GROUP DIVISIBLE DESIGNS

by

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This is to certify that I have examined this copy of a master's thesis by

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and have found that it is complete and satisfactory in all respects,
and that any and all revisions required by the final
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To my parents

ABSTRACT

This thesis is on *Group Divisible Designs* (*GDDs*), one of the main structures in *Combinatorial Design Theory*. *GDDs* were first introduced by Hanani in 1975 and widely used in constructing other combinatorial structures such as *pairwise balanced designs*, *frames* and *balanced incomplete block designs* since then.

A *GDD* is a triple (X, G, B), where X is the point set, which satisfies the following properties:
(1) G is a partition of X into subsets called groups.
(2) B is a set of subsets of X (called blocks) such that a group and a block contain at most one common point.
(3) every pair of points from distinct groups occurs in a unique block.

A $K - GDD$ of type $g_1^{u_1} g_2^{u_2} \dots g_s^{u_s}$ is a *GDD* where every block has size from the set K and there are u_i groups of size g_i, for $i = 1, 2, \dots, s$. In our study all blocks have size k. In this thesis we will present:
(1) $3 - GDDs$ of type:
(i) g^u (uniform $3 - GDDs$)
(ii) $g^u m^1$
(iii) $g^u 1^t$
(iv) $g^1 v^1 1^g$

(2) $4 - GDDs$ of type:
(i) g^u (uniform $4 - GDDs$)
(ii) $g^u m^1$.

ÖZETÇE

Bu tez *Gruplara Bölünebilen Tasarımlar* (GDD'ler) üzerinedir. GDD'ler Kombinatoryel Tasarımlar Teorisindeki ana yapılardan birisidir. GDD'ler ilk defa 1975'de Hanani tarafından tanımlanmış ve o günden bu yana mesala PBD, Frame ve BIBD gibi kombinatoryel yapıların inşaasında sıkça kullanılmıştır.

Bir GDD (X, G, B) üçlüsü olarak tanımlanır öyle ki, X noktaların oluşturduğu küme olmak üzere, aşağıdaki özelliklerin sağlanması gerekir:

(1) G X'in gruplar dediğimiz parçalarından oluşur.

(2) B X'in alt kümlerini (blokları) içeren bir kümedir öyleki bir grup ve bir blok en fazla bir noktada kesişebilir.

(3) Farklı gruplardan seçilen her ikili sadece ve sadece bir blokta görünür.

$g_1^{u_1} g_2^{u_2} \dots g_s^{u_s}$ çeşitindeki bir $K - GDD$ öyle bir GDD'dir ki her bloğun uzunluğu K kümesinin içindedir ve her $i = 1, 2, \dots, s$ için g_i büyüklüğündeki gruptan u_i tane vardır. Bizim çalışmamızda her bloğun uzunluğu sabit bir k'dr. Bu tezde anlatacaklarımız:

(1) $3 - GDDs$ çeşitleri:

(i) g^u (düzenli $3 - GDDs$)

(ii) $g^u m^1$

(iii) $g^u 1^t$

(iv) $g^1 v^1 1^g$

(2) $4 - GDDs$ çeşitleri:

(i) g^u (düzenli $4 - GDDs$)

(ii) $g^u m^1$.

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NOMENCLATURE

GDDs	Group Divisible Designs
RGDDs	Resolvable Group Divisible Designs
DGDDs	Double Group Divisible Designs
MGDDs	Modified Group Divisible Designs
IGDDs	Incomplete Group Divisible Designs
BIBDs	Balanced Incomplete Block Designs
TDs	Transversal Designs
ITDs	Incomplete Transversal Designs
HTDs	Holey Transversal Designs
PBDs	Pairwise Balanced Designs
IPBDs	Incomplete Pairwise Balanced Designs
STS	Steiner Triple System
KTS	Kirkman Triple System
WFC	Wilson's Fundamental Construction

INTRODUCTION

Definition: A *design* is a pair (X, B) such that the following properties are satisfied:

(i) X is set of elements called *points*, and

(ii) A is a collection (i.e *multiset*) of nonempty subsets of X called *blocks*.

Definition: A *group divisible design* (or GDD) is a triple (X, G, B) where X is set of points, G is a partition of X into subsets called groups, and B is a collection of subsets of X called blocks such that any pair of distinct points from X occur together either in some group or in exactly one block, but not both. A K-GDD of type $g_1^{u_1} g_2^{u_2} \ldots g_s^{u_s}$ is a *GDD* in which every block has size from the set K with u_i groups of size g_i, $i = 1, 2, \ldots, s$.

Definition: Let v and k be positive integers such that $v > k \geqslant 2$. A $(v, k)-$ *balanced incomplete block design* (which we abbreviate to (v, k)-BIBD or BIBD(k, v)) is a design (X, B) such that the following properties satisfied:

(i) $|X| = v$,

(ii) each block contains exactly k points and

(iii) every pair of distinct points is contained exactly in one block.

Definition: A *transversal design* is a $k-$GDD of type g^k and it is denoted by TD(k, g). Therefore a TD is a GDD in which all groups has the same size g and the number of groups is equal to the block size k.

Definition: Let K be a subset of positive integers, a *pairwise balanced design* (PBD(K, v)) or $K-$PBD) of order v with block sizes from K is a pair (V, B), where V is a finite set (the *point set*) of cardinality v and B is a family of subsets (*blocks*) which satisfy the following properties:

(i) if $b \in B$ then $|b| \in K$.

(ii) Every pair of distinct element of V occurs in exactly one block of B.

Definition: A GDD, PBD, or BIBD is *resolvable* if the blocks of the design can be partitioned into parallel classes. A *parallel class* is a set of blocks that partitions the set of points. A resolvable GDD is denoted by RGDD. In the case of resolvable BIBD(k, v), RBIBD(k, v) or RB(v, k) is used.

Definition: Let be (X, G, B) a GDD. A *holey parallel class* of the GDD is a set of disjoint blocks that contain each element of the GDD, except those of a group $g_j \in G$ once; no elements of the group g_j appear in any block of the set.

Definition: An *incomplete group divisible design* (IGDD) is a quadruple (X, Y, G, B) where X is a set of points, Y is a subset of X (called a hole), G is a partition of X into groups, and B is a collection of subsets of X (blocks) such that

(i) for each block $b \in B$, $|\, b \cap Y| \leqslant 1$, and

(ii) any pair of points from X which are not in Y occur together in some group or in exactly one block, but not both.

A K-IGDD of type $(g_1, h_1)^{u_1}(g_2, h_2)^{u_2}...(g_s, h_s)^{u_s}$ is an incomplete group divisible design in which every block has size from K and in which there are u_i groups of size g_i, each of which intersects the hole in h_i points for $i = 1, 2, ..., s$.

Definition: Let K be a set of integers. An *incomplete* PBD of order v with a hole of order h ((v,h;K)-IPBD or IPBD(v,h;K)) is a triple (X, H, B) where X is a set of points, H is a subset of X which contains h elements, and B is a family of subsets (*blocks*) of X which satisfy the following properties:

(i) If $b \in B$, then $|b| \in K$;

(ii) Every pair of distinct elements $\{x, y\} \subseteq X \backslash H$ occurs in exactly one block.

(iii) No pair of distinct elements of H occurs in a block.

Definition: A *double group divisible design* is a quadruple (X, H, G, B) where X is a set of points, H and G are partitions of X (into holes and groups, respectively) and B is a collection of subsets of X (blocks) such that

(i) for each block $b \in B$ and each hole $Y \in H$, $|b \cap Y| \leqslant 1$ and

(ii) any pair of points from X which are not in the same hole occur together in some group or in exactly one block, but not both.

A K-DGDD of type $(g_1, h_1^v)^{u_1}, (g_2, h_2^v)^{u_2}, ..., (g_s, h_s^v)^{u_s}$ is a double group divisible design in which every block has size from the set K and in which there are u_i groups of size g_i, each of which intersects each of the v holes in h_i points. (Thus $g_i = h_i v$ for $i = 1, 2, ..., s$. Not every DGDD can be expressed in this way, of course, but this the most general type that we require.)

Definition: A *modified group divisible design* $K-$MGDD of type g^u is a $K-$DGDD of type $(g, 1^g)^u$.

Definition: An *incomplete transversal design* ITD$(k, g; h^v)$ is a $k-$DGDD of type $(g, h^v)^k$. In addition, existence of a ITD$(k, g; h^v)$ is equivalent to the existence of a set of $k-2$ *holey* MOLS of type h^v (see [5]).

Definition: A *Steiner triple system* (STS) is a PBD with all blocks of size 3. For $n \equiv 1, 3$ (*mod* 6), STS(n) exists, see [19].

Definition: Kirkman triple system is a *resolvable*-STS. For $n \equiv 3$ (*mod* 6), KTS(n) exists, see [19].

Chapter 1

PRELIMINARIES

In this chapter we will give some fundamental constructions which we will use commonly in this thesis.

Theorem 1.0.1. *The necessary conditions for the existence of k-GDDs of type g^u are*
(***i***) $u \geq k$
(***ii***) $(u-1)g \equiv 0 \ (mod\ k-1)$
(***iii***) $u(u-1)g^2 \equiv 0 \ (mod\ k(k-1))$

Proof: Let (X, G, B) be a k-GDD of type g^u
(**i**) This result directly comes from definition. Number of groups u must be equal or greater than the block size k.
(**ii**) Let g be a group in G . If we fix a point $x \in g$ and look at the blocks which contain x, then these blocks partition $X \backslash g$. Therefore, $(k-1) \mid (u-1)g$ (it is the replication number of a point).
(**iii**) We will find the total number of blocks. To find this number we will count the pairs in blocks. Total number of pairs on the set X with gu elements is $\binom{gu}{2}$ and the number of pairs on u groups of size g is $u\binom{g}{2}$, so the total number of pairs used in blocks is $\binom{gu}{2} - u\binom{g}{2}$. On the other hand the number of pairs used in one block is $\binom{k}{2}$. Then total number of blocks is $\frac{\binom{gu}{2} - u\binom{g}{2}}{\binom{k}{2}}$. Therefore, $\binom{k}{2} \mid \binom{gu}{2} - u\binom{g}{2}$ gives the result.□

Now we will give some fundamental constructions to construct GDDs using smaller GDDs.

Theorem 1.0.2 (Wilson's Fundamental Construction (WFC), [8])**.** *Let (X, G, B) be a GDD and $G = \{G_1, ..., G_m\}$ be the set of groups. Let $w(v)$ be an integer weight for each*

$v \in X$. Suppose that for each block $b = \{v_1, ..., v_l\}$ in B, there is a K-GDD with l groups, having sizes $w(v_1), ..., w(v_l)$. Then there is a K-GDD whose groups have sizes $\sum_{v \in Gi} w(v)$ for each $i = 1, 2, ..., m$.

Proof: Let (X, G, B) be a GDD. Let $b \in B$ be a block of size l. For each point $v \in b$,we will give the weight $w(v)$. From the assumption, there exists a K-GDD with l groups, having sizes $w(v_1), ..., w(v_l)$, so we will put this GDD on the block b. Similarly, we will put the required GDD to each block in B. Therefore at the end we will obtain the desired K-GDD with groups of sizes $\sum_{v \in Gi} w(v)$ for each $i = 1, 2, ..., m$. □

Remark: Using WFC we could enlarge group sizes while preserving number of groups. Note that the block sizes may change.

Corollary 1.0.3. *If there exists a (K, M)-GDD on v points, blocks have sizes from K and groups have sizes from M, and for each $k \in K$ if there exists a K'-GDD of type m^k, then there exists a K'-GDD on mv points.*

Proof: Let (X, G, B) be a (K, M)-GDD on v points. For each point $x \in X$, if we give the weight m and apply WFC using the given GDDs as fillers, then we will obtain a K'-GDD on mv points. □

Theorem 1.0.4. *If there exists a (K, M)-GDD on v points and for each $m \in M$ there exists a (K, M')-GDD on m points then there exists a (K, M')-GDD on v points.*

Proof: Let (X, G, B) be a (K, M)-GDD on v points. For each group $g \in G$, $|g| = m \in M$, if we put the given proper (K,M')-GDD with m points on the group g, then we will obtain the desired (K, M')-GDD on v points. □

Remark: Using this construction we could obtain GDDs with small group sizes from GDDs with big group sizes without changing the number of elements.

Theorem 1.0.5. *If there exists a (K, M)-GDD on v points and for each $g \in M$, there exists a $(K, \{m\})-$GDD on $g + m$ points, then there exists a $(K, \{m\})$-GDD on $v + m$ points.*

Proof: Let (X, G, B) be a (K, M)-GDD on v points. We will add m new points and take these m points as a new group. For each group $g \in G$ with the new group on

m points, we put the given $(K, \{m\})$-GDD on these $g + m$ points. Then the desired $(K, \{m\})$-GDD on $v + m$ points is obtained. □

Lemma 1.0.6. *Let $v = (k-1)u$. BIBD$(k, v+1)$ is equivalent to $k-$GDD of type $(k-1)^u$.*

Proof: Let (X, B) be a BIBD$(k, v+1)$ and $v = (k-1)u$. Let x be a point in X. If we delete x, take the blocks containing x as our groups and the rest of the blocks in B as blocks of the GDD then we will obtain the desired GDD.

On the hand side, to obtain a BIBD from a given GDD, we will add a new point y and we take each group of GDD together with y as our new blocks. Therefore, combining with the other blocks of GDD we will get the desired BIBD. □

Lemma 1.0.7. *Suppose that a K-GDD of type $g^t u^1$ exits, $g \equiv 0 \pmod s$ and a K-GDD of type $(g/s)^s w^1$ exists. Then there exists a K-GDD of type $(g/s)^{st}(u+w)^1$.*

Proof: Let (X, G, B) be a K-GDD of type $g^t u^1$. We will add w new points to the group with u points. If we put K-GDD of type $(g/s)^s w^1$ on each group $g \in G$ together with the w points, we will obtain the desired K-GDD of type $(g/s)^{st}(u+w)^1$. □

We can also fill in a single group as follows:

Theorem 1.0.8. *If K-GDDs of types $g^t u^1$ and $g^s x^1$ with $u = sg + x$ exist, then there is a K-GDD of type $g^{t+s}x^1$.*

Proof: Let (X, G, B) be a K-GDD of type $g^t u^1$ and $u = sg + x$. If we put the given K-GDD of type $g^s x^1$ on the group with u points we will obtain desired K-GDD of type $g^{t+s}x^1$. □

Chapter 2

3-GDDS

2.1 3-GDDs of type g^u (uniform 3-GDDs)

Theorem 2.1.1. *The necessary and sufficient conditions for the existence of* $3-$*GDDs of type are: g^u are:*
(***i***) $u \geq 3$
(***ii***) $(u-1)g \equiv 0$ $(mod\ 2)$
(***iii***) $u(u-1)g^2 \equiv 0$ $(mod\ 6)$.

We summarize these conditions in the Table 2.1.

Table 2.1: Necessary and sufficient conditions for the existence of uniform $3-$GDDs

g	necessary and sufficient u	constructed by lemmas
$1,5$ $(mod\ 6)$	$1,3$ $(mod\ 6)$ and $u \geq 3$	2.1.3, 2.1.4
$2,4$ $(mod\ 6)$	$0,1$ $(mod\ 3)$ and $u \geq 3$	2.1.3, 2.1.7
3 $(mod\ 6)$	1 $(mod\ 2)$ and $u \geq 3$	2.1.3, 2.1.8
0 $(mod\ 6)$	$u \geq 3$	2.1.3, 2.1.9

Before giving the proof of the theorem, we need some preliminary results.

Theorem 2.1.2 (Theorem 3.1, [1])**.** *For every $r > 0$, a $TD(3, r)$ exists.*

Lemma 2.1.3. *If there exists a* 3*-GDD of type g^u and r is a positive integer, then there exists a* 3*-GDD of type $(gr)^u$.*

Proof: Let (X, G, B) be a 3-GDD of type g^u. To construct the desired GDD we will use WFC. To each point $x \in X$, we will give the weight r, then to each block $b \in B$, we will put a TD$(3, r)$, to get a 3-GDD of type $(gr)^u$. (We use a TD$(3, r)$ which is equivalent to a 3−GDD of type r^3 as fillers and it exists by Theorem 2.1.2) □

Therefore to prove the Theorem 2.1.1 we need only 3-GDDs of types 1^u, 2^u, 3^u and 6^u for each admissible values of u, then the result follows by applying Lemma 2.1.3.

Lemma 2.1.4 (Lemma 5.4, [1]). *If* $u \equiv 1$ *or* 3 $(mod\ 6)$, *then there exits a* $BIBD(3, u)$.

A 3−GDD of type 1^u is equivalent to a BIBD$(3, u)$, so Lemma 2.1.4 covers the case 3−GDD of type 1^u.

Lemma 2.1.5 (Lemma 6.2, [1]). *If* $u \equiv 0$ *or* 1 $(mod\ 3)$, *then there exists a* $PBD(\{3, 4, 6\}, u)$.

Lemma 2.1.6. *If there exists a* $PBD(L, u)$ *and for each* $l \in L$ *there exits a* k−*GDD of type* g^l, *then there exists a* k−*GDD of type* g^u.

Proof: For each point of the PBD we give the weight g and for each block b of size l of the PBD we put the given k-GDD of type g^l. Then we obtain the desired k−GDD of type g^u. □

Lemma 2.1.7. *If* $u \equiv 0$ *or* 1 $(mod\ 3)$, *then there exists a* 3-*GDD of type* 2^u.

Proof: By Lemma 2.1.5 we know that for $u \equiv\ 0$ or 1 $(mod\ 3)$ a $PBD(\{3, 4, 6\}, u)$ exists. If we have 3-GDDs of type 2^3, 2^4 and 2^6, then by Lemma 2.1.6 and giving the weight 2 to each point of the PBD, we have a 3-GDD of type 2^u. The required 3−GDDs exist by [6]. □

Lemma 2.1.8. *If* $u \equiv 1$ $(mod\ 2)$, *then there exits a* 3-*GDD of type* 3^u.

Proof: We take a KTS$(3u)$ ($3u \equiv 3$ $(mod\ 6)$ so KTS$(3u)$ exists) and a parallel class of this KTS as groups of the 3−GDD of type 3^u and other blocks as our blocks of the 3−GDD, then we are done.

Lemma 2.1.9. *For every* $u \geq 3$ *there exists a* 3-*GDD of type* 6^u.

Proof: For every $u \geq 3$, there exists a $PBD(\{3,4,5,6,8\},u)$ (see [1]). Therefore, if we have 3−GDDs of types 6^3, 6^4, 6^5, 6^6 and 6^8, as previously by Lemma 2.1.6, we cover the spectrum of 3−GDDs of type 6^u. We have all the required 3−GDDs. (see [6]) .□

Now we are ready to give the proof of Theorem 2.1.1:

Proof of Theorem 2.1.1:

Using the results of Lemmas 2.1.3, 2.1.4, 2.1.7, 2.1.8 and 2.1.9, we cover the spectrum of the 3−GDDs of type g^u. □

2.2 3-GDDs of type $g^u m^1$, $m > 0$

Theorem 2.2.1. *Let g, u and m be nonnegative integers. There exists a* 3*-GDD of the type* $g^u m^1$ *if and only if the following conditions are all satisfied:*

(***i***) $u \geqslant 3$*, or if* $u = 2$ *then* $m = g$*;*

(***ii***) $m \leqslant g(u-1)$*;*

(***iii***) $g(u-1) + m \equiv 0 \ (mod\ 2)$*;*

(***iv***) $gu \equiv 0 \ (mod\ 2)$*;*

(***v***) $\dfrac{g^2u(u-1)}{2} + gum \equiv 0 \ (mod\ 3)$.

Proof of Necessity Part:

This part is quite straightforward:

(**i**) First, any block intersects exactly three different groups so $u \geqslant 2$. If there are exactly three groups (3−GDD of type g^2m^1), then $g = m$. To see this, let (X, G, B) be a 3-GDD with $|G| = 3$ and let $\mathcal{G} \in G$ be a group. If we fix an element x in the group $\mathcal{G}$, and consider the blocks which contain x, since the block size is 3 and there are exactly three groups, the blocks intersect with both of the remaining groups. Therefore their sizes must be equal. But the group we have chosen in the beginning may vary, so the sizes of all the three groups must be equal.

(**ii**) Let (X, G, B) be a 3-GDD of type $g^u m^1$. If we fix an element x of a group of size g, then we observe that each element in the group of size m (which we refer as the *long* group) must appear in a block containing x. Since x appears at most in $g(u-1)$

such blocks, then the inequality $m \leqslant g(u-1)$ follows.

(**iii**) Let x be an element in a group of size g. Since x appears in $g(u-1)+m$ pairs, the number of blocks containing x should be $\frac{g(u-1)+m}{2}$.

(**iv**) This is similar to (**iii**), this time we will consider an element in the long group.

(**v**) Simply this asserts that the total number of pairs is divisible by 3, since each block contains 3 unique pairs then total number of pairs is $g^2\binom{u}{2}+gum$. □

We give necessary and sufficient conditions for the existence of 3−GDDs of type $g^u m^1$ in Table 2.2. Now we will deal with sufficiency. We recall some preliminary results that settle parts of our problem. We may assume that m is positive, since the existence of 3−GDDs of type g^u was been completely settled by Theorem 2.1.1. In the Table 2.2, we give possible congruence classes for m (*modulo* 6) for each combination of congruence classes of g and u (*modulo* 6).

Doyen and Wilson [10] proved the first essential result settling a part of the sufficiency:

Theorem 2.2.2 (Doyen and Wilsons's Theorem)**.** *Let $v \equiv 1,3\ (mod\ 6)$, $w \equiv 1,3\ (mod\ 6)$, and $v \geqslant 2w+1$. Then there is a Steiner triple system of order v (say $STS(v)$) containing a sub-Steiner triple system of order w.*

Corollary 2.2.3. *There exists a 3-GDD of type $1^u m^1$ for all $m \leqslant u-1$, $m+u \equiv 1,3\ (mod\ 6)$ and $m \equiv 1,3\ (mod\ 6)$.*

Proof: Using Theorem 2.2.2 we will form a STS($u+m$) with sub-STS(m). Then we will delete the blocks of the subdesign and form a group of size m. Therefore, if we take each remaining element as a singleton group together with the group of size m, then we will obtain the required 3-GDD of type $1^u m^1$. □

Remark: This corollary covers the spectrum of 3-GDDs of type $1^u m^1$ except the case (u,m)=$(0,5)$ in modulo 6.

Proof: We will find the spectrum of 3-GDDs of type $1^u m^1$ from the necessary conditions. Taking $g=1$ in (*ii*) we have $m \leqslant u-1$, by (*iii*) we have $2|(m+u-1)$ and by (*iv*) we have $2|u$, so this implies u must be even. By combining (*ii*),(*iii*) and (*iv*),

Table 2.2: Possible values for m (*modulo* 6).

	g (mod 6)					
u (mod 6)	**0**	**1**	**2**	**3**	**4**	**5**
0	0, 2, 4	1, 3, 5	0, 2, 4	1, 3, 5	0, 2, 4	1, 3, 5
	by 2.2.14, 2.2.16	by 2.2.20 iv	by 2.2.20 i, ii, 2.2.4	by 2.2.9, 2.2.17	by 2.2.20 iii, 2.2.23	by 2.2.20 v, 2.2.25
1	0, 2, 4	-	0	-	0	-
	by 2.2.13		by 2.2.20 i, ii, 2.2.24		by 2.2.20 iii, 2.2.23	
2	0, 2, 4	1	2	1, 3, 5	4	5
	by 2.2.14	by 2.2.20 iv	by 2.2.20 i, ii, 2.2.24	by 2.2.9, 2.2.17	by 2.2.20 iii, 2.2.23	by 2.2.20 v, 2.2.25
3	0, 2, 4	-	0, 2, 4	-	0,2,4	-
	by 2.2.13		by 2.2.20 i, ii, 2.2.24		by 2.2.20 iii, 2.2.23	
4	0, 2, 4	3	0	1, 3, 5	0	3
	by 2.2.14, 2.2.15	by 2.2.20 iv ,	by 2.2.20 i, ii, 2.2.24	by 2.2.9, 2.2.17	by 2.2.20 iii, 2.2.23	by 2.2.20 v, 2.2.25
5	0, 2, 4	-	2	-	4	-
	by 2.2.13		by 2.2.20 i, ii, 2.2.24		by 2.2.20 iii, 2.2.23	

we get $2|(m-1)$, so m must be odd. Finally by (v) we get $3|(u(u-1)/2+um)$ and if we combine all of them we will have $(u,m) \in \{(0,1),(0,3),(0,5),(2,1),(4,3)\}$ with respect to modulo 6. (Also we can find these values directly from the table)

Secondly, we will find which parts of the spectrum of 3-GDDs of type $1^u m^1$ is completed by Corollary 2.2.3. We will use $m+u \equiv 1,3$ $(mod\ 6)$ and $m \equiv 1,3$ $(mod\ 6)$. So when $m \equiv 1$ $(mod\ 6)$, we have $u \equiv 0,2$ $(mod\ 6)$ and when $m \equiv 3$ $(mod\ 6)$ we have $u \equiv 0,4$ $(mod\ 6)$. Therefore we have $(u,m) \in \{(0,1),(0,3),(2,1),(4,3)\}$ in modulo 6. What is left only the case $(u,m)=(0,5)$ in modulo 6. □

Corollary 2.2.4. *There exists a* $3-GDD$ *of type* $2^u m^1$ *for all* $m \leqslant 2u-2$, $2u+m \equiv 0,2$ $(mod\ 6)$ *and* $m \equiv 0,2$ $(mod\ 6)$.

Proof: By Theorem 2.2.2 we will form a STS$(2u+m+1)$ with a sub-STS$(m+1)$. We will delete a point of the subdesign to form a 3-GDD of type $2^{u+(m/2)}$. Finally, if we delete all the blocks of the subdesign, then we obtain a 3-GDD of type $2^u m^1$. □

Remark: This corollary covers the spectrum of 3-GDDs of type $2^u m^1$ except the cases $(u,m) \in \{(0,4),(3,4)\}$ in modulo 6.

Proof: From the table we have

$(u,m) \in \{(0,0),(0,2),(0,4),(1,0),(2,2),(3,0),(3,2),(3,4),(4,0),(5,2)\}$ in modulo 6 as necessary conditions on (u,m).

Secondly, by Corollary 2.2.4 we have $m \leqslant 2u-2$, $2u+m \equiv 0,2$ $(mod\ 6)$ and $m \equiv 0,2$ $(mod\ 6)$. If $m \equiv 0$ $(mod\ 6)$, then $u \equiv 0,1,3,4$ $(mod\ 6)$. If $m \equiv 2$ $(mod\ 6)$, then $u \equiv 0,2,3,5$ $(mod\ 6)$. Therefore, these cover the spectrum except the cases $(u,m) \in \{(0,4),(3,4)\}$ in modulo 6. □

Mendelsohn and Rosa [11] established a result for the remaining case:

Lemma 2.2.5. *Let* $v,w \equiv 5$ $(mod\ 6)$ *and* $v \geqslant 2w+1$. *Then there exits a* $(v,w;\{3\})$*-IPBD.*

Corollary 2.2.6. *There exists a* 3*-GDD of type* $1^{6k} m^1$ *for all* $m \equiv 5$ $(mod\ 6)$, $m \leqslant 6k-1$.

Proof: By Lemma 2.2.5 form a $(6k + m, m; \{3\})$-IPBD. If we take the hole of size m as a group and each element not in the hole as a singleton group, then this design gives the required 3-GDD of type $1^{6k}m^1$ for all $m \equiv 5$ $(mod\ 6)$. □

Remark: By this corollary we cover the spectrum of 3-GDD of type $1^u m^1$.

Corollary 2.2.7. *There exists a 3-GDD of type $2^{3k}m^1$ for all $m \equiv 4$ $(mod\ 6)$, $m \leqslant 6k - 2$.*

Proof: By Lemma 2.2.5 form a $(6k+m+1, m+1; \{3\})$-IPBD. If we delete a point from the hole of size $m + 1$, then we form the required 3−GDD of type $2^{3k}m^1$. □

Remark: This corollary covers the spectrum of 3−GDDs of type $2^u m^1$. Therefore, up to this point we have established the sufficiency for 3−GDDs of type $1^u m^1$ and $2^u m^1$.

The case $g = 3$ is also settled. Rees [12] established the following:

Lemma 2.2.8. *Let $m \geqslant 1$ and $0 \leqslant r \leqslant 2m$, $(m, r) \neq (1, 2)$ or $(3, 6)$. There exists a $\{2, 3\}$−GDD of type $(2m)^3$ which is resolvable into r parallel classes of blocks of size 3 and $4m - 2r$ parallel classes of blocks of size 2.*

Corollary 2.2.9. *Let $u \equiv 0$ $(mod\ 2)$, $m \equiv 1$ $(mod\ 2)$, $(m, u) \neq (1, 2)$, and $m \leqslant 3u - 3$. Then there exists a 3−GDD of type $3^u m^1$.*

Proof: Firstly, we will prove our statement for $m \geqslant u - 1$. So assume that $m \geqslant u - 1$ and also assume that $m \neq u - 1$ when $u = 6$. By Lemma 2.2.8 construct a *resolvable* $\{2, 3\}$−GDD of type u^3 having $m - u + 1$ parallel classes of blocks of size 2 and $(3u - m - 1)/2$ parallel classes of blocks of size 3. Now take one parallel class of triples as groups of the 3−GDD of type $3^u m^1$ being constructed. If we turn back to our parallel classes of pairs, we add $m - u + 1$ "ideal" points, one for each parallel class of pairs. Therefore a collection of triples (our blocks in the 3−GDD) is formed in this way. Also we use the triples in parallel classes as our new blocks in the 3−GDD of type $3^u m^1$. We have taken one parallel class of triples as the groups of size 3 in constructing 3−GDD of type $3^u m^1$, but also we need a long group of size m.

So take the $m-u+1$ ideal points in the long group. We need additional $u-1$ points to complete our long group to m. So add $u-1$ points to the long group. However, up to here we have all the blocks of 3−GDD except that the blocks on the groups of $\{2,3\}$−GDD and newly added $u-1$ points. So to obtain these blocks, we will put $(2u-1, u-1; \{3\})$−IPBD on each group of the $\{2,3\}$−GDD and newly added $u-1$ points and align the hole on these $u-1$ points. (Required IPBDs exist, see Theorem 2.2.2 and 2.2.5. To get IPBD in Theorem 2.2.2 we delete the blocks of sub-STS) We only left the case (m,u)=$(5,6)$. For this case we start with an idempotent quasigroup of order 6, then we will apply the same process.

Now we have the case $1 \leqslant m < u-1$. The same strategy using a smaller hole can be used to settle most cases, when m is small, except for the case when $u \equiv 4 \pmod 6$ and $m = 1$. For this last case, we use the standard construction of a STS$(6m+1)$ from a commutative idempotent quasigroup of order $2m+1$ that gives a system with $2m$ disjoint blocks for all $m \geqslant 2$. □

Using Lemma 2.2.8 we can give one more useful corollary:

Corollary 2.2.10. *Let $g \equiv 0 \pmod 2$, $m \equiv 0 \pmod 2$, $u \equiv 3 \pmod 6$ and $m \leqslant g(u-1)$. Then there exists a* 3−*GDD of type* $g^u m^1$.

Proof: We start with the case $u = 3$. That is we will build a 3−GDD of type $g^3 m^1$. By Lemma 2.2.8, form a resolvable $\{2,3\}$−GDD of type g^3 having m parallel classes of pairs. For the 3−GDD, we need a long group with m points. We form this group by adding m ideal points, one to the pairs of each parallel class of pairs. Therefore, with these new triples and by taking triples of $\{2,3\}$−GDD we form all the blocks of the 3−GDD of type $g^3 m^1$.

Next we finish the other cases. Assume that $u = 6n+3$ and form a KTS$(6n+3)$ on the point set E. So we have $3n+1$ parallel classes of triples $P_1, P_2, ..., P_{3n+1}$. Now we will form our 3−GDD of type $g^u m^1$ on elements $(\mathbb{Z}_g \times E\ ) \cup M$ with $|M| = m$. Partition M into sets $M_1, M_2, ..., M_{3n+1}$ with $|M_i| = m_i$ and we will choose m_i's such that $m_i \equiv 0 \pmod 2$, $0 \leqslant m_i \leqslant 2g$ and $m = \sum_{i=1}^{3n+1} m_i$. We will obtain the blocks of

3−GDD by the following method. Think each group of size g of the 3−GDD as points of E (since $|E| = 6n + 3 = u$ this is valid), then for each block $\{x, y, z\}$ in the parallel class P_i, we will put a 3−GDD of type $g^3 m_i^1$ on the set $(\mathbb{Z}_g \times \{x, y, z\}) \cup M_i$.

The case $g = 4$ was also solved in [13]:

Lemma 2.2.11 (Lemma 1.10, [13])**.** *Let* $m \equiv 0\ (mod\ 2)$, $m \leqslant 4u - 4$ *and* $u(2u + 1 + m) \equiv 0\ (mod\ 3)$. *Then there exists a* 3−*GDD of type* $4^u m^1$.

We will use these designs with small g and small u as building blocks in the recursive constructions to be developed. The main recursive constructions is WFC which is Theorem 1.0.2, and also we will use Lemma 1.0.7 and Theorem1.0.8 to develop the recursions.

We have one more useful recursive construction:

Lemma 2.2.12. *Let* (X, G, B) *be a* 3−*GDD with group sizes* $g_1, g_2, ..., g_n$ *and* $u \geqslant 3$. *If there exist* 3−*GDDs of type* $g_i^u m^1$ *for all* $i = 1, 2, ..., n$, *then there exists a* 3−*GDD of type* $(|X|)^u m^1$.

Proof: Let (X, G, B) be a 3−GDD with group sizes $g_1, g_2, ..., g_n$. We will construct a 3−GDD on the set $(X \times \mathbb{Z}_u) \cup M$, with $|M| = m$. So in this set we have u copies of X and u copies of each G_i where $|G_i| = g_i$ and $\bigcup_{i=1}^{n} G_i = X$. First obtain the blocks on the u copies of each G_i with M by placing 3−GDDs of type $g_i^u m^1$ on $(G_i \times \mathbb{Z}_t) \cup M$ for each $1 \leqslant i \leqslant n$. Secondly, for each block $\{x, y, z\} \in B$, place a TD$(3, u)$ on $\{x, y, z\} \times \mathbb{Z}_u$ and align the groups to $\{x\} \times \mathbb{Z}_u$, $\{y\} \times \mathbb{Z}_u$, $\{z\} \times \mathbb{Z}_u$. Here we have obtained the blocks between the copies of the groups, but we also have blocks in X that we do not want. To solve this problem we delete one parallel class of the TD$(3, u)$ aligned on $\{x, y, z\} \times \{i\}$ for each $i \in \mathbb{Z}_u$. □

Now we will give the sufficiency for the case $g \equiv 0\ (mod\ 3)$.

Lemma 2.2.13. *Let* u *be odd,* $x \geqslant 1$, $0 \leqslant m \leqslant 6x(u - 1)$, *and* $m \equiv 0\ (mod\ 2)$. *Then there exits a* 3−*GDD of type* $(6x)^u m^1$.

Proof: First, build a resolvable 3−GDD of type 3^u. Since u is odd, $3u \equiv 3$ $(mod\ 6)$ so a KTS($3u$) exists. If we take one parallel class of triples as the groups of the 3−GDD, then we obtain a resolvable 3−GDD of type 3^u. Since we have $(3u-1)/2$ parallel classes in the KTS, and we use one parallel class as the groups of the 3−GDD, we have $3(u-1)/2$ parallel classes of triples in 3−GDD. If we add one point to each of $3(u-1)/2$ parallel class of the $3-GDD$, then we obtain a 4−GDD of type $3^u(3(u-1)/2)^1$. In this 4-GDD if we give even weights between 0 and $4x$ to every point in the long group satisfying that the sum of the weights gives m and the weight $2x$ to each point not in the long group, and finally apply WFC using 3−GDDs of type $(2x)^3w^1$ for all even w satisfying $0 \leqslant w \leqslant 4x$ as fillers (they exists by Corollary 2.2.10), then we obtain the required 3−GDD of type $(6x)^u m^1$. □

Lemma 2.2.14. *Let* $u \geqslant 8$, *u even, m even,* $x \geqslant 1$, *and* $0 \leqslant m \leqslant 6x(u-1)$. *Then there exits a* 3−*GDD of type* $(6x)^u m^1$.

Proof: We start with a *resolvable* 3−GDD of type $6^{u/2}$. (The required resolvable GDD exists. See [14], [15]) As above we will add $3(u-2)/2$ ideal points to each parallel class and get a 4−GDD of type $6^{u/2}(3(u-2)/2)^1$. Now we will build a $\{4,7\}$−GDD of type $3^u(3(u-2)/2)^1$. To obtain this $\{4,7\}$−GDD, we add a point y to the long group of 4−GDD and connect it with the other groups. So take them as new blocks of size 7 and delete a different point z in the long group of the 4−GDD, and take the blocks containing z as the new groups of size 3. This process preserves the size of the long group. Therefore, the result is a $\{4,7\}$−GDD of type $3^u(3(u-2)/2)^1$ in which only y in the long group belongs to blocks of size 7. We give the weight 0 or $10x$ to y and even weights between 0 and $4x$ to the other points in the long group. Then to all points which are not in the long group, we give the weight $2x$. Now if we apply WFC with proper filler GDDs, then we will obtain the required 3−GDD of type $(6x)^u m^1$. The first kind of filler GDD that we will put on the blocks of size 3 is a 3−GDD of type g^3m^1 and it exists by Corollary 2.2.10. The second type that we will put on the blocks of size 7 is a 3−GDD of type $(2x)^6(10x)^1$ (for this GDD see Lemma 2.2 in [9]). □

The cases $u = 4$ and 6 are handled by methods tailored for these cases:

Lemma 2.2.15. *Let $x \geqslant 1$, m even, and $0 \leqslant m \leqslant 18x$. Then there exist $3-$GDDs of type $(6x)^4m^1$.*

Proof: First, we will handle the case $u \leqslant 12x$. If we remove one point of a BIBD(4, 16) (see [1]), then we get a $4-$GDD of type 3^5 and we can think it as a $4-$GDD of type 3^43^1. We give weights $2x$ to the points in the first 4 groups and even weights between 0 and $4x$ to the each point in the last group. If we apply WFC using $3-$GDDs of type $(2x)^3w^1$ (these exist by Corollary 2.2.10), then we get the required $3-$GDDs of type $(6x)^4m^1$. Secondly, we will handle the remaining case $12x \leqslant m \leqslant 18x$. We take a PBD with 12 elements with four parallel classes of triples and three parallel classes of pairs. (see [12]). We will form a $\{3,4\}-$GDD of type 3^46^1 from this PBD. To build the GDD, we will add six ideal points, one to each parallel class of pairs, and one to each of the three parallel classes of triples of the PBD. So we take the remaining parallel class of triples as groups of size 3 of the GDD and take the added 6 points as the group of size 6 to get a $\{3,4\}-$GDD of type 3^46^1. If we give weight $2x$ to each point not in the long group and also three ideal points which we add to the parallel classes of pairs, and assign even weights between 0 to $4x$ to the remaining three ideal points in the long group, we will get the required GDD. Even this gives more than we want, this satisfies for the cases $6x \leqslant m \leqslant 18x$. □

Lemma 2.2.16. *Let $x \geqslant 1$, m even, and $0 \leqslant u \leqslant 30x$. Then there exist $3-$GDDs of type $(6x)^6m^1$.*

Proof: There exists a PBD on 18 elements having eight parallel classes of triples and one parallel class of pairs (see [15]). If we proceed as Lemma 2.2.15, we complete the cases $2x \leqslant m \leqslant 30x$. For the next case, we will use a $4-$GDD of type 3^46^2 and it exists by [16]. We will form a $\{4,7\}-$GDD of type 3^66^1 from the $4-$GDD, then add an element to a group of size 6 of the $4-$GDD and connect with other groups. Next we delete one element from the same group and take the blocks containing this element as groups of the $\{4,7\}-$GDD. Together with the group of size 6, in which only one point belongs to the blocks of size 7, we obtain the $\{4,7\}-$GDD of type $3^6 6^1$. Unlike the case in Lemma 2.2.14, this point belongs to blocks of size 4. Hence when we apply WFC, we will always assign 0 weight to this point. Next we apply WFC to the $\{4,7\}-$GDD and give weight $2x$ to the points not in the long group. We assign even weights between 0 and $4x$ to the points in the long group (the group of size 6). Therefore, we complete the case $0 \leqslant m \leqslant 20x$ that is even more than we want. □

Now we turn the case for $g \equiv 0\ (mod\ 3)$.

Lemma 2.2.17. *Let $u \geqslant 4$, m odd, $g \equiv 3\ (mod\ 6)$, and $1 \leqslant m \leqslant g(u-1)$. Then there exist $3-$GDDs of type $g^u m^1$.*

Proof: There exits a KTS(g). So we have $(g-1)/2$ parallel class of triples, $P_1, P_2, ..., P_{(g-1)/2}$. Let V be the element set of such a KTS, so $|V| = g$. Now we will build a $3-$GDD of type $g^u m^1$ on the point set $(V \times \mathbb{Z}_u) \cup M$ where $|M| = m$. For $2 \leqslant i \leqslant (g-1)/2$, let $0 \leqslant m_i \leqslant 2u-2$, $m_i \equiv 0\ (mod\ 2)$; let $0 \leqslant m_1 \leqslant 3u-3$, $m_1 \equiv 1\ (mod\ 2)$ so that $m = \sum_{i=1}^{(g-1)/2} m_i$. Let $M_1, M_2, ..., M_{(g-1)/2}$ be disjoint sets so that $|M_i| = m_i$. Let $M = \bigcup_{i=1}^{(g-1)/2} M_i$. For each parallel class P_i, $i = 1, 2, ..., (g-1)/2$, and each block $\{x, y, z\}$ of P_i, on $\{x, y, z\} \times \mathbb{Z}_u$ we put a $3-$GDD of type $u^3(m_i)^1$ omitting a parallel class on $\{x, y, z\} \times \{i\}$ for $i \in \mathbb{Z}_u$. The groups are aligned on $\{x\} \times \mathbb{Z}_u$, $\{y\} \times \mathbb{Z}_u$, $\{z\} \times \mathbb{Z}_u$ and M_i.

Finally, we must treat $\{x, y, z\} \times \mathbb{Z}_u$ for $\{x, y, z\}$ in P_1. Here we place a $3-$GDD of type $3^u(m_1)^1$ (Corollary 2.2.9) with groups aligned on $\{x, y, z\} \times \{i\}$ and on M_1. All required choices for the m_i are possible by Corollaries 2.2.9 and 2.2.10. □

Lemma 2.2.18 (Lemma 3.3, [9])**.** *Let g be even, $u \geqslant 3$, $0 \leqslant m \leqslant u-1$ and $0 \leqslant x \leqslant 2u-2$ with $x \equiv 0\ (mod\ 2)$ if $u \equiv 0\ (mod\ 3)$, $x \equiv 0\ (mod\ 6)$ if $u \equiv 1\ (mod\ 3)$, and $x \equiv 2\ (mod\ 6)$ if $u \equiv 2\ (mod\ 3)$. Then there exists a $3-$GDD of type $g^u((r-1)(u-1)+6m+x)^1$.*

Lemma 2.2.19 (Lemma 3.5, [9])**.** *Let g be odd and u be even. Let $0 \leqslant m \leqslant u-1$ and $0 \leqslant x \leqslant u-1$ where $x \equiv 1\ (mod\ 2)$ if $u \equiv 0\ (mod\ 6)$, $x \equiv 1\ (mod\ 6)$ if $u \equiv 2\ (mod\ 6)$, $x \equiv 3\ (mod\ 6)$ if $u \equiv 4\ (mod\ 6)$. Then there exists a $3-$GDD of type $g^u(r(u-1)+6m+x)^1$.*

The consequences of Lemmas 2.2.18 and 2.2.19 are quite surprising:

Lemma 2.2.20 (Lemma 3.6, [9])**.** *Let g, u and m satisfy the necessary conditions of Theorem 2.2.1. Then a* 3−*GDD of type $g^u m^1$ exists whenever*
(***i***) $g \equiv 2, 8$ (*mod* 24)*;*
(***ii***) $g \equiv 14, 20$ (*mod* 24) *and* $m \geqslant 6u - 6$*;*
(***iii***) $g \equiv 4$ (*mod* 6) *and* $m \geqslant 2u - 2$*;*
(***iv***) $g \equiv 1$ (*mod* 6)*; or*
(***v***) $g \equiv 5$ (*mod* 6) *and* $m \geqslant 4u - 4 + x$*, where* $x = 1$ *if* $u \equiv 0, 2$ (*mod* 6)*, and* $x = 3$ *if* $u \equiv 4$ (*mod* 6)*.*

Lemma 2.2.21 (Lemma 3.7, [9])**.** *Let g, u and m satisfy the necessary conditions of Theorem 2.2.1 with* $g \equiv 2, 4$ (*mod* 6)*,* $u \equiv 1, 2$ (*mod* 3)*, and* $m \geqslant 2g + 2$*. Then there exists a* 3−*GDD of type $g^u m^1$.*

There is a further method for handling some cases when m is large.

Lemma 2.2.22. *Let $u > 4$ be even. If a* 3−*GDD of type* $(2g)^{u/2} u^1$ *exists, then there is a* 3−*GDD of type* $g^u (u + g)^1$*.*

Proof: If we apply Lemma 1.0.7 with $s = 2$, then we obtain the required result. □

Now we deal with large ordinary groups (the groups of size g).

Lemma 2.2.23. *Let* $g \equiv 4$ (*mod* 6) *and let g, u and m satisfy the conditions of Theorem 2.2.1. Then there exists a* 3−*GDD of type $g^u m^1$.*

Proof: By Lemma 2.2.20 (iii) we can restrict our attention to $m < 2u - 2$. We write $g = 6n + 4$ and by Lemma 2.2.11 we can assume $n > 0$. When $n \geqslant 3$, we apply Lemma 2.2.12 to a 3−GDD of type $6^n 4^1$, using 3−GDDs of types $6^u m^1$ and $4^u m^1$. This handles $m \leqslant 4u - 4$ for $g \geqslant 22$, and hence completes these cases.

We apply Lemma 2.2.12 to a 3−GDD of type $2^{3n} 4^1$ using 3−GDDs with $g = 2$. This handles all $m \leqslant 2u - 2$ when $u \equiv 0, 1$ (*mod* 3). For $g = 16$, we apply Lemma

2.2.12 to a 3−GDD of type 4^4 using 3−GDDs of type $4^u m^1$ to handle all the remaining cases. This leaves only the case $g = 10$ when $u \equiv 2$ (*mod* 3).

For this case, if we apply Lemma 2.2.21, we handle all $m \geqslant 22$. In the remaining cases where $u \geqslant 17$, we fill the long group of a 3−GDD of type $10^{u-6}(60+m)^1$ with a 3−GDD of type $10^6 m^1$. For $u \in \{11, 14\}$, we proceed similarly with a 3−GDD of type $10^{u-3}(30+m)^1$ and one of type $10^3 m^1$; this handles when $m \leqslant 16$. Recalling that $m \geqslant 2u-2$ is handled by Lemma 2.2.20, this completes $u \in \{11, 14\}$.

For the remaining cases, $10^8 10^1 = 10^9$ is handled in the case when $m = 0$. Type $10^8 4^1$ is handled by filling a 3−GDD of type $10^5 34^1$ with a 3−GDD of type $10^3 4^1$. The only remaining case is of type $10^5 4^1$, we refer to Lemma 4.1 of [9]. □

Lemma 2.2.24. *Let* $g \equiv 14, 20$ (*mod* 24) *and let g, t and m satisfy the conditions of Theorem 2.2.1. Then there exists a* 3−*GDD of type* $g^u m^1$.

Proof: By Lemma 2.2.20 (ii) we need only considering $m < 6u - 6$. We write $g = 6n + 8$. For $n \geqslant 3$, we apply 2.2.12 to a 3−GDD of type $6^n 8^1$ using 3−GDDs of types $6^u m^1$ and $8^u m^1$. This handles all cases with $g \geqslant 26$, leaving only $g \in \{14, 20\}$.

For u even, $u \neq 4$, we handle $m \geqslant g$ using Lemma 2.2.22. For $u \geqslant 8$, we apply 2.2.12 to a 3−GDD of type 2^{10} or 2^7. For $u \in \{4, 6\}$ and $g = 20$, we apply Lemma 2.2.12 to a 3−GDD of type $4^3 8^1$. This completes all the cases when u is even except for $g = 14$ when $u \in \{4, 6\}$.

For u odd, $g = 20$, Lemma 2.2.21 handles all $m \geqslant 42$ (by Corollary 2.2.10 we may assume that $u \equiv 1, 2$ (*mod* 3)). Then using 3−GDDs of type $20^{(u-6)}(120+m)^1$ and $20^6 m^1$ (Theorem 1.0.8) handles all remaining cases with $u > 13$. Using 3−GDDs of type $20^{(u-3)}(60+m)^1$ and $20^3 m^1$ handles all remaining cases with $u > 7$. For $u \in \{5, 7\}$, it remains to construct $20^5 14^1$ and $20^7 30^1$, see Lemma 2.2.24.

There remains $g = 14$ to handle. For u even, only $u \in \{4, 6\}$ remains. For u odd, we proceed as $g = 20$, leaving only cases with $u \in \{5, 7\}$. The cases $14^5 2^1$ and $14^5 8^1$ can be obtained from Lemma 2.2.12. Thus for $u = 5$ we may assume $m > 14$. For each $u = 4, 5, 6, 7$, we take a TD$(u+1, 7)$ and apply WFC giving every point in the first u groups weight 2. In the last group, give points weights $\equiv 0$ (*mod* 2) when

$u = 6$, weights $\equiv 0$ (*mod* 6) when $u \in \{4, 7\}$ and weights $\equiv 2$ (*mod* 6) when $u = 5$. □

We handle next all remaining cases for g odd when $g > 11$.

Lemma 2.2.25. *Let $g \equiv 5$ (mod 24) and let g, u and m satisfy the conditions of Theorem 2.2.1. Then provided that this theorem holds for $g = 5$ and $g = 11$, there exists a $3-$GDD of type $g^u m^1$.*

Proof: The proof proceeds inductively, we assume the solutions for $g \in \{5, 11\}$ exist. By Lemma 2.2.20, we need only considering $m < 4u - 1$ when $u \equiv 4$ (*mod* 6) and $m < 4u - 3$ otherwise. If $m \geqslant g$ and $u > 4$, we apply Lemma 2.2.22. In general we write $g = 6n+5$, and form a GDD of type $n^6 5^1$ when $n \equiv 3, 5$ (*mod* 6), $(n-1)^6 11^1$ when $n \equiv 0, 4$ (*mod* 6), $(n-2)^6 17^1$ when $n \equiv 1$ (*mod* 6), or $(n-3)^6 23^1$ when $n \equiv 2$ (*mod* 6), and apply Lemma 2.2.12. This gives a complete solution when $n \geqslant 5$.

Now for $g = 29$, use instead a GDD of type $5^4 9^1$ in the application of Lemma 2.2.12. This handles all $m \leqslant 5u - 5$.

When $g = 23$, for $u \geqslant 8$ Lemma 2.2.12 with $3^6 5^1$ handles all $m \leqslant 21$. For $u = 6$, Lemma 2.2.12 applied to the GDD of type $3^6 5^1$ handles all $m \leqslant 15$, leaving $23^6 17^1$ and $23^6 19^1$ to be handled. For $23^6 m^1$ see Lemma 4.3 in [9].

For $g = 23$ and $u = 4$, Lemma 2.2.20 handles $m \geqslant 15$ and Lemma 2.2.12 with $3^6 5^1$ handles $m \leqslant 9$.

The case $g = 17$ is treated using a $3-$GDD of type $3^4 5^1$ in the application of Lemma 2.2.12. □

At this point, we have completed the proof of Theorem 2.2.1 when g is even with no exception. The case for g odd still rests on the completion of $g \equiv 5$ (*mod* 6).

Lemma 2.2.26 (Lemma 5.1, [9])**.** *There exist $3-$GDDs of types $5^4 3^1$, $5^4 9^1$, $11^4 3^1$, and $11^4 9^1$.*

Lemma 2.2.27 (see [22])**.** *Let g, u be positive integers satisfying $u \geqslant 3$, $g \geqslant 3$, $(g-1)(u-1) \equiv 0$ (mod 2) and $gt(g-1)(u-1) \equiv 0$ (mod 3). Then there exists a PBD on $v = gt$*

points having one parallel class of blocks of size g, one parallel class of blocks of size u, and all remaining blocks of size 3.

Corollary 2.2.28. *Let g, u, m satisfy the conditions of Theorem 2.2.1, with g odd, $u \equiv 0, 1$ (mod 3), and $m > 1$. Then there exists a $3-$GDD of type $g^u m^1$.*

Proof: We use Lemma 2.2.27 to form a PBD with one parallel class of g blocks and one parallel class of u blocks on gt points. We take parallel class of g blocks as groups and add m points at infinity. On each u block together with these m points, if we place a 3-GDD of type $1^u m^1$, then we obtain the required $3-$GDD of type $g^u m^1$. □

This completes all cases when $u \equiv 0, 1$ (*mod* 3) with $g \in \{5, 11\}$, with the two exceptions that we treat next:

Lemma 2.2.29. *There exist 3-GDDs of types $11^6 7^1$ and $11^6 9^1$.*

Proof: We use a TD(7, 7), apply weight 5 to six elements of one block, and weight 1 or 3 to the seventh point of the block. We apply weight 1 to all other elements. If we apply the WFC using 3-GDDs of types 1^7, $1^6 3^1$, $1^6 5^1$, $5^6 1^1$ and $5^6 3^1$, the we obtain the required 3-GDDs of types $11^6 7^1$ and $11^6 9^1$. □

If we turn the cases when $u \equiv 2$ (*mod* 3), we first observe that using the $3-$GDDs of types $g^{u-6}(6g + m)^1$ and $g^6 m^1$ handles all cases with $m < g$ for $u \geqslant 14$, leaving only $u = 8$. In this case, we need only treat to $11^8 5^1$.

Lemma 2.2.30 (Lemma 5.5, [9])**.** *There exists a 3-GDD of type $11^8 5^1$.*

This completes the last case for $g \in \{5, 11\}$, and hence the last case of the proof of Theorem 2.2.1.

2.3 3-GDDs of type $g^u 1^t$

Theorem 2.3.1. *Let g, u and t be positive integers. Then there exists a $3-$GDD of type $g^u 1^t$ if and only if the following conditions are satisfied:*
(***i***) $g \equiv 1\ (mod\ 2)$;
(***ii***) $u + t \equiv 1\ (mod\ 2)$;
(***iii***) *if* $u = 1$, *then* $t \geqslant g + 1$;
(***iv***) *if* $u = 2$, *then* $t \geqslant g$;
(***v***) $\binom{t}{2} + ugt + \binom{u}{2} g^2 \equiv 0\ (mod\ 3)$.

Proof: We can establish the necessity as follows. Since $t \geqslant 1$, we consider an element in a group of size 1. It must appear in $\frac{1}{2}\,(ug + t - 1)$ triples and, hence $ug + t - 1 \equiv 0\ (mod\ 2)$. Since $u \geqslant 1$, consider an element in a group of size g. It appears in $\frac{1}{2}\,(g(u-1)+t)$ triples and, hence $ug + t - g \equiv 0\ (mod\ 2)$. For both to hold, g must be odd. When g is odd since $ug + t - 1 \equiv u + t - 1\ (mod\ 2)$, $u + t$ must also be odd. For (*iii*) any element in a singleton group must appear in a triple with each element of the group of size g. The third elements of such triples are all distinct and none of them appear in the large group. Thus $t \geqslant g + 1$. For (*iv*), consider an element x in one of the groups of size g. Consider the triples in which x appears with the elements of the other group of size g. The third element of such triples form g distinct elements, none of which appear in one of the large groups. Thus $t \geqslant g$. Finally, for (*v*) the number of pairs occurring in triples must be divisible by 3. □

We summarize these conditions in the Table 2.3.

Corollary 2.3.2. *The conditions of the Theorem 2.3.1 are sufficient when $u = 1$.*

Proof: From Theorem 2.2.1, there exists a $3-$GDD of type $g^1 1^t$. □

When $g \in \{1, 3\}$ the conditions of the Theorem 2.3.1 are also sufficient. Since when $g = 1$, the existence of a $3-$GDD of type $1^u 1^t$ is equivalent to the existence of a STS$(u + t)$ and when $g = 3$ we use a resolvable-STS and an almost resolvable-STS (see [17]). Henceforth, we treat only the cases when $g \geqslant 5$.

Table 2.3: Possible values for t (*mod* 6)

	g (mod 6)		
u (mod 6)	**1**	**3**	**5**
0	1, 3	1, 3	1, 3
	by 2.3.11	by 2.3.11	by 2.3.13, 2.3.16
1	0, 2	0, 4	0
	by 2.3.11	by 2.3.11	by 2.3.13
2	1, 5	1, 3	-
	by 2.3.11	by 2.3.11	
3	0, 4	0, 4	0, 4
	by 2.3.11	by 2.3.11	by 2.3.13, 2.3.15
4	3, 5	1, 3	3
	by 2.3.11	by 2.3.11	by 2.3.13
5	2, 4	0, 4	-
	by 2.3.11	by 2.3.11	

In general, our strategy in proving the sufficiency for the conditions for GDDs is to develop recursive constructions providing a finite number of small cases that can be produced by direct techniques. The required small 3−GDDs exist by [6] and [18].

Lemma 2.3.3. *The conditions of the Theorem 2.3.1 are sufficient when* $gu + t \leqslant 60$ *and also for* 3−*GDDs of types* $9^7 1^4$, $9^7 1^6$, $11^4 1^{39}$, $11^4 1^{45}$, $11^7 1^6$, $13^4 1^{11}$, $17^4 1^{57}$, $17^4 1^{63}$, $17^4 1^{69}$, *and* $23^7 1^t$ *for* $t \in \{6, 12, 18\}$.

Proof: Sufficiency for $gu + t \leqslant 60$ follows from the main computational results in [6]. For the remaining cases refer to Lemma 1.3 in [18]. □

Now we will develop a construction that is useful for obtaining GDDs with small number of groups.

Lemma 2.3.4. *Suppose that there exists a resolvable* $3-GDD$ *of type* g^u*, having* $p = g(u-1)/2$ *parallel classes. Let* w *be an integer and let* $A(w) = \{x : \exists 3-GDD\ of\ type\ w^3x^1\}$*. Suppose further that there is a* $\{2,3\}-GDD$ *of type* w^3 *in which the blocks of size two have a partition into* $u-3+e$ *parallel classes, and for each of the three groups, one holey parallel class that misses precisely the one of the three groups of size* w*. Finally write* $p = m + l$*, suppose that* $a_1, ..., a_l \in A(w)$ *and that a* $3-GDD$ *of type* $m^u1^{em+\sum_{i=1}^{l} a_i}$ *exists. Then a* $3-GDD$ *of type* $(wg+m)^u1^{em+\sum_{i=1}^{l} a_i}$ *exists.*

Proof: Let $P_1, ..., P_p$ be the parallel classes of a resolvable 3−GDD of type g^u. We give every point of this resolvable 3−GDD, the weight w. For each parallel class P_i, $1 \leqslant i \leqslant l$, we add a_i additional elements. Then for each block of P_i, on the $3w$ corresponding points, together with the a_i additional points, we place the blocks of a 3−GDD of type $w^3a_i^1$. For the parallel classes $P_{l+1}, ..., P_p$ we proceed differently. For each parallel class P_i, $l+1 \leqslant i \leqslant p$, we add one new point to the u groups and e extra new points. (So now each group has $gw+(p-l)$ elements) Then for each block of the parallel class we proceed as follows: we place a $\{2,3\}$−GDD of type w^3 on $3w$ points arising in this block. The $u-3+e$ parallel classes of this $\{2,3\}$−GDD are used to form the triples with each of e extra points and each of $u-3$ new points added to the groups that are not met by the chosen block. Finally, three holey 1-factors (pairs that misses a specified group) are used to form triples with three new points added to the groups that are met by the chosen block. Once all parallel classes have been treated, $m = p - l$ additional elements have been added to each of the u groups. On these, together with em extra elements and $\sum_{i=1}^{l} a_i$ additional elements, we place the blocks of a 3−GDD of type $m^u1^{em+\sum_{i=1}^{l} a_i}$. Therefore, the result is a 3−GDD of type $(wg+m)^u1^{em+\sum_{i=1}^{l} a_i}$. □

The existence of resolvable 3−GDDs has been settled, see [14] and [20].

Lemma 2.3.5 ([20])**.** *There exists a resolvable* 3−*GDD of type* g^u *if and only if* $g(u-1) \equiv 0 \ (mod\ 2)$, $gu \equiv 0 \ (mod\ 3)$ *and* $(g, u) \notin \{(2, 3), (2, 6), (6, 3)\}$.

Now there exists a {2, 3}−GDD of type 2^3 having three holey 1-factors and two triples. Specifically, on groups $\{\{0, 1\}, \{2, 3\}, \{4, 5\}\}$, form the triples $\{\{0, 2, 4\}, \{1, 3, 5\}\}$ and the holey parallel classes $\{\{2, 5\}, \{3, 4\}\}$, $\{\{0, 5\}, \{1, 4\}\}$ and $\{\{0, 3\}, \{1, 2\}\}$. This allows the use of $w = 2$ and $u - 3 + e = 0$ in Lemma 2.3.4

Corollary 2.3.6. *For* $g \equiv 1 \ (mod\ 2)$, $t \equiv 0, 4 \ (mod\ 6)$, *and* $t \leqslant 2g - 6$ *then there exists a* 3−*GDD of type* $g^3 1^t$.

Proof: If $g \in \{1, 3, 5, 13\}$, the required 3−GDDs are from Lemma 2.3.3. Otherwise we will apply Lemma 2.3.4 with $g = (u - 1)/2$, $w = 2$, $u = 3$, $m = 1$ and $e = 0$. A resolvable 3−GDD of type g^3 has g parallel classes. Theorem 2.2.1 gives $A(2) = \{0, 2, 4\}$. Thus $t = \sum_{i=1}^{l} a_i$ where $a_i \in A(2)$ if and only if $t \leqslant 2u - 6$. Furthermore the 3−GDD of type $m^u 1^{em+\sum_{i=1}^{l} a_i}$ required by Lemma 2.3.4 is a STS of order $3 + t$, after a substitution, and it exists since $t \equiv 0, 4 \ (mod\ 6)$. So we have all we needed and have obtained the desired result. □

There exists a {2, 3}-GDD of type 4^3 having three holey parallel classes of pairs, two parallel classes of pairs and eight triples. For construction, see [18]. Now using this GDD and using $w = 4$, $u - 3 + e = 2$ in Lemma 2.3.4 we obtain the following.

Corollary 2.3.7. *Suppose that Theorem 2.3.1 holds for* $u = 5$ *and* $g \in \{7, 9\}$. *If the conditions of the Theorem 2.3.1 are met for* $g, u, t, r = 5$ *and* $t < g$ *then there exists a* 3−*GDD of type* $g^5 1^t$.

Proof: Write $m \equiv g \ (mod\ 12)$, $0 < m < 12$. If we apply Lemma 2.3.4 with $g = (g - m)/4$, $w = 4$, $u = 7$ and $e = 0$, then we get the result. □

There is a {2, 3}-GDD of type 4^3 having three holey parallel classes and four parallel classes of pairs, see [18].

Corollary 2.3.8. *Let $g, u, t, u = 7$ satisfy the conditions of the Theorem 2.3.1 and let $t < g$. Let $m \equiv g$ (mod 12), and $0 < m < 12$. Suppose that Theorem 2.3.1 holds for $m^7 1^t$. Then there is a 3−GDD of type $g^7 1^t$.*

Proof: If $m < 12$, the statement holds by assumption in the corollary. If $g = 23$, the required 3−GDDs are from Lemma 2.3.3 otherwise if we apply Lemma 2.3.4 with $g = (g - m)/4$, $w = 4$, $u = 7$ and $e = 0$, then we get the required 3−GDD. □

Finally we will present a construction for $u \in \{4, 8\}$ using a result of Rees, see Lemma 2.2.8.

Lemma 2.3.9 (Lemma 2.7, [18])**.** *For $g \equiv 1$ (mod 6), $g \geqslant 19$, $t \equiv 5$ (mod 6) and $t < g$, there exits a 3 − GDD of type $g^u 1^t$ for $u = 4$ and 8.*

Now we will do the cases $g \equiv 1, 3$ (*mod* 6). First, we employ Theorem 2.2.1 to settle the majority of the cases.

Lemma 2.3.10. *Let $0 \leqslant s < u$. If a 3−GDD of type $g^{u-s}(sg + t)^1$ and a 3−GDD of type $g^s 1^t$ exists, then a 3−GDD of type $g^u 1^t$ exists.*

Proof: If we fill the group of size $(sg + t)$ with the 3−GDD of type $g^s 1^t$ in the 3−GDD of type $g^{u-s}(sg + t)^1$, then we get the result. □

Theorem 2.3.11. *There exists a 3−GDD of type $g^u 1^t$ whenever $g \equiv 1, 3$ (mod 6) and the conditions of the Theorem 2.3.1 are met.*

Proof: First we establish that if the necessary conditions in the Theorem 2.3.1 are sufficient for $g \equiv 1, 3$ (*mod* 6) when $t < g$, they are sufficient for $g \equiv 1, 3$ (*mod* 6) and all t. If $t \geqslant g$, we write $s = \lfloor t/g \rfloor$. We form a 3−GDD of type $g^{u+s}1^{t-sg}$. If we fill s groups of size g of the 3−GDD of type $g^{u+s}1^{t-sg}$ with the triples of a 3−GDD of type 1^g, then we obtain the required 3−GDD of type $g^u 1^t$. We suppose henceforth that $t < g$.

If (u, g, t) (*mod* 6, 6, 6) is one of (0, 1, 1), (0, 1, 3), (0, 3, 1), (0, 3, 3), (2, 1, 1), (2, 3, 1), (2, 3, 3), (4, 1, 3), (4, 3, 1) or (4, 3, 3) and we apply Lemma 2.3.10 with $s = 0$, then we get the required 3−GDDs. (Theorem 2.2.1 produces the required 3−GDDs.)

If $u = 3$, we apply Corollary 2.3.6. So we suppose that $u \geqslant 4$. If (u, g, t) $(mod\ 6, 6, 6)$ is one of $(1, 1, 0)$, $(1, 3, 0)$, $(1, 3, 4)$, $(3, 1, 0)$, $(3, 1, 4)$, $(3, 3, 0)$, $(3, 3, 4)$, $(5, 1, 4)$, $(5, 3, 0)$ and $(5, 3, 4)$, and $u \geqslant 8$ we apply Lemma 2.3.10 with $s = 3$ (The 3−GDD of type $g^{u-3}(3g + t)^1$ is from the Theorem 2.2.1).

If $u = 4$, $g \equiv 1$ $(mod\ 6)$, $g > 13$, and $t \equiv 5$ $(mod\ 6)$, we apply Lemma 2.3.9, then we get the result. If instead $g \in \{7, 13\}$, we apply Lemma 2.3.3, then this completes the last case when $u = 4$.

If $u = 5$, the cases with $g \in \{7, 9\}$ and $t < g$ are from the Lemma 2.3.10. Now if we apply Corollary 2.3.7, then we settle all the cases with $g > 12$ and $t < g$.

If (u, g, t) $(mod\ 6, 6, 6)$ is $(2, 1, 5)$ or $(4, 1, 5)$ and $u \geqslant 10$, we apply Lemma 2.3.10 with $s = 4$, using Theorem 2.2.1 to provide the 3−GDD of type $g^{u-4}(4g + t)^1$, then we get the result.

If (u, g, t) $(mod\ 6, 6, 6)$ is $(1, 1, 2)$ or $(5, 1, 2)$ and $u \geqslant 12$, we apply Lemma 2.3.10 with $s = 5$, using Theorem 2.2.1 to provide the 3−GDD of type $g^{u-5}(5g + t)^1$, then we get the result.

It remain all cases with $u = 7$, the case when $u = 8$, $(g, t) \equiv (1, 5)$ $(mod\ 6, 6)$ and the case when $u = 11$, $(g, t) \equiv (1, 2)$ $(mod\ 6, 6)$ (in each case under the restriction that $t < g$). For $u = 7$, all 3−GDDs with $g \in \{7, 9\}$ and $t < g$ can be obtained by Lemma 2.3.3. Now if we apply Corollary 2.3.8, then we settle the remaining cases with $t < g$ and $g > 12$. For $u = 8$, if we apply Lemma 2.3.9, then we get the required 3−GDD. For $u = 11$, see Theorem 3.2 in [18]. □

We complete the case $g \equiv 1, 3$ $(mod\ 6)$. Next we will consider the case $g \equiv 5$ $(mod\ 6)$ and $u \equiv 0$ $(mod\ 3)$. In this case, firstly we will give a construction for $u = 3$.

Lemma 2.3.12. *Let $g \equiv 5$ $(mod\ 6)$, $t \equiv 0, 4$ $(mod\ 6)$, $2g - 4 \leqslant t < 6g$. Then there exists a 3−GDD of type $g^3 1^t$.*

Proof: If $g \in \{5, 11\}$ the required 3−GDD exists by Lemma 2.3.3. Otherwise let $x = (g - 2)/3$ and form a 5−GDD of type x^5 (i.e., a TD$(5, x)$). Choose one special block b. In the first three groups of the 5−GDD, we give all points not in the special block b weight 3; and in the last two groups, give all the points not in b weight 3 or 9. We give the points of the special block in the first three groups weight 5 and give the last two points weight 3, and weight 3 and 7. Using 3−GDDs of types $5^3 3^2$, $5^1 3^4$, $5^1 3^3 9^1$, $5^1 3^2 9^2$, 3^5, $3^4 9^1$, $3^3 9^2$, $7^1 5^3 3^1$, $9^1 7^1 3^3$ and $3^4 7^1$ (all from 2.3.3), we obtain a 3−GDD of type $g^3 v^1 w^1$ in which v and w are groups whose sizes are 1 or 3 (mod 6). Filling in the groups of size v and w with STSs constructs the required 3−GDD. □

Theorem 2.3.13. *If $g^u 1^t$ meets the conditions of the Theorem 2.3.1, $g \equiv 5$ (mod 6) and $u \equiv 0$ (mod 3) then there exists a 3−GDD of type $g^u 1^t$.*

Proof: If $u = 3$ and $t \leqslant 2g - 6$ and we apply Lemma 2.3.6, then we get the desired result. If $u = 3$ and $2g - 4 \leqslant t < 3g$ we apply Lemma 2.3.12, then we obtain the result. If $u = 6$ and $t < 3g$ and we apply Lemma 2.3.10 with $s = 0$, then we get the required 3−GDD. In all other cases if we form a 3−GDD of type$(3g)^{u/3} 1^t$ by Theorem 2.3.11 and fill each group with a 3−GDD of type g^3, the we obtain the required 3−GDD so we finish all the cases. □

Now we complete the case $u \equiv 0$ (mod 3) of the Theorem 2.3.1. The only remaining case is $g \equiv 5$ (mod 6) and $u \equiv 1$ (mod 3) and we will consider this case. First we will give a construction for $u = 4$.

Lemma 2.3.14. *Let $g \equiv 5$ (mod 6), $g \geqslant 23$, $t \equiv 3$ (mod 6) and $3g - 6 \leqslant t < 9u - 18$. Then there exists a 3−GDD of type $g^4 1^t$.*

Proof: We write $x = (g-2)/3$, whence x is odd. A TD$(7, x)$ exists for x except possibly when $x \in \{1, 3, 5, 15, 21, 33, 35, 39, 45, 51\}$, see [21] (A TD$(7, x)$ is a 7−GDD of type x^7). First, in one block we apply weights $5, 5, 5, 5, 3, 3$ and 3, and give all remaining points in the first four groups weight 3, and in the remaining last three groups weights 3 or 9. Using 3−GDDs of type $5^4 3^3$, $5^1 3^6$, $5^1 3^5 9^1$, $5^1 3^4 9^2$, $5^1 3^3 9^3$, 3^7, $3^6 9^1$, $3^5 9^2$ and $3^4 9^3$ (all from Lemma 2.3.3), we obtain the required 3−GDD for the specified values of t except when $g \in \{47, 65, 101, 107, 119, 147, 155\}$. If $g \in \{47, 65, 101, 119, 147, 155\}$ we write $x = (g - 8)/3$ and proceed as before using weight 11 in place of weight 5, then we get the required result. This leaves us only the case $g = 107$. For this case, we employ a TD$(7, 31)$. We give weights 11 and 3 on the special block, and choose one disjoint block with weights $9, 9, 9, 9, 3$ or 9, 3 or 9, 3 or 9. In each case obtain a 3−GDD with four groups of size g and three more groups each having $3 \pmod 6$ elements. These three groups can be filled by a 3−GDD having all groups of size 1, so we obtain the required 3−GDD of type $107^4 1^t$. □

Theorem 2.3.15. *If $g^u 1^t$ meets the conditions of the Theorem 2.3.1, $g \equiv 5 \pmod 6$ and $u \equiv 4 \pmod 6$, then there exists a 3−GDD of type $g^u 1^t$.*

Proof: If $t \leqslant (u-1)g$, we apply Lemma 2.3.10 with $s = 0$, then we complete this case. If $t \geqslant ug + 7$, we form a 3−GDD of type $(ug + 3)^1 1^{t-3}$ (from Theorem 2.2.1) fill the group of size $ug + 3$ with a 3−GDD of type $g^u 1^3$, then we get the required 3−GDD. When $u = 4$, if we apply Lemma 2.3.14, then we complete the determination when $g \geqslant 23$ and if we apply Lemma 2.3.3 we complete the case $g \leqslant 17$. Now for $u \geqslant 10$, it remains to treat the cases when $(u - 1)g < t \leqslant ug + 1$. In these cases, if we use a result in the following section Lemma 2.4.1 to form a 3−GDD of type $((u-3)g)^1(t-(u-6)g)^1 1^{(u-3)g}$ and fill the first group with a 3−GDD of type g^{u-3} and the second with a 3−GDD of type $g^3 1^{t-(u-3)g}$ (from Theorem 2.3.13), then we get the result. □

Theorem 2.3.16. *If $g^u 1^t$ meets the conditions of the Theorem 2.3.1, $g \equiv 5 \pmod 6$ and $u \equiv 1 \pmod 6$, then there exists a 3−GDD of type $g^u 1^t$.*

Proof: If $t > ug$, then we form a 3−GDD of type $(ug)^1 1^t$, and fill the group of size ug with a 3−GDD of type g^u, so we get the required 3−GDD. If $(u-3)g \leqslant t \leqslant ug$, when $u = 7$, we use Lemma 2.4.1 to form a 3−GDD of type $(4g+6m+3)^1(3g+6l+4)^1 1^{4g+6m+3}$, where m and l satisfy $t = 4g + 10 + 12m + 6l$, $6m + 3 \leqslant 3g$, $6l + 4 \leqslant 2g$, and $6l + 4 \leqslant g + 6m + 3$. We fill its two large groups with 3−GDDs of type $g^4 1^{6m+3}$ and $g^3 1^{6l+4}$ to obtain the desired 3−GDD. For $u \geqslant 13$, we use again Theorem 2.4.1 to form a 3−GDD of type $((u - 4)g)^1(t - (u - 8)g)^1 1^{(u-4)g}$ and fill the big groups with 3−GDDs of type g^{u-4} and $g^4 1^{t-(u-4)g}$.

When $g < t \leqslant (u - 3)g$, we apply Lemma 2.3.10 with $s = 1$. For $t < g$, if $u \geqslant 13$, we apply Lemma 2.3.10 with $s = 3$; for $u = 7$, first we settle the case $g \in \{5, 11\}$ and

$t < g$ by Lemma 2.3.10, and then apply Corollary 2.3.8 to get the required 3−GDD. Therefore we complete all the cases. □

2.4 3-GDDs of type $g^1v^11^g$

Lemma 2.4.1. *A* 3−*GDD of type* $g^1v^11^g$ *exists if and only if* $(g, v) \equiv (1, 1), (3, 1), (3, 3), (3, 5), (5, 1)$ $(mod\ 6, 6)$ *and* $v \leqslant g$.

Proof: First, we will prove the necessary conditions. If we fix a point in the group of size g and look at the blocks containing this fixed point then $2|(g + v)$. If we fix a singleton group and look at the blocks containing the singleton group, then $2|(2g + v - 1)$ so $2|(v - 1)$. Also we have $2|(g - 1)$. Therefore, both g and v are odd integers. Next, we will count the pairs in the blocks so $3 \mid \left[\binom{2g+v}{2} - \binom{g}{2} - \binom{v}{2}\right]$ and we get $6 \mid g(3g + 4v - 1)$. Both of them give us the necessary conditions.

Secondly, we will prove the sufficiency. Let $g \equiv 3$ $(mod\ 6)$, $X = \{x_1, x_2, ..., x_g\}$ be a set of $g = 2s + 1$ points and let $P_1, P_2, ..., P_s$ be the parallel classes of a KTS on X. Let $Y = \{y_1, y_2, ..., y_g\}$ be the set of g additional elements. For $1 \leqslant i \leqslant (g - v)/2$ we replace each triple $\{x_a, x_b, x_c\}$ of P_i by the triples $\{y_a, x_b, x_c\}$ $\{x_a, y_b, x_c\}$ and $\{x_a, x_b, y_c\}$. The remaining pairs containing an element of X and an element of Y form a v-regular bipartite graph. Form a 1−factorization $F_1, F_2, ..., F_v$ of this graph then add v new elements $\{z_1, z_2, ..., z_v\}$, and for each edge $\{\alpha, \beta\}$ of F_i form the triple $\{z_i, \alpha, \beta\}$. The result is a 3−GDD of type $g^1v^11^g$.

When $g \equiv 1, 5$ $(mod\ 6)$, we form instead a partial cyclic STS on X having $(v-1)/6$ full orbits of triples (using Theorem 2.3.11). Let $d_1, d_2, ..., d_{(g-v)/2}$ be the remaining differences on X. For each such difference d, let f satisfies $2f \equiv d$ $(mod\ g)$ and form the set of g triples as follows $\{\{x_i, x_{i+d}, x_{i+f}\} : 0 \leqslant i < g\}$ subscripts modulo g. Now on $X \cup Y$ what remains a v−regular bipartite graph, so we proceed as before. □

Chapter 3

4-GDDS

3.1 4-GDDs of type g^u (uniform 4-GDDs)

For uniform 4-GDDs, by Theorem 1.0.1 we can easily observe that

Theorem 3.1.1. *The necessary conditions for the existence of* $4-$*GDDs of type* g^u *are:*
(***i***) $u \geq 4$
(***ii***) $(u-1)g \equiv 0$ $(mod\ 3)$
(***iii***) $u(u-1)g^2 \equiv 0$ $(mod\ 6)$.

We summarize the necessary conditions in the Table 3.1.

Theorem 3.1.2 (Theorem 3.5, [1])**.** *For every* $v > 6$ *there exists a* $TD(4, v)$.

Lemma 3.1.3. *If there exists a* 4*-GDD of type* g^u *and* r *is a positive integer, then there exists a* 4*-GDD of type* $(gr)^u$.

Proof: Let (X, G, B) be a 4-GDD of type g^u. To each point in X we will give the weight r, then apply WFC. To each block $b \in B$, if we put a TD$(4, r)$, then we get a 4$-$GDD of type $(gr)^u$. (A TD$(4, r)$ is equivalent to a 4$-$GDD of type r^4 and it exists

Table 3.1: Necessary and sufficient conditions for the existence of uniform 4$-$GDDs

g	necessary and sufficient u	constructed by lemmas
$1, 5$ $(mod\ 6)$	$1, 4$ $(mod\ 12)$ and $u \geq 4$	3.1.3 and 3.1.4
$2, 4$ $(mod\ 6)$	1 $(mod\ 3)$ and $u \geq 4$ *and* $(g, u) \neq (2, 4)$	3.1.3, 3.1.9, and 3.1.11
3 $(mod\ 6)$	$0, 1$ $(mod\ 4)$ and $u \geq 4$	3.1.3 and 3.1.10
0 $(mod\ 6)$	$u \geq 4$ *and* $(g, u) \neq (6, 4)$	3.1.3 and 3.1.14

by Theorem 3.1.2, except TD(4, 2) and TD(4,6)) □

Therefore to prove Theorem 3.1.1 we need only 4−GDDs of type $1^u, 2^u, 3^u, 4^u$ and 6^u. Then by applying Lemma 3.1.3 we complete all the cases of the main theorem. (We cannot build a 4−GDD of type 4^u from a 4−GDD of type 2^u since a TD(4, 2) does not exists, so we need to give a different proof for it.) □

Lemma 3.1.4 (Lemma 5.11, [1])**.** *If $v \equiv 1$ or 4 (mod 12), then a BIBD(4, v) exits.*

A BIBD(4, v) is equivalent to a 4−GDD of type 1^u. We complete the first case.

Lemma 3.1.5 (Lemma 3.24, [2])**.** *For $n > 0$ there exists a TD(5, 4n).*

Lemma 3.1.6. *Let $h \leqslant t$, if there exists a TD(5, t) and a 4−GDD of type m^u on v points where $v \in \{3h + m, 3t + m\}$, then there exists a 4−GDD of type m^w on $3(4t + h) + m$ points.*

Proof: By deleting $t - h$ points from one group of a TD(5,t) (which exists by Lemma 3.1.5), we obtain a {4, 5}−GDD of type t^4h^1 and by [7] 4−GDDs of type 3^4 and 3^5 exist. In the {4, 5}−GDD to the points in each block, we apply the weight 3 and put 4−GDDs of type 3^4 or 3^5 on the blocks of the {4, 5}−GDD, so we obtain a 4−GDD of type $(3t)^4(3h)^1$. Then we add m further points and apply Theorem 1.0.5 using the given 4−GDDs on $3h + m$ and $3t + m$ points to get the desired 4-GDD of type m^w on $3(4t + h) + m$ points. □

Lemma 3.1.7. *Let $v = mu$. If there exists a TD(4, v) and a 4−GDD of type m^u, then there exists a 4−GDD of type m^{4u}.*

Proof: A TD(4, v) is equivalent to a 4−GDD of type v^4. Using 4−GDDs of type m^u as fillers if we apply Theorem 1.0.4, then we will get the desired 4−GDD of type m^{4u}. □

Lemma 3.1.8 (Lemma 6.10, [1])**.** *If $q \equiv 1$ (mod 6) is a prime power, then there exists a 4−GDD of type 2^q.*

Lemma 3.1.9. *If $u \equiv 1$ (mod 3) and $u \neq 4$ then there exists a 4−GDD of type 2^u.*

Proof: Let $2u = 6s + 2$, $s \neq 1$. For $s = 0$ the result is trivial. Considering Lemma 3.1.6 with $t \equiv 0$ $(mod\ 4)$, $h \equiv 0$ $(mod\ 2)$, $m = 2$ and applying Lemma 3.1.5 it suffices to prove our result for $s \in S = \{2, 3, ..., 15, 17, 21, 22, 23, 25, 31, 33, 41\}$. By Lemma 3.1.7 and Theorem 3.1.2, it is sufficient to prove the result for $s \not\equiv 1$ $(mod\ 4)$ and for $s = 5$ and Lemma 3.1.8 proves for $s \in S$ all the cases $s \equiv 0$ $(mod\ 2)$. Consequently it remains to prove the result for $s \in \{3, 5, 7, 11, 15, 23, 31\}$ which is done by Lemma 6.11 in [2]. □

Lemma 3.1.10. *If* $u \equiv 0$ *or* 1 $(mod\ 4)$, *then there exists a* $4-$*GDD of type* 3^u.

Proof: Follows from Lemmas 3.1.4 and 1.0.6. □

Since we do not have a TD(2, 4) we could not build $4-$GDDs of type 4^u from 4-GDDs of type 2^u. Therefore, we need the following construction:

Lemma 3.1.11. *If* $u \equiv 1$ $(mod\ 3)$, *then there exists a* $4-$*GDD of type* 4^u.

Proof: There exists a *resolvable*$-$ BIBD$(4, 4u)$(see [3]). Consider one of the parallel classes in this design as groups of a GDD. □

Lemma 3.1.12 (Lemma 5.18, [1])**.** *For every integer* $v \geq 5$, *there exists a BIBD*(K, v), *where* $K = \{5, 6, ..., 20, 22, 23, 24, 27, 28, 29, 32, 33, 34, 39\}$.

Lemma 3.1.13 (Lemma 6.14, [2])**.** *If* q *is a power of an odd prime and* $q \neq 3$, *then there exists a* $4-$*GDD of type* 6^q.

Lemma 3.1.14. *If* $u > 4$, *then there exists a* $4-$*GDD of type* 6^u.

Proof: By Lemmas 2.1.6 and 3.1.12 it suffices to prove our result for $u \in \{5, 6, ..., 20, 22, 23, 24, 27, 28, 29, 32, 33, 34, 39\}$. If we use Corollary 1.0.3 and Lemma 3.1.9, and fill with $4-$GDDs of type 3^4, 3^5, then we take care of the cases $u \equiv 1$ $(mod\ 3)$, and Lemma 3.1.13 takes care of the cases when u is a prime power of an odd prime. Further by Lemma 3.1.7 and Theorem 3.1.2 we do not have to prove the result for $u \equiv 0$ $(mod\ 4)$, $u \geq 20$. It leaves us with the cases $u \in \{6, 8, 12, 14, 15, 18, 33, 39\}$, the proof of which is given by Lemma 6.15 in [2]. □

Now we are ready to give the proof of 3.1.1:

Proof of Theorem 3.1.1:

By Lemmas 3.1.3, 3.1.4, 3.1.9, 3.1.10, 3.1.11 and 3.1.14, we cover the spectrum for 4−GDDs of type g^u. □

3.2 4-GDDs of type $g^u m^1$

Theorem 3.2.1. *The necessary conditions for the existence of* 4*-GDDs of type* $g^u m^1$ *are:*

(***i***) $g, m > 0$,

(***ii***) $u \geq 4$,

(***iii***) $m \leq g(u-1)/2$,

(***iv***) $gu \equiv 0 \ (mod\ 3)$,

(***v***) $g(u-1) + m \equiv 0 \ (mod\ 3)$,

(***vi***) $\binom{gu+m}{2} - u\binom{g}{2} - \binom{m}{2} \equiv 0 \ (mod\ 6)$.

In [7], Kreher and Stinson have determined almost all group types for 4−GDDs on at most 30 points, leaving possible exceptions $3^5 6^2$, $2^3 5^4$, and $2^2 5^5$. In addition, following types have been constructed.

Lemma 3.2.2. *There exist* 4−*GDDs of types* $6^5 3^1$, $6^5 9^1$, $6^6 3^1$, $6^6 9^1$, $6^6 12^1$, $6^7 3^1$, $6^7 9^1$, $9^4 6^1$, $9^5 6^1$, $3^6 9^2$, $3^7 9^2$, $3^8 6^2$, $3^8 9^4$, $3^{15} 9^2$, $6^4 12^2$, $3^8 6^1 12^1$, $3^9 6^1 12^1$, $3^{11} 6^1 15^1$, $6^4 9^1 12^1$, *and* $6^5 12^1 15^1$.

Theorem 3.2.3. *There exists* 4−*GDDs of types* $1^u m^1$ *or* $3^u m^1$ *if and only if necessary conditions are satisfied.*

Proof: If we delete a point of the hole of $(v, w; 4)$-IPBD, we obtain a 4−GDD of type $3^{(v-w)/3}(w-1)^1$. If we just take the hole as a group, then we produce a 4−GDD of type $1^{v-w} w^1$.

Corollary 3.2.4. *There exists a* 4−*GDD of type* $g^u (g(u-1)/2)^1$ *if and only if necessary conditions are satisfied, with exceptions of types* 2^4, 6^4 *and* $2^6 5^1$, *for which* 4−*GDDs do not exist.*

Table 3.2: Necessary and sufficient conditions for the existence of 4−GDDs of type $g^u m^1$

Necessary Existence Criteria for a 4−GDD of Type $g^u m^1$			
g	**u**	**m**	by Theorems
0 (*mod* 6)	no cond.	0 (*mod* 3)	3.2.29 and 3.2.30 (partially completed)
1 (*mod* 6)	0 (*mod* 12)	1 (*mod* 3)	3.2.30 (partially completed)
	3 (*mod* 12)	1 (*mod* 6)	not completed
	9 (*mod* 12)	4 (*mod* 6)	not completed
2 (*mod* 6)	0 (*mod* 3)	2 (*mod* 3)	3.2.29 and 3.2.30 (partially completed)
3 (*mod* 6)	0 (*mod* 4)	0 (*mod* 3)	3.2.30 (partially completed)
	1 (*mod* 4)	0 (*mod* 6)	not completed
	3 (*mod* 4)	3 (*mod* 6)	not completed
4 (*mod* 6)	0 (*mod* 3)	1 (*mod* 3)	3.2.29 and 3.2.30 (partially completed)
5 (*mod* 6)	0 (*mod* 12)	2 (*mod* 3)	3.2.30 (partially completed)
	3 (*mod* 12)	5 (*mod* 6)	not completed
	9 (*mod* 12)	2 (*mod* 6)	not completed

Proof: There exists a resolvable 3−GDD of type g^u if and only if the necessary conditions $g(u-1)$ is even, $gu \equiv 0$ (*mod* 3), and $u \geqslant 3$ are satisfied, with the three exceptions (g,u)=(2,3), (6,3), and (2,6), see [7]. To obtain the required 4−GDD, we add a new point to each parallel class of triples in the resolvable 3−GDD.

Theorem 3.2.5 (Theorem 1.5, [23])**.** *The necessary and sufficient conditions for the existence of* 4*-GDDs of type* g^4m^1 *are:*

(***i***) *Let* $g \equiv 0$ (*mod* 6) *and* $m > 0$. *There exists a* 4−*GDD of type* g^4m^1 *if and only if* $m \equiv 0$ (*mod* 3) *and* $0 < m \leqslant 3g/2$, *except possibly when* $(g,m) = (18,6)$.

(***ii***) *Let* $g \equiv 3$ (*mod* 6) *and* $m > 0$. *There exists a* 4−*GDD of type* g^4m^1 *if and only if* $m \equiv 0$ (*mod* 3) *and* $0 < m \leqslant (3g-3)/2$, *except possibly when* $(g,m) = (9,3)$.

Remark:

4−GDDs of types $9^4 3^1$ and $18^4 6^1$ have also been constructed later, see Remark 1.12 in [24].

3.2.1 4−GDDs of type $g^u m^1$ with m as large or as small as possible

We will need double group divisible designs (DGDDs) for new constructions. From definition, a modified group divisible design (K−MGDD) of type g^u is a K−DGDD of type $(g, 1^g)^u$. A k−DGDD of type $(g, h^v)^k$ is an incomplete transversal design ITD$(k, g; h^v)$ and is equivalent to a set of $k-2$ holey MOLS of type h^v. First, we will make use of the following existence results of Theorem 1.7 in [24].

Theorem 3.2.6 (Theorem 1.7, [24])**.** (***i***) *An ITD$(4, g; h^v)$ exists if and only if $h \geqslant 1$ and $v \geqslant 4$, except when $(h, v) = (1, 6)$.*
(***ii***) *A 4−MGDD of type g^u exists if and only if $(g-1)(u-1) \equiv 0 \pmod 3$ and $g, u \geqslant 4$, except for $(g, u) = (6, 4)$ and except possibly for $(g, u) \in \{(6, 16), (6, 22), (10, 15), (10, 18)\}$.*

We will make use of the following construction in [24] for DGDDs.

Construction 3.2.7. *Suppose that there is a K−GDD of type $g_1^{u_1}, g_2^{u_2}, ..., g_s^{u_s}$ and that for each $k \in K$ there exits a 4−DGDD of type $(hv, h^v)^k$. Then there exists a 4−DGDD of type $(hvg_1, (hg_1)^v)^{u_1}, (hvg_2, (hg_2)^v)^{u_2}, ..., (hvg_s, (hg_s)^v)^{u_s}$.*

Proof: If we apply weight hv to each point in the K−GDD, replace each block of size k by a 4−DGDD of type $(hv, h^v)^k$, then we get the required 4−DGDD. □

By Theorem 3.2.6 there exist 4−DGDDs of type $(g, 1^v)^u$ and $(g, h^v)^4$. Together with the above construction we get the following:

Corollary 3.2.8. *Suppose that there is a 4−GDD of type $g_1^{u_1}, g_2^{u_2}, ..., g_s^{u_s}$ and that $h \geqslant 1$, $v \geqslant 4$ and $(h, v) \neq (1, 6)$. Then there exists a 4−DGDD of type*
$(hvg_1, (hg_1)^v)^{u_1}, (hvg_2, (hg_2)^v)^{u_2}, ..., (hvg_s, (hg_s)^v)^{u_s}$.

One way of constructing our 4−GDDs will be to “fill the groups” in DGDDs. Therefore we will get a new way of constructing GDDs. First we get DGDDs from

GDDs, then from DGDDs we will get GDDs. For the second part, we will use the Construction 1.10 in [24].

Construction 3.2.9. *Suppose that there is a* $4-$*DGDD of type* $(g_1, h_1^v)^{u_1}, (g_2, h_2^v)^{u_2}, ...,$ $(g_s, h_s^v)^{u_s}$ *and for each* $i = 1, 2, ..., s$ *there is a* $4-$*GDD of type* $h_i^v a^1$ *where* a *is a fixed nonnegative integer. Then there is a* $4-$*GDD of type* $h^v a^1$ *where* $h = \sum_{i=1}^{s} u_i h_i$.

Proof: We adjoin a ideal points to the DGDD, and for each $i = 1, 2, ..., s$, then we construct on each group of size g_i together with a ideal points a $4-$GDD of type $h_i^v a^1$, aligning the groups on the hole of the DGDD and a ideal points. A $4-$GDD of type $h^v a^1$ results. Note that the groups in this GDD correspond to the holes of the original DGDD, together with the a ideal points. □

Theorem 3.2.10. *For each* $g \equiv 3 \pmod 6$ *and* $u \equiv 0 \pmod 4$ *there exists a* $4-$*GDD of type* $g^u((g(u-1)-3)/2)^1$.

Proof: From Theorem 3.2.3 and Theorem 3.2.5, we may assume that $g \geqslant 9$ and $u \geqslant 8$. We will first suppose that $u \geqslant 12$. Let $u = 4t$ and $t \geqslant 3$. From Theorem 3.2.4, there is a $4-$GDD of type $(4g)^t(2g(t-1))^1$. We adjoin $(3g-3)/2$ ideal points and fill in $4-$GDDs of type $g^4((3g-3)/2)^1$ to yield a $4-$GDD of type $g^{4t}(2g(t-1)+(3g-3)/2)^1 \equiv g^u((g(u-1)-3)/2)^1$. We have completed the case $u \geqslant 12$. There remains the case $u = 8$. For this case see Theorem 1.13 in [24]. □

Theorem 3.2.11. *For each* $g \equiv 1$ *or* $5 \pmod 6$ *and each* $u \equiv 0 \pmod{12}$ *there exists a* $4-$*GDD of type* $g^u((g(u-1)-3)/2)^1$.

Proof: Let $u = 12t$. By Theorem 3.2.10 there is a $4-$GDD of type $(3g)^{4t}((3g(4t-1)-3)/2)^1$. We adjoin g ideal points and fill in $4-$GDDs of type g^4 to obtain a $4-$GDD of type $g^{12t}((3g(4t-1)-3)/2+g)^1 \equiv g^u((g(u-1)-3)/2)^1$. □

Now we have completed all the maximum cases. As the next case we will do the small cases of m. In this part we work toward analogues of Corollary 3.2.4 and Theorems 3.2.10, 3.2.11 where m is as small as possible. The theoretical minimums

Table 3.3: The maximum and minimum values of m in a 4−GDD of type $g^u m^1$

Necessary existence criteria for a 4−GDD of type $g^u m^1$				
g	u	m	m_{min}	m_{max}
a. (g, u) satisfy hypothesis of Corollary 3.2.4				
$\equiv 0\ (mod\ 6)$	no condition	$\equiv 0\ (mod\ 3)$	0	$g(u-1)/2$
$\equiv 3\ (mod\ 6)$	$\equiv 1\ (mod\ 4)$	$\equiv 0\ (mod\ 6)$	0	$g(u-1)/2$
	$\equiv 3\ (mod\ 4)$	$\equiv 3\ (mod\ 6)$	3	$g(u-1)/2$
$\equiv 2\ (mod\ 6)$	$\equiv 0\ (mod\ 3)$	$\equiv 2\ (mod\ 3)$	2	$g(u-1)/2$
$\equiv 4\ (mod\ 6)$	$\equiv 0\ (mod\ 3)$	$\equiv 1\ (mod\ 3)$	1	$g(u-1)/2$
$\equiv 1\ (mod\ 6)$	$\equiv 3\ (mod\ 12)$	$\equiv 1\ (mod\ 6)$	1	$g(u-1)/2$
	$\equiv 9\ (mod\ 12)$	$\equiv 4\ (mod\ 6)$	4	$g(u-1)/2$
$\equiv 5\ (mod\ 6)$	$\equiv 3\ (mod\ 12)$	$\equiv 5\ (mod\ 6)$	5	$g(u-1)/2$
	$\equiv 9\ (mod\ 12)$	$\equiv 2\ (mod\ 6)$	2	$g(u-1)/2$
b. (g, u) do not satisfy hypothesis of Corollary 3.2.4				
$\equiv 3\ (mod\ 6)$	$\equiv 0\ (mod\ 4)$	$\equiv 0\ (mod\ 3)$	0	$(g(u-1)-3)/2$
$\equiv 1\ (mod\ 6)$	$\equiv 0\ (mod\ 12)$	$\equiv 1\ (mod\ 3)$	1	$(g(u-1)-3)/2$
$\equiv 5\ (mod\ 6)$	$\equiv 0\ (mod\ 12)$	$\equiv 2\ (mod\ 3)$	2	$(g(u-1)-3)/2$

are given in Table 3.3. Now when $g \equiv 0$ (*mod* 6), or $g \equiv 3$ (*mod* 6) and $u \equiv 0$ or 1 (*mod* 4), Table 3.3 gives $m_{min} = 0$; that is, these GDDs are uniform and so their existence is determined in the previous section.

Theorem 3.2.12. *Let $g \equiv 3$ (mod 6), $g \geqslant 9$ and $u \equiv 3$ (mod 4), $u \geqslant 7$. Then there exists a 4−GDD of type g^u3^1.*

Proof: Suppose first that $g \geqslant 15$. Write $g = 6k + 3$, $k \geqslant 2$. If k is even, then apply Corollary 3.2.8 to a 4−GDD of type 3^{2k+1}, taking $h = 1$ and $v = u$, to obtain a 4−DGDD of type $(3u, 3^u)^{2k+1}$. Now, apply Construction 3.2.9 to this DGDD, taking $a = 3$ to yield a 4−GDD of type $(6k + 3)^u3^1 \equiv g^u3^1$. If k is odd, then apply Construction 3.2.7 to a $\{4, 7\}$−GDD of type 3^{2k+1} (from Theorem 3.2.3 a 4−GDD of type $1^{6k-3}7^1$ exists; delete a point not in the group of size 7), by taking $h = 1$ and $v = u$. We require a 4−DGDD of type $(u, 1^u)^4$ and a 4−DGDD of type $(u, 1^u)^7$ (both exist by Theorem 3.2.6). The result is a 4−DGDD of type $(3u, 3^u)^{2k+1}$. Now proceed as in the case k is even. There remains the case $g = 9$. For this case refer to Theorem 2.2 in [24]. □

Theorem 3.2.13. *Let $g \equiv 2$ (mod 6) and $t \geqslant 2$. Then there exists a 4−GDD of type $g^{3t}2^1$.*

Proof: The case $g = 2$ is settled in the previous section and for $g = 8$, we apply Construction 3.2.9 to an ITD$(4, 6t, 2^{3t})$ ($\equiv$ 4−DGDD of type $(6, 2^{3t})^4$) which exists by Theorem 3.2.6, by taking $a = 2$. We may henceforth assume that $g \geqslant 14$.

We write $g = 6k + 2$, $k \geqslant 2$. If $t \geqslant 3$, then we apply Corollary 3.2.8 to a 4−GDD of type 2^{3k+1}, by taking $h = 1$ and $v = 3t$, to yield a 4−DGDD of type $(6t, 2^{3t})^{3k+1}$. Now apply Construction 3.2.9 to this DGDD, by taking $a = 2$, to obtain a 4−GDD of type $(6k + 2)^{3t}2^1 \equiv g^{3t}2^1$. There remains $t = 2$ to deal with. For this case, we refer to Theorem 2.3 in [24]. □

We now consider 4−GDDs of type $g^{3t}1^1$ where $g \equiv 4$ (*mod* 6).

Theorem 3.2.14 (Theorem 1.15, [24])**.** (***i***) *There exists a 4−GDD of type 2^u5^1 for every $u \equiv 0$ (mod 3) with $u \geqslant 9$.*

(***ii***) *There exists a resolvable 4−GDD of type 3^u if and only if $u \equiv 0$ (mod 4) and $u \geqslant 8$, except possibly when $u \in \{88, 124\}$.*

Lemma 3.2.15. *There exits a* $4-$*GDD of type* $4^{3t}1^1$ *for all* $t \geqslant 2$.

Proof: For $t \neq 22$ or 31, there is a resolvable $4-$GDD of type 3^{4t} (see Theorem 3.2.14); we adjoin one point to complete the groups, and take this point together one of the parallel classes of quadruples in the resolvable GDD as the the groups of the resulting GDD. For $t = 22$, we take a $4-$GDD of type $15^4 6^1$ (Theorem 3.2.5); we apply weight 4 to each point, adjoin one ideal point and fill in $4-$GDDs of types $4^1 51^1$ and $4^6 1^1$. Similarly for $t = 31$, we take a $4-$GDD of type $21^4 9^1$ (Theorem 3.2.5); apply weight 4 to each point, adjoin one ideal point, and fill in $4-$GDDs of types $4^2 11^1$ and $4^9 1^1$. □

Theorem 3.2.16. *Let* $g \equiv 4$ $(mod\ 12)$ *and* $t \geqslant 2$. *Then there exists a* $4-$*GDD of type* $g^{3t}1^1$.

Proof: For $g = 4$ see Lemma 3.2.15. For $g = 16$, we apply Construction 3.2.9 to an ITD$(4, 12t; 4^{3t})$ ($\equiv$ $4-$DGDD of type $(12t, 4^{3t})^4$), which exists by Theorem 3.2.6, by taking $a = 1$ and filling in $4-$GDDs of type $4^{3t}1^1$, we get a $4-$GDD of type $16^{3t}1^1$. We may henceforth assume that $g \geqslant 28$.

We write $g = 12k + 4$, $k \geqslant 2$. We apply Corollary 3.2.8 to a $4-$GDD of type 2^{3k+1}, by taking $h = 2$ and $v = 3t$, we get a $4-$DGDD of type $(12t, 4^{3t})^{3k+1}$. Now we apply Construction 3.2.9 to this DGDD, by taking $a = 1$ and fill in $4-$GDDs of type $4^{3t}1^1$, we obtain a $4-$GDD of type $(12k + 4)^{3t}1^1 \equiv g^{3t}1^1$, as desired. □

Before proceeding to the case $g \equiv 10$ $(mod\ 12)$, we will require the following preliminary result.

Lemma 3.2.17. *Let* $u \geqslant 4$, $u \notin \{10, 12, 12, 13, 14, 15, 17, 18, 19, 23\}$. *Then there exists a* $4-$*GDD of type* $6^u 9^1$.

Proof: For $u = 4$, we refer to Corollary 3.2.4; for $u = 5, 6$ or 7, see Lemma 3.2.2, and for u = 8,9 or 11, see Appendix in [24]. For $u = 16$, we adjoin nine ideal points to a $4-$GDD of type 24^4 and fill in $4-$GDDs of type $6^4 9^1$ to get the desired GDD. For $u = 20$, 21 or 22, we take a $4-$GDD of type $24^4 24^1$, $24^4 30^1$ or $24^4 36^1$ (from Theorem 3.2.5); then we adjoin nine ideal points and fill in $4-$GDDs of types $6^4 9^1$, $6^5 9^1$, $6^6 9^1$

or $6^7 9^1$. Finally, for $u = 4k$, $u = 4k + 1$, $u = 4k + 2$, $u = 4k + 3$ where $k \geqslant 7$, we take 4−GDD of type $(6k-4)^4(6(u-4k+4)+9)^1$ (from Theorem 3.2.5) and fill in the groups with 4−GDDs of types 6^{k-1} and $6^{u-4k+4}9^1$. □

Theorem 3.2.18 (Lemma 2.7, [24]). (***i***) *There exists a* 4−*GDD of type* $10^{3t}1^1$ *and for all* $t \geqslant 2$, *except possibly for* $t = 7$.
(***ii***) *There exists a* 4−*GDD of type* $22^{3t}1^1$ *for all* $t \geqslant 2$, *except possibly for* $t = 7$.

Theorem 3.2.19. *Let* $g \equiv 10$ (*mod* 12) *and* $t \geqslant 2$, $t \neq 7$. *Then there exists a* 4−*GDD of type* $g^{3t}1^1$.

Proof: For $g = 10$ or 22, see Lemma 3.2.18. For $g = 34$ and $t \geqslant 3$, take a 4−GDD of type $4^6 10^1$ (Corollary 3.2.4) and apply Corollary 3.2.8 with $h = 1$ and $v = 3t$ to obtain a 4−DGDD of type $(12, 4^{3t})^6\ (30t, 10^{3t})^1$. Now we apply Construction 3.2.9 to this DGDD, by taking $a = 1$ and filling in with 4−GDDs of type $4^{3t}1^1$ (Lemma 3.2.15) and $10^{3t}1^1$, we obtain a 4−GDD of type $34^{3t}1^1$. For a 4−GDD of type $34^6 1^1$, we refer to Theorem 2.8 of [24].

Now let $g \geqslant 46$, that is $g = 12k + 10$, $k \geqslant 3$. We take a 4−GDD of type $2^{3k}5^1$ (see Theorem 3.2.14) and apply Corollary 3.2.8 with $h = 2$ and $v = 3t$ to obtain a 4−DGDD of type $(12t, 4^{3t})^{3k}(30t, 10^{3t})^1$. We apply Construction 3.2.9 to this DGDD, by taking $a = 1$ and filling in 4−GDDs of type $4^{3t}1^1$ and $10^{3t}1^1$, as above, we yield a 4−GDD of type $(12k+10)^{3t}1^1 \equiv g^{3t}1^1$, as desired. □

By combining Theorems 3.2.16 and 3.2.19 we get the following result.

Theorem 3.2.20. *Let* $g \equiv 4$ (*mod* 6) *and* $t \geqslant 2$. *Then there exists a* 4−*GDD of type* $g^{3t}1^1$ *except possibly when* $g \equiv 10$ (*mod* 12) *and* $t = 7$.

Remark: By considering the proof of Theorem 3.2.19, we see that the possible exceptions in Theorem 3.2.20 can be eliminated by constructing 4−GDDs of types $10^{21}1^1$ and $22^{21}1^1$. Now by Theorem 3.2.6, there exist 4−DGDDs of types $(21, 1^{21})^{10}$ and $(21, 1^{21})^{22}$. By adjoining four ideal points and filling in (25, 4)−BIBDs (in which the ideal points form a block in one BIBD and a hole in each remaining BIBD) we obtain 4−GDDs of types $10^{21}4^1$ and $22^{21}4^1$. Since there is a 4−GDD of type $4^{21}4^1$ ($\equiv$ 4−GDD of type 4^{22}), we obtain (via the proof of the Theorem 3.2.19) a 4−GDD of type $g^{21}4^1$ for every $g \equiv 10 \ (mod\ 12)$.

Finally, we consider $g \equiv 1$ or $5 \ (mod\ 6)$.

Theorem 3.2.21. *Let $g \equiv 1 \ (mod\ 6)$ and $g \geqslant 7$.*
(***i***) *For every $u \equiv 0, 3 \ (mod\ 12)$ with $u \geqslant 12$, there exists a 4−GDD of type $g^u 1^1$.*
(***ii***) *For every $u \equiv 9 \ (mod\ 12)$ with $u \geqslant 9$, there exists a 4−GDD of type $g^u 4^1$.*

Proof: From Theorem 3.2.6, there exist 4−DGDDs of types $(u, 1^u)^g$ for all indicated parameters g, u. For $u \equiv 0$ or $3 \ (mod\ 12)$, we adjoin one ideal point and fill in $(u+1, 4)$−BIBDs, we obtain a 4−GDD of type $g^u 1^1$; for $u \equiv 9 \ (mod\ 12)$ we adjoin four ideal points and fill in $(u+4, 4)$−BIBDs ($\equiv$ 4−GDDs of type $1^u 4^1$) to obtain a 4−GDD of type $g^u 4^1$, as desired. □

Remark: The case $g = 1$ and $u \equiv 0, 3$, or $9 \ (mod\ 12)$, $u \geqslant 3$, is covered by Theorem 3.2.3.

Lemma 3.2.22. *Let $g \equiv 5 \ (mod\ 6)$ and $u \equiv 0 \ (mod\ 12)$. Then there exists a 4−GDD of type $g^u 2^1$, except possibly when $(g, u) = (11, 12)$ or $(17, 12)$.*

Proof: If $u \geqslant 24$, then we write $u = 12t$, $t \geqslant 2$. From Theorem 3.2.13, there is a 4−GDD of type $(4g)^{3t}2^1$. We construct a 4−GDD of type g^4 on each group of size $4g$ to obtain a 4−GDD of type $g^{12t}2^1 \equiv g^u 2^1$.

There remains the case $u = 12$ to deal with. For construction of a 4−GDD of type $5^{12}2^1$ refer to Lemma 2.12 in [24]. Now let $g \geqslant 23$ and write $g = 6k + 5$, $k \geqslant 3$. We apply Corollary 3.2.8 to a 4−GDD of type $2^{3k}5^1$ (which exists by Theorem 3.2.14), by taking $h = 1$ and $v = 12$, we obtain a 4−DGDD of type $(24, 2^{12})^{3k}(60, 5^{12})^1$. Now we apply Construction 3.2.9 with $a = 2$, by filling in 4−GDDs of types 2^{13} and $5^{12}2^1$, we obtain a 4−GDD of type $(6k+5)^u 5^1 \equiv g^{12}2^1$, as desired. □

Lemma 3.2.23. *Let $g \equiv 5 \ (mod\ 6)$, $g \neq 11$ or 17, and $u \equiv 3 \ (mod\ 12)$, $u \geqslant 15$. Then there exists a 4−GDD of type $g^u 5^1$.*

Proof: For $g = 5$, the GDD becomes a uniform 4−GDD so the result follows from previous section. For $g \geqslant 23$ we write $g = 6k+5$, $k \geqslant 3$, and proceed as with the case $u = 12$ in Lemma 3.2.22, replacing "$v = 12$" by "$v = u$" and setting $a = 5$, noting that there exist 4−GDDs of types $2^u 5^1$ (see Theorem 3.2.14 (i)) and $5^u 5^1$. The result is a 4−GDD of type $(6k+5)^u 5^1 \equiv g^u 5^1$. □

Before proceeding to the case $u \equiv 9$ (*mod* 12), we will require the following result.

Lemma 3.2.24 (Lemma 2.14, [24]). *There exists a* 4−*GDD of type* $5^u 2^1$ *for every* $u \equiv 9$ (*mod* 12), *except possibly for* $u = 33$.

Lemma 3.2.25. *Let* $g \equiv 5$ (*mod* 6), $g \neq 11$ *or* 17, *and* $u \equiv 9$ (*mod* 12). *Then there exists a* 4−*GDD of type* $g^u 2^1$, *except possibly for* $u = 33$.

Proof: For $g = 5$, the result follows from Lemma 3.2.24. For $g \geqslant 23$, we write $g = 6k + 5$, $k \geqslant 3$, and proceed exactly as the case $u = 12$ in Lemma 3.2.22. We replace "$v = 12$" by "$v = u$" and set $a = 2$. We fill in groups with 4−GDDs of type 2^{u+1} and $5^u 2^1$ (Lemma 3.2.24). Thus, we obtain a 4−GDD of type $(6k+5)^u 2^1 \equiv g^u 2^1$, as desired. □

Remark: The possible exception $u = 33$ in Lemma 3.2.25 can be eliminated by constructing a 4−GDD of type $5^{33} 2^1$. For this construction see Remark 2.16 in [24].

If we combine Lemmas 3.2.22, 3.2.23, and 3.2.25, then we get the following result for $g \equiv 5$ (*mod* 6).

Theorem 3.2.26. *Let* $g \equiv 5$ (*mod* 6), $g \geqslant 5$.
(***i***) *For every* $u \equiv 0$ *or* 9 (*mod* 12), *there exists a* 4−*GDD of type* $g^u 2^1$ *except possibly when* $g = 11$ *or* 17 *and* $u \equiv 9$ (*mod* 12), *or when* $(g, u) \in \{(11, 12), (17, 12)\}$, *or when* $u = 33$.
(***ii***) *For every* $u \equiv 3$ (*mod* 12), $u \geqslant 15$, *there exists a* 4−*GDD of type* $g^u 5^1$ *except possibly when* $g = 11$ *or* 17.

The following theorem also completes some holes.

Theorem 3.2.27 (Theorem 2.17, [33])**.** (***i***) *Let* $g \equiv 4$ (*mod* 6) *and* $u \equiv 0$ (*mod* 3)*,* $u \geqslant 6$*. Then there exist* 4−*GDDs of type* $g^u 1^1$*.*
(***ii***) *Let* $g \equiv 5$ (*mod* 6) *and* $u \equiv 0$ (*mod* 3)*,* $u \geqslant 9$ *and* $u \not\equiv 6$ *mod* (*mod* 12)*. If* $u \equiv 0$ (*mod* 12) *then there exists* 4−*GDDs of type* $g^u 2^1$*, except possibly for type* $g^{12} 2^1$ *when* $g \in \{11, 17\}$*; if* $u \equiv 3$ (*mod* 12) *then there exists* 4−*GDDs of type* $g^u 5^1$*, except possibly for type* $11^{27} 5^1$*; if* $u \equiv 9$ (*mod* 12) *then there exists* 4−*GDDs of type* $g^u 2^1$*, except possibly for type* $11^{21} 2^1$*.*

The main result of this section is the combination of 3.2.4, 3.2.10, 3.2.11, 3.2.12, 3.2.13, 3.2.20, 3.2.21, 3.2.26, the remarks following to Theorem 3.2.20 and Lemma 3.2.25, and Theorem 3.2.27. Therefore, we get the following theorem:

Theorem 3.2.28. (***i***) *Let* $g \equiv 0$ (*mod* 6) *and* $u \geqslant 4$*. Then there exist* 4−*GDDs of types* g^u *and* $g^u (g(u-1)/2)^1$*, except that there is no* 4−*GDD of type* 6^4*. There is a* 4−*GDD of type* $6^4 3^1$*.*
(***ii***) *Let* $g \equiv 3$ (*mod* 6) *and* $u \geqslant 4$*,* $u \not\equiv 2$ (*mod* 4)*. If* $u \equiv 0$ (*mod* 4) *then there exist* 4−*GDDs of types* g^u *and* $g^u ((g(u-1)-3)/2)^1$*; if* $u \equiv 1$ (*mod* 4) *then there exist* 4−*GDDs of types* g^u *and* $g^u (g(u-1)/2)^1$*; if* $u \equiv 3$ (*mod* 4) *then there exist* 4−*GDDs of types* $g^u 3^1$ *and* $g^u (g(u-1)/2)^1$*.*
(***iii***) *Let* $g \equiv 1$ (*mod* 6) *and* $u \equiv 0$ (*mod* 3)*,* $u \geqslant 9$ *and* $u \not\equiv 6$ (*mod* 12) *. If* $u \equiv 0$ (*mod* 12) *then there exist* 4−*GDDs of types* $g^u 1^1$ *and* $g^u ((g(u-1)-3)/2)^1$*; if* $u \equiv 3$ (*mod* 12) *then there exist* 4−*GDDs of types* $g^u 1^1$ *and* $g^u (g(u-1)/2)^1$*; if* $u \equiv 9$ (*mod* 12) *then there exist* 4−*GDDs of types* $g^u 4^1$ *and* $g^u (g(u-1)/2)^1$*.*
(***iv***) *Let* $g \equiv 4$ (*mod* 6) *and* $u \equiv 0$ (*mod* 3)*,* $u \geqslant 6$*. Then there exist* 4−*GDDs of types* $g^u 1^1$ *and* $g^u (g(u-1)/2)^1$*.*
(***v***) *Let* $g \equiv 2$ (*mod* 6) *and* $u \equiv 0$ (*mod* 3)*,* $u \geqslant 6$*. Then there exist* 4−*GDDs of types* $g^u 2^1$ *and* $g^u (g(u-1)/2)^1$*, except that there is no* 4−*GDD of type* $2^6 5^1$*.*
(***vi***) *Let* $g \equiv 5$ (*mod* 6)*,* $u \equiv 0$ (*mod* 3)*,* $u \geqslant 9$ *and* $u \not\equiv 6$ (*mod* 12)*. If* $u \equiv 0$ (*mod* 12) *then there exist* 4−*GDDs of types* $g^u 2^1$ *and* $g^u ((g(u-1)-3)/2)^1$*, except possibly for type* $g^{12} 2^1$ *when* $g \in \{11, 17\}$*; if* $u \equiv 3$ (*mod* 12) *then there exist* 4−*GDDs of types* $g^u 5^1$ *and* $g^u (g(u-1)/2)^1$*, except possibly for type* $11^{27} 5^1$ *; if* $u \equiv 9$ (*mod* 12) *then there exist* 4−*GDDs of types* $g^u 2^1$ *and* $g^u (g(u-1)/2)^1$*, except possibly for type* $11^{21} 2^1$*.*

Therefore, we completed all the cases of 4−GDDs of type $g^u m^1$ where m is as large or as small as possible.

3.2.2 *Establishing a part of the spectrum of* 4−*GDDs of type* $g^u m^1$

In this subsection we will cover a part of spectrum of 4−GDDs. We will determine for each even g, all values of m, for which a 4−GDD of type $g^u m^1$ exists, for every $u \equiv 0 \ (mod\ 4)$. Similarly, we will determine for each odd $g \neq 11$ or 17, all values of m for which a 4−GDD of type $g^u m^1$ exists, for every $u \equiv 0 \ (mod\ 3)$. Finally, we will establish, up to a finite number of values of u, the spectrum for 4−GDD of type $g^u m^1$ where gu is even and $g \notin \{11, 17\}$ (for the conditions see Table 3.2). The main results will be the following two theorems in [25].

Theorem 3.2.29. (***i***) *Let* $g \equiv 0 \ (mod\ 6)$ *and* $u \equiv 0 \ (mod\ 4)$*, where* $u = 4$ *or* $u \geqslant 12$*. Then there exists a* 4−*GDD of type* $g^u m^1$ *for every* $m \equiv 0 \ (mod\ 3)$ *with* $0 < m \leqslant g(u-1)/2$*, except possibly when* $u = 12$ *and either* $g = 6$ *or* $0 < m < g$*.*
(***ii***) *Let* $g \equiv 2$ *or* $4 \ (mod\ 6)$ *and* $u \equiv 0 \ (mod\ 12)$*. Then there exists a* 4−*GDD of type* $g^u m^1$ *for every* $m \equiv g \ (mod\ 3)$ *with* $0 < m \leqslant g(u-1)/2$*, except possibly* $u = 12$ *and* $0 < m < g$*.*

Theorem 3.2.30. (***i***) *Let* $g \equiv 0 \ (mod\ 6)$ *and* $u \in \{n : n \geqslant 79\} \backslash \{93, 94, 95, 97, 98, 117, 118\}$*. Then there exists a* 4−*GDD of type* $g^u m^1$ *for every* $m \equiv 0 \ (mod\ 3)$ *with* $0 \leqslant m \leqslant g(u-1)/2$*.*
(***ii***) *Let* $g \equiv 3 \ (mod\ 6)$ *and* $u \equiv 0 \ (mod\ 4)$*,* $u \neq 8$*. Then there exists a* 4−*GDD of type* $g^u m^1$ *for every* $m \equiv 0 \ (mod\ 3)$ *with* $0 \leqslant m \leqslant (g(u-1)-3)/2$*, except possibly when* $u = 12$ *and* $0 < m < g$*.*
(***iii***) *Let* $g \equiv 1$ *or* $5 \ (mod\ 6)$*,* $g \notin \{11, 17\}$*,* $u \equiv 0 \ (mod\ 12)$*,* $u \neq 24$*. Then there exits a* 4−*GDD of type* $g^u m^1$ *for every* $m \equiv g \ (mod\ 3)$ *with* $0 \leqslant m \leqslant (g(u-1)-3)/2$*, except possibly when* $u \in \{12, 72, 120, 168\}$ *and* $0 < m < g$*.*
(***iv***) *Let* $g \equiv 2$ *or* $4 \ (mod\ 6)$ *and* $u \in \{n : n \equiv 0 \ (mod\ 3),\ n \geqslant 192\} \backslash \{231, 234, 237\}$*. Then there exits a* 4−*GDD of type* $g^u m^1$ *for every* $m \equiv g \ (mod\ 3)$ *with* $0 \leqslant m \leqslant g(u-1)/2$*, except possibly when* $g = 2$ *and* $u \in \{291, 294, 297, 303, 306, 309, 315, 318, 321, 327, 330, 333\}$*.*

To obtain these results we will use the 4−GDDs of type $g^u m^1$, where m is as small as or as large as possible, that we obtained in the previous section. Secondly, we will give another construction to obtain GDDs but this time we will use IGDDs.

The following construction is modified version of WFC to use on IGDDs.

Construction 3.2.31. *Let (X, Y, G, B) be a $K-$IGDD of type $(g_1, h_1)^{u_1}, (g_2, h_2)^{u_2}, ..., (g_s, h_s)^{u_s}$, and let $w : X \to \mathbb{Z}$ and $d : X \to \mathbb{Z}$ be functions with $w(x) \geqslant d(x) \geqslant 0$ for all $x \in X$. Suppose that for each block $b \in B$ there is a $4-$IGDD of type $\{(w(x), d(x)) : x \in b\}$ and there is a $4-$IGDD of type $\{(\sum_{x\in G_i\cap Y} w(x), \sum_{x\in G_i\cap Y} d(x)) : G_i \in G\}$. Then there is a $4-$IGDD of type $\{(\sum_{x\in G_i} w(x), \sum_{x\in G_i} d(x)) : G_i \in G\}$.*

Remark: By setting $Y = \emptyset$ and $d(x) = 0$ for all $x \in X$, we obtain the WFC for GDDs.

As a first step, we will establish, up to finite number of values of u, the spectrum for $4-$GDDs of type $g^u m^1$ where gu is even, $g \notin \{11, 17\}$. Now from the table of necessary conditions, we see that where g is odd, either $g \equiv 3$ $(mod\ 6)$ and $u \equiv 0$ $(mod\ 4)$, with $0 \leqslant m \leqslant (g(u-1)-3)/2$ and $m \equiv 0$ $(mod\ 3)$; or $g \equiv 1$ or 5 $(mod\ 6)$ and $u \equiv 0$ $(mod\ 12)$, with $0 \leqslant m \leqslant (g(u-1)-3)/2$ and $m \equiv g$ $(mod\ 3)$.

Now we will deal with Theorem 3.2.29. We will use Theorem 3.2.5 and 3.2.28 to establish one quarter of the spectrum for $4-$GDDs of type $g^u m^1$. We begin with $g \equiv 0$ $(mod\ 3)$.

Theorem 3.2.32. *Let $g \equiv 0$ $(mod\ 3)$ and $u \equiv 0$ $(mod\ 4)$, $u \geqslant 16$, where $u \not\equiv 8$ $(mod\ 16)$ when $g \equiv 3$ $(mod\ 6)$. Then there exists a $4-$GDD of type $g^u m^1$ for every $m \equiv 0$ $(mod\ 3)$ with $0 \leqslant m \leqslant g(u-1)/2$.*

Proof: For $g \equiv 0$ $(mod\ 6)$, we write $u = 4t$, $t \geqslant 4$, and write $m = x + y$ where $x = 0$ or $g(t-1)/2$ and $0 \leqslant y \leqslant 3gt/2$, $y \equiv 0$ $(mod\ 3)$. (When $g = 6$ and $t = 4$, we write $x = 3$ or 9.) Now a $4-$GDD of type 6^{16} exists by previous section; otherwise we take a $4-$GDD of type $(gt)^4 y^1$ (Theorem 3.2.5) and adjoin x ideal points, then finally fill in $4-$GDDs of type $g^t x^1$ (see Theorem 3.2.28 (i)). The result is a $4-$GDD of type $g^{4t}(x+y)^1 \equiv g^u m^1$. For $g \equiv 3$ $(mod\ 6)$, we proceed similarly (see Theorem 2.1 in [25]). □

Theorem 3.2.33. (*i*) *Let* $g \equiv 2$ (*mod* 6) *and* $u \equiv 0$ (*mod* 12), $u \geqslant 24$. *Then there exists a* 4−*GDD of type* $g^u m^1$ *for every* $m \equiv 2$ (*mod* 3) *with* $2 \leqslant m \leqslant g(u-1)/2$.
(*ii*) *Let* $g \equiv 4$ (*mod* 3) *and* $u \equiv 0$ (*mod* 12), $u \geqslant 24$. *Then there exists a* 4−*GDD of type* $g^u m^1$ *for every* $m \equiv 1$ (*mod* 3) *with* $1 \leqslant m \leqslant g(u-1)/2$.

Proof: (**i**) We write $u = 12t$, $t \geqslant 2$ and write $m = x + y$ where $x = 2$ or $g(3t-1)/2$ and $0 \leqslant y \leqslant 9gt/2$, $y \equiv 0$ (*mod* 3). (When $g = t = 2$, we take $x = 2$.) Now a 4−GDD of type $2^{24}23^1$ exits by Theorem 3.2.28 (v); otherwise we take a 4−GDD of type 4−GDD of type $(3gt)^4 y^1$ (Theorem 3.2.5) and adjoin x ideal points, then finally fill in 4−GDDs of type $g^{3t}x^1$ (see Theorem 3.2.28 (v)). The result is a 4−GDD of type $g^{12t}(x+y)^1 \equiv g^u m^1$.
(**ii**) This case is similar to the previous case, for the proof, we refer to Theorem 2.2 (ii) in [25]. □

Now we move on the case $g \equiv 1$ or 5 (*mod* 6).

Theorem 3.2.34. *Let* $g \equiv 1$ (*mod* 6) *and* $u \equiv 0$ (*mod* 12), $u \geqslant 36$, *where* $u \not\equiv 24$ (*mod* 48) *when* $g \equiv 3$ (*mod* 6). *Then there exists a* 4−*GDD of type* $g^u m^1$ *for every* $m \equiv 1$ (*mod* 3) *with* $1 \leqslant m \leqslant (g(u-1)-3)/2$.

Proof: We write $u = 12t$, $t \geqslant 3$ and $t \not\equiv 2$ (*mod* 4), and $m = x + y$ where
(i) $x = 1$ or $(g(3t-1)-3)/2$ and $0 \leqslant y \leqslant 9gt/2$, $y \equiv 0$ (*mod* 3), if $t \equiv 0$ (*mod* 4);
(ii) $x = 1$ or $g(3t-1)/2$ and $0 \leqslant y \leqslant (9gt-3)/2$, $y \equiv 0$ (*mod* 3), if $t \equiv 1$ (*mod* 4);
(iii) $x = 4$ or $g(3t-1)/2$ and $0 \leqslant y \leqslant (9gt-3)/2$, $y \equiv 0$ (*mod* 3), if $t \equiv 3$ (*mod* 4).
Now a 4−GDD of type $g^{12t}1^1$ exists by Theorem 3.2.28 (iii); otherwise we take a 4−GDD of type $(3gt)^4 y^1$ (Theorem 3.2.5) and adjoin x ideal points, then finally fill in with 4−GDDs of type $g^{3t}x^1$ (see Theorem 3.2.28 (iii)). The result is a 4−GDD of type $g^{12t}(x+y)^1 \equiv g^u m^1$, as we desire. □

Theorem 3.2.35. *Let* $g \equiv 5$ (*mod* 6), $g \neq 11$ *or* 17, *and* $u \equiv 0$ (*mod* 12), $u \geqslant 36$, *where* $u \not\equiv 24$ (*mod* 48). *Then there exists a* 4−*GDD of type* $g^u m^1$ *for every* $m \equiv 2$ (*mod* 3) *with* $2 \leqslant m \leqslant (g(u-1)-3)/2$.

Proof: We write $u = 12t$, $t \geqslant 3$ and $t \not\equiv 2$ $(mod\ 4)$, and $m = x + y$ where
(i) $x = 2$ or $(g(3t-1)-3)/2$ and $0 \leqslant y \leqslant 9gt/2$, $y \equiv 0$ $(mod\ 3)$, if $t \equiv 0$ $(mod\ 4)$;
(ii) $x = 5$ or $g(3t-1)/2$ and $0 \leqslant y \leqslant (9gt-3)/2$, $y \equiv 0$ $(mod\ 3)$, if $t \equiv 1$ $(mod\ 4)$;
(iii) $x = 4$ or $g(3t-1)/2$ and $0 \leqslant y \leqslant (9gt-3)/2$, $y \equiv 0$ $(mod\ 3)$, if $t \equiv 3$ $(mod\ 4)$.
Now a 4−GDD of type $g^{12t}2^1$ exists by Theorem 3.2.28 (vi). Otherwise, if $t \neq 11$ we take a 4−GDD of type $(3gt)^4y^1$ (Theorem 3.2.5) and adjoin x ideal points, then finally fill in with 4−GDDs of type $g^{3t}x^1$ (see Theorem 3.2.28 (vi)). The result is a 4−GDD of type $g^{12t}(x+y)^1 \equiv g^u m^1$, as we desire.

Now let $t = 11$. By Theorem 3.2.28 (vi), there is a 4−GDD of type $g^{33}8^1$. Thus, for $m \geqslant 8$ we write $m = x + y$, as above, except that $x = 8$ or $g(3t-1)/2$. For $m = 5$, we take a 4−GDD of type $(12g)^{11}$ (uniform 4−GDD, we did in previous section) and adjoin m ideal points, fill in 4−GDDs of type $g^{12}m^1$ (see Lemma 3.2.22, note that as there exists 4−GDDs of types 2^{12} and $5^{16} \equiv 5^{15}5^1$, the construction for $u = 12$ works for $a = 5$). □

Theorem 3.2.36. *Let $g > 0$, $g \neq 6$. Then there exists a 4−GDD of type $g^{12}m^1$ for every $g \leqslant m \leqslant 11g/2$ with $m \equiv g$ $(mod\ 3)$.*

Proof: We assume first that $g \neq 2$. We proceed as follows, we set $m = x + y$ where $x = g$ and $0 \leqslant 9g/2$, $y \equiv 0$ $(mod\ 3)$. We take, a 4−GDD of type $(3g)^4y^1$ (by Theorem 3.2.5), adjoin g ideal points and fill in 4−GDDS of type g^4 to obtain a 4−GDD of type $g^{12}m^1$, as we desire. For the case $g = 2$, refer to Theorem 2.5 in [25].□

If we combine Theorems 3.2.5, 3.2.32, 3.2.33 and 3.2.36, we obtain Theorem 3.2.29. Therefore, we evaluate our first result.

Next we will obtain Theorem 3.2.30. The following Theorem in [25] stronger than combinations of Theorems 3.2.32, 3.2.34 and 3.2.35.

Theorem 3.2.37 (Theorem 3.6, [25])**.** (***i***) *Let $g \equiv 3$ $(mod\ 6)$ and $u \equiv 0$ $(mod\ 4)$, $u \geqslant 16$. Then there exists a 4−GDD of type $g^u m^1$ for every $m \equiv 0$ $(mod\ 3)$ with*
$0 \leqslant m \leqslant (g(u-1)-3)/2$.

(***ii***) *Let* $g \equiv 1$ *or* $5 \pmod 6$, $g \neq 11$ *or* 17, *and* $u \equiv 0 \pmod{12}$, $u \geqslant 24$. *Then there exists a* $4-$*GDD of type* $g^u m^1$ *for every* $m \equiv g \pmod 3$ *with* $0 \leqslant m \leqslant (g(u-1)-3)/2$, *except possibly when* $u \in \{24, 72, 120, 168\}$ *and* $0 < m < g$ *or when* $u = 24$ *and either* $g \in \{35, 55, 77, 85, 95, 119, 175, 295, 335\}$ *or* $g \geqslant 11$ *and* $10g < m < (23g-21)/2$.

The following existence result is then an immediate consequence of Theorems 3.2.5, 3.2.36, and 3.2.37.

Theorem 3.2.38. (***i***) *Let* $g \equiv 3 \pmod 6$ *and* $u \equiv 0 \pmod 4$, $u \notin \{8\}$. *Then there exists a* $4-$*GDD of type* $g^u m^1$ *for every* $m \equiv 0 \pmod 3$ *with* $0 \leqslant m \leqslant (g(u-1)-3)/2$, *except possibly when* $u = 12$ *and* $0 < m < g$.
(***ii***) *Let* $g \equiv 1$ *or* $5 \pmod 6$, $g \notin \{11, 17\}$, *and* $u \equiv 0 \pmod{12}$, $u \notin \{24\}$. *Then there exists a* $4-$*GDD of type* $g^u m^1$ *for every* $m \equiv g \pmod 3$ *with* $0 \leqslant m \leqslant (g(u-1)-3)/2$, *except possibly when* $u \in \{12, 72, 120, 168\}$ *and* $0 < m < g$.

Now, we will prove Theorem 3.2.30. We henceforth assume that g is even. We will use the full force of Construction 3.2.31 to provide us 4−IGDDs with the following rich source of 4−IGDDs.

Theorem 3.2.39. *Let* (X, G, B) *be a* $\{5, 6\}-$*GDD with* $G = \{G_1, G_2, ..., G_s\}$. *Then for every sequence* $n_1, n_2, ..., n_s$ *of integers with* $0 \leqslant n_i \leqslant |G_i|$, $i = 1, 2, ..., s$, *there is a* $4-$*IGDD of type* $\{(6|G_i| + 3n_i, 3n_i) : i = 1, 2, ..., s\}$.

Proof: From [26] and [27] the result is true for GDDs of types 1^5 and 1^6; that is the authors construct 4−IGDDs of type $(9,3)^{k_1}6^{5-k_1}$ and $(9,3)^{k_2}6^{6-k_2}$ for all $0 \leqslant k_1 \leqslant 5$ and $0 \leqslant k_2 \leqslant 6$. Now we apply Construction 3.2.31 to (X, G, B) ($Y = \emptyset$ here), as follows. We write $X = X' \cup X''$, where for each $i = 1, 2, ..., s$, $|X' \cap G_i| = n_i$ (and so $|X'' \cap G_i| = G_i - n_i$). For $x \in X'$, we take $w(x) = 9$ and $d(x) = 3$, while for $x \in X''$, we take $w(x) = 6$ and $d(x) = 0$. For each block $b \in B$ the requisite 4−IGDD exists by the foregoing, since $|b| = 5$ or 6. The result is a 4−IGDD of type $\{(9n_i + (6|G_i| - n_i), 3n_i) : i = 1, 2, ..., s\} \equiv \{(6|G_i| + 3n_i, 3n_i) : i = 1, 2, ..., s\}$ as desired. □

We will be applying Theorem 3.2.39 predominantly to $\{5, 6\}-$GDDs obtained from transversal designs TD$(6, n)$. It is well-known that a TD$(6, n)$ exists if and only if $n \geqslant 5$, except for $n = 6$ and possibly for $n \in \{10, 14, 18, 22\}$. Now again from Table 3.2 we see that where g is even we must have either $g \equiv 0$ $(mod\ 6)$ and $u \geqslant 4$, with $0 \leqslant m \leqslant g(u-1)/2$ and $m \equiv 0$ $(mod\ 3)$, $(g, u, m) \neq (6, 4, 0)$ or $g \equiv 2$ or 4 $(mod\ 6)$ and $u \equiv 0$ $(mod\ 3)$, $u \geqslant 6$, with $0 \leqslant m \leqslant g(u-1)/2$ and $m \equiv g$ $(mod\ 3)$, $(g, u, m) \neq (2, 6, 5)$. We begin with $g \equiv 0$ $(mod\ 6)$, using the following construction for "filling the holes" in [27].

Construction 3.2.40. *Let (X, Y, G, B) be a 4−IGDD of type $(g_1, h_1)^{u_1}, (g_2, h_2)^{u_2}, \ldots, (g_s, h_s)^{u_s}$ and let $a \geqslant 0$. Suppose that for each $i = 1, 2, \ldots, s$ there is a 4−GDD of type with $a + g_i$ points having a group of size a and a group of size h_i. Then there is a 4−GDD of type with $a + \sum_i u_i g_i$ points having a group of size $\sum_i u_i g_i$.*

Proof: We adjoin a set I of a ideal points to (X, Y, G, B). For each $G_j \in G$ construct on $G_j \cup I$ a copy of the relevant 4−GDD, by aligning groups on I and $G_j \cap Y$. The resulting 4−GDD will have groups aligned on I and Y. □

Theorem 3.2.41. *Let $g \equiv 0$ $(mod\ 6)$, $n \equiv -g/6$ $(mod\ 2g/3)$, $n \geqslant 15g/6$, and $w \equiv 0$ $(mod\ g/6)$, $g/2 \leqslant w \leqslant n$. Then for every $m \equiv 0$ $(mod\ 3)$ with $0 \leqslant m \leqslant g(u-1)/2$ there exists a 4−GDD of type $g^u m^1$, where $u = 1 + 6(5n + w)/g$.*

Proof: From previous section since a uniform 4−GDD of type g^u exists, we may assume that $m > 0$. We take a transversal design TD$(6, n)$ and truncate on of its groups to w points. We write $m = 3(n_1 + n_2 + n_3 + n_4 + n_5 + w')$, where $0 \leqslant n_i \leqslant n$ for $i = 1, 2, \ldots, 5$ and $w' = w$ or 0, unless $g = 6$ and $w = 3$ in this case we take $w = 1$ or 3. By Theorem 3.2.39 there is a 4−IGDD of type $(\prod_i (6n + 3n_i, 3n_i)^1)(6w + 3w', 3w')^1$. Now we adjoin a set I of g points to this 4−IGDD and apply Construction 3.2.40, fill in 4−GDDs of type $g^{1+(6n/g)}(3n_i)^1$, $i = 1, 2, \ldots, 5$ and a 4−GDD of type $g^{1+(6w/g)}(3w')^1$. Note that $1 + (6n/g) \equiv 0$ $(mod\ 4)$, $1 + (6n/g) > 16$, and $1 + (6w/g) \equiv 0$ $(mod\ 4)$, $1 + (6w/g) > 16$ and so these GDDs exist by Theorem 2.1 in [25] and Theorem 3.2.28. The result is a 4−GDD of type $g^{1+(20n+6w)/g}(3w' + \sum_i 3n_i)^1 \equiv g^u m^1$, as desired. □

Corollary 3.2.42. *Let $g \equiv 0$ $(mod\ 6)$ and $u \in \{n : n \geqslant 79\} \backslash \{93, 94, 95, 97, 98, 117, 118\}$.*

Then there exists a $4-GDD$ *of type* $g^u m^1$ *for every* $m \equiv 0\ (mod\ 3)$ *with* $0 \leqslant m \leqslant g(u-1)/2$.

Proof: By Theorem 3.2.41 the admissible values of u are those in the set $\{1+6(5n+w)/g : n \equiv -g/6\ (mod\ 2g/3), n \geqslant 15g/6, \text{and } w \equiv 0\ (mod\ g/6), g/2 \leqslant w \leqslant n\}$. Now $6g/n = 4t-1$ and $n \geqslant 15g/6$ implies $6n/g \geqslant 15$, which in turn implies $t \geqslant 4$. On the other hand, $6w/g = t'$ where the condition $g/2 \leqslant w \leqslant n$ translates to $3 \leqslant t' \leqslant 4t-1$. That is, the admissible values of u are those in the set $\{20t-4+t' : t \geqslant 4 \text{ and } 3 \leqslant t' \leqslant 4t-1\} = \{n : n \geqslant 79\} \backslash \{92, 93, 94, 95, 96, 97, 98, 116, 117, 118\}$. Now by Theorem 2.1 in [25] we can remove $n = 92, 96$ and 116 from the foregoing set of excluded values. The result follows. □

In particular in the case $g = 6$, we can get a better result:

Theorem 3.2.43. *There exists a* $4-GDD$ *of type* $6^u m^1$ *for every* $u \geqslant 68$ *and every* $m \equiv 0\ (mod\ 3)$ *with* $0 \leqslant m \leqslant 3(u-1)$.

Proof: By Corollary 3.2.42 we need to consider only $68 \leqslant u \leqslant 78$, $93 \leqslant u \leqslant 98$ and $117 \leqslant u \leqslant 118$. First, $u = 68$ is covered by Theorem 2.1. For $69 \leqslant u \leqslant 78$ we start with a resolvable $(65, 5)-$BIBD and adjoin $u-66$ infinite points to obtain a $\{5, 6\}-$GDD of type $5^{13}(u-66)^1$. We write $m = 3(n_1 + n_2 + ... + n_{13} + w')$ where $0 \leqslant n_i \leqslant 5$ for each $i = 1, 2, ..., 13$, and $w' = 1$ or $u-66$, unless $u = 69$ in which case we take $w' = 1$ or 3 (since we know that a uniform 4−GDD of type g^u exists by previous section we may assume that $m > 0$). By Theorem 3.2.39 there is a 4−IGDD of type $(\prod_i (30+3n_i, 3n_i)^1)(6(u-66)+3w', 3w')^1$. We adjoin 6 ideal points to this IGDD and apply Construction 3.2.40, fill in 4−GDDs of types $6^6(3n_i)^1$, $i = 1, 2, \ldots, 13$ and a 4−GDD of type $6^{u-65}(3w')^1$ (these 4−GDDs exist by Lemma 3.2.2, Theorem 3.2.28 and uniform 4−GDDs in the previous section). The result is a 4−GDD of type $6^u m^1$, as desired. For $93 \leqslant u \leqslant 98$ and $117 \leqslant u \leqslant 118$ we proceed similarly, starting instead with resolvable $(v, 5)-$BIBDs for $v = 85$ and $v = 105$ respectively. The result follows. □

Construction 3.2.44. *Let (X, Y, G, B) be a $4-$IGDD of type $(g_1, h_1)^{u_1}(g_2, h_2)^{u_2}, ..., (g_s, h_s)^{u_s}$ and let $a \geqslant b \geqslant 0$. Suppose that for each $i = 1, 2, ..., s-1$ (and also $i = s$ if $u_s \geqslant 2$) there is a $4-$IGDD (X_i, Y_i, G_i, B_i) on $a + g_i$ points with a group M_i of size $b + h_i$ and a group of N_i of size $a - b$ in which $Y_i = N_i \cup (Y_i \cap M_i)$, $|Y_i| = a$. Suppose further that there is a $4-$GDD on $a + g_s$ points with a group of size $b + h_s$. Then there is a $4-$GDD on $a + \sum_i u_i g_i$ points with a group of size $b + \sum_i u_i h_i$.*

Proof: Let (X, Y, G, B) be a $4-$IGDD of type $(g_1, h_1)^{u_1}(g_2, h_2)^{u_2}, ..., (g_s, h_s)^{u_s}$. We adjoin a set $I = \{1, 2, \ldots, a\}$ of a ideal points to this IGDD. For each group $G_j \in G$ with $|G_j| = g_i$ and $|G_j \cap Y| = h_i$, $i = 1, 2, \ldots, s-1$, we construct a copy of (X_i, Y_i, G_i, B_i) on $G_j \cup I$, align the group M_i on $(G_i \cap Y) \cup \{1, 2, \ldots, b\}$ and the group N_i on $I \backslash \{1, 2, \ldots, b\}$, and align the hole Y_i on I. Then we do the same for all but one of the groups of type (g_s, h_s) in (X, Y, G, B). Finally on the last group G' (g_s, h_s) together with I, we construct a copy of indicated $4-$GDD and align the group of size $b + h_s$ on $(G' \cap Y) \cup \{1, 2, \ldots, b\}$. The resulting $4-$GDD on $a + \sum_i u_i g_i$ points has group of size $b + \sum_i u_i h_i$ aligned on $Y \cup \{1, 2, \ldots, b\}$, as desired. □

We will employ the following class of $4-$IGDDs as 'fillers' in Construction 3.2.44.

Lemma 3.2.45. (***i***) *Let $g \equiv 2 \pmod 6$ and $t \geqslant 2$, and let $x = 2$ or $g(3t-1)/2$, $(g, t, x) \neq (2, 2, 5)$. Then for every $y \equiv 0 \pmod 3$ with $0 \leqslant y \leqslant 9gt/2$ there exits a $4-$IGDD of type $g^{9t}(3tg, 3tg)^1(x + y, x)^1$.*

(***ii***) *Let $g \equiv 4 \pmod 6$ and $t \geqslant 2$, and let $x = 1$ or $g(3t-1)/2$ where $(t, x) \neq (7, 1)$ when $g \equiv 10 \pmod{12}$. Then for every $y \equiv 0 \pmod 3$ with $0 \leqslant y \leqslant 9gt/2$ there exists a $4-$IGDD of type $g^{9t}(3tg, 3tg)^1(x + y, x)^1$.*

Proof: If we consider the constructions in Theorem 2.2 in [25], then all cases satisfying the foregoing hypothesis give a $4-$GDD of type $g^{12t}(x + y)^1$ which has a sub-GDD of type $g^{3t}x^1$. Simply if remove the blocks (but not the points) from this sub-GDD then we obtain the required $4-$IGDD. □

Theorem 3.2.46 (Theorem 4.9, [25])**.** *Let $g \equiv 2$ or $4 \pmod 6$. Let $n \equiv 0 \pmod{3g/2}$, $n \geqslant 6g$ and $w \equiv 0 \pmod{g/2}$, $0 < w \leqslant n$, where $(g, n) \neq (2, 18)$. Let $u = (32n + 6w)/g$.*

Then for every $m \equiv g \ (mod\ 3)$ *with* $0 < m \leqslant g(u-1)/2$ *there exists a* $4-GDD$ *of type* $g^u m^1$*, except possibly when* $g \equiv 10 \ (mod\ 12)$*,* $n = 21g/2$ *and* $0 < m < 83g/2$*.*

Corollary 3.2.47 (Corollary 4.10, [25])**.** *Let* $g \equiv 2$ *or* $4 \ (mod\ 6)$ *and* $u \in \{n \equiv 0 \ (mod\ 3) : n \geqslant 192\} \backslash \{231, 234, 237\}$*. Then there exists a* $4-GDD$ *of type* $g^u m^1$ *for every* $m \equiv g \ (mod\ 3)$ *with* $0 < n \leqslant g(u-1)/2$*, except possibly when* $g = 2$ *and* $u \in \{291, 294, 297, 303, 306, 309, 315, 318, 321, 327, 330, 333\}$*, or when* $g \equiv 10 \ (mod\ 12)$ *and* $0 < m < 83g/2$*, where* $u \in \{345, 351, 354, 357, 363, 366, 369, 375, 378, 381\}$*.*

We now work to remove most of the possible exceptional cases with $u \geqslant 240$ from Corollary 3.2.47. To do this we will use the following special case of the Construction 3.2.44 obtained by setting $a = b$.

Construction 3.2.48. *Let* (X, Y, G, B) *be a* $4-IGDD$ *of type* $(g_1, h_1)^{u_1}, (g_2, h_2)^{u_2}, ..., (g_s, h_s)^{u_s}$ *and let* $b \geqslant 0$*. Suppose that for each* $i = 1, 2, ..., s$ *there is a* $4-GDD$ *with* $b + g_i$ *points having a group of size* $b + h_i$*. Then there is a* $4-GDD$ *with* $b + \sum_i u_i g_i$ *points having a group of size* $b + \sum_i u_i h_i$*.*

Theorem 3.2.49 (Theorem 4.12, [25])**.** *Let* $g \equiv 2$ *or* $4 \ (mod\ 6)$*. Let* $n \equiv 0 \ (mod\ 2g)$*,* $n \geqslant 4g$ *and* $w \equiv 0 \ (mod\ g/2)$*,* $g \leqslant w \leqslant n$*. Then for every* $m \equiv g \ (mod\ 3)$ *with* $0 < m \leqslant g(u-1)/2 - 15\lceil g/6 \rceil - 3x$ *there exists a* $4-GDD$ *of type* $g^u m^1$*, where* $u = 6(5n + w)/g$ *and* $x = 1$ *if* $g = w = 2$ *while* $x = 0$ *otherwise.*

Corollary 3.2.50 (Corollary 4.13, [25])**.** *] Let* $g \equiv 2$ *or* $4 \ (mod\ 6)$ *and* $u \in \{n \equiv 0 \ (mod\ 3) : n \geqslant 126\} \backslash \{147, 150, 153, 159, 162, 165, 171, 174, 177, 183, 219, 222, 225, 231, 234, 237, 243, 291, 297, 300, 303, 363\}$ *. Then there exists a* $4-GDD$ *of type* $g^u m^1$ *for every* $m \equiv g \ (mod\ 3)$ *with* $0 < m \leqslant g(u-1)/2 - 15\lceil g/6 \rceil$*, except possibly for* $g = 2$ *and* $m = u - 16$ *when* $u \equiv 6 \ (mod\ 60)$*.*

If we combine Corollaries 3.2.47 and 3.2.50 we get the following result.

Corollary 3.2.51 (Corollary 4.14, [25])**.** *Let* $g \equiv 2$ *or* $4 \ (mod\ 6)$ *and* $u \in \{n \equiv 0 \ (mod\ 3) : n \geqslant 192\} \backslash \{231, 234, 237\}$*. Then there exists a* $4-GDD$ *of type* $g^u m^1$ *for every* $m \equiv g \ (mod\ 3)$

with $0 < m \leqslant g(u-1)/2$*, except possibly either*

(***i***) $g = 2$ *and* $u \in \{291, 294, 297, 303\}$*, or*

(***ii***) $g = 2$ *and* $m = u - 16$ *where* $u = 306$*, or* $g = 2$, $u - 13 \leqslant m \leqslant u - 4$*, where* $u \in \{306, 309, 315, 318, 321, 327, 330, 333\}$*, or*

(***iii***) $g \equiv 10$ $(mod\ 12)$ *and* $0 < m \leqslant 83g/2$*, where* $u = 363$.

Finally, we can eliminate the possible exceptions of type (iii) in Corollary 3.2.51.

Lemma 3.2.52 (Lemma4.15, [25])**.** *There exists a* $4-GDD$ *of type type* $g^{363}m^1$ *for every* $g \equiv 10$ $(mod\ 12)$ *and every* $m \equiv 1$ $(mod\ 3)$ *with* $1 \leqslant n \leqslant 181g$.

Collecting Corollaries 3.2.42, 3.2.51 and Lemma 3.2.52 gives the following analogue to Theorem 3.2.38 for even g.

Theorem 3.2.53. (***i***) *Let* $g \equiv 0$ $(mod\ 6)$ *and* $u \in \{n : n \geqslant 79\}\backslash\{93, 94, 95, 97, 117, 118\}$. *Then there exists a* $4-GDD$ *of type* $g^u m^1$ *for every* $m \equiv 0$ $(mod\ 3)$ *with* $0 \leqslant m \leqslant g(u-1)/2$.
(***ii***) *Let* $g \equiv 2$ *or* 4 $(mod\ 6)$ *and* $u \in \{n \equiv 0\ (mod\ 3) : n \geqslant 192\}\backslash\{231, 234, 237\}$. *Then there exists a* $4-GDD$ *of type* $g^u m^1$ *for every* $m \equiv g$ $(mod\ 3)$ *with* $0 < m \leqslant g(u-1)/2$*, except possibly when* $g = 2$ *and* $u \in \{291, 294, 297, 303, 306, 309, 315, 318, 321, 327, 330, 333\}$.

Remark: As we say in Corollary 3.2.51 (ii), we can get all but at most five values of m, when $g = 2$ and $306 \leqslant u \leqslant 333$.

Conclusion: If we combine Theorems 3.2.38 and 3.2.53, we get Theorem 3.2.30. Therefore we obtained our second result.

3.2.3 4−GDDs of type $6^u m^1$

In this section we will construct 4−GDDs of type $6^u m^1$ up to 13 possible exceptions so we will prove the following theorem. It is easy to see that the necessary conditions for the existence of a 4−GDD of type $6^u m^1$ are that $u \geqslant 4$, $m \equiv 0$ $(mod\ 3)$, and $0 \leqslant m \leqslant 3m - 3$, $(u, m) \neq (4, 0)$.

Theorem 3.2.54. *There exists a* $4-GDD$ *of type* $6^u m^1$ *for every* $u \geqslant 4$ *and* $m \equiv 0$ $(mod\ 3)$ *with* $0 \leqslant m \leqslant 3u - 3$ *except for* $(u, m) = (4, 0)$ *and except possibly for*

$(u, m) \in \{(7, 15), (11, 21), (11, 24), (11, 27), (13, 27), (13, 33), (17, 39), (17, 42), (19, 45), (19, 48), (19, 51), (23, 60), (23, 63)\}$.

We will give some previous results that we did in previous sections.

Theorem 3.2.55 (Theorem 3.2.28 (i))**.** *Let $u \geqslant 4$. There exist $4-$GDDs of type 6^u and $6^u(3u-3)^1$, except that there is no $4-$GDD of type 6^4. There is a $4-$GDD of type $6^4 3^1$.*

Theorem 3.2.56 (Lemma 3.5, [29])**.** *There exists a $4-$GDD of type $6^u 3^1$ for every $u \geqslant 4$, except possibly for $u \in \{10, 11, 12, 13, 14, 15, 17, 18, 19, 23\}$.*

Theorem 3.2.57 (See Lemma 3.2.17)**.** *There exists a $4-$GDD of type $6^u 9^1$ for every $u \geqslant 4$, except possibly for $u \in \{10, 12, 13, 14, 15, 17, 18, 19, 23\}$*

Theorem 3.2.58 (See Theorem 2.1, [25])**.** *Let $u \equiv 0 \ (mod\ 4)$, $u \geqslant 16$. Then there exists a $4-$GDD of type $6^u m^1$ for every $m \equiv 0 \ (mod\ 3)$ with $0 \leqslant m \leqslant 3u-3$.*

Theorem 3.2.59 (See Theorem 3.2.43)**.** *There exits a $4-$GDD of type $6^u m^1$ for every $u \geqslant 68$ and every $m \equiv 0 \ (mod\ 3)$ with $0 \leqslant m \leqslant 3u-3$.*

Lemma 3.2.60 (See Lemma 2.1, [28])**.** *There exists a $4-$GDD of type $6^u m^1$ for $(u, m) \in \{(8, 12), (8, 15), (9, 12), (10, 3), (10, 9), (10, 12), (10, 15), (10, 18), (10, 21), (10, 24), (11, 3), (11, 12), (11, 15), (11, 18), (12, 3), (12, 9), (12, 12), (12, 18), (12, 21)\}$.*

Lemma 3.2.61 (Lemma 2.2, [28])**.** *Let $4 \leqslant u \leqslant 12$. There exists a $4-$GDD of type $6^u m^1$ for all $m \equiv 0 \ (mod\ 3)$ with $0 \leqslant m \leqslant 3u-3$, except for $(u, m) = (4, 0)$ and except possibly for $(u, m) \in \{(7, 15), (11, 21), (11, 24), (11, 27)\}$.*

Now we will give a result using Theorem 3.2.39 and Construction 3.2.40 that we proved in previous section.

Corollary 3.2.62. *Suppose that there exists a $\{5, 6\}-$GDD (X, G, B) on v points with group sizes from the set $\{4, 5, 7, 8, 9, 11\}$. Then for each $m \equiv 0 \ (mod\ 3)$ with $0 \leqslant m \leqslant 3v$ there exists a $4-$GDD of type $6^{v+1} m^1$.*

Proof: Let $G = \{G_1, G_2, \ldots, G_s\}$ and we write $m = 3(n_1 + n_2 + \ldots + n_s)$ where $0 \leqslant n_i \leqslant |G_i|$ for each $i = 1, 2, \ldots, s$. By Theorem 3.2.39 there is a 4−IGDD of type $\{(6g_i + 3n_i, 3n_i) : i = 1, 2, \ldots, s\}$. Now we adjoin 6 ideal points to this IGDD and apply Construction 3.2.40, then we construct on the ith group together with the ideal points a 4−GDD of type $6^{|G_i|+1}(3n_i)^1$ which exists by Lemma 3.2.61. The result is a 4−GDD of type $6^{v+1}m^1$. □

Remark: Under the hypothesis of Corollary 3.2.62 we can allow one further group $G' \in G$ with size from the set $\{3, 6, 10, 12\}$ and get the same conclusion, as follows. We write $m = 3(n_1 + n_2 + \ldots + n_s + n)$ as above, where $n = 0$ or $|G'|$ (unless $|G'| = 3$, in which case we take $n = 1$ or 3). On the group in the IGDD corresponding to G' together with the ideal points we construct a 4−GDD of type $6^{|G'|+1}(3n)^1$ (see Theorem 3.2.55). This works because $n < \sum_{i=1}^{s} n_i$.

Table 3.4: Source of {5, 6}−GDDs

u	{5, 6}−GDD of type	Source
25,26	$4^6, 5^5$	(25, 5)-BIBD
29,30,31	$5^5 3^1, 5^5 4^1, 5^6$	TD(6, 5)
37	$4^7 8^1$	Resolvable (28, 4)-BIBD
39,41,42,43	$7^5 3^1, 7^5 5^1, 7^5 6^1, 7^6$	TD(6, 7)
45,46,47,49	$8^5 4^1, 8^5 5^1, 8^5 6^1, 8^6$	TD(6, 8)
50,51,53,54,55	$9^5 4^1, 9^5 5^1, 9^5 7^1, 9^5 8^1, 9^6$	TD(6, 9)
57	$4^{12} 8^1$	PBD({5, 9*}, 57), see [31]
59,61,62,63,65	$11^5 3^1, 11^5 5^1, 11^5 6^1, 11^5 7^1, 11^5 9^1$	TD(6, 11)
66,67	$11^5 10^1, 11^6$	

Lemma 3.2.63. *Let* $U = \{n : n \equiv 1, 2 \text{ or } 3 \pmod 4) \text{ and } 25 \leqslant n \leqslant 67\} \backslash \{27, 33, 34, 35, 38, 58\}$.

Then for each $u \in U$ there exists a $4-GDD$ of type $6^u m^1$ for every $m \equiv 0 \pmod 3$ with $0 \leqslant m \leqslant 3u-3$.

Proof: We apply Corollary 3.2.62 and the remark following it, exhibiting for each $u \in U$ a $\{5,6\}-$GDD on $v = u-1$ points with group sizes from the set $\{3,4,...,12\}$ having at most one group with size from the set $\{3,6,10,12\}$ (see Table 3.4). □

There remain the values $u \in \{27,33,34,35,38,58\}$ to deal with.

Lemma 3.2.64. *Let $u \in \{35,38,58\}$. Then for each $m \equiv 0 \pmod 3$ with $0 \leqslant m \leqslant 3u-3$ there exists a $4-GDD$ of type $6^u m^1$.*

Proof: For $u = 38$ we take $TD(6,7)$ and remove five points from a block to obtain a $\{5,6\}-$GDD of type $6^5 7^1$. It is not difficult to see that we can write $m = 3(n_1 + n_2 + n_3 + n_4 + n_5 + n_6)$ where $n_i \in \{0,1,2,\ldots,6\}\setminus\{5\}$ for $i = 1,2,...,5$ and $0 \leqslant n_6 \leqslant 7$. Now by Theorem 3.2.39 there is a $4-$IGDD of type $\{(36+3n_i, 3n_i) : i = 1,2,...,5\} \cup \{(42+3n_6, 3n_6)\}$. If we adjoin 6 ideal points to this IGDD and apply Construction 3.2.40 and fill in $4-$GDDS of type $6^7(3n_i)^1$ (by Lemma 3.2.61), we obtain the result we want.

The other cases of u similar to this case, for these refer to Lemma 3.5 in [28]. □

Lemma 3.2.65 (Lemm 3.6, [28])**.** *Let $u \in \{27,33,34\}$. Then for each $m \equiv 0 \pmod 3$ with $0 \leqslant m \leqslant 3u-3$ there exists a $4-GDD$ of type $6^u m^1$.*

If we collect the Lemmas 3.2.63 - 3.2.65, we get the following result:

Theorem 3.2.66. *For each $u \geqslant 24$ and each $m \equiv 0 \pmod 3$, $0 \leqslant m \leqslant 3u-3$ there exists a $4-GDD$ of type $6^u m^1$.*

Now, we will complete our existence result for $4-$GDDs of type $6^u m^1$. The vast majority of the designs in the range $13 \leqslant u \leqslant 23$ are obtained by direct constructions in [28].

Theorem 3.2.67 (Theorem 4.1, [28])**.** *Let $13 \leqslant u \leqslant 23$. There exists a $4-GDD$ of type $6^u m^1$ for every $m \equiv 0 \pmod 3$ with $0 \leqslant m \leqslant 3u-3$, except possibly for $(u,m) \in \{(13,27),(13,33),(17,39),(17,42),(19,45),(19,48),(19,51),(23,60),(23,63)\}$.*

The main result of this section (Theorem 3.2.54) follows from Lemma 3.2.61, Theorem 3.2.66 and Theorem 3.2.67.

3.2.4 4−GDDs of type $g^u m^1$ for small g

In this section we will continue to investigate the spectrum for 4−GDDs of type $g^u m^1$ with small g. We will show that, for each $g \in \{2, 4, 5, 12, 15\}$ and each admissible pair (u, m), a 4−GDD of type $g^u m^1$ exists with definite exception for $(g, u, m) = (2, 6, 5)$ and five possible exceptions when $g = 2$.

First we will prove that the necessary conditions for the existence of 4−GDDs of type $g^u m^1$ for $g = 4, 12$ are also sufficient. We begin with the case $g = 12$.

Theorem 3.2.68. *There exists a* 4−*GDD of type* $12^u m^1$ *for each* $u \geqslant 4$ *and* $u \equiv 0, 1, 3 \ (mod\ 4)$ *with* $m \in \{3, 6, 9\}$.

Proof: We start with a TD$(5, u)$ and adjoin an infinite point ∞ to the groups, then we delete a finite point to form $\{5, u+1\}$−GDD of type $4^u u^1$. Note that each block of size $u+1$ intersects the group of size u in the infinite point ∞ and each block of size 5 intersects the group of size u, but certainly not in ∞. In the group of size u, we give to ∞, the weight 0 (when $u \equiv 0, 1 \ (mod\ 4)$) or 3 (when $u \equiv 3 \ (mod\ 4)$) and give the remaining points weight 0 or 3. We give all other points in the $\{5, u+1\}$−GDD the weight 3. Finally, we replace the blocks of $\{5, u+1\}$−GDD by 4−GDDs of type 3^u, 3^{u+1}, 3^4, or 3^5 to obtain the 4−GDDs as desired. □

For the following result, we need a new construction on HTDs (Holey transversal designs). A k−DGDD of type $(g, h^v)^k$ is a holey transversal design k−HTD of type h^v and is equivalent to a set of $k-2$ holey MOLS of type h^v(see [5]). The following construction is a slight modification of Theorem 3.5 in [23].

Construction 3.2.69. *Suppose that* $k = 4$ *or* 5 *and there exists a* k−*HTD of type* $h^n u^1$ *whose block set contains* t *disjoint holey parallel classes with the hole of size* u. *Then for each* $0 \leqslant a \leqslant t$, *there exists a* $\{k, k-1\}$−*DGDD of type* $(hn(k-1), (h(k-1)^n))^k$ *whose blocks of size* $k-1$ *can be partitioned into* $u(k-1)^2 + a(k-1)$ *parallel classes.*

In order to apply Construction 3.2.69, we need the following Lemmas on 4−HTDs.

Lemma 3.2.70 (Lemma 2.12, [32]). *There exists an FMOLS*$(1^n u^1)$ *if and only if* $n \geqslant 2u+1$ *except for* $(n, u) = (5, 1)$.

Lemma 3.2.71 (Lemma 2.13, [32]). *For each* $k \geqslant 2$, *there exists an ISOLS*$(1^{2k+3}k^1)$ *containing two disjoint transversals with the hole of size k.*

Lemma 3.2.72 (Lemma 2.14, [32]). *For each* $k \geqslant 2$, *there exists an FMOLS*$(1^{2k}(k-1)^1)$ *containing two disjoint holey transversals with the hole of size k.*

Now we are ready to give our lemma on 4−GDD of type $12^u m^1$.

Lemma 3.2.73. *There exists a* 4−*GDD of type* $12^n m^1$ *for each* $n \geqslant 8$ *and* $n \equiv 0, 1, 3 \ (mod\ 4)$ *with* $m \equiv 0 \ (mod\ 3)$ *and* $12 \leqslant m \leqslant 6(n-1) - 6$.

Proof: For any given $n \geqslant 8$ and $n \equiv 0, 1, 3 \ (mod\ 4)$, by Lemma 3.2.70 we have a 4−HTD of type $1^n u^1$ with $n \geqslant 2u + 1$. If we apply Construction 3.2.69, we obtain $\{3, 4\}$−DGDD of type $(3n, 3^n)^4$, where all the blocks of size 3 can be partitioned into $9u$ parallel classes. We adjoin $9u$ infinite points to complete the parallel classes and then adjoin further k ideal points. Next we fill in a 4−GDDs of type $3^n k^1$ coming from Theorem 3.2.3, then we obtain a 4−GDD of type $12^n(9u + k)^1 \equiv 12^n m^1$, as we desire. □

Lemma 3.2.74. *There exists a* 4−*GDD of type* $12^n(6n - 9)^1$ *for each* $n \geqslant 7$ *and* $n \equiv 0, 1, 3 \ (mod\ 4)$.

Proof: For any given $n \geqslant 7$ and $n \equiv 1, 3 \ (mod\ 4)$, by Lemma 3.2.71, we have a 4−HTD of type $1^n((n-3)/2)^1$ containing two disjoint parallel classes with the hole of size $(n-3)/2$. If we apply Construction 3.2.69 with '$a = t = 2$', we obtain a $\{3, 4\}$−DGDD of type $(3n, 3^n)^4$, where all the blocks of size 3 can be partitioned into $((n-3)/2) \cdot 9 + 2 \cdot 3 = (9n-15)/2$ parallel classes. We adjoin $(9n-15)/2$ infinite points to complete the parallel classes and then we adjoin further $3(n-1)/2$ points.

Finally, if we fill in 4−GDDs of type $3^n(3(n-1)/2)^1$ coming from Theorem 3.2.3, then we obtain a 4−GDD of type $12^n(((9n-15)/2)+(3(n-1)/2))^1 \equiv 12^n(6n-9)^1$, as we desired.

Similarly, for any given $n \geqslant 8$, $n \equiv 0 \ (mod\ 4)$, by Lemma 3.2.72 we have a 4−HTD of type $1^n((n/2)-1)^1$ containing a disjoint holey parallel class with a hole of size $(n/2)-1$. If we apply again Construction 3.2.69 with "$a = t = 1$", we obtain a 4−GDD of type $12^n((n/2)-1)\cdot 9+3+((3(n-1)-3)/2))^1 \equiv 12^n(6n-9)^1$. □

If we combine Theorem 1.2 in [32] and Lemmas 3.2.68-3.2.74, we have the following result.

Lemma 3.2.75. *There exists a* 4−*GDD of type* $12^u m^1$ *for each* $u \geqslant 8$ *and* $u \equiv 0, 1, 3 \ (mod\ 4)$ *with* $m \equiv 0 \ (mod\ 3)$ *and* $12 \leqslant m \leqslant 6(u-1)$.

Since $u = 4$ has been solved by Theorem 3.2.5, we have only the cases for $u = 5, 7$ and $u \equiv 2 \ (mod\ 4)$ to be considered.

Lemma 3.2.76. *There exists a* 4−*GDD of type* $12^u m^1$ *for each* $u \in \{5, 7, 10, 14, 18, 22\}$ *with* $m \equiv 0 \ (mod\ 3)$ *and* $0 \leqslant m \leqslant 6(u-1)$.

Proof: For these GDDs refer to Lemmas from 3.5 to 3.11 in [32]. □

Lemma 3.2.77. *There exists a* 4−*GDD of type* $12^u m^1$ *for each* $u \geqslant 30$, $u \equiv 2 \ (mod\ 4)$ *and* $u \neq 34, 38, 46$ *and* $0 \leqslant m \leqslant 6(u-1)$.

Proof: For $0 \leqslant m \leqslant 9$, suppose that $u = 4s+2$ and $s \geqslant 7$. We take a 4−GDD of type $(12s-12)^4(72+m)^1$ coming from Theorem 3.2.5 and fill in 4−GDDs of type 12^{s-1} and 4−GDDs of type $12^6 m^1$ to obtain the 4−GDDs as desired. For other values of m, we refer to Lemma 3.12 in [32]. □

This leaves $u = 26, 34, 38, 46$ to be considered. With the following Lemma 3.15 in [32] we will complete this case.

Lemma 3.2.78. *There exists a* 4−*GDD of type* $12^u m^1$ *for each* $u \in \{26, 34, 38, 46\}$ *with* $0 \leqslant m \leqslant 6(u-1)$.

If we combine Theorem 3.2.5 and Lemmas 3.2.75-3.2.77 and 3.2.78, we have the following result:

Theorem 3.2.79. *There exists a* $4-GDD$ *of type* $12^u m^1$ *for each* $u \geqslant 4$ *and* $m \equiv 0 \ (mod\ 3)$ *with* $0 \leqslant m \leqslant 6(u-1)$.

With this theorem we complete the case $g = 12$. In the next case we will do $g = 4$.

Lemma 3.2.80 (Lemma 3.17, [32])**.** *There exist* $4-GDDs$ *of types* $4^6 m^1$ *and* $4^9 n^1$ *for all admissible* $m \equiv n \equiv 1 \ (mod\ 3)$.

Lemma 3.2.81. *There exists a* $4-GDD$ *of type* $4^u m^1$ *for each* $u \geqslant 6$, $u \equiv 0 \ (mod\ 3)$ *and* $m \equiv 1 \ (mod\ 3)$ *with* $1 \leqslant m \leqslant 2(u-1)$.

Proof: For any given $u \geqslant 12$, $u \equiv 0 \ (mod\ 3)$ and $m \equiv 1 \ (mod\ 3)$ with $4 \leqslant m \leqslant 2(u-1)$ we take a 4−GDD of type $12^{u/3}(m-4)^1$ coming from Theorem 3.2.79. We adjoin 4 infinite points and fill the holes with a 4−GDD of type 4^4 to obtain a 4−GDD of type $4^u m^1$. If we combine Theorem 3.2.28 and Lemma 3.2.80, then we get the conclusion. □

With this lemma we complete the case $g = 4$. In the next case we will do $g = 5$ and $g = 15$. First we will prove the case when $g = 15$.

Theorem 3.2.82. *A* $4-GDD$ *of type* $15^u m^1$ *exists if and only if either* $u \equiv 0 \ (mod\ 4)$ *and* $m \equiv 0 \ (mod\ 3)$, $0 < m \leqslant (15u-18)/2$*; or* $u \equiv 1 \ (mod\ 4)$ *and* $m \equiv 0 \ (mod\ 6)$, $0 < m \leqslant (15u-15)/2$*; or* $u \equiv 3 \ (mod\ 4)$ *and* $m \equiv 3 \ (mod\ 6)$, $0 < m \leqslant (15u-15)/2$.

Proof: For $u = 4$, the designs come from Theorem 3.2.5. For other values of u, we start with a $TD(6, u)$ and adjoin an infinite point ∞ to the groups, then we delete a finite point so as to form a $\{6, u+1\}$−GDD of type $5^u u^1$. Note that each block of size $u+1$ intersects the group of size u in the infinite point ∞ and each block of size 6 intersects the group of size u, but certainly not in ∞. Now, in the group of size u, we give ∞ weight a, where $a \in \{0, 3, ..., (3u-6, theorem3.1)/2\}$ (when $u \equiv 0 \ (mod\ 4)$), $a \in \{0, 6, ..., (3u-3)/2\}$ (when $u \equiv 1 \ (mod\ 4)$) or $a \in \{3, 9, ..., (3u-3)/2\}$

(when $u \equiv 3 \pmod 4$) and give to remaining points weight 0 or 6. We give to all the other points in the $\{6, u+1\}$−GDD weight 3. We replace the blocks in the $\{6, u+1\}$−GDD by 4−GDDs of type $3^u a^1$, 3^5, or $3^5 6^1$ to obtain 4−GDD as desired. □

Theorem 3.2.83. *A* 4−*GDD of type* $5^u m^1$ *exists if and only if either* $u \equiv 3 \pmod{12}$ *and* $m \equiv 5 \pmod 6$, $5 \leqslant m \leqslant (5u-5)/2$; *or* $u \equiv 9 \pmod{12}$ *and* $m \equiv 2 \pmod 6$, $2 \leqslant m \leqslant (5u-5)/2$; *or* $u \equiv 0 \pmod{12}$ *and* $m \equiv 2 \pmod 3$, $2 \leqslant m \leqslant (5u-8)/2$.

Proof: For $u = 9$, $m = 2$ and $m = 20$, the designs come from Theorem 3.2.28. For $u = 9$, $m = 14$ and for $u = 9$, $m = 8$ refer to Theorem 4.2 in [32].

For any given $u \geqslant 12$, $u \equiv 0, 3, 9 \pmod{12}$ and all admissible $m \geqslant 5$. We take a 4−GDD of type $15^{u/3}(m-5)^1$ coming from Theorem 3.2.82. We adjoin five infinite points and fill the groups with a 4−GDD of type 5^4 to obtain a 4−GDD of type $5^u m^1$ as we want. □

Now we will deal with the case $g = 2$ and prove that necessary conditions for the existence of 4−GDDs of type $2^u m^1$ are also sufficient with at most five possible exceptions.

Lemma 3.2.84. *Let* $u \in \{6, 9, 12, 15, 18\}$. *There exists a* 4−*GDD of type* $2^u m^1$ *for all* $m \equiv 2 \pmod 3$ *with* $2 \leqslant m \leqslant u-1$, *except* $(u, m) = (6, 5)$.

Proof: From Theorem 1.2 in [32] and Theorem 3.2.14, it suffices to consider $(u, m) \in \{(12, 8), (15, 8), (15, 11), (18, 8), (18, 11), (18, 14)\}$. For 4−GDD of types $2^{15}8^1$ and $2^{18}8^1$ refer to Theorem 3.2.67. For 4−GDD of type $2^{12}8^1$ refer to Theorem 2.5 in [33]. Next, we get a 4−GDD of type $2^{18}14^1$ by starting with a 4−GDD of type 12^4 and adjoining two ideal points, and then filling in 4−GDDs of type 2^7 on three of the four groups of size 12 together with the ideal points. Finally for 4−GDDs of types $2^{15}11^1$ and $2^{18}8^1$ refer to Lemma 5.6 in [32]. □

Lemma 3.2.85. *There exists a* 4−*GDD of type* $2^{21}m^1$ *for all* $m \equiv 2 \pmod 3$ *with* $2 \leqslant m \leqslant 20$, *except possibly for* $m = 17$.

Proof: From Theorem 1.2 in [32] and Theorem 3.2.14, it suffices to consider $m \in \{8, 11, 14\}$. For these designs we refer to Lemma 5.7 in [32]. □

Lemma 3.2.86. *Let* $u \geqslant 24$ *and* $u \equiv 0$ $(mod\ 6)$*. There exists a* $4-GDD$ *of type* $2^u m^1$ *for all* $m \equiv 2$ $(mod\ 3)$ *with* $2 \leqslant m \leqslant u-1$*.*

Proof: For any given $u \geqslant 24$ and $u \equiv 0$ $(mod\ 6)$ and all admissible $m \geqslant 2$, we take a 4−GDD of type $12^{u/6}(m-2)^1$ coming from Theorem 3.2.79. We adjoin two points and fill in the groups with a 4−GDD of type 2^7 to obtain a 4−GDD of type $2^u m^1$ as desired. □

After this lemma, we need only to consider the case for $u \equiv 3$ $(mod\ 6)$ and $u \geqslant 27$.

Lemma 3.2.87. *There exists a* $4-GDD$ *of type* $2^u m^1$ *for each* $u \in \{27, 33, 39, 45, 51, 57, 63, 69, 81, 99, 105\}$ *with all* $m \equiv 2$ $(mod\ 3)$ *and* $2 \leqslant m \leqslant u-1$*, except possibly for* $(u, m) \in \{(33, 23), (33, 29), (33, 35)\}$*.*

Lemma 3.2.88 (Lemma 5.10, [32])**.** *There exists a* $4-GDD$ *of type* $2^u m^1$ *for* $(u, m) \in \{(33, 11), (33, 14), (33, 17), (33, 26), (39, 8), (39, 11), (39, 14), (39, 17), (39, 29), (39, 32), (51, 8), (51, 11), (51, 14), (51, 20), (51, 23), (57, 8), (57, 11), (57, 23)\}$*.*

Now, by Theorem 3.2.30 (iv), we only have $u \in U = \{75, 87, 93, 111, 117, 123, 129, 135, 141, 147, 153, 159, 165, 171, 177, 183, 189, 231, 237, 291, 297, 303, 309, 315, 321, 327, 333\}$ to be considered.

Lemma 3.2.89 (Lemma 2.20, [32])**.** *Let* $u \in U$ *then there exists a* $4-GDD$ *of type* $2^u m^1$ *for all* $m \equiv 2$ $(mod\ 3)$ *with* $2 \leqslant m \leqslant u-1$*.*

If we combine Theorem 3.2.30, Lemmas 3.2.84-3.2.89, we have the following result.

Theorem 3.2.90. *There exists a* $4-GDD$ *of type* $2^u m^1$ *for each* $u \geqslant 6$*,* $u \equiv 0$ $(mod\ 3)$ *and* $m \equiv 2$ $(mod\ 3)$ *with* $2 \leqslant m \leqslant u-1$*, except for* $(u, m) = (6, 5)$ *and possibly except* $(u, m) = \{(21, 17), (33, 23), (33, 29), (39, 35), (57, 44)\}$*.*

In this subsection, we have shown that the necessary conditions for the existence of a 4−GDD of type $g^u m^1$ are also sufficient for the small $g = 2, 4, 5, 12, 15$ with 5 possible exceptions when $g = 2$.

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