THIN SPECTRUM AND EXCITATION LIFE TIME IN ATOMIC BOSE-EINSTEIN CONDENSATES

by

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This is to certify that I have examined this copy of a master's thesis by

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and have found that it is complete and satisfactory in all respects,
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I dedicate this thesis to
all who seek to understand what's going on...

ABSTRACT

In this thesis we study two distinct mechanisms of phase diffusion in atomic Bose-Einstein condensates (BEC's). The first one is studied via a so-called *toy-model* and gives a finite life time that the ground state of a Bose-Einstein condensate can remain coherent. This mechanism, which was before used for coherent states, is generalized to squeezed and the thermal-coherent states, which we introduce as states having both coherent and thermal character. Then the recently introduced *thin spectrum* formalism is reviewed and applied to BEC's in order to obtain a decay rate for excitations. Relation of thin spectra with Spontaneous Symmetry Breaking (SSB) is studied and some expansions on the formalism are made.

ÖZETÇE

Bu tezde, atomik Bose-Einstein Yoğunlaşıklarını etkileyen iki faz bozunum mekanizmasını inceliyoruz. Bu mekanizmalardan ilkini, Bose-Einstein Yoğunlaşıklarının temel durumunun özuyumlu kalabildiği sonlu bir ömür öngören ve *basit model* adı verilen bir yöntemle araştırdık. Daha önce özuyumlu durumlar için kullanılan bu yöntemi, sıkıştırılmış-özuyumlu ve termal-özuyumlu durumlara da uyguladık ve bunun için hem termal hem de özuyumlu özellik taşıyan termal-özuyumlu durumları tanımladık. Daha sonra, yakın zamanda bulunmuş olan *ince spektrum* formalizmi hakkında bilgi verdik ve bu formalizmi atomik Bose-Einstein Yoğunlaşıklarına uygulayarak uyarılmış durumlar için bir bozunum hızı elde ettik. İnce spektrumların Kendiliğinden Simetri Bozunumu ile ilişkisini inceledik ve bu formalizme birkaç ekleme yaptık.

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The number of people who have helped me in the course of this thesis and before is too high to have an exact list here. For his reason I will be able to mention only some: I would like to thank my parents for providing me with every opportunity of education and life, and usually working for this harder than me. Then I humbly thank my supervisors in this thesis for guiding me in the deep ocean of physics, in which I quite often feared to drown. All friends of mine, but especially those who were close to me in these last days and sometimes suffered more than me, deserve a thank too, which cannot be quantified but would be quite large in orders of magnitude if it could.

Finally, I would like to thank the members of my thesis committee for examining (and hopefully approving) this thesis.

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PRELIMINARIES

1.1 Introduction

Towards the end of the first quarter of 20^{th} century, it was proposed that if a macroscopic system of indistinguishable bosons is cooled enough, a finite fraction of the bosons will occupy the same quantum state [1]. Although the first observation of superfluid Helium was made shortly after their papers [2], it took 70 years for the cooling and trapping technology to advance enough to obtain Bose-Einstein condensation (BEC) in weakly interacting, dilute atomic gases [3]. Contrary to the case for liquid Helium, it is quite easy to construct a microscopic theory for dilute gases and therefore they prove to be an excellent playground to study the quantum nature of matter. Many interesting phenomena, such as quantized vortices and vortex arrays, have been studied extensively, both experimentally and theoretically, in such systems.

Shortly after the initial discovery of BEC, it was immediately understood that a finite sized condensate suffers from quantum phase diffusion [4, 5], an interaction driven decoherence due to atomic number fluctuations from within the condensate [6], in addition to the usual decoherence from an imperfect isolation from the environment. The reason of this is that Bose-Einstein condensation is a symmetry broken state, in which the gauge symmetry is spontaneously broken. There is always a finite probability of tunneling from a ground state to another degenerate ground state, so the Hamiltonian of the system seeks to restore the broken symmetry.

In this study we study a third source of decoherence, which affects the excitations on a condensate. This mechanism is related with the presence of a *thin spectrum* [7], which is in turn related to spontaneous symmetry breaking, via the Nambu-Goldstone Theorem [8]. The idea of thin spectrum and its relation with decoherence have been proposed recently and is studied extensively for only but one model, namely the Lieb-Mattis model. Our work, besides giving life times for BEC's, is the first application of thin spectrum formalism on another system.

After some preliminary comments, we begin by a review of a toy-model calculation for various ground-states. We then review the idea of thin spectrum and show its relation with decoherence for any quantum system. Then we apply this idea to atomic Bose-Einstein condensates and obtain life times for quasiparticle excitations. Finally we generalize the idea of thin spectrum to a system with more than one spontaneously broken symmetries.

1.2 Spontaneous Symmetry Breaking

The concept of symmetry is not a new one, and indeed it can be traced back to the first philosophers. At this point it is instructive to cite Leibniz, speaking of Archimedes:

"But in order to proceed from mathematics to natural philosophy, another principle is requisite, as I have observed in my Theodicy: I mean, the principle of a sufficient reason, viz. that nothing happens without a reason why it should be so, rather than otherwise. And therefore Archimedes being to proceed from mathematics to natural philosophy, in his book De Aequilibrio, was obliged to make use of a particular case of the great principle of a sufficient reason. He takes it for granted, that if there be a balance, in which everything is alike on both sides, and if equal weights are hung on the two ends of that balance, the whole will be at rest. 'Tis because no reason can be given, why one side should weigh down, rather than the other. Now, by that single principle, viz. that there ought to be a sufficient reason why things should be so, and not otherwise, one may demonstrate the being of God, and all the other parts of metaphysics or natural theology; and even, in some measure, those principles of natural philosophy, that are independent upon mathematics: I mean, the dynamical principles, or the principles of force."

Although Leibniz uses the symmetry concept to obtain metaphysical conclusions, in doing so he outlines the intuitive basics which are related with physical theories. However, this intuitive ideas had to be revised when it was found that asymmetric states can emerge from symmetric Hamiltonians [9]. In many common physical systems, the instability of the system's symmetry is proportional to the number of degrees of freedom [11]. Therefore, in the thermodynamic limit where $N \to \infty$, it is possible to have an asymmetric state emerge from a symmetric Hamiltonian so long as there is an infinitesimal fluctuation; which is always present in quantum physics due to Heisenberg's uncertainty principle. This phenomenon is called the *spontaneous symmetry breaking* (SSB) and is observed in many subfields of physics. It implies that quantitative differences may lead to qualitative ones, and therefore *"More is different"* [9].

1.3 Hamiltonian

We consider a homogenous system of indistinguishable bosonic atoms in no external trapping potential. We denote the state of this system by $|n_0, n_{\vec{k}_1}, n_{\vec{k}_2}, ...\rangle$ where $n_{\vec{k}}$ is the number of atoms with wave vector $\vec{k}$ and n_0 is the number of atoms with zero momentum. The annihilation operator $a_{\vec{k}}$, which annihilates a boson with $\vec{k}$, and its conjugate are defined as

$$a_{\vec{k}}|..., n_{\vec{k}}, ...\rangle = \sqrt{n_{\vec{k}}}|..., n_{\vec{k}} - 1, ...\rangle, \tag{1.1}$$

$$a^{\dagger}_{\vec{k}}|..., n_{\vec{k}}, ...\rangle = \sqrt{n_{\vec{k}} + 1}|..., n_{\vec{k}} + 1, ...\rangle. \tag{1.2}$$

Their commutation relations are

$$[a_{\vec{k}}, a^{\dagger}_{\vec{k}'}] = \delta_{\vec{k},\vec{k}'}, \tag{1.3}$$

$$[a_{\vec{k}}, a_{\vec{k}'}] = [a^{\dagger}_{\vec{k}}, a^{\dagger}_{\vec{k}'}] = 0. \tag{1.4}$$

Unless otherwise necessary, the operator hats on these operators will be omitted.

If there is no interatomic interaction, the energy eigenstates are the plane waves which have energy $E_{\vec{k}} = \hbar^2 k^2/2m$ where m is the mass of the atoms. In this case the second quantized Hamiltonian is

$$\mathcal{H} = \sum_{\vec{k}} E_{\vec{k}} a^{\dagger}_{\vec{k}} a_{\vec{k}}. \tag{1.5}$$

As the gas we consider is dilute and weakly interacting, we assume that there is only a contact (zero range) interaction between the bosons. Such an interaction can be represented by a Dirac delta, that is $V(\vec{r}, \vec{r}') = u\delta(\vec{r} - \vec{r}')$, where u is a constant, which is for our case

$$u = \frac{4\pi\hbar^2 a_s}{m} \tag{1.6}$$

with a_s as the s-wave scattering length. The sign of this constant u determines whether the interaction is attractive or repulsive. Although it is possible to obtain meta-stable condensates with attractive interaction ($a_s < 0$) too, throughout this study we consider only repulsive interactions.

Second quantized form of a general two-body potential $V(\vec{r}, \vec{r}')$ is given by

$$V_{sec} = \frac{1}{2} \int \int \hat{\Psi}^{\dagger}(\vec{r}) \hat{\Psi}^{\dagger}(\vec{r'}) V(\vec{r}, \vec{r}') \hat{\Psi}(\vec{r'}) \hat{\Psi}(\vec{r}) d^d r d^d r' \tag{1.7}$$

where the field operator $\hat{\Psi}(\vec{r})$ has the expansion

$$\hat{\Psi}(\vec{r}) = \sum_{\vec{k}} \psi_{\vec{k}}(\vec{r}) a_{\vec{k}}. \tag{1.8}$$

Here, $\psi_{\vec{k}}(\vec{r})$ is the wave function of the $\vec{k}$th mode and integrals are over the whole d-dimensional space. Substituting the contact potential into this expression, its second quantized form is found as

$$V_{sec} = \frac{\tilde{u}}{2} \sum_{\vec{k},\vec{p},\vec{q}} a^{\dagger}_{\vec{p}+\vec{q}} a^{\dagger}_{\vec{k}-\vec{q}} a_{\vec{k}} a_{\vec{p}} \tag{1.9}$$

in terms of a new interaction constant $\tilde{u} = u/V$ where V is the quantization volume. Including this interaction, the full Hamiltonian becomes:

$$\boxed{\mathcal{H} = \sum_{\vec{k}} E_{\vec{k}} a^{\dagger}_{\vec{k}} a_{\vec{k}} + \frac{\tilde{u}}{2} \sum_{\vec{k},\vec{p},\vec{q}} a^{\dagger}_{\vec{p}+\vec{q}} a^{\dagger}_{\vec{k}-\vec{q}} a_{\vec{k}} a_{\vec{p}}} \tag{1.10}$$

In order to fix the average number of atoms, we also include a chemical potential, μ, in $E_{\vec{k}}$, that is

$$E_{\vec{k}} = \frac{\hbar^2 k^2}{2m} - \mu. \tag{1.11}$$

1.4 Quasiparticles

We consider the ground state $|\Psi\rangle$ of a Bose-Einstein condensed system [10]. Since the interparticle interaction is weak there is macroscopic occupation only in the zero-momentum state.

Let $|\Psi\rangle = |N_0, N_{\vec{k}_1}, N_{\vec{k}_2}, ...\rangle$. Then $N_0 \gg 1$ and $N_{\vec{k}\neq 0} \sim 1$. Therefore

$$\begin{aligned} a_0|\Psi\rangle &= \sqrt{N_0}|N_0 - 1, N_{\vec{k}_1}, N_{\vec{k}_2}, ...\rangle, && (1.12)\\ &\simeq \sqrt{N_0}|N_0, N_{\vec{k}_1}, N_{\vec{k}_2}, ...\rangle, && (1.13)\\ &\simeq \sqrt{N_0}|\Psi\rangle. && (1.14) \end{aligned}$$

This suggests that treating a_0 separately might be useful. Also, since $N_{\vec{k}\neq 0} \ll N_0$, it is possible to omit terms which are 3^{rd} or 4^{th} order in $a_{\vec{k}\neq 0}$. The Hamiltonian becomes:

$$\mathcal{H} = \mathcal{H}_z + \mathcal{H}_e, \tag{1.15}$$

$$\mathcal{H}_z = \frac{\tilde{u}}{2}(\hat{n}_0^2 - \hat{n}_0) - \mu\hat{n}_0 \tag{1.16}$$

$$\mathcal{H}_e = \sum_{k\neq 0}\left[\left(E_{\vec{k}} + 2\tilde{u}\hat{n}_0\right)\hat{n}_{\vec{k}} + \frac{\tilde{u}}{2}\left(a^\dagger_{\vec{k}}a^\dagger_{-\vec{k}}a_0a_0 + h.c.\right)\right] \tag{1.17}$$

where $\hat{n}_{\vec{k}} = a^\dagger_{\vec{k}}a_{\vec{k}}$. Note that these two parts of the Hamiltonian do not commute with each other and

$$[\mathcal{H}_z, \mathcal{H}_e] = \frac{\tilde{u}^2}{2}\sum_{k\neq 0}\left[a^\dagger_{\vec{k}}a^\dagger_{-\vec{k}}\left(-\hat{n}_0a_0^2 - a_0^2\hat{n}_0 + a_0^2 + \frac{2\mu}{\tilde{u}}a_0^2\right) - h.c.\right]. \tag{1.18}$$

Considering the fact that

$$\begin{aligned} [a_0, a_0^\dagger] &= 1 && (1.19)\\ &\ll N_0 && (1.20) \end{aligned}$$

one might replace a_0's and $a_0^\dagger$'s with the c-number $\sqrt{N_0}$ and in this way ignore their non-commutativity:

$$a_0 \to \sqrt{N_0}. \tag{1.21}$$

This prescription is called the Bogoliubov approximation and leads to $[\mathcal{H}_z, \mathcal{H}_e] = 0$. (However, note that instead of $\sqrt{N_0}$, one could use $e^{i\theta}\sqrt{N_0}$ for any real θ. By choosing $\theta = 0$ we explicitly break the phase symmetry.) The excitation Hamiltonian is now

$$\mathcal{H}_e = \sum_{k \neq 0} \left[(E_k + 2u\rho_0)\hat{n}_{\vec{k}} + \frac{u\rho_0}{2}(\hat{a}^\dagger_{\vec{k}}\hat{a}^\dagger_{-\vec{k}} + h.c.) \right] \tag{1.22}$$

where we have defined $\rho_0 = N_0/V$.

According to the usual Bogoliubov theory, a_0's in $\mathcal{H}_z$ are replaced by $\sqrt{N_0}$ too. This corresponds to ignoring the quantum nature of the condensate mode. However, since we have assumed $a_0|\Psi\rangle = \sqrt{N_0}|\Psi\rangle$, the condensate mode must have some phase characteristic and hence shall have nonzero number fluctuations. (The results of various interference experiments achieved by BEC's support this reasoning.) Indeed, the toy model calculations of chapter 2 will show that the zero mode decoheres in time and therefore its quantum nature is important. For this reason we don't make the Bogoliubov replacement for the zero mode Hamiltonian. The total Hamiltonian is

$$\mathcal{H} = \frac{\tilde{u}}{2}(\hat{n}_0^2 - \hat{n}_0) - \mu\hat{n}_0 + \sum_{k \neq 0} \left[(E_k + 2u\rho_0)\hat{n}_{\vec{k}} + \frac{u\rho_0}{2}(\hat{a}^\dagger_{\vec{k}}\hat{a}^\dagger_{-\vec{k}} + h.c.) \right]. \tag{1.23}$$

The excitation Hamiltonian $\mathcal{H}_e$ and zero-mode Hamiltonian $\mathcal{H}_z$ now commute with each other. However, the price paid is that the total number operator $\hat{N} = \sum_{\vec{k}} \hat{n}_{\vec{k}}$ no longer commutes with the total Hamiltonian and the theory is no longer number-conserving.

The excitation Hamiltonian (1.22) has off-diagonal terms $a_{-\vec{k}}a_{\vec{k}}$ and $a^\dagger_{\vec{k}}a^\dagger_{-\vec{k}}$. We seek to find a canonical transformation to define new operators $b_{\vec{k}}$ such that

$$\mathcal{H}_e = \sum_{k \neq 0} \epsilon_k b^\dagger_{\vec{k}} b_{\vec{k}} + constants \tag{1.24}$$

where $\epsilon_{\vec{k}}$ are some constants. Such a transformation has been developed by Bogoliubov [12]. Let

$$a_{\vec{k}} = u_{\vec{k}} b_{\vec{k}} + v_{\vec{k}} b^\dagger_{-\vec{k}} \tag{1.25}$$

with real $u_{\vec{k}}$, $v_{\vec{k}}$ and $u_{\vec{k}} = u_{-\vec{k}}$, $v_{\vec{k}} = v_{-\vec{k}}$. Imposing the canonical commutation relations $[b_{\vec{k}}, b^{\dagger}_{\vec{k}'}] = \delta_{\vec{k}\vec{k}'}$ and $[b_{\vec{k}}, b_{\vec{k}'}] = [b^{\dagger}_{\vec{k}}, b^{\dagger}_{\vec{k}'}] = 0$ implies $u^2_{\vec{k}} - v^2_{\vec{k}} = 1$. So one can parameterize them as $u_{\vec{k}} = \cosh \xi_{\vec{k}}$ and $v_{\vec{k}} = \sinh \xi_{\vec{k}}$. Equating right hand sides of equations (1.22) and (1.24) it is easy to find that

$$\tanh 2\xi_{\vec{k}} = \frac{\rho_0 u}{E_{\vec{k}} + 2u\rho_0} \tag{1.26}$$

and

$$\epsilon_{\vec{k}} = \sqrt{(E_{\vec{k}} + 2u\rho_0)^2 - (u\rho_0)^2}. \tag{1.27}$$

Now that the Hamiltonian has become of the form (1.24) we have a group of non-interacting bosons with energies $\epsilon_{\vec{k}}$. These bosons are called the *quasi*particles and the corresponding operators $b_{\vec{k}}$ and $b^{\dagger}_{\vec{k}}$ are called the quasiparticle operators.

The gaplessness condition, $\epsilon_{\vec{k}} \to 0$ when $\vec{k} \to 0$, can be used to find the chemical potential. It gives $\mu = u\rho_0$.

1.5 Coherent and Squeezed-Coherent States

In this section we briefly review the properties of single-mode coherent [13] and squeezed states. A wider discussion can be found in many textbooks in the field of quantum optics [14].

The coherent state $|\alpha\rangle$ is defined as the right eigenstate of the annihilation operator:

$$a|\alpha\rangle = \alpha|\alpha\rangle. \tag{1.28}$$

It is easy to prove that such an eigenstate exists for every complex number α. But since a is not Hermitian, coherent states are not also left eigenstates a. Indeed, no left eigenstate of the annihilation operator exists since the Fock space is bounded from below but not from above.

Using the defining equation (1.28), Fock state expansion of $|\alpha\rangle$ is found as

$$|\alpha\rangle = \mathrm{e}^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle. \tag{1.29}$$

Coherent states can also be obtained by acting on the ground state by the 'displacement' operator $D(\alpha)$:

$$D(\alpha) = \exp(\alpha a^\dagger - \alpha^* a) \tag{1.30}$$

$$|\alpha\rangle = D(\alpha)|0\rangle \tag{1.31}$$

$D(\alpha)$ is a unitary operator, therefore $D^\dagger(\alpha)D(\alpha) = \mathcal{I}$ where $\mathcal{I}$ is the identity operator. It also satisfies $D^\dagger(\alpha) = D(-\alpha)$ and

$$D(\alpha_1)D(\alpha_2) = \mathrm{e}^{(\alpha_1\alpha_2^* - \alpha_1^*\alpha_2)/2} D(\alpha_1 + \alpha_2). \tag{1.32}$$

Every coherent state is a minimum uncertainty state with equally distributed quadrature variances. This is the reason that they are the most classical quantum states and are also called the 'quasi-classical' states [15]. They do not form an ortho-complete basis but rather an over-complete basis:

$$\langle\alpha_1|\alpha_2\rangle = \mathrm{e}^{-|\alpha_1-\alpha_2|^2/2}\mathrm{e}^{(\alpha_1^*\alpha_2 - \alpha_1\alpha_2^*)/2} \neq \delta^2(\alpha_1 - \alpha_2) \tag{1.33}$$

$$\mathcal{I} = \frac{1}{\pi}\int |\alpha\rangle\langle\alpha| d^2\alpha \tag{1.34}$$

It is possible to construct orhto-complete bases by using one dimensional subspaces of α space; for example by using α's lying on a straight line [16] or a circle [17].

The single mode unitary squeeze operator is defined as

$$S(\gamma) = e^{\frac{\gamma}{2}\hat{a}\hat{a} - \frac{\gamma^*}{2}\hat{a}^\dagger\hat{a}^\dagger}, \tag{1.35}$$

and we introduce the squeezed-coherent state in terms of it:

$$|\alpha,\gamma\rangle = D(\alpha)S(\gamma)|0\rangle \tag{1.36}$$

An alternative definition $S(\gamma)D(\alpha)|0\rangle$ is also common in literature but it leads to states of the same form since

$$D(\alpha)S(\gamma) = S(\gamma)D(\alpha'_+), \tag{1.37}$$

$$S(\gamma)D(\alpha) = D(\alpha'_-)S(\gamma), \tag{1.38}$$

$$\alpha'_{\pm}(\gamma) = \mu\alpha \pm \nu\alpha^*, \tag{1.39}$$

$$\mu = \cosh|\gamma|, \tag{1.40}$$

$$\nu = \frac{\gamma}{|\gamma|}\sinh|\gamma|. \tag{1.41}$$

Henceforth we use the definition given in equation (1.36).

The two Hermitian quadratures $x = (a + a^\dagger)/2$ and $p = i(a - a^\dagger)/2$ have unequal variances in a squeezed state. The difference between the arguments of complex parameters α and γ determine which quadrature is squeezed and which quadrature is stretched. In order to illustrate this fact better, we plot the Q-function of the squeezed state [18],

$$\begin{aligned} Q(\beta,\beta^*) =& \frac{\mathrm{sech}|\gamma|}{\pi}\exp\left\{-\frac{1}{2}\left(|\alpha|^2+|\beta|^2\right)+\beta^*\alpha\,\mathrm{sech}|\gamma|\right\} \\ & \exp\left\{-\frac{1}{2}\left[\mathrm{e}^{i\gamma/|\gamma|}(\beta^{*2}-\alpha^{*2})+\mathrm{e}^{-i\gamma/|\gamma|}(\beta^2-\alpha^2)\right]\tanh|\gamma|\right\}, \end{aligned} \tag{1.42}$$

for various values of γ in figure 1.1.

For future reference, we give the Fock state expansion of $|\alpha,\gamma\rangle$ [19]:

$$\begin{aligned} |\alpha,\gamma\rangle &= \sum_{n=0}^{\infty} A_n(\alpha,\zeta)|n\rangle \\ &= (1-|\zeta|^2)^{1/4} e^{-\frac{(\alpha+\zeta\alpha^*)\alpha^*}{2}} \sum_{n=0}^{\infty}\sqrt{\frac{\zeta^n}{2^n n!}} H_n\left(\frac{\alpha+\zeta\alpha^*}{\sqrt{2\zeta}}\right)|n\rangle \end{aligned} \tag{1.43}$$

Here, H_n is the n-th order Hermite polynomial, and a new parameter

$$\zeta = \gamma\frac{\tanh|\gamma|}{|\gamma|} \tag{1.44}$$

is defined for notational convenience.

1.5.1 Quasiparticle Vacuum and Multimode Squeezing

The unitary multi-mode squeeze operator is defined as [20]

$$S(\vec{r}) = \exp\left[\frac{1}{2}\sum_{k\neq 0}\left(r_{\vec{k}} a^\dagger_{\vec{k}} a^\dagger_{-\vec{k}} - h.c.\right)\right]. \tag{1.45}$$

$\vec{r}$ is a vector whose components $r_{\vec{k}}$ determine the amount of squeezing in $\vec{k}$th mode. The sum is over whole space so each $\vec{k}$ appears twice. Hence

$$S(\vec{r})a_{\vec{k}}S^{\dagger}(\vec{r}) = \exp\left[r_{\vec{k}}a^{\dagger}_{\vec{k}}a^{\dagger}_{-\vec{k}} - h.c.\right] a_{\vec{k}} \exp\left[r^{*}_{\vec{k}}a_{-\vec{k}}a_{\vec{k}} - h.c.\right]. \tag{1.46}$$

Using the Taylor expansions of hyperbolic functions and the Baker-Hausdorff Lemma [21]

$$e^{G}Ae^{-G} = \frac{A}{0!} + \frac{[G,A]}{1!} + \frac{[G,[G,A]]}{2!} + ... \tag{1.47}$$

which is valid for anti-hermitian G, it is trivial to show that

$$S(\vec{r})a_{\vec{k}}S^{\dagger}(\vec{r}) = \cosh r_{\vec{k}}a_{\vec{k}} - \sinh r_{\vec{k}}a^{\dagger}_{-\vec{k}}, \tag{1.48}$$

$$= b_{\vec{k}} \tag{1.49}$$

for real $\vec{r}$. Hence it is seen that Bogoliubov transformation can be achieved by using the multi-mode squeeze operators.

The ground state $|\Psi\rangle$ has no excitations. It is the quasiparticle vacuum, therefore

$$b_k|\Phi\rangle = 0, \tag{1.50}$$

$$= Sa_kS^{-1}|\Phi\rangle \tag{1.51}$$

$$a_kS^{-1}|\Phi\rangle = 0, \tag{1.52}$$

$$|\Phi\rangle = S|vac\rangle. \tag{1.53}$$

Thus it is seen that the quasiparticle vacuum $|\Phi\rangle$ is in fact a squeezed vacuum of particles with nonzero k.

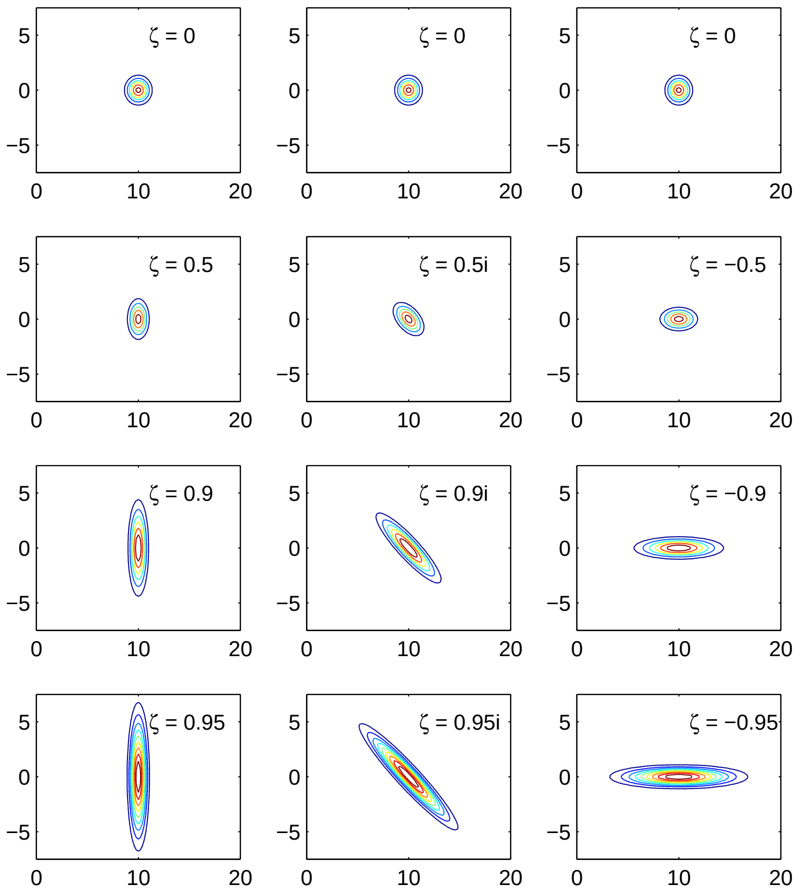

Figure 1.1: Contour plots for Q-functions of various squeezed states with $\alpha = 10$ and different ζ's. Q-function is defined in terms of the system's density matrix ρ as $Q(\beta, \beta^*) = \frac{1}{\pi}\langle\beta|\rho|\beta\rangle$. The horizontal axis corresponds to $\text{Re}(\beta)$ and the vertical one to $\text{Im}(\beta)$.

Chapter 2

TOY MODEL CALCULATIONS

In this chapter we consider the ground state $|\Psi\rangle$ of a Bose-Einstein condensed system in more detail. The Bogoliubov prescription (1.21) entails that the zero mode annihilation operator has a nonzero expectation value in the ground state:

$$\langle\Psi|a_0|\Psi\rangle = \sqrt{N} \tag{2.1}$$

which is a fact related with the broken global phase symmetry. Henceforth we focus our attention only to the zero momentum mode (zero-mode) and omit the '0' subscripts unless otherwise necessary.

The zero-mode ground state, which we denote by $|\text{gr}\rangle$, cannot simply be a Fock state, since in that case $\langle\text{gr}|a|\text{gr}\rangle$ would be equal to zero. A natural candidate which carries this required characteristic of a broken phase symmetry would be a coherent state. The time evolution of a coherent state under the zero mode Hamiltonian

$$\mathcal{H} = \frac{\tilde{u}}{2}(\hat{n}^2 - \hat{n}) - \mu\hat{n} \tag{2.2}$$

is studied before [22, 23] and it is found that since a coherent state is not an energy eigenstate, it suffers phase collapse [4, 5]. In this section we first review this calculation and then consider two other states, namely squeezed and thermal-coherent states.

2.1 Toy Model for Coherent Zero-Mode Occupation

We consider a variational, symmetry breaking ground state, a coherent state satisfying $a|\alpha\rangle = \alpha|\alpha\rangle$. Minimization of the mean free energy $\langle\alpha|\mathcal{H}|\alpha\rangle$ fixes $|\alpha| \simeq \sqrt{N}$ where we assume that there are N atoms in the condensate.

As the order parameter for BEC in this case, we use the expectation value of the annihilation operator. In the Heisenberg picture, the operator $a(t)$ is

$$a(t) = \mathrm{e}^{\frac{i}{\hbar}\mathcal{H}t} a\, \mathrm{e}^{-\frac{i}{\hbar}\mathcal{H}t}. \tag{2.3}$$

In terms of the eigenenergy $E_n = \frac{\tilde{u}}{2}(n^2 - n) - \mu n$ defined through $\mathcal{H}|n\rangle = E_n|n\rangle$ for the n^{th} Fock state $|n\rangle$, one can easily calculate

$$\langle\alpha|a|\alpha\rangle \;\; = \;\; \sqrt{N} \exp\left(N[\mathrm{e}^{-\frac{i}{\hbar}\tilde{u}t} - 1]\right) \mathrm{e}^{\frac{i}{\hbar}\mu t}, \tag{2.4}$$

whose short time behavior is found to be

$$\langle\alpha|a|\alpha\rangle = \sqrt{N}\mathrm{e}^{\frac{i}{\hbar}\mu t}\mathrm{e}^{-i\frac{N\tilde{u}}{\hbar}t}\mathrm{e}^{-\frac{N\tilde{u}^2}{2\hbar^2}t^2}, \tag{2.5}$$

i.e., revealing an exponential decay [4, 5]. At longer time scale it turns out that $\langle\alpha|a|\alpha\rangle$ revives due to discrete and thus periodic nature of the exact time evolution (2.4).

The short time decay defines a collapse-time proportional to $t_c \sim \hbar/\sqrt{N}\tilde{u}$. The ratio of the revival time t_r required for the order parameter scales as $t_r/t_c = \sqrt{N}$ and becomes infinite in the thermodynamic limit, that is when $N \to \infty$, $V \to \infty$ with N/V fixed. In order to get an estimate of this t_c, we introduce characteristic length scale for the harmonic trap potential as $a_{\mathrm{ho}} = \sqrt{\hbar/(M\omega_{\mathrm{tr}})}$ in terms of the harmonic trap frequency ω_{tr}. Denoting the density of condensed atom numbers in the quantization volume as $\rho = N/V$, we find

$$t_c = \frac{\sqrt{N}}{4\pi N_{\mathrm{eff}}}\frac{1}{\omega_{\mathrm{tr}}}, \tag{2.6}$$

where we have defined $N_{\mathrm{eff}} = \rho a_{\mathrm{ho}}^2 a_s$. Assuming a typical situation of current experiments with $N \sim 10^6$, $a_s = 10$ nm, $a_{\mathrm{ho}} = 1$ μm, and $\rho = 10^{21}$ m^{-3}, we get $t_c \simeq 10/\omega_{\mathrm{tr}}$. For a magnetic trap with $\omega_{\mathrm{tr}} = 100$ Hz, this amounts to $t_c \sim 10^{-1}$ seconds, clearly within the regime to be confirmed and studied experimentally [6].

2.2 *Toy Model for Squeezed-Coherent Zero-Mode Occupation*

We now consider a squeezed-coherent state $|\alpha, \gamma\rangle = D(\alpha)S(\gamma)|vac\rangle$ as the zero-mode occupation [24]. As already mentioned, the arguments of γ and α determine which quadrature is squeezed. In particular, if both γ and α are real, then the state is a number squeezed state, with the uncertainty in atom number reduced at the cost of higher uncertainty in the conjugate phase variable. We expect such a state to have a longer life time since the phase collapse speed is generally proportional to ΔN, which is smaller in this case, as have recently observed experimentally [25, 26]. A wide phase distribution, on the other hand, makes the squeezed state more similar to a Fock state which has a uniform phase distribution, and is less influenced by the decoherence effect due to the U(1) symmetry breaking field because of the reduced number fluctuations.

In order to understand the essence of the above discussion, we choose to follow similar arguments as with the coherent state considered previously. We will study the time evolution of the single mode state $|\alpha,\gamma\rangle$ subject to the same $U(1)$ gauge symmetric Hamiltonian (2.2). For notational convenience we define

$$\zeta = \gamma\frac{\tanh(|\gamma|)}{|\gamma|}. \tag{2.7}$$

The Fock state expansion of the squeezed state in terms of this new variable is [19]

$$\begin{aligned} |\alpha,\gamma\rangle &= \sum_{n=0}^{\infty} A_n(\alpha,\zeta)|n\rangle & (2.8) \\ &= (1+|\zeta|^2)^{1/4}\mathrm{e}^{-\frac{(\alpha+\zeta\alpha^*)\alpha^*}{2}}\sum_{n=0}^{\infty}\sqrt{\frac{\zeta^n}{2^n n!}}H_n\left(\frac{\alpha+\zeta\alpha^*}{\sqrt{2\zeta}}\right)|n\rangle, & (2.9) \end{aligned}$$

where H_n is the n-th order Hermite polynomial. In the limit $x\to\infty$, $H_n(x)$ behaves like $2^n x^n$. Hence the squeezed state approaches a coherent state when $\zeta\to 0$. The corresponding expectation value for $a(t)$ now takes the form

$$\langle\alpha,\gamma|a(t)|\alpha,\gamma\rangle = \sum_{n=0}^{\infty}\sqrt{n+1}A_n^* A_{n+1}\mathrm{e}^{\frac{i}{\hbar}(E_n-E_{n+1})t}, \tag{2.10}$$

where the complex nature of $A_n(\alpha,\zeta)$ makes the analytic evaluation of this expression nontrivial. We therefore resort to numerical studies. First we consider the time evolution of (2.10) for $\alpha = 10$ at $\zeta = 0.5$ and 0.9 respectively. The results as shown in Fig. 2.1 clearly display squeezing in the radial (N) direction due to a wide phase

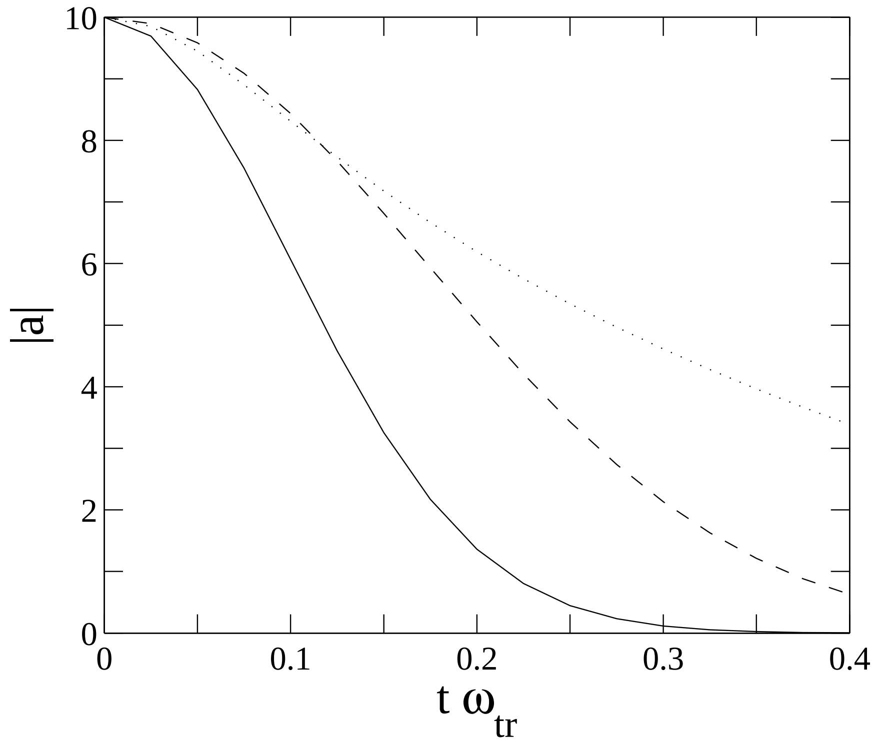

Figure 2.1: The comparison of the short time decay character for a coherent state condensate with that for a squeezed state at $\zeta = 0.5$ and $\zeta = 0.9$. The parameters used are $a_s = 10nm$, $a_{\rm ho} = 1\ \mu$m, $n = 10^{21}$ m^{-3}, but now for $N = 100$. In this case, the dimensionless time is in units of $\hbar/\tilde{u}$ becomes $\hbar/\tilde{u} = \omega_{\rm tr}^{-1}$. The fastest decay (solid line) denotes the result for a coherent state, while the dashed (dotted) line refers to that of a squeezed state with $\zeta = 0.5$ ($\zeta = 0.9$). As expected, the choice of a squeezed state with real parameters α and ζ improves the coherence time.

distribution in the azimuthal direction. This consequently improves the coherence time for the condensate, which is approximately of the same order of magnitude here.

More generally, our discussions can reach beyond the choices of real parameters α and ζ. Consequently different results may be expected, as we illustrate the comparisons between a coherent state and squeezed states with $\zeta = 0.5$, $\zeta = 0.5i$, and $\zeta = -0.5$ in Fig. 2.4. We see that the last two choice of the squeezing parameters lead to reduced coherence times, a result that again can be reasonably understood in terms of the increased uncertainty in the atom number as it causes faster collapse.

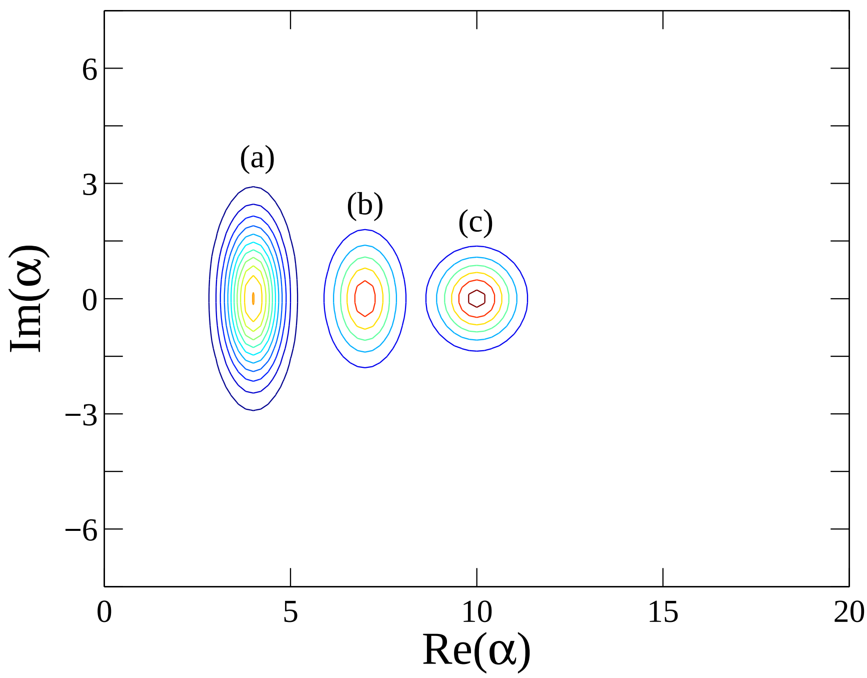

Figure 2.2: The phase space distributions of the initial states used in Fig. 2.1. Curves (a), (b) and (c) correspond to $\zeta = 0.9$, $\zeta = 0.5$ and $\zeta = 0$, respectively. Although all of them should be centered at $\alpha = 10$, they are shifted for convenience.

2.3 Toy Model for Thermal-Coherent Zero-Mode Occupation

To extend the above discussions to finite temperature systems, we now introduce the thermal-coherent state, which possesses both a thermal character and a phase. Consider the following density matrix for a thermal state

$$\begin{aligned}\rho_{\mathrm{th}} &= \mathrm{e}^{-\beta\mathcal{H}} \\ &= \sum_n \mathrm{e}^{-\beta E_n}|n\rangle\langle n|,\end{aligned} \tag{2.11}$$

where $\beta = 1/k_B T$, k_B is the Boltzmann constant and T is the temperature. In this state (2.11), $\mathcal{H}$ is the Hamiltonian operator excluding the chemical potential, $\mathcal{H} = \tilde{u}(a^\dagger a^\dagger a a - a^\dagger a)/2$. E_n is redefined, corresponding to $\mathcal{H}|n\rangle = E_n|n\rangle$. ρ_{th} is a mixed state that has a a thermal character but not a definite phase. Therefore the

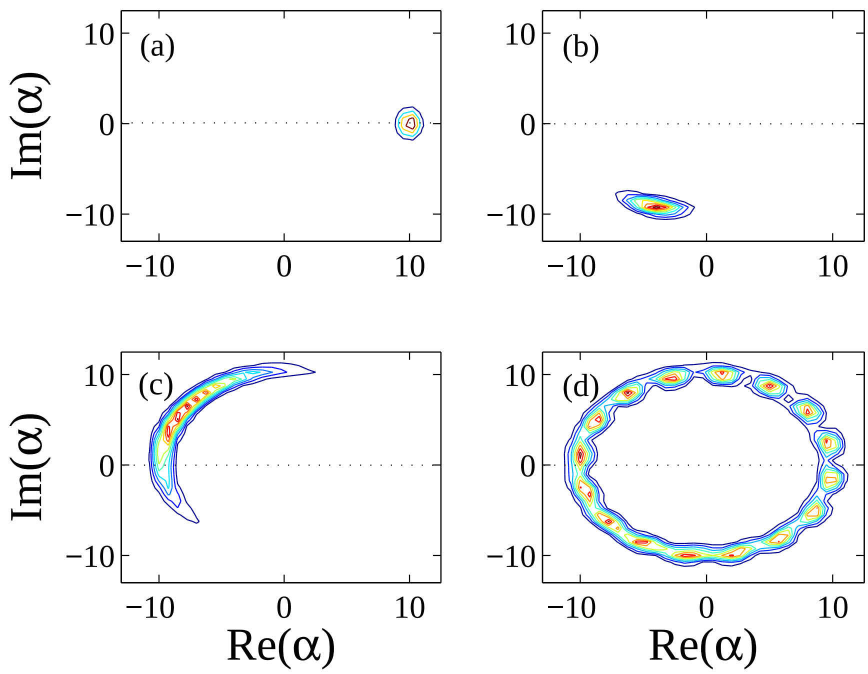

Figure 2.3: Time evolution of the Q-function for squeezed-coherent state with $\alpha = 10$ and $\zeta = 0.5$ for different values of $t\omega_{tr}$. Figures (a), (b), (c) and (d) shows the Q-function distributions for $t\omega_{tr} = 0$, $t\omega_{tr} = 0.02$, $t\omega_{tr} = 0.10$ and $t\omega_{tr} = 0.40$. It is seen that as the order parameter decays, the broken phase symmetry is restored since the Q-function distribution becomes rotationally symmetric.

expectation value of a is zero:

$$\begin{aligned} \langle a \rangle &= \mathrm{Tr}\left(a\rho_{th}\right), &(2.12)\\ &= 0. &(2.13) \end{aligned}$$

In order to introduce a coherent component and also to change the mean number of atoms, we can make use of the displacement operator

$$\rho = D(\alpha)\rho_{\mathrm{th}}D^{\dagger}(\alpha). \tag{2.14}$$

This state we shall call a thermal-coherent state, whose properties can be conveniently

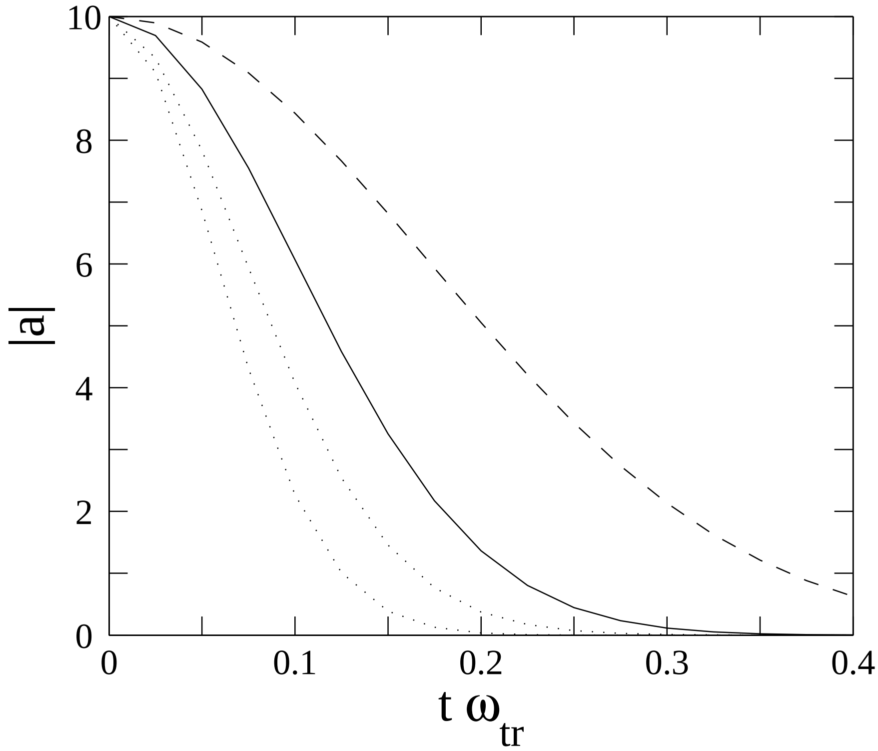

Figure 2.4: Decay of the order parameter for the coherent state and squeezed states of $\zeta = 0.5$, $\zeta = 0.5i$ and $\zeta = -0.5$ as a function of $t\omega_{tr}$. The solid line is the coherent state, the dashed line is the squeezed state with $\zeta = 0.5$, and the dotted ones are the squeezed states with $\zeta = 0.5i$ and $\zeta = -0.5$ Only the state with real squeezing parameter has a longer life time than the coherent state.

studied with the aid of the generalized coherent or displaced number states [27]

$$\begin{aligned} D(\alpha)|n\rangle &= |n, \alpha\rangle \\ &= \sum_{n=0}^{\infty} \mathrm{e}^{-\frac{1}{2}|\alpha|^2} \sqrt{\frac{n!}{m!}} \alpha^{m-n} \mathrm{L}_n^{m-n}(|\alpha|^2)|m\rangle \\ &= \sum_{n=0}^{\infty} C_m(n, \alpha)|m\rangle, \end{aligned} \tag{2.15}$$

where L_k^l is the generalized Laguerre Polynomial. The thermal-coherent density ma-

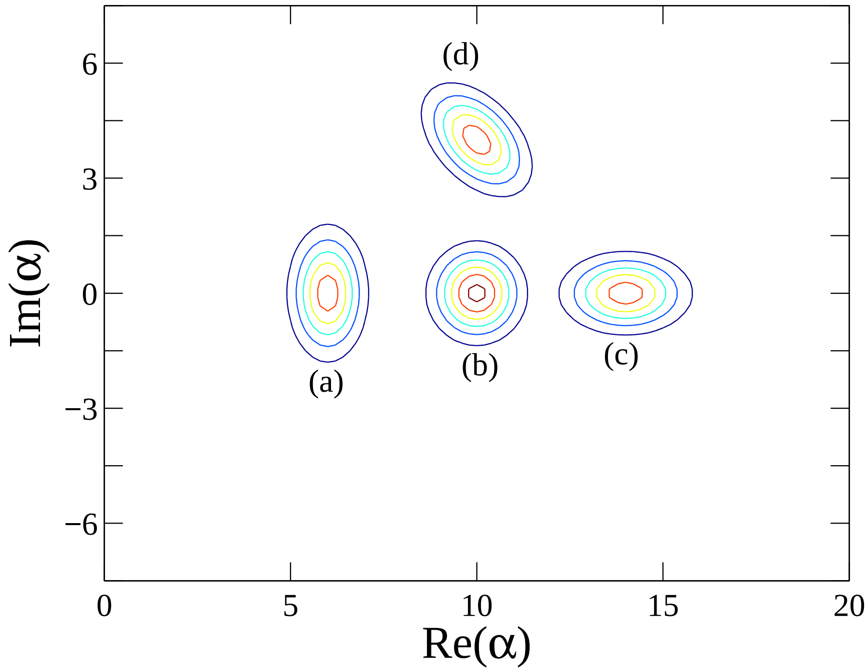

Figure 2.5: The phase space distributions of the initial states used in Fig. 2.4. Curves (a), (b), (c) and (d) correspond to $\zeta = 0.5$, $\zeta = 0$, $\zeta = -0.5$ and $\zeta = 0.5i$, respectively. Although all of them should be centered at $\alpha = 10$, they are shifted for convenience.

trix now becomes

$$\begin{aligned}\rho &= \sum_{n=0}^{\infty} \mathrm{e}^{-\beta E_n} D(\alpha)|n\rangle\langle n|D^{\dagger}(\alpha) \\ &= \sum_{n=0}^{\infty} \mathrm{e}^{-\beta E_n} |n,\alpha\rangle\langle n,\alpha| \\ &= \sum_{nmm'} \mathrm{e}^{-\beta E_n} C_m(n,\alpha) C^*_{m'}(n,\alpha)|m\rangle\langle m'|, \end{aligned} \tag{2.16}$$

with which we can again consider the time evolution of the expectation value of $a(t)$,

$$\langle a(t)\rangle = \sum_{nmm'k} \mathrm{e}^{-\beta E_n} C_m(n,\alpha) C^{\dagger}_{m'}(n,\alpha)\langle k|m\rangle\langle m'|\mathrm{e}^{\frac{i}{\hbar}\mathcal{H}t} a \mathrm{e}^{-\frac{i}{\hbar}\mathcal{H}t}|k\rangle, \tag{2.17}$$

calculated according to $\langle a(t)\rangle = \mathrm{Tr}\,(\rho a(t))$. In the end, we find

$$\langle a(t)\rangle = \sum_{nm} \mathrm{e}^{-\beta E_n} C_{m+1}(n,\alpha) C^*_m(n,\alpha)\sqrt{m}\,\mathrm{e}^{-\frac{i}{\hbar}(E_{m+1}-E_m)t}. \tag{2.18}$$

In general, the phase factors will interfere destructively in the above. The thermal distribution weight $\mathrm{e}^{-\beta E_n}$ determines how many different terms contribute. This implies that the temperature definitely leads to a reduced coherence time for the state. Figure 2.6 illustrates the results from a numerical calculation for 100 atoms at various temperature scales, including some that are in fact not experimentally relevant.

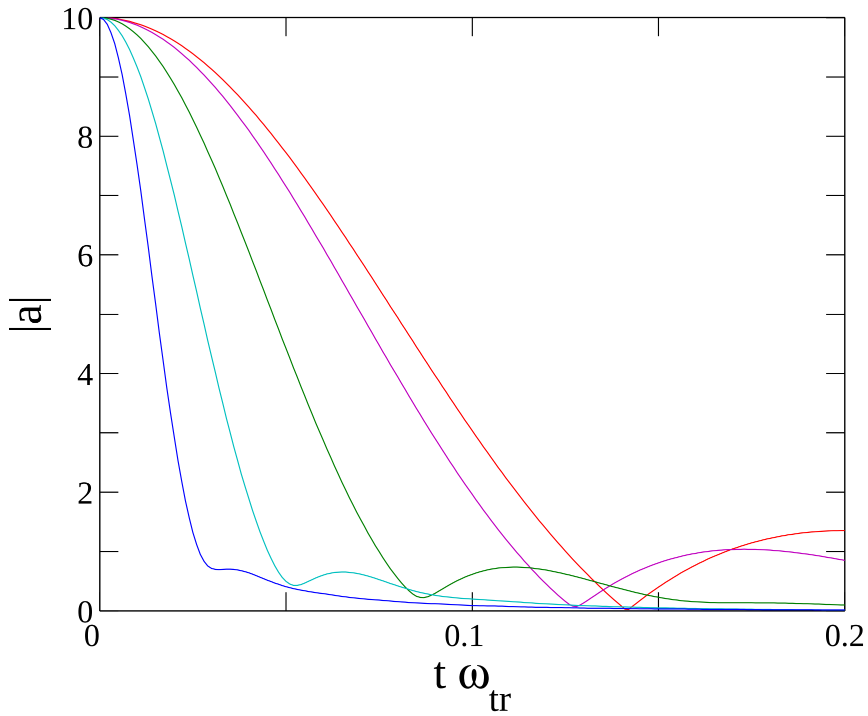

Figure 2.6: The short time decays for thermal-coherent states. The lines correspond respectively to $T = 1000$ nK, 100 nK, 10 nK, 1 nK, and 0.001 nK from left to right. The humps are due to the ground degeneracy $E_0 = E_1$. Even as the temperature approaches zero, the state (2.14) does not approach the ordinary coherent state $D(\alpha)|0\rangle$. Instead, it is a superposition state $D(\alpha)(|0\rangle + |1\rangle)/\sqrt{2}$.

Chapter 3

THIN SPECTRUM FORMALISM

3.1 Introduction

By *thin spectrum* we refer to a group of states, whose energy spacings are so low that they are not controllable in any experiment. In many-body systems, quite often there exist a spectrum with level spacing inversely proportional to the system size. These states with vanishing energy difference in the thermodynamic limit are beyond experimental reach and therefore constitute a thin spectrum. The effect of a thin spectrum on the partition function and decoherence has been studied in Ref. [28] and more extensively in Ref. [7], both in the context of Lieb-Mattis Model [29]. What Wezel *et. al.* have found in Ref. [7] is that the thin spectrum in Lieb-Mattis Model leads to decoherence of excitations in a time scale proportional to $t_c \sim N\hbar/k_B T$ where N is the number of spins. They also claimed that this time scale, being independent of the details of the system, should be valid for other kind of systems too. One of the main results of this theses will be to prove that quasiparticles in BEC's decay with this rate, a result supporting their claim.

In this section, we review the ideas developed in [7], which use two quantum numbers: n and m, to denote the thin spectrum and ordinary states (elementary excitations) respectively. The system is initially prepared at $m = 0$, that is the ground state of the system. However, as the thin spectrum distribution cannot be manipulated, it will be a thermal one. This leads the initial state of the system to be

$$\rho(t=0) = Z^{-1} \sum_n \mathrm{e}^{-\beta E_0^{(n)}} |0, n\rangle\langle 0, n|, \tag{3.1}$$

where $\mathcal{H}|m,n\rangle = E_m^{(n)}|m,n\rangle$. Z is the partition function, $Z = \sum_n \exp\left(-\beta E_0^{(n)}\right)$. An elementary (observable) excitation can be created by a unitary transformation

$|0,n\rangle \rightarrow \sum_m C_m|m,n\rangle$, where $\sum_m |C_m|^2 = 1$. After the system is excited in this manner, the density matrix becomes

$$\rho = Z^{-1} \sum_{nmm'} \mathrm{e}^{-\beta E_0^{(n)}} C_m C_{m'}^* |m,n\rangle\langle m',n|, \tag{3.2}$$

which, after time evolution, evolves to

$$\rho(t>0) = \sum_{nmm'} \frac{\mathrm{e}^{-\beta E_0^{(n)}}}{Z} \mathrm{e}^{-\frac{i}{\hbar}(E_m^{(n)}-E_{m'}^{(n)})t} C_m C_{m'}^* |m,n\rangle\langle m',n|. \tag{3.3}$$

When it is observed, the details of this density matrix cannot be seen since the thin spectrum is assumed to be beyond experimental reach. Therefore, only the reduced density matrix, which is obtained by taking the trace of ρ over the thin spectrum states, is observed. We define the thin spectrum state $|j_{\text{thin}}\rangle$ by $\langle j_{\text{thin}}|m,n\rangle = \delta_{j,n}|m\rangle$ where $|m\rangle$ denotes the ordinary observable state of a system. This then allows us to compute the reduced system density matrix:

$$\begin{aligned} \rho^{(\text{red})} &= \sum_j \langle j_{\text{thin}}|\rho(t>0)|j_{\text{thin}}\rangle \\ &= \sum_{mm'n} \frac{\mathrm{e}^{-\beta E_0^{(n)}}}{Z} \mathrm{e}^{-\frac{i}{\hbar}(E_m^{(n)}-E_{m'}^{(n)})t} C_m C_{m'}^* |m\rangle\langle m'|. \end{aligned} \tag{3.4}$$

While the diagonal elements $\rho_{mm}^{(\text{red})} = |C_m|^2$ experience no time evolution, the off diagonal elements suffer a phase collapse unless $E_m^{(n)} - E_{m'}^{(n)}$ is independent of n. For a two state system ($m = 0,1$), the off-diagonal elements will decay at a rate $\Delta E_{\text{thin}}/E_{\text{thin}}$ with $\Delta E_{\text{thin}} = E_1^{(n)} - E_0^{(n)}$ and $E_{\text{thin}} = E_0^{(n)}$ [7].

3.2 Continuous Symmetry Breaking and the Goldstone theorem

The Nambu-Goldstone Theorem [8] dictates the existence of a gapless mode whenever a continuous symmetry is broken spontaneously. For a ferromagnetic material, this mode is the long wavelength spin waves [30]. For a crystalline structure when the translational symmetry is broken, the Nambu-Goldstone mode (NGM) is the overall motion of the crystal [7]. For an atomic condensate, where the BEC leads to the breaking of the gauge symmetry, the corresponding gapless mode induces phase displacement of the condensate [5, 31].

Consider a diagonal Hamiltonian, which may corresponds to normal mode excitations with different ω's,

$$\mathcal{H} = \sum_k \hbar\omega_k b_k^\dagger b_k, \tag{3.5}$$

where b_k is the annihilation operator for the k-th mode. As usual, the bosonic commutation relations are assumed $[b_{k'}, b_k^\dagger] = \delta_{k,k'}$ and $[b_k, b_{k'}] = [b_k^\dagger, b_{k'}^\dagger] = 0$. If there is a broken symmetry, motion along the axis of this symmetry will experience no restoring force and hence the Hamiltonian of this mode will have the form $p^2/2I$ rather than $a^\dagger a$, where p is the corresponding momentum operator and I is the corresponding inertia parameter. Hence, the Hamiltonian becomes

$$\mathcal{H} = \frac{1}{2I}p^2 + \sum_k \hbar\omega_k b_k^\dagger b_k. \tag{3.6}$$

The Hamiltonians for both a crystal [7] and a condensate [5] can be shown to take this form. In both cases the inertia parameter I depends on the total atom number N and either diverges or vanishes in the thermodynamic limit when $N \to \infty$.

The relationship between the Nambu-Goldstone Theorem and the thin spectrum is that each NGM guarantees the existence of a thin spectrum, since the corresponding momentum p can take arbitrarily small values. Therefore p is always capable of giving rise to thermal fluctuations below the experimental precision and every NGM leads to decoherence.

3.3 Explicit Calculation for the Sample Hamiltonian

The Hamiltonian (3.6) is quite common therefore it is useful and instructive to study its collapse explicitly. We denote the state of the system as $|p, \{N_k\}\rangle$. The two sets of quantum numbers p and $\{N_k\}$ denote thin and elementary excitations respectively. For simplicity, we assume that both p and k are one dimensional quantities. Furthermore only two different states of the system are considered in order to use it as a qubit. Assume that the elementary excitation which brings the system from $\{N_k\}$ to $\{N'_k\}$ has an corresponding energy ϵ. In general, such an excitation may also change the inertia factor I of the p term. For example, an interstitial excitation changes the total mass of the crystal [7]. Similarly, an excitation inside an atomic condensate can change its peak density, which determines the inertia factor in front of the phase coordinate [5, 32]. Such a change is necessary for our mechanism of phase diffusion to occur. We assume that the inertia parameter changes from I to $I(1+\delta)$, where δ is small compared to 1. The off-diagonal element in equation (3.4) evolves in time as

$$\rho_{\rm od}^{(\rm red)} = Z^{-1}\left[\sum_p \mathrm{e}^{-\beta E_0^{(p)}} \mathrm{e}^{-\frac{i}{\hbar}(E_1^{(p)}-E_0^{(p)})t}\right] C_1 C_0^*, \tag{3.7}$$

where $E_0^{(p)} = p^2/2I$ and $E_1^{(p)} = \epsilon + p^2/2I(1+\delta)$. Upon substituting this we find

$$\rho_{\rm od}^{(\rm red)} = Z^{-1}\mathrm{e}^{-\frac{i}{\hbar}\epsilon t}\left[\sum_p \mathrm{e}^{-\left(\frac{\beta}{2I}-\frac{i}{2\hbar}\frac{\delta}{I}t\right)p^2}\right] C_1 C_0^*. \tag{3.8}$$

Since p is continuous, its summation becomes an integral, so

$$\rho_{\rm od}^{(\rm red)} = Z^{-1}\mathrm{e}^{-\frac{i}{\hbar}\epsilon t}\frac{\sqrt{\pi}}{2}\frac{1}{\sqrt{\left(\frac{\beta}{2I}-2\frac{i}{\hbar}\frac{\delta}{I}t\right)}} C_1 C_0^*, \tag{3.9}$$

$$= (\text{const.})\frac{1}{\sqrt{1-4it\delta/\beta\hbar}}, \tag{3.10}$$

and

$$|\rho_{\rm od}^{(\rm red)}|^2 = (\text{const.})\frac{1}{\sqrt{1+16t^2\delta^2/\beta^2\hbar^2}}. \tag{3.11}$$

Thus the off diagonal term decays in a time

$$t_c \sim \hbar/k_B T\delta, \tag{3.12}$$

as seen in Figure 3.1.

To apply the above result to an atomic BEC, we consider the relevant temperature scale at $T \sim 100nK$ and assume that a particular observable excitation have $\delta \sim 10^{-1}$. In this case we see that $t_c \sim 10^{-3}$ seconds, less than the life times of many observed

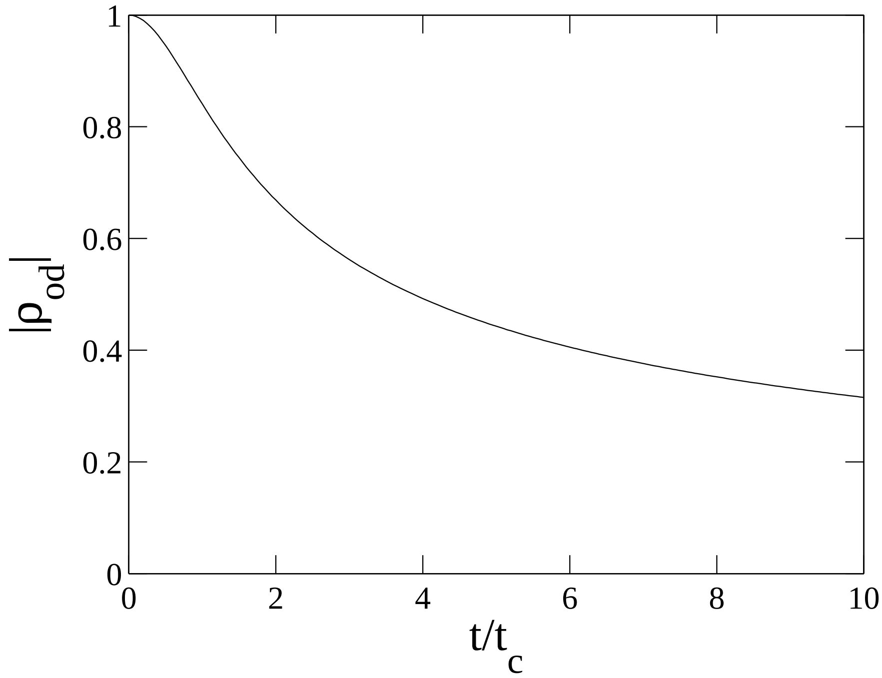

Figure 3.1: The decay of $|\rho_{\rm od}^{\rm (red)}|$ as a function of t/t_c.

ground states. We can also try to obtain an approximation to the coherence time of the condensate ground state. Taking the atom number as $N \sim 10^6$, if the ground state is assumed a coherent state, than the number fluctuations is of the order of $\Delta N = \sqrt{N}$. The inertia parameter I is proportional to $I \sim N^{2/5}$ [5, 32] in the Thomas-Fermi limit, which gives $\delta = \left[(N + \Delta N)^{2/5} - N^{2/5}\right]/N^{2/5} = 2\Delta N/5N$, or $\delta \sim 10^{-3}$. Substituting this in we find $t_c \sim 10^{-1}$ seconds, much larger than for the excited state as to be expected. Furthermore, the result for the ground state life time is in agreement with our previous toy model calculation for the coherent state.

Chapter 4

EXCITATION LIFE TIME IN BOSE-EINSTEIN CONDENSATES

4.1 Thermal Occupation

We now use the thin spectrum formalism of the preceding section to study in detail the life times of quasiparticles in atomic BEC's. We again consider the zero mode Hamiltonian

$$\mathcal{H}_z = \frac{\tilde{u}}{2}(\hat{n}_0^2 - \hat{n}_0) - \mu_0 \hat{n}_0. \tag{4.1}$$

The ground state of such a system will be a number state $|N_0\rangle$ with $N_0 = \mu_0 V/u_0 + 1/2$. We assume that although N_0 and V may fluctuate, their ratio ρ_0 is always constant as in the thermodynamic limit. In this case, $\mathcal{H}_z$ becomes

$$\mathcal{H}_z = \frac{u_0 \rho_0}{2N_0}(\hat{n}_0^2 - \hat{n}_0) - \mu_0 \hat{n}_0. \tag{4.2}$$

Substituting $\mu_0 = u_0\rho_0 - u_0\rho_0/2N_0$ we get

$$\mathcal{H} = \frac{u_0 \rho_0}{2N_0}\hat{n}_0^2 - \rho_0 u_0 \hat{n}_0 + \mathcal{H}_e. \tag{4.3}$$

Now, we consider a state with n atoms in the condensate mode and m quasi-particle excitations at a certain, single k mode, while all other modes are empty. We will denote such a state by $|n, m\rangle$

$$\begin{aligned} \hat{n}_0|n,m\rangle &= n|n,m\rangle, & (4.4)\\ \hat{n}_{k'}|n,m\rangle &= m\,\delta_{k,k'}|n,m\rangle. & (4.5) \end{aligned}$$

In the regime $\hbar^2 k^2/2m \gg \rho_0 u_0$, one can assume that each quasi-particle excitation reduces the number of condensate atoms by one. In this case, the energy of this state can be written as

$$\begin{aligned}\mathcal{H}|n,m\rangle &= E_m^{(n)}|n,m\rangle \\ &= \left[\frac{u_0\rho_0 n^2}{2(N_0-m)} - u_0\rho_0 n + m\omega\right]|n,m\rangle,\end{aligned} \tag{4.6}$$

where we simply denote $\omega = \omega_k$.

Assume the system can be initially prepared with no quasi-particle excitation at all, but is in a Boltzmann weighted distribution over the states $|n,0\rangle$, i.e.,

$$\rho(t=0) \propto \sum_n \mathrm{e}^{-\beta E_0^{(n)}}|n,0\rangle\langle n,0|. \tag{4.7}$$

This state will allow us to study the number fluctuations due to unknown nonzero temperature constituents that make up the occupations of the thin spectrum [7]. The summation index can take any positive integers and therefore the summation should be over $0 \le n < +\infty$. However, we note that the maximum of $E_0^{(n)}$ is at $N_0 \gg 1$ and it becomes extremely small for small values of n, we can extend the summation to be over the full range $-\infty < n < +\infty$ and replace it with an integral in the continuous limit as down in the following.

Excitation of a quasi-particle brings each $|n,0\rangle$ to $|n,1\rangle$, the off diagonal element of the resulting state will evolve according to

$$\begin{aligned}\rho_{\mathrm{od}}(t>0) &\propto \int_{-\infty}^{\infty} \mathrm{e}^{-\beta E_0^{(n)}}\mathrm{e}^{-\frac{i}{\hbar}(E_1^{(n)}-E_0^{(n)})t}dn \\ &\propto \int_{-\infty}^{\infty} \mathrm{e}^{(-\beta u_0\rho_0/2N_0+itu_0\rho_0/2\hbar N_0^2)n^2+\beta\rho_0 u_0 n}dn \\ &\propto \sqrt{\pi}\frac{\exp\left(\frac{\beta^2\rho_0^2u_0^2}{2\beta u_0\rho_0/N_0-2itu_0\rho_0/\hbar N_0^2}\right)}{\sqrt{\beta u_0\rho_0/2N_0-itu_0\rho_0/2\hbar N_0^2}},\end{aligned} \tag{4.8}$$

which gives

$$|\rho_{\mathrm{od}}(t)|^2 \propto \frac{\exp\left(\frac{\beta^3N_0^3u_0\rho_0}{\beta^2N_0^2+t^2/\hbar^2}\right)}{\sqrt{\beta^2+t^2/\hbar^2N_0^2}}, \tag{4.9}$$

after omitting terms with only a phase factor. Although the denominator and the numerator have quite different forms, we find that both decay in a time proportional

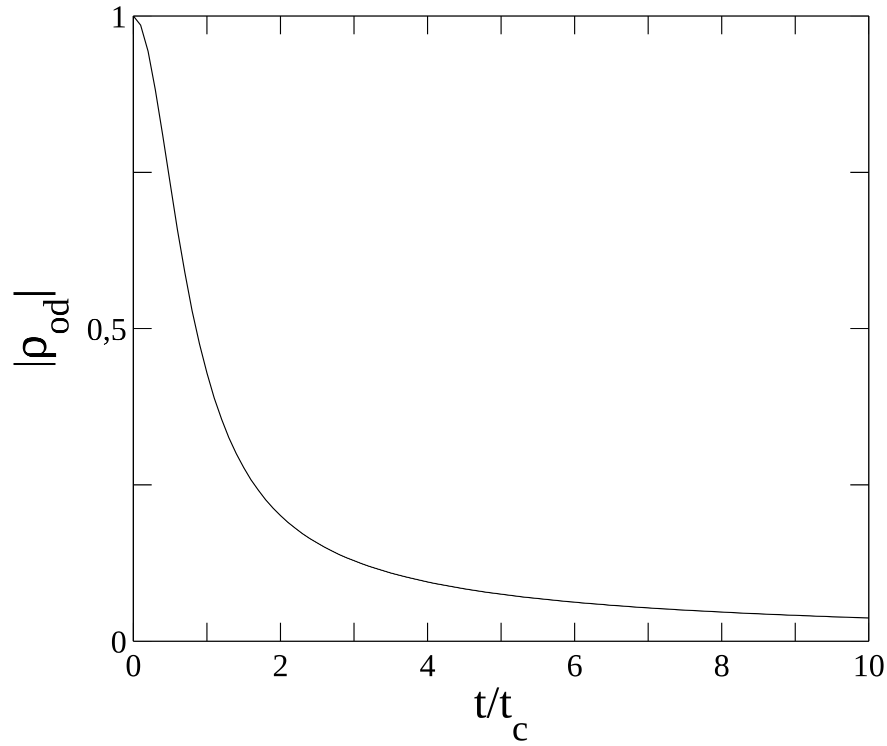

Figure 4.1: The relative decay of the off diagonal element in equation (4.9) as a function of t/t_c for unit values of parameters.

to $t_c \sim \hbar N_0/k_B T$. This is the same result that Wezel *et. al.* have found for a crystal [7]. The decay of this function is plotted in Fig. 4.1 for unit values of parameters.

For an atomic Bose-Einstein condensate, the relevant parameters are $N_0 \sim 10^6 - 10^8$ and $T \sim 10^{-8} - 10^{-7}$ K. These then lead $t_c \sim 10^2 - 10^5$ seconds, which is a time much larger than both theoretical and observed ground state life time. However, this is the life time for a single quasi-particle excitation, i.e., for $m = 1$. It is easy to show that the collapse time is inversely proportional to m for not too large m values. An easily tractable excitation should have $m \sim N_0$ and this gives $t_c \sim 10^{-4} - 10^{-3}$ seconds, much smaller than both the observed and expected ground state life times.

The study of temperature dependence for the damping rates of Bogoliubov excitations of any energy has been carried out before using perturbation theory and a linear temperature was found [33], surprisingly coinciding with the linear dependence found here based on the decoherence of the thin spectrum. Our result clearly would make a quantitative contribution to the total decay of the quasiparticles, although we note that our calculation is limited only to the single-particle excitation regime as we have used $\epsilon_k = E_k \gg u_0\rho_0$. In the phonon branch corresponding to the low-lying collective excitations out of a condensate, more complicated temperature dependencies may occur [34]. In contrast to damping mechanisms based upon excitation collision processes in the condensate, the thin spectrum caused decay rate shows no system specific dependencies, apart from the dependencies on temperature and the number of atoms. It is independent of the interatomic interaction strength or the scattering length, as well as independent of the quasiparticle spectrum. This is due to the fact that thin spectrum emerges as a result of a global symmetry breaking in a quantum system so that local properties of the system do not contribute to the associated decay rate.

4.2 Thermal-Coherent Occupation

We now generalize the above idea to a thermal-coherent occupation of the zero-mode. The initial density matrix in this case becomes

$$\begin{aligned} \rho(0) &= Z^{-1} \sum_{n} \mathrm{e}^{-\beta E_0^{(n)}} D(\alpha)|n,0\rangle\langle n,0|D^\dagger(\alpha) \\ &= Z^{-1} \sum_{nmm'} \mathrm{e}^{-\beta E_0^{(n)}} C_m(n,\alpha) C^*_{m'}(n,\alpha)|m,0\rangle\langle m',0|. \end{aligned} \tag{4.10}$$

The system is now brought into a superposition of no quasi-particle and one quasi-particle state, i.e., $|n,0\rangle \to (|n,0\rangle + |n,1\rangle)/\sqrt{2}$. After further time evolution, the state becomes

$$\begin{aligned} \rho(t) = Z^{-1} \sum_{nmm'} \sum_{kk'=0,1} \frac{\mathrm{e}^{-\beta E_0^{(n)}}}{2} C_m(n,\alpha) C^*_{m'}(n,\alpha) \\ \mathrm{e}^{-\frac{i}{\hbar}(E_1^{(m)} - E_0^{(m')})t}|m,k\rangle\langle m',k'|, \end{aligned} \tag{4.11}$$

giving rise to the reduced density matrix and its off diagonal element below

$$\rho^{(\mathrm{red})} = Z^{-1} \sum_{nl} \sum_{kk'=0,1} \frac{\mathrm{e}^{-\beta E_0^{(n)}}}{2} |C_l(n,\alpha)|^2 \mathrm{e}^{-\frac{i}{\hbar}(E_k^{(l)} - E_{k'}^{(l)})t} |k\rangle\langle k'|, \tag{4.12}$$

$$\rho_{\mathrm{od}}^{(\mathrm{red})} = Z^{-1} \sum_{nl} \frac{\mathrm{e}^{-\beta E_0^{(n)}} |C_l(n,\alpha)|^2}{2} \mathrm{e}^{-\frac{i}{\hbar}(E_1^{(l)} - E_0^{(l)})t}. \tag{4.13}$$

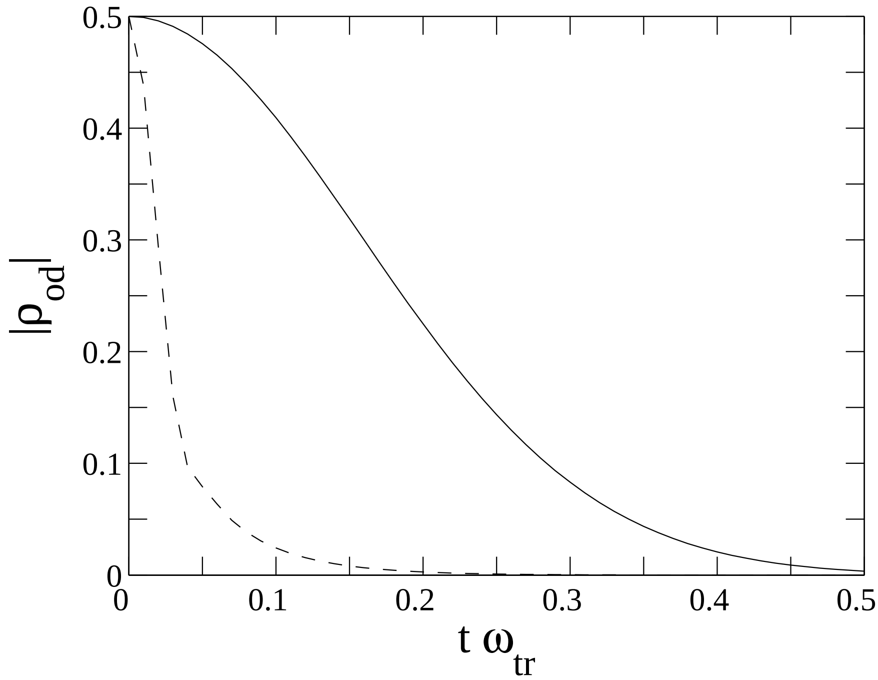

Figure 4.2: Decay of the off diagonal element at $T = 10$ nK as a function of $t\omega_{tr}$. Dashed line shows the decay in the case of the thermal-coherent occupation and the solid line shows that in the case of the thermal occupation.

Figures 4.2 and 4.3 show the early time decay at temperatures of 10 nK and 100 nK respectively. It is seen that the decay time for a thermal occupation, which we have studied in the preceding section, exhibits strong temperature dependence; whereas the decay time for the thermal-coherent occupation is not very much changed by temperature. Therefore, we conclude that for a thermal-coherent occupation, the main reason for the decay of the off diagonal element is the decay of the zero mode

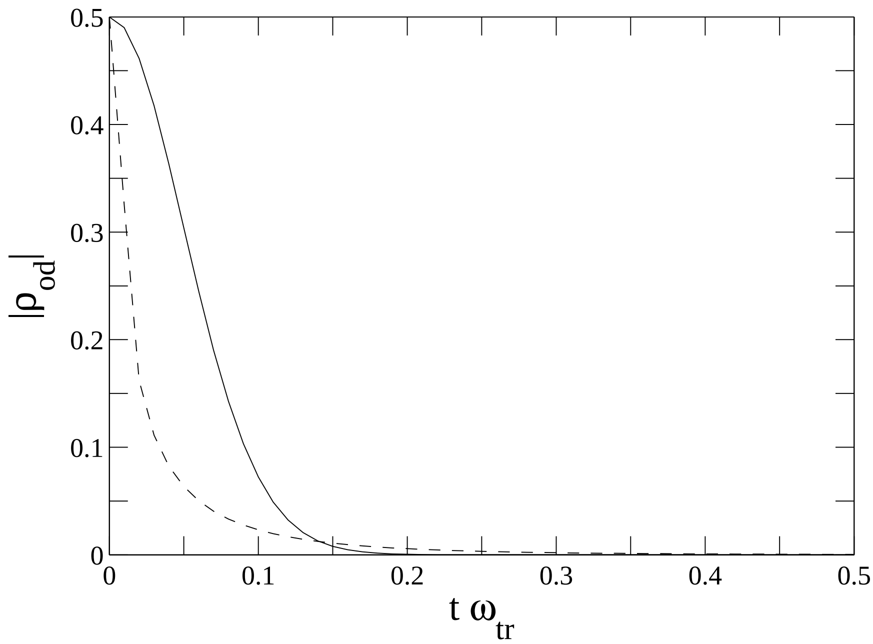

Figure 4.3: Decay of the off diagonal element for $T = 100$ nK and thermal-coherent occupation of the zero mode as a function of $t\omega_{tr}$. Dashed line shows the decay in the case of the thermal-coherent occupation and the solid line shows that in the case of the thermal occupation.

distribution. However, if there is solely thermal occupation, no decay of the zero mode occurs and the off diagonal element decays only due to the temperature.

4.3 Excitation Phase Collapse due to Multiple Broken Symmetries

A system may have more than one spontaneously broken symmetries. For example, in addition to a broken gauge symmetry, the formation of vortices breaks the rotational symmetry of a condensate in a spherically symmetric trap [35]. Rotational symmetry can also be broken for a multi-component [36] or a spinor condensate [37]. When more than one continuous symmetries are broken, there will exist more than one gapless modes, each with its own thin spectrum. In this section, we briefly consider the effect of more than one thin spectrum.

Consider a general effective Hamiltonian with two gapless modes

$$\mathcal{H} = \alpha_1 p_1^2 + \alpha_2 p_2^2 + \alpha_{12} p_1 p_2 + \sum_k \hbar\omega_k b_k^\dagger b_k, \tag{4.14}$$

which after a canonical transformation, reduces to

$$\mathcal{H} = \alpha_1' {p_1'}^2 + \alpha_2' {p_2'}^2 + \sum_k \hbar\omega_k b_k^\dagger b_k. \tag{4.15}$$

Without loss of generality we use this form of the Hamiltonian and henceforth omit the primes. The observable state will be denoted by n, and an easy extension leads to $\mathcal{H}|n, p_1, p_2\rangle = E_n^{(p_1,p_2)}|n, p_1, p_2\rangle$ with $E_n^{(p_1,p_2)} = E_n^{(0,0)} + \alpha_1 p_1^2 + \alpha_2 p_2^2$. More generally, the primary excitation may affect both inertia terms in the two thin spectra, which may themselves be coupled, i.e., $\alpha_1 = \alpha_1(n, p_2)$ and $\alpha_2 = \alpha_2(n, p_1)$. Expanding around the small p_1 and p_2 , we find around $p_j = 0$

$$\begin{aligned} E_n^{(p_1 p_2)} = E_n^{(0,0)} &+ [\alpha_1(n,0) + \alpha_1'(n,0)p_2 + ...]\, p_1^2 + \\ &[\alpha_2(n,0) + \alpha_2'(n,0)p_1 + ...]\, p_2^2 \end{aligned} \tag{4.16}$$

$$E_n^{(p_1 p_2)} \simeq E_n^{(0,0)} + \alpha_1(n,0)p_1^2 + \alpha_2(n,0)p_2^2 \tag{4.17}$$

up to the second orders in p_j. Thus we can safely ignore the inertia terms' dependence on other's thin excitations to the first approximation and let $\alpha_1(n, p_2) = \alpha_1(n)$. Instead of (3.7) we now find

$$\rho_{\text{od}}^{(\text{red})} = Z^{-1} \left[\sum_{p_1 p_2} \mathrm{e}^{-\beta E_0^{(p_1,p_2)}} \mathrm{e}^{-\frac{i}{\hbar}(E_1^{(p_1,p_2)} - E_0^{(p_1,p_2)})t} \right] C_1 C_0^*. \tag{4.18}$$

Upon substituting the approximate forms for the E_js, we find

$$\begin{aligned} \rho_{\text{od}}^{(\text{red})} = Z^{-1} &\mathrm{e}^{-\beta E_0^{(0,0)}} \mathrm{e}^{-\frac{i}{\hbar}(E_1^{(0,0)} - E_0^{(0,0)})t} \\ &\sum_{p_1,p_2} \mathrm{e}^{-\frac{i}{\hbar}[\alpha_1(1)-\alpha_1(0)]p_1^2 t} \mathrm{e}^{-\frac{i}{\hbar}[\alpha_2(1)-\alpha_2(0)]p_2^2 t} \mathrm{e}^{-\beta[\alpha_1(0)p_1^2 + \alpha_2(0)p_2^2]} C_1 C_0^*, \end{aligned} \tag{4.19}$$

$$\rho_{\text{od}}^{(\text{red})} = (\text{const.})\mathrm{e}^{-t/t_c^{(1)}} \mathrm{e}^{-t/t_c^{(2)}}. \tag{4.20}$$

Thus, we see that the collapse due to different thin spectra do not influence each other severely. They combine to give a resulting decay with a simple single decay time

$$t_c = \left(\frac{1}{t_c^{(1)}} + \frac{1}{t_c^{(2)}} \right)^{-1}. \tag{4.21}$$

Chapter 5

CONCLUSIONS

Based on a toy model calculation for the decoherence dynamics of a coherent ground state condensate, we have generalized the calculations of the dephasing times to cases of a squeezed-coherent ground state as well as a thermal-coherent ground state. The numerical results for a squeezed ground state reveal that phase fluctuations increases its coherence lifetime, whereas temperature increases always decrease the lifetimes for ground state quantum coherence.

The dynamics of thin spectrum is shown to lead to decoherence, not just on the ground state, but on quasi-particle excitations, or rather on superpositions of excitations. We have introduced simple approximations that allowed for the calculations of the decoherence lifetime of the condensate ground state as well as its coherence excitations. These calculations make possible the discussion of temperature effects in terms of the thermal and thermal-coherent occupations of the ground state zero mode. We find that the lifetimes for these two cases are of the same order of magnitude, although the lifetime for the latter shows a weak dependence on temperature whereas that of the former displays a strong dependence. This difference may be used to experimentally test whether the thermal-coherent ground state is a good model for the BEC ground state, however the lack of precise experimental data precludes such a possibility for the time being.

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