# MULTIPERIOD PORTFOLIO OPTIMIZATION IN STOCHASTIC MARKETS USING THE MEAN-VARIANCE APPROACH

by

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This is to certify that I have examined this copy of a master's thesis by

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and have found that it is complete and satisfactory in all respects,
and that any and all revisions required by the final
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## ABSTRACT

In this thesis, several multiperiod portfolio optimization problems are considered where the investors try to achieve their goal at the end of the investment horizon. The market consists of a riskless asset and several risky assets whose returns have a mean vector and a covariance matrix both of which depend on the prevailing market conditions described by a Markov chain. The returns are therefore serially correlated with each other via this stochastic market. Various objectives including the safety-first approach, the coefficient of variation and the quadratic utility function are considered. The common feature of the objective functions analyzed in this thesis is that they are formulated using the mean and the variance of the final wealth at the end of the investment horizon, which also corresponds to the final portfolio return if the initial wealth is taken to be one. An auxiliary problem is formulated and solved to find the optimal portfolios and generate the efficient frontier. Optimal portfolio management policies and the implied optimal mean and variance of the final wealth are found analytically for each problem. Illustrative cases are given to demonstrate the solution procedure, and the optimal policies are interpreted for each problem.

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# NOMENCLATURE

| | |
|---|---|
| $E$ | State space of the stochastic market |
| $E_i[\ ]$ | Expectation given that the initial market state is $i$ |
| $P$ | Probability |
| $Q$ | Transition matrix of the stochastic market process |
| $R(i)$ | Return vector for risky assets in state $i$ |
| $r_f(i)$ | Return of the riskless asset in state $i$ |
| $u$ | Vector representing the amounts invested in risky assets |
| $U$ | Utility function |
| $Var_i(\ )$ | Variance given that the initial state is $i$ |
| $X_n$ | Money available for investment at the $n$th period |
| $Y$ | Stochastic market process |
| $\sigma(i)$ | Covariance matrix in state $i$ |

Chapter 1

# INTRODUCTION

Portfolio management is one of the most important subjects in financial engineering. Portfolio selection problem is the problem of an investor who wants to allocate his wealth among different assets within a market according to an objective given by a utility function representing his preferences. The investor's decision about which portion of his wealth to invest in each asset is generally called an investment policy. When all assets had deterministic returns, the optimal investment policy would be simply to invest all the wealth in the asset with the highest return. In real life, however, the returns are stochastic which leads to the problem of portfolio selection. Many factors such as the length of the investment horizon, the characteristics of the market and the objective of the decision maker affect the optimal investment policy. In this thesis, we consider a multiperiod portfolio selection problem where the returns of the assets are modulated by a Markov chain that represents the stochastic market. The main objective is to come up with optimal analytical solutions to several multiperiod formulations with different objective functions that represent the investor's preferences.

The traditional single period mean-variance model developed by Markowitz has been the basis of portfolio theory for more than fifty years. Investors use this model in a variety of financial settings mainly for asset allocation. The mean-variance model is a parametric optimization model for the single period portfolio selection problem which provides analytical solutions for both an investor trying to maximize his expected wealth without exceeding a predetermined risk level and an investor trying to minimize his risk ensuring a predetermined wealth. Despite its wide-ranging success, the single period framework suffers from an important deficiency. It is difficult to apply to long-term investors having goals at particular dates in the future, for which the investment decisions should be evaluated with regard to temporal issues besides the static risk-reward trade-off.

Researchers have tried to adapt the classical mean-variance model or similar models for multiperiod case considering the fact that investors invest continuously rather than for a single period. Most of the time, it is assumed that the return of a specified asset in a specified period is independent of the return of the same asset in previous periods. More realistically, some sort of dependence among the returns should be considered. In this thesis, it is assumed that there exist some macroeconomic factors affecting the asset returns. These factors form the stochastic market which is assumed to be a Markov chain with stationary transition probabilities.

Early portfolio problems were mainly based on expected gain maximization; and the well-known mean-variance problem is one of the oldest approaches. However, some objections were raised to this approach: First of all, it is argued that an ordinary investor has only one occasion to invest and therefore the expected outcome is irrelevant. Moreover, many investors do not have a precise knowledge of all possible outcomes of an action with their probabilities so that they are unable to go through the mathematics of the expected utility theorem. Finally, some other investors put more emphasis on the quality of the investment or the risk of default rather than on the investment's yield so that they want to secure at least a minimal return with a high probability. All these arguments suggest that a simpler and more objective decision model is needed, which leads to another well-known portfolio selection problem, the safety-first problem. The safety-first problem is of practical importance in portfolio selection, and it is one of the problems that are analyzed in this thesis.

The quadratic utility function is a utility function that has been used in the finance literature to describe investor's behavior. This problem is also analyzed here with the objective of maximizing the expectation of the quadratic utility function at the end of the investment horizon. Another problem analyzed in this thesis is related to minimizing the coefficient of variation of the final wealth of an investor at the end of the investment horizon. The coefficient of variation is an objective measure for investors who want to manage their portfolios, and therefore a single optimal solution is obtained on the efficient frontier.

The study is organized as follows: The literature survey on single-period/multiperiod portfolio optimization and various portfolio selection models with different utility functions is given in Chapter 2. Chapter 3 describes the stochastic structure of the market. Equivalent mean-variance problem formulations in generating efficient multiperiod portfolio policies are given in Chapter 4. The solution of the auxiliary problem that is found by dynamic programming is given in Chapter 5. Chapter 6 gives the solution procedure of the multiperiod portfolio problem for the most general case where an arbitrary utility function satisfying certain conditions is given. The quadratic utility model is analyzed in Chapter 7. The coefficient of variation model is defined and solved analytically in Chapter 8. Chapter 9 defines the safety-first problem in the single period setting and then provides the derivation of the analytical solution for it in the multiperiod case. The multiperiod portfolio selection problems are analyzed on a periodical basis in Chapter 10. Illustrative cases for a single risky asset and then for two risky assets demonstrating the application of the analytical solutions are given in Chapters 11 and 12, respectively. Finally, Chapter 13 presents the concluding remarks and possible further research topics.

# Chapter 2

## LITERATURE SURVEY

The concept of portfolio mean-variance optimization introduced by Harry Markowitz [1] forms the backbone of modern portfolio theory. The classical mean-variance approach is the first systematic treatment of investors' conflicting objectives of high return versus low risk. The objective of the single period mean-variance model is maximizing the expected return of a portfolio of assets without exceeding a predetermined risk level or alternatively minimizing the risk of a portfolio of assets ensuring a predetermined return. The problem can be solved explicitly for optimal portfolios, and the set of efficient portfolios can be determined. Markowitz's paper initiated a huge amount of related research on portfolio optimization. His classical mean-variance model is undoubtedly the most celebrated one within the vast area of portfolio management.

Merton [2] deals with the analytic derivation of the mean-variance efficient portfolio frontier. The efficient portfolio frontiers are derived explicitly and the characteristics for them are verified. The mutual fund theorem is proved by showing that any efficient portfolio can be obtained by a linear combination of two other efficient portfolios. The case of having a riskless asset is also analyzed. The separation theorem is stated and proved in the context of a mutual fund theorem. Two mutual funds can be chosen in such a way that one fund holds only the riskless security and the other fund contains only risky assets. He explains that the procedure to find the efficient frontier, when one of the assets is riskless, is to graph the efficient frontier for risky assets only and then to draw a line from the intercept tangent to the efficient frontier.

The safety-first criterion was developed by Roy [3] in 1952, the same year when Markowitz developed his mean-variance approach, as an alternative to the classical mean-variance concept in portfolio selection. His model is considered to be a simpler decision model to understand that concentrates on bad outcomes. In this paper, Roy [3] gives reasons for developing the safety-first approach which has the objective of minimizing the probability that the terminal wealth of an investor is below a preselected amount. He asserts that there are some objections to most widely used "expected gain maximization". First, the ordinary man has only one occasion to invest and the expected outcome is irrelevant. Second, many investors/decision makers do not have a precise knowledge of all possible outcomes of an action with their probabilities so that they are unable to go through the mathematics of the expected utility theorem. Moreover, some of them put more emphasis on the quality of the investment or the risk of default than on the investment's yield and therefore want to secure a minimal return with a high probability. All these arguments suggest that a simpler and more objective decision model is needed which leads to the safety-first problem. A disaster is defined to occur if an individual incurs a net loss as the result of some activity, and the principle of safety-first asserts that it is reasonable for an individual to reduce the chance of such a catastrophe occurring as much as possible. First, the objective function is defined formally where the Chebyshev's inequality is used to find the upper bound of the probability of a disaster. Then, the analysis is applied to the particular problem of holding $n$ assets, and finally the particular case of two assets is examined in more detail. Moreover, Roy independently develops a mean-variance efficient set for his problem similar to that of Markowitz.

Multiperiod portfolio optimization models have been studied by many researchers using different approaches. In this literature survey, we give information about major research papers on portfolio optimization, putting special emphasis on multiperiod formulations and on objectives other than the mean-variance trade-off.

This thesis involves the application of multiperiod portfolio optimization in several problems having different objective functions, and it follows the work of Çakmak and Özekici [4] which is about multiperiod mean-variance portfolio optimization in Markovian markets. The correlation among returns in different periods is constructed by a stochastic market representing underlying economic factors. It is assumed to be a Markov chain with some finite state space and stationary transition matrix. Considering a market with one riskless and $m$ risky assets, a multiperiod mean-variance formulation is developed. An auxiliary problem generating the same efficient frontier as the classical mean-variance formulation is used to eliminate nonseparability in the sense of dynamic programming. The analytical optimal solution is obtained for both the auxiliary problem and the main problem. To demonstrate the application of the solution procedure, some illustrative cases are given at the end.

Modeling a stochastic financial market by a Markov chain is a reasonable approach, and this idea dates back to the paper written by Pye [5]. In this paper, it is assumed that the future one-period interest rates follow a finite Markov chain. In the continuous time setting, Norberg [6] considers an interest rate model that is modulated by a Markov process. The idea of a financial market driven by a continuous time Markov chain is introduced in this paper. Recently, Elliott and Mamon [7] provide a yield curve description of a Markovian interest rate model.

A lot of research on portfolio management has been done so far since the original work of Markowitz. One of the first multiperiod models is the portfolio revision approach developed by Smith [8]. He extends the existing Markowitz model which forms a basis for selecting and revising portfolios. In order to overcome the shortcomings of this model that tells how to select a portfolio only at a single point in time, he proposes a transition model. He asserts that if an investor constantly changes his investment holdings such that his portfolio is efficient, this will result in excessive portfolio turnover so that brokerage fees and taxes will substantially reduce portfolio yield. The suggested technique is an adaptive type of mechanism performed at finite intervals, according to which a transition should be made only if its expected dollar return exceeds the dollar cost of the transition which consists of brokerage fees and the associated taxes that must be paid by the investor. The technique is illustrated over an eight-year period using 150 common stocks. Portfolio yields, which result from the revision procedure, are compared with similar performance measures from unrevised portfolios. The result is that higher portfolio yields can be achieved by revising portfolios using the proposed transition framework.

One of the earliest papers on optimal multiperiod portfolio policies written by Mossin [9] involves dynamic programming that has been widely used to solve multiperiod portfolio selection problems. He asserts that a formulation of the problem in terms of portfolio rate of return causes the role of the absolute size of the portfolio to be underrated. Accordingly, in order for a formulation not to be misleading in multiperiod theory, the development through time of total wealth must be considered. First, the single period problem is analyzed to reveal the problems related to the rate of return formulation. Then, the multiperiod problem is formulated using dynamic programming. It is given that sequential portfolio decisions are contingent upon the outcomes of previous periods and take into account the information regarding future probability distributions. By means of a backward recursion procedure, an optimal first-period decision is determined assuming statistical independence among yields in different periods and without taking transaction costs into account. The analysis covers general and quadratic utility functions and utility functions implying constant asset proportions. The procedure is illustrated in a numerical two-period example with a quadratic utility function and two assets having random yields. Moreover, the author analyzes utility functions allowing myopic decisions, where the investor's sequence of decisions is obtained as a series of single period decisions. He also discusses whether there can exist an optimal stationary portfolio policy and finds that this cannot be the case if yield distributions are not stationary. He finally discusses the time effect in multiperiod portfolio problems, i.e. how the optimal investment depends on the number of periods left, given an initial wealth and a yield distribution.

Chen et al. [10] discuss a portfolio revision process, by which an investor revises his portfolio periodically to adapt to changing conditions. They claim that investment decisions are usually made starting with a portfolio rather than cash, and as a result some assets must be liquidated to permit investment in others. A single period portfolio revision model is formulated which includes the expected transfer costs to be incurred in transition. Investors are assumed to revise their portfolios up to the point that the marginal utility of revision equals the marginal cost of revision. Analytical results are derived and compared to Smith's [8] target portfolio. It is shown that Smith did not consider the multiperiod aspect of the model and thus suggested a controlled transition approach which is inferior to the true optimal solution given in this paper. The model is then extended to the multiperiod case by using a dynamic programming framework and compared to Mossin's [9] dynamic portfolio selection model through numerical examples. One example assumes an investor starting with cash (portfolio selection problem), and the other assumes an investor starting with a portfolio (portfolio revision problem). It is stated that Mossin's model has to be modified since investors starting with a portfolio of earning assets are locked in, and therefore transfer costs have to be taken into account. Finally, the multiperiod multi-asset case is also discussed.

Samuelson [11] formulates and solves a generalized multiperiod portfolio selection model corresponding to lifetime planning of consumption and investment decisions. A stochastic programming problem, which needs to be solved simultaneously for the optimal saving/consumption and the portfolio selection decisions over time, is derived and then the optimal decision as a function of initial wealth is found. The method is applied to a problem with one riskless asset and alternative risky assets. Cases where the utility functions are isoelastic are also analyzed. It is stated that the optimal portfolio decision is independent of wealth at each stage and independent of all saving/consumption decisions for isoelastic marginal utility functions. Moreover, for isoelastic marginal utilities, the same risk tolerance both at the beginning and towards the end of life is valid.

Dumas and Luciano [12] provide an exact analytical solution to the dynamic portfolio choice problem under transaction costs. The investor accumulates wealth without consuming until some terminal point in time and has the objective of maximizing the expected utility from his terminal consumption. A continuous time portfolio selection model is given and then the necessary conditions, which must be satisfied when it is optimal to refrain from trading and which must prevail when trading takes place, are stated. The calculation of the portfolio policy in the form of two control barriers, between which portfolio proportions are allowed to fluctuate, is shown by means of the theory of optimal regulated Brownian motion. Finite horizon versus infinite horizon solutions in the absence and presence of transaction costs are compared which are assumed to be proportional to the value of the trade. Finally, deviations are examined in the dimensions of increasing transaction costs, increasing risk aversion and increasing risk. It is stated that increased transaction costs do not bias the optimal portfolio one way or the other, and that there is very little interaction between transaction costs and risk aversion.

Roy [13] deals with dynamic portfolio choice for survival under uncertainty and develops a discrete time dynamic optimization model where the objective is to maximize the long-run probability of survival through portfolio choice over time. There is a given minimum withdrawal requirement, and the investor survives only if his wealth is large enough to meet this requirement every time period over an infinite horizon. Part of the current wealth is withdrawn and the rest is allocated between a risky and a riskless asset in each time period. If the returns of the risky assets are assumed to be independent and identically distributed with continuous density, the existence of a stationary optimal policy is proved and the dynamic programming equation is given which is used to characterize the maximum survival probability and the stationary optimal policies. There exists a critical level of initial wealth below which survival is impossible, independent of what actions are chosen, and another critical level of initial wealth above which survival can be ensured with probability one by choosing to concentrate all investment on the riskless asset in every period. Between these two critical levels, the maximum survival probability is continuous and strictly increasing in current wealth.

Ehrlich and Hamlen [14] solve the stochastic portfolio consumption control problem under the assumption that individuals follow precommitment strategies over finite intervals of time. The precommitment approach is an alternative to continuous time stochastic dynamic control problem, which assumes instantaneous feedback and costless revisions of choices all along the time path. The investor is allowed to choose the proportion of his wealth to be invested in a riskless asset and the rest to be invested in a portfolio of risky assets corresponding to the market portfolio. It is shown that under precommitment individuals tend to hold portfolios that are a function of their expected risk and return parameters, but are independent of their wealth levels and risk preferences. It is also shown that the intertemporal consumption growth path would be a relatively smooth function of the risk-free rate of return, time preference, and the coefficient of relative risk aversion, and independent of the portfolio's risk parameters. Finally, empirical implications are illustrated for both portfolio holding and consumption patterns.

Bodily and White [15] analyze the optimal consumption and portfolio mixture for a discrete time, discrete state preference model. The investor's preferences for future consumption depend on current wealth and on past consumption experience through a summary descriptor of past consumption. Relations between the optimal consumption and investment decisions are found. The passage of time indicates the correctness of the investor's beliefs while preferences are subjective and represented by a von Neumann-Morgenstern utility function. After stating the investor's problem and developing the model, the authors illustrate the modeling flexibility of this approach by an example. The properties of admissible strategies and policy implications are examined. Following results are found based on certain assumptions: The optimal expected utility over a planning horizon of given length is nondecreasing in initial wealth and nonincreasing in summary descriptor. The optimal consumption level does not decrease as wealth and summary descriptor increase. The optimal fraction of investment in the risky opportunity does not decrease as wealth increases.

Elton and Gruber [16] compare selecting portfolios on the basis of the geometric mean of future multiperiod returns and selecting portfolios on the basis of the expected utility of multiperiod returns. They show that when the ability of the investor to revise his portfolio is considered, each of these rules is only appropriate under a very restrictive set of conditions. The goal of portfolio managers is to maximize the expected utility of the investor's wealth at some terminal time. The analysis is performed both when return distributions are unchanging over time and when they change in a regular pattern. Assuming that the investor has the ability to revise his portfolio at the end of each period, the geometric mean is an appropriate decision criterion when the investor's utility function is logarithmic and the distribution of returns in all future periods are constant over time or the distribution of returns in any period is expected to be the first-period returns multiplied by a constant. The selection of portfolios that maximize the expected value of a utility function in terms of return and risk is appropriate whenever either utility functions are quadratic or returns are normally distributed.

Li and Ng [17] consider the mean-variance formulation in multiperiod portfolio selection and derive an analytical optimal portfolio policy and an analytical expression of the mean-variance efficient frontier, where the starting point is the concept of Markowitz's mean-variance formulation. The model assumes independence of returns over time, and dynamic programming is used for this multiperiod portfolio selection problem. A separable auxiliary problem generating the same efficient frontier with the classical models is used to solve the mean-variance formulation. The special case in which there exists a single riskless asset is discussed. They also consider a more general problem formulation with an arbitrary utility function of the expected value and the variance of the final wealth satisfying certain conditions and propose an efficient solution for that. The paper is concluded with three illustrative examples to demonstrate the efficiency of the solution method derived. Their work represents an extension of the existing literature dealing with risk management in dynamic portfolio selection.

In a very recent paper published in 2004, Leippold et al. [18] present a geometric approach to discrete time multiperiod mean-variance portfolio optimization of assets and liabilities that largely simplifies the mathematical analysis and the economic interpretation of such model settings. Using the geometric approach to dynamic mean-variance optimization, they obtain closed form solutions for portfolios consisting of both assets and liabilities. As a special case, the solution of the mean-variance problem with only one risky asset in Li and Ng [17] follows directly and can be represented in terms of simple products of some single period orthogonal returns. They also illustrate the usefulness of the geometric representation of multiperiod optimal policies and mean-variance frontiers by discussing specific issues related to portfolios of assets and liabilities.

The studies summarized so far do not consider serial correlation between returns. The case where the asset returns over the periods are statistically dependent has received only limited attention due to its apparent complexity.

Hakansson and Liu [19] consider a model where the asset returns are serially correlated. They generalize the capital growth model to the case in which investment returns are statistically dependent on returns in previous periods. Their model uses logarithmic utility function which implies risk aversion so that risk factors are not taken into account directly. The investor makes decisions at discrete points which may be unequally spaced in time. Both riskless and risky assets are available in each period. The returns in a given period depend on the change in the general condition of the economic environment. The transition probabilities are constants which implies that the economy obeys a nonstationary Markov process. The Markov chain formed by the transition probabilities is assumed to be irreducible and ergodic. All investments are realizable in cash at the end of each period, and taxes and conversion costs are proportional to the amount invested. The returns from risky assets in a given period can depend on the state of the economy both at the beginning and end of the period. An optimal investment strategy is obtained on the basis of a slightly generalized and weakened version of the rational criterion that more is preferred to less in the very long run. The optimal policy obtained is myopic and maximizes the long run growth rate.

Hakansson [20] considers Mossin's [9] results to isolate the class of utility functions of terminal wealth. He builds models similar to the model built in his past work [19] for both cases with and without serial correlation of returns. He solves these models, examines the results and shows that Mossin's [9] conclusions are true only in a limited sense even when returns are serially independent. When investment returns in the various periods are statistically dependent, only the logarithmic function provides utility functions of short-run wealth, which are myopic. He assumes stochastically constant returns to scale, perfect liquidity, divisibility of assets at each decision point and absence of transaction costs.

Hakansson [21] extends the standard portfolio selection model to the multiperiod case and analyzes the results obtained from multiperiod mean-variance approach based on average compound return. It is shown that the set of efficient portfolios in any one period decreases as the horizon increases and converges to a single efficient sequence. If an investor wants to maximize the expected average compound return over $N$ periods where $N \geq 2$, then there exists a unique, single period von Neumann-Morgenstern utility function defined on wealth which is consistent with this objective. This utility function implies risk aversion and does not in general produce a mean-variance efficient portfolio in the single period case. When $N$ is large, the set of average compound returns, which are mean-variance efficient, can be exactly or approximately obtained only with a subset of the terminal functions which induce myopic single period utility functions. The growth-optimal portfolio is demonstrated to be efficient in the limit. It is then found that only the riskless portfolio sequence will generally be efficient with respect to both single period and total return as well as the long-run average compound return. The author indicates rapid convergence of the long-run efficient portfolios to $N$-period efficient portfolios and later presents implications of graphic analysis by use of indifference curves. Mean-variance formulations of average compound return over two or more periods imply risk aversion without reference to the variance, are consistent with von Neumann-Morgenstern utility theory, automatically insure investor's survival, and imply that myopic investment behavior is optimal.

Hernández-Hernández and Marcus [22] discuss the existence of optimal stationary policies which maximize the long run average reward for infinite horizon risk sensitive Markov control processes with denumerable state space, unbounded cost function, and long run average cost. Using the vanishing discount approach, it is proved that there exist optimal stationary policies, and then an optimal stationary policy is derived for a given utility function.

Bieleceki et al. [23] extend standard dynamic programming results for the risk sensitive optimal control of discrete time Markov chains to a new class of models. The state space is restricted to be finite and the underlying Markov transition matrix is assumed to be irreducible so that same kinds of dynamic programming results found in the existing discrete time risk sensitive control theory literature still remain valid. The set of possible factor values becomes the state space for the Markov control model. In each period, the action taken is the allocation of wealth among assets. Asset returns, which depend on the factor's state both at the beginning and end of the period, combine with the chosen action to determine the portfolio's return. The optimal trading strategy is characterized in terms of a dynamic programming equation. The results are applied to the financial problem of managing a portfolio of assets which are affected by Markovian microeconomic and macroeconomic factors and where the investor seeks to maximize the portfolio's risk adjusted growth rate. Finally, optimal stationary policies are given and some illustrative cases are presented.

In a review paper, Steinbach [24] takes a look at the mean-variance models in financial portfolio analysis. This paper refers to 208 research papers which shows the diversity of different models and approaches used to analyze this problem. Both single period models and multiperiod models are considered in this paper.

Mulvey et al. [25] extend the static portfolio model to a dynamic multiperiod setting where the framework consists of the three basic elements: a stochastic scenario generator, a policy simulator, and an optimization module. Their basic idea in the process of developing a forward-looking financial planning model in a multiperiod context is to construct a set of multiperiod scenarios, to simulate several policies and then to optimize over these scenarios in conjunction with the investor's unique circumstances such as sets of eligible assets, transaction costs, liabilities, and so on.

In this thesis, special attention is given to the safety-first approach, and therefore a literature review on this approach will follow now.

In their book on portfolio theory, Elton and Gruber [26] introduce other criteria for portfolio selection as an alternative to the classical mean-variance approach. Kataoka [27] modified Roy's approach by prespecifying the acceptable probability of a bad outcome, and choosing between two portfolios the one with the highest critical return at that probability. In this case, prespecifying the probability is equivalent to prespecifying the acceptable number of standard deviations that the critical return can lie below the mean, with the objective of picking the portfolio with the highest critical return. Telser [28] combines the criteria of Roy [3] and Kataoka [27], such that the optimal safety-first portfolio maximizes expected return constrained by a limit to the probability that the return could be less than some prespecified critical return. The constraint can exclude parts of the efficient set, or even eliminate the efficient set, but the optimal portfolio lies on the efficient set or is non-existent.

Pyle and Turnovsky [29] discuss three different safety-first criteria developed so far in the literature and show that these criteria lead to optimization of expressions involving the mean and standard deviation. They discuss the relationship between these three approaches and compare them with the more conventional approach based on expected utility maximization. Levy and Sarnat [30] try to relate the safety-first principle to the expected utility principle. They show that the mean-variance approach emerges as a special case of safety-first criterion under the condition that the disaster level is chosen to be equal to the rate of return of the riskless asset in which case they lead to identical optimal portfolios.

Li et al. [31] extend the safety-first approach to multiperiod portfolio selection problems. In this paper it is assumed that the rates of return of risky assets in a certain period are independent of their values in other periods. A multiperiod safety-first formulation in dynamic portfolio selection is given, and using some transformations the original safety-first problem is embedded into a tractable auxiliary problem. An analytical solution for the multiperiod portfolio problem is achieved and then some illustrative examples are given at the end.

One of the most recent papers related to the safety-first approach was written by Jansen et al. [32] which deals with portfolio choice with limited downside risk in a single period setting. Contrary to the empirical applications of the safety-first principle which used the Chebyshev's inequality, this study characterizes for an unknown distribution of returns the behavior in the tails of the distribution so that one can improve the estimate of the failure probability. Jansen et al. [32] introduced the use of extreme value analysis in portfolio choices with safety-first. The approach can calculate the probability of extreme events, even with no such observations in the sample. Following Arzac and Bawa [33], they use extreme value theory to show that the conventional safety-first criteria can be successfully improved upon by exploiting the fat tail property of asset returns. They show that portfolio selection with limited downside risk includes both the safety-first investor of Roy [3] and Arzac and Bawa [33].

In their very recent paper published in 2004, Haque et al. [34] discuss all approaches proposed so far to solve the safety-first problem and then deal mainly with safety-first portfolio optimization for US investors in emerging global, Asian and Latin American markets. Their study examines the diversification benefits in these markets within a safety-first context. The approach proposed by Jansen et al. [32] making use of the extreme value theory is used in this research.

Finally, general utility functions in single period portfolio selection problems are considered by Levy and Markowitz [35]. They show empirically that the ordering of portfolios by the mean-variance rule is almost identical to the order obtained by using expected utility for various utility functions and historical distributions of returns. The expected utility is approximated by a function of mean and variance that yields a portfolio with almost as great an expected utility as the maximum obtainable expected utility. The comparison was made for a finite number of portfolios in this paper. In their later work, Kroll et al. [36] extend this study to include an infinite number of possible mixtures of a finite number of securities. The expected utility of the optimal portfolio for various utility functions is compared to the expected utility of well-selected portfolios from the mean-variance efficient frontier. It is shown that the best mean-variance efficient portfolio is the portfolio which maximizes expected utility or which at least has a near optimum expected utility.

Chapter 3

# THE STOCHASTIC MARKET

The generally accepted assumption that the return of a specified asset in a certain period is independent of the return of that asset in previous periods simplifies the multiperiod portfolio model and the derivation of the optimal solution. More realistically, a market should be considered where returns are serially correlated.

In this thesis, the market consists of several risky assets and a riskless asset. The returns of assets, except for the riksless one, are assumed to be random. Not the exact distribution of the returns are known, but the factors affecting their distribution, hence their mean, variance and covariance with each other, are assumed to be known. These factors are the underlying economic factors forming the stochastic market and they define the states of a Markov chain. Consequently, the state of the market formed by underlying economic factors in a specified period is dependent on the state in previous period because economic factors changing over time are correlated. The factors affecting the market change randomly over time forming a stochastic market process which is used to construct the serial correlation among returns in different periods. As the state of the market changes over time, the returns will change accordingly.

In this thesis, a process is considered where the state of the market in a period depends only on the state of the last period, which is the well-known property of the Markov chain. We let $Y_n$ denote the state of the market at period $n$ so that $Y = \{Y_n;\ n = 0, 1, 2, \ldots\}$ is a Markov chain with some state space $E$ and transition matrix $Q$. Then, the transition probabilities are given as

$$P\{Y_{n+1} = j \mid Y_n = i\} = Q(i, j). \tag{3.1}$$

It is assumed that the state of any period is known at the beginning of that period.

The construction of the relationship between the stochastic market and the distribution of the returns is as follows: We assume that the distribution of the return of a specified asset in a period depends only on the state of the market in that period. Formally, $R_k$ is the random variable representing the random return of an asset indexed by $k$ and $R_k(i)$ denotes the random return of this asset in any period where $i \in E$ is the state in that period. As a consequence, the expected value and the variance of the return of an asset depend only on the current state of the stochastic market. These assumptions imply that the expected return, the variance and the covariance with other assets of a specified asset in two different periods will be the same when the state of the market in both periods is the same. This approach eliminates the need for generating lots of parameters such as the mean, the variance and the covariance of the returns for all assets and for all periods, which turns out to be an extremely hard work when the investment horizon is long. In this thesis, one should generate these parameters for all assets and only for all market states. It is clear that the number of states of the market will be much less than the number of periods for a long investment horizon.

In the stochastic market process described previously, the market consists of one riskless asset with known return $r_f(i)$ and standard deviation $\sigma_f(i) = 0$ and $m$ risky assets with random returns $R(i) = (R_1(i), R_2(i), ..., R_m(i))$ in state $i$. We let $r_k(i) = E[R_k(i)]$ denote the mean return of the $k$th asset in state $i$ and $\sigma_{kj}(i) = \text{Cov}(R_k(i), R_j(i))$ denote the covariance between $k$th and $j$th asset returns in state $i$. The excess return of the $k$th asset in state $i$ is $R_k^e(i) = R_k(i) - r_f(i)$. It follows that

$$r_k^e(i) = E[R_k^e(i)] = r_k(i) - r_f(i) \quad (3.2)$$

$$\sigma_{kj}(i) = \text{Cov}(R_k(i) - r_f(i), R_j(i) - r_f(i)). \quad (3.3)$$

From the expressions given above, it can be concluded that $r_f(i)$ is a scalar and $r(i) = (r_1(i), r_2(i), ..., r_m(i))$ and $r^e(i) = (r_1^e(i), r_2^e(i), ..., r_m^e(i))$ are column vectors for all $i$. For any column vector $z$, $z'$ denotes the row vector representing its transpose.

We define $X_n$ as the amount of investor's wealth at period $n$ and correspondingly $X_T$ denotes the final wealth. The vector $u = (u_1, u_2, ..., u_m)$ gives the amounts invested in risky assets $(1, 2, ..., m)$ at period $n$. Shortly, $u$ denotes the investment strategy.

Since the model in consideration is multiperiod, it has to include a wealth dynamic equation that keeps track of the money available for investment at the beginning of each period. The amounts invested in each risky asset are multiplied by the corresponding asset returns and the remaining amount is invested in the risk-free asset so that it is multiplied by the prevailing risk-free rate. The wealth dynamic equation, which is encountered as a constraint in each multiperiod model that we will be discussing, is

$$\begin{aligned} X_{n+1}(u) &= R(Y_n)' u + \left(X_n - 1'u\right) r_f(Y_n) \\ &= r_f(Y_n) X_n + (R(Y_n)' - 1' r_f(Y_n)) u \\ &= r_f(Y_n) X_n + R^e(Y_n)' u \end{aligned} \quad (3.4)$$

where $1 = (1, 1, ..., 1)$ is the column vector consisting of 1's.

The assumptions regarding the model formulation can be summarized as follows:

- Unlimited borrowing and lending at the prevailing return of the riskless asset in any period are possible.
- Short selling is allowed for all assets in all periods.
- No capital additions or withdrawals are allowed throughout the investment horizon.
- Transaction costs are negligible.

Chapter 4

# MEAN-VARIANCE MODEL FORMULATIONS

The classical mean-variance approach introduced by Harry Markowitz provides the foundation for single period investment theory and it addresses the trade-off between the expected return and the variance of the return of a portfolio. Markowitz's basic formulation is based on a market with $m$ assets having known expected returns $r = (r_1, r_2, ..., r_m)$ and covariances $\sigma_{ij}$ for $i, j = 1, 2, ..., m$. A portfolio is defined by a set of $m$ weights $u_i$, $i = 1, 2, ..., m$, that sum up to 1. Markowitz's portfolio selection problem is formulated through the following optimization formulation:

$$\begin{aligned} MV(\mu) \quad : \quad \min \quad & \sum_{i,j=1}^{m} u_i u_j \sigma_{ij} \\ \text{s.t.} \quad & \sum_{i=1}^{m} u_i r_i = \mu \\ & \sum_{i=1}^{m} u_i = 1 \end{aligned} \tag{4.1}$$

The problem, given in (4.1), is to find the best allocation of wealth among $m$ assets with the objective of minimizing the portfolio risk while ensuring a predetermined portfolio return $\mu$. This problem can be solved analytically using Lagrange multipliers so that efficient frontiers can be obtained which show how much risk corresponds to a specified return level $\mu$. Alternatively, Markowitz also formulated the problem of an investor who wants to maximize his return while keeping his risk below a predetermined risk level $\sigma$.

In the multiperiod setting, $P1(\sigma)$ and $P2(\mu)$ given in (4.2) and (4.3) respectively are equivalent mean-variance formulations corresponding to Markowitz's portfolio selection problems, given that the initial market state is $i$. We use the notation $E_i[Z] = E[Z \mid Y_0 = i]$ and $\text{Var}_i(Z) = E_i[Z^2] - E_i[Z]^2$ to denote the conditional expectation and variance of any random variable $Z$ given that the initial market state is $i \in E$.

$$\begin{aligned} P1(\sigma) \quad : \quad & \max \quad E_i[X_T] \\ & \text{s.t.} \quad \text{Var}_i(X_T) \leq \sigma \\ & \qquad X_{n+1}(u) = r_f(Y_n) X_n + R^e(Y_n)' u \end{aligned} \tag{4.2}$$

$$\begin{aligned} P2(\mu) \quad : \quad & \min \quad \text{Var}_i(X_T) \\ & \text{s.t.} \quad E_i[X_T] \geq \mu \\ & \qquad X_{n+1}(u) = r_f(Y_n) X_n + R^e(Y_n)' u \end{aligned} \tag{4.3}$$

The multiperiod mean-variance formulations $P1(\sigma)$ and $P2(\mu)$ do not have straightforward solutions as in the single period case. In order to obtain the analytical solutions, dynamic programming has to be used. However, it turns out that these two problems cannot be solved using dynamic programming due to their nonseparability. Li et al. [31] prove that an equivalent formulation to both $P1(\sigma)$ and $P2(\mu)$ in generating efficient multiperiod portfolio policies is

$$\begin{aligned} P3(\omega) \quad : \quad & \max \quad E_i[X_T] - \omega \text{Var}_i(X_T) \\ & \text{s.t.} \quad X_{n+1}(u) = r_f(Y_n) X_n + R^e(Y_n)' u \end{aligned} \tag{4.4}$$

where $\omega > 0$. These problems being equivalent imply that there are one-to-one relationships between their parameters $\sigma$, $\mu$ and $\omega$ such that once $P3(\omega)$ is solved parametrically for $\omega$, it is sufficient to set $\text{Var}_i(X_T) = \sigma^2$ and $E_i[X_T] = \mu$ to identify which $\omega$ gives the optimal solution of $P1(\sigma)$and $P2(\mu)$ respectively. The efficient frontier on the mean versus standard deviation graph ($E_i[X_T]$ versus $\sqrt{\text{Var}_i(X_T)}$ graph) is obtained by changing the value of $\omega$ in the objective function of $P3(\omega)$.

Since $P3(\omega)$ is still not separable in the sense of dynamic programming, it is further embedded into a tractable auxiliary problem $P4(\lambda, \omega)$ which is

$$\begin{aligned} P4(\lambda, \omega) \quad : \quad & \max \quad E_i\left[-\omega X_T^2 + \lambda X_T\right] \\ & \text{s.t.} \quad X_{n+1}(u) = r_f(Y_n) X_n + R^e(Y_n)' u \end{aligned} \tag{4.5}$$

where $\omega$ is a positive parameter so that it can be taken out of the objective function to get the modified formulation

$$\begin{aligned} P4(\lambda,\omega) \quad : \quad & \max \quad \omega E_i\left[-X_T^2 + \frac{\lambda}{\omega}X_T\right] \\ & \text{s.t.} \quad X_{n+1}(u) = r_f(Y_n)\,X_n + R^e(Y_n)'\,u. \end{aligned} \tag{4.6}$$

Analyzing the objective function in (4.6) shows that the optimal policy will be a function of $\lambda/\omega$ which is denoted by $\gamma$. Since $P4(\lambda,\omega)$ is separable in the sense of dynamic programming, its formulation given in (4.5) will be used to solve the dynamic multiperiod portfolio selection problem.

The important relationship between these four formulations is that the optimal solution sets of former problems are included in the optimal solution sets of later formulations so that the solutions of former problems can be obtained from $P4(\lambda,\omega)$. In other words, solving $P4(\lambda,\omega)$ means solving $P3(\omega)$ which in turn means solving both $P1(\sigma)$ and $P2(\mu)$ for $\sigma$ and $\mu$ associated with $\omega$.

Chapter 5

# SOLUTION OF THE AUXILIARY PROBLEM

Dynamic programming is the method used in the derivation of the optimal solution for the multiperiod mean-variance problem, the details of which is given in Çakmak and Özekici [4]. The auxiliary problem $P4(\lambda, \omega)$ involves the maximization of a simple expected utility function using only the first two moments of the terminal wealth at the end of the investment horizon. In order to solve $P4(\lambda, \omega)$, we define $v_n(i,x)$ as the optimal expected utility using the optimal policy given that the market state is $i \in E$ and the amount of money available for investment is $x$ at period $n$. Then, the dynamic programming equation becomes

$$v_n(i,x) = \max_u E\left[v_{n+1}\left(Y_{n+1}, X_{n+1}(u)\right) | Y_n = i\right] \tag{5.1}$$

which can be rewritten as

$$v_n(i,x) = \max_u \sum_{j \in E} Q(i,j)\, E\left[v_{n+1}\left(j, r_f(i)\, x + R^e(i)' u\right)\right] \tag{5.2}$$

for $n = 0, 1, 2, ..., T-1$ with the boundary condition $v_T(i,x) = -\omega x^2 + \lambda x$ for all $i \in E$. The solution for this problem is found by solving the dynamic programming equation recursively.

Some terminology and notation used in the derivation of the optimal solution are as follows: We define the matrix

$$V(i) = E\left[R^e(i)\, R^e(i)'\right] \tag{5.3}$$

for any state $i \in E$. The covariance matrix $\sigma(i)$ is assumed to be positive definite for all $i \in E$ which is a justified assumption since

$$z'\sigma(i)z = E[(z_1 R_1(i) + z_2 R_2(i) + \cdots + z_m R_m(i))^2] \geq 0 \tag{5.4}$$

for any vector $z = (z_1, z_2, ..., z_m)$. This property of $\sigma(i)$ is inherited by $V(i)$ such that for any $i \in E$, $V(i) = \sigma(i) + r^e(i)\, r^e(i)'$ is a positive definite matrix.

We now define $f(i)$, $g(i)$ and $h(i)$ which are functions of asset returns as

$$f(i) = r_f(i)^2 [1 - h(i)] \tag{5.5}$$

$$g(i) = r_f(i) [1 - h(i)] \tag{5.6}$$

where

$$h(i) = r^e(i)' V^{-1}(i) r^e(i). \tag{5.7}$$

It turns out that for any $i \in E$, $f(i), g(i) > 0$ and $0 < h(i) < 1$.

For any matrix $M$ and vector $f$, we define the matrix $M_f$ such that

$$M_f(i,j) = M(i,j) f(j) \tag{5.8}$$

for $i, j \in E$ and the vector $\overline{M}$ such that

$$\overline{M}(i) = \sum_{j \in E} M(i,j). \tag{5.9}$$

Using this notation $M_f^n$ is the $n$th power of $M_f$, and $\overline{M}_f^n$ is simply the vector obtained by adding the columns of the matrix $M_f^n$ for $n \geq 0$. It follows that $\overline{M}_f^0 = 1$ when $n = 0$ and $\overline{M}_f = Mf$ when $n = 1$.

If $a, b$ and $c$ are three vectors, then $(a/b) \bullet c$ denotes the vector where $((a/b) \bullet c)(i) = (a(i)/b(i))c(i)$. Moreover, we define

$$h_n(i) = \frac{\overline{Q}_g^n(i)}{\overline{Q}_f^n(i)} h(i) \tag{5.10}$$

$$\overline{h}_n(i) = \left( \frac{\overline{Q}_g^n(i)}{\overline{Q}_f^n(i)} \right)^2 h(i) \tag{5.11}$$

which will be used in the derivation of the optimal solution.

The main results of Çakmak and Özekici [4] will be provided now without presenting their proofs that can be found in the related paper. We use $x_0$ to denote the initial wealth which is assumed to be known.

The optimal solution of $P4(\lambda, \omega)$ is

$$v_n(i,x) = -\omega_n(i) x^2 + \lambda_n(i) x + \alpha_n(i) \tag{5.12}$$

and the corresponding optimal policy maximizing the objective function is

$$u_n(i,x) = \left[ \frac{1}{2} \left( \frac{\lambda}{\omega} \right) \frac{\overline{Q}_g^{T-n-1}(i)}{\overline{Q}_f^{T-n-1}(i)} - r_f(i) x \right] V^{-1}(i) r^e(i) \tag{5.13}$$

where

$$\omega_n(i) = \omega\overline{Q}_f^{T-n-1}(i) f(i) \quad (5.14)$$

$$\lambda_n(i) = \lambda\overline{Q}_g^{T-n-1}(i) g(i) \quad (5.15)$$

$$\alpha_n(i) = \sum_{k=n+2}^{T} Q^{k-n-2}\overline{Q}_{\bar{\alpha}_k}(i) + \bar{\alpha}_{n+1}(i) \quad (5.16)$$

and

$$\bar{\alpha}_n(i) = \frac{\left(\lambda\overline{Q}_g^{T-n}(i)\right)^2}{4\omega\overline{Q}_f^{T-n}(i)} h(i) \quad (5.17)$$

for $n = 0, 1, ..., T-1$. In (5.16), the summation on the right hand side vanishes if $n = T-1$.

The optimal investment policy $u_n(i, x)$ in (5.13) gives the amount of money that should be invested in each risky asset at period $n$ given the market state $i$ and the current wealth $x$. This formula shows that the amounts invested in risky assets are determined based on investor's attitude toward risk, reflected in the first term inside the parenthesis, and investor's current wealth, reflected in the second term inside the parenthesis. The first term can be calculated before the investment process starts whereas the second term is calculated at every time period when the current wealth is observed. By substituting (5.13) into the wealth dynamic equation given in (3.4) and then taking expectations of $X_n$ and $X_n^2$, we obtain

$$E_i[X_n] = \overline{Q}_g^{n-1}(i) g(i) x_0 + \frac{\lambda}{2\omega}\sum_{k=1}^{n} Q^{k-1}\left(\overline{Q}_g^{n-k} \bullet h_{T-k}\right)(i) \quad (5.18)$$

$$E_i\left[X_n^2\right] = \overline{Q}_f^{n-1}(i) f(i) x_0^2 + \left(\frac{\lambda}{2\omega}\right)^2 \sum_{k=1}^{n} Q^{k-1}\left(\overline{Q}_f^{n-k} \bullet \overline{h}_{T-k}\right)(i) \quad (5.19)$$

for $n = 1, ..., T$.

If we define

$$a_1(i) = \overline{Q}_g^{T-1}(i) g(i) \quad (5.20)$$

$$a_2(i) = \overline{Q}_f^{T-1}(i) f(i) \quad (5.21)$$

$$b(i) = \frac{1}{2}\sum_{k=1}^{T} Q^{k-1}\left(\frac{\left(\overline{Q}_g^{T-k}\right)^2}{\overline{Q}_f^{T-k}} \bullet h\right)(i) \quad (5.22)$$

then the optimal solution satisfies the simplified expressions

$$E_i[X_T] = a_1(i) x_0 + b(i)\gamma \quad (5.23)$$

$$E_i\left[X_T^2\right] = a_2(i) x_0^2 + \frac{1}{2} b(i)\gamma^2 \quad (5.24)$$

where $\gamma = \lambda/\omega$. Consequently, the variance of the terminal wealth is

$$\mathrm{Var}_i\left(X_T\right) = \left(a_2(i) - a_1(i)^2\right) x_0^2 - 2a_1(i)b(i)x_0\gamma + \left(\frac{1}{2} - b(i)\right) b(i)\gamma^2. \tag{5.25}$$

With respect to the multiperiod portfolio optimization problem, $E_i\left[X_T\right]$ is the expected wealth (or the expected return when $x_0 = 1$) at the end of the investment horizon and $\mathrm{Var}_i\left(X_T\right)$ measures the risk of the final wealth. The expectation of $X_T$ versus the standard deviation of $X_T$ given the initial market state corresponds to an optimal point on the mean-variance efficient frontier.

Here, it should be noted that $a_1(i), a_2(i)$ and $b(i)$ have values that are greater than zero. Moreover, $b(i)$ is a quantity smaller than 1/2.

After finding the optimal solution for the auxiliary problem, Çakmak and Özekici [4] focus on the multiperiod mean-variance portfolio problems $P1(\sigma)$ and $P2(\mu)$ given in (4.2) and (4.3) respectively. The optimal solution of $P3(\omega)$ is found to be

$$u_n\left(i, x\right) = \left[\left(\frac{1 + 2\omega a_1(i)x_0}{2\omega\left(1 - 2b(i)\right)}\right) \frac{\overline{Q}_g^{T-n-1}(i)}{\overline{Q}_f^{T-n-1}(i)} - r_f\left(i\right) x\right] V^{-1}\left(i\right) r^e\left(i\right) \tag{5.26}$$

for $n = 0, 1, ..., T - 1$ such that

$$E_i\left[X_T\right] = \frac{a_1(i)x_0}{\left(1 - 2b(i)\right)} + \frac{b(i)}{\omega\left(1 - 2b(i)\right)} \tag{5.27}$$

$$\mathrm{Var}_i\left(X_T\right) = \left(a_2(i) - \frac{a_1(i)^2}{\left(1 - 2b(i)\right)}\right) x_0^2 + \frac{b(i)}{2\omega^2\left(1 - 2b(i)\right)}. \tag{5.28}$$

The optimal solutions of $P1(\sigma)$ and $P2(\mu)$ are obtained from $P3(\omega)$ by taking

$$\omega = \sqrt{\frac{b(i)}{2\left[\left(1 - 2b(i)\right)\sigma - \left[\left(1 - 2b(i)\right)a_2(i) - a_1(i)^2\right] x_0^2\right]}} \tag{5.29}$$

for $P1(\sigma)$, and

$$\omega = \frac{b(i)}{\left(1 - 2b(i)\right)\mu - a_1(i)x_0} \tag{5.30}$$

for $P2(\mu)$.

The solution procedure for $P1(\sigma)$ or $P2(\mu)$ is as follows: One should first calculate $f(i)$, $g(i)$ and $h(i)$ for the given initial state $i$ of the market using (5.5)-(5.7). Then, $a_1(i)$, $a_2(i)$ and $b(i)$ should be computed using (5.20)-(5.22). The associated $\omega$ will be calculated in terms of $\sigma$ or $\mu$ using (5.29) or (5.30). Finally, substituting the calculated $\omega$ into (5.26) yields the optimal multiperiod portfolio policy for $P1(\sigma)$ or $P2(\mu)$ which leads to the expectation and the variance of the final wealth given in (5.27) and (5.28) respectively that are also calculated by substituting $\omega$ into the related equations.

Finally, the mean-variance efficient frontier is found to be

$$\mathrm{Var}_i(X_T) = \left(a_2(i) - \frac{a_1(i)^2}{(1-2b(i))}\right) x_0^2 + \frac{\left[(1-2b(i))\, E_i\left[X_T\right] - a_1(i)x_0\right]^2}{2b(i)\,(1-2b(i))} \tag{5.31}$$

defined for $E_i\left[X_T\right] \geq a_1(i)x_0/\left(1-2b(i)\right)$. The minimum variance point of the efficient frontier is found by minimizing the expression for variance in (5.25) with respect to $\gamma$. This point has a gamma value of

$$\frac{2a_1(i)x_0}{(1-2b(i))} \tag{5.32}$$

which implies that the $\gamma$ value has to be greater than $(2a_1(i)x_0)/(1-2b(i))$ so as to get portfolios on the efficient frontier. The minimum variance portfolio has

$$E_i\left[X_T\right] = \frac{a_1(i)x_0}{(1-2b(i))} \tag{5.33}$$

$$\mathrm{Var}_i\left(X_T\right) = \left(a_2(i) - \frac{a_1(i)^2}{(1-2b(i))}\right) x_0^2. \tag{5.34}$$

This chapter has been a review of Çakmak and Özekici [4], in which the whole derivation of the solution procedure is given. From now on, we will suppress $i$, which denotes the initial market state, in $a_1(i), a_2(i)$ and $b(i)$ to simplify the notation so that $a_1, a_2$ and $b$ are going to be used in place of them. It must be remembered that $a_1, a_2$ and $b$ are still functions of the initial market state $i$.

Chapter 6

# SOLUTION PROCEDURE FOR GENERAL UTILITY FUNCTIONS

All multiperiod portfolio selection problems that are considered in this thesis have objectives that are functions of the mean and variance of the final wealth which implies that the objective of a general utility model can be denoted by $U\left(E_i\left[X_T\right], \mathrm{Var}_i\left(X_T\right)\right)$. The multiperiod portfolio selection problem then takes the form

$$\begin{aligned} U \quad : \quad \max \quad & U\left(E_i\left[X_T\right], \mathrm{Var}_i\left(X_T\right)\right) \\ \text{s.t.} \quad & X_{n+1}(u) = r_f\left(Y_n\right) X_n + R^e\left(Y_n\right)' u. \end{aligned} \tag{6.1}$$

After finding the optimal policy of the multiperiod mean-variance problem and the corresponding expectation and variance of the final wealth by using the auxiliary problem $P4(\lambda, \omega)$, the solution procedure for general multiperiod portfolio problems continues by replacing the corresponding expressions for the mean and variance, given in (5.23) and (5.25) respectively, into the objective function $U\left(E_i\left[X_T\right], \mathrm{Var}_i\left(X_T\right)\right)$ of the specific problem. The only restriction regarding the use of this solution procedure is that the utility function has to be a function of the expected final wealth $E_i\left[X_T\right]$ and the variance of the final wealth $\mathrm{Var}_i\left(X_T\right)$, and that the function has to be increasing with respect $E_i\left[X_T\right]$ and decreasing with respect to $\mathrm{Var}_i\left(X_T\right)$ in order to assure that the auxiliary problem gives equivalent solutions on the efficient frontier for the problem in consideration.

Investors in this thesis are assumed to have an objective of maximizing their final wealth while keeping their risk as low as possible so that the utility function $U\left(E_i\left[X_T\right], \mathrm{Var}_i\left(X_T\right)\right)$ should satisfy

$$\frac{\partial U\left(E_i\left[X_T\right], \mathrm{Var}_i\left(X_T\right)\right)}{\partial E_i\left[X_T\right]} > 0 \tag{6.2}$$

and

$$\frac{\partial U\left(E_i\left[X_T\right], \mathrm{Var}_i\left(X_T\right)\right)}{\partial \mathrm{Var}_i\left(X_T\right)} < 0. \tag{6.3}$$

It should be noted that this utility function can be nonlinear with respect to $E_i\left[X_T\right]$ and $\mathrm{Var}_i\left(X_T\right)$ or can include any fraction of $E_i\left[X_T\right]$ and $\mathrm{Var}_i\left(X_T\right)$, provided that the above conditions hold.

Kroll et al. [36] compared efficient mean-variance portfolios with optimal portfolios from direct utility maximization problem. They showed that the best portfolio obtained from a set of efficient mean-variance portfolios that maximizes $U\left(E_i\left[X_T\right], \mathrm{Var}_i\left(X_T\right)\right)$ and the optimal portfolio obtained by direct utility maximization are highly correlated to each other. In other words, it is found that the best mean-variance efficient portfolio is frequently the portfolio which maximizes the expected utility or at least has a near optimum expected utility. The utility functions analyzed by them are of the form

$$\text{exponential}: \; -e^{-(1+R)},$$

$$\text{power}: \; (1+R)^a \quad (a=0.1, 0.5, 0.9),$$

$$\text{power:} \; (2+R)^a \quad (a=0.1, 0.5)$$

and

$$\text{logarithmic}: \; \ln(i+R) \quad (i=1,2)$$

which are general utility functions that are mostly used to describe investor behavior, where $R$ stands for the rate of return on investment. According to their results, it is logical to focus on utility functions with $\partial U/\partial E_i\left[X_T\right] > 0$ and $\partial U/\partial \mathrm{Var}_i\left(X_T\right) < 0$, which imply mean-variance portfolios on the efficient frontier, since they give near optimal solutions even for other utility functions considered.

Li and Ng [17] prove that problem $U$ given in (6.1) can be embedded into problem $P3(\omega)$ given in (4.4) which further can be embedded into the auxiliary problem $P4(\lambda, \omega)$ given in (4.5) implying that a multiperiod portfolio problem of maximizing $U\left(E_i\left[X_T\right], \mathrm{Var}_i\left(X_T\right)\right)$ can be embedded into $P4(\lambda, \omega)$.

The general solution procedure is as follows: After replacing the optimal values of $E_i\left[X_T\right]$ and $\mathrm{Var}_i\left(X_T\right)$ given in (5.23) and (5.25) into $U\left(E_i\left[X_T\right], \mathrm{Var}_i\left(X_T\right)\right)$, the objective function of the specific problem $U$ is obtained in terms of $\gamma$. The next step is then to obtain the derivative of $U$ with respect to $\gamma$ and solve for the maximum point that will be reached at $\gamma^*$. In general

$$\frac{dU}{d\gamma} = \left(\frac{\partial U}{\partial E_i\left[X_T\right]} - 2E_i\left[X_T\right]\frac{\partial U}{\partial \mathrm{Var}_i\left(X_T\right)}\right)\frac{dE_i\left[X_T\right]}{d\gamma} + \frac{\partial U}{\partial \mathrm{Var}_i\left(X_T\right)}\frac{dE_i\left[X_T^2\right]}{d\gamma} \tag{6.4}$$

where

$$\frac{dE_i\left[X_T\right]}{d\gamma}=b \text{ and } \frac{dE_i\left[X_T^2\right]}{d\gamma}=b\gamma \tag{6.5}$$

which are found from (5.23) and (5.24) respectively. Setting $dU/d\gamma$ equal to zero, the necessary optimality condition for $\gamma$ is obtained

$$\left(\frac{\partial U}{\partial E_i\left[X_T\right]}-2E_i\left[X_T\right]\frac{\partial U}{\partial \mathrm{Var}_i\left(X_T\right)}\right)+\frac{\partial U}{\partial \mathrm{Var}_i\left(X_T\right)}\gamma=0 \tag{6.6}$$

which implies

$$\gamma^*=2E_i\left[X_T\right]-\frac{\partial U}{\partial E_i\left[X_T\right]}/\frac{\partial U}{\partial \mathrm{Var}_i\left(X_T\right)}. \tag{6.7}$$

Furthermore, it has to be verified that the utility function is a concave function of $\gamma$ so that the maximum utility is obtained. The optimal portfolio policy for the related problem will be obtained by substituting $\lambda/\omega$ in (5.13) with the optimal $\gamma^*$. Finally, the expectation and variance of the final wealth are calculated by substituting the optimal $\gamma^*$ into (5.23) and (5.25) respectively.

# Chapter 7

## QUADRATIC UTILITY FUNCTION FORMULATION

The general quadratic function $X_T - AX_T^2$, where $X_T$ denotes the final wealth and $A$ is a positive coefficient, is the utility function that has been used frequently in the economics and finance literature to describe investor behavior. The problem of maximizing the expectation of this utility function corresponds to the auxiliary problem $P4(\lambda, \omega)$ which has the objective function of maximizing $E\left[-\omega X_T^2 + \lambda X_T\right]$. Rearranging this auxiliary objective gives $\lambda E\left[-\frac{\omega}{\lambda} X_T^2 + X_T\right]$, which shows that the quadratic utility problem is a special case of the auxiliary problem with $\lambda = 1$ and $\omega = A$. This means that no initial condition is required to solve this problem since the solution obtained in Chapter 5 is also valid in this case.

The quadratic utility function is a function of $X_T$ and $X_T^2$ which will be a function of $E\left[X_T\right]$ and $E\left[X_T^2\right]$ after taking the expectation of it. By using the definition of the variance

$$E\left[X_T - AX_T^2\right] = E\left[X_T\right] - AE[X_T^2] = E[X_T] - A\left[\text{Var}(X_T) + E\left[X_T\right]^2\right] \tag{7.1}$$

so that the multiperiod objective can be expressed as a function of $E_i\left[X_T\right]$ and $\text{Var}_i\left(X_T\right)$ with

$$U\left(E_i\left[X_T\right], \text{Var}_i\left(X_T\right)\right) = -AE_i\left[X_T\right]^2 + E_i[X_T] - A\text{Var}_i(X_T). \tag{7.2}$$

The multiperiod portfolio problem of an investor having this utility function is

$$\begin{aligned} QU(A) \ : \ \max \quad & -AE_i\left[X_T\right]^2 + E_i[X_T] - A\text{Var}_i(X_T) \\ \text{s.t.} \quad & X_{n+1}\left(u\right) = r_f\left(Y_n\right) X_n + R^e\left(Y_n\right)' u. \end{aligned} \tag{7.3}$$

Given the initial state $i$, the expectation and variance of the final wealth are already given in (5.23) and (5.25). After putting these expressions into the objective function given in (7.3), the objective in terms of $\gamma$ turns out to be

$$U\left(\gamma\right) = -\frac{1}{2}Ab\gamma^2 + b\gamma + a_1x_0 - Aa_2x_0^2. \tag{7.4}$$

Taking the derivative of $U(\gamma)$ with respect to $\gamma$ and equating it to zero

$$\frac{dU}{d\gamma} = -Ab\gamma + b = 0 \tag{7.5}$$

reveals one extreme point so that

$$\gamma^* = \frac{1}{A}. \tag{7.6}$$

This optimal point is also a maximum point since the second derivative of $U(\gamma)$ with respect to $\gamma$ turns out to be negative

$$\frac{d^2U}{d\gamma^2} = -Ab < 0. \tag{7.7}$$

The degree of the risk aversion exhibited by a utility function is related to the magnitude of the bend in the function (the stronger the bend, the greater the risk aversion) which can be quantified in terms of the second derivative of the utility function. This means that investors having the quadratic function as their utility function are becoming more risk averse as $A$ increases since in such a case the quadratic utility function has a higher curvature.

The solution procedure to solve this problem is as follows: The optimal $\gamma^*$ is $1/A$. The optimal policy $u_n(i,x)$ will follow directly from the expression given in (5.13) by replacing the optimal $\gamma^*$ in place of $\lambda/\omega$. The expected value and the variance of the final wealth $X_T$ are then found by replacing $\gamma^*$ into (5.23) and (5.25) respectively. The efficient frontier obtained in this problem corresponds to the same efficient frontier obtained from the mean-variance problem $P3(\omega)$. To get the same $(\sqrt{\text{Var}_i(X_T)}, E_i[X_T])$ pair on the efficient frontiers, the optimal $\gamma^*$ values of both problems are equated. The optimal $\gamma^*$ of $P3(\omega)$ in terms of $\omega$ is given in Çakmak and Özekici [4] as

$$\gamma^* = \frac{1 + 2\omega a_1 x_0}{\omega - 2b\omega}. \tag{7.8}$$

Equating $\gamma^*$ in (7.8) to $\gamma^*$ in (7.6) reveals the relationship between the parameter of $P3(\omega)$ and the parameter of $QU(A)$ which is

$$\omega = \frac{A}{1 - 2b - 2Aa_1x_0}. \tag{7.9}$$

Equation (7.9) shows that by changing the value of parameter $A$ and solving the quadratic utility problem for various values of it, the mean-variance efficient frontier is obtained.

Since the quadratic utility function formulation is a special case of the auxiliary problem $P4(\lambda, \omega)$ that is solved optimally using dynamic programming, the multiperiod quadratic utility problem can also be directly solved without putting any condition on the input parameter of the problem. However, most of the time, investors are assumed to prefer more wealth to less wealth which corresponds to the nonsatiation property, implying that the first derivative of the utility function with respect to the expected final wealth should be positive. If an investor who is consistent with the nonsatiation assumption is to be considered, the following condition

$$\frac{\partial U}{\partial E_i\left[X_T\right]} = -2AE_i\left[X_T\right] + 1 > 0 \tag{7.10}$$

must be placed on $E_i\left[X_T\right]$, which implies

$$E_i\left[X_T\right] < \frac{1}{2A}. \tag{7.11}$$

An additional analysis for this problem is then to find the range of the coefficient $A$ that will assure that $E_i\left[X_T\right] < 1/2A$ which corresponds to nonsatiation. Given the expected final wealth $E_i\left[X_T\right]$ in (5.23), this condition turns into the inequality $a_1x_0 + b\gamma < 1/2A$ where $\gamma$ is taken to be the optimal one which is $1/A$. After some manipulation, the range for the coefficient $A$ turns out to be

$$A < \frac{1-2b}{2a_1x_0} = A^*. \tag{7.12}$$

This range $A \in (0, A^*)$ will ensure that the investor prefers more wealth to less wealth in case that other decision criteria are equal. Nonsatiation property is not necessary to solve the quadratic utility problem. However, given the input parameters, the range of the coefficient $A$ can be found for which the nonsatiation property is satisfied. Then, the solution would be obtained for that $A$ value which is specified by the investor.

Moreover, if the investor is also assumed to exhibit risk aversion, the first derivative of the utility function with respect to the variance of the final wealth

$$\frac{\partial U}{\partial \mathrm{Var}_i\left(X_T\right)} = -A \tag{7.13}$$

is expected to be negative. This condition is already satisfied since $A$ is taken to be positive.

The bounds of the range $(0, A^*)$ correspond to the minimum variance point for $A$ approaching $A^*$ and to the upper end of the efficient frontier for $A$ approaching 0 in the limit. For $A = A^*$, the optimal $\gamma^*$, which is $1/A$ as given in (7.6), is equal to the $\gamma^*$ value of the minimum variance portfolio given in (5.32); and for $A$ approaching 0, the optimal $\gamma^*$

approaches $+\infty$ which implies infinite $E_i[X_T]$ and $\text{Var}_i(X_T)$ from (5.23) and (5.25) respectively. For intermediate values of the parameter $A$, the optimal portfolios move upwards on the efficient frontier as $A$ is decreased from $A^*$ to 0. This result is expected since a higher value of $A$ implies higher risk aversion so that less money is invested in the risky assets which leads to lower expectation and lower variance for the final wealth. If $A$ is further increased above $A^*$, which is possible since the quadratic utility problem does not require an initial condition for the proposed solution procedure, the portfolios obtained are on the minimum-variance set but not on the efficient frontier anymore.

The quadratic utility problem on its own does not have an explicit interpretation except for the fact that it can satisfy the risk-averseness and the nonsatiation property of the investor by putting some constraints on its parameter $A$. However, it is important to note here that there is a utility problem having an explicit interpretation that turns out to have the same objective function as the quadratic utility problem. The objective of this problem can be attached a certain meaning and it is given as

$$\min P\{\mid X_T - \alpha \mid > \epsilon\}. \tag{7.14}$$

This objective aims to get a final wealth $X_T$ which is not significantly different from a specified value $\alpha$, which is logically assumed to be greater than zero. That is, the investor is trying to maximize the probability that $X_T$ is in the vicinity of $\alpha$. Using Markov's inequality

$$P\{(X_T - \alpha)^2 > \epsilon^2\} \leq \frac{E\left[(X_T - \alpha)^2\right]}{\epsilon^2}, \tag{7.15}$$

the objective function in (7.14) turns out to have the upper bound $E\left[X_T^2 - 2\alpha X_T + \alpha^2\right]/\epsilon^2$. Minimizing this upper bound is the same as minimizing $E\left[X_T^2\right] - 2\alpha E\left[X_T\right]$ since both $\alpha^2$ and $\epsilon^2$ are predetermined. Rearranging this expression yields the objective

$$\max 2\alpha \left(E\left[X_T\right] - \frac{1}{2\alpha} E\left[X_T^2\right]\right) \tag{7.16}$$

which is equivalent to

$$\max \left\{E\left[X_T\right] - \frac{1}{2\alpha} E\left[X_T^2\right]\right\} \tag{7.17}$$

since $\alpha$ is taken to be greater than zero.

Comparing (7.17) with (7.1) shows that this problem is the same as the quadratic utility problem with $A = 1/2\alpha$, meaning that the same solution procedure as given above can be used for this problem as well. In order for the nonsatiation property to be satisfied, the condition in (7.12) can be rearranged, by putting $1/2\alpha$ in place of $A$, to give $\alpha > (a_1x_0)/(1-2b) = k^*$ where $k^*$ is constant for a given portfolio problem and an important notation used in the formulation of the safety-first problem.

# Chapter 8

## COEFFICIENT OF VARIATION MODEL FORMULATION

The coefficient of variation is a measure of relative dispersion for a probability distribution and is defined formally as the ratio of the standard deviation over the mean; $\sqrt{\mathrm{Var}\left(X_T\right)}/E\left[X_T\right]$. It is generally expressed as a percentage. The use of the coefficient of variation lies in the fact that the mean and standard deviation tend to change together in many cases so that a knowledge of relative variation is valuable in distributions.

A logical objective function for an investor dealing with the multiperiod portfolio optimization would be to minimize the coefficient of variation of the final wealth. This logic can be turned formally into a multiperiod portfolio problem objective by making use of the reciprocal of the coefficient of variation. Minimizing the coefficient of variation turns into maximizing the utility function given by

$$U\left(E\left[X_T\right], \mathrm{Var}\left(X_T\right)\right) = \frac{E\left[X_T\right]}{\sqrt{\mathrm{Var}\left(X_T\right)}}. \tag{8.1}$$

The corresponding multiperiod problem is then

$$\begin{aligned} CV \quad : \quad & \max \quad \frac{E_i\left[X_T\right]}{\sqrt{\mathrm{Var}_i\left(X_T\right)}} \\ & \text{s.t.} \quad X_{n+1}\left(u\right) = r_f\left(Y_n\right) X_n + R^e\left(Y_n\right)' u. \end{aligned} \tag{8.2}$$

Since the problem does not require any input parameters associated with the investor's preferences, the coefficient of variation problem is perhaps the most objective one among other portfolio selection problems. The derivative of this objective function with respect to the expected final wealth is

$$\frac{\partial U}{\partial E_i\left[X_T\right]} = \frac{1}{\sqrt{\mathrm{Var}_i\left(X_T\right)}} > 0 \tag{8.3}$$

and the derivative of this objective function with respect to the variance of the final wealth is

$$\frac{\partial U}{\partial \mathrm{Var}_i\left(X_T\right)} = -\frac{1}{2}\frac{E_i\left[X_T\right]}{\sqrt{\mathrm{Var}_i\left(X_T\right)^3}} < 0 \tag{8.4}$$

which reveal that the auxiliary problem $P4(\lambda, \omega)$ is applicable to solve this problem. For the expected final wealth to be greater than zero so that the condition in (8.4) is satisfied, $\gamma$ has to be greater than $-a_1 x_0 / b$ which is a negative quantity. Since the optimal $\gamma^*$ given subsequently turns out to be greater than zero, this condition is already satisfied.

Replacing $E_i[X_T]$ and $\mathrm{Var}_i\,(X_T)$ given in (5.23) and (5.25) into the objective function of problem (8.2), we obtain

$$U\left(\gamma\right)=\frac{a_1 x_0 + b\gamma}{\sqrt{\left(a_2 - a_1^2\right) x_0^2 + \left(\frac{b}{2} - b^2\right)\gamma^2 - 2a_1 b x_0 \gamma}}. \tag{8.5}$$

The first derivative of the objective function $U\left(\gamma\right)$ with respect to $\gamma$ is

$$\frac{dU}{d\gamma}=\frac{a_2 b x_0^2 - 0.5 a_1 b x_0 \gamma}{\left[(a_2 - a_1^2)x^2 + (0.5b - b^2)\gamma^2 - 2a_1 b x \gamma\right]^{3/2}}. \tag{8.6}$$

Equating the derivative in (8.6) to zero gives the single optimal point to be

$$\gamma^* = \frac{2a_2 x_0}{a_1}. \tag{8.7}$$

The first derivative in (8.6) reveals that the optimal $\gamma^*$ is a maximum point as required. This is obvious from the fact that the first derivative is positive for $\gamma$ values smaller than $\gamma^*$ and negative for $\gamma$ values greater than $\gamma^*$.

The coefficient of variation problem has only one single solution since the objective function does not involve any parameter which could depend on the investor's preferences. This means that the solution is the same for all investors and it gives a single point on the efficient frontier. This point corresponds to a single $\omega$ value of problem $P3(\omega)$. This value can be found by equating the optimal $\gamma^*$ values of both problems given in (7.8) and (8.7). The relationship turns out to be

$$\omega = \frac{a_1}{2x_0(a_2 - 2a_2 b - a_1^2)}. \tag{8.8}$$

Finally, the optimal policy to this problem can be obtained by replacing the optimal $\gamma^*$ in place of $\lambda/\omega$ in (5.13). The optimal values of $E_i[X_T]$ and $\mathrm{Var}_i\,(X_T)$ are also found by replacing $\gamma^*$ into (5.23) and (5.25) respectively.

# Chapter 9

# SAFETY-FIRST MODEL FORMULATION

## *9.1 Single Period Investment Model*

The objective of the safety-first model is to minimize the downside risk of an investor for the whole investment horizon by investing in portfolios that change periodically. According to the original definition given by Roy [3], it is the minimization of the probability of a disaster or the chance of a dread event which corresponds to an undesired return at the end of the investment horizon. More formally, the safety-first problem deals with the minimization of the probability that the final wealth $X_T$ is smaller than a prespecified disaster level $k$ given by the investor (i.e., $P\{X_T \leq k\}$). This minimization corresponds to the maximization of its negative which in turn is the same problem as

$$\max \{1 - P\{X_T \leq k\}\} \tag{9.1}$$

which is equivalent to

$$\max P\{X_T > k\}. \tag{9.2}$$

According to this result, the safety-first objective can be stated as maximizing the probability that the final wealth $X_T$ is greater than a prespecified level $k$ given by the investor. In other words, this problem also tries to maximize the upside potential.

Roy [3] and many other researchers made use of the Chebyshev's inequality to formulate the safety-first problem. The reason for using this bound is that it is very robust since it does not assume any distribution about the random variable considered. Chebyshev's inequality states that

$$P\{X_T \leq k\} \leq P\{\mid X_T - E\left[X_T\right] \mid \geq E\left[X_T\right] - k\} \leq \frac{\mathrm{Var}\left(X_T\right)}{\left(E\left[X_T\right] - k\right)^2} \tag{9.3}$$

which leads to

$$P\{X_T \leq k\} \leq \frac{\mathrm{Var}\left(X_T\right)}{\left(E\left[X_T\right] - k\right)^2}. \tag{9.4}$$

The objective is then to minimize the upper bound

$$\frac{\mathrm{Var}\,(X_T)}{(E\,[X_T] - k)^2} \tag{9.5}$$

which in turn can be stated as to maximize the ratio

$$\frac{E\,[X_T] - k}{\sqrt{\mathrm{Var}\,(X_T)}} \tag{9.6}$$

so that the final objective of the safety-first portfolio selection problem is maximizing

$$U\,(E\,[X_T]\,, \mathrm{Var}\,(X_T)) = \frac{E\,[X_T] - k}{\sqrt{\mathrm{Var}\,(X_T)}}. \tag{9.7}$$

The version of the safety-first criteria introduced here falls into the category of general mean-standard deviation, because the Chebyshev's inequality uses only the first two moments of the final wealth to substitute the information of the whole distribution. Because the Chebyshev's inequality provides an upper limit on the probability of goal failure, the decisions made based on this version of safety-first criteria may be more conservative than necessary to achieve the original criteria. Therefore, other inequalities that can provide the limit closer to the actual probabilities are studied in the literature. It is well known that the Chebyshev's inequality is not very strict and correspondingly is a rather loose bound. Its relevance remains in its generality, not in its accuracy. Nevertheless, the inaccuracy of this inequality implies that the model is conservative, that is to say that the true risk is smaller than the one considered by the model.

An alternative approach to using the Chebyshev's inequality is based on the assumption that the distribution of $X_T$ can be described by two parameters, namely its mean $\mu$ and variance $\sigma^2$. According to this idea, a standardized variable

$$\frac{X_T - E\,[X_T]}{\sqrt{\mathrm{Var}\,(X_T)}} = \frac{X_T - \mu}{\sigma} \tag{9.8}$$

and its probability distribution function

$$P\{X_T \leq k\} \equiv F\left(\frac{k - \mu}{\sigma}\right) \tag{9.9}$$

are defined. Since minimizing the safety-first objective (i.e., minimizing $P\{X_T \leq k\}$) is the same as minimizing the cumulative probability distribution in (9.9) and therefore its standardized variable $(k - \mu)/\sigma$ due to the monotonic property of the cumulative function, the maximization of the same ratio $(\mu - k)/\sigma$ follows which corresponds to

$$\max\left(\frac{E\left[X_T\right]-k}{\sqrt{\operatorname{Var}\left(X_T\right)}}\right). \tag{9.10}$$

The formulated safety-first objective is a utility function of the expected value and the variance of the final wealth $X_T$;

$$U\left(E\left[X_T\right], \operatorname{Var}\left(X_T\right)\right)=\frac{E\left[X_T\right]-k}{\sqrt{\operatorname{Var}\left(X_T\right)}}. \tag{9.11}$$

This utility has to be an increasing function of the expected value of $X_T$ and a decreasing function of the variance of $X_T$ in order to be able to apply the auxiliary problem $P4(\lambda, \omega)$. The derivative with respect to the expected final wealth

$$\frac{\partial U}{\partial E\left[X_T\right]}=\frac{1}{\sqrt{\operatorname{Var}\left(X_T\right)}} \tag{9.12}$$

is greater than zero for all values of expectation and variance, whereas the derivative with respect to the variance of the final wealth

$$\frac{\partial U}{\partial \operatorname{Var}\left(X_T\right)}=-\frac{1}{2} \frac{E\left[X_T\right]-k}{\sqrt{\operatorname{Var}\left(X_T\right)^3}} \tag{9.13}$$

is smaller than zero for all values of the variance but only for values of expected final wealth that are greater than the disaster level. This means that the solution procedure that will be based on the auxiliary problem $P4(\lambda, \omega)$ is applicable only if the expected final wealth $E\left[X_T\right]$ is greater than the disaster level $k$, implying that $E\left[X_T\right] > k$ is the primal condition of the safety-first problem in order to be able to apply the auxiliary problem. The condition that the expected final wealth should be greater than the disaster level given by the investor is not only a technical constraint but it is also a logical one since the value of $k$ should reflect a disaster which logically should be set smaller than the expected amount of money.

The safety-first approach can be used by investors whose main interest is speculative gain or by investors who are content with modest yields over a long period of time. For example, people may try to ensure that they make a speculative gain of not less than five per cent corresponding to the objective of minimizing $P\{X_T \leq 1.05\}$ or that they prevent an excessive loss of five per cent at the end of the investment horizon corresponding to the objective of minimizing $P\{X_T \leq 0.95\}$.

Overall, the safety-first objective limits portfolio losses associated with infrequent disastrous events and otherwise optimizes performance. Safety-first considers the investor's desire to minimize the chance of large negative returns, and may be appropriate for emerging markets, because their asset distributions are subject to extreme returns. There are many other instances where safety-first consideration would be appropriate for protection from dreadful events that might substantially erode wealth.

The different safety-first criteria which are introduced shortly in the literature survey in Chapter 2 produce equivalent choices for portfolio optimization when distributions are normal, but are more complicated in the absence of normality. In all cases, they require the existence of first and second moments in return distributions and allow for use of Chebyshev's inequality to calculate the maximum probabilities of obtaining outcomes below some prespecified value in the absence of any assumptions concerning return distributions. The complications arising from safety-first problem seem unavoidable when trying to limit the downside risk.

## 9.2 Multiperiod Investment Model

Consistent with the single period objective given in (9.7), the multiperiod safety-first problem is formulated as

$$\begin{aligned} SF(k) \quad : \quad \max \quad & \frac{E_i\left[X_T\right]-k}{\sqrt{\operatorname{Var}_i\left(X_T\right)}} \\ \text{s.t.} \quad & X_{n+1}(u)=r_f\left(Y_n\right)X_n+R^e\left(Y_n\right)'u. \end{aligned} \tag{9.14}$$

After replacing the expressions for the expectation and the variance of the final wealth given in (5.23) and (5.25) respectively into the safety-first objective function, the safety-first utility can be expressed in terms of $\gamma$ as

$$U(\gamma)=\frac{a_1x_0+b\gamma-k}{\sqrt{\left(a_2-a_1^2\right)x_0^2+\left(\frac{b}{2}-b^2\right)\gamma^2-2a_1bx_0\gamma}}. \tag{9.15}$$

The only thing that remains to solve the safety-first problem is then to find the extreme points of this utility function and then to check whether they are minimum or maximum points. It turns out that there is a unique extreme point that satisfies

$$\frac{dU}{d\gamma}=\frac{b\left[a_2x_0^2-a_1x_0k-0.5a_1x_0\gamma+0.5k\gamma-kb\gamma\right]}{\left[(a_2-a_1^2)x_0^2+(0.5b-b^2)\gamma^2-2a_1bx_0\gamma\right]^{3/2}}=0 \tag{9.16}$$

so that

$$\gamma^*=\frac{2a_2x_0^2-2a_1kx_0}{a_1x_0-k+2bk}. \tag{9.17}$$

The sign of the first derivative depends on disaster level $k$. It turns out that the utility function has a maximum point for $k$ values smaller than

$$k^* = \frac{a_1 x_0}{1-2b} \tag{9.18}$$

and a minimum point for $k$ values greater than $k^*$.

Taking the derivative of $\gamma^*$ with respect to $k$ and analyzing it reveals that $\gamma^*$ is an increasing function of $k$; implying that the higher the value of $k$ is, the higher the value of optimal $\gamma^*$ will be. Higher $\gamma^*$ on the other hand leads to higher mean and variance on the efficient frontier for the final wealth which can be verified from (5.23) and (5.25). This result is not unexpected since choosing a higher disaster level requires a bigger portion of money to be invested in risky assets so as not to fall below the now-higher level so that they cause both the mean and the variance of the final wealth to increase. Furthermore, as the disaster level increases, the probability of disaster also increases.

At the end of the previous section which is about the single period safety-first problem, it is stated that $E[X_T] > k$ is the primal condition of the safety-first problem in order to be able to apply the auxiliary problem. To find the allowable range of the disaster level $k$ which will assure that $E_i[X_T] > k$, the optimal $\gamma^*$ given in (9.17) is replaced into (5.23) so that $E_i[X_T]$ is found in terms of the disaster level as

$$E_i[X_T] = \frac{x_0\left[a_1\left(a_1 x_0 - k\right) + 2a_2 b x_0\right]}{a_1 x_0 - k + 2bk}. \tag{9.19}$$

The next step is then to find the relationship between $E_i[X_T]$ in (9.19) and the disaster level $k$. Two cases arise depending on the sign of the denominator in (9.19).

If the denominator in (9.19) is positive corresponding to the case that $k < k^*$, where $k^*$ is already given in (9.18), $E_i[X_T]$ is greater than $k$ if

$$x_0\left[a_1\left(a_1 x_0 - k\right) + 2a_2 b x_0\right] > k(a_1 x_0 - k + 2bk) \tag{9.20}$$

or equivalently if

$$(1-2b)k^2 - 2a_1 x_0 k + a_1^2 x_0^2 + 2a_2 b x_0^2 > 0. \tag{9.21}$$

The real roots of the left-hand-side (LHS) in (9.21) denoted by $k_{1,2}$ are found to be

$$k_{1,2} = k^* \pm \frac{\sqrt{2b}x_0}{(1-2b)}\sqrt{a_1^2 + 2a_2 b - a_2}. \tag{9.22}$$

The expression under the second radical in (9.22), formally $a_1^2 + 2a_2b - a_2$, turns out to be smaller than or equal to zero. This result is obtained with the help of equation (5.34) that gives the formula for $\text{Var}_i(X_T)$ of the minimum variance portfolio for a given multiperiod problem, which is the same as

$$\text{Var}_i(X_T) = \frac{\left(a_2 - 2a_2b - a_1{}^2\right)x_0^2}{(1-2b)} \tag{9.23}$$

after some arrangement. It is known from Çakmak and Özekici [4] that $b < 1/2$ implying that the denominator in (9.23) is positive. In order for the whole expression of $\text{Var}_i(X_T)$ in (9.23) to be greater than or equal to zero, $\left(a_2 - 2a_2b - a_1{}^2\right)$ must be greater than or equal to zero as well, which is exactly the negative of the expression under the second radical in (9.22) implying that

$$a_1^2 + 2a_2b - a_2 \leq 0. \tag{9.24}$$

This means that the LHS in (9.21) either has no real root corresponding to the case that $a_1^2+2a_2b-a_2 < 0$ or one real root, which is $k^*$, corresponding to the case that $a_1^2+2a_2b-a_2 = 0$ .

The LHS in (9.21) is a quadratic function of $k$ with the coefficient $(1-2b)$ in front $k^2$ being positive using $b < 1/2$. This argument combined with the previous results regarding the roots of the given expression proves that the LHS is always greater than or equal to zero, the equality reached for $k = k^*$. Since this analysis is valid for $k < k^*$, the condition (9.21) is satisfied for disaster levels smaller than $k^*$.

If the denominator in (9.19) is negative corresponding to the case that $k > k^*$, $E_i[X_T]$ is greater than $k$ if

$$x_0\left[a_1(a_1x_0 - k) + 2a_2bx_0\right] < k(a_1x_0 - k + 2bk) \tag{9.25}$$

or equivalently if

$$(1-2b)k^2 - 2a_1x_0k + a_1^2x_0^2 + 2a_2bx_0^2 < 0. \tag{9.26}$$

However, it is already shown that the LHS in (9.26) is always greater than or equal to zero so that the condition (9.26) cannot hold, which shows that the safety-first problem cannot be solved for $k > k^*$ using the solution procedure proposed in this thesis.

To sum up, the primal condition of the safety-first problem (i.e., $E_i[X_T] > k$) is satisfied for disaster levels which are smaller than the critical level $k^*$ defined in (9.18). The important feature of this critical level $k^*$ is that if the condition $k \in (-\infty, k^*)$ is satisfied, the safety-first utility function in (9.15) has a well-defined single optimal $\gamma^*$ given in (9.17) that maximizes investor's utility at the end of the investment horizon.

The disaster level has an upper bound so that the investor is advised to require only modest returns for his investment since the main objective should be minimizing the down-side risk and not maximizing the gain. Given an initial wealth of 1 unit, the safety-first problem can be solved for investors requiring a minimal return of up to $k^*$.

The constraint for the disaster level range leads to another argument regarding the efficient portfolios. At the end of Chapter 5, the expected final wealth of the minimum variance point is given to be $(a_1 x_0)/(1-2b)$, and it is stated that $E_i[X_T] \geq (a_1 x_0)/(1-2b)$ so that optimal portfolios on the efficient frontier are obtained. The right-hand-side of this inequality is the same as $k^*$, showing that $E_i[X_T] \geq k^*$ for efficient portfolios. Since disaster levels up to $k^*$ are allowed for the safety-first problem, it is guaranteed for efficient portfolios that the expected final wealth will be greater than the specified disaster level $k$.

The proposed solution procedure for the multiperiod safety-first problem is as follows: First, the range of disaster level $k$ is found for which the auxiliary problem is applicable; that is, for which $E_i[X_T]$ is greater than the disaster level $k$. Then the optimal $\gamma^*$ is calculated using 9.17 and the optimal policy $u_n(i,x)$ will follow directly from the expression given in (5.13) by replacing the optimal $\gamma^*$ in place of $\lambda/\omega$, which leads to the optimal expectation versus standard deviation pair on the efficient frontier. The expected value and the variance of the final wealth $X_T$ are found by replacing $\gamma^*$ given in (9.17) into (5.23) and (5.25) respectively.

The efficient frontier obtained in this problem corresponds to the same efficient frontier obtained from problem $P3(\omega)$. To get the same $(\sqrt{\mathrm{Var}_i(X_T)}, E_i[X_T])$ pair on the efficient frontiers, the optimal $\gamma^*$ of $P3(\omega)$ given in (7.8) and the optimal $\gamma^*$ of safety first problem given in (9.17) are equated to obtain

$$\omega = \frac{a_1 x_0 - k + 2bk}{2x_0^2(a_2 - 2a_2 b - a_1^2)} \tag{9.27}$$

which shows that the selected disaster level corresponds to a certain $\omega$ value. This result implies that solving the safety-first problem for different values of the disaster level will lead to optimal portfolios on the mean-variance efficient frontier.

The bounds of the allowable range $(-\infty, k^*)$ correspond to the minimum variance point for $k$ approaching $-\infty$ and to the upper end of the efficient frontier for $k$ approaching $k^*$ in the limit. For $k$ approaching $-\infty$, the optimal $\gamma^*$ given in (9.17), approaches to the $\gamma^*$ value of the minimum variance portfolio given in (5.32); and for $k$ approaching $k^*$, the optimal $\gamma^*$ approaches $+\infty$ which implies infinite $E_i[X_T]$ and $\text{Var}_i(X_T)$ from (5.23) and (5.25) respectively. For intermediate values of the disaster level, the optimal portfolios move upwards on the efficient frontier as $k$ is increased from $-\infty$ to $k^*$. This result is not unexpected since more money has to be invested in the risky assets if a higher value of $k$ is required so as not to fall below this level which leads to higher expectation and higher variance for the final wealth.

The critical level $k^*$ is greater than zero, and it is shown numerically that it is actually greater than 1 for any multiperiod safety-first problem solved so far. This implies that the safety-first problem can be solved optimally for all disaster levels smaller than 1 since they are smaller than $k^*$ in such a case. This result makes a major contribution to our problem regarding the original meaning given by Roy [3]. Since $k$ levels chosen above 1 cannot actually be thought as a disaster in its real meaning, it is much more logical to choose them to be below 1, which is really a disaster since this implies a loss for the investor at the end of the investment horizon.

From among the models considered in this thesis, especially the safety-first problem is of practical importance. The following results are obtained for the safety-first problem:

- The investor can avoid a loss and, more importantly, secure a minimal return with a high probability by following the optimal safety-first investment policy.
- The realized final wealth at the end of the investment horizon is expected to be greater than the specified disaster level.
- The efficient frontier obtained from safety first approach exactly matches the mean-variance efficient frontier.

As a last remark to the safety-first problem, for investors requiring a return higher than $k^*$, the safety-first problem is not appropriate since its main aim is to minimize the downside risk. But instead, another problem can be used for these types of investors which is already discussed at the end of Chapter 7 and the formulation of which is given in (7.14).

The condition that $\alpha > k^*$ implies that an investor who wants his final wealth to be in the vicinity of $\alpha$, which should be higher than $k^*$, can use this problem and apply the corresponding policy which will better fit his objective of exceeding the critical disaster level at the end of the investment horizon.

Chapter 10

# PERIODIC ANALYSIS OF THE EFFICIENT FRONTIERS

The optimal solutions obtained by our solution procedure, specifically the expected value of the final wealth $E[X_T]$ and the variance of the final wealth $\mathrm{Var}(X_T)$, both of which depend on the initial state of the stochastic market, give final values at the end of the investment horizon which implies that they are not comparable to each other in case that $T$ is different for them. If the length of the investment horizons of given problems are not equal, the expected value and the variance of the final wealth will be based on different scales and therefore a logical comparison cannot be made. Two approaches are considered in this thesis that are used to transform the final results depending on $T$ to a periodic basis.

The periodic return is defined as the return with a constant mean and variance that will lead to the expected final wealth $E[X_T]$ and the variance of the final wealth $\mathrm{Var}(X_T)$ at the end of $T$ periods by investing the initial wealth periodically using that return. More formally,

$$X_T = X_0(1 + r_1)(1 + r_2)...(1 + r_T) \tag{10.1}$$

where $r_j$ denotes the periodic rate of return in period $j$. This equality states that the final wealth $X_T$ at the end of the investment horizon can be obtained by investing the initial wealth $X_0$ at a rate of return $r_j$ that has a certain distribution and that changes periodically over time due to its random nature provided by the mean and variance of the random return distribution.

The periodic rates of return $r_j$ are taken to be independent and identically distributed (IID) so that they have the same mean $r$ and the same variance $\sigma^2$. The assumption that periodic asset returns are IID through time is an acceptable approximation, and the justification for this assumption can be made as follows: If the investment horizon is taken to be long enough, the Markov chain representing the stochastic market will reach its steady state. This means that regardless of the initial state, the probability of reaching a certain state and therefore having a certain return at later time periods will be constant, implying that the periodic returns will be independent and identically distributed. Accordingly, $r$ and $\sigma^2$ denote the periodic mean and the periodic variance that lead to the same expected final wealth and the same variance of the final wealth at the end of the investment horizon.

Assuming that $X_0 = 1$ without loss of generality, $X_T$ corresponds to the compound return over $T$ periods. Taking the expectation of (10.1) while $X_0 = 1$ leads to

$$\begin{aligned} E[X_T] &= E[(1+r_1)(1+r_2)...(1+r_T)] \\ &= E[1+r_1]E[1+r_2]...E[1+r_T] \\ &= (1+r)^T = R^T. \end{aligned} \tag{10.2}$$

Equation (10.2) gives the relationship between the expected final wealth and the mean periodic rate of return $r$ or the mean periodic return $R$. Accordingly, the mean periodic return is found to be

$$R = E[X_T]^{1/T}. \tag{10.3}$$

Taking the variance of (10.1) and assuming that $X_0 = 1$ without loss of generality lead to

$$\text{Var}(X_T) = \text{Var}((1+r_1)(1+r_2)...(1+r_T)). \tag{10.4}$$

By using the definition of the variance

$$\begin{aligned} \text{Var}(X_T) &= E\left[(1+r_1)^2(1+r_2)^2...(1+r_T)^2\right] - E[(1+r_1)(1+r_2)...(1+r_T)]^2 \\ &= E\left[(1+r_j)^2\right]^T - ((1+r)^T)^2 \end{aligned} \tag{10.5}$$

and

$$E\left[(1+r_j)^2\right] = \text{Var}(1+r_j) + E[1+r_j]^2 = \sigma^2 + (1+r)^2. \tag{10.6}$$

Then, the variance for the $T$-period case is found to be

$$\begin{aligned} \text{Var}(X_T) &= \left(\sigma^2 + (1+r)^2\right)^T - ((1+r)^2)^T \\ &= \left(\sigma^2 + R^2\right)^T - (R^2)^T. \end{aligned} \tag{10.7}$$

After finding the mean and variance of the final wealth in terms of $R$ and $\sigma^2$ as given in (10.2) and (10.7) respectively, a system of two equations with two unknowns is obtained where $R$ and $\sigma^2$ are the unknowns whereas $E[X_T]$ and $\text{Var}(X_T)$ are already known after solving the multiperiod portfolio problem. First, the mean periodic return can be found from (10.3). Since the length of the investment horizon $T$ is already known for a given problem, the mean periodic return can be found easily. After finding $R$, (10.7) is used to find $\sigma^2$ where the variance of the final wealth $\text{Var}(X_T)$ is also known from the given solution procedure. After using these equations, the periodic standard deviation $\sigma$ and the periodic mean return $R$ are found to be

$$\left(\sqrt{\left(\text{Var}\left(X_T\right)+E\left[X_T\right]^2\right)^{1/T}-E\left[X_T\right]^{2/T}}, E\left[X_T\right]^{1/T}\right) \tag{10.8}$$

which can then be inserted into the mean-variance graph to get the periodic frontier.

In order to draw a frontier based on periodic analysis, the second form of the mean-variance formulation $P2\left(\mu\right)$ is used. As mentioned in Chapter 4, this problem is equivalent to the auxiliary problem $P4(\lambda,\omega)$ in generating efficient portfolios which is common to all problems discussed so far. This implies that the optimal solutions of all given problems will be on the efficient frontier obtained from $P2\left(\mu\right)$. The reason for using $P2\left(\mu\right)$ is that the efficient frontier on the mean versus standard deviation graph will be traced by specifying the expected value and then solving for the corresponding variance of the final wealth so that we will at least know one coordinate of the optimal point on the efficient frontier which makes the drawing of the frontier more tractable.

A question that arises, before illustrative periodic frontiers are drawn on a periodic mean-variance graph, is whether the transformed periodic frontiers will be efficient or not. Efficient frontiers have the important feature that portfolios on these frontiers provide the best mean-variance combinations for investors which are not dominated by each other in terms of the mean and variance and that any efficient portfolio can be duplicated as a combination of two other efficient portfolios. In order for the portfolios which are periodically invested at the random rate to be efficient, they have to be solutions to an optimization problem that maximizes the periodic mean return and minimizes the periodic variance given in (10.8) and that is formulated as

$$\max\left\{E\left[X_T\right]^{1/T}-\omega\left[\left(\text{Var}\left(X_T\right)+E\left[X_T\right]^2\right)^{1/T}-E\left[X_T\right]^{2/T}\right]\right\} \tag{10.9}$$

where $\omega$ is a positive coefficient. The objective (10.9) has to be an increasing function of $E\left[X_T\right]$ and a decreasing function of $\text{Var}(X_T)$ so that it is maximized by increasing $E\left[X_T\right]$ and decreasing $\text{Var}(X_T)$ which is consistent with the mean-variance trade-off. The derivative with respect to $\text{Var}(X_T)$

$$-\frac{\omega}{T}\left(\text{Var}\left(X_T\right)+E\left[X_T\right]^2\right)^{(1/T)-1} \tag{10.10}$$

turns out to be negative whereas the sign of the derivative with respect to $E\left[X_T\right]$

$$\frac{1}{T}\left\{E\left[X_T\right]^{(1/T)-1}-2\omega E\left[X_T\right]\left(\text{Var}\left(X_T\right)+E\left[X_T\right]^2\right)^{(1/T)-1}+2\omega E\left[X_T\right]^{(2/T)-1}\right\} \tag{10.11}$$

is inconclusive since it can be both positive or negative which means that the periodic frontiers obtained from this approach are not necessarily efficient. In the next chapter, it will be verified that the periodic frontiers obtained from this approach are not efficient since they turn out to be convex rather than concave.

A second approach for calculating the periodic mean returns and variances uses the same equation given in (10.1) for defining the wealth growth. Different from the previous one, this approach assumes that the periodic returns $r_j$ are small so that the product can be expanded and then only the first-order terms be kept to give

$$X_T = X_0(1+r_{end}) \approx X_0(1+r_1+r_2+...+r_T) \tag{10.12}$$

where $r_{end}$ is the rate of return that stands for the whole investment horizon. This relationship leads to the result that $r_{end}$ is approximately equal to the sum of all periodic returns; that is, $r_{end} \approx r_1+r_2+...+r_T = \sum_{j=1}^{T} r_j$. Making the same assumption as in the previous approach (i.e., assuming that the periodic returns are IID so that they have a common mean $r$ and variance $\sigma^2$) the expressions for the mean and variance of $r_{end}$ can be obtained as

$$E\left[r_{end}\right] = rT \tag{10.13}$$

$$\text{Var}\left(r_{end}\right) = \sigma^2 T. \tag{10.14}$$

Taking $X_0 = 1$ as before, $E\left[X_T\right]$ corresponds to the expected return at the end of the investment horizon with $E\left[X_T\right] = 1+E\left[r_{end}\right] = 1+rT$ using (10.12). This shows that the mean periodic rate of return can be obtained as

$$r = \frac{E\left[X_T\right]-1}{T}. \tag{10.15}$$

Moreover, by using (10.12), the variance of the final wealth turns out to be

$$Var\left(X_T\right) = Var\left(r_{end}\right) = \sigma^2 T. \tag{10.16}$$

This shows that the periodic variance can be obtained as

$$\sigma^2 = \frac{\mathrm{Var}\,(X_T)}{T}. \tag{10.17}$$

To sum up, the mean value $r$ and the variance $\sigma^2$ of the periodic return can be obtained by putting the solutions of the multiperiod problem, namely $E\,[X_T]$ and $\mathrm{Var}(X_T)$, into the related equations above. The periodic standard deviation $\sigma$ and the periodic mean return $R$ are found to be

$$\left(\sqrt{\frac{\mathrm{Var}\,(X_T)}{T}}, 1 + \frac{E\,[X_T] - 1}{T}\right) \tag{10.18}$$

which can then be inserted into the mean-variance graph to get the periodic frontier.

As in the previous approach, the periodic frontiers that will be obtained are checked for whether they are efficient or not. In order for the portfolios which are periodically invested at the random rate to be efficient, they have to be solutions to an optimization problem that maximizes the periodic mean return and minimizes the periodic variance in (10.18) and that is formulated as

$$\max\left\{1 + \frac{E\,[X_T] - 1}{T} - \omega\frac{\mathrm{Var}\,(X_T)}{T}\right\} \tag{10.19}$$

where $\omega$ is a positive coefficient. After rearranging, the objective function in (10.19) becomes

$$\max\left\{\frac{E\,[X_T]}{T} - \omega\frac{\mathrm{Var}\,(X_T)}{T} - \frac{1}{T} + 1\right\} \tag{10.20}$$

which is equivalent to

$$\frac{1}{T}\max\left\{E\,[X_T] - \omega\mathrm{Var}\,(X_T)\right\} - \frac{1}{T} + 1 \tag{10.21}$$

in generating optimal solutions. This last objective function finally leads to the same form as the objective function of problem $P3(\omega)$ given in (4.4) since $T$ is constant for a certain investment problem. This result shows that solutions maximizing (10.19) will at the same time maximize $P3(\omega)$. A consequence of this result is that an investor will get the same optimal portfolios for both problems provided that $\omega$ is the same. Moreover, portfolios on the transformed periodic mean-variance graph will be efficient. We can therefore talk about efficient periodic frontiers for this second approach. The interpretation of the periodic mean and variance is then as follows: If an investor wants to earn a return of $R$ each period, then he must incur a minimum risk of $\sigma$ each period. Alternatively, if an investor specifies a certain periodic risk of $\sigma$, the maximum mean return that he will earn periodically is $R$.

# Chapter 11

# ILLUSTRATIVE CASE WITH ONE RISKY ASSET

## *11.1 Solutions for $T = 5$*

In order to illustrate the application of the analytical solutions developed in this thesis, an exemplary case is provided here. For the sake of simplicity and to keep the solution procedure tractable, it is assumed that there exists a stochastic market which is modulated by a Markov chain that has only two states and which consists of a single risky asset and a riskless asset. The stochastic market is assumed to be in state 1 initially, and we consider the problem of an investor who has a unit wealth for investment at the beginning of the investment horizon that is taken to be five periods. The objective is to find the best allocation of investor's wealth among the two assets. The return $r_f$ of the riskless asset, the expected value $r$ and the standard deviation $\sigma$ of the return of the risky asset for each state are given in Table 11.1.

| State $i$ | $r_f(i)$ | $r(i)$ | $\sigma(i)$ |
|---|---|---|---|
| 1 | 1.05 | 1.11 | 0.15 |
| 2 | 1.06 | 1.09 | 0.12 |

Table 11.1: Expected returns and variances for one risky asset case

The transition probability matrix $Q$ of the Markov chain that the stochastic market process follows is given as

$$Q = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}.$$

Given these input data, $V(i)$, the formula of which is given in (5.3), is computed to be

$$V(1) = [0.0261], \; V(2) = [0.0153].$$

One can then calculate the vectors $f(i)$, $g(i)$ and $h(i)$ using the definitions given in (5.5)-(5.7) as follows

$$f(i) = \begin{bmatrix} 0.9504 \\ 1.0575 \end{bmatrix}, \quad g(i) = \begin{bmatrix} 0.9052 \\ 0.9976 \end{bmatrix}, \quad h(i) = \begin{bmatrix} 0.1379 \\ 0.0588 \end{bmatrix}$$

where the first entry of the vectors corresponds to $i = 1$ and the second one to $i = 2$. Once we have these vectors together with the transition probability matrix $Q$, we can use the definitions of $a_1$, $a_2$ and $b$ given in (5.20)-(5.22) to obtain

$$a_1 = \begin{bmatrix} 0.7630 \\ 0.8572 \end{bmatrix}, \quad a_2 = \begin{bmatrix} 0.9963 \\ 1.1321 \end{bmatrix}, \quad b = \begin{bmatrix} 0.2078 \\ 0.1754 \end{bmatrix}$$

which are the only parameters that we need in order to find the optimal analytical solutions for the multiperiod portfolio problems having any type of utility function. Once again, the first entry of the vectors corresponds to $i = 1$ and the second one to $i = 2$.

According to the given input data, the efficient frontier faced by an investor at time zero for $T = 5$ is given in Fig. 11.1 for an initial state of 1, which can be found by using (5.31).

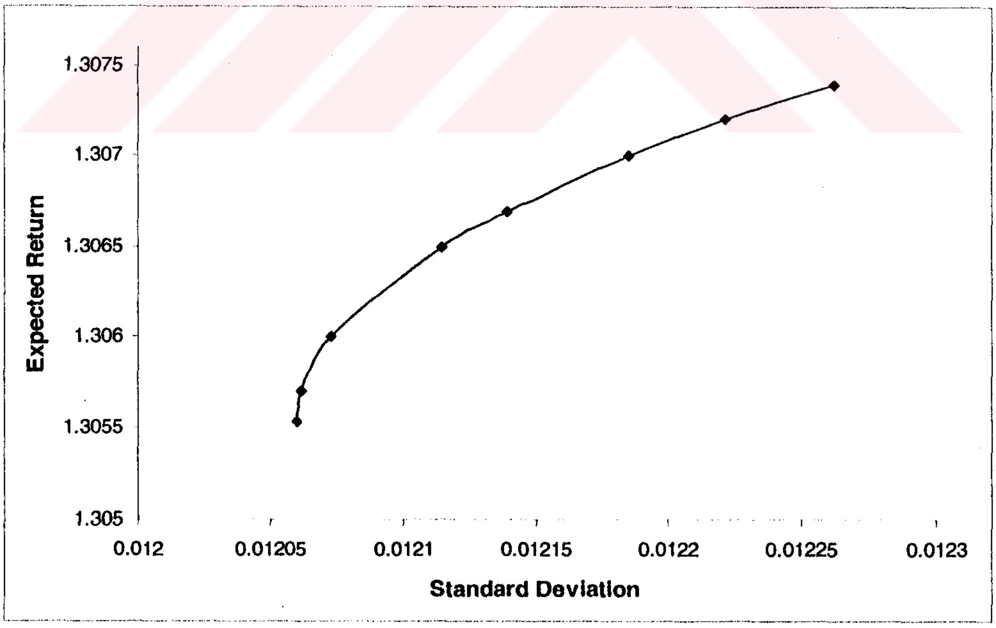


Figure 11.1: Mean-variance efficient frontier for the illustrative case with one risky asset

Obviously, the minimum-variance portfolio has nonzero risk over five periods. In an exact form, the minimum-variance portfolio has an expected final return of 1.3055 with a standard deviation of $0.01206 > 0$ which are found using (5.33) and (5.34). The reason for this observation is that investing in the risk-free asset over several periods has a random return since the return of the risk-free asset depends on the state of the market which changes stochastically over time.

In order to keep track of the optimal investment strategy proposed by each multiperiod problem, a scenario is created in which it is assumed that the Markovian market follows the path $i = 1, 1, 2, 1, 1$ at $n = 0, 1, 2, 3, 4$, and that the expected returns given in Table 11.1 are realized in each period. The scenario analysis includes the computation of the optimal investment policy $u_n(i, x)$ for each period by using (5.13) where corresponding optimal $\gamma^*$ values are going to be used in place of $\lambda/\omega$ for each model. Moreover, the investor's wealth is calculated at the end of each period by using the wealth dynamic equation given in (3.4). This scenario analysis is going to be performed for each problem separately, and then the results are going to be compared with each other.

The multiperiod problem is first solved by assuming that the investor has a quadratic utility function. The range of parameter $A$ turns out to be between 0 and 0.383, where 0.383 corresponds to $A^*$ given in (7.12). For an arbitrary $A$ value such as $A = 0.35$, the utility graph given in Fig. 11.2 and having the formula (7.4) is obtained in terms of $\gamma$.

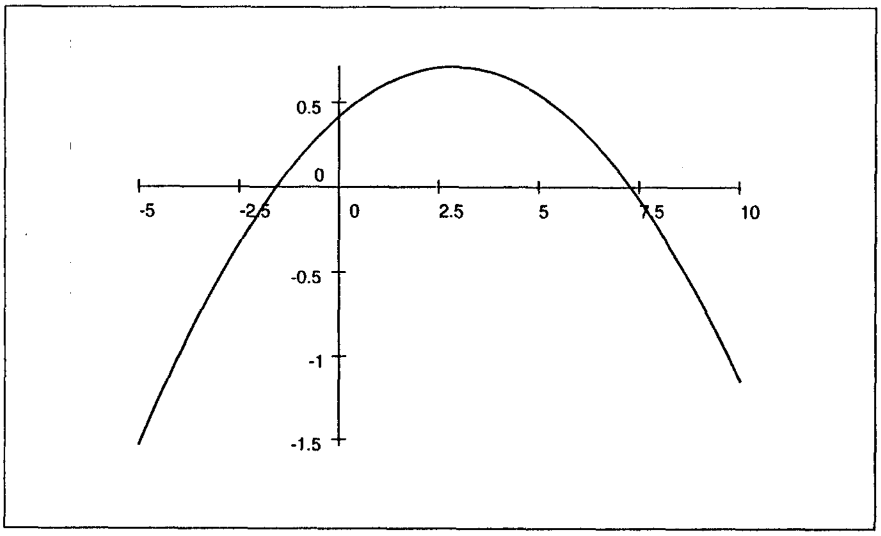


Figure 11.2: Utility graph for quadratic utility problem with $A = 0.35$

The optimal $\gamma^*$ given in (7.6) turns out to be 2.857. The expectation and the standard deviation of the final wealth are then found to be 1.357 and 0.062 respectively using (5.23) and (5.25).

According to the scenario given, the optimal investment strategy proposed by the multiperiod quadratic utility model with $A = 0.35$ and the related wealth at the end of each period are given in Table 11.2 where approximately 20% of available money is invested in the risky asset at each period $n$.

| $n$ | $i$ | $u_n(i,x)$ | $X_{n+1}$ |
|---|---|---|---|
| 0 | 1 | 23% | 1.06 |
| 1 | 1 | 22% | 1.13 |
| 2 | 2 | 16% | 1.20 |
| 3 | 1 | 21% | 1.28 |
| 4 | 1 | 21% | 1.35 |

Table 11.2: Scenario analysis of quadratic utility problem with $A = 0.35$ for one risky asset case

The next problem to solve is the coefficient of variation problem. The optimal $\gamma^*$ given in (8.7) turns out to be 2.611, which can also be seen in the utility graph given in Fig. 11.3 and having the formula (8.5) given in terms of $\gamma$. The expected value of the final wealth is then found to be 1.306 with a standard deviation of 0.012 using (5.23) and (5.25).

According to the scenario given, the optimal investment strategy proposed by the multiperiod coefficient of variation model and the related wealth at the end of each period are given in Table 11.3.

Table 11.3 shows that in order to minimize the relative dispersion of the final wealth, which also corresponds to the classical trade-off between minimizing risk and maximizing return without specifying any additional parameter, almost all of the current wealth has to be invested in the risk-free asset in each period.

The same input data are also used for the safety-first problem. The objective is to minimize the upper bound of the probability that the final wealth is below a preselected disaster level. First, the disaster level range, for which the dynamic solution procedure is

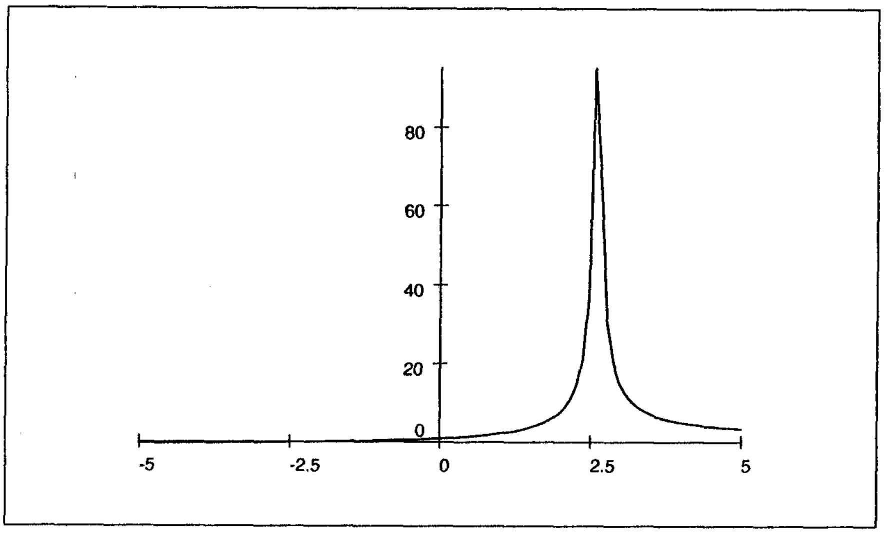


Figure 11.3: Utility graph for coefficient of variation problem

| $n$ | $i$ | $u_n(i,x)$ | $X_{n+1}$ |
|---|---|---|---|
| 0 | 1 | 0% | 1.05 |
| 1 | 1 | 2% | 1.10 |
| 2 | 2 | 0% | 1.17 |
| 3 | 1 | 2% | 1.23 |
| 4 | 1 | 3% | 1.29 |

Table 11.3: Scenario analysis of coefficient of variation problem for one risky asset case

applicable, is found. It turns out that the problem can be solved optimally for an investor requiring a minimal return of up to $k^* = 1.306$ at the end of five periods which is computed using (9.18). For different applicable $k$ values, the problem is solved and then the expected value and the standard deviation of the final wealth are put on a graph shown in Fig. 11.4.

As shown in (9.27), there exists a one-to-one relationship between the efficient frontier obtained from the mean-variance portfolio optimization and the one obtained from safety-first portfolio optimization. This is also verified graphically by comparing the efficient frontiers given in Fig. 11.1 and 11.4. This relationship is not surprising since the same auxiliary problem is used for both cases.

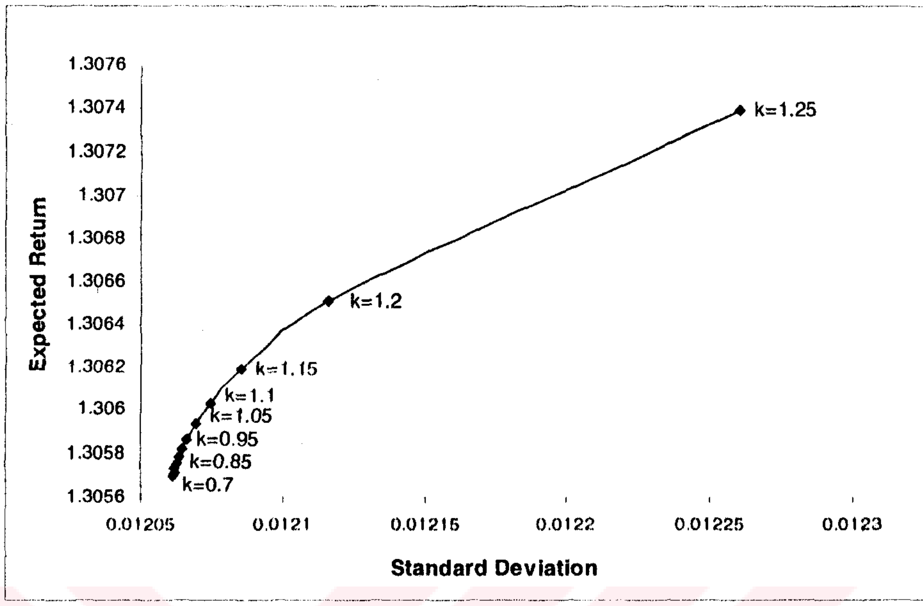


Figure 11.4: Efficient frontier for safety-first problem with one risky asset

The utility graph $U(\gamma)$ having the formula (9.15) in terms of $\gamma$ with $k = 1.3$ is shown in Fig. 11.5. The $k$ value is in the allowable range which is smaller than $k^*$ so that the problem has an explicit solution. For $k = 1.3$, the optimal $\gamma^*$ given in (9.17) is equal to 2.701. The expectation of $X_T$ turns out to be 1.324 using (5.23), and the standard deviation of $X_T$ is found to be 0.025 using (5.25).

According to the scenario given, the optimal investment strategy proposed by the multiperiod safety-first model with $k = 1.3$ and the related wealth at the end of each period are given in Table 11.4.

It can be seen that $6\% - 10\%$ of the available money is periodically invested in the risky asset in order to minimize the probability that the final wealth is below the required return which is 1.3. The disaster level was intentionally selected high so that it is not possible for the investor to reach this wealth level trivially by only investing in the risk-free asset. Moreover, the final wealth is found to be 1.31 which is a little bit more than which was required.

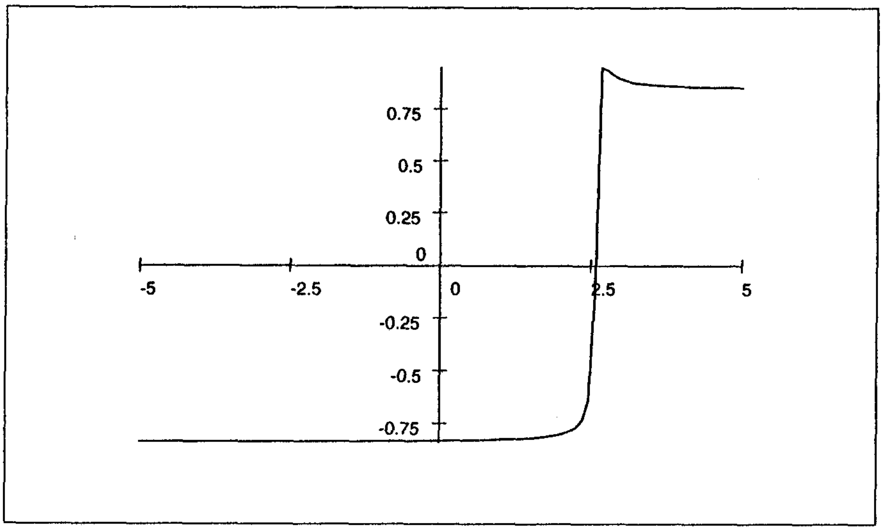


Figure 11.5: Utility graph for safety-first problem with $k = 1.3$

| $n$ | $i$ | $u_n(i,x)$ | $X_{n+1}$ |
|---|---|---|---|
| 0 | 1 | 8% | 1.05 |
| 1 | 1 | 9% | 1.11 |
| 2 | 2 | 6% | 1.18 |
| 3 | 1 | 9% | 1.25 |
| 4 | 1 | 10% | 1.31 |

Table 11.4: Scenario analysis of safety-first problem with $k = 1.3$ for one risky asset case

Before comparing the results of the three types of utility functions, a second disaster level for the safety-first problem is selected for further analysis of the proposed optimal policy. The safety-first utility graph having the formula (9.15) in terms of $\gamma$ with a disaster level $k$ of 1.1 is shown in Fig. 11.6.

For $k = 1.1$, the optimal $\gamma^*$ given in (9.17) is equal to 2.613. The expectation of $X_T$ turns out to be 1.306 and the standard deviation of $X_T$ is 0.012 by using (5.23) and (5.25) respectively. Compared to the previous results for the safety-first problem with $k = 1.3$, this problem reveals a lower expected final wealth because of the decreased disaster level but correspondingly a lower risk for the final wealth.

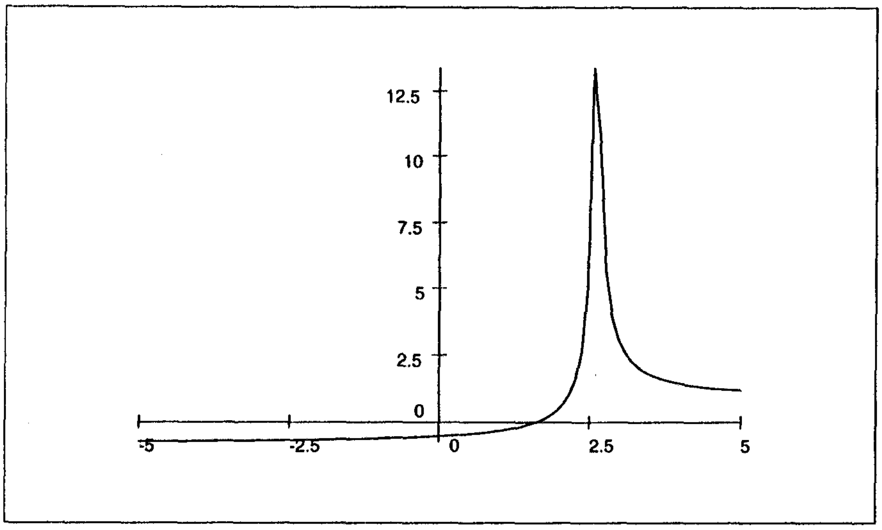


Figure 11.6: Utility graph for safety-first problem with $k = 1.1$

According to the scenario given, the optimal investment strategy proposed by the multiperiod safety-first model with $k = 1.1$ and the related wealth at the end of each period are given in Table 11.5.

| $n$ | $i$ | $u_n(i,x)$ | $X_{n+1}$ |
|---|---|---|---|
| 0 | 1 | 0% | 1.05 |
| 1 | 1 | 2% | 1.10 |
| 2 | 2 | 0% | 1.17 |
| 3 | 1 | 2% | 1.23 |
| 4 | 1 | 3% | 1.29 |

Table 11.5: Scenario analysis of safety-first problem with $k = 1.1$ for one risky asset case

An important observation related to the results given in Table 11.5 is that some portion of the current wealth is still invested in the risky asset even when the required level 1.1 is reached at the end of the second period. If all this money is invested in the risk-free asset for the remaining three periods, the resulting terminal wealth would be equal to $1.1(1.06)(1.05)^2 = 1.286$ under this scenario. Furthermore, this alternative investment policy would eliminate all the uncertainty coming from investing in the risky asset in the remaining periods. This implies that investing in the risk-free asset for the remaining periods is a better policy than the one given in Table 11.5 since it reaches a comparable return at the end of five periods while not increasing the risk at all. The reason for obtaining a policy which does not lead to an optimal result according to our proposed solution procedure is that we are minimizing the upper bound of the undesired probability $P\{X_T \leq 1.1\}$ and not the probability itself which may have a different optimal policy as in this case.

The utility functions given in terms of $\gamma$ and shown above for the three types of problems analyzed in this study have typical shapes for the corresponding problems. That is, there exists a single extreme point maximizing the utility function for all problems. For the quadratic utility (QU) and the safety-first (SF) problem, the exact shape of the utility function depends on the input parameters $A$ and $k$ respectively, whereas for the coefficient of variation (CV) problem there is only one utility function for a given input data set.

Finally, it is worth to compare the optimal policies of the given problems with each other. The results obtained are combined in Table 11.6.

| | | **QU** ($A = 0.35$) | | **CV** | | **SF** ($k = 1.3$) | | **SF** ($k = 1.1$) | |
|---|---|---|---|---|---|---|---|---|---|
| $n$ | $i$ | $u_n(i,x)$ | $X_{n+1}$ | $u_n(i,x)$ | $X_{n+1}$ | $u_n(i,x)$ | $X_{n+1}$ | $u_n(i,x)$ | $X_{n+1}$ |
| 0 | 1 | 23% | 1.06 | 0% | 1.05 | 8% | 1.05 | 0% | 1.05 |
| 1 | 1 | 22% | 1.13 | 2% | 1.10 | 9% | 1.11 | 2% | 1.10 |
| 2 | 2 | 16% | 1.20 | 0% | 1.17 | 6% | 1.18 | 0% | 1.17 |
| 3 | 1 | 21% | 1.28 | 2% | 1.23 | 9% | 1.25 | 2% | 1.23 |
| 4 | 1 | 21% | 1.35 | 3% | 1.29 | 10% | 1.31 | 3% | 1.29 |

Table 11.6: Comparison of optimal policies and investor's wealth for one risky asset case

As mentioned in Chapter 7, higher $A$ value means higher risk aversion. Therefore, the parameter $A$ of the QU problem is chosen to be close to the upper bound 0.383 (i.e., 0.35) so that the investor with this utility is relatively more risk-averse which is also assumed to be the case for investors trying to minimize the coefficient of variation of their final wealth and for safety-first investors. From Table 11.6 it can be easily seen that the highest portion of available wealth is invested in the risky asset for the QU problem which logically leads to the highest expected final wealth at the end of the investment horizon, but also to the highest risk of the final wealth.

The CV problem having a single utility function reveals the same optimal policy as that of the SF problem with $k = 1.1$ investing very little in the risky asset. Comparing the solutions of CV and SF($k = 1.1$) problems in detail reveals that these two problems yield the same values for the expectation and variance of the final wealth as well. This result is not very unexpected if one compares the objective functions of these problems given in (8.2) and (9.14). The objectives differ from each other by the additional term $k/\sqrt{\mathrm{Var}_i\,(X_T)}$ which can be neglected as $k$ becomes smaller and smaller. A further analysis which is done by decreasing the disaster level $k$ until a value of zero is reached shows that the optimal solution of the safety-first problem converges to the optimal solution of the coefficient of variation problem as $k$ goes to zero.

The SF problem with $k = 1.3$ suggests the second highest investment in the risky asset. This is due to the fact that the required level of the final wealth can only be reached when the risky asset having a higher expected return than that of the risk-free asset is used even in small quantities. The optimal policy of the SF problem with $k = 1.1$ on the other hand shows that almost all of the current wealth is invested in the risk-free asset. This result is logical since a final wealth of 1.1 can be reached more easily compared to a final wealth of 1.3 when money is lent at the risk-free rate.

### *11.2 Efficient Frontiers with Different $T$ Values*

Using the same input data for the means, variances and the transition matrix as before, the multiperiod safety-first problem can be solved for different $T$ values. As shown previously, the efficient frontier for a single $T$ value is obtained by changing the disaster level in the safety-first problem and then finding the optimal expectation and variance of the final wealth. Assuming that the initial state is 1, efficient frontiers given in Fig. 11.7 are obtained for different lengths of the investment horizon.

There are some immediate conclusions that can be made about the characteristics of the efficient frontiers in Fig. 11.7 with respect to changing length of the investment horizon. As the length of the investment horizon increases, a much higher return is expected for the same standard deviation since there is more time to invest so that the initial money will accumulate to a higher level without increasing the risk. Moreover, to reach the same level

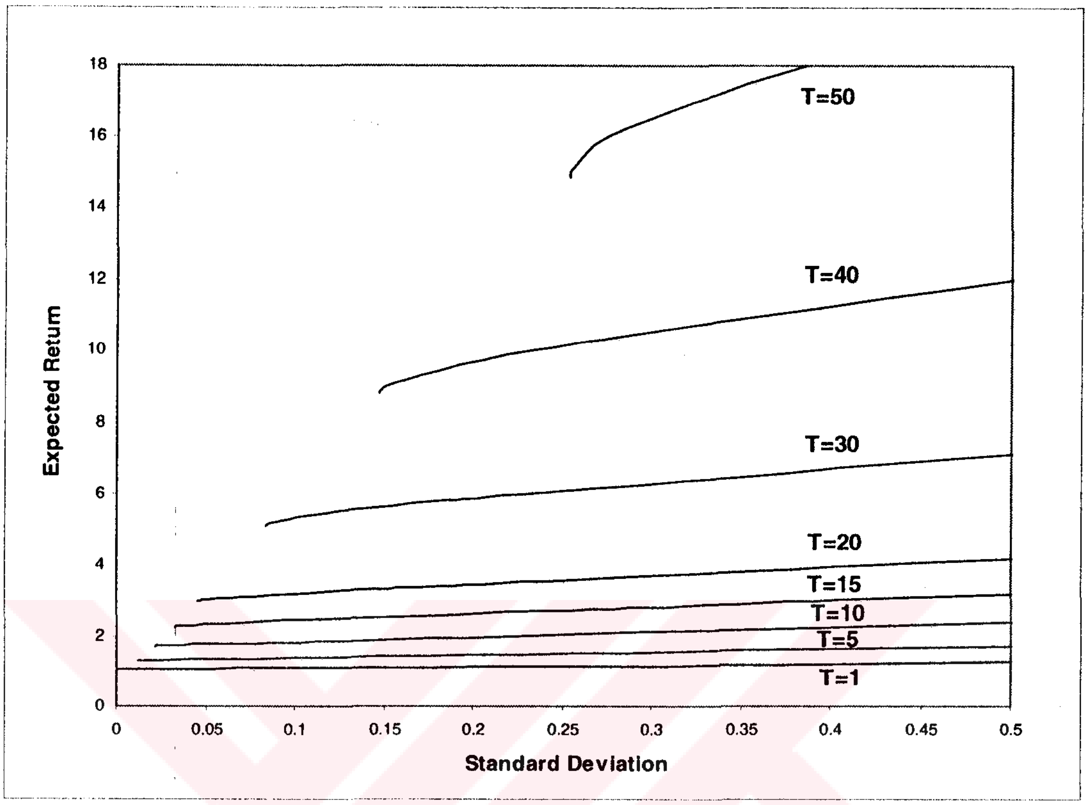


Figure 11.7: Efficient frontiers for different $T$ values

of expected return, a much smaller standard deviation is needed for a longer investment horizon. This is due to the fact that since there is more time to invest, the investment in less risky assets with smaller returns will be enough to reach the required level, whereas more risky assets have to be used in order to reach the same level in a shorter time period.

One particular reason for analyzing the efficient frontiers was to see whether they converge to a common frontier as $T$ increases, which is not the case here as can be seen in Fig. 11.7. If there was such a common frontier, this would imply that there exists a stationary policy that is used for large values of the investment horizon $T$. Therefore, we can conclude that there does not exist such a stationary policy for our multiperiod problems which means that regardless of the value of $T$, each problem with a different $T$ has to be solved independently in order to get the optimal solution.

The minimum variance points on these efficient frontiers correspond to the case where the disaster level $k$ goes to $-\infty$, and the points on the opposite end of the efficient frontiers correspond to the case where $k$ approaches the critical level $k^*$. This means that as $k$ goes from $-\infty$ to the critical level $k^*$, the optimal solutions trace a path forming the efficient frontier beginning with the minimum variance point and going to infinity. The minimum variance points are provided in Fig. 11.8. It is obvious that as the time horizon increases, both the standard deviation and the expectation of the final wealth increase.

The solutions to the quadratic utility problem with $A = 0.35$ for different $T$ values are provided in Fig. 11.9. This quadratic utility problem reveals an interesting pattern; and similar graphs are obtained for other $A$ values as well.

The solutions to the coefficient of variation problem for different $T$ values are provided in Fig. 11.10. As explained before, the coefficient of variation problem gives only one single optimal point on the efficient frontier. It turns out that this point corresponds to a disaster level $k$ equal to zero in the safety-first problem, implying that the investor just wants to avoid a negative wealth at the end of the investment horizon.

The safety-first problem with different $T$ values but the same disaster level is also analyzed where $k$ is taken to be 1.045 in order to ensure that $k < k^*$ for all problems with different investment horizons. The optimal portfolios are shown in Fig. 11.11. It can be seen that although the disaster level is kept constant for different $T$ values, which is actually not a very logical assumption since investors tend to prefer higher returns for longer time horizons, the expectation and the variance of the final wealth increase as $T$ increases.

As given in (7.9), (8.8) and (9.27), there exists a one-to-one relationship between the solutions of all multiperiod problems discussed since they are derived from the same mean-variance formulation, $P1(\sigma)$ or $P2(\mu)$. This means that same efficient frontiers in Fig. 11.7 are obtained for every model considered, except for the case where there is only one efficient point as the single solution of the problem as in the coefficient of variation problem or where there is some condition set on the input parameter of the problem so that only a portion of the efficient frontier is obtained as the solution set. The efficient frontiers with their largest allowable range can be obtained by solving either $P1(\sigma)$ or $P2(\mu)$. Solving the quadratic utility problem with different $A$ values or the safety-first problem with different $k$ values will also yield same efficient frontiers as in Fig. 11.7.

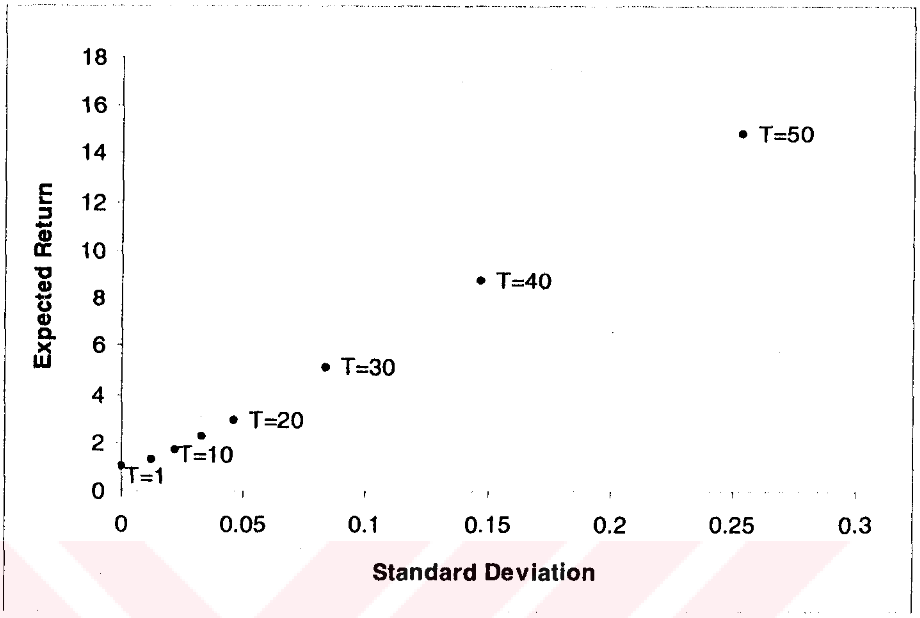


Figure 11.8: Minimum variance points for different $T$ values

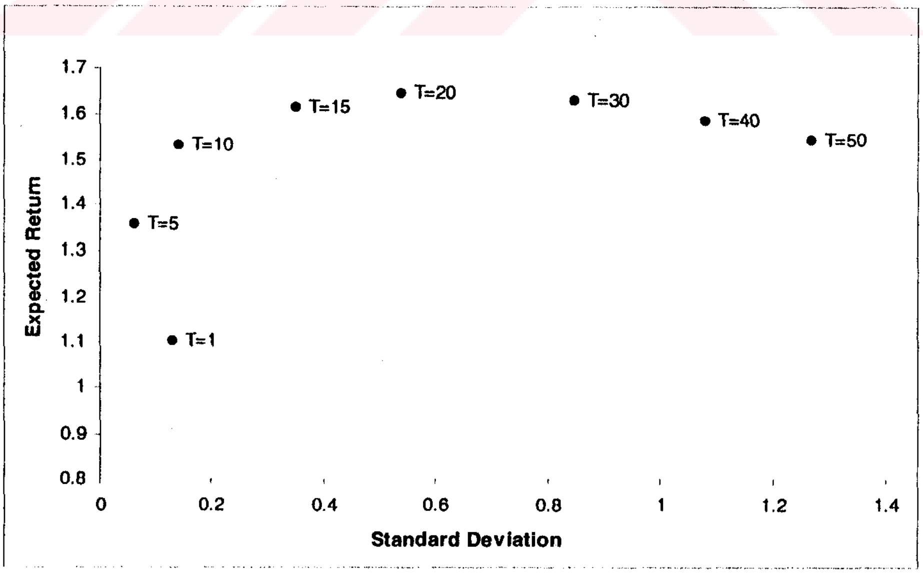


Figure 11.9: Optimal points for quadratic utility problem with $A = 0.35$

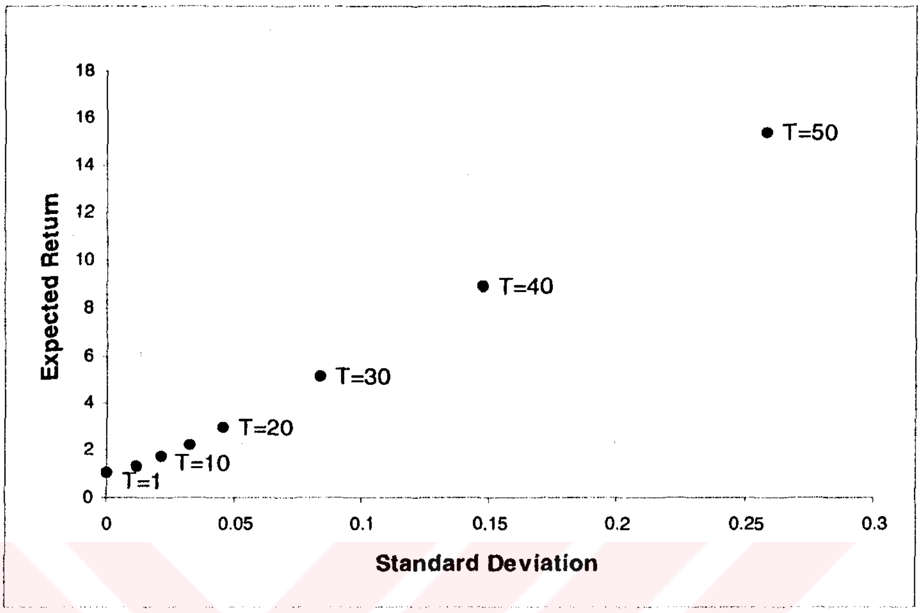


Figure 11.10: Efficient points for coefficient of variation problem

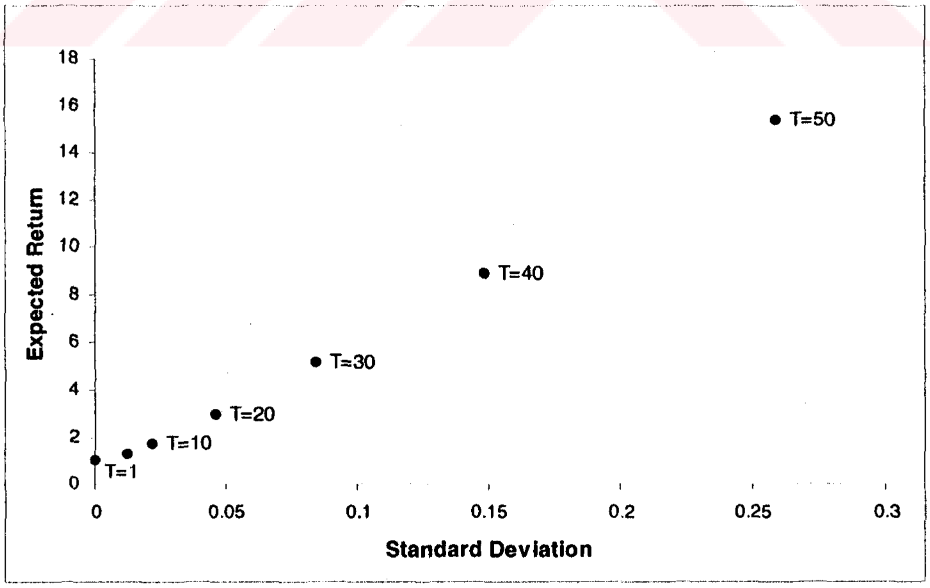


Figure 11.11: Efficient points for safety-first problem with $k = 1.045$

### 11.3 Periodic Analysis of the Efficient Frontiers

The periodic analysis is a more objective way of comparing problems with different investment horizons. In order to compare efficient frontiers belonging to different investment horizons, like the ones in Fig. 11.7, on a periodic basis so that the problems considered have equal time intervals for investment, the frontiers are drawn on a graph which has periodic expected return and periodic standard deviation as its axes. The two approaches explained in Chapter 10 are used here to transform the efficient frontiers to periodic frontiers where the first approach considers geometric growth and the second one arithmetic growth of the random returns.

For the first approach proposed, the formulas of the periodic standard deviation and the periodic mean return are given in (10.8). The following transformed periodic frontiers in Fig. 11.12 are obtained for the exemplary case given at the beginning of this chapter.

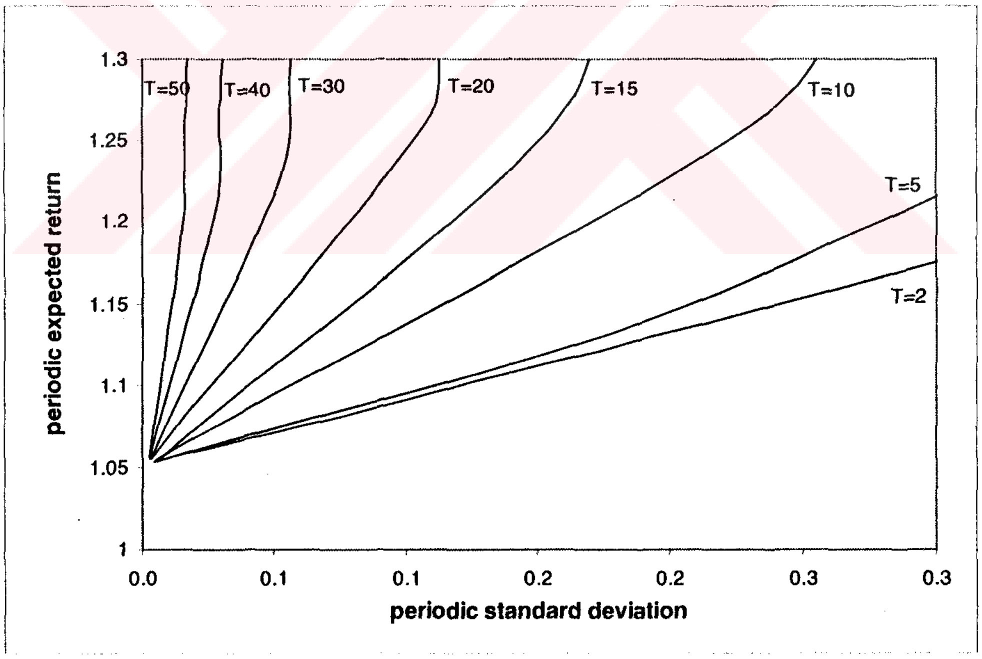


Figure 11.12: Periodic frontiers for different $T$ values using geometric return

The periodic frontiers in Fig. 11.12 are not concave as it is the case for efficient frontiers. However, the concavity of the frontiers is already not expected since the calculated periodic means and standard deviations that are used for drawing are just theoretical values that should lead to the final optimal values at the end of the investment horizon. What is important in this figure is the relative position of the frontiers with respect to each other. As $T$ increases, the frontier is shifted to the left having higher periodic mean return for the same standard deviation and lower periodic standard deviation for the same mean return. This implies that it is more advantageous for investors to invest their money for an investment horizon that is long as much as possible. This conclusion, however, is not realistic since it implies in the limiting case where $T$ goes to infinity that infinite periodic return can be expected while no risk is incurred at all.

A similar graph to that of Fig. 11.12 is obtained if the initial state is changed from $i = 1$ to $i = 2$, and similar patterns are obtained when other input data is used by changing the means, variances, and the transition matrix.

For the second approach proposed, the formulas of the periodic standard deviation and the periodic mean return are given in (10.18). The following transformed periodic frontiers in Fig. 11.13 are obtained for the exemplary case given at the beginning of this chapter.

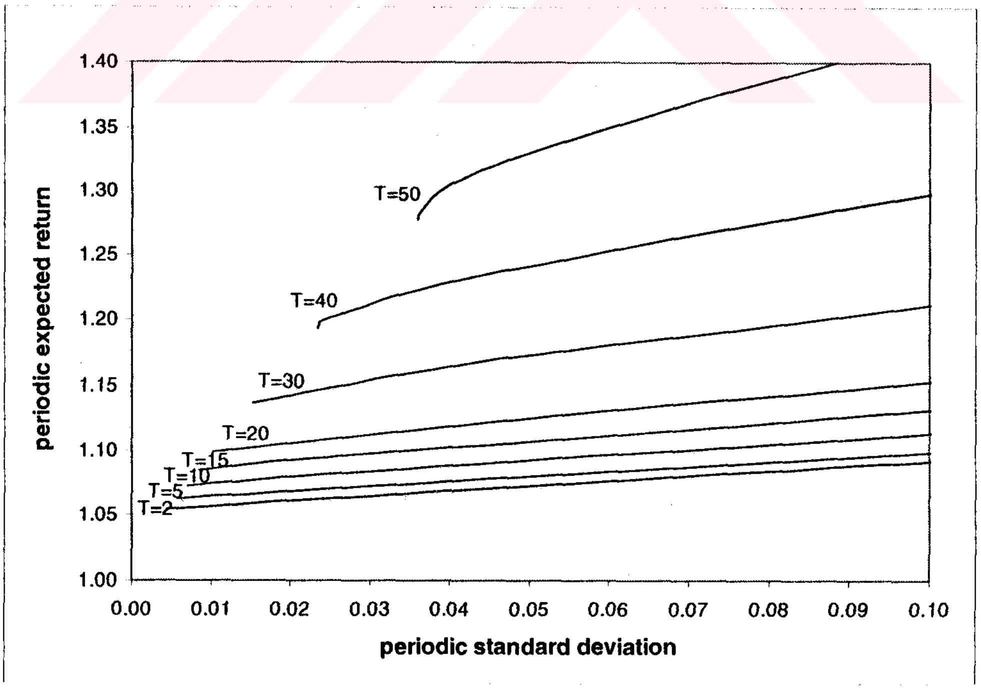


Figure 11.13: Periodic frontiers for different $T$ values using arithmetic return

Compared to Fig. 11.12, Fig. 11.13 seems to be more realistic since there is no limiting case where higher investment horizons lead to infinite periodic return with no risk, as was the case in the first approach. Nevertheless, it is still more advantageous for investors to invest their money for an investment horizon that is long as much as possible, since in such a case they will get higher periodic return for the same periodic risk and alternatively incur lower periodic risk for the same periodic return. The intuition behind this observation could be that investors who are willing to tie up their money for a longer time horizon obtain their reward by getting higher periodic returns with the same periodic risk compared to the investors who prefer to invest their money for a shorter time horizon.

Finally, Fig. 11.14 shows the periodic frontiers for the illustrative problem with $T = 5$ given at the beginning of Chapter 11 which are drawn using the two approaches. As discussed previously, the first approach takes (10.1) as its basis and calculates the periodic mean and variance without making any approximation whereas the second one makes an approximation regarding the values of the periodic returns. Consequently, the first approach corresponds to geometric return whereas the second one corresponds to arithmetic return in Fig. 11.14 which represent the return growth over time.

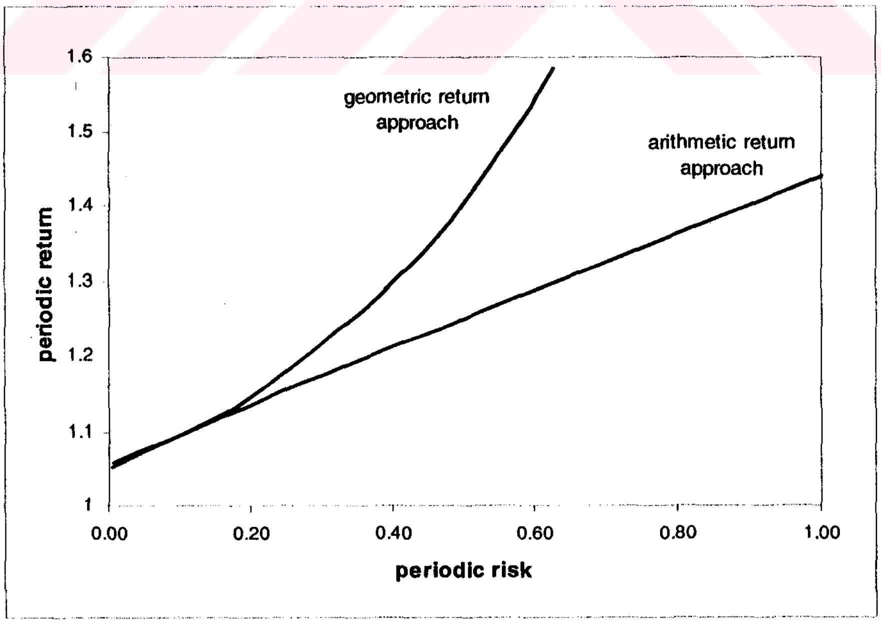


Figure 11.14: Comparison of two approaches for calculating the periodic return ($T = 5$)

Comparing the periodic frontiers in Fig. 11.14 reveals that the two approaches lead to quite different results. Moreover, it is observed that the periodic mean and the periodic variance of the second approach are always greater than those of the first approach under the condition that both of them must yield the same $E[X_T]$ and $\text{Var}(X_T)$ at the end of the investment horizon $T$. This is due to the fact that returns are growing geometrically by multiplying the periodic returns in the first case whereas returns are growing arithmetically by adding the periodic returns in the second case.

### *11.4 Safety-First Problem with Disaster Level Changing with Respect to T*

While analyzing the safety-first problem with different $T$ values, the disaster level is not given much attention except for providing that it does not exceed the critical level $k^*$. However, it seems interesting to change the disaster level with respect to the investment horizon $T$ and analyze the results accordingly, since the determination of a disaster level by the investor will surely depend on the number of periods available for investment. It is assumed in this section that as the investment horizon increases, the investor increases his disaster level because his expectation about the final wealth also increases. More specifically, for an investment horizon of $T$, the disaster level is taken to be $k^T$ where $k$ is a disaster level that does not exceed the critical level $k^*$ even if its $T$ th power is used for higher time periods. For illustrative purposes, the same input data is used as at the beginning of Chapter 11 regarding the means, variances and the transition matrix. The disaster level $k$ is taken to be 1.045, which assures that $k^T$ is always smaller than $k^*$ that is calculated for each time horizon $T$. For example, for $T = 10$ the disaster level is equal to $1.045^{10} = 1.55$ whereas the upper bound $k^*$ is equal to 1.72.

For each time horizon $T$ ranging from 1 to 50, the safety-first problem is solved and then the optimal $E[X_T]$ and $\text{Var}(X_T)$ are found. The expected final wealths at the end of the investment horizons turn out to be greater than the given disaster levels, which is logical since the aim is to minimize the probability of falling below the specified disaster levels. Moreover, solving the safety-first problem for different time periods shows that as $T$ increases, both the expected final wealth and the variance of the final wealth increase, regardless of the initial state of the stochastic market. One question that arises about this problem is whether the means and variances of the periodic returns will be the same if the disaster level is set to be $k^T$ for any investment horizon $T$. The answer to this question will be given now which turns out to be no.

Solving the optimal results $E[X_T]$ and $\text{Var}(X_T)$ for the periodic mean returns and periodic standard deviations based on the first (geometric) approach given in Chapter 10 for different time periods reveals two different patterns on the periodic mean-variance graph depending on the initial state.

Fig. 11.15 shows the case for $i = 1$. In this figure, it turns out that the points corresponding to $T = 1$ and $T = 50$ yield two efficient points on this graph with the first point having no risk but a low return and the second point having higher risk and higher return than that. Another important feature of the second point with $T = 50$ is that it has lowest risk and highest mean return for each period compared to other points corresponding to $T$ values between 1 and 50. This implies that it is more advantageous to invest the money either for a single time period or for a much longer time horizon with a higher disaster level so as to reach the most efficient point on a periodic basis given the same initial conditions.

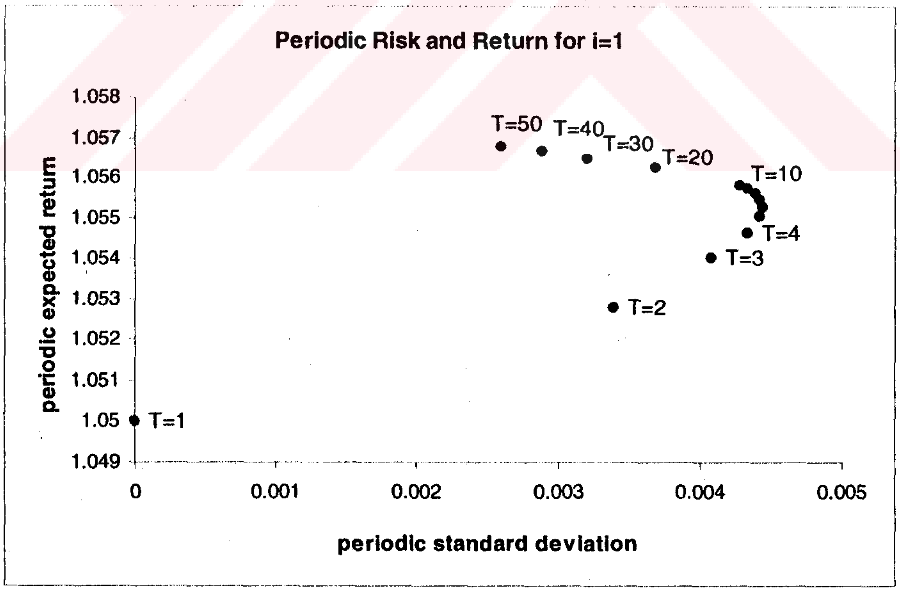


Figure 11.15: Periodic mean and standard deviation for safety-first problem with $k^T$ ($i = 1$)

Fig. 11.16 shows the case for $i = 2$. This graph has only one efficient point that corresponds to $T = 1$ which has a certain return with no risk. According to this graph, the money should be invested in the risk-free asset for a single time point and then reinvested again for longer time horizons.

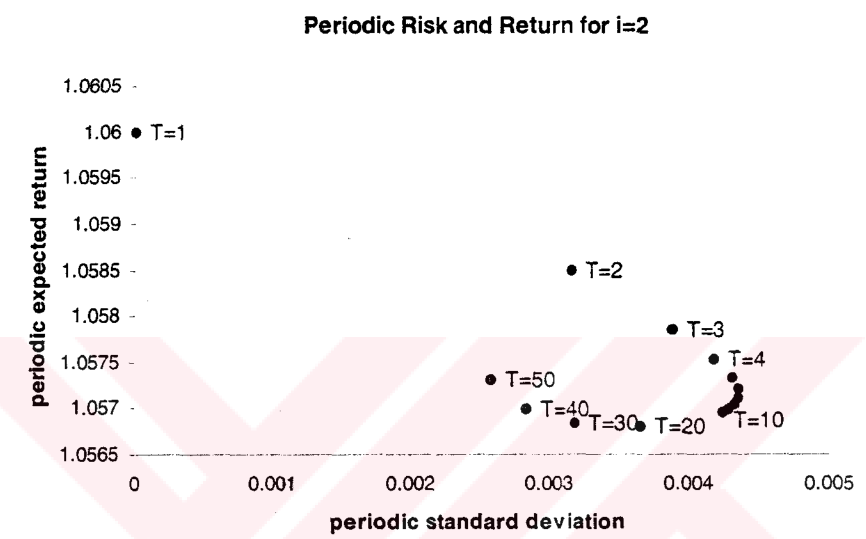


Figure 11.16: Periodic mean and standard deviation for safety-first problem with $k^T$ $(i = 2)$

After analyzing these graphs, other periodic mean-variance graphs are drawn by changing the input data, namely the means and variances of asset returns with respect to the market state and the transition matrix of the stochastic market. It is observed that similar patterns like in the above mentioned figures arise when the optimal solutions are put in the periodic mean-variance graph with respect to changing investment horizon. This means that the investor is advised to either invest in the risk-free asset for a single time period and then reinvest again one period later or invest his money for an investment horizon that is long as much as possible according to the first periodic approach.

The periodic mean returns and periodic standard deviations based on the second (arithmetic) approach given in Chapter 10 are also drawn on a graph for different time periods. Regardless of the initial market state $i$, there is one single pattern which is given in Fig. 11.17 for $i = 1$.

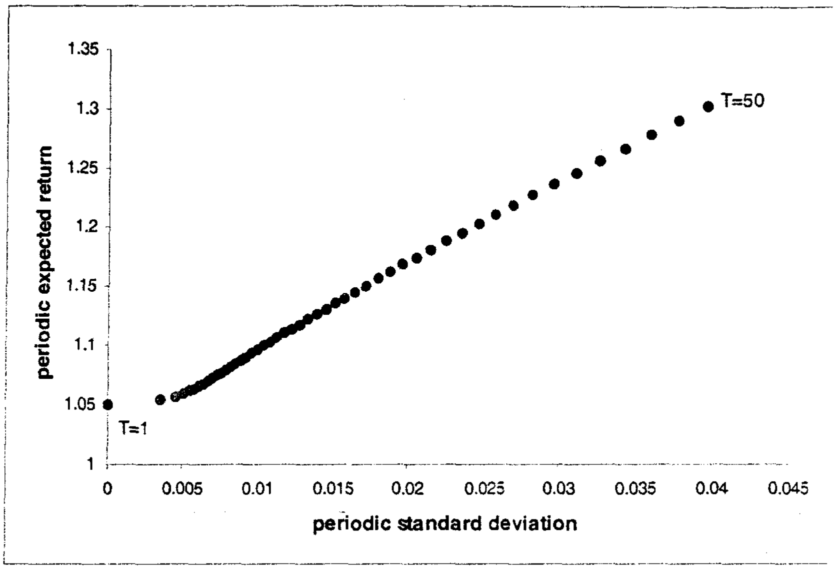


Figure 11.17: Periodic mean and standard deviation for safety-first problem with $k^T$ using arithmetic return approach ($i = 1$)

Fig. 11.17 reveals the same pattern as if the optimal values of $E[X_T]$ and $\mathrm{Var}(X_T)$ are drawn on a mean-variance graph for the safety-first problem having disaster level $k^T$. The periodic values, which are calculated by dividing $E[X_T]$ and $\mathrm{Var}(X_T)$ by $T$ according to the arithmetic approach, are steadily increasing from $T = 1$ to $T = 50$. The interpretation of this graph is obvious: Since the disaster level is increased with increasing investment horizon $T$, more money should be invested in the risky asset so as not to fall below the given disaster level which consequently leads to higher periodic mean return and higher periodic risk.

### *11.5 Optimal Portfolios for Safety-First Problem with One Risky Asset*

Since the safety-first problem is of practical importance for some investors who want to avoid a possible disaster level at the end of the investment horizon, this problem is investigated more deeply by making a sensitivity analysis. This sensitivity analysis is accomplished by changing the mean returns of both the risky and the risk-free asset one by one, and then observing the changes in the optimal policy $u$ which corresponds to the percentage of available money to be invested in the risky asset. In order to calculate the policies, it is assumed that the stochastic market follows the path $i = 1, 1, 2, 1, 1$ at $n = 0, 1, 2, 3, 4$ and that the expected returns given in Table 11.1 are realized in each period. The initial wealth is taken to be 1 and the disaster level 1.2.

Case 1 corresponds to the input data in Table 11.1 and the subsequent transition matrix given at the beginning of Chapter 11. Each case thereafter includes an additional change compared to the previous case. In Case 2, $r_f$ in state 1 is increased from 1.05 to 1.12. In Case 3, $r_f$ in state 2 is increased from 1.06 to 1.1. In Case 4, $r$ in state 1 is decreased from 1.11 to 1.08. In Case 5, $r$ in state 2 is decreased from 1.09 to 1.07. The aim of this analysis is to see how the optimal policy is changing as the the risk-free asset is made more advantageous for investors in both market states. The risk-free asset has no risk and even a better return compared to the risky asset after the input data has been modified in different cases. The optimal policies for these cases are given in Table 11.7.

| | **Case 1** | **Case 2** | **Case 3** | **Case 4** | **Case 5** |
|---|---|---|---|---|---|
| $u_0$ | 0.004007 | -0.00662 | -0.00050 | -0.00205 | -0.00216 |
| $X_1$ | 1.050240 | 1.120066 | 1.120005 | 1.120082 | 1.120086 |
| $u_1$ | 0.018909 | 0.009449 | 0.005316 | 0.020678 | 0.019903 |
| $X_2$ | 1.103887 | 1.25438 | 1.254352 | 1.253665 | 1.253701 |
| $u_2$ | 0.002544 | 0.049079 | -0.00131 | -0.00114 | -0.00520 |
| $X_3$ | 1.170197 | 1.331115 | 1.379801 | 1.379043 | 1.379227 |
| $u_3$ | 0.024625 | 0.015009 | 0.008488 | 0.033098 | 0.030860 |
| $X_4$ | 1.230184 | 1.490699 | 1.545292 | 1.543204 | 1.543499 |
| $u_4$ | 0.037228 | 0.035001 | 0.016148 | 0.060467 | 0.057473 |
| $X_5$ | 1.293927 | 1.669232 | 1.730566 | 1.725970 | 1.726420 |

Table 11.7: Optimal safety-first policies for one risky asset case

The policy formula given in (5.13) depends on the current state of the stochastic market. After the return of the risk-free asset for state 1 is increased in all cases including and after Case 2, the amount invested in risky asset at the beginning of the investment horizon decreases and it even turns out to be negative, meaning that the risky asset is sold short in order to invest more in the risk-free asset. The same effect is also observed in Case 3 where the return of the risk-free asset for state 2 is increased. Since the market state is assumed to be 2 for $n = 2$, the risky asset is again sold short at $n = 2$ to invest more in the risk-free asset in all cases including and after Case 3.

It is important to note here that the realized final wealth $X_5$ at the end of the investment horizon is more than the disaster level which was specified at the beginning of the investment horizon as 1.2. This is due to the fact that while minimizing the probability of falling below 1.2, the safety-first problem equivalently aims to maximize the probability of having a final wealth greater than the prespecified level $k$ given by the investor. Another observation is that in Case 2 and Case 3, $X_5$ increases compared to previous cases since the percentage of available money invested in the risky asset given by $u$ decreases whereas the return of the riskless asset increases which in turn lead to higher final wealths. The realized final wealth does not change much in Case 4 and Case 5 since these cases only involve a decrease in the return of the risky asset which is already not used in big amounts.

Even though the risk-free asset is more advantageous for investors after the returns are modified, there is still some investment in the risky asset. The reason for this observation is that the proposed policy can reach the required level of the final wealth, 1.2 in this case, even when there is an investment in the risky asset. Moreover, since the risky asset can have a return higher than its expected value in reality, this investment has the potential of realizing a higher final wealth while still satisfying the objective of not falling below the disaster level.

# Chapter 12

# ILLUSTRATIVE CASE WITH TWO RISKY ASSETS

## 12.1 Solutions for $T = 5$

In order to further illustrate the application of the analytical solutions, a second exemplary case with two risky assets is provided here. There exists a stochastic market having two states and consisting of two risky assets and a riskless asset. The stochastic market is assumed to be in state 1 initially, and the investor has a unit wealth for investment at the beginning of the investment horizon that is taken to be five periods. The objective is to find the best allocation of this wealth among the three assets. The expected returns of assets, the standard deviations and correlation coefficients of the returns of the risky assets for each state are given in Table 12.1.

| State $i$ | $r_f(i)$ | $r_1(i)$ | $r_2(i)$ | $\sigma_1(i)$ | $\sigma_2(i)$ | $\rho_{12}(i)$ |
|---|---|---|---|---|---|---|
| 1 | 1.05 | 1.11 | 1.10 | 0.15 | 0.14 | 0.94 |
| 2 | 1.06 | 1.09 | 1.12 | 0.12 | 0.13 | 0.62 |

Table 12.1: Expected returns and variances for two risky assets case

The transition matrix $Q$ of the Markov chain that the stochastic market process follows is given as

$$Q = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}.$$

Given these input data, $V(i)$, the formula of which is provided in (5.3), is computed to be

$$V(1) = \begin{bmatrix} 0.0261 & 0.0230 \\ 0.0230 & 0.0225 \end{bmatrix}, \; V(2) = \begin{bmatrix} 0.0153 & 0.0118 \\ 0.0118 & 0.0216 \end{bmatrix}.$$

One can calculate the vectors $f(i)$, $g(i)$ and $h(i)$ using the definitions given in (5.5)-(5.7) as follows

$$f(i) = \begin{bmatrix} 0.9464 \\ 0.9354 \end{bmatrix}, \; g(i) = \begin{bmatrix} 0.9013 \\ 0.8824 \end{bmatrix}, \; h(i) = \begin{bmatrix} 0.1416 \\ 0.1675 \end{bmatrix}$$

where the first entry of the vectors corresponds to $i = 1$ and the second one to $i = 2$. Once we have these vectors together with the transition probability matrix $Q$, we can use the definitions of $a_1$, $a_2$ and $b$ given in (5.20)-(5.22) to obtain

$$a_1 = \begin{bmatrix} 0.5669 \\ 0.5526 \end{bmatrix}, \; a_2 = \begin{bmatrix} 0.7391 \\ 0.7288 \end{bmatrix}, \; b = \begin{bmatrix} 0.2826 \\ 0.2904 \end{bmatrix}$$

which are the only parameters that we need in order to find the optimal analytical solutions for the multiperiod portfolio problems having any type of utility function. The first entry of the vectors corresponds to $i = 1$ and the second one to $i = 2$.

In order to keep track of the optimal investment strategy proposed by each multiperiod problem, the same scenario as in Section 11.1 is created in which it is assumed that the Markovian market follows the path $i = 1, 1, 2, 1, 1$ at $n = 0, 1, 2, 3, 4$, and that the expected returns given in Table 12.1 are realized in each period. The scenario analysis includes the computation of the optimal investment policy for each period by using 5.13 where corresponding optimal $\gamma^*$ values are going to be used in place of $\lambda/\omega$ for each model. Moreover, the investor's wealth is calculated at the end of each period by using the wealth dynamic equation given in (3.4). This scenario analysis is going to be performed for each problem separately, and then the results are going to be compared with each other.

The portfolio selection problem is first solved by assuming that the investor has a quadratic utility function. The range of parameter $A$ turns out to be between 0 and 0.384, where 0.384 corresponds to $A^*$ given in (7.12). For an arbitrary $A$ value such as $A = 0.35$, the optimal $\gamma^*$ given in (7.6) turns out to be 2.857. The expectation and the standard deviation of the final wealth are found to be 1.374 and 0.063 respectively using (5.23) and (5.25). According to the scenario given, the optimal investment strategy proposed by the multiperiod quadratic utility model with $A = 0.35$ and the related wealth at the end of each period are given in Table 12.2 where $u_n^1(i, x)$ denotes the proportion of money invested in the first risky asset at period $n$ and $u_n^2(i, x)$ denotes the proportion of money invested in the second risky asset at period $n$, given the current market state $i$ and the current available money $x$.

| $n$ | $i$ | $u_n^1(i,x)$ | $u_n^2(i,x)$ | $X_{n+1}$ |
|---|---|---|---|---|
| 0 | 1 | 34% | -13% | 1.06 |
| 1 | 1 | 33% | -13% | 1.13 |
| 2 | 2 | -3% | 24% | 1.21 |
| 3 | 1 | 28% | -10% | 1.28 |
| 4 | 1 | 27% | -10% | 1.36 |

Table 12.2: Scenario analysis of quadratic utility problem with $A = 0.35$ for two risky assets case

In Table 12.2, it should be noted that approximately 20% of available money is allocated to the risky assets at each period $n$ which is quite similar to the result obtained in Table 11.2 for the previous case with one risky asset. Moreover, the realized wealths at the end of each period are also nearly equal for both cases which leads to the conclusion that regardless of the number of risky assets in the market, the quadratic utility problem tries to allocate the same proportion of money to the riskless asset.

The next problem to solve is the coefficient of variation problem. The optimal $\gamma^*$ given in (8.7) for this problem is 2.608. It turns out that the expected value of the final wealth is 1.304 with a standard deviation of 0.011 using (5.23) and (5.25). According to the scenario given, the optimal investment strategy proposed by the multiperiod coefficient of variation model and the related wealth at the end of each period are given in Table 12.3 which shows that in order to minimize the relative dispersion of the final wealth, almost all of the current wealth has to be invested in the risk-free asset in each period.

| $n$ | $i$ | $u_n^1(i,x)$ | $u_n^2(i,x)$ | $X_{n+1}$ |
|---|---|---|---|---|
| 0 | 1 | 0% | 0% | 1.05 |
| 1 | 1 | 2% | -1% | 1.10 |
| 2 | 2 | 0% | 0% | 1.17 |
| 3 | 1 | 3% | -1% | 1.23 |
| 4 | 1 | 5% | -2% | 1.29 |

Table 12.3: Scenario analysis of coefficient of variation problem for two risky assets case

The same input data are also used for the safety-first problem. The objective of this problem is to minimize the upper bound of the probability that the final wealth is below a preselected disaster level. This problem can be solved optimally for a safety-first investor requiring a minimal return of up to $k^* = 1.304$ at the end of five periods which is computed using (9.18). For different applicable $k$ values, the problem is solved and then the expected value and the standard deviation of the final wealth are put on a graph shown in Fig. 12.1.

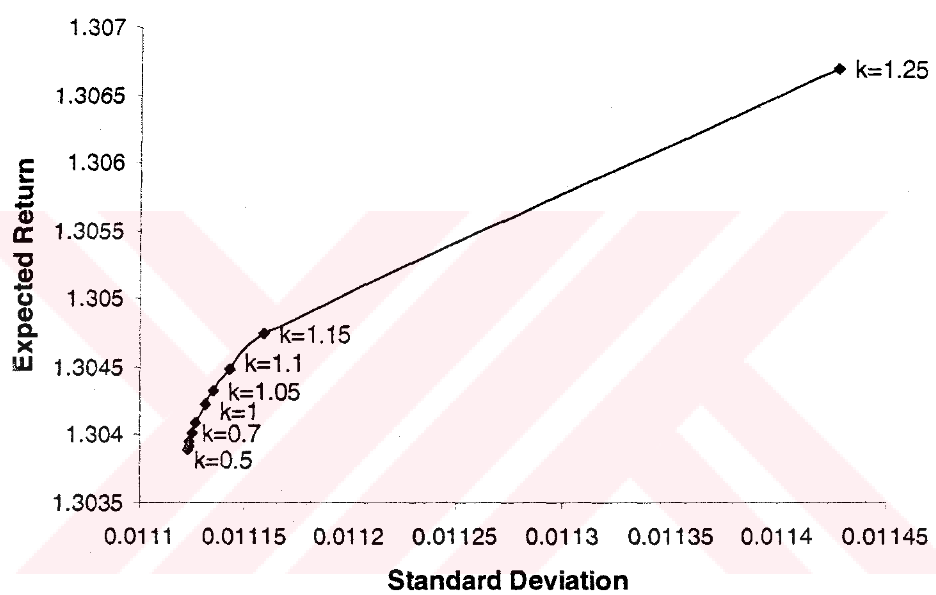


Figure 12.1: Efficient frontier for safety-first problem with two risky assets

For $k = 1.3$, which is in the allowable range, the optimal $\gamma^*$ given in (9.17) is equal to 2.761. The expectation of $X_T$ turns out to be 1.347, and the standard deviation of $X_T$ is 0.040, using (5.23) and (5.25) respectively. According to the scenario given, the optimal investment strategy proposed by the multiperiod safety-first model with $k = 1.3$ and the related wealth at the end of each period are given in Table 12.4.

| $n$ | $i$ | $u_n^1(i,x)$ | $u_n^2(i,x)$ | $X_{n+1}$ |
|---|---|---|---|---|
| 0 | 1 | 21% | -8% | 1.06 |
| 1 | 1 | 21% | -8% | 1.12 |
| 2 | 2 | -2% | 15% | 1.20 |
| 3 | 1 | 18% | -7% | 1.26 |
| 4 | 1 | 19% | -7% | 1.33 |

Table 12.4: Scenario analysis of safety-first problem with $k = 1.3$ for two risky assets case

Before comparing the results of the three types of utility functions, a second disaster level for the safety-first problem is selected for further analysis of the proposed optimal policy. For $k = 1.1$, the optimal $\gamma^*$ given in (9.17) is equal to 2.610. The expectation of $X_T$ turns out to be 1.304 using (5.23), and the standard deviation of $X_T$ is 0.011 using (5.25). According to the scenario given, the optimal investment strategy proposed by the multiperiod safety-first model with $k = 1.1$ and the related wealth at the end of each period are given in Table 12.5.

| $n$ | $i$ | $u_n^1(i,x)$ | $u_n^2(i,x)$ | $X_{n+1}$ |
|---|---|---|---|---|
| 0 | 1 | 0% | 0% | 1.05 |
| 1 | 1 | 2% | -1% | 1.10 |
| 2 | 2 | 0% | 0% | 1.17 |
| 3 | 1 | 3% | -1% | 1.23 |
| 4 | 1 | 5% | -2% | 1.29 |

Table 12.5: Scenario analysis of safety-first problem with $k = 1.1$ for two risky assets case

For all problems discussed in this section, the utility functions $U(\gamma)$ have similar shapes to those functions given in Section 11.1 for the case of one risky asset.

Finally, after obtaining the optimal policies of each problem, they can be compared to each other. The results for the case of two risky assets are combined in Table 12.6.

| | | QU ($A = 0.35$) | | | CV | | | SF ($k = 1.3$) | | | SF ($k = 1.1$) | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $n$ | $i$ | $u_n^1$ | $u_n^2$ | $X_{n+1}$ | $u_n^1$ | $u_n^2$ | $X_{n+1}$ | $u_n^1$ | $u_n^2$ | $X_{n+1}$ | $u_n^1$ | $u_n^2$ | $X_{n+1}$ |
| 0 | 1 | 34% | -13% | 1.06 | 0% | 0% | 1.05 | 21% | -8% | 1.06 | 0% | 0% | 1.05 |
| 1 | 1 | 33% | -13% | 1.13 | 2% | -1% | 1.10 | 21% | -8% | 1.12 | 2% | -1% | 1.10 |
| 2 | 2 | -3% | 24% | 1.21 | 0% | 0% | 1.17 | -2% | 15% | 1.20 | 0% | 0% | 1.17 |
| 3 | 1 | 28% | -10% | 1.28 | 3% | -1% | 1.23 | 18% | -7% | 1.26 | 3% | -1% | 1.23 |
| 4 | 1 | 27% | -10% | 1.36 | 5% | -2% | 1.29 | 19% | -7% | 1.33 | 5% | -2% | 1.29 |

Table 12.6: Comparison of optimal policies and investor's wealth for two risky assets case

As mentioned at the beginning of Chapter 7, higher $A$ value means higher risk aversion. Therefore the parameter $A$ of the quadratic utility problem is chosen to be close to the upper bound 0.383 (i.e., 0.35) so that the investor with this utility is relatively more risk-averse which is also assumed to be the case for investors trying to minimize the coefficient of variation of their final wealth and for safety-first investors. From Table 12.6, it can be easily seen that the highest portion of available wealth is invested in the risky assets for QU problem which logically leads to the highest expected final wealth at the end of the investment horizon. The SF problem with $k = 1.3$ suggests the second highest investment in the risky assets. This is due to the fact that the required level of the final wealth can only be reached when the risky assets with higher expected returns than that of the risk-free asset are used. The optimal policy of the SF problem with $k = 1.1$ on the other hand shows that almost all of the current wealth is invested in the risk-free asset. This result is logical since a final wealth of 1.1 can be reached more easily compared to a final wealth of 1.3 when money is lent at the risk-free rate. The CV problem reveals almost the same optimal policy as that of the SF problem with $k = 1.1$ investing very little in the risky asset.

## 12.2 Optimal Portfolios for Safety-First Problem with Two Risky Assets

The safety-first problem with two risky assets is investigated again by making a sensitivity analysis. This analysis is accomplished by increasing the risk-free return in each state above the returns of the risky assets, each time observing the optimal policy $u$, which gives the percentage of available money to be invested in the risky assets. It is assumed that the stochastic market process follows the path $i = 1.1, 2, 1, 1$ at $n = 0, 1, 2, 3, 4$, and that the expected returns given in Table 12.1 are realized in each period. The initial wealth is taken to be 1 and the disaster level 1.2.

Case 1 corresponds to the input data in Table 12.1, and the subsequent transition matrix $Q$ is used for every case considered here. Each case after Case 1 includes an additional change compared to the previous one. In Case 2, $r_f$ in state 1 is increased from 1.05 to 1.12. In Case 3, $r_f$ in state 2 is increased from 1.06 to 1.13. The aim of this analysis is to see how the optimal policy is changing as the the risk-free asset is made more advantageous for investors in both market states. The risk-free asset has no risk and even a better return compared to the risky assets after the input data is modified in different cases. The optimal policies for these cases are given in Table 12.7.

| | **Case 1** | **Case 2** | **Case 3** |
|---|---|---|---|
| $u_0^1$ | 0.006982 | 0.066992 | 0.001028 |
| $u_0^2$ | -0.00262 | -0.08374 | -0.00128 |
| $X_1$ | 1.050288 | 1.121005 | 1.120015 |
| $u_1^1$ | 0.027708 | -0.06696 | 0.025537 |
| $u_1^2$ | -0.01039 | 0.083694 | -0.03192 |
| $X_2$ | 1.103945 | 1.254521 | 1.254800 |
| $u_2^1$ | -0.0002 | -0.01001 | 0.002092 |
| $u_2^2$ | 0.001848 | 0.094063 | -0.00086 |
| $X_3$ | 1.170287 | 1.335136 | 1.417849 |
| $u_3^1$ | 0.032925 | -0.10995 | 0.038368 |
| $u_3^2$ | -0.01235 | 0.137432 | -0.04796 |
| $X_4$ | 1.23016 | 1.493703 | 1.588567 |
| $u_4^1$ | 0.05071 | -0.26724 | 0.069735 |
| $u_4^2$ | -0.01902 | 0.334055 | -0.08717 |
| $X_5$ | 1.293759 | 1.668939 | 1.780241 |

Table 12.7: Optimal safety-first policies for two risky assets case

As can be seen in Table 12.7, one of the risky assets is always sold short so that the total amount invested in the risky assets is relatively low compared to the amount invested in the risk-free asset. In Case 3, the total percentage invested in the risky assets is mostly a negative quantity meaning that money is borrowed by shortselling risky assets and then invested in the risk-free asset which is more advantageous in terms of expected return.

# Chapter 13

## CONCLUSION

In this thesis, several multiperiod portfolio optimization problems are solved in a stochastic market where the returns of assets are correlated over time. The market consists of several risky assets and a riskless asset whose returns depend on the economic conditions that define the states of a Markov chain and are therefore serially correlated with each other via prevailing market conditions. The main objective is to come up with the optimal solutions to the multiperiod mean-variance formulations.

Various objective functions including the quadratic utility function, the coefficient of variation and the safety-first approach are considered. An auxiliary problem solved with dynamic programming is used to find the optimal portfolios.

First, the stochastic market process is described, and then the mean-variance model formulations are given. The auxiliary problem is summarized, and its main results are provided. The solution procedure for general utility functions is followed by problems with different utility functions, which are formulated and then solved analytically by making use of the solutions of the auxiliary problem. Periodic analysis of the efficient frontiers is given next. Finally, illustrative examples are provided to demonstrate the solution procedure for each problem.

An important note about multiperiod portfolio optimization is that this problem should be solved on a rolling horizon basis. That is, the optimal investment policy is found for the entire investment horizon, but only the decisions for the current period are implemented. In the next period, the information on returns is updated if necessary, and then the problem is resolved and imminent investment decisions guided by the optimal solution are implemented.

A natural extension of the multiperiod portfolio optimization model considered in this thesis is about imperfect information flow which assumes that the stochastic market process is a combination of observed and unobserved market processes so that a hidden Markov model is constructed to solve the problem. Another challenge would be to analyze Bayesian models where the parameters used in our analysis are not known with certainty such that a prior distribution will be imposed on them which can then be updated as more data becomes available.

In this thesis, multiperiod portfolio optimization problems are analyzed that do not include any constraints. But it seems interesting to put some constraints on the original objective, like specifying a minimum expected final wealth or a maximum allowable risk for the final wealth. Moreover, the current problems do not consider transaction costs which could have a significant effect on the optimal solution of the problem. Another interesting issue to analyze can be to find conditions, under which long-term investment strategies having stationary optimal portfolios will exist. Dealing with other utility functions for the multiperiod portfolio optimization is another future research topic. The extension of our model to continuous time is also worth analyzing, considering the fact that many investors change their portfolios continuously rather than at discrete points in time.

Finally, the solution approach proposed in this thesis makes solving multiperiod portfolio optimization problems with objective functions that include a mean-variance trade-off an easy implementation task, in which no distributional assumptions are needed on asset returns.

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