Single Period Stochastic Inventory Problem with Downside Risk Considerations

by

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This is to certify that I have examined this copy of a master's thesis by

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ABSTRACT

In this thesis, we study the single period stochastic inventory (newsvendor) problem with downside risk constraints. The aim in the classical newsvendor problem is maximizing the expected profit. In the classical problem, the risk of earning less than the desired target profit or losing more than an acceptable level due to the randomness of demand is not taken into account. In the literature, there are some models for maximizing the profit while controlling the financial risk or for only controlling the financial risk. In this study, we utilize Value at Risk (VaR) and Conditional Value at Risk (CVaR) as the risk measures in newsvendor framework. In this study, we investigate the multi-product newsvendor problem under VaR and CVaR constraints. Analytical results for two products are obtained. An approximation method for the *N* product problem is also developed. Effects of the system parameters on the optimum product quantities are investigated numerically. In the second part of the thesis, the inventory problem with the objective of minimizing CVaR is analyzed. By using the LP representation of the CVaR problem, this model is solved for *N* products where the profit function has an arbitrary distribution function. The numerical studies are performed for both risk measures.

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NOMENCLATURE

r:	Unit selling price of a product
c:	Unit variable cost of a product
s:	Unit salvage price of a product
r_i:	Unit selling price of product i
c_i:	Unit variable cost of product i
s_i:	Unit salvage price of product i
c_u:	Underage cost that equals r-c
c_o:	Overage cost that equals c-s
c_{u_i}:	Underage cost that equals r_i-c_i
c_{o_i}:	Overage cost that equals c_i-s_i
π_0:	Target profit value
π_{0_β}:	Target profit value for β threshold probability
β:	Threshold probability value of the downside risk constraint
q_i:	Order quantity for product i
D_i:	Demand for product i
μ_i:	Mean demand of product i
cv:	Coefficient of variation
$\pi(q,D)$:	Profit function
Pr (.):	Probability
F (.):	Cumulative distribution function
f (.):	Probability distribution function
$F_D^{-1}(.)$:	Inverse cumulative distribution function

$E\,[.]$:	Expected value of a function
$S\,(q)$:	Satisficing probability function
q_i^*:	Optimum order quantity for product i
q_S^*:	Optimum order quantity for satisficing probability maximization
c_p:	Scalarization constant for the expected profit function in bi-criteria newsvendor model
c_s:	Scalarization constant for the satisficing probability function in bi-criteria newsvendor model
w:	Weight constant in bi-criteria newsvendor model where $0 \leq w \leq 1$
$U(.)$:	Utility function
$\sigma(.)$:	Standard deviation
z_0:	Initial wealth
c^*:	Cost of extra-ordering
z:	Degree of risk aversion
$U'(.)$:	First derivative of the utility function
$U''(.)$:	Second derivative of the utility function
$Var\,[.]$:	Variance
q^0:	Critical order quantity
P_{VaR}:	VaR (Value-at-Risk) probability
$g\,(q_1,q_2,D_1,D_2)$:	Downside risk probability function
π_i:	Profit function in region i for two-product newsvendor problem with a downside risk constraint
$\Pr_{\pi_0}^i$:	Downside risk probability where the random profit is greater than the target profit level π_0 in region i
n:	Number of products
ϕ:	Standard normal function

ϕ^{-1}: Inverse standard normal function

E_π: Expected value of the newsvendor profit function; equals $E[\pi(q_1, q_2, .., q_n, D_1, D_2, .., D_n)]$

σ_π: Standard deviation of the newsvendor profit function; equals $\sqrt{Var[\pi(q_1, q_2, .., q_n, D_1, D_2, .., D_n)]}$

CVaR ($q_1, .., q_m, \pi_0$): Conditional Value-at-Risk (CVaR) function for product order quantities of m products (q_i) and Value-at-Risk (VaR) π_0 value

CVaR$_i$: CVaR function in the Case-i

Chapter 1

INTRODUCTION

The single-period stochastic inventory (newsvendor) model is a well-known problem in inventory management. Expected profit maximization for such a model has received a significant attention in this area. The profit function in this problem is a random variable since demands of the products are modeled as random variables. During the optimization of the expected profit function in the standard formulation of this model, the maximum loss or the minimum gain that could be reached is not taken into consideration. Today, managers and decision makers are also interested in financial risks in managing the supply chain. Examples for such risks are losing more than an acceptable level or gaining less than a minimum desired profit level besides the maximization of the expected profit function. This study is motivated by the need of incorporating financial risk in inventory decisions.

From this point of view, the objective of the problem should be controlling the maximum loss or the minimum gain; or maximizing the expected profit while controlling the financial risk. Various risk measures are available to evaluate the riskiness of a system in the literature. Utility functions, the satisfaction probability function, the scalarization method, Value-at-Risk (VaR) and Conditional Value-at-Risk (CVaR) are among the frequently used risk measures in the finance and operations research literature.

In order to obtain some familiarity with the risk measures frequently used in the literature, let us give some brief information on how the risk concept is used for each one. The detailed analysis of the most frequently risk measures used in the inventory literature is given in Chapter 3. The utility function indicates a decision maker's preference among rewards under risk. The satisfaction probability function is the probability of achieving the target profit value. In the scalarization method, the expected profit function and the satisfaction probability function are scalarized according to their importance for the system. VaR measures the maximum value of the random function or the variable in a β confidence interval. Glasserman [1] expresses VaR as a measure of the maximum potential change in value of a portfolio of financial instruments over a pre-set horizon. VaR answers the question that how much one can lose with x% probability over a given time horizon. If a portfolio is expressed as a 95% one-day VaR of $100 million, this means that there is only a 5% chance that the portfolio will lose more than $100 million over the next day. JP Morgan[1] has placed a VaR calculator on their web page. This gives us how the portfolio could change in value over the next 24 hours, with 95% confidence. CVaR is also a related criterion which measures the conditional expected loss exceeding VaR and accounts for the risks beyond the VaR value.

In a newsvendor setting, using a risk measure in the expected profit maximization is the focus of this study. In the literature, the satisfaction probability maximization model is solved for a single-product case and is investigated for two products; an exact solution has not been presented yet. The satisfaction probability constraint combines the products and creates a close relation between the products. Therefore, solving this model for more than one product is not as easy as solving a newsvendor model for multiple-products.

In order to control the risk of gaining profit in an amount less than a minimum desirable profit value, using the satisfaction probability as a VaR constraint in the newsvendor problem seems to be a reasonable approach. In real life, manufacturers, retailers, decision makers desire to control a system for multiple products instead of a single product. A decision maker would like to decide how much to invest on different assets and know the effects of the system parameters on the decision variables with risk consideration. Our

[1] JP Morgan is a global financial services firm that serves governments, corporations, institutions, individuals and privately held firms with complex financial needs through an integrated range of advisory, financing, trading, investment and related capabilities.

objective is developing a methodology to find a solution for a product portfolio with a given number of products.

For two products, the VaR constraint is examined in disjoint sets according to the random demands and product order quantities. The analytical numerical solutions for two-product case are given in this study. For a larger product portfolio, it is difficult to utilize this method. A normal approximation method is proposed to solve the model for *N*-product portfolio in this thesis.

In the second part of the thesis, we optimize the CVaR function in terms of the newsvendor profit function or loss function for a large product portfolio. By using the solution, we then investigate the effects of parameters on the decision variables and the objective function in this model. The CVaR function for a given loss distribution can be converted to a Linear Programming (LP) model. By using this result, the CVaR optimization problem where the CVaR function defined in terms of the newsvendor's profit or loss is formulated with an LP model for a large product portfolio in this thesis. Both the CVaR optimization and the newsvendor problem with a CVaR constraint are studied.

The previous studies in the literature about different risk measures, the risk-averse newsvendor problem and the other related topics to this thesis are summarized in Chapter 2. The risk measures frequently used in the past studies are analyzed in Chapter 3 in order to introduce the risk concept. The newsvendor problem is analyzed under a VaR constraint; an exact solution methodology for two products and an approximate solution method for *N* products are proposed in Chapter 4. The CVaR optimization and the CVaR constraint formulation and the solution methodology are explained in Chapter 5. In Chapter 6, the numerical experiments, the results and the comments on the results are given. Finally, the concluding remarks, the results are discussed in Chapter 7.

Chapter 2

LITERATURE REVIEW

The issue of optimality criteria in stochastic inventory models has received a significant attention in the literature. In single period models, the most frequently employed criterion is maximizing the expected profit in terms of a newsvendor problem. In recent research, the financial risk in the management of inventory systems is considered from various perspectives. There are many risk measures that are used in risk management in stochastic inventory models such as the satisficing probability maximization, the scalarization method, the utility functions, VaR and CVaR. Below, we explain each of these measures in detail.

2.1 The Satisficing Probability Function

Some authors model the risk concept as a satisficing probability function. This function is defined as the probability of exceeding a prespecified fixed target profit level. In Section 4.2, the satisfaction probability function is used as a constraint of the classical newsvendor problem and the aim is to solve this model for N product case.

Sankarasubramanian and Kumarasamy [2] consider a single-period stochastic inventory problem in which it is required to determine the product order quantity which maximizes the probability of realizing a predetermined level of profit. A condition for deciding the optimal order quantity is found and explicit expressions for the optimal order quantities in three special cases are given.

The satisfaction probability is used as an objective function in a number of studies (Lau [3], Lau and Lau [4], Lau et al. [5], Lau et al. [6] and Parlar and Weng [7]). The aim is to maximize the satisfaction probability function in terms of the product order quantity. Lau [3] solves the satisfaction probability maximization problem for a single product under the assumption of zero salvage value. Parlar and Weng [7] find the condition where the optimum point for the satisfaction probability value maximization is less than the optimum point for the expected profit maximization in the classical newsvendor problem.

Lau and Lau [4] consider the maximization of the probability of achieving a target profit in a two-product newsvendor problem. Solution procedures are developed to find the optimal order quantities of each product that will maximize the probability of achieving the target profit value. Lau et al. [5] and Lau et al. [6] present an analytical solution procedure to maximize the probability of achieving a target profit in a two-product newsvendor problem for uniformly and exponentially distributed demands respectively. Limited analytical results are presented for only restrictive cases.

2.2 The Scalarization Method

In scalarization method, the satisfaction probability function- denotes the riskiness of the newsvendor problem- and the profit maximization-the objective function in the newsvendor problem- are to be balanced according to their importance in the system. Parlar and Weng [7] balance the two different and the conflicting objectives for the newsvendor problem. The first objective is the standard one in the newsvendor model that is the expected profit maximization. The other one is known as the satisficing objective that is the probability of exceeding the expected profit. A scalarization method is used to combine the standard objective with the new objective of maximizing the probability of exceeding the moving target. The aim of the scalarization method is to normalize and control the importance of each objective. This is a bi-criteria newsvendor model where a Vector Optimization Problem (VOP) was formulated and solved.

2.3 The Utility Functions

In literature, risk is also modeled by using utility functions. Lau [3] maximizes the expected utility of the newsvendor problem. The utility function is defined in terms of the expected value of the random profit and its standard deviation. This is the mean-standard deviation trade-off approach. Lau [3] also uses the von Neumann-Morgenstern's utility function as a different approach to the expected utility maximization in newsvendor models.

Eeckhoudt et al. [8] examine the risk aversion in the newsvendor problem by using a utility function of the newsvendor's random profit. As the utility function is an exponential utility function, the riskiness factor is directly located in this definition. Effects of risk, comparative-static effects of changes in the various price and cost parameters are determined and related to the newsvendor's risk aversion. How various parameters in the problem affect the optimum order quantity is thoroughly analyzed. Also, Bouakiz and Sobel [9] examine the risk aversion in the newsvendor problem with an exponential utility function and show that a base-stock policy to be an optimal strategy when a dynamic version of the newsvendor model is optimized with respect to an exponential utility criterion.

Schweitzer and Cachon [10] investigate the decision bias in the newsvendor problem with a known demand distribution. They compare the optimum order quantity for the classical newsvendor problem with the optimum order quantities for the utility maximization orders with risk neutral, risk-averse, risk-seeking, prospect theory, loss-averse, waste-averse, stock out-averse and minimizing ex-post inventory error preferences. Results from these studies show that choices systematically deviate from those that maximize the expected profit. It is concluded that subjects order fewer than optimum order quantities maximizing the expected profit for high-profit products and order more than optimum order quantities maximizing expected profit for low-profit products. (Products are defined as a high-profit product when $(r\text{-}c)/(r\text{-}s) \geq 1/2$ or as a low-profit product otherwise.) In this thesis, the effect of the risk concept on the optimum order quantity is studied in the VaR-constrained single-period stochastic inventory problem for more than one-product.

Anvari [11] examined the use of market valuation models in analyzing the stochastic inventory problems. A newsvendor problem (single-period stochastic inventory problem) is studied using the capital asset pricing model (CAPM). Unlike the other working-capital decisions, the use of CAPM need not imply conflicting assumptions to analyze inventory problems. The resulting optimal policy is characterized and compared with the classical expected benefit maximization framework and it is observed that results are dramatically different from each other.

Up to this point, the only decision variable is the product order quantity. Lau and Lau [12] consider the classical newsvendor problem where a stochastic price-demand relationship exists for the product. Besides the order quantity, price also becomes a decision variable. A flexible approach capable of modeling price-demand relationships at various levels of complexities is presented. Solution procedures are then presented for different optimization objectives such as the maximizing the probability of achieving a target profit level. For the simplest price-demand relationship, analytical solutions are obtainable. For other cases, very efficient numerical procedures are developed. The solutions provided insights on the effects of price sensitivity and demand uncertainty.

Agrawal and Seshadri [13] consider the newsvendor problem in which a risk-averse retailer confronts with an uncertain customer demand and makes a product order quantity and a selling-price decision with the objective of maximizing the expected utility. This problem is similar to the classical newsvendor problem, except two differences. The first difference is that the distribution of demand is a function of the selling price, which is determined by the retailer. The second difference is that the objective of the retailer is to maximize his/her expected utility. The effect of the price on the distribution of the demand is studied in two different ways. Price affects either the scale of the distribution or the location of the distribution. A methodology by which this problem with two decision variables can be reduced to a problem in a single variable is presented. A risk-averse retailer in the first model will determine a higher price and order less in comparison to a risk neutral retailer; whereas a risk-averse retailer in the second will determine a lower price in comparison to a risk-neutral retailer. This study provides a better insight for the pricing behavior of the retailers that would lead to improved price contracts and decision policies.

2.4 The Properties of the Value-at-Risk (VaR) and the Conditional Value-at-Risk (CVaR) Measures

Artzner et al. [14] study both market and non-market risks and discuss the methods of the risk measurements. They present and justify the desirable properties for a risk measure. VaR and CVaR are important risk measures that are emphasized in the literature. They are also concerned with these risk measures and their properties.

2.4.1 The Value-at-Risk (VaR)

In finance, VaR is defined to be the maximum value of the random function or the variable in β confidence interval. VaR is a measure of the maximum potential change in value of a portfolio of financial instruments over a pre-set horizon.

Duffie and Pan [15], Dowd [16], Jorion [17] and Simons [18] give a review of value at risk (VaR). Duffie and Pan [15] describe some of the basic issues involved in measuring the market risk of a financial firm and the list of positions in various instruments that expose the firm to financial risk. While there are many sources of financial risk, concentration is here on the market risk, meaning the risk of unexpected changes in prices, rates or demands in relation to those.

In Tapiero [19] , an asymmetric valuation between ex-ante expected costs above an appropriate target cost and the expected costs below that same target level, provide an explanation for the VaR criterion when it is used as a tool for VaR efficiency design. This approach gives some insights regarding the selection of the VaR probability that turns out to be the ratio of the asymmetric linear cost parameters in this case. This approach is used in a single-period stochastic inventory problem. Also, some numerical studies for a multi-period stochastic inventory problem are presented.

Gan et al. [20] incorporate the VaR concept to a newsvendor problem with a downside risk constraint for a single product. This is a decision making problem of a risk-averse newsvendor subject to the downside risk which is characterized as the probability that the newsvendor's realized profit is less than or equal to his specified target profit. In the first part of this thesis, this model is studied for *N*-product case and the effects of the parameters on the optimum order quantities and the optimum profit are observed.

In this study, a single-period stochastic inventory problem for a large product portfolio is considered with the satisfaction probability constraint where VaR is denoted as the target profit value in the constraint. To solve this problem for *N* products will be meaningful and help decision makers in deciding how much to order from each product while controlling the risk. As the satisfaction probability function combines profit functions corresponding to each product, the constraint create a close relation between the products. Therefore, solving the newsvendor problem with VaR constraint for *N* products is harder than solving the unconstrained newsvendor problem. In this thesis; an exact solution method is given for two-product case and an approximation method is proposed for *N*-product case.

2.4.2 The Conditional Value-at-Risk (CVaR)

CVaR measures the conditional expected loss exceeding VaR and accounts for the risks beyond the VaR value. CVaR was studied in both supply chain and finance literature.

Chen et al. [21] focus on the inventory management for risk-averse retailers. CVaR maximization for one product in profit sense is structured and solved. They propose a general framework for incorporating risk aversion in multi-period inventory models as well as multi-period models that coordinate inventory and pricing decisions for one product case.

Acerbi and Simonetti [23], Coleman et al. [24] and Coleman [25] point out the difficulties encountered if we deal with the VaR and the CVaR optimization and give some insights about how these difficulties could be eliminated by the linearization methods that was first proposed by Rockafellar and Uryasev [27].

Palmquist et al. [26] and Rockafellar and Uryasev [27] focus on the portfolio optimization with the CVaR measure rather than with the VaR measure because CVaR is considered to be more consistent measure of risk than VaR is considered to be. This approach calculates VaR and optimizes CVaR simultaneously in which the calculations often come down to linear programming or non-smooth programming.

Rockafellar and Uryasev [22] derive the fundamental properties of CVaR, as a measure of risk with significant advantages over VaR for loss distributions in finance that can involve discreetness. The CVaR measure is able to quantify dangers beyond the VaR value. Rockafeller and Uryasev [22] modeled the problem as an LP for loss functions that have definite distribution functions. It provides optimization short cuts through the linear programming techniques.

Bertsimas et al. [28] examine the shortfall as a risk measure and define the CVaR measure as the shortfall. They express the advantages of this risk measure over VaR and discuss the optimization of shortfall.

In the second part of our study, the CVaR function for the newsvendor problem is studied for the *N*-product case. The CVaR function used in the finance literature has a given distribution function. The CVaR function which is defined in terms of the newsvendor's profit or loss does not have a given distribution. This function is shown to be convex. Therefore, the CVaR optimization for a large product portfolio where the CVaR function is defined in terms of the newsvendor loss function is solved by utilizing the LP formulation developed in the finance literature.

Tomlin and Wang [29] apply the CVaR measure in studying the mix-flexibility and dual-sourcing literatures by considering the unreliable supply chains that produce multiple products. A firm can invest in product-dedicated resources and totally flexible resources. The product demands are uncertain at the time of investment, and the products can differ in their contribution margins. The resource investments can fail, and the firm may decide to invest in multiple resources for a given product to cope with such failures. The optimization problem is converted to CVaR optimization and simplified to an LP formulation by using the approach initiated by Rockafellar and Uryasev [27] and Rockafellar and Uryasev [22]. A flexible strategy is strictly preferred to a dedicated strategy when the dedicated resources are costlier than the flexible resource if the firm is risk-neutral or if the resource investments are perfectly reliable.

Chapter 3

RISK CONCEPT IN INVENTORY MANAGEMENT

3.1 Introduction

In this thesis, the main focus is on the risk feature of the newsvendor models and the optimization of the newsvendor models for a multi-product portfolio in a risky environment. In both finance and operations research literature, various researchers discussed different risk concepts. In order to have a better understanding on the main risk concepts; these concepts are examined and summarized in this chapter. The main motivation is to present a brief summary on the risk concepts and provide an introduction to the following chapters.

For this purpose, the classical newsvendor problem and its solution are briefly reviewed in section 3.2. The models of the newsvendor problems in a risky environment are discussed in section 3.3 and in its subsections.

3.2 The Classical Newsvendor Problem

The newsvendor problem often referred as the newsvendor problem has received a significant attention in the literature. In a typical newsvendor problem; r the product's unit selling price, c the product's unit variable cost and s the product's unit salvage price are the values where the condition $r>c>s$ is satisfied. In a single period model, the objective is to maximize the expected profit in terms of a newsvendor problem. The expected profit function to be maximized written as:

$$E\,[\pi\,(q,\,D)] = (r\text{-}c)\;q\text{-}(r\text{-}s)\;E\,[\max\,(q\text{-}D,\,0)] \tag{3.1}$$

where D is the random demand and q is the product order quantity.

The solution of the problem was found as, Porteus [30]:

$$q^{*} = F_D^{-1}\left(\frac{r-c}{r-s}\right) \tag{3.2}$$

Generally, managers make their decisions about the products' quantities to be purchased or to be manufactured in order to maximize the firm's expected profit in a random demand environment. This is the generalized classical newsvendor model for N product case in the literature. The project portfolio management, the manufacturing and the buying decisions, the plant capacity decisions, the overbooking, and the target production levels for planned economies are some examples for this type of an optimization problem.

The classical formulation of the newsvendor problem does not take into account various risks that are involved explicitly. More specifically, it does not consider the possible losses or minimum gains as a part of the optimization problem. Increasing the expected profit may cause a raise in the risk. Controlling the risk during the optimization of the firm's profit or only controlling the risk is of interest in this study.

3.3 Models of Newsvendor Problems in a Risky Environment

In the literature on risk modeling, various researchers model the risky environment with different models that include different objectives and constraints. Following subsections briefly introduce these different models and their solution procedures:

3.3.1 The Satisficing Probability

The satisficing probability is defined as the probability of exceeding a target profit level π_0. This model is:

$$\max_{q \geq 0} \Pr(\pi(q, D) \geq \pi_0) \tag{3.3}$$

In a single period stochastic inventory problem, the satisficing probability maximization model was used by Lau and Lau [4], [5] and [6] and Parlar and Weng [7]. Lau and Lau [4] present the optimum order quantity (q^*) that maximizes the satisficing probability value when salvage value is assumed to be zero as:

$$q^* = \frac{\pi_0}{(r - c)} \tag{3.4}$$

There is no analytical solution available for this problem when the salvage value is greater than zero. Parlar and Weng [7] found some bounds and properties for this quantity. The standard assumption $s<c<r$ holds. Random profit versus the order quantity graph can be seen in Figure 3.1. $S(q)$ and $E[\pi(q,D)]$ are satisficing probability and expected profit functions, respectively. $S(q)$ can be written as:

$$S(q) = \Pr[\pi(q, D) \geq E[\pi(q, D)]] = \Pr[x_1(q) \leq X \leq x_2(q)] = \int_{x_1(q)}^{x_2(q)} f(x)dx \tag{3.5}$$

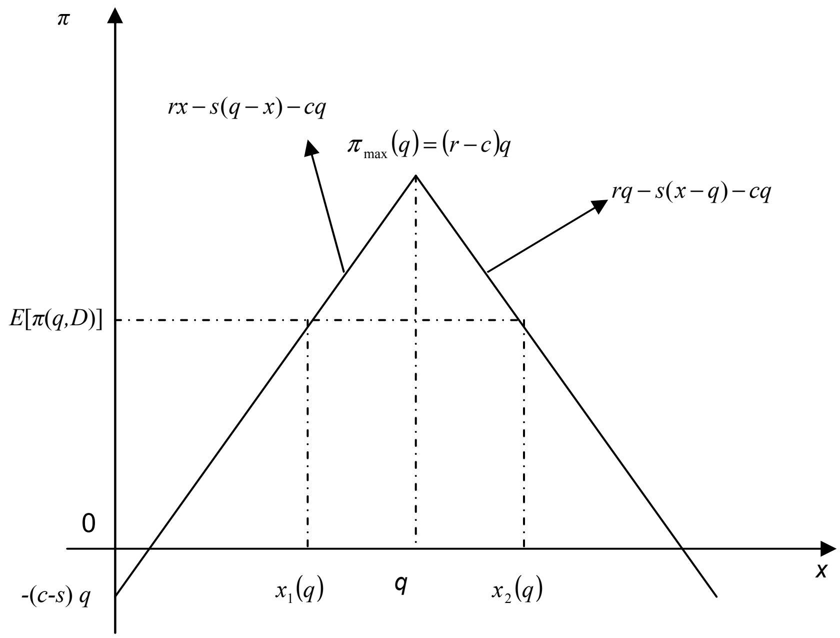

Figure 3.1: Random Profit versus the Decision Variable (Order Quantity)

The difference function Δ (q) $=x_2(q)$-$x_1(q)$ is minimized at a unique point $\tilde{q}$ which satisfies the condition $\tilde{q} < q^*$. For demand densities having the property that $\alpha(q) < \beta(q)$ for all $q \geq \tilde{q}$, where:

$$\alpha(q) = \frac{f[x_2(q)]}{f[x_1(q)]} \text{ and } \beta(q) = \left(\frac{s}{r-c}\right)\frac{1-F(q)}{F(q)} \tag{3.6}$$

The satisficing probability function $S(q)$ is maximized at some point q_S^* that is at least as large as $\tilde{q}$ but smaller than $\tilde{q}$. Under this condition, the optimal solutions q_S^* and q^* have the property that $q_S^* < \tilde{q} < q^*$.

3.3.2 The Scalarization Method

Another model used to control the risk in a newsvendor problem is the bi-criteria newsvendor model. This model is defined by Parlar and Weng [7]. There are two criteria which are the satisficing probability and the expected profit in this model as the name suggests. The model is an unconstrained maximization problem where the objective function is written in terms of these two criteria. These two criteria are scalarized in order to normalize and control the importance of each criterion. The model is:

$$\max_{q\geq 0} c_p wE[\pi(q,D)] + c_s(1-w)S(q), \qquad 0 \leq w \leq 1 \tag{3.7}$$

where $E[\pi(q,D)]$ is the expected profit function and $S(q)$ is the satisficing probability function. The constants c_p (equals $1/E[\pi(q^*,D)]$) and c_s (equals $1/S(q_S^*)$) are introduced to normalize the weighted objective function since the values of the objectives are generally very different from each other. The expected profit function $E[\pi(q,D)]$ may assume any real value whereas the satisficing probability $S(q)$ must take values between zero and one. Weight constant w is regulated according to the importance of each criterion.

This model is a Vector Optimization Problem (VOP). In this problem, a procedure is needed to be developed which generates the set of points in *PS*-plane (*PS*-plane is the solution pairs for expected probability function and satisficing probability function) that are not inferior to any other points. This set of non-inferior points is known as the efficient frontier. (Parlar and Weng [7]):

Definition: The point q^* is a non-inferior solution of VOP if there exists no other feasible solution q such that $E[\pi(q,D)] > E[\pi(q^*,D)]$ and $S(q) > S(q^*)$.

The solution which is the optimum order quantity that maximizes both objectives (criteria) simultaneously is unattainable. For this problem; Parlar and Weng [7] defined the distance in terms of the relative weighted deviations from the ideal solution and solved this optimization problem.

3.3.3 Utility Functions

Another model is maximizing the expected utility of the profit. A utility function is a mathematical expression that assigns a value to all possible choices. In portfolio theory the utility function expresses the preferences of economic entities with respect to perceived risk and expected return. There are several utility functions that are used for the purpose of defining an objective function for the risk-averse newsvendor problem such as the mean-standard deviation trade-off and the exponential utility function.

3.3.3.1 The Mean-Standard Deviation Trade-off Function

This is a popular approach that considers a trade-off between the expected value of the random profit and its standard deviation. The model is as below:

$$\max_{Q} U(\pi(q,D)) \quad \text{where} \quad U(\pi(q,D)) = E[\pi(q,D)] - k\sigma(\pi(q,D)) \tag{3.8}$$

Here, k is the magnitude that reflects an investor's individual degree of risk aversion. This problem could not be solved analytically. Lau [3] found that the optimum order quantity q^* for the classical newsvendor problem is an upper bound for the optimum order quantity found for this utility function.

3.3.3.2 The Exponential Utility Function

Eeckhoudt et al. [8] examined risk aversion by an exponential utility in the newsvendor problem. They consider a utility function of the newsvendor profit and locate the riskiness factor in the utility function. The model is:

$$\max_{Q\geq 0} E[U(\pi(q,D))] \quad \text{where} \quad \pi(q,D) = z_0 + rD - cq + s\max(0, q-D) - c^*\max(0, D-q) \tag{3.9}$$

In this model, z_0 is the initial wealth. The newsvendor is allowed to obtain additional newspapers at c^* if demand exceeds his original order. A natural assumption is that $0 \leq s < c < c^* \leq r$. The utility function is defined to be as $U(\pi) = -\exp(-z\pi)$ where z represents the newsvendor's degree of risk aversion. The degree of absolute risk aversion is given by $z(\pi) = -U''(\pi)/U'(\pi)$. This measure is constant in profit. When the newsvendor is risk neutral ($U''(\pi)=0$), the solution is equal to the well known solution:

$$F(q^*) = \frac{c^* - c}{c^* - s} \tag{3.10}$$

The case of a risk-averse newsvendor implies that:

$$F(q^*) < \frac{c^* - c}{c^* - s} \tag{3.11}$$

3.3.4 The Value-at-Risk (VaR)

The risk of earning less than a threshold value or losing more than a critical value is an important issue. Investors, production planners, in short, all decision makers would aim to maximize their profit and control their risk below a determined level at the same time. The VaR and CVaR are used for these purposes. Some firms consider the risk in terms of inventory on hand that is called the value of inventory at risk that resembles the VaR concept.

VaR is defined to be the β- percentile of the distribution of a random variable-ξ (a smallest value such that the probability that the random variable exceeds or equals to this value is greater than or equal to β). In financial applications, it can be defined as a maximum value in a specified period with some confidence level β. VaR is the maximum loss that could be reached with a confidence interval of β. It quantifies the downside risk compared to variance which is impacted by high returns. VaR is characterized as in the equation and in the graph below:

$$\text{VaR} = \inf \left\{ \pi_0 \mid P(\xi \geq \pi_0) \geq \beta \right\}$$

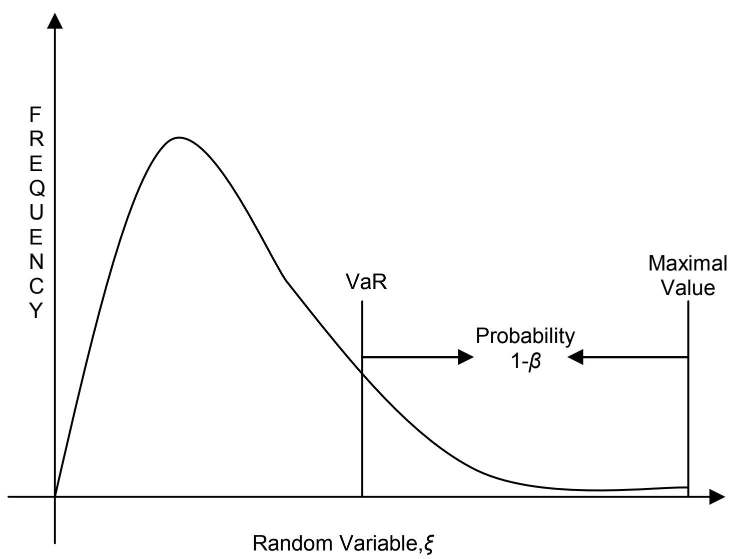

Figure 3.2: VaR of a Loss Distribution

Artzner et al. [14] found that the VaR measure creates a non-convex risk surface. It was difficult to optimize for non-normal distribution because VaR has many extreme points.

The VaR-constrained optimization problem is defined as the maximization of the expected profit with a downside risk constraint in the literature. This model for one product was solved by Gan et al. [20]. The decision problem is as follows:

$$\begin{aligned} &\max_{q\geq 0} \quad E[\pi(q,D)] \\ &\text{subject to} \\ &\Pr(\pi(q,D)\leq \pi_0)\leq \beta \end{aligned} \tag{3.14}$$

where the profit function is $\pi(q,D) = r\min\{q,D\} - cq$ (salvage value is assumed to be zero).

For any target profit level π_0, there is a critical order quantity:

$$q^0 = \frac{\pi_0}{r-c} \tag{3.15}$$

such that for an order quantity $q \leq q^0$, the downside risk probability is one, and for $q > q^0$, the downside risk is:

$$F\left(\frac{\pi_0 + cq}{r}\right) \tag{3.16}$$

The proof which is presented by Gan et al. [20] is as follows:

Proof: If $q \leq q^0$, then $\pi(q,D) = r\min\{q,D\} - cq \leq rq^0 - cq^0 = \pi_0$. Therefore, $\Pr(\pi(q,D) \leq \pi_0) = 1$. If $q > q^0$, then clearly $\Pr[\{\pi(q,D) \leq \pi_0\} \cap \{D > q\}] = 0$ and, furthermore,

$$\Pr(\pi(q,D) \le \pi_0) = \Pr(rD - cq \le \pi_0) = P\left(D \le \frac{\pi_0 + cq}{r}\right) = F\left(\frac{\pi_0 + cq}{r}\right) \tag{3.17}$$

In summary, for the single product newsvendor with risk aversion pair (π_0,β), β>$F(q^0)$, the optimal order quantity is:

$$q^* = \begin{cases} \hat{q} & \text{if} \quad F\left(\frac{\pi_0 + c\hat{q}}{r}\right) \le \beta \\ \frac{rF^{-1}(\beta) - \pi_0}{c} & \text{if} \quad F(q^0) < \beta < F\left(\frac{\pi_0 + c\hat{q}}{r}\right) \end{cases} \tag{3.18}$$

where

$$\hat{q} = F^{-1}\left(\frac{r-c}{r}\right) \tag{3.19}$$

3.3.5 The Conditional Value-at-Risk (CVaR)

CVaR measures the conditional expected loss exceeding VaR and accounts for the risks beyond the VaR value. CVaR for continuous distributions usually coincides with conditional expected loss exceeding VaR (also called Mean Excess Loss or Expected Shortfall). CVaR measures downside risk and accounts for risks beyond VaR. CVaR is the mean loss above the maximum loss could be reached with a β confidence interval. As it is the expected value of the loss function above the VaR value, the CVaR function has a smooth surface. CVaR is applicable to non-symmetric loss distributions. It has a convex risk surface with respect to control variables and has a unique global optimum. CVaR is characterized as in the equation and in the graph below:

$$\text{CVaR} = E[\,\xi \mid \xi \ge \pi_{0_\beta}(\xi)]$$

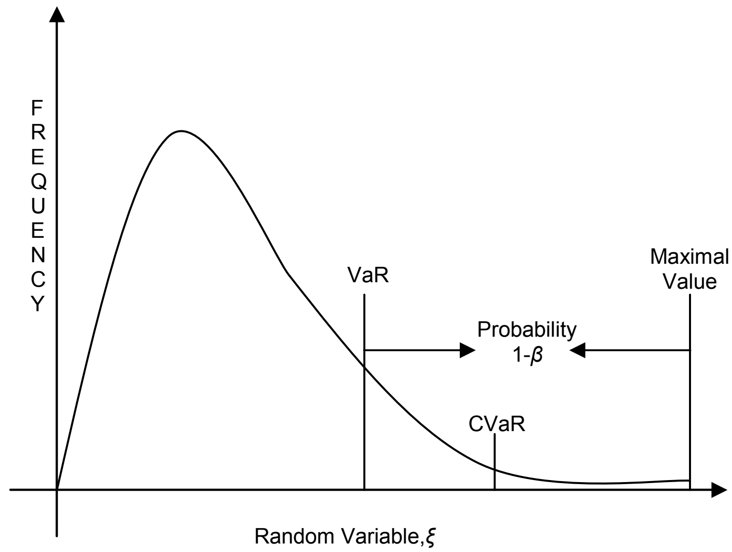

Figure 3.3: CVaR of a Loss Distribution

Chen et al. [21] maximize the CVaR function for a single product newsvendor problem. This CVaR function is defined in terms of profit distribution of a newsvendor problem. The maximization of the CVaR function is more plausible than a maximization of the expected profit function subject to CVaR is less than or equal to CVaR bound because the CVaR bound cannot be calculated in a certain manner, it could only be simulated and used approximately. The model is as follows:

$$\max_{q \geq 0, \pi_0} \quad \mathrm{CVaR}_{\beta}(\pi(q,D)) \tag{3.20}$$

where

$$\mathrm{CVaR}_{\beta}(\pi(q,D)) = \pi_0 + \frac{1}{\beta}\int_{D}\left[(r-c)q-(r-s)(q-D)^{+}-\pi_0\right]^{-} dF(D) \tag{3.21}$$

and $[a]^- = \min(a,0)$.

The optimal product order quantity for the CVaR function is calculated by the equation $q^* = \arg\max_{q \geq 0} \{\max_{\pi_0} CVaR_\beta(\pi(q,D))\}$. For fixed q, the objective function is differentiated in terms of π_0, and then the function is differentiated in terms of q. Finally the optimum order quantity is as below; the optimum order quantity does not depend on π_0:

$$q^* = F_D^{-1}\left(\beta \frac{r-c}{r-s}\right) \tag{3.22}$$

Rockafeller and Uryasev [22] studied general properties of VaR and CVaR in finance literature. Let $f(x,y)$ be the loss associated with the decision vector x and the random vector y and $p(y)$ is the probability distribution of random vector y. The probability of $f(x,y)$ not exceeding a threshold β is given then by:

$$\psi(x,\pi_0) = \int_{f(x,y) \leq \pi_0} p(y)dy. \tag{3.23}$$

The β-VaR and β-CVaR values for the loss random variable associated with x and any specified probability level β in (0,1) will be denoted by $\pi_{0_\beta}(x)$ and $\phi_\beta(x)$. They are given by:

$$\pi_{0_\beta}(x) = \min\{\pi_0 \in IR : \psi(x,\pi_0) \geq \beta\} \tag{3.24}$$

and

$$\varphi_\beta(x) = (1-\beta)^{-1} \int_{f(x,y) \geq \pi_{0_\beta}(x)} f(x,y)p(y)dy \tag{3.25}$$

The optimization problem is minimization of the function below:

$$F_{\beta}(x,\pi_0)=\pi_0+(1-\beta)^{-1}\int_{y\in IR^m}[f(x,y)-\pi_0]^+ p(y)dy \tag{3.26}$$

where $[a]^+ = \max(a,0)$.

Rockafellar and Uryasev [22] show that this optimization problem can be reduced to the following linear programming problem:

$$\begin{aligned}
&\min_{x,\pi_0} \quad \pi_0+(1-\beta)^{-1}\sum_{k=1}^{N} p_k z_k, \\
&\text{subject to} \\
&z_k \geq f(x,y)-\pi_0, \quad k=\overline{1,N}, \\
&z_k \geq 0, \quad k=\overline{1,N}, \quad x\in X,
\end{aligned} \tag{3.27}$$

where

p_k : is the discrete probability value for loss distribution $f(x,y)$ in the k^{th} discretization interval.

k: is the discretization interval which takes values 1 to N.

z_k : the auxiliary variable defined to determine the non-negativity of the function

In this problem, the general loss function $f(x,y)$ is discretized in N intervals and used the discretized value and the corresponding probability values are used in the described model.

3.4 Conclusion

In this chapter, some of the risk measures and the risk models are discussed. In this thesis, we focus on the VaR and CVaR measures in controlling the risk in single period stochastic inventory models.

These measures are easy to interpret by the managers and also provide additional advantages. For example, it is not easy to determine the utility functions to be used in a particular application. Furthermore, VaR and CVaR measure the downside risk of the system, while most of the risk measures in the literature consider both the upside and the downside risk together and measure the risk of deviating from the target. In stochastic inventory models, measuring the downside risk with respect to the system's total profit is more meaningful.

Chapter 4

NEWSVENDOR PROBLEM WITH A VALUE-AT-RISK (VaR) CONSTRAINT

4.1 Introduction

The solution of a single product newsvendor problem for both risk-averse and risk-neutral cases is well known in the literature. The objective of this chapter is to model and to investigate newsvendor problems with a downside risk constraint (VaR constraint) for two or more products. In particular, we present an exact solution for two products. For the multi-product case, due to the difficulty of finding a solution to the problem, we present an approximation method.

The problem description is given in section 4.2. We provide a detailed analysis of the two-product newsvendor problem with the VaR constraint in section 4.3. The analytical solution methods are presented and the results are discussed. We present an approximation solution method for *N*-product case by using the Central Limit Theorem (CLT) approach in the section 4.4. The numerical performances of the approximation, the solutions and discussion are also provided in that section. The conclusion of this chapter is given in the section 4.5.

4.2 Problem Description

In order to introduce the model and the challenges, the newsvendor problem – a newsvendor problem- is considered with a downside risk constraint which is defined as the probability of achieving the target profit value (π_0) is less than or equal to the threshold probability value (β). The demand for a product in the problem is a random variable with a known distribution. This situation makes the profit function a random function with a given distribution function, F. The expected profit function which is expressed in terms of the product quantity q, the unit salvage price s, the unit variable cost c, the unit selling price r and the random demand D was shown in Equation 3.1. We assume that the relationship between the cost and revenue parameters $r>c>s$ holds.

The newsvendor problem with and without a VaR constraint for a single product solution is discussed and the effects of the system parameters are examined in section 6.2.1.

The problem is the maximization of the expected profit function subject to the downside risk constraint. The decision variable is the product order quantity. The optimization problem was given in Equation 3.14.

4.3 Two-Product Case

We first present the extension of the model by Gan et al. [20] for two products. The constraint is the complement of the satisfaction probability function. Here, we present a solution for the expected profit maximization subject to the downside risk constraint for a two-product newsvendor problem. The formulation of the model for two-product newsvendor problem is as below:

$$\begin{aligned}&\max_{q_1,q_2\geq 0} E[\pi(q_1,q_2,D_1,D_2)]\\&\text{subject to}\\&\qquad \Pr(\pi(q_1,q_2,D_1,D_2)\leq \pi_0)\leq \beta\end{aligned} \tag{4.8}$$

where

$$\pi(q_1,q_2,D_1,D_2)=r_1\min(q_1,D_1)-c_1q_1+r_2\min(q_2,D_2)-c_2q_2 \tag{4.9}$$

r_i, c_i, q_i and D_i are revenue values, cost values, quantities ordered and demands for product i respectively.(i=1, 2 in this case). Salvage values are taken to be zero in this part. The risk constraint (4.9) can be rewritten as:

$$1\text{-}g(q_1,q_2)\leq\beta$$

where

$$g(q_1,q_2)=\Pr((r_1\min(q_1,D_1)-c_1q_1+r_2\min(q_2,D_2)-c_2q_2)\geq\pi_0)$$

In order to analyze this function, we consider the function by conditioning on four events as below:

$$\begin{aligned}g(q_1,q_2)=\ &\Pr((r_1D_1-c_1q_1+r_2D_2-c_2q_2)\geq\pi_0 \mid D_1\leq q_1,D_2\leq q_2)\Pr(D_1\leq q_1,D_2\leq q_2)+\\&\Pr((r_1D_1-c_1q_1+r_2q_2-c_2q_2)\geq\pi_0 \mid D_1\leq q_1)\Pr(D_1\leq q_1,D_2>q_2)+\\&\Pr((r_1q_1-c_1q_1+r_2D_2-c_2q_2)\geq\pi_0 \mid D_2\leq q_2)\Pr(D_1>q_1,D_2\leq q_2)+\\&\Pr((r_1q_1-c_1q_1+r_2q_2-c_2q_2)\geq\pi_0)\Pr(D_1>q_1,D_2>q_2)\end{aligned}$$

In order to compute $g(q_1,q_2)$ by using the above expression, all of the the sub-cases for each event must be analyzed. It will be complex to study all these sub-cases in the equation, especially for the first event. Also; if we need to examine this function for demand distributions with bounded probability distributions such as the uniform distribution, there will be additional sub-conditions because of the upper and lower bound of the demand distribution. For this reason, a different method is proposed below to study the equation easily.

Let us define two critical values for demands L_1 and L_2 that are defined as the minimum values of D_1 and D_2 required to achieve π_0 respectively. After defining these critical values L_1 and L_2 for demands, the constraint can be defined in terms of these values in four cases:

<u>Case-I</u>: If L_1 and L_2 are both greater than zero, the satisfaction probability region can be seen in Figure 4.1 and the satisfaction probability function becomes:

$$\Pr(\pi_1 \geq \pi_0) = \Pr^{\mathrm{I}}_{\pi_0} = \int_{L_1}^{q_1} \int_{(\pi_0 + c_1 q_1 + c_2 q_2 - r_1 D_1)/r_2}^{\infty} dF_1(D_1)\, dF_2(D_2) + \int_{q_1}^{\infty} \int_{L_2}^{\infty} dF_1(D_1)\, dF_2(D_2)$$

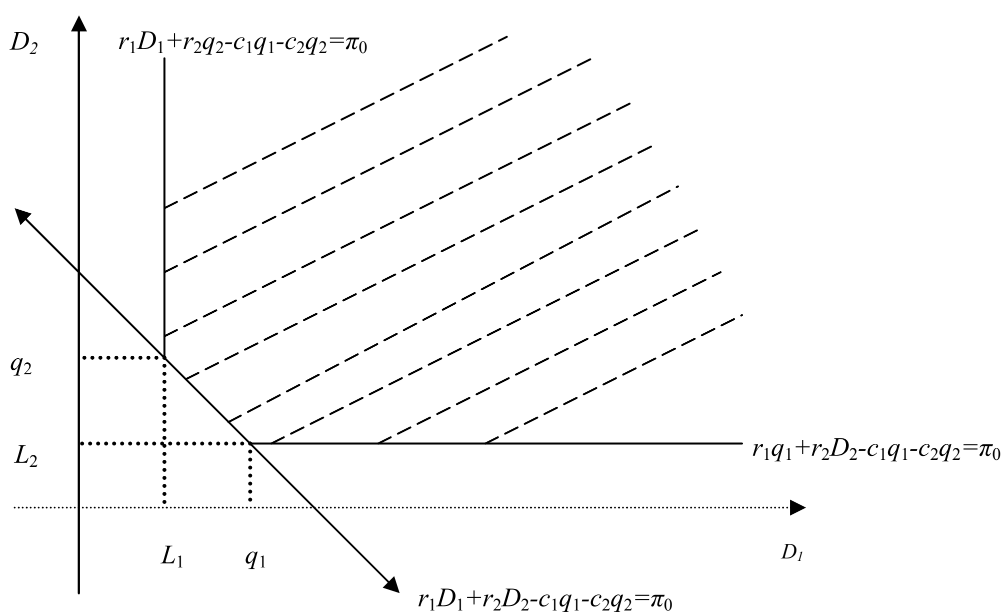

Figure 4.1: Satisfaction Probability Region in Case-I

Up to now, the satisfaction probability function is examined for general demand distributions. If there is a lower and upper bound in the demand distribution e.g. the demand for the first product and the second product are uniformly distributed in the intervals (l_1,u_1) and (l_2,u_2) respectively. As the boundaries on the demand distribution affect the boundaries of the integration in risk probability calculation, there occur four sub-cases in Case-I. Each sub-case is defined by the following conditions (the risk constraint corresponding to each region is calculated - in a similar manner to the above calculation - with their integration bounds):

- Case-I_1: $L_1 > l_1$ and $L_2 > l_2$
- Case-I_2: $L_1 \le l_1$ and $L_2 > l_2$
- Case-I_3: $L_1 > l_1$ and $L_2 \le l_2$
- Case-I_4: $L_1 \le l_1$ and $L_2 \le l_2$

Case-IIa: If L_1 is negative and L_2 is positive, the satisfaction probability region can be seen in Figure 4.2 and the satisfaction probability function becomes:

$$\Pr(\pi_{\mathrm{IIa}} \ge \pi_0) = \Pr_{\pi_0}^{\mathrm{IIa}} = \int_0^{q_1} \int_{(\alpha+c_1q_1+c_2q_2-r_1D_1)/r_2}^{\infty} dF_1(D_1)\, dF_2(D_2) + \int_{q_1}^{\infty} \int_{L_2}^{\infty} dF_1(D_1)\, dF_2(D_2)$$

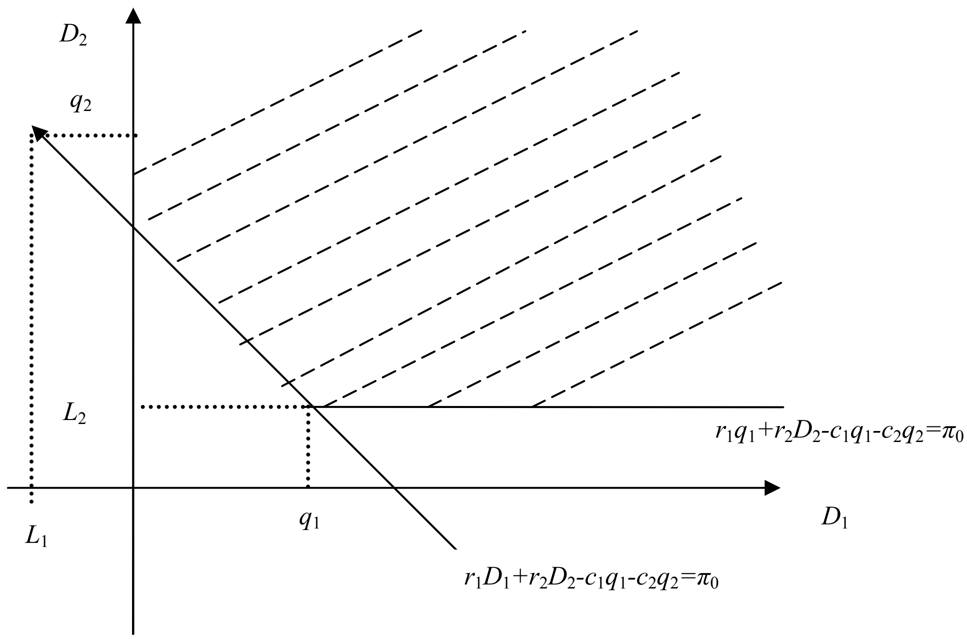

Figure 4.2: Satisfaction Probability Region in Case-IIa

For Case-IIa, there are three sub-cases if there is a lower and upper bound in the demand distribution- (l_1,u_1) and (l_2,u_2) for the first and the second products respectively. Each sub-case is defined by the following conditions (the risk constraint corresponding to each region is calculated - in a similar manner to above calculation - with their integration bounds):

- Case-IIa_1: $L_2 > l_2$
- Case-IIa_2: $L_2 \leq l_2$ and $(\pi_0 + c_1 q_1 + c_2 q_2 - r_1 a)/r_2 > l_2$
- Case-IIa_3: $L_2 \leq l_2$ and $(\pi_0 + c_1 q_1 + c_2 q_2 - r_1 a)/r_2 \leq l_2$

Case-IIb: If L_1 is positive and L_2 is negative, the satisfaction probability region can be seen in Figure 4.3 and the satisfaction probability function becomes:

$$\Pr(\pi_{\text{IIb}} \geq \pi_0) = P_{\pi_0}^{\text{IIb}} = \int_0^{q_2} \int_{(\pi_0+c_1q_1+c_2q_2-r_2D_2)/r_1}^{\infty} dF_1(D_1)\,dF_2(D_2) + \int_{q_2}^{\infty}\int_{L_1}^{\infty} dF_1(D_1)\,dF_2(D_2)$$

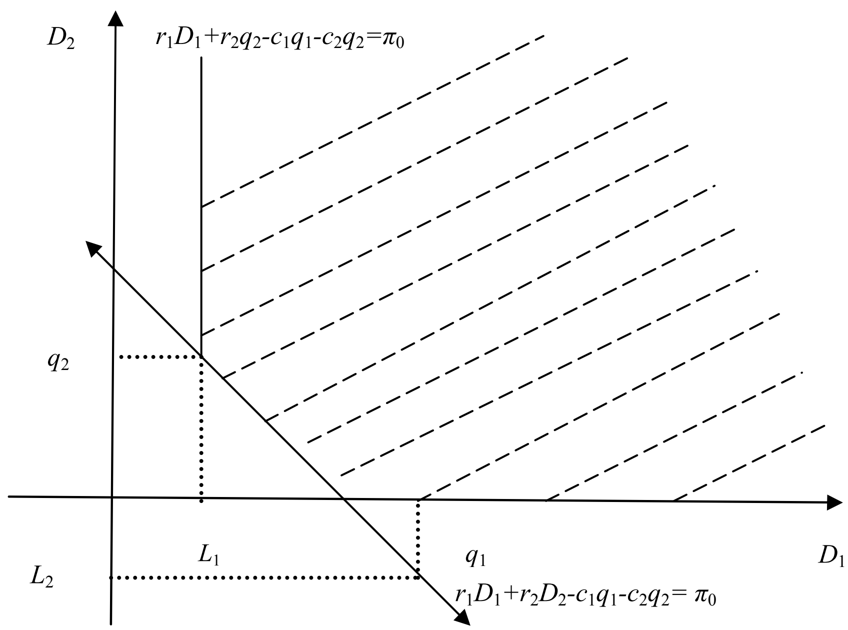

Figure 4.3: Satisfaction Probability Region in Case-IIb

For Case-IIb, there are three sub-cases if there is a lower and upper bound in the demand distribution- (l_1,u_1) and (l_2,u_2) for the first and the second products respectively. Each sub-case is defined by the following conditions (the risk constraint corresponds to each region is calculated - in a similar manner to above calculation - with their integration bounds):

- Case-IIb_1: $L_1>l_1$
- Case-IIb_2: $L_1\leq l_1$ and $(\pi_0+c_1q_1+c_2q_2-r_1a)/r_2>l_2$
- Case-IIb_3: $L_1\leq l_1$ and $(\pi_0+c_1q_1+c_2q_2-r_1a)/r_2\leq l_2$

Case-III: If L_1 is negative and L_2 is negative, the satisfaction probability region can be seen in Figure 4.4 and the satisfaction probability function becomes:

$$\Pr(\pi_{\mathrm{III}} \geq \pi_0) = \Pr_{\pi_0}^{\mathrm{III}} = 1 - \int_0^{(\pi_0+c_1q_1+c_2q_2)/r_2} \int_0^{(\pi_0+c_1q_1+c_2q_2-r_2D_2)/r_1} dF_1(D_1)\, dF_2(D_2)$$

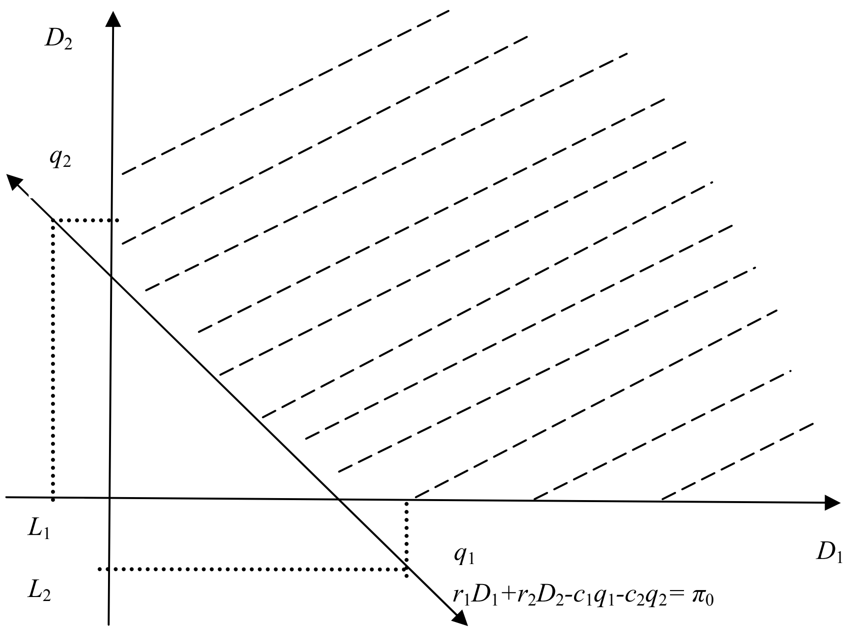

Figure 4.4: Satisfaction Probability Region in Case-III

For Case-III, there are two sub-cases if there is a lower and upper bound in the demand distribution- (l_1,u_1) and (l_2,u_2) for the first and the second products respectively. Each sub-case is defined by the following conditions (the risk constraint corresponds to each region is calculated - in a similar manner to above calculation - with their integration bounds):

- Case-III_1: $(\pi_0+c_1q_1+c_2q_2-r_1a)/r_2>l_2$

- Case-III_2: $\left(\pi_0+c_1q_1+c_2q_2-r_1a\right)/r_2\leq l_2$

The risk constraint that is considered in four cases above could be shown in q_1-q_2 coordinate system below and it can be seen that each case falls into separate regions. When the slope of L_1 is less than the slope of L_2-that means $c_1c_2< (r_1-c_1)(r_2-c_2)$, the graph will be as Figure 4.5a:

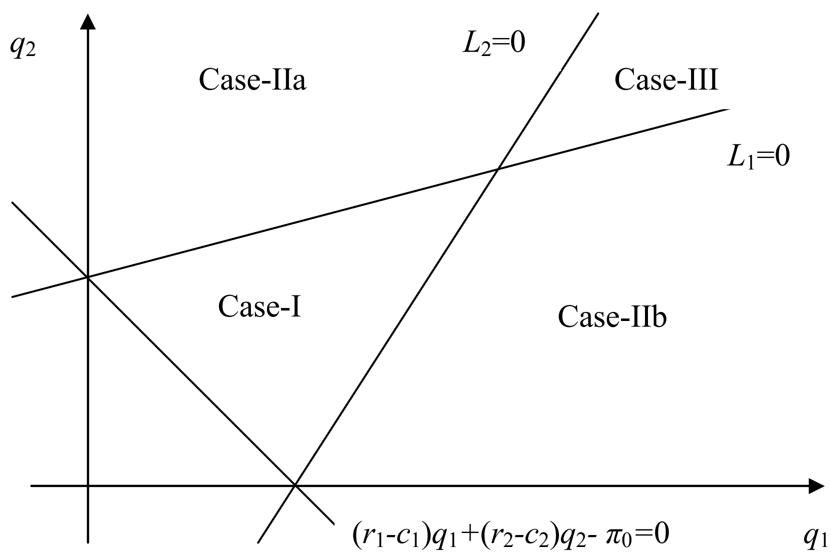

Figure 4.5a: Regions of Satisfaction Probability Constraints

When the slope of L_1 is greater than the slope of L_2-that means $c_1c_2> (r_1-c_1)(r_2-c_2)$, the graph will be as Figure 4.5b:

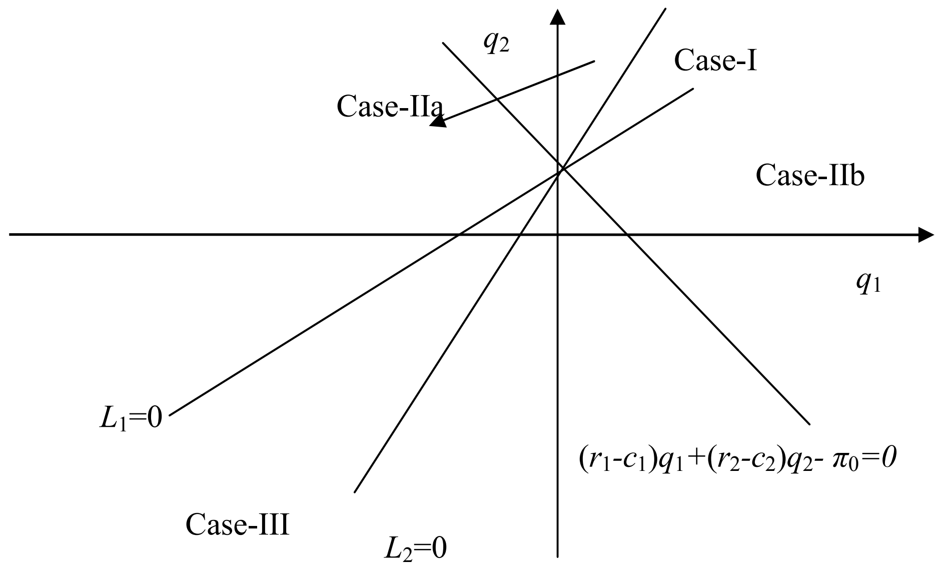

Figure 4.5b: Regions of Satisfaction Probability Constraints

From these two graphs, we have four optimization problems that are each defined in their own regions as shown above:

Case-I:

$$
\begin{aligned}
&\max \mathrm{E}[\pi(q_1,q_2,D_1,D_2)] \\
&\text{subject to} \\
&\qquad 1-\Pr{}^{\mathrm{I}}_{\pi_0} \le \beta \\
&\qquad L_1 \ge 0 \\
&\qquad L_2 \ge 0 \\
&\qquad (r_1-c_1)\,q_1+(r_2-c_2)\,q_2 \ge \pi_0 \\
&\qquad q_1 \ge 0 \\
&\qquad q_2 \ge 0
\end{aligned}
$$

Case-IIa:

$$
\begin{aligned}
&\max \mathrm{E}[\pi(q_1,q_2,D_1,D_2)] \\
&\text{subject to} \\
&\qquad 1-\Pr_{\pi_0}^{\mathrm{IIa}} \le \beta \\
&\qquad L_1 \le 0 \\
&\qquad L_2 \ge 0 \\
&\qquad (r_1-c_1)q_1+(r_2-c_2)q_2 \ge \pi_0 \\
&\qquad q_1 \ge 0 \\
&\qquad q_2 \ge 0
\end{aligned}
$$

Case-IIb:

$$
\begin{aligned}
&\max \mathrm{E}[\pi(q_1,q_2,D_1,D_2)] \\
&\text{subject to} \\
&\qquad 1-\Pr_{\pi_0}^{\mathrm{IIb}} \le \beta \\
&\qquad L_1 \ge 0 \\
&\qquad L_2 \le 0 \\
&\qquad (r_1-c_1)q_1+(r_2-c_2)q_2 \ge \pi_0 \\
&\qquad q_1 \ge 0 \\
&\qquad q_2 \ge 0
\end{aligned}
$$

Case-III:

$$
\begin{aligned}
&\max \mathrm{E}[\pi(q_1,q_2,D_1,D_2)] \\
&\text{subject to} \\
&\qquad 1-\Pr_{\pi_0}^{\mathrm{III}} \le \beta \\
&\qquad L_1 \le 0 \\
&\qquad L_2 \le 0 \\
&\qquad (r_1-c_1)q_1+(r_2-c_2)q_2 \ge \pi_0 \\
&\qquad q_1 \ge 0 \\
&\qquad q_2 \ge 0
\end{aligned}
$$

All cases (and sub-cases for a bounded demand distribution) defined above are added as additional constraints to the problem. The risk probabilities for each of these sub-cases are calculated in terms of q_1 and q_2 for determined cost, revenue and other parameters. This problem with the calculated risk probabilities and region constraints are solved separately by using GAMS software (BARON-Branch-And-Reduce Optimization Navigator) for all sub-cases. The case that gives the maximum profit of all cases is chosen. In order to verify this procedure yields the maximum profit, a simulation test is performed. This test calculates the estimated profit and estimated probability that profit exceeds the target profit. At the end of the numerical results, these two values that are calculated by the non-linear solver and the simulation test are found to be close to each other. Detailed numerical results are presented in section 6.2.

4.4 An Approximation Method for the Multi-product Case

In section 4.3, an exact solution to the 2-product newsvendor problem with a VaR constraint is obtained. As explained in the VaR model section, this model creates four regions in x-y coordinate system in which separate non-linear optimization problems should be solved and the global optimum of these local optimums should be selected. If this exact solution method is tried to be extended even to the three product case, there would be so many regions in the three-coordinate system (one coordinate for each product) and the problem would become would become very complex.

This complexity motivates the necessity of finding an alternative solution to the VaR model. The normal approximation could be a promising approach to the VaR problem provided that the number of products is sufficiently large.

Now, let us introduce the VaR approximation method in detail.

$$\begin{aligned} &\max_{q_1,q_2,..,q_n \geq 0} E\left[\pi\left(q_1,q_2,..,q_n,D_1,D_2,..,D_n\right)\right] \\ &\text{subject to} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (4.10) \\ &\qquad\qquad \Pr\left(\pi\left(q_1,q_2,..,q_n,D_1,D_2,..,D_n\right) \leq \pi_0\right) \leq \beta \end{aligned}$$

where

$$\pi(q_1,..,q_n,D_1,..,D_n)=\sum_{i=1}^{n} c_{u_i} D_i-(c_{u_i}(D_i-q_i)^+ +c_{o_i}(q_i-D_i)^+)$$

The product's demands are taken to be independently distributed random variables. Under the assumption that the conditions for Central Limit Theorem[2] are satisfied, the distribution of the profit function approaches to a normal distribution in a large product portfolio. Then the probabilistic VaR constraint can be expressed:

$$\Pr\left(\pi(q_1,..,q_n,D_1,,..,D_n)\le\pi_0\right)=\phi\left(\frac{\pi_0-E_\pi}{\sigma_\pi}\right)\le\beta \tag{4.11}$$

where

$E_\pi=E\left[\pi\left(q_1,q_2,..,q_n,D_1,D_2,..,D_n\right)\right]$ and $\sigma_\pi=\sqrt{Var\left[\pi\left(q_1,q_2,..,q_n,D_1,D_2,..,D_n\right)\right]}$.

By using the inverse of the normal distribution function, the same condition can be written as:

$$\pi_0-E_\pi\le\phi^{-1}(\beta)\sigma_\pi \tag{4.12}$$

or equivalently,

---

[2] Let x_i be independent random variables and let $E[x_i]=\mu_i$, $E[(x_i-\mu_i)^2]=\sigma_i^2$ for i=1,..,n. $s_n^2=\sum_{i=1}^{n}\sigma_i^2$ .For $s_n\neq 0$ and $E|x_i-\mu_i|^3<\infty$ then,

$$\left|\Pr\left(s_n^{-1}\sum_{i=1}^{n}(x_i-\mu_i)<x\right)-\phi(x)\right|\le 6s_n^{-3}\sum_{i=1}^{n}E|x_i-\mu_i|^3$$. (Sazonov [33] and Thurber [34])

$$\left(E_{\pi}-\pi_{0}\right)^{2}-\left(\phi^{-1}(\beta)\right)^{2}\sigma_{\pi}^{2}\geq 0 \tag{4.13}$$

when $\beta \leq 0.5$ and $\pi_0 \leq E_\pi$.

Since we consider products with independent demand distributions, E_π and σ_π can be calculated as $E_{\pi}=\sum_{i=1}^{n}E\left[\pi\left(q_{i},D_{i}\right)\right]$ and $\sigma_{\pi}^{2}=\sum_{i=1}^{n}Var\left[\pi\left(q_{i},D_{i}\right)\right]$.

Then, the solution of the optimization problem given in the equation (4.10) can be approximated by the solution of the following problem:

$$\begin{aligned}
&\max_{q,q_{2},\ldots,q_{n}\geq 0}\ \sum_{i=1}^{n}E\left[\pi\left(q_{i},D_{i}\right)\right]\\
&\text{subject to}\\
&\qquad\left(\sum_{i=1}^{n}E\left[\pi\left(q_{i},D_{i}\right)\right]-\pi_{0}\right)^{2}-\left(\phi^{-1}(\beta)\right)^{2}\sum_{i=1}^{n}Var\left[\pi\left(q_{i},D_{i}\right)\right]\geq 0\\
&\qquad\pi_{0}\leq E_{\pi}
\end{aligned} \tag{4.14}$$

The normal approximation problem can be solved for the *N*-product case for different demand distributions because the problem becomes a single NLP problem with a single constraint without any complexities caused by the regions.

Let us consider first uniformly distributed demand case. When the demands of the products are uniformly distributed with parameters lower bound a_i and upper bound b_i, then the expected value and the variance of the profit function are given as

$$E\left[\pi\left(q_{i},D_{i}\right)\right]=\frac{1}{2\left(a_{i}-b_{i}\right)}\left(a_{i}^{2}\left(c_{u_{i}}+c_{o_{i}}\right)-2a_{i}c_{o_{i}}q_{i}+q_{i}\left(-2b_{i}c_{u_{i}}+\left(c_{u_{i}}+c_{o_{i}}\right)q_{i}\right)\right) \tag{4.15}$$

$$Var[\pi(q_i,D_i)] = \frac{-\left(a_i^2\left(c_{u_i}+c_{o_i}\right)-2a_i c_{o_i} q_i + q_i\left(-2b_i c_{u_i}+\left(c_{u_i}+c_{o_i}\right)q_i\right)\right)^2}{\left(4(a_i-b_i)^2\right)} +$$

$$\frac{1}{3(a_i-b_i)}\left(a_i^3\left(c_{o_i}^2+c_{u_i}^2\right)-3a_i^2 c_{o_i}^2 q_i + 3a_i c_{o_i}^2 q_i^2 + q_i^2\left(-3b_i c_{u_i}^2+\left(-c_{o_i}+c_{o_i}c_{u_i}+2c_{u_i}^2\right)q_i\right)\right) \quad (4.16)$$

Using these equations in (4.15) and solving the NLP yields the order quantities that maximize the expected profit subject to the VaR constraint.

Similarly, when the demands of the products are exponentially distributed with parameter λ_i, then the expected value and the variance of the function are as below:

$$E[\pi(q_i,D_i)] = \frac{1}{\lambda_i}\left(c_{o_i} - e^{-q_i\lambda_i}c_{o_i} + c_{u_i} - e^{-q_i\lambda_i}c_{u_i} - c_{o_i}q_i\lambda_i\right) \quad (4.17)$$

$$\begin{aligned} Var[\pi(q_i,D_i)] = \frac{e^{-2q_i\lambda_i}}{\lambda_i^2}\Big(&\left(-1+4e^{q_i\lambda_i}+e^{2q_i\lambda_i}q_i\lambda_i\right)c_{u_i}^2 + c_{o_i}^2\left(-1+e^{2q_i\lambda_i}-2e^{q_i\lambda_i}q_i\lambda_i\right) \\ &+2\left(-1+e^{q_i\lambda_i}\right)c_{o_i}c_{u_i}\left(1+e^{q_i\lambda_i}\left(-1+q_i\lambda_i\right)\right)\Big) \end{aligned} \quad (4.18)$$

In order to compare the performance of this approximation method, the solution of the NLP formulation obtained by a non-linear solver and the simulation results for the original VaR problem are compared for different demand distributions.

We solved our approximation model for the products which have independent and exponentially distributed demands with mean 10 and independent and uniformly distributed demands in the interval [0,20]. The results can be seen in Table 4.1a and Table 4.1b. In the second columns of each table, there is an optimum profit value and the optimum product portfolio given as a result of the approximation model for identical products. The trials are done for various sized product portfolios such as 2-product, 5-product, 10-product, 20-product and 30-product. Also, a VaR simulation is run, its outputs are processed and the results are given in the third column. The optimum order quantities reported in the simulation column is gathered through a total enumeration method with an incremental size of 0.1. In simulation, the probability values and the profit values are calculated. The simulated profit and probability values would have some small instabilities with respect to the real values because of the random behavior of the demands and the profit. In order to get rid of these instabilities, the probability values and profit values are fitted to an appropriate function. A clear interpretation could be gathered throughout this method. In the third column, the maximum of these fitted profit values where the fitted probability values satisfy the risk constraint are given.

	Approximation		Simulation		Product Quantity's Percentage Absolute Error
	Profit	Quantity pairs	Profit	Quantity pairs	
2-product	68.19	(9.24,9.24)	68.64	(9.4,9.4)	1.70%
5-product	180	(12,...,12)	180.97	(12,..,12)	0.00%
10-product	360	(12,..,12)	359.92	(12,..,12)	0.00%
20-product	720	(12,..,12)	718.41	(12,..,12)	0.00%
30-product	1080	(12,...,12)	1083.62	(12,..,12)	0.00%

Table 4.1a: The Results of the Approximation Method for Uniformly Distributed Demands (Demands ~U (0,20), π_0=0, c_u =6, c_o =4, β=0.05)

	Approximation		Simulation		Product Quantity's Percentage Absolute Error
	Profit	Quantity pairs	Profit	Quantity pairs	
2-product	39.77	(5.27,5.27)	44.09	(7.5,7.5)	29.72%
5-product	116.68	(8.91,...8.91)	116.61	(8.5,...,8.5)	4.81%
10-product	233.48	(9.16,..,9.16)	234.44	(9.2,...,9.2)	0.40%
20-product	466.77	(9.16,..,9.16)	471.43	(9.2,…,9.2)	0.40%
30-product	700.45	(9.16,...,9.16)	707.55	(9.2,...,9.2)	0.40%

Table 4.1b: The Results of the Approximation Method for Exponentially Distributed Demands
(π_0=0, c_u =6, c_o =4, $1/\lambda$= 10, β=0.05)

We expect that there would be a decrease in the percentage errors as the product amount in the portfolio increases. In both tables, these error percentages decrease as we have expected. Another observation is that, the percentage errors are lower in products that have uniformly distributed demands than in products that have exponentially distributed demands. This is understandable since the profit induced by uniform and exponential demand distribution may converge to a normal distribution at different rates.

In Table 4.1a, the optimum product order quantities percentage error is zero for 5-product, 10-product, 20-product and 30-product because the constraint is not binding for these cases. In Table 4.1b, the optimum product order quantities percentage error is 0.40 for 10-product, 20-product and 30-product because the constraint is not binding for these cases. (This small difference is because of the quantity increment of 0.1 used in the simulation. If the increment size of 0.01 is used in the simulation, the error would be zero.) From these results seen from Table 4.1a and 4.1b, the target profit value should be determined for each size of a product portfolio and this is a managerial problem.

4.5 Conclusion

In this chapter, the risk-averse newsvendor problem is considered where the risk is controlled by the VaR constraint. As stated before, this problem was solved for single product in the literature. The aim is to solve the problem for a large product portfolio and investigate the parameters effect on the decision variables and the objective function. Initially, the problem is considered for two products. The risk constraint is studied in four different disjoint regions because of the nature of the newsvendor problem. Four NLP models are solved for these regions separately and the one which attains the maximum expected profit of all is chosen. For the N product case, this approach would be very complex and cannot be applied. As stated before, decision makers generally desire to solve this model for a large product portfolio and know how much to invest on each product or asset. A normal approximation for large product portfolio is proposed because the profit function for more products converges to a normal distribution –Central Limit Theorem approach. The numerical results demonstrate that the error is small so that the approach could be reasonable for large product portfolios.

Chapter 5

NEWSVENDOR PROBLEM WITH A CONDITIONAL VALUE-AT-RISK (CVAR) CONSTRAINT

5.1 Introduction

In the literature, the classical newsvendor problem was solved for risk-neutral buyers and sellers. In Chapter 3, the questions “What type of risk measures were used in the literature?” and “What kind of properties do these risk measures have?” were answered. In Chapter 4, the newsvendor problem was considered with a VaR constraint. CVaR is defined as the conditional expected loss exceeding the VaR value. VaR and CVaR are popular risk measures in the finance literature. In this chapter, we examine the solution procedure and the results for the newsvendor problem with the CVaR constraint. This motivates the use of CVaR in different applications. This chapter focuses on the investigation of the multi-product newsvendor problem with a CVaR type of risk constraint where CVaR is defined in terms of a random newsvendor’s profit function (the distribution of the CVaR is arbitrary in contrast with the optimization of CVaR in finance literature).

The model description is given in section 5.2. In section 5.3, the CVaR model for two products is studied. Then, an LP formulation of the CVaR optimization problem for the newsvendor problem is given in section 5.4.

5.2 Model Description

As expressed before, the objective of the standard newsvendor problem is the maximization of the expected profit without any constraint for a risk neutral newsvendor. Using the CVaR measure, there are several alternative formulations which are plausible. Our focus is on two main models. The first is the maximization of the CVaR function if it is defined in terms of the newsvendor's profit function (or the minimization of the CVaR function if it is defined in terms of the newsvendor's loss function). The maximization of the CVaR function which is defined in terms of the profit function and the minimization of the CVaR function which is defined in terms of the loss function are as follows:

$$\max \mathrm{CVaR}(q,\pi_0) \text{ where } \mathrm{CVaR}(q,\pi_0) = \pi_0 + \frac{1}{1-\beta}\int_D \left[(r-c)q-(r-s)(q-D)^+ - \pi_0\right]^- dF(D) \quad (5.1)$$

$$\min \mathrm{CVaR}(q,\pi_0) \text{ where } \mathrm{CVaR}(q,\pi_0) = \pi_0 + \frac{1}{1-\beta}\int_D \left[-(r-c)q+(r-s)(q-D)^+ - \pi_0\right]^+ dF(D) \quad (5.2)$$

At this point; let us give some insights about these two CVaR functions above, where the CVaR functions are given in terms of the newsvendor profit or loss function. In equation (5.1); the newsvendor profit function is zero if the product order quantity is zero. In order to have a non-zero integrand in equation (5.1), VaR should be greater than or equal to zero. So, the equation becomes $-\beta\ \pi_0/(1-\beta)$. To attain the optimum value under these circumstances, VaR (π_0) is set to zero and the optimum CVaR is zero. On the other hand, if the integrand is zero where VaR is less than or equal to zero, the equation becomes π_0. To attain the optimum value under these conditions, VaR is set to zero and the optimum CVaR is zero.

Moreover, the newsvendor loss function is zero if the product order quantity is zero in equation (5.2). If the integrand is non-zero in equation (5.2), VaR is less than or equal to zero and the equation becomes $-\beta\ \pi_0/(1-\beta)$. To attain the optimum value under these conditions, VaR is set to zero and the optimum CVaR value is zero. On the other hand; if the integrand is zero where VaR is greater than or equal to zero, the equation becomes π_0. VaR is set to zero and CVaR is found to be zero in the optimum point for this case.

The second model is the maximization of the expected profit function subject to the CVaR constraint where the CVaR function is controlled below a CVaR^0 value. This is similar to the formulation of the VaR-constrained problem studied in Chapter 4. This model is shown as below:

$$
\begin{aligned}
&\max_{q_1,q_2,..,q_n,\pi_0} E[\pi(q_1,q_2,..,q_n,D_1,D_2,..,D_n)] \\
&\text{subject to} \\
&\qquad \text{CVaR} \le \text{CVaR}^0
\end{aligned}
\tag{5.3}
$$

5.3 Two-Product Case

The maximization of the CVaR function, which is defined in terms of the profit function of the two product newsvendor problem, without any constraint is of interest in this part of this study. This is the extension of the single-product model initiated by Chen et al. [21]. This objective function could be examined in different regions for the two products case similar to the VaR problem discussed in Chapter 4. The maximization of the CVaR model and the CVaR function is defined as below:

$$\max \text{CVaR}(q_1,q_2,\pi_0)=\pi_0+\frac{1}{1-\beta}\int_{D_2}\int_{D_1}[K]^- dF(D_1)dF(D_2) \tag{5.4}$$

where

$$K=(r_1-c_1)q_1-(r_1-s_1)(q_1-D_1)^+ +(r_2-c_2)q_2-(r_2-s_2)(q_2-D_2)^+ -\pi_0 \tag{5.5}$$

In order to analyze this function, we consider the function in four regions (salvage value is assumed to be zero) as follows:

$$\begin{aligned}\text{CVaR}\ (q_1,q_2,\pi_0)= &\left(\pi_0+\frac{1}{1-\beta}\int_{D_2}\int_{D_1}[r_1D_1-c_1q_1+r_2D_2-c_2q_2-\pi_0]^- dF(D_1)dF(D_2)\right)\Pr(D_1\le q_1,D_2\le q_2)+\\ &\left(\pi_0+\frac{1}{1-\beta}\int_{D_2}\int_{D_1}[r_1D_1-c_1q_1+r_2q_2-c_2q_2-\pi_0]^- dF(D_1)dF(D_2)\right)\Pr(D_1\le q_1,D_2> q_2)+\\ &\left(\pi_0+\frac{1}{1-\beta}\int_{D_2}\int_{D_1}[r_1q_1-c_1q_1+r_2D_2-c_2q_2-\pi_0]^- dF(D_1)dF(D_2)\right)\Pr(D_1> q_1,D_2\le q_2)+\\ &\left(\pi_0+\frac{1}{1-\beta}\int_{D_2}\int_{D_1}[r_1q_1-c_1q_1+r_2q_2-c_2q_2-\pi_0]^- dF(D_1)dF(D_2)\right)\Pr(D_1> q_1,D_2> q_2)\end{aligned}$$

The expression $[K]^-$ in the CVaR function could be expressed as $K-[K]^+$ where K is given in equation (5.5). The second part of the equation resembles the downside risk constraint in the VaR problem and a similar method used in section 4.3 can be adapted in this calculation.

Similarly; let us define two critical values for demands L_1 and L_2 that are defined as the minimum value of D_1 required to achieve π_0 and the minimum value of D_2 required to achieve π_0 respectively. After defining these critical values L_1 and L_2 for demands, the objective function can be defined in terms of these values in four cases:

<u>Case-I</u>: If L_1 and L_2 are both greater than zero, the CVaR objective function region can be seen in Figure 5.1 and the objective function becomes:

$$\text{CVaR}\left(q_1,q_2,\pi_0\right)=\text{CVaR}_1=\pi_0+\frac{1}{1-\beta}\left[A_1+A_2+A_3+A_4\right]\text{where}$$

$$A_1=\int_{q_2}^{\infty}\int_{0}^{L_1}\left(r_1D_1-c_1q_1+r_2q_2-c_2q_2-\pi_0\right)dF_1\left(D_1\right)dF_2\left(D_2\right)$$

$$A_2=\int_{0}^{q_2}\int_{0}^{L_1}\left(r_1D_1-c_1q_1+r_2D_2-c_2q_2-\pi_0\right)dF_1\left(D_1\right)dF_2\left(D_2\right)$$

$$A_3=\int_{L_1}^{q_1}\int_{0}^{\frac{\pi_0+c_1q_1+c_2q_2-r_1D_1}{r_2}}\left(r_1D_1-c_1q_1+r_2D_2-c_2q_2-\pi_0\right)dF_1\left(D_1\right)dF_2\left(D_2\right)$$

$$A_4=\int_{0}^{L_2}\int_{q_1}^{\infty}\left(r_1q_1-c_1q_1+r_2D_2-c_2q_2-\pi_0\right)dF_1\left(D_1\right)dF_2\left(D_2\right)$$

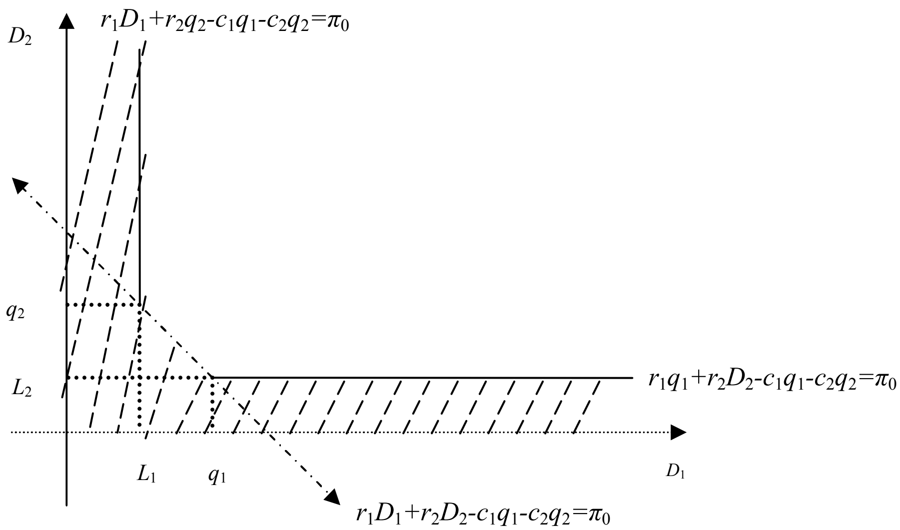

Figure 5.1: CVaR Objective Function Region in Case-I

Case-IIa: If L_1 is negative and L_2 is positive, the CVaR objective function region can be seen in Figure 5.2 and the objective function becomes:

$$\text{CVaR}(q_1,q_2,\pi_0)=\text{CVaR}_{\text{IIa}}=\pi_0+\frac{1}{1-\beta}\left[B_1+B_2\right]$$

where

$$B_1=\int_0^{q_1}\int_0^{\frac{\pi_0+c_1q_1+c_2q_2-r_1D_1}{r_2}}(r_1D_1-c_1q_1+r_2D_2-c_2q_2-\pi_0)\,dF_1(D_1)\,dF_2(D_2)$$

$$B_2=\int_0^{L_2}\int_{q_1}^{\infty}(r_1q_1-c_1q_1+r_2D_2-c_2q_2-\pi_0)\,dF_1(D_1)\,dF_2(D_2)$$

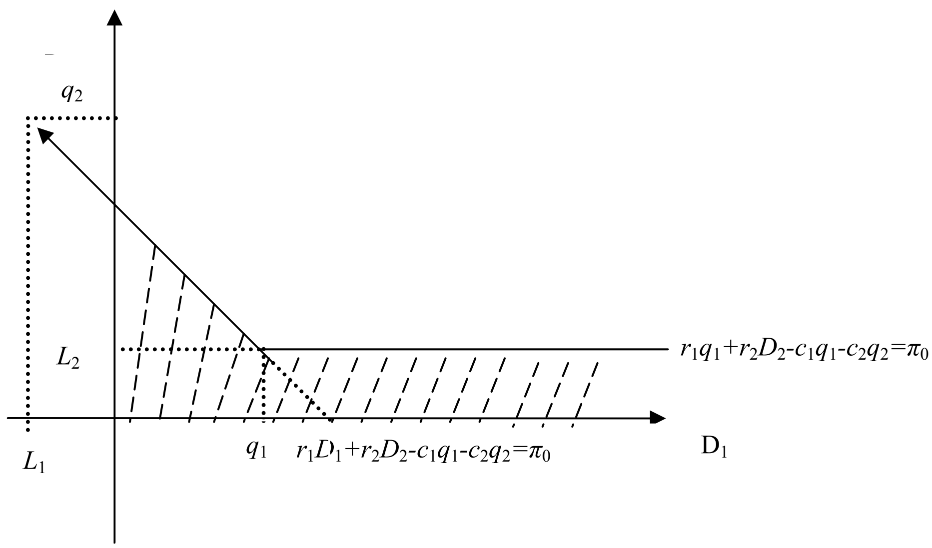

Figure 5.2: CVaR Objective Function Region in Case-IIa

Case-IIb: If L_1 is positive and L_2 is negative, the CVaR objective function region can be seen in Figure 5.3 and the objective function becomes:

$$\mathrm{CVaR}(q_1,q_2,\pi_0)=\mathrm{CVaR}_{\mathrm{IIb}}=\pi_0+\frac{1}{1-\beta}[C_1+C_2+C_3]$$

where

$$C_1=\int_{q_2}^{\infty}\int_{0}^{L_1}(r_1D_1-c_1q_1+r_2q_2-c_2q_2-\pi_0)\,dF_1(D_1)\,dF_2(D_2)$$

$$C_2=\int_{0}^{q_2}\int_{0}^{L_1}(r_1D_1-c_1q_1+r_2D_2-c_2q_2-\pi_0)\,dF_1(D_1)\,dF_2(D_2)$$

$$C_3=\int_{0}^{q_2}\int_{L_1}^{\frac{\pi_0+c_1q_1+c_2q_2-r_2D_2}{r_1}}(r_1D_1-c_1q_1+r_2D_2-c_2q_2-\pi_0)\,dF_1(D_1)\,dF_2(D_2)$$

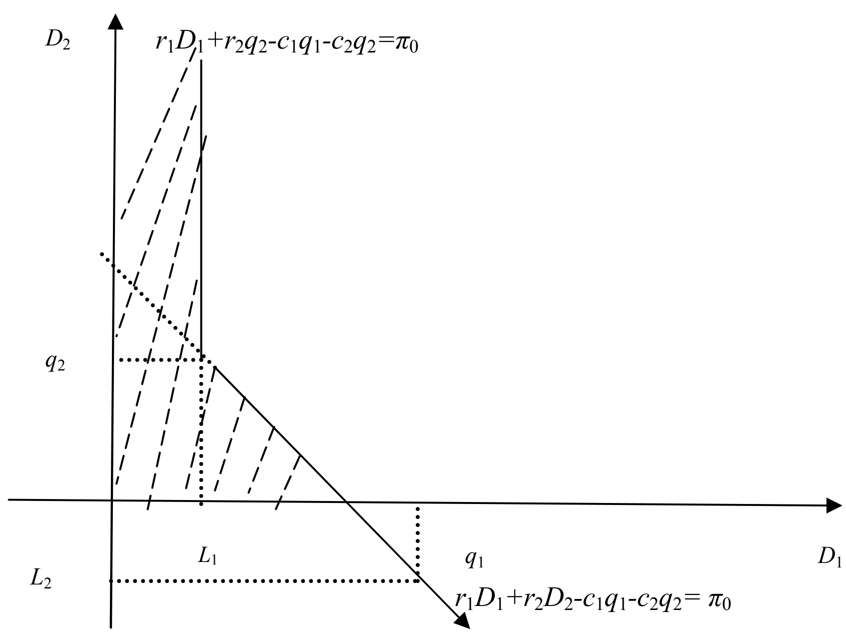

Figure 5.3: CVaR Objective Function Region in Case-IIb

Case-III: If L_1 is negative and L_2 is positive, the CVaR objective function region can be seen in Figure 5.4 and the objective function becomes:

$$\text{CVaR}\left(q_1,q_2,\pi_0\right)=\text{CVaR}_{\text{III}}=\pi_0+\frac{1}{1-\beta}\left[\int_0^{\frac{\pi_0+c_1q_1+c_2q_2}{r_2}}\int_0^{\frac{\pi_0+c_1q_1+c_2q_2-r_2D_2}{r_1}}\left(r_1D_1-c_1q_1+r_2D_2-c_2q_2-\pi_0\right)dF_1\left(D_1\right)dF_2\left(D_2\right)\right]$$

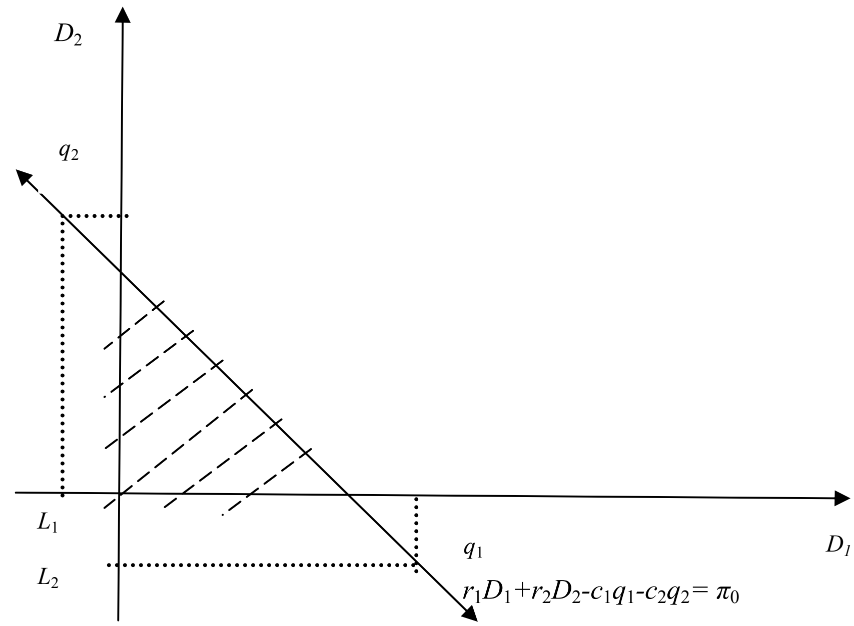

Figure 5.4: CVaR Objective Function Region in Case-III

The CVaR objective function that is considered in the four cases above could be shown in the q_1-q_2 coordinate system below and it can be seen that each case falls into separate regions. When the slope of L_1 is less than the slope of L_2-that means $c_1c_2< (r_1-c_1)(r_2-c_2)$, the graph will be as the Figure 4.5a. When the slope of L_1 is greater than the slope of L_2-that means $c_1c_2> (r_1-c_1)(r_2-c_2)$, the graph will be as the Figure 4.5b.

From these two graphs, we have four optimization problems that are defined in their own regions as shown above-some constraints added to define these regions:

Case-I:

$$
\begin{aligned}
&\max\ \text{CVaR}_{\text{I}}\\
&\text{subject to}\\
&\qquad L_1 \geq 0\\
&\qquad L_2 \geq 0\\
&\qquad (r_1 - c_1)\,q_1 + (r_2 - c_2)\,q_2 \geq \pi_0\\
&\qquad q_1 \geq 0\\
&\qquad q_2 \geq 0\\
&\qquad \pi_0 \geq 0
\end{aligned}
$$

Case-IIa:

$$
\begin{aligned}
&\max\ \text{CVaR}_{\text{IIa}}\\
&\text{subject to}\\
&\qquad L_1 \geq 0\\
&\qquad L_2 \geq 0\\
&\qquad (r_1 - c_1)\,q_1 + (r_2 - c_2)\,q_2 \geq \pi_0\\
&\qquad q_1 \geq 0\\
&\qquad q_2 \geq 0\\
&\qquad \pi_0 \geq 0
\end{aligned}
$$

Case-IIb:

$$
\begin{aligned}
&\max\ \text{CVaR}_{\text{IIb}}\\
&\text{subject to}\\
&\qquad L_1 \geq 0\\
&\qquad L_2 \geq 0\\
&\qquad (r_1 - c_1)\,q_1 + (r_2 - c_2)\,q_2 \geq \pi_0\\
&\qquad q_1 \geq 0\\
&\qquad q_2 \geq 0\\
&\qquad \pi_0 \geq 0
\end{aligned}
$$

Case-III:

$$
\begin{aligned}
&\max \ \text{CVaR}_{\text{III}} \\
&\text{subject to} \\
&\qquad L_1 \geq 0 \\
&\qquad L_2 \geq 0 \\
&\qquad (r_1 - c_1)\, q_1 + (r_2 - c_2)\, q_2 \geq \pi_0 \\
&\qquad q_1 \geq 0 \\
&\qquad q_2 \geq 0 \\
&\qquad \pi_0 \geq 0
\end{aligned}
$$

Here, the procedure is similar to the procedure defined in section 4.3. The CVaR optimization problem with the region constraints are solved separately by using GAMS software (BARON-Branch-And-Reduce Optimization Navigator) for all sub-cases. The case that gives the maximum CVaR value of all cases is chosen. A simulation test is performed in order to verify this value. This test calculates the estimated profit and estimated probability that profit exceeds the target profit. At the end of the numerical results, these two values that are calculated by the non-linear solver and the simulation test are found to be close to each other. The numerical results are deeply analyzed in section 6.3.

5.4 The Case with Multiple Products

If we try to generalize the CVaR model to the *N*-product case, we encounter similar complications with the complications that appear during the generalization of VaR model to the *N* product case with difficult characterizations. There would be more than four regions in more than 2 dimensions. Therefore, as an alternative approach we adapt the LP model which is developed by Rockafellar and Uryasev [22]. As expressed in the section 3.3.5, the original CVaR optimization problem is converted to an LP model.

5.4.1 CVaR Optimization via an LP approach

The minimization of the CVaR function in terms of a general loss distribution was converted to an LP model by Rockafellar and Uryasev [22]. This CVaR function and the LP conversion are given in the equations (3.26) and (3.27). This formulation is motivated by financial problems. The important point in this approach is that the loss function $f(x,y)$ used in the CVaR function has a known probability distribution. In contrast, the loss function in the newsvendor problem is not directly known but has to be calculated using the problem structure.

The result of the LP formulation above is proved to be optimal by Rockafellar and Uryasev [27]. The minimization of the CVaR function with respect to x and ξ can be carried out by first minimizing over ξ for fixed x and then minimizing the result over x. The proof states that if the CVaR function is convex over x and ξ, and the constraints is a convex set, the solution is optimal. Also, convexity is a key property in optimization that eliminates the possibility of a local minimum being different from a global minimum. (Rockafellar [31] and Lemaréchal and Hiriart-Urruty [32])

For this purpose Rockafellar and Uryasev [27] prove that the CVaR function is convex with respect to (x, ξ) whenever the integrand $[\, f(x,\zeta) - \pi_0 \,]^+$ is convex with respect to (x, ξ). For each random variable, $f(x,\zeta) - \pi_0$ is convex. Also, $[t]^+$ is a non-decreasing and convex function. So, $[\, f(x,\zeta) - \pi_0]^+$ can be expressed as a convex function. The convexity of the CVaR function follows from the fact that minimizing of an extended-real-valued convex function of two vector variables with respect to one of those variables, results in a convex function of the remaining variable.

Our CVaR function that consists of a newsvendor's loss function does not have a given profit distribution since the profit function depends on the random demand. The CVaR function is as below:

$$F_\beta(q_1,..,q_m,\pi_0) = \pi_0 + \frac{1}{1-\beta}\iint_D [M]^+ dD$$

where

$$M = -\pi_0 + \sum_{i=1}^{m} -(r_i - c_i)q_i + (r_i - s_i)(q_i - D_i)^+ \tag{5.6}$$

Our function should be examined in terms of convexity. We know that the newsvendor's profit function is concave in terms of the product order quantities (q's). There is a key property that if a function f is concave, another function g which is formed by multiplying f by minus one is a convex function. So; the function in the equation (5.6) is convex in terms of q_i's and π_0. According to the proof in Rockafellar and Uryasev [27] described above; the integrand $[M]^+$ is also convex in terms of q_i's and π_0.

So, the adaptation of the CVaR minimization for the newsvendor's loss function to an LP model would yield an optimum result. We adapt the minimization of the CVaR function for m products to an LP model as follows:

$$\min \quad \frac{1}{1-\beta}\sum_{i_1=1}^{N}\sum_{i_2=1}^{N}\cdots\sum_{i_m=1}^{N} p_{i_1}p_{i_2}..p_{i_m} z_{i_1 i_2 .. i_m} + \pi_0$$

subject to

$$z_{i_1 i_2 .. i_m} \geq -\pi_0 + \sum_{j=1}^{m}\left(r_j - s_j\right)x_{i_j} - \sum_{j=1}^{m}(r_j - c_j)q_j \qquad \text{for all } i_1,..,i_m$$

$$z_{i_1 i_2 .. i_m} \geq 0 \qquad \text{for all } i_1,..,i_m$$

$$x_{i_j} \geq q_j - D_{i_j} \qquad \text{for all } i_j \text{ and } j$$

$$x_{i_j} \geq 0 \qquad \text{for all } i_j \text{ and } j$$

$$q_m \geq 0 \qquad \text{for all } m$$

π_0: unconstrained

Notations:

p_{i_k}: is the discrete probability value for demand of product k where i_m is the discretization intervals.

i_m: is the discretization interval for product m which takes values 1 to N.

$z_{i_1..i_m}$: the auxiliary variable defined to determine the non-negativity of the function $-\pi_0+\sum_{j=1}^{m}(r_j-s_j)x_{i_j}-\sum_{j=1}^{m}(r_j-c_j)q_j$ for all products in each discretization interval.

x_{i_k}: the variable defined to determine the non-negativity of the function $q_k-D_{i_k}$ for product k in each discretization interval.

In this problem, we have a general demand distribution for each product. In the numerical experiments, we have discretized the demand in N intervals and used the discretized demand and the corresponding probability values in the described model.

5.4.2 The CVaR-Constrained Newsvendor Problem via an LP Approach

Recall that we are also interested in the problem of the maximization of the expected profit function subject to a CVaR constraint. The adaptation of this second model to an LP is as follows:

$$\max \sum_{j=1}^{m}(r_j-c_j)q_j+\sum_{i_1=1}^{N}(s_1-r_1)x_{i_1}+\sum_{i_2=1}^{N}(s_2-r_2)x_{i_2}+\ldots+\sum_{i_m=1}^{N}(s_m-r_m)x_{i_m}$$

subject to

$$\frac{1}{1-\beta}\sum_{i_1=1}^{N}\sum_{i_2=1}^{N}\cdots\sum_{i_m=1}^{N} p_{i_1}p_{i_2}..p_{i_m} z_{i_1 i_2 ..i_m} + \pi_0 \leq \text{CVaR}^0$$

$$z_{i_1 i_2 ..i_m} \geq -\pi_0 + \sum_{j=1}^{m}\left(r_j - s_j\right)x_{i_j} - \sum_{j=1}^{m}(r_j - c_j)\, q_j \qquad \text{for all } i_1,..,i_m$$

$z_{i_1 i_2 ..i_m} \geq 0$ for all $i_1,..,i_m$

$x_{i_j} \geq q_j - D_{i_j}$ for all i_j and j

$x_{i_j} \geq 0$ for all i_j and j

$q_m \geq 0$ for all m

π_0: unconstrained

It is difficult to estimate CVaR^0 (upper bound of CVaR) without solving the optimization problem intuitively. The upper bound value CVaR^0 is found by using π_0 value which is used in the VaR-constrained problem and the optimum order quantities (q^*) of this VaR-constrained problem by simulation.

5.5 Conclusion

In this chapter, the risk-averse newsvendor problem is investigated where the financial risk of the system is controlled by the Conditional Value-at-Risk constraint or optimizing the CVaR function in terms of the newsvendor's profit or loss function. For a given CVaR distribution, adapting the CVaR optimization to an LP formulation is known. In this chapter, the CVaR function does not have a given distribution because of the nature of the newsvendor profit or loss function. LP formulations for both CVaR optimization and the expected profit maximization with CVaR constraint where the CVaR function is defined in terms of the newsvendor function are presented. Numerical studies and the results of the experiments on CVaR optimization are presented in section 6.3.

Chapter 6

NUMERICAL RESULTS

6.1 Introduction

Up to now, the newsvendor problem is discussed where the risk behavior of the problem is desired to be controlled. Two different approaches are discussed in the previous chapters. The first approach is the VaR-constrained newsvendor problem and the other one is the CVaR-constrained newsvendor problem or the CVaR optimization. The proposed methods are discussed in Chapter 4 and 5.

In this chapter, the results of the numerical experiments are presented. For the single product case, the problem for each model was solved in the past. The parameters' effects on the decision variables and the objective function are investigated. For two or more product cases, our models are solved and sensitivity analysis is performed. The numerical analysis for the VaR-constrained problem and the CVaR problem are given in sections 6.2 and 6.3 respectively.

6.2 Numerical Analysis for a Newsvendor Problem with a VaR Constraint

In this section, numerical results for the VaR-constrained newsvendor problem are analyzed. In section 6.2.1, the results of the past studies for a single product are analyzed. In section 6.2.2, the two products newsvendor problem with VaR constraint is solved for different instances and numerical results are evaluated. For more products, it was solved in section 4.4.

6.2.1 The Single Product Problem

As stated in the literature review, the unconstrained and VaR-constrained newsvendor problems for a single product were solved by Gan et al. [20]. The solutions for these problems are also stated in the subsection 3.3.4. The effects of the variables on the optimum order quantity will be given in the two sub-sections below for both the risk-averse and the risk-neutral newsvendor problem.

6.2.1.1 Single Product Risk-Neutral Newsvendor Problem

The solution for the classical risk-neutral newsvendor problem where the profit function is $\pi\,(q, D) = (r\text{-}c)\,q\text{-}(r\text{-}s)\,\max\,(q\text{-}D, 0)$ was given in Equation 3.2.

In this problem, c_u is denoted as the underage cost that equals r-c. When the product order quantity is taken to be fewer than the product's demand, this opportunity profit –c_u- is lost. As the underage cost (c_u) increases, the optimum order quantity (q^*) increases. Second derivative of the q^* in terms of c_u is less than zero, q^* versus c_u is a concave function. The proof for these claims for any demand distribution is given in the Appendix A1.

As stated before, c_o is denoted as the overage cost that equals c-s. When the product order quantity is chosen to be more than the product's demand, this cost –c_o- is lost. As c_o increases, the optimum order quantity q^* decreases. The second derivative of the q^* in terms of c_o is greater than zero, q^* is a convex function in c_o. The proof for these claims for any demand distribution is given in the Appendix A2.

There is also an interesting parameter that was an important effect on the optimum order quantity. This is the demand's coefficient of variation (cv). The coefficient of variation of a demand is the ratio of the standard deviation of the demand to the expected value of the demand.

While observing the effect of the demand's cv on the optimum product quantity, the demand's mean should be kept constant in order to isolate demand's mean effect on the optimum product quantity. For uniformly distributed demand – U (a,b), the coefficient of variation is $\frac{(b-a)}{\sqrt{3}(b+a)}$.

Keeping the mean demand constant indicates that the denominator of the cv is also constant. In order to observe the effect of cv on the optimum product quantity, looking at the effect of the difference of the upper bound and the lower bound parameters of the uniform distribution is sufficient:

$$q^* = \frac{c_u}{(c_u+c_o)}(b-a)+\left(\frac{b+a}{2}\right)-\left(\frac{b-a}{2}\right)=\left(\frac{b+a}{2}\right)+(b-a)\left(\frac{c_u-c_o}{2(c_u+c_o)}\right) \tag{6.1}$$

The result shows that if c_u is greater than c_o, increasing the difference results in a raise in q^* with a linear trend; Otherwise, if c_u is less than c_o, increasing the difference results in a fall in q^* with a linear trend for uniformly distributed demand.

For a product that has a normally distributed demand with parameters μ and σ, the optimum order quantity is given as below:

$$q^* = \mu + z^*\sigma \tag{6.2}$$

where $F_Z(z^*) = \frac{c_u}{c_u+c_o}$ and Z is the standard normal random variable.

Keeping the mean demand constant and increasing the standard deviation means increasing the coefficient of variation of a demand. If the cv of the product demand increases and $c_u/(c_u+c_o)$ is greater than 0.5, the optimum order quantity increases. If the cv of the product demand increases and $c_u/(c_u+c_o)$ is less than 0.5, the optimum order quantity decreases. These properties carry over to the standard formulation of the multi-product problem with independent demands.

6.2.1.2 The Single Product Risk-Averse Newsvendor Problem

As stated before, the newsvendor problem with the downside risk constraint with zero salvage value is solved by Gan et al. [20] and the optimum order quantity was summarized with the equations 3.18 and 3.19.

As c_u increases, q^* increases with a linear trend. This result can be observed from the equation for the optimum quantity in the binding constraint region that is manipulated and shown below:

$$q^* = \frac{(c_u + c_o)F^{-1}(\beta) - \pi_0}{c_o} \tag{6.3}$$

In order to observe the effect of c_u for both of the regions, risk-averse and risk-neutral cases are combined and the graph in the Appendix A3 is formed. In this graph, the regions where the problem is infeasible, the risk constraint is binding or not and the behavior of the optimum order quantity according to the underage cost is shown in detail.

The effect of the overage cost c_o on the optimum order quantity has two different behaviors depending on a single condition. If the condition $\pi_0 > c_u F^{-1}(\beta)$ is satisfied, the optimum product quantity q^* is an increasing concave function. If this condition is not satisfied, q^* is a decreasing convex function. These can be shown as below:

$$\frac{\partial q^*}{\partial c_o} = \frac{-c_u F^{-1}(\beta)+\pi_0}{c_o^{\,2}}$$
$$\frac{\partial^2 q^*}{\partial c_o^{\,2}} = \frac{2\left(c_u F^{-1}(\beta)-\pi_0\right)}{c_o^{\,3}} \tag{6.4}$$

In order to observe the effect of c_o for both of the regions, risk-averse and risk-neutral cases are combined and the graphs in the Appendix A4 are formed. In this graph, the regions where the problem is infeasible, the risk constraint is binding or not and the behavior of the optimum order quantity according to the overage cost is shown in detail.

The optimum order quantity versus cv of the demand is an increasing linear function if β is greater than ½ where the demand is uniformly distributed with parameters a and b. If β is less than ½, it is a decreasing linear function. This claim can be observed from the optimum order quantity defined below:

$$q^* = \frac{\mu(c_u + c_o)-\pi_0}{c_o} + \frac{(2\beta-1)(c_u+c_o)}{2c_o}(b-a) \tag{6.5}$$

In this setting, the optimization problem could be either in an infeasible state, or has a binding constraint or does not have a binding constraint. If the problem has a binding constraint, the optimum quantity is equal to the solution of the constrained optimization problem. If the problem does not have a binding constraint, the optimum quantity is equal to the solution of the classical newsvendor problem. The regions defining the problem states and the corresponding problem states can be summarized in the Appendix A5.

If the product demand is normally distributed with parameters μ and σ, the optimum order quantity is shown as below:

$$q^* = \frac{r\left(\phi^{-1}(\beta)\sigma + \mu\right)-\pi_0}{c} \tag{6.6}$$

Keeping the mean demand constant and increasing the standard deviation means increasing the coefficient of variation of a demand. If the cv of the product demand increases and β is greater than 0.5, the optimum order quantity increases. If the cv of the product demand increases and β is greater than 0.5, the optimum order quantity decreases for the single product VaR constrained newsvendor problem.

6.2.2 Two-Product Case

6.2.2.1 The Effect of the Average Demand

The first set of experiments focus on the effect of demand mean in the risk-averse newsvendor problem with a VaR constraint. Initial values are given in the Table-6.1 below:

	Product 1	Product 2
c_o	4	4
c_u	6	6
μ	10	10
Demand	U(0,20)	U(0,20)
π_0=0	β=0.05	

Table-6.1: Initial Values Used in the VaR Numerical Experiments

In order to observe and comment on the results properly, we have to get rid of the variability factor of the demand or somehow keep variability constant. For this purpose, the mean of the second product is changed while its CV is held constant. The results of this experiment can be seen in Table-6.2.

In the second and third columns of Table-6.2; the optimum order quantities and the corresponding optimum profit values are given respectively. In the fourth and fifth columns of Table-6.2; the unconstrained optimum order quantities and the corresponding optimum profit values are given respectively. As the mean demand of the second product increases, the total optimum profit increases, both the first and the second product order quantities increase. But the increasing rate of the second product is higher than the first product, then the second product's percentage in the portfolio increases.

	Optimum Order Quantity Pair	Optimum Profit Value	Unconstrained Optimum Order Quantity Pair	Unconstrained Optimum Profit Value	$\frac{q_2^*}{q_1^* + q_2^*}$
U(0,10)	(6.7,4.5)	45.83	(12,6)	54.1824	0.40
U(0,20)	(7.9,7.9)	63.6	(12,12)	71.98	0.50
U(0,40)	(8.9,13.4)	91.66	(12,24)	107.212	0.60
U(0,80)	(11.8,19.3)	128.51	(12,48)	179.987	0.62
U(0,100)	(11.9,21.4)	141.5	(12,60)	212.498	0.64

Table-6.2: The Effect of the Average Demand on the Product Portfolio Distribution

In this experiment, it can be seen that increasing the mean demand of a product seems to have a diminishing effect on the satisfaction probability level. At the beginning of this experiment, the VaR constraint is binding. As the mean demand of a product is increased, it becomes impossible to catch up with the threshold probability satisfaction level, to make the constraint unbinding and to equalize the optimum profit in the risky newsvendor problem with the unconstrained newsvendor problem's profit.

A similar experiment is performed for exponentially distributed demands. For exponentially distributed demands, there are four regions and four optimization problems corresponding to these regions. There are not any sub-cases in these main regions in contrast to uniformly distributed demand. The maximum of these four cases is taken to be the global optimum and compared to the simulation test results. The mean demand of the first product is the same and the mean demand of the second product is changed.

In this experiment, the unconstrained and VaR constrained newsvendor results are compared. In the Table-6.3, the first column gives experimental demand means $1/\lambda$ and the second column presents the points where the newsvendor problem attains the maximum profit without considering the risk constraint in each trial. The third column displays the unconstrained optimum expected profit. In the fourth column, the satisfaction probability values at the unconstrained optimum point are given in order to realize if the risk constraint is satisfied or not. The VaR constrained problem is solved by the NLP model in four regions; the maximum of the four is chosen and demonstrated in the fifth column of the Table-6.3.

The CV of an exponentially distributed demand stays constant and equals one. Therefore; as the mean demand of the second product decreases, the optimum profit and the fraction of the second product in the portfolio increases. Moreover, Table-6.3 and the associated Figure 6.1 reveal the effects of the change in mean demand in detail.

$1/\lambda_2$	Unconstrained Optimum Point (q_1,q_2)	Unconstrained Optimum Expected Profit	Satisfaction Probability Values at the Unconstrained Optimum Point	Constrained Optimum	
				Profit	(q_1,q_2)
5	(9.16,4.58)	35.02	0.79	25.67	(3.2,3.4)
10	(9.16,9.16)	46.7	0.82	35.17	(4.24,4.24)
20	(9.16,18.32)	70.04	0.79	54.6	(4.8,7.4)
40	(9.16,36.65)	116.74	0.74	73.43	(7.9,10.2)
50	(9.16,45.81)	140.09	0.73	82.6	(8.3,11.9)

Table-6.3: Effect of the Demand Mean on the Optimum Point

(c_{u_1}=6, c_{u_2}=6, c_{o_1}=4, c_{o_2}=4, $1/\lambda_1$=10, π_0=0, β=0.05)

From Table-6.3, the simulated satisfaction probability values decrease in the case of increasing the mean demand as we have expected above. From the Figure 6.1, the rate of increase in the optimum profit without any risk is greater than the rate of increase in the optimum profit with a risk as the demand mean increases. Interestingly, the rate of increase in the optimum profit is reduced by the risk constraint.

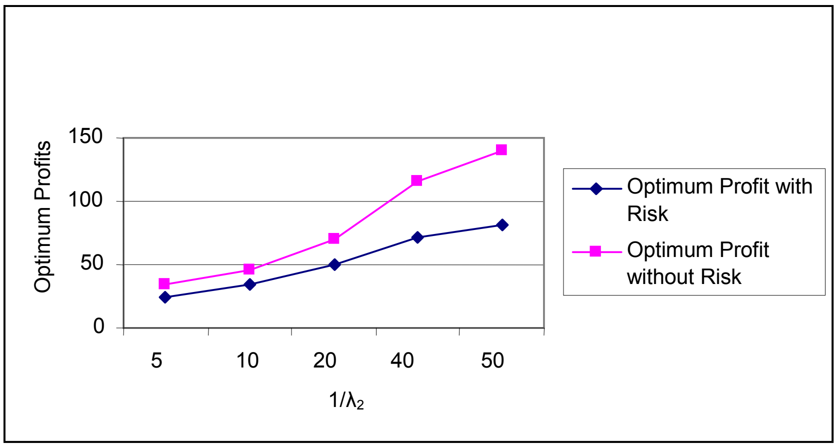

Figure 6.1: Optimum Profit versus Mean Demand

6.2.2.2 Effect of CV (Coefficient of Variation) of Demand

Secondly, let us observe the effect of CV of the demand. In order to see the effect of CV of the demand on the optimum order point and the optimum profit, all the parameters are taken to be same and only the second product's demand distribution is changed. When changing the second demand, the upper and lower parameters of the second demand are changed while the mean of the demand is kept constant. By this way; cv of the demand changes, whereas the mean demand remains constant.

In Table-6.4a, the results to the Non-Linear Programming (NLP) model for the regions of the problem are given. The results are reported in two columns which correspond the optimum expected profit and the corresponding quantity pairs. The outputs of the Case-I, Case-IIb_2 and Case-IIb_3 are all infeasible in this problem and not included in the Table-6.4a. In Table-6.4b, the simulation column indicates the point where the expected profit attains the maximum value while the newsvendor's realized profit is less than or equal to his specified target profit (π_0) with threshold probability value (β). The simulation results are reported with an error less than 0.1 for product order quantities and 0.5 for profit values within a 95% confidence interval. The unconstrained column indicates the optimum point for the unconstrained newsvendor problem for two products.

According to the Table-6.4a and the Table-6.4b, the point that gives the maximum profit of Case-I, Case-IIa, Case-IIb and Case-III and the point that is given as an output of the simulation are very close. The two profits at these points are also very close. All numerical results in these two tables illustrate this same situation. So, solving the risk-averse newsvendor problem for all regions separately and choosing the one that gives the maximum profit of all seems to be a valid optimization approach.

	Case-IIa						Case-IIb		Case-III			
	Case-IIa_1		Case-IIa_2		Case-IIa_3		Case-IIb_1		Case-III_1		Case-III_2	
	Profit	Pair	Profit	Pair	Profit	Pair	Profit	Pair	Profit	Pair	Profit	Pair
U(0,20)	62.4	(6.3,9.5)	Infeasible		Infeasible		62.4	(9.5,6.3)	63.6	(7,9,7.9)	Infeasible	
U(4,16)	56.8	(3.2,14.7)	77.5	(8.4,12.6)	54.4	(3.8,6.3)	78.9	(13.3,8.9)	81.5	(11.4,10.8)	54.3	(4,6)
U(8,12)	Infeasible		84	(8,12)	85.3	(7.4,11)	88.9	(14.5,9.7)	91.2	(12,10.4)	90	(10,10)
U(9,11)	Infeasible		64.1	(9,13.5)	87.1	(7,10.5)	91.4	(14.7,9.8)	93.5	(12,10.2)	93.6	(12,10.2)

Table-6.4a: The Solution of the VaR-Constrained Newsvendor Problem

Demand Distribution of the Second Product	CV of the Second Product's Demand	VaR Constrained Simulation		Unconstrained	
		Profit	Pair	Profit	Pair
U(0,20)	1.73	63.9	(8,8)	72	(12,12)
U(4,16)	1.04	81.96	(11.4,10.8)	82	(12,11.2)
U(8,12)	0.35	91.85	(12,10.4)	92	(12,10.4)
U(9,11)	0.17	93.6	(12,10.2)	93.8	(12,10.2)

Table-6.4b: The Simulated and the Unconstraint Solution to the VaR-constrained Problem

The values in the Table-6.4a and the Table-6.4b indicate that as the CV of the second product's demand decreases, the realized profit and the optimum profit increases. Also; as the CV of the second product demand decreases, the probabilities that the realized profit of the newsvendor achieves the targeted profit increases. So, the maximum profit to be attained with the risk constraint becomes closer to the maximum profit to be attained without any risk constraint. This leads us to a simple conclusion that the risk constraint becomes no more binding and the optimum profit becomes equal to the profit for newsvendor problem without any risk constraint as the CV of the second product's demand decreases.

These results motivate the following question. Although the coefficient of variation of the second product decreases, why does the second product's percentage in the portfolio decrease? Should not the decision maker try to invest more on the less risky asset (that means the product that has lower variability on its demand)?

After more numerical experiments, we observe that the ratio of the underage cost to the overage cost is an important factor in this situation. Figure 6.2a shows that when the ratio of the overage cost to the underage cost is less than one, decreasing the coefficient of variation of a product causes a fall in the percentage of the product in the portfolio. Figure

6.2b shows that if this ratio is greater than one, there occurs an increase in the product's percentage in the portfolio and demand would get the higher portion from the portfolio.

The underage cost means the net profit gained in a possible product selling case and the overage cost means the net loss in a possible over-ordering case. The possible cause for this result is that if the ratio is greater than one (losing money because of over-ordering is more important than losing money because of under-ordering), the product with lower variability has a higher portion in the product portfolio.

Also, the optimum points with the VaR constraint for different overage and underage costs can be seen from the Table-6.5. In both of the scenarios; as the cv of the second product's demand decreases, the quantities for both the first product and the second product increase. A possible reason for this is that the risk in the system decreases in the case of a fall in the cv of the second product's demand. But, as we have expressed above the percentage of the second product in the product portfolio depends on the ratio of the overage cost to the underage cost.

Demand Distribution of the Second Product	CV of the Second Product's Mean	$c_{u_1}=c_{u_2}$=6, $c_{o_1}=c_{o_2}$=4		$c_{u_1}=c_{u_2}$=2, $c_{o_1}=c_{o_2}$=8	
		Optimum with the VaR Constraint		Optimum with the VaR Constraint	
		Profit	Optimum Pair	Profit	Optimum Pair
U(0,60)	0.58	190.86	(23.72,23.72)	9.04	(2.53,2.53)
U(4,56)	0.50	212.64	(26.87,27.29)	17.68	(5.27,6.32)
U(8,52)	0.42	230.6	(30.36,30.26)	20.16	(6.38,11.07)
U(12,48)	0.35	244.37	(34.21,32.53)	17.7	(6.75,12)
U(16,44)	0.27	254. 4	(36,32.8)	5.92	(7.75,16)
U(20,40)	0.19	264	(36,32)	55.37	(9.25,24)

Table-6.5: Results of the VaR-Constrained Problem for Different Cost Values

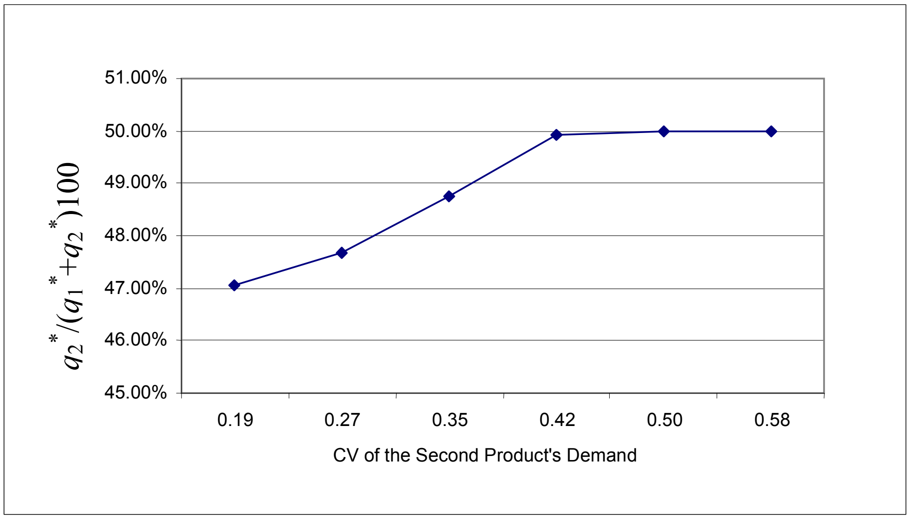

Figure 6.2a: Percentage of Second Product in Portfolio (c_o=4, c_u=6)

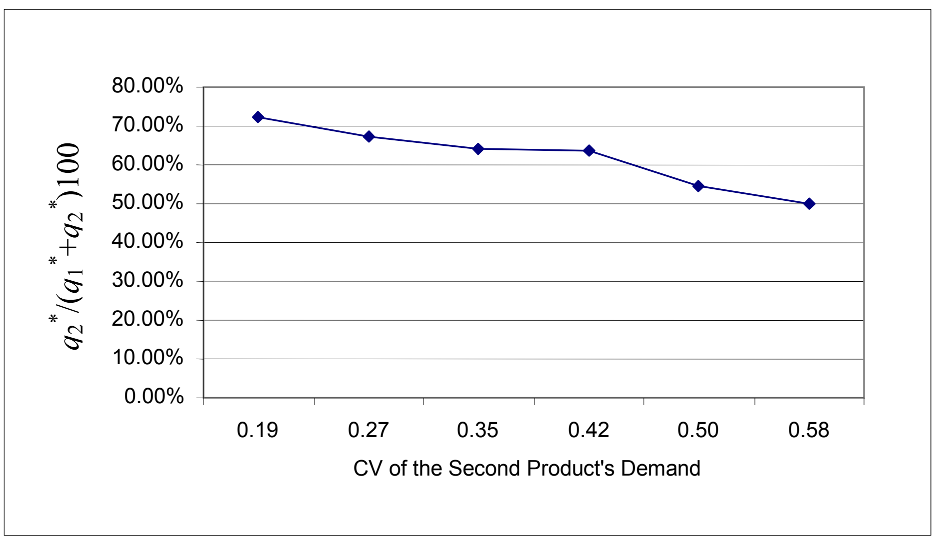

Figure 6.2b: Percentage of Second Product in Portfolio (c_o=8, c_u=2)

6.2.2.3 The Effect of the Overage Cost (c_o)

Now; let us look at the changes when the overage cost (c_{o_2}) - the cost incurred when the product order quantity is greater than the realized demand- changes so that c_{u_2} / c_{o_2} ratio changes. Initial values are given in the Table-6.1.

In this testing, the unconstrained and VaR constrained newsvendor results are compared. In Table-6.6, the first column gives experimental overage costs (c_o) and the second column presents the points where the newsvendor problem attains the maximum profit without considering the risk constraint in each trial. The third column displays the unconstrained optimum expected profit. In the fourth column, the satisfaction probability values at the unconstrained optimum point are given in order to realize if the risk constraint is satisfied or not. The VaR constrained problem is solved by the NLP model in four regions; the maximum of the four is chosen and demonstrated in the fifth column of the Table-6.6.

Obviously, an increase in the overage cost results in a decrease in the optimum profit values. From Table-6.6, it can be seen that as c_{o_2} increases and c_{u_2} / c_{o_2} decreases- that means losing the profit opportunity becomes less important than increasing the cost because of over-ordering-, the cost of the second product increases and the percentage of the second product in the portfolio decreases as we expected. Also, where the cost values and revenue values of products are equal, second product's percentage is near to 50%.

c_{o_2}	Unconstrained Optimum Point (q_1,q_2)	Unconstrained Optimum Expected Profit	Satisfaction Probability Values at the Unconstrained Optimum Point	Constrained Optimum	
				Profit	(q_1,q_2)
2	(9.16,13.86)	55.62	0.83	41.81	(2.79,8.37)
3	(9.16,10.98)	50.39	0.83	39.52	(4.06,5.89)
4	(9.16,9.16)	46.7	0.83	35.17	(4.24,4.24)
5	(9.16,7.88)	43.93	0.83	33.51	(4.47, 3.73)

Table-6.6: Effect of the Overage Cost on the Optimum Point

(c_{u_1}=6, c_{u_2}=6, c_{o_1}=4, $1/\lambda_1$= $1/\lambda_2$=10, π_0=0, β=0.05**)**

In addition, we wonder how changing the overage cost or wholesale price would affect the satisfaction probability levels. In Table-6.6, the satisfaction probability values at the unconstrained optimum points show that satisfaction probability level remains same. From the Figure 6.3, the profit rate decrease with the risk case is close to the profit rate decrease without risk. In the Table-6.6, it can be seen that changing the wholesale price does not affect the rate of change of the expected profit. An interesting result is that as the overage cost of the second product increases, the optimum order quantity of the first product increases and the optimum order quantity of the second product decreases whereas the total product quantity in the portfolio decreases.

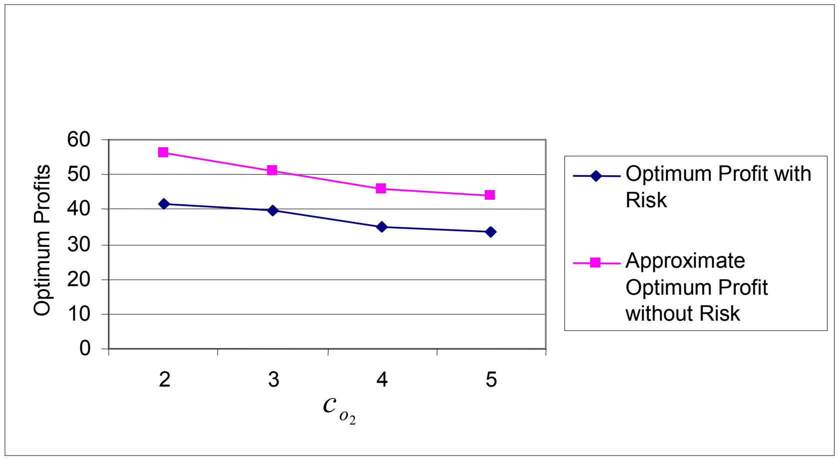

Figure 6.3: Optimum Profit versus the Overage Cost (c_o) of the Second Product

6.2.2.4 The Effect of the Target Profit Level

Another variable to observe its effect on the optimum profit line is the target profit level (π_0) which is called the VaR profit. In this testing, the unconstrained and VaR constrained newsvendor results are compared. In Table-6.7, the first column gives π_o value and the second column presents the points where the newsvendor problem attains the maximum profit without considering the risk constraint in each trial. The third column displays the unconstrained optimum expected profit. In the fourth column, the satisfaction probability values at the unconstrained optimum point are given in order to realize if the risk constraint is satisfied or not. Once again, the VaR-constrained problem is solved by the NLP model in four regions; the maximum of the four is chosen and demonstrated in the fifth column of the Table-6.7.

While the target profit (π_0) is increased, the satisfaction probabilities decrease and profits remain the same. In the Figure 6.4a, the decreasing trend of the satisfaction probabilities is shown. As the VaR profit increases, the optimum profit decreases. For higher VaR profit levels, the optimization problem turns out to be an infeasible problem. The result is comprehensible because the satisfaction probability level decreases as stated in Table-6.7.

For greater VaR profits, it is impossible to get the satisfaction probability level above the threshold probability value-0.95 in our setting. The results are shown in Figure 6.4b.

π_0	Unconstrained Optimum Point (q_1,q_2)	Unconstrained Optimum Expected Profit	Satisfaction Probability Values at the Unconstrained Optimum Point	Constrained Optimum	
				Profit	(q_1,q_2)
0	(9.16,9.16)	46.7	0.83	35.17	(4.24,4.24)
5	(9.16,9.16)	46.7	0.81	31.72	(3.61,3.61)
10	(9.16,9.16)	46.7	0.80	Infeasible	
15	(9.16,9.16)	46.7	0.78	Infeasible	
20	(9.16,9.16)	46.7	0.75	Infeasible	
25	(9.16,9.16)	46.7	0.68	Infeasible	
30	(9.16,9.16)	46.7	0.64	Infeasible	

Table-6.7: Effect of the Target Profit Level on the Optimum Point

(c_{u_1}=6, c_{u_2}=6, c_{o_1}=4, c_{o_2}=4, $1/\lambda_1$=10, $1/\lambda_2$=10, β=0.05)

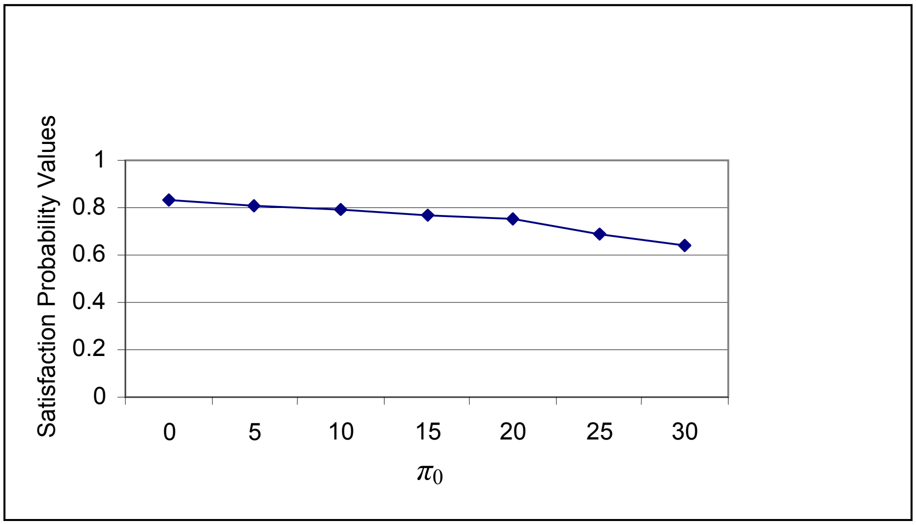

Figure 6.4a: Satisfaction Probability Plot with Different Target Profit Levels

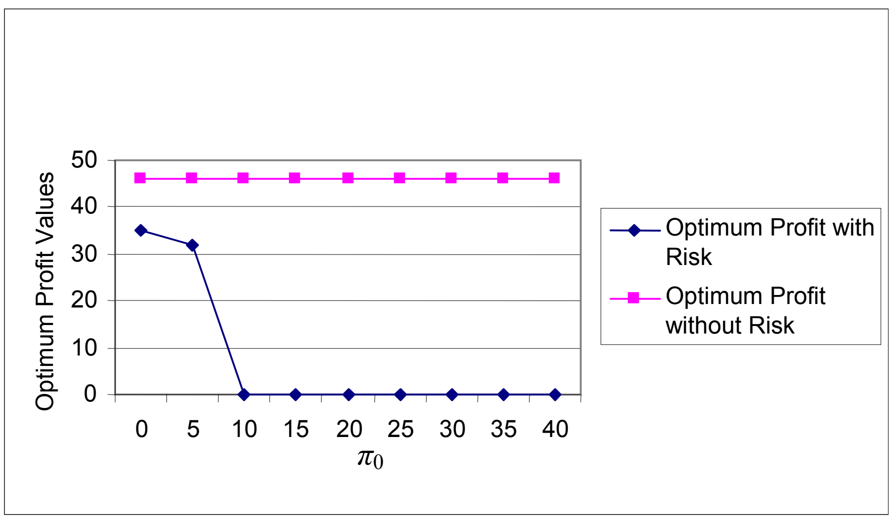

Figure 6.4b: Optimum Profits with and without Risk versus Target Profit Level

6.3 Numerical Analysis for Newsvendor Problem with a CVaR Constraint

The solution methodology for the CVaR optimization problems was given in chapter 5 where the CVaR function is defined in terms of the newsvendor's profit or loss function.

The single-product CVaR optimization problem with newsvendor consideration was solved by Chen et al.[21]. The solution and parametric effects of the solution for single product are presented in section 6.3.1. The experimental results for two products and more products are investigated in section 6.3.2 and 6.3.3 respectively.

6.3.1 The Single Product Analysis

The first model stated above was studied for one product by Chen et al. [21]. The optimum order quantity is given as:

$$q^* = \arg\max_{q\geq 0}\left\{\max_{\pi_0} \mathrm{CVaR}(q,\pi_0)\right\}$$

$$q^* = F^{-1}\left(\beta\frac{r-c}{r-s}\right)$$

The underage cost (c_u) equals *r-c* in the newsvendor problem. The effect of c_u on the optimum order quantity in the CVaR problem is similar to the effect of c_u on the optimum order quantity in the risk-neutral newsvendor problem. As c_u increases, the optimum order quantity increases. The overage cost (c_o) equals *c-s* in the newsvendor problem. Also, effect of c_o on the optimum order quantity is similar to the effect of c_o on the optimum order quantity in the risk-neutral newsvendor problem. As c_o increases, the optimum order quantity decreases.

The effect of coefficient of variation (cv) on the optimum order quantity is also interesting. The mean of the demand should be kept constant to see the effect of cv on the optimum order quantity clearly. For uniformly distributed demand- U(a,b), q^* is as shown below:

$$q^* = \frac{c_u}{(c_u+c_o)}\beta(b-a)+\left(\frac{b+a}{2}\right)-\left(\frac{b-a}{2}\right)=\left(\frac{b+a}{2}\right)+(b-a)\left(\frac{c_u(2\beta-1)-c_o}{2(c_u+c_o)}\right)$$

The result shows that if $c_u(2\beta-1)$ is greater than c_o, increasing the difference results in a raise in q^* in a linear trend. Otherwise if $c_u(2\beta-1)$ is less than c_o, increasing the difference between the distribution parameters results in a fall in q^* in a linear trend.

If the product demand is normally distributed with parameters μ and σ, the optimum order quantity is shown as below:

$$q^* = \phi^{-1}\left(\beta\frac{c_u}{c_u+c_o}\right)\sigma+\mu$$

Keeping the mean demand constant and increasing the standard deviation means increasing the coefficient of variation of a demand. If the cv of the product demand increases and (β $c_u/(c_u+c_o)$) is greater than 0.5, the optimum order quantity increases and if the cv of the product demand increases and (β $c_u/(c_u+c_o)$) is less than 0.5, the optimum order quantity decreases for the single product constrained newsvendor problem.

6.3.2 Two-Product Case

6.3.2.1 CVaR Optimization

In section 5.5.1, the LP model formulation of the CVaR optimization problem is given. As stated in section 5.5.1, the CVaR minimization in terms of the newsvendor's loss function could be solved via an LP formulation. Because of the convexity of our CVaR function, the LP model gives the optimum solution for the problem.

If an LP model is written for the CVaR maximization in terms of the newsvendor's profit function similarly, could we guarantee that the solution is optimal? Let us look at a numerical example for this purpose.

In this experiment, there are two products that each have exponentially distributed demands with same mean value 10. Underage costs and overage costs are same for the products and 6, 4 respectively. In Section 5.4, the solution procedure of CVaR maximization for two-product newsvendor profit function is given. It resembles the solution procedure for two-product VaR-constrained newsvendor problem. The CVaR maximization problem is solved for all regions and the maximum of all is chosen among them. In the Table-6.8, the column called "Result of the CVaR Maximization with NLP in Four Different Regions" displays the optimum CVaR values and the corresponding Value-at-Risk and product order quantity values-(VaR, q_1, q_2) for all four regions. The maximum of four regions is selected as the optimum result of this NLP method. The column called "Result of the CVaR Maximization with LP Formulation" displays the solution for the LP formulated CVaR maximization of the two-product newsvendor problem. In this table, simulation results are also available. The result of the NLP formulation is very close to the simulation results; but the LP formulation result does not seem to be acceptable according to the simulation result. Rockafellar et. al.'s LP formulation is not applicable in CVaR optimization for newsvendor problem. The reason could be the concavity of the CVaR function. This could be a future research.

<table>
<tr><td colspan="6">Result of the CVaR Maximization with NLP in Four Different Regions</td><td colspan="2" rowspan="5">Result of the CVaR Maximization with LP Formulation</td><td colspan="2" rowspan="5">Simulation</td></tr>
<tr><td colspan="2">Case-I</td><td colspan="2">Case-IIb</td><td colspan="2" rowspan="4">Maximum of Four Regions</td></tr>
<tr><td>CVaR</td><td>(VaR, q_1, q_2)</td><td>CVaR</td><td>(VaR, q_1, q_2)</td></tr>
<tr><td>2.88</td><td>(4.88,1.29,1.29)</td><td>2.42</td><td>(3.93,1.5,1.27)</td></tr>
<tr><td colspan="2">Case-IIa</td><td colspan="2">Case-III</td></tr>
<tr><td>CVaR</td><td>(VaR, q_1, q_2)</td><td>CVaR</td><td>(VaR, q_1, q_2)</td><td>CVaR</td><td>(VaR, q_1, q_2)</td><td>CVaR</td><td>(VaR, q_1, q_2)</td><td>CVaR</td><td>(VaR, q_1, q_2)</td></tr>
<tr><td>2.42</td><td>(3.93,1.27,1.5)</td><td>1.97</td><td>(2.97,1.49,1.49)</td><td>2.88</td><td>(4.88,1.29,1.29)</td><td>0.6</td><td>(0.60,0.05,0.05)</td><td>2.89</td><td>(5,1.3,1.3)</td></tr>
</table>

Table-6.8: Comparison of NLP and LP CVaR Maximization Results with Simulation (D_1~Exponential (0.1) and D_2~Exponential (0.1), $c_{u_1}=c_{u_2}$=6, $c_{o_1}=c_{o_2}$=4, β=0.95)

6.3.2.2 The Effect of the Average Demand

Firstly, let us perform a numerical study to see the effect of the average demand in the risk-averse newsvendor problem. The initial values are given in the Table-6.9 below:

	Product 1	Product 2
c_o	4	4
c_u	6	6
μ	10	10
Demand	U(0,20)	U(0,20)
π_0=0	β=0.95	

Table-6.9: Initial Values Used in the CVaR Numerical Experiments

In this experiment; only the second product's distribution is changed. In order to observe the effect of the demand mean purely, we have to get rid of the variability effect of the demand by keeping the coefficient of variation constant. In the Table-6.10, the optimum CVaR values and the corresponding optimum pairs are given for this experiment. As the mean demand of the second product increases, the second product's percentage in the product portfolio increases and the optimum CVaR value is improved. An interesting result in this experiment is the change in the first product optimum order quantity. As the second product's demand mean increases, the optimum order quantity of the first product also increases.

μ_2	min CVaR	
	Optimum CVaR Value	Optimum Triple (VaR, q_1, q_2)
U(0,10)	-4.13	(-7.1,1.8,1.55)
U(0,20)	-5.68	(-10,2.5,2.5)
U(0,40)	-8.26	(-14.2,3.1,3.6)
U(0,80)	-12.72	(-21.8,3.9,5.4)
U(0,100)	-14.81	(-25.6,4.3,6.3)

Table-6.10: Demand Mean Effect on the Optimum Point in CVaR Minimization

6.3.2.3 The Effect of CV (Coefficient of Variation) of the Demand

Secondly, let us observe the effect of the CV of the demand on the optimum point and profit values. In order to see the effect of CV of the demand on the optimum order point and the optimum profit, all the parameters are taken to be same and only the demand distribution of the second product is changed. When changing the second demand, the upper and the lower parameters of the second demand are changed and at the same time the mean of the demand is taken to be the same (CV of demand changes, mean remains constant).

Second Product's Demand Distribution	CV of the Second Product's Demand	min CVaR		Simulation	
		Optimum CVaR Value	Optimum Triple (VaR, q_1, q_2)	Optimum CVaR Value	Optimum Triple (VaR, q_1, q_2)
U(0,20)	1.73	-5.68	(-10,2.5,2.5)	-5.7	(-10,2.5,2.5)
U(4,16)	1.04	-28.47	(-31.68,1.94,5.74)	-28.8	(-31,1.9,5.7)
U(8,12)	0.35	-50.96	(-53.12,1.26,8.86)	-50.84	(-53,1.3,8.9)
U(9,11)	0.17	-56.46	(-58.22,0.96,9.51)	-56.33	(-58,1,9.5)

Table-6.11: CV of the Demand Effect on the Optimum Point in CVaR Minimization

In the Table-6.11, the minimization of the CVaR function is solved via LP formulation. In all of the experimental studies in this chapter, demands are discretized with equal probability values in 100 equal intervals. The optimum CVaR values and the corresponding optimum pairs are given in that table. Also, the optimum CVaR values and the optimum points found via simulation are also available. The simulation results are reported with an error less than 0.1 for product order quantities and 0.5 for VaR and optimum CVaR values within a 95% confidence interval. The results of the LP formulation and the simulation results are close to each other. From this point, examining only the LP results for the CVaR minimization would be sufficient in order to perform an experimental study.

As the CV of the second product's demand decreases, which means the riskiness of the second product demand falls, the percentage of the second product in the portfolio increases and the optimum CVaR value is improved as expected. During this change, the optimum order quantity of the first product is decreased whereas the total quantity in the portfolio increases.

At this point, we wonder if the ratio of the underage cost to the overage cost has an effect or not on the CV experiment on the second product percentage in the product portfolio similar to the VaR experiments. For this purpose, we have performed 3 experiments where (c_u,c_o) pairs are the same for all products and (6,4),(2,8) and (8,2). The first product's demand is uniformly distributed between 0 and 60. The second product's demand is varied in order to observe the effect of the CV while keeping the demand's mean constant. The Figure 6.5 shows that when the ratio of the overage cost to the underage cost increases, second product's percentage in the product portfolio increases for the same input variables. It is reasonable because if the overage cost is more important than the underage cost, the decision maker should avoid over-ordering. So, the decision maker would order more from the less risky product.

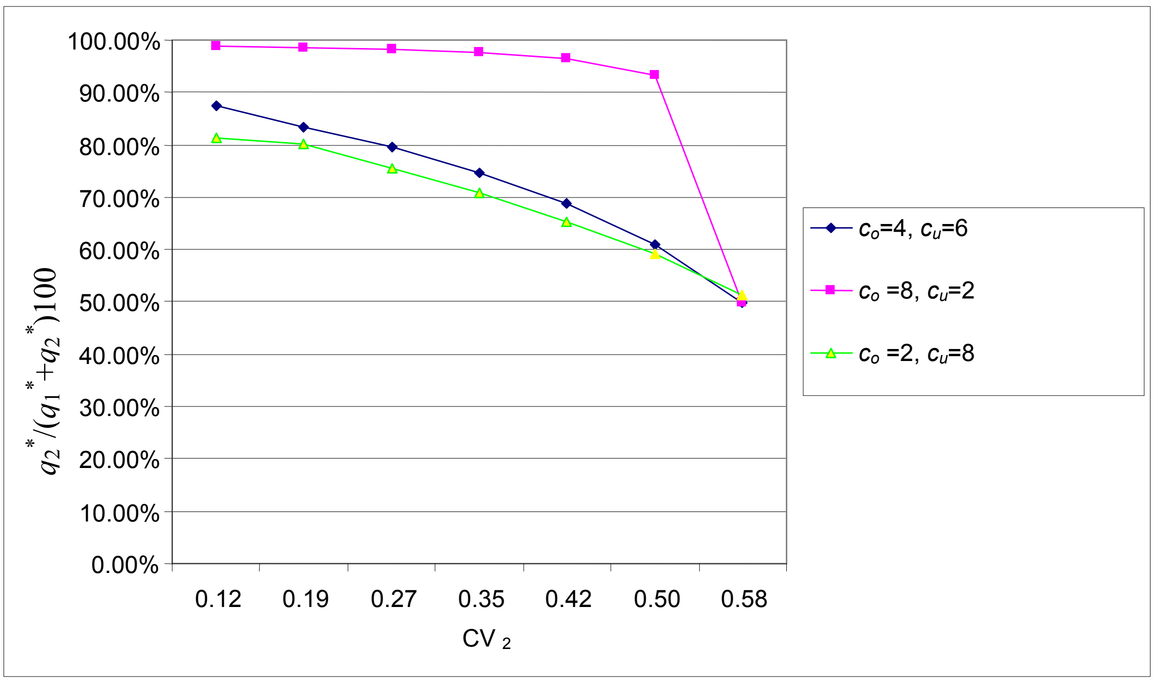

Figure-6.5: CV effect on the Optimum Product Portfolio Distribution

6.3.2.4 The Effect of the Overage Cost (c_o)

Another variable to examine the effects on the optimum product portfolio is the overage cost (c_o). Only, the overage cost of the second product is changed. In Table-6.12, the optimum CVaR values and the corresponding optimum pairs are given for this experiment.

As the second product's overage cost increases, the second product's percentage in the product portfolio decreases and the optimum CVaR value is worsened. Increasing the overage cost means over-ordering causes higher risk for the system. So, in order to avoid this risk it is logical to lower the optimum order quantities. An interesting result in this experiment is the change in the first product optimum order quantity. As the second product's overage cost increases, the optimum order quantity of the first product also decreases.

c_{o_2}	min CVaR	
	Optimum CVaR Value	Optimum Triple (VaR, q_1, q_2)
2	-4.72	(-7.48,1.59,2.06)
3	-3.65	(-5.86,1.44,1.68)
4	-2.92	(-5.15,1.31,1.31)
5	-2.39	(-4.29,1.11,1.03)

Table-6.12: The Effect of the Overage Cost on the Optimum Point in CVaR Minimization ($c_{u_1}=c_{u_2}$=6, c_{o_1}=4, λ_1=λ_2=0.1, β=0.95)

6.3.2.5 The Effect of the Underage Cost (c_u)

Another variable to examine the effects on the optimum product portfolio is the underage (c_u). Only, the underage cost of the second product is changed. In the Table-6.13, the optimum CVaR values and the corresponding optimum pairs are given for this experiment. As the second product's underage cost increases, the second product's optimum order quantity increases as expected. Also, the optimum order quantity of the first product increases. But, the second product's percentage in the product portfolio decreases. However, the optimum CVaR value is improved. Increasing the underage cost means under-ordering becomes more important than over-ordering. It is intuitively anticipated that increasing the underage cost results in an increase in the optimum order quantities.

c_{u_2}	min CVaR	
	Optimum CVaR Value	Optimum Triple (VaR, q_1, q_2)
4	-1.69	(-3.06,0.78,0.91)
5	-2.28	(-4,1.04,1.13)
6	-2.92	(-5.15,1.31,1.31)
7	-3.58	(-6.12,1.45,1.34)

Table-6.13: The Effect of the Underage Cost on the Optimum Point in CVaR Minimization (c_{u_1}=6, $c_{o_1}=c_{o_2}$=4, λ_1=λ_2=0.1, β=0.95)

6.3.2.6 The Effect of the Threshold Probability (*β*)

Let us examine the effects of the threshold probability (β) on the optimum product portfolio. Initial values of the experiment can be seen in the Table-6.9.

In the Table-6.14, the optimum CVaR values and the corresponding pairs are given for various threshold probability values (β). As the β increases, the decision maker cares more about the riskiness of the environment. So; the optimum product order quantities fall and the optimum CVaR value is increased.

Threshold probability (β)	min CVaR	
	Optimum CVaR Value	Optimum Pair (VaR, q_1, q_2)
0.75	-19.17	(-36.4,4.7,4.7)
0.8	-15.89	(-29.6,4.3,4.3)
0.85	-12.61	(-22.8,3.9,3.9)
0.9	-9.24	(-15.6,3.3,3.3)
0.95	-5.68	(-10,2.5,2.5)
0.98	-3.2	(-4.4,1.7,1.7)
0.99	-2.5	(-3.2,1.1,1.1)

Table-6.14: The Effect of the Threshold probability on the Optimum Point in CVaR Minimization

6.3.3 The Case with Multiple Products

The number of products in the portfolio is another variable that we are concerned with its effect on the optimum point in CVaR minimization problem. The results of the experiment are summarized in the Table-6.15 below:

Product Portfolio	Min CVaR		Simulation	
	Optimum Profit Value	Optimum Point (VaR, q_i)	Optimum Profit Value	Optimum Point (VaR, q_i)
Single Product	-1.8	(-3,0.5)	-1.75	(-3,0.5)
2-product	-5.6	(-10,2.5,2.5)	-5.63	(-10,2.5,2.5)
3-product	-18.2	(-25,3,3,3)	-17.52	(-25,3,3,3)
4-product	-29.4	(-43,4.5,4.5,4.5,4.5)	-29.66	(-42,4.5,4.5,4.5,4.5)

Table-6.15: The Effect of the Portfolio Size on the Optimum Point in CVaR Minimization ($c_{u_1}=c_{u_2}$=6, $c_{o_1}=c_{o_2}$=4 Demand=U(0,20) β=0.95)

The experiment is performed for single-product, identical 2-product, identical 3-product and identical 4-product problems. In this experiment, we have taken 20 equal intervals for demand discretization in order to cope with the problem size. Increasing the number of discretization intervals would increase the problem size. In the second and third columns, the optimum profit values and the corresponding optimum points calculated by the LP formulation and the simulation are given respectively. As the number of products in the portfolio is increased, the optimum CVaR value is improved and the optimum order quantities for each product also increase.

6.3.4 CVaR-Constrained Newsvendor Problem

Lastly, we turn our attention to the model where the expected profit is maximized with a CVaR constraint. The results of the experiment are summarized in the Table-6.16. In the second column, the results of the VaR problem for different trials are shown. As explained before, the CVaR constraint controls the CVaR function below an upper bound value ($CVaR^0$). The $CVaR^0$ bound is estimated by using the results of the VaR problem with a simulation. The expected profit maximization with the CVaR constraint is solved and the results are revealed in the third column. The optimum profit values and the corresponding optimum pairs are very close to each other.

Second Product's Demand Distribution	VaR-Constrained Optimization		CVaR-Constrained Optimization	
	Optimum Profit Value	Optimum Pair (q_1, q_2)	Optimum Profit Value	Optimum Triple (VaR, q_1, q_2)
U(0,20)	63.6	(7.9,7.9)	63.56	(-0.88,7.88,7.9)
U(4,16)	81.5	(11.4,10.8)	81.2	(-2.38,10.94,10.66)
U(8,12)	91.2	(12,10.4)	91.2	(-0.09,11.9,10.38)
U(9,11)	93.5	(12,10.2)	93.6	(0,11,12,10.2)

Table-6.16: CVaR-Constrained Newsvendor Problem Result

6.4 Conclusion

In this chapter, the experimental studies and the results for the two main models which are the VaR-constrained newsvendor problems and the CVaR optimization for newsvendor problems are available. According to the numerical results, both of the models lower the optimum order quantities if the risk measure is binding. Otherwise, the decision variables are equal to the decision variables for the unconstrained newsvendor problem.

To summarize, increasing the mean of the demand has the similar effect on the optimum order quantities for both of the models. It increases the optimum order quantity of the product as well as optimum profit value. Increasing the overage cost causes a decrease in the optimum order quantity and the optimum profit value. Increasing the underage cost has the opposite effect on the optimum order quantity and the optimum profit value with increasing the overage cost. In addition, demand pooling effect can be observed from these experimental results. Increasing the number of product types in the product portfolio causes a decrease in the risk of the model and the product order quantities increase in product basis-total product order quantities increase by this way.

Moreover, the product order quantities are found to be closely related even if we work with the independent demands. This reveals the binding property of the VaR and CVaR measures.

Finally, both of the models are found to be input sensitive. The demand distribution of the products and the cost ratios are extremely important. As an example, the coefficient of variation of a product demand effect on the optimum point changes according to the overage-underage cost ratios. The best example is the effect of the coefficient of variation of a demand on the optimum order quantities and profit values. The effect of cv of a demand on the decision variables and the objective function depends on whether the product is a high-profit product or a low-profit product. (As stated before, a product is defined as a high-profit product when $(r\text{-}c)/(r\text{-}s) \geq 1/2$ or as a low-profit product otherwise)

Chapter 7

CONCLUSION

In this thesis, we are primarily interested in the multi-product newsvendor problems with financial risk constraints. The necessity and the importance of controlling the risk in making inventory decisions in a stochastic demand environment are emphasized. Our motivation is to control the risk of earning less than a threshold value or losing more than a critical value while maximizing the expected profit of the system.

Two important and complementing risk measures, VaR and CVaR, are used to control the risk in finance literature. The main aim of this thesis is to use VaR and CVaR measures in controlling the financial risk in the single period stochastic inventory models for a large product portfolio. The VaR-constrained newsvendor problem and the CVaR optimization in a newsvendor problem are solved for the single-product case in the past. It is beneficial to extend these models to N-product case for investors, retailers in short all decision makers. This study gives a solution approach for each model described above. The VaR-constrained newsvendor problem is solved for two products exactly and an approximation method is proposed for more-products case. CVaR optimization is solved by an LP formulation for a large product portfolio where the CVaR function is defined in terms of the newsvendor's profit or loss function.

The focus of the first part of this thesis is on inventory control using the Value-at-Risk concept. As expressed before, the satisfaction probability is the probability of exceeding a prespecified, fixed target profit level. The VaR value is taken to be this target profit value in the complement of the satisfaction probability function in this study. We used the satisfaction probability function as the VaR constraint where the objective is the expected profit maximization in the newsvendor problem. The main objective of this part is to solve this VaR-constrained newsvendor problem for large product portfolios. As the satisfaction probability function is defined in terms of the newsvendor's profit, it combines the products and creates a relation between them. Therefore, a numerical analytical solution is found by solving an NLP formulation for two-product case and an approximation method is proposed for larger product portfolios. By using the Central Limit Theorem for large products, the total profit distribution is approximated with a normal distribution. The errors reported for this approximation method seem adequate.

In the second part of the thesis, we focus on Conditional Value-at-Risk that has a convex risk surface with respect to control variables and has a unique global optimum. In this study, the CVaR value is expressed in terms of the newsvendor's profit or loss function and the aim is to optimize this value by deciding the ordering quantities for each product. For large product portfolios, this CVaR optimization model for the *N*-product newsvendor problem is solved by using a LP formulation.

Many experiments on these VaR and CVaR models are performed and the effects of the parameters on the optimum order quantities and the expected profit values are examined. Some interesting results are gathered from these experiments. First of all, the optimum quantities and the total profit for the VaR-constrained newsvendor problem are lower than for the unconstrained newsvendor problem if the risk constraint is binding. If the risk constraint is not binding, all values are identical with each other. The CVaR optimization problem lowers the optimum order quantities similar to the VaR-constrained newsvendor problem.

Secondly, we work with the products which have independent demands in both VaR constrained newsvendor optimization and CVaR optimization for the newsvendor problem. Although the product demands are independently distributed, it is observed that the product order quantities are closely related.

Moreover, the problem is found to be input sensitive. Its behavior changes according to demand distributions, overage and underage cost ratios. In addition, the demand pooling effect can be observed for both of the models we are concerned with in this study.

As a future research, both of the models could be studied for correlated demands and the effect of the correlation coefficient between the products could be inspected. Extending these two models to the multi-period risk-averse newsvendor problem would also be an interesting extension to this research. For a long time horizon, the financial risk of a stochastic inventory model could be controlled in every time interval separately leading to a Stochastic Dynamic Programming (SDP) problem.

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Appendix A: SENSITIVITY ANALYSIS FOR A SINGLE PRODUCT NEWSVENDOR PROBLEM WITH AND WITHOUT VaR CONSTRAINT

A.1 The Effect of the Underage Cost (c_u) on the Optimum Order Quantity for a Single Product Unconstrained Newsvendor Problem

As the underage cost (c_u) increases, the optimum order quantity (q^*) increases. Second derivative of the q^* in terms of c_u is less than zero, q^* versus c_u is a concave function.

Proof: The optimum order quantity satisfies the equation $F_D(q^*)=c_u/(c_u+c_o)$. The optimum order quantity q^* in terms of c_u is an increasing function. First derivative of this function is in the following:

$$f\left(q^*\right)\frac{\partial q^*}{\partial c_u} = \frac{c_o}{\left(c_u+c_o\right)^2}$$

$$\frac{\partial q^*}{\partial c_u} = \frac{c_o}{\left(c_u+c_o\right)^2}\frac{1}{f\left(q^*\right)} > 0$$

The optimum order quantity in terms c_u is a concave function. Second derivative of this function is in the following:

$$\frac{\partial f\left(q^*\right)}{\partial c_u}\left(\frac{\partial q^*}{\partial c_u}\right)^2 + \frac{\partial^2 q^*}{\partial c_u{}^2} f\left(q^*\right) = \frac{-2c_o}{\left(c_u+c_o\right)^3}$$

$$\frac{\partial^2 q^*}{\partial c_u{}^2} = \frac{-2c_o}{\left(c_u+c_o\right)^3}\frac{1}{f\left(q^*\right)} < 0$$

A.2 The Effect of the Overage Cost (c_o) on the Optimum Order Quantity for a Single Product Unconstraint Newsvendor Problem

As c_o increases, the optimum order quantity q^* decreases. The second derivative of the q^* in terms of c_o is greater than zero, q^* versus c_o is a convex function.

Proof: The optimum order quantity satisfies the equation $F_D(q^*)=c_u/(c_u+c_o)$. The optimum order quantity q^* in terms of c_o is a decreasing function. First derivative of this function is in the following:

$$f(q^*)\frac{\partial q^*}{\partial c_o} = \frac{-c_u}{(c_u+c_o)^2}$$

$$\frac{\partial q^*}{\partial c_u} = \frac{-c_u}{(c_u+c_o)^2}\frac{1}{f(q^*)} < 0$$

The optimum order quantity in terms c_o is a convex function. Second derivative of this function is in the following:

$$\frac{\partial f(q^*)}{\partial c_o}\left(\frac{\partial q^*}{\partial c_o}\right)^2 + \frac{\partial^2 q^*}{\partial c_o^{\,2}} f(q^*) = \frac{2c_u}{(c_u+c_o)^3}$$

$$\frac{\partial^2 q^*}{\partial c_u^{\,2}} = \frac{2c_u}{(c_u+c_o)^3}\frac{1}{f(q^*)} > 0$$

A.3 The Effect of the Underage Cost (c_u) on the Optimum Order Quantity for a Single Product VaR-Constrained Newsvendor Problem

In order to observe the effect of c_u for both of the regions, risk-averse and risk-neutral cases are combined and the graph in the following is formed. In this graph, the regions where the problem is infeasible, the risk constraint is binding or not and the behavior of the optimum order quantity according to the overage cost is shown in detail.

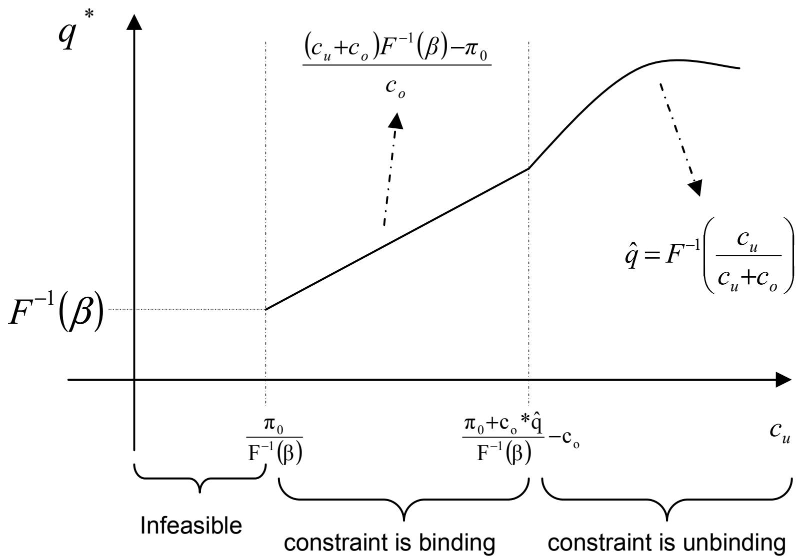

Figure A.1: Effect of Underage Cost (c_u) on the optimum order quantity (q^*)

A.4 The Effect of the Overage Cost (c_o) on the Optimum Order Quantity

In order to observe the effect of c_o for both of the regions, risk-averse and risk-neutral cases are combined and the graph in the following is formed. In this graph, the regions where the problem is infeasible, the risk constraint is binding or not and the behavior of the optimum order quantity according to the overage cost is shown in detail. Note that if the condition

$$c_o \leq \frac{c_u F^{-1}(\beta)-\pi_0}{\hat{q}-F^{-1}(\beta)}$$

is satisfied, the VaR constraint is not binding and the overage cost does not affect the feasibility of the problem. Also, assume that the condition

$$\frac{c_u}{c_u + c_o} \geq \beta$$

is satisfied-the Figure A.2 is drawn where the condition $\alpha > c_u F^{-1}(\beta)$ is satisfied; if that condition is not satisfied the Figure A.3 will be as the second graph:

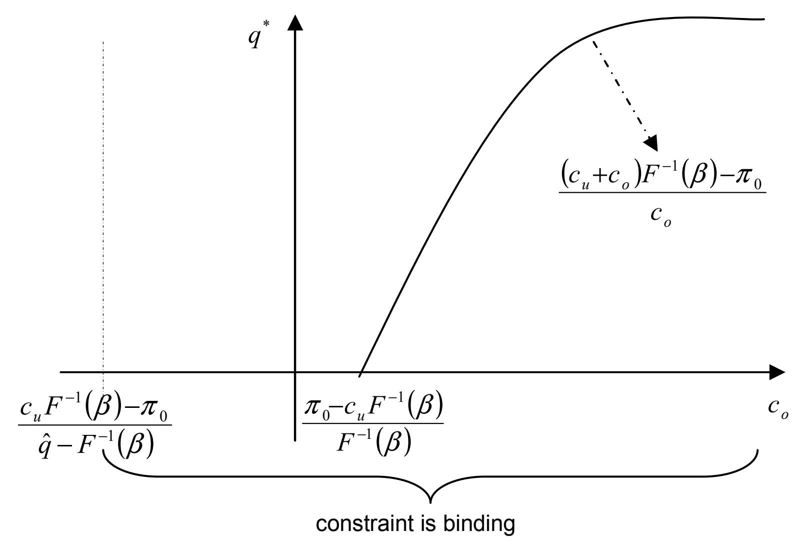

Figure A.2: Effect of Overage Cost (c_o) on the optimum order quantity (q^*)

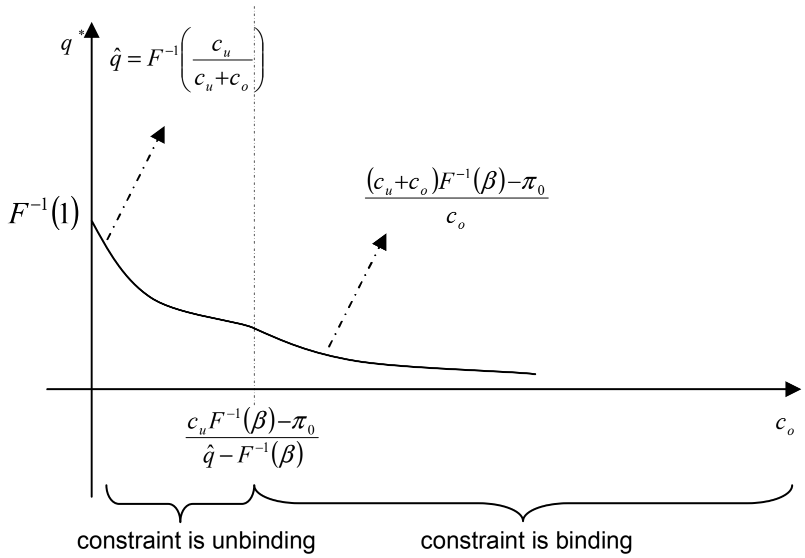

Figure A.3: Effect of Underage Cost (c_o) on the optimum order quantity (q^*)

A.5 The Effect of the Coefficient of Variation (cv) on the Optimum Order Quantity

In this setting, the optimization problem could be either in an infeasible state, or has a binding constraint or does not have a binding constraint. If the problem has a binding constraint, the optimum quantity is equal to the solution of the constraint optimization problem. If the problem does not have a binding constraint, the optimum quantity is equal to the solution of the classical newsvendor problem. The state of the problem depends on two variables β, (b-a) and in which region each variable fall. If β is greater or less than 1/2, there are three different regions that (b-a) can fall into for both two conditions. Each region defines the problem state. The regions defining the problem states and the corresponding problem states can be summarized as follows:

if β>1/2
$$\begin{cases} (b-a) \geq \dfrac{\dfrac{\pi_0 + c_o \hat{q}}{c_u + c_o} - \mu}{\beta - 0.5} & \text{the downside risk constraint is not binding} \\ \dfrac{\dfrac{\pi_0}{c_u} - \mu}{\beta - 0.5} \leq (b-a) \leq \dfrac{\dfrac{\pi_0 + c_o \hat{q}}{c_u + c_o} - \mu}{\beta - 0.5} & \text{the downside risk constraint is binding} \\ (b-a) < \dfrac{\dfrac{\pi_0}{c_u} - \mu}{\beta - 0.5} & \text{the problem is infeasible} \end{cases}$$

If β≤1/2
$$\begin{cases} (b-a) \leq \dfrac{\dfrac{\pi_0 + c_o \hat{q}}{c_u + c_o} - \mu}{\beta - 0.5} & \text{the downside risk constraint is not binding} \\ \dfrac{\dfrac{\pi_0}{c_u} - \mu}{\beta - 0.5} \geq (b-a) \geq \dfrac{\dfrac{\pi_0 + c_o \hat{q}}{c_u + c_o} - \mu}{\beta - 0.5} & \text{the downside risk constraint is binding} \\ (b-a) > \dfrac{\dfrac{\pi_0}{c_u} - \mu}{\beta - 0.5} & \text{the problem is infeasible} \end{cases}$$

VITA

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