2^{n,PR}(k)\leq 3*(1-zs_2^{n,PR}(k))\ \forall k
\end{aligned}\tag{3.22}$$

$$
\begin{aligned}
&\sum_{t=k}^{\min\{K,k+1\}} p_2^{n,PR}(t) - 1 = Y1_2^{n,PR}(k) - Y2_2^{n,PR}(k)\ \forall k \\
&Y1_2^{n,PR}(k) \le zc_2^{n,PR}(k)\ \forall k \\
&Y2_2^{n,PR}(k) \le 1 - zc_2^{n,PR}(k)\ \forall k \\
&B_{1,A}^{n,PR}(k) = 50 * bp_{1,A}^{n,PR}(k)\ \forall k \\
&B_{2,B}^{n,PR}(k) = 15 * BM_{2,B}^{n,PR}(k)\ \forall k \\
&45 * ps_{1,smB}^{n,B}(k) \le B_{1,B}^{n,PR}(k) \le 60 * ps_{1,B}^{n,PR}(k)\ \forall k
\end{aligned}
\tag{3.22}
$$

The variables that are used to model machine B in the second modeling style are listed as follows:

$$p_\theta^{n,PR}(k) = \begin{cases} 1, & \text{if press at node } n \text{ is in mode } \theta \text{ at time } k \\ 0, & \text{otherwise} \end{cases}$$

$$bp_{1,A}^{n,PR}(k) = \begin{cases} 1, & \text{if press at node } n \text{ is in mode 1 produces } A \text{ at time } k \\ 0, & \text{otherwise} \end{cases}$$

$$ps_{2,B}^{n,PR}(k) = \begin{cases} 1, & \text{if press at node } n \text{ is in mode 2 produces } B \text{ at time } k \\ 0, & \text{otherwise} \end{cases}$$

$B_{1,A}^{n,PR}(k)$: amount produced of A in mode 1 at time k at press of node n

$B_{2,B}^{n,PR}(k)$: amount produced of B in mode 2 at time k at press of node n

$BM_{2,B}^{n,PR}(k)$: integer batch size multiplier at time k for product B in mode 2 at press of node n

$$pl_\theta^{n,PR}(k) = \begin{cases} 1, & \text{if } k \text{ is the last day of a production campaign of mode } \theta \text{ at press of node } n \\ 0, & \text{otherwise} \end{cases}$$

$$pf_\theta^{n,PR}(k) = \begin{cases} 1, & \text{if } k \text{ is the first day of a production campaign of mode } \theta \text{ at press of node } n \\ 0, & \text{otherwise} \end{cases}$$

$$f_\theta^{n,PR}(k) = \begin{cases} 1, & \text{if } k \text{ is the time of initiation of mode } \theta \text{ at press at node } n \\ 0, & \text{otherwise} \end{cases}$$

Based on these variables, the model of press by modeling style 2 is created in Eq.(3.23).

$$
\begin{aligned}
& pf_\theta^{n,PR}(1) - p_\theta^{n,PR}(1) \geq 0 \ \forall \theta \\
& pf_\theta^{n,PR}(k) - p_\theta^{n,PR}(k) + p_\theta^{n,PR}(k-1) \geq 0 \ \forall \theta, k \geq 1 \\
& pl_\theta^{n,PR}(K) - p_\theta^{n,PR}(K) \geq 0 \ \forall \theta \\
& pl_\theta^{n,PR}(k) - p_\theta^{n,PR}(k) + p_\theta^{n,PR}(k+1) \geq 0 \ \forall \theta, k \leq K-1 \\
& \sum_\theta p_\theta^{n,PR}(k) \leq 1 \ \forall k \\
& \sum_{t=1}^{k} \left(pf_\theta^{n,PR}(t) - pl_\theta^{n,PR}(t) \right) \leq 1 \ \forall \theta, k \\
& \sum_{t=1}^{k} \left(pf_\theta^{n,PR}(t) - pl_\theta^{n,PR}(t) \right) \geq 0 \ \forall \theta, k \\
& \sum_{\theta' \in \{1,2\}} \sum_{t=\max\left\{1, k-4-CT_{\theta'}^{n,PR}\right\}}^{k} pl_{\theta'}^{n,PR}(t) + pf_1^{n,PR}(k) \leq 1 \ \forall k \geq 1 \\
& \sum_{\theta' \in \{1,2\}} \sum_{t=\max\left\{1, k-3-CT_{\theta'}^{n,PR}\right\}}^{k} pl_{\theta'}^{n,PR}(t) + pf_2^{n,PR}(k) \leq 1 \ \forall k \geq 1 \\
& f_\theta^{n,PR}\left(k - ST_\theta^{n,PR}\right) - pf_\theta^{n,PR}(k) = 0 \ \forall \theta, k \geq ST_\theta^{n,PR} + 1 \\
& f_\theta^{n,PR}(k) = 0 \ \forall \theta, k \leq ST_\theta^{n,PR} \\
& bp_{1,A}^{n,PR}(k) \leq p_1^{n,PR}(k) \ \forall k \\
& ps_{2,B}^{n,PR}(k) \leq p_2^{n,PR}(k) \ \forall k \\
& B_{1,A}^{n,PR}(k) = 50 * bp_{1,A}^{n,PR}(k) \ \forall k \\
& B_{2,B}^{n,PR}(k) = 15 * BM_{2,B}^{n,PR}(k) \ \forall k \\
& 45 * ps_{2,B}^{n,PR}(k) \leq B_{2,B}^{n,PR}(k) \leq 60 * ps_{2,B}^{n,PR}(k) \ \forall k
\end{aligned}
\tag{3.23}
$$

3.1.5.5.4 Inventory Constraints

The variables that are used in modeling inventory are given below:

$q = r \cup s \cup p$

$L = \{W, MA, MB, PR\}$

$I_r^{n,W}(k)$: inventory of raw material r at the warehouse at node n

$I_r^{n,M}(k)$: inventory of raw material r beside machines A and B at node n

$I_s^{n,M}(k)$: inventory of semi finished product s beside machines A and B at node n

$I_s^{n,PR}(k)$: inventory of semi finished product s beside press at node n

$I_p^{n}(k)$: inventory of product p at node n

$y_q^{n,L1,L2,v,t}(k)$: material sent from $L1$ to $L2$ by vehicle type v in transportation mode t at time k

$L1, L2 \in L$

There are two operation modes for the forklift, namely in mode 1 forklift carries rmA; and in mode 2 forklift carries rmB. The constraints that govern the level of inventory at the warehouse are given in Eq.(3.24).

$$I_{rmA}^{n,W}(k+1) = I_{rmA}^{n,W}(k) - \sum_{d\in\{A\}}\sum_{v\in\{F\}}\sum_{t\in\{1\}} y_{rmA}^{n,W,d,v,t}(k)\ \forall k \geq 1$$
$$I_{rmB}^{n,W}(k+1) = I_{rmB}^{n,W}(k) - \sum_{d\in\{B\}}\sum_{v\in\{F\}}\sum_{t\in\{2\}} y_{rmA}^{n,W,d,v,t}(k)\ \forall k \geq 1 \tag{3.24}$$

The constraints that govern the level of raw material inventory at machine A are given in Eq.(3.25).

$$I_{rmA}^{n,MA}(k+1) = I_{rmA}^{n,MA}(k) - \sum_{\theta\in\{1\}}\sum_{R\in\{smA\}} CR_{\theta,rmA_R}^{n,MA} * B_{\theta,R}^{n,MA}(k) + \sum_{s\in\{W\}}\sum_{v\in\{F\}}\sum_{t\in\{1\}} y_{rmA}^{n,s,MA,v,t}(k)\ \forall k$$
$$I_{rmA}^{n,MA}(k) \geq \sum_{\theta\in\{1\}}\sum_{R\in\{smA\}} CR_{\theta,rmA_R}^{n,MA} * B_{\theta,R}^{n,MA}(k)\ \forall k \tag{3.25}$$

where, $CR_{\theta,rmA_{smA}}^{n,A} = 2$.

The constraints that govern the level of raw material inventory at machine B are given Eq.(3.26).

$$I_{rmB}^{n,MB}(k+1) = I_{rmB}^{n,MB}(k) - \sum_{\theta\in\{1\}}\sum_{R\in\{smB\}} CR_{\theta,rmB_R}^{n,MB} * B_{\theta,R}^{n,MB}(k) + \sum_{s\in\{W\}}\sum_{v\in\{F\}}\sum_{t\in\{2\}} y_{rmB}^{n,s,MB,v,t}(k)\ \forall k$$
$$I_{rmB}^{n,MB}(k) \geq \sum_{\theta\in\{1\}}\sum_{R\in\{smB\}} CR_{\theta,rmB_R}^{n,MB} * B_{\theta,R}^{n,MB}(k)\ \forall k \tag{3.26}$$

where, $CR_{\theta,rmB_{smB}}^{n,B} = 1$.

There is only one operation mode for both conveyors, so there is no need to define a material transfer variable which includes the operation mode. The conveyor from machine A to press is referred to as C1 whereas the conveyor from machine B to press is referred to as C2. Based on this, the constraints that govern the semi-finished inventory level at machine A are given in Eq.(3.27).

$$I_{smA}^{n,MA}(k+1) = I_{smA}^{n,MA}(k) - \sum_{d\in\{PR\}}\sum_{v\in\{C1\}} y_{smA}^{n,MA,d,v}(k) + \sum_{\theta\in\{1\}} B_{1,smA}^{n,MA}(k)\ \forall k$$
$$I_{smA}^{n,MA}(k) \geq \sum_{d\in\{PR\}}\sum_{v\in\{C1\}} y_{smA}^{n,MA,d,v}(k)\ \forall k \tag{3.27}$$

The constraints that govern the semi-finished inventory level at machine B are given in Eq.(3.28).

$$
\begin{aligned}
&I_{smB}^{n,MB}(k+1) = I_{smB}^{n,MB}(k) - \sum_{d\in\{PR\}}\sum_{v\in\{C2\}} y_{smB}^{n,MB,d,v}(k) + \sum_{\theta\in\{1\}} B_{1,smB}^{n,MB}(k)\ \forall k \\
&I_{smB}^{n,MB}(k) \geq \sum_{d\in\{PR\}}\sum_{v\in\{C2\}} y_{smB}^{n,MB,d,v}(k)\ \forall k
\end{aligned}
\tag{3.28}
$$

The constraints that govern the semi-finished inventory level at press are given in Eq.(3.29).

$$
\begin{aligned}
&I_{smA}^{n,PR}(k+1) = I_{smA}^{n,PR}(k) - \sum_{\theta\in\{1\}}\sum_{R\in\{A,B\}} CR_{\theta,smA_R}^{n,PR} * B_{\theta,R}^{n,PR}(k) + \sum_{s\in\{MA\}}\sum_{v\in\{C1\}} y_{smA}^{n,s,PR,v}(k)\ \forall k \\
&I_{smA}^{n,PR}(k) = \sum_{\theta\in\{1\}}\sum_{R\in\{A,B\}} CR_{\theta,smA_R}^{n,PR} * B_{\theta,R}^{n,PR}(k)\ \forall k \\
&I_{smB}^{n,PR}(k+1) = I_{smB}^{n,PR}(k) - \sum_{\theta\in\{1\}}\sum_{R\in\{A,B\}} CR_{\theta,smB_R}^{n,PR} * B_{\theta,R}^{n,PR}(k) + \sum_{s\in\{MB\}}\sum_{v\in\{C1\}} y_{smA}^{n,s,PR,v}(k)\ \forall k \\
&I_{smB}^{n,PR}(k) = \sum_{\theta\in\{1\}}\sum_{R\in\{A,B\}} CR_{\theta,smB_R}^{n,PR} * B_{\theta,R}^{n,PR}(k)\ \forall k
\end{aligned}
\tag{3.29}
$$

where, $CR_{\theta,smA_A}^{n,PR} = 3$, $CR_{\theta,smA_B}^{n,PR} = 2$, $CR_{\theta,smB_A}^{n,PR} = 1$ and $CR_{\theta,smB_B}^{n,PR} = 4$.

Finally, the constraints that govern the finished goods inventory of products A and B are modeled in Eq.(3.30).

$$
\begin{aligned}
&I_A^n(k+1) = I_A^n(k) + B_{1,A}^{n,PR}(k)\ \forall k \\
&I_B^n(k+1) = I_B^n(k) + B_{2,B}^{n,PR}(k)\ \forall k
\end{aligned}
\tag{3.30}
$$

3.1.5.5.5 Transportation Models

There are two types of transportation means in this system, namely conveyors and the forklift. The conveyors can be modeled as a pipe. The following parameters are used in modeling the conveyors.

$$
\begin{aligned}
&VI_{C1}^n = 1 \\
&VI_{C2}^n = 1 \\
&f_{smA}^{C1} = 1/30 \\
&f_{smB}^{C2} = 1/45
\end{aligned}
$$

Hence, the constraints that model the conveyors are given in Eq.(3.31).

$$\begin{aligned}
&\sum_{s\in\{smA\}} f_s^{C1} * y_s^{n,MA,PR,C1}(k) \le VI_{C1}^n \ \forall k \\
&\sum_{s\in\{smB\}} f_s^{C2} * y_s^{n,MB,PR,C2}(k) \le VI_{C2}^n \ \forall k
\end{aligned} \tag{3.31}$$

On the other hand, the forklift can be modeled as standard transportation vehicle; the constraints that define the behavior of the forklift are given in Eq.(3.32).

$$\begin{aligned}
VI_F^{n,W}(k+1) &= VI_F^{n,W}(k) - \sum_{d\in\{MA,MB\}}\sum_{t\in\{1,2\}} VS_{F,t}^{n,W,d}(k) \\
&\quad + \sum_{s\in\{MA,MB\}}\sum_{t\in\{1,2\}} VS_{F,t}^{n,s,W}\left(k+1-LT_F^{n,s,W}\right) \forall k \\
VI_F^{n,MA}(k+1) &= VI_F^{n,MA}(k) - \sum_{d\in\{W\}}\sum_{t\in\{1,2\}} VS_{F,t}^{n,MA,d}(k) \\
&\quad + \sum_{s\in\{W\}}\sum_{t\in\{1,2\}} VS_{F,t}^{n,s,MA}\left(k+1-LT_F^{n,s,MA}\right) \forall k \\
VI_F^{n,MB}(k+1) &= VI_F^{n,MB}(k) - \sum_{d\in\{W\}}\sum_{t\in\{1,2\}} VS_{F,t}^{n,MB,d}(k) \\
&\quad + \sum_{s\in\{W\}}\sum_{t\in\{1,2\}} VS_{F,t}^{n,s,MB}\left(k+1-LT_F^{n,s,MB}\right) \forall k
\end{aligned} \tag{3.32}$$

where, $LT_F^{n,s,d} = 1 \ \forall s,d \in \{W,MA,MB\}$.

In order to define the material transfer limits, the relevant parameters are defined as follows:

$$f_{rmA}^F = 1/20$$
$$f_{rmB}^F = 1/15$$

And, the constraints for the material transfer limits are defined in Eq.(3.33).

$$\begin{aligned}
&\sum_{s\in\{rmA\}} f_s^F * y_s^{n,W,MA,F,1}(k) \le VS_{F,1}^{n,W,MA} \ \forall k \\
&\sum_{s\in\{rmB\}} f_s^F * y_s^{n,W,MB,F,2}(k) \le VS_{F,2}^{n,W,MB} \ \forall k
\end{aligned} \tag{3.33}$$

3.1.5.5.6 Model of the Plant

Eq.(3.24) to Eq.(3.33) and the application of one modeling style for each of the machines A , B and the press that are demonstrated through Eq.(3.18) to Eq.(3.23) collectively describe the behavior of the plant. All variables are greater than or equal to zero.

3.1.6 Market Models

In this section marketing decision variables and elementary demand equations are introduced.

3.1.6.1 Marketing Decision Variables

Marketing decision variables are the variables, which can affect the level of demand, under firm's control. These are different than environmental and competitive-action variables, which also affect demand but are not totally and directly controlled by the firm.

Marketing decision variables are classified into "four P's". These are listed below.

i. Price variables
   a. Allowances and deals
   b. Distribution and Retailer Markups
   c. Discount Structure
ii. Product variables
   a. Quality
   b. Models and sizes
   c. Packaging
   d. Brands
   e. Service
iii. Promotion variables
   a. Advertising
   b. Sales promotion
   c. Personal selling
   d. Publicity
iv. Place variables
   a. Channels of distribution
   b. Outlet location

c. Sales territories
d. Warehousing system

Price is the only marketing mix variable which can directly affect the revenue, which makes it one of the most important marketing mix variables. Furthermore, because of the fact that price affects the quantity sold, it affects costs as well.

The factors that must be considered during the pricing decisions are:

i. The objectives of the organization
ii. Consumers' willingness to pay for the product
iii. The costs of producing and marketing the product
iv. Competition
v. Changes of the second, third and fourth factors over time

3.1.6.2 Microeconomic View of Pricing

Simple Monopoly Pricing in a Static Environment

Firm must consider the willingness of the consumers to pay for its product, namely, their reservation prices. This is modeled via specifying a demand curve, which shows the number of units that can be sold at different prices.

$$Q = a - bP \tag{3.34}$$

where, Q is the quantity sold, P is the price, a is the constant denoting the quantity sold when price is 0, and b is the constant denoting the slope of the demand curve.

The slope of the demand curve is negative which models that the consumers will buy less if price gets higher and vice versa. If the demand curve is represented as $P = (a/b) - (1/b)Q$ then $(a/b) - (1/b)Q$ is the maximum price that consumers are willing to pay for the Q^{th} unit.

A popular shape for the demand function is based on constant price elasticity as given in Eq.(3.35).

$$Q = aP^{-b} \tag{3.35}$$

The exponent (b>0) is the price elasticity.

All of the preceding demand models assume the same key points that impose some limitations to their applicability that can be listed as:

i. The only parties to consider in setting the price are the firm's immediate customers
ii. Competition, if any, is assumed to be passive
iii. Price can be set independent of the levels set for the other marketing variables
iv. Buyers tend to react to price changes similarly
v. Competition remains passive over time

Lilien et al. [34] mentions that Kalish developed a framework for studying dynamic pricing in a monopoly. Two distinctions are made regarding the dynamic effects: the effects that are the result of what has happened previously and those that are not. Those dynamic changes are temporally linked, and then the price in one period may affect demand or cost in another period. Hence, the optimal price in one period is not necessarily the price which maximizes the profit for that particular period. At this point, it makes sense to distinguish between demand related and cost related effects.

Factors such as economic conditions, changes in income, taste, seasonality, the legal environment are those that are not inter-temporally linked. These are exogenous elements and pricing actions in a particular period will not affect the following periods' pricing rule. Endogenous changes in demand are those which are induced by the effect of past decisions on the current and the future demands. The positive effects are:

i. Information effects
ii. Network effects
iii. Other positive phenomena such as brand loyalty and reputation

The negative effects are:

i. Saturation effects
ii. Social snob effects

Another important concept in pricing is reference price. "A reference price is an internal price that consumers use to compare to actual prices". If demand function is linearly related to the reference price then a natural demand function is $Q = a - b(P - P^r)$. The explanations differ for this phenomenon each leading to different demand functions [34]:

i. The Weber-Fechner Law: Individuals respond to relative or proportional change in a stimulus
ii. The Adaptation-Level Theory: Individuals respond to a stimulus relative to the preceding stimulus
iii. The Assimilation Contrast Theory: There is a range of acceptable levels for a stimulus, and individuals respond significantly to response outside that range of acceptable levels
iv. Prospect Theory: Individuals react differently to gains and losses relative to a specific "frame" (the reference price here)

The demand functions are summarized in Table 3.3.

If $Q = aP^{-b}$ is to be used as the demand model, there is a special case for $b = 1$. In this case revenue is a constant regardless of the price. Therefore revenue maximization is meaningless. On the other hand, profit maximization becomes cost minimization since revenue is a constant and costs decrease the profit.

In this study, in addition to $Q = aP^{-1}$, any linear demand function is allowed provided that the algorithm given in chapter 4 is used.

Table 3.3. Demand models developed from psychological theories [34]

Theory	**Model**
Standard Demand Model	$Q_t = a + bp_t^0 + cX_t + \varepsilon_t$
Weber-Fechner Models	$Q_t = (a - bd) + b(p_t^0 - f\ln p_t^0) + cX_t + \varepsilon_t$ $Q_t = a + b\ln\left[\frac{(p_t^0 - p_t^r)}{p_t^r}\right] + cX_t + \varepsilon_t \ \forall (p_t^0 - p_t^r) > 0$
Adaptation-Level Theory	$Q_t = a + b(p_t^0 - e^{d+f\ln p_{t-1}^0}) + cX_t + \varepsilon_t$ $Q_t = (a - bd) + b(p_t^0 - fp_{t-1}^0) + cX_t + \varepsilon_t$
Assimilation-Contrast Theory	$Q_t = (a - bd) + b(p_t^0 - fp_t^0 - gp_t^{0^2} - hp_t^{0^3}) + cX_t + \varepsilon_t$
Prospect Theory	$Q_t = a + b_1(p_t^0 - p_t^r)d_1 + b_2(p_t^0 - p_t^r)d_2 + cX_t + \varepsilon_t,$ $where\ d_1 = \begin{cases} 1\ if\ (p_t^0 - p_t^r) > 0 \\ 0\ otherwise \end{cases}$ $d_2 = \begin{cases} 1\ if\ (p_t^0 - p_t^r) < 0 \\ 0\ otherwise \end{cases}$

$$p_t^0 = current\ observed\ price$$
$$p_t^r = current\ reference\ price$$
$$X_t = other\ marketing\ variables$$
$$Q_t = demand$$
$$a, b, c, d, f, g, h = parameters$$

3.2 Modeling of Supply Chain Nodes

3.2.1 Non Production Nodes

In a contemporary supply chain network, non-production nodes are of two types: inventory holding nodes and cross-docking nodes, which do not hold inventory and only serve for transportation purposes. In this study, only inventory holding nodes are described.

A non-production inventory holding node serves for two reasons in a supply chain: first, serving end customers; second, serving as a storage point to serve other supply chain nodes. A classical inventory holding node is shown in Fig.3.2.

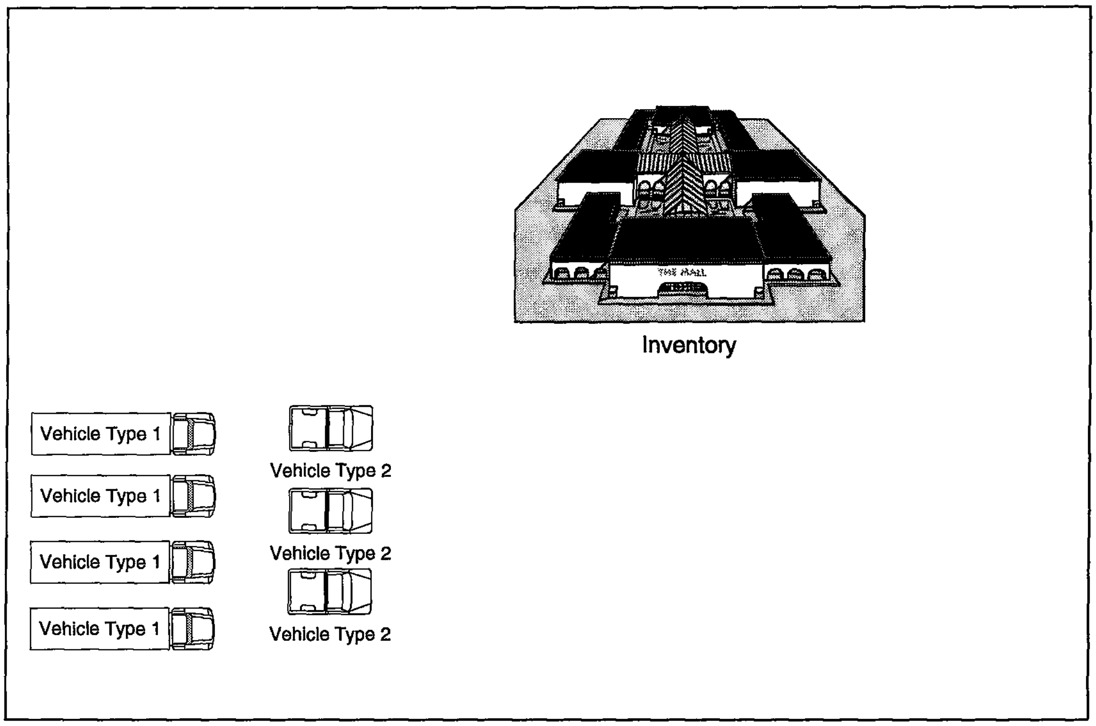

Figure 3.2. A Classical Non Production Node

A non production node is described by inventory balance; transportation means and product transfer limits constraints.

A model is created for the example case shown in Fig.3.3.

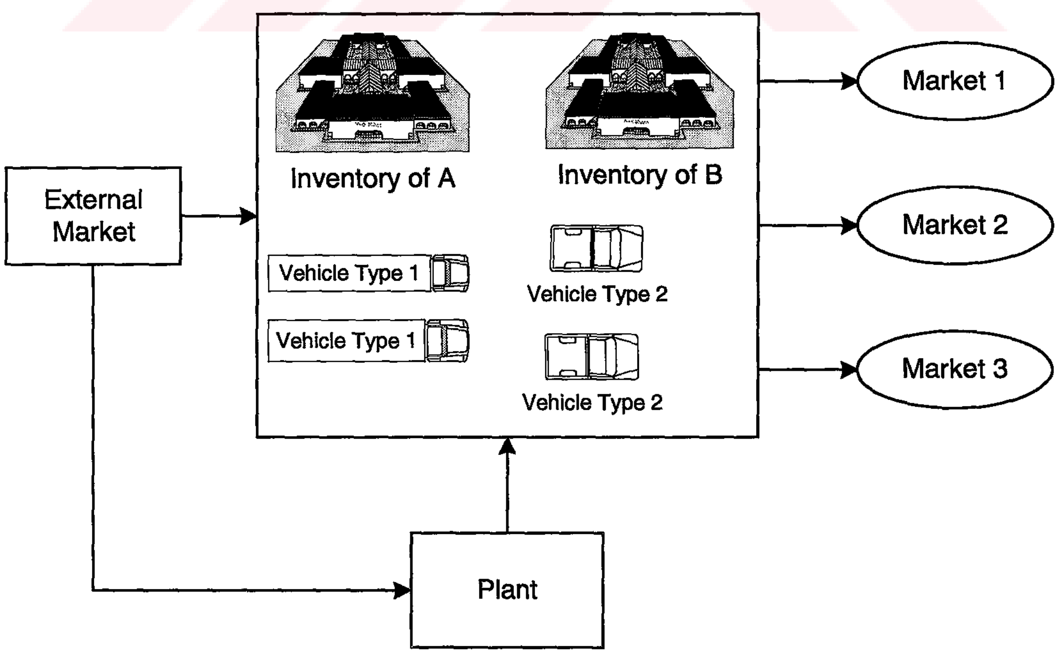

Figure 3.3. Example Non-Production Node Scenario

The non-production node holds inventory of products A and B. There are two types of vehicles with two different transportation modes and respective capacities as given in Table 3.4. The vehicles can carry each product to the amount given in Table 3.4, irrespective of the amount the other product carried.

Table 3.4. The operational modes and respective capacities of vehicles

Vehicle	Mode	A	B
V1	1	50	30
	2	30	80
V2	1	100	80
	2	80	120

There is an external market, EM that supplies products A and B without requiring a vehicle from the non-production node. However, it is possible to transport product A from the external market to the non-production node using vehicle type 1. The plant, which is denoted by PN, can supply products A and B by using both transportation vehicles and also the transportation of product A from the plant to the non-production node can be outsourced. The time to travel between the plant and the non-production node for all types of vehicles is 1. The time to travel from the non-production node to market and the time to go from market to non-production node is 1. The agreed lead time of transportation from the external market to the non-production node is 3. The time to travel between the non-production node and the external market with vehicle type 1 is 2.

There are three markets that are only served by the non-production node. The first market demands product A and accepts product B as a substitute of A, two units of product B can be used to satisfy one unit of demand for product A, and the demand is related to price linearly. The second market demands product A, and the demand is not related to price. This market requires the products to be delivered by vehicles. The third market demands B and there is a linear relationship between price and demand. The set of markets are denoted by $M = \{M1, M2, M3\}$.

3.2.1.1 Inventory Constraints

The inventory constraints are given in Eq.(3.36).

$$
\begin{aligned}
I_A^n(k+1) &= I_A^n(k) - \sum_{d\in\{M1\}} y_A^{n,d}(k) - \sum_{d\in\{M2\}}\sum_{v\in\{v1,v2\}}\sum_{t\in\{1,2\}} y_A^{n,d,v,t}(k) \\
&\quad + \sum_{s\in\{PN\}}\sum_{v\in\{v1,v2\}}\sum_{t\in\{1,2\}} y_A^{s,n,v,t}\left(k+1-LT_v^{s,n}\right) + \sum_{s\in\{PN\}} y_A^{s,n}\left(k+1-LT_v^{s,n}\right) \\
&\quad + \sum_{s\in\{EM\}}\sum_{v\in\{v1\}}\sum_{t\in\{1,2\}} y_A^{s,n,v,t}\left(k+1-LT_v^{s,n}\right) + \sum_{s\in\{EM\}} y_A^{s,n}\left(k+1-ALT_v^{s,n}\right)\ \forall k \\
I_A^n(k) &\geq \sum_{d\in\{M1\}} y_A^{n,d}(k) + \sum_{d\in\{M2\}}\sum_{v\in\{v1,v2\}}\sum_{t\in\{1,2\}} y_A^{n,d,v,t}(k)\ \forall k \\
I_B^n(k+1) &= I_B^n(k) - \sum_{d\in\{M1\}} y_B^{n,d}(k) - \sum_{d\in\{M3\}} y_B^{n,d}(k) \\
&\quad + \sum_{s\in\{PN\}}\sum_{v\in\{v1,v2\}}\sum_{t\in\{1,2\}} y_B^{s,n,v,t}\left(k+1-LT_v^{s,n}\right) + \sum_{s\in\{EM\}} y_B^{s,n}\left(k+1-ALT_v^{s,n}\right)\ \forall k \\
I_B^n(k) &\geq \sum_{d\in\{M1\}} y_B^{n,d}(k) + \sum_{d\in\{M3\}} y_B^{n,d}(k)\ \forall k
\end{aligned}
\tag{3.36}
$$

The terms that include material transfer variables with lead times are included only when their time index $k+1-LT_v^{s,n}$ or $k+1-ALT_v^{s,n}$ is greater than or equal to 1. An example is provided in Eq.(3.37).

$$
\begin{aligned}
I_A^n(3) &= I_A^n(2) - \sum_{d\in\{M1\}} y_A^{n,d}(2) - \sum_{d\in\{M2\}}\sum_{v\in\{v1,v2\}}\sum_{t\in\{1,2\}} y_A^{n,d,v,t}(2) \\
&\quad + \sum_{s\in\{PN\}}\sum_{v\in\{v1,v2\}}\sum_{t\in\{1,2\}} y_A^{s,n,v,t}(2) + \sum_{s\in\{PN\}} y_A^{s,n}(2) \\
&\quad + \sum_{s\in\{EM\}}\sum_{v\in\{v1\}}\sum_{t\in\{1,2\}} y_A^{s,n,v,t}(1)
\end{aligned}
\tag{3.37}
$$

Since $3-ALT_v^{EM,n}$ evaluates to 0, $y_A^{EM,n}\left(3-ALT_v^{s,n}\right)$ is not included in the inventory balance equation.

3.2.1.2 Transportation Models

All transportation means in the non-production node can be modeled as standard transportation vehicles. The constraints are given in Eq.(3.38).

$$
\begin{aligned}
VI_{v1}^{n}(k+1) &= VI_{v1}^{n}(k) - \sum_{d\in\{PN,EM,M2\}}\sum_{t\in\{1,2\}} VS_{v1,t}^{n,d}(k) + \sum_{s\in\{PN\}}\sum_{t\in\{1,2\}} VS_{v1,t}^{s,n}\left(k+1-LT_{v1}^{s,n}\right) \\
&\quad + \sum_{d\in\{EM,M2\}}\sum_{t\in\{1,2\}} VS_{v1,t}^{n,d}\left(k+1-LT_{v1}^{n,d}-LT_{v1}^{d,n}\right)\ \forall k \\
VI_{v1}^{n}(k+1) &\geq \sum_{d\in\{PN,EM,M2\}}\sum_{t\in\{1,2\}} VS_{v1,t}^{n,d}(k)\ \forall k \\
VI_{v2}^{n}(k+1) &= VI_{v2}^{n}(k) - \sum_{d\in\{PN,M2\}}\sum_{t\in\{1,2\}} VS_{v2,t}^{n,d}(k) + \sum_{s\in\{PN\}}\sum_{t\in\{1,2\}} VS_{v2,t}^{s,n}\left(k+1-LT_{v2}^{s,n}\right) \\
&\quad + \sum_{d\in\{M2\}}\sum_{t\in\{1,2\}} VS_{v2,t}^{n,d}\left(k+1-LT_{v2}^{n,d}-LT_{v2}^{d,n}\right)\ \forall k \\
VI_{v2}^{n}(k+1) &\geq \sum_{d\in\{PN,M2\}}\sum_{t\in\{1,2\}} VS_{v2,t}^{n,d}(k)\ \forall k
\end{aligned}
\tag{3.38}
$$

Like inventory balance equations, the vehicle sending terms that include lead time in the time index are included in a specific equation when their time index evaluates to a value greater than or equal to 1. An example is provided in Eq.(3.39).

$$
VI_{v1}^{n}(2) = VI_{v1}^{n}(1) - \sum_{d\in\{PN,EM,M2\}}\sum_{t\in\{1,2\}} VS_{v1,t}^{n,d}(1) + \sum_{s\in\{PN\}}\sum_{t\in\{1,2\}} VS_{v1,t}^{s,n}(1)
\tag{3.39}
$$

The material transfer limit constraints are defined in Eq.(3.40).

$$
\begin{aligned}
&\sum_{p\in\{A\}} f_{p}^{v1,t} * y_{p}^{EM1,n,v1,t}(k) \leq VS_{v1,t}^{n,EM}\left(k-LT_{v1}^{n,EM}\right)\ \forall t\in\{1,2\}, k\geq LT_{v1}^{n,EM}+1 \\
&y_{p}^{EM1,n,v1,t}(k) = 0\ \forall t\in\{1,2\}, k\leq LT_{v1}^{n,EM} \\
&\sum_{p\in\{A\}} f_{p}^{v1,t} * y_{p}^{n,M2,v1,t}(k) \leq VS_{v1,t}^{n,M2}\left(k-LT_{v1}^{n,M2}\right)\ \forall t,\in\{1,2\}, k\geq LT_{v1}^{n,M2}+1 \\
&y_{p}^{n,M2,v1,t}(k) = 0\ \forall t, k\leq LT_{v1}^{n,M2} \\
&\sum_{p\in\{A\}} f_{p}^{v2,t} * y_{p}^{n,M2,v2,t}(k) \leq VS_{v2,t}^{n,M2}\left(k-LT_{v2}^{n,M2}\right)\ \forall t\in\{1,2\}, k\geq LT_{v2}^{n,M2}+1 \\
&y_{p}^{n,M2,v2,t}(k) = 0\ \forall t\in\{1,2\}, k\leq LT_{v2}^{n,M2}
\end{aligned}
\tag{3.40}
$$

where, $f_A^{v1,1} = 1/50$, $f_A^{v1,2} = 1/30$, $f_A^{v2,1} = 1/100$, $f_A^{v2,2} = 1/80$, $f_B^{v1,1} = 1/30$, $f_A^{v1,2} = 1/80$, $f_B^{v2,1} = 1/80$, $f_B^{v2,2} = 1/120$.

3.2.1.3 Model of the Non-Production Node

Eq.(3.36), Eq.(3.38) and Eq.(3.40) model the non-production node, where all variables must be greater than or equal to zero.

3.2.2 Production Nodes

A production node can be defined as a supply chain node where products are converted from one form to another. The conversion can be from finished goods to raw materials (recycling); or from raw materials and semi-finished goods to other semi-finished or finished goods. In most general terms, the difference of a PN from an NPN is only the conversion ability and a PN can have all properties of an NPN. The conversion operations are carried out by a production complex, including at least one plant. A classical PN is shown in Fig.3.4.

It must also be noted that, if there is more than one plant in a production complex, the plants can have different capabilities. It is possible to model very complex production environments with this building block.

A production node is described by inventory balance; transportation means and product transfer limits constraints and one or more plant models.

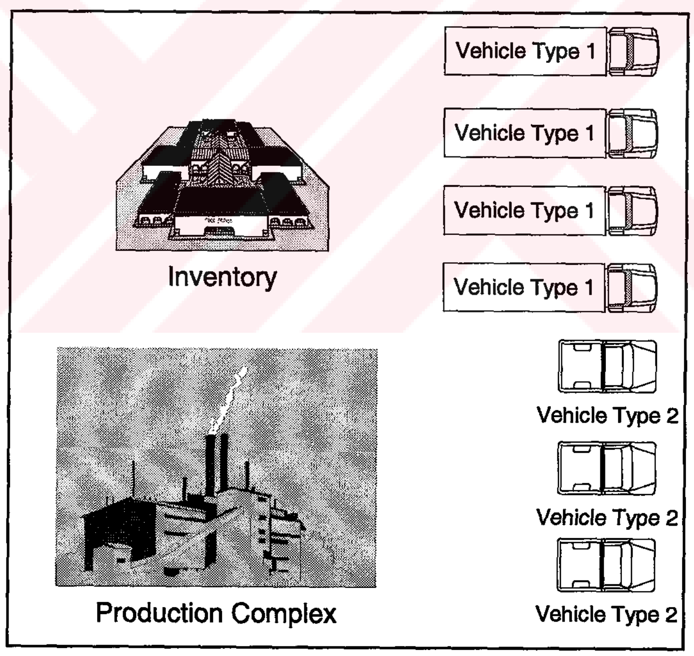

Figure 3.4. A classical Production Node

A model is created for the example case given in Fig.3.5.

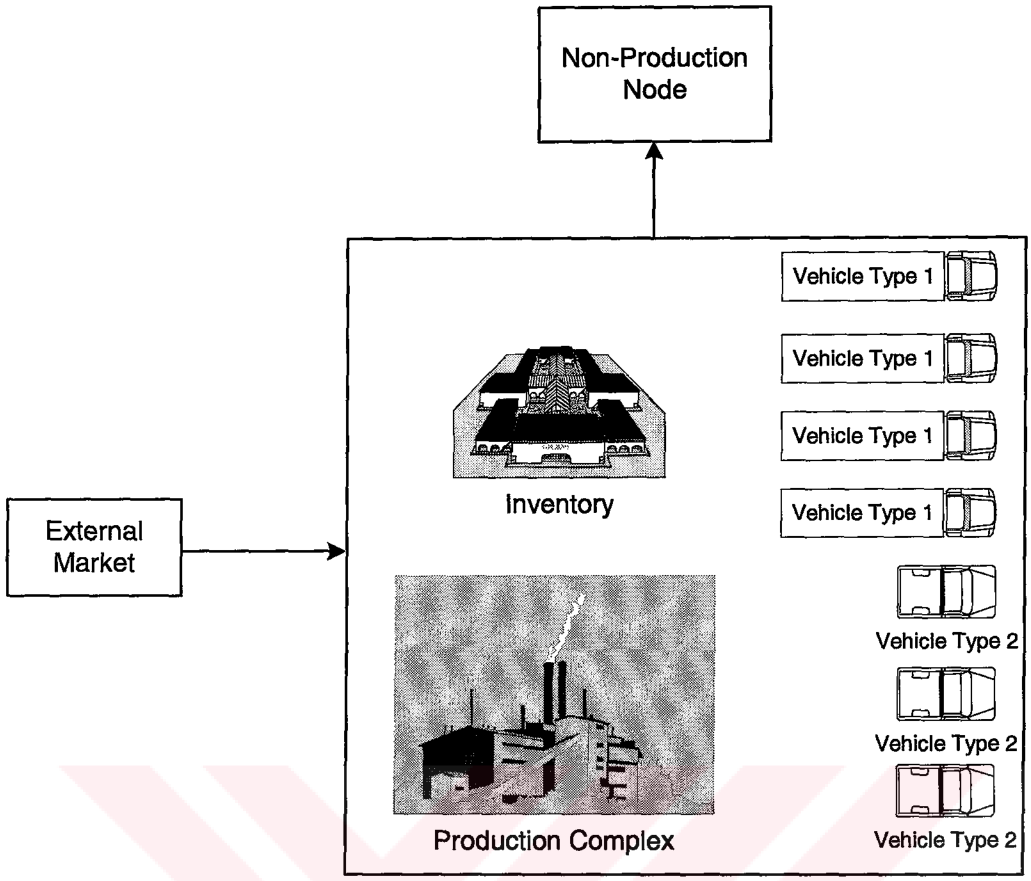

Figure 3.5. Example Production Node Scenario

The production complex is the plant modeled in section 3.1.5.5. The vehicles, their capacities and the lead times between non-production node and the plant are the same as given in section 3.2.1. External market can supply raw materials rmA and rmB to the production node with an agreed lead time of 1. The vehicles transport products A and B from the production node to the non-production node, which is denoted as NPN. Transportation of product A from the production node to the non-production node can be outsourced.

In order to model this production node, the raw material inventory constraints at the warehouse and finished good inventories given in equations 3.24 and 3.30, respectively, of the plant model given in section 3.1.5.5 must be changed and material transfer limits as well as vehicle balance equations must be add for inter-node material transfers.

3.2.2.1 Inventory Constraints

Only raw material and finished good inventory constraints are going to be derived. The other inventory constraints remain the same as demonstrated in Eq.(3.25) through Eq.(3.29). The raw material inventory constraints are defined in Eq.(3.41).

$$\begin{aligned} I_{rmA}^{n}(k+1) &= I_{rmA}^{n}(k) - \sum_{d\in\{A\}}\sum_{v\in\{F\}}\sum_{t\in\{1\}} y_{rmA}^{n,W,d,v,t}(k) + \sum_{s\in\{EM\}} y_{rmA}^{s,n}\left(k+1-ALT_{rmA}^{s,n}\right) \ \forall k \geq 1 \\ I_{rmB}^{n}(k+1) &= I_{rmB}^{n}(k) - \sum_{d\in\{B\}}\sum_{v\in\{F\}}\sum_{t\in\{2\}} y_{rmA}^{n,W,d,v,t}(k) + \sum_{s\in\{EM\}} y_{rmB}^{s,n}\left(k+1-ALT_{rmB}^{s,n}\right) \ \forall k \geq 1 \end{aligned} \tag{3.41}$$

The finished goods inventory constraints are defined in 3.42.

$$\begin{aligned} I_{A}^{n}(k+1) &= I_{A}^{n}(k) + B_{1,A}^{n,PR}(k) - \sum_{d\in\{NPN\}}\sum_{v\in\{v1,v2\}}\sum_{t\in\{1,2\}} y_{A}^{n,d,v,t}(k) - \sum_{d\in\{NPN\}} y_{A}^{n,d}(k) \ \forall k \\ I_{A}^{n}(k+1) &\geq \sum_{d\in\{NPN\}}\sum_{v\in\{v1,v2\}}\sum_{t\in\{1,2\}} y_{A}^{n,d,v,t}(k) + \sum_{d\in\{NPN\}} y_{A}^{n,d}(k) \ \forall k \\ I_{B}^{n}(k+1) &= I_{B}^{n}(k) + B_{2,B}^{n,PR}(k) - \sum_{d\in\{NPN\}}\sum_{v\in\{v1,v2\}}\sum_{t\in\{1,2\}} y_{B}^{n,d,v,t}(k) \ \forall k \\ I_{B}^{n}(k+1) &\geq \sum_{d\in\{NPN\}}\sum_{v\in\{v1,v2\}}\sum_{t\in\{1,2\}} y_{B}^{n,d,v,t}(k) \ \forall k \end{aligned} \tag{3.42}$$

3.2.2.2 Transportation Models

The vehicles that transport products A and B to non-production node can be modeled as standard transportation vehicles. The constraints are given in Eq.(3.43).

$$\begin{aligned} VI_{v1}^{n}(k+1) &= VI_{v1}^{n}(k) - \sum_{d\in\{NPN\}}\sum_{t\in\{1,2\}} VS_{v1,t}^{n,d}(k) + \sum_{s\in\{NPN\}}\sum_{t\in\{1,2\}} VS_{v1,t}^{s,n}\left(k+1-LT_{v1}^{s,n}\right) \ \forall k \\ VI_{v1}^{n}(k+1) &\geq \sum_{d\in\{NPN\}}\sum_{t\in\{1,2\}} VS_{v1,t}^{n,d}(k) \ \forall k \\ VI_{v2}^{n}(k+1) &= VI_{v2}^{n}(k) - \sum_{d\in\{NPN\}}\sum_{t\in\{1,2\}} VS_{v2,t}^{n,d}(k) + \sum_{s\in\{NPN\}}\sum_{t\in\{1,2\}} VS_{v2,t}^{s,n}\left(k+1-LT_{v2}^{s,n}\right) \ \forall k \\ VI_{v2}^{n}(k+1) &\geq \sum_{d\in\{NPN\}}\sum_{t\in\{1,2\}} VS_{v2,t}^{n,d}(k) \ \forall k \end{aligned} \tag{3.43}$$

And the constraints for material transfer limits are defined in Eq.(3.44).

$$\begin{aligned}
&\sum_{p\in\{A\}} f_p^{v1,t} * y_p^{n,NPN,v1,t}(k) \le VS_{v1,t}^{n,NPN}(k)\ \forall t,\in\{1,2\}, k \ge LT_{v1}^{n,NPN}+1\\
&y_A^{n,NPN,v1,t}(k)=0\ \forall t, k \le LT_{v1}^{n,NPN}\\
&\sum_{p\in\{B\}} f_p^{v1,t} * y_p^{n,NPN,v1,t}(k) \le VS_{v1,t}^{n,NPN}(k)\ \forall t,\in\{1,2\}, k \ge LT_{v1}^{n,NPN}+1\\
&y_B^{n,NPN,v1,t}(k)=0\ \forall t, k \le LT_{v1}^{n,NPN}\\
&\sum_{p\in\{A\}} f_p^{v2,t} * y_p^{n,NPN,v2,t}(k) \le VS_{v2,t}^{n,NPN}(k)\ \forall t\in\{1,2\}, k \ge LT_{v2}^{n,NPN}+1\\
&y_A^{n,NPN,v2,t}(k)=0\ \forall t\in\{1,2\}, k \le LT_{v2}^{n,NPN}\\
&\sum_{p\in\{B\}} f_p^{v2,t} * y_p^{n,NPN,v2,t}(k) \le VS_{v2,t}^{n,NPN}(k)\ \forall t\in\{1,2\}, k \ge LT_{v2}^{n,NPN}+1\\
&y_B^{n,NPN,v2,t}(k)=0\ \forall t\in\{1,2\}, k \le LT_{v2}^{n,NPN}
\end{aligned} \tag{3.42}$$

where, $f_A^{v1,1}=1/50$, $f_A^{v1,2}=1/30$, $f_A^{v2,1}=1/100$, $f_A^{v2,2}=1/80$, $f_B^{v1,1}=1/30$, $f_A^{v1,2}=1/80$, $f_B^{v2,1}=1/80$, $f_B^{v2,2}=1/120$.

3.2.2.3 Model of the Production Node

Eq.(3.25) to Eq.(3.29), Eq.(3.41) to Eq.(3.44) and the application of the one modeling style for each machine as demonstrated in through Eq.(3.18) to Eq.(3.23) model the production node. All variables must be greater than or equal to zero.

3.2.3 Markets

In this framework, a market is a supply chain node which neither stores nor produces any product, but is a demand sink for products which is specified by

i. A possibly price related demand function for a specific product
ii. A set of acceptable substitute products, each with an associated penalty and effective number of units or amount of this product required to satisfy the demand for one unit of the specifically desired product in the market
iii. The availability of backordering and backordering cost if appropriate
iv. The requirement of vehicles for the delivery of products

Therefore, the mathematical specification of a market requires possibly an order balance and a price related demand function which are created with respect to the other attributes of the market. If backordering is not allowed then the material transfer limit with respect to demand must be incorporated as well. The requirement of vehicles for the delivery of products must be incorporated by constraints that govern the nodes supplying this market, which was demonstrated in section 3.2.1.2. Models of three markets that are depicted in Fig.3.6 are demonstrated.

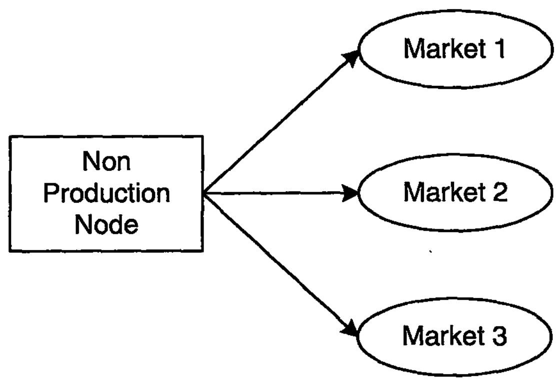

Figure 3.6. Example Market Scenario

First market is specified by:

i. Linearly price related demand for A
ii. B is accepted as a substitute for A, with two units of B substituting one unit of A
iii. Backordering is allowed

Second market is specified by:

i. Price unrelated demand for A
ii. Requires product delivery, the vehicles are the two vehicles introduced in section 3.2.1
iii. Backordering is allowed

Third market is specified by:

i. Linearly price related demand for B
ii. Backordering is not allowed

The market models are derived in Eq.(3.45).

$$
\begin{aligned}
&O_A^{M1}(k+1) = O_A^{M1}(k) - \sum_{n\in\{NPN\}} \sum_{j\in\{A,B\}} \alpha_j^A * y_{j,A}^{n,d}(k) + d_A^{M1}(k) \ \forall k \\
&d_A^{M1}(k) = a^{M1} - b^{M1} * p_A^{M1}(k) \ \forall k \\
&O_A^{M2}(k+1) = O_A^{M2}(k) - \sum_{n\in\{NPN\}} \sum_{v\in\{v1,v2\}} \sum_{t\in\{1,2\}} y_A^{n,M2,v,t}\left(k - LT_v^{n,M2}\right) + d_A^{M1}(k) \ \forall k \\
&d_B^{M3}(k) = a^{M3} - b^{M3} * p_B^{M3}(k) \ \forall k \\
&\sum_{n\in\{NPN\}} y_A^{n,M3}(k) \le d_B^{M3}(k) \ \forall k
\end{aligned} \tag{3.45}
$$

Naturally, all variables must be greater than or equal to zero, and $\alpha_A^A = 1$, $\alpha_B^A = 1/2$. A dumping node can be described as a market whose capacity defined as a parameter and generates negative revenue per unit of product delivered with no backordering.

3.2.4 External Markets

External markets are suppliers of raw materials, semi-finished products and finished products. They can supply materials to all production and non-production nodes in a limited amount. The limit of maximum supply each time period is a parameter. The transportation of materials can either be done by the external market or by the vehicles sent by the ordering supply chain node, and the implied vehicle related constraints must be incorporated to the ordering supply chain node as demonstrated in section 3.2.1.2. Therefore, in their most basic form, the external markets are simply sources of materials. It is also straightforward to devise lower and upper bounding constraints on the amount of material transferred.

Hence, external markets are simply modeled as material transfer limit constraints. An example external market is provided in Fig.3.7.

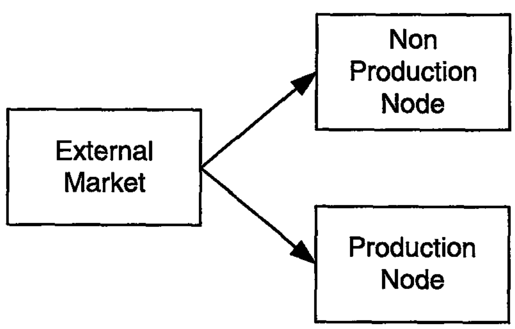

Figure 3.7. Example External Market Scenario

The non-production node is the one described in section 3.2.1 and the production node is the one described in section 3.2.2. Therefore, the external market that was serving both the production node and the non-production node is the same. The model of this external market is given in Eq.(3.46).

$$\begin{aligned}
&\sum_{d\in\{NPN\}}\sum_{v\in\{v1\}}\sum_{t\in\{1,2\}} y_A^{EM,d,v,t}(k)+\sum_{d\in\{NPN\}} y_A^{EM,d}(k)\le I_A^{EM}(k)\ \forall k\\
&\sum_{d\in\{NPN\}} y_B^{EM,d}(k)\le I_B^{EM}(k)\ \forall k\\
&\sum_{d\in\{NPN\}} y_{rmA}^{EM,d}(k)\le I_{rmA}^{EM}(k)\ \forall k\\
&\sum_{d\in\{NPN\}} y_{rmB}^{EM,d}(k)\le I_{rmB}^{EM}(k)\ \forall k
\end{aligned} \tag{3.46}$$

where, $I_p^{EM}(k), p\in\{A,B,rmA,rmB\}$ are parameters.

3.2.5 Objective Function

It is a well-known fact that while the constraints define the feasible region of the optimization, the objective function defines the direction and the purpose of the optimization. Different organizations may have different purposes depending on the environment they exist. In this section the most general of these objectives and their effects to the supply chain optimization problem are going to be discussed.

3.2.5.1 Cost Minimization

In our framework costs are specified as constants, which specify the fixed and variable costs. Therefore, specifying the objective as cost minimization casts the supply chain optimization problem as a mixed integer linear optimization formulation, whose solution algorithms are well-developed and pose no significant problem.

3.2.5.2 Revenue Maximization

For revenue maximization there are two possibilities in terms of model structure. If price values are specified as independent parameters, then the supply chain optimization problem becomes a mixed integer linear optimization. On the other hand, if at least one price value is specified as a variable, then the supply chain optimization problem becomes a bilinear mixed integer optimization problem, which is a special but difficult case of mixed integer nonlinear optimization. Bilinear functions are non-convex and therefore do not necessarily maximized or minimized at critical points. There are commercially available nonlinear and mixed integer nonlinear solvers; however they do not guarantee global optimality.

3.2.5.3 Profit Maximization

The structure of the profit maximization problem is dictated by the structure of the revenue terms in the objective function. If revenue terms do not impose bilinear terms then the supply chain optimization problem becomes a mixed integer linear optimization; whereas if the revenue terms impose bilinear terms then the supply chain optimization problem becomes bilinear mixed integer optimization.

3.2.5.4 Other Objective Function Types

As stated before different organization may have different purposes depending on their business vision. Some of these may be minimizing average inventory level, stabilizing inventory levels at a target value, or maximizing the order fill rate. Depending on the objective function the solution and the solution algorithm of the supply chain optimization problem may change. If the objective function contains only mixed integer linear terms then any commercially available mixed integer linear solver can be used. On the other hand, if the objective function contains bilinear or quadratic terms then the algorithm provided in chapter 4 can be used.

3.2.6 An Example Supply Chain Model

An example supply chain system is given in Fig.3.8.

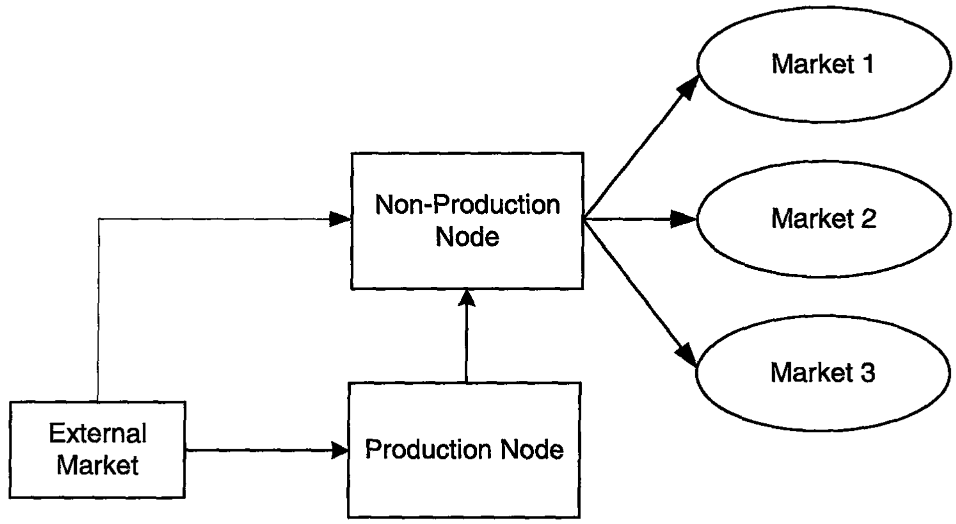

Figure 3.8. Example Supply Chain Scenario

The non production node, production node, external market and the markets are the ones modeled in sections 3.2.1.3, 3.2.2.3, 3.2.3, and 3.2.4, respectively. The n parameter for the model given in 3.2.1.3 is NPN and for the model given in 3.2.2.3 is PN. For the sake of simplicity we set the objective function is to revenue maximization. The objective function is formulated in Eq.(3.47).

$$\begin{aligned} revenue = & \sum_{k} \sum_{j \in \{A,B\}} p_A^{M1}(k) * \alpha_j^A * y_{j,A}^{NPN,M1}(k) \\ & + \sum_{k} \sum_{v \in \{v1,v2\}} \sum_{t \in \{1,2\}} p_A^{M2} * y_A^{NPN,M2,v,t}(k) \\ & + \sum_{k} p_B^{M3}(k) * y_B^{NPN,M3}(k) \end{aligned} \tag{3.47}$$

Hence, the optimization is specified in Eq.(3.48).

$$
\begin{aligned}
&\textit{Maximize revenue} = \sum_{k} \sum_{j \in \{A,B\}} p_A^{M1}(k) * \alpha_j^A * y_{j,A}^{NPN,M1}(k) \\
&\quad + \sum_{k} \sum_{v \in \{v1,v2\}} \sum_{t \in \{1,2\}} p_A^{M2} * y_A^{NPN,M2,v,t}(k) \\
&\quad + \sum_{k} p_B^{M3}(k) * y_B^{NPN,M3}(k) \\
&\textit{subject to} \\
&\text{Eq.(3.41) to Eq.(3.44)} \\
&\text{one of Eq.(3.18) and Eq.(3.19)} \\
&\text{one of Eq.(3.20) and Eq.(3.21)} \\
&\text{one of Eq.(3.22) and Eq.(3.23)} \\
&\text{Eq.(3.25) to Eq.(3.29)} \\
&\text{Eq.(3.34), Eq.(3.35)} \\
&\text{Eq.(3.38)} \\
&\text{Eq.(3.43), Eq.(3.44)} \\
&\text{All variables are greater than or equal to zero}
\end{aligned}
\tag{3.48}
$$

3.3 Advanced Modeling Techniques

In a supply chain there are many different types of operations; some of the operations in a supply chain may be of more importance compared to rest and therefore must be planned more precise. For example, a plant in a node may be producing short-in-supply and action critical components and therefore, that plant must be planned in higher resolution in terms of time. Furthermore, some of product inventories may be more critical and demand intensive than others and consequently must be watched closely. For example, it may be the case that product A is shipped monthly and product B is shipped weekly at node X whereas product A is shipped weekly and product B is shipped monthly at node Y. This implies that, actually, there are several times scales in the supply chain that operations are carried on and controlled.

The fact that there are processes in the supply chain that are carried on and controlled on different time scales creates a problem of defining variables, namely, if the variables are defined on different time scales then the constraints governing the processes that make up the supply chain cannot be stated as demonstrated in chapter 3 up to here. Usually the solution to this problem is to take the greatest common divisor of all these time scales and define all of the time indexed variables in terms of the greatest common divisor. Clearly, this approach increases the number of variables unnecessarily which leads to greater memory usage and a probable increase in solution time during optimization.

Whenever the time resolution for an operation does not change throughout the planning horizon, there are two issues associated with time scales.

i. Resolution differences between time scales
ii. Synchronization differences between time scales

The first issue addresses the need for different time resolution needs for different operations. This phenomenon is illustrated in Fig.3.9.

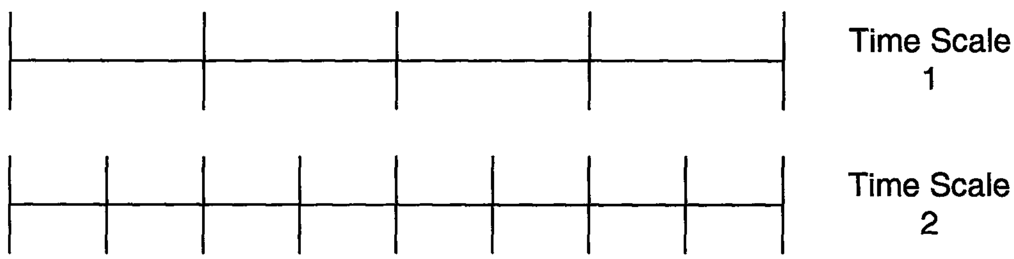

Figure 3.9. Resolution Differences in Time Scales

Whenever a new period starts in the time scale with less resolution, there is a corresponding new starting period in the time scale with more resolution as given in Fig.3.9. However, in some situations this may not be the case as illustrated in Fig.3.10.

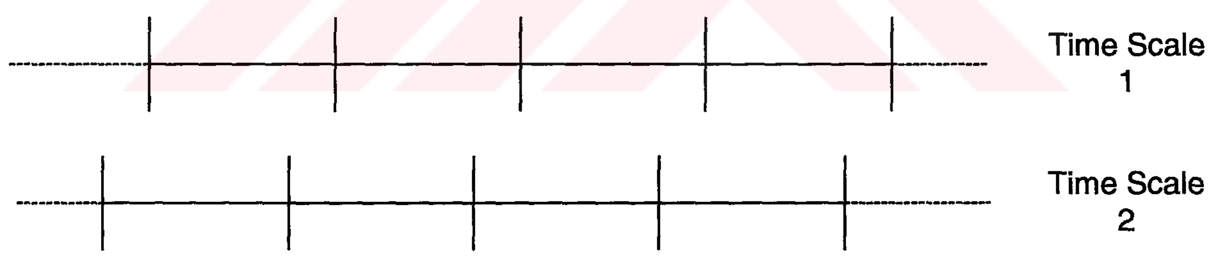

Figure 3.10. Synchronization Differences in Time Scales

Two time scales in Fig.3.10 are not synchronized although the resolutions are the same. Two time scales which are not synchronized and of different resolutions are demonstrated in Fig.3.11.

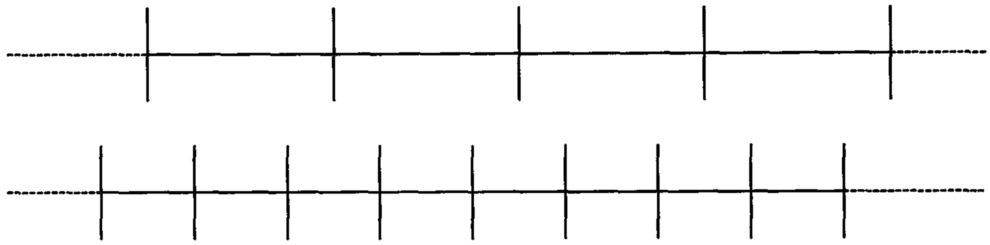

Figure 3.11. Synchronization and Resolution Differences in Time Scales

A novel approach to solve time scale problem is to associate different time scales to supply chain components where necessary and map these time scales to each other. Basically, the mapping must provide a relation for synchronizing information and material flow in different supply chain nodes. Kallrath and Timpe [4] utilized this concept in their modeling practices, but they just differentiated between commercial and production time scales. Their production and distribution model also allows for defining non-homogeneous time scales just to increase the optimization horizon. That is, their first few periods may be defined in terms of days, next few periods in terms of months and the rest of the periods in terms of years. Kallrath and Timpe basically worked on time scales with only resolution differences. A natural improvement on their work is to account for synchronization differences as well and also allowing all of the variables to be defined in their own time scale to increase the reality of the model and precision. Basically, the concept states that every inventory and production variable is defined in terms its own time resolution, not the greatest common divisor, and then these variables are linked by mappings defined between their time scales. The mapping is going to take care of both resolution and synchronization issues.

As stated before, a mapping must provide a relation for information and material flow in different supply chain nodes. The material flow can be of two types in a supply chain, namely, discrete and continuous flow. Discrete flow occurs whenever materials are transported by a vehicle like a truck. On the other hand, continuous flow occurs whenever materials are transported through pipes and devices alike. On the other hand, information flow is always discrete. Therefore, the mapping relations to be used must be able to deal with discrete and continuous flows.

One of the problematic issues in using different time scales is the unit of lead time. When there is a material flow between two supply chain units which use different time scales, the time scale of lead time must be specified. That is, if one of the nodes uses a weekly based time scale and the other uses a monthly based time scale, the meaning of lead time's magnitude, i.e. just saying "3" creates an ambiguity. In specifying the mapping between time scales, specifying the lead time in the time scale of the sending supply chain unit is the accepted convention from now on.

3.3.1 Mapping Time Scales for Material Transfer

Any mapping of time scales must provide which periods are mapped to each other and the ratio of the material transferred from one period to another for both discrete and continuous flows. Based on these requirements, the following mapping convention can be devised:

i. For each point X on a time scale specify the points N of the other time scale such that some part of the interval $(N-1,N]$ has an intersection with interval $(X-1,X]$
ii. For each period that is going on as explained in "i", provide two parameters:
    a. The ratio of material flow if continuous material transfer is taken as basis, this number is between 0 and 1, both are inclusive, call CM
    b. The ratio of material flow if discrete material transfer is taken as basis, this number is either 0 or 1, call DM

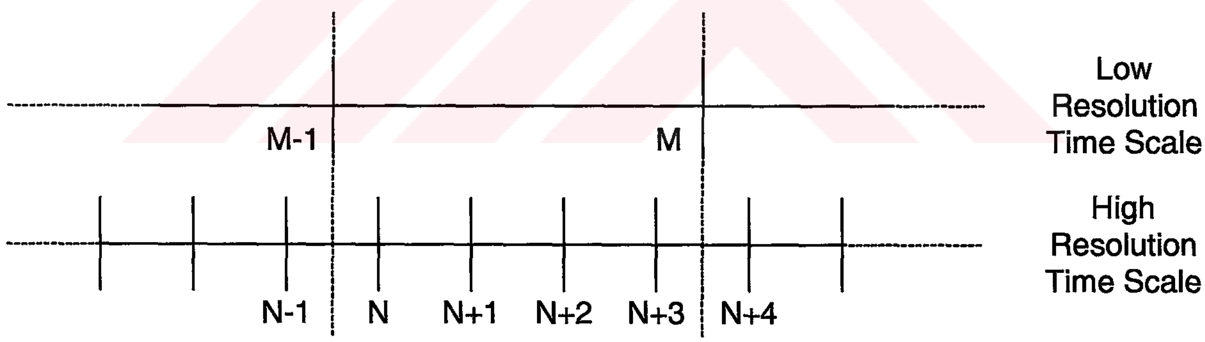

Figure 3.12. Mapping Time Scales with Synchronization and Resolution Differences

The high resolution time scale periods that map to low resolution time scale period M are N to N+4 as shown in Fig.3.12. The parameters of material transfer for this mapping for continuous material flow are 0.5, 1, 1, 1, and 0.5 for N to N + 4, respectively, assuming linear flow. The parameters for material transfer for this mapping for discrete material flow are 1, 1, 1, 1 and 0 for periods N to N+4, respectively. The low resolution time scale periods that map to high resolution time scale period N are M-1 and M. The parameters of material transfer for this mapping for continuous material flow are 0.125 and 0.125, assuming linear flow. The parameters for material transfer for this mapping for discrete material flow are 1 and 0. The low resolution time scale period that maps to high resolution time scale period N+1 is only M. The ratio of material flow is 0.25 for continuous flow assuming linear flow and 0 for discrete flow, since period M does not end in N+1.

Another example is constructed for the time scales given in Fig.3.13.

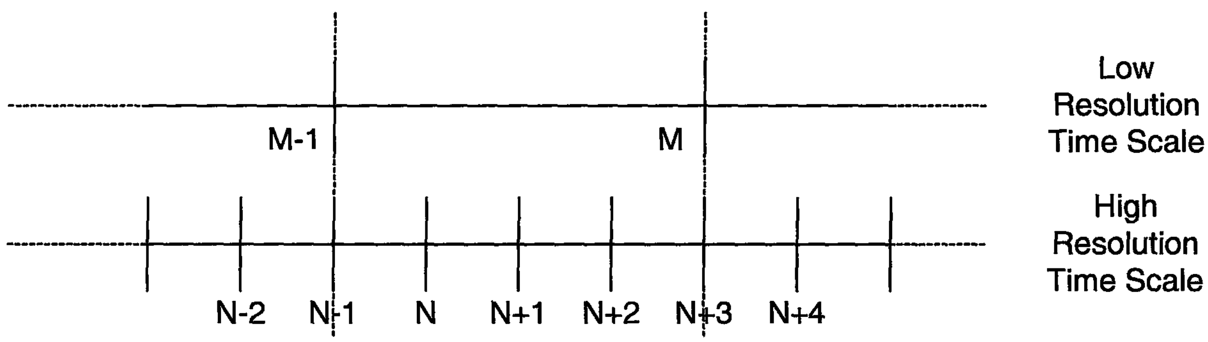

Figure 3.13. Mapping Time Scales with Resolution Differences

The high resolution time scale periods that map to low resolution time scale period M are periods N to N+3. The ratio of material flows are 1 for all periods from N to N+3 for both continuous and discrete material flows. The low resolution time scale period that map to high resolution time scale period N-1 is only M-1 by convention. The ratio of material flows for continuous and discrete material flows are 0.25 and 1, respectively. The low resolution time scale period that map to high resolution time scale period N is M. The ratio of material flow is 0.25 for continuous flow and 0 for discrete flow.

This convention of mapping can be mathematically described by making use of the following parameters.

KS : The ordered set of periods which contain the period to which other periods are going to be mapped

KT : The ordered set of periods which are mapped to the periods in KS

$K_{KS}^{KT}(ks)$: The ordered subset of periods of KT which are mapped to the period ks of KS

$CM_{KS}^{KT}(ks,kt)$: The ratio of material flow of period kt of KT which is mapped to the period ks of KS for continuous mapping

$DM_{KS}^{KT}(ks,kt)$: The ratio of material flow of period kt of KT which is mapped to the period ks of KS for discrete mapping

Since the periods that go on concurrently are mapped together with the ratios of material flow for both continuous and discrete material flows, the inventory balance equations, which are affected by the use of different time scales, can now be modified to take into account of this fact. It is worth noting that since there is no parameter associated with inventory update, the time scale associated with inventory update need not be homogeneous in resolution as long as the mapping between time scales is done correctly. The inventory balance constraint can be expressed as demonstrated in Eq.(3.49).

$$
\begin{aligned}
I_{j1_i}^{n}(ks+1) = I_{j1_i}^{n}(ks) &- \sum_{d\in S_n}\sum_{j2_i}\sum_{v}\sum_{t} y_{j1_i,j2_i}^{n,d,v,t}(ks) - \sum_{d\in S_n}\sum_{j2_i}\sum_{p} y_{j1_i,j2_i}^{n,d,p}(ks) - \sum_{d\in S_n}\sum_{j2_i} y_{j1_i,j2_i}^{n,d}(ks) \\
&- \sum_{d\in N}\sum_{v}\sum_{t} y_{j1_i}^{n,d,v,t}(ks) - \sum_{d\in N}\sum_{j2_i}\sum_{p} y_{j1_i}^{n,d,p}(ks) - \sum_{d\in N} y_{j1_i}^{n,d}(ks) \\
&+ \sum_{s\in N}\sum_{v}\sum_{t}\sum_{kt\in K_{KS}^{KT}(ks+1)} DM_{KS}^{KT}(ks+1,kt) * y_{j1_i}^{s,n,v,t}\left(kt - LT_{v}^{s,n}\right) \\
&+ \sum_{s\in N}\sum_{p}\sum_{kt\in K_{KS}^{KT}(ks+1)} CM_{KS}^{KT}(ks+1,kt) * y_{j1_i}^{s,n,p}(kt-1) \\
&+ \sum_{s\in N}\sum_{kt\in K_{KS}^{KT}(ks+1)} DM_{KS}^{KT}(ks+1,kt) * y_{j1_i}^{s,n}\left(kt - LT^{s,n}\right) \\
&+ \sum_{e\in E_n}\sum_{v}\sum_{t}\sum_{kt\in K_{KS}^{KT}(ks+1)} DM_{KS}^{KT}(ks+1,kt) * y_{j1_i}^{e,n,v,t}\left(kt - LT_{v}^{e,n}\right) \\
&+ \sum_{e\in E_n}\sum_{v}\sum_{t}\sum_{kt\in K_{KS}^{KT}(ks+1)} DM_{KS}^{KT}(ks+1,kt) * y_{j1_i}^{e,n}\left(kt - ALT_{j1_i}^{e,n}\right) \\
&+ \sum_{f}\sum_{\theta}\sum_{kt\in K_{KS}^{KT}(ks+1)} CM_{KS}^{KT}(ks+1,kt) * B_{\theta,j1_i}^{n,f}(kt-1) \\
&+ \sum_{f}\sum_{\theta}\sum_{kt\in K_{KS}^{KT}(ks+1)} DM_{KS}^{KT}(ks+1,kt) * B_{\theta,j1_i}^{n,f}(kt-1) \\
&- \sum_{f}\sum_{\theta}\sum_{r_{j1_i}} CR_{\theta,r_{j1_i}}^{n,f} * B_{\theta,r_{j1_i}}^{n,f}(ks) \quad \forall ks \geq 1
\end{aligned}
\tag{3.49}
$$

Note that expressing the inventory balance equation as demonstrated in Eq.(3.49) allows the use of different time scales for the parts of production system that increase the level of corresponding product inventory as well as allowing the use of different time scales for other inventory variables. Furthermore, this equation also makes it possible to specify the production batches as continuous and discrete amounts. For example, the production of a ton of oil may be continuous and the produced amount can be used before the production of the whole batch is completed; whereas the production of a car is certainly discrete and the car cannot be used before it is fully completed.

In addition to the updates made to the inventory balance equations, order balance equations may also be modified to take into account of this improvement. Note that this modification makes it possible to define time scales for markets. That is, the demand of a market may be specified monthly whereas the demand of another market may be specified daily; and this situation is indeed what is observed as reality. The application of the approach is similar.

Since material usage and material transportation out of the inventory location variables have the same time scale of the corresponding inventory variable there is no need to update material transfer limit constraints described in Eq.(3.17).

3.3.2 Mapping Time Scales for Vehicle Balance and Vehicle Capacity Constraints

Although the inventory balance equations are regulated with respect to different time scales, the vehicle balance equations, and the vehicle capacity restriction constraints on the material transfer are not yet addressed. For transportation by pipes, there is no problem with capacity restriction constraints due to time scales since capacity is specified as a parameter. Moreover, there is no vehicle balance equation for pipes. However, since it is possible that a discrete transportation vehicle such as a truck can be used by several nodes in a supply chain, and may be able to carry all types of products, it may be the case that a vehicle type operates on some different time scales present in the supply chain. This leads to obvious problems while defining vehicle balance and capacity restriction equations. In order to solve these problems three approaches are developed.

3.3.2.1 Solution Approach-1

A basic solution to this problem is that all vehicles are assigned a common vehicle time scale such that this common vehicle time scale has high enough resolution and is synchronized such that whenever an inventory time scale has a period starting then there exist a period of the common vehicle time scale starting at the same time as well. This concept is illustrated in Fig.3.14. In this figure, time scales one, two and three are the time scales that exist naturally in the supply chain. Vehicle time scale is the generated common time scale. For each starting period of the first three time scales, there exists a starting period of generated common vehicle time scale.

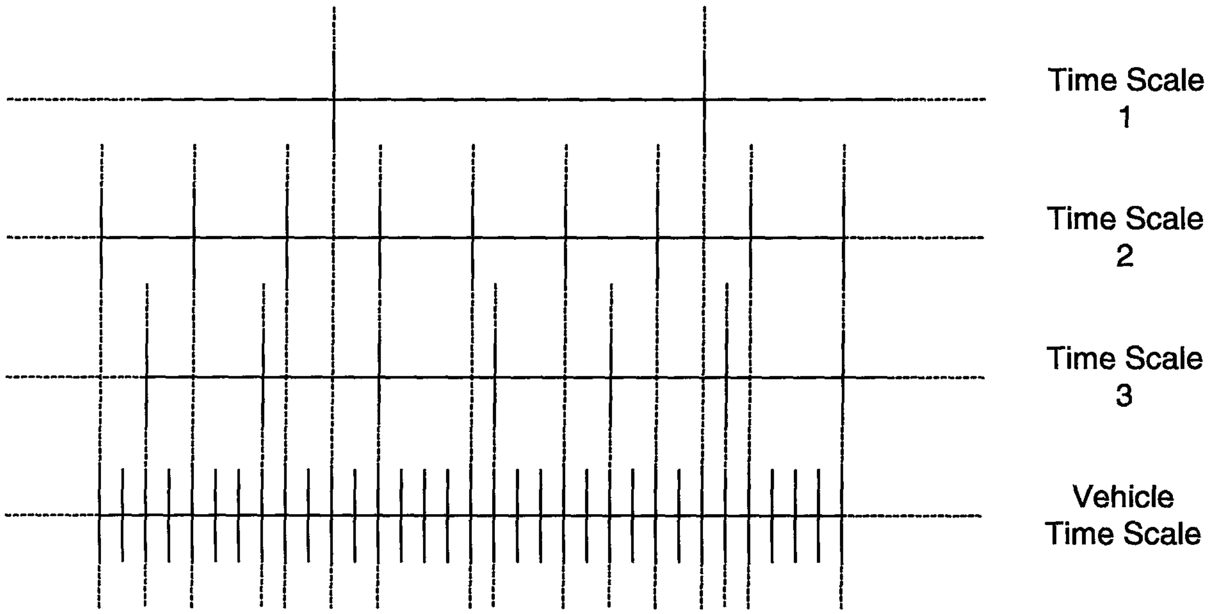

Figure 3.14. Basic Solution to Vehicle Time Scale Problem

With this solution, all vehicles operate on the same generated time scale. A mapping can be defined from each inventory time scale to generated common vehicle as explained in the previous section, with the exception that the mapping only provides the simultaneously starting period of common vehicle time scale for a given period of an inventory time scale. Therefore, in this way, vehicle balance equations can be written in terms of generated common vehicle time scale and vehicle capacity restriction constraints can be defined via the mapping between the common vehicle time scale and the scale on which material is transferred. While defining vehicle balance equations, the lead times must be converted to the units of common vehicle time scale. Based on these, vehicle balance equations and vehicle capacity restriction constraints can be mathematically specified as follows.

KV : Generated common vehicle time scale

$SK_{KS}^{KV}(ks)$: The simultaneously starting period of KV for the given period ks of KS, where KS is an inventory time scale

$LT_{v,KV}^{n,d}$: Lead time specified in terms of time scale KV

Vehicle balance equation can be specified as demonstrated in Eq.(3.50).

$$
\begin{aligned}
VI_v^n(kv) = VI_v^n(kv) - \sum_{d \in N \cup M \cup E_n} \sum_t VS_{v,t}^{n,d}(kv) + \sum_{s \in N} \sum_t VS_{v,t}^{s,n}\left(kv + 1 - LT_{v,KV}^{s,n}\right) \\
+ \sum_e \sum_t VS_{v,t}^{n,e}\left(kv + 1 - LT_{v,KV}^{n,e} - LT_{v,KV}^{e,n}\right) \\
+ \sum_m \sum_t VS_{v,t}^{n,m}\left(kv + 1 - LT_{v,KV}^{n,m} - LT_{v,KV}^{m,n}\right) \ \forall kv \geq 1
\end{aligned}
\tag{3.50}
$$

Capacity restriction constraint is modified as well, which is expressed in Eq.(3.51).

$$
\begin{aligned}
& VS_{v,t}^{n,d}\left(SK_{KS}^{KV}(ks)\right) \geq \sum_{j2_i \in J_i} f_{j1_i}^{v,t} * \left(y_{j1_i}^{n,d,v,t}(ks) + y_{j1_i,j2_i}^{n,d,v,t}(ks)\right) \ \forall ks \in KS \\
& VS_{v,t}^{n,e}\left(SK_{KS}^{KV}(ks) - LT_{v,KV}^{n,e}\right) \geq \sum_{j2_i \in J_i} f_{j1_i}^{v,t} * y_{j1_i}^{e,n,v,t}(ks) \ \forall ks \in KS \ni SK_{KS}^{KV}(ks) - LT_{v,KV}^{n,e} \geq 1 \\
& y_{j1_i}^{e,n,v,t}(ks) = 0 \ \forall ks \in KS \ni SK_{KS}^{KV}(ks) - LT_{v,KV}^{n,e} \leq 0
\end{aligned}
\tag{3.51}
$$

3.3.2.2 Solution Approach-2

This approach is a refined version of the former solution approach. Defining a common vehicle time scale for all vehicles in the supply chain may lead to a time scale which has a very high resolution, which leads to a high number of $VI_v^n(kv)$ and $VS_{v,t}^{n,d}(kv)$ variables; and a high number of vehicle balance equations. Furthermore, many of these variables and constraints may be unnecessary since some of the vehicle types may not be using all nodes and among the nodes it visits, may not be using all of the existing time scales at those nodes. Therefore, for each set of vehicles which use the exactly the same time scales throughout the entire supply chain, it is possible to generate a different common vehicle time scale, which is going to eliminate the unnecessary constraints and variables. This approach can be mathematically described as follows.

TV : Set of set of type of vehicles which use the same inventory time scales
$KV(tv)$: Generated common vehicle time scale for $tv \in TV$
$SK_{KS}^{KV(tv)}(ks)$: The simultaneously starting period of $KV(tv)$ for the given period ks of KS where KS is an inventory time scale

Implied vehicle balance equation is specified in Eq.(3.52).

$$
\begin{aligned}
VI_v^n(kv+1) = VI_v^n(kv) - \sum_{d\in N\cup M\cup E_n}\sum_t VS_{v,t}^{n,d}(kv) + \sum_{s\in N}\sum_t VS_{v,t}^{s,n}\left(kv+1-LT_{v,KV(tv)}^{s,n}\right) \\
+\sum_e\sum_t VS_{v,t}^{n,e}\left(kv+1-LT_{v,KV(tv)}^{n,e}-LT_{v,KV(tv)}^{e,n}\right) \\
+\sum_m\sum_t VS_{v,t}^{n,m}\left(kv+1-LT_{v,KV(tv)}^{n,m}-LT_{v,KV(tv)}^{m,n}\right)\ \forall kv\in KV(tv)
\end{aligned}
\tag{3.52}
$$

Capacity restriction constraint is demonstrated in Eq.(3.53).

$$
\begin{aligned}
& VS_{v,t}^{n,d}\left(SK_{KS}^{KV(tv)}(ks)\right) \geq \sum_{j2_i\in J_i} f_{j1_i}^{v,t} * \left(y_{j1_i}^{n,d,v,t}(ks) + y_{j1_i,j2_i}^{n,d,v,t}(ks)\right)\ \forall ks\in KS \\
& VS_{v,t}^{n,e}\left(SK_{KS}^{KV(tv)}(ks) - LT_{v,KV(tv)}^{n,e}\right) \geq \\
& \qquad \sum_{j2_i\in J_i} f_{j1_i}^{v,t} * y_{j1_i}^{e,n,v,t}(ks)\ \forall ks\in KS \ni SK_{KS}^{KV(tv)}(ks) - LT_{v,KV(tv)}^{n,e} \geq 1 \\
& y_{j1_i}^{e,n,v,t}(ks) = 0\ \forall ks\in KS \ni SK_{KS}^{KV(tv)}(ks) - LT_{v,KV(tv)}^{n,e} \leq 0
\end{aligned}
\tag{3.53}
$$

3.3.2.3 Solution Approach-3

Not every node in the supply chain has an obligation to coordinate the operation of vehicles in the same time resolution and in synchronization. For example, a node may coordinate the operation of trucks on an hourly basis whereas another may coordinate that on a 12-hour basis. Furthermore, it must be noted that a single node in the supply chain is not obliged to coordinate all of the vehicles it uses in the same resolution and synchronization. That is, if a supply chain node transfers the same material both by ships and trucks; it may be the case that ships operate on a monthly basis whereas trucks operate on an hourly basis. Creating a time scale for ships and another for trucks is going to eliminate the unnecessary variables and vehicle balance constraints for ships.

The most general solution to this problem is to create vehicle time scales at each node and associate vehicle types that are used at a particular node with a vehicle time scale effective in that node. Furthermore, in order to implement capacity restriction constraints on material transfer levels, shipments from a node are defined on the time scale of vehicles that transport the products. Then, a mapping is defined between inventory control time scales and vehicle time scales. Finally, the inventory balance equations are implemented with respect to these updates. Exactly the same concept can be applied for the materials that are updated by production operations, as a raw material. Therefore, production related operations that consume a product are also allowed to be carried on a different time scale than that of the inventory time scale of the consumed. A vehicle type can only be associated with a single time scale in a specific node. This concept is illustrated in Fig.3.15.

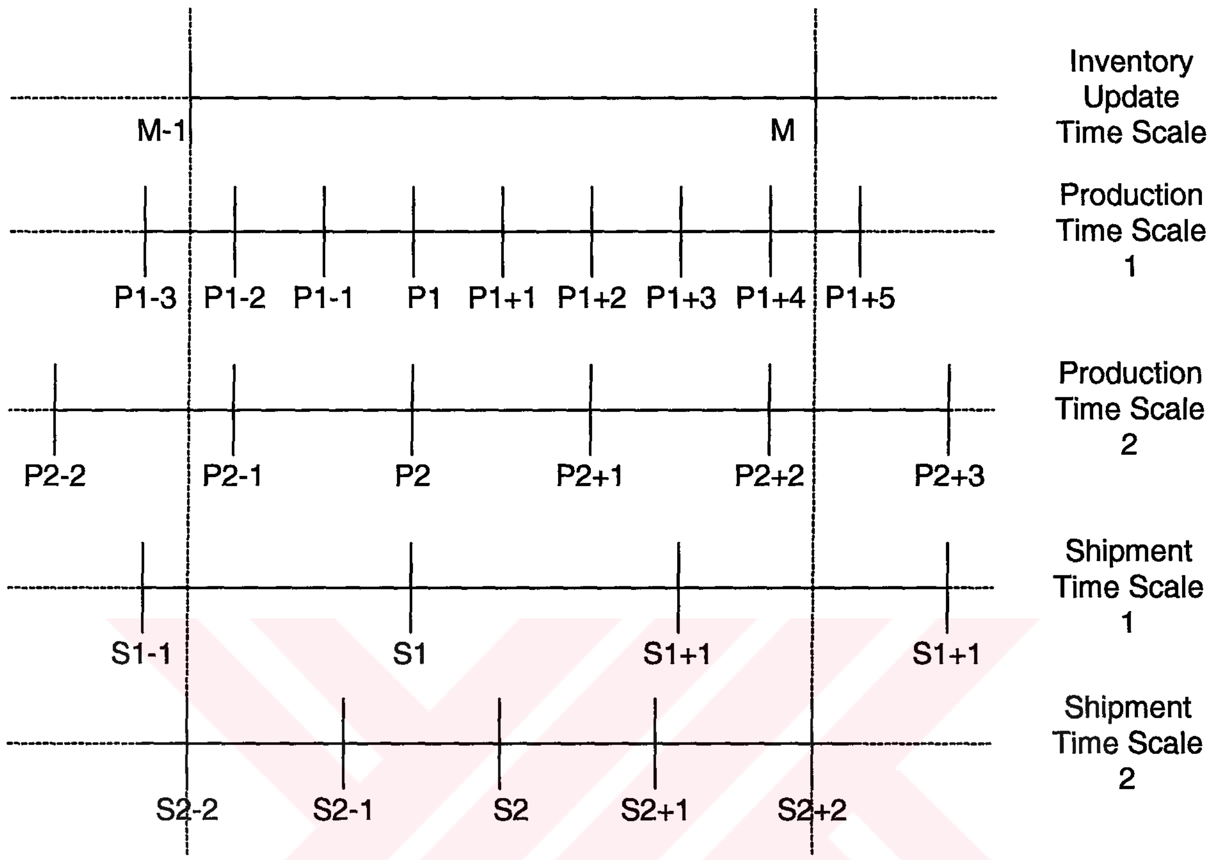

Figure 3.15. Different Time Scales for Inventory Control, Shipment and Production

The previous solution approaches generated common time scales for vehicles which have at least as high resolution as that of time scale of a product inventory that the vehicles carries the product of. In this solution approach, the vehicle time scale's resolution is not necessarily as high as that of product inventory. For example, the inventory of a product may be watched daily whereas a ship which carries this product overseas may be coordinated monthly. Conversely, although a particular inventory is coordinated in a daily basis, the vehicles associated with that inventory may be coordinated on an hourly basis to send empty vehicles to other nodes.

The following parameters are used to model the material transfer at different time scales:

KS : An inventory time scale at node n
TV^n : Set of set of type of vehicles which use the same vehicle time scale at node n
TP^n : Set of set of type of pipes which use the same pipe time scale at node n
TO^n : Set of set of outsourcing or market selling time scales at node n
$KV(tv^n)$: Vehicle time scale for $tv^n \in TV^n$
$KP(tp^n)$: Pipe time scale for $tp^n \in TP^n$
$KO(to^n)$: Outsourcing or market selling time scale for $to^n \in TO^n$
$DM_{K1}^{K2}(k1,k2)$: The ratio of material flow of period $k2$ of $K2$ which is mapped to the period $k1$ of $K1$ for discrete mapping
$CM_{K1}^{K2}(k1,k2)$: The ratio of material flow of period $k2$ of $K2$ which is mapped to the period $k1$ of $K1$ for continuous mapping
$K_{K1}^{K2}(k1)$: The subset of periods of $K2$ which are mapped to the period $k1$ of $K1$
$ML_{K1}^{K2}(k1)$: The subset of periods of $K2$ which are mapped to the period $k1$ of $K1$ regarding material transfer limit
$DML_{K1}^{K2}(k1,k2)$: The ratio of material flow of period $k2$ of $K2$ which is mapped to the period $k1$ of $K1$ for discrete mapping regarding material transfer limit
$CML_{K1}^{K2}(k1,k2)$: The ratio of material flow of period $k2$ of $K2$ which is mapped to the period $k1$ of $K1$ for continuous mapping regarding material transfer limit

The mapping convention for ML can be specified as follows:

i. For each point X on a time scale specify the points N of the other time scale such that some part of the interval $[N,N+1)$ has an intersection with interval $[X,X+1)$
ii. For period that is going on as explained in "i", provide two parameters:
   a. The ratio of material flow if continuous material transfer is taken as basis, this number is between 0 and 1, both are inclusive, call CML
   b. The ratio of material flow if discrete material transfer is taken as basis, this number is either 0 or 1, call DML

Consider the example shown in Fig.3.12. $ML_{LR}^{HR}(M-1)$ is given by points N-1 to N+3. DML parameters for this mapping are 0, 1, 1, 1 and 1 whereas CML parameters for this mapping are 0.5, 1, 1, 1 and 0.5 assuming linear flow. $ML_{HR}^{LR}(N-1)$ is M-2 and M-1. DML

parameters for this mapping are 0 and 1; *CML* parameters for this mapping are 0.125 and 0.125 assuming linear flow. Finally $ML_{HR}^{LR}(N)$ is M-1. *DML* parameter is 0 and *CML* parameter is 0.125.

Consider the example depicted in figure 3.13. $ML_{LR}^{HR}(M-1)$ is N-1 to N+2. *DML* and *CML* parameters for this mapping are all 1. $ML_{HR}^{LR}(N-1)$ and $ML_{HR}^{LR}(N)$ is M-1. *DML* parameters for these mappings are 1 and 0 whereas *CML* parameters for these mappings are 0.25 for both assuming linear flow.

The inventory balance equation is specified in Eq.(3.54). Vehicle balance equation is specified in Eq.(3.55). Vehicle capacity restriction constraint is specified in Eq.(3.56). And product transfer limit constraint is specified in Eq.(3.57).

$$
\begin{aligned}
I_{j1_i}^{n}(ks+1) = I_{j1_i}^{n}(ks) \\
& -\sum_{d\in S_n}\sum_{j2_i}\sum_{tv^n\in TV^n}\sum_{v\in tv^n}\sum_{t}\sum_{kv^n\in K_{KS}^{KV(tv^n)}(ks+1)} DM_{KS}^{KV(tv^n)}(ks+1,kv^n)*y_{j1_i,j2_i}^{n,d,v,t}(kv^n-1) \\
& -\sum_{d\in S_n}\sum_{j2_i}\sum_{tp^n\in TP^n}\sum_{p\in tp^n}\sum_{kp^n\in K_{KS}^{KP(tp^n)}(ks+1)} CM_{KS}^{KV(tp^n)}(ks+1,kp^n)*y_{j1_i,j2_i}^{n,d,p}(kp^n-1) \\
& -\sum_{d\in S_n}\sum_{j2_i}\sum_{to\in TO^n}\sum_{ko^n\in K_{KS}^{KO(to^n)}(ks+1)} DM_{KS}^{KO(to^n)}(ks+1,ko^n)*y_{j1_i,j2_i}^{n,d}(ko^n-1) \\
& -\sum_{d\in N}\sum_{tv^n\in TV^n}\sum_{v\in tv^n}\sum_{t}\sum_{kv^n\in K_{KS}^{KV(tv^n)}(ks+1)} DM_{KS}^{KV(tv^n)}(ks+1,kv^n)*y_{j1_i}^{n,d,v,t}(kv^n-1) \\
& -\sum_{d\in N}\sum_{j2_i}\sum_{tp^n\in TP^n}\sum_{p\in tp^n}\sum_{kp^n\in K_{KS}^{KP(tp^n)}(ks+1)} CM_{KS}^{KV(tp^n)}(ks+1,kp^n)*y_{j1_i}^{n,d,p}(kp^n-1) \\
& -\sum_{d\in N}\sum_{to\in TO^n}\sum_{ko^n\in K_{KS}^{KO(to^n)}(ks+1)} DM_{KS}^{KO(to^n)}(ks+1,ko^n)*y_{j1_i}^{n,d}(ko^n-1) \\
& +\sum_{s\in N}\sum_{v}\sum_{t}\sum_{kt\in K_{KS}^{KT}(ks+1)} DM_{KS}^{KT}(ks+1,kt)*y_{j1_i}^{s,n,v,t}(kt-LT_v^{s,n}) \\
& +\sum_{s\in N}\sum_{p}\sum_{kt\in K_{KS}^{KT}(ks+1)} CM_{KS}^{KT}(ks+1,kt)*y_{j1_i}^{s,n,p}(kt-1) \\
& +\sum_{s\in N}\sum_{kt\in K_{KS}^{KT}(ks+1)} DM_{KS}^{KT}(ks+1,kt)*y_{j1_i}^{s,n}(kt-LT^{s,n}) \\
& +\sum_{e\in E_n}\sum_{v}\sum_{t}\sum_{kt\in K_{KS}^{KT}(ks+1)} DM_{KS}^{KT}(ks+1,kt)*y_{j1_i}^{e,n,v,t}(kt-LT_v^{e,n}) \\
& +\sum_{e\in E_n}\sum_{v}\sum_{t}\sum_{kt\in K_{KS}^{KT}(ks+1)} DM_{KS}^{KT}(ks+1,kt)*y_{j1_i}^{e,n}(kt-ALT_{j1_i}^{e,n}) \\
& +\sum_{f}\sum_{\theta}\sum_{kt\in K_{KS}^{KT}(ks+1)} CM_{KS}^{KT}(ks+1,kt)*B_{\theta,j1_i}^{n,f}(kt-1) \\
& +\sum_{f}\sum_{\theta}\sum_{kt\in K_{KS}^{KT}(ks+1)} DM_{KS}^{KT}(ks+1,kt)*B_{\theta,j1_i}^{n,f}(kt-1) \\
& -\sum_{f}\sum_{\theta}\sum_{r_{j1_i}}\sum_{kt\in K_{KS}^{KT}(ks+1)} CM_{KS}^{KT}(ks+1,kt)*CR_{\theta,r_{j1_i}}^{n,f}*B_{\theta,r_{j1_i}}^{n,f}(kt-1) \\
& -\sum_{f}\sum_{\theta}\sum_{r_{j1_i}}\sum_{kt\in K_{KS}^{KT}(ks+1)} DM_{KS}^{KT}(ks+1,kt)*CR_{\theta,r_{j1_i}}^{n,f}*B_{\theta,r_{j1_i}}^{n,f}(kt-1)\ \forall ks\ge 1
\end{aligned}
\tag{3.54}
$$

$$
\begin{aligned}
VI_v^n(kv^n+1) = VI_v^n(kv^n) - \sum_{d\in N\cup M\cup E_n}\sum_{t} VS_{v,t}^{n,d}(kv^n) \\
& +\sum_{s\in N}\sum_{t}\sum_{tv^s\in TV^s}\sum_{kv^s\in K_{KV(tv^n)}^{KV(tv^s)}(kv^n+1)} VS_{v,t}^{s,n}\left(kv^s-LT_{v,KV(tv^s)}^{s,n}\right) \\
& +\sum_{e}\sum_{t} VS_{v,t}^{n,e}\left(kv^n+1-LT_{v,KV(tv^n)}^{e,n}-LT_{v,KV(tv^n)}^{n,e}\right) \\
& +\sum_{m}\sum_{t} VS_{v,t}^{n,m}\left(kv^n+1-LT_{v,KV(tv^n)}^{n,m}-LT_{v,KV(tv^n)}^{m,n}\right)\ \forall kv^n\in KV(tv^n)
\end{aligned}
\tag{3.55}
$$

$$VS_{v,t}^{n,d}\left(kv^n\right) \geq \sum_{j2_i \in J_i} f_{j1_i}^{v,t} * \left(y_{j1_i}^{n,d,v,t}\left(kv^n\right) + y_{j1_i,j2_i}^{n,d,v,t}\left(kv^n\right)\right) \; \forall kv^n$$

$$VS_{v,t}^{n,e}\left(kv^n - LT_{v,KV\left(tv^n\right)}^{n,e}\right) \geq \sum_{j2_i \in J_i} f_{j1_i}^{v,t} * y_{j1_i}^{e,n,v,t}\left(kv^n\right) \; \forall kv^n \ni kv^n - LT_{v,KV\left(tv^n\right)}^{n,e} \geq 1 \tag{3.56}$$

$$y_{j1_i}^{e,n,v,t}\left(kv^n\right) = 0 \; \forall kv^n \ni kv^n - LT_{v,KV\left(tv^n\right)}^{n,e} \leq 0$$

$$\begin{aligned}
I_{j1_i}^n(ks) \geq & \sum_{d \in S_n} \sum_{j2_i} \sum_{tv^n \in TV^n} \sum_{v \in tv^n} \sum_{t} \sum_{kv^n \in ML_{KS}^{KV\left(tv^n\right)}(ks)} DML_{KS}^{KV\left(tv^n\right)}\left(ks, kv^n\right) * y_{j1_i,j2_i}^{n,d,v,t}\left(kv^n\right) \\
& + \sum_{d \in S_n} \sum_{j2_i} \sum_{tp^n \in TP^n} \sum_{p \in tp^n} \sum_{kp^n \in ML_{KS}^{KP\left(tp^n\right)}(ks)} CML_{KS}^{KV\left(tp^n\right)}\left(ks, kp^n\right) * y_{j1_i,j2_i}^{n,d,p}\left(kp^n\right) \\
& + \sum_{d \in S_n} \sum_{j2_i} \sum_{to \in TO^n} \sum_{ko^n \in ML_{KS}^{KO\left(to^n\right)}(ks)} DML_{KS}^{KO\left(to^n\right)}\left(ks, ko^n\right) * y_{j1_i,j2_i}^{n,d}\left(ko^n\right) \\
& + \sum_{d \in N} \sum_{tv^n \in TV^n} \sum_{v \in tv^n} \sum_{t} \sum_{kv^n \in ML_{KS}^{KV\left(tv^n\right)}(ks)} DML_{KS}^{KV\left(tv^n\right)}\left(ks, kv^n\right) * y_{j1_i}^{n,d,v,t}\left(kv^n\right) \\
& + \sum_{d \in N} \sum_{j2_i} \sum_{tp^n \in TP^n} \sum_{p \in tp^n} \sum_{kp^n \in ML_{KS}^{KP\left(tp^n\right)}(ks)} CML_{KS}^{KV\left(tp^n\right)}\left(ks, kp^n\right) * y_{j1_i}^{n,d,p}\left(kp^n\right) \\
& + \sum_{f} \sum_{\theta} \sum_{r_{j1_i}} \sum_{kt \in ML_{KS}^{KT}(ks)} DML_{KS}^{KT}\left(ks, kt\right) * CR_{\theta,r_{j1_i}}^{n,f} * B_{\theta,r_{j1_i}}^{n,f}\left(kt\right) \\
& + \sum_{f} \sum_{\theta} \sum_{r_{j1_i}} \sum_{kt \in ML_{KS}^{KT}(ks)} CML_{KS}^{KT}\left(ks, kt\right) * CR_{\theta,r_{j1_i}}^{n,f} * B_{\theta,r_{j1_i}}^{n,f}\left(kt\right) \; \forall ks
\end{aligned} \tag{3.57}$$

3.3.3 An Example Application

Consider an inventory of a semi-finished product as illustrated in Fig.3.16.

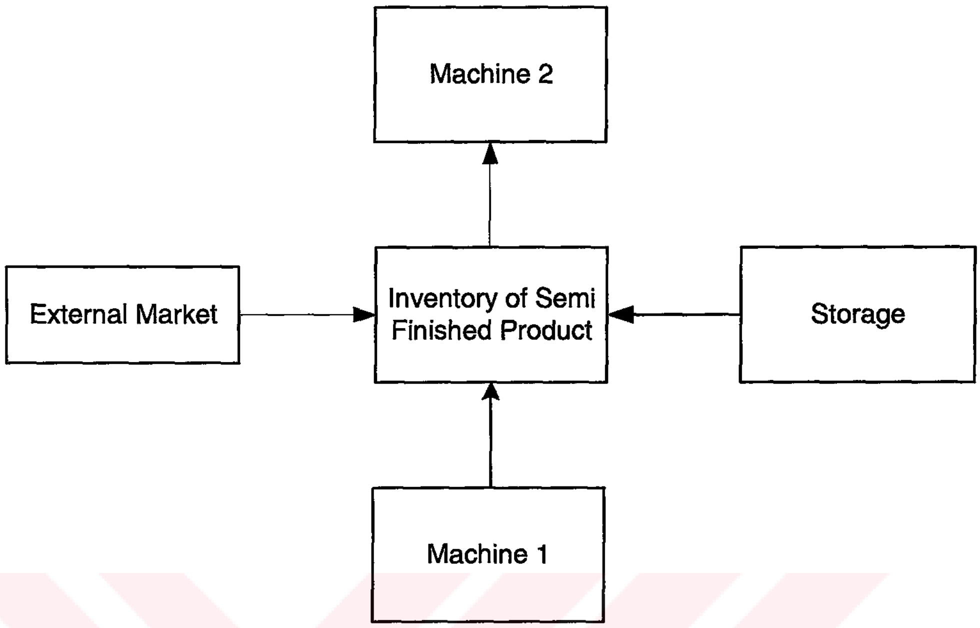

Figure 3.16. Different Time Scales Scenario

Machine 1 produces to the inventory of the semi-finished product, and coordinates its schedule with a resolution of an hour. Machine 2 consumes the inventory of the semi-finished product at a unit rate of three in each production mode and coordinates its schedule with a resolution of half a day, that is, machine 2 can have at most two different production modes a day. Since production rate of machine 2 is very high, a vehicle transports the semi-finished product to the consumption area of machine 2. The vehicle transports semi-finished product from both the storage and the external market. The schedule of the vehicle is coordinated in a resolution of two hours at the storage area whereas it is coordinated in a resolution of an hour at the inventory of the semi-finished product area. The time to travel between any two locations is two hours. The external market specifies its capacity in a daily basis. The level of inventory of the semi-finished product is updated half a day. The volume factor of the product is 1 for all vehicles and all modes of them. The time scales are demonstrated in Fig.3.17.

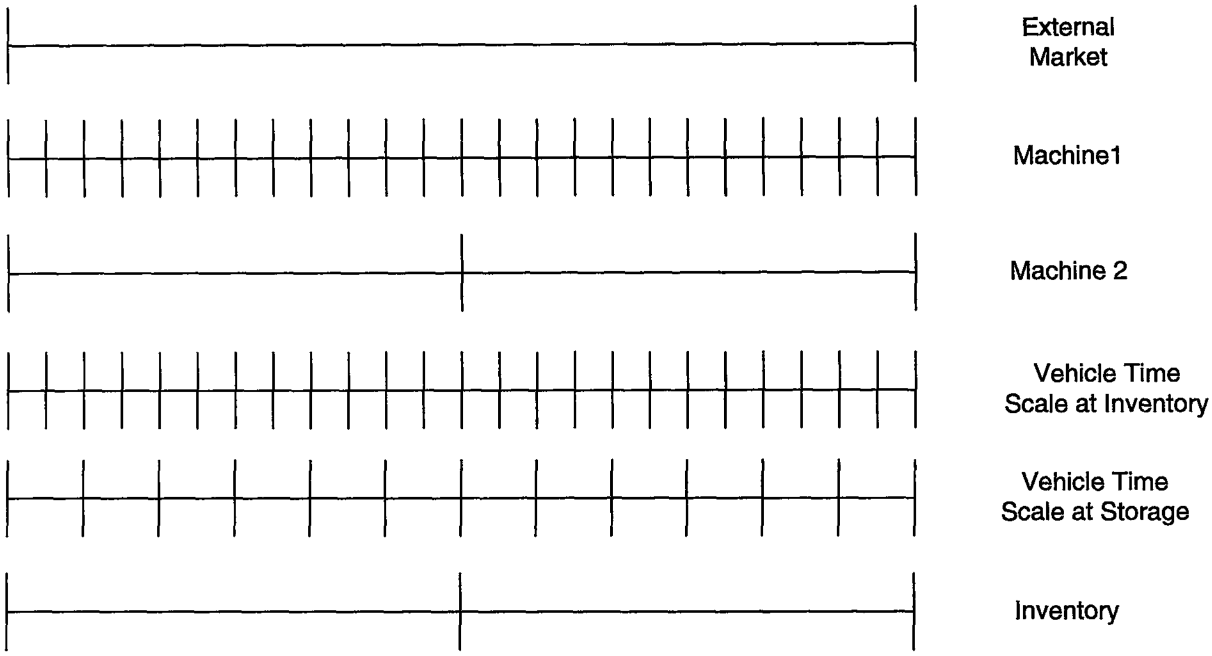

Figure 3.17. Time Scales Present in Scenario

In order to model this system a simplified form of the notation demonstrated in 3.3.2.3 is used. Since all flows are discrete and synchronized, there is no need to use *CM*, *DM*, *CML* and *DML* parameters, the mappings are going to be sufficient for these purposes.

ke : Time scale of external market
$km1$: Time scale of machine 1
$km2$: Time scale of machine 2
kv^s : Vehicle time scale at storage
kv^i : Vehicle time scale at inventory of the semi-finished product
ki : Time scale of the inventory of the semi-finished product

Based on this notation the maps are generated in Eq.(3.58).

$$
\begin{aligned}
&K_{ki}^{km1}(ki) : (ki-2)*12+1 < km1 \le (ki-1)*12+1 \\
&K_{ki}^{km2}(ki) : km2 = ki \\
&ML_{ki}^{km2}(ki) : km2 = ki \\
&K_{ki}^{kv^s}(ki) : (ki-2)*6+1 < kv^s \le (ki-1)*6+1 \\
&K_{ki}^{kv^i}(ki) : (ki-2)*12+1 < kv^i \le (ki-1)*12+1
\end{aligned}
\qquad (3.58)
$$

$$\begin{aligned}
&K_{ke}^{kv^i}(ke):(ke-2)*24+1<kv^i\le(ke-1)*24+1\\
&ML_{ke}^{kv^i}(ke):(ke-1)*24+1\le kv^i\le ke*24\\
&K_{kv^s}^{kv^i}(kv^s):(kv^s-2)*2+1<kv^i\le(kv^s-1)*2+1\\
&K_{kv^i}^{kv^s}(kv^i):(kv^i-1)/2<kv^s\le(kv^i+1)/2
\end{aligned}\tag{3.58}$$

Note that all time scales start from index value of 1. Based on these maps, the inventory balance equation of the semi finished product is derived from Eq.(3.54) and is demonstrated in Eq.(3.59).

$$\begin{aligned}
I_p^I(ki+1)=I_p^I(ki)&-\sum_{\theta}\sum_{km2\in K_{ki}^{km2}(ki+1)}3*B_\theta^{M2}(km2-1)\\
&+\sum_{\theta}\sum_{km1\in K_{ki}^{km1}(ki+1)}B_\theta^{M1}(km1-1)\\
&+\sum_{kv^s\in K_{ki}^{kv^s}(ki+1)}y_p^{S,n}(kv^s-1)+\sum_{kv^i\in K_{ki}^{kv^i}(ki+1)}y_{j1_i}^{EM,I}(kv^i-2)\ \forall ki
\end{aligned}\tag{3.59}$$

The lead time to go from storage area to semi-finished inventory location is 1 in terms of the vehicle time scale at storage location whereas it is 2 in terms of the vehicle time scale at semi-finished inventory location as illustrated in Eq.(3.59).

The material transfer limit of the semi-finished product inventory is derived from Eq.(3.57) and is illustrated in Eq.(3.60).

$$I_p^I(ki)\ge\sum_{\theta}\sum_{km2\in ML_{ki}^{km2}(ki)}3*B_\theta^{M2}(km2)\ \forall ki\tag{3.60}$$

The inventory limit of the external market is again derived from Eq.(3.57) and is illustrated in Eq.(3.61).

$$I_p^{EM}(ki)\ge\sum_{kv^i\in ML_{ki}^{kv^i}(ki)}y_{j1_i}^{EM,I}(kv^i)\ \forall ki\tag{3.61}$$

The vehicle balance equations are derived from Eq.(3.55) and are illustrated in Eq.(3.62).

$$
\begin{aligned}
VI_v^I\left(kv^i+1\right) &= VI_v^I\left(kv^i\right) - \sum_{d\in\{S,EM\}}\sum_t VS_{v,t}^{I,d}\left(kv^i\right) \\
&+ \sum_t \sum_{kv^s\in K_{kv^i}^{kv^s}(kv^i+1)} VS_{v,t}^{S,I}\left(kv^s-1\right) \\
&+ \sum_e\sum_t VS_{v,t}^{I,e}\left(kv^i-1\right) \\
VI_v^S\left(kv^s+1\right) &= VI_v^S\left(kv^s\right) - \sum_{d\in\{I\}}\sum_t VS_{v,t}^{S,d}\left(kv^s\right) \\
&+ \sum_t \sum_{kv^i\in K_{kv^s}^{kv^i}(kv^s+1)} VS_{v,t}^{I,S}\left(kv^i-2\right)
\end{aligned}
\tag{3.62}
$$

The vehicle capacity restriction constraints are derived from Eq.(3.56) and are illustrated in Eq.(3.63).

$$
\begin{aligned}
& VS_{v,t}^{S,I}\left(kv^s\right) \geq y_p^{S,I,v,t}\left(kv^s\right)\ \forall kv^s \\
& VS_{v,t}^{I,EM}\left(kv^i-2\right) \geq y_p^{EM,I,v,t}\left(kv^n\right)\ \forall kv^i \ni kv^i-2\geq 1 \\
& y_p^{EM,I,v,t}\left(kv^i\right) = 0\ \forall kv^i \ni kv^i-2\leq 0
\end{aligned}
\tag{3.63}
$$

CHAPTER 4 – A NOVEL GLOBAL OPTIMIZATION ALGORITHM FOR BILINEAR PROGRAMMING PROBLEMS

The supply chain optimization problem posed in the previous chapters may include bilinear terms in the objective function; and general integer variables in both constraints and objective function, even in bilinear terms. The algorithm reformulates the given optimization problem into a mixed-integer linear problem provided that an acceptable amount of error is specified.

4.1 Profit and Revenue Maximization with Variable Price

When price is a variable then since revenue is the multiplication of price and the amount of material sold, revenue becomes a bilinear term. Since bilinear terms are not convex, the supply chain optimization problem introduced in the previous chapters reduces to the maximization of a nonconvex objective function subject to a convex feasible region with integer variables.

Commercially available solvers can handle nonlinear and mixed-integer nonlinear optimization problems. However, they require the nonlinear terms to be convex. In the case that the nonlinear terms are not convex, these solvers do not guarantee global optimality of the solutions they provide. Furthermore, these solvers usually handle general integer variables by converting them into binary variables that slows down the branch and bound process considerably as it is shown in section 4.5.

4.2 Bilinear Optimization Algorithms

Optimization problems that involve bilinear terms in the objective function are encountered in many fields in science such as game theory, location theory and risk management problems. For bilinear optimization problems that involve only continuous variables there are several algorithms. Al-Khayyal and Falk [27] devised an algorithm for bilinear optimization problems when a variable appears in a bilinear term at most once. Sherali and Alameddine [28] developed a reformulation-linearization technique based optimization algorithm for general bilinear programming problems. There exist also other competing nonlinear and mixed integer nonlinear optimization algorithms as described in section 1.2.

Although all of these algorithms solve bilinear programming problems when the variables are continuous, none of them address the use of integer variables. Therefore, in order to solve the supply chain problem posed in previous chapters, a new algorithm is developed.

4.3 An Algorithm for Bilinear & Mixed Integer Bilinear Optimization

The problem of interest can be given as in Eq.(4.1).

$$
\begin{aligned}
&(Maximize)\ Minimize\ \mathrm{z} \\
&subject\ to \\
&\mathrm{z}\text{-}\mathrm{c}_1\mathrm{x}\text{-}\mathrm{x}^{\mathrm{T}}\mathrm{Ay}\text{-}c_2 y = 0 \\
&B\begin{pmatrix} x \\ y \end{pmatrix} \le b \\
&0 \le l_i^x \le x_i,\ x_i \in \mathbb{R} \vee x_i \in \mathbb{Z} \\
&0 \le l_j^y \le y_j,\ y_j \in \mathbb{R} \vee y_j \in \mathbb{Z} \\
&x_i \le u_i^x < \infty\ if\ \exists A_{ij} \ne 0 \\
&y_j \le u_j^y < \infty\ if\ \exists A_{ij} \ne 0 \\
&A_{ij} \ge 0\ \forall i, j
\end{aligned}
\tag{4.1}
$$

The algorithm makes use of convex overestimators and underestimators for bilinear terms and the next two sections introduce the overestimating and underestimating procedures for bilinear terms. Underestimators for bilinear terms are used for minimization problems whereas overestimators for bilinear terms are used for maximization problems.

Although it is straightforward to generalize the algorithm also for a matrix including $A_{ij} \le 0$; for the sake of the simplicity of this study, the algorithm is devised only for matrices such that all $A_{ij} \ge 0$.

For overestimating and underestimating, Taylor approximations of first degree are used. Let $f(x, y) = kxy$ and $k \in \mathbb{R}^+$, then the Taylor approximation of first degree around point (x_0, y_0) is given by $kx_0 y + ky_0 x - kx_0 y_0$.

4.3.1 A Convex Underestimator for Bilinear Terms

4.3.1.1 Error as a Number

When error tolerance is expressed as a real number $\varepsilon > 0$, the following function can be defined:

$$g(x,y) = kxy - kx_0 y - ky_0 x + kx_0 y_0 \tag{4.2}$$

Now, the region for which the inequality $g(x,y) \le \varepsilon$ is satisfied is identified. Note that,

$$g(x,y) = k(x - x_0)(y - y_0) \tag{4.3}$$

So, $g(x,y) \le \varepsilon$ is actually $(x - x_0)(y - y_0) \le \varepsilon/k$. Such a region is shown in Fig.4.1.

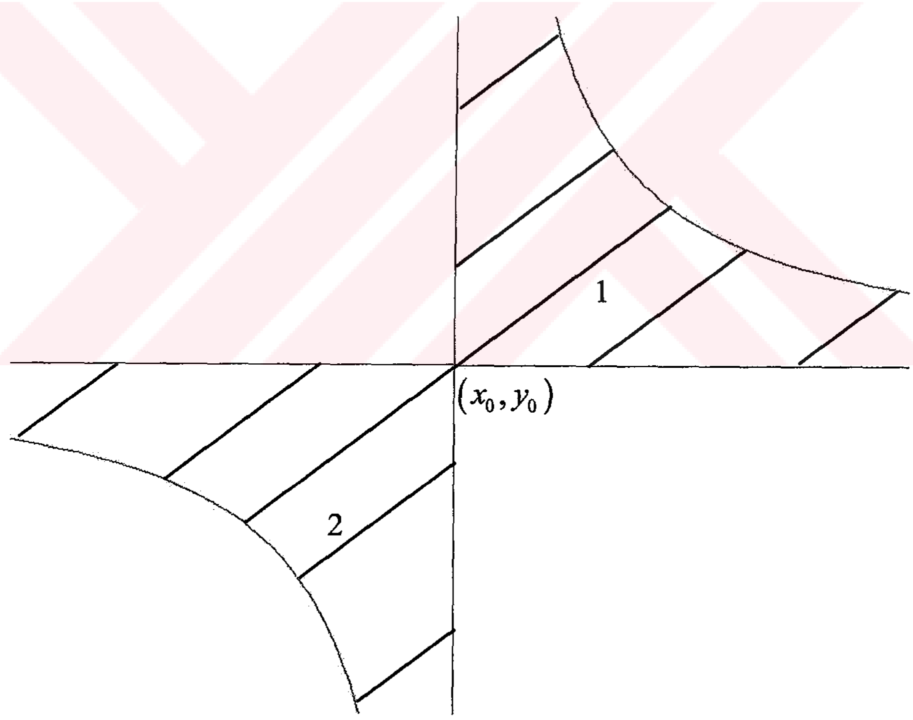

Figure 4.1. $(x - x_0)(y - y_0) \le \varepsilon/k$

Regions 1 and 2 between the axes lines form the region described by $(x - x_0)(y - y_0) \le \varepsilon/k$. Therefore, first degree Taylor approximation written at point (x_0, y_0) underestimates the bilinear term kxy by error tolerance ε in regions 1 and 2.

4.3.1.2 Error as a Ratio

In order to define error as a ratio, the estimated value must be nonzero; otherwise error is undefined because of division by zero. This implies that when any of the variables in the bilinear term has a lower bound of zero, error cannot be defined as a ratio directly. However, this problem can be overcome by defining a change of variables. Namely, whenever a variable v_i with a lower bound of 0 and an upper bound of u_i exists in a bilinear term, a new variable can be defined ($v_i^{'} = v_i + 1$) with lower bound of 1 and upper bound of $u_i + 1$; then, every occurrence of v_i can be replaced by $v_i^{'} - 1$. Therefore, the optimization problem is transformed into a form where error can be defined as a ratio. As an example, let the bilinear optimization problem be given as described in Eq.(4.4).

$$\begin{aligned} &\text{Minimize } xy + pq \\ &\text{subject to} \\ &x + 2y \leq 12 \\ &3p + q \leq 17 \\ &0 \leq x, y, p \leq 5 \\ &1 \leq q \leq 10 \end{aligned} \tag{4.4}$$

In this problem x, y and p must be replaced by new variables in order to express the error term as a ratio. Let,

$$\begin{aligned} &x^{'} = x + 1 \\ &y^{'} = y + 1 \\ &p^{'} = p + 1 \\ &1 \leq x^{'}, y^{'}, p^{'} \leq 6 \end{aligned} \tag{4.5}$$

The corresponding equivalent problem can be formulated as demonstrated in Eq.(4.6).

$$\begin{aligned} &\text{Minimize } \left(x^{'} - 1\right)\left(y^{'} - 1\right) + \left(p^{'} - 1\right)q \\ &\text{subject to} \\ &\left(x^{'} - 1\right) + 2\left(y^{'} - 1\right) \leq 12 \\ &3\left(p^{'} - 1\right) + q \leq 17 \\ &1 \leq x^{'}, y^{'}, p^{'} \leq 6 \\ &1 \leq q \leq 10 \end{aligned} \tag{4.6}$$

Or in more explicit form,

$$\begin{aligned}
&\textit{Minimize } x'y' - x' - y' + p'q - q + 1\\
&\textit{subject to}\\
&x' + 2y' \le 15\\
&3p' + q \le 20\\
&1 \le x', y', p' \le 6\\
&1 \le q \le 10
\end{aligned} \tag{4.7}$$

Therefore, expressing the error as a ratio is not a problem after making the necessary change of variables for a given problem. When error tolerance is expressed as a ratio $0 < \varepsilon < 1$ and $x_0, y_0 > 0$, the following function is defined,

$$g(x,y) = kxy - kx_0 y - ky_0 x + kx_0 y_0 \tag{4.8}$$

Now, the region for which inequality $g(x,y) \le \varepsilon(kx_0 y + ky_0 x - kx_0 y_0)$ is satisfied is identified. Such a definition implies that whenever the value of the underestimator is replaced by the corresponding value of the bilinear term, the error incurred is not more than a ratio of ε. Note that,

$$\begin{aligned}
&g(x,y) \le \varepsilon(kx_0 y + ky_0 x - kx_0 y_0)\\
&\Leftrightarrow g(x,y) = kxy - kx_0 y - ky_0 x + kx_0 y_0 \le \varepsilon(kx_0 y + ky_0 x - kx_0 y_0)\\
&\Leftrightarrow xy - x_0 y - y_0 x + x_0 y_0 \le \varepsilon(x_0 y + y_0 x - x_0 y_0)\\
&\Leftrightarrow xy - (1+\varepsilon)x_0 y - (1+\varepsilon)y_0 x + (1+\varepsilon)x_0 y_0 \le 0\\
&\Leftrightarrow xy - (1+\varepsilon)x_0 y - (1+\varepsilon)y_0 x \le -(1+\varepsilon)x_0 y_0\\
&\Leftrightarrow xy - (1+\varepsilon)x_0 y - (1+\varepsilon)y_0 x + (1+\varepsilon)^2 x_0 y_0 \le (1+\varepsilon)^2 x_0 y_0 - (1+\varepsilon)x_0 y_0\\
&\Leftrightarrow (x - (1+\varepsilon)x_0)(y - (1+\varepsilon)y_0) \le \varepsilon(1+\varepsilon)x_0 y_0
\end{aligned} \tag{4.9}$$

Then, $(x - x_0')(y - y_0') \le \dfrac{\varepsilon x_0' y_0'}{1+\varepsilon}$ when $x_0' = (1+\varepsilon)x_0$ and $y_0' = (1+\varepsilon)y_0$.

So, $g(x,y) \le \varepsilon(kx_0 y + ky_0 x - kx_0 y_0)$ is actually $(x - x_0')(y - y_0') \le \dfrac{\varepsilon x_0' y_0'}{1+\varepsilon}$. Such a region is shown in Fig.4.2.

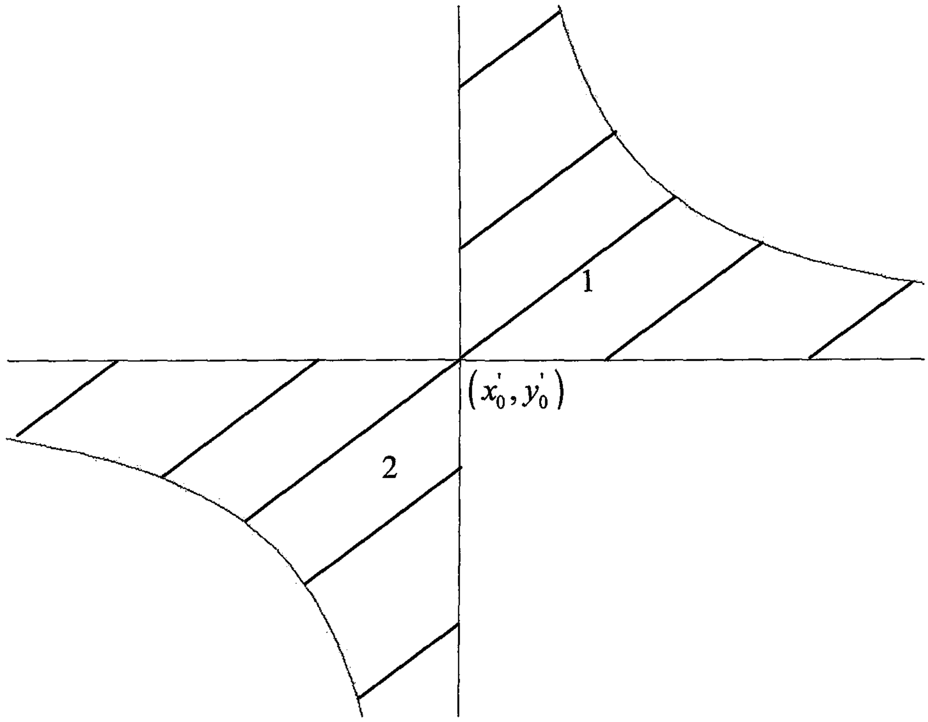

Figure 4.2. $(x - x_0')(y - y_0') \le \frac{\varepsilon x_0' y_0'}{1+\varepsilon}$

Regions 1 and 2 between the axes lines form the region described by $(x - x_0')(y - y_0') \le \frac{\varepsilon x_0' y_0'}{1+\varepsilon}$. Therefore, first degree Taylor approximation written at point (x_0, y_0) underestimates the bilinear term kxy by error tolerance ε in regions 1 and 2.

4.3.1.3 Iterative Underestimating Procedure

In this section an iterative underestimating procedure is developed for both types of underestimators for a given error tolerance either as a number or a ratio, lower and upper bounds on variables involved in bilinear terms. As illustrated in the previous two sections, regardless of the way the error is expressed, the underestimation of a bilinear term by a first degree Taylor polynomial is only possible in regions 1 and 2.

Consider a bilinear term kxy for the rectangular region $x^l \le x \le x^u$ and $y^l \le y \le y^u$. Let $x_0 = x^l$ and $y_0 = y^l$. Given ε, a point (x_1, y_1) can be identified such that the underestimator written at point (x_0, y_0) underestimates kxy with a maximum error of ε. As illustrated in

Fig.4.3, such a point (x_1, y_1) is given by the intersection of the parallel lines to the coordinate axes drawn from the intersection of the error boundary with the upper bounds of x and y.

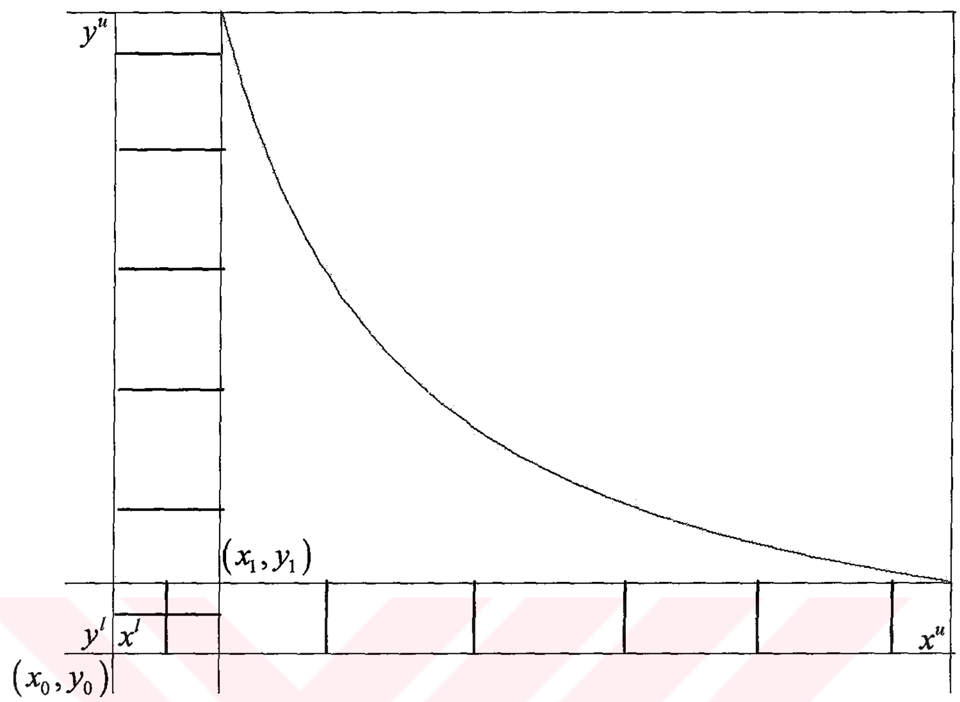

Figure 4.3. Identification of next underestimation point

When (x_1, y_1) is identified, it is known that for the regions $x_0 \le x \le x^u, y_0 \le y \le y_1$ and $x_0 \le x \le x_1, y_0 \le y \le y^u$ first degree Taylor polynomial written at (x_0, y_0) is within the given error bound. Furthermore, the remaining region that must be underestimated is another rectangular region given by $x_1 \le x \le x^u, y_1 \le y \le y^u$, where the same underestimation procedure can be applied.

Given a point (x_0, y_0),

i. If ε is a number, then the next point (x_1, y_1) can be calculated as

a. $$x_1 = x_0 + \frac{\varepsilon/k}{y^u - y_0} \tag{4.10}$$

b. $$y_1 = y_0 + \frac{\varepsilon/k}{x^u - x_0} \tag{4.11}$$

ii. If ε is a ratio, then the next point (x_1, y_1) can be calculated as

a. $$x_1 = (1+\varepsilon)x_0 + \frac{\varepsilon(1+\varepsilon)x_0 y_0}{y^u - (1+\varepsilon)y_0} \tag{4.12}$$

b. $$y_1 = (1+\varepsilon)y_0 + \frac{\varepsilon(1+\varepsilon)x_0 y_0}{x^u - (1+\varepsilon)x_0} \tag{4.13}$$

These points are calculated for both cases of ε by evaluating the curves $(x-x_0)(y-y_0) \le \varepsilon/k$ and $(x-x_0')(y-y_0') \le \frac{\varepsilon x_0' y_0'}{1+\varepsilon}$ at y^u and x^u for formulating x_1 and y_1.

Given these formulations, the algorithm to underestimate kxy with error allowance ε is as follows:

1. Initialize $x_0 = x^l$, $y_0 = y^l$
2. Calculate x_1 and y_1 if $x_0 \le x_1 < x^u$ and $y_0 \le y_1 < y^u$ then go to 3, else go to 5
3. Estimate regions $x_0 \le x \le x^u, y_0 \le y \le y_1$ and $x_0 \le x \le x_1, y_0 \le y \le y^u$ by $kx_0 y + ky_0 x - kx_0 y_0$: the first of these rectangles is called region 1 and the second of these rectangles is called region 2; these two rectangles together are called a partition
4. Set $x_0 = x_1$ and $y_0 = y_1$, go to 2
5. Estimate $x_0 \le x \le x^u, y_0 \le y \le y^u$ by $kx_0 y + ky_0 x - kx_0 y_0$, STOP

The correctness of this algorithm is trivial since the number of partitions is finite, and the number of partitions is the number of points generated including points generated including (x_0, y_0) and (x^u, y^u) minus 1.

4.3.2 A Convex Overestimator for Bilinear Terms

4.3.2.1 Error as a Number

When error tolerance is expressed as a number such that $\varepsilon > 0$, the following function can be defined:

(4.14)

$g(x,y) = kx_0 y + ky_0 x - kx_0 y_0 - kxy$

The region for which the inequality $g(x,y) \le \varepsilon$ is satisfied can be identified. Note that,

$$g(x,y) = -k(x - x_0)(y - y_0) \tag{4.15}$$

Therefore, $g(x,y) \le \varepsilon$ is actually $(x - x_0)(y - y_0) \ge -\varepsilon/k$ as shown in Fig.4.4.

Regions 1 and 2 between the axes lines form the region is described by $(x - x_0)(y - y_0) \ge -\varepsilon/k$. Therefore, the first degree Taylor approximation written at point (x_0, y_0) overestimates the bilinear term kxy by error tolerance ε in regions 1 and 2.

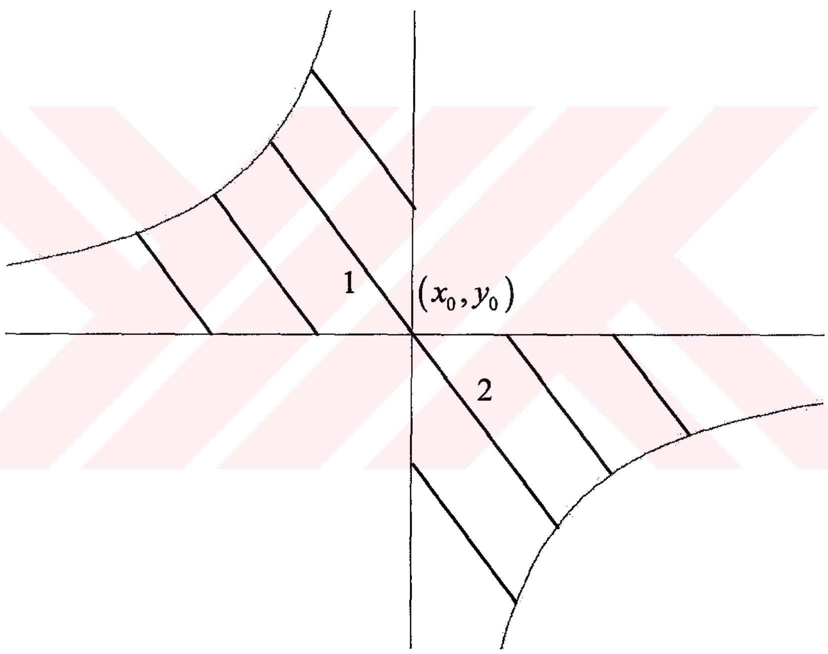

Figure 4.4. $(x - x_0)(y - y_0) \ge -\varepsilon/k$

4.3.2.2 Error as a Ratio

Similar to the underestimation case, variables with lower bound of 0 create a problem for the definition of error as a ratio. However, the same change of variables procedure as explained in section 4.3.1.3 overcomes this problem.

When error tolerance is expressed as a ratio $0 < \varepsilon < 1$ and $x_0, y_0 > 0$, the following function can be defined:

(4.16)

$g(x,y) = kx_0 y + ky_0 x - kx_0 y_0 + kxy$

The region for which the inequality $g(x,y) \le \varepsilon(kx_0 y + ky_0 x - kx_0 y_0)$ is satisfied can be identified. Such a definition implies that whenever the value of the overestimator is replaced by the corresponding value of the bilinear term, the error incurred is not more than a ratio of ε. Note that,

$$
\begin{aligned}
& g(x,y) \le \varepsilon(kx_0 y + ky_0 x - kx_0 y_0) \\
\Leftrightarrow\ & g(x,y) = kx_0 y - ky_0 x - kx_0 y_0 - kxy \le \varepsilon(kx_0 y + ky_0 x - kx_0 y_0) \\
\Leftrightarrow\ & x_0 y + y_0 x - x_0 y_0 - xy \le \varepsilon(x_0 y + y_0 x - x_0 y_0) \\
\Leftrightarrow\ & xy - (1-\varepsilon)x_0 y - (1-\varepsilon)y_0 x \ge -(1-\varepsilon)x_0 y_0 \\
\Leftrightarrow\ & xy - (1-\varepsilon)x_0 y - (1-\varepsilon)y_0 x + (1-\varepsilon)^2 x_0 y_0 \ge (1-\varepsilon)^2 x_0 y_0 - (1-\varepsilon)x_0 y_0 \qquad (4.17) \\
\Leftrightarrow\ & y(x - (1-\varepsilon)x_0) - (1-\varepsilon)y_0(x - (1-\varepsilon)x_0) \ge -\varepsilon(1-\varepsilon)x_0 y_0 \\
\Leftrightarrow\ & (x - (1-\varepsilon)x_0)(y - (1-\varepsilon)y_0) \le -\varepsilon(1-\varepsilon)x_0 y_0
\end{aligned}
$$

Let $x_0' = (1-\varepsilon)x_0$ *and* $y_0' = (1-\varepsilon)y_0$*. Then,*

$$(x - x_0')(y - y_0') \le \frac{-\varepsilon x_0' y_0'}{1-\varepsilon}$$

So, $g(x,y) \le \varepsilon(kx_0 y + ky_0 x - kx_0 y_0)$ is actually $(x - x_0')(y - y_0') \le \frac{-\varepsilon x_0' y_0'}{1-\varepsilon}$ as shown in Fig.4.5.

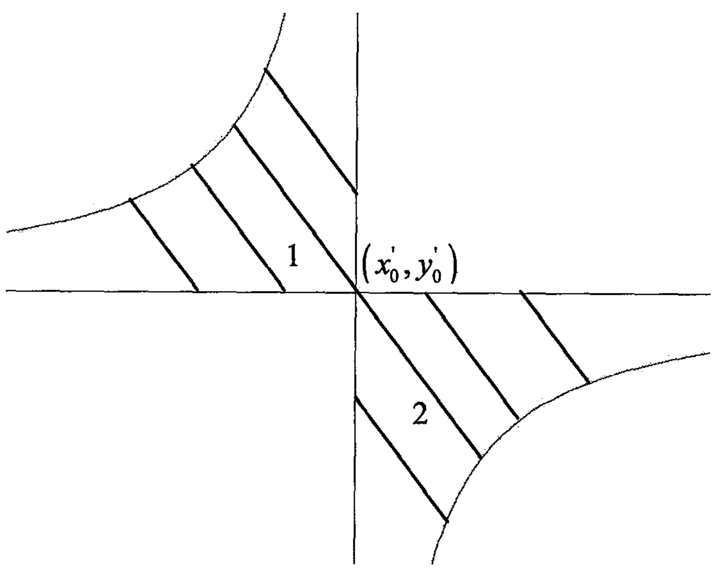

Figure 4.5. $(x - x_0')(y - y_0') \le \frac{-\varepsilon x_0' y_0'}{1-\varepsilon}$

Regions 1 and 2 between the axes lines form the region described by $(x-x_0^{'})(y-y_0^{'}) \le \frac{-\varepsilon x_0^{'} y_0^{'}}{1-\varepsilon}$. Therefore, the first degree Taylor approximation written at point (x_0, y_0) overestimates the bilinear term kxy by error tolerance ε in regions 1 and 2.

4.3.2.3 Iterative Overestimating Procedure

In this section an iterative overestimating procedure is described for both types of overestimators for a given error tolerance either as a number or a ratio, lower and upper bounds on variables involved in bilinear terms. As illustrated in the previous two sections, regardless of the way the error is expressed, the overestimation of a bilinear term by a first degree Taylor polynomial is only possible in regions 1 and 2.

Consider a bilinear term kxy for the rectangular region $x^l \le x \le x^u$ and $y^l \le y \le y^u$ and let $x_0 = x^l$ and $y_0 = y^l$. For a given ε, a point (x_1, y_1) can be identified such that the overestimator written at point (x_0, y_0) overestimates kxy with a maximum error of ε. As illustrated in Fig.4.6, such a point (x_1, y_1) is given by the intersection of the parallel lines to the coordinate axes drawn from the intersection of the error boundary with the upper bounds of x and y.

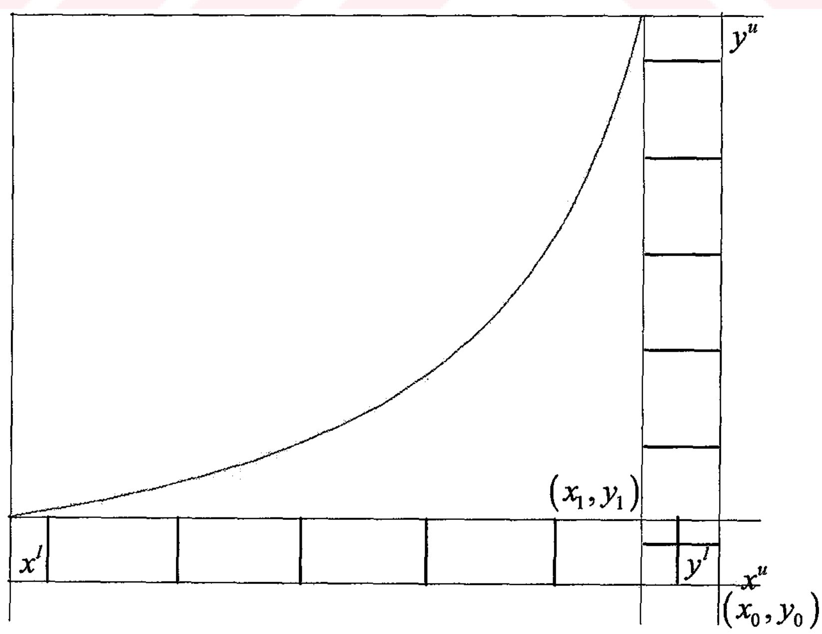

Figure 4.6. Identification of next overestimation point

When (x_1, y_1) is identified, it is known that for the regions $x_0 \le x \le x^u, y_0 \le y \le y_1$ and $x_0 \le x \le x_1, y_0 \le y \le y^u$ the first degree Taylor polynomial written at (x_0, y_0) is within the given largest error. Furthermore, the remaining region that must be overestimated is another rectangular region given by $x_1 \le x \le x^u, y_1 \le y \le y^u$, where the same overestimation procedure can be applied.

Given a point (x_0, y_0),

i. If ε is taken as a number, then the next point (x_1, y_1) can be calculated as

a. $$x_1 = x_0 - \frac{\varepsilon/k}{y^u - y_0} \tag{4.18}$$

b. $$y_1 = y_0 + \frac{\varepsilon/k}{x_0 - x^l} \tag{4.19}$$

ii. If ε is taken as a ratio, then the next point (x_1, y_1) can be calculated as

a. $$x_1 = (1-\varepsilon)x_0 - \frac{\varepsilon(1-\varepsilon)x_0 y_0}{y^u - (1-\varepsilon)y_0} \tag{4.20}$$

b. $$y_1 = (1-\varepsilon)y_0 - \frac{\varepsilon(1-\varepsilon)x_0 y_0}{x^l - (1-\varepsilon)x_0} \tag{4.21}$$

For both cases of ε, these points are calculated by evaluating the curves $(x - x_0)(y - y_0) \ge -\varepsilon/k$ and $(x - x_0')(y - y_0') \le \frac{-\varepsilon x_0' y_0'}{1-\varepsilon}$ at y^u and x^l for formulating x_1 and y_1.

Given these formulations, the algorithm to overestimate kxy with error allowance ε is as follows:

1. Initialize $x_0 = x^u$, $y_0 = y^l$
2. Calculate x_1 and y_1 if $x_0 \le x_1 < x^l$ and $y_0 \le y_1 < y^u$ then go to 3, else go to 5
3. Estimate regions $x_0 \le x \le x^l, y_0 \le y \le y_1$ and $x_1 \le x \le x_0, y_0 \le y \le y^u$ by $kx_0 y + ky_0 x - kx_0 y_0$: the first of these rectangles is called region 1 and the second

of these rectangles is called region 2; these two rectangles together are called a partition

4. Set $x_0 = x_1$ and $y_0 = y_1$, go to 2
5. Estimate $x_0 \le x \le x^l, y_0 \le y \le y^u$ by $kx_0 y + ky_0 x - kx_0 y_0$, STOP

The correctness of this algorithm is trivial. The number of partitions is the number of points generated including points generated including (x_0, y_0) and (x^l, y^u) minus 1.

4.3.3 The Global Optimization Algorithm

In this section, a novel mixed-integer linear formulation is presented which underestimates or overestimates the bilinear optimization problem depending on the direction of optimization. If the bilinear optimization problem is a maximization problem, then overestimation is used whereas underestimation is used for minimization problems.

4.3.3.1 A Mixed Integer Linear Underestimating Formulation

For each $A_{ij} \neq 0$,

i. Let the number of partitions created by an underestimation procedure for a given error bound ε for x_i and y_j be P_{ij} and the associated index be p_{ij}
ii. Define $x_i D_{ij}(p_{ij})$ and $y_j D_{ij}(p_{ij})$ that contain the coordinates of the points generated by the underestimation procedure used
iii. Let $k \in \{1,2\}$ be the index of rectangular regions in a partition
iv. Define $z_{ij}(p_{ij},k)\begin{cases}1, \text{ if } x_i \text{ and } y_j \text{ assumes a value in region } k \text{ of partition } p_{ij} \\ 0, \text{ otherwise}\end{cases}$
v. Define the underestimated value of the bilinear term of x_i and y_j as t_{ij}
vi. Define $x_i t_{ij}(p_{ij},k)$ and $y_j t_{ij}(p_{ij},k)$ that hold the value of x_i and y_j if they are in the region k of partition p_{ij}, otherwise is equal to 0

The underestimator written for the regions of a specific partition $x_0 \le x \le x^u, y_0 \le y \le y_1$ and $x_0 \le x \le x_1, y_0 \le y \le y^u$ can be tightened by using two more first degree Taylor polynomials. Given the bounds on variables x^u and y^u; (x_0, y_0) and (x_1, y_1), consider the two rectangular regions $x_0 \le x \le x^u, y_0 \le y \le y_1$ and $x_0 \le x \le x_1, y_0 \le y \le y^u$. Constructing a first degree Taylor polynomial at (x^u, y_1) for the first rectangle and another first degree Taylor polynomial at (x_1, y^u) for the second rectangle generates underestimators for the respective rectangles. For a specific rectangle of a partition p_{ij}, the value of the bilinear term is better bounded by the maximum of the underestimator written at (x_0, y_0) and the underestimator written at (x^u, y_1) in the first rectangle; and the underestimator written at (x_0, y_0) and the underestimator written at (x_1, y^u) in the second rectangle. This type of underestimating was proven to be the tightest convex underestimator of a bilinear term for a rectangular region by Al-Khayyal and Falk [27].

Based on these, the ε underestimating mixed integer linear formulation is given in Eq.(4.22).

Minimize z

subject to

$$z - c_1 x - \sum_i \sum_j t_{ij} - c_2 y = 0$$

$$B\begin{pmatrix} x \\ y \end{pmatrix} \le b$$

$$x_i - \sum_{p_{ij}} \sum_k x_i t_{ij}\left(p_{ij}, k\right) = 0 \ \forall i, j \ni A_{ij} \ne 0$$

$$y_j - \sum_{p_{ij}} \sum_k y_j t_{ij}\left(p_{ij}, k\right) = 0 \ \forall i, j \ni A_{ij} \ne 0$$

$$x_i D_{ij}\left(p_{ij}\right) * z_{ij}\left(p_{ij}, 1\right) \le x_i t_{ij}\left(p_{ij}, 1\right) \le u_i^x * z_{ij}\left(p_{ij}, 1\right) \ \forall i, j, p_{ij}$$

$$x_i D_{ij}\left(p_{ij}\right) * z_{ij}\left(p_{ij}, 2\right) \le x_i t_{ij}\left(p_{ij}, 2\right) \le x_i D_{ij}\left(p_{ij} + 1\right) * z_{ij}\left(p_{ij}, 2\right) \ \forall i, j, p_{ij}$$

$$y_j D_{ij}\left(p_{ij}\right) * z_{ij}\left(p_{ij}, 1\right) \le y_j t_{ij}\left(p_{ij}, 1\right) \le y_j D_{ij}\left(p_{ij} + 1\right) * z_{ij}\left(p_{ij}, 1\right) \ \forall i, j, p_{ij}$$

$$y_j D_{ij}\left(p_{ij}\right) * z_{ij}\left(p_{ij}, 2\right) \le y_j t_{ij}\left(p_{ij}, 2\right) \le u_j^y * z_{ij}\left(p_{ij}, 2\right) \ \forall i, j, p_{ij}$$

$$\sum_{p_{ij}} \sum_k z_{ij}\left(p_{ij}, k\right) = 1$$

$$t_{ij} \ge A_{ij} * \left[\sum_{p_{ij}} \sum_k \begin{pmatrix} y_j D_{ij}\left(p_{ij}\right) * x_i t_{ij}\left(p_{ij}, k\right) \\ + x_i D_{ij}\left(p_{ij}\right) * y_j t_{ij}\left(p_{ij}, k\right) \\ - x_i D_{ij}\left(p_{ij}\right) * y_j D_{ij}\left(p_{ij}\right) * z_{ij}\left(p_{ij}, k\right) \end{pmatrix} \right] \ \forall i, j \ni A_{ij} \ne 0 \tag{4.22}$$

$$t_{ij} \ge A_{ij} * \left[\begin{array}{l} \sum_{p_{ij}}^{P_{ij}-1} \begin{pmatrix} y_j D_{ij}\left(p_{ij} + 1\right) * x_i t_{ij}\left(p_{ij}, 1\right) \\ + u_i^x * y_j t_{ij}\left(p_{ij}, 1\right) \\ - u_i^x * y_j D_{ij}\left(p_{ij}\right) * z_{ij}\left(p_{ij}, 1\right) \end{pmatrix} \\ + \\ \sum_{p_{ij}}^{P_{ij}-1} \begin{pmatrix} u_j^y * x_i t_{ij}\left(p_{ij}, 2\right) \\ + x_i D_{ij}\left(p_{ij} + 1\right) * y_j t_{ij}\left(p_{ij}, 2\right) \\ - x_i D_{ij}\left(p_{ij} + 1\right) * u_j^y * z_{ij}\left(p_{ij}, 2\right) \end{pmatrix} \\ + \\ \sum_k \begin{pmatrix} u_j^y * x_i t_{ij}\left(P_{ij}, k\right) \\ + u_i^x * y_j t_{ij}\left(P_{ij}, k\right) \\ - u_i^x * u_j^y * z_{ij}\left(P_{ij}, k\right) \end{pmatrix} \end{array} \right] \ \forall i, j \ni A_{ij} \ne 0$$

$$0 \le l_i^x \le x_i, \ x_i \in \mathbb{R} \vee x_i \in \mathbb{Z} \ \forall \mathrm{i}$$

$$0 \le l_j^y \le y_j, \ y_j \in \mathbb{R} \vee y_j \in \mathbb{Z} \ \forall j$$

$$x_i t_{ij}\left(p_{ij}, k\right), x_i t_{ij}\left(p_{ij}, k\right), t_{ij} \ge 0 \ \forall i, j, k \ni A_{ij} \ne 0$$

$$z_{ij}\left(p_{ij}, k\right) \in \{0,1\} \ \forall i, j, k \ni A_{ij} \ne 0$$

4.3.3.2 A Mixed Integer Linear Overestimating Formulation

For each $A_{ij} \neq 0$,

i. Let the number of partitions created by an overestimation procedure for a given error bound ε for x_i and y_j be P_{ij} and the associated index be p_{ij}

ii. Define $x_iD_{ij}\left(p_{ij}\right)$ and $y_jD_{ij}\left(p_{ij}\right)$ that contain the coordinates of the points generated by the overestimation procedure used

iii. Let $k \in \{1,2\}$ be the index of rectangular regions in a partition

iv. Define $z_{ij}\left(p_{ij},k\right)\begin{cases}1, \; \textit{if } x_i \textit{ and } y_j \textit{ assumes a value in region } k \textit{ of partition } p_{ij} \\ 0, \; \textit{otherwise}\end{cases}$

v. Define the overestimated value of the bilinear term of x_i and y_j as t_{ij}

vi. Define $x_it_{ij}\left(p_{ij},k\right)$ and $y_jt_{ij}\left(p_{ij},k\right)$ that hold the value of x_i and y_j if they are in the region k of partition p_{ij}, otherwise is equal to 0

The overestimator written for the regions of a specific partition $x_0 \le x \le x^l, y_0 \le y \le y_1$ and $x_1 \le x \le x_0, y_0 \le y \le y^u$ can be tightened by using two more first degree Taylor polynomials. Given the bounds on variables x^l and y^u; (x_0, y_0) and (x_1, y_1), consider the two rectangular regions $x^{\acute{y}} \le x \le x_0, y_0 \le y \le y_1$ and $x_1 \le x \le x_0, y_0 \le y \le y^u$. Constructing a first degree Taylor polynomial at $\left(x^l, y_1\right)$ for the first rectangle and another first degree Taylor polynomial at $\left(x_1, y^u\right)$ for the second rectangle generates overestimators for the respective rectangles. For a specific rectangle of a partition p_{ij}, the value of the bilinear term is better bounded by the maximum of the overestimator written at $\left(x_0, y_0\right)$ and the overestimator written at $\left(x^l, y_1\right)$ in the first rectangle; and the overestimator written at $\left(x_0, y_0\right)$ and the overestimator written at $\left(x_1, y^u\right)$ in the second rectangle. This type of overestimating was proven to be the tightest convex overestimator of a bilinear term for a rectangular region by Al Khayyal and Falk [27].

Based on these, the ε overestimating mixed integer linear formulation is given in Eq.(4.23).

$$
\begin{aligned}
&\textit{Maximize } z \\
&\textit{subject to} \\
&z - c_1 x - \sum_i \sum_j t_{ij} - c_2 y = 0 \\
&B\begin{pmatrix} x \\ y \end{pmatrix} \le b \\
&x_i - \sum_{p_{ij}} \sum_k x_i t_{ij}(p_{ij},k) = 0 \ \forall i,j \ni A_{ij} \ne 0 \\
&y_j - \sum_{p_{ij}} \sum_k y_j t_{ij}(p_{ij},k) = 0 \ \forall i,j \ni A_{ij} \ne 0 \\
&l_i^x * z_{ij}(p_{ij},1) \le x_i t_{ij}(p_{ij},1) \le x_i D_{ij}(p_{ij}) * z_{ij}(p_{ij},1) \ \forall i,j,p_{ij} \\
&x_i D_{ij}(p_{ij}+1) * z_{ij}(p_{ij},2) \le x_i t_{ij}(p_{ij},2) \le x_i D_{ij}(p_{ij}) * z_{ij}(p_{ij},2) \ \forall i,j,p_{ij} \\
&y_j D_{ij}(p_{ij}) * z_{ij}(p_{ij},1) \le y_j t_{ij}(p_{ij},1) \le y_j D_{ij}(p_{ij}+1) * z_{ij}(p_{ij},1) \ \forall i,j,p_{ij} \\
&y_j D_{ij}(p_{ij}) * z_{ij}(p_{ij},2) \le y_j t_{ij}(p_{ij},2) \le u_j^y * z_{ij}(p_{ij},2) \ \forall i,j,p_{ij} \\
&\sum_{p_{ij}} \sum_k z_{ij}(p_{ij},k) = 1 \\
&t_{ij} \le A_{ij} * \left[\sum_{p_{ij}} \sum_k \begin{pmatrix} y_j D_{ij}(p_{ij}) * x_i t_{ij}(p_{ij},k) \\ + x_i D_{ij}(p_{ij}) * y_j t_{ij}(p_{ij},k) \\ - x_i D_{ij}(p_{ij}) * y_j D_{ij}(p_{ij}) * z_{ij}(p_{ij},k) \end{pmatrix} \right] \ \forall i,j \ni A_{ij} \ne 0 \\
&t_{ij} \le A_{ij} * \left[\begin{array}{l} \sum_{p_{ij}}^{P_{ij}-1} \begin{pmatrix} y_j D_{ij}(p_{ij}+1) * x_i t_{ij}(p_{ij},1) \\ + l_i^x * y_j t_{ij}(p_{ij},1) \\ - l_i^x * y_j D_{ij}(p_{ij}+1) * z_{ij}(p_{ij},1) \end{pmatrix} \\ + \\ \sum_{p_{ij}}^{P_{ij}-1} \begin{pmatrix} u_j^y * x_i t_{ij}(p_{ij},2) \\ + x_i D_{ij}(p_{ij}+1) * y_j t_{ij}(p_{ij},2) \\ - x_i D_{ij}(p_{ij}+1) * u_j^y * z_{ij}(p_{ij},2) \end{pmatrix} \\ + \\ \sum_k \begin{pmatrix} u_j^y * x_i t_{ij}(P_{ij},k) \\ + l_i^x * y_j t_{ij}(P_{ij},k) \\ - l_i^x * u_j^y * z_{ij}(P_{ij},k) \end{pmatrix} \end{array} \right] \ \forall i,j \ni A_{ij} \ne 0 \\
&0 \le l_i^x \le x_i,\ x_i \in \mathbb{R} \vee x_i \in \mathbb{Z} \ \forall i \\
&0 \le l_j^y \le y_j,\ y_j \in \mathbb{R} \vee y_j \in \mathbb{Z} \ \forall j \\
&x_i t_{ij}(p_{ij},k), x_i t_{ij}(p_{ij},k), t_{ij} \ge 0 \ \forall i,j,k \ni A_{ij} \ne 0 \\
&z_{ij}(p_{ij},k) \in \{0,1\} \ \forall i,j,k \ni A_{ij} \ne 0
\end{aligned}
\tag{4.23}
$$

4.3.3.3 Improving the Bounds of Variables Participating in Bilinear Terms

When a lower (upper) bound on the objective function value of a maximization (minimization) bilinear problem is known, it is possible to improve the lower and upper bounds of variables participating in bilinear terms. This, in turn, makes it possible to generate tighter underestimators or overestimators for bilinear terms with possibly using less number of partitions. This implies that if the lower and upper bounds of variables participating in bilinear terms are tightened, less number of binary variables is necessary compared to the original lower and upper bounds. Furthermore, the underestimating (overestimating) mixed integer linear model can be used for this purpose by simply changing the objective function and adding a single constraint.

Let the constraint set of an underestimating (overestimating) mixed integer linear formulation as introduced in sections 4.3.3.1 and 4.3.3.2 be CS. Let the bound on the objective function value be z_b. Then, the bound updating formulation is given in Eq.(4.24).

$$\begin{aligned}
&\textit{Maximize or Minimize } x_i \textit{ or } y_j \\
&\textit{subject to} \\
&\begin{cases} z \ge z_b \textit{ if the underlying bilinear problem is Maximization} \\ z \le z_b \textit{ if the underlying bilinear problem is Minimization} \end{cases} \\
&CS
\end{aligned} \tag{4.24}$$

The maximization of a variable value determines the new upper bound whereas the minimization of the variable value determines the new lower bound.

4.3.3.4 A Novel Global Optimization Algorithm for Bilinear Optimization

In this section a novel global optimization algorithm is proposed for the general bilinear optimization problems. The algorithm is based on generating tighter underestimating (overestimating) mixed-integer linear formulations while shrinking the difference between the lower and upper bounds of the variables participating in bilinear terms.

In order to verify optimality two error tolerances are going to be used. Basically, one of the tolerances specifies the error in terms of ratio, whereas the other one is used to specify the error in an absolute sense. Let z be the optimal solution of an underestimating (overestimating) mixed-integer linear formulation and (x, y) specify the optimal value of the variables of the original bilinear optimization problem. $z_c = c_1 x + x^T Ay + c_2 y$ is the corresponding solution of the original bilinear optimization problem which provides an upper/lower bound on the value of the optimal solution of the original problem for minimization and maximization directions respectively. Specifically, the suggested termination condition is,

$$\begin{aligned} &\text{if } z = 0 \text{ then } \left|z - z_c\right| \le \varepsilon_t^a \\ &\text{else } \left|\frac{z - z_c}{z}\right| \le \varepsilon_t^r \text{ or } \left|z - z_c\right| \le \varepsilon_t^a \end{aligned} \tag{4.25}$$

where $\varepsilon_t^a > 0$ and $\varepsilon_t^r > 0$ are user specified error tolerances. (ε_t^a is called absolute error tolerance and ε_t^r is called ratio error tolerance.)

The steps of the global optimization algorithm is given as follows.

1. Modify the variables of the optimization problem whose lower bound is zero as explained in sections 4.3.1.2 and 4.3.2.2; let the modified problem be represented by,

$$\begin{aligned} &\textit{Maximize or Minimize } \mathrm{z} \\ &\textit{subject to} \\ &\mathrm{z\text{-}c_1x\text{-}x^TAy\text{-}}c_2 y - K = 0 \\ &B\begin{pmatrix} x \\ y \end{pmatrix} \le b \\ &0 < l_i^x \le x_i \le u_i^x < \infty,\ x_i \in \mathbb{R} \vee x_i \in \mathbb{Z} \textit{ if } \exists A_{ij} \ne 0 \\ &0 < l_j^y \le y_j \le u_j^y < \infty,\ y_j \in \mathbb{R} \vee y_j \in \mathbb{Z} \textit{ if } \exists A_{ij} \ne 0 \\ &0 \le l_i^x \le x_i,\ x_i \in \mathbb{R} \vee x_i \in \mathbb{Z} \\ &0 \le l_j^y \le y_j,\ y_j \in \mathbb{R} \vee y_j \in \mathbb{Z} \\ &A_{ij} \ge 0\ \forall i, j \\ &K \in R \end{aligned} \tag{4.26}$$

2. Determine the target error bounds ε_t^a and ε_t^r for the objective function of the modified problem; and the underestimation/overestimation method, error either as a ratio or a number
3. Set $0 < \varepsilon_0^{ij}$ for each bilinear term

4. Generate and solve ε_0^{ij} underestimating (overestimating) mixed-integer linear formulation for minimization (maximization) respectively
    a. If the underestimating (overestimating) mixed-integer linear formulation is infeasible, then so is the underlying bilinear optimization problem, STOP
    b. If the objective function value of the underestimating (overestimating) mixed-integer linear formulation is unbounded, then so is the underlying bilinear optimization problem, STOP
    c. If the underestimating (overestimating) mixed-integer linear formulation has a bounded optimal solution, then let the objective value be z, and the solution be given by (x, y), GOTO 5
5. Evaluate $z_c = c_1 x + x^t A y + c_2 y + K$
6. Check for termination conditions
    a. If $z = 0$ then if $|z - z_c| \leq \varepsilon_t^a$ STOP
    b. Else if $\left|\frac{z - z_c}{z}\right| \leq \varepsilon_t^r$ or $|z - z_c| \leq \varepsilon_t^a$ then STOP, otherwise GOTO 7
7. Set $\varepsilon_1^{ij} \leq \varepsilon_0^{ij}$, with the condition that $\varepsilon_1^{\alpha\beta} < \varepsilon_0^{\alpha\beta}$ such that
    a. If error is specified as a ratio for underestimation (overestimation) procedure then

$$\underset{i,j}{Max}\left\{\left|\left(t_{ij} - A_{ij}x_i y_j\right)/t_{ij}\right|\right\} = \left|\left(t_{\alpha\beta} - A_{\alpha\beta}x_\alpha y_\beta\right)/t_{\alpha\beta}\right| \tag{4.27}$$

    b. If error is specified as a number for underestimation/overestimation procedure then

$$\underset{i,j}{Max}\left\{\left|t_{ij} - A_{ij}x_i y_j\right|\right\} = \left|t_{\alpha\beta} - A_{\alpha\beta}x_\alpha y_\beta\right| \tag{4.28}$$

8. Update the bounds of the desired variables by taking the lower (upper) bound on the objective function value as z_c, preferably for x_i and y_j such that $\varepsilon_1^{ij} < \varepsilon_0^{ij}$
9. Set $\varepsilon_0^{ij} = \varepsilon_1^{ij}$, GOTO 4

The algorithm is trivially correct provided that the value of $\varepsilon_1^{\alpha\beta}$ in the seventh step is determined by a procedure which eventually sets each ε_1^{ij} small enough so that either

$\left|(z - z_c)/z\right| \le \varepsilon_t^r$ or $\left|z - z_c\right| \le \varepsilon_t^a$ is satisfied. One simple such procedure to be applied at seventh step is $\varepsilon_1^{\alpha\beta} = \varepsilon_0^{\alpha\beta}/q \ni q>1$.

If the bilinear optimization problem is infeasible or unbounded, this procedure can be applied only once and if the problem has a bounded optimal solution then steps 4.a and 4.b are applied only in the first iteration of this procedure.

If underestimators (overestimators) are used for generating underestimating (overestimating) mixed-integer formulations when error is specified as a number, then the corresponding bilinear optimization problem can be solved to the desired accuracy in one iteration. On the other hand, if underestimators (overestimators) when error is specified as a ratio are used for generating underestimating (overestimating) mixed-integer formulations, then the corresponding bilinear optimization problem can be solved to the desired accuracy in at most two iterations. As a matter of fact, the upper or lower bound on the optimal result of the corresponding bilinear optimization problem is found in the first iteration when error is specified as a ratio. When the upper or lower bound on the optimal solution is known, one can easily specify a ratio error tolerance for each bilinear term which will meet the target absolute error tolerance.

4.4 Illustrative Example

Consider the optimization problem given in Eq.(4.29) that was proposed by Al-Khayyal and Falk [27].

$$\begin{aligned}
&\textit{Minimize } -x+xy-y\\
&\textit{subject to}\\
&-6x+8y\le 3\\
&3x-y\le 3\\
&0\le x,y\le 5
\end{aligned} \tag{4.29}$$

The underestimators of the bilinear terms are generated by expressing error as a ratio.

4.4.1 Initialization

Since the lower bounds on the variables are zero, the model must be changed as described in section 4.3.1.2. This step constitutes the first step of the algorithm described in section

4.3.3.4. The new variables and the corresponding equivalent optimization problem are given in Eq.(4.30).

$$\begin{aligned}
&\textit{Minimize } x'y' - 2x' - 2y' + 3\\
&\textit{subject to}\\
&-6x' + 8y' \le 5\\
&3x' - y' \le 5\\
&1 \le x', y' \le 6\\
&\textit{where } x = x' - 1,\ y = y' - 1
\end{aligned} \tag{4.30}$$

In order to tighten the bounds on the variables, the optimization problem presented in Eq.(4.31) is solved.

$$\begin{aligned}
&\textit{Minimize / Maximize } x', y'\\
&\textit{subject to}\\
&-6x' + 8y' \le 5\\
&3x' - y' \le 5\\
&1 \le x', y' \le 6
\end{aligned} \tag{4.31}$$

The solution of the optimization problems presented in Eq.(4.31) updates the bounds as shown in Table 4.1.

Table 4.1. Updated Bounds of Variables for Modified Bilinear Problem

Variable	Lower Bound	Upper Bound
x'	1	2.5
y'	1	2.5

Therefore, the optimization problem formulated in Eq.(4.30) can now be modified as shown in Eq.(4.32).

$$\begin{aligned}
&\textit{Minimize } x'y' - 2x' - 2y' + 3\\
&\textit{subject to}\\
&-6x' + 8y' \le 5\\
&3x' - y' \le 5\\
&1 \le x', y' \le 2.5
\end{aligned} \tag{4.32}$$

4.4.2 Iteration 1

The ratio error is chosen as 0.03. The underestimation procedure introduced in section 4.3.1.3 is applied to generate the partitions. The results of this procedure are tabulated in Table 4.2.

Table 4.2. Underestimation Partition Points for Iteration 1

	0	1	2	3	4	5	6	7	8	9	10	11	12
x'	1	1.05	1.11	1.17	1.24	1.31	1.40	1.49	1.61	1.76	1.95	2.24	2.5
y'	1	1.05	1.11	1.17	1.24	1.31	1.40	1.49	1.61	1.76	1.95	2.24	2.5

The values in table 4.2 are written to two decimals for simplicity; in reality, these values are calculated to nine decimals while solving the problem on the computer. The parameters of the underestimating optimization problem are provided as follows.

$$
\begin{aligned}
&p = 0..11 \\
&P = 11 \\
&k \in \{1,2\} \\
&xD(p) = [1, 1.05, 1.11, 1.17, 1.24, 1.31, 1.40, 1.49, 1.61, 1.76, 1.95, 2.24, 2.5] \\
&yD(p) = [1, 1.05, 1.11, 1.17, 1.24, 1.31, 1.40, 1.49, 1.61, 1.76, 1.95, 2.24, 2.5] \\
&u^x = u^y = 2.5 \\
&l^x = l^y = 1
\end{aligned}
\tag{4.33}
$$

The underestimating optimization problem is formulated in Eq.(4.34).

$$
\begin{aligned}
& \textit{Minimize } t - 2x' - 2y' + 3 \\
& \textit{subject to} \\
& -6x' + 8y' \le 5 \\
& 3x' - y' \le 5 \\
& x' - \sum_p \sum_k xt(p,k) = 0 \\
& y' - \sum_p \sum_k yt(p,k) = 0 \\
& xD(p) * z(p,1) \le xt(p,1) \le u^x * z(p,1) \ \forall p \\
& xD(p) * z(p,2) \le xt(p,2) \le xD(p+1) * z(p,2) \ \forall p \\
& yD(p) * z(p,1) \le yt(p,1) \le yD(p+1) * z(p,1) \ \forall p \\
& yD(p) * z(p,2) \le yt(p,2) \le u^y * z(p,2) \ \forall p \\
& \sum_p \sum_k z(p,k) = 1 \\
& t \ge \left[\sum_p \sum_k \begin{pmatrix} yD(p) * xt(p,k) \\ +xD(p) * yt(p,k) \\ -xD(p) * yD(p) * z(p,k) \end{pmatrix} \right] \\
& t \ge \left[\begin{matrix} \sum_{p_{ij}}^{P_{ij}-1} \begin{pmatrix} yD(p+1) * xt(p,1) \\ +u^x * yt(p,1) \\ -u^x * yD(p) * z(p,1) \end{pmatrix} + \sum_p^{P-1} \begin{pmatrix} u^y * xt(p,2) \\ +xD(p+1) * yt(p,2) \\ -xD(p+1) * u^y * z(p,2) \end{pmatrix} \\ +\sum_k \begin{pmatrix} u^y * xt(P,k) \\ +u^x * yt(P,k) \\ -u^x * u^y * z(P,k) \end{pmatrix} \end{matrix} \right] \\
& xt(p,k), yt(p,k), t \ge 0 \\
& l^x \le x' \le u^x \\
& l^x \le y' \le u^y \\
& z(p,k) \in \{0,1\}
\end{aligned}
\tag{4.34}
$$

The solution of the problem given in Eq.(4.34) with the parameters specified as declared gives a solution with underestimated value of -1.1085, $x' = 2.1774$ and $y' = 1.5321$. The real objective value corresponding to the given values of x' and y' is -1.083. Therefore, $z = -1.1085$ and $z_c = -1.083$. Assume that this much of accuracy does not satisfy the error tolerances, therefore another iteration is necessary. Before proceeding to the next iteration, the bounds of the variables are updated. In order to do this, the optimization formulation presented in Eq.(4.24) is be used. The required bound updating formulation is provided in Eq.(4.35).

$$
\begin{aligned}
& Minimize\,/\,Minimize\ x^{'}\ or\ y^{'} \\
& subject\ to \\
& t-2x^{'}-2y^{'}+3\leq -1.083 \\
& -6x^{'}+8y^{'}\leq 5 \\
& 3x^{'}-y^{'}\leq 5 \\
& x^{'}-\sum_{p}\sum_{k}xt(p,k)=0 \\
& y^{'}-\sum_{p}\sum_{k}yt(p,k)=0 \\
& xD(p)*z(p,1)\leq xt(p,1)\leq u^{x}*z(p,1)\ \forall p \\
& xD(p)*z(p,2)\leq xt(p,2)\leq xD(p+1)*z(p,2)\ \forall p \\
& yD(p)*z(p,1)\leq yt(p,1)\leq yD(p+1)*z(p,1)\ \forall p \\
& yD(p)*z(p,2)\leq yt(p,2)\leq u^{y}*z(p,2)\ \forall p \\
& \sum_{p}\sum_{k}z(p,k)=1 \\
& t\geq\left[\sum_{p}\sum_{k}\begin{pmatrix} yD(p)*xt(p,k) \\ +xD(p)*yt(p,k) \\ -xD(p)*yD(p)*z(p,k)\end{pmatrix}\right] \\
& t\geq\left[\begin{matrix}\sum_{p_{ij}}^{P_{ij}-1}\begin{pmatrix} yD(p+1)*xt(p,1) \\ +u^{x}*yt(p,1) \\ -u^{x}*yD(p)*z(p,1)\end{pmatrix}+\sum_{p}^{P-1}\begin{pmatrix} u^{y}*xt(p,2) \\ +xD(p+1)*yt(p,2) \\ -xD(p+1)*u^{y}*z(p,2)\end{pmatrix} \\ +\sum_{k}\begin{pmatrix} u^{y}*xt(P,k) \\ +u^{x}*yt(P,k) \\ -u^{x}*u^{y}*z(P,k)\end{pmatrix}\end{matrix}\right] \\
& xt(p,k),yt(p,k),t\geq 0 \\
& l^{x}\leq x^{'}\leq u^{x} \\
& l^{y}\leq y^{'}\leq u^{y} \\
& z(p,k)\in\{0,1\}
\end{aligned}
\tag{4.35}
$$

The updated bounds of the variables are illustrated in Table 4.3.

Table 4.3. Updated Bounds of Variables for Modified Bilinear Problem

Variable	Lower Bound	Upper Bound
x'	2.0837	2.2347
y'	1.2578	1.7041

4.4.3 Iteration 2

The ratio error is chosen to be 0.001 for this iteration with the condition that the error bound for this iteration is less than that of the previous iteration. The parameters for this iteration are calculated as follows.

$$
\begin{aligned}
&p = 0..11\\
&P = 11\\
&k \in \{1,2\}\\
&xD(p) = [2.08, 2.09, 2.10, 2.11, 2.12, 2.13, 2.14, 2.15, 2.16, 2.18, 2.20, 2.23]\\
&yD(p) = [1.26, 1,28, 1.30, 1.32, 1.34, 1.37, 1.40, 1.43, 1.47, 1.52, 1.58, 1.70]\\
&l^x = 2.0837\ u^x = 2.2347\\
&l^y = 1.2578\ u^y = 1.7041
\end{aligned}
\tag{4.36}
$$

The optimization formulation presented in Eq.(4.34) is solved with these parameters. The optimal solution is calculated to be $z = \text{-}1.0837$, $x' = 2.1619$, $y' = 1.4857$, $z_c = \text{-}1.08327$.

The analytically calculated global optimal solution of this problem is $z = \text{-}1.0833$ with $x = 7/6$ and $y = 1/2$.

4.5 Performance Evaluation using Benchmark Problems

Sherali and Alameddine [28] present a list of nine bilinear optimization problems with no integer variables in the literature. They solved these problems with their method "Reformulation-Linearization Technique" and reported the CPU time. They used is MINOS 5.1 for solving the sub-problems on an IBM 3090 computer. Among those nine problems we were able to find and test three of them for benchmarking. The results are shown in Table 4.4.

Table 4.4. Benchmarking Problems in the Literature

Problem	Reformulation-Linearization*	Proposed Algorithm **
AF0	14 sec	0.29 sec
AF1	12 sec	0.30 sec
AF2	15 sec	0.43 sec

* IBM 3090, MINOS 5.1
** IBM R40 Laptop, CPLEX 8.1

It is worth noting that the algorithm developed by Sherali and Alameddine [28] does not handle integer variables unlike the algorithm developed in this chapter however the algorithm of Adams and Sherali [29] handles integer variables.

CHAPTER 5 – EXAMPLE

In this chapter, the application of the modeling methodology for supply chain management problems described in chapter 3 and the performance of the global optimization algorithm developed in chapter 4 will be illustrated. The performance of the developed algorithm is also evaluated against a commercial MINLP solver for the same problem.

5.1 An Example Complex Supply Chain Management Problem

In order to illustrate the effectiveness of the modeling methodology developed in this thesis, a complex supply chain management problem shown in Fig.5.1 is considered.

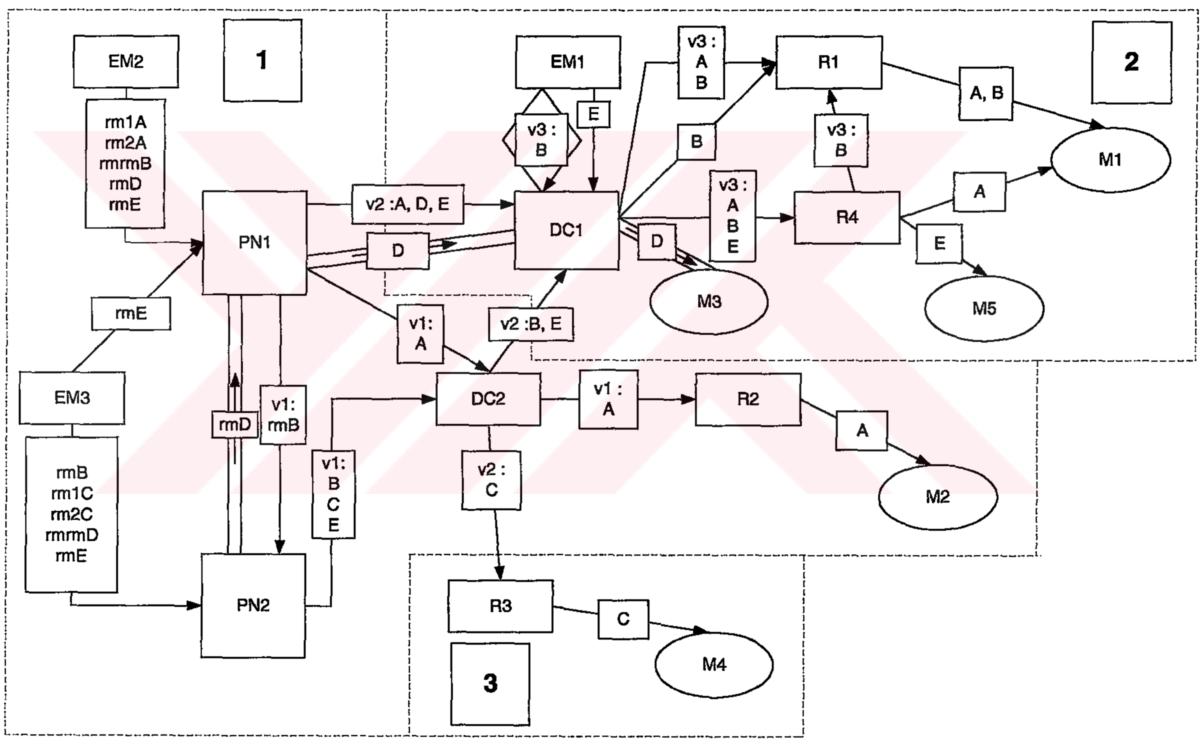

Figure 5.1. Example Complex Supply Chain Management Problem

Production nodes are denoted by PN, distribution centers are denoted by DC, retailers are denoted by R, external markets are denoted by EM and markets are denoted by M. Two parallel lines with an arrow in between symbolize a pipe. For example there is a pipe between PN2 and PN1. The direction of the arrow shows the direction of material flow.

There are three regions in the supply chain, which are labeled by bold region numbers in rectangles. The borders of the regions are marked by dashed lines. The regions determine the movement areas of vehicle types and also determine the areas where there are synchronization differences with respect to region 1. That is, between region 1 and region 2 there is a synchronization difference, and between region 1 and region 3 there is a synchronization difference. The synchronization differences between regions are summarized in Table 5.1. Table 5.2 basically states that region 1 is 12 hours east to region2 and the relation with region 3 is similar.

Table 5.1. Synchronization differences between regions

	Region 2	Region 3
Region 1	+12 hours	+6 hours

There are three types of vehicles in this supply chain system, v1, v2 and v3. v1 can operate only in the first region, v3 can operate only in the second region and v2 can operate between first and second; and between first and third regions, but never in any region. v1 has two transportation modes whereas v2 and v3 have only one transportation mode. The modes and the associated capacities of the vehicles are listed in Table 5.2. The products and raw materials that each vehicle can carry are listed in Table 5.3. As an example, v2's time scale at PN1, DC1, DC2 and RE3 has resolution values of 24 hours, 24 hours, 12 hours, and 12 hours, respectively.

Table 5.2. Vehicle capacities in associated modes

	Mode 1	Mode 2
v1	100	120
v2	1000	-
v3	80	-

Table 5.3. Product Transportation Capabilities of Vehicles

	Mode 1	Mode 2
v1	A, rmB, E	B, C, E
v2	A, B, C, D, E	-
v3	A, B, E	-

There are two production nodes in the supply chain, PN1 and PN2. Of the two, PN1 is going to be explained in detail. PN2 consists of a single production unit. On the other hand, PN1 consists of three machines, a forklift, and two conveyors, making it exactly the same as the plant described in section 3.1.5.5 and in figure 3.1, with only difference being the number of modes and associated products for machines. The modes of the machines and associated products for PN1 are illustrated in Table 5.4. The time scales associated with machines A, B and Press has resolutions of 2, 4, and 8 hours, respectively. The forklift operating in the production site has a resolution of 2 hours. The inventory of raw material rm1A at machine A has a time scale of resolution of 4 hours as an example of work in process inventory time scales.

Table 5.4. Machine modes and associated products at PN1

	Mode 1	Mode 2	Mode 3
A	sfA	sfrmB	-
B	sfD	E	-
C	A	rmB	D

The raw material requirements of the end products are shown in Fig.5.2.

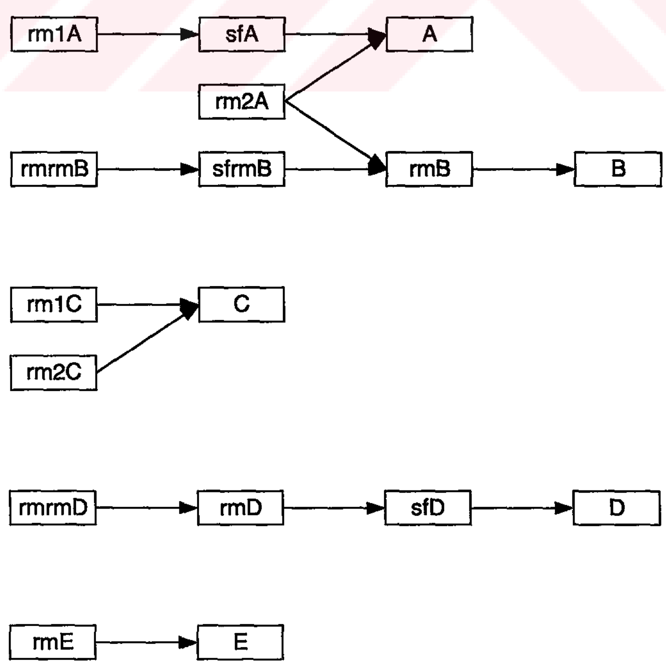

Figure 5.2. Raw Material Requirements of End Products

There are two distribution centers and four retailers in the supply chain. As an example, DC2 stores products A, B, C, and E with inventory variables of resolutions 24 hours, 12 hours, 12 hours, and 12 hours respectively. The schedule of v1 and v2 are coordinated in resolutions of 24 and 12 hours, respectively.

There are five markets that the supply chain serves. All markets have backordering. Markets M1, M2 and M5 have price related demand. Market M1 also accepts a substitute product. The markets are summarized in Table 5.5.

Table 5.5. Markets and associated properties in the supply chain

	Maximum Demand	Slope of the Demand curve	Demand Generation Resolution	Desired Product	Substitute Products
M1	1000	20	1 day	B	A
M2	2000	50	3 days	A	-
M3	-	-	3 days	D	-
M4	-	-	2 days	C	-
M5	3000	60	1 day	E	-

Demand generation resolution symbolizes the frequency of generating the next demand value.

Planning horizon length of 30 days is considered in this problem. As explained before, some of the supply chain units have a resolution of 2 hours, and there are synchronization differences between regions as well. This hypothetical supply chain is modeled in GAMS with the modeling concepts described in chapter 3 and objective is set to maximize the profit.

The complex supply chain management problem is modeled in GAMS [35] that has DICOPT [36] as the MINLP solver. DICOPT is a local optimization solver that does not guarantee global optimality and requires an NLP solver and an MILP solver. We used CONOPT3 [37] and CPLEX version 9.0 [38] as NLP and MILP solvers respectively. The same problem is reformulated to effectively apply our proposed global optimization algorithm. Unlike DICOPT, our algorithm uses an MILP solver (we used CPLEX version 9.0) and guarantees global optimality within a specified error, ε. The statistics of the problem size for DICOPT and our algorithm are given in Table 5.6.

Table 5.6. Size of the Optimization Problems

	Constraints	Continuous Variables	Discrete Variables	Nonlinear Terms
DICOPT	25324	23266	10803	224
Proposed Algorithm	43537	25022	16223	0

The best solution DICOPT has found had an objective value of 125772.3205, and this solution was found in more that 2,341,841.25 seconds of CPU time. DICOPT has abruptly terminated after 2,982,820 seconds without improving the objective value because of reaching resource limits. Proposed algorithm has found a solution with a corresponding objective value (z_c) of 127,717.8610 in 51485.20 seconds of CPU time as shown in Table 5.7. On the other hand DICOPT's solution at 51485.20 seconds of CPU time was 119519.7775.

Table 5.7. Solution Statistics

	DICOPT	Proposed Algorithm	% difference
Objective Function Value	125,772.3205	127,717.8610	1.55
CPU Time (sec)	2,982,820	51485.20	5693.55

CHAPTER 6 – CONCLUSIONS AND FUTURE RESEARCH

6.1 Conclusions and Discussions

As the globalization makes the world markets more competitive and as the trade regulations converge to each other around the globe, accurate and precise planning becomes more and more important to extract the maximum performance of any industrial system. This implies that, any component of the system cannot be neglected and the planning must be as precise as possible.

Market models are neglected so far in supply chain models because of the fact that the resulting problem becomes a nonconvex MINLP whenever revenue terms are included in the objective function and backordering is allowed. However, Chen and Chu [3] demonstrated that matching demand and production rate can offer significant benefits in terms of profit, and also the industry practices such as extended enterprise approach implies that firms are moving in this direction. Therefore, the inclusion of detailed market models in the supply chain planning and scheduling is certainly necessary provided that the effort to solve this problem is acceptable. On the other hand, material transportation activities are also an important part of the supply chain planning obviously because of feasibility and cost reasons. However, such activities are also neglected because of the increased complexity of the resulting problem. But, as stated before, the exclusion of these may result in infeasibilities and/or high transportation costs. Therefore, the inclusion of detailed market models and transportation means is likely to provide significant benefits to supply chain systems.

Extraction of the maximum performance from industrial systems is becoming more important as the competition gets fiercer. As a consequence, detailed and more accurate models for supply chain systems are required. However, with classical modeling techniques increased precision implies an unnecessary increase in the number of variables and constraints. The modeling approach described in chapter 3 is an effort to overcome the unnecessary increase in number of both variables and constraints, while introducing detailed and accurate supply chain system models.

It is also straightforward to introduce time scales with changing resolution even when there are parameters associated with variables defined on that time scale by defining the parameters for each time point on the time scale. This allows for optimizing a supply chain system when the planning precision is very high at the beginning and then gradually gets less and less precise for the latter periods. Such a concept is introduced by Timpe and Kallrath [4].

Furthermore, cross-docking nodes can also be easily modeled based on the model of non-production nodes. Simply stated, in a cross-docking node each vehicle type and vehicle mode must be associated with an inventory variable. Whenever a vehicle in a certain mode is sent from the cross-docking node to another node it can pick from any inventory held in any type of the vehicle while the inventories associated with vehicle types in their modes must be less than or equal to the corresponding available capacity. This means that a vehicle can load materials from other vehicles and other vehicles can load from the former one.

Bilinear optimization problems persistently occur in many engineering problems. Supply chain modeling techniques introduced in this thesis may also require the solution of a bilinear mixed-integer problem. A novel global optimization algorithm for problems with bilinear terms in the objective function is introduced in chapter 4. The algorithm's performance is proved to be satisfactory on benchmark problems and on a large scale supply chain optimization problem.

The contributions of this thesis can be listed as follows:

i. Introduction of the detailed market models to the supply chain optimization problem and models of different transportation means operating in the supply chain
ii. Development of advanced modeling techniques in section 3.3 to model the supply chain system with better precision without unnecessarily increasing the number of variables and constraints
iii. Development of the global optimization algorithm for bilinear and mixed integer bilinear problems which can be used to solve various important engineering problems including the supply chain optimization problem introduced in this thesis

6.2 Future Research Directions

The research directions are also related to four main issues.

i. The performance advantages in terms of profit with the inclusion of detailed market models and transportation systems can be quantified. That is, real systems with as much realistic as possible parameters can be modeled and the performance advantages can be measured. Furthermore, the performance of the supply chain system when other types of objectives such as meeting an order fill rate, minimizing the level of inventory while realizing a target revenue, or keeping the inventory levels at a target value while meeting an order fill rate is also a research direction which may shed light on the path to the competitiveness of the industrial systems.

ii. The advantages obtained from applying the advanced modeling techniques introduced in section 3.3 can be evaluated. The possible advantages are of two types, namely, the operational benefits in terms of performance of the high precision modeled industrial system and the CPU time benefits in terms of solving the same system in the same precision modeled in classical techniques.

iii. A decision support system based on the supply chain model developed in this thesis can be formally modeled. Such a system requires the specification of the software architecture and GUI designs.

iv. It is also straightforward to generalize the algorithm specified in chapter 4 to handle constraints involving bilinear terms with little effort, which also occurs in many engineering problems. Furthermore, it is also possible to generalize the algorithm to handle bilinear terms with relaxing the condition that the variables should be finitely bounded. And, based on the philosophy used while approximating bilinear terms, it may be possible to approximate higher degree polynomials. Since any twice differentiable function can safely be approximated by a first degree Taylor polynomial, it seems possible to devise a general global optimization algorithm that can handle any twice differentiable function and integer variables in the optimization formulation.

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