GENERAL FRAMEWORK
TO PERFORM SENSITIVITY ANALYSIS ON MARKOVIAN QUEUEING AND INVENTORY SYSTEMS

by

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ABSTRACT

This thesis focuses on two well-known areas in operations research: queueing and inventory management. The objective in both of these problems is to compensate the losses caused by the mismatches that arise between supply (service) and demand processes. The class of problems we concentrate on comprises controlled queueing systems and production/ inventory systems modeled by make-to-stock queues. In particular this thesis focuses on three dynamic control policies: pricing control, admission control and stock rationing. As the aim of sensitivity analysis is to understand the effects of changes in system parameters on the optimal solution of a problem, we can understand how systems respond to these changes by sensitivity analysis. Moreover, the insight about the system obtained by the result of sensitivity analysis can be used in the decision procedure. Since there are only a few studies on the effects of the system parameters on the optimal policies and the results of sensitivity analysis are crucial to understand the behavior of the system when parameters change, we aim to establish a general framework to perform sensitivity analysis on the optimal policies of queueing and inventory control problems. Our framework generalizes several existing results on specific models in the literature, and provides a structural methodology to perform sensitivity analysis for any control problem considered here.

ÖZETÇE

Bu tezde, yöneylem araştırmasının önemli konularından kuyruk ve envanter yönetimi üzerinde çalıştık. Her iki problemde de amaç; talep ve arz süreçleri arasında ortaya çıkan uyumsuzluğun neden olduğu kayıpları gidermektir. Üzerinde çalıştığımız problemler kuyruk ve üretim sistemleri denetimini kapsamaktadır. Bu problemlerden özellikle fiyatlama, giriş denetimi ve stok paylama problemleri üzerinde durduk. Duyarlılık analizinin amacı sistem parametrelerinde meydana gelen değişikliklerin en iyi kararlar üzerindeki etkilerini araştırmak olduğu için bir sistemin bu tür değişikliklere verdiği tepkiyi duyarlılık analizi sayesinde anlayabiliriz. Ayrıca, sistem hakkında kazanılan bu öngörü çeşitli karar destek süreçlerinde de kullanılabilir. Duyarlılık analizinin sonuçlarının öneminden ve üzerinde yapılan çalışma sayısı çok az olduğundan, bu tezde kuyruk ve envanter denetimi problemlerinde duyarlılık analizi yapabilmek için genel bir yapı oluşturmayı hedefledik. Bu yapı hali hazırda bulunan sonuçları genellediği gibi üzerinde durduğumuz denetim problemlerinde duyarlılık analizi yapabilmek için yapısal bir metodoloji sunmaktadır.

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NOMENCLATURE

$\mathbb{S}$	Sample Space of the system
A	Set of allowable actions
A_s	Set of allowable actions for state s
$W^t(i,a)$	Immediate reward gained at time t for state i and action a
$w^t(i,a)$	Reward rate at time t for state i and action a
$P^t_{ij}(a)$	Transition probability from state i to j at time t for action a
$F^t_{ij}(T\|a)$	Probability that the transition from state i to j occurs before T at time t for action a
β	Discount (Exponential Failure) rate
x	Current amount of inventory (The number of customers in the system)
$v^n(x)$	Optimal expected total discounted profit of the finite horizon problem with initial state x and n transitions remaining
$v^n_i(x)$	Optimal expected total discounted profit of the finite horizon problem with initial state x and n transitions remaining in Model i
$v(x)$	Optimal expected total discounted profit over infinite horizon with initial state x
$v_i(x)$	Optimal expected total discounted profit over infinite horizon with initial state x in Model i
λ	Arrival rate
λ_i	Demand rate in Model i
μ	Service (Production) rate
μ_i	Service (Production) rate in Model i
θ	Fictitious transition rate
M	Time scale parameter
p	Price

R	Reservation price
$\bar{F}_R(p)$	Probability of buying when price of p is announced
$p_i^*(x)$	Optimal price for the state x in Model i
R_j	Reward obtained by satisfying class-j customers
R_{j_i}	Reward obtained by satisfying class-j customers in Model i
p_{j_B}	Probability that an arriving batch consists of B class-j customers
B	The number of customers in an arriving batch
κ_j	The number of class-j customers to be satisfied from an arriving batch
$\kappa_{j_i}^*(x)$	Optimal number of class-j customers to be admitted from an arriving batch for the state x in Model i
$l_{j_i}^*$	Optimal threshold level for class-j in Model i
π	Used portion of the service rate
c_π	Cost of the used portion of the service rate
τ	Production cost
τ_i	Production cost in Model i
S_i^*	Optimal base-stock level in Model i
$a(x)$	Probability that an arriving customer joins the system for the state x in the arrival operator
$b(x)$	Probability of a service completion for the state x in the departure operator
c	Number of servers
K	Capacity of the system
$h(x)$	Holding cost rate for the state x
$h_i(x)$	Holding cost rate for the state x in Model i
E	State space of the Markov Chain characterizing the environment
e	Current environment state
λ_e	Arrival rate for environment e
λ_{ei}	Arrival rate for environment e in Model i
Q_{ej}	Environment state transition rate from environment e to j

(x, e)	Current amount of inventory(The number of customers in the system) for the environment e
$v_i^n(x, e)$	Optimal expected total discounted profit of the finite horizon problem with initial state (x, e) and n transitions remaining in Model i
$v_i(x, e)$	Optimal expected total discounted profit over infinite horizon with initial state (x, e) in Model i
$p_i^*(x, e)$	Optimal price for the state (x, e) in Model i

Chapter 1

INTRODUCTION

This thesis focuses on two well-known areas in operations research: queueing and inventory management. The objective in both of these problems is to compensate the losses caused by the mismatches that arise between supply (service) and demand processes. To achieve this aim, it may be desirable to employ a control on the production (service) or the demand arrivals. As a result of the rapid evolution of information technologies, applicability of control policies in practical problems increases, and thus control problems in queueing and inventory systems have attracted very much attention in the past few decades. The possibility of collecting valuable information about the demand, inventory level, etc. and processing it in real time allows the decision makers to adjust any variable under control dynamically. Therefore, dynamic control policies which depend on time are mostly employed in queueing and inventory problems rather than static policies which do not depend on time.

The class of problems we concentrate on comprises controlled queueing systems and production/inventory systems modeled by make-to-stock queues. In particular this thesis focuses on three dynamic control policies: pricing control, admission control and stock rationing. We view these policies as basic mechanisms for arrival and service control. In the pricing problem, the decision maker changes the prices according to the number of the customers or the amount of inventory in the system to adjust the load of the system for maximizing his profit. On the other hand, he achieves the same objective by admitting or rejecting the demand in admission control and stock rationing problems. There is a lot of research considering different aspects of these policies including structural results.

In general, the system parameters such as service and arrival rates in production and service systems can not be calculated exactly, and frequently these parameters are either represented by estimators or considered as random variables. In both cases, parameters may change over time. As the aim of sensitivity analysis is to understand the effects of changes in system parameters on the optimal solution of a problem, we can understand how systems respond to these changes by sensitivity analysis. Moreover, the insight about the system obtained by the result of sensitivity analysis can be used in the decision procedure. Sensitivity analysis can of course be performed numerically by solving the underlying control problem several times. However, this is time consuming and requires a lot of effort. Using our results, decision makers can directly find out the impact of changes in problem parameters without resorting to numerical experiments. At the same time, to our knowledge there are only a few and specific studies which concentrate on sensitivity analysis in spite of its importance and value.

Since there are only a few studies on the effects of the system parameters on the optimal policies and the results of sensitivity analysis are crucial to understand the behavior of the system when parameters change, we aim to establish a general framework to perform sensitivity analysis on the optimal policies of queueing and inventory control problems in this thesis.

In order to observe the effects of the parameters on the optimal decisions, the models that we consider have to possess some structural properties (e.g. monotonicity, concavity). In other words, we need to show the structure of the models before performing sensitivity on the optimal policies of these models. Puterman [42], Smith and McCradle [44], Veatch and Wein [50], Koole [26] provide frameworks that can be used to determine the structure of optimal stochastic dynamic policies. We use the event-based dynamic programming technique introduced by Koole [26] within these frameworks. In this technique, Koole establishes that if certain event operators have some structural properties (monotonicity, concavity, convexity) under given assumptions, then the value function of the system constructed by using these operators also has the same structural properties. Our main contribution is to extend this approach to observe the effects of the parameters on the optimal decisions. Since one have to study the structure of a model before its sensitivity analysis, our investigation not only introduces a general framework to perform sensitivity analysis but also examines the structure of the models and the optimal policies.

Even though our modeling approach is fairly general, there are certain limitations. For instance, the state space of the underlying control problem has to be one dimensional and the approach only extend to multi-dimensional structures under restrictive conditions. To gain a better understanding of these limitations we also present a multi-dimensional example in detail and explain the challenges.

The rest of the thesis is organized as follows. In Chapter 2, we provide the necessary background and literature review on the queueing and inventory control problems.

In Chapter 3, we introduce the basic definitions of Markov Decision Processes and present the methods used to obtain the analytical results in this thesis, e.g. event-based dynamic programming technique, on a simplified stock rationing and replenishment problem.

In Chapter 4, we first define certain event operators which represent the events occurred in Markovian stochastic control problems. Then, we work on whether these operators preserve some structural properties of a function on which they are applied or not in Chapter 5. By this way, we also show that the models constructed by our operators keep the same structural properties that the operators preserve. After that, we observe the behavior of the operators when the system parameters change in order to use the results while investigating the effects of system parameters on the optimal decisions.

In Chapter 6, we illustrate our framework by performing sensitivity analysis on the existing and structurally characterized queuing and inventory control problems: dynamic pricing, admission control and stock rationing problems. As a pricing problem, we perform sensitivity analysis on the joint dynamic pricing and replenishment problem studied by Gayon et. al. [20]. Then, we study the admission control problem introduced by Örmeci and Burnetas [37]. The last problem that we work on is the stock rationing and replenishment problem presented by Ha [22]. We not only perform sensitivity analysis on the model but also extend it by considering batch arrivals. By these examples, we also show that our results can be used in many real-life applications where stochastic control policies are employed. Rental businesses, call centers, job-shop manufacturing systems, make-to-stock production systems, airlines, hotels, tolls in high-ways are some of the example systems in which our results can be used.

In Chapter 7, we consider a two-dimensional model as a natural extension of our study. We determine the structure of the optimal policy of the model but due to the limitations of our approach, we can not perform a sensitivity analysis by using our framework. This highlights some of the difficulties that may be encountered to extend our results to general multi-dimensional state spaces.

In the last chapter, we shortly summarize the performed study, and mention the future research perspectives.

Chapter 2

LITERATURE REVIEW

2.1 Introduction

As we mentioned in the previous chapter, determining the structure of the optimal policy is necessary before performing sensitivity analysis. Therefore, we first focus on the paper providing general framework to establish the structure of the corresponding value functions and optimal policies. Puterman [42] considers a general methodology to prove certain structural properties of the optimal policy of an infinite horizon Markov Decision Process. In his methodology, he provides some conditions which ensure the optimality of monotone policies. Smith and McCardle [44] work on stochastic dynamic programs and presents several fundamental results to prove the existence of monotonicity and concavity of the value functions of the models considered. They show that the value functions satisfy property P if the reward function satisfy the property P and the transition probabilities satisfies the probabilistic version of this property. Veatch and Wein [50] considers a general queueing control problem. They present some monotonicity theorems, and then by using these theorems they show the optimality of transition-monotone policies. In a similar paper, Koole [26] not only provides a general framework to prove the structural properties of value functions and optimal policies but also proposes to define distinct event operators and examine the structural properties of the operators rather than the whole value function because he shows that if the event operators preserve some structural properties then the value function which is the combination of these operators will also satisfy the same properties.

After the papers on the structural analysis, we now review the limited number of papers which concentrate on the effects of the changes in the parameters on the optimal policies.

One of the papers including a complete sensitivity analysis is Ku and Jordan [27]. In this study, they consider an admission control problem of two multi-server loss queues in series with two different demand classes. They show that under appropriate conditions the optimal admission policy maximizing the expected total discounted reward over an infinite horizon is given by a switching curve. They also characterize the form and shape of this curve and its variation according to system parameters.

Ziya et. al [54] study the static pricing problem of a single-server queueing system with finite waiting room capacity. They observe the effects of the customer willingness to pay, the system parameters (service and arrival rates), and the waiting room capacity on the optimal static pricing policies. Although it contains a complete sensitivity analysis, this study is not directly in our scope because we only focus on dynamic control policies in our study.

Another extensive sensitivity analysis is performed by Gans and Savin [19] for a joint admission and dynamic pricing problem of a multi-server loss system. In this paper, they not only establish the structure of the optimal policies maximizing the expected long-run average reward but also investigate the effects of the parameters on these policies. The problem in this study can be modeled by using the operators that we introduce in the subsequent chapters, and the results of the original model and our redefined model coincide. Moreover, in the redefined model, one can additionally observe the effects of the number of servers on the optimal policies. Hence, [19] is a specific case of our general framework.

The last papers that we review as a sensitivity analysis are Aktaran-Kalaycı and Ayhan [1] and Çil et.al. [7]. Both of the studies consider a dynamic pricing problem of a multi-server queueing system with infinite waiting room capacity and examine the effects of the parameters on the optimal prices. The models in these two independent studies are the same, and the only slight difference between them is that the objective is maximizing the expected total discounted and long-run average profit in [7] whereas the objective is only maximizing the expected long-run average profit in [1]. As [19], [1] and [7] can be modeled by using our operators, and thus they are also specific cases of our general framework.

In the remaining part of the literature review, we focus on pricing control, admission control and rationing policies which are studied in the literature. Pricing, admission and rationing problems have been studied since 1960s, and for comprehensive reviews, we refer to Stidham [47], Altman [2], Bitran and Caldentey [5], Yano and Gilbert [53], Chan et. al. [9], and Elmaghraby and Keskinocak [15] as a summary of the research papers on these control problems which covers different aspects of the problems. The main difference of the following research and our thesis is that none of them consider a complete sensitivity analysis (both theoretical and numerical). Moreover, some of them can be modeled by using our operators, and therefore they are specific cases of our framework.

2.2 Pricing Control Problem

The pricing control problem has been studied by many researchers in both queueing and inventory literature. In the queueing context, Naor [36] is the first to investigate the pricing problem by presenting quantitative arguments based on a single server queueing system with finite waiting room capacity. Then, Knudsen [25] extends Naor's study to multiserver queuing systems. The major difference between these two studies and our framework is that the objective of the problem is to find the optimal static pricing policy in both of these studies.

On the other hand, there is also some research on the dynamic pricing problem of queueing systems. Low [31] first focuses on the dynamic pricing problem of a multiserver queue with finite waiting room capacity. In this study, he proves the monotonicity of the optimal prices in the state of the system. He then extends his results to multiserver queue with infinite waiting room capacity [32]. Paschalidis and Tsitsiklis [41] consider the pricing problem of a service provider, which provides access to communication network, by modeling the problem as a dynamic pricing problem of multiserver loss system with N different customers. As an important result, they show the monotonicity of the optimal prices in the number of customers in the system. Another dynamic pricing problem is introduced by Chen and Frank [10]. They consider a queuing system where a monopolist can see the length of the queue and charges a fee depending on the number of customers in the system. They prove the concavity of the expected total discounted reward and establish the existence of monotone optimal prices. Moreover, they investigate the effects of the arrival rate on the optimal prices through a numerical example. All of the above studies are specific cases of our general framework because they can be redefined by using our operators. Therefore, it is possible to perform sensitivity analysis on their optimal policies as we will illustrate in the subsequent chapters.

The pricing control problem is also an attractive topic in the inventory control literature. Federgruen and Heching [16], Thomas [48], Chen and Simchi-Levi [11], and Feng and Chen [17] examine the pricing control problem by considering various aspects of the problem (periodic reviews, continuous reviews, set-up costs) and present the characteristics of the optimal policies of the corresponding models. However, only a few of these papers explicitly model production capacity constraints that induce endogenous and random lead times. In the periodic review case, Federgruen and Heching [16] discuss these extensions to their basic model but their main focus is on the uncapacitated zero-lead time case.

Li [29] seems to be the first paper whose focus is a production/inventory system with joint pricing and replenishment considerations. He considers a continuous review model with price-sensitive demand where the cumulative production and demand are modeled by Poisson processes with controllable intensities. The demand is a continuous function of the price. When there is demand in excess, sales are lost. The production and holding costs are linear and the demand intensity is controlled through the price. When the prices are set dynamically over time, Li shows that a base-stock policy is optimal. Moreover, it is shown that the optimal price is a non-increasing function of the inventory level. In a recent study, Gayon et. al. [20] extend [29] by considering the demand arrivals as a Markov Modulated Poisson process. As we focus on the joint dynamic pricing and replenishment problem in our general framework, [29] and [20] can be modeled by using our operators. Actually, [20] is one of the example models that we use to illustrate our general framework.

2.3 Admission Control Problem

Buffer capacity control in production and service systems addresses optimal allocation of fixed buffer resources to different demand segments. Since this objective can be achieved by admission control policies which determine when to accept or reject different segments, this class of control problems has received a lot of attention in the queueing literature.

Most of the earlier studies especially focus on systems where customers arrive individually. Altman et. al. [3], Örmeci et. al. [39] and Savin et. al. [43] work on an admission control problem in a loss system receiving single arrivals from two classes of jobs which demand exponential services with different rates. Altman et. al. [3] show that the optimal policies are of threshold type. Örmeci et. al. [39] and Savin et. al. [43] analyze the issue of the preferred class. Örmeci et. al. [39] also establish the monotonicity of the thresholds under certain conditions. Furthermore, Örmeci and van der Wal [40] extend several of these results to systems with renewal arrivals as opposed to Poisson arrivals; in particular the existence of optimal thresholds, and sufficient conditions to have preferred class(es). Besides the studies on the loss system, a number of studies analyze admission control in queues with infinite waiting room receiving single arrivals. Ghoneim and Stidham [21], Stidham [45] and [46], Blanc et. al. [6] investigate the structure of the optimal decisions in queueing systems with infinite waiting room capacity and prove that the optimal admission policies are of threshold type when the customers arrive individually.

Admission control problems in queueing systems receiving batch arrivals have also been investigated in the literature as a natural extension of the single arrivals. Moreover, considering batch arrivals allows to study the system when customers may request more than one resources. In models with batch arrivals, either a partial acceptance policy in which some of the jobs in a batch can be admitted while the rest are rejected or a batch acceptance policy where the system can either admit or reject the whole batch is employed as the control policy.

The partial acceptance policy has been studied more than the batch acceptance policy both in queueing systems with finite and infinite waiting room capacity since the policy has a well-behaved structure and thus it is possible to prove some structural results such as concavity and the existence of thresholds. As an instance on systems with finite capacity, Örmeci and Burnetas [37] consider a partial acceptance problem of a Markovian multi-server loss system receiving jobs which arrive in different batch sizes, bring different rewards, and demand exponential services with the same rate. They establish the existence of an optimal sequential threshold policy with monotone thresholds. Then, Örmeci and Burnetas [38] analyze partial acceptance policies in a multi-server loss system with two classes of customers whose batch sizes, service times and rewards are different. There are also some studies on the partial acceptance problem of infinite capacity queues. In [45], Stidham considers the batch arrivals in addition to the single arrival case. He examines the structure of the partial acceptance policy in a $GI/M/1$ queue and proves that the optimal policy is still threshold policy after assuming the batch arrivals. Similar to this study, Langen [28] and Helm and Waldmann [24] observe the partial acceptance policy in a $GI/M/c$ queue and shows the existence of monotone optimal decision rules. For the batch acceptance policy, Çil et. al. [8] perform a complete study on the problem, where they show that the optimal policy keeps some structural properties only under some specific conditions.

As we mentioned before, in order to perform sensitivity analysis using our framework, the optimal policy needs to have a structure. Therefore, in the scope of this thesis, we only work on the partial acceptance policy and take [37] as an example for admission control problems to illustrate our framework.

2.4 Stock Rationing Problem

The rationing problem appears when different customer classes whose economic importance varies have to be satisfied from a common stock. Since demand classes have different values, it may be more important to satisfy one class than the others in anticipation of future demand from the more valuable classes. The basic stock rationing problem has been studied widely in the inventory literature. Topkis [49] studies optimal ordering and rationing policies for a single-product inventory system with several demand classes. He shows that a base-stock policy is optimal for ordering and the optimal rationing policy reserves items in stock for future demands of more valuable customers. Nahmias and Demmy [35] observe the cost improvements due to stock rationing for inventory systems with two demand classes. Cohen et. al. [12] and Frank et. al. [18] consider the rationing problem for two classes of customers where one of the class has high priority and show the complexity of obtaining the structure of optimal rationing policy.

In all of the above papers, one of the important characteristics of the problem, the limited capacity of the system, is not considered. However, there is also some research that models the limited production capacity. Ha [22], [23], and de Vèricourt et. al. [14] examine the inventory systems with limited production capacity by employing queueing based models. Ha [22] considers the stock rationing problem in a make-to-stock production system with several demand classes and lost sales and characterizes the optimal policy. After that, Ha [23] studies the rationing problem in a make-to-stock system with two demand classes where unsatisfied demand are back-ordered. In this study, Ha can not obtain a general structure for the optimal policy. However, de Vèricourt et. al. [14] complete the characterization of the optimal policy for the two demand class rationing problem and extend the model in [23] by considering multiple-demand classes. In the scope of this thesis, we concentrate on the stock rationing problem with limited capacity and lost sales. Thus, we study the model introduced in [22] as an example rationing problem.

Chapter 3

MARKOV DECISION PROCESSES

3.1 Introduction

In this chapter, we present the approach we will use in subsequent chapters to construct and analyze Markov Decision Process optimality equations for certain systems with pricing, admission, rationing and replenishment decisions. Markov Decision Processes (MDP) are used to model the systems where decisions are made sequentially under uncertainty, and the MDP models provide the opportunity to compare the immediate gain of current decisions and the possible outcomes of future decision making opportunities [42].

The modern study of stochastic sequential decision problems begins with Wald's work on sequential statistical problems during the Second World War. He later published his studies in his book [51]. Also, Pierre Massé, minister in charge of French electrical planning, introduced many of the basic concepts in his analysis of water resource management models (1946). Many investigators studied sequential problems after the works of these two pioneers, and Bellman introduced the common ingredients to these problems as states, actions, transition probabilities, and developed the fundamental equations to determine the optimal policies [4]. He is considered as the first one to develop the mathematical foundations of dynamic programming.

3.2 Markov Decision Problems

In this section, we introduce the basic properties of the Markovian Decision problems that we considered in the scope of this thesis. In Markov Decision problems, the decision maker has to choose an action to influence the future performance of the system at some time points which are called *decision epochs*. The set of decision epochs can be either discrete or continuous. If it is discrete, then the decisions are made at each decision epoch; otherwise the decisions could be made at random points in time when certain events occur, at the time chosen by the decision maker, or continuously. In the problems discussed in this thesis, the decisions are made at random points of time when certain events as arrivals, replenishments, service completions or environment state changes occur. Therefore, the Markov Decision problems that we analyzed can be described by the following five elements:

1. $\mathbb{S}$ is the set of all possible states of the system.

2. A is the set of allowable actions a decision maker can choose at each decision epoch. $A = \cup_{s \in \mathbb{S}} A_s$ where A_s is the set of allowable actions for each state s. It is assumed that $\mathbb{S}$ and A do not vary with time.

3. $C^t(i,a)$ ($W^t(i,a)$) is the immediate cost incurred (reward received) and $c^t(i,a)$ ($w^t(i,a)$) is the cost (reward) rate imposed from time t until the next transition occurs when action a is chosen at state i at time t. That is, if a transition occurs after T units, then the total cost incurred is given by $C^t(i,a) + Tc^t(i,a)$. $C^t(i,a)$ ($W^t(i,a)$) and $c^t(i,a)$ ($w^t(i,a)$) together constitute the cost (reward) structure of the model.

4. $P_{ij^t(a)}$ is the probability the next state will be j when the initial state is i and action a is chosen at time t.

5. $F^t_{ij}(T|a)$ is the probability that the transition from i to j will occur before T time units when action a is chosen at time t.

In the problems considered here, it is assumed that the costs (rewards), transition probability and transition time distributions are independent of time. Therefore, we drop the superscript "t" hereafter. The performance criterion for optimality can be either the discounted cost (reward) over an infinite horizon or the long-run average cost (reward) criterion. The objective of the Markov Decision problem may be minimizing the expected total discounted cost over an infinite horizon (maximize the expected total discounted reward); or minimizing the expected long-run average cost (maximize the expected long-run average reward). In all of the problems studied in this thesis, the objective is to maximize the profit. Thus, we will focus on maximization problems in all of the MDP formulations presented.

If the discounted reward criterion is used, then the optimal expected total discounted reward with initial state i and discount rate β ($\beta > 0$) is denoted by $v_\beta(i)$ and is presented as below:

$$v_\beta(i) = \max_{a \in A_i} \{\bar{W}_\beta(i,a) + \sum_{j=0}^{\infty} P_{ij}(a) \int_0^\infty e^{-\beta T} v_\beta(j) dF_{ij}(T|a)\}, \tag{3.1}$$

where

$$\bar{W}_\beta(i,a) = W(i,a) + \sum_{j=0}^{\infty} P_{ij}(a) \int_0^\infty \int_0^T e^{-\beta s} w(i,a) ds dF_{ij}(T|a).$$

When $F_{ij}(.|a)$ are exponential for all i, j, a, and A, $\mathbb{S}$ and the reward structure satisfy certain conditions, it is well-known that there exists an optimal Markovian deterministic policy which depends on the previous system states and actions through the current state of the system (memoryless) and chooses an action with certainty (See Theorems 5.5.1, 8.1.2, 8.4.2, and Proposition 6.2.1 in Puterman). The problems we will consider in this thesis satisfy these conditions, so we restrict ourselves to the set of Markovian deterministic policies.

3.3 The Example Model

We will now formulate an example model to motivate the concepts of Markov decision theory and illustrate our approach event-based dynamic programming. As an example model, we consider a make-to-stock production system with Poisson arrivals and exponential production times in which back-order is not allowed. We let the arrival rate be λ and the production rate be μ. Items are produced one by one where the inventory capacity is limited to K and the unit production cost is τ. Then, we assume that at any time, the decision maker has to decide whether or not to satisfy an arriving demand (rationing decision) and whether or not to produce (production decision) if the capacity is available. A reward of R

is obtained if an arriving customer is satisfied and the customer is lost otherwise. Moreover, h is the holding cost per unit per unit time.

We define the current state as $X(t)$, the amount of inventory at time t, such that $0 \leq X(t) \leq K$. Due to the exponential transition times, it is clear that observing only the current state is sufficient so that we do not need to keep the historical information of the process. Therefore, we simply denote the current state of the system as x without any reference to the time.

Let $a = (a_1, a_2)$ be the action taken at the state x such that a_1 is the production decision (0:not produce, 1:produce) and a_2 is the rationing decision (0:not satisfy, 1:satisfy). Then, if the action is $(0, 0)$, the only possible event is that a demand arrival occurs and the transition time will be exponential with rate λ. Since the rationing decision is not to satisfy the demand, the next state will be (x), and no reward will be obtained. Similarly, if the action is $(0, 1)$, the only possible event is that a demand arrival occurs and the transition time will be exponential with rate λ but the next state will be $(x - 1)$ and a reward of R is obtained as the rationing decision is to satisfy the demand. On the other hand, if the action is $(1, 0)$, the transition time will be exponential with rate $(\lambda + \mu)$ and there will be two possible events: replenishment and an arrival with respective transition probabilities $[\mu/(\lambda+\mu)]$ and $[\lambda/(\lambda+\mu)]$. If a replenishment occurs before an arrival occurs, the next state will be $(x+1)$ and production cost τ will be incurred. Otherwise, the next state will be (x) and no reward will be obtained as a result of rationing decision. Likewise, the next state will be $(x + 1)$ and production cost τ will be incurred when a replenishment occurs first if the action is $(1, 1)$ but the next state will be $(x - 1)$ and a reward of R will be obtained when an arrival occurs first.

From now on, the discount rate will be fixed and taken as β, and we drop the subscript denoting the discount rate hereafter. Thus, the optimal expected total discounted profit with initial inventory amount x, $v(x)$, can be represented as:

$$v(x) = max\{A_1(x), A_2(x), A_3(x), A_4(x)\}, \tag{3.2}$$

where,

$$
\begin{aligned}
A_1(x) &= -\int_0^\infty \int_0^t e^{-\beta s} hxds\lambda e^{-\lambda t}dt + \int_0^\infty e^{-\beta t} v(x)\lambda e^{-\lambda t}dt, \\
A_2(x) &= -\int_0^\infty \int_0^t e^{-\beta s} hxds\lambda e^{-\lambda t}dt + \int_0^\infty e^{-\beta t} [R + v(x-1)]\lambda e^{-\lambda t}dt, \\
A_3(x) &= -\int_0^\infty \int_0^t e^{-\beta s} hxds(\lambda+\mu) e^{-(\lambda+\mu)t}dt \\
&\quad + \frac{\mu}{\lambda+\mu}\int_0^\infty e^{-\beta t}[-\tau + v(x+1)](\lambda+\mu)e^{-(\lambda+\mu)t}dt \\
&\quad + \frac{\lambda}{\lambda+\mu}\int_0^\infty e^{-\beta t} v(x)(\lambda+\mu)e^{-(\lambda+\mu)t}dt, \\
A_4(x) &= -\int_0^\infty \int_0^t e^{-\beta s} hxds(\lambda+\mu) e^{-(\lambda+\mu)t}dt \\
&\quad + \frac{\mu}{\lambda+\mu}\int_0^\infty e^{-\beta t}[-\tau + v(x+1)](\lambda+\mu)e^{-(\lambda+\mu)t}dt \\
&\quad + \frac{\lambda}{\lambda+\mu}\int_0^\infty e^{-\beta t}[R + v(x-1)](\lambda+\mu)e^{-(\lambda+\mu)t}dt.
\end{aligned}
$$

When we compute the integrals, we obtain that,

$$v(x) = max\{A_1(x), A_2(x), A_3(x), A_4(x)\},$$

where,

$$
\begin{aligned}
A_1(x) &= \left[\frac{-hx}{\beta+\lambda} + \frac{\lambda}{\beta+\lambda}v(x)\right], \\
A_2(x) &= \left[\frac{\lambda R - hx}{\beta+\lambda} + \frac{\lambda}{\beta+\lambda}v(x-1)\right], \\
A_3(x) &= \left[\frac{-\mu\tau - hx}{\beta+\lambda+\mu} + \frac{\mu}{\beta+\lambda+\mu}v(x+1) + \frac{\lambda}{\beta+\lambda+\mu}v(x)\right], \\
A_4(x) &= \left[\frac{\lambda R - \mu\tau - hx}{\beta+\lambda+\mu} + \frac{\mu}{\beta+\lambda+\mu}v(x+1) + \frac{\lambda}{\beta+\lambda+\mu}v(x-1)\right].
\end{aligned}
$$

The discount rate β will be considered as the exponential failure rate where P [failure by time t]=$1 - e^{-\beta t}$, and P [not failure by time t]=$e^{-\beta t}$. Thus, if the production decision is not to produce, $a = (0,0)$ or $a = (0,1)$, the expected transition time will be $[1/(\beta+\lambda)]$ unit times and the probability that an arrival occurs before the exponential failure will be $[\lambda/(\beta+\lambda)]$. Then, the expected profit will be $[-hx]$ and $[\lambda R - hx]$ times the expected transition time if the action is $(0,0)$ and $(0,1)$ respectively. Here, λR is the revenue rate and hx is the holding cost rate. On the other hand, if the production decision is to produce, $a = (1,0)$ or $a = (1,1)$, the expected transition time will be $[1/(\beta + \lambda + \mu)]$ unit times and the probability that an arrival, replenishment or exponential failure occurs first will be $[\lambda/(\beta + \lambda + \mu)]$, $[\mu/(\beta + \lambda + \mu)]$, and $[\beta/(\beta + \lambda + \mu)]$, respectively. Moreover, the expected profit will be $[-\mu\tau - hx]$ times the expected transition time if the action is $(1,0)$ where $\mu\tau$ is the production cost rate, and it will be $[\lambda R - \mu\tau - hx]$ if the action is $(1,1)$.

3.4 Methods of Uniformization and Value Iteration

Uniformization and value iteration are commonly used tools to determine optimal policies in queueing and inventory control problems. These methods can also be used to establish the structure of optimal policies of the models constructed by using the operators we introduced and the effects of parameters on these policies. Therefore, we present the methods of uniformization and value iteration on the example model described above in this section.

As our example model evolves as a continuous time Markov chain and it is difficult to track the revenues in such a system, we prefer to track the expected revenues in a discrete time equivalent of the example model. To achieve this aim, we uniformize the system by ensuring that the long-run average of the revenues and costs earned at discrete event epochs equals to the long-run average of revenues and costs that would have earned in continuous time (Lippman,1975). Then, instead of analyzing the optimality equation in the previous section, we analyze the discrete time equivalent of the system by assuming that the time between two consecutive transitions is exponentially distributed with rate $\lambda + \mu + \theta \leq 1$, and we rescale the time by taking $\beta + \lambda + \mu + \theta = 1$. θ is the fictitious event (an event that does not change the state of the system) rate and it is introduced to ensure that time scale is not affected by the changes in the parameters during sensitivity analysis. To illustrate, when we observe the effects of the arrival rate, λ, if we increase (decrease) λ, then we will decrease (increase) θ, and we ensure that time scale remains the same. After applying the uniformization method on the example model, the transition probabilities for the state x depending on the action are:

$a = (a1, a2)$	$a = (0,0)$	$a = (0,1)$	$a = (1,0)$	$a = (1,1)$
$P_{x,x}$	$\mu + \lambda + \theta$	$\mu + \theta$	$\lambda + \theta$	θ
$P_{x,x+1}$	0	0	μ	μ
$P_{x,x-1}$	0	λ	0	λ
$P_{failure}$	β	β	β	β

As we mentioned before, we will determine the structure of the optimal policy of a system which operates over an infinite horizon. For this purpose, we first prove the structure of the optimal policy with the objective of maximizing the expected total β-discounted reward for a finite number of transitions, n. The finite horizon problems allow us to use induction to prove the structural properties for all finite n. We denote the maximum expected total β-discounted reward of a system starting in state x when n transitions remain by $v_n(x)$ conferring a terminal reward of v_0, and we specify the terminal reward v_0 as $v_0(x) = 0$ for all states x. Then, we present the optimality equation of the finite horizon problem as:

$$v_{n+1}(x) = \begin{cases} \mu \max\{-\tau + v_n(x+1), v_n(x)\} + (\lambda + \theta)v_n(x) + \beta 0 - hx & \text{if } x = 0 \\ \mu \max\{-\tau + v_n(x+1), v_n(x)\} & \\ +\lambda \max\{R + v_n(x-1), v_n(x)\} + \theta v_n(x) + \beta 0 - hx & \text{if } 0 < x < K \\ (\mu + \theta)v_n(x) + \lambda \max\{R + v_n(x-1), v_n(x)\} + \beta 0 - hx & \text{if } x = K \end{cases} \tag{3.3}$$

We use the method of value iteration to relate the finite horizon problem with the infinite horizon problems with the objective of maximizing the discounted and the long-run average profits. This method is widely used when the Markov Decision problem considered satisfies the assumptions below:

1. Rewards, transition probability and transition time distributions are stationary.

2. Rewards are bounded: $|\bar{W}(i,a)| \leq M < \infty$ for all $a \in A_i$ and $i \in \mathbb{S}$.

3. Future rewards are discounted according to a discount factor α, with $0 \leq \alpha < 1$.

4. $\mathbb{S}$ is discrete.

5. A_i is finite for each $i \in \mathbb{S}$, or A_i is compact, $\bar{W}(i,a)$ is continuous in a for each $i \in \mathbb{S}$, and for each $i, j \in \mathbb{S}$, $P_{ij}(a)$ is continuous in a.

Consider the real-valued functions $\mathbf{L}$ defined on $\mathbb{S}$. The following theorem constructs the basis of the method of value iteration. (Proof of Theorem and Corollary could be found in Chapter 6 of Puterman).

Theorem 1 *If the above assumptions are satisfied, then there exists an optimal deterministic policy d such that v^* is the unique solution to the equation $v = \max_{d\in\mathbf{D}}\{r_d + P_d v\} = Lv$ where L is an operator in $\mathbf{L}$ and $\mathbf{D}$ denotes the set of Markovian Decision rules.*

Corollary 1 *If $v_0 \in \mathbf{L}$, v_n defined by equation $v_{n+1} = Lv_n$ converges to v^* such that $v_{n+1} = \sup\{\bar{W}(i,a) + \sum_{j\in\mathbb{S}} P_{ij}(a)v_n(j)\}$ converges to $v^*(i)$ for all $i \in \mathbb{S}$.*

In our example model, the state space is finite for every value of n, the action space is finite ($a = (0,0)$, $a = (0,1)$, $a = (1,0)$ or $a = (1,1)$) and rewards are bounded. Therefore, there exists an optimal deterministic policy for the finite horizon problem by Theorem 1 and $v_n(x)$ converges to $v(x)$, the maximum expected total discounted profit for the infinite horizon problem, by Corollary 1, i.e. $v(x) = \lim_{n\to\infty} v_n(x)$. Hence, the structural results obtained for the finite horizon problem also apply for the infinite horizon discounted and long-run average problems. Moreover, we present the optimality equation for the long-run average case as follows, where g^* is the optimal average gain and $v'(x)$ is the relative value function.

$$v'(x) = \mu\max\{-\tau + v'(x+1), v'(x)\} + \lambda\max\{R + v'(x-1), v'(x)\} + \theta v'(x) - hx$$

3.5 The Optimal Policy and Event-Based Dynamic Programming

In the inventory control problems, concavity of the value function, $v(x)$, implies the optimality of the base-stock policy where it is optimal to produce if the amount of inventory on-hand is lower than a base-stock level and not to produce otherwise. To illustrate the relationship between the concavity and the base-stock policy, we assume that $v(x)$ is concave in x and it is optimal to produce at the state $x+1$ whereas the optimal action is not to produce at the state x, i.e. the optimal replenishment policy is not a base-stock policy. Then we have that:

$$v(x) \geq v(x+1) - \tau,$$

$$v(x+2) - \tau \geq v(x+1).$$

When we combine these equations, we obtain that $v(x) - v(x+1) \geq v(x+1) - v(x+2)$, and it obviously contradicts with the concavity of $v(x)$. Therefore, our assumption is not correct, and if the value function, $v(x)$, is concave in x, then the optimal replenishment policy will be a base-stock policy. Hence, we focus on the concavity of the value function in this section. We prove the concavity of $v(x)$ by using the event-based dynamic programming technique to illustrate how the technique can be used.

In order to prove the concavity, we first work on the finite horizon problem and show that $v_n(x)$ is concave in x for all finite n by induction. The initial condition of the induction (for $n = 0$) is obviously true since we specify the terminal reward v_0 as $v_0(x) = 0$ for all states x. Then, we need to show that $v_{n+1}(x)$ is concave in x if $v_n(x)$ is concave. To simplify the notation, we define $\Delta v(x) = v(x) - v(x+1)$, and thus we have to prove $\Delta v_{n+1}(x) \leq \Delta v_{n+1}(x+1)$, i.e. the following inequality:

$$\begin{bmatrix} \mu \max\{-\tau + \Delta v_n(x+1), \Delta v_n(x)\} \\ +\lambda \max\{R + \Delta v_n(x-1), \Delta v_n(x)\} \\ +\theta \Delta v_n(x) + h \end{bmatrix} \leq \begin{bmatrix} \mu \max\{-\tau + \Delta v_n(x+2), \Delta v_n(x+1)\} \\ +\lambda \max\{R + \Delta v_n(x), \Delta v_n(x+1)\} \\ +\theta \Delta v_n(x+1) + h \end{bmatrix} \tag{3.4}$$

In the traditional approach, we have to consider all of the possible action cases ($a = (0, 0)$, $a = (0, 1)$, $a = (1, 0)$ or $a = (1, 1)$) and write the above equation for each case. Hopefully, there are only 2 production and rationing decisions but in more complicated cases there may be more than 2 cases for each decision. For example, if there are 2 production and 5 rationing decisions then the total number of possible actions will be 10. However, production and rationing decisions are independent. Therefore, we can work on each maximization separately and in this case we have to deal with 7 possible cases. Using this idea, Koole [26] propose to divide the whole value function into distinct event operators and work on these operators rather than working on the whole function.

Now, we demonstrate how one can use the event-based dynamic programming technique by redefining the value function of the example model as the combination of certain event operators. After introducing the appropriate event operators, the value function is:

$$v_{n+1}(x) = T_{COST}(\mu T_{PRD} v_n(x) + \lambda T_{RAT} v_n(x) + \theta T_{FIC} v_n(x))$$

where,

$$T_{PRD}v(x) = \max\{v(x), v(x+1) - \tau\},$$

$$T_{RAT}v(x) = \begin{cases} \max\{v(x), v(x-1) + R\} & \text{if } x > 0 \\ v(0) & \text{if } x = 0, \end{cases}$$

$$T_{FIC}v(x) = v(x)$$

$$T_{COST}v(x) = v(x) - hx.$$

Then, Equation 3.4 becomes:

$$\begin{bmatrix} \mu\Delta T_{PRD} v_n(x)\} \\ +\lambda\Delta T_{RAT} v_n(x)\} \\ +\theta\Delta T_{FIC} v_n(x) \\ +h \end{bmatrix} \leq \begin{bmatrix} \mu\Delta T_{PRD} v_n(x+1)\} \\ +\lambda\Delta T_{RAT} v_n(x+1)\} \\ +\theta\Delta T_{FIC} v_n(x+1) \\ +h \end{bmatrix}.$$

At this point, we assume that all of the operators, T_{PRD}, T_{RAT}, T_{COST}, preserve the concavity of $v(x)$, i.e. if $v(x)$ is concave in x, then $Tv(x)$ will also be concave in x for all operators. By using this assumption, it is obvious that the first three line of the above inequality is true, and the last line is also true since both the left and right hand sides are equal. Thus, Equation 3.4 holds, and $v_n(x)$ is concave in x for all finite n. Furthermore, $v(x)$ is also concave in x because $v_n(x)$ converges to $v(x)$ as we mentioned before. As a direct implication of the concavity, the optimal replenishment policy is the base-stock policy and it is optimal to satisfy the demand always when it is possible.

As in this example, it will be easy to prove that the value function of a model has some properties, say $Prop_1, \ldots, Prop_k$, if all of the operators which builds the model preserve the same properties, $Prop_1, \ldots, Prop_k$, of the function on which they are applied. Hence, the most important issues are to introduce the proper event operators and prove that they preserve certain properties. Therefore, we define some event operators to model the events in Markovian systems and examine their structures in the subsequent chapter.

Chapter 4

THE OPERATORS

In this chapter, we define some event operators to model certain events in Markovian queueing and inventory systems. By using these operators, various models can be constructed and then the structure of the control policies such as dynamic pricing, admission control and stock rationing can be investigated.

In the operator definition, we consider $v(x)$ as the value function of the models and x as the state of the system (the number of customer in the queue or the amount of inventory). We define the following operators for one-dimensional systems in which the state space is unbounded, i.e. systems with infinite capacity, and the objective is to maximize the profit. However, the operators can be used for systems with finite capacity by specifying that the operator does not change the state of the system when the system reaches its capacity, i.e. $Tv(x) = v(x)$ if $x = K$ where K is the capacity of the system. Moreover, the operators can also be used for Markov Modulated models although these models may consist of two or more states because the environment state in Markov Modulated models is exogenous and we do not have any control on this state. To illustrate the usage in Markov Modulated models, let a certain operator T be defined as $Tv(x) = v(x + 1)$, then this operator can be used in a Markov Modulated model by redefining the operator as $Tv(x, e) = v(x+1, e)$ where e is the state of the exogenous environment. Our focus in this thesis is on the properties of the operators and the value functions, not on the existence of the optimal policies and the validity of the value iteration. In the case of a finite state and action spaces, existence and validity are guaranteed for all models (See Puterman [42]). For the problems with infinite state space, it is necessary to check the existence of the optimal policy and the validity of the value iteration before using our framework.

The operators that we consider are:

The *cost operator* represents the systems incurring a holding cost, $h(x)$, which is a function of the state of the system:

$$T_{COST}v(x) = v(x) + h(x).$$

The *arrival operator* represents the arrival of a new customer to the system, where the arrival rate may depend on the state of the system. We denote the probability that an arriving customer joins the system when there are x customers by the function $a(x)$, which is non-increasing and convex in x. Then:

$$T_{ARR}v(x) = a(x)v(x+1) + [1 - a(x)]v(x).$$

The *departure operator* represents the departure of an existing customer from the system, where the service rate may depend on the state of the system. We denote the probability of a service completion when the system has x customers by the function $b(x)$, which is non-decreasing and concave in x. Thus:

$$T_{DEP}v(x) = b(x)v(x-1) + [1 - b(x)]v(x).$$

The *controlled departure* and the *controlled production* operators, T_{CD} and T_{C_PRD}, represent the choice of the best service rate in queuing and inventory systems, respectively. π is the used portion of the service rate and c_π is the cost of this portion with $c_\pi \leq 0$. We assume that $c_0 = max_{\pi \in [0,1]} c_\pi$. Then:

$$T_{CD}v(x) = \begin{cases} max_{\pi \in [0,1]} \{c_\pi + \pi v(x-1) + (1-\pi)v(x)\} & \text{if } x > 0 \\ c_0 + v(x) & \text{if } x = 0, \end{cases}$$
$$T_{C\_PRD}v(x) = \max_{\pi \in [0,1]} \{c_\pi + \pi v(x+1) + (1-\pi)v(x)\}.$$

The *queue pricing* and the *inventory pricing* operators, T_{Q_PRC} and T_{I_PRC} represent the optimal price to be charged for the arriving customers in queuing and inventory systems, respectively. $F_R(.)$ is the cumulative distribution function of the reservation price of an arriving customer, R, where R is the maximum price a customer is willing to pay. Hence:

$$T_{Q\_PRC}v(x) = \max_p \left\{\bar{F}_R(p)[v(x+1) + p] + F_R(p)v(x)\right\},$$

$$T_{I\_PRC}v(x) = \begin{cases} \max_p \left\{\bar{F}_R(p)[v(x-1) + p] + F_R(p)v(x)\right\} & \text{if } x > 0 \\ v(x) & \text{if } x = 0, \end{cases}$$

where $\bar{F}_R(p) = 1 - F_R(p)$.

The *batch admission* and the *batch rationing* operators, $T_{B\_ADM_i}$ and $T_{B\_RATIO_i}$, represent the choice of the number of the customers to be admitted from an arriving batch of class-i customers in queuing and inventory systems, respectively. Partial admission meaning that a number of customers from an arriving batch can be accepted whereas the others are rejected is permitted. B is the size of an arriving batch, κ_i is the number of class-i customers admitted from this batch, and R_i is the reward obtained by admitting one class-i customer. Therefore:

$$T_{B\_ADM_i}v(x) = \max_{\kappa_i \leq B} \left\{ \kappa_i R_i + v(x + \kappa_i) \right\},$$
$$T_{B\_RATIO_i}v(x) = \max_{\kappa_i \leq min\{x,B\}} \left\{ \kappa_i R_i + v(x - \kappa_i) \right\}.$$

The *fictitious*, T_{FIC}, operator represents the fictitious service completions and productions, which affect neither the state nor the reward of the system:

$$T_{FIC}v(x) = v(x).$$

The *environment*, $T_{ENV(j)}$, operator represents the transition between the exogenous environment e and the exogenous environment j in the Markov Modulated models. Then:

$$T_{ENV(j)}v(x, e) = v(x, j).$$

A number of different control problems can now be described by using combinations of the above operators. For instance, consider the inventory control problem introduced in Chapter 3. This problem consists of the following operators: T_{COST}, T_{C_PRD} where $\pi \in \{0, 1\}$ and $c_\pi = -\pi\tau$, and T_{B_RATIO} where $B = 1$.

The next chapter investigates the structural properties for each of the operators introduced here.

Chapter 5

STRUCTURE OF THE OPERATORS

5.1 Introduction

In this chapter, we first prove that the operators introduced in Chapter 4 preserve some structural properties such as monotonicity and concavity under certain assumptions. This will enable us to conclude that the models constructed by using these operators also have the same properties. Actually, some of our operators are similar to the operators in Koole [26] such as arrival, departure and admission control, and Koole has already shown structural properties of these operators. However, we generalize some of these operators: We consider the state dependent arrival and departure rates in the arrival and the departure operators, and the batch arrivals in the admission control operator. Therefore, we also investigate the structure of the operators which are similar to Koole's operators as the operators which have not been considered before. After determining the structure of the operators, we observe the effects of the system parameters on the operators in subsection 5.3 where we focus on supermodularity and submodularity properties and introduce a new term that turns out to be critical for sensitivity analysis. We named this extra term as "the extra gain".

5.2 Structural Properties

The first property on which we focus is the monotonicity. We define monotonicity as the value function is non-increasing in x for all states x, i.e., $v(x) \geq v(x+1)$. This property implies a positive burden of an additional customer or an additional unit of inventory in the system. We refer to this burden as the opportunity cost of an additional customer or an additional inventory, and denote it by $\Delta v(x)$ as $\Delta v(x) = v(x) - v(x+1)$. For most plausible queueing systems, this burden is indeed positive. However, in inventory models, opportunity costs are not always positive, so that and the monotonicity of the inventory operators may not hold. Moreover, in many queuing and inventory problems, the opportunity costs affect optimal decisions and monotonicity of the opportunity costs (e.g. concavity of the value function because of the maximization problem), implies the monotonicity of the optimal policies. For instance, threshold policies are optimal in admission control problems due to the monotonicity of opportunity costs. Therefore, we also study the monotonicity of opportunity costs in addition to the monotonicity of the value function.

Besides monotonicity and concavity, upper and lower bounds on the opportunity costs, $\Delta v(x)$, are considered in queueing and inventory problems. The existence of upper bounds in queueing systems implies that the opportunity cost of a new customer, $\Delta v(x)$, may be lower than the reward of one or more demand classes for all x, and thus these classes are admitted to the system whenever it is possible. Similarly, the existence of lower bounds in inventory systems implies that opportunity cost of an additional inventory, $\Delta v(x)$, may be higher than the reward of one or more demand classes for all x, and thus the demand from these classes are satisfied whenever it is possible. Such classes are defined to be preferred classes in [43] and [39], a terminology we adapt here. Hence, these bounds imply the existence of preferred class(es). Because of these relationships, we investigate the upper and lower bounds on the opportunity costs as well as the monotonicity and concavity of the value function. To denote the bounds, we define new properties "Lower-Bounded Differences (LBD)" and "Upper-Bounded Differences (UBD)" such that f is a LBD function if $v(x) - v(x+1) \geq LB$ for all x and it is a UBD function if $v(x) - v(x+1) \leq UB$ for all x, where UB, $LB \in \mathbb{R}$.

The results on the monotonicity, concavity and bounds of the operators are presented in the following lemma. In Lemma 1, we use a similar notation to the one in [26]: For a certain event operator, T, $Prop_1, \ldots, Prop_k \rightarrow Prop$ denotes that if the function v has the properties $Prop_1, \ldots, Prop_k$ then Tv has the property $Prop$. However, while considering the cost operator, T_{COST}, the holding cost h should be non-decreasing and convex to prove the monotonicity and concavity of $T_{COST}v$, respectively. In addition, for $R \geq 0$;

$$\begin{aligned}
Non-Inc(x) &: v(x) \geq v(x+1), \\
Conc(x) &: \Delta v(x) \leq \Delta v(x+1), \\
LBD(R) &: v(x) - v(x+1) \geq -R \\
UBD(R) &: v(x) - v(x+1) \leq R
\end{aligned}$$

We use $-R$ in the LBD property because in inventory models the opportunity cost of an additional inventory may be less than 0.

Lemma 1

- T_{COST}: $Non{-}Inc(x) \rightarrow Non{-}Inc(x)$; $LBD(R) \rightarrow LBD(R)$; $Conc(x) \rightarrow Conc(x)$.

- T_{ARR}: $Non - Inc(x) \rightarrow Non - Inc(x)$; $UBD(R) \rightarrow UBD(R)$; $Non - Inc(x)$, $Conc(x) \rightarrow Conc(x)$.

- T_{DEP} : $Non - Inc(x) \rightarrow Non - Inc(x)$; $UBD(R) \rightarrow UBD(R)$; $Non - Inc(x)$, $Conc(x) \rightarrow Conc(x)$.

- T_{CD} : $Non - Inc(x) \rightarrow Non - Inc(x)$; $UBD(R) \rightarrow UBD(R)$; $Conc(x) \rightarrow Conc(x)$.

- T_{C_PRD} : $LBD(R) \rightarrow LBD(R)$; $Conc(x) \rightarrow Conc(x)$.

- T_{Q_PRC} : $Non - Inc(x) \rightarrow Non - Inc(x)$; $UBD(R) \rightarrow UBD(R)$; $Conc(x) \rightarrow Conc(x)$.

- T_{I-PRC} : $LBD(R) \rightarrow LBD(R)$; $Conc(x) \rightarrow Conc(x)$.

- $T_{B\_ADM_i}$: $Non - Inc(x) \rightarrow Non - Inc(x)$; $UBD(R) \rightarrow UBD(R)$; $Conc(x) \rightarrow Conc(x)$.

- $T_{B\_RATIO_i}$: $LBD(R) \rightarrow LBD(R)$; $Conc(x) \rightarrow Conc(x)$.

- $T_{ENV(j)}$: $Non - Inc(x) \rightarrow Non - Inc(x)$; $UBD(R) \rightarrow UBD(R)$; $LBD(R) \rightarrow LBD(R)$; $Conc(x) \rightarrow Conc(x)$.

- T_{FIC} : $Non{-}Inc(x) \rightarrow Non{-}Inc(x)$; $UBD(R) \rightarrow UBD(R)$; $LBD(R) \rightarrow LBD(R)$; $Conc(x) \rightarrow Conc(x)$.

The proof of the lemma is trivial for T_{FIC} since it is the same as $v(x)$, and the proof for $T_{ENV(j)}$ is obvious when we assume that $v(x, e)$ has some structural properties for all demand environments. The proofs of the monotonicity and LBD for T_{COST} are obvious when the holding cost is assumed to be non-decreasing, and similarly the proof of the concavity for T_{COST} is also trivial when h is assumed to be convex. For the remaining operators, the proofs can be found in the Appendix.

In the following technical remark,we discuss why T_{COST} does not preserve the UBD property and the implication of this structural property.

Remark 1 *Unlike monotonicity, LBD and concavity, T_{COST} does not preserve the UBD property of the function on which it is applied. To illustrate, we let $v(x)$, the value function, be a UBD function, and then try to prove that $T_{COST}v(x)$ is also a UBD function. To achieve this aim, we have to prove that $\Delta v(x) - \Delta h(x) \leq R$ is true. We know that $\Delta v(x) \leq R$ by our assumption, so that we need to show $\Delta h(x) \geq 0$, i.e. the holding cost is non-increasing. However, a non-increasing holding cost function is not a realistic assumption in a queueing system in which the objective is to maximize the expected profit because if the holding cost is non-increasing in the number of customers in the system then, there will not be a positive burden of a new customer to the system. Therefore, the cost operator does not preserve the UBD property of $v(x)$, and this structural property implies that there does not exist any preferred class in queueing systems incurring holding cost.*

5.3 Effects of Parameters

In the beginning of the paper, we mentioned the importance of the information about the effects of the changes in the parameters on the optimal decisions. Therefore, in this subsection, we first focus on the behavior of the operators when the parameters change as a starting point of the sensitivity analysis since we use the event-based dynamic programming technique. The parameters we examine are the service rate or the production rate, μ, the arrival rate, λ, the number of servers, c, and the waiting room capacity (including the servers), K.

In order to perform sensitivity analysis on the optimal policy, we increase α, the parameter whose effect we would like to observe, by ε and compare the opportunity costs in the systems with parameters α and $\alpha + \varepsilon$. For this purpose we use supermodularity and submodularity properties. These properties are generally used in models with two dimensional state spaces to indicate the effects of one dimension on the other one. However, here we say that the value function, $v(x)$, is supermodular with respect to α and x if the opportunity cost is non-increasing in α, and it is submodular if the opportunity cost is non-decreasing in α. Formally, Equations 5.1 and 5.2 represent the supermodularity and submodularity of $v(x)$, respectively, where we denote the value function of the system with parameter $\alpha + \varepsilon$

by $\tilde{v}(x)$.

$$\Delta v(x) \geq \Delta \tilde{v}(x) \tag{5.1}$$

$$\Delta v(x) \leq \Delta \tilde{v}(x) \tag{5.2}$$

Supermodularity and submodularity properties are not sufficient to investigate the effects of the changes in a parameter on the opportunity cost. In particular, when we increase the parameter α whose effects we would like to observe by ε, the occurrence rate of the event related with the parameter increases whereas the occurrence rate of the fictitious event decreases. Therefore, we obtain an extra term of $T\tilde{v}(x) - \tilde{v}(x)$ in the optimality equation where T is the operator related with α. We call this term as "the extra gain" in this paper. For example, consider the example model in Chapter 3 and its optimality equation. If the arrival rate, λ, increases by ε in this model then θ will decrease by ε to ensure that $\mu + \lambda + \theta = 1$ and thus, the optimality equation becomes:

$$\begin{aligned}
\tilde{v}(x) &= T_{COST}(\mu T_{PRD}\tilde{v}(x) + \lambda T_{RAT}\tilde{v}(x) + \theta T_{FIC}\tilde{v}(x) + \varepsilon[T_{RAT}\tilde{v}(x) - \tilde{v}(x)]) \text{ whereas,} \\
v(x) &= T_{COST}(\mu T_{PRD}v(x) + \lambda T_{RAT}v(x) + \theta T_{FIC}v(x)).
\end{aligned}$$

Here, the extra gain, $T_{RAT}\tilde{v}(x) - \tilde{v}(x)$, emerges as a result of an increase in the occurrence rate of the arrival event. While comparing the opportunity costs ($\Delta v(x)$ and $\Delta\tilde{v}(x)$) during sensitivity analysis, it is obvious that the first three terms can be compared by using supermodularity or submodularity but we still have to deal with the remaining term, $T_{RAT}\tilde{v}(x) - \tilde{v}(x)$ in order to complete the comparison. Therefore, we investigate the structure of this latter term besides supermodularity and submodularity properties of the other terms.

5.3.1 Supermodularity and Submodularity

According to our literature survey, a complete study on the supermodularity and submodularity of the operators we consider has not been performed before. Although Gans and Savin [19] and Ku and Jordan [27] prove the monotonicity of optimal actions in the system parameters for certain loss systems, they do not work on the supermodularity and submodularity of the operators used in their models. Thus, we prove supermodularity and submodularity of all operators in this subsection.

While considering the service (production) rate, μ, and the arrival rate, λ, all operators preserve supermodularity and submodularity of the value function, $v(x)$ because the changes in these parameters do not affect the definition of the operators.

While considering the waiting room capacity, K, T_{ARR} preserves supermodularity and submodularity of $v(x)$ according to the effects of K on the function $a(x)$, join probability, because K may affect the definition of the function $a(x)$, and thus the definition of the arrival operator. For instance, consider a system in which $a(x)$ is defined as $a(x) = (K - x)/K$. In this system, $a(x)$ will be $a(x) = (K + 1 - x)/(K + 1)$ if the capacity is increased by 1. In a system where K affects $a(x)$, one should check whether the conditions in lemmas hold or not to show the supermodularity and submodularity of $T_{ARR}v(x)$. The conditions in Lemma 2, the supermodularity lemma, ensure that the join probability for each state decreases by an increase in K and the amount of decrease in the join probability is a non-decreasing function of the state x. In other words, these conditions imply that the load of the system will decrease with an increasing rate if the capacity increases, and thus the opportunity cost of a new customer will decrease. Similarly, the conditions in Lemma 3 imply that the load of the system will increase with an increasing rate if the capacity increases, and thus the opportunity cost of a new customer will increase.

For capacitated systems, it is specified that the operators, T_{Q_PRC}, $T_{B\_ADM_i}$ and T_{C_PRD}, do not change the state of the system when the system reaches its capacity. Therefore the capacity, K, affects the definition of the operators. As a result of this effect, these operators can only preserve the supermodularity of the value function. The intuition behind the non-submodular property is clear: In a queueing system with waiting room capacity K, new arrivals are not allowed to enter the system when there are K customers in the system whether it is optimal or not. If it is optimal to reject the arrivals, the optimal policy does not change when the capacity increases. On the other hand, if the arrivals are rejected because of the capacity, a new customer may be accepted when there are K customers after an increase in the capacity. Thus, it is obvious that an increase in the capacity can not lead to rejecting the customers who are already accepted before the capacity increases. In other words, the opportunity cost of a new customer can not increase by an increase in the capacity.

Since the operators T_{DEP}, T_{CD}, T_{I_PRC}, $T_{B\_RAT_i}$, $T_{ENV(j)}$, T_{FIC} and T_{COST} are not affected by an increase in the capacity, these operators preserve both supermodularity and submodularity of the value function with respect to K and x.

The effects of the number of servers, c, is similar to the effects of K. We only examine the effects of the number of servers on queueing related operators because multi-server systems are mostly used in queueing problems. Like the effects of K on $a(x)$, an increase in c affects the definition of the function $b(x)$. Therefore, T_{DEP} preserves supermodularity and submodularity of the value function according to the effects of c on the function $b(x)$. The remaining operators, T_{Q_PRC}, $T_{B\_ADM_i}$, T_{DEP}, T_{CD}, $T_{ENV(j)}$, T_{FIC} and T_{COST} preserve both supermodularity and submodularity of $v(x)$ with respect to c and x since they are not affected by an increase in the number of server. However, if $K = c$ the arrival related operators, T_{Q_PRC} and $T_{B\_ADM_i}$, only preserves supermodularity of $v(x)$ by the same intuition we stated while considering K. We do not examine the effect of c on the controlled departure operator, T_{CD}, because systems constructed by using T_{CD} are single-server systems.

We summarize our results about the effects of the parameters on the operators in the following lemmas. We use the same notation in Lemma 1. In addition, we define $SuperM(\alpha, x)$ and $SubM(\alpha, x)$ to denote the supermodularity and submodularity with respect to α and x.

The supermodularity lemma is:

Lemma 2

- T_{COST}:*$SuperM(\alpha, x) \rightarrow SuperM(\alpha, x)$, for all $\alpha \in \{\mu, \lambda, c, K\}$.*

- T_{ARR} *: $Non-Inc(x)$, $Conc(x)$, $SuperM(\alpha, x) \rightarrow SuperM(\alpha, x)$. Given that $a(x)$ is non-increasing in α and submodular with respect to α and x, for all $\alpha \in \{\mu, \lambda, c, K\}$.*

- T_{DEP} *: $Non-Inc(x)$, $Conc(x)$, $SuperM(\alpha, x) \rightarrow SuperM(\alpha, x)$. Given that $b(x)$ is non-decreasing in α and supermodular with respect to α and x, for all $\alpha \in \{\mu, \lambda, c, K\}$.*

- T_{CD} *: $SuperM(\alpha, x) \rightarrow SuperM(\alpha, x)$, for all $\alpha \in \{\mu, \lambda, K\}$.*

- T_{C_PRD} *: $SuperM(\alpha, x) \rightarrow SuperM(\alpha, x)$, for all $\alpha \in \{\mu, \lambda, K\}$.*

- T_{Q_PRC} *: $SuperM(\alpha, x) \rightarrow SuperM(\alpha, x)$, for all $\alpha \in \{\mu, \lambda, c, K\}$.*

- T_{I_PRC} *: $SuperM(\alpha, x) \rightarrow SuperM(\alpha, x)$, for all $\alpha \in \{\mu, \lambda, K\}$.*

- $T_{B\_ADM_i}$: *Conc*(x), *SuperM*$(\alpha, x) \rightarrow$ *SuperM*(α, x), *for all* $\alpha \in \{\mu, \lambda, c, K\}$.

- $T_{B\_RATIO_i}$: *Conc*(x), *SuperM*$(\alpha, x) \rightarrow$ *SuperM*(α, x), *for all* $\alpha \in \{\mu, \lambda, K\}$.

- $T_{ENV(j)}$: *SuperM*(α, x) *for all demand environments* $\rightarrow$ *SuperM*(α, x) *for all demand environments, for all* $\alpha \in \{\mu, \lambda, c, K\}$.

- T_{FIC}:*SuperM*$(\alpha, x) \rightarrow$ *SuperM*(α, x), *for all* $\alpha \in \{\mu, \lambda, c, K\}$.

The submodularity lemma is:

Lemma 3
- T_{COST}:*SubM*$(\alpha, x) \rightarrow$ *SubM*(α, x), *for all* $\alpha \in \{\mu, \lambda, c, K\}$.

- T_{ARR} : *Non* $-$ *Inc*(x), *Conc*(x), *SubM*$(\alpha, x) \rightarrow$ *SubM*(α, x). *Given that* $a(x)$ *is non-decreasing in* α *and supermodular with respect to* α *and* x, *for all* $\alpha \in \begin{cases} \{\mu, \lambda, c, K\} & \textit{if } K \neq c \\ \{\mu, \lambda, c\} & \textit{if } K = c \end{cases}$.

- T_{DEP} : *Non* $-$ *Inc*(x), *Conc*(x), *SubM*$(\alpha, x) \rightarrow$ *SubM*(α, x). *Given that* $b(x)$ *is non-increasing in* α *and submodular with respect to* α *and* x, *for all* $\alpha \in \{\mu, \lambda, c, K\}$.

- T_{CD} : *SubM*$(\alpha, x) \rightarrow$ *SubM*(α, x), *for all* $\alpha \in \{\mu, \lambda, K\}$.

- T_{C_PRD} : *SubM*$(\alpha, x) \rightarrow$ *SubM*(α, x).$\alpha \in \{\mu, \lambda\}$.

- T_{Q_PRC} : *SubM*$(\alpha, x) \rightarrow$ *SubM*(α, x), *for all* $\alpha \in \begin{cases} \{\mu, \lambda, c\} & \textit{if } K \neq c \\ \{\mu, \lambda\} & \textit{if } K = c \end{cases}$.

- T_{I_PRC} : *SubM*$(\alpha, x) \rightarrow$ *SubM*(α, x), *for all* $\alpha \in \{\mu, \lambda, K\}$.

- $T_{B\_ADM_i}$: *Conc*(x), *SubM*$(\alpha, x) \rightarrow$ *SubM*(α, x), *for all* $\alpha \in \begin{cases} \{\mu, \lambda, c\} & \textit{if } K \neq c \\ \{\mu, \lambda\} & \textit{if } K = c \end{cases}$.

- $T_{B\_RATIO_i}$: *Conc*(x), *SubM*$(\alpha, x) \rightarrow$ *SubM*(α, x), *for all* $\alpha \in \{\mu, \lambda, K\}$.

- $T_{ENV(j)}$: *SubM*(α, x) *for all demand environments* $\rightarrow$ *SubM*(α, x) *for all demand environments, for all* $\alpha \in \{\mu, \lambda, c, K\}$.

- T_{FIC}: *SubM*$(\alpha, x) \rightarrow$ *SubM*(α, x), *for all* $\alpha \in \{\mu, \lambda, c, K\}$.

5.3.2 The Extra Gain, $Tv(x) - v(x)$

As we mentioned before, the effect of a parameter on the opportunity cost is closely related with the structure of the extra gain obtained by the operator related to the parameter. Simply the relation is: If the extra gain is non-decreasing in x then the opportunity cost will be non-increasing in the parameter and if the extra gain is non-increasing in x then the opportunity cost will be non-decreasing in x. To illustrate this relation , let us concentrate on the example model that we discussed in Chapter 3 and assume that all of the operators preserve supermodularity of the value function and $T_{RAT}v(x) - v(x)$ is non-decreasing in x. Then, we obtain

$$\Delta v(x) = \begin{bmatrix} \mu \Delta T_{PRD} v(x) \\ +\lambda \Delta T_{RAT} v(x) \\ +\theta \Delta T_{FIC} v(x) \\ \\ \end{bmatrix} \geq \begin{bmatrix} \mu \Delta T_{PRD} \tilde{v}(x) \\ +\lambda \Delta T_{RAT} \tilde{v}(x) \\ +\theta \Delta T_{FIC} \tilde{v}(x) \\ +\varepsilon \Delta [T_{RAT} \tilde{v}(x) - \tilde{v}(x)] \end{bmatrix} = \Delta \tilde{v}(x)$$

because of the supermodularity and the structure of the extra gain: The first three lines hold by the supermodularity and the last line holds by the monotone structure of $T_{RAT}v(x) - v(x)$. As it can be easily seen, this equation implies that the opportunity cost of an additional inventory decreases when the arrival rate increases in our example model.

The monotone structure of the extra gain depends on the characteristics of the event and the context of the problem. In the queueing control context, the departure of an existing customer is more valuable when there is one more customer in the system where as the arrival of a new customer is more valuable when there is one less customer. Therefore, $Tv(x) - v(x)$ is non-decreasing in x for the departure operators and non-increasing in x for the arrival related operators. However, while considering T_{ARR}, this intuition may not work if the join probability function, $a(x)$ is not constant because the join probability decreases while the number of customers in the system is increasing. In other words, the system faces conflicting effect: Since the load of the system becomes higher, the arrival of a new customer has a positive burden on the system but a new customer also relieves this burden because of a decrease in the join probability. Hence, we only consider the arrival operator given that $a(x)$ is constant while observing the structure of $T_{ARR}v(x) - v(x)$.

On the other hand, in the inventory control context, $Tv(x) - v(x)$ is non-decreasing in x for the arrival related operators and non-increasing in x for the production operator because if there is much more inventory on-hand, then the arrival of a new customer will be more valuable and it will not be worth producing a new product. We state the results on the extra gain in Lemma 4. The notation is similar to Lemma 1 but in Lemma 4, $Prop_1, \ldots, Prop_k \rightarrow Prop_j$ denotes that if the function f has the properties $Prop_1, \ldots, Prop_k$ then $Tf - f$ has the property $Prop_j$. Furthermore, $Non - Dec(x)$ represents $v(x) \leq v(x+1)$.

Lemma 4

- T_{ARR}: *$Non - Inc(x)$, $Conc(x)$ $\rightarrow$ $Non - Inc(x)$. Given that $a(x)$ is constant.*
- T_{DEP} *: $Non - Inc(x)$, $Conc(x) \rightarrow Non - Dec(x)$.*
- T_{CD} *: $Conc(x) \rightarrow Non - Dec(x)$.*
- T_{C_PRD} *: $Conc(x) \rightarrow Non - Inc(x)$.*
- T_{Q_PRC} *: $Conc(x) \rightarrow Non - Inc(x)$.*
- T_{I-PRC} *: $Conc(x) \rightarrow Non - Dec(x)$.*
- $T_{B\_ADM_i}$ *: $Conc(x) \rightarrow Non - Inc(x)$.*
- $T_{B\_RATIO_i}$ *: $Conc(x) \rightarrow Non - Dec(x)$.*

Chapter 6

SENSITIVITY ANALYSIS EXAMPLES

6.1 Introduction

In this chapter, we illustrate our framework by performing sensitivity analysis on queueing and inventory control problems that have been studied in literature. We first work on three basic problems discussed in Lippman [30]: the admission control problem of Miller [34], the pricing control problem of Low [31], and the service rate control problem of Crabill [13]. We redefine each of the models considered in these problems by using our operators. We use the batch admission operator, T_{B_ADM}, in [34], the queue pricing operator, T_{Q_PRC}, in [31], and the controlled departure operator, T_{CD}, in [13]. At the end of this investigation, we show that the thresholds are non-decreasing in the service rate and non-increasing in the arrival rate, the optimal prices are non-increasing in the service rate and non-decreasing in the arrival rate, and the optimal used portion of server is non-increasing in the service rate and non-decreasing in the arrival rate.

Then, we concentrate on more complex and recent queueing and inventory control problems. We start with the joint dynamic pricing and replenishment problem introduced by Gayon et. al. [20]. Then, we observe the effects of system parameters on the model in Örmeci and Burnetas [37] where the partial acceptance problem in a multi-server loss system with batch arrivals is considered. Finally, we extend the stock rationing problem studied by Ha [22] and obtain new structural results and perform sensitivity analysis on the extended model.

Since we analyze three different models, we use an index m with $m \in \{1, 2, 3\}$ to distinguish the parameters, functions and variables defined in each model. For example, R_{im} denotes the reward obtained by satisfying a class-i customer in model m.

6.2 Pricing Control Problem

Dynamic pricing for inventory management has received a lot of attention in recent research. Since the model introduced in [20] is one of the most recent and interesting ones that fall into our framework, this model is the first example to illustrate the approach. Gayon et. al. [20] consider a joint dynamic pricing and replenishment problem of a supplier who produces a single part at a single facility. The processing time is exponentially distributed with mean $1/\mu$ and the completed items are placed in a finished goods inventory. The supplier incurs a unit variable cost, τ, per part and $h(x)$, a convex function of the inventory level, is the inventory holding cost per unit time. The demand arrivals occur according to a Markov Modulated Poisson process (MMDP) with potential arrival rate λ_e depending on the state of the demand environment e. Then, an arriving customer buys the part if his reservation price, i.e., the maximum willingness to pay, is higher than the announced price, p, otherwise he leaves the system. The reservation price, R, is assumed as a random variable with the cumulative distribution function $F_R(.)$. The demand environment state is a continuous time Markov Chain with state space $E = \{1, \ldots, n\}$ and transition rates q_{ej} from state e to $j \neq e$. In the original model, Gayon et. al. prove that a base stock policy is the optimal production control policy and optimal prices are non-increasing in the inventory level. Furthermore, they numerically show that dynamic pricing is significantly more beneficial than the static pricing only when the demand environment fluctuations are high.

We use T_{I_PRC}, T_{C_PRD} and T_{FIC} to denote the pricing control, the production control and the fictitious events, respectively. To use the controlled production operator, T_{C_PRD}, we specify that π_1, the used portion of the production rate in Model 1, can be either 0 (not to produce) or 1 (to produce), and $c_{\pi_1} = -\pi_1\tau_1$, i.e., the cost of the used portion is τ_1 (unit variable cost in Model 1) if $\pi_1 = 1$, and it is 0 otherwise. In addition, we use the environment operator, $T_{ENV(j)}$, to represent the transition between states e and $j \neq e$.

At any time, the decision maker has to decide whether to produce or not and choose a price $p \in \mathcal{P}$. $\mathcal{P}$ may be either discrete or continuous. When $\mathcal{P}$ is continuous, it is assumed to be a compact subset of $\mathbb{R}^+$. As in the original model, we assume that the arrivals occur according to a Markov Modulated Poisson process, and denote the potential arrival rate of the demand environment e in Model 1 by λ_{e1}. We also use the same assumption on the reservation price as in the original model, i.e. $F_R(.)$ is the cumulative distribution function of the reservation price, R.

Then, the current state of the system is described by the state variable (x, e) with x as the stock level and e as the environment state. (x, e) belongs to the state space $\mathbb{S}_1 \times E$ such that $\mathbb{S}_1 = \{x : 0 \leq x < \infty\}$ and $E = \{1, \ldots, n\}$.

The objective of the problem is to maximize the expected total β-discounted reward over an infinite horizon. To achieve this aim, we build the corresponding discrete-time MDP of the model by using uniformization and normalization since the potential transition rate is finite. Then, we assume that the time between two consecutive transition is exponentially distributed with rate $\mu_1 + \sum \lambda_{e1} + \sum_e \sum_{j \neq e} q_{ej} + \beta + \theta$ and using the appropriate time scale, assume that $\mu_1 + \sum \lambda_{e1} + \sum_e \sum_{j \neq e} q_{ej} + \beta + \theta = 1$. We introduce the new parameter θ to ensure that time scale remains the same after the parameters μ_1 and λ change during the sensitivity analysis. For example, if the arrival rate, λ, increases by ε then, θ will decrease by ε and time scale will not change.

As we will describe some structural properties (concavity, supermodularity, etc.) of the systems which operate over an infinite horizon in this study, we first prove these structural properties with the objective of maximizing the expected total β-discounted reward for a finite number of transitions, n. The finite horizon problems allow us to use the induction to prove the structural properties for all finite n. We let $v_1^n(x, e)$ be the maximum expected total β-discounted reward of a system starting in state (x, e) with n transitions remaining in the future, and specify the terminal function v_1^0 as $v_1^0(x, e) = 0$ for all states (x, e) to start the induction. Then, we present the optimality equations of the finite horizon problem as:

$$\begin{aligned} v_1^n(x,e) \quad = T_{COST}(\quad & \mu_1 T_{C\_PRD} v_1^{n-1}(x,e) + \lambda_{e1} T_{I\_PRC} v_1^{n-1}(x,e) \\ & + \sum_{j \neq e} q_{ej} T_{ENV(j)} v_1^{n-1}(x,e) + (\sum_{i \neq e} \lambda_i + \sum_{i \neq e} \sum_{j \neq i} q_{ij} + \theta) T_{FIC} v_1^{n-1}(x,e)), \end{aligned}$$

where the holding cost function is $h_1(x, e)$, which is non-decreasing and convex in x.

By using the standard arguments of Markov decision theory (See Puterman [42]), the structural results for the finite horizon problem can be extended to the infinite horizon problem with the objective of maximizing the discounted reward as well as the expected average reward. We denote the value function of the system for the infinite horizon expected discounted reward criterion by $v_1(x, e)$, the gain and the relative value function of the system for the average reward criterion by g_1 and $v_1'(x, e)$, respectively. Thus, for $\beta > 0$,

$v_1(x,e) = \lim_{n\to\infty} v_1^n(x,e)$ and the optimality equation for the average reward criterion is:

$$\begin{aligned} g_1 + v_1'(x,e) &= T_{COST}(\quad \mu_1 T_{C\_PRD} v_1'(x,e) + \lambda_{e1} T_{I\_PRC} v_1'(x,e) \\ &\quad + \sum_{j\neq e} q_{ej} T_{ENV(j)} v_1'(x,e) + (\sum_{i\neq e} \lambda_{i1} + \sum_{i\neq e}\sum_{j\neq i} q_{ij} + \theta) T_{FIC} v_1'(x,e)). \end{aligned}$$

To this end, we define the value function of the model. In order to use Lemmas 2, 3, and 4, we have to show the basic structural properties (monotonicity, concavity) of the value function. However, these properties have already been proven in the original model so that, we only focus on the effects of the parameters on the opportunity cost, $\Delta v_1(x,e)$, and the optimal policy. The parameters that we consider in the sensitivity analysis are the production rate, μ_1, and the arrival rate in a demand environment k, λ_{k1}, for all k.

6.2.1 Effects of the Parameters: μ_1 and λ_{k1}

Intuitively, if the production rate, μ, increases, then the system will process faster and the opportunity cost of an additional inventory will increase because it is not necessary to keep more inventory when the system processes faster. On the other hand, the opportunity cost of an additional inventory decreases by an increase in the arrival rate, λ, because the higher the arrival rate the faster the inventory on hand is consumed. Equation 5.1 and 5.2 implies the intuitions about the effects of the arrival rate and the production rate on the opportunity cost, respectively, and thus, we work on the supermodularity of the value function, $v_1(x,e)$, with respect to λ_{k1} and x, and the submodularity of $v_1(x,e)$ with respect to μ_1 and x.

To prove the supermodularity and submodularity of the value function, $v_1(x,e)$, we first prove these properties of $v_1^n(x,e)$ for all finite n by induction. Both proofs are similar and we only present the proof of the supermodularity of $v_1(x,e)$ with respect to λ_{k1} and x. The initial condition of the induction holds by the specification that $v_1^0(x,e) = 0$ for all states (x,e). Then, we assume that $v_1^{n-1}(x,e)$ is supermodular with respect to λ_{k1}, and x, and show the supermodularity of $v_1^n(x,e)$ with respect to λ_{k1} and x, for all k. Since we observe the supermodularity for all k, we have to investigate the cases: $e = k$ and $e \neq k$. For $e = k$, we can write the supermodularity equation, Equation 5.1, for $v_1^n(x,e)$ by using the optimality equation as follows:

$$\begin{array}{ccc}
\mu_1 \Delta T_{C_PRD} v_1^{n-1}(x,e) & & \mu_1 \Delta T_{C_PRD} \tilde{v}_1^{n-1}(x,e) \\
+\lambda_{e1} \Delta T_{I_PRC} v_1^{n-1}(x,e) & & +\lambda_{e1} \Delta T_{I_PRC} \tilde{v}_1^{n-1}(x,e) \\
+\sum_{j\neq e} q_{ej} \Delta T_{ENV(j)} v_1^{n-1}(x,e) & \geq & +\sum_{j\neq e} q_{ej} \Delta T_{ENV(j)} \tilde{v}_1^{n-1}(x,e) \\
+[\sum_{i\neq e}(\lambda_{i1}+\sum_{j\neq i} q_{ij})+\theta] \Delta T_{FIC} v_1^{n-1}(x,e) & & +[\sum_{i\neq e}(\lambda_{i1}+\sum_{j\neq i} q_{ij})+\theta] \Delta T_{FIC} \tilde{v}_1^{n-1}(x,e) \\
 & & +\varepsilon\left[\Delta T_{I_PRC} \tilde{v}_1^{n-1}(x) - \Delta \tilde{v}_1^{n-1}(x)\right].
\end{array} \tag{6.1}$$

It is obvious that the third line is true by the assumption on $v_1^{n-1}(x,e)$ and the first, the second and the fourth lines are true due to Lemma 2. Therefore, we only need to show that the last line is smaller than or equal to 0. As a result of Lemma 4, $T_{I\_PRC} f(x) - f(x)$ is non-decreasing in x, i.e., $\Delta T_{I\_PRC} f(x) - \Delta f(x) \leq 0$. Hence, the last line is also true and we complete the proof for the case $e = k$.

Similar to the case $e = k$, we can write Equation 5.1 for $v_1^n(x,e)$ where $e \neq k$ as follows:

$$\begin{array}{ccc}
\mu_1 \Delta T_{C_PRD} v_1^{n-1}(x,e) & & \mu_1 \Delta T_{C_PRD} \tilde{v}_1^{n-1}(x,e) \\
+\lambda_{e1} \Delta T_{I_PRC} v_1^{n-1}(x,e) & & +\lambda_{e1} \Delta T_{I_PRC} \tilde{v}_1^{n-1}(x,e) \\
+\sum_{j\neq e} q_{ej} \Delta T_{ENV(j)} v_1^{n-1}(x,e) & \geq & +\sum_{j\neq e} q_{ej} \Delta T_{ENV(j)} \tilde{v}_1^{n-1}(x,e) \\
+[\sum_{i\neq e}(\lambda_{i1}+\sum_{j\neq i} q_{ij})+\theta] \Delta T_{FIC} v_1^{n-1}(x,e) & & +[\sum_{i\neq e}(\lambda_{i1}+\sum_{j\neq i} q_{ij})+\theta] \Delta T_{FIC} \tilde{v}_1^{n-1}(x,e),
\end{array} \tag{6.2}$$

and Equation 6.2 is true by the induction hypothesis and Lemma 2 as Equation 6.1. Thus, we show the supermodularity of $v_1^n(x,e)$ with respect to λ_{k1} and x for all finite n by considering the cases $e = k$ and $e \neq k$. As a result of the relationship between the finite horizon problem and the infinite horizon problem, $v_1(x,e)$ and $v_1'(x,e)$ are also supermodular with respect to λ_{k1} and x. We state the effects of the production rate and the arrival rate on the value function and the optimal actions in the following theorem and corollary, respectively. The proof for the effects of the parameters on the optimal actions can be found in the Appendix.

Theorem 2 *In this model:*

- *$v_1(x,e)$ and $v_1'(x,e)$ are submodular with respect to μ_1 and x.*

- *$v_1(x, e)$ and $v_1'(x, e)$ are supermodular with respect to λ_{k1} and x, whether $e = k$ or not.*

Corollary 2 *The optimal base-stock level for the demand environment e, $S_1^*(e)$, and the optimal price for the state (x, e), $p_1^*(x, e)$, are non-increasing in μ_1 and non-decreasing in λ_{k1}, for all k.*

6.3 Admission Control Problem

Admission control is one of the control problems in queueing control, and we use the model in Örmeci and Burnetas [37] to illustrate our framework. In this paper, Örmeci and Burnetas consider a partial acceptance problem, where some of the jobs in a batch can be admitted while the remaining ones are rejected, in a loss system which consists of c identical parallel servers with no waiting room and N classes of jobs. Arrivals occur according to a Poisson process with rate μ. At each arrival epoch, a random number of jobs from each class arrive at the system. They denote an arriving batch by $j = (j_1, \ldots, j_N)$ where j_i is the number of class-i jobs in an arriving batch. The system receives a batch $j = (j_1, \ldots, j_N)$ with probability p_j. A reward of $R_i \geq 0$ is obtained if a class-i job is admitted and each admitted job requires an exponential service time with rate μ. In the original model, Örmeci and Burnetas show that it is always optimal to accept class-1 jobs and there exists an optimal sequential threshold policy for the remaining classes if the classes are ordered with respect to their rewards, i.e., $R_1 \geq \cdots \geq R_N$.

Before redefining the model by using our operators, we assume that each arriving batch consists of only one class of jobs and the probability that B class-i jobs arrive in a batch is p_{iB}. These assumptions do not change the nature of the problem and our results about the redefined model can also be shown for the original model in [37]. With the assumption, we use the batch admission operator, $T_{B\_ADM_i}$, the departure operator T_{DEP}, where the probability of service completion is $b_2(x) = x/M$, and the fictitious operator, T_{FIC} to denote the arrivals and control policy, the departures and the fictitious service completions, respectively. The new parameter M, which is greater than all c considered during the sensitivity analysis, is introduced to ensure that time scale remains the same after the number of servers is changed.

At each arrival epoch, the decision maker has to decide the number of class-i jobs to be admitted from an arriving batch, κ_{i2}. Then, whenever a class-i customer is admitted, a reward of $R_{i2} \geq 0$ is obtained. We define the state of the system as x, the number of customers in the system, and x belongs to the state space $\mathbb{S}_2$, such that $\mathbb{S}_2 = \{x : 0 \leq x \leq c\}$.

Our objective in this problem is finding the optimal policy that maximizes the expected total β-discounted reward over an infinite horizon. Therefore, we build the corresponding discrete-time MDP of the model by using uniformization and normalization. After the uniformization and normalization, we assume that the time between two consecutive transition is exponentially distributed with rate $M\mu_2 + \lambda_2 + \beta + \theta = 1$.

As in the previous example model, we denote the maximum expected total β-discounted reward of the system starting in state x after n transitions by $v_2^n(x)$, where $v_2^0(x) = 0$ for all states x, the value function of the system for the infinite horizon expected discounted reward criterion by $v_2(x)$, and the gain and the relative value function of the system for the average reward criterion by g_2 and $v_2'(x)$, respectively. Then, we present the optimality equations of the finite horizon problem and the infinite horizon problem for the average reward criterion as in Equations 6.3 and 6.4, respectively. Finally, by using the standard arguments of Markov decision theory, we state that for $\beta > 0$, $v_2(x) = \lim_{n\to\infty} v_2^n(x)$ and all structural results obtained for $v_2^n(x)$ is also valid for $v_2(x)$ and $v_2'(x)$.

$$v_2^n(x) = M\mu_2 T_{DEP} v_2^{n-1}(x) + \lambda_2 \sum_{i=1}^{N} \sum_{B} p_{iB} T_{B\_ADM_i} v_2^{n-1}(x) + \theta T_{FIC} v_2^{n-1}(x), \tag{6.3}$$

$$g_2 + v_2'(x) = M\mu_2 T_{DEP} v_2'(x) + \lambda_2 \sum_{i=1}^{N} \sum_{B} p_{iB} T_{B\_ADM_i} v_2'(x) + \theta T_{FIC} v_2'(x). \tag{6.4}$$

6.3.1 *Effects of the Parameters: μ_2, λ_2, and c*

The effects of the service and arrival rates on a queueing system are the reverse of the effects on the inventory system: The congestion in the system decreases and thus the opportunity cost of an additional customer decreases by an increase in the service rate, μ_2, whereas the congestion and the opportunity increases by an increase in the arrival rate, λ_2. Therefore, we work on the supermodularity of $v_2(x)$ with respect to μ_2 and x and the submodularity of $v_2(x)$ with respect to λ_2 and x. In addition to μ_2 and λ_2, we also examine the effects of the number of servers, c. Intuitively, if the number of servers increases, then the congestion in the system will decrease since the occurrence probability of the service completion increases. Thus, we focus on the supermodularity of the value function as in the case of the service rate.

The proofs of the supermodularity and submodularity of the value function, $v_2(x)$, are similar to the proof in the previous example. We first prove the properties for $v_2^n(x)$ by using Lemmas 2, 3, and 4 and then, extend the results for the infinite horizon problem. The following theorem summarizes our results about the effects of the parameters on the value function and the optimal decisions in the partial acceptance problem.

Theorem 3 *In this partial acceptance problem:*

- $v_2(x)$ *and* $v_2'(x)$ *are supermodular with respect to* μ_2 *and* x*.*
- $v_2(x)$ *and* $v_2'(x)$ *are submodular with respect to* λ_2 *and* x*.*
- $v_2(x)$ *and* $v_2'(x)$ *are supermodular with respect to* c *and* x*.*

Corollary 3 *The optimal number of class-i customers to be admitted from an arriving batch for the state* x*,* $\kappa_{i2}^*(x)$*, is non-decreasing in the service rate and the number of servers and non-increasing in the arrival rate.*

6.4 Stock Rationing Problem

The final example we present is on the stock rationing problem of a make-to-stock production system. As an example rationing problem to perform sensitivity analysis, we choose the model introduced by Ha [22]. In this study, Ha considers a make-to-stock production system that produces a single product with N demand classes and lost sales. When a demand arises, it is either satisfied from on-hand inventory or rejected. A rejected demand is lost and a lost sale cost of c_i with $c_1 \geq \cdots \geq c_N$ is incurred. Demand arrivals occur according to a Poisson process with rate λ and each customer requests one unit of product. The production time is exponential with mean $1/\mu$. Moreover, the inventory holding cost per unit time, h, is a non-decreasing and convex function of the on-hand inventory.

In real life applications, it is plausible that some classes of customers may demand more than one unit of product at a time. However, as a result of the assumption that each demand requests one unit of product, the original model in [22] does not cover multiple demand requests. Therefore, we not only redefine the model but also extend it by considering batch arrivals. As in Model 2, an arriving batch consists of only one class of customers and p_{iB} is the probability that there are B class-i customers in the batch. Furthermore, we assume that the variable production cost is τ_3, which is set to 0 in [22]. Then, we denote the arrivals and rationing policy by $T_{B\_RATIO_i}$, the production control by T_{C_PRD}, where $\pi_3 \in \{0, 1\}$ and $c_{\pi_3} = -\pi_3\tau_3$, and the fictitious production by T_{FIC}.

At any time, the decision maker has to decide whether to produce or not and the number of class-i customers to be satisfied from an arriving batch, κ_{i3}. Since we concentrate on profit maximization rather than cost minimization, we assume that a reward of $R_{i3} > 0$ is obtained if a class-i customer is admitted, and without loss of generality we assume that $R_{13} \geq \cdots \geq R_{N3}$

As in the previous example models, we build the discrete-time MDP of the system. We denote the current state of the system by x, the amount of on-hand inventory, the maximum expected total β-discounted reward of the system starting in state x with n remaining transitions by $v_3^n(x)$, where $v_3^n(x) = 0$ for all x, the value function for the expected β-discounted criterion by $v_3(x)$, and the relative value function for the average reward criterion by $v'(x)$. In this setting, for $\beta > 0$, $v_3(x) = \lim_{n\to\infty} v_3^n(x)$ and the optimality equations of the finite horizon problem for the expected β-discounted criterion and infinite horizon problem for the average reward criterion are as in Equations 6.5 and 6.6.

$$v_3^n(x) = T_{COST}(\mu_3 T_{C\_PRD} v_3^{n-1}(x) + \lambda_3 \sum_{i=1}^{N} \sum_{B} p_{iB} T_{B\_RATIO_i} v_3^{n-1}(x) + \theta T_{FIC} v_3^{n-1}), \quad (6.5)$$

$$g_3 + v_3'(x) = T_{COST}(\mu_3 T_{C\_PRD} v_3'(x) + \lambda_3 \sum_{i=1}^{N} \sum_{B} p_{iB} T_{B\_RATIO_i} v_3'(x) + \theta T_{FIC} v_3'(x)), \quad (6.6)$$

where the holding cost function is $h_3(x)$, which is non-decreasing and convex in x.

6.4.1 Structure of the Optimal Policy

In the original model [22], Ha proves the convexity of the value function and then show that the optimal production control and rationing policies are of threshold type and class-1, which has the largest lost sale cost, is the preferred class, i.e., it is always optimal to satisfy class-1 whenever it is possible. We investigate whether similar structures can be proven for the extended model or not. We first concentrate on the concavity of the value function because we seek profit maximization whereas the objective in [22] is cost minimization. To prove the concavity $v_3(x)$, we first prove the concavity of $v_3^n(x)$ for all finite n by induction. The initial condition of the induction holds as a result of the specification on v_3^0. Then, we assume that $v_3^{n-1}(x)$ is concave in x and show $v_3^n(x)$ is also concave in x. In other words, we show the following inequality is true.

$$\begin{array}{c} \mu_3 \Delta T_{C\_PRD} v_3^{n-1}(x) \\ +\lambda_3 \sum_{i=1}^{N} \sum_{B} p_{iB} \Delta T_{B\_RATIO_i} v_3^{n-1}(x) \\ +\Delta\theta T_{FIC} v_3^{n-1}(x) \\ +\Delta h_3(x) \end{array} \leq \begin{array}{c} \mu_3 \Delta T_{C\_PRD} v_3^{n-1}(x+1) \\ +\lambda_3 \sum_{i=1}^{N} \sum_{B} p_{iB} \Delta T_{B\_RATIO_i} v_3^{n-1}(x+1) \\ +\Delta\theta T_{FIC} v_3^{n-1}(x+1) \\ +\Delta h_3(x+1) \end{array} \tag{6.7}$$

First three lines are true due to Lemma 1 and the last line is true by the concavity of $h_3(x)$. Hence, $v_3^n(x)$ is concave in x for all finite n. By using the relationship between the finite and infinite horizon problems, $v_3(x)$ and $v_3'(x)$ are also concave in x.

Now, we state the effects of the concavity of $v_3(x)$ on the optimal actions: Define the values S_3^* and l_{i3}^* for each class-i such that $S_3^* = min\{x : v_3(x) - v_3(x+1) > -\tau_3\}$ and $l_{i3}^* = max\{x : v_3(x-1) - v_3(x) < -R_i\}$, where we set $l_{i3}^* = 0$ if there is no such x. Because of the concavity of $v_3(x)$, for all states $x \geq S_3^*$, the opportunity cost of an additional inventory exceeds the production cost, $-\tau_3$ and thus, it is not worth producing a new unit. Similarly, for all states $x \leq l_{i3}^*$ the optimal rationing policy is rejecting the entire batch which consists of class-i customers. For state $x = l_{i3}^* + 1$, we know that $v(l_{i3}^*) + R_{i3} \geq v(l_i + 1_3^*)$ by the definition of l_{i3}. Therefore, satisfying one class-i customer is better than rejecting the whole batch. Then, assume satisfying two class-i customer is better than satisfying one customer, i.e. $v(l_{i3}^* - 1) + 2R_{i3} \geq v(l_{i3}^*) + R_{i3}$. If we arrange this inequality, we obtain that $v(l_{i3}^* - 1) - v(l_{i3}^*) \geq -R_{i3}$ which contradict with the definition of l_{i3}^*. Thus, the optimal number of class-i customers to be satisfied from an arriving batch for the state $x = l_{i3}^* + 1$, $\kappa_{i3}^*(x)$, is 1. When we iterate this reasoning, $\kappa_{i3}^*(x) = x - l_{i3}^*$ for the states $x > l_{i3}^*$. However, for the states $x \geq l_{i3}^* + B$, $\kappa_{i3}^*(x) = B$ since we can not satisfy more than the number of customers in a batch. Briefly, the optimal rationing policy is to partially satisfy the demand if $l_{i3}^* < x < l_{i3}^* + B$, and satisfy the entire batch if $x \geq l_{i3}^* + B$.

Moreover, if the reward obtained by admitting a class-i batch is higher than the reward of a class-j batch, then the optimal threshold of class-i will be lower than that of class-j as a result of the definition of l_{i3}^* and it is optimal to satisfy class-1 whenever it is possible. The proof of the existence of the preferred class can be seen in the Appendix. The same results are also valid for the average reward criterion. We summarizes our results on Model 3 in the following theorem.

Theorem 4 *In this stock rationing problem,* $\Delta v_3(x) \leq \Delta v_3(x+1)$ *and* $\Delta v_3'(x) \leq \Delta v_3'(x+1)$.

Corollary 4

- *A base-stock policy is the optimal production control policy.* S_3^* *is the critical inventory level such that it is optimal to produce if the on-hand inventory is below* S_3^*.
- *The optimal rationing policy is a sequential threshold policy for each demand class, where the optimal number of class-i customers to be satisfied from an arriving batch for the state* x, $\kappa_{i3}^*(x)$, *is as follows,* $\kappa_{i3}^*(x) = \begin{cases} \min\{B, x - l_{i3}^*\} & \text{if } x > l_{i3}^* \\ 0 & \text{if } x \leq l_{i3}^* \end{cases}$.
- *Moreover,* l_{i3}^**'s are monotone in i and class-1 is the preferred class, i.e.,* $l_{N3}^* \geq \cdots \geq l_{13}^* = 0$

6.4.2 Effects of the Parameters: μ_3, λ_3

In this model, we examine the effects of the changes in the production rate, μ_3, and the arrival rate, λ_3, on the value function and the optimal policies. As in the joint dynamic pricing and replenishment problem in [20], the opportunity cost of an additional inventory is non-decreasing in μ_3 and non-increasing in λ_3. Therefore, we work on the submodularity of $v_3(x)$ with respect to μ_3 and x and the supermodularity of $v_3(x)$ with respect to λ_3 and x. The proof of the supermodularity and the submodularity of the value function and the proof of the effects of the parameters on the optimal actions are similar to previous proofs. We present the results of the sensitivity analysis in the following theorem.

Theorem 5 *In this model:*

- $v_3(x)$ *and* $v_3'(x)$ *are submodular with respect to* μ_3 *and* x.

- $v_3(x)$ *and* $v_3'(x)$ *are supermodular with respect to* λ_3 *and* x.

Corollary 5

- *The optimal base-stock level,* S_3^*, *and the optimal rationing thresholds,* l_{i3}^*, *are non-increasing in* μ_3 *and non-decreasing in* λ_3.

- *The optimal number of class-i customers to be satisfied from an arriving batch for the state* x, $\kappa_{i3}^*(x)$, *is non-decreasing in* μ_3 *and non-increasing in* λ_3.

Chapter 7

2-DIMENSIONAL MODEL

7.1 Introduction

In this section, we work on extending our findings on the sensitivity analysis of one-dimensional models to two-dimensional models. To achieve this aim, we consider a dynamic pricing problem of a queueing system with 2 classes of customers differing in their holding costs. Maglaras [33] studies a similar problem by assuming multiple classes. In this study, he can not prove the structure of the optimal policy but he determine the optimal policy by fluid approximations.

As in one-dimension, it is essential to show the structure of the model before sensitivity analysis. Therefore, we first focus on the structural properties of two-dimensional model that we consider. Then, we investigate the effects of the parameters on the optimal decisions. Unfortunately, we can not prove the results that we expect about the effects of the parameters. This seems to be a shortcoming of using event-based dynamic programming. Hence, the scope of this section is to show the limitations of our approach while considering two-dimensional models.

7.2 The Model

We consider a single server queue with infinite waiting room capacity and 2 classes of customers. Arrivals occur according to a Poisson process with rate λ. At each arrival epoch, the probability that an arriving customer is a class-j customer with $j = 1, 2$ is p_j with $p_1 + p_2 = 1$. Whenever a class-j customer arrives, he either enters the system if his reservation price, R_j, is higher than the announced price or leaves the system without bringing any reward. We assumed that R_j's are random variables with cumulative distribution function of $F_{R_j}(.)$. The service times of all customers are exponentially distributed with mean $1/\mu$ regardless of the class of customers. Moreover, the queue owner incurs a holding cost per unit time, h_j, and without loss of generality it is assumed that $h_1 > h_2$. In this model, we are interested in dynamic pricing policies that maximize the total expected discounted profit with a continuous discount rate β over an infinite horizon as well as the long run average profit.

At any time, the decision maker has to decide which class of customer is served and choose a price from a discrete (but arbitrarily large) set $[p_{min}, p_{max}]$. Under any given feasible scheduling and pricing policy, π, the system evolves as a continuous time Markov chain with state $(X_1(t), X_2(t))$, where $X_j(t)$ is the number of class-j customers in the system. Due to the Markovian property, it is clear that we observe only the current state and do not need to refer to the time, and thus we simply denote the current state of the system by (x_1, x_2), where $(x_1, x_2) \in \mathbb{Z}^2$.

In order to find the optimal policy π^* that maximizes the total expected discounted profit, we build the discrete time equivalent of the original system by using uniformization and normalization. To this end, we assume that the time between two consecutive transitions is exponentially distributed with rate $\mu + \lambda + \theta + \beta$, and using the appropriate time scale, assume that $\mu + \lambda + \theta + \beta = 1$. θ is a fictitious transition (i.e. state of the system does not change) rate and introduced to ensure that the time scale is not affected by the changes in the parameters.

As we will describe some structural properties of a system which operate over an infinite horizon in this example, we first prove these structural properties with the objective of maximizing the expected total β-discounted reward for a finite number of transitions, n. The finite horizon problems allow us to use the induction to prove the structural properties for all finite n. To start the induction we specify the initial function $v_0(x_1, x_2)$ as $v_0(x_1, x_2) = 0$ for all states (x_1, x_2). $v_n(x_1, x_2)$ is the maximum expected total β-discounted reward of the system starting in state (x_1, x_2) with n transitions remaining in the future and the optimality equation of the finite horizon problem is:

$$v_{n+1}(x_1,x_2) = \mu T_{DEP2} v_n(x_1,x_2) + \lambda \sum_{j=1,2} p_j T_{PRC_j} v_n(x_1,x_2) + \theta T_{FIC} v_n(x_1,x_2) - h_1x_1 - h_2x_2$$

where,

$$T_{DEP2}v(x_1,x_2) = \begin{cases} max\left\{v(x_1-1,x_2), v(x_1,x_2-1)\right\} & \text{if } x_1>0,\ x_2>0 \\ max\left\{v(x_1-1,0), v(x_1,0)\right\} & \text{if } x_1>0,\ x_2=0 \\ max\left\{v(0,x_2-1), v(0,x_2)\right\} & \text{if } x_1=0,\ x_2>0 \\ v(0,0) & \text{if } x_1=0,\ x_2=0, \end{cases}$$

$$T_{PRC_1}v(x_1,x_2) = \max_p \left\{\bar{F}_{R_1}(p)[v(x_1+1,x_2)+p] + F_{R_1}(p)v(x_1,x_2)\right\},$$

$$T_{PRC_2}v(x_1,x_2) = \max_p \left\{\bar{F}_{R_2}(p)[v(x_1,x_2+1)+p] + F_{R_2}(p)v(x_1,x_2)\right\},$$

$$T_{FIC}v(x_1,x_2) = v(x_1,x_2).$$

By using the standard arguments of Markov Decision theory (See Puterman [42]), there exists an optimal stationary policy for the infinite horizon problem and $v(x_1,x_2) = \lim_{n\to\infty} v_n(x_1,x_2)$ whenever $\beta > 0$. $v(x_1,x_2)$ is the value function of the infinite horizon problem. Therefore, structural results obtained for $v_n(x_1,x_2)$ hold for $v(x_1,x_2)$. Moreover, these structural results are also true for the average reward criterion as a result of the conditions introduced by Weber and Stidham [52]. The optimality equation for the average reward criterion is as follows, where g^* is the optimal expected revenue per unit time and $v'(x_1,x_2)$ is the relative value function:

$$g^* + v'(x_1,x_2) = \mu T_{DEP2} v'(x_1,x_2) + \lambda \sum_{j=1,2} p_j T_{PRC_j} v'(x_1,x_2) + \theta T_{FIC} v'(x_1,x_2) - h_1x_1 - h_2x_2$$

7.3 Structure of the Model

As in the one-dimensional models, we first focus on the monotonicity property to determine the existence of opportunity cost. However, in two-dimensional models, we need to observe three monotonicity properties: monotonicity in x_1, monotonicity in x_2 and monotonicity on the diagonal. We denote these properties as:

$$\text{Monotonicity in } x_1: \quad v(x_1,x_2) \geq v(x_1+1,x_2) \tag{7.1}$$

$$\text{Monotonicity in } x_2: \quad v(x_1,x_2) \geq v(x_1,x_2+1) \tag{7.2}$$

$$\text{Monotonicity on the diagonal:} \quad v(x_1,x_2+1) \geq v(x_1+1,x_2) \tag{7.3}$$

Equation B.1 implies that when a new class-1 customer enters the system, the expected discounted profit decreases. In other words, it implies a positive opportunity cost of an additional class-1 customer. Similarly, Equation B.2 implies a positive opportunity cots of an additional class-2 customer. Besides these equations, diagonal monotonicity, Equation B.3, means that when a class-2 customer is changed by a class-1 customer, system incurs a positive burden because of the higher holding cost of class-1 customers, so that this property implies the opportunity cost of an additional expensive, class-1, customer instead of a cheaper, class-2, customer. For the sake of simplicity, we define

$$\begin{aligned} \Delta_1 v(x_1, x_2) &= v(x_1, x_2) - v(x_1 + 1, x_2), \\ \Delta_2 v(x_1, x_2) &= v(x_1, x_2) - v(x_1, x_2 + 1), \\ \Delta_D v(x_1, x_2) &= v(x_1, x_2 + 1) - v(x_1 + 1, x_2). \end{aligned}$$

Then, we denote the opportunity cost of class-1 by $\Delta_1 v(x_1, x_2)$, the opportunity cost of class-2 by $\Delta_2 v(x_1, x_2)$ and the opportunity cost of an additional expensive customer instead of a cheaper one.

After monotonicity properties, we focus on concavity properties because concavity represents the monotonicity of opportunity costs and optimal decisions are directly related with the monotonicity of opportunity costs as we mentioned for the one-dimensional models. Since we have two dimensions, we work on concavity in both x_1 and x_2. In the following equations, Equation 7.4 implies the concavity of $v(x_1, x_2)$ in x_1, i.e., the opportunity cost of a class-1 customer is non-decreasing in x, and Equation 7.5 implies the concavity of $v(x_1, x_2)$ in x_2:

$$\Delta_1 v(x_1, x_2), \leq \Delta_1 v(x_1 + 1, x_2) \tag{7.4}$$

$$\Delta_2 v(x_1, x_2) \leq \Delta_2 v(x_1, x_2 + 1). \tag{7.5}$$

Although concavity properties are quite intuitive, it is difficult to prove these equations directly. Therefore, we use some supporting properties in order to prove concavity. These supporting properties are submodularity and subconcavity. The most significant advantage of these properties is that their sum implies the concavity equations, and thus we can prove concavity of the model by proving both submodularity and subconcavity. Here, the definition of submodularity is the same as the one we used in sensitivity analysis but it means that the opportunity cost of a class-1 (class-2) customer is non-decreasing in x_2 (x_1) rather than a system parameter. On the other hand, subconcavity is the monotonicity of the opportunity cost on the diagonal. Since we have two classes of customers, subconcavity consists of two conditions: The first condition says that the opportunity cost of a class-1 customer is non-decreasing on the diagonal whereas the second condition says that the opportunity cost of a class-2 customer is non-increasing on the diagonal. We present submodularity and subconcavity as:

$$\begin{aligned}
\text{Submodularity:} \quad & \Delta_1 v(x_1, x_2) \le \Delta_1 v(x_1, x_2+1) \text{ or } \Delta_2 v(x_1, x_2) \le \Delta_2 v(x_1+1, x_2) \\
\text{Subconcavity } 1^{st} \text{ condition:} \quad & \Delta_1 v(x_1, x_2+1) \le \Delta_1 v(x_1+1, x_2) \\
\text{Subconcavity } 2^{nd} \text{ condition:} \quad & \Delta_2 v(x_1+1, x_2) \le \Delta_2 v(x_1, x_2+1)
\end{aligned}$$

The following figure illustrates the relationship between submodularity, subconcavity and concavity:

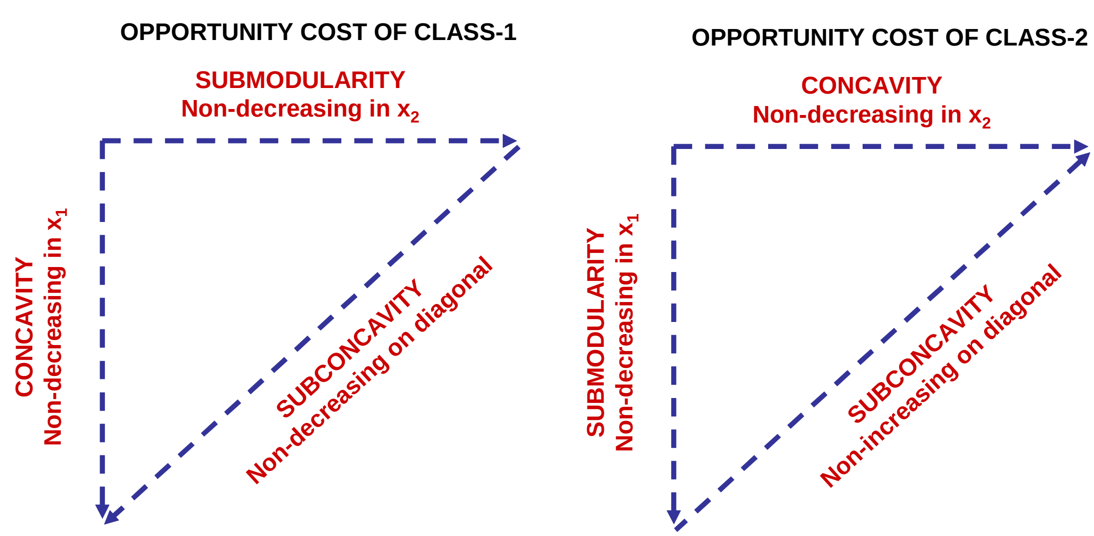

Figure 7.1: The relationship between submodularity, subconcavity and concavity

Before studying the whole value function, we first focus on the operators and show that all operators preserve the monotonicity and concavity properties of a function, $v(x_1, x_2)$, on which they are applied. The proof for the fictitious operator is obvious and the proofs for the other operators can be seen in the appendix. We present the results about the operators in the following lemma by using a similar notation to the previous lemmas: For a certain event operator, T, $Prop_1, \ldots, Prop_k \rightarrow Prop_j$ denotes that if the function v has the properties $Prop_1, \ldots, Prop_k$ then Tv has the property $Prop_j$. We denote monotonicity in x_1, monotonicity in x_2, monotonicity on the diagonal, submodularity, subconcavity 1^{st} condition and subconcavity 2^{nd} by $Monx_1$, $Monx_2$, $MonD$, $SubM$, $SubC_1$ and $SubC_2$, respectively.

Lemma 5

- *T_{DEP2}: $Monx_1$, $Monx_1$, $MonD \rightarrow Monx_1$, $Monx_1$, $MonD$.*
- *T_{PRC_1}: $Monx_1 \rightarrow Monx_1$; $Monx_2 \rightarrow Monx_2$; $MonD \rightarrow MonD$.*
- *T_{PRC_2}: $Monx_1 \rightarrow Monx_1$; $Monx_2 \rightarrow Monx_2$; $MonD \rightarrow MonD$.*
- *T_{FIC}: $Monx_1 \rightarrow Monx_1$; $Monx_2 \rightarrow Monx_2$; $MonD \rightarrow MonD$*
- *T_{DEP2}: $Monx_1$, $Monx_1$, $MonD$, $SubM \rightarrow SubM$.*
- *T_{DEP2}: $Monx_1$, $Monx_1$, $MonD$, $SubM$, $SubC_1$, $SubC_2 \rightarrow SubC_1$, $SubC_2$.*
- *T_{PRC_1}: $SubM \rightarrow SubM$; $SubM$, $SubC_1$, $SubC_2 \rightarrow SubC_1$, $SubC_2$.*
- *T_{PRC_2}: $SubM \rightarrow SubM$; $SubM$, $SubC_1$, $SubC_2 \rightarrow SubC_1$, $SubC_2$.*
- *T_{FIC}: $SubM \rightarrow SubM$; $SubC_1 \rightarrow SubC_1$; $SubC_2 \rightarrow SubC_2$.*

7.3.1 Monotonicity of $v(x_1, x_2)$

Now, we prove the monotonicity of the value function, i.e. show that monotonicity equations, Equation B.1, B.2 and B.3, are true for the value function, $v(x_1, x_2)$. For this purpose, we first work on the finite horizon value function, $v_n(x_1, x_2)$, and show the monotonicity properties of $v_n(x_1, x_2)$ for all finite n by induction.

Initial condition of the induction is true because we set $v_0(x_1, x_2) = 0$ for all (x_1, x_2). Then, we show that $v_{n+1}(x_1, x_2)$ is monotone in x_1, x_2 and on the diagonal by assuming $v_n(x_1, x_2)$ is monotone in x_1, x_2 and on the diagonal. Monotonicity equations for

$v_{n+1}(x_1,x_2)$ can be written according to the optimality equation as follows, respectively:

$$\begin{array}{ccc} \mu T_{DEP2} v_n(x_1,x_2) & & \mu T_{DEP2} v_n(x_1+1,x_2) \\ +\lambda \sum\limits_{j=1,2} p_j T_{PRC_j} v_n(x_1,x_2) & \geq & +\lambda \sum\limits_{j=1,2} p_j T_{PRC_j} v_n(x_1+1,x_2) \\ +\theta T_{FIC} v_n(x_1,x_2) & & +\theta T_{FIC} v_n(x_1+1,x_2) \\ -h_1 x_1 - h_2 x_2 & & -h_1(x_1+1) - h_2 x_2 \end{array}$$

$$\begin{array}{ccc} \mu T_{DEP2} v_n(x_1,x_2) & & \mu T_{DEP2} v_n(x_1,x_2+1) \\ +\lambda \sum\limits_{j=1,2} p_j T_{PRC_j} v_n(x_1,x_2) & \geq & +\lambda \sum\limits_{j=1,2} p_j T_{PRC_j} v_n(x_1,x_2+1) \\ +\theta T_{FIC} v_n(x_1,x_2) & & +\theta T_{FIC} v_n(x_1,x_2+1) \\ -h_1 x_1 - h_2 x_2 & & -h_1 x_1 - h_2(x_2+1) \end{array}$$

$$\begin{array}{ccc} \mu T_{DEP2} v_n(x_1,x_2+1) & & \mu T_{DEP2} v_n(x_1+1,x_2) \\ +\lambda \sum\limits_{j=1,2} p_j T_{PRC_j} v_n(x_1,x_2+1) & \geq & +\lambda \sum\limits_{j=1,2} p_j T_{PRC_j} v_n(x_1+1,x_2) \\ +\theta T_{FIC} v_n(x_1,x_2+1) & & +\theta T_{FIC} v_n(x_1+1,x_2) \\ -h_1 x_1 - h_2(x_2+1) & & -h_1(x_1+1) - h_2 x_2 \end{array}$$

In all of these three equations, the last lines are true due to holdings costs, h_1 and h_2, and the assumption of $h_1 > h_2$. Moreover, the first three lines are true by Lemma 5. Hence, we show that if $v_n(x_1,x_2)$ has all of the three monotonicity properties, $v_{n+1}(x_1,x_2)$ will also have these properties, and thus $v_n(x_1,x_2)$ is monotone in x_1, x_2 and on the diagonal for all finite n. Since $v(x_1,x_2) = \lim_{n\to\infty} v_n(x_1,x_2)$, $v(x_1,x_2)$ is also monotone in x_1, x_2 and on the diagonal. The same properties are also true for $v'(x_1,x_2)$, the relative value function. Since $v(x_1,x_2+1) \geq v(x_1+1,x_2)$ by the monotonicity of the value function on the diagonal, serving a class-1 customer is always more preferable than serving a class-2 customer, i.e. class-1 has priority on service. The following theorem and corollary summarize our results about the monotonicity of the value function and the structure of the optimal service policy.

Theorem 6 *In this model, $v(x_1,x_2)$ and $v'(x_1,x_2)$ are non-increasing in x_1, x_2 and on the diagonal.*

Corollary 6 *The optimal service policy is to serve class-1 first whenever there is a class-1 customer in the queue.*

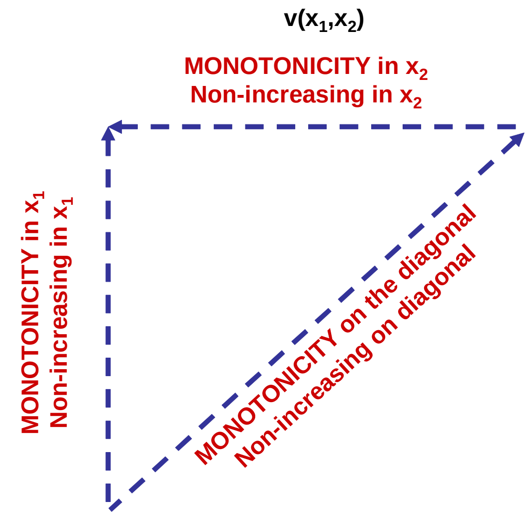

Figure 7.2: The illustration of Theorem 6

7.3.2 Submodularity, Subconcavity and Concavity of $v(x_1, x_2)$

As a result of the monotonicity properties of $v(x_1, x_2)$, we determine the optimal service policy. However, we still need to determine the optimal pricing policy. Therefore, we work on the concavity of the value function since it implies the monotonicity of the opportunity cost, and also implies the monotonicity of the optimal prices. Instead of proving the concavity of $v(x_1, x_2)$, we concentrate on the submodularity and subconcavity of value function because submodularity and subconcavity together add up to concavity. Since the proofs of submodularity and subconcavity is so similar, we only show the proof of submodularity.

As in the previous proofs, we start with considering the finite horizon problem and prove the submodularity of $v_n(x_1, x_2)$ for all finite n by induction. Initial condition, $\Delta_1 v_0(x_1, x_2) \leq \Delta_1 v_0(x_1, x_2+1)$, is true because of the specification that $v_0(x_1, x_2) = 0$ for all states (x_1, x_2). Then, we assume the submodularity of $v_n(x_1, x_2)$ and show that of $v_{n+1}(x_1, x_2)$. We can write the submodularity equation for $v_{n+1}(x_1, x_2)$ as:

$$\begin{array}{ccc} \mu\Delta_1 T_{DEP2} v_n(x_1, x_2) & & \mu\Delta_1 T_{DEP2} v_n(x_1, x_2+1) \\ +\lambda \sum\limits_{j=1,2} p_j \Delta_1 T_{PRC_j} v_n(x_1, x_2) & \leq & +\lambda \sum\limits_{j=1,2} p_j \Delta_1 T_{PRC_j} v_n(x_1, x_2+1) \\ +\theta\Delta_1 T_{FIC} v_n(x_1, x_2) & & +\theta\Delta_1 T_{FIC} v_n(x_1, x_2+1). \end{array} \tag{7.6}$$

We know that $v_n(x_1, x_2)$ is monotone in x_1, x_2 and on the diagonal by Theorem 6, and assume that $v_n(x_1, x_2)$ is submodular as the induction hypothesis. Thus, Equation 7.6 is true by Lemma 5 and $v_n(x_1, x_2)$ is submodular, i.e. $\Delta_1 v_n(x_1, x_2) \leq \Delta_1 v_n(x_1, x_2 + 1)$, for all finite n. Then, $v(x_1, x_2)$ and $v'(x_1, x_2)$ are also submodular as $v_n(x_1, x_2)$ converges to $v(x_1, x_2)$. By a similar proof, $v(x_1, x_2)$ and $v'(x_1, x_2)$ satisfy both of the subconcavity properties. Furthermore, when we add submodularity and subconcavity 1^{st} condition, we obtain the concavity of $v(x_1, x_2)$ in x_1, and similarly when we add submodularity and subconcavity 2^{nd} condition, we obtain the concavity of $v(x_1, x_2)$ in x_2. As in the one-dimensional models, monotone opportunity costs lead to monotone optimal prices in this model and we present the structure of the optimal prices in the following theorem. Theorem 7 and Corollary 7 summarizes the structure of the value function and the optimal pricing policy, respectively. The proof of the structure of the optimal prices can be seen in the appendix.

Theorem 7 *In this model, $v(x_1, x_2)$ and $v'(x_1, x_2)$ satisfy submodularity, subconcavity 1^{st} and subconcavity 2^{nd} conditions, and they are concave in both x_1 and x_2.*

Corollary 7 *The optimal prices for class-1 customers, $p_1^*(x_1, x_2)$, are non-decreasing in x_1, x_2 and on the diagonal, whereas the optimal prices for class-2 customers, $p_2^*(x_1, x_2)$, are non-decreasing in x_1, x_2 and non-increasing on the diagonal.*

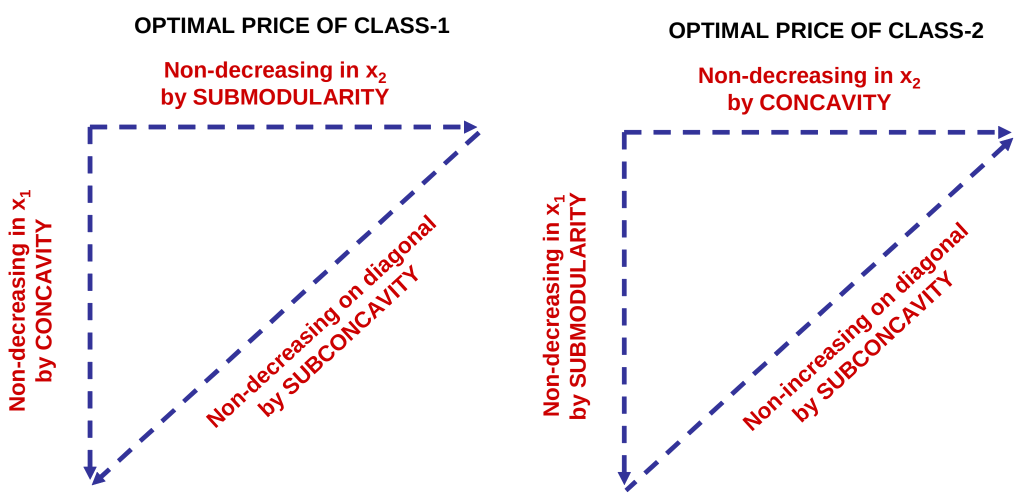

Figure 7.3: Structure of the optimal prices in the 2-dimensional model

7.4 Sensitivity Analysis

We aim to investigate the effects of changes in the system parameters, the service rate (μ) and the arrival rate (λ), on the optimal prices in this subsection. As we did in one-dimensional models, we compare the opportunity costs in a system with parameter α and the opportunity costs in a system with parameter $\alpha+\varepsilon$. Intuitively, the opportunity cost of a class-1 customer, $\Delta_1 v(x_1,x_2)$, the opportunity cost of a class-2 customer, $\Delta_2 v(x_1,x_2)$ and the opportunity cost of an additional expensive customer instead of an additional cheap customer, $\Delta_D v(x_1,x_2)$, decreases by an increase in the service rate and increases by an increase in the arrival rate. The intuition is clear because if the system serves the existing customers faster, then the expected average waiting time will decrease, and thus the burden of the customers on the system will decrease. In order to represent this intuition, we define two set of properties $C_1(\alpha)$ and $C_2(\alpha)$, such that:

If $v(x_1,x_2) \in C_1(\alpha)$, then

$$\Delta_1 v(x_1,x_2) \geq \Delta_1 \tilde{v}(x_1,x_2)$$
$$\Delta_2 v(x_1,x_2) \geq \Delta_2 \tilde{v}(x_1,x_2)$$
$$\Delta_D v(x_1,x_2) \geq \Delta_D \tilde{v}(x_1,x_2)$$

If $v(x_1,x_2) \in C_2(\alpha)$, then

$$\Delta_1 v(x_1,x_2) \leq \Delta_1 \tilde{v}(x_1,x_2)$$
$$\Delta_2 v(x_1,x_2) \leq \Delta_2 \tilde{v}(x_1,x_2)$$
$$\Delta_D v(x_1,x_2) \leq \Delta_D \tilde{v}(x_1,x_2),$$

where $\tilde{v}(x_1,x_2)$ is the value function of the system with parameter $\alpha+\varepsilon$.

By using this definition, we will show that $v(x_1,x_2) \in C_1(\mu)$ and $v(x_1,x_2) \in C_2(\lambda)$ to prove our intuition. As we use event-based dynamic programming, we need to focus on the operators before the whole value function and have to prove that all of the operators preserve the same set of properties that the function on which they are applied, $v(x_1,x_2)$, keeps. However, we encounter a contradiction while working on the pricing operator.

To illustrate this contradiction, let us consider the price set $\{p_L,p_H\}$ where $F_R(p_L)=1$ and $F_R(p_H)$=0 and concentrate on the case where the optimal price of class-1 for the states (x_1,x_2) and (x_1,x_2+1) are p_L and p_H, respectively, before and after an increase in the service rate. In other words, the optimal price for the states x and $x+1$ are not affected by an increase in the service rate. This might be one of the possible cases, and in this case, we have that:

$$T_{PRC_1}v(x_1,x_2) = p_L + v(x_1+1,x_2), \tag{7.7}$$

$$T_{PRC_1}v(x_1,x_2+1) = v(x_1,x_2+1), \tag{7.8}$$

$$T_{PRC_1}\tilde{v}(x_1,x_2) = p_L + \tilde{v}(x_1+1,x_2), \tag{7.9}$$

$$T_{PRC_1}\tilde{v}(x_1,x_2+1) = \tilde{v}(x_1,x_2+1). \tag{7.10}$$

According to our intuition, we have $v(x_1,x_2) \in C_1(\mu)$, so that we need to show if $v(x_1,x_2) \in C_1(\mu)$, then $Tv(x_1,x_2) \in C_1(\mu)$ for all operators. When we specifically focus on the second condition of $C_1(\mu)$ for the pricing operator, $\Delta_2 Tv(x_1,x_2) \geq \Delta_2 T\tilde{v}(x_1,x_2)$, by considering our example case, we see that this condition can be written as follows by using Equations 7.7-7.10:

$$v(x_1+1,x_2) - v(x_1,x_2+1) \geq \tilde{v}(x_1+1,x_2) - \tilde{v}(x_1,x_2+1),$$

and it definitely contradicts with the third condition of the set $C_1(\mu)$. Hence, we can not prove that T_{PRC_1} preserves all of the properties of $C_1(\mu)$, and our approach, based on event-based dynamic programming, fails during the sensitivity analysis of this model. We think that similar problems will arise in other 2-dimensional models.

We can also illustrate the same contradiction by a numerical example. In this example, we set the parameters as: $\mu = 150$, $\lambda = 100$, $\theta = 50$, $p_1 = 0.5$, $p_2 = 0.5$, $h_1 = 1$, $h_2 = 0.8$. Moreover, we specify the price set as $\{100, 200\}$ where $F_R(100) = 1$ and F_R(200)=0. To see the effects of an increase in the service rate, we increase μ from 150 to 151.

We exhibits the result of this numerical study for the states $(1,20)$, $(1,21)$, $(2,20)$ and $(2,21)$ in Table 7.1. The value functions and the optimal pricing policy for this states can be seen in this table. Then, using the value functions, we calculate $T_{PRC_1}v(x_1,x_2)$ values for the states $(1,20)$, $(1,21)$, $(2,20)$ and $(2,21)$ as in Table 7.2. We also illustrate that although the value function keeps all of the properties of the set $C_1(\mu)$, $T_{PRC_1}v(x_1,x_2)$ may not have the second property of this set in Table 7.2. Hence, we can not prove the effects of the parameters on the optimal prices by using event-based dynamic programming technique. However, this does not mean that these effects can not be proven by another technique because we numerically observe the monotonicity of the optimal prices in the system parameters as in the one-dimensional models.

$v(x_1, x_2)$: value function of the system with $\mu = 150$

(x_1, x_2)	20	21
1	27120.04	27021.67
2	27020.78	26920.8

$\tilde{v}(x_1, x_2)$: value function of the system with $\mu = 151$

(x_1, x_2)	20	21
1	26944.62	26847.03
2	26846.15	26746.77

$p_1^*(x_1, x_2)$: optimal price for class-1 in the system with $\mu = 150$

(x_1, x_2)	20	21
1	100	200
2	200	200

$p_1^*(x_1, x_2)$: optimal price for class-1 in the system with $\mu = 151$

(x_1, x_2)	20	21
1	100	200
2	200	200

Table 7.1: Numerical results of the effects of the service rate on the value function and the optimal prices of the example 2-dimensional model

$T_{PRC_1} v(1, 20)$=27120.78	$\Delta_2 T_{PRC_1} v_n(1, 20) = 99.11 \leq \Delta_2 T_{PRC_1} \tilde{v}_n(1, 20) = 99.12$
$T_{PRC_1} v(1, 21)$=27021.67	$\Delta_1 v(1, 20) = 99.26 \geq 98.47 = \Delta_1 \tilde{v}(1, 20)$
$T_{PRC_1} \tilde{v}(1, 20)$=26946.15	$\Delta_2 v(1, 20) = 98.37 \geq 97.59 = \Delta_2 \tilde{v}(1, 20)$
$T_{PRC_1} \tilde{v}(1, 21)$=26847.03	$\Delta_D v(1, 20) = 0.89 \geq 0.88 = \Delta_D \tilde{v}(1, 20)$.

Table 7.2: Counter example to show that T_{PRC_1} does not preserve all of the properties of the set $C_1(\mu)$

Chapter 8

CONCLUSION

In the scope of this thesis, we aimed to establish a general framework to perform sensitivity analysis on a class of Markovian queueing and inventory systems. We consider dynamic control policies employed in these systems because the applications of such policies are gaining popularity both in the literature and industry. Specifically, we consider pricing control, admission control and stock rationing problems.

Since our objective is to understand the behavior of the optimal policies when system parameters change, we use event-based dynamic programming as an approach to prove the structure of the models and optimal policies. To use this approach, we first define certain event operators to represent the events occurring in Markovian queueing and inventory systems such as, arrival, service completion, replenishment, pricing, rationing, etc. Then, we show that the operators preserve monotonicity, concavity, supermodularity and submodularity properties of the function on which they are applied.

After studying the properties of individual operators, we focus on some existing queueing and inventory problems in the literature. We first work on the joint pricing and replenishment problem of a make-to-stock inventory system in which the demand arrival process is assumed to be a Markov Modulated Poisson process. As a result of sensitivity analysis, we establish that the optimal base-stock level and the optimal prices are non-increasing in the production rate and non-decreasing in the arrival rate. The intuition behind this result is: When the system can process items faster, the opportunity cost of inventory on-hand increases, so that we do not want to produce as much as we produced before, and we encourage demand arrivals by decreasing the price. Similarly, when the arrival rate increases, the opportunity cost of inventory on-hand decreases since we expect more customers. Then, obviously, we want to produce more and increase the price to obtain more profit.

The second problem that we consider is the admission control of a queueing system with customers arriving in batches. For this problem, we observe that the optimal number of customers to be admitted from an arriving batch is non-decreasing in the service rate and non-increasing in the arrival rate. The intuition is similar to the previous one but there are some differences since the model is a queueing system. When the service rate increases, the opportunity cost of a new customer decreases because we can serve the existing customers faster, and thus we can admit more customers. On the other hand, the opportunity cost of a new customer increases by an increasing in the arrival rate since the system load increases. Therefore, we do not want to admit customers as much as we did before.

While considering the third problem, the stock rationing problem of a make-to-stock inventory system with multiple arrivals and lost sales, we not only perform a sensitivity analysis but also extend the original model by allowing multiple demand requests. Since we provide some extensions, we determine the structure of the optimal policy before the sensitivity analysis. We establish that the optimal replenishment policy is a base-stock policy and the optimal rationing policy is a sequential threshold policy. Then, we work on the effects of parameters on the optimal policies. As a result, we obtain that the optimal base-stock level and the thresholds are non-increasing in the production rate and non-decreasing in the arrival rate. The intuition of the result is very similar to the first model.

After investigating several one-dimensional models, we consider a two-dimensional model which is a pricing problem of a queueing system with 2 classes of customers whose holding costs are different. We prove that the optimal prices for each class is monotone in the number of customers in the system. However, we can not establish the structure of the optimal prices when system parameters change. This manifests the limitation of our approach for general multi-dimensional systems.

In conclusion, we constructed a general framework to analyze the effects of the changes in system parameters on the optimal policies. As we illustrated, our framework can be used to perform sensitivity analysis on a class of Markovian queueing and inventory systems. In addition to the three models that we consider, sensitivity analysis can also be performed on some of the papers mentioned in the literature review. Our framework generalizes several existing results on specific models in the literature, and provides a structural methodology to perform sensitivity analysis for any problem that can be described by the operators described here.

A natural extension of this study is to establish a similar framework for non-Markovian queueing and inventory systems, especially for systems where the arrival times are generally distributed. Since most of the structural properties preserved by Markovian systems continue to be valid for non-Markovian systems, this extension seems promising. There could also be other interesting extensions on particular operators and different problems. For instance, one possible path might be to investigate the effects of changes in the reservation price distribution in pricing control problems, to allow batch production in the inventory problems, to add perishability to the inventory models by considering inventory as decaying over time or to assume that customers have a patience time and abandon the system if they are not served until this time.

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Appendix A

ONE- DIMENSIONAL MODELS

A.1 Proof: Monotonicity of the Operators

In this proof, we show the monotonicity of a certain operator, T, under the assumption on $f(x)$. In other words, we prove the following equation for all queueing related operators (T_{ARR}, T_{DEP}, T_{CD}, T_{Q_PRC}, $T_{B\_ADM_i}$) where $f(x)$ is a non-increasing function of x. We show all of the proofs for the incapacitated queues and note that the proofs are also valid for the capacitated ones.

$$Tf(x) \geq Tf(x+1) \tag{A.1}$$

A.1.1 Monotonicity of T_{ARR}

We can write Equation A.1 for T_{ARR} as follows by using the definition of the operator:

$$a(x)f(x+1) + [1-a(x)]f(x) \geq a(x+1)f(x+2) + [1-a(x+1)]f(x+1).$$

When we rearrange this equation, we obtain that,

$$\begin{array}{ccc} [1-a(x)]f(x) & \geq & [1-a(x)]f(x+1) \\ +a(x+1)f(x+1) & & +a(x+1)f(x+2) \end{array} . \tag{A.2}$$

Both of the lines are true by the assumption on $f(x)$ and thus, we complete the proof of the monotonicity of $T_{ARR}f(x)$. For the capacitated case, we need to observe the boundary effects, i.e., the state $x = K - 1$, where K is the waiting room capacity of the system (including the servers). For this state, $a(x+1) = 0$ because the system can not admit new arrivals when there are already K customers in the system. Therefore, the last line of Equation A.2 becomes 0 and then Equation A.1 still holds for the capacitated case.

A.1.2 Monotonicity of T_{DEP}

Similar to the previous proof, we can write and rearrange Equation A.1 as follows:

$$\begin{array}{c} b(x)f(x-1) \\ +[1-b(x+1)]f(x) \end{array} \geq \begin{array}{c} b(x)f(x) \\ +[1-b(x+1)]f(x+1) \end{array},$$

and the equation is true by the monotonicity of $f(x)$. Therefore, the departure operator,$T_{DEP}f(x)$, will be non-increasing in x if $f(x)$ is non-increasing in x. Since the waiting room capacity of the system does not affect the departures, we do not work on the proof of capacitated case.

A.1.3 Monotonicity of T_{CD}

Let π_x and π_{x+1} be the optimal service rates for the states x and $x+1$, respectively. Then, we can write Equation A.1 for T_{CD} as follows:

$$c_{\pi_x} + \pi_x f(x-1) + (1-\pi_x)f(x) \geq c_{\pi_{x+1}} + \pi_{x+1}f(x) + (1-\pi_{x+1})f(x+1). \tag{A.3}$$

By the definition of the operator and the monotonicity of $f(x)$, we have,

$$\begin{aligned} c_{\pi_x} + \pi_x f(x-1) + (1-\pi_x)f(x) &\geq c_{\pi_{x+1}} + \pi_{x+1}f(x-1) + (1-\pi_{x+1})f(x), \textit{and} \\ c_{\pi_{x+1}} + \pi_{x+1}f(x-1) + (1-\pi_{x+1})f(x) &\geq c_{\pi_{x+1}} + \pi_{x+1}f(x) + (1-\pi_{x+1})f(x+1). \end{aligned}$$

When we combine these two equations, we obtain that Equation A.3 holds and thus, $T_{CD}f(x)$ is non-increasing in x. As in the departure operator, we do not observe the capacitated case.

A.1.4 Monotonicity of T_{Q_PRC}

Let p_x and p_{x+1} be the optimal prices for the states x and $x+1$, respectively. Then, Equation A.1 for T_{Q_PRC} is as follows:

$$\bar{F}_R(p_x)[f(x+1)+p_x] + F_R(p_x)f(x) \geq \bar{F}_R(p_{x+1})[f(x+2)+p_{x+1}] + F_R(p_{x+1})f(x+1) \tag{A.4}$$

As in the monotonicity of the controlled departure, we have the following equations by the definition of the operator and the monotonicity of $f(x)$.

$$\begin{aligned} \bar{F}_R(p_x)[f(x+1)+p_x] + F_R(p_x)f(x) &\geq \bar{F}_R(p_{x+1})[f(x+1)+p_{x+1}] + F_R(p_{x+1})f(x) \\ \bar{F}_R(p_{x+1})[f(x+1)+p_{x+1}] + F_R(p_{x+1})f(x) &\geq \bar{F}_R(p_{x+1})[f(x+2)+p_{x+1}] + F_R(p_{x+1})f(x+1). \end{aligned}$$

Then, when we combine these equations, we obtain that Equation A.4 holds and thus, T_{Q_PRC} is non-increasing in x. The pricing operator can also be used in the capacitated queues. In this case, we need to observe the monotonicity of the operator for the state $x = K - 1$. However, since we use the optimality of p_x, the foregoing proof is still true for the capacitated queues.

A.1.5 Monotonicity of $T_{B\_ADM_i}$

Let κ_x and κ_{x+1} be the optimal number of class-i customers to be admitted form an arriving batch. Then, we can write Equation A.1 for the operators as follows:

$$\kappa_x R_i + f(x + \kappa_x) \geq \kappa_{x+1} R_i + f(x + 1 + \kappa_{x+1}) \tag{A.5}$$

Since κ_x is the optimal action for the state x and $f(x)$ is non-increasing in x, we have,

$$\begin{aligned} \kappa_x R_i + f(x + \kappa_x) \quad &\geq \quad \kappa_{x+1} R_i + f(x + \kappa_{x+1}), \mathit{and} \\ \kappa_{x+1} R_i + f(x + \kappa_{x+1}) \quad &\geq \quad \kappa_{x+1} R_i + f(x + 1 + \kappa_{x+1}). \end{aligned}$$

As in the previous proofs, when we combine these equations we complete the proof. Therefore, $T_{B\_ADM_i} f(x)$ is non-increasing in x if $f(x)$ is a non-increasing function of x. As in the pricing operator, the proof of the monotonicity for the capacitated queues are the same as the incapacitated queues since we use the optimality of κ_x.

A.2 Proof: Upper-Bounded Difference, UBD

In this proof, we show that if $f(x)$ is an UBD function then $Tf(x)$ will also be an UBD function for a certain event operator, T. In other words, we prove the following equation for each operator under the assumption that $f(x) - f(x+1) \leq R$.

$$Tf(x) - Tf(x+1) \leq R \tag{A.6}$$

Since the UBD property of $f(x)$ is preserved only by the queueing related operators, we only work on T_{ARR}, T_{DEP}, T_{CD}, T_{Q_PRC} and $T_{B\_ADM_i}$. As in the monotonicity proof, we prove Equation A.6 for the incapacitated queues and then, note that the proofs are also valid for the capacitated cases. We only focus on the arrival related operators while considering the capacitated cases because the waiting room capacity does not affect the departures.

A.2.1 UBD of T_{ARR}

We can write Equation A.6 for the arrival operator as follows:

$$\begin{array}{c} a(x)f(x+1)+[1-a(x)]f(x) \\ -a(x+1)f(x+2)-[1-a(x+1)]f(x+1) \end{array} \leq R. \tag{A.7}$$

Since $a(x) \geq a(x+1)$ by the definition of the function a and by the assumption that $f(x) - f(x+1) \leq R$, we have,

$$\begin{array}{ccc} \begin{array}{c} a(x)f(x+1)+[1-a(x)]f(x) \\ -a(x+1)f(x+2)-[1-a(x+1)]f(x+1) \end{array} & = & \begin{array}{c} a(x+1)[f(x+1)-f(x+2)] \\ +[1-a(x)][f(x)-f(x+1)] \end{array} \\ & \leq & [1-a(x)+a(x+1)]R \\ & \leq & R. \end{array}$$

Thus, we complete the proof and $T_{ARR}f(x) - T_{ARR}f(x+1) \leq R$ if $f(x) - f(x+1) \leq R$. For the capacitated case, we observe the state $x = K - 1$ and for this state $a(x+1) = 0$. Therefore, $[1-a(x)+a(x+1)]R$ is still less than or equal to R and $T_{ARR}f(x) - T_{ARR}f(x+1) \leq R$ is also true for the capacitated case.

A.2.2 UBD of T_{DEP}

Equation A.6 for this operator can be written as follows:

$$\begin{array}{c} b(x)f(x-1)+[1-b(x)]f(x) \\ -b(x+1)f(x)-[1-b(x)]f(x+1) \end{array} \leq R. \tag{A.8}$$

When we rearrange the left hand side, we obtain that,

$$\begin{array}{c} b(x)f(x-1)+[1-b(x)]f(x) \\ -b(x+1)f(x)-[1-b(x)]f(x+1) \end{array} = \begin{array}{c} b(x)[f(x-1)-f(x)] \\ +[1-b(x+1)][f(x)-f(x+1)] \end{array},$$

and by the definition of $b(x)$ and the assumption on $f(x)$,

$$\begin{array}{c} b(x)[f(x-1)-f(x)] \\ +[1-b(x+1)][f(x)-f(x+1)] \end{array} \leq [1+b(x)-b(x+1)]R \leq R.$$

Therefore, Equation A.8 is true and $T_{DEP}f(x) - T_{DEP}f(x+1) \leq R$ if $f(x) - f(x+1) \leq R$.

A.2.3 UBD of T_{CD}

Let π_x be the optimal service rate for the state x. Then, we can write Equation A.6 for T_{CD} as follows:

$$\begin{array}{c} c_{\pi_x} + \pi_x f(x-1) + (1-\pi_x) f(x) \\ -c_{\pi_{x+1}} - \pi_{x+1} f(x) - (1-\pi_{x+1}) f(x+1) \end{array} \leq R. \tag{A.9}$$

As a result of the optimality of π_{x+1}, we have that,

$$\begin{array}{c} c_{\pi_x} + \pi_x f(x-1) + (1-\pi_x) f(x) \\ -c_{\pi_{x+1}} - \pi_{x+1} f(x) - (1-\pi_{x+1}) f(x+1) \end{array} \leq \begin{array}{c} c_{\pi_x} + \pi_x f(x-1) + (1-\pi_x) f(x) \\ -c_{\pi_x} - \pi_x f(x) - (1-\pi_x) f(x+1) \end{array}.$$

By rearranging the right hand side,

$$\begin{array}{c} c_{\pi_x} + \pi_x f(x-1) + (1-\pi_x) f(x) \\ -c_{\pi_{x+1}} - \pi_{x+1} f(x) - (1-\pi_{x+1}) f(x+1) \end{array} \leq \begin{array}{c} \pi_x [f(x-1) - f(x)] \\ +(1-\pi_x)[f(x) - f(x+1)]. \end{array}.$$

Since we assume that $f(x) - f(x+1) \leq R$, it is obvious that Equation A.9 is true and thus, $T_{CD} f(x) - T_{CD} f(x+1) \leq R$.

A.2.4 UBD of T_{Q_PRC}

Let p_x be the optimal price for the state x. Then, Equation A.6 for T_{Q_PRC}, where R is the maximum price, is:

$$\begin{array}{c} \bar{F}_R(p_x)[f(x+1) + p_x] + F_R(p_x) f(x) \\ -\bar{F}_R(p_{x+1})[f(x+2) + p_{x+1}] - F_R(p_{x+1}) f(x+1) \end{array} \leq R. \tag{A.10}$$

As in the proof of T_{CD}, by the optimality of p_{x+1} we have that,

$$\begin{array}{c} \bar{F}_R(p_x)[f(x+1) + p_x] + F_R(p_x) f(x) \\ -\bar{F}_R(p_{x+1})[f(x+2) + p_{x+1}] - F_R(p_{x+1}) f(x+1) \end{array} \leq \begin{array}{c} \bar{F}_R(p_x)[f(x+1) - f(x+2))] \\ F_R(p_x)[f(x) - f(x+1))] \end{array}.$$

Since we assume that $f(x) - f(x+1) \leq R$, Equation A.10 is true and thus, $T_{Q\_PRC} f(x) - T_{Q\_PRC} f(x+1) \leq R$. In addition, we need to observe the state $x = K-1$ for the capacitated queues. For this state, Equation A.10 turns out to:

$$\bar{F}_R(p_x) p_x + F_R(p_x)[f(x) - f(x+1)] \leq R,$$

and this equation is true since $f(x) - f(x+1) \leq R$ and R is the maximum price. Therefore, $T_{Q\_PRC} f(x) - T_{Q\_PRC} f(x+1) \leq R$ also holds for the capacitated cases.

Cases	$\bar{\kappa^*} = (\kappa^*_x, \kappa^*_{x+1})$	Rewritten form of Equation A.11
Case I	(0,0)	$f(x) - f(x+1) \leq R_i$
Case II	(a+1,a)	$R_i \leq R_i$
Case III	(B,B)	$f(x+B) - f(x+B+1) \leq R_i$

Table A.1: Possible optimal actions in states x, $x+1$

A.2.5 UBD of $T_{B\_ADM_i}$

Let $\bar{\kappa} = (\kappa^*_x, \kappa^*_{x+1})$ be the optimal action vector and κ^*_x be the optimal number of customers to be admitted form an arriving batch for the state x. Then, we show that the following equation will be true if $f(x) - f(x+1) \leq R_i$ for all possible $\bar{\kappa}$.

$$\begin{array}{c} \kappa^*_x R_i + f(x+\kappa^*_x) \\ -\kappa^*_{x+1} R_i - f(x+1+\kappa^*_{x+1}) \end{array} \leq R_i. \tag{A.11}$$

Although there are more possible $\bar{\kappa}$, it is enough to consider the cases: $(0,0)$, $(a+1,a)$, and (B,B), where $0 \leq a < B$, as a result of the concavity of $f(x)$. We rewrite Equation A.11 for each case in Table A.1. Case II is obviously true and Cases I and III are also true by the assumption that $f(x) - f(x+1) \leq R_i$. Thus, Equation A.11 is true for all cases and then $T_{B\_ADM_i} f(x) - T_{B\_ADM_i} f(x+1) \leq R_i$. While considering the capacitated queues, we need to observe the states $x \leq K - B$ in order to see the boundary effects. For these states, Case I and II ares still possible but Case III is not. Since there is not any additional cases on the boundaries and it is already proven that Equation A.11 is true for Cases I and II, $T_{B\_ADM_i} f(x) - T_{B\_ADM_i} f(x+1) \leq R_i$ also holds for the capacitated queues.

A.3 Proof: Lower-Bounded Difference, *LBD*

In this proof, we show that if $f(x)$ is an LBD function then the following equation holds for the operators: T_{C_PRD}, T_{I_PRC} and $T_{B\_RATIO_i}$.

$$Tf(x) - Tf(x+1) \geq -R. \tag{A.12}$$

A.3.1 LBD of T_{I_PRC}

Let p_x be the optimal price for the state x. Then, Equation A.12 for T_{I_PRC}, where R is the maximum price, is:

$$\begin{array}{c} \bar{F}_R(p_x)[f(x-1)+p_x]+F_R(p_x)f(x) \\ -\bar{F}_R(p_{x+1})[f(x)+p_{x+1}]+F_R(p_{x+1})f(x+1) \end{array} \geq -R. \tag{A.13}$$

As a result of the optimality of p_x, we have that,

$$\begin{array}{c} \bar{F}_R(p_x)[f(x-1)+p_x]+F_R(p_x)f(x) \\ -\bar{F}_R(p_{x+1})[f(x)+p_{x+1}]-F_R(p_{x+1})f(x+1) \end{array} \geq \begin{array}{c} \bar{F}_R(p_{x+1})[f(x-1)+p_{x+1}]+F_R(p_{x+1})f(x) \\ -\bar{F}_R(p_{x+1})[f(x)+p_{x+1}]-F_R(p_{x+1})f(x+1) \end{array}.$$

When we rearrange this equation, we obtain that,

$$\begin{array}{c} \bar{F}_R(p_x)[f(x-1)+p_x]+F_R(p_x)f(x) \\ -\bar{F}_R(p_{x+1})[f(x)+p_{x+1}]+F_R(p_{x+1})f(x+1) \end{array} \geq \begin{array}{c} \bar{F}_R(p_{x+1})[f(x-1)-f(x)] \\ +F_R(p_{x+1})[f(x)-f(x+1)] \end{array}.$$

This equation is obviously true since we assume that $f(x)-f(x+1) \geq -R$ and thus $T_{I\_PRC}f(x)-T_{I\_PRC}f(x+1) \geq -R$. However, this proof is only valid for $x>0$ because we can not use the optimality of p_0. p_0 is equal to the maximum price in order to ensure that the system does not satisfy the new arrival when there is no inventory on hand. Therefore, we have to observe the state $x=0$. For this state, we can write and rearrange Equation A.13 as follows:

$$-\bar{F}_R(p_1)p_1+F_R(p_1)[f(0)-f(1)] \geq -R,$$

and this equation is also true because of the assumptions that $f(0)-f(1) \geq -R$ and $p_1 \leq R$. Thus, we also prove that $T_{I\_PRC}f(x)-T_{I\_PRC}f(x+1) \geq -R$ for $x=0$ and complete the proof.

The proof of LBD of T_{C_PRD} is similar to this proof and it is not necessary to investigate the state $x=0$ for T_{C_PRD}.

A.3.2 LBD of $T_{B\_RATIO_i}$

Let $\bar{\kappa}=(\kappa_x^*,\kappa_{x+1}^*)$ be the optimal action vector and κ_x^* be the optimal number of customers to be satisifed form an arriving batch for the state x. Then, we show that the following equation will be true if $f(x)-f(x+1) \geq -R_i$ for all possible $\bar{\kappa}$.

$$\begin{array}{c} \kappa_x^* R_i+f(x-\kappa_x^*) \\ -\kappa_{x+1}^* R_i-f(x+1-\kappa_{x+1}^*) \end{array} \leq R_i. \tag{A.14}$$

Cases	$\bar{\kappa^*} = (\kappa^*_x, \kappa^*_{x+1})$	Rewritten form of Equation A.14
Case I	(0,0)	$f(x) - f(x+1) \geq -R_i$
Case II	(a,a+1)	$-R_i \geq -R_i$
Case III	(B,B)	$f(x-B) - f(x+1-B) \geq -R_i$

Table A.2: Possible optimal actions in states x, $x+1$

Although there are more possible $\bar{\kappa}$, it is enough to consider the cases: $(0,0)$, $(a, a+1)$, and (B, B), where $0 \leq a < B$, as a result of the concavity of $f(x)$. We rewrite Equation A.14 for each case in Table A.2. Case II is obviously true and Cases I and III are also true by the assumption that $f(x) - f(x+1) \geq -R_i$. Thus, Equation A.14 is true for all cases and then $T_{B\_RATIO_i} f(x) - T_{B\_RATIO_i} f(x+1) \geq -R_i$. As a remark, the last case, (B, B), is not possible for the states $x < B$. We only consider the first two cases for these states and Equation A.14 is also true for the states $x < B$ in Cases I and II.

A.4 *Proof: Concavity*

In this proof, we show that a certain operator, T, is concave in x under certain assumptions on $f(x)$. We prove that Equation holds for T_{ARR} and T_{DEP} if $f(x)$ is non-increasing and concave in x and holds for the remaining ones if $f(x)$ is concave in x. As in the previous proofs, we first show the proofs for the incapacitated cases and then note that they are still valid for the capacitated cases.

$$\Delta T f(x) \leq \Delta T f(x+1) \tag{A.15}$$

A.4.1 Concavity of T_{ARR}

We can write the concavity equation for the operator as follows:

$$\begin{matrix} a(x)f(x+1) + [1-a(x)]f(x) \\ -a(x+1)f(x+2) - [1-a(x+1)]f(x+1) \end{matrix} \leq \begin{matrix} a(x+1)f(x+2) + [1-a(x+1)]f(x+1) \\ -a(x+2)f(x+3) - [1-a(x+2)]f(x+3). \end{matrix}$$

When we rearrange the equation, we obtain that,

$$\begin{matrix} [1-a(x)][f(x) - f(x+1)] \\ +[2a(x+1) - a(x)][f(x+1) - f(x+2)] \end{matrix} \leq \begin{matrix} [1-a(x)][f(x+1) - f(x+2)] \\ +a(x+2)[f(x+2) - f(x+3)]. \end{matrix} \tag{A.16}$$

The first line is true due to the concavity of $f(x)$. Therefore, we have to focus on the second line. We have the following equation as a result of the definition of the function a and the assumptions on $f(x)$.

$$\begin{aligned} 2a(x+1)-a(x) &\leq a(x+2) \\ a(x) &\geq 0 \\ f(x)-f(x+1) &\geq 0 \\ \Delta f(x) &\leq \Delta f(x+1) \end{aligned}$$

When we combine these equations, we obtain that the second line is also true and thus we complete the proof of the concavity of $T_{ARR}f(x)$ in x. While considering the capacitated queues, $a(x+2)=0$ for the state $x=K-2$ and thus Equation A.16 is still true. Therefore $T_{ARR}f(x)$ is also concave in x for the capacitated queues.

A.4.2 Concavity of T_{DEP}

Similar to the concavity of T_{ARR}, we can write and rearrange Equation A.15 for T_{DEP} as follows:

$$\begin{array}{c} b(x)[f(x-1)-f(x)] \\ +[1-2b(x+1)+b(x)][f(x)-f(x+1)] \end{array} \leq \begin{array}{c} b(x)[f(x)-f(x+1)] \\ +[1-b(x+2)][f(x+1)-f(x+2)]. \end{array}$$

As in the previous proof, Equation A.17 is true by the definition of $b(x)$ and the assumptions on $f(x)$. Thus, $T_{DEP}f(x)$ is concave in x if $f(x)$ is non-increasing and concave in x.

A.4.3 Concavity of T_{CD}

Let π_x, π_{x+1} and π_{x+2} be the optimal service rates for the states x, $x+1$ and $x+2$. then, the concavity equation of the operator is:

$$\begin{array}{c} c_{\pi_x}+\pi_x f(x-1)+(1-\pi_x)f(x) \\ -c_{\pi_{x+1}}-\pi_{x+1}f(x)-(1-\pi_{x+1})f(x+1) \end{array} \leq \begin{array}{c} c_{\pi_{x+1}}+\pi_{x+1}f(x)+(1-\pi_{x+1})f(x+1) \\ -c_{\pi_{x+2}}-\pi_{x+2}f(x+1)-(1-\pi_{x+2})f(x+2) \end{array}. \tag{A.17}$$

Since π_{x+1} is the optimal service rate for the state $x+1$, we have that,

$$\begin{array}{c} c_{\pi_x}+\pi_x f(x-1)+(1-\pi_x)f(x) \\ -c_{\pi_{x+1}}-\pi_{x+1}f(x)-(1-\pi_{x+1})f(x+1) \end{array} \leq \begin{array}{c} c_{\pi_x}+\pi_x f(x-1)+(1-\pi_x)f(x) \\ -c_{\pi_x}-\pi_x f(x)-(1-\pi_x)f(x+1) \end{array}.$$

After rearranging the right hand side,

$$\begin{array}{ccc} c_{\pi_x} + \pi_x f(x-1) + (1-\pi_x) f(x) & \leq & f(x) - f(x+1) \\ -c_{\pi_{x+1}} - \pi_{x+1} f(x) - (1-\pi_{x+1}) f(x+1) & & +\pi_x \left[[f(x-1) - f(x)] - [f(x) - f(x+1)] \right] \end{array} .$$

By the concavity of $f(x)$, $[f(x-1) - f(x)] - [f(x) - f(x+1)] \leq 0$ and then,

$$\begin{array}{ccc} c_{\pi_x} + \pi_x f(x-1) + (1-\pi_x) f(x) & \leq & f(x) - f(x+1) \\ -c_{\pi_{x+1}} - \pi_{x+1} f(x) - (1-\pi_{x+1}) f(x+1) & & \end{array} . \tag{A.18}$$

Similarly, we have that,

$$\begin{array}{ccc} c_{\pi_{x+1}} + \pi_{x+1} f(x) + (1-\pi_{x+1}) f(x+1) & \geq & f(x) - f(x+1) \\ -c_{\pi_{x+2}} - \pi_{x+2} f(x+1) - (1-\pi_{x+2}) f(x+2) & & \end{array} . \tag{A.19}$$

If Equations A.18 and A.19 are combined, it is obvious that Equation A.17 is true. Hence, $T_{CD} f(x)$ preserves concavity of $f(x)$ in x.

Proofs of the concavity of T_{C_PRD}, T_{Q_PRC} and T_{I_PRC} are similar to this proof. However, while considering the capacitated queues, we need to observe the concavity of T_{Q_PRC} for the state $x = K-2$. Since we use the optimal action for the state $x+1$, i.e., $K-1$, and it is not affected by the waiting room capacity, the foregoing proof is still valid for the capacitated cases.

A.4.4 Concavity of $T_{B\_ADM_i}$

Let $\bar{\kappa^*} = (\kappa^*_x, \kappa^*_{x+1}, \kappa^*_{x+2})$ be the optimal action vector and κ^*_x be the optimal number of customers admitted form an arriving batch for the state x. Then, we prove that the batch admission operator will be concave in x if $f(x)$ is concave in x. In other words, we show that the following equation is true for all possible $\bar{\kappa^*}$.

$$\begin{array}{ccc} \kappa^*_x R_i + f(x + \kappa^*_x) & \leq & \kappa^*_{x+1} R_i + f(x+1+\kappa^*_{x+1}) \\ -\kappa^*_{x+1} R_i - f(x+1+\kappa^*_{x+1}) & & -\kappa^*_{x+2} R_i - f(x+2+\kappa^*_{x+2}) \end{array} . \tag{A.20}$$

However, we do not need to observe all possible optimal action permutation, i.e., all possible $\bar{\kappa^*}$, because of the concavity of $f(x)$ and it is enough to consider only the cases: $(0,0,0)$, $(1,0,0)$, $(a+2, a+1, a)$, $(B, B, B-1)$ and (B, B, B). We rewrite Equation A.20 for each case in Table A.3. Case III is obviously true and cases I and IV are true due to the concavity of $f(x)$. In case II, the optimal action is rejecting the entire batch for the state

Cases	$\bar{\kappa^*} = (\kappa^*_x, \kappa^*_{x+1}, \kappa^*_{x+2})$	Rewritten form of Equation A.20
Case I	(0,0,0)	$f(x) - f(x+1) \leq f(x+1) - f(x+2)$
Case II	(1,0,0)	$R_i \leq f(x+1) - f(x+2)$
Case III	(a+2,a+1,a)	$R_i \leq R_i$
Case IV	(B,B,B-1)	$f(x+B) - f(x+B+1) \leq R_i$
Case V	(B,B,B)	$f(x+B) - f(x+B+1) \leq f(x+B+1) - f(x+B+2)$

Table A.3: Possible optimal actions in states x, $x+1$ and $x+2$

$x+1$ and it implies that $f(x+1) \geq R_i + f(x+2)$. When we arrange this inequality, we obtain that $R_i \leq f(x+1) - f(x+2)$ and thus, Equation A.20 is true in case II. In a similar manner, Equation A.20 is also true in case IV by the optimal action in the state $x+1$. Therefore, we complete the proof and show that $T_{B\_ADM_i} f(x)$ is concave in x when $f(x)$ is concave in x. For the capacitated queues, we need to focus on the states $x = K - B - 1$ and $x \geq K - B$ to investigate the boundary effect. For $x = K - B - 1$, Case V is not possible because while considering the state $x + 2 = K - B + 1$, admitting B customer is not possible, so that we only work on the first four cases. Equation A.20 holds for all these cases as it is shown for the incapacitated queues. On the other hand, for $x \geq K - B$, Case IV is also not possible beside Case V because while considering the state $x = K - B$, $B - 1$ can not be optimal action for the state $x + 2 = K - B + 2$. Thus, we only work on the first three cases for the states $x \geq K - B$ and Equation A.20 is also true for all these cases. Therefore, $T_{B\_ADM_i} f(x)$ is concave in x for the capacitated queues.

The proof of the concavity of $T_{B\_RATIO_i} f(x)$ is similar to this proof.

A.5 Proof: Supermodularity

In this proof, we prove that a certain operator, T, is supermodular with respect to the parameter α whose effects we would like to observe and x under certain assumptions mentioned in Lemma 2. We denote the function f, a, and b by $\tilde{f}$, $\tilde{a}$, and $\tilde{b}$ after the parameter α increases, respectively and write the supermodularity equation for a certain operator, T, as follows:

$$\Delta T f(x) \geq \Delta T \tilde{f}(x). \tag{A.21}$$

A.5.1 Supermodularity of T_{ARR}

We can write the supermodularity equation, Equation A.21, for this operator as follows:

$$\begin{array}{c} a(x)f(x+1)+[1-a(x)]f(x) \\ -a(x+1)f(x+2)-[1-a(x+1)]f(x+1) \end{array} \geq \begin{array}{c} \tilde{a}(x)\tilde{f}(x+1)+[1-\tilde{a}(x)]\tilde{f}(x) \\ -\tilde{a}(x+1)\tilde{f}(x+2)-[1-\tilde{a}(x+1)]\tilde{f}(x+1) \end{array}. \tag{A.22}$$

When we rearrange the equation, we obtain that,

$$\begin{array}{c} a(x+1)[f(x+1)-f(x+2)] \\ +[1-a(x)][f(x)-f(x+1)] \\ \\ \\ \end{array} \geq \begin{array}{c} a(x+1)[\tilde{f}(x+1)-\tilde{f}(x+2)] \\ +[1-a(x)][\tilde{f}(x)-\tilde{f}(x+1)] \\ +[a(x)-\tilde{a}(x)][\tilde{f}(x)-\tilde{f}(x+1)] \\ -[a(x+1)-\tilde{a}(x+1)][\tilde{f}(x+1)-\tilde{f}(x+2)] \end{array}. \tag{A.23}$$

The first two lines are true by the supermodularity of $f(x)$ with respect to α and x. Then, if we show that the third line is smaller than or equal to the last line we will complete the proof. We have the following equations as a result of the assumptions on $a(x)$ and $f(x)$.

$$\begin{array}{rcl} \tilde{f}(x)-\tilde{f}(x+1) & \geq & 0 \\ \tilde{f}(x)-\tilde{f}(x+1) & \leq & \tilde{f}(x+1)-\tilde{f}(x+2) \\ a(x)-\tilde{a}(x) & \geq & 0 \\ a(x)-\tilde{a}(x) & \leq & a(x+1)-\tilde{a}(x+1) \end{array}$$

When we combine these equations, we have that,

$$[a(x)-\tilde{a}(x)][\tilde{f}(x)-\tilde{f}(x+1)] \leq [a(x+1)-\tilde{a}(x+1)][\tilde{f}(x+1)-\tilde{f}(x+2)].$$

Thus, the third line in Equation A.23 is smaller than or equal to the last line and we complete the proof of the supermodularity of $T_{ARR}f(x)$ with respect to α and x.

A.5.2 Supermodularity of T_{DEP}

As in the supermodularity of T_{ARR}, we can write and rearrange the supermodularity equation for this operator as follows:

$$\begin{array}{c} b(x)[f(x-1)-f(x)] \\ +[1-b(x+1)][f(x)-f(x+1)] \\ \\ \\ \end{array} \geq \begin{array}{c} b(x)[\tilde{f}(x-1)-\tilde{f}(x)] \\ +[1-b(x+1)][\tilde{f}(x)-\tilde{f}(x+1)] \\ +[\tilde{b}(x)-b(x)][\tilde{f}(x-1)-\tilde{f}(x)] \\ -[\tilde{b}(x+1b(x+1)][\tilde{f}(x)-\tilde{f}(x+1)] \end{array}. \tag{A.24}$$

The first two lines are true due to the supermodularity of $f(x)$. Therefore,if we show that the third line is smaller than or equal to the last line we will complete the proof. By the assumptions on $b(x)$ and $f(x)$, we have the following equations:

$$\begin{aligned}
\tilde{f}(x) - \tilde{f}(x+1) &\geq 0, \\
\tilde{f}(x) - \tilde{f}(x+1) &\leq \tilde{f}(x+1) - \tilde{f}(x+2), \\
\tilde{b}(x) - b(x) &\geq 0, \\
\tilde{b}(x) - b(x) &\leq \tilde{b}(x+1) - b(x+1).
\end{aligned}$$

When we combine these equations, we can obtain that,

$$[\tilde{b}(x) - b(x)][\tilde{f}(x-1) - \tilde{f}(x)] \leq [\tilde{b}(x+1b(x+1)][\tilde{f}(x) - \tilde{f}(x+1)].$$

Hence, we show that the third line in Equation A.24 is smaller than or equal to the last line and $T_{DEP}f(x)$ is supermodular with respect to α and x under the certain assumptions mentioned in Lemma 2.

A.5.3 Supermodularity of T_{CD}

Let π_x and $\pi_{\tilde{x}}$ be the optimal service rates for the state x before and after the parameter α increases, respectively. Then, we show that the following supermodularity equation is true for T_{CD}.

$$\begin{array}{c} c_{\pi_x} + \pi_x f(x-1) + (1-\pi_x) f(x) \\ -c_{\pi_{x+1}} - \pi_{x+1} f(x) - (1-\pi_{x+1}) f(x+1) \end{array} \geq \begin{array}{c} c_{\pi_{\tilde{x}}} + \pi_{\tilde{x}} \tilde{f}(x-1) + (1-\pi_{\tilde{x}}) \tilde{f}(x) \\ -c_{\pi_{\tilde{x+1}}} - \pi_{\tilde{x+1}} \tilde{f}(x) - (1-\pi_{\tilde{x+1}}) \tilde{f}(x+1) \end{array}. \tag{A.25}$$

As a result of the optimality of π_x for the state x, we have that,

$$\begin{array}{c} c_{\pi_x} + \pi_x f(x-1) + (1-\pi_x) f(x) \\ -c_{\pi_{x+1}} - \pi_{x+1} f(x) - (1-\pi_{x+1}) f(x+1) \end{array} \geq \begin{array}{c} c_{\pi_{\tilde{x}}} + \pi_{\tilde{x}} f(x-1) + (1-\pi_{\tilde{x}}) f(x) \\ -c_{\pi_{x+1}} - \pi_{x+1} f(x) - (1-\pi_{x+1}) f(x+1) \end{array}.$$

Then, if we rearrange the right hand side, we can obtain,

$$\begin{array}{c} c_{\pi_x} + \pi_x f(x-1) + (1-\pi_x) f(x) \\ -c_{\pi_{x+1}} - \pi_{x+1} f(x) - (1-\pi_{x+1}) f(x+1) \end{array} \geq \begin{array}{c} c_{\pi_{\tilde{x}}} - c_{\pi_{x+1}} \\ +\pi_{\tilde{x}}[f(x-1) - f(x)] \\ +(1-\pi_{x+1})[f(x) - f(x+1)] \end{array}. \tag{A.26}$$

Cases	$\bar{\kappa^*} = (\kappa^*_x, \kappa^*_{x+1}, \kappa^*_{\tilde{x}}, \kappa^*_{\tilde{x+1}})$	Rewritten form of Equation A.28
Case I	$(0,0,0,0)$	$f(x) - f(x+1) \geq \tilde{f}(x) - \tilde{f}(x+1)$
Case II	$(0,0,a+1,a)$	$f(x) - f(x+1) \geq R_i$
Case III	$(0,0,B,B)$	$f(x) - f(x+1) \geq \tilde{f}(x+B) - \tilde{f}(x+B+1)$
Case IV	$(a+1,a,a+d+1,a+d)$	$R_i \geq R_i$
Case V	$(a+1,a,B,B)$	$R_i \geq \tilde{f}(x+B) - \tilde{f}(x+B+1)$
Case VI	(B,B,B,B)	$f(x+B) - f(x+B+1) \geq \tilde{f}(x+B) - \tilde{f}(x+B+1)$

Table A.4: Possible optimal actions in states x, $x+1$, $\tilde{x}$ and $\tilde{x+1}$

Similarly, we can also obtain that,

$$\begin{array}{c} c_{\pi_{\tilde{x}}} + \pi_{\tilde{x}} \tilde{f}(x-1) + (1-\pi_{\tilde{x}})\tilde{f}(x) \\ -c_{\pi_{\tilde{x+1}}} - \pi_{\tilde{x+1}} \tilde{f}(x) - (1-\pi_{\tilde{x+1}})\tilde{f}(x+1) \end{array} \leq \begin{array}{c} c_{\pi_{\tilde{x}}} - c_{\pi_{x+1}} \\ +\pi_{\tilde{x}}[\tilde{f}(x-1) - \tilde{f}(x)] \\ +(1-\pi_{x+1})[\tilde{f}(x) - \tilde{f}(x+1)] \end{array} . \tag{A.27}$$

When we combine Equations A.26 and A.27, it is obvious that the supermodularity equation for this operator, Equation A.25, is true and thus, we complete the proof of the supermodularity of $T_{CD}f(x)$ with respect to α and x.

Proofs of the supermodularity of T_{C_PRD}, T_{Q_PRC} and T_{I_PRC} are similar to this proof.

A.5.4 Supermodularity of $T_{B\_ADM_i}$

Let $\bar{\kappa^*} = (\kappa^*_x, \kappa^*_{x+1}, \kappa^*_{\tilde{x}}, \kappa^*_{\tilde{x+1}})$ be the optimal action vector, where κ^*_x and $\kappa^*_{\tilde{x}}$ are the optimal number of customers to be admitted form an arriving batch for the state x before and after the parameter α increases, respectively. Then, we show that the following supermodularity equation is true for the batch admission operator.

$$\begin{array}{c} \kappa^*_x R_i + f(x+\kappa^*_x) \\ -\kappa^*_{x+1} R_i - f(x+1+\kappa^*_{x+1}) \end{array} \leq \begin{array}{c} \kappa^*_{\tilde{x}} R_i + \tilde{f}(x+\kappa^*_{\tilde{x}}) \\ -\kappa^*_{\tilde{x+1}} R_i - \tilde{f}(x+1+\kappa^*_{\tilde{x+1}}) \end{array} . \tag{A.28}$$

We have to consider all possible optimal action vector to prove the supermodularity of $T_{B\_ADM_i}$. However, it is enough to consider only the cases: $(0,0,0,0)$, $(0,0,a+1,a)$, $(0,0,B,B)$, $(a+1,a,a+d+1,a+d)$, $(a+1,a,B,B)$ and (B,B,B,B), where $0 \leq a \leq B-1$ and $0 \leq d \leq B-a-1$, because of the concavity and supermodularity of $f(x)$. We rewrite Equation A.28 for each case in Table A.4. It is obvious that Case IV is true and Cases I and VI are true by the supermodularity of $f(x)$. In Case II, it is optimal to reject the entire batch for the state x. As a result of this action, we have that $f(x) \geq f(x+1) + R_i$. If we rearrange this inequality we can obtain that $f(x) - f(x+1) \geq R_i$, which is the rewritten form of the supermodularity equation for this case. Therefore, Equation A.28 is true in Case II. Similarly, in Case V, the optimal action is admitting the whole batch for the state $x \tilde{+} 1$ and it implies that $BR_i + \tilde{f}(x + B1) \geq (B-1)R_i + \tilde{f}(x+B)$. Since this inequality is the rewritten form of the supermodularity equation, Equation A.28 is true in Case V. Finally, Equation A.28 is also true in Case III due to the optimal actions for the states x and $x \tilde{+} 1$. Thus, we show that Equation A.28 is true for all of the six cases and $T_{B\_ADM_i} f(x)$ is supermodular with respect to α and x if $f(x)$ is concave in x and supermodular with respect to α and x.

The proof of $T_{B\_RATIO_i}$ is similar to this proof.

A.6 Proof: Submodularity

The proofs of the submodularity of the operators are similar to the proofs of the supermodularity so that, we do not show these proofs here. In this section we work on the "non-submodular" property of the queueing related operators while considering the waiting room capacity and the number of server. The intuition behind the "non-submodular" property can be explained by the following example: Let consider a capacitated system with total capacity K. In this system, new arrivals are not allowed to enter the system when there are K customers in the system whether it is optimal or not. If it is optimal to reject the arrivals, the optimal policy does not change when the capacity increases. On the other hand, if the arrivals are rejected because of the capacity, a new customer may be accepted when there are K customers after an increase in the capacity. Thus, it is obvious that an increase in the capacity can not lead to reject the customers who are already accepted before the capacity increases. In other words, the opportunity cost of a new customer can not increase by an increase in the capacity. The intuition for the increase in the number of servers is the same.

Since we intuitively know that the queueing related operators are not submodular while considering the waiting room capacity and the number of servers, we disproof the submodularity of the operators by a counter example. We consider a loss system consisting of 10

# of customers	$\Delta f(x)$	$\Delta\tilde{f}(x)$	$\Delta T_{DEP}f(x)$	$\Delta T_{DEP}\tilde{f}(x)$	$\Delta T_{B\_ADM_1}f(x)$	$\Delta T_{B\_ADM_1}\tilde{f}(x)$
0	7.969376	7.20371	7.438084	6.723463	8.396632	7.589944
1	8.396632	7.589944	7.808373	7.058198	8.869616	8.017516
2	8.869616	8.017516	8.215243	7.426005	9.395681	8.493074
3	9.395681	8.493074	8.664089	7.831757	9.983758	9.024689
4	9.983758	9.024689	9.161354	8.281279	10.8597	9.622239
5	10.8597	9.622239	9.843738	8.781573	11.94845	10.39721
6	11.94845	10.39721	10.71638	9.394077	13.26869	11.4722
7	13.26869	11.4722	11.768	10.20573	14.87671	12.8293
8	14.87671	12.8293	13.02732	11.25023	17.7874	14.52129
9	17.7874	14.52129	14.85516	12.53801	25	17.46946

Table A.5: The relative value function and the operators before and after the number of servers increases

servers with no waiting room capacity and 5 different customer classes. Customers arrive according to a Poisson process with a rate of 140 customers per hour and the service times are exponentially distributed with a mean of 1/6 hour. At each arrival epoch, an arriving customer is a class-i customer with probability of p_i, where $p_1 = p_2 = p_3 = p_4 = p_5 = 0.2$, and the decision maker has to decide whether to admit or reject the arriving customer. Whenever, a class-i customer is admitted, a reward of R_i is obtained such that $R_1 = 25$, $R_2 = 20$, $R_3 = 15$, $R_4 = 10$, and $R_5 = 5$. The objective of the problem is maximizing the total expected average reward over an infinite horizon. We denote the relative value function and the average reward by $f(x)$ and g, respectively and present the optimality equation as follows:

$$g + f(x) = M\mu T_{DEP}f(x) + \lambda \sum_{i=1}^{5} p_i T_{B\_ADM_i} f(x),$$

where $M\mu + \lambda = 1$ after normalization, $b(x) = x/M$ in T_{DEP} and $B = 1$ in $T_{B\_ADM_i}$. M is introduced to ensure that time scale is not affected by an increase in the number of servers and equal to 15.

We increase the number of servers (c) from 10 to 11 in order to see the effects of the parameter on the relative value function and the operators. We solve the optimality equation by value iteration and obtain the relative value function and the operators as in Table A.5, where $\tilde{f}(x)$ represents the relative value function after the number of servers increases. Since the batch admission operator for each class is the same, we only present the operator for class-1. As it can be seen in the table, none of the $\Delta f(x)$, $\Delta T_{DEP} f(x)$, and $\Delta T_{B\_ADM_1} f(x)$ values are increasing in the number of servers. Thus, the departure and batch admission operators are not submodular with respect to the capacity and the state of the system. There are similar examples to disproof the submodularity of the operators: T_{ARR}, T_{CD}, and T_{Q_PRC}.

A.7 Proof: Structure of $Tf(x) - f(x)$

In this proof, we show that if $f(x)$ has some structural properties then, $Tf(x) - f(x)$ will be either non-increasing or non-decreasing in x according to the characteristics of the operator T. As we mentioned in the paper, while considering queueing problems, $Tf(x) - f(x)$ is non-decreasing in x for the departure related operators and non-increasing in x for the arrival related operators. On the other hand, $Tf(x) - f(x)$ is non-increasing in x for the production operator and non-decreasing in x for the arrival operators.

We use the following equations to denote the structure of $Tf(x) - f(x)$: Equation A.29 represents that $Tf(x) - f(x)$ is non-decreasing in x, whereas Equation A.30 represents that $Tf(x) - f(x)$ is non-increasing in x.

$$Tf(x) - f(x) \leq Tf(x+1) - f(x+1), \tag{A.29}$$

$$Tf(x) - f(x) \geq Tf(x+1) - f(x+1). \tag{A.30}$$

A.7.1 $T_{ARR} f(x) - f(x)$

Since we use the arrival operator in queueing systems, we are expecting that $T_{ARR} f(x) - f(x)$ is non-increasing in x. However, as we mentioned in the paper, the arrival operator holds this property only under the assumptions that $a(x)$ is constant and the buffer capacity is infinite. Then, under these assumptions, we write Equation A.30 for the operator as:

$$f(x+1) - f(x) \geq f(x+2) - f(x+1).$$

This equation is true by the concavity of $f(x)$, so that the proof is completed.

A.7.2 $T_{DEP}f(x) - f(x)$

Unlike the arrival operator, we are expecting that $T_{DEP}f(x) - f(x)$ is non-decreasing in x and Equation A.29 for the departure operator can be written as:

$$\begin{array}{ccc} b(x)f(x-1) & \leq & b(x)f(x) \\ +[1-b(x+1)]f(x) - f(x) & & +[1-b(x+1)]f(x+1) - f(x+1). \end{array}$$

When we arrange this equation, we have that,

$$\begin{array}{ccc} b(x)[f(x-1) - f(x)] & \leq & b(x)[f(x) - f(x+1)] \\ & & +[b(x+1) - b(x)][f(x) - f(x+1)]. \end{array}$$

It is obvious that the first line is true by the concavity of $f(x)$. Then, we complete the proof if we show that the last line is greater than or equal to 0. Since $[b(x+1) - b(x)] \geq 0$ by the definition of the function b and $[f(x) - f(x+1)] \geq 0$ by the monotonicity of $f(x)$, the last line is greater than or equal to 0 and thus we complete the proof.

As we mentioned in the previous proofs, the foregoing proof is still valid for the capacitated systems because the departure related operators are not affected by the boundary effects.

A.7.3 $T_{CD}f(x) - f(x)$

Let π_x be the optimal service rates for the state x. As in the departure operator, $T_{CD}f(x) - f(x)$ is non-decreasing in x intuitively and we write Equation A.29 for this operator as:

$$\begin{array}{ccc} c_{\pi_x} + \pi_x f(x-1) & \leq & c_{\pi_{x+1}} + \pi_{x+1} f(x) \\ +(1-\pi_x)f(x) - f(x) & & +(1-\pi_{x+1})f(x+1) - f(x+1). \end{array} \tag{A.31}$$

Since π_{x+1} is the optimal action for the state $x+1$,

$$\begin{array}{ccc} c_{\pi_{x+1}} + \pi_{x+1} f(x) & \geq & c_{\pi_x} + \pi_x f(x) \\ +(1-\pi_{x+1})f(x+1) - f(x+1) & & +(1-\pi_x)f(x+1) - f(x+1). \end{array} \tag{A.32}$$

Moreover, by using the concavity of $f(x)$,

$$\begin{array}{ccccc} c_{\pi_x} + \pi_x f(x) & = & c_{\pi_x} & \geq & c_{\pi_x} \\ +(1-\pi_x)f(x+1) - f(x+1) & & +\pi_x[f(x) - f(x+1)] & & +\pi_x[f(x-1) - f(x)], \end{array} \tag{A.33}$$

and by arranging the right hand side,

$$\begin{array}{ccc} c_{\pi_x} & & c_{\pi_x} + \pi_x f(x-1) \\ +\pi_x[f(x-1) - f(x)] & = & +(1-\pi_x)f(x) - f(x). \end{array} \tag{A.34}$$

By combining Equations A.32, A.33, and A.34, Equation A.31 is true and $T_{CD}f(x) - f(x)$ is non-decreasing in x.

A.7.4 $T_{C\_PRD}f(x) - f(x)$

The proof is similar to the previous proof but intuitively $T_{C\_PRD}f(x) - f(x)$ is non-increasing in x because of its characteristics. Then, let π_x be the optimal service rates for the state x and write Equation A.30 for the operator as:

$$\begin{array}{ccc} c_{\pi_x} + \pi_x f(x+1) & \geq & c_{\pi_{x+1}} + \pi_{x+1} f(x+2) \\ +(1-\pi_x)f(x) - f(x) & & +(1-\pi_{x+1})f(x+1) - f(x+1). \end{array} \tag{A.35}$$

Since π_x is the optimal action for the state x,

$$\begin{array}{ccc} c_{\pi_x} + \pi_x f(x+1) & \geq & c_{\pi_{x+1}} + \pi_{x+1} f(x+1) \\ +(1-\pi_x)f(x) - f(x) & & +(1-\pi_{x+1})f(x) - f(x) \end{array} \tag{A.36}$$

As in the previous proof, by using the concavity of $f(x)$ and arranging the right hand side, we have that,

$$\begin{array}{ccc} c_{\pi_{x+1}} + \pi_{x+1} f(x+1) & \geq & c_{\pi_{x+1}} + \pi_{x+1} f(x+2) \\ +(1-\pi_{x+1})f(x) - f(x) & & +(1-\pi_{x+1})f(x+1) - f(x+1) \end{array} \tag{A.37}$$

By combining Equations A.36 and A.37, Equation A.35 is true and thus, $T_{C\_PRD}f(x) - f(x)$ is non-decreasing in x.

A.7.5 $T_{Q\_PRC}f(x) - f(x)$

Since the operator, $T_{Q\_PRC}f(x)$, is an arrival related queueing operator, we are expecting that $T_{Q\_PRC}f(x) - f(x)$ is non-increasing in x. As a result of this intuition, we write Equation A.30 for this operator as follows by letting p_x is the optimal price for the state x:

$$\begin{array}{ccc} \bar{F}_R(p_x)[f(x+1) + p_x] & \geq & \bar{F}_R(p_{x+1})[f(x+2) + p_{x+1}] \\ +F_R(p_x)f(x) - f(x) & & +F_R(p_{x+1})f(x+1) - f(x+1) \end{array} \tag{A.38}$$

Since p_x is the optimal price, we have that,

$$\begin{array}{c} \bar{F}_R(p_x)[f(x+1)+p_x] \\ +F_R(p_x)f(x)-f(x) \end{array} \geq \begin{array}{c} \bar{F}_R(p_{x+1})[f(x+1)+p_{x+1}] \\ +F_R(p_{x+1})f(x)-f(x) \end{array} = \begin{array}{c} \bar{F}_R(p_{x+1})p_{x+1} \\ +\bar{F}_R(p_{x+1})[f(x+1)-f(x)]. \end{array} \tag{A.39}$$

Moreover, by the concavity of $f(x)$,

$$\begin{array}{c} \bar{F}_R(p_{x+1})p_{x+1} \\ +\bar{F}_R(p_{x+1})[f(x+1)-f(x)] \end{array} \geq \begin{array}{c} \bar{F}_R(p_{x+1})p_{x+1} \\ +\bar{F}_R(p_{x+1})[f(x+2)-f(x+1)] \end{array} = \begin{array}{c} \bar{F}_R(p_{x+1})[f(x+2)+p_{x+1}] \\ +F_R(p_{x+1})f(x+1)-f(x+1). \end{array} \tag{A.40}$$

By combining Equations A.39 and A.40, Equation A.38 is true and thus, $T_{Q\_PRC}f(x)-f(x)$ is non-decreasing in x. In the capacitated case, we need to observe Equation A.38 for the state $x = K-1$. However, since we use the optimality of p_x, the foregoing proof is still true for the capacitated queues.

A.7.6 $T_{I\_PRC}f(x) - f(x)$

Since the operator, $T_{I\_PRC}f(x)$, is an arrival related inventory operator, we are expecting that $T_{I\_PRC}f(x) - f(x)$ is non-decreasing in x. As a result of this intuition, we write Equation A.29 for this operator as follows by letting p_x is the optimal price for the state x:

$$\begin{array}{c} \bar{F}_R(p_x)[f(x-1)+p_x] \\ +F_R(p_x)f(x)-f(x) \end{array} \leq \begin{array}{c} \bar{F}_R(p_{x+1})[f(x)+p_{x+1}] \\ +F_R(p_{x+1})f(x+1)-f(x+1) \end{array} \tag{A.41}$$

Since p_{x+1} is the optimal price, we have that,

$$\begin{array}{c} \bar{F}_R(p_{x+1})[f(x)+p_{x+1}] \\ +F_R(p_{x+1})f(x+1)-f(x+1) \end{array} \geq \begin{array}{c} \bar{F}_R(p_x)[f(x)+p_x] \\ +F_R(p_x)f(x+1)-f(x+1) \end{array} = \begin{array}{c} \bar{F}_R(p_x)p_x \\ +\bar{F}_R(p_x)[f(x)-f(x+1)]. \end{array} \tag{A.42}$$

Moreover, by the concavity of $f(x)$,

$$\begin{array}{c} \bar{F}_R(p_x)p_x \\ +\bar{F}_R(p_x)[f(x)-f(x+1)] \end{array} \geq \begin{array}{c} \bar{F}_R(p_x)p_x \\ +\bar{F}_R(p_x)[f(x-1)-f(x)] \end{array} = \begin{array}{c} \bar{F}_R(p_x)[f(x-1)+p_x] \\ +F_R(p_x)f(x)-f(x). \end{array} \tag{A.43}$$

By combining Equations A.42 and A.43, Equation A.41 is true and thus, $T_{Q\_PRC}f(x)-f(x)$ is non-decreasing in x.

Cases	$\bar{\kappa^*} = (\kappa_x^*, \kappa_{x+1}^*, \kappa_{x+2}^*)$	Rewritten form of Equation A.44
Case I	(0,0)	$0 \geq 0$
Case II	(a+1,a)	$R_i \geq f(x) - f(x+1)$
Case III	(B,B)	$f(x+B) - f(x) \geq f(x+B+1) - f(x+1)$

Table A.6: Possible optimal actions in states x and $x+1$

A.7.7 $T_{B\_ADM} f(x) - f(x)$

Let $\bar{\kappa^*} = (\kappa_x^*, \kappa_{x+1}^*)$ be the optimal action vector and κ_x^* be the optimal number of customers admitted form an arriving batch for the state x. Then, we prove that $T_{B\_ADM} f(x) - f(x)$ is non-increasing in x since it is an arrival related queueing operator. In other words, we show that the following equation is true for all possible $\bar{\kappa^*}$.

$$\kappa_x R_i + f(x+\kappa_x) - f(x) \geq \kappa_{x+1} R_i + f(x+1+\kappa_{x+1}) - f(x+1) \tag{A.44}$$

However, we do not need to observe all possible optimal action permutation, i.e., all possible $\bar{\kappa^*}$, because of the concavity of $f(x)$ and it is enough to consider only the cases: $(0,0)$, $(a+1,a)$ and (B,B). We rewrite Equation A.44 for each case in Table A.6. Case I is obviously true and cases III is true due to the concavity of $f(x)$. In case II, the optimal action is admitting $a+1$ customer from an arriving batch for the state x and it implies that $R_i \geq f(x+a) - f(x+a+1)$. Moreover, by the concavity of $f(x)$, we have that $f(x) - f(x+1) \leq f(x+a) - f(x+a+1)$. When we combine these two inequalities, we obtain that $R_i \geq f(x) - f(x+1)$ and thus, Equation A.44 is true in case II. Therefore, we complete the proof and show that $T_{B\_ADM_i} f(x) - f(x)$ is non-increasing in x when $f(x)$ is concave in x. For the capacitated queues, we need to focus on the states $x \geq K - B$ in order to investigate the boundary effect. For these states, Case III is not possible because the optimal action can not be admitting B customer for the state $x+1$. Thus, we only work on Cases I and II, and the proof is the same as the proof of Cases I and II in the incapacitated case.

The proof of the structure of $T_{B\_RATIO_i} f(x) - f(x)$ is similar to this proof.

A.8 Proof: Effects of the Parameters on the Optimal Actions in Model 1

In this proof, we show the effects of the service rate and the arrival rate on the base-stock level and the optimal prices. We first work on the effects of μ on the base-stock level. Let S_e and $\tilde{S}_e$ be the optimal base-stock levels for the demand environment e before and after the service rate increases, respectively, and assume that $S_e < \tilde{S}_e$. Therefore, we have that it is optimal to produce when there are S_e units of inventory on hand after the service rate increases whereas it is not optimal to produce for this state before the service rate increases, i.e.,

$$\begin{aligned} v_1(S_e, e) - v_1(S_e + 1, e) &\geq -\tau_1, \ \textit{and} \\ \tilde{v}_1(S_e, e) - \tilde{v}_1(S_e + 1, e) &\leq -\tau_1. \end{aligned}$$

When we combine these equations, we obtain that $v_1(S_e, e) - v_1(S_e + 1, e) \geq \tilde{v}_1(S_e, e) - \tilde{v}_1(S_e + 1, e)$ and this inequality contradicts with the submodularity of $v_1(x, e)$ with respect to μ and x. Therefore, the assumption of $S_e < \tilde{S}_e$ is not correct and the base-stock level, $S_1^*(e)$, is non-increasing in the service rate. The proof of the effects of λ on the base-stock level is similar to this proof.

Now, we show the effects of the service rate on the optimal prices. Let p and $\tilde{p}$ be the optimal price for the state x before and after the service rate increases and assume that $p < \tilde{p}$. Then, we have the following equations due to the optimality of p and $\tilde{p}$.

$$\begin{aligned} \bar{F}(p)[v_1(x-1, e) + p] + F(p)v_1(x, e) &\geq \bar{F}(\tilde{p})[v_1(x-1, e) + \tilde{p}] + F(\tilde{p})v_1(x, e), \\ \bar{F}(\tilde{p})[\tilde{v}_1(x-1, e) + \tilde{p}] + F(\tilde{p})\tilde{v}_1(x, e) &\geq \bar{F}(p)[\tilde{v}_1(x-1, e) + p] + F(p)\tilde{v}_1(x, e). \end{aligned}$$

When we combine these equations, we obtain that,

$$[\bar{F}(p) - \bar{F}(\tilde{p})][[v_1(x-1, e) - v_1(x, e)] - [\tilde{v}_1(x-1, e) - \tilde{v}_1(x, e)]] \geq 0. \tag{A.45}$$

$\bar{F}(p) - \bar{F}(\tilde{p}) > 0$ and $[v_1(x-1, e) - v_1(x, e)] - [\tilde{v}_1(x-1, e) - \tilde{v}_1(x, e)] \leq 0$ by the assumption on the optimal prices and the submodularity of $v_1(x, e)$ so that, Equation A.45 can not be true. In other words, there is a contradiction because of our assumption and thus, the optimal price for the state x, $p_1^*(x, e)$, is non-increasing in the service rate.The proof of the effects of λ on the the optimal prices is similar to this proof.

A.9 Proof: Effects of the Parameters on the Optimal Actions in Model 2

Let κ and $\tilde{\kappa}$ be the optimal number of customers to be admitted from an arriving batch and assume that $\kappa > \tilde{\kappa}$. Then, by the definition of the operator,

$$\begin{aligned} \kappa R_2 + v_2(x+\kappa) &\geq \tilde{\kappa} R_2 + v_2(x+\tilde{\kappa}), \text{ and} \\ \tilde{\kappa} R_2 + \tilde{v}_2(x+\tilde{\kappa}) &\geq \kappa R_2 + \tilde{v}_2(x+\kappa). \end{aligned}$$

When we combine these equations, we obtain that, $v_2(x+\tilde{\kappa}) - v_2(x+\kappa) \leq \tilde{v}_2(x+\tilde{\kappa}) - \tilde{v}_2(x+\kappa)$, which contradicts with the supermodularity of $v_2(x)$. Hence, the assumption on the optimal actions is not correct and the optimal number of customers to be admitted from an arriving batch, $\kappa_{i2}^*(x)$, is non-decreasing in the service rate. The proof of the effects of λ and c on the the optimal admission policy is similar to this proof.

A.10 Proof: Existence of a Preferred Class in Model 3

In this proof, we show that class-1 is a preferred class in Model 3, i.e., $\Delta v_3(x) \geq -R_1$ for all x. To prove this equation, we first prove the following equation for all finite n by induction.

$$\Delta v_3^n(x) \geq -R_1. \tag{A.46}$$

The initial condition of the induction, $\Delta v_3^0(x) \geq -R_1$, holds by the specification on $v_3^0(x)$. Then, we rewrite Equation A.46 by using the optimality equation as follows:

$$\begin{aligned} &\mu \Delta T_{C\_PRD} v_3^{n-1}(x) \\ &+\lambda \sum_{i=1}^{N} \sum_{B} p_{iB} \Delta T_{B\_RATIO_i} v_3^{n-1}(x) \quad \geq -R_1 \\ &+\theta \Delta T_{FIC} v_3^{n-1}(x) \end{aligned}$$

This equation is true as a result of $LBD(R)$ property of the operators (See Lemma 1) and thus, $\Delta v_3^n(x) \geq -R_1$ for all finite n. By using the relationship between $v_3^n(x)$ and $v_3(x)$, $v_3(x)$ has also $LBD(R)$ property. This result implies that $v_3(x) + R_1 \geq v_3(x+1)$, i.e., satisfying class-1 customers is optimal whenever it is possible.

Appendix B

TWO- DIMENSIONAL MODEL

B.1 Proof: Monotonicity of the Operators

In this proof, we prove that the operators that we consider in the 2-dimensional model, T_{DEP2}, T_{PRC_1} and T_{PRC_2}, preserve the monotonicity properties of the function on which they are applied, $v(x_1, x_2)$. In other words, we prove that the following monotonicity equations are true for these operators under certain assumption mentioned in the lemma.

$$\text{Monotonicity in } x_1: \qquad Tv(x_1, x_2) \geq Tv(x_1 + 1, x_2) \tag{B.1}$$

$$\text{Monotonicity in } x_2: \qquad Tv(x_1, x_2) \geq Tv(x_1, x_2 + 1) \tag{B.2}$$

$$\text{Monotonicity on the diagonal:} \qquad Tv(x_1, x_2 + 1) \geq Tv(x_1 + 1, x_2) \tag{B.3}$$

B.1.1 Monotonicity of T_{DEP2}

During the proof of the monotonicity of $T_{DEP2}v(x_1, x_2)$, we assume that $v(x_1, x_2 + 1) \geq v(x_1 + 1, x_2)$ and this property implies that serving the expensive customer is more valuable than serving a cheap customer. Therefore, we can redefine this operator by using this fact as:

$$T_{DEP2}v(x_1, x_2) = \begin{cases} v(x_1 - 1, x_2) & \text{if } x_1 > 0,\ x_2 > 0 \\ v(0, x_2 - 1) & \text{if } x_1 = 0,\ x_2 > 0 \\ v(0, 0) & \text{if } x_1 = 0,\ x_2 = 0. \end{cases} \tag{B.4}$$

Then, we investigate whether the departure operator keeps all of the three monotonicity properties. Since the operator is partially defined, we consider each possible cases: ($x_1 > 0$, $x_2 > 0$), ($x_1 = 0$, $x_2 > 0$) and ($x_1 = 0$, $x_2 = 0$), separately for each property.

Monotonicity in x_1

We can write the first monotonicity equation for T_{DEP2} as follows for each cases:

Cases	$T_{DEP2}v(x_1, x_2) \geq T_{DEP2}v(x_1 + 1, x_2)$
$(x_1 > 0,\, x_2 > 0)$	$v(x_1 - 1, x_2) \geq v(x_1, x_2)$
$(x_1 = 0,\, x_2 > 0)$	$v(0, x_2 - 1) \geq v(0, x_2)$
$(x_1 = 0,\, x_2 = 0)$	$v(0, 0) \geq v(0, 0)$

It is obvious that three cases are true: the first case is true by the monotonicity of $v(x_1, x_2)$ in x_1, the second case is true by the monotonicity of $v(x_1, x_2)$ in x_2, and the left hand side and the right hand side is equal in the third case. Thus, the departure operator preserves the first monotonicity property of $v(x_1, x_2)$.

Monotonicity in x_2

Similar to first property, we can write the second monotonicity equation for T_{DEP2} as follows for each cases:

Cases	$T_{DEP2}v(x_1, x_2) \geq T_{DEP2}v(x_1, x_2 + 1)$
$(x_1 > 0,\, x_2 > 0)$	$v(x_1 - 1, x_2) \geq v(x_1 - 1, x_2 + 1)$
$(x_1 = 0,\, x_2 > 0)$	$v(0, x_2 - 1) \geq v(0, x_2)$
$(x_1 = 0,\, x_2 = 0)$	$v(0, 0) \geq v(0, 0)$

First and second cases are true by the monotonicity of $v(x_1, x_2)$ in x_2 and the third case is obvious. Therefore, the departure operator preserves the second monotonicity property of $v(x_1, x_2)$.

Monotonicity on the diagonal

The third monotonicity equation for T_{DEP2} as follows for each cases:

Cases	$T_{DEP2}v(x_1, x_2 + 1) \geq T_{DEP2}v(x_1 + 1, x_2)$
$(x_1 > 0,\, x_2 > 0)$	$v(x_1 - 1, x_2 + 1) \geq v(x_1, x_2)$
$(x_1 = 0,\, x_2 > 0)$	$v(0, x_2) \geq v(0, x_2)$
$(x_1 = 0,\, x_2 = 0)$	$v(0, 0) \geq v(0, 0)$

In this monotonicity property, all of the three cases are also true: the first one is true by the monotonicity of $v(x_1, x_2)$ on the diagonal and the remaining ones are true as in the previous monotonicity properties. Hence, the departure operator also preserves the third monotonicity property of $v(x_1, x_2)$, and it preserves all of the monotonicity properties of $v(x_1, x_2)$.

B.1.2 Monotonicity of T_{PRC_1}

Monotonicity in x_1

Let p_1^* and p_2^* be the optimal prices for the states (x_1, x_2) and (x_1+1, x_2), respectively. Then, we show that T_{PRC_1} preserves the monotonicity of $v(x_1, x_2)$ in x_2. Equation B.1 for this operator can be written as:

$$\bar{F}(p_1)[p_1 + v(x_1+1, x_2)] + F(p_1)v(x_1, x_2) \geq \bar{F}(p_2)[p_2 + v(x_1+2, x_2)] + F(p_2)v(x_1+1, x_2). \tag{B.5}$$

Since p_1^* is the optimal price for the state (x_1, x_2), we have that,

$$\bar{F}(p_1)[p_1 + v(x_1+1, x_2)] + F(p_1)v(x_1, x_2) \geq \bar{F}(p_2)[p_2 + v(x_1+1, x_2)] + F(p_2)v(x_1, x_2), \tag{B.6}$$

and by the monotonicity of $v(x_1, x_2)$ in x_1,

$$\bar{F}(p_2)[p_2 + v(x_1+1, x_2)] + F(p_2)v(x_1, x_2) \geq \bar{F}(p_2)[p_2 + v(x_1+2, x_2)] + F(p_2)v(x_1+1, x_2). \tag{B.7}$$

When we combine Equations B.6 and B.7, it is obvious that Equation B.5 is true and thus T_{PRC_1} preserves the monotonicity of $v(x_1, x_2)$ in x_1.

Monotonicity in x_2

Similar to the proof of the monotonicity in x_2, we let p_1^* and p_2^* be the optimal prices for the states (x_1, x_2) and (x_1+1, x_2), respectively, and write Equation B.2 for the pricing operator as:

$$\bar{F}(p_1)[p_1 + v(x_1+1, x_2)] + F(p_1)v(x_1, x_2) \geq \bar{F}(p_2)[p_2 + v(x_1+1, x_2+1)] + F(p_2)v(x_1, x_2+1). \tag{B.8}$$

Since p_1^* is the optimal price for the state (x_1, x_2), we have that,

$$\bar{F}(p_1)[p_1 + v(x_1+1, x_2)] + F(p_1)v(x_1, x_2) \geq \bar{F}(p_2)[p_2 + v(x_1+1, x_2)] + F(p_2)v(x_1, x_2), \tag{B.9}$$

and by the monotonicity of $v(x_1, x_2)$ in x_2,

$$\bar{F}(p_2)[p_2 + v(x_1+1, x_2)] + F(p_2)v(x_1, x_2) \geq \bar{F}(p_2)[p_2 + v(x_1+1, x_2+1)] + F(p_2)v(x_1, x_2+1). \tag{B.10}$$

When we combine Equations B.9 and B.10, it is obvious that Equation B.8 is true and thus T_{PRC_1} preserves the monotonicity of $v(x_1, x_2)$ in x_2.

Monotonicity on the diagonal

Similar to the previous monotonicity proofs of the pricing operator, we let p_1^* and p_2^* be the optimal prices for the states $(x_1, x_2 + 1)$ and $(x_1 + 1, x_2)$, respectively, and write Equation B.3 for the pricing operator as:

$$\bar{F}(p_1)[p_1+v(x_1+1,x_2+1)]+F(p_1)v(x_1,x_2+1) \geq \bar{F}(p_2)[p_2+v(x_1+2,x_2)]+F(p_2)v(x_1+1,x_2). \tag{B.11}$$

Since p_1^* is the optimal price for the state $(x_1, x_2 + 1)$, we have that,

$$\bar{F}(p_1)[p_1+v(x_1+1,x_2+1)]+F(p_1)v(x_1,x_2+1) \geq \bar{F}(p_2)[p_2+v(x_1+1,x_2+1)]+F(p_2)v(x_1,x_2+1), \tag{B.12}$$

and by the monotonicity of $v(x_1, x_2)$ on the diagonal,

$$\bar{F}(p_2)[p_2+v(x_1+1,x_2+1)]+F(p_2)v(x_1,x_2+1) \geq \bar{F}(p_2)[p_2+v(x_1+2,x_2)]+F(p_2)v(x_1+1,x_2). \tag{B.13}$$

When we combine Equations B.12 and B.13, it is obvious that Equation B.11 is true and thus T_{PRC_1} preserves the monotonicity of $v(x_1, x_2)$ on the diagonal.

The monotonicity proofs for T_{PRC_2} is similar to these proofs.

B.2 Proof: Submodularity of the Operators

In this proof, we prove that T_{DEP2}, T_{PRC_1} and T_{PRC_2} preserve the submodularity of a function on which they are applied $v(x_1, x_2)$. In other words, we show that the below equation is true for T_{DEP2}, T_{PRC_1} and T_{PRC_2} under the assumptions mentioned in the lemma.

$$\Delta_1 Tv(x_1, x_2) \leq \Delta_1 Tv(x_1, x_2 + 1) \tag{B.14}$$

B.2.1 Submodularity of T_{DEP2}

While considering the departure operator, we assume that $v(x_1, x_2)$ is non-increasing in x_1, x_2 and on the diagonal, and it satisfies the submodularity and subconcavity equations, i.e. $v(x_1, x_2)$ is concave in x_1 and x_2. Since we assume the monotonicity of $v(x_1, x_2)$, we can use the redefined version of the departure operator, i.e. Equation B.4. Then, we need to examine the submodularity equation for the three possible cases: $(x_1 > 0, x_2 > 0)$, $(x_1 = 0,$

$x_2 > 0$) and ($x_1 = 0$, $x_2 = 0$), separately. Equation B.14 can be written as follows for each cases:

Cases	$\Delta_1 T_{DEP2} v(x_1, x_2) \leq \Delta_1 T_{DEP2} v(x_1, x_2 + 1)$
$(x_1 > 0,\, x_2 > 0)$	$v(x_1 - 1, x_2) - v(x_1, x_2) \leq v(x_1 - 1, x_2 + 1) - v(x_1, x_2 + 1)$
$(x_1 = 0,\, x_2 > 0)$	$v(0, x_2 - 1) - v(0, x_2) \leq v(0, x_2) - v(0, x_2 + 1)$
$(x_1 = 0,\, x_2 = 0)$	$v(0, 0) - v(0, 0) \leq v(0, 0) - v(0, 1)$

The first case is true by the submodularity of $v(x_1, x_2)$, the second case is true by the concavity of $v(x_1, x_2)$ in x_2, and the last case is true by the monotonicity of $v(x_1, x_2)$ in x_2. Hence, T_{DEP2} preserves the submodularity of $v(x_1, x_2)$ under given assumptions.

B.2.2 Submodularity of T_{PRC_1}

We let the optimal prices for the states (x_1, x_2), $(x_1 + 1, x_2)$, $(x_1, x_2 + 1)$ and $(x_1 + 1, x_2 + 1)$ as follows:

$p^*(.,.)$	x_2	$x_2 + 1$
x_1	$p_{1,1}$	$p_{1,2}$
$x_1 + 1$	$p_{2,1}$	$p_{2,2}$

Then, we write the submodularity equation for the pricing operator as:

$$\begin{array}{c} \bar{F}(p_{1,1})[p_{1,1} + v(x_1 + 1, x_2)] \\ +F(p_{1,1})v(x_1, x_2) \\ -\bar{F}(p_{2,1})[p_{2,1} + v(x_1 + 2, x_2)] \\ -F(p_{2,1})v(x_1 + 1, x_2) \end{array} \leq \begin{array}{c} \bar{F}(p_{1,2})[p_{1,2} + v(x_1 + 1, x_2 + 1)] \\ +F(p_{1,2})v(x_1, x_2 + 1) \\ -\bar{F}(p_{2,2})[p_{2,2} + v(x_1 + 2, x_2 + 1)] \\ -F(p_{2,2})v(x_1 + 1, x_2 + 1). \end{array} \tag{B.15}$$

Since $p_{2,1}$ is the optimal price for the state $(x_1 + 1, x_2)$, we have that,

$$\begin{array}{c} \bar{F}(p_{1,1})[p_{1,1} + v(x_1 + 1, x_2)] \\ +F(p_{1,1})v(x_1, x_2) \\ -\bar{F}(p_{2,1})[p_{2,1} + v(x_1 + 2, x_2)] \\ -F(p_{2,1})v(x_1 + 1, x_2) \end{array} \leq \begin{array}{c} \bar{F}(p_{1,1})[p_{1,1} + v(x_1 + 1, x_2)] \\ +F(p_{1,1})v(x_1, x_2) \\ -\bar{F}(p_{2,2})[p_{2,2} + v(x_1 + 2, x_2)] \\ -F(p_{2,2})v(x_1 + 1, x_2). \end{array}$$

If we rearrange the right hand side, we can obtain that,

$$\begin{array}{c} \bar{F}(p_{1,1})[p_{1,1} + v(x_1 + 1, x_2)] \\ +F(p_{1,1})v(x_1, x_2) \\ -\bar{F}(p_{2,1})[p_{2,1} + v(x_1 + 2, x_2)] \\ -F(p_{2,1})v(x_1 + 1, x_2) \end{array} \leq \begin{array}{c} \bar{F}(p_{1,1})p_{1,1} - \bar{F}(p_{2,1})p_{2,2} \\ +F(p_{1,1})[v(x_1, x_2) - v(x_1 + 1, x_2)] \\ +\bar{F}(p_{2,2})[v(x_1 + 1, x_2) - v(x_1 + 2, x_2)]. \end{array} \tag{B.16}$$

Similarly, since $p_{1,2}$ is the optimal price for the state (x_1, x_2+1),

$$\begin{array}{c} \bar{F}(p_{1,2})[p_{1,2}+v(x_1+1,x_2+1)] \\ +F(p_{1,2})v(x_1,x_2+1) \\ -\bar{F}(p_{2,2})[p_{2,2}+v(x_1+2,x_2+1)] \\ -F(p_{2,2})v(x_1+1,x_2+1) \end{array} \geq \begin{array}{c} \bar{F}(p_{1,1})p_{1,1}-\bar{F}(p_{2,1})p_{2,2} \\ +F(p_{1,1})[v(x_1,x_2+1)-v(x_1+1,x_2+1)] \\ +\bar{F}(p_{2,2})[v(x_1+1,x_2+1)-v(x_1+2,x_2+1)]. \end{array} \tag{B.17}$$

Moreover, by using the submodularity of $v(x_1, x_2)$, we have that,

$$\begin{array}{c} \bar{F}(p_{1,1})[p_{1,1}-p_{2,2}] \\ +F(p_{1,1})[v(x_1,x_2)-v(x_1+1,x_2)] \\ +\bar{F}(p_{2,2})[v(x_1+1,x_2)-v(x_1+2,x_2)] \end{array} \leq \begin{array}{c} \bar{F}(p_{1,1})p_{1,1}-\bar{F}(p_{2,1})p_{2,2} \\ +F(p_{1,1})[v(x_1,x_2+1)-v(x_1+1,x_2+1)] \\ +\bar{F}(p_{2,2})[v(x_1+1,x_2+1)-v(x_1+2,x_2+1)]. \end{array} \tag{B.18}$$

Finally, when we combine Equations B.16, B.17 and B.18, it is obvious that Equation B.15 holds and T_{PRC_1} preserves the submodularity of $v_(x_1, x_2)$. The proof for T_{PRC_2} is similar.

B.3 Proof: Subconcavity of the Operators

In this proof, we prove that T_{DEP2}, T_{PRC_1} and T_{PRC_2} preserve both of the subconcavity conditions of a function on which they are applied $v(x_1, x_2)$. In other words, we show that the below equations are true for T_{DEP2}, T_{PRC_1} and T_{PRC_2} under the assumptions mentioned in the lemma.

$$\Delta_1 Tv(x_1, x_2+1) \leq \Delta_1 Tv(x_1+1, x_2) \tag{B.19}$$

$$\Delta_2 Tv(x_1+1, x_2) \leq \Delta_2 Tv(x_1, x_2+1) \tag{B.20}$$

B.3.1 Subconcavity of T_{DEP2}

We can write the first condition of subconcavity for each of the cases as follows:

Cases	$\Delta_1 T_{DEP2} v(x_1,x_2+1) \leq \Delta_1 T_{DEP2} v(x_1+1,x_2)$
$(x_1 > 0,\ x_2 > 0)$	$v(x_1-1,x_2+1)-v(x_1,x_2+1) \leq v(x_1,x_2)-v(x_1+1,x_2)$
$(x_1 = 0,\ x_2 > 0)$	$v(0,x_2)-v(0,x_2+1) \leq v(0,x_2)-v(1,x_2)$
$(x_1 = 0,\ x_2 = 0)$	$v(0,0)-v(0,1) \leq v(0,0)-v(1,0)$

The first case is true by the first condition of subconcavity and the last two cases are true by the monotonicity of $v(x_1, x_2)$ on the diagonal, i.e. $v(x_1, x_2 + 1) \geq v(x_1 + 1, x_2)$. Therefore, T_{DEP2} preserves the first condition of subconcavity.

Similar to the first condition, we can write the second condition of subconcavity as:

Cases	$\Delta_2 T_{DEP2} v(x_1 + 1, x_2) \leq \Delta_2 T_{DEP2} v(x_1, x_2 + 1)$
$(x_1 > 0,\ x_2 > 0)$	$v(x_1, x_2) - v(x_1, x_2 + 1) \leq v(x_1 - 1, x_2 + 1) - v(x_1 - 1, x_2 + 2)$
$(x_1 = 0,\ x_2 > 0)$	$v(0, x_2) - v(0, x_2 + 1) \leq v(0, x_2) - v(0, x_2 + 1)$
$(x_1 = 0,\ x_2 = 0)$	$v(0, 0) - v(0, 1) \leq v(0, 0) - v(, 1)$

The last two cases are obviously true and the first case is true by the second condition of subconcavity. Hence, T_{DEP2} also preserves the second condition of subconcavity.

B.3.2 Subconcavity of T_{PRC_1}

In order to prove both of the subconcavity conditions, we let the optimal prices for the states $(x_1, x_2 + 1)$, $(x_1, x_2 + 2)$, $(x_1 + 1, x_2)$, $(x_1 + 1, x_2 + 1)$ and $(x_1 + 2, x_2)$ as follows:

$p^*(.,.)$	x_2	$x_2 + 1$	$x_2 + 2$
x_1		$p_{1,2}$	$p_{1,3}$
$x_1 + 1$	$p_{2,1}$	$p_{2,2}$	
$x_1 + 2$	$p_{3,1}$		

and then focus on the subconcavity equations.

1^{st}*Condition*

We can write the subconcavity first condition for the pricing operator as:

$$\begin{array}{c} \bar{F}(p_{1,2})[p_{1,2} + v(x_1 + 1, x_2 + 1)] \\ +F(p_{1,2})v(x_1, x_2 + 1) \\ -\bar{F}(p_{2,2})[p_{2,2} + v(x_1 + 2, x_2 + 1)] \\ -F(p_{2,2})v(x_1 + 1, x_2 + 1) \end{array} \leq \begin{array}{c} \bar{F}(p_{2,1})[p_{2,1} + v(x_1 + 2, x_2)] \\ +F(p_{2,1})v(x_1 + 1, x_2) \\ -\bar{F}(p_{3,1})[p_{3,1} + v(x_1 + 3, x_2)] \\ -F(p_{3,1})v(x_1 + 2, x_2). \end{array} \tag{B.21}$$

Since $p_{2,2}$ is the optimal price for the state $(x_1 + 1, x_2 + 1)$, we have that,

$$\begin{array}{c} \bar{F}(p_{1,2})[p_{1,2} + v(x_1 + 1, x_2 + 1)] \\ +F(p_{1,2})v(x_1, x_2 + 1) \\ -\bar{F}(p_{2,2})[p_{2,2} + v(x_1 + 2, x_2 + 1)] \\ -F(p_{2,2})v(x_1 + 1, x_2 + 1) \end{array} \leq \begin{array}{c} \bar{F}(p_{1,2})[p_{1,2} + v(x_1 + 1, x_2 + 1)] \\ +F(p_{1,2})v(x_1, x_2 + 1) \\ -\bar{F}(p_{3,1})[p_{3,1} + v(x_1 + 2, x_2 + 1)] \\ -F(p_{3,1})v(x_1 + 1, x_2 + 1). \end{array}$$

If we rearrange the right hand side, we can obtain that,

$$\begin{array}{c} \bar{F}(p_{1,2})[p_{1,2}+v(x_1+1,x_2+1)] \\ +F(p_{1,2})v(x_1,x_2+1) \\ -\bar{F}(p_{2,2})[p_{2,2}+v(x_1+2,x_2+1)] \\ -F(p_{2,2})v(x_1+1,x_2+1) \end{array} \leq \begin{array}{c} \bar{F}(p_{1,2})p_{1,2}-\bar{F}(p_{3,1})p_{3,1} \\ +F(p_{1,2})[v(x_1,x_2+1)-v(x_1+1,x_2+1)] \\ +\bar{F}(p_{3,1})[v(x_1+1,x_2+1)-v(x_1+2,x_2+1)]. \end{array} \tag{B.22}$$

Similarly, by using the optimality of $p_{2,1}$,

$$\begin{array}{c} \bar{F}(p_{2,1})[p_{2,1}+v(x_1+2,x_2)] \\ +F(p_{2,1})v(x_1+1,x_2) \\ -\bar{F}(p_{3,1})[p_{3,1}+v(x_1+3,x_2)] \\ -F(p_{3,1})v(x_1+2,x_2) \end{array} \geq \begin{array}{c} \bar{F}(p_{1,2})p_{1,2}-\bar{F}(p_{3,1})p_{3,1} \\ +F(p_{1,2})[v(x_1+1,x_2)-v(x_1+2,x_2)] \\ +\bar{F}(p_{3,1})[v(x_1+2,x_2)-v(x_1+3,x_2)]. \end{array} \tag{B.23}$$

Moreover, by using the subconcavity first condition, we have that

$$\begin{array}{c} \bar{F}(p_{1,2})p_{1,2}-\bar{F}(p_{3,1})p_{3,1} \\ +F(p_{1,2})[v(x_1,x_2+1)-v(x_1+1,x_2+1)] \\ +\bar{F}(p_{3,1})[v(x_1+1,x_2+1)-v(x_1+2,x_2+1)] \end{array} \leq \begin{array}{c} \bar{F}(p_{1,2})p_{1,2}-\bar{F}(p_{3,1})p_{3,1} \\ +F(p_{1,2})[v(x_1+1,x_2)-v(x_1+2,x_2)] \\ +\bar{F}(p_{3,1})[v(x_1+2,x_2)-v(x_1+3,x_2)]. \end{array} \tag{B.24}$$

Finally, when we combine Equations B.22, B.23 and B.24, it is obvious that Equation B.21 holds, and thus we complete the proof for the first condition.

2^{nd} Condition

The proof of the second condition is not as trivial as the first condition because second condition is related with the opportunity costs of class-2 customers and the pricing operator is defined for class-1 customers. Therefore, we work on the second condition in two cases: ($p_{2,1} \geq p_{1,3}$) and ($p_{2,1} < p_{1,3}$). The idea of this case by case analysis comes from our computational studies. In these studies, we observe that for some holding cost parameters $p_{2,1} \geq p_{1,3}$ whereas for some other parameters $p_{2,1} < p_{1,3}$. This result implies that the opportunity cost of an additional class-1 customer at state (x_1+1, x_2) may or not may not be higher than the opportunity cost of an additional class-1 customer at state (x_1, x_2+2) according to the cost parameters. The intuition behind this result is the ratio of the holding cost of an expensive customer and a cheap customer. When this ratio is very high, i.e. cost of an expensive customer is much more higher than cost of a cheap one, having 2 more class-1 customer may be more expensive than having one expensive and two cheap customers, and thus $p_{2,1} \geq p_{1,3}$.

Then, we write Equation B.20 for the pricing operator as:

$$\begin{array}{c} \bar{F}(p_{2,1})[p_{2,1} + v(x_1+2, x_2)] \\ +F(p_{2,1})v(x_1+1, x_2) \\ -\bar{F}(p_{2,2})[p_{2,2} + v(x_1+2, x_2+1)] \\ -F(p_{2,2})v(x_1+1, x_2+1) \end{array} \leq \begin{array}{c} \bar{F}(p_{1,2})[p_{1,2} + v(x_1+1, x_2+1)] \\ +F(p_{1,2})v(x_1, x_2+1) \\ -\bar{F}(p_{1,3})[p_{1,3} + v(x_1+1, x_2+2)] \\ -F(p_{1,3})v(x_1, x_2+2). \end{array} \tag{B.25}$$

Case 1: $(p_{2,1} \geq p_{1,3})$

Since $p_{2,2}$ is the optimal price for the state (x_1, x_2), we have that,

$$\begin{array}{c} \bar{F}(p_{2,1})[p_{2,1} + v(x_1+2, x_2)] \\ +F(p_{2,1})v(x_1+1, x_2) \\ -\bar{F}(p_{2,2})[p_{2,2} + v(x_1+2, x_2+1)] \\ -F(p_{2,2})v(x_1+1, x_2+1) \end{array} \leq \begin{array}{c} \bar{F}(p_{2,1})[p_{2,1} + v(x_1+2, x_2)] \\ +F(p_{2,1})v(x_1+1, x_2) \\ -\bar{F}(p_{2,1})[p_{2,1} + v(x_1+2, x_2+1)] \\ -F(p_{2,1})v(x_1+1, x_2+1). \end{array}$$

When we rearrange the right hand side,

$$\begin{array}{c} \bar{F}(p_{2,1})[p_{2,1} + v(x_1+2, x_2)] \\ +F(p_{2,1})v(x_1+1, x_2) \\ -\bar{F}(p_{2,2})[p_{2,2} + v(x_1+2, x_2+1)] \\ -F(p_{2,2})v(x_1+1, x_2+1) \end{array} \leq \begin{array}{c} \bar{F}(p_{2,1})[v(x_1+2, x_2) - v(x_1+2, x_2+1)] \\ +F(p_{2,1})[v(x_1+1, x_2) - v(x_1+1, x_2+1)] \end{array} \tag{B.26}$$

Similarly, by using the optimality of $p_{1,2}$, we have that

$$\begin{array}{c} \bar{F}(p_{1,2})[p_{1,2} + v(x_1+1, x_2+1)] \\ +F(p_{1,2})v(x_1, x_2+1) \\ -\bar{F}(p_{1,3})[p_{1,3} + v(x_1+1, x_2+2)] \\ -F(p_{1,3})v(x_1, x_2+2) \end{array} \geq \begin{array}{c} \bar{F}(p_{1,3})[v(x_1+1, x_2+1) - v(x_1+1, x_2+2)] \\ +F(p_{1,3})[v(x_1, x_2+1) - v(x_1, x_2+2)]. \end{array} \tag{B.27}$$

Now, we focus on the right hand side of the Equations B.26 and B.27 and show that the following inequality holds.

$$\begin{array}{c} \bar{F}(p_{2,1})[v(x_1+2, x_2) - v(x_1+2, x_2+1)] \\ +F(p_{2,1})[v(x_1+1, x_2) - v(x_1+1, x_2+1)] \end{array} \leq \begin{array}{c} \bar{F}(p_{1,3})[v(x_1+1, x_2+1) - v(x_1+1, x_2+2)] \\ +F(p_{1,3})[v(x_1, x_2+1) - v(x_1, x_2+2)]. \end{array} \tag{B.28}$$

As we know that $(p_{2,1} \geq p_{1,3})$, we also have that $\bar{F}(p_{1,3}) = \bar{F}(p_{2,1}) + \xi$, where $\xi > 0$. Then, Equation B.28 becomes:

$$\begin{array}{c} \bar{F}(p_{2,1})[v(x_1+2,x_2) - v(x_1+2,x_2+1)] \\ +F(p_{2,1})[v(x_1+1,x_2) - v(x_1+1,x_2+1)] \end{array} \leq \begin{array}{c} \bar{F}(p_{2,1})[v(x_1+1,x_2+1) - v(x_1+1,x_2+2)] \\ +F(p_{2,1})[v(x_1,x_2+1) - v(x_1,x_2+2)] \\ \xi \left[\begin{array}{c} [v(x_1+1,x_2+1) - v(x_1+1,x_2+2)] \\ -[v(x_1,x_2+1) - v(x_1,x_2+2)] \end{array} \right] \end{array}.$$

Here, first two lines are true by the second condition of subconcavity and the last line is true by the submodularity of $v(x_1, x_2)$. Therefore, Equation B.28 is true. When we combine Equations B.26, B.27 and B.28, it is obvious that Equation B.25 holds for the first case.

Case 2: $(p_{2,1} < p_{1,3})$

We first rearrange Equation B.25 as:

$$\begin{array}{c} \bar{F}(p_{2,1})[p_{2,1} + v(x_1+2,x_2)] \\ +F(p_{2,1})v(x_1+1,x_2) \\ -\bar{F}(p_{1,2})[p_{1,2} + v(x_1+1,x_2+1)] \\ +F(p_{1,2})v(x_1,x_2+1) \end{array} \leq \begin{array}{c} \bar{F}(p_{2,2})[p_{2,2} + v(x_1+2,x_2+1)] \\ -F(p_{2,2})v(x_1+1,x_2+1) \\ -\bar{F}(p_{1,3})[p_{1,3} + v(x_1+1,x_2+2)] \\ -F(p_{1,3})v(x_1,x_2+2). \end{array} \tag{B.29}$$

Then, by using the optimality of $p_{1,2}$, we have that

$$\begin{array}{c} \bar{F}(p_{2,1})[p_{2,1} + v(x_1+2,x_2)] \\ +F(p_{2,1})v(x_1+1,x_2) \\ -\bar{F}(p_{1,2})[p_{1,2} + v(x_1+1,x_2+1)] \\ +F(p_{1,2})v(x_1,x_2+1) \end{array} \leq \begin{array}{c} \bar{F}(p_{2,1})[p_{2,1} + v(x_1+2,x_2)] \\ +F(p_{2,1})v(x_1+1,x_2) \\ -\bar{F}(p_{2,1})[p_{2,1} + v(x_1+1,x_2+1)] \\ +F(p_{2,1})v(x_1,x_2+1). \end{array}$$

When we work on the right hand side, we obtain that,

$$\begin{array}{c} \bar{F}(p_{2,1})[p_{2,1} + v(x_1+2,x_2)] \\ +F(p_{2,1})v(x_1+1,x_2) \\ -\bar{F}(p_{1,2})[p_{1,2} + v(x_1+1,x_2+1)] \\ +F(p_{1,2})v(x_1,x_2+1) \end{array} \leq \begin{array}{c} \bar{F}(p_{2,1})[v(x_1+2,x_2) - v(x_1+1,x_2+1)] \\ +F(p_{2,1})[v(x_1+1,x_2) - v(x_1,x_2+1)] \end{array} \tag{B.30}$$

Similarly, by using the optimality of $p_{2,2}$,

$$\begin{array}{c} \bar{F}(p_{2,2})[p_{2,2} + v(x_1+2,x_2+1)] \\ -F(p_{2,2})v(x_1+1,x_2+1) \\ -\bar{F}(p_{1,3})[p_{1,3} + v(x_1+1,x_2+2)] \\ -F(p_{1,3})v(x_1,x_2+2) \end{array} \geq \begin{array}{c} \bar{F}(p_{1,3})[v(x_1+2,x_2+1) - v(x_1+1,x_2+2)] \\ +F(p_{1,3})[v(x_1+1,x_2+1) - v(x_1,x_2+2)] \end{array} \tag{B.31}$$

As in the previous case, we show that the following equation is true:

$$\begin{array}{rcl} \bar{F}(p_{2,1})[v(x_1+2,x_2)-v(x_1+1,x_2+1)] & \leq & \bar{F}(p_{1,3})[v(x_1+2,x_2+1)-v(x_1+1,x_2+2)] \\ +F(p_{2,1})[v(x_1+1,x_2)-v(x_1,x_2+1)] & & +F(p_{1,3})[v(x_1+1,x_2+1)-v(x_1,x_2+2)] \end{array} \tag{B.32}$$

Since $(p_{2,1} \geq p_{1,3})$, we have that $\bar{F}(p_{2,1}) = \bar{F}(p_{1,3}) + \xi$, where $\xi > 0$. Then, Equation B.32 becomes:

$$\begin{array}{rcl} \bar{F}(p_{1,3})[v(x_1+2,x_2)-v(x_1+1,x_2+1)] & & \bar{F}(p_{1,3})[v(x_1+2,x_2+1)-v(x_1+1,x_2+2)] \\ +F(p_{1,3})[v(x_1+1,x_2)-v(x_1,x_2+1)] & \leq & +F(p_{1,3})[v(x_1+1,x_2+1)-v(x_1,x_2+2)] \\ +\xi\left[\begin{array}{c} [v(x_1+2,x_2)-v(x_1+1,x_2+1)] \\ -[v(x_1+1,x_2)-v(x_1,x_2+1)] \end{array}\right] & & . \end{array}$$

Here, first two lines are true by the second condition of subconcavity and the last line is true by the first condition of subconcavity. Therefore, Equation B.32 is true. When we combine Equations B.30, B.31 and B.32, it is obvious that Equation B.25 holds for the second case. Therefore, we prove that T_{PRC_1} preserves the second condition of the subconcavity for both of the cases.

In conclusion, we show that $T_{PRC_1}v(x_1, x_2$ will keep both conditions of the subconcavity if the necessary assumptions are satisfied. The proof for T_{PRC_2} is similar to this proof. While considering T_{PRC_2}, we need to work on the first condition in two cases: $(p_{1,2}) \geq p_{3,1}$ and $(p_{1,2}) < p_{3,1}$.

B.4 Proof: Monotonicity of the Optimal Prices

Let p_1^* and p_2^* be the optimal prices for class-1 for the states (x_1, x_2) and (x_1+1, x_2), respectively, and assume that $p_1^* > p_2^*$. Then, we have the following equation as a result of the optimality of p_1^* and p_2^*:

$$\begin{array}{rcl} \bar{F}(p_1)[p_1+v(x_1+1,x_2)]+F(p_1)v(x_1,x_2) & \geq & \bar{F}(p_2)[p_2+v(x_1+1,x_2)]+F(p_2)v(x_1,x_2) \\ \bar{F}(p_2)[p_2+v(x_1+2,x_2)]+F(p_2)v(x_1+1,x_2) & \geq & \bar{F}(p_1)[p_1+v(x_1+2,x_2)]+F(p_1)v(x_1+1,x_2) \end{array}$$

When we combine these equations, we obtain that,

$$[F(p_1)-F(p_2)][[v(x_1,x_2)-v(x_1+1,x_2)]-[v(x_1+1,x_2)-v(x_1+2,x_2)]] \geq 0 \tag{B.33}$$

Since $F(p_1) - F(p_2) > 0$ by our assumption on the optimal prices and $[v(x_1, x_2) - v(x_1 + 1, x_2)] - [v(x_1 + 1, x_2) - v(x_1 + 2, x_2)] \leq 0$ by the concavity of $v(x_1, x_2)$, Equation B.33 can not be true. Therefore,there is contradiction and our assumption on the optimal prices are not correct. Hence, the optimal prices for class-1, $p_1^*(x_1, x_2)$, are non-decreasing in the number of class-1 customer in the system. The monotonicity of the optimal prices for class-1 in x_2 and on the diagonal can be proven in a similar manner. Moreover, the monotonicity of the optimal prices for class-2 can also be proven by a similar proof.

VITA

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