Stochastic Production Planning and Sourcing Problems with Service Level Constraints

by

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This is to certify that I have examined this copy of a master's thesis by

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ABSTRACT

We study stochastic multiperiod production planning problems of a manufacturer with single/multiple plant(s) and/or subcontractors. Each source, i.e. each plant and subcontractor, has a different production cost, capacity, and lead time. The manufacturer has to meet the demand for single/multiple product(s) according to the service level requirements set by a retailer. The demand for each product in each period is random. We present a methodology that a manufacturer can utilize to make its production and sourcing decisions, i.e., to decide how much to produce, when to produce, where to produce, how much inventory to carry, etc. This methodology is based on a mathematical programming approach. Stochasticity in the problem that comes from random demand and service level constraints is integrated in a deterministic mathematical program by adding a number of additional linear constraints. Solving this deterministic equivalent problem yields the an approximation to the solution of the stochastic problem. We justify the equivalencies between the base stock model and the deterministic equivalent model with modified service level constraints solved on a rolling horizon basis in the single product single production facility setting. For the multiple plants setting without lead time, we show that the deterministic equivalent model gives good enough solutions to the threshold subcontracting model. Finally, motivated by a production planning and sourcing problem in the textile-apparel-retail channel, we use the proposed methodology to perform some numerical experiments to get insights regarding the interaction among the cost, lead time, and variability of demand and how they affect the sourcing decisions.

Keywords: stochastic production planning, service level constraints, subcontracting

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NOMENCLATURE

h_t	The inventory holding cost for the specific product at time t
d_t	The demand for the specific product at time t
X_t	The production quantity of the specific product at time t
I_t	The inventory level of the specific product at the end of time period t
$g(X_t)$	All possible costs that depend on the production amount of the specific products in each period, X_t
$v(.)$	The vector function that defines the inequalities
$w(.)$	The vector function that defines the equalities
α_t	The service level requirement for the specific product at time t
l_t	The minimum cumulative production quantity for the specific product at time t
$\overline{d_t}$	The mean demand for the specific product in period t
$F_t(.)$	The cumulative distribution function of the random sum: $\sum_{\tau=1}^{t} d_\tau$
$f_t(.)$	The probability density function of the random sum: $\sum_{\tau=1}^{t} d_\tau$.
S_1	The base stock level in the base stock policy without lead time
LT	The production lead time of the specific production facility
SR_t	The scheduled receipts for the specific product at time t
S_2	The base stock level in the base stock policy including lead time
S	The inventory level below which the in-house production facility operates
Z	The inventory level below which subcontracting option is utilized

$p_{i,j}$	The cost of producing one unit of product *i* at plant *j*
l_i	The minimum cumulative production quantity for product *i*
C_j	The production capacity of plant *j*
$X_{i,j}$	The production quantity of product *i* at plant *j*
LT_j	The production lead time of plant *j*
$\overline{d_{i,t}}$	The mean demand for product *i* in period *t*
$l_{i,t}$	The minimum cumulative production quantity for product *i* in period *t*
$C_{j,t}$	The production capacity of plant *j* in period *t*
h_i	The inventory holding cost for product *i*
$I_{i,t}$	The inventory level of product *i* at the end of time period *t*
$SR_{i,t}$	The scheduled receipts for product *i* in period *t*
$X_{i,j,t}$	The production quantity of product *i* at plant *j* in period *t*

Chapter 1

INTRODUCTION

The challenges faced by the firms operating in the competitive manufacturing environment of today emphasize the importance of the firm's capability to react to the changing conditions immediately and optimally by making the right decisions at the right time. The mathematical models for optimizing inventory management and sourcing decisions address questions such as: when to produce or when to subcontract, and how much to produce or how much to subcontract and how much inventory to carry. The assumptions made about the demand, cost structure and physical characteristics of the system determine the complexity of these models.

One of the main challenges in these models is incorporating the uncertainty. The uncertainty might be related to the production or demand variability. The unexpected events that cause delays in the manufacturing environments and the randomness in demand make it difficult to anticipate the ultimate effects of the performed actions. Since the manufacturer's capability to compete is determined by the degree of responsiveness to the customer demand, there has been a lot of work in the literature for the formulation and solution of the stochastic production planning problems.

Some stochastic models consider possible scenarios of an unknown future. It is known that the optimal solution to stochastic production planning problems can be obtained by stochastic dynamic or sequential-stochastic programming formulations. However, it is very challenging to construct and solve such models. Studies for even small sized problems require substantial amount of time and effort.

Therefore, in order to address sourcing and production planning problems, it is, in general, preferable to utilize deterministic-based approximations that are solved on a rolling horizon basis and heuristic solutions are proposed. In these models, the uncertainties are not explicitly treated since all inputs are taken as deterministic. The heuristics that offer speed and tractability are preferred to optimality in the most current supply chain planning practices. However, utilization of these heuristics influences the decision makers to make sub-optimal decisions which might result in lower than desired performance for the supply chain.

While making production decisions, manufacturers must determine the planned level of production for each specific product to be produced in each time period during the planning horizon, and must make a trade-off among capacity acquisition, inventory holding and stock out costs in order to maximize profitability. Maintaining a relatively constant production rate and holding inventory to satisfy peak demands might be an alternative for responding to changes in demand. Another alternative might be synchronization of the production rate with the demand rate by varying the production capacity and therefore following the demand closely. The production capacity might be increased in the long run by investing in new facilities, hiring new workers, etc. or it might be increased in the short run by overtime of workers, subcontracting, etc. Not surprisingly, mixed strategies might be utilized whenever it is more profitable.

Recently, subcontracting to third parties has become a commonly adopted approach across many industries. According to Day [1], subcontracting refers to the case in which the prime contractor procures an item or service that the firm is normally capable of economic production in its own facilities and which requires the contractor to make specifications available to the supplier. Although a manufacturer can increase its responsiveness by investing in an additional capacity or by carrying higher inventory, these alternatives are costly and also risky in volatile market conditions. In broad sense, subcontracting can satisfy the manufacturer's needs with the gains of specialization and low cost of production. Additional costs of utilizing subcontractors can be justified by reducing the inventory levels and its associated costs. Adoption of subcontracting practices provides many advantages to the firms. It enables greater production flexibility and allows better supervision of the production process and greater efficiency in the use of plant and machinery, which improve the firm's responsiveness and ability to manage highly variable customer demand in supply chains, but comes at higher costs relative to the in-house production costs.

As mentioned previously, responding to and satisfying customer demand in a timely manner is very crucial in manufacturing environments. In order to evaluate the effectiveness of inventory management policies, service measures are often used in many practical applications. Although there are a number of different definitions of service measures, they generally refer to the probability that a demand or a proportion of demands is met.

The two most commonly utilized service levels are named as *Type* 1 and *Type* 2. *Type* 1 *Service Level* is defined to be the fraction of periods in which there is no stock out. It can be viewed as the plant's no-stock out frequency. It is also called the *cycle service level* or the *ready rate*. This service level measures whether or not a backorder occurs but is not concerned with the size of the backorder. *Type* 2 *Service Level* is defined to be the proportion of demand that must be satisfied from inventory on hand. This measure is also known as the *fill rate*. This service level considers not only the probability of a stock out but also the size of the backorder.

In real life applications, it is preferred to limit the number of backorders by the service level requirements. The stock out costs contain both tangible and intangible components.

Tangible components might include the lost or deferred profit from sales, or the bookkeeping expenses of keeping track of unsatisfied orders. Intangible components might include loss of customer goodwill, or potential delays to other parts of the system. Since it is very difficult to accurately estimate the unit stock out costs, models with service level constraints are attractive for managers.

In this study, we focus on stochastic production planning and sourcing problems with service level constraints where the demand is assumed to be random. Service level constraints are expressed as probabilistic statements.

The objective of our study presented in this thesis can be summarized as follows: First, we would like to develop a methodology that a manufacturer can utilize to make its production and sourcing decisions, i.e. to decide how much to produce, when to produce, where to produce, how much inventory to carry, etc. Second, once such a tool is developed, we would like to evaluate the performance of the tool by comparing the results of it with those of the benchmarks chosen and validate our proposed approach. Finally, motivated by a frequently encountered problem in the textile-apparel-retail channel, we would like to obtain insights regarding the interaction among the cost, lead time, and variability of demand and how they affect the sourcing decisions.

One of the main contributions of this thesis is the systematic analysis of the integration of the deterministic mathematical programming approach for a manufacturer's production and sourcing problem with randomness arising from stochastic demand and service level constraints. Stochastic demand and probabilistic service level constraints can be transformed into a set of constraints in the deterministic mathematical programming. It is shown that this approach is valid for several kinds of service level definitions.

Another important contribution of the study is the justification of the equivalencies between the base stock model and the deterministic equivalent model with modified service level constraints solved on a rolling horizon basis in the single product single production facility setting. In addition to this, it is shown that the deterministic equivalent model with modified service level constraints solved on a rolling horizon basis performs as well as the threshold subcontracting model in the single product multiple production facility setting.

The final contribution of this thesis lies in the insights regarding the interaction among the cost, lead time, and variability of demand and their effects on the sourcing decisions. It is declared that sourcing decisions of a manufacturer depends on many parameters. The analytical findings and the numerical experiments show that the situation is much more complicated and context dependent in general.

The organization of the remaining part of the study is as follows: Chapter 2 provides the literature review on production planning and inventory management issues.

A systematic approach that enables the randomness in demand and the desired service levels to be incorporated in a mathematical programming framework is presented in Chapter 3.

Chapter 4 includes the assessment of the performance of the proposed deterministic equivalent model with modified service level constraints. The results of the deterministic equivalent model are compared with those of the benchmarks created and the results are interpreted.

Chapter 5 focuses on a numerical study motivated by a sourcing problem in the textile-apparel-retail channel. The objective of the study explained in this chapter is to come up with the production and sourcing decisions of the manufacturer by utilizing the methodology described in detail in previous chapters and then to get insights regarding the interaction among the cost, lead time, and variability of demand and their effects on the sourcing decisions.

Chapter 6 presents the summary of our study, concluding remarks and possible future research topics.

Some of the lengthy proofs and analytical developments are presented in the Appendices. Appendix A proves Proposition 4.2, Appendix B focuses on the M/M/1 dual source model and Appendix C proves Proposition 5.1.

Chapter 2

LITERATURE SURVEY

The literature that includes relevant work with our study can be categorized under three topics: general production planning problems, production planning problems with stochastic demand and production planning problems with subcontracting options.

2.1 General Production Planning Problems

The classical deterministic production planning problem, its mathematical programming formulations and solution methodologies have received a lot of attention for many years (see Candea and Hax [2] for a number of well-known models). Thomas and McClain [3] give a complete overview of production planning, review some literature and discuss some planning problems related to the use of operations research tools. Shapiro [4] presents mathematical models and solution methods that have been applied to or that are promising to be implemented for practical situations. Shapiro [5] extends the general mathematical programming framework for supply chain planning problems. Zipkin [6] focuses on the formulation, analysis and use of mathematical models of inventory systems and covers most of the classical inventory theory models. Chand, Hsu and Sethi [7] provide a summary of the research papers in the area of forecast, solution and rolling horizons in operations management problems by focusing on five dimensions that identify the horizon type, the model type, the sources of horizon, the method used to obtain horizon results, and the subject area of the paper.

2.2 Production Planning Problems with Stochastic Demand

Bitran and Yanasse [8] study a stochastic production planning problem with a service level requirement. They focus on a service level type which forces the probability of having a stock out to be less than or equal to a prespecified value in each period. They provide non-sequential, sequential stochastic and deterministic equivalent formulations of the model aiming to minimize the costs related with production, overtime and inventory holding. They derive relative error bounds for non-sequential and sequential production planning problems and show that the relative error bounds are very small for some of the commonly used probability distributions. Bitran and Sarkar [9] extend this study and provide better upper bounds and they focus on mostly error bounds, as well.

Beyer and Ward [10] examine a production and inventory problem of Hewlett-Packard's Network Server Division, which manufactures a major subassembly of network servers in Singapore and ships it to its distribution centers. The authors mention that no other previous work simultaneously investigates all of the complicating factors as the presence of high non-stationary demand with large random fluctuations, use of different shipment modes with different associated cost and lead times, short product lifecycles, rapid depreciation, high risk of obsolescence. The performance of the system is measured by a Type 2 service level for each of the products across all distribution centers. However, it is not desired to observe any significant imbalances in service levels either between the distribution centers or over time. Therefore, they propose to use Type 1 service level goal in each time period for both the warehouse and the distribution centers. They determine the order-up-to levels from the probabilistic service level constraints by the utilization of Fast

Fourier Transforms. Given the demand distribution information and the service level requirements, they simulate their model for a number of different demand scenarios in order to see how shipment decisions would be made. They average out the resulting costs and the realized service levels over all runs and take them as approximations to the expected cost and service level values of the given set of inventory targets. Then, among the inventory targets that satisfy the service level requirements of the warehouse and the distribution center simultaneously, they determine the one that leads to minimum cost value. By developing a tool that integrates the above mentioned steps, they show that their proposed heuristic performs well in reducing inventory and shipment related costs and provide some insights.

Bitran, Haas and Matsuo [11] present a model that is motivated by a case in consumer electronics and textile-apparel industry. An approximate solution for the stochastic production planning problem is given by a hierarchical approach. In this model, the stochastic problem is transformed into a deterministic one by replacing the random demand with their average values. Then, the solution of the transformed problem gives answers to the questions of what to produce and when to produce. The complete solution is obtained by determining how much to produce from a newsboy-type formulation based on the solution of the deterministic problem.

Feiring and Sastri [12] focuses on production smoothing plans with rolling horizon strategies and confidence levels for the demand, which are set by the production planners. The probabilistic constraints in the demand-driven scheduling model are revised by Bayesian procedures and are transformed into deterministic constraints by inverse transformation of normally distributed demand. The model provides solutions for the cases in which decisions for regular/overtime, hiring/firing, single/multiple products, etc. should be made effectively.

Zapfel [13] claims that MRP II systems can be inadequate for the solution of production planning problems with uncertain demand because of the insufficiently supported aggregation/disaggregation process. While solving the hierarchical model proposed for the uncertain demand, it is assumed that the upper and lower bounds of the end product quantities and the aggregate demand of all product groups are known with certainty. The paper proposes a procedure which will help finding robust aggregate plans and consistent disaggregate plans for the MPS.

Kelle, Clendenen and Dardeau [14] extend the economic lot scheduling problem for the single-machine, multi-product case with random demands. This study is motivated by a problem of a large chemical company. Their objective is to find the optimal length of production cycles that minimizes the sum of set-up costs and inventory holding costs per unit of time and satisfies the demand of products at the required service levels.

Clay and Grossman [15], motivated by the chemical processing industry, focus on a two-stage fixed-recourse problem with stochastic Right-Hand-Side terms and stochastic cost coefficients and propose a sensitivity-based successive disaggregation algorithm.

Sox and Muckstadt [16] present a model for the finite-horizon, discrete-time, capacitated production planning problem with random demand for multiple products. The proposed model includes backorder cost in the objective function rather than enforcing service level constraints. A subgradient optimization algorithm is developed for the solution of the proposed model by using Lagrangian relaxation and some satisfactory computational results are provided.

Albritton, Shapiro and Spearman [17] study a production planning problem with random demand and limited information. In this paper; two solution methods, a simulation based optimization method and a discrete simulation based optimization method, are proposed.

A hierarchical production planning and scheduling problem motivated by the fibre industry is studied by Qiu and Burch [18]. In the proposed procedure, the concept of expected set-up costs is introduced, an optimization model that uses logic of expert systems is developed and significant savings are obtained.

Van Delft and Vial [19] consider a multi-period supply chain contracts with options. In order to analyze the contracts, they propose a methodology to formulate the deterministic equivalent from the base deterministic model and from an event tree representation of the stochastic process and solve the stochastic linear program by discretizing demand under the backlog assumption.

2.3 Production Planning Problems with Subcontracting Options

Atamturk and Hochbaum [20] examine the trade-offs between capacity acquisition, subcontracting, production and inventory holding decisions to satisfy non-stationary demand over a multi-period planning horizon. They analyze these decisions not in isolation; instead they optimize these interrelated decisions simultaneously. They also identify the forecast-robustness of the optimal solutions to the capacity acquisition and subcontracting models.

Van Mieghem [21] addresses coordinating capacity, subcontracting and production decisions. They present a two-stage stochastic investment decision model of a manufacturer and a subcontractor. In the first stage, the sources decide on their capacity investment levels separately but simultaneously. After observation of the market demands, both sources decide on their production levels. They then analyze outsourcing conditions for different contract types and present the outcomes of the study.

Bradley [22] focuses on the optimal dual base stock and capacity policies for a dual source M/M/1 system. This study not only proves the structure of the optimal control policy of the M/M/1 dual source model, but also presents exact closed-form expressions for the optimal base stock parameters. Numerical studies of this study investigate the situations in which subcontracting is profitable and useful in reducing in-house capacity and inventory holding costs and conclude that the subcontractor's role is more crucial in reducing capacity rather than inventory. Bradley [23] constructs a tractable Brownian motion approximation to the optimal dual base stock and capacity policies for a dual source M/M/1 model. Bradley and Glynn [24] extend the above mentioned Brownian motion approximation by jointly optimizing the capacity and inventory holding decisions in a single product single plant make-to-stock manufacturing system

Van der Wal [25] develops an analytical approach for threshold subcontracting in a make-to-order job-shop type production system. The decision to accept or to subcontract is made based on a threshold structure. If the subcontracting cost is above a threshold level depending on the number of jobs in the shop, the jobs are accepted to the shop. Otherwise, the jobs are subcontracted. Assuming that the job shop is always in steady state, the optimal strategy is proven to be the threshold strategy.

Abernathy et al. [26] focuses on a problem that is frequently observed in the textile-apparel-retail channel. Rapid changing styles, product proliferation, uncertain customer demand and longer lead times make it more difficult to estimate the demand accurately in this channel. In order to cope with the changes in the environment and in order to minimize the associated risks, most of the retailers are adopting lean retailing practices. As a result, the risks associated with inventory shift to manufacturers from retailers. Manufacturers may then produce to stock or increase their capacities in order to respond to orders quickly.

When manufacturers have limited capacity and the demand for products is highly variable, subcontracting option can be utilized. Abernathy et al. [27] focus on a multiperiod production planning and sourcing problem in the above mentioned channel. Without a formal basis, they propose that a local short-cycle manufacturer can used for products with high demand variability and an offshore manufacturer can be used for products with low variability.

Tan and Gershwin [28] and Tan [29] focus on production and subcontracting strategies. In order to get insights, they formulate the problem in a stochastic optimal control framework.

Due to the complexity of the problem, instead of modeling it analytically, a simulation model has also been developed by Yang, Lee, and Ho [30]. The authors use a simulation-based optimization technique which is referred as ordinal optimization to determine the parameters of a production and inventory control policy that gives a good-enough solution approximately.

Chapter 3

A FRAMEWORK FOR STOCHASTIC PRODUCTION PLANNING AND SOURCING PROBLEMS WITH SERVICE LEVEL CONSTRAINTS

3.1 Introduction

The objective of the study presented in this chapter is to develop a methodology that a manufacturer utilizes to make its production and sourcing decisions, i.e. to decide how much to produce, when to produce, where to produce, how much inventory to carry, etc. Since our objective is to build a planning tool that incorporates most of the assumptions and the features, we propose a *mathematical programming* approach. Stochasticity in the problem that comes from random demand and service level constraints is integrated in a deterministic mathematical program by adding a number of additional linear constraints. We show that solving this deterministic equivalent problem yields the same result as the solution of the stochastic problem. Therefore, the proposed methodology of determining the deterministic equivalent problem can easily be integrated with the Advanced Planning and Optimization tools, such as the products of i2, Manugistics, etc., that are commonly used in practice.

The main contribution of this chapter is to develop a production and sourcing planning methodology that has the power of mathematical programming and that also incorporates demand variability in an equivalent deterministic model.

The approach utilized in this chapter is summarized in Section 3.2. In Section 3.3, the general methodology of determining the additional constraints on minimum production quantities for different service levels is presented. Finally, conclusions are presented in Section 3.4.

3.2 The Model and the Approach

Figure 3.1 below depicts the system we consider in this study. The manufacturer has multiple plants and also works with a number of subcontractors. Each production source has a different cost of production for each product, different lead time and capacity. One possible setting might be a manufacturer that produces at its own plants close to the market and also subcontracts a portion of its orders to low-cost subcontractors that have longer lead times. The manufacturer faces a demand from a retailer for a number of different products and promises a service level for each product and period.

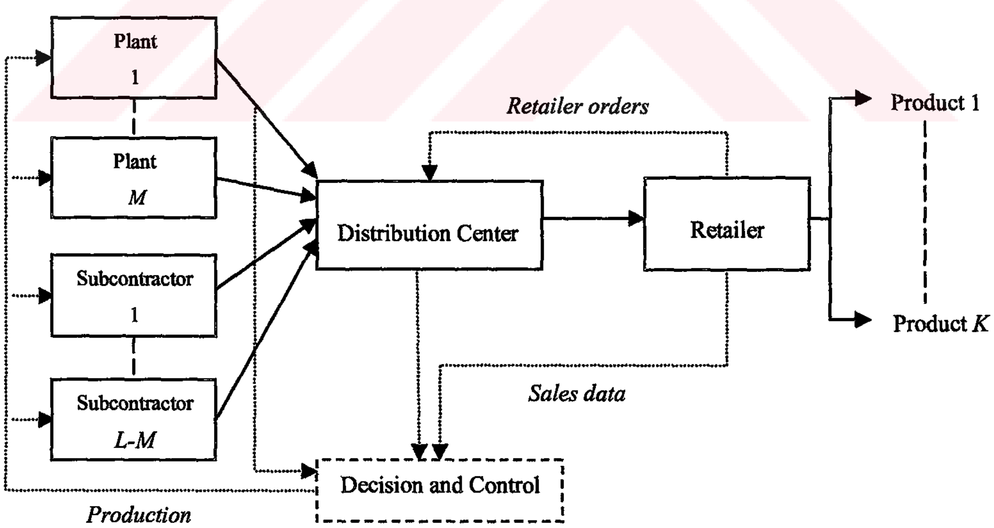

Figure 3.1: A Manufacturer with multiple plants that sells multiple products to a retailer

The basic costs are the production costs and the inventory holding costs. There are production capacity constraints associated with each source. The manufacturer minimizes the total cost of production for the planning period by deciding on the production quantities of each product at each production source in each time period.

The above problem can easily be modeled as a mathematical program. If the demand for each product in each time period were deterministic, the resulting mathematical program would also be deterministic and could easily be solved by using commercially available solvers.

Our approach is to transform the probabilistic service level constraints into a number of additional deterministic constraints and then forming a deterministic mathematical program to solve the resulting model. Figure 3.2 summarizes this approach.

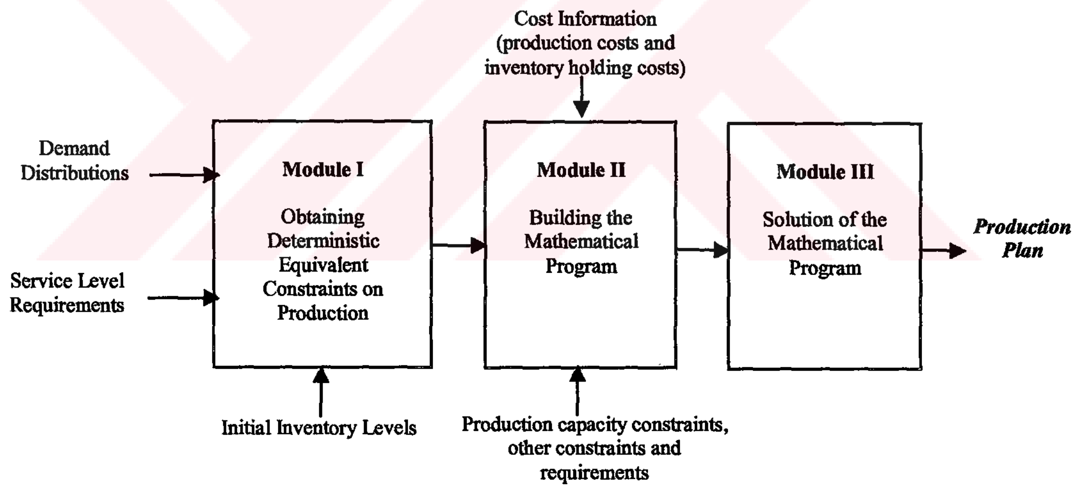

Figure 3.2: Block diagram of the methodology

In the rest of this chapter, we focus essentially on Module I, i.e., obtaining deterministic equivalent constraints on production when the demand distributions, service level requirements, and the initial inventory levels are given. When these additional constraints are obtained, the final mathematical program can be constructed by incorporating production capacity constraints and other requirements, as well as by forming the objective function with the given cost information in the second module. Finally, the production plan is obtained from the solution of the mathematical program by using a commercially available LP solver. If an Advanced Planning and Optimization module is already available for the deterministic version of the problem, the random nature of the demand and probabilistic service level constraints can easily be incorporated into the existing program.

3.3 The General Methodology

3.3.1 General Model of the Stochastic Production Planning Problem with Service Level Constraints

There has been a lot of work in the literature for the formulation and the solution of the stochastic production planning problem. In this section, we will begin with a simplified formulation of the planning problem that is frequently observed in production environments and then, extend the model by including the service level constraints and other features.

First, assume that there is a single product to be produced at a single plant in each time period. The demand for this specific product at time t, d_t is random. The main decision variable is the production quantity at time t, X_t. The inventory level at the end of time period t is denoted by I_t. The number of periods in the planning horizon is T. The inventory holding cost per unit per unit time is h_t and $g(X_t)$ defines all other possible costs that depend on the production amount in each period, X_t.

The constraints that define inequalities are given by the vector function $v(.)$ and the constraints that define the equalities are given by the vector function $w(.)$.

The Stochastic Production Planning Problem (SP) can be defined as:

$$Z^*(\mathrm{SP}) = \mathrm{Min}\sum_{t=1}^{T}\left\{g(X_t) + E[h_t(I_t)^+]\right\}$$

subject to

$$I_t = I_{t-1} + X_t - d_t,\ t = 1,...,T; \tag{3.1}$$

$$v(X_t) \le 0,\ t = 1,...,T; \tag{3.2}$$

$$w(X_t) = 0,\ t = 1,...,T; \tag{3.3}$$

$$X_t \ge 0,\ t = 1,...,T. \tag{3.4}$$

where $(I_t)^+ = \mathrm{Max}\{0, I_t\},\ t = 1,...,T.$

The objective of the problem SP is to minimize the total expected cost, which is the expected value of the sum of the inventory holding costs and all other costs relevant to production in the planning horizon.

The first constraint set defines the inventory balance equations for each time period whereas the second and third constraint sets define all possible other constraints that depend on production quantity in each period. Service level constraints, limits on production quantities, etc. can be counted as examples for these kinds of constraints.

The above formulation can be extended to the lost sales case. In the lost sales case, the inventory balance equation for each time period takes the form of $I_t = \left(I_{t-1} + X_t - d_t\right)^+$. Although conceptually similar to the backorder case, the lost sales case is analytically more difficult and will not be investigated in this study. From now on, we focus our attention on the backorder case (i.e. all unsatisfied demand is backordered at no additional cost).

The optimal solution of the above model answers the questions of when to produce, how much to produce and how much product inventory to carry in each time period throughout the planning horizon. Clearly, a better solution can be obtained by a model that is to be solved in each time period by incorporating the realized demand and inventory on hand information, sequentially. This sequential stochastic planning problem (see Bitran and Yanasse [8] for details) is a stochastic dynamic program which suffers from the well-known curse of dimensionality. An exact solution of this formulation cannot be obtained except for very special cases.

3.3.2 Dealing with Service Level Constraints

As mentioned previously, required service levels impose constraints on the performance (related to backorders) of the system. The service level constraints of the problem can be constructed in several different ways based on different definitions of service levels. In this study, service definitions are taken to be the most commonly used ones.

Throughout the literature, *Type* 1 *Service Level* is defined to be the fraction of periods in which there is no stock out. It can be viewed as the plant's no-stock-out frequency. It is also called the *cycle service level* or the *ready rate*. This service level measures whether or not a backorder occurs but is not concerned with the size of the backorder. Let α be the service level requirement of the planning horizon. $\mathbf{1}_A$ is the indicator function for event A, $\mathbf{1}_A=1$ if A is true and $\mathbf{1}_A=0$ otherwise. Then, the Type 1 service level constraint can be constructed as:

$$\frac{1}{T}E\left[\sum_{t=1}^{T}\mathbf{1}_{\{I_t\geq 0\}}\right]\geq\alpha \qquad (3.5)$$

Modified Type 1 *Service Level* forces the probability of having no stock out to be greater than or equal to a service level requirement in each period. Therefore, it is tighter than Type 1 Service Level. Let α_t be the service level requirement in period *t*. Then, the Modified Type 1 Service Level constraint can be expressed as:

$$P\{I_t \geq 0\} \geq \alpha_t,\ t = 1,...,T. \tag{3.6}$$

Type 2 *Service Level* is defined to be the proportion of demand that must be satisfied from inventory on hand. This measure is also known as the *fill rate*. This service level considers not only the probability of a stock-out but also the size of the backorder. Let α be the service level requirement of the planning horizon. The Type 2 service level constraint can be written as:

$$1 - \frac{E\left[\left(\sum_{t=1}^{T} d_t - \sum_{t=1}^{T} X_t - I_0\right)^+\right]}{E\left[\sum_{t=1}^{T} d_t\right]} \geq \alpha \tag{3.7}$$

where the numerator of the ratio is the expected shortage of the specific product in the T-period-long planning horizon and the denominator is the expected demand of the specific product in the T-period-long planning horizon.

Modified Type 2 *Service Level* or *Modified Fill Rate* is defined as the proportion of demand that is satisfied from inventory on hand in each time period. An alternative definition is the following: the Modified Fill Rate is 1 minus the ratio of the average backlog at the end of a period and the mean demand per period. Let α_t be the service level requirement in period t. Then, the corresponding constraints can be expressed as:

$$1 - \frac{E\left[\left(\sum_{\tau=1}^{t} d_\tau - \sum_{\tau=1}^{t} X_\tau - I_0\right)^+\right]}{E[d_t]} \geq \alpha_t,\ t = 1,...,T. \tag{3.8}$$

3.3.3 Obtaining Solutions with Service Level Constraints

The deterministic equivalent problem with service level constraints that has been mentioned in the previous sections can be modeled as below:

Deterministic Equivalent Production Planning Problem (DEP):

$$Z^*(\text{DEP}) = \text{Min}\sum_{t=1}^{T}\{g(X_t)+h_t(I_t)^+\}$$

subject to

$$I_t = I_{t-1} + X_t - \overline{d_t},\ t=1,...,T; \tag{3.9}$$

$$v(X_t) \le 0,\ t=1,...,T; \tag{3.10}$$

$$w(X_t) = 0,\ t=1,...,T; \tag{3.11}$$

$$\sum_{\tau=1}^{t} X_\tau + I_0 \ge l_t,\ t=1,...,T; \tag{3.12}$$

$$X_t \ge 0,\ t=1,...,T. \tag{3.13}$$

where $\overline{d_t}$ is a prefixed value for the demand in period t and l_t denotes the minimum cumulative production quantity in period t which is determined according to the service level constraints by using a methodology explained in the next section. Note that $v(.) \le 0$ and $w(.)=0$ are functions of deterministic variables in this case.

If $l_t \ge \sum_{\tau=1}^{t} \overline{d_\tau},\ t=1,...,T$ then $I_t \ge 0,\ t=1,...,T$ and the problem can be modified as:

Modified Deterministic Equivalent Production Planning Problem (MDEP):

$$Z^*(\text{MDEP}) = \text{Min}\sum_{t=1}^{T}\left\{g(X_t)+h_t(I_0+\sum_{\tau=1}^{t}X_\tau-\sum_{\tau=1}^{t}\overline{d_\tau})^+\right\}$$

subject to

$$v(X_t) \le 0,\ t=1,...,T; \tag{3.14}$$

$$w(X_t) = 0,\ t = 1,\ldots,T; \qquad (3.15)$$

$$\sum_{\tau=1}^{t} X_\tau + I_0 \geq l_t,\ t = 1,\ldots,T; \qquad (3.16)$$

$$X_t \geq 0,\ t = 1,\ldots,T. \qquad (3.17)$$

As noted by Bitran and Yanasse [8], for any prespecified $\overline{d_\tau}$ satisfying $l_t \geq \sum_{\tau=1}^{t} \overline{d_\tau},\ t = 1,\ldots,T$, the optimal solution to MDEP is the same as the solution of DEP. Because the constraints of MDEP are not functions of $\overline{d_\tau}$, the objective function value of MDEP is the same as that of DEP except for a constant term.

Both DEP and MDEP replace the probabilistic service level constraints by deterministic linear constraints (3.12) and (3.16) respectively. There is an important issue to be clarified in this general approach: how to find the appropriate value of the parameter l_t appearing in the right hand side of equations (3.12) and (3.16) that will result in a mathematical program equivalent to the original problem SP. This issue will be addressed in the next section.

3.3.4 Construction of Deterministic Equivalent Constraints based on Service Level Requirements

In order to utilize a deterministic mathematical programming model such as DEP or MDEP, the probabilistic service level constraints should be transformed into equivalent deterministic ones. We investigate this transformation in this section. It is assumed that the demand for the specific product follows a general continuous distribution which may vary between periods. The following subsections illustrate how the transformation is performed for the modified service level types.

3.3.4.1 Modified Type 1 Service Level

It is instructive to outline the transformation from the probabilistic constraint to a corresponding deterministic constraint for the Modified Type 1 service level (Bitran and Yanasse [8]).

Proposition 3.1: Let l_t denote the (deterministic equivalent) minimum cumulative production quantity in period t which is calculated by solving the probabilistic inequality $P\left\{\sum_{\tau=1}^{t} d_\tau \le l_t\right\} = \alpha_t,\ t=1,\ldots,T$ for l_t ($t=1,\ldots,T$).

$l_t = F_t^{-1}(\alpha_t),\ t=1,\ldots,T$ where $F_t(.)$ is the cumulative distribution function of the random sum: $\sum_{\tau=1}^{t} d_\tau$. Then, the probabilistic constraint $P\{I_t \ge 0\} \ge \alpha_t,\ t=1,\ldots,T$ can be expressed equivalently by:

$$\sum_{\tau=1}^{t} X_\tau + I_0 \ge l_t,\ t=1,\ldots,T \qquad (3.18)$$

Proof:

$$P\left\{\sum_{\tau=1}^{t} X_t - \sum_{\tau=1}^{t} d_\tau + I_0 \ge 0\right\} \ge F_t(l_t) = \alpha_t,\ t=1,\ldots,T$$

$$P\left\{\sum_{\tau=1}^{t} d_\tau \le \sum_{\tau=1}^{t} X_\tau + I_0\right\} \ge F_t(l_t) = \alpha_t,\ t=1,\ldots,T$$

$$P\left\{\sum_{\tau=1}^{t} d_\tau \le \sum_{\tau=1}^{t} X_\tau + I_0\right\} \ge P\left\{\sum_{\tau=1}^{t} d_\tau \le l_t\right\} = \alpha_t,\ t=1,\ldots,T \quad \blacksquare$$

3.3.4.2 Modified Type 2 Service Level

In a similar manner, the deterministic equivalent constraint for the Modified Type 2 service level can be obtained by solving the following inequality for the minimum $z_t = \sum_{\tau=1}^{t} X_\tau + I_0,\ t = 1,...,T$ values and then replacing the probabilistic constraint with $z_t = \sum_{\tau=1}^{t} X_\tau + I_0 \geq l_t,\ t = 1,...,T$.

$$1 - \frac{E\left[(d_t - I_{t-1} - X_t)^+\right]}{E[d_t]} \geq \alpha_t,\ t = 1,...,T$$

$$1 - \frac{1}{E[d_t]} \cdot E\left[\text{Max}\left\{0, \sum_{\tau=1}^{t} d_\tau - \sum_{\tau=1}^{t} X_\tau - I_0\right\}\right] = 1 - \frac{1}{E[d_t]} \cdot \int_{\sum_{\tau=1}^{t} X_\tau + I_0}^{\infty} \left(y - \sum_{\tau=1}^{t} X_\tau - I_0\right) f_t(y)dy \geq \alpha_t,\ t = 1,...,T$$

$$1 - \frac{1}{E[d_t]} \cdot \left\{ E\left[\sum_{\tau=1}^{t} d_\tau\right] - \left(\sum_{\tau=1}^{t} X_\tau + I_0\right) - \int_{0}^{\sum_{\tau=1}^{t} X_\tau + I_0} y f_t(y)dy + \left(\sum_{\tau=1}^{t} X_\tau + I_0\right) F_t\left(\sum_{\tau=1}^{t} X_\tau + I_0\right) \right\} \geq \alpha_t,\ t = 1,...,T$$

$$1 - \frac{1}{E[d_t]} \cdot \left\{ E\left[\sum_{\tau=1}^{t} d_\tau\right] - \left(\sum_{\tau=1}^{t} X_\tau + I_0\right) - \int_{0}^{\sum_{\tau=1}^{t} X_\tau + I_0} \left(y - \sum_{\tau=1}^{t} X_\tau - I_0\right) f_t(y)dy \right\} \geq \alpha_t,\ t = 1,...,T \quad (3.19)$$

where $f_t(.)$ denotes the probability density function of the random sum: $\sum_{\tau=1}^{t} d_\tau$.

Whether we can extract an inequality for the cumulative production quantity in the planning horizon in the form of: $z_t = \sum_{\tau=1}^{t} X_\tau + I_0 \geq l_t,\ t = 1,...,T$; from the above inequality, is not very trivial. The next proposition ensures that this can be done:

Proposition 3.2: Let $z_t = \sum_{\tau=1}^{t} X_\tau + I_0$, $t = 1,...,T$. Then, the equation (3.18) takes the form: $z_t \geq l_t$, $t = 1,...,T$. If $f_t(y)$ is continuous and (strictly) positive for all $y > 0$, there is a unique z_t and therefore l_t, $t = 1,...,T$ for which the constraint (3.19) is satisfied with equality.

Proof: For any constant t=1,…,T; assume that $z_1 < z_2$ and $f_t(y) > 0$, then

$$\int_{z_1}^{\infty} (y - z_1) f_t(y) dy = \int_{z_1}^{\infty} y f_t(y) dy - \int_{z_1}^{\infty} z_1 f_t(y) dy$$

$$\int_{z_2}^{\infty} (y - z_2) f_t(y) dy = \int_{z_2}^{\infty} y f_t(y) dy - \int_{z_2}^{\infty} z_2 f_t(y) dy = \int_{z_2}^{\infty} y f_t(y) dy - \int_{z_2}^{\infty} z_1 f_t(y) dy - \int_{z_2}^{\infty} (z_2 - z_1) f_t(y) dy$$

$$\int_{z_1}^{\infty} (y - z_1) f_t(y) dy - \int_{z_2}^{\infty} (y - z_2) f_t(y) dy = \int_{z_1}^{z_2} y f_t(y) dy - \int_{z_1}^{z_2} z_1 f_t(y) dy + \int_{z_2}^{\infty} (z_2 - z_1) f_t(y) dy > 0$$

Hence, the realized service level:

$$1 - \frac{1}{E[d_t]} \cdot \int_{\sum_{\tau=1}^{t} X_\tau + I_0}^{\infty} \left(y - \sum_{\tau=1}^{t} X_\tau - I_0\right) f_t(y) dy$$

is strictly increasing and continuous in $z_t = \sum_{\tau=1}^{t} X_\tau + I_0$. This ensures that a unique minimum z_t value can be found such that the realized service level is greater than or equal to the pre-specified service level target α_t. Therefore, it can be concluded that an inequality of the form $z_t = \sum_{\tau=1}^{t} X_\tau + I_0 \geq l_t$, $t = 1,...,T$ can be extracted from (3.19). ■

3.3.5 A Procedure to determine the Minimum Cumulative Production Quantities when only the Two Moments of the Demand Distribution are available

After the transformation of the probabilistic constraints, we obtain linear deterministic constraints in the form of (3.18) for Modified Type 1 and Modified Type 2 service levels.

The procedure to calculate the minimum cumulative production target levels in each period is summarized below.

For applications, the complete demand distribution for a given product is unknown. We therefore focus on a procedure where only the mean and the variance of the distribution are available. Since the demand distribution that we focus on is assumed to be unimodal (i.e. have a single peak), we choose to fit a Weibull distribution with the desired mean and variance. The selection of a Weibull distribution enables us to model situations where the Coefficient of Variation (CV) of demand can be extremely variable (from very small to very large). The probability density function of this distribution is given as:

$f(x)=\frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1} e^{-(x/\beta)^{\alpha}}$ where $x \geq 0,\ \alpha>0, \beta>0$ and α is the shape, β is the scale parameter.

After providing the mean and the coefficient of variation values, the appropriate Weibull distribution parameters α and β with the desired mean and CV can be calculated.

In order to calculate minimum cumulative production quantities, the distribution of cumulative demand in each period is required next. We perform the convolution operation by discretizing the demand distribution. Obviously, if the complete probability mass function of the initial demand distributions is provided as input data, there is no need to fit a Weibull distribution initially. Once the discrete convolutions are obtained, the minimum cumulative production quantities can be found using a direct numerical search to determine l_t values in (3.18).

3.3.6 Obtaining Lower Bounds on the Objective Function Value

A useful property of the deterministic mathematical programming approach is that it can yield simple bounds for the objective function value of the corresponding stochastic problem. In particular, it is shown for the Modified Type 1 service level that the objective function value of the problem MDEP obtained by replacing the random demand in each period by its average yields a lower bound on the objective function value of the problem SP (Bitran and Yanasse [8]). The proposition below extends the results for Modified Type 1 service level to Modified Type 2 service levels.

Proposition 3.3: Let Z^*(MDEP) be the objective function value of the problem MDEP obtained by replacing d_t by $E[d_t]$, then $Z^*(\text{MDEP}) \le Z^*(\text{SP})$.

Proof: The proof exploits an application of Jensen's Inequality: if f is a convex function and X is a random variable, then $E[f(X)] \ge f(E[X])$.

Noting that, $h_t(\sum_{\tau=1}^{t} X_\tau - \sum_{\tau=1}^{t} d_\tau + I_0)^+$ is a convex function in $\sum_{\tau=1}^{t} d_\tau$, by Jensen's Inequality,

$$E\left\{\sum_{\tau=1}^{t}\left[g(X_t) + h_t(\sum_{\tau=1}^{t} X_\tau - \sum_{\tau=1}^{t} d_\tau + I_0)^+\right]\right\} \ge \sum_{\tau=1}^{t}\left\{g(X_t) + h_t(\sum_{\tau=1}^{t} X_\tau - \mu_t + I_0)^+\right\}$$

where $\mu_t = E\left[\sum_{\tau=1}^{t} d_\tau\right] = \sum_{\tau=1}^{t} E[d_\tau]$. Then,

$$Z^*(\text{SP}) = \text{Min}\left\{E\left\{\sum_{\tau=1}^{t}\left[g(X_t) + h_t(\sum_{\tau=1}^{t} X_\tau - \sum_{\tau=1}^{t} d_\tau + I_0)^+\right]\right\}\right\}$$

$$\geq \text{Min} \left\{ \sum_{\tau=1}^{t} \left\{ g(X_t) + h_t (\sum_{\tau=1}^{t} X_\tau - \mu_t + I_0)^+ \right\} \right\} = Z^*(\text{MDEP})$$

where both minimums are taken over the feasible region as defined in (3.1), (3.2), (3.3), and (3.4). ■

Note that the objective function value of the optimal solution of MDEP in Proposition 3.3, $Z^*(\text{MODP})$ is not the actual cost value that is realized, but only the output of the mathematical model formulated replacing random variables by their mean values. Bitran and Yanasse [8] show, through numerical examples, that this bound is fairly tight for the Modified Type 1 service level. We do not pursue the evaluation of the bound here because our main interest is in obtaining a production/outsourcing plan under service level constraints. Nevertheless, Proposition 3.3 could be useful in other contexts as a quick approximation of the expected realized cost.

3.3.7 Observations on Minimum Cumulative Production Quantities

It is interesting to examine how the minimum cumulative production quantities change over time; as the mean demand, the coefficient of variation of the demand or the service level requirements change. To this end, we set a 26-period numerical example with a single product whose demand is stationary and calculate the minimum cumulative production quantities using the approach outlined above for Modified Type 1 and Modified Type 2 service levels. For our numerical experiments, the demand random variables are discretized with an interval length of one unit and the upper tail of the distribution is truncated at 6 standard deviations away from the mean.

In Figure 3.3, we report the required production levels (i.e. the quantities l_t-l_{t-1}) for each period for different demand distributions and different service levels. It can be seen in

Figure 3.3 that the production levels are decreasing as a function of time and they converge to the mean demand in time. For the initial periods of the planning horizon, satisfying the desired service levels requires keeping significant buffer inventories in addition to the mean demand. Towards the end of the horizon, the CV of the cumulative demand decreases, approaches to zero and there is little need to constitute a safety inventory and producing the mean demand is sufficient. Other anticipated orderings can also be observed from the figure: increased mean demand or CV leads to higher production levels. Increased service levels also have the same effect. For the same demand level, however, the effects of coefficient of variation and the service level selection subside after a number of initial periods.

3.4 Conclusion

In many practical situations, mathematical models of production planning/subcontracting problems have to deal with the randomness in demand. Frequently, randomness is dealt with in ad hoc manner (by replacing a random variable by a single point estimate value or by its mean). In this chapter, we present a systematic approach that enables the randomness in demand and the desired service levels to be incorporated in a mathematical programming framework.

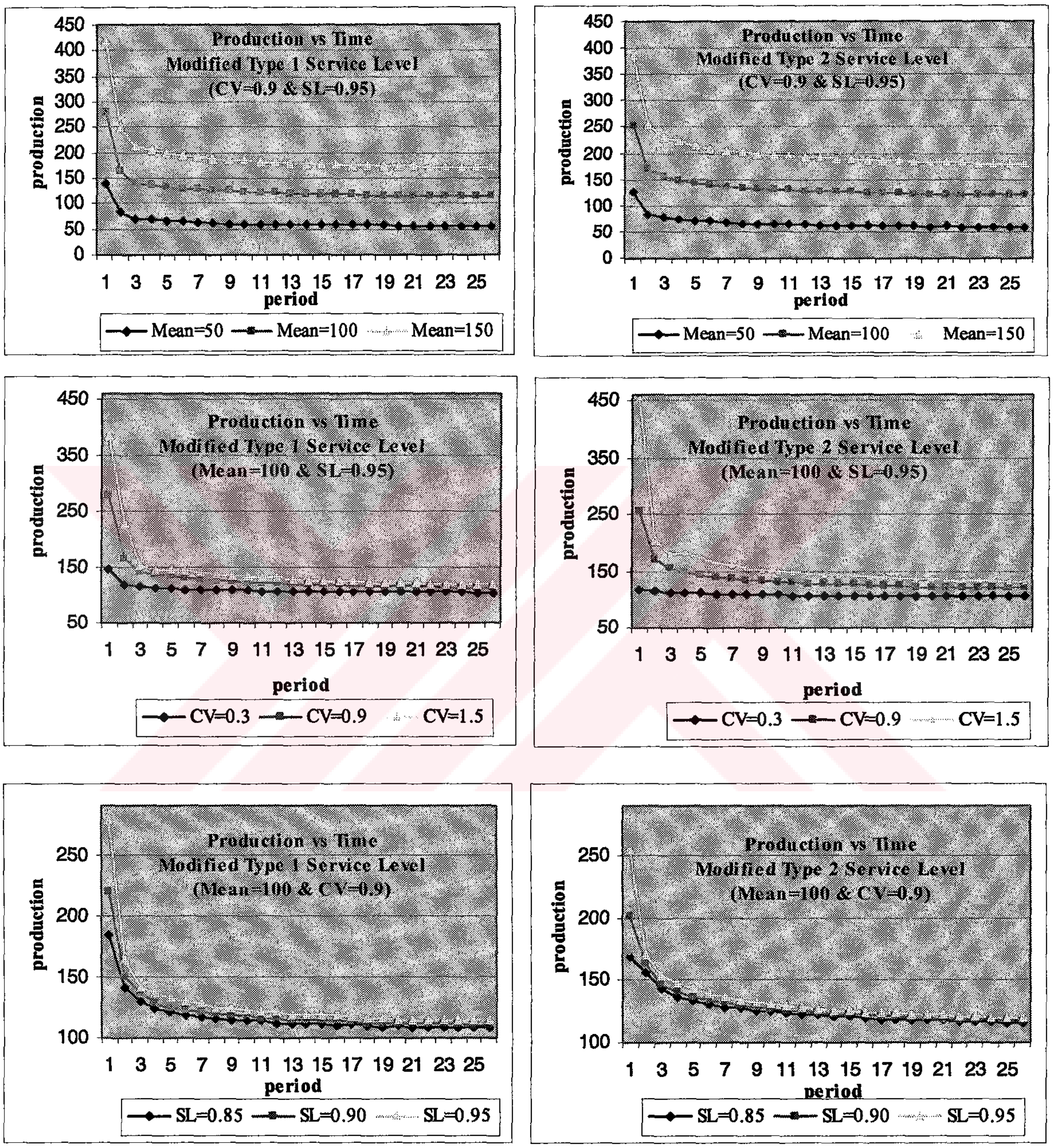

Figure 3.3: Production levels in each time period (l_t-l_{t-1}) for Modified Type 1 and Modified Type 2 service levels

Chapter 4

PERFORMANCE EVALUATION OF THE DETERMINISTIC EQUIVALENT MODEL

4.1 Introduction

In this chapter, we aim to assess the performance of the proposed deterministic equivalent model with modified service level constraints. Can we be sure that the proposed approach gives better solutions than some of the commonly utilized policies, or does it perform worse than our expectations? In order to gain some insights, we need to find some benchmarks with which we can compare the results of deterministic equivalent model.

In Section 4.2, the base stock model is chosen to be the benchmark for the single plant setting. The similarities between the base stock model and the deterministic equivalent model with modified service level constraints solved on a rolling horizon basis in the single product single production facility setting are examined in detail. The equivalencies of these two models are shown for two cases: for the case in which there is a production lead time and for the case in which there is not.

In Section 4.3, the threshold subcontracting model is chosen to be the benchmark for the multiple plants setting without lead time. The control parameters of the policy are determined by using a simulation-based optimization procedure. The results of the benchmark are then compared with those of the deterministic equivalent model.

4.2 Performance Evaluation of the Model for the Single Plant Setting

It is worth examining the similarities and differences between the base stock policy and the deterministic equivalent model. The base stock policy is widely known and can be easily utilized in many applications. It is an inventory policy, with a single parameter which is a reorder level and a base lot size of one unit. It aims to maintain a prespecified inventory level. In the base stock policy, the sequence of events is as follows; the system starts with a prespecified base stock level in the finished goods inventory. The arrival of the customer demand triggers the consumption of an end-item from the inventory and issuing of a replenishment order to the production facility. The resulting policy is the base stock policy. Under this type of policy, an order is placed (or the manufacturing facility operates) if and only if the inventory level drops below the base stock level. The comparison of these two models is performed for two cases. Subsection 4.2.1 discusses the case without production lead time and subsection 4.2.2 examines the case including the production lead time.

4.2.1 The Case without Lead Time

In this first scenario, there is a single product to be produced by a single production facility. It is assumed that the demand of this specific product stays stationary over the planning horizon. We propose that solving the deterministic equivalent model with modified service level constraints on a rolling horizon basis is equivalent to operating the system under the base stock policy. The next proposition establishes this equivalence:

Proposition 4.1: When the production facility has no lead time and the demand is stationary, using a base stock policy is equivalent to solving the deterministic equivalent model with service level constraints on a rolling horizon basis (either Modified Type 1 or Modified Type 2).

Assume that the base stock level in the base stock policy equals I_0(BS)=S_1 and the initial inventory level in the deterministic equivalent problem equals I_0(DEP)=l_1. If S_1=l_1, then the equivalent base stock policy gives the same total expected cost value, yields the same production plan and results in the same service level with the deterministic equivalent model with modified service level constraints solved on a rolling horizon basis.

Proof: We use induction to show that;

i. If the inventory levels at the beginning of the first period are equal, I_0(BS)=I_0(DEP)=S_1=l_1, then production quantities in the first period and the inventory at the end of first period for both policies become equal, i.e. X_1(BS)=X_1(DEP)=0 and I_1(BS)=I_1(DEP)=l_1-d_1;

and

ii. If the inventory levels at the end of period t are equal, I_t(BS)=I_t(DEP)=S_1-d_t=l_1-d_t, then the production quantities in period (t+1) and the inventory levels at the end of period (t+1) for both policies become equal; i.e. X_{t+1}(BS)=X_{t+1}(DEP)=d_t and I_{t+1}(BS)=I_{t+1}(DEP)=l_1-d_{t+1}.

Assume that the initial inventory levels are equal such that I_0(BS)=S_1, I_0(DEP)=l_1 and S_1=l_1. In the base stock policy, each demand observed is produced in the next period; therefore there is no production in the first period, X_1(BS)=0. In the deterministic equivalent approach, the production quantity in the first period is determined according to the constraint $X_1(\text{DEP}) + I_0(\text{DEP}) = X_1(\text{DEP}) + l_1 \geq l_1$ and therefore $X_1(\text{DEP}) \geq 0$. Since the problem is of minimization type, the production quantity in the first period equals zero, X_1(DEP)=0. Next, a customer demand of d_1 arrives. The end of period inventory for the base stock policy becomes I_1(BS)=I_0(BS)+X_1(BS)-d_1=S_1+0-d_1=S_1-d_1 and the end of period inventory for the deterministic equivalent approach becomes I_1(DEP)=I_0(DEP)+X_1(DEP)-d_1=l_1+0-d_1=l_1-d_1. Since we know that S_1=l_1, I_1(BS)=I_1(DEP).

Now assume that at the end of any period t, the ending inventory levels for both policies are such that I_t(BS)=S_1-d_t, I_t(DEP)=l_1-d_t and S_1=l_1. In period (t+1), the base stock policy produces an amount equal to the demand of the previous period, i.e. X_{t+1}(BS)=d_t. Now, assume that we solve the deterministic equivalent model for the modified service level types repeatedly in each time period by incorporating the realized demand and inventory on hand information. The demand is assumed to be stationary over the planning horizon. Although solving the model on a rolling horizon basis throughout the planning horizon requires integration of the minimum cumulative production quantites for the number of periods in the rolling horizon into the model, only the minimum cumulative production quantity of the first period, l_1, is fully utilized. The production quantity in period t is determined according to the constraint $X_{t+1}(\text{DEP})+I_t(\text{DEP})=X_{t+1}(\text{DEP})+l_1-d_t\geq l_1$ and therefore $X_{t+1}(\text{DEP})\geq d_t$. Since the model is of minimization type X_{t+1}(DEP)=d_t. Next, a customer demand of d_{t+1} arrives. The end of period inventory for the base stock policy becomes I_{t+1}(BS)=I_t(BS)+X_{t+1}(BS)-d_t=S_1-d_t+d_t-d_{t+1}=S_1-d_{t+1} and the end of period inventory for the deterministic equivalent approach becomes I_1(DEP)=I_t(DEP)+X_{t+1}(DEP)-d_t=l_1-d_t+d_t-d_{t+1}=l_1-d_{t+1}. Since we know that S_1=l_1, I_{t+1}(BS)=I_{t+1}(DEP). This proves our proposition.■

Remark: It is worth mentioning that if we set the initial inventory level to be S_1=l_1, the resulting production plan is the same with that of the base stock policy which starts with a base stock level of S_1=l_1. Although the base stock policy does not guarantee the assurance of the service levels, since we know that the deterministic equivalent model satisfies the required service levels and the two policies are equivalent, we can say that the base stock level S_1=l_1 ensures that the resulting production plan satisfies the required service levels. □

4.2.2 The Case including Lead Time

The deterministic equivalent model with service level constraints (DEP) can be extended to a case in which the production facility has a production lead time. Assume that there is a production lead time of LT periods and the initial scheduled receipts are denoted by $SR_t, t = 1,...,LT$. Then, the problem can be modeled in the following way:

Deterministic Equivalent Production Planning Problem including Lead Time (DEPLT):

$$Z^*(\text{DEPLT}) = \text{Min}\sum_{t=1}^{T}\{g(X_t) + h_t(I_t)^+\}$$

subject to

$$I_t = I_{t-1} + SR_t - \overline{d_t},\ t = 1,...,LT; \quad (4.1)$$

$$I_t = I_{t-1} + X_{t-LT} - \overline{d_t},\ t = (LT+1),...,T; \quad (4.2)$$

$$v(X_{t-LT}) \leq 0,\ t = (LT+1),...,T; \quad (4.3)$$

$$w(X_{t-LT}) = 0,\ t = (LT+1),...,T; \quad (4.4)$$

$$\sum_{\tau=LT+1}^{t} X_{\tau-LT} + \sum_{\tau=1}^{LT} SR_\tau + I_0 \geq l_t,\ t = (LT+1),...,T; \quad (4.5)$$

$$X_t \geq 0,\ t = 1,...,T. \quad (4.6)$$

After showing the equivalence for the case in which there is no production delay in the previous section and formulating the deterministic equivalent model with service level constraints including lead time in this section, now we can show the equivalence for the case in which the production facility has a specific lead time. Proposition 4.2 and its proof state this equivalence.

Proposition 4.2: When the production facility has a specific lead time, the demand is stationary and there are no scheduled receipts initially, using a base stock policy is equivalent to solving the deterministic equivalent model with service level constraints on a rolling horizon basis (either Modified Type 1 or Modified Type 2).

Assume that the base stock level in the base stock policy including lead time equals I_0(BSLT)=S_2 and the initial inventory level in the deterministic equivalent model including lead time equals I_0(DEPLT)=l_{LT+1}. If S_2=l_{LT+1},then the equivalent base stock policy gives the same total expected cost value, yields the same production plan and results in the same service level with the deterministic equivalent model with service level constraints solved on a rolling horizon basis.

Proof: The proof of proposition 4.2 is quite lengthy and therefore, is presented in Appendix A. ■

4.3 Performance Evaluation of the Deterministic Equivalent Model for the Multiple Plants Setting

In order to get insights from the deterministic equivalent model constructed in the previous chapter and in order to evaluate its performance based on modified service levels for the multiple plants setting without lead time, we need to find an appropriate benchmark and compare the results of the deterministic equivalent model with those of the benchmark.

Bradley [22] proves that the optimal control policy structure for continuous cases is a dual-base stock policy (see Appendix B for the M/M/1 dual-source model), therefore we decide to choose the threshold subcontracting model as a benchmark to our deterministic equivalent model and as the optimal structure has not been proven for discrete cases, we only assume that the threshold subcontracting model we create might be a good approximation to the optimal solution.

In this case, there is a single product to be produced either by the original in-house production facility or the subcontractor. It is assumed that the production sources have no lead time. The order arrivals are governed by a Poisson process with rate 10 products per period. The in-house facility has a finite capacity whereas the subcontractor has an infinite capacity. The production cost is assumed to be $4 per product for the in-house facility. The initial inventory level of the specific product is set to be zero. The service level requirement is set to be 95% for both Modified Type 1 and Modified Type 2 service levels.

The deterministic equivalent model is created by OPL Studio and is solved for a rolling horizon of 10 periods repeatedly throughout a planning horizon of 1000 periods. 5000 sample demand streams are generated and the realized inventory levels are integrated in the model accordingly. The production plans and the realized cost values between periods 451 and 550 are observed. All cost values are calculated on a per period basis. The model is modified for each service level by integrating the deterministic equivalent of the relevant probabilistic service level constraint.

In order to evaluate the performance of the deterministic equivalent model in a multiple-plant setting, we propose a threshold subcontracting model. In this benchmark model, the in-house production facility operates if the inventory level drops below the target level S and stops producing when the inventory level again reaches S. When the inventory level decreases to a threshold level of Z, a subcontractor with an infinite capacity is utilized. Assume that the inventory level drops below S, but is still above Z. The in-house facility produces to cover the shortfall with respect to S, if possible. However, if there is not sufficient capacity to cover the whole shortfall, the in-house facility operates at full capacity and the portion of demand that cannot be satisfied is backlogged.

Let $X_{1,t}$ and $X_{2,t}$ denote the production amounts of the in-house facility and the subcontractor in period t respectively, and let C be the capacity of the in-house facility. Then, the production amounts of each production facility in each time period can be determined for the threshold subcontracting model in the following way:

$$X_{1,t} = \text{Min}\{S-Z, S-I_{t-1}, C\},\ t=1,\ldots,T; \tag{4.7}$$

$$X_{2,t} = \text{Max}\{0, Z-I_{t-1}\},\ t=1,\ldots,T. \tag{4.8}$$

This threshold subcontracting model is performed by a direct numerical search and is coded in MATLAB. It is assumed that there are 1000 periods in the planning horizon and the same 5000 sample demand streams are utilized. The service level requirement is modified such that we create an upper confidence limit for the service level with a confidence coefficient of 0.95. In other words, we would like to be 95% sure that the required service level lies within the one-sided confidence interval we create. The underlying reasoning behind making this modification in service levels is that, the sample size we utilize might not be sufficient enough to make the realized service level equal exactly to the required one. Among the base stock and threshold levels that satisfy the relevant service level requirements, the model aims to find the one with minimum total cost. The calculations are performed for periods between 451 and 550.

The comparison between the deterministic equivalent model and the threshold subcontracting model is performed for nine combinations of subcontracting cost to in-house production cost, holding cost to in-house production cost and capacity to mean demand ratios. The combinations of subcontracting costs, holding costs and the in-house production capacities and therefore, the combinations of relevant subcontracting cost to in-house production cost, holding cost to in-house production cost and capacity to mean demand ratios for which the comparisons are made can be observed in Table 4.1.

Table 4.1: The possible scenarios for which comparisons are made

Subcontracting Cost	Holding Cost	In-house Production Capacity	Subcontracting Cost / In-house Prod. Cost	Holding Cost / In-house Prod. Cost	In-house Prod. Capacity / Mean Demand
4	16	8	1	4	0.8
4	16	12	1	4	1.2
4	16	20	1	4	2
6	1	8	1.5	0.25	0.8
6	1	12	1.5	0.25	1.2
6	1	20	1.5	0.25	2
6	4	8	1.5	1	0.8
6	4	12	1.5	1	1.2
6	4	20	1.5	1	2

For each of the above mentioned problem settings, the base stock and threshold levels observed in the threshold subcontracting model for each modified service level type can be found in Table 4.2 below:

Table 4.2: Base stock and threshold levels observed in each scenario for each modified service level type

Subcontracting Cost	Holding Cost	In-house Production Capacity	Modified Type 1		Modified Type 2	
			Base Stock	Threshold	Base Stock	Threshold
4	16	8	15	7	12	3
4	16	12	15	3	12	-3
4	16	20	15	-∞	12	-∞
6	1	8	17	7	14	3
6	1	12	16	0	12	-3
6	1	20	15	-∞	12	-∞
6	4	8	15	7	12	3
6	4	12	15	3	12	-3
6	4	20	15	-∞	12	-∞

Note that, in some of the cases, the base stock and threshold pairs are observed to be the same. The reasoning behind this is, these pairs lead to the same average inventory levels and minimum cost values in these settings.

While comparing the two models, total expected cost, average production cost, average inventory holding cost values and the assignment of production to the plants (in percentages) are the key elements we focus on. Table 4.3 summarizes the total expected cost values of the deterministic equivalent model (DEM) and the threshold subcontracting model (TSM) for the nine different scenarios for each modified service level type:

Table 4.3: The comparison of total expected cost values observed in each scenario for each modified service level type

Subcont. Cost	Holding Cost	In-house Prod. Cap.	Modified Type 1			Modified Type 2		
			DEM	TSM	Percentage Difference	DEM	TSM	Percentage Difference
4	16	8	121.66	121.66	0.00	80.50	72.11	11.62
4	16	12	121.66	121.66	0.00	80.50	74.85	7.54
4	16	20	121.66	121.62	0.03	80.50	80.47	0.04
6	1	8	49.97	49.89	0.16	47.34	46.51	1.79
6	1	12	46.16	45.65	1.12	43.58	42.53	2.49
6	1	20	45.10	45.10	0.02	42.53	42.52	0.02
6	4	8	65.33	65.33	0.00	55.04	52.61	4.61
6	4	12	61.47	61.47	0.00	51.18	49.06	4.32
6	4	20	60.42	60.40	0.03	50.13	50.11	0.03

The above figures display that the deterministic equivalent model gives good enough solutions when compared with the threshold subcontracting model for both types of the modified levels. The deterministic equivalent model results in total expected cost values equal to or a little bit larger than those of the threshold subcontracting model. However, the percentage differences between the deterministic equivalent model and the threshold subcontracting model for the Modified Type 1 service level are smaller than those for the Modified Type 2 service level. Therefore, it is worth mentioning that for our set of numerical experiments, the deterministic equivalent model gives closer results to the threshold subcontracting model when the service level requirement is of Modified Type 1.

Tables 4.4 and 4.5 display the comparison of average production and holding cost values for the two models for each modified service level type. As can be seen, no general structure can be observed in these figures and we cannot conclude that either one of the models performs better than the other in any of these two comparisons.

Table 4.4: The comparison of average production cost values observed in each scenario for each modified service level type

Subcont. Cost	Holding Cost	In-house Prod. Cap.	Modified Type 1			Modified Type 2		
			DEM	TSM	Percentage Difference	DEM	TSM	Percentage Difference
4	16	8	39.99	39.99	0.00	39.99	39.99	0.00
4	16	12	39.99	39.99	0.00	39.99	39.99	0.00
4	16	20	39.99	39.99	0.00	39.99	39.99	0.00
6	1	8	44.06	44.36	-0.68	44.02	44.22	-0.45
6	1	12	41.05	40.24	2.03	41.05	40.35	1.75
6	1	20	40.00	39.97	0.06	40.00	39.99	0.01
6	4	8	44.91	44.91	0.00	44.91	44.58	0.74
6	4	12	41.05	41.05	0.00	41.05	40.35	1.75
6	4	20	40.00	39.99	0.01	40.00	39.99	0.01

Table 4.5: The comparison of average holding cost values observed in each scenario for each modified service level type

Subcont. Cost	Holding Cost	In-house Prod. Cap.	Modified Type 1			Modified Type 2		
			DEM	TSM	Percentage Difference	DEM	TSM	Percentage Difference
4	16	8	81.67	81.67	0.00	40.50	32.12	26.10
4	16	12	81.67	81.67	0.00	40.50	34.86	16.19
4	16	20	81.67	81.63	0.05	40.50	40.47	0.07
6	1	8	5.92	5.53	6.91	3.32	2.29	45.02
6	1	12	5.10	5.41	-5.67	2.53	2.18	16.19
6	1	20	5.10	5.10	0.05	2.53	2.53	0.07
6	4	8	20.42	20.42	0.00	10.13	8.03	26.09
6	4	12	20.42	20.42	0.00	10.13	8.71	16.19
6	4	20	20.42	20.41	0.05	10.13	10.12	0.07

Table 4.6 summarizes the percentage of production assigned to the in-house production facility for both the deterministic equivalent model and the threshold subcontracting model. The results suggest that the production assignments of the deterministic model follow a similar pattern with the benchmark chosen.

Table 4.6: The percentage of production assignments to the in-house production facility observed in each scenario for each modified service level type

Subcontracting Cost	Holding Cost	In-house Production Capacity	Modified Type 1		Modified Type 2	
			Base Stock	Threshold	Base Stock	Threshold
4	16	8	75.45	75.40	75.45	77.06
4	16	12	94.73	94.70	94.73	98.23
4	16	20	99.97	100.00	99.97	100.00
6	1	8	79.76	78.17	79.95	78.88
6	1	12	94.73	98.78	94.73	98.23
6	1	20	99.97	100.00	99.97	100.00
6	4	8	75.45	75.40	75.45	75.40
6	4	12	94.73	94.70	94.73	98.23
6	4	20	99.97	100.00	99.97	100.00

Based on these figures, we can conclude that the proposed deterministic equivalent model solved on a rolling horizon basis performs as well as the threshold subcontracting model solved on a simulation-based optimization technique for both types of the modified service levels. The total expected cost values of deterministic equivalent models for all nine different cases are equal to or a little bit larger than those of the threshold subcontracting model. However, we cannot reach the same conclusion for the average production and holding cost values. Deterministic equivalent model performs either worse for some cases or better for some other cases when the comparison is based on average production or holding cost values. However, the sum of these two terms, the total expected cost, is equal to a little bit larger than that of the threshold subcontracting model. Moreover, the proportion of production assigned to the in-house facility in the deterministic equivalent model resembles that in the simulation based threshold subcontracting model.

Remarks: It is worth mentioning that the sample size utilized in the above numerical comparisons, 5000, might not be large enough to satisfy the service level requirements in each time period that the modified service level definitions necessitate. The coefficient of variation in the realized service level values might be larger than expected. To handle this problematic issue, we introduced one-sided confidence intervals. Although, the threshold subcontracting model constitutes a lower bound in terms of total expected cost values for our set of numerical examples, it can not be generalized from our examples that the deterministic equivalent model always gives solutions worse than those of the threshold subcontracting model.

Moreover, the solutions found above might not be the optimal solutions to the problem defined and the nine possible problem settings might not be adequate to come up with generalizations. Although numerical experiments have been performed for a larger number of problem settings, since the running of these algorithms take considerable amount of time, the models are solved for a smaller sample size resulting in higher coefficient of variations in the realized service levels. However, it is assumed that if large enough sample sizes for the two models were utilized, the two models would give better and much closer results. □

4.4 Conclusion

In order to state the validity of our proposed approach, the performance of the methodology should be compared with some benchmarks created.

In this chapter, the base stock model is chosen to be the benchmark for the single plant setting. The equivalencies of the base stock model and the deterministic equivalent model with modified service level constraints solved on a rolling horizon basis in the single product single production facility setting are shown.

Moreover, the threshold subcontracting model solved on a simulation-based optimization technique is chosen to be the benchmark for the multiple plants setting without lead time. It is concluded that the deterministic equivalent model performs as well as the threshold subcontracting model and the threshold subcontracting model might constitute a lower bound to the deterministic equivalent model. The shortcomings of the numerical experiments are also mentioned.

Chapter 5

APPLICATION OF THE METHODOLOGY FOR MULTIPLE PRODUCTS IN A STOCHASTIC PRODUCTION PLANNING AND SOURCING PROBLEM WITH SERVICE LEVEL CONSTRAINTS

5.1 Introduction

In this chapter, we consider a production planning problem encountered in the textile-apparel-retail channel. In particular, a sourcing problem in this channel forms the motivation of this chapter. New styles are being introduced very fast as fashion changes very quickly in the textile-apparel-retail channel. Product proliferation and uncertain customer demand make it more difficult to estimate the demand accurately. As a result, most of the retailers are beginning to adopt lean retailing practices in order to minimize the associated risks (Abernathy et al. [26]). Lean retailers transform the basis of competition for all suppliers by reducing the amount of time manufacturers have to respond to orders, which means that suppliers must be able to provide more frequent deliveries, in smaller quantities, of more diverse products.

Adoption of lean retailing practices and rapid replenishment programs by the retailers force manufacturers to build capabilities to respond quickly to changes in customer demand. In order to compete with other manufacturers and attain satisfactory profit levels, a manufacturer needs to make its production and inventory planning decisions in the best way possible.

Although a manufacturer can increase its responsiveness by investing in an additional capacity or by carrying higher inventory, these alternatives are costly and also risky in volatile market conditions. In recent years, manufacturers use subcontracting as an alternative to increase their capacities temporarily whenever it is needed. Subcontracting is the procurement of an item or service that a firm is normally capable of economic production in its own facilities and which requires the prime contractor to make specifications available to the supplier. However, when to subcontract and how much to subcontract are the challenging questions to be answered.

In this chapter, we focus on a production planning problem of a manufacturer with multiple plants and subcontractors. Each source, i.e. each plant and subcontractor, has a different production cost, capacity, and lead time. The manufacturer has to meet the demand for multiple products according to the service level requirements set by a retailer. The demand for each product in each period is random. Although the demand is assumed to be stationary and random, the manufacturer has historical data that are used to estimate demand probability distributions or at least their means and variances in each period. A qualitative discussion of this problem can be found in Abernathy et al. [27].

The objective of the study explained in this chapter is to come up with the production and sourcing decisions of the manufacturer, i.e. to aid the manufacturer in deciding how much to produce, when to produce, where to produce, how much inventory to carry, etc. by utilizing the methodology described in detail in previous chapters and then to get insights regarding the interaction among the cost, lead time, and variability of demand and how they affect the sourcing decisions.

Simplified versions of the above mentioned problem have been investigated in the past by using different methodologies. For example, Tan and Gershwin [28] and Tan [29] formulate a simplified version of the problem in a stochastic optimal control framework by focusing on the question of when and how to use a subcontractor to get insights rather than developing a tool that can be used by a manufacturer on a daily-basis. In these two studies, there is no lead time associated with the production sources, there is only one product, and the demand is stationary.

Due to the complexity of the problem, instead of modeling it analytically, a simulation model has also been developed (Yang, Lee and Ho [30]). Then, a simulation-based optimization technique that is referred to as ordinal optimization has been used to determine the parameters of a production and inventory control policy that gives a good-enough solution approximately. However, one needs to set a specific production and inventory control policy in the simulation. In addition to the difficulty of setting a plausible policy in a complicated case, as the number of sources and products increase, the number of parameters to be optimized also increases. As a result, finding a good-enough solution requires a considerable time.

In Section 5.2, we focus on the analytical solution of a single-period production planning problem in which multiple products are produced by multiple plants, and then, in Section 5.3 we focus on the formulation of a multi-product multi-period production planning problem. Finally, in Section 5.4, based on a two-product two-plant example, we try to interpret and get insights from the analytical solutions and the numerical observations, and come up with general results.

5.2 The Single Period Production Planning Problem with Service Level Constraints

In this section, we turn our attention to a single period problem with multiple products and multiple sources. The question that has received significant research interest in this environment is when and how much of which product to produce in which plant. The volatility in the demand of the products in each period, the lead times of the plants, the unit production costs of the products in the plants, the capacity of the plants and many other factors affect the assignment of the production of the products to the plants.

Now, let us construct the single period multiple product multiple source production planning problem. Assume that there are K different products to be produced by L different production facilities in a single period. We try to find answers to the questions of where to produce and how much of which product to produce in a single period.

Let l_i be the deterministic equivalent demand of product i calculated based on a service level requirement of α_i for each product i, $p_{i,j}$ be the cost of producing one unit of product i at plant j, C_j be the production capacity of plant j and $X_{i,j}$ be the production quantity of product i at plant j. I_i, the beginning inventory level of product i is set to be zero for both of the two products

The Single Period Deterministic Equivalent Production Planning Problem (SPDEP) with service level constraints can be defined as:

$$Z^*(\text{SPDEP}) = \text{Min} \sum_{i=1}^{K} \sum_{j=1}^{L} p_{i,j} X_{i,j}$$

subject to

$$\sum_{j=1}^{L} X_{i,j} \geq l_i,\ i = 1,\ldots,K; \tag{5.1}$$

$$\sum_{i=1}^{K} X_{i,j} \leq C_j, j = 1,\ldots,L; \tag{5.2}$$

$$X_{i,j} \geq 0;\ i = 1,\ldots,K,\ j = 1,\ldots,L. \tag{5.3}$$

Note that in the above formulation, the probabilistic service level constraints are transformed into the linear deterministic equivalent constraints as described in Chapter 3.

After describing the Single Period Deterministic Equivalent Production Planning Problem, we now concentrate on deriving the production plan of a manufacturer which has the options of utilizing two production facilities (a cheaper and an expensive facility) while satisfying the demand of two different products. The formulation of this mathematical problem is as follows:

$$Z^* = \text{Min} \sum_{i=1}^{2} \sum_{j=1}^{2} p_{i,j} X_{i,j}$$

subject to

$$\sum_{j=1}^{2} X_{i,j} \geq l_i,\ i = 1,2; \tag{5.4}$$

$$\sum_{i=1}^{2} X_{i,j} \leq C_j, j = 1,2; \tag{5.5}$$

$$X_{i,j} \geq 0,\ i = 1,2;\ j = 1,2. \tag{5.6}$$

Now, assume that the first plant is the expensive production facility and the second plant is the cheaper production facility which means $p_{1,1}>p_{1,2}$ and $p_{2,1}>p_{2,2}$. We assume that the first product has a higher minimum production quantity whereas the second product has a lower minimum cumulative production quantity; i.e. $l_1>l_2$. Note that, in order to find a feasible solution, the total production capacity should be large enough to meet the total production requirements of the two products. Otherwise, the problem would be infeasible. Therefore, we assume that $C_1+C_2 \geq l_1+l_2$ at the very beginning. Then, we make the following proposition:

Proposition 5.1: The Single Period Production Planning Problem, in which two products are to be produced either by a cheaper or by an expensive plant or by both of the two plants, can result in different optimal solutions based on different combinations of $p_{i,j}$, C_j and l_i values. For the specified parameter conditions, all possible optimal solutions can be observed below:

If $p_{1,1}-p_{1,2} \leq p_{2,1}-p_{2,2}$ and if:

- $l_1 + l_2 \leq C_2$, then the optimal solution is $X_{1,1} = 0$, $X_{1,2} = l_1$, $X_{2,1} = 0$ and $X_{2,2} = l_2$ with $Z^* = p_{1,2} \cdot l_1 + p_{2,2} \cdot l_2$;

- $l_2 \le C_2 \le l_1 + l_2$, then the optimal solution is $X_{1,1} = (l_1 + l_2) - C_2$, $X_{1,2} = C_2 - l_2$, $X_{2,1} = 0$ and $X_{2,2} = l_2$ with $Z^* = p_{1,1} \cdot [(l_1 + l_2) - C_2] + p_{1,2} \cdot (C_2 - l_2) + p_{2,2} \cdot l_2$;
- $C_2 \le l_2$, the optimal solution is $X_{1,1} = l_1$, $X_{1,2} = 0$, $X_{2,1} = l_2 - C_2$ and $X_{2,2} = C_2$ with $Z^* = p_{1,1} \cdot l_1 + p_{2,1} \cdot (l_2 - C_2) + p_{2,2} \cdot C_2$.

If $p_{1,1}$-$p_{1,2} \ge p_{2,1}$-$p_{2,2}$ and if:

- $l_1 + l_2 \le C_2$, then the optimal solution is $X_{1,1} = 0$, $X_{1,2} = l_1$, $X_{2,1} = 0$ and $X_{2,2} = l_2$ with $Z^* = p_{1,2} \cdot l_1 + p_{2,2} \cdot l_2$;
- $l_1 \le C_2 \le l_1 + l_2$, then the optimal solution is $X_{1,1} = 0, X_{1,2} = l_1$, $X_{2,1} = (l_1 + l_2) - C_2$ and $X_{2,2} = C_2 - l_1$ with $Z^* = p_{1,1} \cdot l_1 + p_{2,1} \cdot [(l_1 + l_2) - C_2] + p_{2,2} \cdot (C_2 - l_1)$;
- $C_2 \le l_1$, the optimal solution is $X_{1,1} = l_1 - C_2$, $X_{1,2} = C_2$, $X_{2,1} = l_2$ and $X_{2,2} = 0$ with $Z^* = p_{1,1} \cdot (l_1 - C_2) + p_{1,2} \cdot C_2 + p_{2,1} \cdot l_2$.

Proof: The proof of proposition 5.1 is quite lengthy and therefore, is presented in Appendix C. ■

The above solutions prove that the Single Period Production Planning Problem with service level constraints, in which two products are to be produced either by a cheaper or by an expensive plant or by both of the two plants, can result in different optimal solutions based on different combinations of parameters: $p_{i,j}$, C_j and l_i for each i=1,2 and j=1,2.

The insights that can be gained from the above proposition are as follows:

Since the production costs of both products at the second plant is always lower than or equal to those at the first plant, it is always cost-advantageous and therefore, it is always preferred to produce the production requirements of both products in the first and the cheaper plant.

If the capacity of the cheaper plant is large enough to produce the required quantities of products 1 and 2, then both products are produced only in the cheaper plant.

If the capacity of the cheaper plant is unable to satisfy the requirements of both products, the relative cost reductions obtained by producing the products at the cheaper plant ($p_{1,1}$-$p_{1,2}$ and $p_{2,1}$-$p_{2,2}$ values) should be taken into consideration. The product with higher cost reduction value is the cost-advantageous product and is preferred to be produced in the cheaper plant.

Moreover, if the production requirement of the cost-advantageous product exceeds the capacity of the cheaper plant, then the cheaper plant dedicates its whole capacity to that product. The remaining portion of the cost-advantageous and the other products' production requirements are then produced in the expensive plant.

If the capacity of the cheaper plant is unable to satisfy the total production requirements of the two products, but if it is greater than or equal to the production requirement of the cost-advantageous product, then the second and cheaper plant gives priority to the production of the product with the greater cost advantage. The cheaper plant first satisfies the requirement of the cost-advantageous product. Then, the remaining capacity is dedicated to the production of the other product. The unsatisfied portion of the other product's requirement is then produced in the expensive plant.

Note that if the product with the greater cost advantage is altered, the whole production plan changes. This indicates the criticality of the relative cost reductions obtained by producing the products at the cheaper plant ($p_{1,1}$-$p_{1,2}$ and $p_{2,1}$-$p_{2,2}$ values).

Once we obtain a solution, we can perform comparative statics analysis and examine the change in the production plan when we vary a specific parameter ceteris paribus.

For instance; let product 1 be the cost-advantageous product (i.e. $p_{1,1}$-$p_{1,2}$≥$p_{2,1}$-$p_{2,2}$) and assume that the capacity of the cheaper plant is unable to satisfy the total production requirements of the two products, but it is still greater than or equal to the production requirement of product 1 (i.e. $l_1 \leq C_2 \leq l_1 + l_2$).

Note that, the change in l_1 value can be performed by altering either the coefficient of variation or the required service level of the first product. If we increase l_1, three different cases might be observed. The optimal solution might stay the same as long as $l_1 \leq C_2$. But if $C_2 \leq l_1$ and $C_1+C_2 \geq l_1+l_2$, the optimal solution changes to $X_{1,1} = l_1 - C_2$, $X_{1,2} = C_2$, $X_{2,1} = l_2$ and $X_{2,2} = 0$ with an objective function value of $Z^* = p_{1,1} \cdot (l_1 - C_2) + p_{1,2} \cdot C_2 + p_{2,1} \cdot l_2$. If we increase l_1 further and if $C_1+C_2<l_1+l_2$, the problem becomes infeasible. If we decrease l_1, the optimal solution either stays the same if $C_2 \leq l_1 + l_2$ or changes to $X_{1,1} = 0$, $X_{1,2} = l_1$, $X_{2,1} = 0$ and $X_{2,2} = l_2$ with $Z^* = p_{1,2} \cdot l_1 + p_{2,2} \cdot l_2$ if $l_1 + l_2 \leq C_2$.

Moreover, the increase in l_2 value does not affect the optimal solution if $C_1+C_2 \geq l_1+l_2$, it stays the same. But if $C_1+C_2<l_1+l_2$, the problem becomes infeasible. The decrease in l_2 value might change the optimal solution. It might stay the same as long as $C_2 \leq l_1 + l_2$, or it might change to $X_{1,1} = 0$, $X_{1,2} = l_1$, $X_{2,1} = 0$ and $X_{2,2} = l_2$ with $Z^* = p_{1,2} \cdot l_1 + p_{2,2} \cdot l_2$ if $l_1 + l_2 \leq C_2$.

The change in the capacity of the expensive plant, C_1, has no effect on the optimal solution as long as $C_1+C_2 \geq l_1+l_2$. The problem becomes infeasible otherwise.

If we increase the production capacity of the cheaper plant, C_2, the solution might stay the same if $C_2 \leq l_1 + l_2$ or might change to $X_{1,1} = 0$, $X_{1,2} = l_1$, $X_{2,1} = 0$ and $X_{2,2} = l_2$ with $Z^* = p_{1,2} \cdot l_1 + p_{2,2} \cdot l_2$ if $l_1 + l_2 \leq C_2$. The decrease in C_2 does not change optimal solution if

$l_1 \leq C_2$. However, if $C_2 \leq l_1$ and $C_1+C_2 \geq l_1+l_2$, the optimal solution changes to $X_{1,1} = l_1 - C_2$, $X_{1,2} = C_2$, $X_{2,1} = l_2$ and $X_{2,2} = 0$ with $Z^* = p_{1,1} \cdot (l_1 - C_2) + p_{1,2} \cdot C_2 + p_{2,1} \cdot l_2$. If we decrease C_2 further and $C_1+C_2<l_1+l_2$, the problem turns out to be infeasible.

In the end, if we change the cost-advantageous plant to be the second plant, i.e. $p_{1,1}-p_{1,2} \leq p_{2,1}-p_{2,2}$, the solution changes to be $X_{1,1} = (l_1 + l_2) - C_2$, $X_{1,2} = C_2 - l_2$, $X_{2,1} = 0$ and $X_{2,2} = l_2$ with $Z^* = p_{1,1} \cdot [(l_1 + l_2) - C_2] + p_{1,2} \cdot (C_2 - l_2) + p_{2,2} \cdot l_2$.

This analysis can be performed to other optimal solutions as well and the new optimal solutions might be observed.

5.3 The Multi-Period Production Planning Problem with Service Level Constraints

In this section, we extend our focus in the previous section by concentrating on multiple periods. The parameters of the multi-product multi-source multi-period production planning problem can be defined as:

T = the number of periods in the planning horizon

K= the number of products

L = the number of plants

LT_j=lead time of plant j

$LT_{\min} = \underset{j \in \{1,..,L\}}{\text{Min}} \{LT_j\}$

$LT_{\max} = \underset{j \in \{1,..,L\}}{\text{Max}} \{LT_j\}$

$\overline{d_{i,t}}$ =mean demand of product i in period t

$l_{i,t}$= minimum cumulative production quantity for product i in period t

$C_{j,t}$= production capacity of plant j in period t

$p_{i,j}$ = unit production cost of product i in plant j

h_i = inventory holding cost of product i per unit per period

$I_{i,0}$ = initial inventory level of product i

$SR_{i,t}$= the scheduled receipts for product i for periods $t=1,.., LT_{max}$

The decision variables of the model can be defined as the following:

$X_{i,j,t}$ = production quantity of product i at plant j in period t

$I_{i,t}$ = inventory level of product i at the end of period t

The Multi-Period Deterministic Equivalent Production Planning Problem (MPDEP) can be formulated as follows:

$$Z^*(\text{MPDEP}) = \text{Min} \sum_{t=1}^{T}\left(\sum_{i=1}^{K}\sum_{j=1}^{L} p_{i,j} X_{i,j,t} + \sum_{i=1}^{K} h_i \left(I_{i,t}\right)^+\right)$$

subject to

$$I_{i,t} = I_{i,t-1} + SR_{i,t} - \bar{d}_{i,t},\ i=1,\dots,K;\ t=1,\dots,LT_{\min}; \tag{5.7}$$

$$I_{i,t} = I_{i,t-1} + \sum_{j=1}^{L} X_{i,j,t-LT_j} + SR_{i,t} - \bar{d}_{i,t},\ i=1,\dots,K;$$

$$j \in \{1,\dots,L\} : LT_j \le t;\ t = (LT_{\min}+1),\dots,LT_{\max}; \tag{5.8}$$

$$I_{i,t} = I_{i,t-1} + \sum_{j=1}^{L} X_{i,j,t-LT_j} - \overline{d_{i,t}},\ i=1,\dots,K;\ t=(LT_{\max}+1),\dots,T; \tag{5.9}$$

$$\sum_{i=1}^{K} X_{i,j,t} \le C_{j,t}\ ,j=1,\dots,L\ ;\ t=1,\dots,T; \tag{5.10}$$

$$I_{i,0} + \sum_{\tau=1}^{t} SR_{i,\tau} + \sum_{\tau=LT_{\min}+1}^{t}\sum_{j=1}^{L} X_{i,j,\tau-LT_j} \ge l_{i,t},\ i=1,\dots,K;$$

$$j \in \{1,\dots,L\} : LT_j \le t;\ t=(LT_{\min}+1),\dots,LT_{\max}; \tag{5.11}$$

$$I_{i,0} + \sum_{\tau=1}^{LT_{\max}} SR_{i,\tau} + \sum_{\tau=LT_{\max}+1}^{t} \sum_{j=1}^{L} X_{i,j,\tau-LT_j} \geq l_{i,t},$$

$$i = 1,...,K;\ j = 1,...,L;\ t = (LT_{\max}+1),...,T; \quad (5.12)$$

$$X_{i,j,t} \geq 0,\ i = 1,...,K;\ j = 1,...,L;\ t = 1,...,T. \quad (5.13)$$

where $\left(I_{i,t}\right)^{+} = \text{Max}\left\{0, I_{i,t}\right\},\ t = 1,...,T.$

The objective function of the model aims to minimize the sum of total production and total inventory holding costs over all periods in the planning horizon.

Constraint set (5.8) defines the inventory balance equations until the maximum lead time among plants and ensures that the inventory level of a product i is determined by the inventory carried from the previous period, the total production of that product in the current period if possible, scheduled receipts of the product to the current period and the mean demand of the product in the same period. Constraint set (5.9) is the inventory balance equations for the rest of the periods in the planning horizon. The inventory level of a product i is determined by the inventory carried from the previous period, the total production of that product in the current period and the mean demand of the product in the same period. Constraint set (5.10) is the capacity constraints for each production source.

Although there might be demand for product i in periods $t=1,..,LT_{min}$, it is assumed that the production of either plant is not possible. Therefore, we cannot affect or make any changes in the production plan in the first LT_{min} periods nor can we construct any deterministic equivalent service level constraints up to period (LT_{min}+1). The demand during lead time can only be satisfied through the initial scheduled receipts. If it is desired to meet the service level requirements during the lead time, the parameters $I_{i,0}$ and $SR_{i,t}$ for product i and for periods $t=1,..,LT_{min}$ should be set such that:

$$I_{i,0} + \sum_{\tau=1}^{t} SR_{i,\tau} \geq l_{i,t}\ ,\ i = 1,...,K\ ;\ t = 1,...,LT_{\min}$$

To satisfy the minimum cumulative production quantity constraints, this initialization should be made and the right hand side of this constraint can only be satisfied by initial inventory plus the scheduled receipts in those periods.

Constraint sets (5.11) and (5.12) are the minimum cumulative production constraints for the rest of the periods. Finally, Constraint set (5.13) ensures the non-negativity of the production quantities.

5.4 A Two-product Two-plant Example

In our illustrative example, there are two products and two production sources with different lead times and production costs. The closer facility is assumed to have a shorter lead time at a higher production cost whereas the remote facility is assumed to have a longer lead time but at a lower production cost. Moreover, it is assumed that one of the products has a smaller mean demand with a higher coefficient variation value whereas the other product is assumed to have a larger mean demand with a lower coefficient variation value. The initial inventories and initial scheduled receipts are assumed to be zero. We try to decide where and how much of which product to produce in a multiperiod planning horizon to satisfy the demand.

It is assumed that the more responsive and expensive plant has a lead time of 1 period whereas the slower and cheaper plant has a lead time of 3 periods. There are no scheduled receipts for the first three periods and the initial inventories are set to be zero. In order not to observe any infeasibilities in the first three periods where the simultaneous production of the two plants is not possible, it is assumed that no demand for any of the products is observed. In each of the upcoming 12 periods, the first product is assumed to have a stationary demand of mean 25 whereas the second product is assumed to have a stationary demand of mean 100. It is also assumed that the CV's of the products are also known. Note that we have no other information about the demand distribution of the products. The service level requirement is 95% for the Modified Type 1 service level. The numerical experiments can be performed for the Modified Type 2 service level in a similar way. The only difference would be the l_t values to be utilized.

We solve the deterministic equivalent model once for each problem setting (not on a rolling horizon basis) and would like to observe the effects of the changes in production and holding costs, lead times and the coefficient of variations on the production assignments in the following subsections.

5.4.1 Effect of Production Costs on Production Assignments

The possible optimal solutions to the Single Period Production Planning Problem suggests that there is a relationship between the amount of production assigned to each plant for each product and the ratio of the difference between the production costs at the two plants of a specific product to that of the other product, i.e. $X_{i,j}$ values depend on the ratio $(p_{1,1}-p_{1,2})/(p_{2,1}-p_{2,2})$ for each i=1,2 and j=1,2. The optimal solution also depends on the $C_{j,t}$ and $l_{i,t}$ values for i=1,2, j=1,2 and t=1,…,12. However, the effect of the changes in any of these parameters is not so obvious and a careful examination of their effect on the optimal solution is necessary. Therefore, we focus our studies on how the production assignment to each plant for each product changes when we alter the ratio $(p_{1,1}-p_{1,2})/(p_{2,1}-p_{2,2})$. This ratio gives us clues about producing which product in the cheaper plant is more advantageous for us. It is the ratio of the relative cost reduction obtained by producing the first product at the cheaper plant to that obtained by producing the second product at the cheaper plant. If it is greater than 1, producing the first product in the second and cheaper plant is more cost-advantageous and if it is less than 1, producing the second product in the second and cheaper plant is more cost-advantageous. If this ratio equals 1, both products have equal cost advantage. Assume that the coefficients of variations are 2 and 0.25 for products 1 and 2 respectively. We examine the cases in which the ratio $(p_{1,1}-p_{1,2})/(p_{2,1}-p_{2,2})$ equals 3, 2, 1, 1/2, and 1/3. We are also interested in the change in the production assignments when the capacities of the plants are varied between 77 (the minimum production capacity that does not cause any infeasibility), 80, 90, 100, 110, 120, 130, 140 and 150. Moreover, keeping the ratio constant, we observe how the changes in the production costs affect the production assignments.

Table 5.1 displays how the assignment of the total production quantities over the horizon to the plants takes place under different combinations of plant capacities when $(p_{1,1}-p_{1,2})/(p_{2,1}-p_{2,2})$ equals 3 or 2, assuming that the holding costs are 0.1 and 0.05 for products 1 and 2 respectively. In each case, the production capacities of the two plants are assumed to be equal. The observations start with 88; the minimum capacity that does not result in any infeasibilities in the model and end with 150; a capacity greater than the minimum production quantity of the second product in the third period which is the first period having the demand different than zero. Keeping the ratio constant and equal to 3, the analysis is performed for the cases; for $p_{1,1}$=6, $p_{1,2}$=3, $p_{2,1}$=3, $p_{2,2}$=2 and for $p_{1,1}$=5, $p_{1,2}$=2, $p_{2,1}$=3, $p_{2,2}$=2. In these two cases, the majority of the high variability product is produced in the slower plant whereas the majority of the low variability product is produced in the quicker plant. In addition to this, 100% of the high variability product, and the majority of the low variability product are produced in the cheaper and slower plant if possible. That is because; when the production capacities get scarce, the importance of the fact that it is relatively more advantageous for the high variability product to be produced in the cheaper plant becomes more significant. Moreover, we can say that the individual production costs have no significance by themselves; instead it is the ratio $(p_{1,1}-p_{1,2})/(p_{2,1}-p_{2,2})$ that has a major impact. The analysis is repeated for $p_{1,1}$=6, $p_{1,2}$=4, $p_{2,1}$=3, $p_{2,2}$=2 and for $p_{1,1}$=4,

$p_{1,2}$=2, $p_{2,1}$=3, $p_{2,2}$=2 when $(p_{1,1}-p_{1,2})/(p_{2,1}-p_{2,2})=2$. As a result, the same production assignments are observed.

Table 5.1: The effect of production costs on the percentage of total production assigned to each plant for each product when $(p_{1,1}-p_{1,2})/(p_{2,1}-p_{2,2})$=3 or 2

Capacity of each plant	Percentage of Product 1 produced in		Percentage of Product 2 produced in	
	Plant 1	Plant 2	Plant 1	Plant 2
77	5.47	94.53	72.23	27.77
80	4.96	95.04	69.76	30.24
90	3.25	96.75	61.53	38.47
100	1.54	98.46	53.29	46.71
110	0.00	100.00	44.99	55.02
120	0.00	100.00	36.00	64.00
130	0.00	100.00	27.02	72.98
140	0.00	100.00	18.04	81.96
150	0.00	100.00	13.02	86.98

Table 5.2 summarizes the results of the same analysis performed for $(p_{1,1}-p_{1,2})/(p_{2,1}-p_{2,2})=1$. The analysis is repeated for $p_{1,1}$=6, $p_{1,2}$=4, $p_{2,1}$=3, $p_{2,2}$=1 and for $p_{1,1}$=6, $p_{1,2}$=5, $p_{2,1}$=3, $p_{2,2}$=2. In these cases, the majority of the low variability product is produced in the slower plant whereas the majority of the high variability product is produced in the quicker plant. If possible, 100% of the low variability product, and the majority of the high variability product are produced in the cheaper and slower plant. This highlights the significance of the ratio $(p_{1,1}-p_{1,2})/(p_{2,1}-p_{2,2})$ rather than the individual $p_{i,j}$'s.

Table 5.2: The effect of production costs on the percentage of total production assigned to each plant for each product when $(p_{1,1}-p_{1,2})/(p_{2,1}-p_{2,2})=1$

Capacity of each plant	Percentage of Product 1 produced in		Percentage of Product 2 produced in	
	Plant 1	Plant 2	Plant 1	Plant 2
77	94.53	5.47	33.23	66.77
80	95.04	4.96	30.31	69.69
90	96.75	3.25	20.58	79.42
100	98.46	1.54	10.85	89.15
110	95.90	4.10	2.99	97.01
120	78.80	21.20	1.50	98.50
130	60.00	40.00	0.75	99.25
140	41.20	58.80	0.00	100.00
150	29.74	70.26	0.00	100.00

Table 5.3 summarizes the results of the same analysis performed for $(p_{1,1}-p_{1,2})/(p_{2,1}-p_{2,2})=1/2$ and $1/3$. The analysis is repeated for $p_{1,1}$=6, $p_{1,2}$=5, $p_{2,1}$=3, $p_{2,2}$=1 and for $p_{1,1}$=4, $p_{1,2}$=3,and for $p_{2,1}$=3, $p_{2,2}$=2 when $(p_{1,1}-p_{1,2})/(p_{2,1}-p_{2,2})=1/2$; and for $p_{1,1}$=7, $p_{1,2}$=6, $p_{2,1}$=5, $p_{2,2}$=2 and for $p_{1,1}$=8, $p_{1,2}$=6, $p_{2,1}$=7, $p_{2,2}$=1 when $(p_{1,1}-p_{1,2})/(p_{2,1}-p_{2,2})=1/3$. In these cases, as in the case $(p_{1,1}-p_{1,2})/(p_{2,1}-p_{2,2})=1$, the majority of the low variability product is produced in the slower plant whereas the majority of the high variability product is produced in the quicker plant. In addition to this, 100% of the low variability product, and the majority of the high variability product are produced in the cheaper and slower plant if possible. That is because; when the production capacities get scarce, the importance of the fact that it is relatively more advantageous for the low variability product to be produced in the cheaper plant becomes more significant. Moreover, we can repeat the fact that the production costs have no importance in quantity by themselves; instead the ratio $(p_{1,1}-p_{1,2})/(p_{2,1}-p_{2,2})$ is of significance.

Table 5.3: The effect of production costs on the percentage of total production assigned to each plant for each product when $(p_{1,1}-p_{1,2})/(p_{2,1}-p_{2,2})$=1/2 or 1/3

Capacity of each plant	Percentage of Product 1 produced in		Percentage of Product 2 produced in	
	Plant 1	Plant 2	Plant 1	Plant 2
77	100.00	0.00	30.84	69.16
80	100.00	0.00	28.14	71.86
90	100.00	0.00	19.16	80.84
100	100.00	0.00	10.18	89.82
110	95.90	4.10	2.99	97.01
120	78.80	21.20	1.50	98.50
130	60.00	40.00	0.75	99.25
140	41.20	58.80	0.00	100.00
150	29.74	70.26	0.00	100.00

5.4.2 Effect of Holding costs on Production Assignments

Next, we investigate how the production assignment to each plant for each product changes when we alter the holding cost. Assume that the coefficients of variations are 2 and 0.25 for products 1 and 2 respectively. Table 5.4 displays the results when the capacities of the two plants equal 120 and when $(p_{1,1}-p_{1,2})/(p_{2,1}-p_{2,2})=2$ for the case for $p_{1,1}$=6, $p_{1,2}$=4, $p_{2,1}$=3, $p_{2,2}$=2. In this case, no matter what the combination of holding costs of the products is, the majority of both products is produced in the slower plant.

Table 5.4: The effect of holding costs on the percentage of total production assigned to each plant for each product when $(p_{1,1}-p_{1,2})/(p_{2,1}-p_{2,2})=2$

Holding cost for		Percentage of Product 1 produced in		Percentage of Product 2 produced in	
Product 1	**Product 2**	**Plant 1**	**Plant 2**	**Plant 1**	**Plant 2**
0.1	0.05	0.00	100.00	36.00	64.00
0.1	0.1	0.00	100.00	36.00	64.00
0.1	0.15	0.00	100.00	36.00	64.00
1	1	0.00	100.00	36.00	64.00
1	0.05	0.00	100.00	36.00	64.00
0.05	1	0.00	100.00	36.00	64.00
4	1	0.00	100.00	36.00	64.00
5	1	0.00	100.00	36.00	64.00
6	1	0.00	100.00	36.00	64.00
0.05	1.05	1.54	98.46	35.33	64.67
0.05	2	1.54	98.46	35.33	64.67
0.05	6	1.54	98.46	35.33	64.67
4	6	1.54	98.46	35.33	64.67
6	6	0.00	100.00	36.00	64.00

Table 5.5 displays the results when the capacities of the two plants equal 120 and it is assumed that $p_{1,1}$=6, $p_{1,2}$=4, $p_{2,1}$=3, $p_{2,2}$=1 when $(p_{1,1}-p_{1,2})/(p_{2,1}-p_{2,2})=1$ and $p_{1,1}$=6, $p_{1,2}$=5, $p_{2,1}$=3, $p_{2,2}$=1 when $(p_{1,1}-p_{1,2})/(p_{2,1}-p_{2,2})=1/2$. In this case, no matter what the combination of holding costs of the products is, the majority of the low variability product is produced in the slower plant whereas the majority of the high variability product is produced in the quicker plant. The reasoning behind this is the fact the deterministic equivalent service level constraints are always binding.

Table 5.5: The effect of holding costs on the percentage of total production assigned to each plant for each product when $(p_{1,1}-p_{1,2})/(p_{2,1}-p_{2,2})=1$ or 1/2

Holding cost for		Percentage of Product 1 produced in		Percentage of Product 2 produced in	
Product 1	Product 2	Plant 1	Plant 2	Plant 1	Plant 2
0.1	0.05	78.80	21.20	1.50	98.50
0.1	0.1	78.80	21.20	1.50	98.50
0.1	0.15	78.80	21.20	1.50	98.50
1	1	78.80	21.20	1.50	98.50
1	0.05	78.80	21.20	1.50	98.50
0.05	1	78.80	21.20	1.50	98.50
4	1	78.80	21.20	1.50	98.50
5	1	78.80	21.20	1.50	98.50
6	1	78.80	21.20	1.50	98.50
0.05	1.05	78.80	21.20	1.50	98.50
0.05	2	78.80	21.20	1.50	98.50
0.05	6	78.80	21.20	1.50	98.50
4	6	78.80	21.20	1.50	98.50
6	6	78.80	21.20	1.50	98.50

5.4.3 Effect of the Length of the Lead Time on Production Assignments

It is interesting to understand how the production of the two products will be distributed among the two plants, when the lead times of the plants vary. Assume that the coefficients of variations are 2 and 0.25 for products 1 and 2 respectively. Moreover, let the lead time of the first plant be constant and be 1 period. We examine the changes in the production assignments when the lead time of the second plant varies between 1 and 5 periods. The planning period is assumed to be LT_{max} periods added to the 12 periods. In the first LT_{max} periods where the simultaneous production of the two plants is not possible, it is assumed that no demand for any of the products is observed. For each case, the production capacity of each plant is assumed to be equal and is taken to be the minimum capacity that does not cause any infeasibilities.

Table 5.6 summarizes these results when $(p_{1,1}-p_{1,2})/(p_{2,1}-p_{2,2})$=2. In these two cases, 100% of the high variability product, and the majority of the low variability product are produced in the slower plant if possible. Otherwise, the majority of the high variability product is produced in the slower plant whereas the majority of the low variability product is produced in the quicker plant.

Table 5.6: The effect of the length of the lead time on the percentage of total production assigned to each plant for each product when $(p_{1,1}-p_{1,2})/(p_{2,1}-p_{2,2})$=2

Capacity of each plant	Lead time of		Percentage of Product 1 produced in		Percentage of Product 2 produced in	
	Plant 1	Plant 2	Plant 1	Plant 2	Plant 1	Plant 2
125	1	1	0.00	100.00	31.51	68.49
88	1	2	3.59	96.41	63.17	36.83
77	1	3	5.47	94.53	72.23	27.77
72	1	4	6.32	93.68	76.35	23.65
69	1	5	7.18	92.82	78.67	21.33

Table 5.7 and Table 5.8 summarizes the results when $(p_{1,1}-p_{1,2})/(p_{2,1}-p_{2,2})$=1 and 1/2, respectively. In both cases, the majority of the high variability product is produced in the quicker plant whereas the majority of the low variability product is produced in the slower plant.

Table 5.7: The effect of the length of the lead time on the percentage of total production assigned to each plant for each product when $(p_{1,1}-p_{1,2})/(p_{2,1}-p_{2,2})=1$

Capacity of each plant	Lead time of		Percentage of Product 1 produced in		Percentage of Product 2 produced in	
	Plant 1	Plant 2	Plant 1	Plant 2	Plant 1	Plant 2
125	1	1	69.40	30.60	1.12	98.88
88	1	2	96.41	3.59	22.53	77.47
77	1	3	94.53	5.47	33.23	66.77
72	1	4	93.68	6.32	38.10	61.90
69	1	5	92.82	7.18	41.17	58.83

Table 5.8: The effect of the length of the lead time on the percentage of total production assigned to each plant for each product when $(p_{1,1}-p_{1,2})/(p_{2,1}-p_{2,2})=1/2$

Capacity of each plant	Lead time of		Percentage of Product 1 produced in		Percentage of Product 2 produced in	
	Plant 1	Plant 2	Plant 1	Plant 2	Plant 1	Plant 2
125	1	1	69.40	30.60	1.12	98.88
88	1	2	100.00	0.00	20.96	79.04
77	1	3	100.00	0.00	30.84	69.16
72	1	4	100.00	0.00	35.33	64.67
69	1	5	100.00	0.00	38.02	61.98

5.4.4 Effect of Coefficient of Variations on Production Assignments

Now, we would like to examine how the changes in the coefficients of variations of the two products affect the production decisions. Assume that the lead times are 1 and 3 periods for plants 1 and 2 respectively. The coefficient of the first product is set to be constant and 2. The coefficient of variation of the second product varies between 0 and 2. The capacities of the plants are set to be the minimum capacity values that do not produce any infeasibilities. Table 5.9 summarize the results when $(p_{1,1}-p_{1,2})/(p_{2,1}-p_{2,2})$=2. In this case, the majority of the high variability product is produced in the slower and cheaper plant whereas the majority of the low variability product is produced in the quicker and more expensive plant. Moreover, if the production of the two products by the slower and cheaper plant is possible and feasible, this option is utilized.

Table 5.9: The effect of the coefficient of variations on the percentage of total production assigned to each plant for each product when $(p_{1,1}-p_{1,2})/(p_{2,1}-p_{2,2})=2$

Capacity of each Plant	Coefficient of Variation of		Percentage of Product 1 produced in		Percentage of Product 2 produced in	
	Product 1	Product 2	Plant 1	Plant 2	Plant 1	Plant 2
70	2.00	0.00	6.84	93.16	75.44	24.56
77	2.00	0.25	5.47	94.53	72.23	27.77
86	2.00	0.50	3.93	96.07	68.41	31.59
97	2.00	0.75	2.05	97.95	64.14	35.86
109	2.00	1.00	0.00	100.00	60.06	39.94
122	2.00	1.25	0.00	100.00	55.27	44.73
133	2.00	1.50	0.00	100.00	52.04	47.96
142	2.00	1.75	0.00	100.00	49.96	50.04
150	2.00	2.00	0.00	100.00	48.36	51.64

Tables 5.10 and 5.11 summarize the results when $(p_{1,1}-p_{1,2})/(p_{2,1}-p_{2,2})=1$ and 1/2 respectively. In these cases the majority of the high variability product is produced in the quicker and the more expensive plant whereas the majority of the low variability product is produced in the slower and cheaper plant.

Table 5.10: The effect of the coefficient of variations on the percentage of total production assigned to each plant for each product when $(p_{1,1}-p_{1,2})/(p_{2,1}-p_{2,2})=1$

Capacity of each Plant	Coefficient of variation of		Percentage of Product 1 produced in		Percentage of Product 2 produced in	
	Product 1	Product 2	Plant 1	Plant 2	Plant 1	Plant 2
70	2.00	0.00	93.16	6.84	33.39	66.61
77	2.00	0.25	94.53	5.47	33.23	66.77
86	2.00	0.50	96.07	3.93	32.19	67.81
97	2.00	0.75	97.95	2.05	30.10	69.90
109	2.00	1.00	100.00	0.00	27.73	72.27
122	2.00	1.25	100.00	0.00	25.50	74.50
133	2.00	1.50	100.00	0.00	24.29	75.71
142	2.00	1.75	98.63	1.37	24.15	75.85
150	2.00	2.00	93.16	6.84	25.20	74.80

Table 5.11: The effect of the coefficient of variations on the percentage of total production assigned to each plant for each product when $(p_{1,1}-p_{1,2})/(p_{2,1}-p_{2,2})=1/2$

Capacity of each Plant	Coefficient of Variation of		Percentage of Product 1 produced in		Percentage of Product 2 produced in	
	Product 1	Product 2	Plant 1	Plant 2	Plant 1	Plant 2
70	2.00	0.00	100.00	0.00	30.06	69.94
77	2.00	0.25	100.00	0.00	30.84	69.16
86	2.00	0.50	100.00	0.00	30.65	69.35
97	2.00	0.75	100.00	0.00	29.37	70.63
109	2.00	1.00	100.00	0.00	27.73	72.27
122	2.00	1.25	100.00	0.00	25.50	74.50
133	2.00	1.50	100.00	0.00	24.29	75.71
142	2.00	1.75	98.63	1.37	24.15	75.85
150	2.00	2.00	93.16	6.84	25.20	74.80

5.5 Conclusion

In the existing literature, the assignment of low variability products to cheaper sources with long lead times and of high variability products to faster sources is frequently suggested as a heuristic without a formal basis. For instance, Abernathy, Dunlop, Hammond and Weil [26] show that a local short-cycle manufacturer is more appropriate for items with high variability whereas an offshore manufacturer can be utilized for items with low variability. However, the situation is much more complicated and context dependent in general. Our numerical results performed for Modified Type 1 service level, only confirm the validity of this heuristic in certain cases but suggests a different criterion in general. When $(p_{1,1}-p_{1,2})/(p_{2,1}-p_{2,2})>1$, 100% of the high variability product, and the majority of the low variability product are produced in the slower plant if possible. Otherwise, the majority of the high variability product is produced in the slower plant whereas the majority of the low variability product is produced in the quicker plant. In these cases, as in the case $(p_{1,1}-p_{1,2})/(p_{2,1}-p_{2,2})\leq 1$, 100% of the low variability product, and the majority of the high variability product are produced in the slower plant if possible. Otherwise, the majority of the low variability product is produced in the slower plant whereas the majority of the high variability product is produced in the quicker plant. Moreover, it is observed that the change in the holding costs do not affect the production assignments when the ratio $(p_{1,1}-p_{1,2})/(p_{2,1}-p_{2,2})$ is kept constant. While observing the effect of the length of the lead times of the plants on the production decisions, when $(p_{1,1}-p_{1,2})/(p_{2,1}-p_{2,2})=2$, 100% of the high variability product, and the majority of the low variability product are produced in the slower plant if possible. Otherwise, the majority of the high variability product is produced in the slower plant whereas the majority of the low variability product is produced in the quicker plant. When

$(p_{1,1}-p_{1,2})/(p_{2,1}-p_{2,2})$=1 or 1/2, the majority of the high variability product is produced in the quicker plant whereas the majority of the low variability product is produced in the slower plant. While observing the effect of the change in coefficient of variations of the products on the production assignments to the plants, when $(p_{1,1}-p_{1,2})/(p_{2,1}-p_{2,2})$=2, the majority of the high variability product is produced in the slower and cheaper plant whereas the majority of the low variability product is produced in the quicker and more expensive plant. Moreover, it is possible and feasible to produce the two products by the slower and cheaper plant, this option is utilized. When $(p_{1,1}-p_{1,2})/(p_{2,1}-p_{2,2})$=1 or 1/2, the majority of the high variability product is produced in the quicker and the more expensive plant whereas the majority of the low variability product is produced in the slower and cheaper plant.

Chapter 6

CONCLUSIONS

In this thesis, stochastic production planning and sourcing problems with service level constraints are examined in detail. The randomness in demand should be incorporated in mathematical models of production planning and sourcing problems in many practical situations. In the existing literature, most of the time, randomness is handled by replacing a random variable by a single point estimate value or by its mean. In this thesis, we develop a framework that explicitly addresses the random components.

In Chapter 3, a systematic approach that enables the randomness in demand and the desired service levels is proposed. This enables the incorporation of random demand and the probabilistic service level constraints in a mathematical programming framework leading to a methodology that aids manufacturers in deciding how much to produce, when to produce, where to produce, how much inventory to carry.

To establish the validity of the proposed approach, the performance of the methodology is compared with some benchmarks created in Chapter 4. The proposed benchmark in the single production facility problem is the well-known base stock inventory policy. For both the case including production lead time and for the case without lead time, the equivalencies between the base stock model and the deterministic equivalent model with modified service level constraints solved on a rolling horizon basis are shown in the single product single production facility setting. Moreover, for the multiple plants setting without lead time, the threshold subcontracting model is chosen to be the benchmark. The control parameters of the policy are determined by using a simulation-based optimization procedure. The results of both the proposed model and the benchmark are then observed and compared with each other. As a result, it is concluded that the proposed approach gives promising solutions.

In Chapter 5, the optimal sourcing strategies for the single period production planning problem, in which two products are to be produced either by a cheaper or by an expensive plant or by both of the two plants, are derived and the results are interpreted. It is emphasized that if the product with the greater cost advantage is altered, the whole production plan changes and the criticality of the relative cost reductions obtained by producing the products at the cheaper plant is shown. Motivated by a production planning and sourcing problem in the textile-apparel-retail channel, the proposed methodology is applied for multiple products setting in a multiperiod stochastic production planning and sourcing problem with service level constraints. It is found out that the situation is context dependent and based on different capacity, minimum cumulative production quantity, or production cost parameter settings, different optimal solutions might be observed. Based on a two-product two plant numerical example, by focusing mainly on the relative cost reduction ratio of the two products, different solutions are observed and the insights gained are presented.

The contributions of this thesis can be summarized as:

- The integration of the deterministic mathematical programming approach for a manufacturer's production and sourcing problem with randomness arising from stochastic demand and service level constraints.
- The justification of the equivalencies between the base stock model and the deterministic equivalent model with modified service level constraints solved on a rolling horizon basis in the single product single production facility setting.

- The presentation of the similarities between the results of the deterministic equivalent model with modified service level constraints solved on a rolling horizon basis and the threshold subcontracting model solved on a simulation-based optimization technique in the single product multiple plants setting.
- The insights obtained regarding the interaction among the cost, lead time, and variability of demand and their effects on the sourcing decisions.

There are several other significant issues that are worth investigating from a methodological point of view. The effect of rolling horizon procedures and frozen planning periods can be investigated. The effect of forecast updates and their incorporation in mathematical programming formulations are also interesting issues for future research.

Moreover, the numerical experiments for the single product multiple plants setting can be performed for larger sample sizes for a larger number of scenarios. It would be worth investigating whether the threshold subcontracting model always constructs a lower bound on total expected cost for the deterministic equivalent model.

Appendix A

PROOF OF PROPOSITION 4.2

i. If the inventory levels at the beginning of the first period are equal, I_0(BSLT)=I_0(DEPLT)=l_{LT+1}, then production quantities in the first period and the inventory at the end of first period for both policies become equal, i.e. X_1(BSLT)=X_1(DEPLT)=0 and I_1(BSLT)=I_1(DEPLT)=l_{LT+1}-d_1;

ii. If the inventory levels at the end of period t_1 such that $t_1 \leq LT$ are equal, $I_{t_1}(\text{BSLT}) = I_{t_1}(\text{DEPLT}) = l_{LT+1} - \sum_{\tau=1}^{t_1} d_\tau$, then the production quantities in period (t_1+1) and the inventory levels at the end of period (t_1+1) for both policies become equal; i.e. $X_{t_1+1}(\text{BSLT}) = X_{t_1+1}(\text{DEPLT}) = d_{t_1}$ and $I_{t_1+1}(\text{BSLT}) = I_{t_1+1}(\text{DEPLT}) = l_{LT+1} - \sum_{\tau=1}^{t_1+1} d_\tau$.

and

iii. If the inventory levels at the end of period (LT+1) are equal, I_{LT+1}(BSLT)=I_{LT+1}(DEPLT)=$l_{LT+1} - \sum_{\tau=1}^{LT+1} d_\tau$, then production quantities in period (LT+2) and the inventory levels at the end of period (LT+2) for both policies become equal, i.e. X_{LT+2}(BSLT)=X_{LT+2}(DEPLT)=d_{LT+1} and I_{LT+2}(BSLT) =I_{LT+2}(DEPLT)= $=l_{LT+1}-\sum_{\tau=2}^{LT+2} d_\tau$;

iv. If the inventory levels at the end of period t_2 such that $t_2 \geq LT$ are equal, $I_{t_2}(\text{BSLT})=I_{t_2}(\text{DEPLT})=l_{LT+1}-\sum_{\tau=t_2-LT}^{t_2} d_\tau$, then the production quantities in period (t_2+1) and the inventory levels at the end of period (t_2+1) for both policies become equal; i.e. $X_{t_2+1}(\text{BSLT})=X_{t_2+1}(\text{DEPLT})=d_{t_2}$ and $I_{t_2+1}(\text{BSLT})=I_{t_2+1}(\text{DEPLT})=l_{LT+1}-\sum_{\tau=t_2+1-LT}^{t_2+1} d_\tau$.

Assume that the initial inventory levels are equal such that I_0(BSLT)=S_2, I_0(DEPLT)=l_{LT+1} and S_2=l_{LT+1}. In the base stock policy, each demand observed is produced in the next period; therefore there is no production in the first period, X_1(BSLT)=0. In the deterministic equivalent approach, the production quantity in the first period is determined according to the constraint $X_1(\text{DEPLT})+\sum_{\tau=1}^{LT} SR_\tau(\text{DEPLT})+I_0(\text{DEPLT})=X_1(\text{DEPLT})$ $+0+l_{LT+1}\geq l_{LT+1}$ and therefore, $X_1(\text{DEPLT})\geq 0$. Since the problem is of minimization type, the production quantity in the first period equals zero, i.e. X_1(DEPLT)=0. Next, a customer demand of d_1 arrives. The end of period inventory for the base stock policy becomes I_1(BSLT)=I_0(BSLT)+SR_1(BSLT)=S_2+0-d_1=S_2-d_1 and the end of period inventory for the deterministic equivalent approach becomes I_1(DEPLT)=I_0(DEPLT) +SR_1(DEPLT)-d_1=l_{LT+1}+0-d_1=l_{LT+1}-d_1. Since we know that S_2=l_{LT+1}, I_1(BSLT)=I_1(DEPLT).

In the second period, the base stock policy produces the demand of the first period, i.e. X_2(BSLT)=d_1. At the beginning of the second period, the deterministic equivalent model is rerun since it is solved on a rolling horizon basis. The demand is assumed to be stationary over the planning horizon. Although solving the model on a rolling horizon basis throughout the planning horizon requires integration of the minimum cumulative production quantites for the number of periods in the rolling horizon into the model, only the minimum cumulative production quantity of period (LT+1), l_{LT+1}, is fully utilized. The production quantity of the deterministic equivalent model in the second period is determined by $X_2(\text{DEPLT}) + \sum_{\tau=2}^{LT+1} SR_\tau(\text{DEPLT}) + I_1(\text{DEPLT}) = X_2(\text{DEPLT}) + X_1(\text{DEPLT})$ $+I_1(\text{DEPLT}) = X_2(\text{DEPLT}) + 0 + l_{LT+1} - d_1 \geq l_{LT+1}$; therefore, $X_2(\text{DEPLT}) \geq d_1$. In order to minimize the production costs, the production quantity in the second period equals the demand of the first period, i.e. $X_2(\text{DEPLT}) = d_1$. After the arrival of a customer demand of d_2, the end of period inventory for the base stock policy becomes I_2(BSLT)=I_1(BSLT)+SR_2(BSLT)-d_2=S_2-d_1-d_2 and the end of period inventory for the deterministic equivalent approach becomes I_2(DEPLT)=I_1(DEPLT)+SR_2(DEPLT)-d_2= l_{LT+1} -d_1-d_2. Since S_2=l_{LT+1}, we can say that I_2(BSLT)=I_2(DEPLT).

Since demand during lead time cannot be satisfied no sooner than (LT+1) periods of time, the inventory levels at the end of any period t_1 such that $t_1 \leq (LT-1)$ can be written as $I_{t_1}(\text{BSLT}) = S_2 - \sum_{\tau=1}^{t_1} d_\tau$, $I_{t_1}(\text{DEP}) = l_{LT+1} - \sum_{\tau=1}^{t_1} d_\tau$ and S_2=l_{LT+1}. In period (t_1+1), the base stock policy produces $X_{t_1+1}(\text{BSLT}) = d_{t_1}$. In the deterministic equivalent approach, the production quantity is determined by the constraint $X_{t_1+1}(\text{DEPLT}) + \sum_{\tau=t_1+1}^{t_1+LT} SR_\tau(\text{DEPLT})$ $+I_{t_1}(\text{DEPLT}) = X_{t_1+1}(\text{DEPLT}) + \sum_{\tau=1}^{t_1} X_\tau(\text{DEPLT}) + I_{t_1}(\text{DEPLT}) = X_{t_1+1}(\text{DEPLT}) + \sum_{\tau=1}^{t_1-1} d_\tau + l_{LT+1}$ $-\sum_{\tau=1}^{t_1} d_\tau \geq l_{LT+1}$; therefore, $X_{t_1+1}(\text{DEPLT}) \geq d_{t_1}$. Since the problem is of minimization type, $X_{t_1+1}(\text{DEPLT}) = d_{t_1}$. Then, a customer demand of d_{t_1+1} is observed. The end of period inventory for the base stock policy becomes $I_{t_1+1}(\text{BSLT}) = I_{t_1}(\text{BSLT}) + SR_{t_1+1}(\text{BSLT})$ $-d_{t_1+1} = S_2 - \sum_{\tau=1}^{t_1} d_\tau - d_{t_1+1} = S_2 - \sum_{\tau=1}^{t_1+1} d_\tau$ and the end of period inventory for the deterministic equivalent approach becomes $I_{t_1+1}(\text{DEPLT}) = I_{t_1}(\text{DEPLT}) + SR_{t_1+1}(\text{DEPLT}) - d_{t_1+1} = l_{LT+1}$ $-\sum_{\tau=1}^{t_1} d_\tau - d_{t_1+1} = l_{LT+1} - \sum_{\tau=1}^{t_1+1} d_\tau$. Since S_2=l_{LT+1}, $I_{t_1+1}(\text{BSLT}) = I_{t_1+1}(\text{DEPLT})$.

Similarly, d_{LT+1} is produced by the base stock policy in period (LT+1), i.e. X_{LT+1}=d_{LT+1}. The constraint $X_{LT+1}(\text{DEPLT}) + \sum_{\tau=LT+1}^{2LT} SR_\tau(\text{DEPLT}) + I_{LT}(\text{DEPLT}) = X_{LT+1}(\text{DEPLT}) + \sum_{\tau=1}^{LT} X_\tau$ $+I_{LT}(\text{DEPLT}) = X_{LT+1}(\text{DEPLT}) + \sum_{\tau=1}^{LT-1} d_\tau + l_{LT+1} - \sum_{\tau=1}^{LT} d_\tau \ge l_{LT+1}$; i.e. $X_{LT+1}(\text{DEPLT}) \ge d_{LT}$ determines the production quantity of the deterministic equivalent model in period (LT+1). Then, $X_{LT+1}(\text{DEPLT}) = d_{LT}$. Next, a customer demand of d_{LT+1} arrives. The end of period inventory for the base stock policy becomes $I_{LT+1}(\text{BSLT}) = I_{LT}(\text{BSLT}) + SR_{LT+1}(\text{BSLT})$ $-d_{LT+1} = S_2 - \sum_{\tau=1}^{LT} d_\tau + X_1(\text{BSLT}) - d_{LT+1} = S_2 - \sum_{\tau=1}^{LT} d_\tau + 0 - d_{LT+1} = S_2 - \sum_{\tau=1}^{LT+1} d_\tau$ and the end of period inventory for the deterministic equivalent approach becomes $I_{LT+1}(\text{DEPLT}) = I_{LT}(\text{DEPLT}) + SR_{LT+1}(\text{DEPLT}) - d_{LT+1} = l_{LT+1} - \sum_{\tau=1}^{LT} d_\tau + X_1(\text{DEPLT}) - d_{LT+1}$ $= l_{LT+1} - \sum_{\tau=1}^{LT} d_\tau + 0 - d_{LT+1} = l_{LT+1} - \sum_{\tau=1}^{LT+1} d_\tau$ .Since S_2=l_{LT+1}, $I_{LT+1}(\text{BSLT}) = I_{LT+1}(\text{DEPLT})$.

In period (LT+2), the base stock policy produces X_{LT+1}(BSLT)=d_{LT+2}. For the deterministic equivalent approach, we know that $X_{LT+2}(\text{DEPLT}) + \sum_{\tau=LT+2}^{2LT+1} SR_\tau(\text{DEPLT})$ $I_{LT+1}(\text{DEPLT}) = X_{LT+2}(\text{DEPLT}) + \sum_{\tau=2}^{LT+1} X_\tau + I_{LT+1}(\text{DEPLT}) = X_{LT+2}(\text{DEPLT}) + \sum_{\tau=1}^{LT} d_\tau + l_{LT+1}$

$-\sum_{\tau=1}^{LT+1} d_\tau \geq l_{LT+1}$; i.e. $X_{LT+2}(\text{DEPLT}) \geq d_{LT+1}$ and then, $X_{LT+2}(\text{DEPLT}) = d_{LT+1}$. After the arrival of d_{LT+2}, the following end of period inventory levels are observed $I_{LT+2}(\text{BSLT}) = I_{LT+1}(\text{BSLT}) + SR_{LT+2}(\text{BSLT}) - d_{LT+2} = S_2 - \sum_{\tau=1}^{LT+1} d_\tau + X_2(\text{BSLT}) - d_{LT+2} = S_2 - \sum_{\tau=1}^{LT+1} d_\tau + d_1 - d_{LT+2} = S_2 - \sum_{\tau=2}^{LT+2} d_\tau$ and $I_{LT+2}(\text{DEPLT}) = I_{LT+1}(\text{DEPLT}) + SR_{LT+2}(\text{DEPLT}) - d_{LT+2} = l_{LT+1} - \sum_{\tau=1}^{LT+1} d_\tau + X_2(\text{DEPLT}) - d_{LT+2} = l_{LT+1} - \sum_{\tau=1}^{LT+1} d_\tau + d_1 - d_{LT+2} = l_{LT+1} - \sum_{\tau=2}^{LT+2} d_\tau$.

Since we know that $S_2 = l_{LT+1}$, $I_{LT+2}(\text{BSLT}) = I_{LT+2}(\text{DEPLT})$.

Now assume that at the end of any period t_2 such that $t_2 \geq (LT+1)$, $I_{t_2}(\text{BSLT}) = S_2 - \sum_{\tau=t_2-LT}^{t_2} d_\tau$, $I_{t_2}(\text{DEPLT}) = l_{LT+1} - \sum_{\tau=t_2-LT}^{t_2} d_\tau$ and $S_2 = l_{LT+1}$. In period (t_2+1), $X_{t_2+1}(\text{BSLT}) = d_{t_2}$ and $X_{t_2+1}(\text{DEPLT})$ is determined by the constraint $X_{t_2+1}(\text{DEPLT}) + \sum_{\tau=t_2+1}^{t_2+LT} SR_\tau(\text{DEPLT}) + I_{t_2}(\text{DEPLT}) = X_{t_2+1}(\text{DEPLT}) + \sum_{\tau=1}^{t_2} X_\tau + I_{t_2}(\text{DEPLT}) = X_{t_2+1}(\text{DEPLT}) + \sum_{\tau=1}^{t_2-1} d_\tau + l_{LT+1} - \sum_{\tau=1}^{t_2} d_\tau \geq l_{LT+1}$; $X_{t_2+1}(\text{DEPLT}) \geq d_{t_2}$ and since the model is of minimization type $X_{t_2+1}(\text{DEPLT}) = d_{t_2}$. Next, a customer demand of d_{t_2+1} arrives. The end of period inventory levels for both policies become $I_{t_2+1}(\text{BSLT}) = I_{t_2}(\text{BSLT}) + SR_{t_2+1}(\text{BSLT}) - d_{t_2+1} = S_2 - \sum_{\tau=t_2-LT}^{t_2} d_\tau + X_{t_2+1-LT}(\text{BSLT}) - d_{t_2+1} = S_2 - \sum_{\tau=t_2-LT}^{t_2} d_\tau + d_{t_2-LT} - d_{t_2+1} = S_2 - \sum_{\tau=t_2+1-LT}^{t_2+1} d_\tau$ and $I_{t_2+1}(\text{DEPLT}) = I_{t_2}(\text{DEPLT}) + SR_{t_2+1}(\text{DEPLT}) - d_{t_2+1} = S_2 - \sum_{\tau=t_2-LT}^{t_2} d_\tau + X_{t_2+1-LT}(\text{DEPLT})$

$-d_{t_2+1} = S_2 - \sum_{\tau=t_2-LT}^{t_2} d_\tau + d_{t_2-LT} - d_{t_2+1} = S_2 - \sum_{\tau=t_2+1-LT}^{t_2+1} d_\tau$. Since we know that $S_2 = l_{LT+1}$, $I_{t_2+1}(\text{BSLT}) = I_{t_2+1}(\text{DEPLT})$. This proves our proposition. ■

Appendix B

M/M/1 DUAL SOURCE MODEL

We focus on a single stage, single server, make-to-stock production system in which a single product is manufactured. It is assumed that the order arrivals are governed by a Poisson process and that the order processing times of the manufacturing facility are exponentially distributed. The production system is controlled by a dual base stock policy with a base stock level of s and a threshold level of z. The manufacturing facility operates if and only if the finished goods inventory level drops below the target level s, i.e. whenever the amount of shortfall with respect to the base stock is positive. The facility stops producing when the finished goods inventory level again reaches the target level s. If the number of items stored in the inventory is positive, orders are fulfilled from the finished goods inventory. Otherwise, an order that cannot be satisfied is backlogged. Moreover, when the inventory on hand decreases to a threshold level of z, the subcontracting option is utilized. It is assumed that the processing times of the subcontractor are also exponentially distributed and both the in-house manufacturing facility and the subcontractor have finite capacities. Note that at time zero, there are s items in the finished goods inventory.

For modeling we define the following notation:

$A(t)$: Number of order arrivals at time t

$D(t)$: Number of items produced at time t

$K(t)$: Number of items subcontracted at time t

$R(t)$: Number of orders fulfilled at time t

$R(t)=\text{Min}\{s+D(t)+K(t), A(t)\}$

$I(t)$: Number of items in the inventory at time t

$I(t)=s+D(t)+K(t)-R(t)$

$B(t)$: Number of items backlogged at time t

$B(t)=A(t)-R(t)$

$X(t)$: Finished goods inventory level at time t (either positive or negative)

$X(t)=I(t)-B(t)=s-[A(t)-D(t)-K(t)]$

$N(t)$: Amount of shortfall with respect to the base stock level s at time t

$N(t)=s-X(t)=A(t)-D(t)-K(t)$

$C(t)$: $C(t)=\text{Min}\{A(t)-D(t)-K(t), s\}$

λ: Order arrival rate

μ: Processing rate of the in-house manufacturing facility

μ_z: Processing rate of the subcontractor

s: Base stock level

z: Threshold level for the finished goods inventory below which subcontracting option is utilized

k: The difference between base stock and threshold subcontracting levels

$k=s-z$

The system under consideration can be modeled as an M/M/1 queuing system.The inventory related calculations depend on the cases for which $z>0$ and $z<0$. The first case that should be examined in detail is the one in which $z>0$. Figure B.1 displays the queuing process of the finished goods inventory level at time t, $X(t)$.

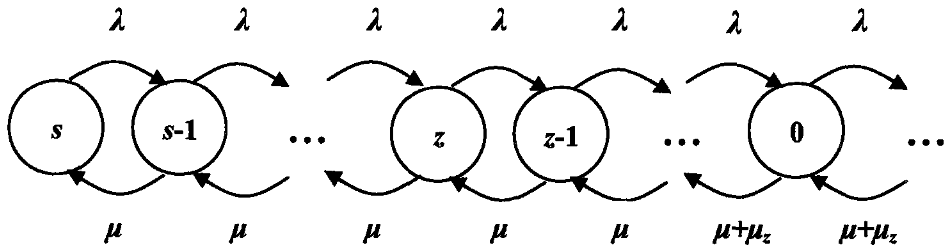

Figure B.1: $X(t)$ Process when $z>0$

Compared with the queueing model in which $X(t)$ is the underlying queuing process as described above, the model in which $N(t)$ is the underlying queuing process is easier to deal with. In fact, these two queueing models are equivalent. The only difference is $N(t)$ is the amount of shortfall with respect to s; i.e. $N(t)=s-X(t)$. Therefore, from now on, we will model the production system as an M/M/1 queueing system in which $N(t)$ is the underlying queuing process as displayed in Figure B.2.

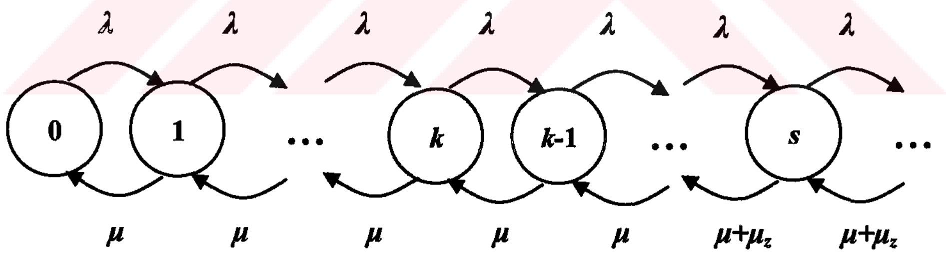

Figure B.2: $N(t)=s-X(t)$ Process when $z>0$

In order to have a stable system, the total processing capacity, which is the sum of the capacities of the in-house manufacturing facility and the subcontractor, should be sufficient enough to meet the order arrivals; i.e. the total utilization of the system, $\mu_f=\mu+\mu_z$, should be greater than or equal to the order arrival rate λ.

The other case that should be examined in detail is the one in which z<0. Figure B.3 below visualizes the $X(t)$ process:

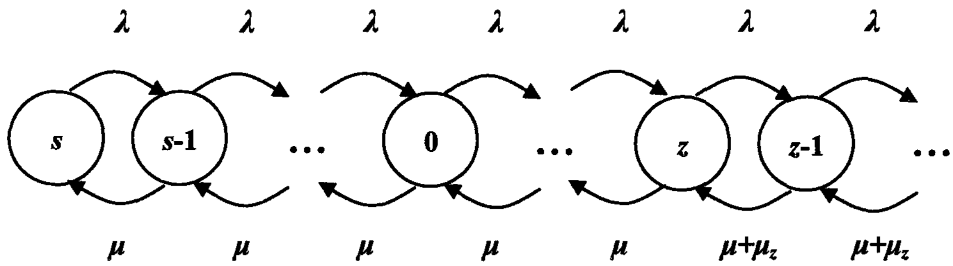

Figure B.3: $X(t)$ Process when z<0

And the queueing process that is easier to deal with can be constructed as shown in Figure B.4 below:

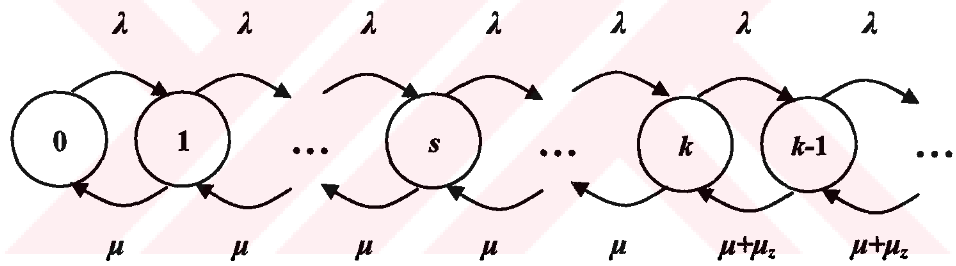

Figure B.4: $N(t)=s-X(t)$ Process when z<0

The steady state probabilities of the $N(t)$ process can be calculated in the following way:

$$P_0 = \begin{cases} \dfrac{(\lambda - \mu)(\lambda - \mu_t)\mu^k}{\mu^{k+1}(\mu_t - \lambda) - \lambda^{k+1}\mu_z}, & \text{if } \lambda \neq \mu \ \& \ \lambda < \mu_t \\ \dfrac{\mu_t - \lambda}{(k+2)\mu_t - (k+1)\lambda}, & \text{if } \lambda = \mu \ \& \ \lambda < \mu_t \end{cases}$$

$$P_i = (\frac{\lambda}{\mu})^i P_0, \ i = 1,..,k$$

$$P_{k+i} = (\frac{\lambda}{\mu})^k (\frac{\lambda}{\mu_t})^i P_0, \ i = 1,..,\infty$$

The objective of the model proposed is to minimize the long-run average operating costs of the manufacturing facility; which consist of the inventory holding cost, backlogging cost, subcontracting cost and in-house production cost. Let h be the inventory holding cost per item per period, b be the backlogging cost per item per period, c be the subcontracting cost per item per period and d be the in-house production cost per item per period. Assume that $E[I]$ denotes the expected inventory level, $E[B]$ denotes the expected number of backlogged items, $E[K]$ denotes the expected number of items subcontracted, and $E[H]$ denotes the expected number of items produced in the in-house facility. Then, the total expected cost, $E[TC]$, can be calculated as:

$$E[TC]=h.E[I]+b.E[B]+c.E[K]+d.E[H]$$

We first focus on the case $z>0$. Then, the probability distribution of the number of items in the inventory can be calculated as follows:

$$I(t) = \text{Max}\{s - N(t), 0\}$$

$$P\{I=0\} = P\{N \geq s\} = \sum_{i=s-k}^{\infty} (\frac{\lambda}{\mu})^k (\frac{\lambda}{\mu_t})^i P_0 = (\frac{\lambda}{\mu})^k \frac{(\frac{\lambda}{\mu_t})^{s-k}}{1-\frac{\lambda}{\mu_t}}$$

$$P\{I=n\} = P\{N = s-n\} = \begin{cases} (\frac{\lambda}{\mu})^{s-n} P_0, \text{ if n} \geq \text{z} \\ (\frac{\lambda}{\mu})^k (\frac{\lambda}{\mu_t})^{s-n-k} P_0, \text{ if n<z} \end{cases}$$

Moreover, we know that;

$$C(t) = \text{Min}\{N(t), s\}$$

$$P\{C=n\} = P\{N=n\} = \begin{cases} (\frac{\lambda}{\mu})^n P_0, \text{ if } n \leq s - z = k \\ (\frac{\lambda}{\mu})^k (\frac{\lambda}{\mu_t})^{n-k} P_0, \text{ if } n > s - z = k \end{cases}$$

$$P\{C=s\} = P\{N \geq s\} = (\frac{\lambda}{\mu})^k \frac{(\frac{\lambda}{\mu_t})^{s-k}}{1-\frac{\lambda}{\mu_t}} P_0$$

We now focus on the case when $\lambda \neq \mu$ & $\lambda < \mu_t$. Then, it seems easier to calculate $E[I]$, expected inventory level in the long run, from $E[C]$ since we know that $E[I]+E[C]=s$.

$$E[C] = \sum_{n=0}^{s} nP\{C=n\}$$

$$=\sum_{i=0}^{k-1} i(\frac{\lambda}{\mu})^i P_0 + \sum_{i=0}^{s-k-1}(i+k)(\frac{\lambda}{\mu})^k(\frac{\lambda}{\mu_t})^i P_0 + s(\frac{\lambda}{\mu})^k \frac{(\frac{\lambda}{\mu_t})^{s-k}}{1-\frac{\lambda}{\mu_t}} P_0$$

$$=\frac{\{\lambda^{k+1}[\mu_t^{s-k}-\lambda^{s-k}]+k\lambda^k\mu_t^{s-k}[\mu_t-\lambda]\}(\mu-\lambda)}{\mu_t^{s-k-1}(\mu_t-\lambda)\{\mu^{k+1}(\mu_t-\lambda)-\lambda^{k+1}\mu_z\}}+\frac{\{k\lambda^k(\lambda-\mu)+\lambda(\mu^k-\lambda^k)\}(\mu_t-\lambda)\mu}{(\mu-\lambda)\{\mu^{k+1}(\mu_t-\lambda)-\lambda^{k+1}\mu_z\}}$$

$$E[I]=s-E[C]$$

$$=s-\frac{\{\lambda^{k+1}[\mu_t^{s-k}-\lambda^{s-k}]+k\lambda^k\mu_t^{s-k}[\mu_t-\lambda]\}(\mu-\lambda)}{\mu_t^{s-k-1}(\mu_t-\lambda)\{\mu^{k+1}(\mu_t-\lambda)-\lambda^{k+1}\mu_z\}}-\frac{\{k\lambda^k(\lambda-\mu)+\lambda(\mu^k-\lambda^k)\}(\mu_t-\lambda)\mu}{(\mu-\lambda)\{\mu^{k+1}(\mu_t-\lambda)-\lambda^{k+1}\mu_z\}}$$

In order to calculate the expected number of backlogged items, we perform the following operations:

$$B(t)=\text{Max}\{N(t)-s,0\}$$

$$P\{B=0\}=P\{N\le s\}=\sum_{i=0}^{k}(\frac{\lambda}{\mu})^i P_0+\sum_{i=1}^{s-k}(\frac{\lambda}{\mu})^k(\frac{\lambda}{\mu_t})^{s-k}P_0=\left[\frac{1-(\frac{\lambda}{\mu})^{k+1}}{1-\frac{\lambda}{\mu}}+(\frac{\lambda}{\mu})^k\frac{\lambda}{\mu_t}\frac{1-(\frac{\lambda}{\mu_t})^{s-k}}{1-\frac{\lambda}{\mu_t}}\right]P_0$$

$$P\{B=n\}=P\{N=s+n\}=(\frac{\lambda}{\mu})^k(\frac{\lambda}{\mu_t})^{s+n-k}P_0$$

$$E[B]=\sum_{n=0}^{\infty}nP\{B=n\}=\frac{\lambda^{s+1}(\mu-\lambda)}{\mu_t^{s-k-1}(\mu_t-\lambda)[\mu^{k+1}(\mu_t-\lambda)-\lambda^{k+1}\mu_z]}$$

The expected number of items subcontracted and produced are calculated as follows:

$$E[K] = \mu_z \sum_{n=1}^{\infty} P\{N = k+n\} = \frac{\lambda^{k+1}\mu_z}{\mu^k(\mu_t - \lambda)} P_0$$

$$E[H] = \mu \left[\sum_{n=1}^{k} (\frac{\lambda}{\mu})^n P_0 + \sum_{n=1}^{\infty} (\frac{\lambda}{\mu})^k (\frac{\lambda}{\mu_t})^n P_0 \right] = \frac{\lambda(\mu^k - \lambda^k)(\mu_t - \lambda) + \lambda^{k+1}(\mu - \lambda)}{\mu^{k-1}(\mu - \lambda)(\mu_t - \lambda)}$$

Moreover, we know that the demand is satisfied either through in-house production or subcontracting. Therefore, $E[K]+E[H]=\lambda$ which is exactly the same in our case.

Assuming that k, the difference between base stock and threshold subcontracting levels, is fixed, we can write the total expected cost as a function of s, the base stock level:

$$\begin{aligned} E[TC(s)] = {} & h\left(s - \frac{\mu\left(k\lambda^k(\lambda-\mu) + \lambda(-\lambda^k + \mu^k)\right)(-\lambda+\mu_t)}{(-\lambda+\mu)\left(-\mu_z\lambda^{1+k} + \mu^{1+k}(-\lambda+\mu_t)\right)} \right) \\ & -h\frac{(-\lambda+\mu)\mu_t^{1+k-s}\left(k\lambda^k\mu_t^{-k+s}(-\lambda+\mu_t) + \lambda^{1+k}\left(-\lambda^{-k+s} + \mu_t^{-k+s}\right)\right)}{(-\lambda+\mu_t)\left(-\mu_z\lambda^{1+k} + \mu^{1+k}(-\lambda+\mu_t)\right)} \\ & +b\frac{\lambda^{1+s}(-\lambda+\mu)\mu_t^{1+k-s}}{(-\lambda+\mu_t)\left(-\mu_z\lambda^{1+k} + \mu^{1+k}(-\lambda+\mu_t)\right)} + c\frac{\mu_z\lambda^{1+k}(\lambda-\mu)(\lambda-\mu_t)}{(-\lambda+\mu_t)\left(-\mu_z\lambda^{1+k} + \mu^{1+k}(-\lambda+\mu_t)\right)} \\ & +d\frac{(\lambda-\mu)\mu(\lambda-\mu_t)\left(\lambda^{1+k}(-\lambda+\mu) + \lambda\left(-\lambda^k + \mu^k\right)(-\lambda+\mu_t)\right)}{(-\lambda+\mu)(-\lambda+\mu_t)\left(-\mu_z\lambda^{1+k} + \mu^{1+k}(-\lambda+\mu_t)\right)} \end{aligned}$$

It can be observed that the subcontracting and in-house production costs turn out to be constant terms since k is assumed to be fixed. Note that, the second term of the inventory holding cost turns out to be constant as k is fixed, too. Therefore, these three terms can be ignored while performing minimization and the optimization calculations are focused on the portion of the total expected cost function which is variable in s.

Lemma B.1: $E[TC(s)]$ is convex in s.

Proof:

$$\begin{aligned}\Delta E[TC(s)] &= E[TC(s+1)] - E[TC(s)] \\ &= h - \frac{(b+h)\lambda^{s+1}(\lambda-\mu)\mu_t^{k-s}}{\mu_z\lambda^{k+1}+\mu^{k+1}(\lambda-\mu_t)} \\ &= h-(h+b)\frac{\lambda\mu_t^k(\lambda-\mu)}{\mu_z\lambda^{k+1}+\mu^{k+1}(\lambda-\mu_t)}\left(\frac{\lambda}{\mu_t}\right)^s\end{aligned}$$

It is clear that as s increases, the difference function $\Delta E[TC(s)]$ increases. (Note that $\frac{\lambda}{\mu_t}<1$). Therefore, $\Delta E[TC(s)]$ is non-decreasing in s and it can be concluded that $\Delta E[TC(s)]$ is convex in s.

Proposition B.1: The global optimizer s^* that minimizes the total cost equals:

$$s^*(k) = \left\lfloor \frac{Log\left[\frac{h\{\mu_z\lambda^{k+1}+\mu^{k+1}(\lambda-\mu_t)\}}{(h+b)\lambda\mu_t^k(\lambda-\mu)}\right]}{Log\left(\frac{\lambda}{\mu_t}\right)} \right\rfloor$$

where $s^*(k)$ is the optimal s that minimizes the total cost when k is constant.

Proof: Using Lemma B.1, we calculate the root of the difference equation and obtain the above mentioned global optimizer in the following way:

$$\Delta E[TC(s)] = h-(h+b)\frac{\lambda\mu_t^k(\lambda-\mu)}{\mu_z\lambda^{k+1}+\mu^{k+1}(\lambda-\mu_t)}\left(\frac{\lambda}{\mu_t}\right)^s = 0$$

Having calculated the optimal s, the next step in our study is to calculate the optimal k that minimizes the total expected cost. Therefore, $E[TC(s^*(k)]$ is constructed such that:

$$E\left[TC\left(s^*(k)\right)\right]=h\frac{Log\left[\frac{h}{b+h}\right]+Log\left[\frac{\mu_z\lambda^{1+k}+\mu^{1+k}(\lambda-\mu_t)\mu_t^{-k}}{\lambda(\lambda-\mu)}\right]}{Log\left[\frac{\lambda}{\mu_t}\right]}$$

$$+h\frac{\mu\left(-\lambda\mu^k+\lambda^k(\lambda-k\lambda+k\mu)\right)(\lambda-\mu_t)}{(-\lambda+\mu)\left(\mu_z\lambda^{1+k}+\mu^{1+k}(\lambda-\mu_t)\right)}$$

$$+h\frac{-\lambda^{\frac{Log\left[\frac{h}{b+h}\right]+Log\left[\frac{\lambda}{\mu_t}\right]+Log\left[\frac{\mu_z\lambda^{1+k}+\mu^{1+k}(\lambda-\mu_t)\mu_t^{-k}}{\lambda(\lambda-\mu)}\right]}{Log\left[\frac{\lambda}{\mu_t}\right]}}\mu_t^{\frac{Log\left[\frac{h}{b+h}\right]-kLog\left[\frac{\lambda}{\mu_t}\right]+Log\left[\frac{\mu_z\lambda^{1+k}+\mu^{1+k}(\lambda-\mu_t)\mu_t^{-k}}{\lambda(\lambda-\mu)}\right]}{Log\left[\frac{\lambda}{\mu_t}\right]}}(\lambda-\mu)\mu_t}{\left(\mu_z\lambda^{1+k}+\mu^{1+k}(\lambda-\mu_t)\right)(\lambda-\mu_t)}$$

$$+h\frac{(\lambda-\mu)\mu_t\lambda^k(\lambda-k\lambda+k\mu_t)}{\left(\mu_z\lambda^{1+k}+\mu^{1+k}(\lambda-\mu_t)\right)(\lambda-\mu_t)}$$

$$+b\frac{\lambda^{1+\frac{Log\left[\frac{h}{b+h}\right]+Log\left[\frac{\mu_z\lambda^{1+k}+\mu^{1+k}(\lambda-\mu_t)\mu_t^{-k}}{\lambda(\lambda-\mu)}\right]}{Log\left[\frac{\lambda}{\mu_t}\right]}}(-\lambda+\mu)\mu_t^{1+k\frac{Log\left[\frac{h}{b+h}\right]+Log\left[\frac{\mu_z\lambda^{1+k}+\mu^{1+k}(\lambda-\mu_t)\mu_t^{-k}}{\lambda(\lambda-\mu)}\right]}{Log\left[\frac{\lambda}{\mu_t}\right]}}}{\left(\mu_z\lambda^{1+k}+\mu^{1+k}(\lambda-\mu_t)\right)(\lambda-\mu_t)}$$

$$+c\frac{\mu_z\lambda^{1+k}(\lambda-\mu)}{\mu_z\lambda^{1+k}+\mu^{1+k}(\lambda-\mu_t)}+d\frac{\lambda\mu\left(\mu^k(\lambda-\mu_t)+\lambda^k(-\mu+\mu_t)\right)}{\mu_z\lambda^{1+k}+\mu^{1+k}(\lambda-\mu_t)}$$

However, we cannot say anything about the convexity of the above function due to the complexity of the expressions. Therefore, a closed form for the optimal k that minimizes the total expected cost function cannot be derived. Still we can find the optimal k by performing numerical analysis and calculate the optimal threshold subcontracting value $z^*(k^*)$ by just subtracting the k^* value from $s^*(k^*)$, i.e. $z^*(k^*)=s^*(k^*)-k^*=s^*(k^*)-s^*(k^*)+z^*(k^*)$. For instance, if we take $\lambda=9$, $\mu=10$, $\mu_z=10$, $h=1$, $b=50$, $c=5$, $d=2$, the optimal $s^*(k)$ values for the given constant k values and the resulting total expected cost values can be observed in Table B.1:

Table B.1: $s^{*}(k)$ and $E[TC(s^{*}(k))]$ values observed for the given constant k values

k	$s^{*}(k)$	$E[TC(s^{*}(k))]$
0	3	37.4842
1	4	31.6109
3	5	28.3309
5	6	28.0775
7	8	27.2811
8	9	27.2377
9	10	27.3493
11	11	28.6676
13	13	29.1135
15	15	29.8933
17	16	31.7035

When we plot the $TC(s(k))$ for each k and for each s, we observe the below figure:

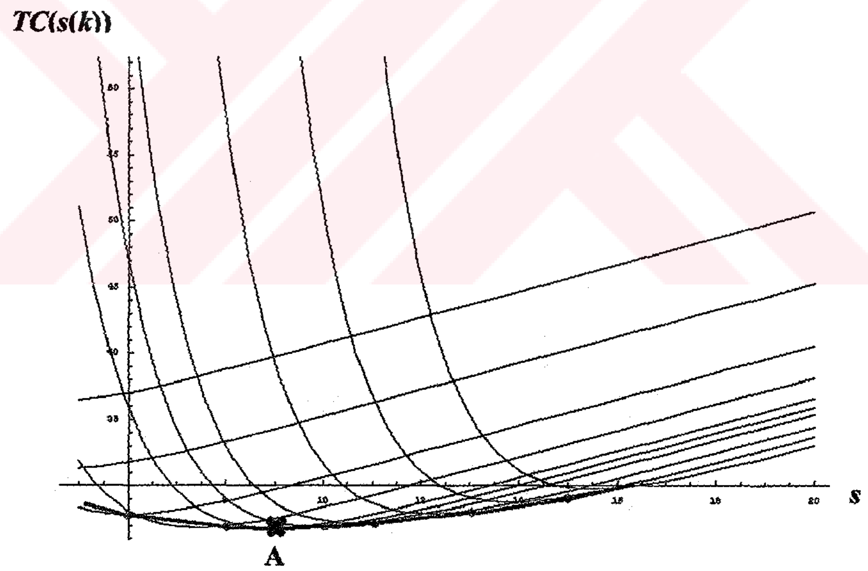

Figure B.5: $TC(s(k))$ vs. s drawn for each k displayed in Table B.1

As can be seen in Figure B.5, the total expected cost function displays a convex function structure in k. The optimal s and k values that minimize the total cost can be found by enveloping the minimum values of the total cost functions plotted for each k. Then the optimal z value can be calculated accordingly. For our specific numerical example, point A where $s^*=9$, $k^*=8$, $z^*=s^*-k^*=1$ results in the minimum total cost value of 27.2377.

Note that, all of the above calculations and observations are performed for the case in which z>0 and $\lambda \neq \mu$ & $\lambda < \mu_t$. The calculations should be repeated for the case in which $z>0$ and $\lambda = \mu$ & $\lambda < \mu_t$, and then for the case in which $z<0$.

Independently from our studies, Bradley [22] has looked at a similar problem motivated by the optimal control. Bradley's model also uses in-house capacity costs and arrives at similar results to ours.

Appendix C

PROOF OF PROPOSITION 5.1

We utilize the Simplex Method in order to find the optimal solution to the Single Period Production Planning Problem, in which two products are to be produced either by a cheaper or by an expensive plant or by both of the two plants. Note that we assume $p_{1,1}>p_{1,2}$, $p_{2,1}>p_{2,2}$, $l_1>l_2$, $(C_1+C_2)-(l_1+l_2)\geq 0$ at the very beginning. First, we formulate the problem as a minimization problem as follows:

$$Z^{**} = \text{Max}\left(-(p_{1,1}X_{1,1} + p_{1,2}X_{1,2} + p_{2,1}X_{2,1} + p_{2,2}X_{2,2})\right)$$

subject to

$$X_{1,1} + X_{1,2} \geq l_1;$$
$$X_{2,1} + X_{2,2} \geq l_2;$$
$$X_{1,1} + X_{2,1} \leq C_1;$$
$$X_{1,2} + X_{2,2} \leq C_2;$$
$$X_{i,j} \geq 0;\ i=1,2;\ j=1,2.$$

In order to apply the simplex method, we need to transform the above maximization problem into the standard form. Let S_m denote the slack variable for constraint m=1,..,4. Then, the problem takes the form:

$$Z^{**} = \text{Max}\left(-(p_{1,1}X_{1,1} + p_{1,2}X_{1,2} + p_{2,1}X_{2,1} + p_{2,2}X_{2,2})\right)$$

subject to

$$-X_{1,1} - X_{1,2} + s_1 = -l_1;$$
$$-X_{2,1} - X_{2,2} + s_2 = -l_2;$$

$$X_{1,1}+X_{2,1}+s_3=C_1;$$
$$X_{1,2}+X_{2,2}+s_4=C_2;$$
$$X_{i,j}\geq 0;\ i=1,2;\ j=1,2.$$

The problem can be represented in the initial simplex tableau as:

Table C.1: Initial Tableau

	Z^{**}	$X_{1,1}$	$X_{1,2}$	$X_{2,1}$	$X_{2,2}$	S_1	S_2	S_3	S_4	RHS
Z^{**}	1	$p_{1,1}$	$p_{1,2}$	$p_{2,1}$	$p_{2,2}$	0	0	0	0	0
S_1	0	-1	-1	0	0	1	0	0	0	$-l_1<0$
S_2	0	0	0	-1	-1	0	1	0	0	$-l_2<0$
S_3	0	1	0	1	0	0	0	1	0	$C_1>0$
S_4	0	0	1	0	1	0	0	0	1	$C_2>0$

In many situations, it is easier to solve a linear program by beginning with a simplex tableau in which each variable in row zero has a non-negative coefficient (dual feasible) and at least one constraint has a negative right-hand-side (RHS) (primal infeasible). Starting with our initial tableau which corresponds to a dual feasible and primal infeasible solution, we utilize the Dual Simplex Method in order to find the optimal solution.

Since $l_1>l_2$, we know that $-l_1<-l_2$. Therefore, S_1 should leave the basis displayed in Table 5.2. By applying the Minimum Ratio Test, Min $\{p_{1,1}\ ,\ p_{1,2}\}=p_{1,2}$ (since plant 2 is the cheaper plant), we see that $X_{1,2}$ should leave the basis. Then, the tableau takes the following form:

Table C.2: S_1 leaves, $X_{1,2}$ enters in Table C.1

	Z^{**}	$X_{1,1}$	$X_{1,2}$	$X_{2,1}$	$X_{2,2}$	S_1	S_2	S_3	S_4	RHS
Z^{**}	1	$p_{1,1}-p_{1,2}$	0	$p_{2,1}$	$p_{2,2}$	$p_{1,2}$	0	0	0	$-p_{1,2}.l_1$
$X_{1,2}$	0	1	1	0	0	-1	0	0	0	$l_1>0$
S_2	0	0	0	-1	-1	0	1	0	0	$-l_2<0$
S_3	0	1	0	1	0	0	0	1	0	$C_1>0$
S_4	0	-1	0	0	1	1	0	0	1	C_2-l_1?

Depending on the signs and the relations of the right-hand-side values of the basic variables, different variables might leave and enter the basis. There are three different possibilities:

<u>Case 1:</u> **If C_2-l_1≥0;** S_2 should leave and $X_{2,2}$ should enter the basis and the resulting tableau can be found below:

Table C.3: S_2 leaves, $X_{2,2}$ enters in Table C.2

	Z^{**}	$X_{1,1}$	$X_{1,2}$	$X_{2,1}$	$X_{2,2}$	S_1	S_2	S_3	S_4	RHS
Z^{**}	1	$p_{1,1}-p_{1,2}$	0	$p_{2,1}-p_{2,2}$	0	$p_{1,2}$	$p_{2,2}$	0	0	$-p_{1,2}.l_1-p_{2,2}.l_2$
$X_{1,2}$	0	1	1	0	0	-1	0	0	0	$l_1>0$
$X_{2,2}$	0	0	0	1	1	0	-1	0	0	$l_2>0$
S_3	0	1	0	1	0	0	0	1	0	$C_1>0$
S_4	0	-1	0	-1	0	1	1	0	1	$C_2-(l_1+l_2)$?

We still cannot be sure whether we have reached the optimal tableau. There are three different possibilities for Case 1.

<u>Case 1.1:</u> **If $C_2-l_1\geq 0$ and $C_2-(l_1+l_2)\geq 0$;** since each variable in row zero has a non-negative coefficient and each constraint has a positive right-hand-side value, we have reached a dual and a primal feasible solution. Therefore, we can say that optimal solution is found with an objective function value of $Z^{*}=-Z^{**}=p_{1,2}.l_1+p_{2,2}.l_2$ and the production amounts of $X_{1,2}=l_1$ and $X_{2,2}=l_2$.

<u>Case 1.2:</u> **If $C_2-l_1\geq 0$, $C_2-(l_1+l_2)<0$ and $p_{1,1}-p_{1,2}\leq p_{2,1}-p_{2,2}$;** S_4 should leave and $X_{1,1}$ should enter the basis resulting in the following tableau:

Table C.4: S_4 leaves, $X_{1,1}$ enters in Table C.3

	Z^{**}	$X_{1,1}$	$X_{1,2}$	$X_{2,1}$	$X_{2,2}$	S_1	S_2	S_3	S_4	RHS
Z^{**}	1	0	0	$(p_{2,1}-p_{2,2})$ $-(p_{1,1}-p_{1,2})$	0	$p_{1,1}$	$(p_{1,1}-p_{1,2})$ $+p_{2,2}$	0	$p_{1,1}-p_{1,2}$	$-p_{1,2}.l_1-p_{2,2}.l_2$ $+(p_{1,1}-p_{1,2}).[C_2-(l_1+l_2)]$
$X_{1,2}$	0	0	1	-1	0	0	1	0	1	$C_2-l_2>0$
$X_{2,2}$	0	0	0	1	1	0	-1	0	0	$l_2>0$
S_3	0	0	0	0	0	1	1	1	1	$(C_1+C_2)-(l_1+l_2)\geq 0$
$X_{1,1}$	0	1	0	1	0	-1	-1	0	-1	$(l_1+l_2)-C_2>0$

We know that $C_2-l_1\geq 0$, therefore $C_2-l_2\geq 0$ since $l_1>l_2$. Since we have reached dual and primal feasibility, we can say that, we come up with an optimal solution with $Z^*=p_{1,2}.l_1+p_{2,2}.l_2-(p_{1,1}-p_{1,2}).[C_2-(l_1+l_2)]$; $X_{1,1}=(l_1+l_2)-C_2$; $X_{1,2}=C_2-l_2$ and $X_{2,2}=l_2$.

Case 1.3: **If $C_2-l_1\geq 0$, $C_2-(l_1+l_2)<0$ and $p_{1,1}-p_{1,2}\geq p_{2,1}-p_{2,2}$;** S_4 should leave and $X_{2,1}$ should enter the basis resulting in the following tableau:

Table C.5: S_4 leaves, $X_{2,1}$ enters in Table C.3

	Z^{**}	$X_{1,1}$	$X_{1,2}$	$X_{2,1}$	$X_{2,2}$	S_1	S_2	S_3	S_4	RHS
Z^{**}	1	$(p_{1,1}-p_{1,2})$ $-(p_{2,1}-p_{2,2})$	0	0	0	$(p_{2,1}-p_{2,2})$ $+p_{1,2}$	$p_{2,1}$	0	$p_{2,1}-p_{2,2}$	$-p_{1,2}.l_1-p_{2,2}.l_2$ $+(p_{2,1}-p_{2,2}).[C_2-(l_1+l_2)]$
$X_{1,2}$	0	1	1	0	0	-1	0	0	0	$l_1>0$
$X_{2,2}$	0	-1	0	0	1	1	0	0	1	$C_2-l_1\geq 0$
S_3	0	0	0	0	0	1	1	1	1	$(C_1+C_2)-(l_1+l_2)\geq 0$
$X_{2,1}$	0	1	0	1	0	-1	-1	0	-1	$(l_1+l_2)-C_2\geq 0$

Since we have reached dual and primal feasibility, we can say that the optimal solution to the problem is $Z^*=p_{1,2}.l_1+p_{2,2}.l_2-(p_{2,1}-p_{2,2}).[C_2-(l_1+l_2)]$; $X_{1,2}=l_1$; $X_{2,1}=(l_1+l_2)-C_2$ and $X_{2,2}=C_2-l_1$.

Case 2: **If $C_2-l_1<0$ and $C_2-l_1\leq -l_2$;** S_4 should leave and $X_{1,1}$ should enter the basis and the resulting tableau is:

Table C.6: S_4 leaves, $X_{1,1}$ enters in Table C.2

	Z^{**}	$X_{1,1}$	$X_{1,2}$	$X_{2,1}$	$X_{2,2}$	S_1	S_2	S_3	S_4	RHS
Z^{**}	1	0	0	$p_{2,1}$	$(p_{1,1}-p_{1,2})+p_{2,2}$	$p_{1,1}$	0	0	$p_{1,1}-p_{1,2}$	$-p_{1,2}.l_1+(p_{1,1}-p_{1,2}).(C_2-l_1)$
$X_{1,2}$	0	0	1	0	1	0	0	0	1	$C_2>0$
S_2	0	0	0	-1	-1	0	1	0	0	$-l_2<0$
S_3	0	0	0	1	1	1	0	1	1	$(C_1+C_2)-l_1$?
$X_{1,1}$	0	1	0	0	-1	-1	0	0	-1	$l_1-C_2\geq 0$

We have mentioned that $(C_1+C_2)-(l_1+l_2)\geq 0$ not to observe any infeasibilities; which also means that $(C_1+C_2)-l_1\geq 0$. Then, S_2 should leave the basis and either $X_{2,1}$ or $X_{2,2}$ should enter the basis.

<u>**Case 2.1:**</u> **If $C_2-l_1<0$, $C_2-l_1\leq -l_2$ and $p_{1,1}-p_{1,2}\geq p_{2,1}-p_{2,2}$;** S_2 should leave and $X_{2,1}$ should enter the basis resulting in the following tableau:

Table C.7: S_2 leaves, $X_{2,1}$ enters in Table C.6

	Z^{**}	$X_{1,1}$	$X_{1,2}$	$X_{2,1}$	$X_{2,2}$	S_1	S_2	S_3	S_4	RHS
Z^{**}	1	0	0	0	$(p_{1,1}-p_{1,2})-(p_{2,1}-p_{2,2})$	$p_{1,1}$	$p_{2,1}$	0	$p_{1,1}-p_{1,2}$	$-p_{1,2}.l_1-p_{2,1}.l_2+(p_{1,1}-p_{1,2}).(C_2-l_1)$
$X_{1,2}$	0	0	1	0	1	0	0	0	1	$C_2>0$
$X_{2,1}$	0	0	0	1	1	0	-1	0	0	$l_2>0$
S_3	0	0	0	0	0	1	1	1	1	$(C_1+C_2)-(l_1+l_2)\geq 0$
$X_{1,1}$	0	1	0	0	-1	-1	0	0	-1	$l_1-C_2>0$

Since each variable in row zero has a non-negative coefficient and each constraint has a positive right-hand-side value, we have reached a dual and primal feasible solution. Therefore, we can say that optimal solution is found with $Z^{*}=p_{1,2}.l_1+p_{2,1}.l_2-(p_{1,1}-p_{1,2}).(C_2-l_1)$; $X_{1,1}=l_1-C_2$; $X_{1,2}=C_2$ and $X_{2,1}=l_2$.

<u>**Case 2.2:**</u> **If $C_2-l_1<0$, $C_2-l_1\leq -l_2$ and $p_{1,1}-p_{1,2}\leq p_{2,1}-p_{2,2}$;** S_2 should leave and $X_{2,2}$ should enter the basis resulting in the following tableau:

Table C.8: S_2 leaves, $X_{2,2}$ enters in Table C.6

	Z^{**}	$X_{1,1}$	$X_{1,2}$	$X_{2,1}$	$X_{2,2}$	S_1	S_2	S_3	S_4	RHS
Z^{**}	1	0	0	$(p_{2,1}-p_{2,2})$ $-(p_{1,1}-p_{1,2})$	0	$p_{1,1}$	$(p_{1,1}-p_{1,2})$ $+p_{2,2}$	0	$p_{1,1}-p_{1,2}$	$-p_{1,2}.l_1$ $+(p_{1,1}-p_{1,2}).(C_2-l_1)$ $-[(p_{1,1}-p_{1,2})+p_{2,2}].l_2$
$X_{1,2}$	0	0	1	-1	0	0	1	0	1	C_2-l_2?
$X_{2,2}$	0	0	0	1	1	0	-1	0	0	$l_2>0$
S_3	0	0	0	0	0	1	1	1	1	$(C_1+C_2)-(l_1+l_2)\geq 0$
$X_{1,1}$	0	1	0	1	0	-1	-1	0	-1	$(l_1+l_2)-C_2$?

Case 2.2.1: **If $C_2-l_1<0$, $C_2-l_1\leq -l_2$, $p_{1,1}-p_{1,2}\leq p_{2,1}-p_{2,2}$ and $C_2-l_2\geq 0$;** since $C_2-l_1<0$, we know that $l_1-C_2>0$, then $(l_1+l_2)-C_2>0$. Since we have reached dual and primal feasibility, we can say that optimal solution is found with $Z^*=p_{1,2}.l_1-(p_{1,1}-p_{1,2}).(C_2-l_1)+[(p_{1,1}-p_{1,2})+p_{2,2}].l_2$; $X_{1,1}=(l_1+l_2)-C_2$; $X_{1,2}=C_2-l_2$ and $X_{2,2}=l_2$.

Case 2.2.2: **If $C_2-l_1<0$, $C_2-l_1\leq -l_2$, $p_{1,1}-p_{1,2}\leq p_{2,1}-p_{2,2}$ and $C_2-l_2<0$;** if $C_2-l_2<0$, we know that $l_2-C_2>0$ and then $(l_1+l_2)-C_2>0$. Therefore, $X_{1,2}$ should leave the basis and $X_{2,1}$ should enter the basis which results in the tableau below:

Table C.9: $X_{1,2}$ leaves, $X_{2,1}$ enters in Table C.8

	Z^{**}	$X_{1,1}$	$X_{1,2}$	$X_{2,1}$	$X_{2,2}$	S_1	S_2	S_3	S_4	RHS
Z^{**}	1	0	$(p_{2,1}-p_{2,2})$ $-(p_{1,1}-p_{1,2})$	0	0	$p_{1,1}$	$p_{2,1}$	0	$p_{2,1}-p_{2,2}$	$-p_{1,2}.l_1+(p_{1,1}-p_{1,2}).(C_2-l_1)$ $-[(p_{1,1}-p_{1,2})+p_{2,2}].l_2$ $+[(p_{2,1}-p_{2,2})-(p_{1,1}-p_{1,2})].(C_2-l_2)$
$X_{2,1}$	0	0	-1	1	0	0	-1	0	-1	$l_2-C_2>0$
$X_{2,2}$	0	0	1	0	1	0	0	0	1	$C_2>0$
S_3	0	0	0	0	0	1	1	1	1	$(C_1+C_2)-(l_1+l_2)\geq 0$
$X_{1,1}$	0	1	1	0	0	-1	0	0	0	$l_1>0$

Since we have reached dual and primal feasibility, we can say that optimal solution is found with $Z^*=p_{1,2}.l_1-(p_{1,1}-p_{1,2}).(C_2-l_1)+[(p_{1,1}-p_{1,2})+p_{2,2}].l_2-[(p_{2,1}-p_{2,2})-(p_{1,1}-p_{1,2})].(C_2-l_2)$; $X_{1,1}=l_1$; $X_{2,1}=l_2-C_2$ and $X_{2,2}=C_2$.

<u>Case 3:</u> If C_2-l_1<0 and C_2-l_1≥-l_2; S_2 should leave and $X_{2,2}$ should enter the basis and results in the below tableau:

Table C.10: S_2 leaves, $X_{2,2}$ enters in Table C.2

	Z^{**}	$X_{1,1}$	$X_{1,2}$	$X_{2,1}$	$X_{2,2}$	S_1	S_2	S_3	S_4	RHS
Z^{**}	1	$p_{1,1}-p_{1,2}$	0	$p_{2,1}-p_{2,2}$	0	$p_{1,2}$	$P_{2,2}$	0	0	$-p_{1,2}.l_1-p_{2,2}.l_2$
$X_{1,2}$	0	1	1	0	0	-1	0	0	0	$l_1>0$
$X_{2,2}$	0	0	0	1	1	0	-1	0	0	$l_2>0$
S_3	0	1	0	1	0	0	0	1	0	$C_1>0$
S_4	0	-1	0	-1	0	1	1	0	1	$C_2-(l_1+l_2)<0$

In this case, S_4 should leave and either $X_{1,1}$ or $X_{2,1}$ should enter the basis according to the relationship between $p_{1,1}$-$p_{1,2}$ and $p_{2,1}$-$p_{2,2}$ values.

<u>Case 3.1:</u> If C_2-l_1<0, C_2-l_1≥-l_2 and $p_{1,1}$-$p_{1,2}$≤$p_{2,1}$-$p_{2,2}$; S_4 should leave and either $X_{1,1}$ should enter the basis resulting in the following tableau:

Table C.11: S_4 leaves, $X_{1,1}$ enters in Table C.10

	Z^{**}	$X_{1,1}$	$X_{1,2}$	$X_{2,1}$	$X_{2,2}$	S_1	S_2	S_3	S_4	RHS
Z^{**}	1	0	0	$(p_{2,1}-p_{2,2})$ $-(p_{1,1}-p_{1,2})$	0	$p_{1,1}$	$(p_{1,1}-p_{1,2})$ $+p_{2,2}$	0	$p_{1,1}-p_{1,2}$	$-p_{1,2}.l_1-p_{2,2}.l_2$ $+(p_{1,1}-p_{1,2}).[C_2-(l_1+l_2)]$
$X_{1,2}$	0	0	1	-1	0	0	1	0	1	C_2-l_2?
$X_{2,2}$	0	0	0	1	1	0	-1	0	0	$l_2>0$
S_3	0	0	0	0	0	1	1	1	1	$(C_1+C_2)-(l_1+l_2)\geq 0$
$X_{1,1}$	0	1	0	1	0	-1	-1	0	-1	$(l_1+l_2)-C_2>0$

Since C_2-l_1<0, we know that l_1-C_2>0 and then we can say that (l_1+l_2)-C_2>0. Therefore, the optimal solution depends on the sign of C_2-l_2.

<u>Case 3.1.1:</u> If C_2-l_1<0, C_2-l_1≥-l_2, $p_{1,1}$-$p_{1,2}$≤$p_{2,1}$-$p_{2,2}$ and C_2-l_2≥0; since each variable in row zero has a non-negative coefficient and each constraint has a positive right-hand-side value, we have reached a dual and primal feasible solution. Therefore, we have found the optimal solution with $Z^{*}=p_{1,2}.l_1+p_{2,2}.l_2-(p_{1,1}-p_{1,2}).[C_2-(l_1+l_2)]$; $X_{1,1}=(l_1+l_2)-C_2$; $X_{1,2}=C_2-l_2$ and $X_{2,2}=l_2$.

<u>**Case 3.1.2:**</u> **If $C_2-l_1<0$, $C_2-l_1\geq-l_2$, $p_{1,1}-p_{1,2}\leq p_{2,1}-p_{2,2}$ and $C_2-l_2<0$;** $X_{1,2}$ should leave and $X_{2,1}$ should enter the basis resulting in the following tableau:

Table C.12: $X_{1,2}$ leaves, $X_{2,1}$ enters in Table C.11

	Z^{**}	$X_{1,1}$	$X_{1,2}$	$X_{2,1}$	$X_{2,2}$	S_1	S_2	S_3	S_4	RHS
Z^{**}	1	0	$(p_{2,1}-p_{2,2})$ $-(p_{1,1}-p_{1,2})$	0	0	$p_{1,1}$	$p_{2,1}$	0	$p_{2,1}-p_{2,2}$	$-p_{1,2}.l_1-p_{2,2}*l_2$ $+(p_{1,1}-p_{1,2}).[C_2-(l_1+l_2)]$ $+[(p_{2,1}-p_{2,2})-(p_{1,1}-p_{1,2})].(C_2-l_2)$
$X_{2,1}$	0	0	-1	1	0	0	-1	0	-1	$l_2-C_2>0$
$X_{2,2}$	0	0	1	0	1	0	0	0	1	$C_2>0$
S_3	0	0	0	0	0	1	1	1	1	$(C_1+C_2)-(l_1+l_2)\geq 0$
$X_{1,1}$	0	1	1	0	0	-1	0	0	0	$l_1>0$

Since we have reached dual and primal feasibility, we can say that optimal solution is found with $Z^{*}=p_{1,2}.l_1+p_{2,2}.l_2-(p_{1,1}-p_{1,2}).[C_2-(l_1+l_2)]-[(p_{2,1}-p_{2,2})-(p_{1,1}-p_{1,2})].(C_2-l_2)$; $X_{1,1}=l_1$; $X_{2,1}=l_2-C_2$ and $X_{2,2}=C_2$.

<u>**Case 3.2:**</u> **If $C_2-l_1<0$, $C_2-l_1\geq-l_2$ and $p_{1,1}-p_{1,2}\geq p_{2,1}-p_{2,2}$;** S_4 should leave and either $X_{2,1}$ should enter the basis resulting in the following tableau:

Table C.13: S_4 leaves, $X_{2,1}$ enters in Table C.10

	Z^{**}	$X_{1,1}$	$X_{1,2}$	$X_{2,1}$	$X_{2,2}$	S_1	S_2	S_3	$S_{\dot{y}}$	RHS
Z^{**}	1	$(p_{1,1}-p_{1,2})$ $-(p_{2,1}-p_{2,2})$	0	0	0	$(p_{2,1}-p_{2,2})$ $+p_{1,2}$	$p_{2,1}$	0	$p_{2,1}-p_{2,2}$	$-p_{1,2}.l_1-p_{2,2}.l_2$ $+(p_{2,1}-p_{2,2}).[C_2-(l_1+l_2)]$
$X_{1,2}$	0	1	1	0	0	-1	0	0	0	$L_1>0$
$X_{2,2}$	0	-1	0	0	1	1	0	0	1	$C_2-l_1<0$
S_3	0	0	0	0	0	1	1	1	1	$(C_1+C_2)-(l_1+l_2)\geq 0$
$X_{2,1}$	0	1	0	1	0	-1	-1	0	-1	$(l_1+l_2)-C_2>0$

Since C_2-l_1<0, we know that l_1-C_2>0 and then we can say that (l_1+l_2)-C_2>0. Therefore, $X_{2,2}$ should leave and $X_{1,1}$ should enter the basis.

Table C.14: $X_{2,2}$ leaves, $X_{1,1}$ enters in Table C.10

	Z^{**}	$X_{1,1}$	$X_{1,2}$	$X_{2,1}$	$X_{2,2}$	S_1	S_2	S_3	S_4	RHS
Z^{**}	1	0	0	0	$(p_{1,1}-p_{1,2})$ $-(p_{2,1}-p_{2,2})$	$p_{1,1}$	$p_{2,1}$	0	$p_{1,1}-p_{1,2}$	$-p_{1,2}.l_1-p_{2,2}.l_2$ $+[(p_{1,1}-p_{1,2})-(p_{2,1}-p_{2,2})].(C_2-l_1)$ $+(p_{2,1}-p_{2,2}).[C_2-(l_1+l_2)]$
$X_{1,2}$	0	0	1	0	1	0	0	0	1	C_2>0
$X_{1,1}$	0	1	0	0	-1	-1	0	0	-1	l_1-C_2>0
S_3	0	0	0	0	0	1	1	1	1	$(C_1+C_2)-(l_1+l_2)\geq 0$
$X_{2,1}$	0	0	0	1	1	0	-1	0	0	l_2>0

Since we have reached dual and primal feasibility, we can say that optimal solution is found with $Z^{*}=p_{1,2}.l_1+p_{2,2}.l_2-[(p_{1,1}-p_{1,2})-(p_{2,1}-p_{2,2})].(C_2-l_1)-(p_{2,1}-p_{2,2}).[C_2-(l_1+l_2)]$; $X_{1,1}=l_1-C_2$; $X_{1,2}=C_2$ and $X_{2,1}=l_2$.

Note that some cases are sequenced under more than one case instance. For instance; the case in which C_2-$l_1\geq 0$, $C_2-(l_1+l_2)<0$ and $p_{1,1}-p_{1,2}=p_{2,1}-p_{2,2}$ can be classified either in Case 1.2 or Case 1.3. That is why alternative optimal solutions might be observed.

Different optimal solutions can be observed based on different combinations of $p_{i,j}$, C_j and l_i values. Which parameter condition results in which optimal solution is presented in the above iterations. ■

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