Inventory Pricing and Replenishment for a Make-to-Stock Production System with Fluctuating Demand and Lost Sales

by

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ABSTRACT

In this thesis, a continuous-review, infinite horizon inventory pricing and replenishment problem with capacitated supply is analysed. Demand is modeled as a Markov-modulated Poisson process where the potential demand rate depends on an external environment process. The supply side is modeled by a single server with exponential processing times. The structures of the optimal replenishment policies are demonstrated for two pricing models applied in business. One of these is the static pricing method where the price of the item remains fixed over time, and the other is the dynamic pricing method where the price may change over time depending on the current inventory level and the external environment. It is found that the optimal replenishment policy is an environment-dependent base-stock policy for both pricing models. Moreover, these two methods are compared in a numerical study and it is concluded that the dynamic pricing method would result in a limited improvement on the firm's profit compared to static pricing when both methods are applied optimally, and the benefit of dynamic pricing tends to increase with the demand variability.

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NOMENCLATURE

S	Sample space of the system
A	Set of allowable actions
A_S	Set of allowable actions for state *s*
P	Probability
$W^t(i,a)$	Immediate reward gained at time t for state *i* and action *a*
$w^t(i,a)$	Reward rate at time t for state *i* and action *a*
$P_{ij}^t(a)$	Transition probability from *i* to *j* at time *t* for action *a*
$F_{ij}^t(T \mid a)$	Probability that the transition from *i* to *j* occurs before *T* at time *t* for action *a*
β	Discount (exponential failure) rate
$V(x)$	Optimal total discounted profit in infinite horizon with inventory amount *x*
λ	Demand rate
μ	Production rate
p	Price
R	Reservation price
$\overline{F}(p)$	Probability of buying when a price of *p* is offered
λ_p	Buying rate with price *p* and demand rate λ
x	Current amount of ınventory
$h(x)$	Inventory holding cost rate for inventory amount *x*
c	Production cost
a	Optimal replenishment action

$V_n(x)$	Optimal total discounted profit of the n-stage problem with ınitial inventory amount x
M	Markov chain characterizing the environment
E	State space of M
Q	Infinitesimal generator of M
e	Current environment state
λ_e	Demand rate for environment e
Q_{ej}	Environment state transition rate from environment e to j
$V(x,e)$	Optimal total discounted profit in ınfinite horizon with inventory amount x and environment e
$V_n(x,e)$	Optimal total discounted profit of the n-stage problem with initial inventory amount x and environment e
p_{xe}	Optimal price for inventory amount x and environment state e
tp	Transition probability
tr	Transition rates

Chapter 1

INTRODUCTION

The objective of inventory management is to reduce the losses caused by the mismatches that arise between supply and demand processes. With the advances in computers and communication technology, this discipline has improved very much in the past few decades. Today, the role of inventory management has changed from cost control to value creation. Therefore, the issues inventory management studies now include both the traditional decisions such as inventory replenishment and in addition the strategic decisions made by the firm such as pricing. Since the joint optimization of pricing and replenishment decisions results in significant improvements on the firm’s profit [1], most of the current research study the integrated pricing and replenishment decisions rather than sequential optimization of these two decisions. The inspiring results obtained on this topic so far encouraged us to analyse an inventory pricing and replenishment problem.

The structure of the demand and supply processes is the main characteristic of the inventory system studied, and it has a direct effect on the structure of the optimal pricing and replenishment policy. In the inventory system studied in this thesis, the demand is time-varying, price-sensitive and stochastic, and the capacitated supply process is also stochastic. A single item is produced in a fluctuating environment. The uncertainty in demand and supply is a common element of many inventory management problems in literature. However, the problems studied recently also take into account unpredictable variations in demand and so does the problem analysed here. Moreover, the price sensitivity of demand is taken into account since price has become a very effective management tool to control the demand. With the increasing popularity of E-commerce applications, price changes are easy to implement on the Internet, and therefore the firms are interested to try innovative pricing strategies such as dynamic pricing (where the price changes over time).

Dynamic pricing models have been attaining great popularity in both production and service industries. Airlines and hotels charge different prices for the perishable and non-renewable set of resources they offer to their customers [2], retail chains apply markdown pricing and discount the prices to clear out the inventory before the end of season, and companies selling goods with short or long life-cycle such as Coca Cola and Amazon.com has also tried to apply dynamic pricing model [3], [4]. Because of the increasing interest on dynamic pricing model, the structure of dynamic pricing policies with replenishment decisions are analysed here along with the traditional (static) pricing model with replenishment where the price remains fixed over time.

In this thesis, we study a continuous review, infinite horizon inventory pricing and replenishment problem with capacitated supply where the demand is modeled as a Markov-modulated Poisson process and there is a single server with exponential processing time. There is no set up cost, and the production cost is linear. Moreover, the inventory holding cost is convex and nondecreasing in the inventory level.

In Chapter 2, we provide the necessary background and literature review on the inventory models with integrated pricing and replenishment decisions and with similar system definitions.

In Chapter 3, we introduce the basic definitions of Markov Decision Processes and present the method used to obtain the analytical results in this thesis on a simplified inventory pricing and replenishment problem with stationary Poisson demand where static pricing is applied.

In Chapter 4, we analyse the static pricing method with replenishment and describe the structure of the optimal replenishment policy for a given price. Moreover, we investigate whether the optimal policy reflects any interesting characteristics.

In Chapter 5, we study the dynamic pricing method with replenishment and again characterize the optimal replenishment policy. Moreover, we compare the static and dynamic pricing methods in a numerical study using relative value iteration method.

The thesis is concluded with a short summary of the performed study and future research work.

Chapter 2

LITERATURE SURVEY

Pricing models and integrated pricing and production decisions have been studied since 1960s, and the excellent reviews of Yano and Gilbert [5], Chan et al. [4], Elmaghraby and Keskinocak [3], and Bitran and Caldentey [2] provide a summary of the research papers on this area by focusing on different aspects of the problem. The characteristics of the general model which is the focus of Yano and Gilbert's review is the one mostly similar to the inventory pricing and replenishment problem studied in this thesis. Therefore, this literature survey is organized in a similar manner to this review.

The characterization of the optimal inventory pricing and replenishment policy mostly depends on the assumptions on demand and cost structure of the problem. Here we will discuss the models with stochastic demand with one exception which is the work of Kunreuther and Schrage [6] since they assume the demand is deterministic. They develop an algorithm for determining the pricing and ordering decisions for a firm producing one product to satisfy deterministic, time-varying, price-sensitive demand in finite horizon. The time-varying demand, although it is stochastic with a specific definition of the process, is the most important assumption of our model, and the work of Kunreuther and Schrage is one of the earliest introducing this concept to the problem.

They assume that unsatisfied demand is lost and develop a two-step algorithm which finds the optimal production schedule for a given price and then chooses the optimal price based on the marginal costs associated with the production schedule of each price. Their algorithm gives bounds on the optimal price, but it does not necessarily give the optimal solution.

One of the popular approaches for defining the stochastic demand as a function of price is to define the demand as a random perturbation plus a decreasing function of price (additive demand), $d = \eta + u(p)$, or multiply the perturbation with the demand function (multiplicative demand), $d = \eta\ u(p)$. Zabel [7] studies first the multiplicative demand case for a single-period problem with convex production and linear inventory holding cost. He also assumes that unsatisfied demand is lost and demonstrates that there is a unique optimal policy for every initial stock level, and when the inventory level is sufficiently low, the optimal price is less than the optimal price with deterministic demand.

Zabel [8] then extends those results to the multi-period problem with both additive and multiplicative demand cases where the demand for period n, d_n, is given such that $d_n = \eta_n\ u(p_n)$ for the multiplicative demand and $d_n = \eta_n + u(p_n)$ for the additive demand. He finds that the firm should produce when the initial inventory is below a critical point, x_n, and the price is a decreasing function of the inventory. Moreover, he demonstrates that the optimal replenishment level is decreasing in the initial amount of inventory for the additive demand model, and the optimal price, the optimal replenishment level and the critical point of inventory decrease in the number of periods left.

Zabel is one of the first to characterize the optimal inventory replenishment policy as a critical number policy in a multi-period model without set up cost. Many researchers studied inventory problems without set up cost and with different assumptions on the demand and supply processes, and they also found that the critical number policy, the so-called "base-stock" policy, is in fact the optimal inventory replenishment policy when there is no set up cost. We will now discuss the other models without set up cost and with similarities on the other assumptions with our model.

Thowsen [9] studies the case of price-sensitive nonstationary demand consisting of a general function of price with an additive stochastic component. The unit production cost is linear, inventory holding and shortage costs are convex, and backlogging, partial-backlogging and lost sales assumptions are considered along with the assumption of deteriorating inventory with a deterministic fraction. He conjectures that a period-dependent base-stock list price (y, p) policy, similar to critical number inventory policies, is optimal, and specifies the conditions for its optimality. The (y, p) policy is defined such that: For the n^{th} period, if the inventory level is below y_n, then order up to y_n, and do not replenish otherwise. Then, charge the price $p_n(I)$ on the optimal price trajectory depending on the current inventory I.

Federgruen and Heching [10] study a periodic-review inventory pricing and replenishment problem where the demand is stochastic and the unsatisfied demand is backlogged. They assume that one-period expected inventory holding cost function is jointly concave in order-up-to-level and price, production cost is linear and inventory holding cost is convex. They demonstrate that a base-stock list price policy is optimal for both average and discounted profit in finite or infinite horizon. Their results also extend to the case with production capacity limits.

Chan et al. [11] study a finite horizon inventory pricing and replenishment problem with limited capacity, lost sales and the possibility of rationing. The demand is stochastic and price-sensitive such that the random error may depend upon the selected price. The production and inventory holding costs are linear, and all parameters are nonstationary. They study control policies in which either a pricing or a production decision is made at the beginning of the horizon; this decision is held fixed throughout the horizon and decisions in the other category are allowed to change in each period.

In the scenario where they fix the price at the beginning of the horizon and determine the production quantity before the demand is realized at each period n (Delayed Production

Scenario), they demonstrate that the optimal policy has an optimal order-up-to (Y_n) and save-up-to (S_n) level in each period such that if the inventory level is below Y_n, then order up to Y_n, otherwise do not replenish. Moreover, the rationing is allowed to keep the inventory level at the amount S_n, so the firm may reject customers to obey the optimal policy and make sure the inventory level at the end of the period is at least S_n.

In the Delayed Pricing scenario where production decision is made at the beginning of the horizon and pricing decisions are made at the beginning of each period, they show that the optimal prices are not necessarily monotonic in the inventory levels. Moreover, from a computational study, they conclude that the performance of the dynamic pricing policy tends to increase with demand seasonality and tightness of capacity, and the benefit of dynamic pricing under increasing uncertainty is less clear.

The demand uncertainty is reflected in our model with the Markov-modulation concept, and in our numerical study we investigate the benefit of dynamic pricing compared to static pricing for one of the possible uncertainty cases that is not included in the computational study of Chan et al.

The models discussed until now are all finite horizon and periodic-review problems. Our problem is an infinite horizon and continuous-review problem, and now we will discuss the problems with these assumptions and still without set up cost. These studies also conclude that the base-stock policy is the optimal replenishment policy when there is no set up cost.

Li [12] studies a continuous-review model where the demand and production are Poisson counting processes, and the demand is price-sensitive. He assumes that unsatisfied demand is lost. The production and holding costs are linear, and the capacity decision is made at the beginning of the horizon. Li shows that for both cases in which a single price is chosen and in the case where price may be changed dynamically, a base-stock policy is optimal. Moreover, he shows that the price is a non-increasing function of the inventory level in the dynamic pricing case.

Gayon [13] studies a single-server make-to-stock production system producing one item to satisfy price-sensitive Poisson demand. The production time is exponential. He shows that for the infinite horizon model with either the discounted or average cost criterion, the optimal policy is a base-stock policy with prices decreasing with the current inventory level. We generalize his results to the systems operating in a fluctuating environment.

Now we will discuss the problems with set up cost where the structure of the optimal replenishment policy slightly changes. The only change in the optimal policy is the addition of a second inventory level, s, which is the threshold value to give an order and the optimal policy is to give an order up to the so-called "order-up-to-level", S, when inventory falls below the threshold value.

Thomas [14] studies the periodic-review, finite horizon model with incapacitated supply and stochastic, price-sensitive demand. He assumes that unsatisfied demand is backlogged. He proposes a simple period-dependent (s, S, p) policy. Moreover, he claims that price depends on the initial inventory level at the beginning of the period. Then, Thomas also provides a counterexample for this policy and he conjectures that an (s, S, p) policy is optimal when prices satisfy certain conditions.

Chen and Simchi-Levi [15] study a periodic-review, finite horizon model where the demand is a general function of the form $d_n = \alpha_n D_n(p_n) + \beta_n$ for the n^{th} period. The perturbations α_n and β_n are random variables satisfying $E[\alpha_n] = 1$ and $E[\beta_n] = 0$. All the unsatisfied demand is backlogged. They demonstrate that in the case of additive demand the base-stock list price policy (s, S, p) is optimal where the price is determined according to the inventory at the beginning of the period, and for more general demand functions, this policy is not necessarily optimal. Moreover, Chen and Simchi-Levi [16] extend their results to the infinite horizon case and show that (s, S, p) policy is optimal for both the discounted and average profit models with the demand function defined above.

Feng and Chen [17] study an infinite-horizon continuous-review model where the demand follows Poisson processes that are parameterized with prices (the demand rate for p_i is λ_i). There is a fixed set up cost and a variable production cost, and the holding and shortage costs are jointly determined by a quasi-convex and unbounded function depending on the current inventory level. The replenishment can be done instantaneously and there are no capacity limits. Unsatisfied demand is fully backlogged. The pricing decision is to choose the price to charge from a finite set ($p_1>p_2>\ldots>p_N$), and Feng and Chen introduce the concept of a *maximum increasing concave envelope* via which dominated prices can be excluded. The envelope is actually a way of creating a lower bound on the revenue rate ($\lambda_i p_i$) corresponding to each price where $\lambda_1< \lambda_2<\ldots<\lambda_N$.

They show that the optimal policy has the form ($s=d_n$, d_{n-1}, …, d_1, D_1, …, D_{n-1}, $D_n=S$), where s is the reorder level, S is the order-up-to point, and d_{n-1}, …, d_1, D_1, …, D_{n-1} define the upper and lower inventory levels for each price such that it is optimal to charge price p_i when the inventory level is between d_i and d_{i-1} or D_{i-1} and D_i. The former inventory range to charge price p_i may appear counterintuitive since it contradicts the common intuition to charge higher when the inventory is low. However, it comes from the fact that it is optimal to stimulate demand by lowering the price and deplete enough inventories to trigger an order to reduce the shortage costs. This happens when the inventory levels are relatively low or some of the d_i values are negative.

The work of Feng and Chen uses a demand model very similar to ours except that we include a fluctuating environment concept with continuous pricing, as opposed to their discrete prices, in our model. The difference between the optimal replenishment policies of the two models can be expressed as due to the differences in model assumptions.

Although it is not a pricing model, we find it worthwhile to discuss the work of Ha [18] on inventory rationing since the demand and supply processes have similarities with ours, and the lost sales cost can be taken as the price. Ha studies the problem of inventory rationing for a make-to-stock production system producing a single product to satisfy several demand classes. Unsatisfied demand is assumed to be lost with a cost charged at each lost sale. The demand is a Poisson process and the production is exponential. Ha shows that the optimal policy can be characterized by a sequence of monotone stock rationing levels. That is, there is a critical inventory level for each demand class such that it is optimal to start rejecting this customer class in anticipation of future arrivals of higher priority customers when the inventory level falls below the critical point.

Besides searching for the optimal pricing policy for a specific problem, a comparison of static and dynamic pricing method is done by Chen et al. [1]. They study an infinite horizon inventory pricing and replenishment problem where the demand is price-sensitive and modeled by Brownian motion. They consider both the long-run average and discounted objectives. They show that the joint optimization of both decisions may result in significant profit improvement over the traditional way of making sequential optimization. Moreover, they also show that changing price with the inventory level (dynamic pricing) will only result in a limited profit improvement over static pricing when both methods are optimally applied. This result is compatible with the results of our numerical study.

The studies reviewed so far include the joint optimization of pricing and replenishment policies and/or the structure of the optimal replenishment (pricing) policy for a given pricing (replenishment) method. In none of the models, the possibility of unpredictable shifts in demand is included since none of these studies include the fluctuating demand environment in an inventory pricing (static or dynamic) and replenishment model. However, there are many research studies including the possibility of unpredictable shifts in demand in an inventory replenishment problem. A widely-used approach to insert the effect of fluctuating environment is defining the demand process to be driven by an exogenous Markov chain. The demand process defined as such is called a Markov-modulated demand process. Now we will discuss mainly the models with Markov-modulated demand and various assumptions on the other elements of the model.

Before introducing the studies with Markov-modulated demand, it is worthwhile to note the early work of Karlin [19]. Karlin is one of the first to introduce to the dynamic inventory model the changes in the demand distribution from period to period such that the demand constitutes a sequence of independent random variables over successive periods which may not be identically distributed. He assumes that purchase cost is linear and holding and shortage costs are convex. There are no time lags in delivery, and the unsatisfied demand is lost. Karlin shows that a base-stock policy is optimal at each period but could vary since the demand distribution may change at each period.

Moreover, there are studies which provide algorithms to obtain the steady state values of the systems they consider. For the inventory system we now discuss there are two algorithms as such to our knowledge. Kalpakam and Arivarignan [20] discuss an (s, S) inventory model with lost sales and Markov-modulated supply and demand rates. They provide an efficient algorithm to evaluate the steady state values and also obtain the transient and limiting values of the mean reorder and shortage rates. Feldman [21] derives the steady-state distribution of the inventory position for a continuous review (s, S) inventory system where the demand is a discrete-valued Markov-modulated compound Poisson process and the orders are replenished instantaneously.

A special case for the Markov-modulated demand is the stochastic periodic demand case where the demand distribution changes in cycles. Zipkin [22] studies a periodic-review, infinite-horizon inventory problem with stochastic periodic demand and no set up cost where unsatisfied demand is fully backlogged. He shows that a periodic base-stock policy is optimal. That is, the base-stock level depends on the state of the cycle the system currently occupies. Kapuscinski and Tayur [23] extend this result to the case of capacitated supply for the finite-horizon, the discounted infinite-horizon and the infinite-horizon average cost criteria.

We use the Markov-modulated, price-sensitive Poisson demand to model the demand process in this thesis, and the demand process we use is the same with the one used by Song and Zipkin [24] except that we include price sensitivity.

Song and Zipkin [24] present an inventory model that includes a fluctuating demand environment where the demand rate varies with an underlying state-of-the-world variable. They model the world as a continuous-time Markov chain with a discrete state space. In their model, the world affects demand as follows: When the world is in state i, demand follows a Poisson process with rate λ_i. Thus, the overall demand process is a Markov-modulated Poisson process.

The other components of the model are: a fixed or stochastic order lead time, inventory holding and backorder costs, and a positive discount rate. Hence, the overall model becomes a continuous-time, discrete-state dynamic program with two state variables, the world and the inventory position.

They show that if the production cost is linear, a world state-dependent base-stock policy is optimal. In the case where there is a fixed cost to place an order, Song and Zipkin show that a world-dependent (s, S) policy is optimal. They also show that when the problem data are ordered in a certain natural way, the optimal base-stock levels are ordered in the same way. Our results are compatible with the results of this study for the linear production cost case.

We will continue to discuss the studies with this demand model and various system assumptions in the chronological order, and the structure of the optimal policy does not change in any of these. That is, the base-stock policy is optimal when the production cost is linear, and an (*s*, *S*) policy is optimal when there is a fixed set up cost.

Beyer and Sethi [25] prove the existence of state-dependent (*s*, *S*) policies for infinite-horizon average cost criterion of a stochastic inventory problem with Markovian demand, fixed ordering cost, and convex surplus cost. They assume that the unsatisfied demand is backlogged.

Sethi and Cheng [26] generalizes the results obtained for the classical inventory problem with fixed ordering cost by including real life restrictions such as no ordering periods, shelf capacity and service level constraints. The demand is periodic and has a general distribution dependent on a finite-state Markov chain, and they also include the case of cyclic or seasonal demand. They assume the unsatisfied demand is backlogged and the inventory/backlog cost is convex and state-dependent. Sethi and Cheng show that a state-dependent (*s*, *S*) policy is optimal for the finite-horizon and infinite-horizon nonstationary problem they concern.

Özekici and Parlar [27] study a periodic-review inventory model with fixed ordering cost where demand, supply and cost parameters change with respect to a randomly changing environment. They assume that unsatisfied demand is backlogged, and the planning horizon is infinite. The environmental process follows a time-homogeneous Markov chain, and the demand, supplier availability, fixed ordering, unit purchase, unit holding and shortage cost is modulated by the time-homogeneous Markov chain controlling the environment. The supplier availability is defined as up or down where the order is fully received if supplier availability is up and the order could not be received if supplier availability is down. Özekici and Parlar show that environment-dependent base-stock policy is optimal when the order cost is linear in order quantity, and environment-dependent (*s*, *S*) policy is optimal when there is a fixed cost of ordering.

Cheng and Sethi [28] show that state-dependent (*s*, *S*) policies are optimal for the inventory model with Markovian demand they presented. They consider the finite horizon problem where the demand at one period is a nonnegative random variable depending on the current demand state which changes according to the fluctuating external conditions.

Chen and Song [29] consider a serial production/distribution system where random demand arises at Stage 1, Stage 1 orders from Stage 2, etc. and Stage *N* orders from an outside supplier with unlimited stock. The demand process is driven by an exogenous Markov Chain. Excess demand is fully backlogged, linear holding costs are incurred at every stage, and linear backorder costs are incurred at Stage 1. The ordering costs are also linear. The objective is to minimize the long run average costs in the system. They show that echelon base-stock policies with state dependent order-up-to levels are optimal for the system. They also provide an efficient algorithm to compute an optimal policy.

The work of Gallego and Hu [30] is the most recent one on this problem; they study a discrete-time, single-item, single-location, periodic-review production/inventory system with Markov-modulated demands and yields and with finite capacity. The yield is determined as the proportion of usable supply after the order is received. The demand and supply processes are driven by two independent, discrete-time, finite-state and time-homogeneous Markov chains. They prove that a modified, state-dependent, inflated base-stock policy (See p.392 of [31]) is optimal for the single-period, multi-period, and infinite period problems, and the finite-horizon solution converges to the infinite-horizon solution.

The research about two different problems is surveyed in this chapter; the inventory pricing (static or dynamic) and replenishment problem and the inventory replenishment problem in a fluctuating environment. Our model combines these two features with capacitated supply where the capacitated supply will be a new challenge on this topic.

Moreover, the relative benefit of dynamic pricing with respect to static pricing is an open area which we aim to provide guidelines in this thesis.

Chapter 3

MARKOV DECISION PROCESSES

3.1 Introduction

In this chapter, we present the approach we will use in subsequent chapters to construct and analyse the Markov Decision Process optimality equations for certain systems with pricing and replenishment decisions. Markov Decision Processes [MDP] are used to model the systems where decisions are made sequentially under uncertainty, and the MDP models provide the opportunity to compare the immediate gain of current decisions and the possible outcomes of future decision making opportunities [32].

The modern study of stochastic sequential decision problems began with Wald’s work on sequential statistical problems during the Second World War. He later published his studies in his book [33]. Also, Pierre Massé, minister in charge of French electrical planning, introduced many of the basic concepts in his analysis of water resource management models (1946). Many investigators studied sequential problems after the works of these two pioneers, and Bellman introduced the common ingredients to these problems as states, actions, transition probabilities and developed the fundamental equations to determine the optimal policies [34]. He is considered as the first one to develop the mathematical foundations of dynamic programming.

3.2 Markov Decision Problems

In this section, we will construct and analyse a simplified version of the problems studied in subsequent chapters to present the formulation of Markov Decision Processes and the solution method applied in this thesis.

The time points at which the decision maker has to choose an action to influence the future performance of the system are called *decision epochs*. The set of decision epochs can be either discrete or continuous. If it is discrete, then the decisions are made at each decision epoch; otherwise the decisions could be made at random points in time when certain events occur, at the times chosen by the decision maker, or continuously. In the problems discussed in this thesis, the decisions are made at random points of time when certain events as arrivals, replenishments or environment state changes occur. Therefore, the Markov Decision problems analysed here can be described by the five elements below:

1. S is the set of all possible states of the system.
2. A is the set of allowable actions a decision maker could choose at each decision epoch. $A = \cup_{s \in S} A_s$ where A_s is the set of allowable actions for each state s. It is assumed that S and A do not vary with time.
3. $C^t(i,a)$ ($W^t(i,a)$) is the immediate cost incurred (reward received) and $c^t(i,a)$ ($w^t(i,a)$) is the cost (reward) rate imposed from time t until the next transition occurs when action a is chosen in state i at time t. That is, if a transition occurs after T units, then the total cost incurred is given by $C^t(i,a)+Tc^t(i,a)$. $C^t(i,a)$ ($W^t(i,a)$) and $c^t(i,a)$ ($w^t(i,a)$) together constitute the cost (reward) structure of the model.
4. $P_{ij}^{t}(a)$ is the probability the next state will be j when the state is i and action a is chosen at time t.

5. $F_{ij}^{t}(T \mid a)$ is the probability the transition from *i* to *j* will occur before *T* time units when action *a* is chosen at time *t*.

In the problems studied here, it is assumed that the costs (rewards), transition probability and transition time distributions are independent of time. Hence, we drop the superscript "*t*" hereafter. The performance criterion for optimality can be either the discounted cost (reward) over an infinite horizon or long-run average cost (reward) criterion. If the discounted cost (reward) criterion is taken, the objective of the Markov Decision problem is to minimize the expected total discounted cost over an infinite horizon (maximize the expected total discounted reward); otherwise the objective is to minimize the expected long-run average cost (maximize the expected long-run average reward). In all the problems studied in this thesis, the objective is to maximize the profit. Therefore, we will concentrate on maximization problems in all the MDP formulations presented here.

If the discounted reward criterion is used, then the optimal expected total discounted reward with initial state *i* and discount rate β ($\beta > 0$) is denoted by $V_\beta(i)$ and is represented as below [35]:

$$V_\beta(i) = \max_{a \in A_i} \{\overline{W}_\beta(i,a) + \sum_{j=0}^{\infty} P_{ij}(a) \int_0^\infty e^{-\beta T} V_\beta(j) dF_{ij}(T \mid a)\},$$

where (3.1)

$$\overline{W}_\beta(i,a) = W(i,a) + \sum_{j=0}^{\infty} P_{ij}(a) \int_0^\infty \int_0^T e^{-\beta s} w(i,a) ds dF_{ij}(T \mid a).$$

When $F_{ij}(. \mid a)$ are exponential for all *i*, *j*, *a*, and *A*, *S*, and the reward structure satisfy certain conditions, it is well-known that there exists an optimal Markovian deterministic policy (See Theorems 5.5.1, 8.1.2., 8.4.1, and Proposition 6.2.1. in [32]). The problems we will describe in this thesis satisfy these conditions, so we restrict ourselves to the set of Markovian deterministic policies.

We will now formulate a simplified version of the problem analysed in subsequent chapters. We have a production facility producing a single product to stock. There is a single demand class, and customers arrive at the system according to a Poisson process with rate λ. The price is denoted as p, and it is determined in the very beginning of the whole process. We model the random structure denoting the choice of the customer to buy or not to buy in the demand process formulation with the reservation price concept. The reservation price is defined as the maximum price a customer is willing to pay for one unit of a good or service, and we represent the probability a customer buys the product when a price of p is offered as $P(R \geq p)=\overline{F}(p)$ where R is a random variable denoting the reservation price of the customer. Therefore, the buying rate with price p (λ_p) would be $\lambda\overline{F}(p)$. It is assumed that the reservation price distribution is known in advance and λ_p is strictly decreasing in p. In addition, the buying rate is bounded from below by "0" and from above by "λ". A single resource processes one item at a time, and the processing time is exponentially distributed with mean $1/\mu$.

Let $X(t)$ be the amount of stock at time t denoting the state of the system at time t and $h(X(t))$ be the inventory holding cost function. The decision maker has to decide whether to produce at any time, and the decisions depend on the current amount of stock because we consider Markovian policies. Due to the exponential transition times, it is clear that we observe only the current state and do not need the historical information of the process; therefore we simply denote the current amount of inventory as x without any reference to the time point at which the decision is made. The holding cost $h(x)$ is assumed to be nondecreasing and convex. Moreover, *h(x)* is finite for every finite *x*. Production is done one by one, and we also include a fixed production cost of *c* where $c-p\leq 0$.

If the state is *x*, *h(x)* is the cost rate imposed until the next transition occurs. Then, if the action is not to produce ($a = 0$), the only possible event is that a sale occurs. Therefore, the next state will be obviously (x-1). Revenue of price *p* will be gained, and the transition time will be exponential with rate λ_p. However, if the action is to produce ($a = 1$), the transition time will be exponential with ($\lambda_p + \mu$) and there are two possible states for the next transition; (x+1) and (x-1) with respective transition probabilities $[\lambda_p/(\lambda_p+\mu)]$ and $[\mu/(\lambda_p+\mu)]$. If the next state is (x+1), production cost of *c* will be incurred. Otherwise, revenue of price *p* will be gained. From now on, the discount rate will be fixed and taken as β, and we drop the subscript denoting the discount rate hereafter. Thus, the optimal expected total discounted profit with initial inventory amount *x*, *V*(*x*), can be represented as:

$$V(x)=\max\left\{\left[-\int_0^\infty\int_0^t e^{-\beta s}h(x)ds\lambda_p e^{-\lambda_p t}dt+\int_0^\infty e^{-\beta t}[p+V(x-1)]\lambda_p e^{-\lambda_p t}dt\right],\right. \quad (3.2)$$

$$\left[\left(\frac{\lambda_p}{\lambda_p+\mu}[-\int_0^\infty\int_0^t e^{-\beta s}h(x)ds(\lambda_p+\mu)e^{-(\lambda_p+\mu)t}dt+\int_0^\infty e^{-\beta t}[p+V(x-1)](\lambda_p+\mu)e^{-(\lambda_p+\mu)t}dt]\right)\right.$$

$$\left.\left.+\left(\frac{\mu}{\lambda_p+\mu}[-\int_0^\infty\int_0^t e^{-\beta s}h(x)ds(\lambda_p+\mu)e^{-(\lambda_p+\mu)t}dt+\int_0^\infty e^{-\beta t}[-c+V(x+1)](\lambda_p+\mu)e^{-(\lambda_p+\mu)t}dt]\right)\right]\right\}$$

$$= \max\left\{\left[-\frac{h(x)}{\beta+\lambda_p}+\frac{\lambda_p}{\beta+\lambda_p}[p+V(x-1)]\right],\right.$$

$$\left.\left[-\frac{h(x)}{\beta+\lambda_p+\mu}+\frac{\lambda_p}{\beta+\lambda_p+\mu}[p+V(x-1)]+\frac{\mu}{\beta+\lambda_p+\mu}[-c+V(x-1)]\right]\right\}$$

$$= \max\left\{\left[\frac{\lambda_p p-h(x)}{\beta+\lambda_p}+\frac{\lambda_p}{\beta+\lambda_p}V(x-1)\right],\right.$$

$$\left.\left[\frac{\lambda_p p-\mu c-h(x)}{\beta+\lambda_p+\mu}+\frac{\lambda_p}{\beta+\lambda_p+\mu}V(x-1)+\frac{\mu}{\beta+\lambda_p+\mu}V(x+1)\right]\right\}$$

The discount rate β will be considered as the exponential failure rate hereafter where P [failure by time t] = 1- $e^{-\beta t}$, and P [not failure by time t] = $e^{-\beta t}$. Thus, if the action is not to produce (a = 0), the expected transition time will be $[1/\beta+\lambda_p]$ unit times. Hence, the expected profit will be $[\lambda_p p\text{-}h(x)]$ times the expected transition time where $\lambda_p p$ is the revenue rate. Moreover, the probability a sales occurs before the exponential failure will be$[\lambda_p/\beta+\lambda_p]$. Otherwise ($a$ = 1), the expected profit will be $[\lambda_p p\text{-}\mu c\text{-}h(x)]$ times the expected transition time which is $[1/\beta+\mu+\lambda_p]$ since μc is the production cost rate. Moreover, the probability a sales, replenishment or exponential failure occurs first will be$[\lambda_p/\beta+\mu+\lambda_p]$, $[\mu/\beta+\mu+\lambda_p]$, and $[\beta/\beta+\mu+\lambda_p]$ respectively.

3. 3 Method of Uniformization and Value Iteration

We will use the method of uniformization and value iteration to determine the optimal policies of the models stated in subsequent chapters. We therefore present our solution approach on the simplified problem described above in this section.

Instead of analysing the optimality equation above, we use Lippman's uniformization [36] to analyse the system in discrete time with uniformization rate $\gamma = \lambda + \mu \leq 1$, and we rescale the time by taking $\gamma + \beta = 1$.

We then use the method of value iteration to find the structure of the optimal policy maximizing the discounted (v^*) and long-run average rewards. This method is widely used when the Markov Decision problem considered satisfies the assumptions below:

1. Rewards, transition probability and transition time distributions are stationary.
2. Rewards are bounded; $|\overline{W}(i,a)| \leq M < \infty$ for all $a \in A_i$ and $i \in S$.
3. Future rewards are discounted according to a discount factor α, with $0 \leq \alpha < 1$.
4. S is discrete.
5. A_i is finite for each $i \in S$, or A_i is compact, $\overline{W}(i,a)$ is continuous in a for each $i \in S$, and, for each $i, j \in S$, $P_{ij}(a)$ is continuous in a.

Consider the set of real-valued functions $\boldsymbol{L}$ defined on S. The following theorem constructs the basis of the method of value iteration (Proof of Theorem 1 and Corollary 1 could be found in Chapter 6 of [32]; Theorem 6.1.1, 6.2.2, 6.2.3, 6.2.6, 6.2.10).

Theorem 1. If the above assumptions are satisfied, then there exists an optimal deterministic policy d such that v^* is the unique solution to the equation $v = \max_{d \in D} \{r_d + P_d v\} = Lv$ where L is an operator in $\boldsymbol{L}$ and $\boldsymbol{D}$ denotes the set of Markovian Decision rules.

Corollary 1. If $v^0 \in \boldsymbol{L}$, v^n defined by the equation $v^{n+1} = Lv^n$ converges to v^* such that $v^{n+1}(i) = \sup_{a \in A_i} \{\overline{W}(i,a) + \sum_{j \in S} P_{ij}(a) v^n(j)\}$ converges to $v^*(i)$ for all $i \in S$.

The probability of exponential failure is not included here since the process will terminate with the reward of "0" in this case. Hence, $\sum_{j \in S} P_{ij}(a) \leq 1$ in the above equation.

It is worthwhile to note that v^n is the maximum expected total discounted reward of the n-stage problem conferring a terminal reward of v^0. If it is proven that the state space is finite for every value of n in the finite period problem, then it can be concluded that the state space of the infinite-horizon problem is finite and the model is unichain. In our model, we will show that the inventory amount is always finite and so is the inventory holding cost. Moreover, rewards are bounded. Hence, the results obtained here applies for both the discounted and long-run average reward cases since the action space is finite (a = 0 or a = 1), (Theorem 6.2.10 and 8.4.5. in [32]). After applying the uniformization method on the simplified problem presented here, the transition probabilities for state x depending on the action chosen (a = 0 or a = 1) are:

$P_{x,x-1}(0) = \lambda\overline{F}(p)$, $P_{x,x}(0) = \lambda[1-\overline{F}(p)] + \mu$, and

$P_{x,x-1}(1) = \lambda\overline{F}(p), P_{x,x+1}(1) = \mu, P_{x,x}(1) = \lambda[1-\overline{F}(p)]$.

The probability of the exponential failure is equal to β for both actions. Therefore, we have the optimality equation:

$$\begin{aligned} V(x) = -h(x) + \lambda\overline{F}(p)\{V(x-1) + p\} + \lambda[1-\overline{F}(p)]V(x) \\ +\mu \max\left\{V(x), V(x+1) - c\right\} + \beta.0 \end{aligned} \tag{3.3}$$

Let $\Delta V(x)$ be the operator defined as $\Delta V(x) = V(x) - V(x-1)$. Then,

Lemma 1.

a. For every $x \geq 0$, $V(x)$ is concave in x, i. e. $\Delta V(x) - \Delta V(x+1) \geq 0$.

b. $p-\Delta V(x+1) \geq 0$; for all $x \geq 0$.

Proof. We will now use the method of value iteration and generate a sequence of optimal value functions for the n-period problems, $V_n(x)$. Then, we will prove the lemma for the finite period problem with the induction method, and the lemma follows from Theorem 1 and Corollary 1 because we will prove that the inventory state space is finite for every value of n in the finite period problem so that the rewards are bounded for every state and action since the inventory holding cost is bounded at each state. Hence, $V(x)=\lim_{n\to\infty} V_n(x)$ [32]. Moreover, the results obtained here applies for both the discounted and long-run average reward cases since the action space is finite (a = 0 or a = 1), (Theorem 6.2.10 and 8.4.5. in [32]). Set $V_0(x)=0$, for all x.

For $n \geq 1$,

$$\begin{aligned} V_n(x) &= -h(x) + \lambda \bar{F}(p)p + \lambda \bar{F}(p) V_{n-1}(x-1) + \lambda[1-\bar{F}(p)]V_{n-1}(x) \\ &\quad + \mu \max\{V_{n-1}(x), V_{n-1}(x+1)-c\} \end{aligned} \tag{3.4}$$

The lemma is trivially true for all x when n=0. Assume it is true for n-1 for all x. We will now show that it is also true for n. We need to consider the cases x = 1 and $x \geq 2$ separately due to the boundary condition. Then, for $x \geq 2$,

$$\begin{aligned} &\Delta V_n(x) - \Delta V_n(x+1) \\ &= [V_n(x) - V_n(x-1)] - [V_n(x+1) - V_n(x)] \\ &= \{-h(x) - [-h(x-1)]\} - \{-h(x+1) - [-h(x)]\} + \lambda \bar{F}(p)\{\Delta V_{n-1}(x-1) - \Delta V_{n-1}(x)\} \\ &+ \lambda[1-\bar{F}(p)]\{\Delta V_{n-1}(x) - \Delta V_{n-1}(x+1)\} \\ &+ \mu\{\max[V_{n-1}(x), V_{n-1}(x+1)-c] - \max[V_{n-1}(x-1), V_{n-1}(x)-c]\} \\ &- \mu\{\max[V_{n-1}(x+1), V_{n-1}(x+2)-c] - \max[V_{n-1}(x), V_{n-1}(x+1)-c]\} \geq 0 \end{aligned} \tag{3.5}$$

Before discussing the inequality above, a brief explanation is needed about the consequences of the replenishment decisions made at each state considered. It is clear that the replenishment decision is not to replenish at a state x if and only if $((V_{n-1}(x) \geq V_{n-1}(x+1)-c) \equiv (c \geq \Delta V_{n-1}(x+1)))$. Then, if the optimal action is not to replenish at a state x meaning that $c \geq \Delta V_{n-1}(x+1)$, this inequality will also hold for the states with greater amount of inventory by the first part of the lemma since it is assumed to be true for n-1 for all x. Hence, if it is not optimal to replenish at state x, it will not be optimal to replenish at states (x+1), (x+2), and so on. Therefore, a summary of all the possible replenishment decisions related with (3.5) and the inequalities determining those decisions are given in the table below:

Table 1. Possible cases for the replenishment decision for states x-1, x, x+1 in stationary environment

Repl :1 No Repl:0	States and Inequalities			
Cases	x-1	x	x+1	Inequalities
1	0	0	0	$c \geq \Delta V_{n-1}(x) \geq \Delta V_{n-1}(x+1) \geq \Delta V_{n-1}(x+2)$
2	1	0	0	$\Delta V_{n-1}(x) \geq c \geq \Delta V_{n-1}(x+1) \geq \Delta V_{n-1}(x+2)$
3	1	1	0	$\Delta V_{n-1}(x) \geq \Delta V_{n-1}(x+1) \geq c \geq \Delta V_{n-1}(x+2)$
4	1	1	1	$\Delta V_{n-1}(x) \geq \Delta V_{n-1}(x+1) \geq \Delta V_{n-1}(x+2) \geq c$

We can now return to (3.5), this inequality holds since $h(x)$ is convex and Lemma 1 is assumed to be true for n-1 for all x, and

$$\begin{aligned}&\{\max[V_{n-1}(x), V_{n-1}(x+1)-c] - \max[V_{n-1}(x-1), V_{n-1}(x)-c]\}\\&-\{\max[V_{n-1}(x+1), V_{n-1}(x+2)-c] - \max[V_{n-1}(x), V_{n-1}(x+1)-c]\} \geq 0\end{aligned} \tag{3.6}$$

We will show that (3.6) holds for all the cases in Table 1. The inequalities that must be satisfied when the replenishment decisions in Table 1 are applied to (3.6) is listed below for each case with the verifications:

Case 1: $\{V_{n-1}(x) - V_{n-1}(x-1)\} - \{V_{n-1}(x+1) - V_{n-1}(x)\} = \Delta V_{n-1}(x) - \Delta V_{n-1}(x+1) \geq 0$

This inequality holds since the first part of the lemma holds for n-1 for all x by the induction assumption.

Case 2: $\{V_{n-1}(x)-(V_{n-1}(x)-c)\}-\{V_{n-1}(x+1)-V_{n-1}(x)\}=c-\Delta V_{n-1}(x+1)\geq 0$

This inequality holds since it is not optimal to replenish at state x in this case.

Case 3:

$$\{V_{n-1}(x+1)-c-(V_{n-1}(x)-c)\}-\{V_{n-1}(x+1)-(V_{n-1}(x+1)-c)\}=\Delta V_{n-1}(x+1)-c\geq 0$$

This inequality holds since it is optimal to replenish at state x in this case.

Case 4:

$$\{V_{n-1}(x+1)-c-(V_{n-1}(x)-c)\}-\{V_{n-1}(x+2)-c-(V_{n-1}(x+1)-c)\}$$
$$=\Delta V_{n-1}(x+1)-\Delta V_{n-1}(x+2)\geq 0$$

This inequality holds since the first part of the lemma holds for n-1 for all x by the induction assumption. Thus, the lemma is true for $x\geq 2$.

For x = 1,

$$\begin{aligned}
&\Delta V_n(1)-\Delta V_n(2)\\
&=[V_n(1)-V_n(0)]-[V_n(2)-V_n(1)]\\
&=[(-h(1)+\lambda\overline{F}(p)[p+V_{n-1}(0)]+\lambda[1-\overline{F}(p)]V_{n-1}(1)+\mu\max[V_{n-1}(1),V_{n-1}(2)-c])\\
&-(-h(0)+\lambda\overline{F}(p)V_{n-1}(0)+\lambda[1-\overline{F}(p)]V_{n-1}(0)+\mu\max[V_{n-1}(0),V_{n-1}(1)-c])]\\
&-[(-h(2)+\lambda\overline{F}(p)[p+V_{n-1}(1)]+\lambda[1-\overline{F}(p)]V_{n-1}(2)+\mu\max[V_{n-1}(2),V_{n-1}(3)-c])\\
&-(-h(1)+\lambda\overline{F}(p)[p+V_{n-1}(0)]+\lambda[1-\overline{F}(p)]V_{n-1}(1)+\mu\max[V_{n-1}(1),V_{n-1}(2)-c])]\\
&=\{-h(1)-[-h(0)]\}-\{-h(2)-[-h(1)]\}+\lambda\overline{F}(p)p-\lambda\overline{F}(p)\Delta V_{n-1}(1)\\
&+\lambda[1-\overline{F}(p)]\{\Delta V_{n-1}(1)-\Delta V_{n-1}(2)\}\\
&+\mu\{\max[V_{n-1}(1),V_{n-1}(2)-c]-\max[V_{n-1}(0),V_{n-1}(1)-c]\}\\
&-\mu\{\max[V_{n-1}(2),V_{n-1}(3)-c]-\max[V_{n-1}(1),V_{n-1}(2)-c]\}\geq 0
\end{aligned} \tag{3.7}$$

The inequality above holds since $h(x)$ is convex, $V_{n-1}(x)$ is concave for all x and $p-\Delta V_{n-1}(1)\geq 0$ by the induction assumption, and

$$\begin{aligned}&\{\max[V_{n-1}(1),V_{n-1}(2)-c]-\max[V_{n-1}(0),V_{n-1}(1)-c]\}\\&-\{\max[V_{n-1}(2),V_{n-1}(3)-c]-\max[V_{n-1}(1),V_{n-1}(2)-c]\}\geq 0\end{aligned} \tag{3.8}$$

since this inequality holds for all the cases summarized in Table 1 for x=1. Thus, the proof of the first part of the lemma is complete.

For the second part of the lemma, we again need to consider the cases x = 0 and $x \geq 1$ separately due to the boundary condition. Then, for $x \geq 1$,

$$\begin{aligned}&\Delta V_n(x+1)-p\\&=[V_n(x+1)-V_n(x)]-p\\&=\{-h(x+1)-[-h(x)]\}+\lambda\overline{F}(p)\Delta V_{n-1}(x)+\lambda[1-\overline{F}(p)]\Delta V_{n-1}(x+1)\\&+\mu\{\max[V_{n-1}(x+1),V_{n-1}(x+2)-c]-\max[V_{n-1}(x),V_{n-1}(x+1)-c]\}-p\leq 0\end{aligned} \tag{3.9}$$

Since

$p = p(\mu+\lambda\overline{F}(p)+\lambda[1-\overline{F}(p)]+\beta)$ by uniformization we can write (3.9) as follows:

$$\begin{aligned}&\Delta V_n(x+1)-p\\&=[V_n(x+1)-V_n(x)]-p\\&=\{-h(x+1)-[-h(x)]\}+\lambda\overline{F}(p)\{\Delta V_{n-1}(x)-p\}+\lambda[1-\overline{F}(p)]\{\Delta V_{n-1}(x+1)-p\}\\&+\mu\{\max[V_{n-1}(x+1),V_{n-1}(x+2)-c]-\max[V_{n-1}(x),V_{n-1}(x+1)-c]-p\}-\beta p\leq 0\end{aligned}$$

The inequality above holds since $h(x)$ is nondecreasing, $\Delta V_{n-1}(x)-p \leq 0$ for all x by the second part of the induction assumption, and

$$\{\max[V_{n-1}(x+1), V_{n-1}(x+2)-c] - \max[V_{n-1}(x), V_{n-1}(x+1)-c] - p\} \leq 0 . \quad (3.10)$$

This inequality holds for all the cases summarized in the table below by the induction assumption and the inequality c-p≤0. The possible cases are determined by the fact that if the optimal action is not to replenish at a state x meaning $c \geq \Delta V_{n-1}(x+1)$, this inequality will also hold for the states with greater amount of inventory since the lemma is assumed to be true for n-1 for all x.

Table 2. Possible cases for the replenishment decision for states x, x+1 in stationary environment

Repl: 1, No Repl: 0 / Cases	x	x+1	Inequalities
1	0	0	$c \geq \Delta V_{n-1}(x+1) \geq \Delta V_{n-1}(x+2)$
2	1	0	$\Delta V_{n-1}(x+1) \geq c \geq \Delta V_{n-1}(x+2)$
3	1	1	$\Delta V_{n-1}(x+1) \geq \Delta V_{n-1}(x+2) \geq c$

We will now show that (3.10) holds for all the cases in Table 2. The inequalities that must be satisfied when the replenishment decisions in Table 2 are applied to (3.10) is listed below for each case with the verifications:

Case 1: $\{[V_{n-1}(x+1)] - [V_{n-1}(x)] - p\} = \Delta V_{n-1}(x+1) - p \leq 0$

This inequality holds since the second part of the lemma holds for n-1 for all x by the induction assumption.

Case 2: $\{[V_{n-1}(x+1)] - [V_{n-1}(x+1) - c] - p\} = c - p \leq 0$

This inequality holds by the assumptions of our system.

Case 3: $\{[V_{n-1}(x+2)-c-(V_{n-1}(x+1)-c)\}]-p\}=\Delta V_{n-1}(x+2)-p\leq 0$

This inequality holds since the second part of the lemma holds for $V_{n-1}(x)$ for all x by the induction assumption. Thus, the lemma is true for $x \geq 1$.

For $x = 0$,

$$
\begin{aligned}
&\Delta V_n(1)-p \\
&=[V_n(1)-V_n(0)]-p \\
&=[(-h(1)+\lambda\overline{F}(p)[p+V_{n-1}(0)]+\lambda[1-\overline{F}(p)]V_{n-1}(1)+\mu\max[V_{n-1}(1),V_{n-1}(2)-c]) \\
&-(-h(0)+\lambda\overline{F}(p)V_{n-1}(0)+\lambda[1-\overline{F}(p)]V_{n-1}(0)+\mu\max[V_{n-1}(0),V_{n-1}(1)-c])]-p \\
&=\{-h(1)-[-h(0)]\}+\lambda\overline{F}(p)p+\lambda[1-\overline{F}(p)]\Delta V_{n-1}(1) \\
&+\mu\{\max[V_{n-1}(1),V_{n-1}(2)-c]-\max[V_{n-1}(0),V_{n-1}(1)-c]\}-p\leq 0
\end{aligned}
\tag{3.11}
$$

Since

$p=p(\mu+\lambda\overline{F}(p)+\lambda[1-\overline{F}(p)]+\beta)$ by uniformization we can write (3.11) as follows:

$$
\begin{aligned}
&\Delta V_n(1)-p \\
&=[V_n(1)-V_n(0)]-p \\
&=\{-h(1)-[-h(0)]\}+\lambda\overline{F}(p)\{p-p\}+\lambda[1-\overline{F}(p)]\{\Delta V_{n-1}(1)-p\} \\
&+\mu\{\max[V_{n-1}(1),V_{n-1}(2)-c]-\max[V_{n-1}(0),V_{n-1}(1)-c]-p\}-\beta p\leq 0
\end{aligned}
$$

The inequality (3.11) holds since $h(x)$ is nondecreasing, $\Delta V_{n-1}(x)-p \leq 0$ for all x by the induction assumption, and

$$\{\max[V_{n-1}(1),V_{n-1}(2)-c]-\max[V_{n-1}(0),V_{n-1}(1)-c]-p\}\leq 0\,. \tag{3.12}$$

This inequality holds for all the cases summarized in Table 2 for x=0 by the induction assumption and c-p≤0. Thus, the proof of the lemma is complete.

Theorem 2. The optimal replenishment policy is a base-stock policy. Thus, there is an optimal base-stock level S^*, where it is optimal to replenish until the amount of inventory reaches S^*, and not to replenish when $x \geq S^*$.

Proof. Theorem 2 directly follows from Lemma 1.

In this chapter, we presented our solution approach by proving that the optimal replenishment policy is a base-stock policy for the inventory replenishment problem with static pricing in a nonfluctuating environment. In subsequent chapters, we will generalize this result for the replenishment problem with static pricing in a fluctuating environment, and we will also analyse models with dynamic pricing in a fluctuating environment.

Chapter 4

ANALYSIS of REPLENISHMENT POLICIES COMBINED with STATIC PRICING for a MAKE-TO-STOCK PRODUCTION SYSTEM with MARKOV-MODULATED DEMAND and LOST SALES

4.1 Introduction

The widely known results in inventory control model the randomness of demand by using a random component with a well-known density in the definition of the demand process. However, the environmental factors could affect the density of the demand distribution unpredictably, and the focus in the recent studies of inventory control has been shifting to model the impact of fluctuating demand on the optimal replenishment policy.

In particular, changes in the demand distribution might be caused by economic factors such as interest rates, or they might be caused by the changes in business environment conditions such as progress in the product-life-cycle or the consequences of rivals' actions on the market.

In the model we present here, the effect of all the external factors controlling the demand distribution is represented by an underlying variable describing the environment state, and we model the state of the environment as a continuous time, finite-state, homogeneous Markov Chain. The demand rate would depend only on this variable. We assume that there is no backlogging, and the unsatisfied demand is lost forever. In these circumstances, the replenishment policy must adapt to the fluctuating environmental factors; and the structure of the optimal replenishment policy maximizing the expected infinite-horizon discounted profit along with the effect of definite monotonicity patterns of environmental parameters on the optimal policy is analysed in this chapter.

Our main results characterize the structure of the optimal replenishment policy as a base stock policy depending on the state of the environment meaning that there is an optimal base stock level corresponding to each environmental state and the optimal policy requires ordering up to that level. We also show that when the transition rates between different states of the environment reflect a definite monotonicity pattern, the optimal base stock levels of these environmental states have the same order with the demand rates of these states.

4.2 Model Formulation

The underlying Markov Chain representing the environment and the demand rates determined by the states of this chain is defined here. Let $M = \{M(t) : t \geq 0\}$ be the underlying Markov Chain, characterizing the environment; and E be the state space of M. E is assumed to be discrete and finite. Denote by

$Q = \left(q_{ej}\right)_{e,\, j \in E}$, the infinitesimal generator of M; where

$$q_{ee} = -\sum_{j \neq e} q_{ej};\, e, j \in E.$$

We model the demand as a Markov-modulated Poisson process with rate λ_e, where λ_e is the demand rate when $M(t) = e$.

We consider a production facility producing a single product to stock where a single resource processes one item at a time, and the processing time is exponentially distributed with mean $1/\mu$. The price is denoted as p and is determined in the very beginning of the whole process. We model the random structure denoting the choice of the customer to buy or not to buy in the demand process formulation with the reservation price concept. The reservation price is defined as the maximum price a customer is willing to pay for one unit of a good or service, and we represent the probability a customer buys the product when a price of p is offered as $P(R \geq p) = \overline{F}(p)$ where R is a random variable denoting the reservation price of the customer. Therefore, the buying rate in state e with price p would be $\lambda_e \overline{F}(p)$. It is assumed that the reservation price distribution is known in advance and $\overline{F}(p)$ is strictly decreasing in p. In addition, the buying rate is bounded from below by "0" and from above by "λ_e".

Let $X(t)$ be the amount of stock at time t and $h(X(t))$ be the inventory holding cost function. The decision maker has to decide whether to produce at each decision epoch, and the decisions depend on the current amount of stock and the current state of the environment because we consider Markovian policies. Due to exponential transition times, it is clear that we observe only the current state and do not need the historical information of the process; therefore we simply denote the current amount of inventory as x and the current state of the environment by e without any reference to the time point the decision is made. Therefore, we denote the current state of the system by (x, e). The holding cost $h(x)$ is assumed to be nondecreasing and convex. Moreover, $h(x)$ is finite for every finite x. The products are produced one at a time, and we also include a fixed production cost of c where $c\text{-}p \leq 0$.

Let E be the set of environment states 1, 2, …, N where $\lambda_N \geq \lambda_{N-1} \geq \ldots \geq \lambda_e \geq \lambda_{e-1} \geq \ldots \geq \lambda_1$. If the state is (x, e), $h(x)$ is the cost rate imposed until the next transition occurs. Then, if the action is not to produce (*a* = 0), the possible events are occurrence of a sale and environment state transition, and the transition time will be exponential with ($\lambda_e \overline{F}(p) + \sum_{j \neq e} q_{ej}$). Therefore, the next state can be [(*x*-1), *e*] or one of the states (*x*, *j*), $j \neq e$ with respective transition probabilities: $[\lambda_e \overline{F}(p) / \lambda_e \overline{F}(p) + \sum_{j \neq e} q_{ej}]$ and $[q_{ej} / \lambda_e \overline{F}(p) + \sum_{j \neq e} q_{ej}]$. If the next state is [(*x*-1), *e*], a revenue of price *p* will be gained. Otherwise, there will be no immediate revenue or cost. However, if the action is to produce (*a* = 1), the transition time will be exponential with $(\lambda_e \overline{F}(p) + \mu + \sum_{j \neq e} q_{ej})$ and the possible events are occurrence of a sale, replenishment of order and environment state transition. Therefore, the next state can be [(*x*-1), *e*], [(*x*+1), *e*], or one of the states (*x*, *j*), $j \neq e$ with respective transition probabilities: $[\lambda_e \overline{F}(p) / \lambda_e \overline{F}(p) + \mu + \sum_{j \neq e} q_{ej}]$, $[\mu / \lambda_e \overline{F}(p) + \mu + \sum_{j \neq e} q_{ej}]$ and $[q_{ej} / \lambda_e \overline{F}(p) + \mu + \sum_{j \neq e} q_{ej}]$. If the next state is [(*x*-1), *e*], a revenue of price *p* will be gained. If the next state is [(*x*+1), *e*], production cost of *c* will be incurred. Otherwise, there will be no immediate cost or revenue. Thus, the optimal expected total discounted profit of our problem with initial state (*x*, *e*) can be represented as:

$$
\begin{aligned}
V(x,e) = \max\Bigg\{ & \left(\frac{\lambda_e \overline{F}(p)}{\lambda_e \overline{F}(p) + \sum_{j\neq e} q_{ej}} \left[-\int_0^{\infty}\int_0^{t} e^{-\beta s} h(x)\,ds \left(\lambda_e \overline{F}(p) + \sum_{j\neq e} q_{ej}\right) e^{-(\lambda_e \overline{F}(p) + \sum_{j\neq e} q_{ej})t}\,dt \right.\right. \\
& \left.\left. + \int_0^{\infty} e^{-\beta t}[p + V(x-1,e)]\left(\lambda_e \overline{F}(p) + \sum_{j\neq e} q_{ej}\right) e^{-(\lambda_e \overline{F}(p) + \sum_{j\neq e} q_{ej})t}\,dt \right] \right) \\
& + \sum_{j\neq e} \left(\frac{q_{ej}}{\lambda_e \overline{F}(p) + \sum_{j\neq e} q_{ej}} \left[-\int_0^{\infty}\int_0^{t} e^{-\beta s} h(x)\,ds \left(\lambda_e \overline{F}(p) + \sum_{j\neq e} q_{ej}\right) e^{-(\lambda_e \overline{F}(p) + \sum_{j\neq e} q_{ej})t}\,dt \right.\right. \\
& \left.\left. + \int_0^{\infty} e^{-\beta t} V(x,j)\left(\lambda_e \overline{F}(p) + \sum_{j\neq e} q_{ej}\right) e^{-(\lambda_e \overline{F}(p) + \sum_{j\neq e} q_{ej})t}\,dt \right] \right), \\
& \left(\frac{\lambda_e \overline{F}(p)}{\lambda_e \overline{F}(p) + \mu + \sum_{j\neq e} q_{ej}} \left[-\int_0^{\infty}\int_0^{t} e^{-\beta s} h(x)\,ds \left(\lambda_e \overline{F}(p) + \mu + \sum_{j\neq e} q_{ej}\right) e^{-(\lambda_e \overline{F}(p) + \mu + \sum_{j\neq e} q_{ej})t}\,dt \right.\right. \\
& \left.\left. + \int_0^{\infty} e^{-\beta t}[p + V(x-1,e)]\left(\lambda_e \overline{F}(p) + \mu + \sum_{j\neq e} q_{ej}\right) e^{-(\lambda_e \overline{F}(p) + \mu + \sum_{j\neq e} q_{ej})t}\,dt \right] \right) \\
& + \left(\frac{\mu}{\lambda_e \overline{F}(p) + \mu + \sum_{j\neq e} q_{ej}} \left[-\int_0^{\infty}\int_0^{t} e^{-\beta s} h(x)\,ds \left(\lambda_e \overline{F}(p) + \mu + \sum_{j\neq e} q_{ej}\right) e^{-(\lambda_e \overline{F}(p) + \mu + \sum_{j\neq e} q_{ej})t}\,dt \right.\right. \\
& \left.\left. + \int_0^{\infty} e^{-\beta t}[-c + V(x+1,e)]\left(\lambda_e \overline{F}(p) + \mu + \sum_{j\neq e} q_{ej}\right) e^{-(\lambda_e \overline{F}(p) + \mu + \sum_{j\neq e} q_{ej})t}\,dt \right] \right) \\
& + \sum_{j\neq e} \left(\frac{q_{ej}}{\lambda_e \overline{F}(p) + \mu + \sum_{j\neq e} q_{ej}} \left[-\int_0^{\infty}\int_0^{t} e^{-\beta s} h(x)\,ds \left(\lambda_e \overline{F}(p) + \mu + \sum_{j\neq e} q_{ej}\right) e^{-(\lambda_e \overline{F}(p) + \mu + \sum_{j\neq e} q_{ej})t}\,dt \right.\right. \\
& \left.\left. + \int_0^{\infty} e^{-\beta t} V(x,j)\left(\lambda_e \overline{F}(p) + \mu + \sum_{j\neq e} q_{ej}\right) e^{-(\lambda_e \overline{F}(p) + \mu + \sum_{j\neq e} q_{ej})t}\,dt \right] \right) \Bigg\}
\end{aligned}
\tag{4.1}
$$

$$= \max\left\{ \left(\frac{\lambda_e \overline{F}(p)p - h(x)}{\beta + \lambda_e \overline{F}(p) + \sum_{j\neq e} q_{ej}} + \frac{\lambda_e \overline{F}(p)}{\beta + \lambda_e \overline{F}(p) + \sum_{j\neq e} q_{ej}} V(x-1,e) + \frac{\sum_{j\neq e} q_{ej}}{\beta + \lambda_e \overline{F}(p) + \sum_{j\neq e} q_{ej}} V(x,j) \right), \right.$$

$$\left(\frac{\lambda_e \overline{F}(p)p - \mu c - h(x)}{\beta + \lambda_e \overline{F}(p) + \mu + \sum_{j\neq e} q_{ej}} + \frac{\lambda_e \overline{F}(p)}{\beta + \lambda_e \overline{F}(p) + \mu + \sum_{j\neq e} q_{ej}} V(x-1,e) + \frac{\mu}{\beta + \lambda_e \overline{F}(p) + \mu + \sum_{j\neq e} q_{ej}} V(x+1,e) \right.$$

$$\left. \left. + \frac{\sum_{j\neq e} q_{ej}}{\beta + \lambda_e \overline{F}(p) + \mu + \sum_{j\neq e} q_{ej}} V(x,j) \right) \right\}$$

If the action is not to produce (a = 0), the expected transition time will be $[1/\beta + \lambda_e \overline{F}(p) + \sum_{j\neq e} q_{ej}]$ time units. Hence, the expected profit will be $[\lambda_e \overline{F}(p)p - h(x)]$ times the expected transition time where $\lambda_e \overline{F}(p)p$ is the revenue rate. Moreover, the probability that a sales or environment state transition occurs is $[\lambda_e \overline{F}(p) / \beta + \lambda_e \overline{F}(p) + \sum_{j\neq e} q_{ej}]$, and $[\sum_{j\neq e} q_{ej} / \beta + \lambda_e \overline{F}(p) + \sum_{j\neq e} q_{ej}]$, respectively. Otherwise (a = 1), the expected profit will be $[\lambda_e \overline{F}(p)p - \mu c - h(x)]$ times the expected transition time which is $[1/\beta + \lambda_e \overline{F}(p) + \mu + \sum_{j\neq e} q_{ej}]$ since μc is the production cost rate. Moreover, the probability that a sales, replenishment, environment state transition or exponential failure occurs first will be $[\lambda_e \overline{F}(p) / \beta + \lambda_e \overline{F}(p) + \mu + \sum_{j\neq e} q_{ej}]$,

$[\mu/\beta+\lambda_e\overline{F}(p)+\mu+\sum_{j\neq e} q_{ej}]$, $[\sum_{j\neq e} q_{ej}/\beta+\lambda_e\overline{F}(p)+\mu+\sum_{j\neq e} q_{ej}]$ and $[\beta/\beta+\lambda_e\overline{F}(p)+\mu+\sum_{j\neq e} q_{ej}]$ respectively.

To construct the Markov Decision Process optimality equations in discrete time we use Lippman's uniformization [36] with rate :

$\gamma=\sum_{e=1}^{N}\lambda_e+\mu+\sum_{e=1}^{N}\sum_{j\neq e} q_{ej}$, and without loss of generality, we can rescale the time by taking $\gamma+\beta=1$ where β is the discount rate. The discounted systems are equivalent to the systems with exponential failure when the discount rate is equal to the exponential failure rate of the corresponding system [37], therefore β will be considered as the exponential failure rate hereafter.

After applying the uniformization method on our problem presented here, the transition probabilities for state (x, e) depending on the action chosen ($a = 0$ or $a = 1$) are:

$P_{(x,e),(x-1,e)}(0)=\lambda_e\overline{F}(p)$,

$P_{(x,e),(x,e)}(0)=\lambda_e[1-\overline{F}(p)]+\mu+\sum_{i\neq e}\lambda_i+\sum_{i\neq e}\sum_{j\neq i} q_{ij}$, $P_{(x,e),(x,j)}(0)=\sum_{j\neq e} q_{ej}$ and

$P_{(x,e),(x-1,e)}(1)=\lambda_e\overline{F}(p)$, $P_{(x,e),(x+1,e)}(1)=\mu$

$P_{(x,e),(x,e)}(1)=\lambda_e[1-\overline{F}(p)]+\sum_{i\neq e}\lambda_i+\sum_{i\neq e}\sum_{j\neq i} q_{ij}$, $P_{(x,e),(x,j)}(1)=\sum_{j\neq e} q_{ej}$ and

the probability of the exponential failure is equal to β for both actions. Therefore, we have the optimality equation:

$$\begin{aligned} V(x,e) = &-h(x) + \lambda_e \overline{F}(p)\{V(x-1,e) + p\} + \lambda_e[1-\overline{F}(p)]V(x,e) \\ &+\mu \max\left\{V(x,e), V(x+1,e)-c\right\} + \sum_{j\neq e} q_{ej} V(x,j) \\ &+(\sum_{i\neq e} \lambda_i + \sum_{i\neq e}\sum_{j\neq i} q_{ij}) V(x,e) \end{aligned} \tag{4.2}$$

which can be written as follows

$$\begin{aligned} V(x,e) = &-h(x) + \lambda_e \overline{F}(p)p + \lambda_e \overline{F}(p)V(x-1,e) \\ &+\lambda_e[1-\overline{F}(p)]V(x,e) + \mu \max\{V(x,e), V(x+1,e)-c\} \\ &+\sum_{j\neq e} q_{ej} V(x,j) \\ &+(\sum_{i\neq e} \lambda_i + \sum_{i\neq e}\sum_{j\neq i} q_{ij})]V(x,e) \end{aligned}$$

4.3 Structure of the Optimal Policy

In this section, we prove that the optimal value function V is concave. This implies that the optimal replenishment policy is of base-stock type. Let $\Delta V(x,e)$ be the operator defined as $\Delta V(x,e) = V(x,e) - V(x-1,e)$.

Lemma 2.

a. For every $x \geq 0$ and $e \in E$, $V(x, e)$ is concave in x, i. e. $\Delta V(x,e) - \Delta V(x+1,e) \geq 0$

b. p-$\Delta V(x+1,e) \geq 0$; for all $x \geq 0$ and $e \in E$.

Proof. We will use the method of value iteration and generate a sequence of optimal value functions, V_n (x, e), representing the maximum expected total discounted profit of the n-stage problem. Then, we will prove the lemma for the finite period problem with the induction method, and the lemma follows from Theorem 1 and Corollary 1 because we will prove that the inventory state space is finite for every value of n in the finite period problem so that the rewards are bounded for every state and action since the inventory holding cost is bounded at each state. Hence, $V(x,e) = \lim_{n\to\infty} V_n(x,e)$ [32]. Moreover, the results obtained here applies for both the discounted and long-run average reward cases since the action space is finite (a = 0 or a = 1), (Theorem 6.2.10 and 8.4.5. in [32]). Set $V_0(x,e) = 0,$ for all x, e.

For $n \geq 1$,

$$\begin{aligned} V_n(x,e) = &-h(x) + \lambda_e \overline{F}(p)p + \lambda_e \overline{F}(p) V_{n-1}(x-1,e) \\ &+ \lambda_e [1-\overline{F}(p)] V_{n-1}(x,e) + \mu \max\{V_{n-1}(x,e), V_{n-1}(x+1,e) - c\} \\ &+ \sum_{j\neq e} q_{ej} V_{n-1}(x,j) + (\sum_{i\neq e} \lambda_i + \sum_{i\neq e}\sum_{j\neq i} q_{ij})] V_{n-1}(x,e) \end{aligned} \tag{4.3}$$

Then, V (x, e) is concave in x if V_n (x, e) is concave in x for all n by Corollary 1. The lemma is trivially true for all e and x when n=0. Assume it is true for n-1 for all e and x. We need to consider the cases x = 1 and $x \geq 2$ separately due to the boundary conditions. Then, for $x \geq 2$ and all e,

$$\begin{aligned}
&\Delta V_n(x,e) - \Delta V_n(x+1,e) \qquad (4.4)\\
&= [V_n(x,e) - V_n(x-1,e)] - [V_n(x+1,e) - V_n(x,e)]\\
&= \{-h(x) - [-h(x-1)]\} - \{-h(x+1) - [-h(x)]\} + \lambda_e \overline{F}(p)\{\Delta V_{n-1}(x-1,e) - \Delta V_{n-1}(x,e)\}\\
&+\lambda_e[1-\overline{F}(p)]\{\Delta V_{n-1}(x,e) - \Delta V_{n-1}(x+1,e)\}\\
&+\mu\{\max[V_{n-1}(x,e), V_{n-1}(x+1,e) - c] - \max[V_{n-1}(x-1,e), V_{n-1}(x,e) - c]\}\\
&-\mu\{\max[V_{n-1}(x+1,e), V_{n-1}(x+2,e) - c] - \max[V_{n-1}(x,e), V_{n-1}(x+1,e) - c]\}\\
&+\sum_{j\neq e} q_{ej}\{\Delta V_{n-1}(x,j) - \Delta V_{n-1}(x+1,j)\} + (\sum_{i\neq e}\lambda_i + \sum_{i\neq e}\sum_{j\neq i} q_{ij})\{\Delta V_{n-1}(x,e) - \Delta V_{n-1}(x+1,e)\} \geq 0
\end{aligned}$$

Before discussing the inequality above, a brief explanation is needed about the consequences of the replenishment decisions made at each state considered. It is clear that the replenishment decision is not to replenish at a state (*x, e*) if and only if $((V_{n-1}(x,e) \geq V_{n-1}(x+1,e) - c) \equiv (c \geq \Delta V_{n-1}(x+1,e))$. Then, if the optimal action is not to replenish at a state (*x, e*) meaning that $c \geq \Delta V_{n-1}(x+1,e)$, this inequality will also hold for the states with greater amount of inventory in the same environment state since the lemma is assumed to be true for *n*-1 for all *x, e*. Hence, if it is not optimal to replenish at state (*x, e*), it will not be optimal to replenish at states [(*x*+1), e], [(*x*+2), e], and so on. Therefore, a summary of all the possible replenishment decisions related with (4.4) and the inequalities determining those decisions are given in the table below:

Table 3. Possible cases for the replenishment decision for inventory amounts x-1, x, x+1 for environment e

Repl :1 No Repl:0	Inventory Amounts for Environment State e and Inequalities			
Cases	x-1	x	x+1	Inequalities
1	0	0	0	$c \geq \Delta V_{n-1}(x,e) \geq \Delta V_{n-1}(x+1,e) \geq \Delta V_{n-1}(x+2,e)$
2	1	0	0	$\Delta V_{n-1}(x,e) \geq c \geq \Delta V_{n-1}(x+1,e) \geq \Delta V_{n-1}(x+2,e)$
3	1	1	0	$\Delta V_{n-1}(x,e) \geq \Delta V_{n-1}(x+1,e) \geq c \geq \Delta V_{n-1}(x+2,e)$
4	1	1	1	$\Delta V_{n-1}(x,e) \geq \Delta V_{n-1}(x+1,e) \geq \Delta V_{n-1}(x+2,e) \geq c$

We can now return to (4.4), this inequality holds since $h(x)$ is convex and Lemma 2 is assumed to be true for n-1 for all x, e, and

$$\begin{aligned}&\{\max[V_{n-1}(x,e),V_{n-1}(x+1,e)-c]-\max[V_{n-1}(x-1,e),V_{n-1}(x,e)-c]\}\\&-\{\max[V_{n-1}(x+1,e),V_{n-1}(x+2,e)-c]-\max[V_{n-1}(x,e),V_{n-1}(x+1,e)-c]\}\geq 0\end{aligned} \tag{4.5}$$

We will show that (4.5) holds for all the cases in Table 3. The inequalities that must be satisfied when the replenishment decisions in Table 3 are applied to (4.5) is listed below for each case with the verifications:

Case 1:

$$\{V_{n-1}(x,e)-V_{n-1}(x-1,e)\}-\{V_{n-1}(x+1,e)-V_{n-1}(x,e)\}=\Delta V_{n-1}(x,e)-\Delta V_{n-1}(x+1,e)\geq 0$$

This inequality holds since the first part of the lemma holds for n-1 for all x and e by the induction assumption.

Case 2:

$$\{V_{n-1}(x,e)-(V_{n-1}(x,e)-c)\}-\{V_{n-1}(x+1,e)-V_{n-1}(x,e)\}=c-\Delta V_{n-1}(x+1,e)\geq 0$$

This inequality holds since it is not optimal to replenish at state (x, e) in this case.

Case 3:

$$\{V_{n-1}(x+1,e)-c-(V_{n-1}(x,e)-c)\}$$
$$-\{V_{n-1}(x+1,e)-(V_{n-1}(x+1,e)-c)\}=\Delta V_{n-1}(x+1,e)-c\geq 0$$

This inequality holds since it is optimal to replenish at state (x, e) in this case.

Case 4:

$$\{V_{n-1}(x+1,e)-c-(V_{n-1}(x,e)-c)\}-\{V_{n-1}(x+2,e)-c-(V_{n-1}(x+1,e)-c)\}$$
$$=\Delta V_{n-1}(x+1,e)-\Delta V_{n-1}(x+2,e)\geq 0$$

This inequality holds since the first part of the lemma holds for n-1 for all x and e by the induction assumption. Thus, the lemma is true for $x \geq 2$ and all e.

For x = 1 and all e,

$$\begin{aligned}
&\Delta V_n(1,e)-\Delta V_n(2,e) \\
&=[V_n(1,e)-V_n(0,e)]-[V_n(2,e)-V_n(1,e)] \\
&=[(-h(1)+\lambda_e\overline{F}(p)[p+V_{n-1}(0,e)]+\lambda_e[1-\overline{F}(p)]V_{n-1}(1,e)+\mu\max[V_{n-1}(1,e),V_{n-1}(2,e)-c] \\
&+\sum_{j\neq e}q_{ej}V_{n-1}(1,j)+(\sum_{i\neq e}\lambda_i+\sum_{i\neq e}\sum_{j\neq i}q_{ij})V_{n-1}(1,e)) \\
&-(-h(0)+\lambda_e\overline{F}(p)V_{n-1}(0,e)+\lambda_e[1-\overline{F}(p)]V_{n-1}(0,e)+\mu\max[V_{n-1}(0,e),V_{n-1}(1,e)-c] \\
&+\sum_{j\neq e}q_{ej}V_{n-1}(0,j)+(\sum_{i\neq e}\lambda_i+\sum_{i\neq e}\sum_{j\neq i}q_{ij})V_{n-1}(0,e))] \\
&-[(-h(2)+\lambda_e\overline{F}(p)[p+V_{n-1}(1,e)]+\lambda_e[1-\overline{F}(p)]V_{n-1}(2,e)+\mu\max[V_{n-1}(2,e),V_{n-1}(3,e)-c] \\
&+\sum_{j\neq e}q_{ej}V_{n-1}(2,j)+(\sum_{i\neq e}\lambda_i+\sum_{i\neq e}\sum_{j\neq i}q_{ij})V_{n-1}(2,e)) \\
&-(-h(1)+\lambda_e\overline{F}(p)[p+V_{n-1}(0,e)]+\lambda_e[1-\overline{F}(p)]V_{n-1}(1,e)+\mu\max[V_{n-1}(1,e),V_{n-1}(2,e)-c] \\
&+\sum_{j\neq e}q_{ej}V_{n-1}(1,j)+(\sum_{i\neq e}\lambda_i+\sum_{i\neq e}\sum_{j\neq i}q_{ij})V_{n-1}(1,e))]
\end{aligned} \tag{4.6}$$

$$
\begin{aligned}
&= \{-h(1) - [-h(0)]\} - \{-h(2) - [-h(1)]\} + \lambda_e \overline{F}(p)p - \lambda_e \overline{F}(p)\Delta V_{n-1}(1,e) \\
&+ \lambda_e [1 - \overline{F}(p)]\{\Delta V_{n-1}(1,e) - \Delta V_{n-1}(2,e)\} \\
&+ \mu\{\max[V_{n-1}(1,e), V_{n-1}(2,e) - c] - \max[V_{n-1}(0,e), V_{n-1}(1,e) - c]\} \\
&- \mu\{\max[V_{n-1}(2,e), V_{n-1}(3,e) - c] - \max[V_{n-1}(1,e), V_{n-1}(2,e) - c]\} \\
&+ \sum_{j \neq e} q_{ej} \{\Delta V_{n-1}(1,j) - \Delta V_{n-1}(2,j)\} \\
&+ (\sum_{i \neq e} \lambda_i + \sum_{i \neq e} \sum_{j \neq i} q_{ij})\{\Delta V_{n-1}(1,e) - \Delta V_{n-1}(2,e)\} \\
&= \{-h(1) - [-h(0)]\} - \{-h(2) - [-h(1)]\} + \lambda_e \overline{F}(p)\{p - \Delta V_{n-1}(1,e)\} \\
&+ \lambda_e [1 - \overline{F}(p)]\{\Delta V_{n-1}(1,e) - \Delta V_{n-1}(2,e)\} \\
&+ \mu\{\max[V_{n-1}(1,e), V_{n-1}(2,e) - c] - \max[V_{n-1}(0,e), V_{n-1}(1,e) - c]\} \\
&- \mu\{\max[V_{n-1}(2,e), V_{n-1}(3,e) - c] - \max[V_{n-1}(1,e), V_{n-1}(2,e) - c]\} \\
&+ \sum_{j \neq e} q_{ej} \{\Delta V_{n-1}(1,j) - \Delta V_{n-1}(2,j)\} \\
&+ (\sum_{i \neq e} \lambda_i + \sum_{i \neq e} \sum_{j \neq i} q_{ij})\{\Delta V_{n-1}(1,e) - \Delta V_{n-1}(2,e)\} \geq 0
\end{aligned}
$$

The inequality above holds since $h(x)$ is convex, $\Delta V_{n-1}(1,e) - \Delta V_{n-1}(2,e) \geq 0$ and $p - \Delta V_{n-1}(1,e) \geq 0$ for all e by the induction assumptions, and

$$
\begin{aligned}
&\{\max[V_{n-1}(1,e), V_{n-1}(2,e) - c] - \max[V_{n-1}(0,e), V_{n-1}(1,e) - c]\} \\
&-\{\max[V_{n-1}(2,e), V_{n-1}(3,e) - c] - \max[V_{n-1}(1,e), V_{n-1}(2,e) - c]\} \geq 0
\end{aligned}
\tag{4.7}
$$

since this inequality holds for all the cases summarized in Table 3 for x=1. Thus, the proof of the first part of the lemma is complete.

For the second part of the lemma, we again need to consider the cases $x = 0$ and $x \geq 1$ separately due to the boundary conditions. Then, for $x \geq 1$ and all e,

$$\begin{aligned}
&\Delta V_n(x+1,e)-p \\
&=[V_n(x+1,e)-V_n(x,e)]-p \\
&=\{-h(x+1)-[-h(x)]\}+\lambda_e\overline{F}(p)\Delta V_{n-1}(x,e)+\lambda_e[1-\overline{F}(p)]\Delta V_{n-1}(x+1,e) \\
&+\mu\{\max[V_{n-1}(x+1,e),V_{n-1}(x+2,e)-c]-\max[V_{n-1}(x,e),V_{n-1}(x+1,e)-c]\} \\
&+\sum_{j\neq e} q_{ej}\Delta V_{n-1}(x+1,j)+(\sum_{i\neq e}\lambda_i+\sum_{i\neq e}\sum_{j\neq i} q_{ij})\Delta V_{n-1}(x+1,e)-p\leq 0
\end{aligned} \tag{4.8}$$

Since

$p=p(\mu+\lambda_e\overline{F}(p)+\lambda_e[1-\overline{F}(p)]+\sum_{i\neq e}\lambda_i+\sum_{j\neq e} q_{ej}+\sum_{i\neq e}\sum_{j\neq i} q_{ij}+\beta)$ by uniformization,

we can write (4.8) as follows:

$$\begin{aligned}
&\Delta V_n(x+1,e)-p \\
&=[V_n(x+1,e)-V_n(x,e)]-p \\
&=\{-h(x+1)-[-h(x)]\}+\lambda_e\overline{F}(p)\{\Delta V_{n-1}(x,e)-p\}+\lambda_e[1-\overline{F}(p)]\{\Delta V_{n-1}(x+1,e)-p\} \\
&+\mu\{\max[V_{n-1}(x+1,e),V_{n-1}(x+2,e)-c]-\max[V_{n-1}(x,e),V_{n-1}(x+1,e)-c]-p\} \\
&+\sum_{j\neq e} q_{ej}\{\Delta V_{n-1}(x+1,j)-p\}+(\sum_{i\neq e}\lambda_i+\sum_{i\neq e}\sum_{j\neq i} q_{ij})\{\Delta V_{n-1}(x+1,e)-p\}-\beta p\leq 0
\end{aligned}$$

The inequality above holds since $h(x)$ is nondecreasing, $\Delta V_{n-1}(x, e)-p \leq 0$ for all x, e by the second part of the induction assumption, and

$$\{\max[V_{n-1}(x+1,e),V_{n-1}(x+2,e)-c]-\max[V_{n-1}(x,e),V_{n-1}(x+1,e)-c]-p\}\leq 0\,. \tag{4.9}$$

This inequality holds for all the cases summarized in the table below by the induction assumption and the inequality *c-p≤0*. The possible cases are determined by the fact that if the optimal action is not to replenish at a state (*x*, *e*) meaning $c \geq \Delta V_{n-1}(x+1, e)$, this inequality will also hold for the states with greater amount of inventory since the first part of the lemma is assumed to be true for *n*-1 for all *x*.

Table 4. Possible cases for the replenishment decision for inventory amounts *x, x*+1 for environment *e*

Repl: 1, No Repl: 0 / Cases	x	x+1	Inequalities
1	0	0	$c \geq \Delta V_{n-1}(x+1, e) \geq \Delta V_{n-1}(x+2, e)$
2	1	0	$\Delta V_{n-1}(x+1, e) \geq c \geq \Delta V_{n-1}(x+2, e)$
3	1	1	$\Delta V_{n-1}(x+1, e) \geq \Delta V_{n-1}(x+2, e) \geq c$

We will now show that (4.9) holds for all the cases in Table 4. The inequalities that must be satisfied when the replenishment decisions in Table 4 are applied to (4.9) are listed below for each case with the verifications:

Case 1: $\{[V_{n-1}(x+1, e)] - [V_{n-1}(x, e)] - p\} = \Delta V_{n-1}(x+1, e) - p \leq 0$

This inequality holds since the second part of the lemma holds for *n*-1 for all *x, e* by the induction assumption.

Case 2: $\{[V_{n-1}(x+1, e)] - [V_{n-1}(x+1, e) - c] - p\} = c - p \leq 0$

This inequality holds by the assumptions of our system.

Case 3: $\{[V_{n-1}(x+2) - c - (V_{n-1}(x+1) - c)] - p\} = \Delta V_{n-1}(x+2) - p \leq 0$

This inequality holds since the second part of the lemma holds for *n*-1 for all *x, e* by the induction assumption. Thus, the lemma is true for $x \geq 1$.

For $x = 0$ and all e,

$$\begin{aligned}
&\Delta V_n(1,e) - p \\
&= [V_n(1,e) - V_n(0,e)] - p \qquad (4.10)\\
&= [(-h(1) + \lambda_e \overline{F}(p)[p + V_{n-1}(0,e)] + \lambda_e[1 - \overline{F}(p)]V_{n-1}(1,e) \\
&+ \mu \max[V_{n-1}(1,e), V_{n-1}(2,e) - c] + \sum_{j \neq e} q_{ej} V_{n-1}(1,j) + (\sum_{i \neq e} \lambda_i + \sum_{i \neq e} \sum_{j \neq i} q_{ij}) V_{n-1}(1,e)) \\
&-(-h(0) + \lambda_e \overline{F}(p) V_{n-1}(0,e) + \lambda_e[1 - \overline{F}(p)]V_{n-1}(0,e) \\
&+ \mu \max[V_{n-1}(0,e), V_{n-1}(1,e) - c] + \sum_{j \neq e} q_{ej} V_{n-1}(0,j) + (\sum_{i \neq e} \lambda_i + \sum_{i \neq e} \sum_{j \neq i} q_{ij}) V_{n-1}(0,e))] - p \\
&= \{-h(1) - [-h(0)]\} + \lambda_e \overline{F}(p) p + \lambda_e[1 - \overline{F}(p)] \Delta V_{n-1}(1,e) \\
&+ \mu \{\max[V_{n-1}(1,e), V_{n-1}(2,e) - c] - \max[V_{n-1}(0,e), V_{n-1}(1,e) - c]\} \\
&+ \sum_{j \neq e} q_{ej} \Delta V_{n-1}(1,j) + (\sum_{i \neq e} \lambda_i + \sum_{i \neq e} \sum_{j \neq i} q_{ij}) \Delta V_{n-1}(1,e) - p \leq 0
\end{aligned}$$

Since

$$p = p(\mu + \lambda_e \overline{F}(p) + \lambda_e[1 - \overline{F}(p)] + \sum_{i \neq e} \lambda_i + \sum_{j \neq e} q_{ej} + \sum_{i \neq e} \sum_{j \neq i} q_{ij} + \beta)$$

by uniformization we can write (4.10) as follows:

$$\begin{aligned}
&\Delta V_n(1,e) - p \\
&= \{-h(1) - [-h(0)]\} + \lambda_e \overline{F}(p)\{p - p\} + \lambda_e[1 - \overline{F}(p)]\{\Delta V_{n-1}(1,e) - p\} \\
&+ \mu \{\max[V_{n-1}(1,e), V_{n-1}(2,e) - c] - \max[V_{n-1}(0,e), V_{n-1}(1,e) - c] - p\} \\
&+ \sum_{j \neq e} q_{ej} \{\Delta V_{n-1}(1,j) - p\} + (\sum_{i \neq e} \lambda_i + \sum_{i \neq e} \sum_{j \neq i} q_{ij})\{\Delta V_{n-1}(1,e) - p\} - \beta p \leq 0
\end{aligned}$$

The inequality above holds since $h(x)$ is nondecreasing, $\Delta V_{n-1}(1, e)-p \leq 0$ for all x, e by the induction assumption, and

$$\{\max[V_{n-1}(1,e),V_{n-1}(2,e)-c]-\max[V_{n-1}(0,e),V_{n-1}(1,e)-c]-p\} \leq 0 \qquad (4.11)$$

This inequality holds for all the cases summarized in Table 4 for x=0 by the induction assumption and c-p≤0. Thus, the proof of the lemma is complete.

Theorem 3. The optimal replenishment policy is a base-stock policy for every state of the environment. Thus, when the environment state is e, there is an optimal base-stock level S_e^* where it is optimal to replenish until the amount of inventory reaches S_e^*, and not to replenish when x ≥ S_e^*.

Proof. Theorem 3 directly follows from Lemma 2.

Remark 1. The proof of Theorem 3 follows from the concavity of the optimal expected total discounted profit function, $V(x, e)$, in x (Lemma 2) for each e, and therefore the behaviour of the optimal policy is investigated according to the changes in the inventory amount (x) at a specific environment e. Since the optimality equation at a certain environment is examined at the proof of Lemma 2, the optimal policy is again an environment-dependent base-stock policy when the assumptions of the model are generalized to include environment-dependent inventory holding cost function ($h_e(x)$), reservation price distribution ($\overline{F}_e(p)$), production cost per item (c_e), and production rate (μ_e) given that the inventory holding cost function is convex and nondecreasing for all e and the price is at least as much as the maximum value of the production cost that is possible ($p-\max_e\{c_e\} \geq 0$).

4.4 Monotonicity of the Base Stock Levels

In this section, we show that the base stock levels of environmental states have the same ordering with the demand rates of these states. Note that E is the set of environment states 1, 2, …, N where $\lambda_N \geq \lambda_{N-1} \geq ... \geq \lambda_{e+1} \geq \lambda_e \geq \lambda_{e-1} \geq ... \geq \lambda_1$ and β is the discount rate.

Condition 1

a. $q_{ej} \geq q_{e+1,j}$, $\forall e$ where j = 1, 2, …, e-1.

b. $q_{e+1,j} \geq q_{ej}$, $\forall e$ where j = e+2, e+3, …, N.

Condition 1 presents the relationships that the transition rates of the Markov chain characterizing the environment must satisfy for the following lemma and the theorem to be true. We will now explain Condition 1 in more detail.

The first part of Condition 1 states that for every state having lower demand rate than state e, the transition rate from e to these states must be greater than the transition rate from e+1 to these states. The transitions from e+1 to states 1, 2, …, e-1 could be considered as more drastic changes in the environment state than the transitions from e to states 1, 2, …, e-1 since there will be at least one more intermediate state between the current and next state in the former transitions, and intuitively this is not very unrealistic since more drastic changes in the state of the environment are caused by more extraordinary so rarer situations

The second part of Condition 1 states that for every state having higher demand rate than state e+1, the transition rate from e+1 to these states must be greater than the transition rate from e to these states since the transitions from e to states e+2, e+3, …, N could be considered as more drastic changes in the environment state than the transitions from e+1 to states e+2, e+3, …, N.

Lemma 3. $\Delta V(x,e+1) \geq \Delta V(x,e)$, for all x and for $1 \leq e \leq$ N-1.

Proof. We will again use the induction method and show that $\Delta V_n(x,e+1) \geq \Delta V_n(x,e)$ for all x and for $1 \leq e \leq N$-1. The lemma is trivially true for n=0, suppose that it is true for n-1. We again consider x = 1 and $x \geq 2$ separately. Then for $x \geq 2$ and all e,

$$
\begin{aligned}
&\Delta V_n(x,e+1)-\Delta V_n(x,e)\\
&=[V_n(x,e+1)-V_n(x-1,e+1)]-[V_n(x,e)-V_n(x-1,e)]\\
&=[[-h(x)+\lambda_{e+1}\overline{F}(p)p+\lambda_{e+1}\overline{F}(p)V_{n-1}(x-1,e+1)+\lambda_{e+1}[1-\overline{F}(p)]V_{n-1}(x,e+1)\\
&+\mu\max\{V_{n-1}(x,e+1),V_{n-1}(x+1,e+1)-c\}\\
&+\sum_{j\neq e+1}q_{e+1,j}V_{n-1}(x,j)+(\sum_{i\neq e+1}\lambda_i+\sum_{i\neq e+1}\sum_{j\neq i}q_{ij})V_{n-1}(x,e+1)]\\
&-[-h(x-1)+\lambda_{e+1}\overline{F}(p)p+\lambda_{e+1}\overline{F}(p)V_{n-1}(x-2,e+1)+\lambda_{e+1}[1-\overline{F}(p)]V_{n-1}(x-1,e+1)\\
&+\mu\max\{V_{n-1}(x-1,e+1),V_{n-1}(x,e+1)-c\}\\
&+\sum_{j\neq e+1}q_{e+1,j}V_{n-1}(x-1,j)+(\sum_{i\neq e+1}\lambda_i+\sum_{i\neq e+1}\sum_{j\neq i}q_{ij})V_{n-1}(x-1,e+1)]]\\
&-[[-h(x)+\lambda_e\overline{F}(p)p+\lambda_e\overline{F}(p)V_{n-1}(x-1,e)+\lambda_e[1-\overline{F}(p)]V_{n-1}(x,e)\\
&+\mu\max\{V_{n-1}(x,e),V_{n-1}(x+1,e)-c\}\\
&+\sum_{j\neq e}q_{e,j}V_{n-1}(x,j)+(\sum_{i\neq e}\lambda_i+\sum_{i\neq e}\sum_{j\neq i}q_{ij})V_{n-1}(x,e)]\\
&-[-h(x-1)+\lambda_e\overline{F}(p)p+\lambda_e\overline{F}(p)V_{n-1}(x-2,e)+\lambda_e[1-\overline{F}(p)]V_{n-1}(x-1,e)\\
&+\mu\max\{V_{n-1}(x-1,e),V_{n-1}(x,e)-c\}\\
&+\sum_{j\neq e}q_{e,j}V_{n-1}(x-1,j)+(\sum_{i\neq e}\lambda_i+\sum_{i\neq e}\sum_{j\neq i}q_{ij})V_{n-1}(x-1,e)]]\\
&=\lambda_{e+1}\overline{F}(p)\Delta V_{n-1}(x-1,e+1)-\lambda_e\overline{F}(p)\Delta V_{n-1}(x-1,e)\\
&+\lambda_{e+1}[1-\overline{F}(p)]\Delta V_{n-1}(x,e+1)-\lambda_e[1-\overline{F}(p)]\Delta V_{n-1}(x,e)\\
&+\mu\{\max[V_{n-1}(x,e+1),V_{n-1}(x+1,e+1)-c]-\max[V_{n-1}(x-1,e+1),V_{n-1}(x,e+1)-c]\}\\
&-\mu\{\max[V_{n-1}(x,e),V_{n-1}(x+1,e)-c]-\max[V_{n-1}(x-1,e),V_{n-1}(x,e)-c]\}\\
&+\sum_{j\neq e+1}q_{e+1,j}\Delta V_{n-1}(x,j)-\sum_{j\neq e}q_{ej}\Delta V_{n-1}(x,j)\\
&+(\sum_{i\neq e+1}\lambda_i+\sum_{i\neq e+1}\sum_{j\neq i}q_{ij})\Delta V_{n-1}(x,e+1)-(\sum_{i\neq e}\lambda_i+\sum_{i\neq e}\sum_{j\neq i}q_{ij})\Delta V_{n-1}(x,e)
\end{aligned}
\tag{4.12}
$$

$$
\begin{aligned}
&= \lambda_{e+1}\overline{F}(p)\Delta V_{n-1}(x-1,e+1) - \lambda_e \overline{F}(p)\Delta V_{n-1}(x-1,e) \\
&+\lambda_{e+1}[1-\overline{F}(p)]\Delta V_{n-1}(x,e+1) - \lambda_e[1-\overline{F}(p)]\Delta V_{n-1}(x,e) \\
&+\mu\{\max[V_{n-1}(x,e+1),V_{n-1}(x+1,e+1)-c] - \max[V_{n-1}(x-1,e+1),V_{n-1}(x,e+1)-c]\} \\
&-\mu\{\max[V_{n-1}(x,e),V_{n-1}(x+1,e)-c] - \max[V_{n-1}(x-1,e),V_{n-1}(x,e)-c]\} \\
&+(\sum_{j=1}^{e-1} q_{e+1,j} + \sum_{j=e+2}^{N} q_{e+1,j})\Delta V_{n-1}(x,j) + q_{e+1,e}\Delta V_{n-1}(x,e) \\
&-[(\sum_{j=1}^{e-1} q_{e,j} + \sum_{j=e+2}^{N} q_{e,j})\Delta V_{n-1}(x,j) + q_{e,e+1}\Delta V_{n-1}(x,e+1)] \\
&+ \sum_{k\neq e,e+1} \lambda_k\{\Delta V_{n-1}(x,e+1) - \Delta V_{n-1}(x,e)\} + \lambda_e\Delta V_{n-1}(x,e+1) - \lambda_{e+1}\Delta V_{n-1}(x,e) \\
&+(\sum_{i=1}^{e-1}\sum_{j\neq i} q_{ij} + \sum_{i=e+2}^{N}\sum_{j\neq i} q_{ij})\{\Delta V_{n-1}(x,e+1) - \Delta V_{n-1}(x,e)\} \\
&+(\sum_{j=1}^{e-1} q_{e,j} + \sum_{j=e+2}^{N} q_{e,j})\Delta V_{n-1}(x,e+1) + q_{e,e+1}\Delta V_{n-1}(x,e+1) \\
&-[(\sum_{j=1}^{e-1} q_{e+1,j} + \sum_{j=e+2}^{N} q_{e+1,j})\Delta V_{n-1}(x,e) + q_{e+1,e}\Delta V_{n-1}(x,e)]
\end{aligned}
$$

We can regroup the terms in (4.12) as follows:

$$
(1)\quad \begin{aligned}
&\lambda_{e+1}\overline{F}(p)\Delta V_{n-1}(x-1,e+1) - \lambda_e\overline{F}(p)\Delta V_{n-1}(x-1,e) + \lambda_e\Delta V_{n-1}(x,e+1) - \lambda_{e+1}\Delta V_{n-1}(x,e) \\
&+\lambda_{e+1}[1-\overline{F}(p)]\Delta V_{n-1}(x,e+1) - \lambda_e[1-\overline{F}(p)]\Delta V_{n-1}(x,e)
\end{aligned}
$$

$$(2)\quad \begin{aligned}&\mu\{\max[V_{n-1}(x,e+1),V_{n-1}(x+1,e+1)-c]-\max[V_{n-1}(x-1,e+1),V_{n-1}(x,e+1)-c]\}\\&-\mu\{\max[V_{n-1}(x,e),V_{n-1}(x+1,e)-c]-\max[V_{n-1}(x-1,e),V_{n-1}(x,e)-c]\}\\&+\sum_{k\neq e,e+1}\lambda_k\{\Delta V_{n-1}(x,e+1)-\Delta V_{n-1}(x,e)\}\\&+(\sum_{i=1}^{e-1}\sum_{j\neq i}q_{ij}+\sum_{i=e+2}^{N}\sum_{j\neq i}q_{ij})\{\Delta V_{n-1}(x,e+1)-\Delta V_{n-1}(x,e)\}\end{aligned}$$

$$(3)\quad \begin{aligned}&(\sum_{j=1}^{e-1}q_{e+1,j}+\sum_{j=e+2}^{N}q_{e+1,j})\Delta V_{n-1}(x,j)-[(\sum_{j=1}^{e-1}q_{e,j}+\sum_{j=e+2}^{N}q_{e,j})\Delta V_{n-1}(x,j)]\\&+(\sum_{j=1}^{e-1}q_{e,j}+\sum_{j=e+2}^{N}q_{e,j})\Delta V_{n-1}(x,e+1)-[(\sum_{j=1}^{e-1}q_{e+1,j}+\sum_{j=e+2}^{N}q_{e+1,j})\Delta V_{n-1}(x,e)]\end{aligned}$$

We will now analyse each group separately:

(1)

$$\begin{aligned}&\lambda_{e+1}\overline{F}(p)\Delta V_{n-1}(x-1,e+1)-\lambda_e\overline{F}(p)\Delta V_{n-1}(x-1,e)+\lambda_e\Delta V_{n-1}(x,e+1)-\lambda_{e+1}\Delta V_{n-1}(x,e)\\&+\lambda_{e+1}[1-\overline{F}(p)]\Delta V_{n-1}(x,e+1)-\lambda_e[1-\overline{F}(p)]\Delta V_{n-1}(x,e)\\&=(\lambda_{e+1}-\lambda_e)\overline{F}(p)\Delta V_{n-1}(x-1,e+1)+\lambda_e\overline{F}(p)\{\Delta V_{n-1}(x-1,e+1)-\Delta V_{n-1}(x-1,e)\}\\&+(\lambda_{e+1}-\lambda_e)[1-\overline{F}(p)]\Delta V_{n-1}(x,e+1)+\lambda_e[1-\overline{F}(p)]\{\Delta V_{n-1}(x,e+1)-\Delta V_{n-1}(x,e)\}\\&+\lambda_e\Delta V_{n-1}(x,e+1)-\lambda_{e+1}\Delta V_{n-1}(x,e)\\&\geq(\lambda_{e+1}-\lambda_e)\overline{F}(p)\Delta V_{n-1}(x,e+1)+\lambda_e\overline{F}(p)\{\Delta V_{n-1}(x-1,e+1)-\Delta V_{n-1}(x-1,e)\}\\&+(\lambda_{e+1}-\lambda_e)[1-\overline{F}(p)]\Delta V_{n-1}(x,e+1)+\lambda_e[1-\overline{F}(p)]\{\Delta V_{n-1}(x,e+1)-\Delta V_{n-1}(x,e)\}\\&+\lambda_e\Delta V_{n-1}(x,e+1)-\lambda_{e+1}\Delta V_{n-1}(x,e)\\&=\lambda_{e+1}\{\Delta V_{n-1}(x,e+1)-\Delta V_{n-1}(x,e)\}+\lambda_e\overline{F}(p)\{\Delta V_{n-1}(x-1,e+1)-\Delta V_{n-1}(x-1,e)\}\\&+\lambda_e[1-\overline{F}(p)]\{\Delta V_{n-1}(x,e+1)-\Delta V_{n-1}(x,e)\}\geq 0\end{aligned}$$

The first inequality holds since $V_n\,(x, e)$ is concave in x for all e by Lemma 2, and therefore $\Delta V_{n-1}(x-1,e+1) \geq \Delta V_{n-1}(x,e+1)$; the second inequality holds since Lemma 3 is assumed to be true for n-1. Thus, the first part of the inequality holds.

(2)
$$\begin{aligned}
&\mu\{\max[V_{n-1}(x,e+1),V_{n-1}(x+1,e+1)-c]-\max[V_{n-1}(x-1,e+1),V_{n-1}(x,e+1)-c]\}\\
&-\mu\{\max[V_{n-1}(x,e),V_{n-1}(x+1,e)-c]-\max[V_{n-1}(x-1,e),V_{n-1}(x,e)-c]\}\\
&+\sum_{k\neq e,e+1}\lambda_k\{\Delta V_{n-1}(x,e+1)-\Delta V_{n-1}(x,e)\}\\
&+(\sum_{i=1}^{e-1}\sum_{j\neq i}q_{ij}+\sum_{i=e+2}^{N}\sum_{j\neq i}q_{ij})\{\Delta V_{n-1}(x,e+1)-\Delta V_{n-1}(x,e)\}\geq 0
\end{aligned}$$

The inequality above holds by the induction assumption and since

$$\begin{aligned}
&\{\max[V_{n-1}(x,e+1),V_{n-1}(x+1,e+1)-c]-\max[V_{n-1}(x-1,e+1),V_{n-1}(x,e+1)-c]\}\\
&-\{\max[V_{n-1}(x,e),V_{n-1}(x+1,e)-c]-\max[V_{n-1}(x-1,e),V_{n-1}(x,e)-c]\}\geq 0
\end{aligned} \tag{4.13}$$

Before discussing the inequality above, a brief explanation is needed about the consequences of the replenishment decisions made at each state considered. It is clear that the replenishment decision is not to replenish at a state (x, e) if $((V_{n-1}(x,e)\geq V_{n-1}(x+1,e)-c)\equiv(c\geq \Delta V_{n-1}(x+1,e))$. Then, if the optimal action is not to replenish at a state (x, e) meaning that $c\geq \Delta V_{n-1}(x+1,e)$, this inequality will also hold for the states with greater amount of inventory by Lemma 2. Hence, if it is not optimal to replenish at state (x, e), it will not be optimal to replenish at states (x+1, e), (x+2, e), and so on. This information is needed to list all the possible replenishment decisions for an environment state considered above.

Moreover, since Lemma 3 is assumed to be true for all states, if the optimal action is not to replenish at a state (x, e+1) meaning that $c \geq \Delta V_{n-1}(x+1, e+1)$, this inequality will also hold for the states (x, e), (x, e-1) and the action will also be not to replenish at these states. Therefore, a summary of all the possible replenishment decisions related with (4.13) and the inequalities determining those decisions are given in the table below:

Table 5. Possible cases for the replenishment decision for states (x-1, e), (x, e), (x-1, e+1), (x, e+1) in nonstationary environment

Rep :1 No Rep :0	States and Inequalities					
	Env:e+1		Env: e		Inequality for the Case Env: e+1	Inequality for the Case Env: e
Cases	x-1	x	x-1	x		
1	0	0	0	0	$c \geq \Delta V_{n-1}(x,e+1) \geq \Delta V_{n-1}(x+1,e+1)$	$c \geq \Delta V_{n-1}(x,e) \geq \Delta V_{n-1}(x+1,e)$
2	1	0	0	0	$\Delta V_{n-1}(x,e+1) \geq c \geq \Delta V_{n-1}(x+1,e+1)$	$c \geq \Delta V_{n-1}(x,e) \geq \Delta V_{n-1}(x+1,e)$
3	1	0	1	0	$\Delta V_{n-1}(x,e+1) \geq c \geq \Delta V_{n-1}(x+1,e+1)$	$\Delta V_{n-1}(x,e) \geq c \geq \Delta V_{n-1}(x+1,e)$
4	1	1	0	0	$\Delta V_{n-1}(x,e+1) \geq \Delta V_{n-1}(x+1,e+1) \geq c$	$c \geq \Delta V_{n-1}(x,e) \geq \Delta V_{n-1}(x+1,e)$
5	1	1	1	0	$\Delta V_{n-1}(x,e+1) \geq \Delta V_{n-1}(x+1,e+1) \geq c$	$\Delta V_{n-1}(x,e) \geq c \geq \Delta V_{n-1}(x+1,e)$
6	1	1	1	1	$\Delta V_{n-1}(x,e+1) \geq \Delta V_{n-1}(x+1,e+1) \geq c$	$\Delta V_{n-1}(x,e) \geq \Delta V_{n-1}(x+1,e) \geq c$

We will show that (4.13) holds for all the cases in Table 5.

Case 1: $\{\Delta V_{n-1}(x,e+1) - \Delta V_{n-1}(x,e)\} \geq 0$

This inequality holds since the lemma holds for all (x, e) by the induction assumption.

Case 2: $\{c - \Delta V_{n-1}(x,e)\} \geq 0$

This inequality holds since it is not optimal to replenish at state (x-1, e) in this case.

Case 3: $\{c - c\} = 0$.

Case 4: $\{\Delta V_{n-1}(x+1,e+1) - \Delta V_{n-1}(x,e)\} \geq 0$

This inequality holds since it is optimal to replenish at state (*x, e*+1) and not to replenish at state (*x*-1*, e*) in this case.

Case 5:

$\{\Delta V_{n-1}(x+1, e+1) - c\} \geq 0$

This inequality holds since it is optimal to replenish at state (*x, e*+1) in this case.

Case 6:

$\{\Delta V_{n-1}(x+1, e+1) - \Delta V_{n-1}(x+1, e)\} \geq 0$

This inequality holds since the lemma is holds for all (*x, e*) by the induction assumption.

Thus, the second part of the inequality holds.

(3)

$$
\begin{aligned}
&(\sum_{j=1}^{e-1} q_{e+1,j} + \sum_{j=e+2}^{N} q_{e+1,j})\Delta V_{n-1}(x,j) - [(\sum_{j=1}^{e-1} q_{e,j} + \sum_{j=e+2}^{N} q_{e,j})\Delta V_{n-1}(x,j)] \\
&+(\sum_{j=1}^{e-1} q_{e,j} + \sum_{j=e+2}^{N} q_{e,j})\Delta V_{n-1}(x,e+1) - [(\sum_{j=1}^{e-1} q_{e+1,j} + \sum_{j=e+2}^{N} q_{e+1,j})\Delta V_{n-1}(x,e)] \\
&= (\sum_{j=1}^{e-1} q_{e+1,j}[\Delta V_{n-1}(x,j) - \Delta V_{n-1}(x,e)] + \sum_{j=e+2}^{N} q_{e+1,j}[\Delta V_{n-1}(x,j) - \Delta V_{n-1}(x,e)]) \\
&+(\sum_{j=1}^{e-1} q_{e,j}[\Delta V_{n-1}(x,e+1) - \Delta V_{n-1}(x,j)] + \sum_{j=e+2}^{N} q_{e,j}[\Delta V_{n-1}(x,e+1) - \Delta V_{n-1}(x,j)]) \geq 0
\end{aligned}
$$

We will now analyse the inequality above in two groups.

$$(i)(\sum_{j=1}^{e-1} q_{e,j}[\Delta V_{n-1}(x,e+1) - \Delta V_{n-1}(x,j)] + \sum_{j=1}^{e-1} q_{e+1,j}[\Delta V_{n-1}(x,j) - \Delta V_{n-1}(x,e)])$$

$$(ii)(\sum_{j=e+2}^{N} q_{e+1,j}[\Delta V_{n-1}(x,j) - \Delta V_{n-1}(x,e)]) + \sum_{j=e+2}^{N} q_{e,j}[\Delta V_{n-1}(x,e+1) - \Delta V_{n-1}(x,j)])$$

We will now present the relationships between the terms of these two inequalities. It is known that $\Delta V_{n-1}(x,e+1) - \Delta V_{n-1}(x,j) \geq 0$ and $\Delta V_{n-1}(x,j) - \Delta V_{n-1}(x,e) \leq 0$ for j = 1, 2, …, e-1 by the induction assumption. Thus, the first term above is nonnegative if the absolute value of the first summation is greater than or equal to that of the second one.

$$\begin{aligned} &|\Delta V_{n-1}(x,e+1) - \Delta V_{n-1}(x,j)| - |\Delta V_{n-1}(x,j) - \Delta V_{n-1}(x,e)| \\ &= [\Delta V_{n-1}(x,e+1) - \Delta V_{n-1}(x,j)] + [\Delta V_{n-1}(x,j) - \Delta V_{n-1}(x,e)] \\ &= [\Delta V_{n-1}(x,e+1) - \Delta V_{n-1}(x,e)] \geq 0 \end{aligned} \tag{4.14}$$

where j = 1, 2, …, e-1 by the induction assumption. Therefore, when the transition rate multiplied by $[\Delta V_{n-1}(x,e+1) - \Delta V_{n-1}(x,j)]$ is greater than or equal to the transition rate multiplied by $[\Delta V_{n-1}(x,j) - \Delta V_{n-1}(x,e)]$ in (i), i. e., when $(q_{e,j} \geq q_{e+1,j})$ for all j = 1, 2, …, e-1, inequality (4.14) holds. But this is guaranteed by the first part of Condition 1.

It is known that $\Delta V_{n-1}(x,j) - \Delta V_{n-1}(x,e) \geq 0$ and $\Delta V_{n-1}(x,e+1) - \Delta V_{n-1}(x,j) \leq 0$ for j = e+2, e+3, …, N by the induction assumption. Thus, the second term above is nonnegative if the absolute value of the first summation is greater than or equal to that of the second one.

$$\begin{aligned}
&|\Delta V_{n-1}(x,j)-\Delta V_{n-1}(x,e)|-|\Delta V_{n-1}(x,e+1)-\Delta V_{n-1}(x,j)| \\
&=[\Delta V_{n-1}(x,j)-\Delta V_{n-1}(x,e)]+[\Delta V_{n-1}(x,e+1)-\Delta V_{n-1}(x,j)] \\
&=[\Delta V_{n-1}(x,e+1)-\Delta V_{n-1}(x,e)]\geq 0
\end{aligned} \tag{4.15}$$

where $j = e+2, e+3, \ldots, N$ by the induction assumption. Therefore, when the transition rate multiplied by $[\Delta V_{n-1}(x,j)-\Delta V_{n-1}(x,e)]$ is greater than or equal to the transition rate multiplied by $[\Delta V_{n-1}(x,e+1)-\Delta V_{n-1}(x,j)]$ in (*ii*), i. e., when ($q_{e+1,j} \geq q_{e,j}$) for all $j = e+2, e+3, \ldots, N$, inequality (4.15) holds. But this is guaranteed by the second part of Condition 1. Thus, we have shown that (4.12) is nonnegative and therefore Lemma 3 is true for $x \geq 2$, given that Condition 1 holds.

For x=1 and all e,

$$\begin{aligned}
&\Delta V_n(1,e+1)-\Delta V_n(1,e) \\
&=[V_n(1,e+1)-V_n(0,e+1)]-[V_n(1,e)-V_n(0,e)] \\
&=[[(-h(1)+\lambda_{e+1}\overline{F}(p)p+\lambda_{e+1}\overline{F}(p)V_{n-1}(0,e+1) \\
&+\lambda_{e+1}[1-\overline{F}(p)]V_{n-1}(1,e+1)+\mu\max\{V_{n-1}(1,e+1),V_{n-1}(2,e+1)-c\} \\
&+\sum_{j\neq e+1} q_{e+1,j}V_{n-1}(1,j)+(\sum_{i\neq e+1}\lambda_i+\sum_{i\neq e+1}\sum_{j\neq i}q_{ij})V_{n-1}(1,e+1)] \\
&-[(-h(0)+\lambda_{e+1}\overline{F}(p)V_{n-1}(0,e+1)+\lambda_{e+1}[1-\overline{F}(p)]V_{n-1}(0,e+1) \\
&+\mu\max\{V_{n-1}(0,e+1),V_{n-1}(1,e+1)-c\} \\
&+\sum_{j\neq e+1} q_{e+1,j}V_{n-1}(0,j)+(\sum_{i\neq e+1}\lambda_i+\sum_{i\neq e+1}\sum_{j\neq i}q_{ij})V_{n-1}(0,e+1)]] \\
&-[[(-h(1)+\lambda_e\overline{F}(p)p+\lambda_e\overline{F}(p)V_{n-1}(0,e)+\lambda_e[1-\overline{F}(p)]V_{n-1}(1,e) \\
&+\mu\max\{V_{n-1}(1,e),V_{n-1}(2,e)-c\} \\
&+\sum_{j\neq e} q_{e,j}V_{n-1}(1,j)+(\sum_{i\neq e}\lambda_i+\sum_{i\neq e}\sum_{j\neq i}q_{ij})V_{n-1}(1,e)] \\
&-[(-h(0)+\lambda_e\overline{F}(p)V_{n-1}(0,e)+\lambda_e[1-\overline{F}(p)]V_{n-1}(0,e)+\mu\max\{V_{n-1}(0,e),V_{n-1}(1,e)-c\} \\
&+\sum_{j\neq e} q_{e,j}V_{n-1}(0,j)+(\sum_{i\neq e}\lambda_i+\sum_{i\neq e}\sum_{j\neq i}q_{ij})V_{n-1}(0,e)]]
\end{aligned} \tag{4.16}$$

$$
\begin{aligned}
&= (\lambda_{e+1} - \lambda_e)\overline{F}(p)p + \lambda_{e+1}[1-\overline{F}(p)]\Delta V_{n-1}(1,e+1) - \lambda_e[1-\overline{F}(p)]\Delta V_{n-1}(1,e) \\
&+\mu\{\max[V_{n-1}(1,e+1), V_{n-1}(2,e+1)-c] - \max[V_{n-1}(0,e+1), V_{n-1}(1,e+1)-c]\} \\
&-\mu\{\max[V_{n-1}(1,e), V_{n-1}(2,e)-c] - \max[V_{n-1}(0,e), V_{n-1}(1,e)-c]\} \\
&+\sum_{j\neq e+1} q_{e+1,j}\Delta V_{n-1}(1,j) - \sum_{j\neq e} q_{ej}\Delta V_{n-1}(1,j) \\
&+(\sum_{i\neq e+1}\lambda_i + \sum_{i\neq e+1}\sum_{j\neq i} q_{ij})\Delta V_{n-1}(1,e+1) - (\sum_{i\neq e}\lambda_i + \sum_{i\neq e}\sum_{j\neq i} q_{ij})\Delta V_{n-1}(1,e) \\
&= (\lambda_{e+1} - \lambda_e)\overline{F}(p)p + (\lambda_{e+1} - \lambda_e)[1-\overline{F}(p)]\Delta V_{n-1}(1,e+1) \\
&+\lambda_e[1-\overline{F}(p)]\{\Delta V_{n-1}(1,e+1) - \Delta V_{n-1}(1,e)\} \\
&+\mu\{\max[V_{n-1}(1,e+1), V_{n-1}(2,e+1)-c] - \max[V_{n-1}(0,e+1), V_{n-1}(1,e+1)-c]\} \\
&-\mu\{\max[V_{n-1}(1,e), V_{n-1}(2,e)-c] - \max[V_{n-1}(0,e), V_{n-1}(1,e)-c]\} \\
&+(\sum_{j=1}^{e-1} q_{e+1,j} + \sum_{j=e+2}^{N} q_{e+1,j})\Delta V_{n-1}(1,j) + q_{e+1,e}\Delta V_{n-1}(1,e) \\
&-[(\sum_{j=1}^{e-1} q_{e,j} + \sum_{j=e+2}^{N} q_{e,j})\Delta V_{n-1}(1,j) + q_{e,e+1}\Delta V_{n-1}(1,e+1)] \\
&+\sum_{k\neq e,e+1}\lambda_k\{\Delta V_{n-1}(1,e+1) - \Delta V_{n-1}(1,e)\} + \lambda_e\Delta V_{n-1}(1,e+1) - \lambda_{e+1}\Delta V_{n-1}(1,e) \\
&+(\sum_{i=1}^{e-1}\sum_{j\neq i} q_{ij} + \sum_{i=e+2}^{N}\sum_{j\neq i} q_{ij})\{\Delta V_{n-1}(1,e+1) - \Delta V_{n-1}(1,e)\} \\
&+(\sum_{j=1}^{e-1} q_{e,j} + \sum_{j=e+2}^{N} q_{e,j})\Delta V_{n-1}(1,e+1) + q_{e,e+1}\Delta V_{n-1}(1,e+1) \\
&-[(\sum_{j=1}^{e-1} q_{e+1,j} + \sum_{j=e+2}^{N} q_{e+1,j})\Delta V_{n-1}(1,e) + q_{e+1,e}\Delta V_{n-1}(1,e)]
\end{aligned}
$$

We can regroup the terms in (4.16) as follows:

$$
(1)\quad
\begin{aligned}
&(\lambda_{e+1} - \lambda_e)\overline{F}(p)p + (\lambda_{e+1} - \lambda_e)[1-\overline{F}(p)]\Delta V_{n-1}(1,e+1) \\
&+\lambda_e[1-\overline{F}(p)]\{\Delta V_{n-1}(1,e+1) - \Delta V_{n-1}(1,e)\} + \lambda_e\Delta V_{n-1}(1,e+1) - \lambda_{e+1}\Delta V_{n-1}(1,e)
\end{aligned}
$$

$$(2)\quad \begin{aligned}&\mu\{\max[V_{n-1}(1,e+1),V_{n-1}(2,e+1)-c]-\max[V_{n-1}(0,e+1),V_{n-1}(1,e+1)-c]\}\\&-\mu\{\max[V_{n-1}(1,e),V_{n-1}(2,e)-c]-\max[V_{n-1}(0,e),V_{n-1}(1,e)-c]\}\\&+\sum_{k\neq e,e+1}\lambda_k\{\Delta V_{n-1}(1,e+1)-\Delta V_{n-1}(1,e)\}\\&+(\sum_{i=1}^{e-1}\sum_{j\neq i}q_{ij}+\sum_{i=e+2}^{N}\sum_{j\neq i}q_{ij})\{\Delta V_{n-1}(1,e+1)-\Delta V_{n-1}(1,e)\}\end{aligned}$$

$$(3)\quad \begin{aligned}&(\sum_{j=1}^{e-1}q_{e+1,j}+\sum_{j=e+2}^{N}q_{e+1,j})\Delta V_{n-1}(1,j)-[(\sum_{j=1}^{e-1}q_{e,j}+\sum_{j=e+2}^{N}q_{e,j})\Delta V_{n-1}(1,j)]\\&+(\sum_{j=1}^{e-1}q_{e,j}+\sum_{j=e+2}^{N}q_{e,j})\Delta V_{n-1}(1,e+1)-[(\sum_{j=1}^{e-1}q_{e+1,j}+\sum_{j=e+2}^{N}q_{e+1,j})\Delta V_{n-1}(1,e)]\end{aligned}$$

We will now analyse each group separately:

(1)

$$\begin{aligned}&(\lambda_{e+1}-\lambda_e)\overline{F}(p)p+(\lambda_{e+1}-\lambda_e)[1-\overline{F}(p)]\Delta V_{n-1}(1,e+1)+\lambda_e[1-\overline{F}(p)]\{\Delta V_{n-1}(1,e+1)-\Delta V_{n-1}(1,e)\}\\&+\lambda_e\Delta V_{n-1}(1,e+1)-\lambda_{e+1}\Delta V_{n-1}(1,e)\\&=(\lambda_{e+1}-\lambda_e)\overline{F}(p)p+(\lambda_{e+1}-\lambda_e)\Delta V_{n-1}(1,e+1)-(\lambda_{e+1}-\lambda_e)\overline{F}(p)\Delta V_{n-1}(1,e+1)\\&+\lambda_e[1-\overline{F}(p)]\{\Delta V_{n-1}(1,e+1)-\Delta V_{n-1}(1,e)\}\\&+\lambda_e\Delta V_{n-1}(1,e+1)-\lambda_{e+1}\Delta V_{n-1}(1,e)\\&=(\lambda_{e+1}-\lambda_e)\overline{F}(p)p-(\lambda_{e+1}-\lambda_e)\overline{F}(p)\Delta V_{n-1}(1,e+1)\\&+(\lambda_{e+1}-\lambda_e)\Delta V_{n-1}(1,e+1)+\lambda_e\Delta V_{n-1}(1,e+1)-\lambda_{e+1}\Delta V_{n-1}(1,e)\\&+\lambda_e[1-\overline{F}(p)]\{\Delta V_{n-1}(1,e+1)-\Delta V_{n-1}(1,e)\}\\&=(\lambda_{e+1}-\lambda_e)\overline{F}(p)\{p-\Delta V_{n-1}(1,e+1)\}+\lambda_{e+1}\{\Delta V_{n-1}(1,e+1)-\Delta V_{n-1}(1,e)\}\\&+\lambda_e[1-\overline{F}(p)]\{\Delta V_{n-1}(1,e+1)-\Delta V_{n-1}(1,e)\}\geq 0\end{aligned}$$

The inequality above holds since Lemma 3 is assumed to be true for all x and e and $\{p-\Delta V_{n-1}(1,e+1)\geq 0\}$ by the second part of Lemma 2.

$$
(2)\quad \begin{aligned}
&\mu\{\max[V_{n-1}(1,e+1),V_{n-1}(2,e+1)-c]-\max[V_{n-1}(0,e+1),V_{n-1}(1,e+1)-c]\}\\
&-\mu\{\max[V_{n-1}(1,e),V_{n-1}(2,e)-c]-\max[V_{n-1}(0,e),V_{n-1}(1,e)-c]\}\\
&+\sum_{k\neq e,e+1}\lambda_k\{\Delta V_{n-1}(1,e+1)-\Delta V_{n-1}(1,e)\}\\
&+(\sum_{i=1}^{e-1}\sum_{j\neq i}q_{ij}+\sum_{i=e+2}^{N}\sum_{j\neq i}q_{ij})\{\Delta V_{n-1}(1,e+1)-\Delta V_{n-1}(1,e)\}\geq 0
\end{aligned}
$$

The inequality above holds since Lemma 3 is assumed to be true for all x and e and

$$
\begin{aligned}
&\{\max[V_{n-1}(1,e+1),V_{n-1}(2,e+1)-c]-\max[V_{n-1}(0,e+1),V_{n-1}(1,e+1)-c]\}\\
&-\{\max[V_{n-1}(1,e),V_{n-1}(2,e)-c]-\max[V_{n-1}(0,e),V_{n-1}(1,e)-c]\}\geq 0
\end{aligned}
\qquad (4.17)
$$

for all the cases summarized in Table 5 for x=1.

$$
(3)\quad \begin{aligned}
&(\sum_{j=1}^{e-1}q_{e+1,j}+\sum_{j=e+2}^{N}q_{e+1,j})\Delta V_{n-1}(1,j)-[(\sum_{j=1}^{e-1}q_{e,j}+\sum_{j=e+2}^{N}q_{e,j})\Delta V_{n-1}(1,j)]\\
&+(\sum_{j=1}^{e-1}q_{e,j}+\sum_{j=e+2}^{N}q_{e,j})\Delta V_{n-1}(1,e+1)-[(\sum_{j=1}^{e-1}q_{e+1,j}+\sum_{j=e+2}^{N}q_{e+1,j})\Delta V_{n-1}(1,e)]
\end{aligned}
$$

As before, we will now analyse the inequality above in two groups.

$$(iii)(\sum_{j=1}^{e-1} q_{e,j}[\Delta V_{n-1}(1,e+1)-\Delta V_{n-1}(1,j)]+\sum_{j=1}^{e-1} q_{e+1,j}[\Delta V_{n-1}(1,j)-\Delta V_{n-1}(1,e)])$$

$$(iv)(\sum_{j=e+2}^{N} q_{e+1,j}[\Delta V_{n-1}(1,j)-\Delta V_{n-1}(1,e)])+\sum_{j=e+2}^{N} q_{e,j}[\Delta V_{n-1}(1,e+1)-\Delta V_{n-1}(1,j)])$$

We will now present the relationships between the terms of these two inequalities. It is known that $\Delta V_{n-1}(1,e+1)-\Delta V_{n-1}(1,j)\geq 0$ and $\Delta V_{n-1}(1,j)-\Delta V_{n-1}(1,e)\leq 0$ for j = 1, 2, …, e-1 by the induction assumption. Thus, the first term above is nonnegative if the absolute value of the first summation is greater than or equal to that of the second one.

$$\begin{aligned}&|\Delta V_{n-1}(1,e+1)-\Delta V_{n-1}(1,j)|-|\Delta V_{n-1}(1,j)-\Delta V_{n-1}(1,e)|\\&=[\Delta V_{n-1}(1,e+1)-\Delta V_{n-1}(1,j)]+[\Delta V_{n-1}(1,j)-\Delta V_{n-1}(1,e)]\\&=[\Delta V_{n-1}(1,e+1)-\Delta V_{n-1}(1,e)]\geq 0\end{aligned} \tag{4.18}$$

where j = 1, 2, …, e-1 by the induction assumption. Therefore, when the transition rate multiplied by $[\Delta V_{n-1}(1,e+1)-\Delta V_{n-1}(1,j)]$ is greater than or equal to the transition rate multiplied by $[\Delta V_{n-1}(1,j)-\Delta V_{n-1}(1,e)]$ in (*iii*), i. e., when $(q_{e,j}\geq q_{e+1,j})$ for all j = 1, 2, …, e-1, inequality (4.18) holds. But this is guaranteed by the first part of Condition 1.

It is known that $\Delta V_{n-1}(1,j)-\Delta V_{n-1}(1,e)\geq 0$ and $\Delta V_{n-1}(1,e+1)-\Delta V_{n-1}(1,j)\leq 0$ for j = e+2, e+3, …, N by the induction assumption. Thus, the second term above is nonnegative if the absolute value of first summation is greater than or equal to that of the second one.

$$\begin{aligned} &|\Delta V_{n-1}(1,j)-\Delta V_{n-1}(1,e)|-|\Delta V_{n-1}(1,e+1)-\Delta V_{n-1}(1,j)| \\ &=[\Delta V_{n-1}(1,j)-\Delta V_{n-1}(1,e)]+[\Delta V_{n-1}(1,e+1)-\Delta V_{n-1}(1,j)] \\ &=[\Delta V_{n-1}(1,e+1)-\Delta V_{n-1}(1,e)]\geq 0 \end{aligned} \tag{4.19}$$

where $j = e+2, e+3, \ldots, N$ by the induction assumption. Therefore, when the transition rate multiplied by $[\Delta V_{n-1}(1,j)-\Delta V_{n-1}(1,e)]$ is greater than or equal to the transition rate multiplied by $[\Delta V_{n-1}(1,e+1)-\Delta V_{n-1}(1,j)]$ in (*iv*), i. e., when ($q_{e+1,j} \geq q_{e,j}$) for all $j = e+2, e+3, \ldots, N$, inequality (4.19) holds. But this is guaranteed by the second part of Condition 1. Thus, we have shown that (4.16) is nonnegative and therefore Lemma 3 is true for x = 1 given that Condition 1 holds, and the proof is complete.

Theorem 4. The base stock levels for each environmental state e (S_e^*) have the same ordering with the demand rates corresponding to these states given that Condition 1 holds. That is, $S_N^* \geq S_{N-1}^* \geq \ldots \geq S_e^* \geq S_{e-1}^* \geq \ldots \geq S_1^*$ where $\lambda_N \geq \lambda_{N-1} \geq \ldots \geq \lambda_e \geq \lambda_{e-1} \geq \ldots \geq \lambda_1$.

Proof. The theorem is directly implied by Theorem 3 and Lemma 3.

Remark 2. Condition 1 is directly satisfied by definitions of the system when a birth-and-death process represents the environment, i. e., the infinitesimal generator of M is denoted by $Q=\left(q_{ej}\right)_{e,j\in E}$ where $q_{ee}=-\sum_{j\neq e} q_{ej}, e\in E$ and $q_{ej}=0$, for all $j \neq e+1$, e-1. The reason is that $q_{ej}=q_{e+1,j}=0$ for j = 1, 2, …, e-2 and $q_{e,e-1}\geq q_{e+1,e-1}$ since $q_{e+1,e-1}=0$. Hence, the first part of Condition 1 is directly satisfied by definitions of the system when a birth-and-death process represents the environment. Moreover, $q_{e+1,j}=q_{ej}=0$ for j = e+3, e+4, …, N and $q_{e+1,e+2}\geq q_{e,e+2}$ since $q_{e,e+2}=0$. Hence, the second part of Condition 1 is also directly satisfied by definitions of the system. Therefore, the base stock levels for each environmental state e (S_e^*) have the same ordering with the demand rates corresponding to these states without any condition to be satisfied on the values of the environment state transition rates when the continuous-time Markov chain characterizing the environment is a birth-and-death process. (See Appendix A for more detail.)

Remark 3. Condition 1 is automatically satisfied by definitions of the system when the environment has only two states; 1 and 2 where $\lambda_2 \geq \lambda_1$, and there are only two environment state transition rates; q_{12} and q_{21}. The reason is that the inequalities represented by Condition 1 are not necessary since there are not any environment states corresponding to (e-1) and (e+2) when e = 1 and to (e+1) and (e+2) when e = 2. Therefore, the base stock levels for environment states 1 and 2 (S_1^*, S_2^*) have the same ordering with the demand rates corresponding to these states. That is, $S_2^* \geq S_1^*$. (See Appendix A for more detail.)

Remark 4. The proof of Theorem 4 follows from the monotonicity of $\Delta V(x,e)$ in e (Lemma 3) for each e, and therefore the $\Delta V(x,e)$ values of the two environments, e and e+1, are compared with each other. Since the two expressions ($\Delta V(x,e+1)$, $\Delta V(x,e)$) are compared term by term, the base stock levels for each environment have the same ordering with the demand rates of these states given that Condition 1 holds when the assumptions of the model are generalized to include environment-dependent inventory holding cost function ($h_e(x)$) and production cost per item (c_e), given that the marginal inventory holding cost ($h_e(x)$- $h_e(x$-1)) and the production cost per item for each environment have the same ordering with the demand rates of these environments. Thus, if $[h_e(x)\text{-}h_e(x\text{-}1)] \geq [h_e(x)\text{-}h_e(x\text{-}1)]$ and $c_N \geq c_{N-1} \geq ... \geq c_e \geq c_{e-1} \geq ... \geq c_1$ then the base stock levels are monotonic in the demand rates.

Moreover, we provide an illustrative example for the monotonicity of the base-stock levels for a system of four environment states. (See Appendix B for more detail.)

Chapter 5

ANALYSIS of DYNAMIC PRICING POLICIES and COMPARISON with STATIC PRICING

5.1 Introduction

Price is one of the most effective variables that a firm can control, and pricing policies are critical in inventory management since the demand can be encouraged or discouraged by changing the price. The developments in information technologies have increased the use of Internet, and the increasing popularity of E-commerce applications provides firms many advantages such as flexibility in the price changes, easier data collection from the customer, and more accurate information on inventory levels. These advantages influence the firms to try new strategies for pricing, and one of the popular pricing models used in E-commerce is dynamic pricing, where the price of an item may change over time.

Many service industries such as airlines and hotels have been using dynamic pricing for years in revenue management applications, and dynamic pricing is also becoming popular in production and retail environments as many firms investigate the benefits of this pricing model in these sectors. In this chapter, we will first analyse the structure of the optimal replenishment policy when dynamic pricing method is applied on the problem analysed in this thesis and demonstrate that a state-dependent base-stock policy is again optimal in this case. Moreover, we will also show that the optimal prices decrease in the amount of inventory. Next, we will compare the structure of optimal pricing and replenishment policies for dynamic and static pricing methods along with the infinite horizon average profit obtained for these two pricing models.

The average profit criterion is used to compare the static and dynamic pricing policies with replenishment decisions, and relative value iteration method is used to calculate the average profit for the systems in the data range considered ([32], Chapter 8, Section 8.5.5).

5.2 Dynamic Pricing Model Formulation

The system considered in this chapter operates in the same environment described in Chapter 4 by *M(t)* to model the demand arrivals as a Markov-modulated Poisson process with rate λ_e, where λ_e is the demand rate when $M(t) = e$. As before, the demand rates corresponding to the states of *M(t)* are ordered such that $\lambda_N \geq \lambda_{N-1} \geq ... \geq \lambda_e \geq \lambda_{e-1} \geq ... \geq \lambda_1$. Moreover, the service process as well as the reward and cost structure of the problem remains the same. The only difference between the two models is in the action space. Now we need to decide on the price of the item as well as whether to replenish or not to replenish. More explicitly, we are allowed to change the price depending on the environment state and current inventory level, which is commonly referred to as dynamic pricing. Since both the arrival and service processes are the same as in Chapter 4, we can still use uniformization, which gives the following optimality equations for the system with dynamic pricing:

$$V(x,e) = -h(x) + \max_{p}\{\lambda_e \overline{F}(p)[V(x-1,e)+p] + \lambda_e[1-\overline{F}(p)]V(x,e)\} \\ +\mu \max\{V(x,e), V(x+1,e)\} + \sum_{j\neq e} q_{ej}V(x,j) + (\sum_{i\neq e}\lambda_i + \sum_{i\neq e}\sum_{j\neq i} q_{ij})V(x,e) \tag{5.1}$$

where x is the current inventory level and e is the current environment state.

5.3 Structure of the Optimal Policy

In this section, we prove that the optimal value function V is concave; hence the optimal replenishment policy is of base-stock type. Let $\Delta V(x,e)$ be the operator defined as $\Delta V(x,e) = V(x,e) - V(x-1,e)$. Then,

Lemma 4. For every $x \geq 0$ and $e \in E$, $V(x, e)$ is concave in x, i. e. $\Delta V(x,e) - \Delta V(x+1,e) \geq 0$.

Proof. We will use the method of value iteration and generate a sequence of optimal value functions, $V_n(x, e)$, representing the maximum expected total discounted profit of the n-stage problem. Then, we will prove the lemma for the finite period problem with the induction method, and the lemma follows from Theorem 1 and Corollary 1 because we will prove that the inventory state space is finite for every value of n in the finite period problem so that the rewards are bounded for every state and action since the inventory holding cost is bounded at each state and the optimal price is bounded by definitions of the reservation price distribution. Hence, $V(x,e) = \lim_{n\to\infty} V_n(x,e)$ [32]. Moreover, the results obtained here apply for both the discounted and long-run average reward cases since the action space for every state (A_s) is compact because the set of prices that can be offered is a compact set and possible replenishment decisions set is finite (a=0 and a=1), (Theorem 6.2.10 and 8.4.7. in [32]). Set $V_0(x,e) = 0$, for all x, e.

For $n \geq 1$,

$$V_n(x,e) = -h(x) + \max_p\{\lambda_e \overline{F}(p)[V_{n-1}(x-1,e)+p] + \lambda_e[1-\overline{F}(p)]V_{n-1}(x,e)\} \quad (5.2)$$
$$+\mu \max\{V_{n-1}(x,e), V_{n-1}(x+1,e)\} + \sum_{j\neq e} q_{ej}V_{n-1}(x,j)$$
$$+(\sum_{i\neq e}\lambda_i + \sum_{i\neq e}\sum_{j\neq i} q_{ij})V_{n-1}(x,e)$$

Then, $V(x, e)$ will be concave in x if $V_n(x, e)$ is concave in x for all n. The lemma is trivially true for all e and x when $n=0$. Assume it is true for n-1 for all e and x. We need to consider the cases $x = 1$ and $x \geq 2$ separately due to the boundary conditions. Let $p_{x,e}$ be the optimal price for inventory amount x and environment e, which maximizes $V_n(x, e)$, and p_{max} be the maximum price that can be offered. Then, for $x \geq 2$ and all e,

$$\begin{aligned}
&\Delta V_n(x,e)-\Delta V_n(x+1,e) \qquad (5.3)\\
&=[V_n(x,e)-V_n(x-1,e)]-[V_n(x+1,e)-V_n(x,e)]\\
&=\{-h(x)-[-h(x-1)]\}-\{-h(x+1)-[-h(x)]\}\\
&+\{\lambda_e\overline{F}(p_{x,e})[V_{n-1}(x-1,e)+p_{x,e}]+\lambda_e[1-\overline{F}(p_{x,e})]V_{n-1}(x,e)\}\\
&-\{\lambda_e\overline{F}(p_{x-1,e})[V_{n-1}(x-2,e)+p_{x-1,e}]+\lambda_e[1-\overline{F}(p_{x-1,e})]V_{n-1}(x-1,e)\}\\
&-\{\lambda_e\overline{F}(p_{x+1,e})[V_{n-1}(x,e)+p_{x+1,e}]+\lambda_e[1-\overline{F}(p_{x+1,e})]V_{n-1}(x+1,e)\}\\
&+\{\lambda_e\overline{F}(p_{x,e})[V_{n-1}(x-1,e)+p_{x,e}]+\lambda_e[1-\overline{F}(p_{x,e})]V_{n-1}(x,e)\}\\
&+\mu\{\max[V_{n-1}(x,e),V_{n-1}(x+1,e)-c]-\max[V_{n-1}(x-1,e),V_{n-1}(x,e)-c]\}\\
&-\mu\{\max[V_{n-1}(x+1,e),V_{n-1}(x+2,e)-c]-\max[V_{n-1}(x,e),V_{n-1}(x+1,e)-c]\}\\
&+\sum_{j\neq e}q_{ej}\{\Delta V_{n-1}(x,j)-\Delta V_{n-1}(x+1,j)\}\\
&+\Big(\sum_{i\neq e}\lambda_i+\sum_{i\neq e}\sum_{j\neq i}q_{ij}\Big)\{\Delta V_{n-1}(x,e)-\Delta V_{n-1}(x+1,e)\}\overset{?}{\geq}0
\end{aligned}$$

The pricing operator is the only difference between (5.3) and (4.4), and therefore if

$$\begin{aligned}
&\{\lambda_e\overline{F}(p_{x,e})[V_{n-1}(x-1,e)+p_{x,e}]+\lambda_e[1-\overline{F}(p_{x,e})]V_{n-1}(x,e)\} \qquad (5.4)\\
&-\{\lambda_e\overline{F}(p_{x-1,e})[V_{n-1}(x-2,e)+p_{x-1,e}]+\lambda_e[1-\overline{F}(p_{x-1,e})]V_{n-1}(x-1,e)\}\\
&-\{\lambda_e\overline{F}(p_{x+1,e})[V_{n-1}(x,e)+p_{x+1,e}]+\lambda_e[1-\overline{F}(p_{x+1,e})]V_{n-1}(x+1,e)\}\\
&+\{\lambda_e\overline{F}(p_{x,e})[V_{n-1}(x-1,e)+p_{x,e}]+\lambda_e[1-\overline{F}(p_{x,e})]V_{n-1}(x,e)\}\overset{?}{\geq}0,
\end{aligned}$$

then (5.3) also holds. We will now show that (5.4) holds for $x \geq 2$ and all e.

$$
\begin{aligned}
&\{\lambda_e \overline{F}(p_{x,e})[V_{n-1}(x-1,e)+p_{x,e}]+\lambda_e[1-\overline{F}(p_{x,e})]V_{n-1}(x,e)\} \\
&-\{\lambda_e \overline{F}(p_{x-1,e})[V_{n-1}(x-2,e)+p_{x-1,e}]+\lambda_e[1-\overline{F}(p_{x-1,e})]V_{n-1}(x-1,e)\} \\
&-\{\lambda_e \overline{F}(p_{x+1,e})[V_{n-1}(x,e)+p_{x+1,e}]+\lambda_e[1-\overline{F}(p_{x+1,e})]V_{n-1}(x+1,e)\} \\
&+\{\lambda_e \overline{F}(p_{x,e})[V_{n-1}(x-1,e)+p_{x,e}]+\lambda_e[1-\overline{F}(p_{x,e})]V_{n-1}(x,e)\} \\
&\geq \{\lambda_e \overline{F}(p_{x-1,e})[V_{n-1}(x-1,e)+p_{x-1,e}]+\lambda_e[1-\overline{F}(p_{x-1,e})]V_{n-1}(x,e)\} \\
&-\{\lambda_e \overline{F}(p_{x-1,e})[V_{n-1}(x-2,e)+p_{x-1,e}]+\lambda_e[1-\overline{F}(p_{x-1,e})]V_{n-1}(x-1,e)\} \\
&-\{\lambda_e \overline{F}(p_{x+1,e})[V_{n-1}(x,e)+p_{x+1,e}]+\lambda_e[1-\overline{F}(p_{x+1,e})]V_{n-1}(x+1,e)\} \\
&+\{\lambda_e \overline{F}(p_{x+1,e})[V_{n-1}(x-1,e)+p_{x+1,e}]+\lambda_e[1-\overline{F}(p_{x+1,e})]V_{n-1}(x,e)\} \\
&= \{\lambda_e \overline{F}(p_{x-1,e})\Delta V_{n-1}(x-1,e)+\lambda_e[1-\overline{F}(p_{x-1,e})]\Delta V_{n-1}(x,e)\} \\
&-\{\lambda_e \overline{F}(p_{x+1,e})\Delta V_{n-1}(x,e)+\lambda_e[1-\overline{F}(p_{x+1,e})]\Delta V_{n-1}(x+1,e)\} \\
&\geq \{\lambda_e \overline{F}(p_{x-1,e})\Delta V_{n-1}(x,e)+\lambda_e[1-\overline{F}(p_{x-1,e})]\Delta V_{n-1}(x,e)\} \\
&-\{\lambda_e \overline{F}(p_{x+1,e})\Delta V_{n-1}(x,e)+\lambda_e[1-\overline{F}(p_{x+1,e})]\Delta V_{n-1}(x,e)\} = 0
\end{aligned}
$$

where the first inequality holds by the optimality of $p_{x,e}$ in state (x, e) and the second inequality is true by the induction hypothesis. Thus, the lemma is true for n for $x \geq 2$ and all e.

For $x = 1$ and all e,

$$\begin{aligned}
&\Delta V_n(1,e) - \Delta V_n(2,e) \qquad (5.5)\\
&= [V_n(1,e) - V_n(0,e)] - [V_n(2,e) - V_n(1,e)]\\
&= \{-h(1) - [-h(0)]\} - \{-h(2) - [-h(1)]\}\\
&+\lambda_e F(p_{1,e})p_{1,e} + \lambda_e F(p_{1,e})V_{n-1}(0,e) + \lambda_e[1 - F(p_{1,e})]V_{n-1}(1,e)\\
&-\lambda_e V_{n-1}(0,e)\\
&-\lambda_e F(p_{2,e})p_{2,e} - \lambda_e F(p_{2,e})V_{n-1}(1,e) - \lambda_e[1 - F(p_{2,e})]V_{n-1}(2,e)\\
&+\lambda_e F(p_{1,e})p_{1,e} + \lambda_e F(p_{1,e})V_{n-1}(0,e) + \lambda_e[1 - F(p_{1,e})]V_{n-1}(1,e)\\
&+\mu\{\max[V_{n-1}(1,e), V_{n-1}(2,e) - c] - \max[V_{n-1}(0,e), V_{n-1}(1,e) - c]\}\\
&-\mu\{\max[V_{n-1}(2,e), V_{n-1}(3,e) - c] - \max[V_{n-1}(1,e), V_{n-1}(2,e) - c]\}\\
&+\sum_{j\neq e} q_{ej}\{\Delta V_{n-1}(1,j) - \Delta V_{n-1}(2,j)\}\\
&+(\sum_{i\neq e}\lambda_i + \sum_{i\neq e}\sum_{j\neq i} q_{ij})\{\Delta V_{n-1}(1,e) - V_{n-1}(2,e)\} \overset{?}{\geq} 0
\end{aligned}$$

The pricing operator is the only difference between (5.5) and (4.6), and therefore if

$$\begin{aligned}
&\{\lambda_e \overline{F}(p_{1,e})[V_{n-1}(0,e) + p_{1,e}] + \lambda_e[1 - \overline{F}(p_{1,e})]V_{n-1}(1,e)\}\\
&-\lambda_e V_{n-1}(0,e) \qquad (5.6)\\
&-\{\lambda_e \overline{F}(p_{2,e})[V_{n-1}(1,e) + p_{2,e}] + \lambda_e[1 - \overline{F}(p_{2,e})]V_{n-1}(2,e)\}\\
&+\{\lambda_e \overline{F}(p_{1,e})[V_{n-1}(0,e) + p_{1,e}] + \lambda_e[1 - \overline{F}(p_{1,e})]V_{n-1}(1,e)\} \overset{?}{\geq} 0,
\end{aligned}$$

then (5.5) also holds. We will now show that (5.6) holds for x = 1 and all e.

$$
\begin{aligned}
&\{\lambda_e \overline{F}(p_{1,e})[V_{n-1}(0,e)+p_{1,e}]+\lambda_e[1-\overline{F}(p_{1,e})]V_{n-1}(1,e)\} \\
&-\lambda_e V_{n-1}(0,e) \\
&-\{\lambda_e \overline{F}(p_{2,e})[V_{n-1}(1,e)+p_{2,e}]+\lambda_e[1-\overline{F}(p_{2,e})]V_{n-1}(2,e)\} \\
&+\{\lambda_e \overline{F}(p_{1,e})[V_{n-1}(0,e)+p_{1,e}]+\lambda_e[1-\overline{F}(p_{1,e})]V_{n-1}(1,e)\} \\
&\geq \{\lambda_e \overline{F}(p_{\max})[V_{n-1}(0,e)+p_{\max}]+\lambda_e[1-\overline{F}(p_{\max})]V_{n-1}(1,e)\} \\
&-\lambda_e V_{n-1}(0,e) \\
&-\{\lambda_e \overline{F}(p_{2,e})[V_{n-1}(1,e)+p_{2,e}]+\lambda_e[1-\overline{F}(p_{2,e})]V_{n-1}(2,e)\} \\
&+\{\lambda_e \overline{F}(p_{2,e})[V_{n-1}(0,e)+p_{2,e}]+\lambda_e[1-\overline{F}(p_{2,e})]V_{n-1}(1,e)\} \\
&= \lambda_e\{\Delta V_{n-1}(1,e)-\Delta V_{n-1}(2,e)\} \\
&-\{\lambda_e \overline{F}(p_{2,e})[V_{n-1}(1,e)-V_{n-1}(2,e)]\}+\{\lambda_e \overline{F}(p_{2,e})[V_{n-1}(0,e)-V_{n-1}(1,e)]\} \\
&= \lambda_e\{\Delta V_{n-1}(1,e)-\Delta V_{n-1}(2,e)\} \\
&+\{\lambda_e \overline{F}(p_{2,e})[\Delta V_{n-1}(2,e)-\Delta V_{n-1}(1,e)]\} \\
&= [\lambda_e-\lambda_e \overline{F}(p_{2,e})]\{\Delta V_{n-1}(1,e)-\Delta V_{n-1}(2,e)\} \geq 0
\end{aligned}
$$

where the first inequality is true by definitions of $p_{1,e}$, $p_{2,e}$, and p_{max}, and the second inequality holds since $\overline{F}(p_{\max})=0$ by definition of reservation price distribution. Thus, the lemma is true for n for x = 1 and all e.

Lemma 5. Let x_1 and x_2 be two nonnegative inventory levels such that $x_1<x_2$, and p_{1e} and p_{2e} be the optimal price for these inventory levels at environment e, respectively. Then, $p_{1e}>p_{2e}$.

Proof. Suppose $p_{2e}<p_{1e}$. Then, $\overline{F}(p_{1e}) > \overline{F}(p_{2e})$ and for every $e \in E$

$$\begin{aligned}
&\max_p\{\lambda_e \overline{F}(p)[V(x_2-1,e)+p]+\lambda_e[1-\overline{F}(p)]V(x_2,e)\} \\
&=\{\lambda_e \overline{F}(p_{2e})[V(x_2-1,e)+p_{2e}]+\lambda_e[1-\overline{F}(p_{2e})]V(x_2,e)\} \\
&=\{\lambda_e \overline{F}(p_{2e})[V(x_1-1,e)+p_{2e}]+\lambda_e[1-\overline{F}(p_{2e})]V(x_1,e)\} \\
&+\lambda_e \overline{F}(p_{2e})[\Delta V(x_1,e)-\Delta V(x_2,e)]+\lambda_e[V(x_2,e)-V(x_1,e)] \\
&<\{\lambda_e \overline{F}(p_{1e})[V(x_1-1,e)+p_{1e}]+\lambda_e[1-\overline{F}(p_{1e})]V(x_1,e)\} \\
&+\lambda_e \overline{F}(p_{1e})[\Delta V(x_1,e)-\Delta V(x_2,e)]+\lambda_e[V(x_2,e)-V(x_1,e)] \\
&=\{\lambda_e \overline{F}(p_{1e})[V(x_2-1,e)+p_{1e}]+\lambda_e[1-\overline{F}(p_{1e})]V(x_2,e)\}
\end{aligned} \tag{5.7}$$

Thus,

$$\begin{aligned}
&\max_p\{\lambda_e \overline{F}(p)[V(x_2-1,e)+p]+\lambda_e[1-\overline{F}(p)]V(x_2,e)\} \\
&<\{\lambda_e \overline{F}(p_{1e})[V(x_2-1,e)+p_{1e}]+\lambda_e[1-\overline{F}(p_{1e})]V(x_2,e)\}
\end{aligned}$$

However, it contradicts our definitions. Therefore, the assumption $p_{2e}<p_{1e}$ is false and we conclude that $p_{1e}>p_{2e}$.

Theorem 5. For every state of the environment, there is an optimal base-stock level S_e^* where it is optimal to replenish until the amount of inventory reaches S_e^*, and not to replenish when x ≥ S_e^*. Moreover, the optimal price is decreasing in inventory amount, i. e. $\forall e$, there exists a set of optimal prices $\{p_{x,e}\}_{\substack{x=1,\dots,S \\ e=1,\dots,N}}$ where $P_{x,e} \geq P_{x+1,e}$.

Proof. Theorem 5 directly follows from Lemma 4 and Lemma 5.

Finally, we conclude that the optimal replenishment policy for both the static and dynamic pricing method has the same structure; the optimal policy is a base-stock policy for both models. Moreover, in the dynamic pricing method, the optimal prices decrease in the amount of inventory. The reason is that when we have higher amount of stock, the possibility of having lost sales soon is smaller, and therefore the firm wants to sell as much as it can. Thus, the optimal policy decreases the price. However, when the inventory amount is low, the possibility of having lost sales soon increases, and it makes the inventory left more "valuable" in the sense that the firm wants to sell this product to the customers who can pay more. Thus, the optimal policy increases the price.

The dynamic pricing problem here is also analysed in stationary environment by Gayon [13] where he shows that a base-stock policy is also optimal in this case.

Remark 5. The proof of Theorem 3 follows from the concavity of the optimal expected total discounted profit function, $V(x, e)$, in x (Lemma 2) for each e, and therefore the behaviour of the optimal policy is investigated according to the changes in the inventory amount (x) at a specific environment e. Since the optimality equation at a certain environment is examined at the proof of Lemma 2, the optimal policy is again an environment-dependent base-stock policy when the assumptions of the model are generalized to include environment-dependent inventory holding cost function ($h_e(x)$), reservation price distribution ($\overline{F}_e(p)$), production cost per item (c_e), and production rate (μ_e) given that the inventory holding cost function is convex and nondecreasing for all e and the price is at least as much as the maximum value of the production cost that is possible ($p - \max_e\{c_e\} \geq 0$). It is worthwhile to note that the maximum and minimum prices and therefore the price range can be changed depending on the environment ($p_{\max,e}$ and $p_{\min,e}$) since the reservation price distribution depends on the environment.

5.4 Numerical Analysis and Comparison of Dynamic and Static Pricing Policies

In this section, we will present the results obtained by generating sample runs with the parameters described below. The objective of this numerical study is to observe the changes in the structure of the optimal pricing and replenishment policy for both pricing models, and to gain insights about the relative profit increase dynamic pricing method brings compared to the static pricing in stationary and nonstationary environment. Here we have two environment states, namely Low(L) and High(H), where the demand rate of the High(H) state is greater than the demand rate of the Low(L) state. The average profit criterion is used to analyse and compare the static and dynamic pricing policies with replenishment decisions, and relative value iteration method is used to calculate the average profit for the systems in the data range considered ([32], Chapter 8, Section 8.5.5). The value iteration algorithm given in the book of Puterman [32] is coded in C.

5.4.1 Parameter Values Used in the Numerical Analysis

The values of the parameters used to generate the sample runs are taken as below:

- $h(x) = h\,x$, where h is taken to be the inventory holding cost per item per unit time and $h = 0.01$.
- $\overline{F}(p) = b - ap,$ with $a = 1$, $b = 1$.
- In this numerical study, the net profit margin obtained from each product (p-c) is used instead of using price p and the constant unit production cost of c separately since the amount sold will be equal to the amount produced in the long-run. Therefore, a net profit margin of p' is used to represent the net revenue obtained from each sale in the sample runs generated, and the reservation price distribution is shifted by changing a to $a' = a(p/p\text{-}c)$ where $\overline{F}(p) = b - ap = \overline{F'}(p') = b - a'p'$.

- Pricing decision for static pricing is to choose the optimum profit margin from the discrete set p' = 0, 0.01, 0.02, 0.03, ..., 1 with a minimum of 0 and a maximum of 1.
- Pricing decison for dynamic pricing is to choose the optimum margin from the compact set [0, 1] which maximizes the pricing operator in V_n, which is

$$\max_{p'}\{\lambda_e \overline{F}(p')(V_{n-1}(x-1,e)+p')+\lambda_e[1-\overline{F}(p')]V_{n-1}(x,e)\} \tag{5.8}$$

for the inventory level x and environment e. The value iteration method calculates the optimal expected average reward when there are n transitions left for the system to terminate using the V_{n-1} values already calculated in the previous iteration. Therefore, the V_{n-1} values are treated as constants in the derivations below.

- $\lambda_L = (\lambda_{Average} - \epsilon)$, $\lambda_H = (\lambda_{Average} + \epsilon)$, where $\lambda_{Average} = 1$, and ϵ = 0, 0.1, 0.2, 0.3, ..., 0.9; with linear reservation price distribution, and the pricing operator (PO) can be represented as:

$$\begin{aligned} PO &= \max_{p'}\{\lambda_e \overline{F}(p')(V_{n-1}(x-1,e)+p')+\lambda_e[1-\overline{F}(p')]V_{n-1}(x,e)\} \\ &= \max_{p'}\{\lambda_e(b-ap')(V_{n-1}(x-1,e)+p')+\lambda_e[1-(b-ap')]V_{n-1}(x,e)\} \end{aligned}$$

where

$$\frac{d[PO]}{dp'} = b - 2ap' + a\Delta V_{n-1}(x,e) \tag{5.9}$$

$$\frac{d^2(PO)}{d(p')^2} = -2a$$

Thus, PO is concave with the maximizing margin

$$p'_{opt} = \frac{b}{2a} + \frac{\Delta V_{n-1}(x,e)}{2} \tag{5.10}$$

Note that monetary scale (*MS*) can be changed by dividing or multiplying the margin (p') and the inventory holding cost per item per unit time (h) with a certain amount (ms) (the value a' should be rescaled by multiplying the old value with the reciprocal of the amount used to rescale the monetary values, i. e. $a' \rightarrow a'/ms$).

Then, changing the money scale will correspond to multiplying or dividing the expected rewards obtained at each transition. Hence, changing the money scale does not change the optimal policy.

Thus, we actually observe infinitely many systems that can be created by changing the money scale, but we do not include any money scale change in our results since it does not cause any change in the optimal policy.

- To generate samples for different systems, we change the utilization rate (ρ) by changing the production rate (μ) from 0.1 to 1.0 by incrementing μ by 0.01 for every value of ϵ. Hence, μ = 0.1, 0.11, 0.12, ..., 1.00.

  Since the transition probabilities (tp) are the ratios of transition rates (tr) to the normalization rate ($nr = \Sigma tr$), ($tp = tr / nr$), dividing or multiplying all the transition rates changes the time scale without causing any change in the optimal policy, and therefore the change in the time scale only causes a change in the average profit such that the average profit of the new system can directly be obtained by dividing or multiplying the average profit of the old system with the same amount used to change the time scale.

Thus, we actually observe infinitely many systems that can be created by changing the time scale for every value of ρ, but we do not include any time scale change in our results since it does not cause any change in the optimal policy.

- The transition rates are taken as $q_{LH} = q_{HL} = \alpha = 0.01$.

Even though the values of these parameters might seem restricted, they actually cover a complete set of problems for different possible values of a, b, and λ values due to scaling properties. Gayon [13] gives a detailed explanation.

Thus, the system could be described generally in the figure below:

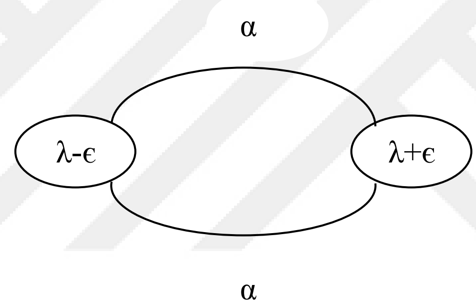

Figure 1. The transition rates for the system with two environment states

5.4.2 Results

In this section, we will present the results obtained from the sample runs generated. First, we will compare the structure of the optimal static and dynamic pricing policies combined with replenishment decisions by comparing the base stock levels and optimal prices for different values of demand variability (ϵ), and then we will compare the average profit obtained by dynamic pricing with the profit obtained by applying static pricing for each system to understand the benefit of dynamic pricing model.

The optimal base-stock values, optimal prices and optimal average profit are obtained for different ϵ values of 0, 0.1, ..., 0.9, and for the values ϵ = 0, 0.3, 0.8, the results are presented in the graphics below:

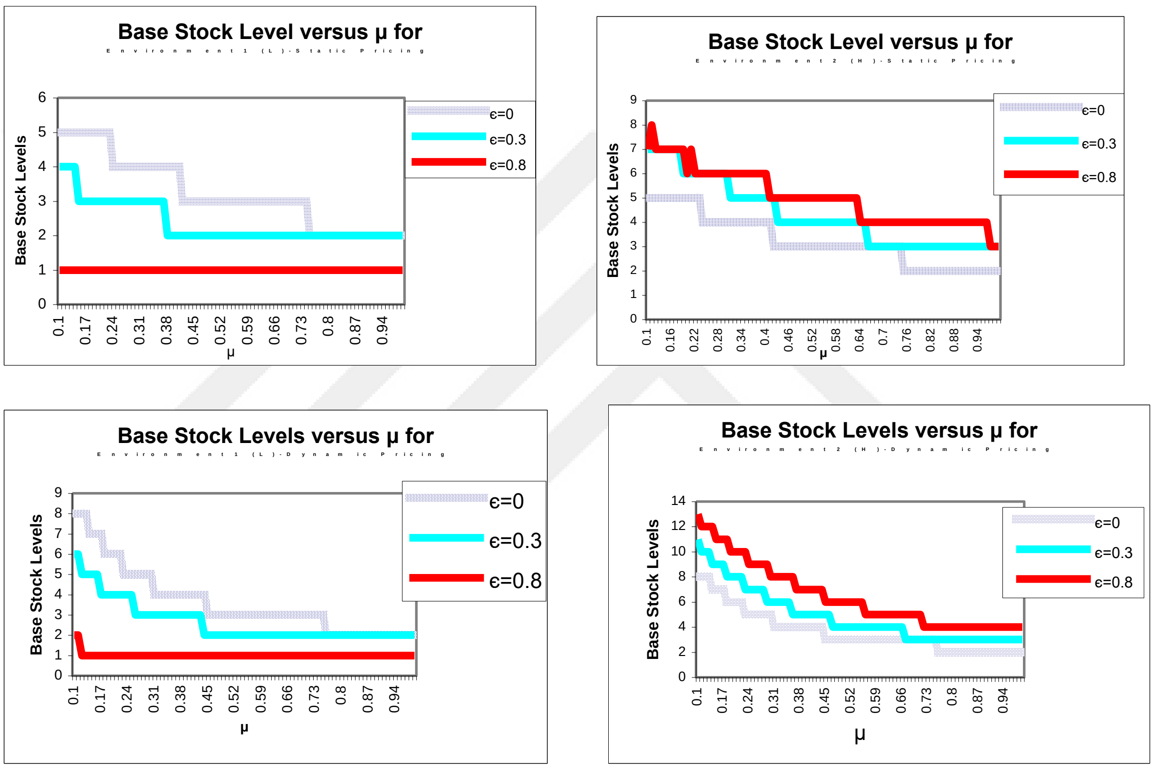

Figure 2. The base-stock levels for static and dynamic pricing for the Low(L) and High(H) environment states at each system and for different demand variabilities

- The base-stock levels of both environments (L and H) decrease in the production rate (μ) since the firm needs less stock when it can produce faster.
- The base-stock level of environment (L) decreases in the demand variability since the demand rate of Low state (λ-ϵ) decreases in ϵ by definition. The firm needs less amount of inventory when the demand is lower. However, the base-stock level of environment

(H) increases in the demand variability since the demand rate of High state ($\lambda+\epsilon$) increases in ϵ by definition. The firm needs greater amount of inventory when the demand is higher.

- The base-stock level determined by dynamic pricing method is higher than the base-stock level determined by static pricing in both environment states (L and H) for all demand variabilities since dynamic pricing gives the firm the opportunity of getting rid of excess stock by decreasing the prices because the firm can control the price and price has a direct effect upon demand. It is optimal to try to avoid lost sales by increasing the amount of stock especially when the capacity is low since the minimum realistic profit margin to offer (0.01) is equal to the holding cost per item per unit time.

To create a benchmark for the optimal profit margin ($p' = p-c$) for static pricing, we choose the maximum margins offered by the dynamic pricing model at both environment states. p'_{1L} and p'_{1H} are the maximum margins charged in Low and High environments, respectively, since the optimal prices decrease in the inventory level (Lemma 5).

In Figures 3a and 3b, we present the optimal margins for both pricing models separately in order to gain insights about the structure of the optimal pricing policy for both methods in stationary and nonstationary environment. In Figure 4, we compare the optimal static and dynamic margins for the cases where $\epsilon = 0$ and $\epsilon = 0.8$. Note that the case with $\epsilon = 0$ corresponds to the stationary environment, which will be denoted by (SE).

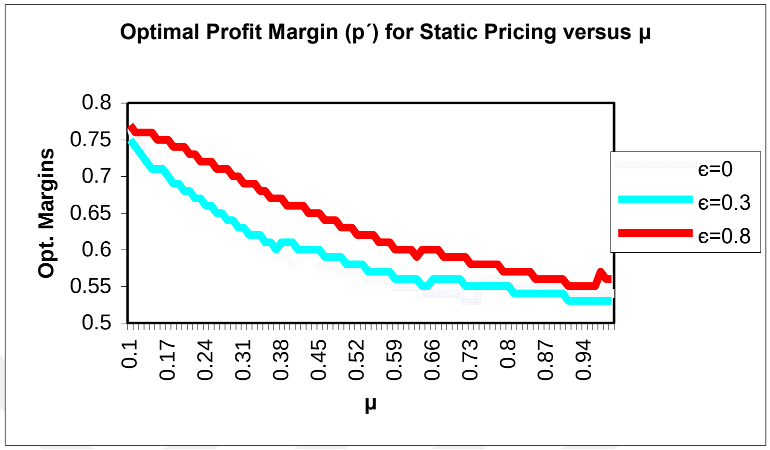

Figure 3a. The optimal profit margins for static pricing for each system with different demand variabilities

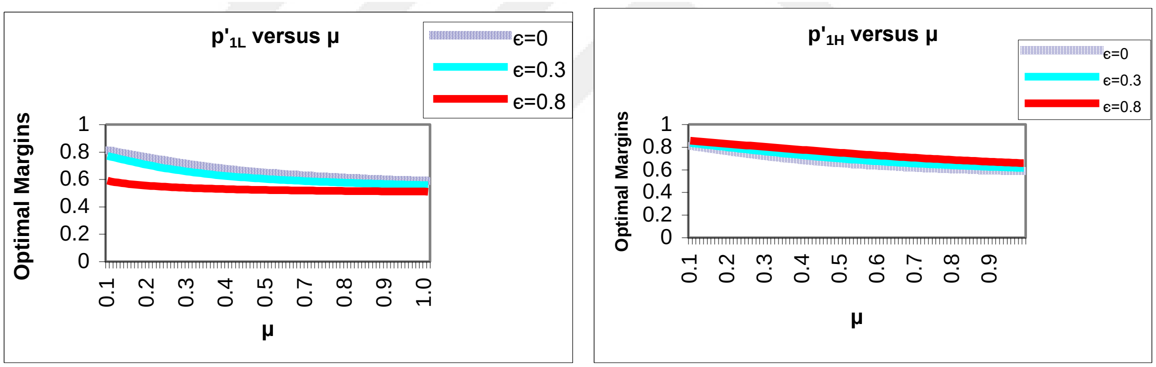

Figure 3b. The maximum values of optimal dynamic profit margins for each system with different demand variabilities

- The optimal static and dynamic pricing margins decrease in production rate (μ) meaning that it is optimal to try to increase the amount of sales as much as possible by decreasing the margin if there is enough capacity rather than increasing the margin obtained from each sale and causing a decrease in the total amount of sales.

- The optimal static pricing margin increases in demand variability meaning that the fluctuating enviroment influences the optimal policy to charge higher prices and earn more from each sale to compensate the uncertainty of demand.
- The maximum optimal dynamic pricing margin offered at the Low state decreases in demand variability whereas the maximum optimal dynamic pricing margin offered at the High state increases in demand variability since the demand rate at the Low state (λ- ϵ) decreases in ϵ whereas the demand rate at the High state (λ+ ϵ) increases in ϵ.

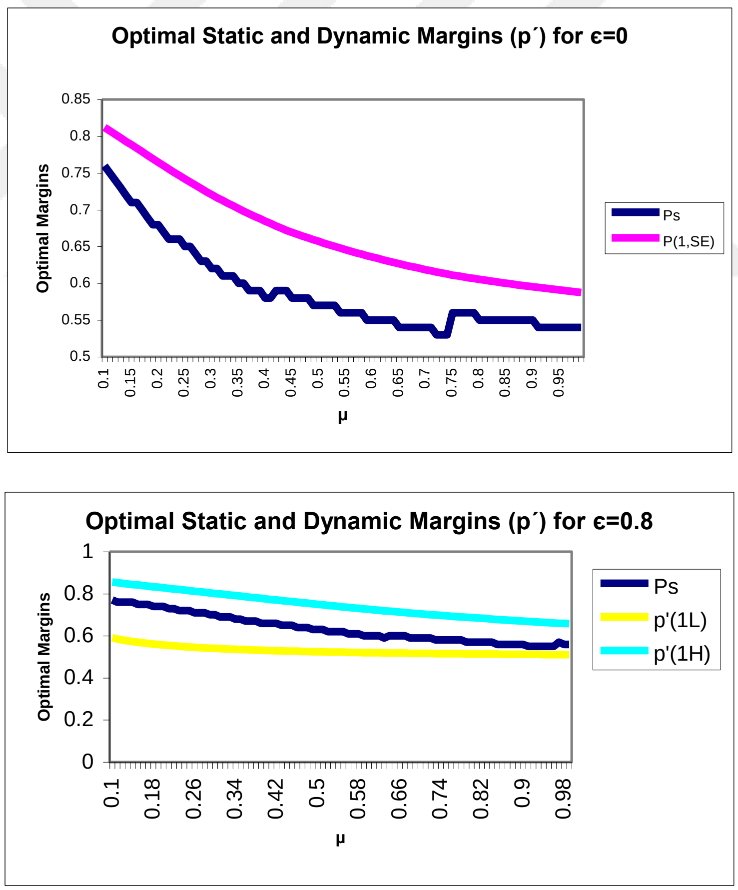

Figure 4. Comparison of the optimal static and maximum dynamic pricing margins for the cases $\epsilon = 0$ and $\epsilon = 0.8$ where SE denotes the stationary environment

- The dynamic pricing method charges higher prices than static pricing in stationary environment. Moreover, the maximum optimal dynamic price in the environment with lower demand rate,which is charged when inventory amount is 1,decreases and falls below the optimal static price as the demand variability increases whereas the maximum optimal dynamic price in the environment with higher demand rate increases above the optimal static price.

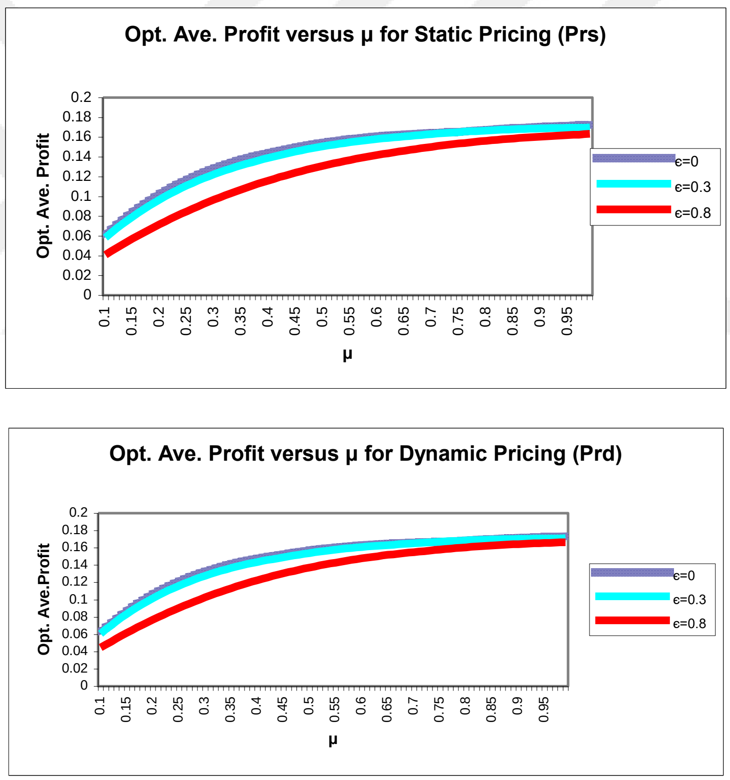

Figure 5a. Optimal average profit for static and dynamic pricing

- The optimal average profit of both pricing policies increases in (μ) and is concave in (μ) since it is not possible to obtain a lower profit with increased capacity and excess capacity brings relatively small increase in profit.

- The optimal average profit of both pricing policies decreases in demand variability (ϵ) since the fluctuating environment and the demand uncertainty it brings decrease the revenue of the firm. It is worthwhile to note that the optimal static prices increase as the demand variability increases, but the optimal average profit decreases in the demand variability meaning that the nonstationary environment is not beneficial for neither the firm nor the customer when static pricing is applied.

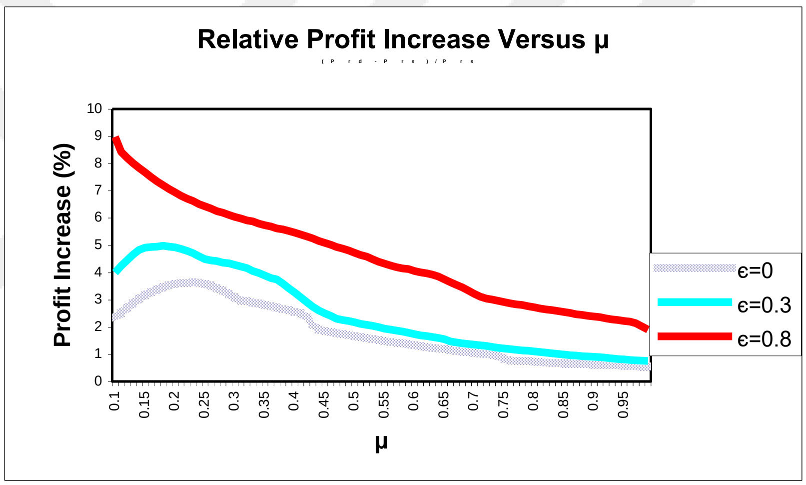

Figure 5b. Relative profit increase obtained with dynamic pricing

- Gayon [13] generates sample runs for his model and finds that the maximum relative profit increase that could be obtained by applying dynamic pricing with linear reservation price distribution is 3.78891% which is obtained when both pricing models are applied optimally, μ is taken to be 0.25 and the holding cost is $h = 0.01$. Our results for stationary environment ($\epsilon = 0$) are compatible with his results.
- The maximum relative profit increase we obtained by applying dynamic pricing with linear reservation price distribution is 3.65914% when μ is taken to be 0.23. Since we

also observed that the dynamic pricing method brings less than 4% increase compared to static pricing when both methods are applied optimally as Chen et al. [1] and Gayon [13] did, we checked if this is the same for the case of nonstationary environment. However, we observe that the relative profit increase of dynamic pricing method can goes up to 10% since the maximum increase we obtained in this numerical study is 10.41474% where ϵ is taken to be 0.7 and μ is 0.12.

Thus, we conclude that dynamic pricing method is more beneficial in fluctuating environments, and the relative profit increase of dynamic pricing tends to increase in demand variability although a perfect monotonicity is not observed for this tendency. Since the firm has more authority on price than it has in the static pricing method, the managers could react more effectively to the environmental fluctuations by changing the price in dynamic pricing method, and this advantage brings more revenues than static pricing.

Moreover, it is observed that the relative profit increase of dynamic pricing increases as the production rate (μ) decreases. The reason is that the firm does not have many options to apply as a replenishment policy when the supply process is very restrictive, and the firm's financial performance depends mainly on its pricing strategy, and the advantages of dynamic pricing gain more importance and become more effective in these cases.

5.4.3 Additional Numerical Examples

In this section, we test the sensitivity of our previous results with respect to the cases with exponential reservation price distribution ($\overline{F}(p) = ae^{-bp}$) and different values of demand variability.

Our first example is for the case with environment transition rates (α) taken to be 0.1. We generated sample runs for the two different cases with α = 0.1 and *a* = *b* = 1. The results are shown in the figure below:

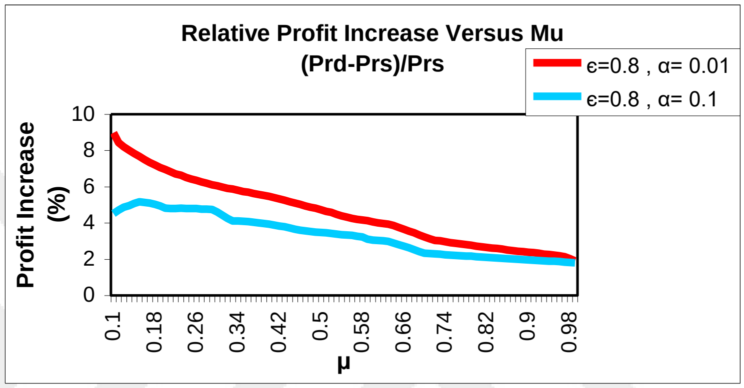

Figure 6. Comparison of the Relative Profit Increase with Dynamic Pricing for α = 0.01 and α = 0. 1

From Figure 6, it is concluded that as the frequency of environment state transitions increase when compared to the demand rate, the benefit of dynamic pricing tends to decrease. The reason is that the pricing strategy cannot be applied long enough to properly gain all the possible benefit when the environment fluctuates frequently compared to the demand arrival.

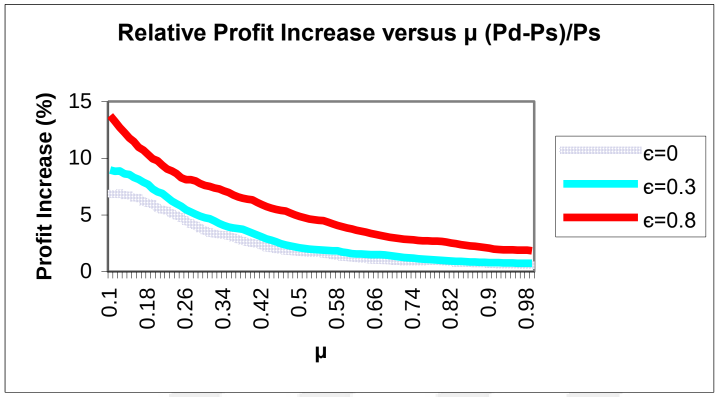

Figure 7. Relative Profit Increase with Dynamic Pricing for $\overline{F}(p) = e^{-p}$

The results we obtained for exponential reservation price distribution are totally compatible with the results we obtained for the linear case. Therefore, we do not present all the graphics here. The relative profit increase for the exponential case was slightly higher than the linear case which is shown in the figure above. The highest increase in profit obtained by applying dynamic pricing in this numerical study was "13.74931%", and it obtained for the exponential case with $\epsilon = 0.8$.

5.4.4 Summary of the Numerical Results

From the studies completed so far, we are able to anticipate the structure of the optimal replenishment policy when static pricing or dynamic pricing is applied in stationary environment. Moreover, the previous numerical studies of Chen et al. [1] and Gayon [13] point that the dynamic pricing method does not bring a higher enough (less than 4%) increase compared to static pricing in stationary environment when both pricing models are applied optimally. However, the structure of the optimal policies for both pricing models and the relative profit increase that can be obtained by applying dynamic pricing was unclear in nonstationary environment.

We aimed to gain insights about this case in this numerical study, and our results show that the structure of the optimal replenishment policy is a state-dependent base-stock policy as it is expected and shown to be true in Lemma 6. The behavior of the optimal pricing and replenishment policies according to the demand variability is observed and demonstrated with figures and explanations. Moreover, it is observed that dynamic pricing is more beneficial in nonstationary environment and with capacitated supply. Chan et al. [11] conclude that the performance of dynamic pricing tends to increase with demand seasonality and tightness of capacity in their numerical study, and we observed the same result in uncertain environment where the uncertainty is reflected with the unpredictable shifts in the demand rate and the tightness of capacity is determined by the amount of the production rate.

Chapter 6

CONCLUSION

In this thesis, a continuous-review, infinite horizon inventory pricing and replenishment problem with capacitated supply is analysed. Demand is modeled as a Markov-modulated Poisson process where the potential demand rate depends on an external environment process. The supply side is modeled by a single server with exponential processing times. The main objective is to demonstrate the structure of optimal policies and also to compare two pricing models applied in business. One of them is the static pricing method where the price of the item remains fixed over time and the other is the dynamic pricing method where the price changes over time depending on the current inventory level and the external environment.

First, the static pricing model with replenishment is analysed and it is found that an environment-dependent base-stock policy is optimal for a given price, and the base-stock levels for each environment reflect the same monotonicity pattern with the corresponding demand rates when the environment transition rates satisfy certain conditions. From the firm's perspective, this result means that the capability of the management to observe the fluctuations in the environment and adjusting the current inventory policy according to these fluctuations is crucially important. However, it should be noted that to achieve this objective, the firm must have an effective demand forecasting model and it should have effective information channels to follow the changes in the economic conditions and detect the unpredictable shifts in the demand as soon as possible. Since determining the values of the base-stock levels is a complicated problem, the results obtained here can be a guideline for the researchers working on this topic. For instance, it is shown that a higher base-stock level should be kept for a higher demand rate when environment transition rates satisfy certain conditions.

Second, the dynamic pricing model with replenishment is analysed and again it is found that the optimal replenishment policy is an environment-dependent base-stock policy once again emphasizing the need for effective demand forecasting techniques and following the changes in economic conditions to get a high performance in inventory management when dynamic pricing is applied. Moreover, it is shown that the optimal prices decrease in the inventory amount. This result comes from the fact that when the firm has lower amount of inventory, the possibility of having lost sales after a short while increases and the firm will rationally want to gain as much as possible from each sale in this situation in order to compensate for any lost sales that might occur in a short time.

Finally, the two pricing models are compared in a numerical study. The objective of this numerical study was to observe the structure of the optimal pricing and replenishment policies and to compare the relative profit increase of these two pricing models for different values of demand variability in a fluctuating environment where there are two environment states. It was observed that the base-stock levels determined by the dynamic pricing method are higher than the ones determined by static pricing. This can be explained by the fact that the firm has an opportunity of getting rid of excess stock by decreasing the price and hence increasing the demand rate. Thus, the firm keeps more inventory to decrease the possibility of having a lost sale. The ability to control demand by changing the price becomes more important especially when the capacity is tight, and applying dynamic pricing gives the firm the opportunity to keep more stock without incurring a certain decrease in profit when compared to the static pricing.

Moreover, the results show that the optimal static prices increase as the demand variability increases. This result gives interesting insights since the demand variability represents the situation where the difference between the demand rates of the two environments increases, so the lower demand rate decreases more and the higher demand increases more. Intuitively, it seems rational to charge lower prices when the demand decreases but, in the static pricing case, a higher price is charged in the environment with lower demand rate. This contradiction comes from the fact that the firm cannot adapt to the changes in the environment, and the customers pay for the cost of this inability of the firm. Also, the optimal average profit decreases in the demand variability for both pricing methods. Thus, for static pricing we conclude that when the environment and therefore the demand fluctuates, both the customer and the firm lose money since the customer has to pay a higher price and the firm gains a lower profit compared to the case of stationary environment.

The most important choice that has to be made here is whether to use static pricing or dynamic pricing. Although dynamic pricing method provides the firm the great advantage of controlling the demand, it might have some drawbacks. For instance, changing the price stickers or catalogs can result in a high increase in transaction costs or the firm might lose its reputation if the customers react negatively to the price changes. If there is a high possibility that the firm's profit might be damaged from these drawbacks, the profit increase dynamic pricing brings might not compensate for these losses. This might happen especially in a stationary demand environment since it is concluded that the dynamic pricing method results in a limited improvement on the firm's profit compared to static pricing when both models are applied optimally. The maximum profit increase obtained in this numerical study for this case is less than 4%. Therefore, we advise to prefer static pricing in a stationary demand environment given that this method is applied optimally. The results show that the relative increase in profit obtained by applying dynamic pricing tends to increase with the demand variability and tightness of the capacity. The reason is that the great advantage of controlling the demand the dynamic pricing method provides the firms becomes more effective on the firm's profit since the fluctuating environment conditions causes unpredictable shifts in demand and the tightness of the capacity restricts the replenishment decision. Therefore, the firms need to control the demand more strongly in these cases.

Hence, we conclude that dynamic pricing can bring a high enough increase to compansate for its drawbacks where the demand variability is high and capacity is tight since we observed relative profit increase values are approximately 10% in these cases.

A natural extension of the inventory pricing and replenishment problem studied in this thesis is to add an assumption of set-up cost to the cost structure where it is expected that the optimal replenishment policy would be an (*s*, *S*) policy with threshold value *s* and order-up-to-level *S* determining the amount of inventory to replenish since all the orders will be given to increase the inventory amount up to the level "*S*". Another challenge would be to investigate whether the optimal prices reflect any monotonicity pattern for the dynamic pricing method.

The dynamic pricing model with replenishment analysed here considers to optimize the prices at each environment state and for each inventory amount. However, it seems interesting to construct another model with dynamic pricing where the optimal price depends on only the environment state and to compare its effect on the firm's financial performance. Moreover, the assumptions of demand and supply processes can be modified by allowing batch arrivals and batch production. Also, the assumptions of our model can be generalized to include modulated prices, supply, etc. For instance, the case of Markov-modulated reservation price distribution would not cause a change in the optimal policy since it has the same effect with changing the demand rate by definitions of our problem.

Another attractive issue might be to add perishability to our model by considering the inventory as decaying over time.

Appendix A

MONOTONICITY of the BASE STOCK LEVELS at SPECIAL CASES

We will now show that the base stock levels for each environmental state e (S_e^*) have the same ordering with the demand rates corresponding to these states without any condition to be satisfied on the values of the environment state transition rates since Condition 1 is directly satisfied by definitions of the system when the continuous-time Markov chain characterizing the environment is a birth-and-death process. In addition, Condition 1 is not necessary when there are only two environment states.

A.1 Monotonicity of the Base Stock Levels When a Birth-and-Death Process Represents the Environment

Note that E is the set of environment states 1, 2, …, N where $\lambda_N \geq \lambda_{N-1} \geq \ldots \geq \lambda_{e+1} \geq \lambda_e \geq \lambda_{e-1} \geq \ldots \geq \lambda_1$ and β is the discount rate. Moreover, the underlying Markov Chain characterizing the environment, $M = \{M(t) : t \geq 0\}$, is a birth-and-death process in this case. That is, the infinitesimal generator of M is denoted by

$Q = \left(q_{ej}\right)_{e,\, j \in E}$ where

$q_{ee} = -\sum_{j \neq e} q_{ej}, e \in E.$ and

$q_{ej} = 0$, for all j $\neq$ e+1, e-1.

We will now show that the base stock levels for each environmental state e (S_e^*) have the same ordering with the demand rates corresponding to these states without any condition to be satisfied on the values of the environment state transition rates since Condition 1 is directly satisfied by definitions of the system. We present the optimality equations for environment states e = 2, 3, …, N-2, e = 1, and $e = N$ separately due to the differences between the transitions possible from each environment state considered. For e = 2, 3, …, N-2, we have the following optimality equation:

$$\begin{aligned} V(x,e) &= -h(x) + \lambda_e \overline{F}(p)\{p + V(x-1,e)\} + \lambda_e[1-\overline{F}(p)]V(x,e) + \mu \max\{V(x,e), V(x+1,e) - c\} \\ &+ q_{e,e+1}V(x,e+1) + q_{e,e-1}V(x,e-1) + \Big(\sum_{i\neq e} \lambda_i + \sum_{\substack{j=1 \\ j\neq e}}^{N-1} q_{j,j+1} + \sum_{\substack{k=2 \\ k\neq e}}^{N} q_{k,k-1}\Big)V(x,e) \end{aligned} \tag{A.1}$$

For e = 1, we have the following optimality equation:

$$\begin{aligned} V(x,1) &= -h(x) + \lambda_1 \overline{F}(p)\{p + V(x-1,1)\} + \lambda_1[1-\overline{F}(p)]V(x,1) + \mu \max\{V(x,1), V(x+1,1) - c\} \\ &+ q_{1,2}V(x,2) + \Big(\sum_{i\neq 1} \lambda_i + \sum_{j=2}^{N-1} q_{j,j+1} + \sum_{k=2}^{N} q_{k,k-1}\Big)V(x,1) \end{aligned} \tag{A.2}$$

For $e = N$, we have the following optimality equation:

$$\begin{aligned} V(x,N) &= -h(x) + \lambda_N \overline{F}(p)\{p + V(x-1,N)\} + \lambda_N[1-\overline{F}(p)]V(x,N) + \mu \max\{V(x,N), V(x+1,N) - c\} \\ &+ q_{N,N-1}V(x,N-1) + \Big(\sum_{i\neq N} \lambda_i + \sum_{j=1}^{N-1} q_{j,j+1} + \sum_{k=2}^{N-1} q_{k,k-1}\Big)V(x,N) \end{aligned} \tag{A.3}$$

Lemma A.1. $\Delta V(x,e+1) \geq \Delta V(x,e)$, for all x and for $1 \leq e \leq$ N-1.

Proof. We will use the induction method and show that $\Delta V_n(x,e+1) \geq \Delta V_n(x,e)$ for all x, $1 \leq e \leq N$-1, and for all n. The lemma is trivially true for n=0, suppose that it is true for n-1. We again consider x = 1 and $x \geq 2$ separately. Moreover, we will consider the cases e = 2, 3, …, N-2, e = 1, and e = N-1 separately. Then, for $x \geq 2$ and e = 2, 3, …, N-2,

$$\Delta V_n(x,e+1) - \Delta V_n(x,e) \tag{A.4}$$

$$= [V_n(x,e+1) - V_n(x-1,e+1)] - [V_n(x,e) - V_n(x-1,e)]$$

$$= [[(-h(x) + \lambda_{e+1}\overline{F}(p)p + \lambda_{e+1}\overline{F}(p)V_{n-1}(x-1,e+1) + \lambda_{e+1}[1-\overline{F}(p)]V_{n-1}(x,e+1) + \mu\max\{V_{n-1}(x,e+1), V_{n-1}(x+1,e+1) - c\}$$

$$+q_{e+1,e+2}V_{n-1}(x,e+2) + q_{e+1,e}V_{n-1}(x,e) + (\sum_{i\neq e+1}\lambda_i + \sum_{\substack{j=1\\ j\neq e+1}}^{N-1} q_{j,j+1} + \sum_{\substack{k=2\\ k\neq e+1}}^{N} q_{k,k-1})V_{n-1}(x,e+1)]$$

$$-[(-h(x-1) + \lambda_{e+1}\overline{F}(p)p + \lambda_{e+1}\overline{F}(p)V_{n-1}(x-2,e+1) + \lambda_{e+1}[1-\overline{F}(p)]V_{n-1}(x-1,e+1) + \mu\max\{V_{n-1}(x-1,e+1), V_{n-1}(x,e+1) - c\}$$

$$+q_{e+1,e+2}V_{n-1}(x-1,e+2) + q_{e+1,e}V_{n-1}(x-1,e) + (\sum_{i\neq e+1}\lambda_i + \sum_{\substack{j=1\\ j\neq e+1}}^{N-1} q_{j,j+1} + \sum_{\substack{k=2\\ k\neq e+1}}^{N} q_{k,k-1})V_{n-1}(x-1,e+1)]]$$

$$-[[(-h(x) + \lambda_e\overline{F}(p)p + \lambda_e\overline{F}(p)V_{n-1}(x-1,e) + \lambda_e[1-\overline{F}(p)]V_{n-1}(x,e) + \mu\max\{V_{n-1}(x,e), V_{n-1}(x+1,e) - c\}$$

$$+q_{e,e+1}V_{n-1}(x,e+1) + q_{e,e-1}V_{n-1}(x,e-1) + (\sum_{i\neq e}\lambda_i + \sum_{\substack{j=1\\ j\neq e}}^{N-1} q_{j,j+1} + \sum_{\substack{k=2\\ k\neq e}}^{N} q_{k,k-1})V_{n-1}(x,e)]$$

$$-[(-h(x-1) + \lambda_e\overline{F}(p)p + \lambda_e\overline{F}(p)V_{n-1}(x-2,e) + \lambda_e[1-\overline{F}(p)]V_{n-1}(x-1,e) + \mu\max\{V_{n-1}(x-1,e), V_{n-1}(x,e) - c\}$$

$$+q_{e,e+1}V_{n-1}(x-1,e+1) + q_{e,e-1}V_{n-1}(x-1,e-1) + (\sum_{i\neq e}\lambda_i + \sum_{\substack{j=1\\ j\neq e}}^{N-1} q_{j,j+1} + \sum_{\substack{k=2\\ k\neq e}}^{N} q_{k,k-1})V_{n-1}(x-1,e)]]$$

$$= \lambda_{e+1}\overline{F}(p)\Delta V_{n-1}(x-1,e+1) - \lambda_e\overline{F}(p)\Delta V_{n-1}(x-1,e)$$

$$+\lambda_{e+1}[1-\overline{F}(p)]\Delta V_{n-1}(x,e+1) - \lambda_e[1-\overline{F}(p)]\Delta V_{n-1}(x,e)$$

$$+\mu\{\max[V_{n-1}(x,e+1), V_{n-1}(x+1,e+1) - c] - \max[V_{n-1}(x-1,e+1), V_{n-1}(x,e+1) - c]\}$$

$$-\mu\{\max[V_{n-1}(x,e), V_{n-1}(x+1,e) - c] - \max[V_{n-1}(x-1,e), V_{n-1}(x,e) - c]\}$$

$$+q_{e+1,e+2}\Delta V_{n-1}(x,e+2) + q_{e+1,e}\Delta V_{n-1}(x,e) + (\sum_{i\neq e+1}\lambda_i + \sum_{\substack{j=1\\ j\neq e+1}}^{N-1} q_{j,j+1} + \sum_{\substack{k=2\\ k\neq e+1}}^{N} q_{k,k-1})\Delta V_{n-1}(x,e+1)$$

$$-[q_{e,e+1}\Delta V_{n-1}(x,e+1)+q_{e,e-1}\Delta V_{n-1}(x,e-1)+(\sum_{i\neq e}\lambda_i+\sum_{\substack{j=1\\ j\neq e}}^{N-1}q_{j,j+1}+\sum_{\substack{k=2\\ k\neq e}}^{N}q_{k,k-1})\Delta V_{n-1}(x,e)]$$

$$=\lambda_{e+1}\overline{F}(p)\Delta V_{n-1}(x-1,e+1)-\lambda_e\overline{F}(p)\Delta V_{n-1}(x-1,e)$$

$$+\lambda_{e+1}[1-\overline{F}(p)]\Delta V_{n-1}(x,e+1)-\lambda_e[1-\overline{F}(p)]\Delta V_{n-1}(x,e)$$

$$+\mu\{\max[V_{n-1}(x,e+1),V_{n-1}(x+1,e+1)-c]-\max[V_{n-1}(x-1,e+1),V_{n-1}(x,e+1)-c]\}$$

$$-\mu\{\max[V_{n-1}(x,e),V_{n-1}(x+1,e)-c]-\max[V_{n-1}(x-1,e),V_{n-1}(x,e)-c]\}$$

$$+q_{e+1,e+2}\Delta V_{n-1}(x,e+2)+q_{e+1,e}\Delta V_{n-1}(x,e)+(\sum_{i\neq e+1}\lambda_i+\sum_{\substack{j=1\\ j\neq e+1,e}}^{N-1}q_{j,j+1}+\sum_{\substack{k=2\\ k\neq e+1,e}}^{N}q_{k,k-1})\Delta V_{n-1}(x,e+1)$$

$$+(q_{e,e+1}+q_{e,e-1})\Delta V_{n-1}(x,e+1)$$

$$-[q_{e,e+1}\Delta V_{n-1}(x,e+1)+q_{e,e-1}\Delta V_{n-1}(x,e-1)+(\sum_{i\neq e}\lambda_i+\sum_{\substack{j=1\\ j\neq e,e+1}}^{N-1}q_{j,j+1}+\sum_{\substack{k=2\\ k\neq e,e+1}}^{N}q_{k,k-1})\Delta V_{n-1}(x,e)$$

$$+(q_{e+1,e+2}+q_{e+1,e})\Delta V_{n-1}(x,e)]$$

$$=\lambda_{e+1}\overline{F}(p)\Delta V_{n-1}(x-1,e+1)-\lambda_e\overline{F}(p)\Delta V_{n-1}(x-1,e)$$

$$+\lambda_{e+1}[1-\overline{F}(p)]\Delta V_{n-1}(x,e+1)-\lambda_e[1-\overline{F}(p)]\Delta V_{n-1}(x,e)$$

$$+\mu\{\max[V_{n-1}(x,e+1),V_{n-1}(x+1,e+1)-c]-\max[V_{n-1}(x-1,e+1),V_{n-1}(x,e+1)-c]\}$$

$$-\mu\{\max[V_{n-1}(x,e),V_{n-1}(x+1,e)-c]-\max[V_{n-1}(x-1,e),V_{n-1}(x,e)-c]\}$$

$$+q_{e+1,e+2}\Delta V_{n-1}(x,e+2)+q_{e+1,e}\Delta V_{n-1}(x,e)+(\sum_{i\neq e+1,e}\lambda_i+\sum_{\substack{j=1\\ j\neq e+1,e}}^{N-1}q_{j,j+1}+\sum_{\substack{k=2\\ k\neq e+1,e}}^{N}q_{k,k-1})\Delta V_{n-1}(x,e+1)$$

$$+(q_{e,e+1}+q_{e,e-1})\Delta V_{n-1}(x,e+1)+\lambda_e\Delta V_{n-1}(x,e+1)$$

$$-[q_{e,e+1}\Delta V_{n-1}(x,e+1)+q_{e,e-1}\Delta V_{n-1}(x,e-1)+(\sum_{i\neq e,e+1}\lambda_i+\sum_{\substack{j=1\\ j\neq e,e+1}}^{N-1}q_{j,j+1}+\sum_{\substack{k=2\\ k\neq e,e+1}}^{N}q_{k,k-1})\Delta V_{n-1}(x,e)$$

$$+(q_{e+1,e+2}+q_{e+1,e})\Delta V_{n-1}(x,e)+\lambda_{e+1}\Delta V_{n-1}(x,e)]$$

$$=(\lambda_{e+1}-\lambda_e)\overline{F}(p)\Delta V_{n-1}(x-1,e+1)+\lambda_e\overline{F}(p)\{\Delta V_{n-1}(x-1,e+1)-\Delta V_{n-1}(x-1,e)\}$$

$$+(\lambda_{e+1}-\lambda_e)[1-\overline{F}(p)]\Delta V_{n-1}(x,e+1)+\lambda_e[1-\overline{F}(p)]\{\Delta V_{n-1}(x,e+1)-\Delta V_{n-1}(x,e)\}$$

$$+\lambda_e\Delta V_{n-1}(x,e+1)-\lambda_{e+1}\Delta V_{n-1}(x,e)$$

$$+\mu\{\max[V_{n-1}(x,e+1),V_{n-1}(x+1,e+1)-c]-\max[V_{n-1}(x-1,e+1),V_{n-1}(x,e+1)-c]\}$$
$$-\mu\{\max[V_{n-1}(x,e),V_{n-1}(x+1,e)-c]-\max[V_{n-1}(x-1,e),V_{n-1}(x,e)-c]\}$$
$$+(\sum_{i\neq e+1,e}\lambda_i+\sum_{\substack{j=1\\ j\neq e+1,e}}^{N-1}q_{j,j+1}+\sum_{\substack{k=2\\ k\neq e+1,e}}^{N}q_{k,k-1})\{\Delta V_{n-1}(x,e+1)-\Delta V_{n-1}(x,e)\}$$
$$+(q_{e,e-1})\{\Delta V_{n-1}(x,e+1)-\Delta V_{n-1}(x,e-1)\}+q_{e+1,e+2}\{\Delta V_{n-1}(x,e+2)-\Delta V_{n-1}(x,e)\}$$
$$\geq(\lambda_{e+1}-\lambda_e)\overline{F}(p)\Delta V_{n-1}(x,e+1)+\lambda_e\overline{F}(p)\{\Delta V_{n-1}(x-1,e+1)-\Delta V_{n-1}(x-1,e)\}$$
$$+(\lambda_{e+1}-\lambda_e)[1-\overline{F}(p)]\Delta V_{n-1}(x,e+1)+\lambda_e[1-\overline{F}(p)]\{\Delta V_{n-1}(x,e+1)-\Delta V_{n-1}(x,e)\}$$
$$+\lambda_e\Delta V_{n-1}(x,e+1)-\lambda_{e+1}\Delta V_{n-1}(x,e)$$
$$+\mu\{\max[V_{n-1}(x,e+1),V_{n-1}(x+1,e+1)-c]-\max[V_{n-1}(x-1,e+1),V_{n-1}(x,e+1)-c]\}$$
$$-\mu\{\max[V_{n-1}(x,e),V_{n-1}(x+1,e)-c]-\max[V_{n-1}(x-1,e),V_{n-1}(x,e)-c]\}$$
$$+(\sum_{i\neq e+1,e}\lambda_i+\sum_{\substack{j=1\\ j\neq e+1,e}}^{N-1}q_{j,j+1}+\sum_{\substack{k=2\\ k\neq e+1,e}}^{N}q_{k,k-1})\{\Delta V_{n-1}(x,e+1)-\Delta V_{n-1}(x,e)\}$$
$$+(q_{e,e-1})\{\Delta V_{n-1}(x,e+1)-\Delta V_{n-1}(x,e-1)\}+q_{e+1,e+2}\{\Delta V_{n-1}(x,e+2)-\Delta V_{n-1}(x,e)\}$$
$$=\lambda_{e+1}\{\Delta V_{n-1}(x,e+1)-\Delta V_{n-1}(x,e)\}+\lambda_e\overline{F}(p)\{\Delta V_{n-1}(x-1,e+1)-\Delta V_{n-1}(x-1,e)\}$$
$$+\lambda_e[1-\overline{F}(p)]\{\Delta V_{n-1}(x,e+1)-\Delta V_{n-1}(x,e)\}$$
$$+\mu\{\max[V_{n-1}(x,e+1),V_{n-1}(x+1,e+1)-c]-\max[V_{n-1}(x-1,e+1),V_{n-1}(x,e+1)-c]\}$$
$$-\mu\{\max[V_{n-1}(x,e),V_{n-1}(x+1,e)-c]-\max[V_{n-1}(x-1,e),V_{n-1}(x,e)-c]\}$$
$$+(\sum_{i\neq e+1,e}\lambda_i+\sum_{\substack{j=1\\ j\neq e+1,e}}^{N-1}q_{j,j+1}+\sum_{\substack{k=2\\ k\neq e+1,e}}^{N}q_{k,k-1})\{\Delta V_{n-1}(x,e+1)-\Delta V_{n-1}(x,e)\}$$
$$+(q_{e,e-1})\{\Delta V_{n-1}(x,e+1)-\Delta V_{n-1}(x,e-1)\}+q_{e+1,e+2}\{\Delta V_{n-1}(x,e+2)-\Delta V_{n-1}(x,e)\}$$
$$=\lambda_{e+1}\{\Delta V_{n-1}(x,e+1)-\Delta V_{n-1}(x,e)\}+\lambda_e\overline{F}(p)\{\Delta V_{n-1}(x-1,e+1)-\Delta V_{n-1}(x-1,e)\}$$
$$+\lambda_e[1-\overline{F}(p)]\{\Delta V_{n-1}(x,e+1)-\Delta V_{n-1}(x,e)\}$$
$$+\mu\{\max[V_{n-1}(x,e+1),V_{n-1}(x+1,e+1)-c]-\max[V_{n-1}(x-1,e+1),V_{n-1}(x,e+1)-c]\}$$
$$-\mu\{\max[V_{n-1}(x,e),V_{n-1}(x+1,e)-c]-\max[V_{n-1}(x-1,e),V_{n-1}(x,e)-c]\}$$

$$+\Big(\sum_{i\neq e+1,e}\lambda_i+\sum_{\substack{j=1\\ j\neq e+1,e}}^{N-1}q_{j,j+1}+\sum_{\substack{k=2\\ k\neq e+1,e}}^{N}q_{k,k-1}\Big)\{\Delta V_{n-1}(x,e+1)-\Delta V_{n-1}(x,e)\}$$

$$+(q_{e,e-1})\{[\Delta V_{n-1}(x,e+1)-\Delta V_{n-1}(x,e)]+[\Delta V_{n-1}(x,e)-\Delta V_{n-1}(x,e-1)]\}$$

$$+q_{e+1,e+2}\{[\Delta V_{n-1}(x,e+2)-\Delta V_{n-1}(x,e+1)]+[\Delta V_{n-1}(x,e+1)-\Delta V_{n-1}(x,e)]\}\geq 0$$

The inequality above holds since V_n (x, e) is concave in x for all e and n ($\Delta V_{n-1}(x-1,e+1)\geq \Delta V_{n-1}(x,e+1)$) by Lemma 2, Lemma A.1 is assumed to be true for n-1, and since

$$\begin{aligned}&\{\max[V_{n-1}(x,e+1),V_{n-1}(x+1,e+1)-c]-\max[V_{n-1}(x-1,e+1),V_{n-1}(x,e+1)-c]\}\\ &-\{\max[V_{n-1}(x,e),V_{n-1}(x+1,e)-c]-\max[V_{n-1}(x-1,e),V_{n-1}(x,e)-c]\}\geq 0\end{aligned} \tag{A.5}$$

(A.5) holds by the same discussion summarized in Table 5 for e = 2, 3, …, N-2. Thus, Lemma A.1 is true for $x\geq$ 2 and e = 2, 3, …, N-2. Then, for $x\geq$ 2 and e = 1,

$$\Delta V_n(x,2)-\Delta V_n(x,1)$$

$$=[V_n(x,2)-V_n(x-1,2)]-[V_n(x,1)-V_n(x-1,1)] \tag{A.6}$$

$$=[[(-h(x)+\lambda_2\overline{F}(p)p+\lambda_2\overline{F}(p)V_{n-1}(x-1,2)+\lambda_2[1-\overline{F}(p)]V_{n-1}(x,2)+\mu\max\{V_{n-1}(x,2),V_{n-1}(x+1,2)-c\}$$

$$+q_{2,3}V_{n-1}(x,3)+q_{2,1}V_{n-1}(x,1)+(\sum_{i\neq 2}\lambda_i+\sum_{\substack{j=1\\ j\neq 2}}^{N-1}q_{j,j+1}+\sum_{k=3}^{N}q_{k,k-1})V_{n-1}(x,2)]$$

$$-[(-h(x-1)+\lambda_2\overline{F}(p)p+\lambda_2\overline{F}(p)V_{n-1}(x-2,2)+\lambda_2[1-\overline{F}(p)]V_{n-1}(x-1,2)+\mu\max\{V_{n-1}(x-1,2),V_{n-1}(x,2)-c\}$$

$$+q_{2,3}V_{n-1}(x-1,3)+q_{2,1}V_{n-1}(x-1,1)+(\sum_{i\neq 2}\lambda_i+\sum_{\substack{j=1\\ j\neq 2}}^{N-1}q_{j,j+1}+\sum_{k=3}^{N}q_{k,k-1})V_{n-1}(x-1,2)]]$$

$$-[[(-h(x)+\lambda_1\overline{F}(p)p+\lambda_1\overline{F}(p)V_{n-1}(x-1,1)+\lambda_1[1-\overline{F}(p)]V_{n-1}(x,1)+\mu\max\{V_{n-1}(x,1),V_{n-1}(x+1,1)-c\}$$

$$+q_{1,2}V_{n-1}(x,2)+(\sum_{i\neq 1}\lambda_i+\sum_{j=2}^{N-1}q_{j,j+1}+\sum_{k=2}^{N}q_{k,k-1})V_{n-1}(x,1)]$$

$$-[(-h(x-1)+\lambda_1\overline{F}(p)p+\lambda_1\overline{F}(p)V_{n-1}(x-2,1)+\lambda_1[1-\overline{F}(p)]V_{n-1}(x-1,1)+\mu\max\{V_{n-1}(x-1,1),V_{n-1}(x,1)-c\}$$

$$+q_{1,2}V_{n-1}(x-1,2)+(\sum_{i\neq 1}\lambda_i+\sum_{j=2}^{N-1}q_{j,j+1}+\sum_{k=2}^{N}q_{k,k-1})V_{n-1}(x-1,1)]]$$

$$= \lambda_2\overline{F}(p)\Delta V_{n-1}(x-1,2)-\lambda_1\overline{F}(p)\Delta V_{n-1}(x-1,1)$$

$$+\lambda_2[1-\overline{F}(p)]\Delta V_{n-1}(x,2)-\lambda_1[1-\overline{F}(p)]\Delta V_{n-1}(x,1)$$

$$+\mu\{\max[V_{n-1}(x,2),V_{n-1}(x+1,2)-c]-\max[V_{n-1}(x-1,2),V_{n-1}(x,2)-c]\}$$

$$-\mu\{\max[V_{n-1}(x,1),V_{n-1}(x+1,1)-c]-\max[V_{n-1}(x-1,1),V_{n-1}(x,1)-c]\}$$

$$+q_{2,3}\Delta V_{n-1}(x,3)+q_{2,1}\Delta V_{n-1}(x,1)+(\sum_{i\neq 2}\lambda_i+\sum_{\substack{j=1\\ j\neq 2}}^{N-1}q_{j,j+1}+\sum_{k=3}^{N}q_{k,k-1})\Delta V_{n-1}(x,2)$$

$$-[q_{1,2}\Delta V_{n-1}(x,2)+(\sum_{i\neq 1}\lambda_i+\sum_{j=2}^{N-1}q_{j,j+1}+\sum_{k=2}^{N}q_{k,k-1})\Delta V_{n-1}(x,1)]$$

$$=\lambda_2\overline{F}(p)\Delta V_{n-1}(x-1,2)-\lambda_1\overline{F}(p)\Delta V_{n-1}(x-1,1)$$

$$+\lambda_2[1-\overline{F}(p)]\Delta V_{n-1}(x,2)-\lambda_1[1-\overline{F}(p)]\Delta V_{n-1}(x,1)$$

$$+\mu\{\max[V_{n-1}(x,2),V_{n-1}(x+1,2)-c]-\max[V_{n-1}(x-1,2),V_{n-1}(x,2)-c]\}$$

$$-\mu\{\max[V_{n-1}(x,1),V_{n-1}(x+1,1)-c]-\max[V_{n-1}(x-1,1),V_{n-1}(x,1)-c]\}$$

$$+q_{2,3}\Delta V_{n-1}(x,3)+q_{2,1}\Delta V_{n-1}(x,1)+(\sum_{i\neq 2}\lambda_i+\sum_{j=3}^{N-1}q_{j,j+1}+\sum_{k=3}^{N}q_{k,k-1})\Delta V_{n-1}(x,2)$$

$$+(q_{1,2})\Delta V_{n-1}(x,2)$$

$$-[q_{1,2}\Delta V_{n-1}(x,2)+(\sum_{i\neq 1}\lambda_i+\sum_{j=3}^{N-1}q_{j,j+1}+\sum_{k=3}^{N}q_{k,k-1})\Delta V_{n-1}(x,1)+q_{2,3}\Delta V_{n-1}(x,1)+q_{2,1}\Delta V_{n-1}(x,1)]$$

$$=(\lambda_2-\lambda_1)\overline{F}(p)\Delta V_{n-1}(x-1,2)+\lambda_1\overline{F}(p)\{\Delta V_{n-1}(x-1,2)-\Delta V_{n-1}(x-1,1)\}$$

$$+(\lambda_2-\lambda_1)[1-\overline{F}(p)]\Delta V_{n-1}(x,2)+\lambda_1[1-\overline{F}(p)]\{\Delta V_{n-1}(x,2)-\Delta V_{n-1}(x,1)\}$$

$$+\lambda_1\Delta V_{n-1}(x,2)-\lambda_2\Delta V_{n-1}(x,1)$$

$$+\mu\{\max[V_{n-1}(x,2),V_{n-1}(x+1,2)-c]-\max[V_{n-1}(x-1,2),V_{n-1}(x,2)-c]\}$$

$$-\mu\{\max[V_{n-1}(x,1),V_{n-1}(x+1,1)-c]-\max[V_{n-1}(x-1,1),V_{n-1}(x,1)-c]\}$$

$$+(\sum_{i\neq 2,1}\lambda_i+\sum_{j=3}^{N-1}q_{j,j+1}+\sum_{k=3}^{N}q_{k,k-1})\{\Delta V_{n-1}(x,2)-\Delta V_{n-1}(x,1)\}$$

$$+q_{2,3}\{\Delta V_{n-1}(x,3)-\Delta V_{n-1}(x,1)\}$$

$$\geq (\lambda_2-\lambda_1)\overline{F}(p)\Delta V_{n-1}(x,2)+\lambda_1\overline{F}(p)\{\Delta V_{n-1}(x-1,2)-\Delta V_{n-1}(x-1,1)\}$$

$$+(\lambda_2-\lambda_1)[1-\overline{F}(p)]\Delta V_{n-1}(x,2)+\lambda_1[1-\overline{F}(p)]\{\Delta V_{n-1}(x,2)-\Delta V_{n-1}(x,1)\}$$

$$+\lambda_1\Delta V_{n-1}(x,2)-\lambda_2\Delta V_{n-1}(x,1)$$

$$+\mu\{\max[V_{n-1}(x,2),V_{n-1}(x+1,2)-c]-\max[V_{n-1}(x-1,2),V_{n-1}(x,2)-c]\}$$

$$-\mu\{\max[V_{n-1}(x,1),V_{n-1}(x+1,1)-c]-\max[V_{n-1}(x-1,1),V_{n-1}(x,1)-c]\}$$

$$+(\sum_{i\neq 2,1}\lambda_i+\sum_{j=3}^{N-1}q_{j,j+1}+\sum_{k=3}^{N}q_{k,k-1})\{\Delta V_{n-1}(x,2)-\Delta V_{n-1}(x,1)\}$$

$$+q_{2,3}\{\Delta V_{n-1}(x,3)-\Delta V_{n-1}(x,1)\}$$

$$=\lambda_2\{\Delta V_{n-1}(x,2)-\Delta V_{n-1}(x,1)\}+\lambda_1\overline{F}(p)\{\Delta V_{n-1}(x-1,2)-\Delta V_{n-1}(x-1,1)\}$$

$$+\lambda_1[1-\overline{F}(p)]\{\Delta V_{n-1}(x,2)-\Delta V_{n-1}(x,1)\}$$

$$+\mu\{\max[V_{n-1}(x,2),V_{n-1}(x+1,2)-c]-\max[V_{n-1}(x-1,2),V_{n-1}(x,2)-c]\}$$

$$-\mu\{\max[V_{n-1}(x,1),V_{n-1}(x+1,1)-c]-\max[V_{n-1}(x-1,1),V_{n-1}(x,1)-c]\}$$

$$+(\sum_{i\neq 2,1}\lambda_i+\sum_{j=3}^{N-1}q_{j,j+1}+\sum_{k=3}^{N}q_{k,k-1})\{\Delta V_{n-1}(x,2)-\Delta V_{n-1}(x,1)\}$$

$$+q_{2,3}\{\Delta V_{n-1}(x,3)-\Delta V_{n-1}(x,1)\}$$

$$=\lambda_2\{\Delta V_{n-1}(x,2)-\Delta V_{n-1}(x,1)\}+\lambda_1\overline{F}(p)\{\Delta V_{n-1}(x-1,2)-\Delta V_{n-1}(x-1,1)\}$$

$$+\lambda_1[1-\overline{F}(p)]\{\Delta V_{n-1}(x,2)-\Delta V_{n-1}(x,1)\}$$

$$+\mu\{\max[V_{n-1}(x,2),V_{n-1}(x+1,2)-c]-\max[V_{n-1}(x-1,2),V_{n-1}(x,2)-c]\}$$

$$-\mu\{\max[V_{n-1}(x,1),V_{n-1}(x+1,1)-c]-\max[V_{n-1}(x-1,1),V_{n-1}(x,1)-c]\}$$

$$+(\sum_{i\neq 2,1}\lambda_i+\sum_{j=3}^{N-1}q_{j,j+1}+\sum_{k=3}^{N}q_{k,k-1})\{\Delta V_{n-1}(x,2)-\Delta V_{n-1}(x,1)\}$$

$$+q_{2,3}\{[\Delta V_{n-1}(x,3)-\Delta V_{n-1}(x,2)]+[\Delta V_{n-1}(x,2)-\Delta V_{n-1}(x,1)]\}\geq 0$$

The inequality above holds since V_n (x, e) is concave in x for all e and n ($\Delta V_{n-1}(x-1,2) \geq \Delta V_{n-1}(x,2)$) by Lemma 2, Lemma A.1 is assumed to be true for n-1, and since

$$\begin{aligned}&\{\max[V_{n-1}(x,2),V_{n-1}(x+1,2)-c]-\max[V_{n-1}(x-1,2),V_{n-1}(x,2)-c]\}\\&-\{\max[V_{n-1}(x,1),V_{n-1}(x+1,1)-c]-\max[V_{n-1}(x-1,1),V_{n-1}(x,1)-c]\}\geq 0\end{aligned} \tag{A.7}$$

(A.7) holds by the same discussion summarized in Table 5 for e = 1. Thus, Lemma A.1 is true for $x \geq$ 2 and e = 1. Then, for $x \geq$ 2 and e = N-1,

$$\begin{aligned}&\Delta V_n(x,N)-\Delta V_n(x,N-1)\\&=[V_n(x,N)-V_n(x-1,N)]-[V_n(x,N-1)-V_n(x-1,N-1)]\\&=[[(-h(x)+\lambda_N\overline{F}(p)p+\lambda_N\overline{F}(p)V_{n-1}(x-1,N)+\lambda_N[1-\overline{F}(p)]V_{n-1}(x,N)+\mu\max\{V_{n-1}(x,N),V_{n-1}(x+1,N)-c\}\\&+q_{N,N-1}V_{n-1}(x,N-1)+(\sum_{i\neq N}\lambda_i+\sum_{j=1}^{N-1}q_{j,j+1}+\sum_{k=2}^{N-1}q_{k,k-1})V_{n-1}(x,N)]\\&-[(-h(x-1)+\lambda_N\overline{F}(p)p+\lambda_N\overline{F}(p)V_{n-1}(x-2,N)+\lambda_N[1-\overline{F}(p)]V_{n-1}(x-1,N)+\mu\max\{V_{n-1}(x-1,N),V_{n-1}(x,N)-c\}\\&+q_{N,N-1}V_{n-1}(x-1,N-1)+(\sum_{i\neq N}\lambda_i+\sum_{j=1}^{N-1}q_{j,j+1}+\sum_{k=2}^{N-1}q_{k,k-1})V_{n-1}(x-1,N)]]\\&-[[(-h(x)+\lambda_{N-1}\overline{F}(p)p+\lambda_{N-1}\overline{F}(p)V_{n-1}(x-1,N-1)+\lambda_{N-1}[1-\overline{F}(p)]V_{n-1}(x,N-1)+\mu\max\{V_{n-1}(x,N-1),V_{n-1}(x+1,N-1)-c\}\\&+q_{N-1,N}V_{n-1}(x,N)+q_{N-1,N-2}V_{n-1}(x,N-2)+(\sum_{i\neq N-1}\lambda_i+\sum_{j=1}^{N-2}q_{j,j+1}+\sum_{\substack{k=2\\k\neq N-1}}^{N}q_{k,k-1})V_{n-1}(x,N-1)]\\&-[(-h(x-1)+\lambda_{N-1}\overline{F}(p)p+\lambda_{N-1}\overline{F}(p)V_{n-1}(x-2,N-1)+\lambda_{N-1}[1-\overline{F}(p)]V_{n-1}(x-1,N-1)+\mu\max\{V_{n-1}(x-1,N-1),V_{n-1}(x,N-1)-c\}\\&+q_{N-1,N}V_{n-1}(x-1,N)+q_{N-1,N-2}V_{n-1}(x-1,N-2)+(\sum_{i\neq N-1}\lambda_i+\sum_{j=1}^{N-2}q_{j,j+1}+\sum_{\substack{k=2\\k\neq N-1}}^{N}q_{k,k-1})V_{n-1}(x-1,N-1)]]\\&=\lambda_N\overline{F}(p)\Delta V_{n-1}(x-1,N)-\lambda_{N-1}\overline{F}(p)\Delta V_{n-1}(x-1,N-1)\\&+\lambda_N[1-\overline{F}(p)]\Delta V_{n-1}(x,N)-\lambda_{N-1}[1-\overline{F}(p)]\Delta V_{n-1}(x,N-1)\end{aligned} \tag{A.8}$$

$$+\mu\{\max[V_{n-1}(x,N),V_{n-1}(x+1,N)-c]-\max[V_{n-1}(x-1,N),V_{n-1}(x,N)-c]\}$$
$$-\mu\{\max[V_{n-1}(x,N-1),V_{n-1}(x+1,N-1)-c]-\max[V_{n-1}(x-1,N-1),V_{n-1}(x,N-1)-c]\}$$
$$+q_{N,N-1}\Delta V_{n-1}(x,N-1)+(\sum_{i\neq N}\lambda_i+\sum_{j=1}^{N-1}q_{j,j+1}+\sum_{k=2}^{N-1}q_{k,k-1})\Delta V_{n-1}(x,N)$$
$$-[q_{N-1,N}\Delta V_{n-1}(x,N)+q_{N-1,N-2}\Delta V_{n-1}(x,N-2)+(\sum_{i\neq N-1}\lambda_i+\sum_{j=1}^{N-2}q_{j,j+1}+\sum_{\substack{k=2\\k\neq N-1}}^{N}q_{k,k-1})\Delta V_{n-1}(x,N-1)]$$
$$=\lambda_N\overline{F}(p)\Delta V_{n-1}(x-1,N)-\lambda_{N-1}\overline{F}(p)\Delta V_{n-1}(x-1,N-1)$$
$$+\lambda_N[1-\overline{F}(p)]\Delta V_{n-1}(x,N)-\lambda_{N-1}[1-\overline{F}(p)]\Delta V_{n-1}(x,N-1)$$
$$+\mu\{\max[V_{n-1}(x,N),V_{n-1}(x+1,N)-c]-\max[V_{n-1}(x-1,N),V_{n-1}(x,N)-c]\}$$
$$-\mu\{\max[V_{n-1}(x,N-1),V_{n-1}(x+1,N-1)-c]-\max[V_{n-1}(x-1,N-1),V_{n-1}(x,N-1)-c]\}$$
$$+q_{N,N-1}\Delta V_{n-1}(x,N-1)+(\sum_{i\neq N}\lambda_i+\sum_{j=1}^{N-2}q_{j,j+1}+\sum_{k=2}^{N-2}q_{k,k-1})\Delta V_{n-1}(x,N)$$
$$+q_{N-1,N}\Delta V_{n-1}(x,N)+q_{N-1,N-2}\Delta V_{n-1}(x,N)$$
$$-[q_{N-1,N}\Delta V_{n-1}(x,N)+q_{N-1,N-2}\Delta V_{n-1}(x,N-2)+(\sum_{i\neq N-1}\lambda_i+\sum_{j=1}^{N-2}q_{j,j+1}+\sum_{k=2}^{N-2}q_{k,k-1})\Delta V_{n-1}(x,N-1)$$
$$+q_{N,N-1}\Delta V_{n-1}(x,N-1)]$$
$$=\lambda_N\overline{F}(p)\Delta V_{n-1}(x-1,N)-\lambda_{N-1}\overline{F}(p)\Delta V_{n-1}(x-1,N-1)$$
$$+\lambda_N[1-\overline{F}(p)]\Delta V_{n-1}(x,N)-\lambda_{N-1}[1-\overline{F}(p)]\Delta V_{n-1}(x,N-1)$$
$$+\mu\{\max[V_{n-1}(x,N),V_{n-1}(x+1,N)-c]-\max[V_{n-1}(x-1,N),V_{n-1}(x,N)-c]\}$$
$$-\mu\{\max[V_{n-1}(x,N-1),V_{n-1}(x+1,N-1)-c]-\max[V_{n-1}(x-1,N-1),V_{n-1}(x,N-1)-c]\}$$
$$+q_{N,N-1}\Delta V_{n-1}(x,N-1)+(\sum_{i\neq N,N-1}\lambda_i+\sum_{j=1}^{N-2}q_{j,j+1}+\sum_{k=2}^{N-2}q_{k,k-1})\Delta V_{n-1}(x,N)$$
$$+q_{N-1,N}\Delta V_{n-1}(x,N)+q_{N-1,N-2}\Delta V_{n-1}(x,N)+\lambda_{N-1}\Delta V_{n-1}(x,N)$$
$$-[q_{N-1,N}\Delta V_{n-1}(x,N)+q_{N-1,N-2}\Delta V_{n-1}(x,N-2)+(\sum_{i\neq N-1,N}\lambda_i+\sum_{j=1}^{N-2}q_{j,j+1}+\sum_{k=2}^{N-2}q_{k,k-1})\Delta V_{n-1}(x,N-1)$$
$$+q_{N,N-1}\Delta V_{n-1}(x,N-1)+\lambda_N\Delta V_{n-1}(x,N-1)]$$
$$=(\lambda_N-\lambda_{N-1})\overline{F}(p)\Delta V_{n-1}(x-1,N)+\lambda_{N-1}\overline{F}(p)\{\Delta V_{n-1}(x-1,N)-\Delta V_{n-1}(x-1,N-1)\}$$
$$+(\lambda_N-\lambda_{N-1})[1-\overline{F}(p)]\Delta V_{n-1}(x,N)+\lambda_{N-1}[1-\overline{F}(p)]\{\Delta V_{n-1}(x,N)-\Delta V_{n-1}(x,N-1)\}$$

$$+\lambda_{N-1}\Delta V_{n-1}(x,N)-\lambda_N\Delta V_{n-1}(x,N-1)$$
$$+\mu\{\max[V_{n-1}(x,N),V_{n-1}(x+1,N)-c]-\max[V_{n-1}(x-1,N),V_{n-1}(x,N)-c]\}$$
$$-\mu\{\max[V_{n-1}(x,N-1),V_{n-1}(x+1,N-1)-c]-\max[V_{n-1}(x-1,N-1),V_{n-1}(x,N-1)-c]\}$$
$$+(\sum_{i\neq N,N-1}\lambda_i+\sum_{j=1}^{N-2}q_{j,j+1}+\sum_{k=2}^{N-2}q_{k,k-1})\{\Delta V_{n-1}(x,N)-\Delta V_{n-1}(x,N-1)\}$$
$$+(q_{N-1,N-2})\{\Delta V_{n-1}(x,N)-\Delta V_{n-1}(x,N-2)\}$$
$$\geq(\lambda_N-\lambda_{N-1})\overline{F}(p)\Delta V_{n-1}(x,N)+\lambda_{N-1}\overline{F}(p)\{\Delta V_{n-1}(x-1,N)-\Delta V_{n-1}(x-1,N-1)\}$$
$$+(\lambda_N-\lambda_{N-1})[1-\overline{F}(p)]\Delta V_{n-1}(x,N)+\lambda_{N-1}[1-\overline{F}(p)]\{\Delta V_{n-1}(x,N)-\Delta V_{n-1}(x,N-1)\}$$
$$+\lambda_{N-1}\Delta V_{n-1}(x,N)-\lambda_N\Delta V_{n-1}(x,N-1)$$
$$+\mu\{\max[V_{n-1}(x,N),V_{n-1}(x+1,N)-c]-\max[V_{n-1}(x-1,N),V_{n-1}(x,N)-c]\}$$
$$-\mu\{\max[V_{n-1}(x,N-1),V_{n-1}(x+1,N-1)-c]-\max[V_{n-1}(x-1,N-1),V_{n-1}(x,N-1)-c]\}$$
$$+(\sum_{i\neq N,N-1}\lambda_i+\sum_{j=1}^{N-2}q_{j,j+1}+\sum_{k=2}^{N-2}q_{k,k-1})\{\Delta V_{n-1}(x,N)-\Delta V_{n-1}(x,N-1)\}$$
$$+(q_{N-1,N-2})\{\Delta V_{n-1}(x,N)-\Delta V_{n-1}(x,N-2)\}$$
$$=\lambda_N\{\Delta V_{n-1}(x,N)-\Delta V_{n-1}(x,N-1)\}+\lambda_{N-1}\overline{F}(p)\{\Delta V_{n-1}(x-1,N)-\Delta V_{n-1}(x-1,N-1)\}$$
$$+\lambda_{N-1}[1-\overline{F}(p)]\{\Delta V_{n-1}(x,N)-\Delta V_{n-1}(x,N-1)\}$$
$$+\mu\{\max[V_{n-1}(x,N),V_{n-1}(x+1,N)-c]-\max[V_{n-1}(x-1,N),V_{n-1}(x,N)-c]\}$$
$$-\mu\{\max[V_{n-1}(x,N-1),V_{n-1}(x+1,N-1)-c]-\max[V_{n-1}(x-1,N-1),V_{n-1}(x,N-1)-c]\}$$
$$+(\sum_{i\neq N,N-1}\lambda_i+\sum_{j=1}^{N-2}q_{j,j+1}+\sum_{k=2}^{N-2}q_{k,k-1})\{\Delta V_{n-1}(x,N)-\Delta V_{n-1}(x,N-1)\}$$
$$+(q_{N-1,N-2})\{\Delta V_{n-1}(x,N)-\Delta V_{n-1}(x,N-2)\}$$
$$=\lambda_N\{\Delta V_{n-1}(x,N)-\Delta V_{n-1}(x,N-1)\}+\lambda_{N-1}\overline{F}(p)\{\Delta V_{n-1}(x-1,N)-\Delta V_{n-1}(x-1,N-1)\}$$
$$+\lambda_{N-1}[1-\overline{F}(p)]\{\Delta V_{n-1}(x,N)-\Delta V_{n-1}(x,N-1)\}$$
$$+\mu\{\max[V_{n-1}(x,N),V_{n-1}(x+1,N)-c]-\max[V_{n-1}(x-1,N),V_{n-1}(x,N)-c]\}$$
$$-\mu\{\max[V_{n-1}(x,N-1),V_{n-1}(x+1,N-1)-c]-\max[V_{n-1}(x-1,N-1),V_{n-1}(x,N-1)-c]\}$$

$$+(\sum_{i\neq N,N-1}\lambda_i+\sum_{j=1}^{N-2}q_{j,j+1}+\sum_{k=2}^{N-2}q_{k,k-1})\{\Delta V_{n-1}(x,N)-\Delta V_{n-1}(x,N-1)\}$$
$$+(q_{N-1,N-2})\{[\Delta V_{n-1}(x,N)-\Delta V_{n-1}(x,N-1)]+[\Delta V_{n-1}(x,N-1)-\Delta V_{n-1}(x,N-2)]\}\geq 0$$

The inequality above holds since V_n (x, e) is concave in x for all e and n ($\Delta V_{n-1}(x-1,N)\geq \Delta V_{n-1}(x,N)$) by Lemma 2, Lemma A.1 is assumed to be true for n-1, and since

$$\{\max[V_{n-1}(x,N),V_{n-1}(x+1,N)-c]-\max[V_{n-1}(x-1,N),V_{n-1}(x,N)-c]\}$$
$$-\{\max[V_{n-1}(x,N-1),V_{n-1}(x+1,N-1)-c]-\max[V_{n-1}(x-1,N-1),V_{n-1}(x,N-1)-c]\}\geq 0 \quad \text{(A.9)}$$

(A.9) holds by the same discussion summarized in Table 5 for $e = N$-1.

Thus, Lemma A.1 is true for $x\geq 2$. We will now analyse the cases with x=1 and e = 2, 3, …, N-2, e = 1, and e = N-1.

For x=1 and e = 2, 3, …, N-2,

$$\Delta V_n(1,e+1)-\Delta V_n(1,e) \quad \text{(A.10)}$$
$$=[V_n(1,e+1)-V_n(0,e+1)]-[V_n(1,e)-V_n(0,e)]$$
$$=[[(-h(1)+\lambda_{e+1}\overline{F}(p)p+\lambda_{e+1}\overline{F}(p)V_{n-1}(0,e+1)+\lambda_{e+1}[1-\overline{F}(p)]V_{n-1}(1,e+1)+\mu\max\{V_{n-1}(1,e+1),V_{n-1}(2,e+1)-c\}$$
$$+q_{e+1,e+2}V_{n-1}(1,e+2)+q_{e+1,e}V_{n-1}(1,e)+(\sum_{i\neq e+1}\lambda_i+\sum_{\substack{j=1\\ j\neq e+1}}^{N-1}q_{j,j+1}+\sum_{\substack{k=2\\ k\neq e+1}}^{N}q_{k,k-1})V_{n-1}(1,e+1)]$$
$$-[(-h(0)+\lambda_{e+1}\overline{F}(p)V_{n-1}(0,e+1)+\lambda_{e+1}[1-\overline{F}(p)]V_{n-1}(0,e+1)+\mu\max\{V_{n-1}(0,e+1),V_{n-1}(1,e+1)-c\}$$
$$+q_{e+1,e+2}V_{n-1}(0,e+2)+q_{e+1,e}V_{n-1}(0,e)+(\sum_{i\neq e+1}\lambda_i+\sum_{\substack{j=1\\ j\neq e+1}}^{N-1}q_{j,j+1}+\sum_{\substack{k=2\\ k\neq e+1}}^{N}q_{k,k-1})V_{n-1}(0,e+1)]]$$
$$-[[(-h(1)+\lambda_e\overline{F}(p)p+\lambda_e\overline{F}(p)V_{n-1}(0,e)+\lambda_e[1-\overline{F}(p)]V_{n-1}(1,e)+\mu\max\{V_{n-1}(1,e),V_{n-1}(2,e)-c\}$$

$$+q_{e,e+1}V_{n-1}(1,e+1)+q_{e,e-1}V_{n-1}(1,e-1)+(\sum_{i\neq e}\lambda_i+\sum_{\substack{j=1\\ j\neq e}}^{N-1}q_{j,j+1}+\sum_{\substack{k=2\\ k\neq e}}^{N}q_{k,k-1})V_{n-1}(1,e)]$$

$$-[(-h(0)+\lambda_e\overline{F}(p)V_{n-1}(0,e)+\lambda_e[1-\overline{F}(p)]V_{n-1}(0,e)+\mu\max\{V_{n-1}(0,e),V_{n-1}(1,e)-c\}$$

$$+q_{e,e+1}V_{n-1}(0,e+1)+q_{e,e-1}V_{n-1}(0,e-1)+(\sum_{i\neq e}\lambda_i+\sum_{\substack{j=1\\ j\neq e}}^{N-1}q_{j,j+1}+\sum_{\substack{k=2\\ k\neq e}}^{N}q_{k,k-1})V_{n-1}(0,e)]]$$

$$=(\lambda_{e+1}-\lambda_e)\overline{F}(p)p+\lambda_{e+1}[1-\overline{F}(p)]\Delta V_{n-1}(1,e+1)-\lambda_e[1-\overline{F}(p)]\Delta V_{n-1}(1,e)$$

$$+\mu\{\max[V_{n-1}(1,e+1),V_{n-1}(2,e+1)-c]-\max[V_{n-1}(0,e+1),V_{n-1}(1,e+1)-c]\}$$

$$-\mu\{\max[V_{n-1}(1,e),V_{n-1}(2,e)-c]-\max[V_{n-1}(0,e),V_{n-1}(1,e)-c]\}$$

$$+q_{e+1,e+2}\Delta V_{n-1}(1,e+2)+q_{e+1,e}\Delta V_{n-1}(1,e)+(\sum_{i\neq e+1}\lambda_i+\sum_{\substack{j=1\\ j\neq e+1}}^{N-1}q_{j,j+1}+\sum_{\substack{k=2\\ k\neq e+1}}^{N}q_{k,k-1})\Delta V_{n-1}(1,e+1)$$

$$-[q_{e,e+1}\Delta V_{n-1}(1,e+1)+q_{e,e-1}\Delta V_{n-1}(1,e-1)+(\sum_{i\neq e}\lambda_i+\sum_{\substack{j=1\\ j\neq e}}^{N-1}q_{j,j+1}+\sum_{\substack{k=2\\ k\neq e}}^{N}q_{k,k-1})\Delta V_{n-1}(1,e)]$$

$$=(\lambda_{e+1}-\lambda_e)\overline{F}(p)p+(\lambda_{e+1}-\lambda_e)[1-\overline{F}(p)]\Delta V_{n-1}(1,e+1)+\lambda_e[1-\overline{F}(p)]\{\Delta V_{n-1}(1,e+1)-\Delta V_{n-1}(1,e)\}$$

$$+\lambda_e\Delta V_{n-1}(1,e+1)-\lambda_{e+1}\Delta V_{n-1}(1,e)$$

$$+\mu\{\max[V_{n-1}(1,e+1),V_{n-1}(2,e+1)-c]-\max[V_{n-1}(0,e+1),V_{n-1}(1,e+1)-c]\}$$

$$-\mu\{\max[V_{n-1}(1,e),V_{n-1}(2,e)-c]-\max[V_{n-1}(0,e),V_{n-1}(1,e)-c]\}$$

$$+q_{e+1,e+2}\Delta V_{n-1}(1,e+2)+q_{e+1,e}\Delta V_{n-1}(1,e)+(\sum_{i\neq e+1,e}\lambda_i+\sum_{\substack{j=1\\ j\neq e+1,e}}^{N-1}q_{j,j+1}+\sum_{\substack{k=2\\ k\neq e+1,e}}^{N}q_{k,k-1})\Delta V_{n-1}(1,e+1)+(q_{e,e+1}+q_{e,e-1})\Delta V_{n-1}(1,e+1)$$

$$-[q_{e,e+1}\Delta V_{n-1}(1,e+1)+q_{e,e-1}\Delta V_{n-1}(1,e-1)+(\sum_{i\neq e,e+1}\lambda_i+\sum_{\substack{j=1\\ j\neq e,e+1}}^{N-1}q_{j,j+1}+\sum_{\substack{k=2\\ k\neq e,e+1}}^{N}q_{k,k-1})\Delta V_{n-1}(1,e)+(q_{e+1,e+2}+q_{e+1,e})\Delta V_{n-1}(1,e)]$$

$$=(\lambda_{e+1}-\lambda_e)\overline{F}(p)\{p-\Delta V_{n-1}(1,e+1)\}+(\lambda_{e+1}-\lambda_e)\Delta V_{n-1}(1,e+1)+\lambda_e[1-\overline{F}(p)]\{\Delta V_{n-1}(1,e+1)-\Delta V_{n-1}(1,e)\}$$

$$+\lambda_e\Delta V_{n-1}(1,e+1)-\lambda_{e+1}\Delta V_{n-1}(1,e)$$

$$+\mu\{\max[V_{n-1}(1,e+1),V_{n-1}(2,e+1)-c]-\max[V_{n-1}(0,e+1),V_{n-1}(1,e+1)-c]\}$$

$$-\mu\{\max[V_{n-1}(1,e),V_{n-1}(2,e)-c]-\max[V_{n-1}(0,e),V_{n-1}(1,e)-c]\}$$

$$+q_{e+1,e+2}\Delta V_{n-1}(1,e+2)+(\sum_{i\neq e+1,e}\lambda_i+\sum_{\substack{j=1\\ j\neq e+1,e}}^{N-1}q_{j,j+1}+\sum_{\substack{k=2\\ k\neq e+1,e}}^{N}q_{k,k-1})\Delta V_{n-1}(1,e+1)+(q_{e,e-1})\Delta V_{n-1}(1,e+1)$$

$$-[q_{e,e-1}\Delta V_{n-1}(1,e-1)+(\sum_{i\neq e,e+1}\lambda_i+\sum_{\substack{j=1\\ j\neq e,e+1}}^{N-1}q_{j,j+1}+\sum_{\substack{k=2\\ k\neq e,e+1}}^{N}q_{k,k-1})\Delta V_{n-1}(1,e)+(q_{e+1,e+2})\Delta V_{n-1}(1,e)]$$

$$=(\lambda_{e+1}-\lambda_e)\overline{F}(p)\{p-\Delta V_{n-1}(1,e+1)\}+\lambda_{e+1}\{\Delta V_{n-1}(1,e+1)-\Delta V_{n-1}(1,e)\}$$
$$+\lambda_e[1-\overline{F}(p)]\{\Delta V_{n-1}(1,e+1)-\Delta V_{n-1}(1,e)\}$$
$$+\mu\{\max[V_{n-1}(1,e+1),V_{n-1}(2,e+1)-c]-\max[V_{n-1}(0,e+1),V_{n-1}(1,e+1)-c]\}$$
$$-\mu\{\max[V_{n-1}(1,e),V_{n-1}(2,e)-c]-\max[V_{n-1}(0,e),V_{n-1}(1,e)-c]\}$$
$$+(\sum_{i\neq e,e+1}\lambda_i+\sum_{\substack{j=1\\ j\neq e,e+1}}^{N-1}q_{j,j+1}+\sum_{\substack{k=2\\ k\neq e,e+1}}^{N}q_{k,k-1})\{\Delta V_{n-1}(1,e+1)-\Delta V_{n-1}(1,e)\}$$
$$+q_{e+1,e+2}\{\Delta V_{n-1}(1,e+2)-\Delta V_{n-1}(1,e)\}+(q_{e,e-1})\{\Delta V_{n-1}(1,e+1)-\Delta V_{n-1}(1,e-1)\}$$
$$=(\lambda_{e+1}-\lambda_e)\overline{F}(p)\{p-\Delta V_{n-1}(1,e+1)\}+\lambda_{e+1}\{\Delta V_{n-1}(1,e+1)-\Delta V_{n-1}(1,e)\}$$
$$+\lambda_e[1-\overline{F}(p)]\{\Delta V_{n-1}(1,e+1)-\Delta V_{n-1}(1,e)\}$$
$$+\mu\{\max[V_{n-1}(1,e+1),V_{n-1}(2,e+1)-c]-\max[V_{n-1}(0,e+1),V_{n-1}(1,e+1)-c]\}$$
$$-\mu\{\max[V_{n-1}(1,e),V_{n-1}(2,e)-c]-\max[V_{n-1}(0,e),V_{n-1}(1,e)-c]\}$$
$$+(\sum_{i\neq e,e+1}\lambda_i+\sum_{\substack{j=1\\ j\neq e,e+1}}^{N-1}q_{j,j+1}+\sum_{\substack{k=2\\ k\neq e,e+1}}^{N}q_{k,k-1})\{\Delta V_{n-1}(1,e+1)-\Delta V_{n-1}(1,e)\}$$
$$+q_{e+1,e+2}\{[\Delta V_{n-1}(1,e+2)-\Delta V_{n-1}(1,e+1)]+[\Delta V_{n-1}(1,e+1)-\Delta V_{n-1}(1,e)]\}$$
$$+q_{e,e-1}\{[\Delta V_{n-1}(1,e+1)-\Delta V_{n-1}(1,e)]+[\Delta V_{n-1}(1,e)-\Delta V_{n-1}(1,e-1)]\}\geq 0$$

The inequality above holds since Lemma A.1 is assumed to be true for *n*-1, $\{p-\Delta V_{n-1}(1,e+1)\geq 0\}$ by the second part of Lemma 2, and since

$$\{\max[V_{n-1}(1,e+1),V_{n-1}(2,e+1)-c]-\max[V_{n-1}(0,e+1),V_{n-1}(1,e+1)-c]\}$$
$$-\{\max[V_{n-1}(1,e),V_{n-1}(2,e)-c]-\max[V_{n-1}(0,e),V_{n-1}(1,e)-c]\}\geq 0 \qquad \text{(A.11)}$$

(A.11) holds by the same discussion summarized in Table 5 for x=1 and e = 2, 3, …, N-2.

For x=1 and e = 1,

$$\Delta V_n(1,2)-\Delta V_n(1,1) \qquad \text{(A.12)}$$

$$=[V_n(1,2)-V_n(0,2)]-[V_n(1,1)-V_n(0,1)]$$

$$=[[(-h(1)+\lambda_2\overline{F}(p)p+\lambda_2\overline{F}(p)V_{n-1}(0,2)+\lambda_2[1-\overline{F}(p)]V_{n-1}(1,2)+\mu\max\{V_{n-1}(1,2),V_{n-1}(2,2)-c\}$$

$$+q_{2,3}V_{n-1}(1,3)+q_{2,1}V_{n-1}(1,1)+(\sum_{i\neq 2}\lambda_i+\sum_{\substack{j=1\\ j\neq 2}}^{N-1}q_{j,j+1}+\sum_{k=3}^{N}q_{k,k-1})V_{n-1}(1,2)]$$

$$-[(-h(0)+\lambda_2\overline{F}(p)V_{n-1}(0,2)+\lambda_2[1-\overline{F}(p)]V_{n-1}(0,2)+\mu\max\{V_{n-1}(0,2),V_{n-1}(1,2)-c\}$$

$$+q_{2,3}V_{n-1}(0,3)+q_{2,1}V_{n-1}(0,1)+(\sum_{i\neq 2}\lambda_i+\sum_{\substack{j=1\\ j\neq 2}}^{N-1}q_{j,j+1}+\sum_{k=3}^{N}q_{k,k-1})V_{n-1}(0,2)]]$$

$$-[[(-h(1)+\lambda_1\overline{F}(p)p+\lambda_1\overline{F}(p)V_{n-1}(0,1)+\lambda_1[1-\overline{F}(p)]V_{n-1}(1,1)+\mu\max\{V_{n-1}(1,1),V_{n-1}(2,1)-c\}$$

$$+q_{1,2}V_{n-1}(1,2)+(\sum_{i\neq 1}\lambda_i+\sum_{j=2}^{N-1}q_{j,j+1}+\sum_{k=2}^{N}q_{k,k-1})V_{n-1}(1,1)]$$

$$-[(-h(0)+\lambda_1\overline{F}(p)V_{n-1}(0,1)+\lambda_1[1-\overline{F}(p)]V_{n-1}(0,1)+\mu\max\{V_{n-1}(0,1),V_{n-1}(1,1)-c\}$$

$$+q_{1,2}V_{n-1}(0,2)+(\sum_{i\neq 1}\lambda_i+\sum_{j=2}^{N-1}q_{j,j+1}+\sum_{k=2}^{N}q_{k,k-1})V_{n-1}(0,1)]]$$

$$=(\lambda_2-\lambda_1)\overline{F}(p)p+\lambda_2[1-\overline{F}(p)]\Delta V_{n-1}(1,2)-\lambda_1[1-\overline{F}(p)]\Delta V_{n-1}(1,1)$$

$$+\mu\{\max[V_{n-1}(1,2),V_{n-1}(2,2)-c]-\max[V_{n-1}(0,2),V_{n-1}(1,2)-c]\}$$

$$-\mu\{\max[V_{n-1}(1,1),V_{n-1}(2,1)-c]-\max[V_{n-1}(0,1),V_{n-1}(1,1)-c]\}$$

$$+q_{2,3}\Delta V_{n-1}(1,3)+q_{2,1}\Delta V_{n-1}(1,1)+(\sum_{i\neq 2}\lambda_i+\sum_{\substack{j=1\\ j\neq 2}}^{N-1}q_{j,j+1}+\sum_{k=3}^{N}q_{k,k-1})\Delta V_{n-1}(1,2)$$

$$-[q_{1,2}\Delta V_{n-1}(1,2)+(\sum_{i\neq 1}\lambda_i+\sum_{j=2}^{N-1}q_{j,j+1}+\sum_{k=2}^{N}q_{k,k-1})\Delta V_{n-1}(1,1)]$$

$$=(\lambda_2-\lambda_1)\overline{F}(p)p+(\lambda_2-\lambda_1)[1-\overline{F}(p)]\Delta V_{n-1}(1,2)+\lambda_1[1-\overline{F}(p)]\{\Delta V_{n-1}(1,2)-\Delta V_{n-1}(1,1)\}$$

$$+\mu\{\max[V_{n-1}(1,2),V_{n-1}(2,2)-c]-\max[V_{n-1}(0,2),V_{n-1}(1,2)-c]\}$$

$$-\mu\{\max[V_{n-1}(1,1),V_{n-1}(2,1)-c]-\max[V_{n-1}(0,1),V_{n-1}(1,1)-c]\}$$

$$+q_{2,3}\Delta V_{n-1}(1,3)+q_{2,1}\Delta V_{n-1}(1,1)+(\sum_{i\neq 2,1}\lambda_i+\sum_{j=3}^{N-1}q_{j,j+1}+\sum_{k=3}^{N}q_{k,k-1})\Delta V_{n-1}(1,2)+\lambda_1\Delta V_{n-1}(1,2)+q_{1,2}\Delta V_{n-1}(1,2)$$

$$-[q_{1,2}\Delta V_{n-1}(1,2)+(\sum_{i\neq 1,2}\lambda_i+\sum_{j=3}^{N-1}q_{j,j+1}+\sum_{k=3}^{N}q_{k,k-1})\Delta V_{n-1}(1,1)]+\lambda_2\Delta V_{n-1}(1,1)+q_{2,3}\Delta V_{n-1}(1,1)+q_{2,1}\Delta V_{n-1}(1,1)$$

$$=(\lambda_2-\lambda_1)\overline{F}(p)\{p-\Delta V_{n-1}(1,2)\}+(\lambda_2-\lambda_1)\Delta V_{n-1}(1,2)+\lambda_1[1-\overline{F}(p)]\{\Delta V_{n-1}(1,2)-\Delta V_{n-1}(1,1)\}$$

$$+\lambda_1\Delta V_{n-1}(1,2)-\lambda_2\Delta V_{n-1}(1,1)$$

$$+\mu\{\max[V_{n-1}(1,2),V_{n-1}(2,2)-c]-\max[V_{n-1}(0,2),V_{n-1}(1,2)-c]\}$$
$$-\mu\{\max[V_{n-1}(1,1),V_{n-1}(2,1)-c]-\max[V_{n-1}(0,1),V_{n-1}(1,1)-c]\}$$
$$+(\sum_{i\neq 2,1}\lambda_i+\sum_{j=3}^{N-1}q_{j,j+1}+\sum_{k=3}^{N}q_{k,k-1})\{\Delta V_{n-1}(1,2)-\Delta V_{n-1}(1,1)\}$$
$$+q_{2,3}\{\Delta V_{n-1}(1,3)-\Delta V_{n-1}(1,1)\}\geq 0$$

The inequality above holds since Lemma A.1 is assumed to be true for *n*-1, $\{p-\Delta V_{n-1}(1,2)\geq 0\}$ by the second part of Lemma 2, and since

$$\begin{aligned}&\{\max[V_{n-1}(1,2),V_{n-1}(2,2)-c]-\max[V_{n-1}(0,2),V_{n-1}(1,2)-c]\}\\&-\{\max[V_{n-1}(1,1),V_{n-1}(2,1)-c]-\max[V_{n-1}(0,1),V_{n-1}(1,1)-c]\}\geq 0\end{aligned}\tag{A.13}$$

(A.13) holds by the same discussion summarized in Table 5 for x = 1 and e = 1. Thus, Lemma A.1 is true for x = 1 and e = 1. For x = 1 and e = N-1,

$$\Delta V_n(1,N)-\Delta V_n(1,N-1)$$
$$=[V_n(1,N)-V_n(0,N)]-[V_n(1,N-1)-V_n(0,N-1)]\tag{A.14}$$
$$=[[(-h(1)+\lambda_N\overline{F}(p)p+\lambda_N\overline{F}(p)V_{n-1}(0,N)+\lambda_N[1-\overline{F}(p)]V_{n-1}(1,N)+\mu\max\{V_{n-1}(1,N),V_{n-1}(2,N)-c\}$$
$$+q_{N,N-1}V_{n-1}(1,N-1)+(\sum_{i\neq N}\lambda_i+\sum_{j=1}^{N-1}q_{j,j+1}+\sum_{k=2}^{N-1}q_{k,k-1})V_{n-1}(1,N)]$$
$$-[(-h(0)+\lambda_N\overline{F}(p)V_{n-1}(0,N)+\lambda_N[1-\overline{F}(p)]V_{n-1}(0,N)+\mu\max\{V_{n-1}(0,N),V_{n-1}(1,N)-c\}$$
$$+q_{N,N-1}V_{n-1}(0,N-1)+(\sum_{i\neq N}\lambda_i+\sum_{j=1}^{N-1}q_{j,j+1}+\sum_{k=2}^{N-1}q_{k,k-1})V_{n-1}(0,N)]]$$
$$-[[(-h(1)+\lambda_{N-1}\overline{F}(p)p+\lambda_{N-1}\overline{F}(p)V_{n-1}(0,N-1)+\lambda_{N-1}[1-\overline{F}(p)]V_{n-1}(1,N-1)+\mu\max\{V_{n-1}(1,N-1),V_{n-1}(2,N-1)-c\}$$
$$+q_{N-1,N}V_{n-1}(1,N)+q_{N-1,N-2}V_{n-1}(1,N-2)+(\sum_{i\neq N-1}\lambda_i+\sum_{j=1}^{N-2}q_{j,j+1}+\sum_{\substack{k=2\\k\neq N-1}}^{N}q_{k,k-1})V_{n-1}(1,N-1)]$$
$$-[(-h(0)+\lambda_1\overline{F}(p)V_{n-1}(0,N)+\lambda_1[1-\overline{F}(p)]V_{n-1}(0,N)+\mu\max\{V_{n-1}(0,N),V_{n-1}(1,N)-c\}$$

$$+q_{N-1,N}V_{n-1}(0,N)+q_{N-1,N-2}V_{n-1}(0,N-2)+(\sum_{i\neq N-1}\lambda_i+\sum_{j=1}^{N-2}q_{j,j+1}+\sum_{\substack{k=2\\k\neq N-1}}^{N}q_{k,k-1})V_{n-1}(0,N-1)]]$$

$$=(\lambda_N-\lambda_{N-1})\overline{F}(p)p+\lambda_N[1-\overline{F}(p)]\Delta V_{n-1}(1,N)-\lambda_{N-1}[1-\overline{F}(p)]\Delta V_{n-1}(1,N-1)$$

$$+\mu\{\max[V_{n-1}(1,N),V_{n-1}(2,N)-c]-\max[V_{n-1}(0,N),V_{n-1}(1,N)-c]\}$$

$$-\mu\{\max[V_{n-1}(1,N-1),V_{n-1}(2,N-1)-c]-\max[V_{n-1}(0,N-1),V_{n-1}(1,N-1)-c]\}$$

$$+q_{N,N-1}\Delta V_{n-1}(1,N-1)+(\sum_{i\neq N}\lambda_i+\sum_{j=1}^{N-1}q_{j,j+1}+\sum_{k=2}^{N-1}q_{k,k-1})\Delta V_{n-1}(1,N)$$

$$-[q_{N-1,N}\Delta V_{n-1}(1,N)+q_{N-1,N-2}\Delta V_{n-1}(1,N-2)+(\sum_{i\neq N-1}\lambda_i+\sum_{j=1}^{N-2}q_{j,j+1}+\sum_{\substack{k=2\\k\neq N-1}}^{N}q_{k,k-1})\Delta V_{n-1}(1,N-1)]$$

$$=(\lambda_N-\lambda_{N-1})\overline{F}(p)p+(\lambda_N-\lambda_{N-1})[1-\overline{F}(p)]\Delta V_{n-1}(1,N)+\lambda_{N-1}[1-\overline{F}(p)]\{\Delta V_{n-1}(1,N)-\Delta V_{n-1}(1,N-1)\}$$

$$+\mu\{\max[V_{n-1}(1,N),V_{n-1}(2,N)-c]-\max[V_{n-1}(0,N),V_{n-1}(1,N)-c]\}$$

$$-\mu\{\max[V_{n-1}(1,N-1),V_{n-1}(2,N-1)-c]-\max[V_{n-1}(0,N-1),V_{n-1}(1,N-1)-c]\}$$

$$+q_{N,N-1}\Delta V_{n-1}(1,N-1)+(\sum_{i\neq N,N-1}\lambda_i+\sum_{j=1}^{N-2}q_{j,j+1}+\sum_{k=2}^{N-2}q_{k,k-1})\Delta V_{n-1}(1,N)+\lambda_{N-1}\Delta V_{n-1}(1,N)+q_{N-1,N}\Delta V_{n-1}(1,N)+q_{N-1,N-2}\Delta V_{n-1}(1,N)$$

$$-[q_{N-1,N}\Delta V_{n-1}(1,N)+q_{N-1,N-2}\Delta V_{n-1}(1,N-2)+(\sum_{i\neq N-1,N}\lambda_i+\sum_{j=1}^{N-2}q_{j,j+1}+\sum_{k=2}^{N-2}q_{k,k-1})\Delta V_{n-1}(1,N-1)]+\lambda_N\Delta V_{n-1}(1,N-1)+q_{N,N-1}\Delta V_{n-1}(1,N-1)$$

$$=(\lambda_N-\lambda_{N-1})\overline{F}(p)\{p-\Delta V_{n-1}(1,N)\}+(\lambda_N-\lambda_{N-1})\Delta V_{n-1}(1,N)+\lambda_{N-1}[1-\overline{F}(p)]\{\Delta V_{n-1}(1,N)-\Delta V_{n-1}(1,N-1)\}$$

$$+\lambda_{N-1}\Delta V_{n-1}(1,N)-\lambda_N\Delta V_{n-1}(1,N-1)$$

$$+\mu\{\max[V_{n-1}(1,N),V_{n-1}(2,N)-c]-\max[V_{n-1}(0,N),V_{n-1}(1,N)-c]\}$$

$$-\mu\{\max[V_{n-1}(1,N-1),V_{n-1}(2,N-1)-c]-\max[V_{n-1}(0,N-1),V_{n-1}(1,N-1)-c]\}$$

$$+(\sum_{i\neq N,N-1}\lambda_i+\sum_{j=1}^{N-2}q_{j,j+1}+\sum_{k=2}^{N-2}q_{k,k-1})\{\Delta V_{n-1}(1,N)-\Delta V_{n-1}(1,N-1)\}+q_{N-1,N-2}\{\Delta V_{n-1}(1,N)-\Delta V_{n-1}(1,N-2)\}$$

$$=(\lambda_N-\lambda_{N-1})\overline{F}(p)\{p-\Delta V_{n-1}(1,N)\}+\lambda_{N-1}[1-\overline{F}(p)]\{\Delta V_{n-1}(1,N)-\Delta V_{n-1}(1,N-1)\}$$

$$+\lambda_N\{\Delta V_{n-1}(1,N)-\Delta V_{n-1}(1,N-1)\}$$

$$+\mu\{\max[V_{n-1}(1,N),V_{n-1}(2,N)-c]-\max[V_{n-1}(0,N),V_{n-1}(1,N)-c]\}$$

$$-\mu\{\max[V_{n-1}(1,N-1),V_{n-1}(2,N-1)-c]-\max[V_{n-1}(0,N-1),V_{n-1}(1,N-1)-c]\}$$

$$+(\sum_{i\neq N,N-1}\lambda_i+\sum_{j=1}^{N-2}q_{j,j+1}+\sum_{k=2}^{N-2}q_{k,k-1})\{\Delta V_{n-1}(1,N)-\Delta V_{n-1}(1,N-1)\}+q_{N-1,N-2}\{\Delta V_{n-1}(1,N)-\Delta V_{n-1}(1,N-2)\}$$

The inequality above holds since Lemma A.1 is assumed to be true for *n*-1, $\{p-\Delta V_{n-1}(1,N)\geq 0\}$ by the second part of Lemma 2, and since

$$\begin{aligned}&\{\max[V_{n-1}(1,N),V_{n-1}(2,N)-c]-\max[V_{n-1}(0,N),V_{n-1}(1,N)-c]\}\\&-\{\max[V_{n-1}(1,N-1),V_{n-1}(2,N-1)-c]-\max[V_{n-1}(0,N-1),V_{n-1}(1,N-1)-c]\}\end{aligned} \tag{A.15}$$

(A.15) holds by the same discussion summarized in Table 5 for $x = 1$ and $e = N$-1. Thus, Lemma A.1 is true for $x = 1$, and the proof is complete.

Proposition A.1. The base stock levels for each environmental state e (S_e^*) have the same ordering with the demand rates corresponding to these states. That is, $S_N^* \geq S_{N-1}^* \geq \ldots \geq S_e^* \geq S_{e-1}^* \geq \ldots \geq S_1^*$ where $\lambda_N \geq \lambda_{N-1} \geq \ldots \geq \lambda_e \geq \lambda_{e-1} \ldots \geq \lambda_1$.

Proof. The proposition is directly implied by Theorem 2 and Lemma A.1.

A.2 Monotonicity of the Base Stock Levels with Only Two Environment States

In this section, we have only two environment states: Low (L) and High (H) where $\lambda_H \geq \lambda_L$, and the transition rates between these two states are denoted by q_{LH} and q_{HL}. Here, we deal with a very simple Markov chain to characterize the environment, and the optimality equations are presented below:

$$\begin{aligned}V(x,H) = &-h(x)+\lambda_H\overline{F}(p)p+\lambda_H\overline{F}(p)V(x-1,H)+\lambda_H[1-\overline{F}(p)]V(x,H)+\mu\max\{V(x,H),V(x+1,H)-c\}\\&+q_{H,L}V(x,L)+(\lambda_L+q_{L,H})V(x,H)\end{aligned} \tag{A.16}$$

$$\begin{aligned}V(x,L) = &-h(x)+\lambda_L\overline{F}(p)p+\lambda_L\overline{F}(p)V(x-1,L)+\lambda_L[1-\overline{F}(p)]V(x,L)+\mu\max\{V(x,L),V(x+1,L)-c\}\\&+q_{L,H}V(x,H)+(\lambda_H+q_{H,L})V(x,L)\end{aligned} \tag{A.17}$$

Lemma A.2. $\Delta V(x,H) \geq \Delta V(x,L)$, for all x.

Proof. We will use the induction method and show that $\Delta V_n(x,H) \geq \Delta V_n(x,L)$ for all x and for all n. The lemma is trivially true for n=0, suppose that it is true for n-1. We again consider x = 1 and $x \geq$ 2 separately.

For $x \geq 2$,

$$\Delta V_n(x,H) - \Delta V_n(x,L) \tag{A.18}$$

$$= [V_n(x,H) - V_n(x-1,H)] - [V_n(x,L) - V_n(x-1,L)]$$

$$= [[(-h(x) + \lambda_H \overline{F}(p)p + \lambda_H \overline{F}(p)V_{n-1}(x-1,H) + \lambda_H[1-\overline{F}(p)]V_{n-1}(x,H) + \mu \max\{V_{n-1}(x,H), V_{n-1}(x+1,H)-c\}$$

$$+q_{H,L}V_{n-1}(x,L) + (\lambda_L + q_{L,H})V_{n-1}(x,H)]$$

$$-[(-h(x-1) + \lambda_H \overline{F}(p)p + \lambda_H \overline{F}(p)V_{n-1}(x-2,H) + \lambda_H[1-\overline{F}(p)]V_{n-1}(x-1,H) + \mu \max\{V_{n-1}(x-1,H), V_{n-1}(x,H)-c\}$$

$$+q_{H,L}V_{n-1}(x-1,L) + (\lambda_L + q_{L,H})V_{n-1}(x-1,H)]]$$

$$-[[(-h(x) + \lambda_L \overline{F}(p)p + \lambda_L \overline{F}(p)V_{n-1}(x-1,L) + \lambda_L[1-\overline{F}(p)]V_{n-1}(x,L) + \mu \max\{V_{n-1}(x,L), V_{n-1}(x+1,L)-c\}$$

$$+q_{L,H}V_{n-1}(x,H) + (\lambda_H + q_{H,L})V_{n-1}(x,L)]$$

$$-[(-h(x-1) + \lambda_L \overline{F}(p)p + \lambda_L \overline{F}(p)V_{n-1}(x-2,L) + \lambda_L[1-\overline{F}(p)]V_{n-1}(x-1,L) + \mu \max\{V_{n-1}(x-1,L), V_{n-1}(x,L)-c\}$$

$$+q_{L,H}V_{n-1}(x-1,H) + (\lambda_H + q_{H,L})V_{n-1}(x-1,L)]]$$

$$= \lambda_H \overline{F}(p)\Delta V_{n-1}(x-1,H) - \lambda_L \overline{F}(p)\Delta V_{n-1}(x-1,L)$$

$$+\lambda_H[1-\overline{F}(p)]\Delta V_{n-1}(x,H) - \lambda_L[1-\overline{F}(p)]\Delta V_{n-1}(x,L)$$

$$+\mu\{\max[V_{n-1}(x,H), V_{n-1}(x+1,H)-c] - \max[V_{n-1}(x-1,H), V_{n-1}(x,H)-c]\}$$

$$-\mu\{\max[V_{n-1}(x,L), V_{n-1}(x+1,L)-c] - \max[V_{n-1}(x-1,L), V_{n-1}(x,L)-c]\}$$

$$+q_{H,L}\Delta V_{n-1}(x,L) - q_{L,H}\Delta V_{n-1}(x,H) + (\lambda_L + q_{L,H})\Delta V_{n-1}(x,H) - (\lambda_H + q_{H,L})\Delta V_{n-1}(x,L)$$

$$= (\lambda_H - \lambda_L)\overline{F}(p)\Delta V_{n-1}(x-1,H) - \lambda_L \overline{F}(p)\{\Delta V_{n-1}(x-1,H) - \Delta V_{n-1}(x-1,L)\}$$

$$+(\lambda_H - \lambda_L)[1-\overline{F}(p)]\Delta V_{n-1}(x,H) - \lambda_L[1-\overline{F}(p)]\{\Delta V_{n-1}(x,H) - \Delta V_{n-1}(x,L)\}$$

$$+\mu\{\max[V_{n-1}(x,H), V_{n-1}(x+1,H)-c] - \max[V_{n-1}(x-1,H), V_{n-1}(x,H)-c]\}$$

$$-\mu\{\max[V_{n-1}(x,L), V_{n-1}(x+1,L)-c] - \max[V_{n-1}(x-1,L), V_{n-1}(x,L)-c]\}$$

$$+q_{H,L}\Delta V_{n-1}(x,L)-q_{L,H}\Delta V_{n-1}(x,H)+(\lambda_L+q_{L,H})\Delta V_{n-1}(x,H)-(\lambda_H+q_{H,L})\Delta V_{n-1}(x,L)$$

$$\geq(\lambda_H-\lambda_L)\overline{F}(p)\Delta V_{n-1}(x,H)-\lambda_L\overline{F}(p)\{\Delta V_{n-1}(x-1,H)-\Delta V_{n-1}(x-1,L)\}$$
$$+(\lambda_H-\lambda_L)[1-\overline{F}(p)]\Delta V_{n-1}(x,H)-\lambda_L[1-\overline{F}(p)]\{\Delta V_{n-1}(x,H)-\Delta V_{n-1}(x,L)\}$$
$$+\mu\{\max[V_{n-1}(x,H),V_{n-1}(x+1,H)-c]-\max[V_{n-1}(x-1,H),V_{n-1}(x,H)-c]\}$$
$$-\mu\{\max[V_{n-1}(x,L),V_{n-1}(x+1,L)-c]-\max[V_{n-1}(x-1,L),V_{n-1}(x,L)-c]\}$$
$$+q_{H,L}\Delta V_{n-1}(x,L)-q_{L,H}\Delta V_{n-1}(x,H)+\lambda_L\Delta V_{n-1}(x,H)+q_{L,H}\Delta V_{n-1}(x,H)-\lambda_H\Delta V_{n-1}(x,L)-q_{H,L}\Delta V_{n-1}(x,L)$$

$$=(\lambda_H-\lambda_L)\Delta V_{n-1}(x,H)-\lambda_L\overline{F}(p)\{\Delta V_{n-1}(x-1,H)-\Delta V_{n-1}(x-1,L)\}$$
$$+\lambda_L\Delta V_{n-1}(x,H)-\lambda_H\Delta V_{n-1}(x,L)-\lambda_L[1-\overline{F}(p)]\{\Delta V_{n-1}(x,H)-\Delta V_{n-1}(x,L)\}$$
$$+\mu\{\max[V_{n-1}(x,H),V_{n-1}(x+1,H)-c]-\max[V_{n-1}(x-1,H),V_{n-1}(x,H)-c]\}$$
$$-\mu\{\max[V_{n-1}(x,L),V_{n-1}(x+1,L)-c]-\max[V_{n-1}(x-1,L),V_{n-1}(x,L)-c]\}$$
$$+q_{H,L}\Delta V_{n-1}(x,L)-q_{L,H}\Delta V_{n-1}(x,H)+q_{L,H}\Delta V_{n-1}(x,H)-q_{H,L}\Delta V_{n-1}(x,L)$$

$$=\lambda_H\{\Delta V_{n-1}(x,H)-\Delta V_{n-1}(x,L)\}-\lambda_L\overline{F}(p)\{\Delta V_{n-1}(x-1,H)-\Delta V_{n-1}(x-1,L)\}$$
$$-\lambda_L[1-\overline{F}(p)]\{\Delta V_{n-1}(x,H)-\Delta V_{n-1}(x,L)\}$$
$$+\mu\{\max[V_{n-1}(x,H),V_{n-1}(x+1,H)-c]-\max[V_{n-1}(x-1,H),V_{n-1}(x,H)-c]\}$$
$$-\mu\{\max[V_{n-1}(x,L),V_{n-1}(x+1,L)-c]-\max[V_{n-1}(x-1,L),V_{n-1}(x,L)-c]\}\geq 0$$

The inequality above holds since for all n ($\Delta V_{n-1}(x-1,H)\geq\Delta V_{n-1}(x,H)$) by Lemma 2, Lemma A.2 is assumed to be true for n-1, and since

$$\begin{aligned}&\{\max[V_{n-1}(x,H),V_{n-1}(x+1,H)-c]-\max[V_{n-1}(x-1,H),V_{n-1}(x,H)-c]\}\\&-\{\max[V_{n-1}(x,L),V_{n-1}(x+1,L)-c]-\max[V_{n-1}(x-1,L),V_{n-1}(x,L)-c]\}\geq 0\end{aligned}\quad\text{(A.19)}$$

(A.19) holds by the same discussion summarized in Table 5 for $e\equiv$ L and e+1 $\equiv$ H where $x\geq 2$. Thus, Lemma A.2 is true for $x\geq 2$.

For $x = 1$,

$$\Delta V_n(1,H)-\Delta V_n(1,L)$$

$$=[V_n(1,H)-V_n(0,H)]-[V_n(1,L)-V_n(0,L)] \tag{A.20}$$

$$=[[(-h(1)+\lambda_H\overline{F}(p)p+\lambda_H\overline{F}(p)V_{n-1}(0,H)+\lambda_H[1-\overline{F}(p)]V_{n-1}(1,H)+\mu\max\{V_{n-1}(1,H),V_{n-1}(2,H)-c\}$$

$$+q_{H,L}V_{n-1}(1,L)+(\lambda_L+q_{L,H})V_{n-1}(1,H)]$$

$$-[(-h(0)+\lambda_H\overline{F}(p)V_{n-1}(0,H)+\lambda_H[1-\overline{F}(p)]V_{n-1}(0,H)+\mu\max\{V_{n-1}(0,H),V_{n-1}(1,H)-c\}$$

$$+q_{H,L}V_{n-1}(0,L)+(\lambda_L+q_{L,H})V_{n-1}(0,H)]]$$

$$-[[(-h(1)+\lambda_L\overline{F}(p)p+\lambda_L\overline{F}(p)V_{n-1}(0,L)+\lambda_L[1-\overline{F}(p)]V_{n-1}(1,L)+\mu\max\{V_{n-1}(1,L),V_{n-1}(2,L)-c\}$$

$$+q_{L,H}V_{n-1}(1,H)+(\lambda_H+q_{H,L})V_{n-1}(1,L)]$$

$$-[(-h(0)+\lambda_L\overline{F}(p)V_{n-1}(0,L)+\lambda_L[1-\overline{F}(p)]V_{n-1}(0,L)+\mu\max\{V_{n-1}(0,L),V_{n-1}(1,L)-c\}$$

$$+q_{L,H}V_{n-1}(0,H)+(\lambda_H+q_{H,L})V_{n-1}(0,L)]]$$

$$=(\lambda_H-\lambda_L)\overline{F}(p)p+\lambda_H[1-\overline{F}(p)]\Delta V_{n-1}(1,H)-\lambda_L[1-\overline{F}(p)]\Delta V_{n-1}(1,L)$$

$$+\mu\{\max[V_{n-1}(1,H),V_{n-1}(2,H)-c]-\max[V_{n-1}(0,H),V_{n-1}(1,H)-c]\}$$

$$-\mu\{\max[V_{n-1}(1,L),V_{n-1}(2,L)-c]-\max[V_{n-1}(0,L),V_{n-1}(1,L)-c]\}$$

$$+q_{H,L}\Delta V_{n-1}(1,L)-q_{L,H}\Delta V_{n-1}(1,H)+(\lambda_L+q_{L,H})\Delta V_{n-1}(1,H)-(\lambda_H+q_{H,L})\Delta V_{n-1}(1,L)$$

$$=(\lambda_H-\lambda_L)\overline{F}(p)p+(\lambda_H-\lambda_L)[1-\overline{F}(p)]\Delta V_{n-1}(1,H)+\lambda_L[1-\overline{F}(p)]\{\Delta V_{n-1}(1,H)-\Delta V_{n-1}(1,L)\}$$

$$+\mu\{\max[V_{n-1}(1,H),V_{n-1}(2,H)-c]-\max[V_{n-1}(0,H),V_{n-1}(1,H)-c]\}$$

$$-\mu\{\max[V_{n-1}(1,L),V_{n-1}(2,L)-c]-\max[V_{n-1}(0,L),V_{n-1}(1,L)-c]\}$$

$$+q_{H,L}\Delta V_{n-1}(x,L)-q_{L,H}\Delta V_{n-1}(x,H)+\lambda_L\Delta V_{n-1}(x,H)+q_{L,H}\Delta V_{n-1}(x,H)-\lambda_H\Delta V_{n-1}(x,L)-q_{H,L}\Delta V_{n-1}(x,L)$$

$$=(\lambda_H-\lambda_L)\overline{F}(p)\{p-\Delta V_{n-1}(1,H)\}+(\lambda_H-\lambda_L)\Delta V_{n-1}(1,H)+\lambda_L[1-\overline{F}(p)]\{\Delta V_{n-1}(1,H)-\Delta V_{n-1}(1,L)\}$$

$$+\lambda_L\Delta V_{n-1}(x,H)-\lambda_H\Delta V_{n-1}(x,L)$$

$$+\mu\{\max[V_{n-1}(1,H),V_{n-1}(2,H)-c]-\max[V_{n-1}(0,H),V_{n-1}(1,H)-c]\}$$

$$-\mu\{\max[V_{n-1}(1,L),V_{n-1}(2,L)-c]-\max[V_{n-1}(0,L),V_{n-1}(1,L)-c]\}$$

$$+q_{H,L}\Delta V_{n-1}(x,L)-q_{L,H}\Delta V_{n-1}(x,H)+q_{L,H}\Delta V_{n-1}(x,H)-q_{H,L}\Delta V_{n-1}(x,L)$$

$$=(\lambda_H-\lambda_L)\overline{F}(p)\{p-\Delta V_{n-1}(1,H)\}+\lambda_L[1-\overline{F}(p)]\{\Delta V_{n-1}(1,H)-\Delta V_{n-1}(1,L)\}$$

$$+\lambda_H\{\Delta V_{n-1}(x,H)-\Delta V_{n-1}(x,L)\}$$

$$+\mu\{\max[V_{n-1}(1,H),V_{n-1}(2,H)-c]-\max[V_{n-1}(0,H),V_{n-1}(1,H)-c]\}$$
$$-\mu\{\max[V_{n-1}(1,L),V_{n-1}(2,L)-c]-\max[V_{n-1}(0,L),V_{n-1}(1,L)-c]\}\geq 0$$

The inequality above holds since for all n ($p-\Delta V_{n-1}(1,H)\geq 0$) by the second part of Lemma 2, Lemma A.2 is assumed to be true for n-1, and since

$$\begin{aligned}&\{\max[V_{n-1}(1,H),V_{n-1}(2,H)-c]-\max[V_{n-1}(0,H),V_{n-1}(1,H)-c]\}\\&-\{\max[V_{n-1}(1,L),V_{n-1}(2,L)-c]-\max[V_{n-1}(0,L),V_{n-1}(1,L)-c]\}\geq 0\end{aligned} \quad \text{(A.21)}$$

(A.21) holds by the same discussion summarized in Table 5 for $e \equiv$ L and e+1 $\equiv$ H where x = 1. Thus, Lemma A.2 is true for x = 1, and the proof is complete.

Proposition A.2. The base stock levels for environmental states Low and High (S_L^*, S_H^*) have the same ordering with the demand rates corresponding to these states. That is, $S_H^* \geq S_L^*$ where $\lambda_H \geq \lambda_L$.

Proof. The proposition is directly implied by Theorem 2 and Lemma A.2.

Appendix B

An ILLUSTRATIVE EXAMPLE-MONOTONICITY of the BASE STOCK LEVELS for a SYSTEM of FOUR ENVIRONMENT STATES

In this section, we show that the base stock levels of the four environmental states induce the same ordering with the demand rates of these states. Note that E is the set of environment states 1, 2, 3, 4 where $\lambda_4 \geq \lambda_3 \geq \lambda_2 \geq \lambda_1$ and β is the discount rate.

Condition B.1

a. $q_{ej} \geq q_{e+1,j}$, $\forall e$ where j = 1, …, e-1.

b. $q_{e+1,j} \geq q_{e,j}$, $\forall e$ where j = e+2, …, 4.

Condition 2 states that the relationships listed below must hold for Theorem 4 to be true for the system studied here:

For e = 1; $q_{2,3} \geq q_{1,3}$, $q_{2,4} \geq q_{1,4}$,

For e = 2; $q_{2,1} \geq q_{3,1}$, $q_{3,4} \geq q_{2,4}$,

For e = 3; $q_{3,1} \geq q_{4,1}$, $q_{3,2} \geq q_{4,2}$.

Lemma B.1. $\Delta V(x,4) \geq \Delta V(x,3) \geq \Delta V(x,2) \geq \Delta V(x,1)$, for all x.

Proof. We will use the induction method and show that $\Delta V_n(x,4) \geq \Delta V_n(x,3) \geq \Delta V_n(x,2) \geq \Delta V_n(x,1)$ for all x if Condition 2 is satisfied. The lemma is trivially true for n=0, suppose that it is true for n-1. We again consider the inequalities $\Delta V(x,4) \geq \Delta V(x,3)$, $\Delta V(x,3) \geq \Delta V(x,2)$, and $\Delta V(x,2) \geq \Delta V(x,1)$ for x = 1 and $x \geq 2$ separately.

For $x \geq 2$,

(1) $\Delta V_n(x,4) - \Delta V_n(x,3)$

$$= [V_n(x,4) - V_n(x-1,4)] - [V_n(x,3) - V_n(x-1,3)] \tag{B.1}$$

$$= [[(-h(x) + \lambda_4 \overline{F}(p)p + \lambda_4 \overline{F}(p)V_{n-1}(x-1,4) + \lambda_4[1-\overline{F}(p)]V_{n-1}(x,4) + \mu \max\{V_{n-1}(x,4), V_{n-1}(x+1,4) - c\}$$

$$+q_{41}V_{n-1}(x,1) + q_{42}V_{n-1}(x,2) + q_{43}V_{n-1}(x,3) + (\sum_{k=1}^{3} \lambda_k + \sum_{i=1}^{3}\sum_{j \neq i} q_{ij})V_n(x,4)]$$

$$-[(-h(x-1) + \lambda_4 \overline{F}(p)p + \lambda_4 \overline{F}(p)V_{n-1}(x-2,4) + \lambda_4[1-\overline{F}(p)]V_{n-1}(x-1,4) + \mu \max\{V_{n-1}(x-1,4), V_{n-1}(x,4) - c\}$$

$$+q_{41}V_{n-1}(x-1,1) + q_{42}V_{n-1}(x-1,2) + q_{43}V_{n-1}(x-1,3) + (\sum_{k=1}^{3} \lambda_k + \sum_{i=1}^{3}\sum_{j \neq i} q_{ij})V_n(x-1,4)]]$$

$$-[[(-h(x) + \lambda_3 \overline{F}(p)p + \lambda_3 \overline{F}(p)V_{n-1}(x-1,3) + \lambda_3[1-\overline{F}(p)]V_{n-1}(x,3) + \mu \max\{V_{n-1}(x,3), V_{n-1}(x+1,3) - c\}$$

$$+q_{31}V_{n-1}(x,1) + q_{32}V_{n-1}(x,2) + q_{34}V_{n-1}(x,4) + (\sum_{k \neq 3} \lambda_k + \sum_{i \neq 3}\sum_{j \neq i} q_{ij})V_n(x,3)]$$

$$-[(-h(x-1) + \lambda_3 \overline{F}(p)p + \lambda_3 \overline{F}(p)V_{n-1}(x-2,3) + \lambda_3[1-\overline{F}(p)]V_{n-1}(x-1,3) + \mu \max\{V_{n-1}(x-1,3), V_{n-1}(x,3) - c\}$$

$$+q_{31}V_{n-1}(x-1,1) + q_{32}V_{n-1}(x-1,2) + q_{34}V_{n-1}(x-1,4) + (\sum_{k \neq 3} \lambda_k + \sum_{i \neq 3}\sum_{j \neq i} q_{ij})V_n(x-1,3)]]$$

$$= \lambda_4 \overline{F}(p)\Delta V_{n-1}(x-1,4) - \lambda_3 \overline{F}(p)\Delta V_{n-1}(x-1,3)$$

$$+\lambda_4[1-\overline{F}(p)]\Delta V_{n-1}(x,4) - \lambda_3[1-\overline{F}(p)]\Delta V_{n-1}(x,3)$$

$$+\mu\{\max[V_{n-1}(x,4), V_{n-1}(x+1,4) - c] - \max[V_{n-1}(x-1,4), V_{n-1}(x,4) - c]\}$$

$$-\mu\{\max[V_{n-1}(x,3), V_{n-1}(x+1,3) - c] - \max[V_{n-1}(x-1,3), V_{n-1}(x,3) - c]\}$$

$$+q_{41}\Delta V_{n-1}(x,1) + q_{42}\Delta V_{n-1}(x,2) + q_{43}\Delta V_{n-1}(x,3) - \{q_{31}\Delta V_{n-1}(x,1) + q_{32}\Delta V_{n-1}(x,2) + q_{34}\Delta V_{n-1}(x,4)\}$$

$$+(\sum_{i \neq 4} \lambda_i + \sum_{i \neq 4}\sum_{j \neq i} q_{ij})\Delta V_{n-1}(x,4) - (\sum_{i \neq 3} \lambda_i + \sum_{i \neq 3}\sum_{j \neq i} q_{ij})\Delta V_{n-1}(x,3)$$

$$= \lambda_4 \overline{F}(p)\Delta V_{n-1}(x-1,4) - \lambda_3 \overline{F}(p)\Delta V_{n-1}(x-1,3)$$

$$+\lambda_4[1-\overline{F}(p)]\Delta V_{n-1}(x,4) - \lambda_3[1-\overline{F}(p)]\Delta V_{n-1}(x,3)$$

$$+\mu\{\max[V_{n-1}(x,4), V_{n-1}(x+1,4) - c] - \max[V_{n-1}(x-1,4), V_{n-1}(x,4) - c]\}$$

$$-\mu\{\max[V_{n-1}(x,3), V_{n-1}(x+1,3) - c] - \max[V_{n-1}(x-1,3), V_{n-1}(x,3) - c]\}$$

$$+q_{41}\Delta V_{n-1}(x,1)+q_{42}\Delta V_{n-1}(x,2)+q_{43}\Delta V_{n-1}(x,3)-\{q_{31}\Delta V_{n-1}(x,1)+q_{32}\Delta V_{n-1}(x,2)+q_{34}\Delta V_{n-1}(x,4)\}$$

$$+\sum_{i\neq 4}\lambda_i\Delta V_{n-1}(x,4)-\sum_{i\neq 3}\lambda_i\Delta V_{n-1}(x,3)+\sum_{i=1}^{2}\sum_{j\neq i}q_{ij}(\Delta V_{n-1}(x,4)-\Delta V_{n-1}(x,3))$$

$$+q_{31}\Delta V_{n-1}(x,4)+q_{32}\Delta V_{n-1}(x,4)+q_{34}\Delta V_{n-1}(x,4)-\{q_{41}\Delta V_{n-1}(x,3)+q_{42}\Delta V_{n-1}(x,3)+q_{43}\Delta V_{n-1}(x,3)\}$$

$$= (\lambda_4-\lambda_3)\overline{F}(p)\Delta V_{n-1}(x-1,4)+\lambda_3\overline{F}(p)\Delta V_{n-1}(x-1,4)-\lambda_3\overline{F}(p)\Delta V_{n-1}(x-1,3)$$

$$+(\lambda_4-\lambda_3)[1-\overline{F}(p)]\Delta V_{n-1}(x,4)+\lambda_3[1-\overline{F}(p)]\Delta V_{n-1}(x,4)-\lambda_3[1-\overline{F}(p)]\Delta V_{n-1}(x,3)$$

$$+\mu\{\max[V_{n-1}(x,4),V_{n-1}(x+1,4)-c]-\max[V_{n-1}(x-1,4),V_{n-1}(x,4)-c]\}$$

$$-\mu\{\max[V_{n-1}(x,3),V_{n-1}(x+1,3)-c]-\max[V_{n-1}(x-1,3),V_{n-1}(x,3)-c]\}$$

$$+q_{41}\{\Delta V_{n-1}(x,1)-\Delta V_{n-1}(x,3)\}+q_{42}\{\Delta V_{n-1}(x,2)-\Delta V_{n-1}(x,3)\}$$

$$+q_{31}\{\Delta V_{n-1}(x,4)-\Delta V_{n-1}(x,1)\}+q_{32}\{\Delta V_{n-1}(x,4)-\Delta V_{n-1}(x,2)\}$$

$$+(\lambda_1+\lambda_2+\sum_{i=1}^{2}\sum_{j\neq i}q_{ij})\{\Delta V_{n-1}(x,4)-\Delta V_{n-1}(x,3)\}+\lambda_3\Delta V_{n-1}(x,4)-\lambda_4\Delta V_{n-1}(x,3)$$

$$\geq (\lambda_4-\lambda_3)\overline{F}(p)\Delta V_{n-1}(x,4)+\lambda_3\overline{F}(p)\{\Delta V_{n-1}(x-1,4)-\Delta V_{n-1}(x-1,3)\}$$

$$+(\lambda_4-\lambda_3)[1-\overline{F}(p)]\Delta V_{n-1}(x,4)+\lambda_3[1-\overline{F}(p)]\{\Delta V_{n-1}(x,4)-\Delta V_{n-1}(x,3)\}$$

$$+\mu\{\max[V_{n-1}(x,4),V_{n-1}(x+1,4)-c]-\max[V_{n-1}(x-1,4),V_{n-1}(x,4)-c]\}$$

$$-\mu\{\max[V_{n-1}(x,3),V_{n-1}(x+1,3)-c]-\max[V_{n-1}(x-1,3),V_{n-1}(x,3)-c]\}$$

$$+q_{41}\{\Delta V_{n-1}(x,1)-\Delta V_{n-1}(x,3)\}+q_{42}\{\Delta V_{n-1}(x,2)-\Delta V_{n-1}(x,3)\}$$

$$+q_{31}\{\Delta V_{n-1}(x,4)-\Delta V_{n-1}(x,1)\}+q_{32}\{\Delta V_{n-1}(x,4)-\Delta V_{n-1}(x,2)\}$$

$$+(\lambda_1+\lambda_2+\sum_{i=1}^{2}\sum_{j\neq i}q_{ij})\{\Delta V_{n-1}(x,4)-\Delta V_{n-1}(x,3)\}+\lambda_3\Delta V_{n-1}(x,4)-\lambda_4\Delta V_{n-1}(x,3)$$

$$= \lambda_4\{\Delta V_{n-1}(x,4)-\Delta V_{n-1}(x,3)\}+\lambda_3\overline{F}(p)\{\Delta V_{n-1}(x-1,4)-\Delta V_{n-1}(x-1,3)\}$$

$$+\lambda_3[1-\overline{F}(p)]\{\Delta V_{n-1}(x,4)-\Delta V_{n-1}(x,3)\}$$

$$+\mu\{\max[V_{n-1}(x,4),V_{n-1}(x+1,4)-c]-\max[V_{n-1}(x-1,4),V_{n-1}(x,4)-c]\}$$

$$-\mu\{\max[V_{n-1}(x,3),V_{n-1}(x+1,3)-c]-\max[V_{n-1}(x-1,3),V_{n-1}(x,3)-c]\}$$

$$+(\lambda_1+\lambda_2+\sum_{i=1}^{2}\sum_{j\neq i}q_{ij})\{\Delta V_{n-1}(x,4)-\Delta V_{n-1}(x,3)\}$$

$$+q_{41}\{\Delta V_{n-1}(x,1)-\Delta V_{n-1}(x,3)\}+q_{42}\{\Delta V_{n-1}(x,2)-\Delta V_{n-1}(x,3)\}$$

$$+q_{31}\{\Delta V_{n-1}(x,4)-\Delta V_{n-1}(x,1)\}+q_{32}\{\Delta V_{n-1}(x,4)-\Delta V_{n-1}(x,2)\}\geq 0$$

The inequality above holds since $\Delta V_{n-1}(x-1,4)-\Delta V_{n-1}(x,4)\geq 0$ for all x by Lemma 2, by the induction assumption, and

$$\begin{aligned}&\{\max[V_{n-1}(x,4),V_{n-1}(x+1,4)-c]-\max[V_{n-1}(x-1,4),V_{n-1}(x,4)-c]\}\\&-\{\max[V_{n-1}(x,3),V_{n-1}(x+1,3)-c]-\max[V_{n-1}(x-1,3),V_{n-1}(x,3)-c]\}\geq 0\end{aligned} \quad \text{(B.2)}$$

by the inequalities in Table 5 for e = 3 given that Condition 2 holds for e = 3. That is, the following inequalities must hold: $q_{3,1} \geq q_{4,1}$, $q_{3,2} \geq q_{4,2}$.

(2) $\Delta V_n(x,3)-\Delta V_n(x,2)$ (B.3)

$$\begin{aligned}
&=[V_n(x,3)-V_n(x-1,3)]-[V_n(x,2)-V_n(x-1,2)]\\
&=[[(-h(x)+\lambda_3\overline{F}(p)p+\lambda_3\overline{F}(p)V_{n-1}(x-1,3)+\lambda_3[1-\overline{F}(p)]V_{n-1}(x,3)+\mu\max\{V_{n-1}(x,3),V_{n-1}(x+1,3)-c\}\\
&+q_{31}V_{n-1}(x,1)+q_{32}V_{n-1}(x,2)+q_{34}V_{n-1}(x,4)+(\sum_{k\neq 3}\lambda_k+\sum_{i\neq 3}\sum_{j\neq i}q_{ij})V_n(x,3)]\\
&-[(-h(x-1)+\lambda_3\overline{F}(p)p+\lambda_3\overline{F}(p)V_{n-1}(x-2,3)+\lambda_3[1-\overline{F}(p)]V_{n-1}(x-1,3)+\mu\max\{V_{n-1}(x-1,3),V_{n-1}(x,3)-c\}\\
&+q_{31}V_{n-1}(x-1,1)+q_{32}V_{n-1}(x-1,2)+q_{34}V_{n-1}(x-1,4)+(\sum_{k\neq 3}\lambda_k+\sum_{i\neq 3}\sum_{j\neq i}q_{ij})V_n(x-1,3)]]\\
&-[[(-h(x)+\lambda_2\overline{F}(p)p+\lambda_2\overline{F}(p)V_{n-1}(x-1,2)+\lambda_2[1-\overline{F}(p)]V_{n-1}(x,2)+\mu\max\{V_{n-1}(x,2),V_{n-1}(x+1,2)-c\}\\
&+q_{21}V_{n-1}(x,1)+q_{23}V_{n-1}(x,3)+q_{24}V_{n-1}(x,4)+(\sum_{k\neq 2}\lambda_k+\sum_{i\neq 2}\sum_{j\neq i}q_{ij})V_n(x,2)]\\
&-[(-h(x-1)+\lambda_2\overline{F}(p)p+\lambda_2\overline{F}(p)V_{n-1}(x-2,2)+\lambda_2[1-\overline{F}(p)]V_{n-1}(x-1,2)+\mu\max\{V_{n-1}(x-1,2),V_{n-1}(x,2)-c\}\\
&+q_{21}V_{n-1}(x-1,1)+q_{23}V_{n-1}(x-1,3)+q_{24}V_{n-1}(x-1,4)+(\sum_{k\neq 2}\lambda_k+\sum_{i\neq 2}\sum_{j\neq i}q_{ij})V_n(x-1,2)]]\\
&=\lambda_3\overline{F}(p)\Delta V_{n-1}(x-1,3)-\lambda_2\overline{F}(p)\Delta V_{n-1}(x-1,2)\\
&+\lambda_3[1-\overline{F}(p)]\Delta V_{n-1}(x,3)-\lambda_2[1-\overline{F}(p)]\Delta V_{n-1}(x,2)\\
&+\mu\{\max[V_{n-1}(x,3),V_{n-1}(x+1,3)-c]-\max[V_{n-1}(x-1,3),V_{n-1}(x,3)-c]\}\\
&-\mu\{\max[V_{n-1}(x,2),V_{n-1}(x+1,2)-c]-\max[V_{n-1}(x-1,2),V_{n-1}(x,2)-c]\}
\end{aligned}$$

$$+q_{3,1}\Delta V_{n-1}(x,1)+q_{3,2}\Delta V_{n-1}(x,2)+q_{3,4}\Delta V_{n-1}(x,4)$$
$$-\{q_{2,1}\Delta V_{n-1}(x,1)+q_{2,3}\Delta V_{n-1}(x,3)+q_{2,4}\Delta V_{n-1}(x,4)\}$$
$$+(\sum_{i\neq 3}\lambda_i+\sum_{i\neq 3}\sum_{j\neq i}q_{ij})\Delta V_{n-1}(x,3)-(\sum_{i\neq 2}\lambda_i+\sum_{i\neq 2}\sum_{j\neq i}q_{ij})\Delta V_{n-1}(x,2)$$
$$=\lambda_3\overline{F}(p)\Delta V_{n-1}(x-1,3)-\lambda_2\overline{F}(p)\Delta V_{n-1}(x-1,2)$$
$$+\lambda_3[1-\overline{F}(p)]\Delta V_{n-1}(x,3)-\lambda_2[1-\overline{F}(p)]\Delta V_{n-1}(x,2)$$
$$+\mu\{\max[V_{n-1}(x,3),V_{n-1}(x+1,3)-c]-\max[V_{n-1}(x-1,3),V_{n-1}(x,3)-c]\}$$
$$-\mu\{\max[V_{n-1}(x,2),V_{n-1}(x+1,2)-c]-\max[V_{n-1}(x-1,2),V_{n-1}(x,2)-c]\}$$
$$+q_{3,1}\Delta V_{n-1}(x,1)+q_{3,2}\Delta V_{n-1}(x,2)+q_{3,4}\Delta V_{n-1}(x,4)$$
$$-\{q_{2,1}\Delta V_{n-1}(x,1)+q_{2,3}\Delta V_{n-1}(x,3)+q_{2,4}\Delta V_{n-1}(x,4)\}$$
$$+\sum_{i\neq 3}\lambda_i\Delta V_{n-1}(x,3)-\sum_{i\neq 2}\lambda_i\Delta V_{n-1}(x,2)+\sum_{i=1,4}\sum_{j\neq i}q_{ij}(\Delta V_{n-1}(x,3)-\Delta V_{n-1}(x,2))$$
$$+q_{2,1}\Delta V_{n-1}(x,3)+q_{2,3}\Delta V_{n-1}(x,3)+q_{2,4}\Delta V_{n-1}(x,3)-\{q_{3,1}\Delta V_{n-1}(x,2)+q_{3,2}\Delta V_{n-1}(x,2)+q_{3,4}\Delta V_{n-1}(x,2)\}$$
$$=(\lambda_3-\lambda_2)\overline{F}(p)\Delta V_{n-1}(x-1,3)+\lambda_2\overline{F}(p)\Delta V_{n-1}(x-1,3)-\lambda_2\overline{F}(p)\Delta V_{n-1}(x-1,2)$$
$$+(\lambda_3-\lambda_2)[1-\overline{F}(p)]\Delta V_{n-1}(x,3)+\lambda_2[1-\overline{F}(p)]\Delta V_{n-1}(x,3)-\lambda_2[1-\overline{F}(p)]\Delta V_{n-1}(x,2)$$
$$+\mu\{\max[V_{n-1}(x,3),V_{n-1}(x+1,3)-c]-\max[V_{n-1}(x-1,3),V_{n-1}(x,3)-c]\}$$
$$-\mu\{\max[V_{n-1}(x,2),V_{n-1}(x+1,2)-c]-\max[V_{n-1}(x-1,2),V_{n-1}(x,2)-c]\}$$
$$+q_{3,1}\{\Delta V_{n-1}(x,1)-\Delta V_{n-1}(x,2)\}+q_{3,4}\{\Delta V_{n-1}(x,4)-\Delta V_{n-1}(x,2)\}$$
$$+q_{2,1}\{\Delta V_{n-1}(x,3)-\Delta V_{n-1}(x,1)\}+q_{2,4}\{\Delta V_{n-1}(x,4)-\Delta V_{n-1}(x,3)\}$$
$$+(\lambda_1+\lambda_4+\sum_{i=1,4}\sum_{j\neq i}q_{ij})\{\Delta V_{n-1}(x,3)-\Delta V_{n-1}(x,2)\}+\lambda_2\Delta V_{n-1}(x,3)-\lambda_3\Delta V_{n-1}(x,2)$$
$$\geq(\lambda_3-\lambda_2)\overline{F}(p)\Delta V_{n-1}(x,3)+\lambda_2\overline{F}(p)\{\Delta V_{n-1}(x-1,3)-\Delta V_{n-1}(x-1,2)\}$$
$$+(\lambda_3-\lambda_2)[1-\overline{F}(p)]\Delta V_{n-1}(x,3)+\lambda_2[1-\overline{F}(p)]\{\Delta V_{n-1}(x,3)-\Delta V_{n-1}(x,2)\}$$
$$+\mu\{\max[V_{n-1}(x,3),V_{n-1}(x+1,3)-c]-\max[V_{n-1}(x-1,3),V_{n-1}(x,3)-c]\}$$
$$-\mu\{\max[V_{n-1}(x,2),V_{n-1}(x+1,2)-c]-\max[V_{n-1}(x-1,2),V_{n-1}(x,2)-c]\}$$
$$+q_{3,1}\{\Delta V_{n-1}(x,1)-\Delta V_{n-1}(x,2)\}+q_{3,4}\{\Delta V_{n-1}(x,4)-\Delta V_{n-1}(x,2)\}$$

$$+q_{2,1}\{\Delta V_{n-1}(x,3)-\Delta V_{n-1}(x,1)\}+q_{2,4}\{\Delta V_{n-1}(x,4)-\Delta V_{n-1}(x,3)\}$$

$$+(\lambda_1+\lambda_4+\sum_{i=1,4}\sum_{j\neq i}q_{ij})\{\Delta V_{n-1}(x,3)-\Delta V_{n-1}(x,2)\}+\lambda_2\Delta V_{n-1}(x,3)-\lambda_3\Delta V_{n-1}(x,2)$$

$$=\lambda_3\{\Delta V_{n-1}(x,3)-\Delta V_{n-1}(x,2)\}+\lambda_2\overline{F}(p)\{\Delta V_{n-1}(x-1,3)-\Delta V_{n-1}(x-1,2)\}$$

$$+\lambda_2[1-\overline{F}(p)]\{\Delta V_{n-1}(x,3)-\Delta V_{n-1}(x,2)\}$$

$$+\mu\{\max[V_{n-1}(x,3),V_{n-1}(x+1,3)-c]-\max[V_{n-1}(x-1,3),V_{n-1}(x,3)-c]\}$$

$$-\mu\{\max[V_{n-1}(x,2),V_{n-1}(x+1,2)-c]-\max[V_{n-1}(x-1,2),V_{n-1}(x,2)-c]\}$$

$$+(\lambda_1+\lambda_4+\sum_{i=1,4}\sum_{j\neq i}q_{ij})\{\Delta V_{n-1}(x,3)-\Delta V_{n-1}(x,2)\}$$

$$+q_{3,1}\{\Delta V_{n-1}(x,1)-\Delta V_{n-1}(x,2)\}+q_{3,4}\{\Delta V_{n-1}(x,4)-\Delta V_{n-1}(x,2)\}$$

$$+q_{2,1}\{\Delta V_{n-1}(x,3)-\Delta V_{n-1}(x,1)\}+q_{2,4}\{\Delta V_{n-1}(x,4)-\Delta V_{n-1}(x,3)\}\geq 0$$

The inequality above holds since $\Delta V_{n-1}(x-1,3)-\Delta V_{n-1}(x,3)\geq 0$ for all x by Lemma 2, by the induction assumption, and

$$\begin{aligned}&\{\max[V_{n-1}(x,3),V_{n-1}(x+1,3)-c]-\max[V_{n-1}(x-1,3),V_{n-1}(x,3)-c]\}\\&-\{\max[V_{n-1}(x,2),V_{n-1}(x+1,2)-c]-\max[V_{n-1}(x-1,2),V_{n-1}(x,2)-c]\}\geq 0\end{aligned} \quad \text{(B.4)}$$

by the inequalities in Table 5 for e = 2 given that Condition 2 holds for e = 2. That is, the following inequalities must hold: $q_{2,1} \geq q_{3,1}$, $q_{3,4} \geq q_{2,4}$.

(3) $\Delta V(x,2)\geq \Delta V(x,1)$

$$=[V_n(x,2)-V_n(x-1,2)]-[V_n(x,1)-V_n(x-1,1)] \quad \text{(B.5)}$$

$$=[[(-h(x)+\lambda_2\overline{F}(p)p+\lambda_2\overline{F}(p)V_{n-1}(x-1,2)+\lambda_2[1-\overline{F}(p)]V_{n-1}(x,2)+\mu\max\{V_{n-1}(x,2),V_{n-1}(x+1,2)-c\}$$

$$+q_{21}V_{n-1}(x,1)+q_{23}V_{n-1}(x,3)+q_{24}V_{n-1}(x,4)+(\sum_{k\neq 2}\lambda_k+\sum_{i\neq 2}\sum_{j\neq i}q_{ij})V_n(x,2)]$$

$$-[(-h(x-1)+\lambda_2\overline{F}(p)p+\lambda_2\overline{F}(p)V_{n-1}(x-2,2)+\lambda_2[1-\overline{F}(p)]V_{n-1}(x-1,2)+\mu\max\{V_{n-1}(x-1,2),V_{n-1}(x,2)-c\}$$

$$+q_{21}V_{n-1}(x-1,1)+q_{23}V_{n-1}(x-1,2)+q_{24}V_{n-1}(x-1,4)+(\sum_{k\neq 2}\lambda_k+\sum_{i\neq 2}\sum_{j\neq i}q_{ij})V_n(x-1,2)]]$$

$$-[[(-h(x)+\lambda_1\overline{F}(p)p+\lambda_1\overline{F}(p)V_{n-1}(x-1,1)+\lambda_1[1-\overline{F}(p)]V_{n-1}(x,1)+\mu\max\{V_{n-1}(x,1),V_{n-1}(x+1,1)-c\}$$

$$+q_{12}V_{n-1}(x,2)+q_{13}V_{n-1}(x,3)+q_{14}V_{n-1}(x,4)+(\sum_{k\neq 1}\lambda_k+\sum_{i\neq 1}\sum_{j\neq i}q_{ij})V_n(x,1)]$$

$$-[(-h(x-1)+\lambda_1\overline{F}(p)p+\lambda_1\overline{F}(p)V_{n-1}(x-2,1)+\lambda_1[1-\overline{F}(p)]V_{n-1}(x-1,1)+\mu\max\{V_{n-1}(x-1,1),V_{n-1}(x,1)-c\}$$

$$+q_{12}V_{n-1}(x-1,2)+q_{13}V_{n-1}(x-1,3)+q_{14}V_{n-1}(x-1,4)+(\sum_{k\neq 1}\lambda_k+\sum_{i\neq 1}\sum_{j\neq i}q_{ij})V_n(x-1,1)]]$$

$$=\lambda_2\overline{F}(p)\Delta V_{n-1}(x-1,2)-\lambda_1\overline{F}(p)\Delta V_{n-1}(x-1,1)$$

$$+\lambda_2[1-\overline{F}(p)]\Delta V_{n-1}(x,2)-\lambda_1[1-\overline{F}(p)]\Delta V_{n-1}(x,1)$$

$$+\mu\{\max[V_{n-1}(x,2),V_{n-1}(x+1,2)-c]-\max[V_{n-1}(x-1,2),V_{n-1}(x,2)-c]\}$$

$$-\mu\{\max[V_{n-1}(x,1),V_{n-1}(x+1,1)-c]-\max[V_{n-1}(x-1,1),V_{n-1}(x,1)-c]\}$$

$$+q_{2,1}\Delta V_{n-1}(x,1)+q_{2,3}\Delta V_{n-1}(x,3)+q_{2,4}\Delta V_{n-1}(x,4)$$

$$-\{q_{1,2}\Delta V_{n-1}(x,2)+q_{1,3}\Delta V_{n-1}(x,3)+q_{1,4}\Delta V_{n-1}(x,4)\}$$

$$+(\sum_{i\neq 2}\lambda_i+\sum_{i\neq 2}\sum_{j\neq i}q_{ij})\Delta V_{n-1}(x,2)-(\sum_{i\neq 1}\lambda_i+\sum_{i\neq 1}\sum_{j\neq i}q_{ij})\Delta V_{n-1}(x,1)$$

$$=\lambda_2\overline{F}(p)\Delta V_{n-1}(x-1,2)-\lambda_1\overline{F}(p)\Delta V_{n-1}(x-1,1)$$

$$+\lambda_2[1-\overline{F}(p)]\Delta V_{n-1}(x,2)-\lambda_1[1-\overline{F}(p)]\Delta V_{n-1}(x,1)$$

$$+\mu\{\max[V_{n-1}(x,2),V_{n-1}(x+1,2)-c]-\max[V_{n-1}(x-1,2),V_{n-1}(x,2)-c]\}$$

$$-\mu\{\max[V_{n-1}(x,1),V_{n-1}(x+1,1)-c]-\max[V_{n-1}(x-1,1),V_{n-1}(x,1)-c]\}$$

$$+q_{2,1}\Delta V_{n-1}(x,1)+q_{2,3}\Delta V_{n-1}(x,3)+q_{2,4}\Delta V_{n-1}(x,4)$$

$$-\{q_{1,2}\Delta V_{n-1}(x,2)+q_{1,3}\Delta V_{n-1}(x,3)+q_{1,4}\Delta V_{n-1}(x,4)\}$$

$$+\sum_{i\neq 2}\lambda_i\Delta V_{n-1}(x,2)-\sum_{i\neq 1}\lambda_i\Delta V_{n-1}(x,1)+\sum_{i=3,4}\sum_{j\neq i}q_{ij}(\Delta V_{n-1}(x,2)-\Delta V_{n-1}(x,1))$$

$$+q_{1,2}\Delta V_{n-1}(x,2)+q_{1,3}\Delta V_{n-1}(x,2)+q_{1,4}\Delta V_{n-1}(x,2)-\{q_{2,1}\Delta V_{n-1}(x,1)+q_{2,3}\Delta V_{n-1}(x,1)+q_{2,4}\Delta V_{n-1}(x,1)\}$$

$$=(\lambda_2-\lambda_1)\overline{F}(p)\Delta V_{n-1}(x-1,2)+\lambda_1\overline{F}(p)\Delta V_{n-1}(x-1,2)-\lambda_1\overline{F}(p)\Delta V_{n-1}(x-1,1)$$

$$+(\lambda_2-\lambda_1)[1-\overline{F}(p)]\Delta V_{n-1}(x,2)+\lambda_1[1-\overline{F}(p)]\Delta V_{n-1}(x,2)-\lambda_1[1-\overline{F}(p)]\Delta V_{n-1}(x,1)$$

$$+\mu\{\max[V_{n-1}(x,2),V_{n-1}(x+1,2)-c]-\max[V_{n-1}(x-1,1),V_{n-1}(x,1)-c]\}$$

$$-\mu\{\max[V_{n-1}(x,1),V_{n-1}(x+1,1)-c]-\max[V_{n-1}(x-1,1),V_{n-1}(x,1)-c]\}$$
$$+q_{2,3}\{\Delta V_{n-1}(x,3)-\Delta V_{n-1}(x,1)\}+q_{2,4}\{\Delta V_{n-1}(x,4)-\Delta V_{n-1}(x,1)\}$$
$$+q_{1,3}\{\Delta V_{n-1}(x,2)-\Delta V_{n-1}(x,3)\}+q_{1,4}\{\Delta V_{n-1}(x,2)-\Delta V_{n-1}(x,4)\}$$
$$+(\lambda_3+\lambda_4+\sum_{i=3,4}\sum_{j\neq i}q_{ij})\{\Delta V_{n-1}(x,2)-\Delta V_{n-1}(x,1)\}+\lambda_1\Delta V_{n-1}(x,2)-\lambda_2\Delta V_{n-1}(x,1)$$
$$\geq (\lambda_2-\lambda_1)\overline{F}(p)\Delta V_{n-1}(x,2)+\lambda_1\overline{F}(p)\Delta V_{n-1}(x-1,2)-\lambda_1\overline{F}(p)\Delta V_{n-1}(x-1,1)$$
$$+(\lambda_2-\lambda_1)[1-\overline{F}(p)]\Delta V_{n-1}(x,2)+\lambda_1[1-\overline{F}(p)]\Delta V_{n-1}(x,2)-\lambda_1[1-\overline{F}(p)]\Delta V_{n-1}(x,1)$$
$$+\mu\{\max[V_{n-1}(x,2),V_{n-1}(x+1,2)-c]-\max[V_{n-1}(x-1,1),V_{n-1}(x,1)-c]\}$$
$$-\mu\{\max[V_{n-1}(x,1),V_{n-1}(x+1,1)-c]-\max[V_{n-1}(x-1,1),V_{n-1}(x,1)-c]\}$$
$$+q_{2,3}\{\Delta V_{n-1}(x,3)-\Delta V_{n-1}(x,1)\}+q_{2,4}\{\Delta V_{n-1}(x,4)-\Delta V_{n-1}(x,1)\}$$
$$+q_{1,3}\{\Delta V_{n-1}(x,2)-\Delta V_{n-1}(x,3)\}+q_{1,4}\{\Delta V_{n-1}(x,2)-\Delta V_{n-1}(x,4)\}$$
$$+(\lambda_3+\lambda_4+\sum_{i=3,4}\sum_{j\neq i}q_{ij})\{\Delta V_{n-1}(x,2)-\Delta V_{n-1}(x,1)\}+\lambda_1\Delta V_{n-1}(x,2)-\lambda_2\Delta V_{n-1}(x,1)$$
$$= \lambda_2\{\Delta V_{n-1}(x,2)-\Delta V_{n-1}(x,1)\}+\lambda_1\overline{F}(p)\{\Delta V_{n-1}(x-1,2)-\Delta V_{n-1}(x-1,1)\}$$
$$+\lambda_1[1-\overline{F}(p)]\{\Delta V_{n-1}(x,2)-\Delta V_{n-1}(x,1)\}$$
$$+\mu\{\max[V_{n-1}(x,2),V_{n-1}(x+1,2)-c]-\max[V_{n-1}(x-1,1),V_{n-1}(x,1)-c]\}$$
$$-\mu\{\max[V_{n-1}(x,1),V_{n-1}(x+1,1)-c]-\max[V_{n-1}(x-1,1),V_{n-1}(x,1)-c]\}$$
$$+(\lambda_3+\lambda_4+\sum_{i=3,4}\sum_{j\neq i}q_{ij})\{\Delta V_{n-1}(x,2)-\Delta V_{n-1}(x,1)\}$$
$$+q_{2,3}\{\Delta V_{n-1}(x,3)-\Delta V_{n-1}(x,1)\}+q_{2,4}\{\Delta V_{n-1}(x,4)-\Delta V_{n-1}(x,1)\}$$
$$+q_{1,3}\{\Delta V_{n-1}(x,2)-\Delta V_{n-1}(x,3)\}+q_{1,4}\{\Delta V_{n-1}(x,2)-\Delta V_{n-1}(x,4)\}$$

The inequality above holds since $\Delta V_{n-1}(x-1,2)-\Delta V_{n-1}(x,2)\geq 0$ for all x by Lemma 2, by the induction assumption, and

$$\begin{aligned}&\{\max[V_{n-1}(x,2),V_{n-1}(x+1,2)-c]-\max[V_{n-1}(x-1,2),V_{n-1}(x,2)-c]\}\\&-\{\max[V_{n-1}(x,1),V_{n-1}(x+1,1)-c]-\max[V_{n-1}(x-1,1),V_{n-1}(x,1)-c]\}\geq 0\end{aligned} \quad \text{(B.6)}$$

by the inequalities in Table 5 for e = 1 given that Condition 2 holds for e = 1. That is, the following inequalities must hold $q_{2,3} \geq q_{1,3}$, $q_{2,4} \geq q_{1,4}$. Thus, we have shown that Lemma 4 is true for $x \geq 2$.

For $x = 1$,

$$
\begin{aligned}
&(1)\, \Delta V_n(1,4) - \Delta V_n(1,3) \\
&= [V_n(1,4) - V_n(0,4)] - [V_n(1,3) - V_n(0,3)] \qquad \text{(B.7)} \\
&= [[(-h(1) + \lambda_4 \overline{F}(p)p + \lambda_4 \overline{F}(p)V_{n-1}(0,4) + \lambda_4[1-\overline{F}(p)]V_{n-1}(1,4) + \mu \max\{V_{n-1}(1,4), V_{n-1}(2,4) - c\} \\
&+ q_{41}V_{n-1}(1,1) + q_{42}V_{n-1}(1,2) + q_{43}V_{n-1}(1,3) + (\sum_{k=1}^{3} \lambda_k + \sum_{i=1}^{3} \sum_{j \neq i} q_{ij})V_n(1,4)] \\
&-[(-h(0) + \lambda_4 \overline{F}(p)V_{n-1}(0,4) + \lambda_4[1-\overline{F}(p)]V_{n-1}(0,4) + \mu \max\{V_{n-1}(0,4), V_{n-1}(1,4) - c\} \\
&+ q_{41}V_{n-1}(0,1) + q_{42}V_{n-1}(0,2) + q_{43}V_{n-1}(0,3) + (\sum_{k=1}^{3} \lambda_k + \sum_{i=1}^{3} \sum_{j \neq i} q_{ij})V_n(0,4)]] \\
&-[[(-h(1) + \lambda_3 \overline{F}(p)p + \lambda_3 \overline{F}(p)V_{n-1}(0,3) + \lambda_3[1-\overline{F}(p)]V_{n-1}(1,3) + \mu \max\{V_{n-1}(1,3), V_{n-1}(2,3) - c\} \\
&+ q_{31}V_{n-1}(1,1) + q_{32}V_{n-1}(1,2) + q_{34}V_{n-1}(1,4) + (\sum_{k \neq 3} \lambda_k + \sum_{i \neq 3} \sum_{j \neq i} q_{ij})V_n(1,3)] \\
&-[(-h(0) + \lambda_3 \overline{F}(p)V_{n-1}(0,3) + \lambda_3[1-\overline{F}(p)]V_{n-1}(0,3) + \mu \max\{V_{n-1}(0,3), V_{n-1}(1,3) - c\} \\
&+ q_{31}V_{n-1}(0,1) + q_{32}V_{n-1}(0,2) + q_{34}V_{n-1}(0,4) + (\sum_{k \neq 3} \lambda_k + \sum_{i \neq 3} \sum_{j \neq i} q_{ij})V_n(0,3)]] \\
&= (\lambda_4 - \lambda_3)\overline{F}(p)p \\
&+ \lambda_4[1-\overline{F}(p)]\Delta V_{n-1}(1,4) - \lambda_3[1-\overline{F}(p)]\Delta V_{n-1}(1,3) \\
&+ \mu\{\max[V_{n-1}(1,4), V_{n-1}(2,4) - c] - \max[V_{n-1}(0,4), V_{n-1}(1,4) - c]\} \\
&- \mu\{\max[V_{n-1}(1,3), V_{n-1}(2,3) - c] - \max[V_{n-1}(0,3), V_{n-1}(1,3) - c]\} \\
&+ q_{41}\Delta V_{n-1}(1,1) + q_{42}\Delta V_{n-1}(1,2) + q_{43}\Delta V_{n-1}(1,3) + (\sum_{k=1}^{3} \lambda_k + \sum_{i=1}^{3} \sum_{j \neq i} q_{ij})\Delta V_n(1,4) \\
&-[q_{31}\Delta V_{n-1}(1,1) + q_{32}\Delta V_{n-1}(1,2) + q_{34}\Delta V_{n-1}(1,4) + (\sum_{k \neq 3} \lambda_k + \sum_{i \neq 3} \sum_{j \neq i} q_{ij})\Delta V_n(1,3)] \\
&= (\lambda_4 - \lambda_3)\overline{F}(p)p
\end{aligned}
$$

$$+(\lambda_4-\lambda_3)[1-\overline{F}(p)]\Delta V_{n-1}(1,4)+\lambda_3[1-\overline{F}(p)]\{\Delta V_{n-1}(1,4)-\Delta V_{n-1}(1,3)\}$$
$$+\mu\{\max[V_{n-1}(1,4),V_{n-1}(2,4)-c]-\max[V_{n-1}(0,4),V_{n-1}(1,4)-c]\}$$
$$-\mu\{\max[V_{n-1}(1,3),V_{n-1}(2,3)-c]-\max[V_{n-1}(0,3),V_{n-1}(1,3)-c]\}$$
$$+q_{41}\Delta V_{n-1}(1,1)+q_{42}\Delta V_{n-1}(1,2)+q_{43}\Delta V_{n-1}(1,3)+(\sum_{k=1}^{3}\lambda_k+\sum_{i=1}^{2}\sum_{j\neq i}q_{ij})\Delta V_n(1,4)+q_{31}\Delta V_{n-1}(1,4)+q_{32}\Delta V_{n-1}(1,4)+q_{34}\Delta V_{n-1}(1,4)$$
$$-[q_{31}\Delta V_{n-1}(1,1)+q_{32}\Delta V_{n-1}(1,2)+q_{34}\Delta V_{n-1}(1,4)+(\sum_{k\neq 3}\lambda_k+\sum_{i=1}^{2}\sum_{j\neq i}q_{ij})\Delta V_n(1,3)+q_{41}\Delta V_{n-1}(1,3)+q_{42}\Delta V_{n-1}(1,3)+q_{43}\Delta V_{n-1}(1,3)]$$

$$=(\lambda_4-\lambda_3)\overline{F}(p)\{p-\Delta V_{n-1}(1,4)\}+\lambda_3\Delta V_{n-1}(1,4)-\lambda_4\Delta V_{n-1}(1,3)$$
$$+(\lambda_4-\lambda_3)\Delta V_{n-1}(1,4)+\lambda_3[1-\overline{F}(p)]\{\Delta V_{n-1}(1,4)-\Delta V_{n-1}(1,3)\}$$
$$+\mu\{\max[V_{n-1}(1,4),V_{n-1}(2,4)-c]-\max[V_{n-1}(0,4),V_{n-1}(1,4)-c]\}$$
$$-\mu\{\max[V_{n-1}(1,3),V_{n-1}(2,3)-c]-\max[V_{n-1}(0,3),V_{n-1}(1,3)-c]\}$$
$$+q_{41}\Delta V_{n-1}(1,1)+q_{42}\Delta V_{n-1}(1,2)+q_{43}\Delta V_{n-1}(1,3)+(\sum_{k=1}^{2}\lambda_k+\sum_{i=1}^{2}\sum_{j\neq i}q_{ij})\Delta V_n(1,4)+q_{31}\Delta V_{n-1}(1,4)+q_{32}\Delta V_{n-1}(1,4)+q_{34}\Delta V_{n-1}(1,4)$$
$$-[q_{31}\Delta V_{n-1}(1,1)+q_{32}\Delta V_{n-1}(1,2)+q_{34}\Delta V_{n-1}(1,4)+(\sum_{k=1}^{2}\lambda_k+\sum_{i=1,2}\sum_{j\neq i}q_{ij})\Delta V_n(1,3)+q_{41}\Delta V_{n-1}(1,3)+q_{42}\Delta V_{n-1}(1,3)+q_{43}\Delta V_{n-1}(1,3)]$$

$$=(\lambda_4-\lambda_3)\overline{F}(p)\{p-\Delta V_{n-1}(1,4)\}+\lambda_4\{\Delta V_{n-1}(1,4)-\Delta V_{n-1}(1,3)\}$$
$$+\lambda_3[1-\overline{F}(p)]\{\Delta V_{n-1}(1,4)-\Delta V_{n-1}(1,3)\}$$
$$+\mu\{\max[V_{n-1}(1,4),V_{n-1}(2,4)-c]-\max[V_{n-1}(0,4),V_{n-1}(1,4)-c]\}$$
$$-\mu\{\max[V_{n-1}(1,3),V_{n-1}(2,3)-c]-\max[V_{n-1}(0,3),V_{n-1}(1,3)-c]\}$$
$$+q_{41}\Delta V_{n-1}(1,1)+q_{42}\Delta V_{n-1}(1,2)+(\sum_{k=1}^{2}\lambda_k+\sum_{i=1}^{2}\sum_{j\neq i}q_{ij})\Delta V_n(1,4)+q_{31}\Delta V_{n-1}(1,4)+q_{32}\Delta V_{n-1}(1,4)$$
$$-[q_{31}\Delta V_{n-1}(1,1)+q_{32}\Delta V_{n-1}(1,2)+(\sum_{k=1}^{2}\lambda_k+\sum_{i=1,2}\sum_{j\neq i}q_{ij})\Delta V_n(1,3)+q_{41}\Delta V_{n-1}(1,3)+q_{42}\Delta V_{n-1}(1,3)]$$

$$=(\lambda_4-\lambda_3)\overline{F}(p)\{p-\Delta V_{n-1}(1,4)\}+\lambda_4\{\Delta V_{n-1}(1,4)-\Delta V_{n-1}(1,3)\}$$
$$+\lambda_3[1-\overline{F}(p)]\{\Delta V_{n-1}(1,4)-\Delta V_{n-1}(1,3)\}$$
$$+\mu\{\max[V_{n-1}(1,4),V_{n-1}(2,4)-c]-\max[V_{n-1}(0,4),V_{n-1}(1,4)-c]\}$$
$$-\mu\{\max[V_{n-1}(1,3),V_{n-1}(2,3)-c]-\max[V_{n-1}(0,3),V_{n-1}(1,3)-c]\}$$
$$+(\sum_{k=1}^{2}\lambda_k+\sum_{i=1}^{2}\sum_{j\neq i}q_{ij})\{\Delta V_n(1,4)-\Delta V_n(1,3)\}$$

$$+q_{41}\{\Delta V_{n-1}(1,1)-\Delta V_{n-1}(1,3)\}+q_{42}\{\Delta V_{n-1}(1,2)-\Delta V_{n-1}(1,3)\}$$
$$+q_{31}\{\Delta V_{n-1}(1,4)-\Delta V_{n-1}(1,1)\}+q_{32}\{\Delta V_{n-1}(1,4)-\Delta V_{n-1}(1,2)\}\geq 0$$

The inequality above holds since by the second part of Lemma 2, by the induction assumption, and

$$\begin{aligned}&\{\max[V_{n-1}(1,4),V_{n-1}(2,4)-c]-\max[V_{n-1}(0,4),V_{n-1}(1,4)-c]\}\\&-\{\max[V_{n-1}(1,3),V_{n-1}(2,3)-c]-\max[V_{n-1}(0,3),V_{n-1}(1,3)-c]\}\geq 0\end{aligned} \tag{B.8}$$

by the inequalities in Table 5 for e = 3 given that Condition 2 holds for e = 3. That is, the following inequalities must hold $q_{3,1}\geq q_{4,1}$, $q_{3,2}\geq q_{4,2}$.

(2) $\Delta V_n(1,3)-\Delta V_n(1,2)$

$$=[V_n(1,3)-V_n(0,3)]-[V_n(1,2)-V_n(0,2)] \tag{B.9}$$

$$=[[(-h(1)+\lambda_3\overline{F}(p)p+\lambda_3\overline{F}(p)V_{n-1}(0,3)+\lambda_3[1-\overline{F}(p)]V_{n-1}(1,3)+\mu\max\{V_{n-1}(1,3),V_{n-1}(2,3)-c\}$$
$$+q_{31}V_{n-1}(1,1)+q_{32}V_{n-1}(1,2)+q_{34}V_{n-1}(1,4)+(\sum_{k\neq 3}\lambda_k+\sum_{i\neq 3}\sum_{j\neq i}q_{ij})V_n(1,3)]$$
$$-[(-h(0)+\lambda_3\overline{F}(p)V_{n-1}(0,3)+\lambda_3[1-\overline{F}(p)]V_{n-1}(0,3)+\mu\max\{V_{n-1}(0,3),V_{n-1}(1,3)-c\}$$
$$+q_{31}V_{n-1}(0,1)+q_{32}V_{n-1}(0,2)+q_{34}V_{n-1}(0,4)+(\sum_{k\neq 3}\lambda_k+\sum_{i\neq 3}\sum_{j\neq i}q_{ij})V_n(0,3)]]$$
$$-[[(-h(1)+\lambda_2\overline{F}(p)p+\lambda_2\overline{F}(p)V_{n-1}(0,2)+\lambda_2[1-\overline{F}(p)]V_{n-1}(1,2)+\mu\max\{V_{n-1}(1,2),V_{n-1}(2,2)-c\}$$
$$+q_{21}V_{n-1}(1,1)+q_{23}V_{n-1}(1,3)+q_{24}V_{n-1}(1,4)+(\sum_{k\neq 2}\lambda_k+\sum_{i\neq 2}\sum_{j\neq i}q_{ij})V_n(1,2)]$$
$$-[(-h(0)+\lambda_2\overline{F}(p)V_{n-1}(0,2)+\lambda_2[1-\overline{F}(p)]V_{n-1}(0,2)+\mu\max\{V_{n-1}(0,2),V_{n-1}(1,2)-c\}$$
$$+q_{21}V_{n-1}(0,1)+q_{23}V_{n-1}(0,3)+q_{24}V_{n-1}(0,4)+(\sum_{k\neq 2}\lambda_k+\sum_{i\neq 2}\sum_{j\neq i}q_{ij})V_n(0,2)]]$$
$$=(\lambda_3-\lambda_2)\overline{F}(p)p$$
$$+\lambda_3[1-\overline{F}(p)]\Delta V_{n-1}(1,3)-\lambda_2[1-\overline{F}(p)]\Delta V_{n-1}(1,2)$$

$$+\mu\{\max[V_{n-1}(1,3),V_{n-1}(2,3)-c]-\max[V_{n-1}(0,3),V_{n-1}(1,3)-c]\}$$
$$-\mu\{\max[V_{n-1}(1,2),V_{n-1}(2,2)-c]-\max[V_{n-1}(0,2),V_{n-1}(1,2)-c]\}$$
$$+q_{31}\Delta V_{n-1}(1,1)+q_{32}\Delta V_{n-1}(1,2)+q_{34}\Delta V_{n-1}(1,4)+(\sum_{k\neq 3}\lambda_k+\sum_{i\neq 3}\sum_{j\neq i}q_{ij})\Delta V_n(1,3)$$
$$-[q_{21}\Delta V_{n-1}(1,1)+q_{23}\Delta V_{n-1}(1,3)+q_{24}\Delta V_{n-1}(1,4)+(\sum_{k\neq 2}\lambda_k+\sum_{i\neq 2}\sum_{j\neq i}q_{ij})\Delta V_n(1,2)]$$
$$=(\lambda_3-\lambda_2)\overline{F}(p)p+\lambda_2\Delta V_{n-1}(1,3)-\lambda_3\Delta V_{n-1}(1,2)$$
$$+(\lambda_3-\lambda_2)\Delta V_{n-1}(1,3)-(\lambda_3-\lambda_2)\overline{F}(p)\Delta V_{n-1}(1,3)+\lambda_2[1-\overline{F}(p)]\{\Delta V_{n-1}(1,3)-\Delta V_{n-1}(1,2)\}$$
$$+\mu\{\max[V_{n-1}(1,3),V_{n-1}(2,3)-c]-\max[V_{n-1}(0,3),V_{n-1}(1,3)-c]\}$$
$$-\mu\{\max[V_{n-1}(1,2),V_{n-1}(2,2)-c]-\max[V_{n-1}(0,2),V_{n-1}(1,2)-c]\}$$
$$+q_{31}\Delta V_{n-1}(1,1)+q_{32}\Delta V_{n-1}(1,2)+q_{34}\Delta V_{n-1}(1,4)+(\sum_{k=1,4}\lambda_k+\sum_{i=1,4}\sum_{j\neq i}q_{ij})\Delta V_n(1,3)+q_{21}\Delta V_n(1,3)+q_{23}\Delta V_n(1,3)+q_{24}\Delta V_n(1,3)$$
$$-[q_{21}\Delta V_{n-1}(1,1)+q_{23}\Delta V_{n-1}(1,3)+q_{24}\Delta V_{n-1}(1,4)+(\sum_{k=1,4}\lambda_k+\sum_{i=1,4}\sum_{j\neq i}q_{ij})\Delta V_n(1,2)+q_{31}\Delta V_n(1,2)+q_{32}\Delta V_n(1,2)+q_{34}\Delta V_n(1,2)]$$
$$=(\lambda_3-\lambda_2)\overline{F}(p)\{p-\Delta V_{n-1}(1,3)\}+\lambda_2\Delta V_{n-1}(1,3)-\lambda_3\Delta V_{n-1}(1,2)$$
$$+(\lambda_3-\lambda_2)\Delta V_{n-1}(1,3)+\lambda_2[1-\overline{F}(p)]\{\Delta V_{n-1}(1,3)-\Delta V_{n-1}(1,2)\}$$
$$+\mu\{\max[V_{n-1}(1,3),V_{n-1}(2,3)-c]-\max[V_{n-1}(0,3),V_{n-1}(1,3)-c]\}$$
$$-\mu\{\max[V_{n-1}(1,2),V_{n-1}(2,2)-c]-\max[V_{n-1}(0,2),V_{n-1}(1,2)-c]\}$$
$$+q_{31}\Delta V_{n-1}(1,1)+q_{34}\Delta V_{n-1}(1,4)+(\sum_{k=1,4}\lambda_k+\sum_{i=1,4}\sum_{j\neq i}q_{ij})\Delta V_n(1,3)+q_{21}\Delta V_n(1,3)+q_{24}\Delta V_n(1,3)$$
$$-[q_{21}\Delta V_{n-1}(1,1)+q_{24}\Delta V_{n-1}(1,4)+(\sum_{k=1,4}\lambda_k+\sum_{i=1,4}\sum_{j\neq i}q_{ij})\Delta V_n(1,2)+q_{31}\Delta V_n(1,2)+q_{34}\Delta V_n(1,2)]$$
$$=(\lambda_3-\lambda_2)\overline{F}(p)\{p-\Delta V_{n-1}(1,3)\}+\lambda_3\{\Delta V_{n-1}(1,3)-\Delta V_{n-1}(1,2)\}$$
$$+\lambda_2[1-\overline{F}(p)]\{\Delta V_{n-1}(1,3)-\Delta V_{n-1}(1,2)\}$$
$$+\mu\{\max[V_{n-1}(1,3),V_{n-1}(2,3)-c]-\max[V_{n-1}(0,3),V_{n-1}(1,3)-c]\}$$
$$-\mu\{\max[V_{n-1}(1,2),V_{n-1}(2,2)-c]-\max[V_{n-1}(0,2),V_{n-1}(1,2)-c]\}$$
$$+(\sum_{k=1,4}\lambda_k+\sum_{i=1,4}\sum_{j\neq i}q_{ij})\{\Delta V_n(1,3)-\Delta V_n(1,2)\}$$
$$+q_{31}\{\Delta V_{n-1}(1,1)-\Delta V_{n-1}(1,2)\}+q_{34}\{\Delta V_{n-1}(1,4)-\Delta V_{n-1}(1,2)\}+q_{21}\{\Delta V_n(1,3)-\Delta V_n(1,1)\}+q_{24}\{\Delta V_n(1,3)-\Delta V_n(1,4)\}\geq 0$$

The inequality above holds by the second part of Lemma 2, by the induction assumption and

$$\begin{aligned}&\{\max[V_{n-1}(1,3),V_{n-1}(2,3)-c]-\max[V_{n-1}(0,3),V_{n-1}(1,3)-c]\}\\&-\{\max[V_{n-1}(1,2),V_{n-1}(2,2)-c]-\max[V_{n-1}(0,2),V_{n-1}(1,2)-c]\}\geq 0\end{aligned} \quad \text{(B.10)}$$

by the inequalities in Table 5 for $e = 2$ given that Condition 2 holds for $e = 2$. That is, the following inequalities must hold $q_{2,1} \geq q_{3,1}$, $q_{3,4} \geq q_{2,4}$.

(3) $\Delta V_n(1,2) - \Delta V_n(1,1)$

$$= [V_n(1,2) - V_n(0,2)] - [V_n(1,1) - V_n(0,1)] \quad \text{(B.11)}$$

$$= [[(-h(1) + \lambda_2 \overline{F}(p)p + \lambda_2 \overline{F}(p)V_{n-1}(0,2) + \lambda_2[1-\overline{F}(p)]V_{n-1}(1,2) + \mu\max\{V_{n-1}(1,2), V_{n-1}(2,2)-c\}$$

$$+q_{21}V_{n-1}(1,1) + q_{23}V_{n-1}(1,3) + q_{24}V_{n-1}(1,4) + (\sum_{k\neq 2}\lambda_k + \sum_{i\neq 2}\sum_{j\neq i} q_{ij})V_n(1,2)]$$

$$-[(-h(0) + \lambda_2\overline{F}(p)V_{n-1}(0,2) + \lambda_2[1-\overline{F}(p)]V_{n-1}(0,2) + \mu\max\{V_{n-1}(0,2), V_{n-1}(1,2)-c\}$$

$$+q_{21}V_{n-1}(0,1) + q_{23}V_{n-1}(0,3) + q_{24}V_{n-1}(0,4) + (\sum_{k\neq 2}\lambda_k + \sum_{i\neq 2}\sum_{j\neq i} q_{ij})V_n(0,2)]]$$

$$-[[(-h(1) + \lambda_1 \overline{F}(p)p + \lambda_1 \overline{F}(p)V_{n-1}(0,1) + \lambda_1[1-\overline{F}(p)]V_{n-1}(1,1) + \mu\max\{V_{n-1}(1,1), V_{n-1}(2,1)-c\}$$

$$+q_{12}V_{n-1}(1,2) + q_{13}V_{n-1}(1,3) + q_{14}V_{n-1}(1,4) + (\sum_{k\neq 1}\lambda_k + \sum_{i\neq 1}\sum_{j\neq i} q_{ij})V_n(1,1)]$$

$$-[(-h(0) + \lambda_1\overline{F}(p)V_{n-1}(0,1) + \lambda_1[1-\overline{F}(p)]V_{n-1}(0,1) + \mu\max\{V_{n-1}(0,1), V_{n-1}(1,1)-c\}$$

$$+q_{12}V_{n-1}(0,2) + q_{13}V_{n-1}(0,3) + q_{14}V_{n-1}(0,4) + (\sum_{k\neq 1}\lambda_k + \sum_{i\neq 1}\sum_{j\neq i} q_{ij})V_n(0,1)]]$$

$$= (\lambda_2 - \lambda_1)\overline{F}(p)p$$

$$+\lambda_2[1-\overline{F}(p)]\Delta V_{n-1}(1,2) - \lambda_1[1-\overline{F}(p)]\Delta V_{n-1}(1,1)$$

$$+\mu\{\max[V_{n-1}(1,2), V_{n-1}(2,2)-c] - \max[V_{n-1}(0,2), V_{n-1}(1,2)-c]\}$$

$$-\mu\{\max[V_{n-1}(1,1), V_{n-1}(2,1)-c] - \max[V_{n-1}(0,1), V_{n-1}(1,1)-c]\}$$

$$+q_{21}\Delta V_{n-1}(1,1)+q_{23}\Delta V_{n-1}(1,3)+q_{24}\Delta V_{n-1}(1,4)+(\sum_{k\neq 2}\lambda_k+\sum_{i\neq 2}\sum_{j\neq i}q_{ij})\Delta V_n(1,2)$$

$$-[q_{12}\Delta V_{n-1}(1,2)+q_{13}\Delta V_{n-1}(1,3)+q_{14}\Delta V_{n-1}(1,4)+(\sum_{k\neq 1}\lambda_k+\sum_{i\neq 1}\sum_{j\neq i}q_{ij})\Delta V_n(1,1)]$$

$$=(\lambda_2-\lambda_1)\overline{F}(p)p+\lambda_1\Delta V_{n-1}(1,2)-\lambda_2\Delta V_{n-1}(1,1)$$

$$+(\lambda_2-\lambda_1)\Delta V_{n-1}(1,2)-(\lambda_2-\lambda_1)\overline{F}(p)\Delta V_{n-1}(1,2)+\lambda_1[1-\overline{F}(p)]\{\Delta V_{n-1}(1,2)-\Delta V_{n-1}(1,1)\}$$

$$+\mu\{\max[V_{n-1}(1,2),V_{n-1}(2,2)-c]-\max[V_{n-1}(0,2),V_{n-1}(1,2)-c]\}$$

$$-\mu\{\max[V_{n-1}(1,1),V_{n-1}(2,1)-c]-\max[V_{n-1}(0,1),V_{n-1}(1,1)-c]\}$$

$$+q_{21}\Delta V_{n-1}(1,1)+q_{23}\Delta V_{n-1}(1,3)+q_{24}\Delta V_{n-1}(1,4)+(\sum_{k=3,4}\lambda_k+\sum_{i=3,4}\sum_{j\neq i}q_{ij})\Delta V_n(1,2)+q_{12}\Delta V_n(1,2)+q_{13}\Delta V_n(1,2)+q_{14}\Delta V_n(1,2)$$

$$-[q_{12}\Delta V_{n-1}(1,2)+q_{13}\Delta V_{n-1}(1,3)+q_{14}\Delta V_{n-1}(1,4)+(\sum_{k=3,4}\lambda_k+\sum_{i=3,4}\sum_{j\neq i}q_{ij})\Delta V_n(1,1)+q_{21}\Delta V_n(1,1)+q_{23}\Delta V_n(1,1)+q_{24}\Delta V_n(1,1)]$$

$$=(\lambda_2-\lambda_1)\overline{F}(p)\{p-\Delta V_{n-1}(1,2)\}+\lambda_1\Delta V_{n-1}(1,2)-\lambda_2\Delta V_{n-1}(1,1)$$

$$+(\lambda_2-\lambda_1)\Delta V_{n-1}(1,2)+\lambda_1[1-\overline{F}(p)]\{\Delta V_{n-1}(1,2)-\Delta V_{n-1}(1,1)\}$$

$$+\mu\{\max[V_{n-1}(1,2),V_{n-1}(2,2)-c]-\max[V_{n-1}(0,2),V_{n-1}(1,2)-c]\}$$

$$-\mu\{\max[V_{n-1}(1,1),V_{n-1}(2,1)-c]-\max[V_{n-1}(0,1),V_{n-1}(1,1)-c]\}$$

$$+q_{23}\Delta V_{n-1}(1,3)+q_{24}\Delta V_{n-1}(1,4)+(\sum_{k=3,4}\lambda_k+\sum_{i=3,4}\sum_{j\neq i}q_{ij})\Delta V_n(1,2)+q_{13}\Delta V_n(1,2)+q_{14}\Delta V_n(1,2)$$

$$-[q_{13}\Delta V_{n-1}(1,3)+q_{14}\Delta V_{n-1}(1,4)+(\sum_{k=3,4}\lambda_k+\sum_{i=3,4}\sum_{j\neq i}q_{ij})\Delta V_n(1,1)+q_{23}\Delta V_n(1,1)+q_{24}\Delta V_n(1,1)]$$

$$=(\lambda_2-\lambda_1)\overline{F}(p)\{p-\Delta V_{n-1}(1,2)\}+\lambda_2\{\Delta V_{n-1}(1,2)-\Delta V_{n-1}(1,1)\}$$

$$+\lambda_1[1-\overline{F}(p)]\{\Delta V_{n-1}(1,2)-\Delta V_{n-1}(1,1)\}$$

$$+\mu\{\max[V_{n-1}(1,2),V_{n-1}(2,2)-c]-\max[V_{n-1}(0,2),V_{n-1}(1,2)-c]\}$$

$$-\mu\{\max[V_{n-1}(1,1),V_{n-1}(2,1)-c]-\max[V_{n-1}(0,1),V_{n-1}(1,1)-c]\}$$

$$+(\sum_{k=3,4}\lambda_k+\sum_{i=3,4}\sum_{j\neq i}q_{ij})\{\Delta V_n(1,2)-\Delta V_n(1,1)\}$$

$$+q_{13}\{\Delta V_{n-1}(1,2)-\Delta V_{n-1}(1,3)\}+q_{14}\{\Delta V_{n-1}(1,2)-\Delta V_{n-1}(1,4)\}+q_{23}\{\Delta V_n(1,3)-\Delta V_n(1,1)\}+q_{24}\{\Delta V_n(1,4)-\Delta V_n(1,1)\}\geq 0$$

The inequality above holds by the second part of Lemma 2, by the induction assumption and

$$\begin{aligned}&\{\max[V_{n-1}(1,2),V_{n-1}(2,2)-c]-\max[V_{n-1}(0,2),V_{n-1}(1,2)-c]\}\\&-\{\max[V_{n-1}(1,1),V_{n-1}(2,1)-c]-\max[V_{n-1}(0,1),V_{n-1}(1,1)-c]\}\geq 0\end{aligned} \quad \text{(B.12)}$$

by the inequalities in Table 5 for e = 1 given that Condition 2 holds for e = 1. That is, the following inequalities must hold $q_{2,3} \geq q_{1,3}$, $q_{2,4} \geq q_{1,4}$.

Proposition 1 The base stock levels of the four environmental states induce the same ordering with the demand rates of these states. That is, $(S_4^* \geq S_3^* \geq S_2^* \geq S_1^*)$ where $\lambda_4 \geq \lambda_3 \geq \lambda_2 \geq \lambda_1$.

Proof. The proposition is directly implied by Theorem 2 and Lemma 4.

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