Inventory Models with Imperfect Information and Random Supply

by

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This is to certify that I have examined this copy of a master's thesis by

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ABSTRACT

The aim of this thesis is to analyze single-item inventory models with random supply and imperfect information in a random environment. We assume that the randomness in supply is attributed to the random capacity of the producer and/or random availability of the transporter. Moreover, the random environment which modulates demand, supply and all cost parameters is modeled as a Markov chain.

In the first part of the thesis, we assume that the random environment is fully observed. Through the first model in this part, we analyze the inventory problem with random supply due to random capacity of the producer and random availability of the transporter. The optimal policy for single, multiple and infinite planning periods is shown to be an environment-dependent base-stock policy. As the second model, we analyze the inventory problem with fixed ordering cost and random capacity only. And we present a counter example showing that environment-dependent (s, S) policy is no longer optimal for this type of inventory problems.

In the second part of the thesis, we assume that the random environment is only partially observed. Therefore, we model the random environment by two processes: an unobserved process which is a Markov chain and observed process which is not necessarily a Markov chain. In the first model of this part, we analyze same inventory problem as the first one in the first part; however, the random environment is assumed to be partially observable. Here, we assume that the random capacity as well as random availability is modulated by the unobserved (real) environmental process whereas all costs are modulated by observed environmental process. Then, we present a counter example showing that base-stock policy is not necessarily optimal for this type of inventory problems. By the second model in this part, we show that base-stock policy is optimal in single and multiple period settings if the capacity process is observed as costs. However, for this model, analyzing the infinite period problem is not practical because the observed process is not a Markov chain. Therefore, as a third model, we analyze the same inventory problem using sufficient statistics. In this case,

we show that state-dependent base-stock policy is still optimal in single, multiple and infinite period settings. As a fourth model, we analyze inventory problems with unrelaible suppliers and fixed ordering cost by using sufficient statistics. We show that state-dependent (s, S) policy is optimal for this type of inventory problems if the availability process of supplier is observable. Finally, we also analyze inventory problems with finite capacity and random yield in a partially-observed random environment and show that state-dependent modified inflated base-stock policy is optimal in single, multiple and infinite planning periods.

ÖZET

Bu tezin amacı rassal çevrede rassal tedarikli, eksik bilgili ve tek ürünlü envanter modellerini incelemektir. Tedarikteki rassallığın, üreticinin kapasitesindeki rassallıktan ve/veya nakliyecinin varoluşundaki rassallıktan kaynaklandığını varsayıyoruz. Ek olarak, talebi, tedariği ve bütün maliyet parametrelerini etkileyen çevreyi Markov zinciri olarak modelliyoruz.

Tezin ilk bölümünde, rassal çevrenin tam olarak gözlendiğini varsayıyoruz. Bu bölümdeki ilk modelle, rassal kapasiteli üretici ve rassal varoluşlu nakliyeciden kaynaklanan rassal tedarikli envanter problemini analiz ediyoruz. Tek, çok ve sonsuz planlama periyotları için optimal envanter politikasının çevreye bağlı temel stok politikası olduğu gösteriliyor. İkinci model olarak, sabit sipariş maliyetli ve sadece rassal kapasiteli envanter problemini inceliyoruz. Ve bu tür envanter problemleri için çevreye bağlı (s, S) politikasının artık optimal olmadığını gösteren bir örnek sunuyoruz.

Tezin ikinci bölümünde rassal çevrenin yarı gözlenebilir olduğunu varsayıyoruz. Bu nedenle, rassal çevreyi biri Markov zinciri olan ancak gözlenemeyen ve diğeri gözlenebilen ancak Markov zinciri olmayan iki farklı süreci kullanarak modelliyoruz. Bu bölümdeki ilk modelle rassal çevrenin yarı gözlenebildiğini varsayarak birinci bölümün ilk modeline benzer bir envanter problemini analiz ediyoruz. Burada, maliyetlerin gözlenebilir çevre tarafından yönlendirildiğini varsayarken rassal varoluş gibi rassal kapasitenin de gözlenemeyen (gerçek) çevre tarafından yönlendirildiğini varsayıyoruz. Daha sonra bu tür envanter problemleri için temel stok politikasının artık optimal olmadığını gösteren bir örnek sunuyoruz. Bu bölümdeki ikinci modelle kapasite süreci maliyetler gibi gözlenebiliyorsa temel stok politikasının tek ve çoklu planlama periyotları için optimal olduğunu gösteriyoruz. Fakat gözlenebilen çevre bir Markov zinciri olmadığı için bu modelin sonsuz planlama periyodu analizini yapmak pratik değil. Bu nedenle üçüncü model olarak aynı envanter problemini yeterli istatistikleri kullanarak inceliyoruz. Bu durumda, duruma bağlı temel

stok politikasının tek, çok ve sonsuz planlama periyotları için optimal olduğunu gösteriyoruz. Dördüncü model olarak da sabit sipariş maliyetli ve güvenilir olmayan tedarikçili envanter problemlerini yeterli istatistikleri kullanarak analiz ediyoruz. Tedarikçinin varoluş sürecinin gözlenebildiğini varsaydığımızda duruma bağlı (s, S) politikasının bu tür envanter problemleri için optimal olduğunu gösteriyoruz. Son olarak yarı izlenebilir çevrelerde sabit kapasiteli ve rassal getirili envanter problemlerini inceliyoruz ve duruma bağlı, değiştirilmiş ve artırılmış temel stok politikasının tek, çok ve sonsuz planlama periyotları için optimal olduğunu gösteriyoruz.

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TABLE OF CONTENTS

LIST OF FIGURES

NOMENCLATURE

IM	:	Inventory Manager
HMM	:	Hidden Markov Model
POMDP	:	Partially Observed Markov Decision Process
MCT	:	Monotone Conevrgence Theorem
$\mathbb{R}$	:	Set of all real numbers ($\mathbb{R}$ =$(-\infty, \infty)$)
N	:	Number of decision epochs in the planning horizon
Z_n	:	State of the real environment at time n
Y_n	:	State of observed environment at time n
Z	:	Real environmental process $\left(Z = \{Z_n; n = 0, 1, 2, 3, ...\}\right)$
Y	:	Observed environmental process $\left(Y = \{Y_n; n = 0, 1, 2, 3, ...\}\right)$
$\mathbb{F}$	:	State space of Z process in the partial observation case $\Big(\mathbb{F} = \{a, b, ...\}$, $\mathbb{F}^{n+1} = \mathbb{F} \times \mathbb{F} \times .... \times \mathbb{F}$ =$\{(a_0, a_1, a_2, \cdots, a_n) : a_m \in \mathbb{F}\}\Big)$
$\mathbb{E}$	:	State space of Y (Z) process in the partial (full) observation case $\Big(\mathbb{E} = \{i, j, ...\}$, $\mathbb{E}^{n+1} = \mathbb{E} \times \mathbb{E} \times \cdots \times \mathbb{E} = \{(i_0, i_2, \cdots, i_n) : i_m \in \mathbb{E}\}\Big)$
$\mathfrak{D}\,(\mathbb{F})$	:	Set of all probability distributions defined on state space $\mathbb{F}$
$\bar{Z}_n$	:	All states of real environment until time n $\left(\bar{Z}_n = (Z_0, Z_1, Z_2, ..., Z_n)\right)$
$\bar{Y}_n$	:	All observations until time n $\left(\bar{Y}_n = (Y_0, Y_1, Y_2, ..., Y_n)\right)$
a_n	:	Realization of real environmental state at time n
i_n	:	Realization of observation at time n
$\bar{a}_n$	:	All realizations of the real environmental process until time n $\left(\bar{a}_n = (a_0, a_1, a_2, \cdots, a_n) \in \mathbb{F}^{n+1}\right)$
$\bar{\imath}_n$	:	All realizations of observed environmental process until time n $\left(\bar{\imath}_n = (i_0, i_1, i_2, \cdots, i_n) \in \mathbb{E}^{n+1}\right)$
P^k	:	$k-$step time-homogenous transition probability matrix of Z process in the first part $\left(P^k(i, j) = P[Z_k = j \mid Z_0 = i],\ P^1 = P\right)$

Q_n^k : $k-$step time-dependent transition probability matrix of Z process in the second part $\left(Q_n^k(a,b) = P[Z_{n+k} = b \mid Z_n = a],\, Q_n^1 = Q_n\right)$

P_n^k : $k-$step time-dependent transition probability matrix of Y process in the second part $\left(P_n^k(\bar{\imath}_n, j) = P[Y_{n+k} = j \mid \bar{Y}_n = \bar{\imath}_n],\right.$ $\left.P_n^k((\pi_n, i), j) = P\left[Y_{n+k} = j \left| \bar{Y}_n, Y_n = i\right.\right],\, P_n^1 = P_n\right)$

E_n : Time-dependent emission matrix $\left(E_n(i,a) = P[Y_n = i \mid Z_n = a]\right)$

I : Identity matrix

$\Psi_n^k(a,j)$: $P\left[Y_{n+k} = j | Z_n = a, \bar{Y}_n = \bar{\imath}_n\right]$ $(\Psi_n^1 = \Psi_n)$

$O_n(\bar{\imath}_n, a)$: $P\left[Z_n = a | \bar{Y}_n = \bar{\imath}_n\right]$ for HMM formulation

π_n^a : $P\left[Z_n = a | \bar{Y}_n = \bar{\imath}_n\right]$ for sufficient statistics formulation

π_n : The conditional distribution vector of Z at time n given $\bar{Y}_n$ $\left(\pi_n = \left[\pi_n^a, \pi_n^b, ...\right]\right)$

$T_b(\pi_n | j)$: $P\left[Z_{n+1} = a | \bar{Y}_n, Y_{n+1} = j\right]$

$T(\pi_n | j)$: The conditional distribution vector of Z at time $n+1$ given π_n and $Y_{n+1} = j$

α : Periodic discount factor $(0 < \alpha < 1)$

R_m^α : $\alpha-$potential matrix of Z process in the first part during m transitions $\left(R_m^\alpha(i,j) = \sum_{k=0}^{m-1} \alpha^k P^k(i,j),\, R^\alpha = R_\infty^\alpha\right)$

$R_{n,m}^\alpha$: $\alpha-$potential matrix of Y process in the second part during m transitions $\left(R_{n,m}^\alpha(\bar{\imath}_n, j) = \sum_{n=0}^{m-1} \alpha^n P_n^k(\bar{\imath}_n, j),\, R_{n,m}^\alpha((\pi, i), j) = \sum_{k=0}^{m-1} \alpha^k P_n^k((\pi, i), j),\right.$ $\left.R_n^\alpha = R_{n,\infty}^\alpha,\, R^\alpha = R_0^\alpha\right)$

x_n : Inventory level at time n

y_n : Order-up-to level at time n

D_n : Random variable denoting total demand in period $n = 1, 2, ...$

C_n : Random variable denoting total capacity in period $n = 1, 2, ...$

A : Finite capacity

U_n : Random variable denoting proportional yield in period $n = 1, 2, ...$

M_i : Conditional cumulative density function of demand

m_i : Conditional probability density function of demand

F_i : Conditional cumulative density function of capacity (yield) in models considering random capacity (yield)

f_i	:	Conditional probability density function of capacity (yield) in models considering random capacity (yield)
u_i	:	Probability that the transporter or the supplier is available $(0 \leq u_i \leq 1)$
μ_a	:	Expected proportional yield in real environment a $\left(\mu_a = E\left[U_{n+1} \middle\| Z_n = a\right]\right)$
$\bar{\mu}_\pi$	:	Expected proportional yield overall real environments $\left(\bar{\mu}_\pi = \sum_{a \in F} \pi^a \mu_a\right)$
K_i	:	Fixed ordering cost per order
c_i	:	Purchasing cost per unit
p_i	:	Shortage cost per unit per period
h_i	:	Holding cost per unit per period
L	:	One-period holding/shortage cost
v_n	:	Finite-horizon optimal expected total discounted cost function
v	:	Infinite-horizon optimal expected total discounted cost function

Chapter 1

INTRODUCTION

In an inventory system, there are several sources of uncertainty. In the literature, the main source of uncertainty is assumed to be the randomness in demand. However, supply may also be random and this randomness contributes to the uncertainty of an inventory system. The randomness in the supply side may be attributed to several reasons. If we think that the supplier is composed of a producer and a transporter, then the supply may be uncertain because of the randomness in the capacity of producer and/or transporter. There are several reasons for the randomness in producer's capacity such as long machine downtimes due to unplanned maintenance, strikes, seconds and scraps in a production run and lack of raw material. Not only the capacity of the producer may be random but the capacity of the transporter may also be random. For example, accidents may cause the transporter to lose some portion of the produced amount. In addition to accidents, quality of transportation and environmental factors like temperature and humidity are highly influential on the amount delivered by the transporter. Starting from 1960s, many researchers noticed the impact of supply side on the uncertainty. Karlin (1958 a,b) published the first papers modeling the fact that the quantity received is not necessarily equal to quantity ordered. Research considering the randomness in supply increased further after late 1970s.

Today, all inventory systems are open to outside effects since we do not live in an isolated world. In other words, something happening in one part of the world occurs as a reaction to another in another part of the world. Therefore, assuming a stationary environment is not realistic. Realistic inventory models must consider the possible effects of changing economic conditions, market conditions and exogenous environmental factors on both demand and supply.

Clearly, demand is affected by outside factors. For example, certain basic economic variables such as GNP, inflation and interest rate are highly influential on demand, i.e., demand increases as GNP increases or demand decreases as inflation decreases. Moreover, demand for certain products are subject to significant changes throughout product life cycle; however, stationary demand environment cannot make use of such information (Song and Zipkin, 1993). In inventory literature, first studies considering fluctuations in demand environment are Karlin and Fabens (1959) and Iglehart and Karlin (1962). Later, Song and Zipkin (1993) modeled the outside world as a Markov chain and focused only on demand side. They assume that demand in successive periods are dependent on a Markov chain representing the environment. In a related paper, Sethi and Cheng (1997) also incorporate fluctuating environment into their model using Markov chain approach as Song and Zipkin (1993). Sethi and Cheng (1997) found the most general setting under which an environment-dependent (s, S) policy is optimal.

In addition to demand, supply is also sensitive to outside world. For example, there is a strong relationship between production and weather conditions, and production and product life cycle. Furthermore, transportation costs are affected by oil price which is subject to various political and economic factors. Hence, possible effects of environmental fluctuations on supply and all cost parameters must also be considered. In another paper, using again Markov chain approach, Song and Zipkin (1996) incorporated the effect of fluctuating environment on supply into their model. They show that the optimal policy has the same structure as in standard models, but its parameters change dynamically to reflect current supply conditions. In this paper, Song and Zipkin (1996) also analyze the case where there is a fluctuating environment on which demand, supply and order costs are dependent. As a result of their analysis, they show that environment-dependent (s, S) (base-stock) policy is optimal for inventory problems with (without) fixed-ordering cost. Another paper which considers the possible effect of fluctuating environment on demand, supply and cost parameters is Özekici and Parlar (1999). They assume that the supplier is either available or unavailable when the order is given so that ordered amount is either totally satisfied or nothing is received by the retailer. In addition, Erdem and Özekici (2002) extend Özekici and Parlar (1999) and assume that the supplier is always available but its capacity is random and depends on the state of the environment.

Most inventory models considering random environment assume that observations of the inventory manager (IM) regarding the true environmental state is perfect. However, this is not realistic. In general, observations are incomplete because information available is limited. Clearly, without perfect information, it is impossible to make perfect observations. As an example, we can consider an IM. Certainly, the IM must consider every single factor affecting the environment in order to determine the true environmental state. However, we are living in a world where everything changes in a second and there are hundreds or thousands of factors which have some sort of effect on our environment. Therefore, it is impossible either to be fast enough to follow these rapid changes or to consider every single detail that are influential on our environment. As a result, most observations about the environment are incomplete. But this is not to say that they are useless. Obviously, they give some partial information about the environment; however, treating them as perfect is erroneous. Hence, realistic inventory models must consider the fact that information about the random environment is imperfect. Models of this type, where the random environment is represented by a Markov chain and the true state of this Markov chain cannot be observed directly (however, there is another process which gives partial information about the true state) are called “Partially Observed Markov Decision Processes” (POMDP). Although there is an extensive research on POMDPs, there is not much direct application of them in inventory literature. In a recent paper, Treharne and Sox (2002) assume that the demand environment is random and it is represented by a Markov chain; however, the state of this Markov chain is only partially observed. They do not consider the supply side so they assume that the capacity of the supplier is infinite. As a result of their analysis in finite time horizon, they show that state-dependent base-stock policy is optimal, where the state is assumed the true environmental state and inventory position. Another paper which applies the POMDP concept in inventory control is Bensoussan et al. (2005 a). They study three different models: information delay, filtered newsvendor and zero balance walk. As a result of their analysis, they show that state-dependent base-stock policy is optimal for information delay model whereas optimal feedback policy is optimal for remaining two models.

This thesis is motivated by the fact that demand, supply and all cost parameters are affected by the random environment and it is not possible to directly observe the state of this environment in general. In order to create a more general and realistic inventory model, we bring these concepts together. This thesis can be divided into two main parts based on the observation of the random environment. In the first part, we assume that the random environment which modulates demand, supply and all cost parameters is directly observable. On the other hand, the random environment is assumed to be partially observable in the second part.

The organization of this thesis is as follows. The next part contains a review of relevant models in the literature and how our models relate to them. Then in Chapter 3, we focus on inventory problems with random supply in a fully-observed random environment. Next in Chapter 4, we try to characterize the optimal policy structure for inventory problems with random supply in random environment with *imperfect information.* Finally in Chapter 5, we give a general summary of the thesis and provide some direction on future research.

Chapter 2

LITERATURE REVIEW

The literature on inventory models can be categorized based on the information about the system. Main body of the literature on inventory models assume that the IMs have complete information about the system. However, few researchers realized that it is not possible to have complete information in certain situations; therefore, inventory models must be modified so that they are applicable when there is not complete information. So we categorize the literature as inventory models with perfect and imperfect information. In Section 2.1, we review inventory literature with perfect information. Then, we give a review of literature about POMDPs and their application on inventory control in Section 2.2.

2.1 Inventory models with perfect information

In this section, we consider the inventory models with random supply and perfect information. Yano and Lee (1995) divides inventory literature on random supply into two main categories: continuous-time models and discrete-time models. Since we consider a discrete-time model, we present a brief review of the literature on periodic-review random supply inventory models here. Among others, some of the papers analyzing continuous-time inventory models are Silver (1976), Shih (1980), Kalro and Gohil (1982), Noori and Keller (1986), Ehrhardt and Taube (1987) and Parlar and Berkin (1991). Detailed analysis of both discrete and continuous-time random supply inventory models can be found in Yano and Lee (1995) and recent advances are summarized in Grosfeld-Nir and Gerchak (2004).

We divide the literature on periodic-review random supply inventory models into two main categories: inventory models in a stationary environment and inventory models in a random environment. By stationary environment, we mean the case where fluctuations in environment are not taken into consideration. In a sense, those inventory models assume that all inventory systems are in an isolated world so that parameters of demand and supply distributions, and all cost parameters are independent of outside environment. On the other hand, by random environment, we mean the case where possible changes in outside world are considered. In this type of inventory models, the effect of outside world on the parameters of an inventory system is considered.

Furthermore, we divide the literature on periodic-review random supply inventory models in stationary and random environment into three subcategories: proportional yield, random capacity and random supply. Proportional yield inventory models assume that the supplier delivers a random proportion of ordered quantity. On the other hand, random capacity inventory models assume that the capacity of supplier is random; therefore, the supplier can produce all order unless the supplier's capacity is less than it. Finally, random supply inventory models contain both random yield and random capacity. In random supply inventory models, it is assumed that the supplier produces a random amount depending on its capacity; moreover, the supplier can deliver the retailer a random proportion of the produced amount.

2.1.1 Inventory models in stationary environment

Research on periodic-review inventory models with random yield in a stationary environment continues since 1950s. Henig and Gerchak (1990) give a detailed analysis of inventory problems with random proportional yield. They assume that the amount received by the retailer is a random proportion of quantity ordered. A detailed analysis of the problem in single, multiple and infinite planning periods show that "nonorder-up-to" policy is optimal under all settings. "Nonorder-up-to" is a policy structure where there is a predetermined inventory level under which an order is always given; however, unlike the base-stock policy, this order does not necessarily bring the inventory level up to a constant base-stock level. Moreover, it increases the inventory level above the predetermined inventory level. Because of this particular structure of "nonorder-up-to" policy, Zipkin (2000, p.392) calls it as "inflated" base-stock policy.

In literature, before characterizing optimal policy structure for inventory problems with random capacity, many researchers focused on inventory problems with deterministic capacity constraints. Two of them are Federgruen and Zipkin (1986 a, b) who assume that the capacity of the supplier is finite or fixed so that the supplier delivers all of the quantity ordered by the retailer unless it is more than capacity. Federgruen and Zipkin (1986 a) analyze the inventory problem with random demand and finite capacity according to the average-cost criterion whereas Federgruen and Zipkin (1986 b) analyze the same problem according to the discounted cost criterion. As a result of their analysis, Federgruen and Zipkin (1986 a, b) show that a "modified" base-stock policy is the optimal policy structure. In such a situation, it is optimal to order up to the critical parameter if the fixed capacity is sufficient, if not, one should order as much as possible. An original study considering random demand and random capacity is Ciarallo et al. (1994) which is similar to Federgruen and Zipkin (1986 a, b); however, the capacity of supplier is random, not finite. They analyze the problem in finite and infinite planning periods and show that base-stock policy is optimal.

Wang and Gerchak (1996) incorporate both random proportional yield and random capacity into their model. In a sense, it is a combination of both Henig and Gerchak (1990) and Ciarallo et al. (1994). As a result of their analysis, Wang and Gerchak (1996) prove that the inflated base-stock policy is optimal in single, multiple and infinite period settings as in Henig and Gerchak (1990).

2.1.2 Inventory models in random environment

A detailed study incorporating the effect of fluctuating environment on supply is Özekici and Parlar (1999). In their paper, Özekici and Parlar (1999) develop an infinite horizon, periodic-review inventory model with unreliable suppliers in a random environment that affects not only the demand but also the supply and the cost parameters. They assume that supplier is either available or unavailable at any particular instant so that the retailer either receives all of its order or receives nothing. They show that an environment-dependent (s, S) (base-stock) policy is optimal for inventory problems with (without) fixed ordering cost. We prefer to categorize Özekici and Parlar (1999) as a random availability model since the supplier is randomly available. Clearly, random availability models are special cases of random yield models where yield is either 0 or 1.

Another paper considering the effect of changing environment conditions on supply as well as demand and cost parameters is Erdem and Özekici (2002). They further extend Ciarallo et al. (1994) to allow random environment and analyze single, multiple and infinite-period problems and show that base-stock policy is still optimal when the environment is random. In addition, Erdem and Özekici (2002) make a comparison between base-stock levels of infinite capacity and random capacity inventory models. They show that base-stock levels do not change with random capacity in single period whereas they increase with random capacity in multiple and infinite periods.

In a recent paper, Gallego and Hu (2004) analyze inventory problems with random proportional yield and finite capacity constraint in a random environment. They assume that capacity of the supplier is finite as in Federgruen and Zipkin (1986 b) and that the retailer receives a random proportion of amount produced as in Henig and Gerchak (1990). In addition, as Erdem and Özekici (2002), Gallego and Hu (2004) consider the effect of fluctuating environment on both demand and supply. However, unlike Erdem and Özekici (2002), they distinguish between demand and supply environments. For this purpose, they use two Markov chains: one for the demand environment and one for the supply environment. Another important difference between Erdem and Özekici (2002) and Gallego and Hu (2004) is that Gallego and Hu (2004) do not consider the effect of the fluctuating random environment on cost parameters. As a result of a detailed analysis, Gallego and Hu (2004) show that modified inflated base-stock policy is optimal in single, multiple and infinite horizons.

Models analyzed in the first part of this thesis are in this category of inventory literature since we assume that the environment is random. In particular, the first model in the first part is a combination of Özekici and Parlar (1999) and Erdem and Özekici (2002) since we bring random capacity and random availability concepts together. Moreover, the second model in the same part can be classified as random capacity model since we analyze random capacity inventory models with fixed ordering cost. In this respect, this model is an extension of Erdem and Özekici (2002) since we consider the case where there is economies of scale.

2.2 Inventory models with imperfect information

The most basic assumption in the main body of inventory literature is that the inventory system is fully observed. However, this is not the case in several real life situations. For example, demand may not be observed directly in some cases. In such cases, the sales can be used to predict the demand. However, sales cannot provide complete information regarding the demand process because unmet demand is excluded. Therefore, the demand can only be partially predicted in this case. In such situations, classical Markov decision process (MDP) formulation of the inventory problem is not appropriate. A generalization of MDP which allows uncertainty regarding the state of a Markov process and state information acquisition is necessary in this case and this is done by POMDP formulation.

POMDP has a wide range of application areas. Among many others, a few of these application areas are machine maintenance and replacement, human learning and instruction, medical diagnosis and decision-making, and search for moving objects (Smallwood and Sondik, 1973). Although there are some researchers studying some form of MDP with imperfect information (i.e., Dynkin (1965) and Sirjaev (1966)), the first explicit POMDP is developed by Drake (1962). From 1960s onward, finite horizon POMDPs are formulated in the stochastic control context. Smallwood and Sondik (1973) were the first solving computational difficulties regarding POMDPs. In this study, they assume that the core process is a finite-state Markov chain and formulated a finite-horizon discounted POMDP problem. In their formulation, the system state is assumed to be the conditional distribution of state of the core process. By this formulation, they transform a finite-state POMDP problem into an infinite-state MDP problem. Then, they show that the value function is piecewise-linear in this system state. Moreover, they developed an algorithm which solves POMDP problems by exploiting special structure of the finite-horizon value function. Later, Sondik (1978) formulated discounted infinite-horizon POMDP problem. In this paper, he showed that discounted infinite-horizon POMDP problems can be solved by a generalization of classical policy iteration technique. Furthermore, he developed an algorithm for solving them. White (1976) extended POMDP by allowing a semi-Markov core process. In addition, he developed the algorithm in Smallwood and Sondik (1973) for finite-horizon POMDPs with semi-Markov core process. A detailed discussion of papers on POMDPs can be found in Monahan (1982). Our main concern in this study is not the computational issues regarding the solution of POMDP problems; however, interested readers are referred to Lovejoy (1991) for a detailed analysis of algorithmic methods.

Although study of POMDPs have started in 1950s, few researchers applied this concept in inventory context. To our knowledge, the first study which directly applies POMDP concept in inventory control is Treharne and Sox (2002). In this paper, they assume that there are a finite number of demand states and all cost parameters are independent of these demand states. Furthermore, they assume that the procurement lead time is positive and constant. They formulate the finite-horizon problem by assuming that the system state is the conditional distribution of true demand state and the inventory position. As a result of their analysis, they show that a state-dependent base-stock policy is optimal for finite-horizon inventory problem. In the remaining of this paper, they compare algorithmic methods for solving finite-horizon problem. Another paper applying POMDP concept in inventory control is Bensoussan et al. (2005 a). In this paper, they analyze three different models. In the first model, which they call filtered newsvendor model, they consider the case where demand is observed via sales. As in Treharne and Sox (2002), they model the real demand as a Markov process. Since unmet demand is lost, the inventory manager cannot have exact information about the true demand state. Moreover, they assume that the excess inventory is salvaged at the end of each period; therefore, their problem is like multi-period newsvendor problem with partially observed demand. In the second model which they call zero balance walk model, they focus on the situation where the inventory manager cannot observe the inventory level due to several reasons. The inventory level is observed only when there is no physical inventory. Moreover, they assume that unmet demand is lost and demand has a known distribution. Finally, they show that optimal feedback policy is optimal for filtered newsvendor and zero balance walk models. Through the last model, which they call information delay model, they analyze the case where the inventory manager cannot observe the current inventory level due to information delay. Instead, he can observe the inventory level of a prior period. Unlike other models, they assume that unmet demand is backordered. As a result of their analysis, they show that base-stock policy is optimal for the information delay model.

Notice that both Treharne and Sox (2002) and Bensoussan et al. (2005 a) consider the partially observed demand environment only. None of them consider the supply side. In most real life situations, the supply can only be observed partially so that partially observed supply environment must also be considered. Therefore, in the second part of this thesis, we extend Treharne and Sox (2002) by allowing random supply and partial observation of the supply environment. However, we assume that there is no delay in procurement so that lead times are zero in all cases. Through our analysis in the second part, we introduce the random supply and partially observed supply environment concepts to the inventory literature.

Chapter 3

INVENTORY MODELS WITH PERFECT INFORMATION

All systems in real life are in touch with each other; therefore, one thing happening in one of these systems has an effect on another one. Hence, every inventory system must also consider this fact. As a result, the fluctuating environment phenomenon must be incorporated into every inventory model for the sake of a more realistic model. This clearly increases the complexity of the system; however, it also allows the IMs to be more flexible and adaptive.

In this part of the thesis, we focus on inventory problems with random demand and supply in a fully observed random environment. A reason for the randomness of supply is random capacity. The retailer may or may not receive all of the ordered quantity depending on the capacity of the supplier; therefore, the quantity received may vary. If we ignore the randomness in capacity, the supply may still be random due to transportation problems. Hence, another reason for random supply is randomness in the capacity or availability of transporter. Therefore, total supply or quantity received by the retailer depends on both capacity of the supplier and transporter.

The inventory system considered here is composed of two main entities: retailer and supplier. For the first model, we assume that there is no economies of scale and the supplier is represented by the producer and the transporter. The retailer orders directly from the producer; however, the producer's capacity is random so that all retailer order may not be satisfied. On the other hand, the transporter is responsible to deliver produced amount; however, we assume that the transporter is either available or unavailable at any particular instant. Therefore, the retailer receives either all of the produced amount or receives nothing. For the second model, we model the supplier as an entity which has a random capacity; moreover, we assume that the supplier is always available and there is fixed ordering cost. Therefore, amount received by the retailer or total supply in both models is random. Moreover, we represent the fluctuating environment by a Markov chain and assume that both demand and supply distributions are modulated by this Markov chain.

A related study in this area is Özekici and Parlar (1999) where an infinite horizon, periodic-review inventory model with unreliable suppliers in a random environment is analyzed. They assume that the random environment affects demand, supply and all cost parameters. As a result of their analysis, they show that environment-dependent base-stock policy is optimal when there is no fixed cost of ordering. Moreover, they also analyze the case with fixed ordering cost and show that environment-dependent (s, S) policy is optimal in this case. Our first model in this part analyzes a similar problem as Özekici and Parlar (1999) since we both consider the random availability and random environment. Moreover, we also incorporate random capacity into our model whereas Özekici and Parlar (1999) do not. Furthermore, we analyze single and multiple planning problems as well as infinite planning problems.

Another related paper is Erdem and Özekici (2002) where periodic-review inventory model with available suppliers having random capacity in a random environment is analyzed. As Özekici and Parlar (1999), they assume that random environment affects demand, supply and all cost parameters. They analyze single, multiple and infinite planning period problems when there is no fixed cost of ordering. They show that the optimal policy structure is environment-dependent base-stock policy. Their study is also similar to our first model since we both consider the random capacity and random environment. As Erdem and Özekici (2002), we analyze single, multiple and infinite planning period problems. However, our model has an important difference since we also introduce a randomly available transporter as well as a producer having random capacity into our model.

In our linear cost model, we analyze a discrete-time, single-item, single-location, periodic-review inventory system with random production capacity and random transporter availability where demand, supply and all cost parameters are modulated by a Markov chain representing fluctuating environment. If we consider an inventory system composed of a retailer, a producer and a transporter, then the supply is random due not only to random production capacity but also to random transporter availability. In this respect, we incorporate random availability and random capacity inventory models. In our fixed cost model, we study exactly the same model as in linear cost model; however, we now assume that the supplier is always available but it has random capacity; moreover, there is fixed ordering cost.

This part is organized as follows. Section 3.1 includes the discussion on the inventory models with random capacity and random availability in a random environment which is fully observed when there is no fixed ordering cost. Next in Section 3.2, we study the same inventory problem in Section 3.1; however, there is fixed ordering cost in this case. Finally, in Section 3.3, we summarize the implications of our analysis.

3.1 Linear cost model with random supply

In this section, we assume that $K_i = 0$ for all i so that there is no fixed ordering cost. In inventory literature, this assumption of no fixed ordering cost generally leads to the optimality of base-stock policies which are of a control-limit type specified by a single number. In our model with environment dependent demand, supply and cost parameters, the optimality of base-stock policies remains to be valid. However, the base-stock level depends on the state of the environment.

The remainder of this section is organized as follows. We present our notation and assumptions in the next section. In Section 3.1.2, we study the problem in a single-period setting. Then in Section 3.1.3, we develop a general finite horizon inventory model and analyze it. Moreover, in Section 3.1.4, we present the results of our analysis for the same problem in infinite-horizon. Finally, in Section 3.1.5, we compare the base-stock levels in multiple and infinite planning periods for inventory problems with random capacity only and for inventory problems with random supply.

3.1.1 Model and assumptions

We consider a single product inventory system which is inspected periodically over a planning horizon of length N. The state of the environment observed at time n is represented by Z_n and we assume that state of environment does not change during a period. In addition, we assume that $Z = \{Z_n; n \geq 0\}$ is a time-homogeneous Markov chain on a discrete state space $\mathbb{E}$ with transition matrix

$$P(i, j) = P[Z_{n+1} = j \mid Z_n = i].$$

And P^n denotes n step transition matrix of Z so that

$$P^n(i,j) = P[Z_n = j \mid Z_0 = i].$$

Clearly, $P^0 = I$ so that

$$P^0(i,j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} .$$

If D_n denotes the total demand in period n, then the demand process $D = \{D_n; n \geq 1\}$ is modulated by the Markov chain Z so that its conditional cumulative distribution function is

$$M_i(z) = P[D_{n+1} \leq z \mid Z_n = i]$$

for all i and $z \geq 0$; moreover, it is differentiable so that m_i is the probability density function of demand. Therefore, the demand distribution is completely specified by the environment.

Let C_n denote the random capacity of supplier in period n. Then, the capacity process $C = \{C_n; n \geq 1\}$ is also modulated by Markov chain Z so that its conditional cumulative distribution function is

$$F_i(z) = P[C_{n+1} \leq z \mid Z_n = i]$$

for all i and $z \geq 0$, and it is also assumed to be differentiable so that f_i is the probability density function of capacity. As a result, capacity distribution is also completely specified by the environment. We suppose that $P[C_{n+1} > z | Z_n = i] = 1 - F_i(z) > 0$ for all $z \geq 0$ which implies that the random capacity has no upper bound so that it is possible to receive all of ordered quantity.

In our setting, we let U_n denote the proportion of produced amount which is received by the retailer in period n. Then $1 - U_n$ denotes the proportion of produced amount lost during transportation in period n. Here we assume that either $U_n = 1$ or $U_n = 0$. In other words, retailer either receives all of what is produced by the supplier, or all of the produced amount is lost during transportation. Then, an order is immediately delivered if the capacity is enough and $U_n = 1$. On the other hand, if the capacity is not enough but $U_n = 1$, then the retailer receives as much as capacity. In the remaining two cases, the retailer receives nothing. In addition, we assume that capacity C_n and the transportation yield U_n are independent. If x_n denotes the inventory level observed at the beginning of period n and an order is placed so as to increase the inventory level up to $y_n \geq x_n$, then the amount received by the retailer is $U_{n+1} \min \{y_n - x_n, C_{n+1}\}$. Therefore, supply is random. If demand in a period is not satisfied, then it is completely backlogged in that period and satisfied in the next period. As in the demand and capacity structure, we assume that the transportation yield process or the capacity process of the transporter $U = \{U_n; n \geq 1\}$ depends on the environment so that

$$P[U_{n+1} = 1 \mid Z_n = i] = u_i$$

for some $0 < u_i \leq 1$ and all i. Thus, u_i is the probability that the transporter is available in environment i. Therefore, the reliability of transporter in different environments is given by $\{u_i\}$. Notice that $u_i > 0$ for all i, which means that there is always positive probability for retailer to receive all of what is produced. Otherwise, it is illogical to order.

Moreover, we assume that all cost parameters depend on the state of the environment. Given that state of environment is i, c_i is the purchase cost per item, h_i is the holding cost per item per period, and p_i is the shortage cost per item per period. Both holding and shortage costs are incurred at the end of the period. Moreover, we assume that $p_i > c_i$, $h_i > c_i$ and $c_i > 0$. And, we assume that all cost parameters are finite. Finally, we let α denote periodic discount factor and assume that $0 < \alpha < 1$.

3.1.2 Single-period model

Here, we assume that there is only one period so that $N = 1$. Assuming inventory level at the beginning of period is x and state of the environment at the beginning of period is i, we let $v_0(i, x)$ denote the single-period minimum cost function at the beginning of first period. Moreover, we assume that $v_1(j, x_1) = 0$ for all j and x_1. Then, $v_0(i, x)$ satisfies

$$v_0(i,x) = \min\nolimits_{y \geq x} J_0(i, x, y) \tag{3.1}$$

for all i and x, where y is the order-up-to level and

$$
\begin{aligned}
J_0(i,x,y) &= u_i \int_0^{y-x} G_0(i,x+z)dF_i(z) + u_i G_0(i,y)\left[1-F_i(y-x)\right] \\
&\quad + (1-u_i)\, G_0(i,x) - c_i x && (3.2) \\
G_0(i,y) &= c_i y + L(i,y) && (3.3) \\
L(i,y) &= h_i \int_0^{y} dM_i(z)(y-z) + p_i \int_y^{\infty} dM_i(z)(z-y). && (3.4)
\end{aligned}
$$

We can easily show that

$$
\begin{aligned}
L'(i,y) &= \frac{\partial L(i,y)}{\partial y} = (p_i+h_i)\, M_i(y) - p_i && (3.5) \\
L''(i,y) &= \frac{\partial^2 L(i,y)}{\partial y^2} = (p_i+h_i)\, m_i(y). && (3.6)
\end{aligned}
$$

It is obvious from (3.6) that L is convex in y because L'' is always nonnegative.

Expected cost in a single period is the sum of expected purchase cost, and expected holding and shortage cost. Let $y_0(i,x)$ denote the optimal order up to level which minimizes the expected discounted cost in (3.2) when the state of environment is i and inventory level is x. In addition, let $v_0'(i,x)$ denote the first derivative of $v_0(i,x)$ with respect to x.

Theorem 1 *The optimal ordering policy for the single-period model is an environment-dependent base-stock policy*

$$
y_0(i,x) = \begin{cases} S_0^i & x \le S_0^i \\ x & x > S_0^i \end{cases} \tag{3.7}
$$

where S_0^i satisfies

$$
c_i + L'\left(i, S_0^i\right) = 0
$$

for all π and i. In addition, $J_0(i,x,y)$ is quasi-convex in y for all i and $x \le y$. The optimal cost is

$$
v_0(i,x) = \begin{cases} J_0\left(i,x,S_0^i\right) & x \le S_0^i \\ L(i,x) & x > S_0^i \end{cases} \tag{3.8}
$$

for all π, i and x. Moreover, $v_0(i,x)$ is convex in x, $\lim_{x\uparrow\infty} v_0'(i,x) = h_i$ and $\lim_{x\downarrow-\infty} v_0'(i,x) = -p_i$ for all i.

Proof. We need to find $y \geq x$ minimizing $J_0(i,x,y)$. Let us define J_0' and J_0'' as the first and the second derivatives of J_0 with respect to y respectively. Then,

$$J_0'(i,x,y) = u_i[1-F_i(y-x)]\left(c_i + L'(i,y)\right) \tag{3.9}$$

$$J_0''(i,x,y) = u_i[1-F_i(y-x)]L''(i,y) - \left(c_i + L'(i,y)\right) f_i(y-x) \tag{3.10}$$

for all $y \geq x$. By our assumption, the first multiplicand in (3.9), $u_i[1-F_i(y-x)] > 0$ for all i and x. Hence, whether $J_0(i,x,y)$ is decreasing or increasing depends on the sign of $c_i + L'(i,y)$. Let S_0^i be the smallest y such that

$$B_0(i,y) = c_i + L'(i,y) = 0. \tag{3.11}$$

Notice that, $\lim_{y\uparrow\infty} B_0(i,y) = c_i + h_i > 0$ and $\lim_{y\downarrow-\infty} B_0(i,y) = c_i - p_i < 0$. Moreover, $B_0(i,y)$ is continuous and nondecreasing in y since L is convex. Then, this implies that there exists a finite S_0^i satisfying (3.11). In addition, $B_0(i,y) < 0$ for $y < S_0^i$ and $B_0(i,y) \geq 0$ for $y > S_0^i$ since $B_0(i,y)$ is nondecreasing in y.

From (3.10), it is obvious that the first term is always nonnegative. Now, consider the two cases: $x \leq S_0^i$ and $x > S_0^i$.

(i) $x \leq S_0^i$: It is obvious from (3.9) that $J_0'(i,x,y) < 0$ for all y in $\left[x, S_0^i\right)$ since $u_i[1-F_i(y-x)] > 0$ for all i and $B_0(i,y) < 0$ for all i and $y < S_0^i$; therefore, $J_0(i,x,y)$ is decreasing for all i and y in $\left[x, S_0^i\right)$. By (3.9), we can also say that $J_0'(i,x,y) \geq 0$ for all y on $\left[S_0^i, \infty\right)$ since $u_i[1-F_i(y-x)] > 0$ for all i and $B_0(i,y) \geq 0$ for all i and $y \geq S_0^i$; therefore, $J_0(i,x,y)$ is nondecreasing for all i and y in $\left[S_0^i, \infty\right)$. In addition, the second term in (3.10) is always nonnegative for $y \in \left[x, S_0^i\right)$ since $B_0(i,y) < 0$ for $y < S_0^i$; as a result, (3.10) is nonnegative. Therefore, $J_0(i,x,y)$ is convex decreasing in y on $\left[x, S_0^i\right)$. Moreover, the second term in (3.10) continues to be nonnegative for y close to S_0^i; therefore, $J_0(i,x,y)$ is convex nondecreasing for y close to S_0^i. However, the second term in (3.10) turns out to be negative for large values of $y > S_0^i$ since $\lim_{y\uparrow\infty}[1-F_i(y-x)] = 0$ and $\lim_{y\uparrow\infty}(c_i + L'(i,y)) = c_i + h_i > 0$. Hence, $J_0(i,x,y)$ is concave nondecreasing for sufficiently large values of $y \geq x$.

(ii) $x > S_0^i$: It is clear that $J_0'(i,x,y) > 0$ so that $J_0(i,x,y)$ is increasing in y on $[x,\infty)$. In addition, the second term in (3.10) is always nonpositive; however, (3.10) continues to be nonnegative for values of y close to S_0^i. Then, this implies that $J_0(i,x,y)$ is convex increasing for values of y close to S_0^i. But for large values of y, (3.10) turns out to be negative since $\lim_{y\uparrow\infty}[1-F_i(y-x)] = 0$ and $\lim_{y\uparrow\infty}(c_i + L'(i,y)) = c_i + h_i > 0$. Hence, $J_0(i,x,y)$ is concave nondecreasing for sufficiently large values of $y \geq x$.

This analysis shows that $J_0(i,x,y)$ is a function satisfying all conditions in Lemma 28; therefore, it is quasi-convex. Hence, $y = S_0^i$ is a global minimum of $J_0(i,x,y)$ for $x \leq S_0^i$ and $y = x$ is a global minimum of $J_0(i,x,y)$ for $x > S_0^i$. This implies that S_0^i is the optimal order-up-to level when $x \leq S_0^i$ and that it is optimal not to order when $x > S_0^i$. As a result, (3.7) gives the optimal ordering policy.

Because base-stock policy in (3.7) is optimal, the optimal cost is

$$v_0(i,x) = \begin{cases} J_0\left(i,x,S_0^i\right) & x \leq S_0^i \\ J_0(i,x,x) & x > S_0^i \end{cases}$$

which leads to (3.8) since $J_0(i,x,x) = L(i,x)$.

Now, we prove that $v_0(i,x)$ is convex. First, we show that $v_0(i,x)$ is convex for $x < S_0^i$ and $x > S_0^i$, separately. Then, we show that convexity is not violated at $x = S_0^i$.

(i) $x < S_0^i$: Using (3.8), the first and second derivatives of $v_0(i,x)$ are

$$\begin{aligned} v_0'(i,x) &= u_i \int_0^{S_0^i - x} L'(i,x+z)dF_i(z) + (1-u_i)L'(i,x) - u_i c_i[1 - F_i(S_0^i - x)] \\ &\quad -u_i c_i[1 - F_i(S_0^i - x)] \qquad (3.12) \\ v_0''(i,x) &= u_i \int_0^{S_0^i - x} L''(i,x+z)dF_i(z) + (1-u_i)L''(i,x) \\ &\quad -u_i f_i\left(S_0^i - x\right)\left(c_i + L'(i,S_0^i)\right). \qquad (3.13) \end{aligned}$$

In (3.13), first and second terms are always nonnegative because L is convex and $u_i > 0$. Moreover, by (3.11), the third term in (3.13) is zero. As a result, (3.13) is always positive so that $v_0(i,x)$ is convex in x for all $x < S_0^i$.

(ii) $x > S_0^i$: Using (3.8), v_0 is convex because L is a convex function.

(iii) $x = S_0^i$: We now show that convexity of v_0 is not violated at $x = S_0^i$. For v_0 to be convex at $x = S_0^i$, the following condition must hold

$$\lim_{x\uparrow S_0^i} v_0'(i,x) \leq \lim_{x\downarrow S_0^i} v_0(i,x) \qquad (3.14)$$

and $v_0(i,x)$ must be continuous at $x = S_0^i$. Using (3.12),

$$\begin{aligned}\lim_{x\uparrow S_0^i} v_0(i,x) &= \lim_{x\uparrow S_0^i}\left\{u_i\int_0^{S_0^i - x} L'(i,x+z)dF_i(z) + (1-u_i)\,L'(i,x)\right.\\ &\qquad \left.-u_i c_i[1-F_i(S_0^i - x)]\right\}\\ &= u_i L'(i,S_0^i)F_i(0) + (1-u_i)\,L'(i,S_0^i) - u_i c_i[1-F_i(0)]\\ &= -c_i\end{aligned}$$

The last equality follows from (3.11). In addition, using (3.8) and (3.11),

$$\lim_{x\downarrow S_0^i} v_0(i,x) = -c_i.$$

Therefore, the condition in (3.14) is satisfied as an equality. Moreover, $v_0(i,x)$ is continuous at $x = S_0^i$ since

$$\lim_{x\uparrow S_0^i} v_0(i,x) = \lim_{x\downarrow S_0^i} v_0(i,x) = J_0\left(i, S_0^i, S_0^i\right).$$

As a result, $v_0(i,x)$ is convex in x for all i. Furthermore, using (3.8) and the fact that $\lim_{x\uparrow\infty} L'(i,x) = h_i$,

$$\lim_{x\uparrow\infty} v_0'(i,x) = \lim_{x\uparrow\infty} L'(i,x) = h_i$$

for all i. And using (3.12) and the fact that $\lim_{x\downarrow-\infty} L'(i,x) = -p_i$,

$$\lim_{x\downarrow-\infty} v_0'(i,x) = -p_i$$

for all i. This completes our proof. ■

Theorem 1 implies that the optimal order-up-to level is independent of the initial level of inventory. Moreover, (3.5) and (3.11) imply that the optimal order-up-to level in single-period is the minimal S_0^i which satisfies

$$S_0^i = M_i^{-1}\left(\frac{p_i - c_i}{p_i + h_i}\right). \tag{3.15}$$

The optimal policy is to order if current level of inventory $x \le S_0^i$, and do not order if $x > S_0^i$.

Moreover, it is obvious from (3.15) that the base-stock level is independent of the capacity distribution F_i and availability probability u_i. Hence, for a single-period problem, the optimal base-stock level will be exactly the same as when there is no random capacity and/or random availability.

3.1.3 Multi-period model

In this section, we assume that there are N periods to plan for. We define $v_n(i,x)$ as the expected cost at the beginning of period $n+1$ using the optimal policy for the periods $n+1, n+2, \cdots, N$ given that the inventory level is x and the environment is i at the beginning of period $n+1$. Set $v_N(j, x_N) = 0$ for all j and x_N; moreover, assume that demand in period $n+1$ is denoted by D and the periodic discount factor is α. Then, v_n satisfies the dynamic programming equation

$$v_n(i,x) = \min_{y \geq x} J_n(i,x,y) \tag{3.16}$$

for all i and x, where y is the order-up-to level in period $n+1$ and

$$\begin{aligned} J_n(i,x,y) &= u_i \int_0^{y-x} G_n(i,x+z)dF_i(z) + u_i G_n(i,y)\left[1 - F_i(y-x)\right] \\ &\quad + (1-u_i)\, G_n(i,x) - c_i x \qquad (3.17) \\ G_n(i,y) &= c_i y + L(i,y) + \alpha \sum_{j \in \mathbb{E}} P(i,j) E_D^i \left[v_{n+1}(j, y-D)\right] \qquad (3.18) \end{aligned}$$

with $L(i,y)$ given in (3.4).

Let $y_n(i,x)$ denote the optimal order-up-to level of the minimization problem in (3.16) when the state of environment is i and inventory level is x at time n. And let $v_n'(i,x)$ denote the derivative of $v_n(i,x)$ with respect to x. In addition, we use the same notation of Section A.3 in the appendix so that $R_m^\alpha(i,j) = \sum_{n=0}^{m-1} \alpha^n P^n(i,j)$. Finally, we assume that h and p are holding cost and shortage cost vectors, respectively.

Theorem 2 *The optimal ordering policy for N-period model is an environment-dependent base-stock policy*

$$y_n(i,x) = \begin{cases} S_n^i & x \leq S_n^i \\ x & x > S_n^i \end{cases} \tag{3.19}$$

where S_n^i satisfies

$$c_i + L'(i, S_n^i) + \alpha \sum_{j \in \mathbb{E}} P(i,j) E_D^i [v_{n+1}'(j, S_n^i - D)] = 0$$

for all π and i. In addition, $J_n(i,x,y)$ is quasi-convex in y for all i and $x \leq y$. The optimal cost is

$$v_n(i,x) = \begin{cases} J_n\left(i,x,S_n^i\right) & x \leq S_n^i \\ L(i,x) + \alpha \sum_{j \in \mathbb{E}} P(i,j) E_D^i \left[v_{n+1}(j, x-D)\right] & x > S_n^i \end{cases} \tag{3.20}$$

for all π, i *and* x. *Moreover,* $v_n(i,x)$ *is convex in* x, $\lim_{x\uparrow\infty} v_n'(i,x) = R_{N-n}^{\alpha} h(i)$ *and* $\lim_{x\downarrow-\infty} v_n'(i,x) = -R_{N-n}^{\alpha} p(i)$ *for all* i.

Proof. We will prove the theorem by induction. Clearly, by Theorem 1, Theorem 2 is valid for $n = N-1$. Now suppose that induction hypothesis holds for times $n+1, n+2, ..., N-1$. Next, we will show that Theorem 2 is still valid for time n. For this purpose, let us analyze the objective function $J_n(i,x,y)$. We define J_n' and J_n'' as the first and the second derivatives of J_n with respect to y respectively. Then,

$$J_n'(i,x,y) = u_i[1-F_i(y-x)]\left(c_i + L'(i,y) + \alpha\sum_{j\in\mathbb{E}} P(i,j)E_D^i[v_{n+1}'(j,y-D)]\right) \quad (3.21)$$

$$\begin{aligned} J_n''(i,x,y) &= u_i[1-F_i(y-x)]\left(L''(i,y) + \alpha\sum_{j\in\mathbb{E}} P(i,j)E_D^i[v_{n+1}''(j,y-D)]\right) \\ &\quad -u_i f_i(y-x)\left(c_i + L'(i,y) + \alpha\sum_{j\in\mathbb{E}} P(i,j)E_D^i[v_{n+1}'(j,y-D)]\right) \quad (3.22)\end{aligned}$$

for all $y \geq x$. Note that, by our assumption, $u_i[1-F_i(y-x)] > 0$ for all i and $y \geq x$. Let S_n^i be the smallest y such that

$$B_n(i,y) = c_i + L'(i,y) + \alpha\sum_{j\in\mathbb{E}} P(i,j)E_D^i[v_{n+1}'(j,y-D)] = 0. \quad (3.23)$$

Using Monotone Convergence Theorem (MCT) and the induction hypothesis that $\lim_{x\uparrow\infty} v_{n+1}'(j,x) = R_{N-n-1}^{\alpha} h(j)$ for all j,

$$\lim_{y\uparrow\infty} B_n(i,y) = c_i + h_i + \alpha\sum_{j\in\mathbb{E}} P(i,j)R_{N-n-1}^{\alpha} h(j)$$

And, using (A.5),

$$\lim_{y\uparrow\infty} B_n(i,y) = c_i + R_{N-n}^{\alpha} h(i) > 0.$$

Moreover, again using MCT and assumptions that $\lim_{x\downarrow-\infty} v_{n+1}'(j,x) = -R_{N-n-1}^{\alpha} p(j)$ for all j and $c_i < p_i$ for all i,

$$\lim_{y\downarrow-\infty} B_n(i,y) = c_i - p_i - \alpha\sum_{j\in\mathbb{E}} P(i,j)R_{N-n-1}^{\alpha} p(j).$$

And, using (A.5),

$$\lim_{y\downarrow-\infty} B_n(i,y) = c_i - R_{N-n}^{\alpha} p(i) < 0.$$

Furthermore, $B_n(i, y)$ is continuous and nondecreasing in y since L and v_{n+1} are convex. Then, this implies that there exists a finite S_n^i satisfying (3.23). In addition, $B_n(i, y) < 0$ for $y < S_n^i$ and $B_n(i, y) \geq 0$ for $y > S_n^i$ since $B_n(i, y)$ is nondecreasing in y. Now, we consider the two cases: $x \leq S_n^i$ and $x > S_n^i$.

(i) $x \leq S_n^i$: The derivative (3.21) is nonpositive, and (3.22) is nonnegative so that $J_n(i, x, y)$ is convex decreasing on $y \in [x, S_n^i)$. Also (3.21) is nonnegative, and (3.22) is nonnegative for values of y close to S_n^i so that $J_n(i, x, y)$ is convex nondecreasing for y close to S_n^i on $[S_n^i, +\infty)$. However, (3.22) is negative for sufficiently large values of y because $\lim_{y\uparrow\infty}[1 - F_i(y - x)] = 0$ and $\lim_{y\uparrow\infty} B_n(i, y) > 0$ so that $J_n(i, x, y)$ is concave increasing for sufficiently large y on $[S_n^i, +\infty)$. Clearly, S_n^i is the global minimum and $y_n(i, x) = S_n^i$ is the order-up-to level when $x < S_n^i$.

(ii) $x > S_n^i$: The derivative (3.21) is nonnegative, and (3.22) is nonnegative for values of y close to S_n^i so that $J_n(i, x, y)$ is convex nondecreasing for y close to S_n^i on $(S_n^i, +\infty]$. However, (3.22) is negative for sufficiently large values of y because $\lim_{y\uparrow\infty}[1 - F_i(y - x)] = 0$ and $\lim_{y\uparrow\infty} B_n(i, y) > 0$ so that $J_n(i, x, y)$ is concave increasing for sufficiently large y on $[S_n^i, +\infty)$. Then, x is the global minimum and $y_n(i, x) = x$ is the order-up-to level when $x > S_n^i$.

This analysis shows that $J_n(i, x, y)$ is quasi-convex since it satisfies all conditions in Lemma 28. Therefore, optimal ordering policy is the environment-dependent base-stock policy defined by (3.19). It follows from that the optimal cost is

$$v_n(i, x) = \begin{cases} J_n\left(i, x, S_n^i\right) & x \leq S_n^i \\ J_n(i, x, x) & x > S_n^i \end{cases}$$

which leads to (3.20) by using (3.17) and (3.18).

Now, we prove that $v_n(i, x)$ is convex. First, we show that $v_n(i, x)$ is convex for $x < S_n^i$ and $x > S_n^i$ separately. Then, we show that convexity is not violated at $x = S_n^i$.

(i) $x < S_n^i$: Using (3.20), the first and second derivatives of $v_n(i,x)$ are

$$v_n'(i,x) = u_i \int_0^{S_n^i - x} G_n'(i, x+z) dF_i(z) + (1-u_i) G_n'(i,x) - c_i \tag{3.24}$$

$$v_n''(i,x) = u_i \int_0^{S_n^i - x} G_n''(i, x+z) dF_i(z) + (1-u_i) G_n''(i,x) - u_i f_i\left(S_n^i - x\right) G_n'(i, S_n^i) \tag{3.25}$$

where

$$G_n'(i,x) = c_i + L'(i,x) + \alpha \sum_{j \in \mathbb{E}} P(i,j) E_D^i \left[v_{n+1}'(j, x-D)\right] \tag{3.26}$$

$$G_n''(i,x) = L''(i,x) + \alpha \sum_{j \in \mathbb{E}} P(i,j) E_D^i \left[v_{n+1}''(j, x-D)\right]. \tag{3.27}$$

In (3.25), first and second terms are always nonnegative because L and v_{n+1} are convex. Moreover, by (3.23), the third term in (3.25) is zero. As a result, (3.25) is always nonnegative so that $v_n(i,x)$ is convex in x for all $x < S_n^i$.

(ii) $x > S_n^i$: Using (3.20), v_n is convex because L and v_{n+1} are convex functions. Moreover, for $x > S_n^i$, first derivative of $v_n(i,x)$ is

$$v_n'(i,x) = L'(i,x) + \alpha \sum_{j \in \mathbb{E}} P(i,j) E_D^i \left[v_{n+1}'(j, x-D)\right]. \tag{3.28}$$

(iii) $x = S_n^i$: We now show that convexity of v_n is not violated at $x = S_n^i$. For v_n to be convex at $x = S_n^i$, the following condition must hold

$$\lim_{x \uparrow S_n^i} v_n'(i,x) \leq \lim_{x \downarrow S_n^i} v_n'(i,x) \tag{3.29}$$

and $v_n(i,x)$ must be continuous at $x = S_n^i$. Using (3.24),

$$\begin{aligned} \lim_{x \uparrow S_n^i} v_n'(i,x) &= \lim_{x \uparrow S_n^i} \left\{ u_i \int_0^{S_n^i - x} G_n'(i, x+z) dF_i(z) + (1-u_i) G_n'(i,x) - c_i \right\} \\ &= u_i G_n'(i, S_n^i) F_i(0) + (1-u_i) G_n'(i, S_n^i) - c_i \\ &= -c_i. \end{aligned} \tag{3.30}$$

The last equality follows from (3.23). In addition, using (3.28) and (3.23),

$$\lim_{x \downarrow S_n^i} v_n'(i,x) = -c_i. \tag{3.31}$$

Therefore, the condition in (3.29) is satisfied as an equality. Moreover, $v_n(i,x)$ is continuous at $x = S_n^i$ since

$$\lim_{x \uparrow S_n^i} v_n(i,x) = \lim_{x \downarrow S_n^i} v_n(i,x) = J_n\left(i, S_n^i, S_n^i\right).$$

As a result, $v_n(i,x)$ is convex for all x. Notice that, by the induction hypothesis, $\lim_{x\uparrow\infty} v'_{n+1}(j,x) = R^{\alpha}_{N-n-1}h(j)$ for all j. Then, using (3.28) and MCT

$$\lim_{x\uparrow\infty} v'_n(i,x) = h_i + \alpha \sum_{j\in\mathbb{E}} P(i,j) R^{\alpha}_{N-n-1}h(j).$$

And, using (A.5),

$$\lim_{x\uparrow\infty} v'_n(i,x) = R^{\alpha}_{N-n}h(i).$$

Moreover, by the induction hypothesis, $\lim_{x\downarrow-\infty} v'_{n+1}(j,x) = -R^{\alpha}_{N-n-1}p(j)$ for all j. Then, using (3.24) and MCT

$$\lim_{x\downarrow-\infty} v'_n(i,x) = -p_i - \alpha \sum_{j\in\mathbb{E}} P(i,j) R^{\alpha}_{N-n-1}p(j).$$

And using (A.5),

$$\lim_{x\downarrow-\infty} v'_n(i,x) = -R^{\alpha}_{N-n}p(i).$$

This completes our proof. ■

By Theorem 2, we see that environment-dependent base-stock policy is still optimal in multiple periods. And, by (3.23), we see that order-up-to level is independent of initial inventory level. Moreover, the objective function J_n is quasi-convex as in single-period model; therefore, S_n^i satisfying (3.23) is the global minimum.

Using Theorem 2 and (A.2), we get that $\lim_{x\uparrow\infty} v'_n(i,x) = E[\sum_{n=0}^{N-n-1} \alpha^n h_{Z_n} \Big| Z_0 = i]$ and $\lim_{x\downarrow-\infty} v'_n(i,x) = -E[\sum_{n=0}^{N-n-1} \alpha^n p_{Z_n} \Big| Z_0 = i]$. These results are intuitively understandable. Consider the case where inventory level at time n is very large. Clearly the inventory level in the remaining periods is also very large so that there are not any stock-outs. In such a case, increasing inventory level by one means that the retailer holds this extra unit until the end of the planning horizon. As a result, from time n until time $N-1$, the retailer incurs extra holding cost which depends on the state of environment at that time. Under discounting, the expected present worth of increase in minimum cost given the initial state i is $E[\sum_{n=0}^{N-n-1} \alpha^n h_{Z_n} \Big| Z_0 = i]$. Similarly, consider the case where inventory level at time n is negative and very large in absolute value. This implies that the retailer is out of stock. In such a case, a unit increase in inventory level implies that the retailer's inventory level is one more in the remaining periods provided that everything is the same. As a result, in the remaining periods, the retailer stocks out one unit less compared to previous case. This implies that the retailer pays p_{Z_n} less at each period until time $N-1$. Then, given initial state i, expected total decrease in the optimal cost is $E[\sum_{n=0}^{N-n-1} \alpha^n p_{Z_n} \Big| Z_0 = i]$.

Our results for multiple periods agree with those of Erdem and Özekici (2002) which analyzes the inventory models in a random environment with random supply due to random capacity of the supplier only. In Erdem and Özekici (2002), it is shown that the recursive cost function is unimodal and optimal policy is base-stock policy where base-stock level is independent of current inventory level when there is random capacity and random environment. In our model, we incorporate random availability as well and show that results obtained by Erdem and Özekici (2002) are still valid.

Note that we assume that there is always a positive probability of receiving fully what we order, i.e., $P[C_{n+1} > y - x | Z_n = i] = 1 - F_i(y - x) > 0$ for all $y \geq x$. However, we can easily extend our results for the cases where the capacity is not random but limited. It has been shown by Federgruen and Zipkin (1986 b) that modified base-stock policy is optimal for inventory problems with random capacity in a certain environment. Moreover, Erdem and Özekici (2002) analyze limited capacity case in multiple periods, and they show that environment-dependent modified base-stock policy is optimal when capacity is not random but finite. Our analysis indicates us that environment-dependent modified base-stock policy is still optimal for inventory problems with limited capacity and random availability.

3.1.4 Infinite-period model

In this section, we study infinite-period inventory problem with transportation yield and random capacity in a random environment. Here, we show that environment-dependent base-stock policy is still optimal for infinite-period problem. In addition, we analyze the convergence and uniqueness properties of the optimal policy in infinite periods. By assuming that $k = N - n$ denotes the number of periods from time n until time N, we use the notation $v_{n,k}$ for the finite horizon optimal cost v_n in the remaining part of this section. Here, we show that, as k increases to infinity, the finite-horizon optimal cost function $v_{0,k}$ in (3.16) converges to the infinite-horizon optimal cost function v that satisfies

$$v(i,x) = \min_{y \geq x} J(i,x,y) \tag{3.32}$$

for all i and x, where y is the order-up-to level and

$$\begin{aligned} J(i,x,y) &= u_i \int_0^{y-x} G(i,x+z) dF_i(z) + u_i G(i,y) \left[1 - F_i(y-x)\right] \\ &\quad + (1-u_i) G(i,x) - c_i x \end{aligned} \tag{3.33}$$

$$G(i,y) = c_i y + L(i,y) + \alpha \sum_{j \in \mathbb{E}} P(i,j) E_D^i \left[v(j, y-D)\right] \tag{3.34}$$

with $L(i,y)$ given in (3.4).

For any real valued function $f : \mathbb{E} \times \mathbb{R} \to \mathbb{R}$, where $\mathbb{R} = (-\infty, +\infty)$, define the mapping $\mathcal{T}$ as

$$\mathcal{T} f(i,x) = \min_{y \geq x} J(i,x,y) \tag{3.35}$$

where $J(i,x,y)$ is given in (3.33) with

$$G(i,y) = c_i y + L(i,y) + \alpha \sum_{j \in \mathbb{E}} P(i,j) E_D^i \left[f(j, y-D)\right]. \tag{3.36}$$

Using (3.1), $\mathcal{T} f$ can be interpreted as the optimal cost function for the one-period problem where the terminal cost function is $\alpha \sum_{j \in \mathbb{E}} P(i,j) E_D^i \left[f(j, .)\right]$. Then, $\mathcal{T}^k$ denotes the composition of the mapping $\mathcal{T}$ with itself k times; that is, for all $k \geq 1$

$$\mathcal{T}^k f(i,x) = \mathcal{T} \mathcal{T}^{k-1} f(i,x) \tag{3.37}$$

with $\mathcal{T}^0 f = f$. Using (3.16), we can interpret $\mathcal{T}^k f$ as the optimal cost function for the $k-$period $\alpha-$discounted problem. Then, using (3.35) and (3.37),

$$\mathcal{T}^k f(i,x) = \min_{y \geq x} J_k(i,x,y) \tag{3.38}$$

where J_k is given in (3.33) with G replaced by

$$G_k(i,y) = c_i y + L(i,y) + \alpha \sum_{j \in \mathbb{E}} P(i,j) E_D^i \left[\mathcal{T}^{k-1} f(j, y-D)\right]. \tag{3.39}$$

Let $f_0(i,x) = 0$ for all i and x. For our analysis in previous sections, we always assume that the terminal cost function is zero. Suppose that the initial cost function is $f_0(i,x)$

so that $\mathcal{T}^0 f(i,x) = f_0(i,x)$ for all i and x. Then, $k-$period optimal cost function is $v_{n,k}(i,x) = \mathcal{T}^k f_0(i,x)$ for all i and x.

Let $f_*(i,x)$ denote the optimal cost over infinite horizon and let

$$f_\infty(i,x) = \lim_{k\uparrow\infty} \mathcal{T}^k f_0(i,x). \tag{3.40}$$

Notice that f_∞ is well-defined provided we allow the possibility that f_∞ can take the value ∞. Our main aim in this section is to show that the finite-horizon optimal cost converges to the infinite-horizon optimal cost as the length of the planning horizon gets longer. In other words, we aim to show that $f_*(i,x) = f_\infty(i,x)$ for all i and x. As stated in Bertsekas (2000 b), it is analytically and computationally important to show that $f_*(i,x) = f_\infty(i,x)$ because if we know that $f_*(i,x) = \lim_{k\uparrow\infty} \mathcal{T}^k f_0(i,x)$, then we can infer the properties of $f_*(i,x)$ from the properties of $k-$period optimal cost functions $\mathcal{T}^k f_0(i,x)$.

Let $\mathcal{Z}_k$ denote the sets

$$\mathcal{Z}_k(i,x,\lambda) = \{y \geq x \,|\, J_k(i,x,y) \leq \lambda\} \tag{3.41}$$

for all i and $x, \lambda \in \mathbb{R}$. According to Proposition 1.7 in Bertsekas (2000 b, p. 148), if we show that the sets in (3.41) are compact for all i, x and λ, then $f_*(i,x) = f_\infty(i,x)$. By the following lemma, we accomplish this task.

Lemma 3 *Assume that* $\lim_{y\uparrow\infty} J_k(i,x,y) = \infty$ *for all* i, x *and* k. *The sets in (3.41) are compact subsets of the Euclidean space for all* i, x *and* λ.

Proof. We need to show that the sets in (3.41) are both bounded and closed in order to show that they are compact. Let us first show that the sets in (3.41) are bounded. Note that J_k is expected discounted cost when there are k periods until the end of planning horizon. Therefore, it is exactly the same as J_n in Section 3.1.3 where $n = N - k$. In multi-period analysis, we showed that $J_k(i,x,y)$ is nonincreasing for $y \in [x, S_k^i]$ and nondecreasing for $y \in [S_k^i, +\infty)$. Then, because we assume that $\lim_{y\uparrow\infty} J_k(i,x,y) = \infty$ for all i, x and k, the sets $\{\mathcal{Z}_k(i,x,\lambda)\}$ in (3.41) are bounded for all i, x and λ. Moreover, the sets $\{\mathcal{Z}_k(i,x,\lambda)\}$ are closed since $J_k(i,x,y)$ is continuous for $y \geq x$ and it is real-valued. Thus, the sets in (3.41) are compact subsets of Euclidean space for all i, x and λ. This completes our proof. ■

One of the cases when our assumption in Lemma 3 is satisfied is when $\lim_{y\uparrow\infty}[1 - F_i(y-x)] > 0$ for all i and x. Clearly, if $\lim_{y\uparrow\infty}[1 - F_i(y-x)] > 0$ for all i and x, then $\lim_{y\uparrow\infty} J_k(i,x,y) = \infty$ since $\lim_{y\uparrow\infty} G_n(i,y) = \infty$ and $u_i > 0$ for all i and n. Notice that this is not a restrictive requirement and it is only technically necessary. However, not all continuous distribution functions satisfy this requirement. As an example, assuming that the capacity distribution is exponential, cumulative distribution of capacity is $F_i(y-x) = 1 - e^{-\mu(y-x)}$ where $1/\mu$ is mean capacity. Then, it is obvious that $\lim_{y\uparrow\infty}[1 - F_i(y-x)] = 0$ for all i and x. Therefore, for probability distributions where $\lim_{y\uparrow\infty}[1 - F_i(y-x)] = 0$, we can use approximations such that $\lim_{y\uparrow\infty}[1 - F_i(y-x)] = \varepsilon > 0$ but it is very small like $\varepsilon = 10^{-10}$. In addition, we can also truncate the distribution F_i at a very large value and use this truncated distribution in place of F_i. Then, our assumption in Lemma 3 is clearly satisfied so that the sets $\{\mathcal{Z}_k(i,x,\lambda)\}$ are compact.

The following proposition tells that f_∞ is a fixed point of the mapping $\mathcal{T}$; moreover, finite-horizon optimal cost function converges to the infinite-horizon optimal cost function.

Proposition 4 *The limit f_∞ is a fixed point of the mapping $\mathcal{T}$ so that*

$$f_\infty(i,x) = \mathcal{T} f_\infty(i,x) \tag{3.42}$$

for all i and x. Moreover,

$$f_\infty(i,x) = f_*(i,x) \tag{3.43}$$

for all i and x. Furthermore, there exists a stationary optimal policy.

Proof. By Lemma 3, the sets in (3.41) are compact subsets of the Euclidean space for all i, x and λ. Then, using Proposition 1.7 in Bertsekas (2000 b, p.148), f_∞ is a fixed point of $\mathcal{T}$ so that (3.42) is valid and there exists a stationary optimal policy. In addition, notice that

$$f_0 \leq \mathcal{T} f_0 \leq ... \leq \mathcal{T}^k f_0 \leq ... \leq f_*$$

because expected cost per period is nonnegative. From this, we get $\lim_{k\uparrow\infty} \mathcal{T}^k f_0(i,x) \leq f_*(i,x)$ so that $f_\infty(i,x) \leq f_*(i,x)$. By (3.42), we know that f_∞ is a fixed point of $\mathcal{T}$. Then, by Proposition 1.2 in Bertsekas (2000 b, p.140), we get that $f_*(i,x) \leq f_\infty(i,x)$. It follows that $f_\infty(i,x) = f_*(i,x)$. This completes our proof. ■

Notice that Proposition 4 implies also that f_∞, the optimal cost function that the finite-horizon cost function converges, satisfies the Bellman's equation since $f_\infty(i,x) = \mathcal{T} f_\infty(i,x)$ by (3.42). Hence,

$$f_\infty(i,x) = \min_{y \geq x} J(i,x,y) \tag{3.44}$$

for all i and x, where $J(i,x,y)$ is given in (3.33) with

$$G(i,y) = c_i y + L(i,y) + \alpha \sum_{j \in \mathbb{E}} P(i,j) E_D^i \left[f_\infty(j, y - D) \right]. \tag{3.45}$$

As stated in Proposition 1.2 in Bertsekas (2000 b, p. 140), f_∞ is not necessarily the unique optimal solution to Bellman's equation because single-period costs are not bounded under positivity assumption; however, f_∞ is the smallest fixed point of $\mathcal{T}$ since $f_\infty = f_*$.

Notice that, for a finite n, k goes to infinity as N goes to infinity. Then, above analysis shows us that, $\lim_{k\uparrow\infty} v_{0,k}(i,x) = v(i,x)$. Moreover, $v(i,x)$ satisfies (3.32) and there exists a stationary optimal policy $y(i,x)$ which minimizes the infinite-period total cost. However, notice that $J_n(i,x,y)$ is not bounded for $y \geq x$; therefore, v is not necessarily unique. Then, we take v as the minimal fixed point of (3.32). In other words, if $f = \mathcal{T} f$, then $v \leq f$. Moreover, we also know that the optimal solution v is that fixed point of $\mathcal{T}$ which can be obtained as $v = \lim_{k\uparrow\infty} \mathcal{T}^k f_0$ with $f_0 = 0$.

Assuming i and x are current environmental state and inventory level respectively, we let $y(i,x)$ denote the optimal order-up-to level of the minimization problem in (3.32). In addition, let $v'(i,x)$ denote the derivative of $v(i,x)$ with respect to x. Here, we again use the same notation of Section A.3 in the Appendix so that $R^\alpha(i,j) = \sum_{n=0}^{\infty} \alpha^n P^n(i,j)$ and we let h and p denote holding cost and shortage cost vectors respectively.

Theorem 5 *The optimal ordering policy for the infinite-period model is an environment-dependent base-stock policy*

$$y(i,x) = \begin{cases} S^i & x \leq S^i \\ x & x > S^i \end{cases} \tag{3.46}$$

where S^i satisfies

$$c_i + L'(i,S^i) + \alpha \sum_{j \in \mathbb{E}} P(i,j) E_D^i [v'(j, S^i - D)] = 0$$

for all π and i. In addition, $J(i,x,y)$ is quasi-convex in y for all i and $x \leq y$. The optimal cost is

$$v(i,x) = \begin{cases} J\left(i,x,S^i\right) & x \leq S^i \\ L(i,x) + \alpha \sum_{j\in\mathbb{E}} P(i,j) E_D^i\left[v(j,x-D)\right] & x > S^i \end{cases} \tag{3.47}$$

for all π, i and x. Moreover, $v(i,x)$ is convex in x, $\lim_{x\uparrow\infty} v'(i,x) = R^\alpha h(i)$, and $\lim_{x\downarrow-\infty} v'(i,x) = -R^\alpha p(i)$ for all i.

Proof. Since $v(i,x) = \lim_{k\uparrow\infty} v_{0,k}(i,x)$ and limit of a convex function is also convex, $v(i,x)$ is a convex function in x for all i. And, we know by Theorem 2 that $\lim_{x\uparrow\infty} v'_{0,k}(i,x) = R_k^\alpha h(i)$, and $\lim_{x\downarrow-\infty} v'_{0,k}(i,x) = -R_k^\alpha p(i)$ for all i. According to Heyman and Sobel (1984), $v'(i,x) = \lim_{k\uparrow\infty} v'_{0,k}(i,x)$ when $v_{0,k}$ is differentiable for all i and x. As a result, $v'(i,x) = \lim_{k\uparrow\infty} v'_{0,k}(i,x)$ for all i and x. Then, $\lim_{x\uparrow\infty} v'(i,x) = R^\alpha h(i)$, and $\lim_{x\downarrow-\infty} v'(i,x) = -R^\alpha p(i)$ for all i.

As in single and multiple period models, we need to analyze (3.33) in order to find the optimal base-stock levels in infinite period. Similarly, we define J' to be the first derivative of J with respect to y. Then,

$$J'(i,x,y) = u_i[1-F_i(y-x)]\left(c_i + L'(i,y) + \alpha \sum_{j\in\mathbb{E}} P(i,j) E_D^i[v'(j,y-D)]\right) \tag{3.48}$$

for all i and $y \geq x$. By our assumption, $u_i[1-F_i(y-x)] > 0$ for all i. Then, whether $J(i,x,y)$ is increasing or decreasing depends on the sign of the expression inside the parenthesis in (3.48). Moreover, the expression is a nondecreasing function of y since L and v are convex. Therefore, $J(i,x,y)$ is nonincreasing if (3.48) is nonpositive, and it is increasing otherwise. Let S^i be the smallest y satisfying

$$B(i,y) = c_i + L'(i,y) + \alpha \sum_{j\in\mathbb{E}} P(i,j) E_D^i[v'(j,y-D)] = 0. \tag{3.49}$$

Then, using MCT and the fact that $\lim_{x\uparrow\infty} v'(j,x) = R^\alpha h(j)$,

$$\lim_{y\uparrow\infty} B(i,y) = c_i + h_i + \alpha \sum_{j\in\mathbb{E}} P(i,j) R^\alpha h(j) > 0.$$

And, using MCT and the fact that $\lim_{x\downarrow-\infty} v'(j,x) = -R^\alpha p(j)$,

$$\lim_{y\downarrow-\infty} B(i,y) = c_i - p_i - \alpha \sum_{j\in\mathbb{E}} P(i,j) R^\alpha p(j) < 0.$$

Because $\lim_{y\uparrow\infty} B(i,y) > 0$, $\lim_{y\downarrow-\infty} B(i,y) < 0$ and $B(i,y)$ is a nondecreasing continuous function of y, there exists a finite S^i satisfying (3.49). Moreover, by a similar discussion as in Section 3.1.2 and Section 3.1.3, we can show that $J(i,x,y)$ is decreasing for all $y < S^i$ and nondecreasing for all $y \geq S^i$. Therefore, by Lemma 28, it is quasi-convex. Then, environment-dependent base-stock policy defined in (3.46) is optimal. Using (3.46), expected optimal discounted cost corresponding to this optimal policy is

$$v(i,x) = \begin{cases} J\left(i,x,S^i\right) & x \leq S^i \\ J(i,x,x) & x > S^i \end{cases} \tag{3.50}$$

for all i and x. Finally, using (3.33) and (3.34), (3.50) becomes exactly the same as (3.47). This completes our proof. ■

By Theorem 5, we see that environment-dependent base-stock policy is still optimal in infinite period; moreover, base-stock level is independent of the current inventory level. However, this optimal policy is not necessarily unique but it is stationary.

Moreover, by Theorem 5 and (A.3), we have $\lim_{x\uparrow\infty} v'(i,x) = E\left[\sum_{n=0}^{\infty} \alpha^n h_{Z_n} \middle| Z_0 = i\right]$ and $\lim_{x\downarrow-\infty} v'(i,x) = -E\left[\sum_{n=0}^{\infty} \alpha^n p_{Z_n} \middle| Z_0 = i\right]$. These results are also intuitively understandable. If we consider the case where inventory level at time n is very large, it is easy to see through a similar explanation as in multiple period case that, from time n until time ∞, one unit increase in inventory level cause the retailer to incur extra holding cost at each period. Under discounting, expected present worth of increase in total cost given the initial state i is $E\left[\sum_{n=0}^{\infty} \alpha^n h_{Z_n} \middle| Z_0 = i\right]$. Similarly, if we consider the case where inventory level at time n is negative and very large in absolute value, the retailer pays p_{Z_n} less at each period until time ∞ because of unit increase in inventory level. Then, given that initial state is i, expected total decrease in the optimal cost is $E\left[\sum_{n=0}^{\infty} \alpha^n p_{Z_n} \middle| Z_0 = i\right]$.

Furthermore, according to the Corollary 1.8 in Çınlar (1975, p. 197), if the state space $\mathbb{E}$ is finite and $\alpha \in [0,1)$, then $R^\alpha = (I - \alpha P)^{-1}$ where I is the identity matrix; $\lim_{x\uparrow\infty} v'(i,x) = (I - \alpha P)^{-1} h(i)$ and $\lim_{x\downarrow-\infty} v'(i,x) = -(I - \alpha P)^{-1} p(i)$ for all i. Therefore, if $\mathbb{E}$ is finite, then using the relation in (A.4), we can see that $\lim_{x\uparrow\infty} v'_{n,k}(i,x) = (I - \alpha^k P^k)(I - \alpha P)^{-1} h(i)$ and $\lim_{x\downarrow-\infty} v'_{n,k}(i,x) = -(I - \alpha^k P^k)(I - \alpha P)^{-1} p(i)$ for all i, where $k = N - n$.

3.1.5 Comparison of base-stock levels

By above analysis, we show that base-stock policy is still optimal for inventory problems with random supply in a random environment; moreover, we show that base-stock level depends on the state of the environment. In this section, we further question the effect of random availability on base-stock levels. In other words, we compare the base-stock levels for inventory problems with random capacity only in a random environment and base-stock levels for our model.

As stated earlier, Erdem and Özekici (2002) analyze inventory models with random capacity and show that base-stock policy is optimal for these kinds of inventory problems. Therefore, we use base-stock levels in their study for comparison purposes. Let $\bar{S}_n^i$ be the optimal order-up-to level for state i when there is perfect availability; but, the production capacity is random. In addition, for technical reasons, we require one further assumption on demand distribution in this section. We assume that demand distribution M_i is strictly increasing so that $m_i(z) > 0$ for all $z \geq 0$. Then, by (3.6), L is a strictly convex function.

From our analysis in Section 3.1.2, we know that random capacity and/or random availability have no effect on the base-stock levels in single-period. Erdem and Özekici (2002) also showed that randomness in capacity has no effect on base-stock levels in single-period. This clearly implies that base-stock levels for our model and for their model are the same in single-period. In other words, letting $\bar{S}_0^i$ denote the base-stock level for inventory problems with random capacity, we have

$$\bar{S}_0^i = S_0^i \tag{3.51}$$

for all i.

First of all, assume that the transporter is always available so that the retailer receives all of the produced quantity. Since there is only one period, ordering more results in excess inventory when the producer's capacity is already sufficient to produce $\bar{S}_0^i - x$. This means an increase in the holding cost. Moreover, if the capacity of the producer is not enough to produce $\bar{S}_0^i - x$ already, ordering more than $\bar{S}_0^i - x$ is still illogical since receiving more than $\bar{S}_0^i - x$ is impossible. In addition, when there is only one period to plan for and the transporter is always available, our model reduces to the model considered and solved by Erdem and Özekici (2002). Then, this implies that $\bar{S}_0^i$ is the optimal order-up-to level for our model in single period when the transporter is always available. Therefore, ordering less is also illogical.

Next suppose that the transporter is unavailable so that the retailer receives nothing. In this case, there is no difference between ordering more or less than $\bar{S}_0^i - x$ since the retailer does not receive anything. As a result, when there is only one period to plan for, it is intuitively understandable that the optimal base-stock level is exactly the same as when there is no random availability. However, this won't be the case when the planning horizon consists of multiple periods. The comparison of base-stock levels in multiple and infinite period cases will be conducted in the following two subsections.

Base-stock levels in multi-period model

By Erdem and Özekici (2002), we know that environment-dependent base-stock policy is still optimal in multi-period when there is only random capacity. Define $\bar{S}_n^i$ to be the base-stock level in period n for random capacity only. Again, from Erdem and Özekici (2002), we know that $\bar{S}_n^i$ satisfies

$$c_i + L'\left(i, \bar{S}_n^i\right) + \alpha \sum_{j \in \mathbb{E}} P\left(i, j\right) E_D^i \left[\bar{v}_{n+1}'\left(j, \bar{S}_n^i - D\right)\right] = 0 \tag{3.52}$$

where

$$\bar{v}_n\left(i, x\right) = \begin{cases} \int_0^{\bar{S}_n^i - x} G_n(i, x+z) dF_i\left(z\right) + G_n(i, \bar{S}_n^i)\left[1 - F_i\left(\bar{S}_n^i - x\right)\right] & x \le \bar{S}_n^i \\ \quad -c_i x & \\ L\left(i, x\right) + \alpha \sum_{j \in \mathbb{E}} P\left(i, j\right) E_D^i\left[\bar{v}_{n+1}\left(j, x - D\right)\right] & x > \bar{S}_n^i \end{cases} \tag{3.53}$$

with $G_n(i, x)$ given in (3.18) and v_n is replaced by $\bar{v}_n$. Moreover, $\bar{v}_n$ is convex.

The following theorem states that the base-stock level for inventory problems with random capacity and availability is always greater than or equal to base-stock level for inventory problems with random capacity only.

Theorem 6 *In the $N-$period model,*

$$\bar{S}_n^i \le S_n^i \tag{3.54}$$

for all n and i. Furthermore,

$$v_n'\left(i, x\right) \le \bar{v}_n'\left(i, x\right) \tag{3.55}$$

for all n, i and x.

Proof. The proof proceeds by induction. First, we show that the induction hypothesis is true for $n = N-1$. We know from Section 3.1.2 that $\bar{S}^i_{N-1} = S^i_{N-1}$ for all i. Therefore, (3.54) is valid for $n = N-1$. The first derivative of $\bar{v}_{N-1}(i,x)$ with respect to x for $x \leq \bar{S}^i_{N-1} = S^i_{N-1}$ is

$$\bar{v}'_{N-1}(i,x) = \int_0^{\bar{S}^i_{N-1}-x} L'(i,x+z)\,dF_i(z) - c_i\left[1 - F_i\left(\bar{S}^i_{N-1} - x\right)\right]$$

for all i, and the first derivative of $v_{N-1}(i,x)$ with respect to x for $x \leq \bar{S}^i_{N-1} = S^i_{N-1}$ is given in (3.12) where $N = 1$. Then, for $x \leq \bar{S}^i_{N-1} = S^i_{N-1}$

$$\begin{aligned} v'_{N-1}(i,x) - \bar{v}'_{N-1}(i,x) &= (1-u_i)\Bigg(-\int_0^{\bar{S}^i_{N-1}-x} L'(i,x+z)\,dF_i(z) + L'(i,x) \\ &\qquad + c_i\left[1 - F_i\left(\bar{S}^i_{N-1} - x\right)\right]\Bigg) \\ &\leq (1-u_i)\left(c_i + L'(i,x)\right)\left[1 - F_i\left(\bar{S}^i_{N-1} - x\right)\right]. \end{aligned} \quad (3.56)$$

The first inequality follows from the fact that L is strictly convex so that $-L'(i,x+z) < -L'(i,x)$ for $z > 0$. Notice that $\bar{S}^i_{N-1} = S^i_{N-1}$ satisfies (3.11). Then, it follows that $c_i + L'(i,x) \leq 0$ for $x \leq \bar{S}^i_{N-1} = S^i_{N-1}$; therefore, right-hand side of the inequality in (3.56) is nonpositive for $x \leq \bar{S}^i_{N-1} = S^i_{N-1}$. This implies that $v'_{N-1}(i,x) \leq \bar{v}'_{N-1}(i,x)$ for $x \leq \bar{S}^i_{N-1} = S^i_{N-1}$. Moreover, the derivative of $\bar{v}_{N-1}(i,x)$ and $v_{N-1}(i,x)$ with respect to x for $x > \bar{S}^i_{N-1} = S^i_{N-1}$ is exactly the same, where

$$v'_{N-1}(i,x) = \bar{v}'_{N-1}(i,x) = L'(i,x)$$

for all i. Therefore, (3.55) is valid for $n = N-1$.

Now suppose that the hypothesis is true for $n+1, n+2, ..., N-1$ so that $\bar{S}^j_{n+1} \leq S^j_{n+1}$ and $v'_{n+1}(j,x) \leq \bar{v}'_{n+1}(j,x)$ for all j and x. Next, we prove that it is also valid for n. First of all, we show that $\bar{S}^i_n \leq S^i_n$. Assume that the converse is true so that $\bar{S}^i_n > S^i_n$. It follows that

$$\alpha \sum_{j\in\mathbb{E}} P(i,j) E^i_D[v'_{n+1}(j, S^i_n - D)] \leq \alpha \sum_{j\in\mathbb{E}} P(i,j) E^i_D[\bar{v}'_{n+1}(j, S^i_n - D)] \quad (3.57)$$

for all i since, by the induction hypothesis, $v'_{n+1}(j,x) \leq \bar{v}'_{n+1}(j,x)$ for all j and x. Then, it follows from (3.57) and (3.23) that

$$\begin{aligned} 0 &\leq c_i + L'\left(i, S_n^i\right) + \alpha \sum_{j\in\mathbb{E}} P(i,j) E_D^i[\bar{v}'_{n+1}(j, S_n^i - D)] \\ &< c_i + L'\left(i, \bar{S}_n^i\right) + \alpha \sum_{j\in\mathbb{E}} P(i,j) E_D^i[\bar{v}'_{n+1}(j, \bar{S}_n^i - D)]. \end{aligned} \tag{3.58}$$

The second inequality above comes from the assumptions that $\bar{S}_n^i > S_n^i$ and L is strictly convex. However, (3.58) is a clear contradiction to the fact that $\bar{S}_n^i$ satisfies (3.52). Hence, (3.54) is valid. Finally, we now show that $v'_n(i,x) \leq \bar{v}'_n(i,x)$ for all i and x. Clearly there are three cases to be analyzed: $x \leq \bar{S}_n^i \leq S_n^i$, $\bar{S}_n^i < x \leq S_n^i$ and $\bar{S}_n^i \leq S_n^i < x$.

(i) $x \leq \bar{S}_n^i \leq S_n^i$: By (3.53), the derivative of $\bar{v}_n(i,x)$ for $x \leq \bar{S}_n^i$ is

$$\begin{aligned} \bar{v}'_n(i,x) &= \int_0^{\bar{S}_n^i - x} \left(c_i + L'(i, x+z) + \alpha \sum_{j\in\mathbb{E}} P(i,j) E_D^i[\bar{v}'_{n+1}(j, x+z-D)] \right) dF_i(z) \\ &\quad -c_i \\ &\geq \int_0^{\bar{S}_n^i - x} \left(c_i + L'(i, x+z) + \alpha \sum_{j\in\mathbb{E}} P(i,j) E_D^i[v'_{n+1}(j, x+z-D)] \right) dF_i(z) \\ &\quad -c_i \\ &= \int_0^{\bar{S}_n^i - x} G'_n(i, x+z)\, dF_i(z) - c_i \end{aligned}$$

where G'_n is exactly the same as in (3.26). The inequality above follows from assumption that $v'_{n+1}(j,x) \leq \bar{v}'_{n+1}(j,x)$ for all j and x. Moreover, $v'_n(i,x)$ is given in (3.24). Then, for all i and $x \leq \bar{S}_n^i \leq S_n^i$,

$$\begin{aligned} v'_n(i,x) - \bar{v}'_n(i,x) &\leq u_i \int_0^{S_n^i - x} G'_n(i,x+z) dF_i(z) + (1-u_i) G'_n(i,x) \\ &\quad - \int_0^{\bar{S}_n^i - x} G'_n(i,x+z) dF_i(z) \\ &\leq u_i \int_0^{S_n^i - x} G'_n(i,x+z) dF_i(z) + (1-u_i) G'_n(i,x) \\ &\quad - \int_0^{S_n^i - x} G'_n(i,x+z) dF_i(z) \\ &= (1-u_i) \left(- \int_0^{S_n^i - x} G'_n(i,x+z) dF_i(z) + G'_n(i,x) \right) \\ &\leq (1-u_i) \left(- \int_0^{S_n^i - x} G'_n(i,x) dF_i(z) + G'_n(i,x) \right) \\ &= (1-u_i) G'_n(i,x) \left[1 - F_i\left(S_n^i - x\right) \right] \end{aligned} \tag{3.59}$$

Second inequality comes from the fact that, by (3.23), $G_n'(i,x) \le 0$ for all $x \le S_n^i$ and $\bar{S}_n^i \le S_n^i$. Moreover, the third inequality follows from the fact that $G_n'(i,x) \le G_n'(i,x+z)$ for $z > 0$ and for all $x \le \bar{S}_n^i \le S_n^i$, since L and v_{n+1} are convex. Finally, notice that (3.59) is nonpositive since $G_n'(i,x) \le 0$ for all $x \le S_n^i$. Therefore, $v_n'(i,x) - \bar{v}_n'(i,x) \le 0$ so that (3.55) for all i and $x \le \bar{S}_n^i \le S_n^i$.

(ii) $\bar{S}_n^i < x \le S_n^i$: From (3.24),

$$\begin{aligned} v_n'(i,\bar{S}_n^i) &= u_i \int_0^{S_n^i - \bar{S}_n^i} G_n'(i,\bar{S}_n^i + z) dF_i(z) + (1-u_i) G_n'\left(i,\bar{S}_n^i\right) - c_i \\ &< u_i \int_0^{S_n^i - \bar{S}_n^i} G_n'(i,S_n^i) dF_i(z) + (1-u_i) G_n'\left(i,S_n^i\right) - c_i \\ &= -c_i \end{aligned} \tag{3.60}$$

The inequality follows from the fact that $G_n'(i,x)$ is increasing and $\bar{S}_n^i < S_n^i$. And, the last equality comes from that, by (3.23), $G_n'(i,S_n^i) = 0$ for all i. Then, (3.60) implies that $v_n'(i,\bar{S}_n^i) < -c_i$. In addition, by (3.30) and (3.31), we know that $v_n'(i,S_n^i) = -c_i$. Moreover, $v_n'(i,x)$ is increasing since L and v_{n+1} are convex. This clearly implies $v_n'(i,x) \le -c_i$ for all i and x in $\left(\bar{S}_n^i, S_n^i\right]$. Furthermore, we know that $\bar{v}_n'(i,\bar{S}_n^i) = -c_i$ and $\bar{v}_n'(i,x)$ is increasing since L and $\bar{v}_{n+1}$ are convex. Hence, $\bar{v}_n'(i,x) \ge -c_i$ for all i and x in $\left(\bar{S}_n^i, S_n^i\right]$. As a result, $v_n'(i,x) \le \bar{v}_n'(i,x)$ so that (3.55) is valid for all i and x in $\left(\bar{S}_n^i, S_n^i\right]$.

(iii) $\bar{S}_n^i \le S_n^i < x$: From (3.20) and (3.53),

$$v_n'(i,x) - \bar{v}_n'(i,x) = \alpha \sum_{j \in \mathbb{E}} P(i,j) \left(E_D^i[v_{n+1}'(j,x-D)] - E_D^i[\bar{v}_{n+1}'(j,x-D)]\right)$$

for all i and $x > S_n^i \ge \bar{S}_n^i$. Notice that $v_{n+1}'(j,x) \le \bar{v}_{n+1}'(j,x)$ for all j and x by assumption.

Therefore, $v_n'(i,x) - \bar{v}_n'(i,x) \le 0$ so that (3.55) is valid for all i and $x > S_n^i \ge \bar{S}_n^i$. This completes our proof. ■

Theorem 6 states that the retailer will order more by increasing the base-stock level if the transporter is randomly available. If the transporter is always available, it is optimal to order $\bar{S}_n^i - x$ if the current inventory level in period n is $x \le \bar{S}_n^i$; moreover, $E\left[\min\left(\bar{S}_n^i - x, C_n\right)\right]$ is the expected amount received by the retailer. But, if the transporter is randomly available, there is a positive probability of receiving nothing; therefore, the expected amount received by the retailer is $u_i E\left[\min\left(\bar{S}_n^i - x, C_n\right)\right]$. This clearly means that expected amount received by the retailer decreases with random availability of the transporter. If we continue to order as the transporter is always available but the capacity of the supplier is random, it is highly probable that we receive less than the amount produced by the producer. Clearly, this leads more shortages; as a result, total expected shortage cost increases.

But, if we increase order-up-to level to $S_n^i > \bar{S}_n^i$ so that the order is more than the certainly available transporter case, the expected amount received by the retailer is $u_i E\left[\min\left(S_n^i - x, C_n\right)\right]$. Then, this will reduce the expected shortages so that total cost of shortages will be less. In addition, ordering more also increases the expected inventory leftover. However, this inventory leftover can be used to reduce shortages in later periods due to random availability. Hence, ordered quantity must be increased when the transporter is randomly available.

Base-stock levels in infinite-period model

Erdem and Özekici (2002) also show that the optimal policy in infinite period setting for inventory models with random capacity only in a random environment is still environment-dependent base-stock policy. Let us denote the base-stock level in this model by $\bar{S}^i$. Then, $\bar{S}^i$ satisfies

$$c_i + L'(i, S^i) + \alpha \sum_{j \in \mathbb{E}} P(i,j) E_D^i[\bar{v}'(j, S^i - D)] = 0 \tag{3.61}$$

and the optimal cost is

$$\bar{v}\left(i,x\right) = \begin{cases} \int_0^{\bar{S}^i - x} G(i, x+z) dF_i\left(z\right) + G(i, \bar{S}^i)\left[1 - F_i\left(\bar{S}^i - x\right)\right] - c_i x & x \leq \bar{S}^i \\ L\left(i,x\right) + \alpha \sum_{j \in \mathbb{E}} P\left(i,j\right) E_D^i\left[\bar{v}\left(j, x - D\right)\right] & x > \bar{S}^i \end{cases} \tag{3.62}$$

with $G(i,x)$ is given in (3.34) where y is replaced by x and v is replaced by $\bar{v}$. Moreover, $\bar{v}$ is convex.

The following theorem states that the base-stock level for inventory problems with random capacity and availability is always greater than or equal to base-stock level for inventory problems with random capacity only.

Theorem 7 *In the infinite-period model,*

$$\bar{S}^i \leq S^i \tag{3.63}$$

for all i.

Proof. Here, we again use $v_{n,k}$ in place of v_n. Then, by Theorem 4, we know that $v(i,x) = \lim_{k\uparrow\infty} v_{0,k}(i,x)$ and $\bar{v}(i,x) = \lim_{k\uparrow\infty} \bar{v}_{0,k}(i,x)$ for all i and x. Then, this clearly implies that $v'(i,x) = \lim_{k\uparrow\infty} v'_{0,k}(i,x)$ and $\bar{v}'(i,x) = \lim_{k\uparrow\infty} \bar{v}'_{0,k}(i,x)$. Moreover, by Theorem 6, we know that $v'_{0,k}(i,x) \leq \bar{v}'_{0,k}(i,x)$ for all i and x.

To prove that $\bar{S}^i \leq S^i$, assume that the converse is true so that $\bar{S}^i > S^i$. Then, by (3.49),

$$\begin{aligned} 0 &= c_i + L'\left(i, S^i\right) + \alpha \sum_{j\in\mathbb{E}} P(i,j) E_D^i[v'(j, S^i - D)] \\ &\leq c_i + L'\left(i, S^i\right) + \alpha \sum_{j\in\mathbb{E}} P(i,j) E_D^i[\bar{v}'(j, S^i - D)] \\ &< c_i + L'\left(i, \bar{S}^i\right) + \alpha \sum_{j\in\mathbb{E}} P(i,j) E_D^i[\bar{v}'(j, \bar{S}^i - D)]. \end{aligned} \tag{3.64}$$

The first inequality in (3.64) follows from the fact that $v' < \bar{v}'$. And, the last inequality comes from the fact that $\bar{S}^i > S^i$ and L is strictly convex and $\bar{v}$ is convex so that $L'\left(i, S^i\right) < L'\left(i, \bar{S}^i\right)$ and $\bar{v}'(j, S^i - D) \leq \bar{v}'(j, \bar{S}^i - D)$. However, (3.64) is a clear contradiction to the fact that $\bar{S}^i$ satisfies (3.61). Therefore, $\bar{S}^i \leq S^i$ for all i. This completes our proof. ■

3.2 Fixed-cost model with random capacity

Özekici and Parlar (1999) showed that an environment-dependent (s, S) policy is optimal for inventory problems with randomly available suppliers and fixed ordering cost in a random environment. In other words, when the supplier produces either all of the ordered quantity or none of the ordered quantity, an environment-dependent (s, S) policy is optimal. Here, we question whether an environment-dependent (s, S) policy is still optimal when the supplier is always available but has random capacity.

In this section, we study the inventory model with fixed ordering cost and random capacity in a random environment. Here, we assume that $U_n = 1$ for all $n \geq 0$ so that $u_i = 1$ for all i and there is no loss during transportation. Moreover, we assume that $K_i > 0$

so that there is fixed cost of ordering. Using notations and assumptions in Section 3.1.1, we formulate the problem and present a counter example showing that (s, S) policy is not necessarily optimal in multiple periods.

Here we assume that the length of the planning horizon is N and we also assume that the expected optimal discounted cost at the beginning of period $N+1$, $v_N(j, x_N) = 0$ for all j and x_N . Assuming state of environment and inventory level at time n are i and x respectively, minimum cost function satisfies

$$v_n(i,x) = \min_{y \geq x} \{K_i \delta(y-x) + J_n(i,x,y)\} \tag{3.65}$$

where y is order-up-to level in period $n+1$, $\delta(z)$ is the indicator function which is equal to 1 only if $z > 0$ and 0 otherwise, and

$$J_n(i,x,y) = \int_0^{y-x} G_n(i,x+z) dF_i(z) + G_n(i,y)[1 - F_i(y-x)] - c_i x \tag{3.66}$$

$$G_n(i,y) = c_i y + L(i,y) + \alpha \sum_{j \in \mathbb{E}} P(i,j) E_D^i [v_{n+1}(j, y-D)] \tag{3.67}$$

with L as given in (3.4).

The analysis of the problem in (3.65) is more difficult because of the complex structure of the cost function. Hence, we study the problem in multi-period setting via numerical examples and show that the environment dependent (s, S) policy is not necessarily optimal.

Remark 1 *The optimal ordering policy for the N-period model with fixed ordering cost is not necessarily an environment-dependent (s,S) policy.*

Example 1 *We assume that the environment has two states so that $\mathbb{E} = \{1, 2\}$, where state 1 represents a "good" environment and state 2 represents a "bad" environment. Moreover, we assume that environment changes state according to the transition matrix*

$$P = \begin{bmatrix} 0.9 & 0.1 \\ 0.1 & 0.9 \end{bmatrix}.$$

All distributions and parameters depend on the state of the environment.

Demand distribution is assumed to be Poisson with mean $\lambda = [10, 5]$ and it is truncated at $[22, 13]$. In other words, we assume that the maximum demand can be at most 22 in the good state and 13 in the bad state. In addition, capacity distribution is assumed to be geometric

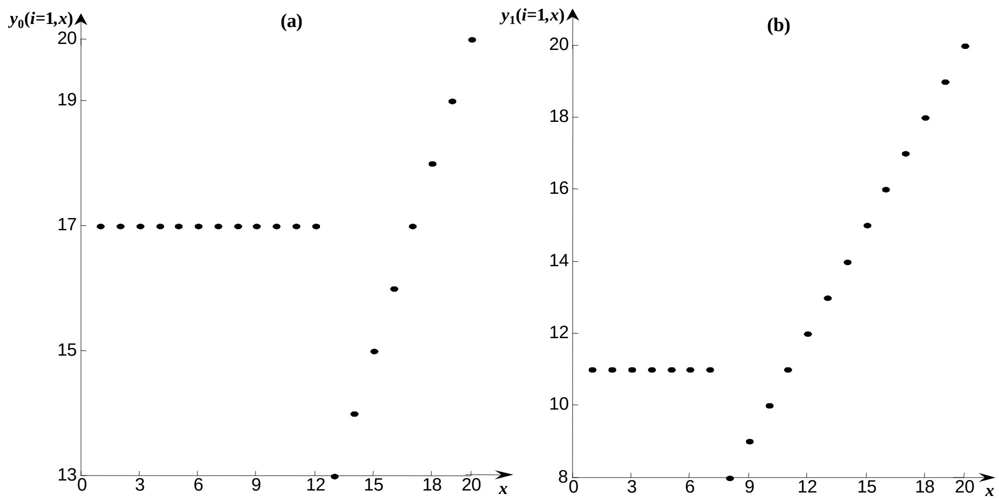

Figure 3.1: Optimal order-up-to levels (if initial state is good)

with probability of failure $q = [0.9, 0.8]$*. Demand and capacity distributions are Poisson and geometric in both states with the given parameters. We assume that both* λ *and* q *change by the state of the environment so that we ensure nonstationarity of probability distributions. We have computed the optimal policy with* $\alpha = 0.95$*,* $c = [1, 2], h = [3, 6], p = [9, 18]$ *and* $K = [10, 20]$*. Moreover, everything gets worse in bad environmental state, i.e., cost parameters double, demand and capacity decrease.*

For $N = 2$ *and given parameters, we solve the minimization problem in (3.65) via backward recursive dynamic programming. As the solution algorithm, we used value iteration algorithm. The value iteration algorithm is coded in MATLAB and is run for* $N = 2$*. The results are shown in Figure 3.1 and Figure 3.2.*

From Figure 3.1, it is obvious that an (s, S) *policy is optimal for both periods if the initial state of environment is good. Figure 3.1 (b) shows that if the state of environment in the last period is good, it is optimal to order up to* 11 *if the inventory level in the last period is less than or equal to* 7 *so that* $s_1^1 = 7$ *and* $S_1^1 = 11$*. In addition, Figure 3.1 (a) shows that when the state of environment in the last period is good, it is optimal to order up to* 17 *if the initial inventory level is less than or equal to* 12 *so that* $s_0^1 = 12$ *and* $S_0^1 = 17$*.*

However, the optimality of (s, S) *is violated if the initial state of the environment is bad*

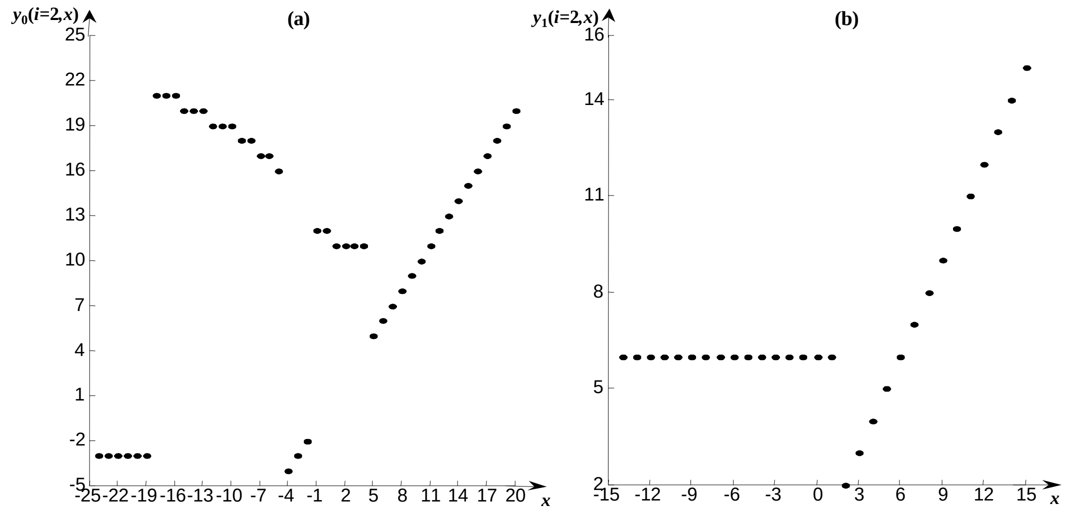

Figure 3.2: Optimal order-up-to levels (if initial state is bad)

as shown in Figure 3.2. From Figure 3.2 (b), it is clear that (s, S) *policy is still optimal in the last period if the state of environment in the last period is bad. And it is optimal to order up to* 6 *in the last period when the state of environment is bad and inventory level is less than or equal to* 1 *so that* $s_1^2 = 1$ *and* $S_1^2 = 6$*. But* (s, S) *policy is no more optimal for the first period if the initial state of environment is bad as shown in Figure 3.2 (a) where there are multiple* s_0^2 *and multiple* S_0^2*. This result is similar to the result obtained by Gallego and Wolf (2000).*

Another observation from Figure 3.1 (b) and Figure 3.2 (b) is that s *and* S *values are environment-dependent. For example,* $s_1^1 = 7$ *and* $s_1^2 = 1$*. This counter example shows us that* (s, S) *is not an optimal policy structure for inventory problems with fixed ordering cost, random capacity and random demand in a random environment. Moreover, it is obvious from Figure 3.2 (a) that there does not exist a fairly simple optimal policy structure for those problems.*

3.3 Summary of results

The aim of our study in Section 3.1 is to characterize the optimal policy structure for inventory problems with random capacity and random availability in a random environment. The state of environment follows a time-homogenous Markov chain which affects all the costs and the distribution of demand and supply. The results in single, multiple and infinite planning periods show that base-stock policy is optimal; however, order-up-to levels depend on the state of the environment. Our results are similar to several studies in the literature where demand, supply and the environment are random. For example, Özekici and Parlar (1999) prove that an "environment-dependent" base-stock policy is optimal for a similar problem when there is randomly available supplier, i.e., proportional yield is either 0 or 1, in a random environment. In addition, Erdem and Özekici (2002) show that the environment-dependent base-stock policy is still optimal when the supplier is always available but has random capacity. In all of these models, the ordering cost is linear in quantity ordered and there is no fixed cost of ordering.

Our study considers an inventory system with a producer having random capacity and a transporter which is randomly available. Therefore, our study is a combination of both Özekici and Parlar (1999) and Erdem and Özekici (2002). First, we try to find out how randomly available transporter in a random environment affects the optimal policy structure. As a result of our analysis, we see that environment-dependent base-stock policy is still optimal. Erdem and Özekici (2002) show that environment-dependent base-stock policy is optimal for random capacity inventory models in a random environment. We introduce randomly available transporter to this model and see that an environment-dependent base-stock policy is still optimal.

Next, we analyze how the randomly available transporter affects base-stock levels. For this purpose, we compare the base-stock levels in Erdem and Özekici (2002) and in this study. As a result of our analysis, we see that randomly available transporter has no effect on base-stock levels in single period. However, base-stock levels in multiple and infinite planning periods increase because of randomly available transporter. Through a similar analysis, Erdem and Özekici (2002) show that base-stock levels in multiple and infinite planning periods increase compared to infinite capacity case; however, they stay the same in a single period. Therefore, we can claim that base-stock levels in multiple and infinite planning periods increase compared to infinite capacity case for inventory models with randomly available transporter and producer having random capacity in a random environment. However, base-stock levels in single planning period stay the same.

Here, we should note that the optimal policy may change due to the supply structure. If we assume that the producer has random capacity and the transporter delivers a random proportion (between 0 and 1) of the produced amount, then it is highly probable that environment-dependent inflated base-stock policy is optimal as in Wang and Gerchak (1996).

In Section 3.2, we extend our study in Section 3.1 by introducing fixed cost of ordering and assuming that the transporter is always available. In general, the optimal policy structure for inventory models with fixed ordering cost is an (s, S) policy. Therefore, it may be expected that the optimal policy would be environment-dependent (s, S) policy when there is fixed ordering cost. However, we show via a counter example that environment-dependent (s, S) policy is not necessarily optimal for inventory problems with fixed ordering cost and random capacity in a random environment. This result is consistent with similar studies in the literature. For example, Shaoxiang and Lambrecht (1996) show that modified (s, S) policy is not optimal even in a stationary environment when the capacity of the producer is finite. Since introduction of fixed ordering cost into our model is more general compared to the case in Shaoxiang and Lambrecht (1996), we can expect that environment-dependent (s, S) policy is not optimal. Actually, we show this through a numerical example. Shaoxiang and Lambrecht (1996) partially characterize the optimal solution for inventory problems with fixed capacity and fixed ordering cost in a stationary environment. They show that there is a point x below which it is optimal to order and there is another point y above which it is optimal not to order. However, they cannot define what to do in between x and y. They prefer to call this policy structure as $x - y$ band. In a related paper, Gallego and Wolf (2000) define possible actions in between x and y. However, this is still a partial characterization of optimal policy structure. Therefore, further research can aim to make a full characterization of the optimal policy in this case.

Chapter 4

INVENTORY MODELS WITH IMPERFECT INFORMATION

In the previous part, our main assumption is that the IM has full information about the inventory system. For example, we assume that the IM knows the state of environment so that availability probability, demand and capacity distributions are all known. In addition, we further assume that the IM observes the inventory level perfectly. However, it is not possible to observe the real environmental process as well as the inventory level in many situations.

First of all, the real environmental process affecting the demand and supply is not always perfectly observed. As stated in Chapter 1, observations regarding the true environmental state is not perfect because of the limited data that is available. As a result, the state that we observe using available data may not be the real one. However, it definitely gives some information regarding the true state. In such cases, we can make use of our observations to make inferences about the real environmental process. Let us make the concept of partial observation clear by considering our previous models. In our previous models, we represent the random environment by a Markov chain and assume that costs, demand and supply depend on states of this Markov chain. In a sense, via demand and supply distributions, we incorporate the random environment into our model. As for any distribution function, observed data is used to compute supply and demand distributions that are specific to different environmental states. Through data, we find distribution type as well as its parameters. However, computed distributions and their parameters are only estimates about the real distribution function. If we recall the correspondence between distribution types and environmental states, we can think computed distribution as observed state and the real distribution as true state. Most of the time, computed distributions (observed environmental states) are not true, but they give some information regarding the real distributions (true environmental state). Therefore, full observation or perfect information assumption is erroneous and misleading. To create more general and realistic models, we must incorporate into our models the fact that our observations are partial. This complicates the formulation of the problem; however, it also enables IM to take more flexible and adaptive decisions.

In inventory models with perfect information, we also assume that the inventory level is fully observed. However, there are many real life situations where this is not possible. For example, inventories are frequently misplaced and/or stolen. Moreover, it is not always possible to notice these misplaced or stolen inventories. Therefore, they are unobservable for the IM. In such cases, real inventory is only partially observed through the observed inventory level. In addition, spoilage and production yield may also cause the difference between observed and real inventory levels. Interested readers are referred to Bensoussan et al. (2004, 2005 b) for a detailed explanation of the factors leading partial observation of inventory level.

In this part of the thesis, we analyze five different models. The first model analyzes exactly the same problem as the first model in the first part. However, we now assume that the random environment is partially observed whereas the inventory level is fully observed. Therefore, there is another process which gives partial information about the Markov chain representing the real environment. In addition, the first model assumes that demand, producer's capacity and transporter availability are modulated by the real environment whereas all costs are dependent on the observed environment. Then, in the second model, we analyze exactly the same problem as in the first model. However, we now assume that the producer's capacity is modulated by the observed environment. Notice that the state of our system in the first two models is assumed to consist of all observations until current time and the current inventory level. As the length of the planning horizon gets longer, the state space of our system increases without bounds. Therefore, the infinite period problem cannot be analyzed using this setting.

Instead, we change our formulation in the third model and assume that our system state is the conditional distribution of the true environmental state and the inventory level at the current time. This change enables us to analyze single, multiple and infinite planning period problems. There are two related studies in this area. In the first study, Treharne and Sox (2002) analyze inventory problems with partially observed demand in finite-horizon. Treharne and Sox (2002) and our third model are similar since we both assume that the demand environment is random and partially observed. However, our model is different in several respects. First of all, we assume that supply and all cost parameters are modulated by random environment. In addition, we study infinite-horizon problem; however, they consider finite-horizon problem only. Finally, they assume that there is a constant lead time in delivery while we assume that the delivery is instantaneous. Second related study is Bensoussan et al. (2005 a) which analyze three different models: filtered newsvendor, zero balance walk and information delay. Filtered newsvendor and our third model are similar since we both assume that the demand environment is random and partially observed. However, they are different in several respects. Firstly, we assume that the inventory level is perfectly observable whereas Bensoussan et al. (2005 a) assume that it is only partially observable in all three models. Secondly, they do not consider supply side as Treharne and Sox (2002) while we incorporate the randomness in supply into our model. Finally, we assume that all cost parameters are environment-dependent while Bensoussan et al. (2005 a) consider stationary cost parameters.

In the fourth model, we assume that the supplier is either available or unavailable at any particular instant; however, its capacity is infinite. In this model, we assume that demand is dependent on the true environmental state while availability process of supplier and all cost parameters are modulated by the observed environment. Here, we still use the same formulation as in the third model so that the state of our system is the conditional distribution of real environmental state and current inventory level. Moreover, we also assume that the fixed ordering cost is positive. Our fourth model is very much related with Özekici and Parlar (1999). However, we extend it by assuming that the random environment is partially observable.

Finally, in the last model, we analyze inventory problems with finite capacity and random proportional yield in a partially-observed random environment. In this model, we also assume that the system state is the current inventory level and the conditional distribution of real environment. Our last model is very similar to Gallego and Hu (2004) since we both assume that demand is random and environment dependent. Moreover, we both assume that the supply is random due to fixed supplier capacity and proportional random yield, and it is environment-dependent. However, we assume that there is only one environment modulating both demand and supply whereas they assume that supply and demand environments are different. In addition, they assume that demand and supply environments are fully observed whereas we relax this assumption and assume that random environment is only partially observed. Furthermore, we assume that all costs are environment dependent whereas their cost parameters are stationary.

This part is organized as follows. In the next section, we present our hidden Markov model, notation and assumptions. Then, in Section 4.2, we show via a numerical example that state-dependent base-stock policy is not necessarily optimal for inventory problems when the capacity of supplier is modulated by the real environment. However, in Section 4.3, we relax our assumption and assume that the capacity is modulated by the observed environment. In this section, we analyze single and multiple planning period problems only. In order to analyze infinite planning period problem, we change our formulation and use sufficient statistics in the remaining part of the thesis. We show in Section 4.4 that conditional distribution of true environment is a sufficient statistic for the past history of observed environment. Then, in Section 4.5, we reformulate our problem in Section 4.3 using sufficient statistics. After that, using sufficient statistic formulation, we study inventory problems with unreliable suppliers and fixed ordering cost in Section 4.6. In Section 4.7, we consider inventory problems with fixed capacity and random proportional yield in a partially observed random environment. Finally, in Section 4.8, we summarize main results of our analyses in the second part.

4.1 Hidden Markov model (HMM)

The main focus of this section is on imperfect information available to determine optimal inventory policy in a stochastically changing environment. We let Z_n denote the state of the stochastic environment at time n, and assume that $Z = \{Z_n;\ n = 0, 1, 2, 3, \ldots\}$ is a Markov chain with some time-dependent transition matrix $Q_n(a, b) = P\left[Z_{n+1} = b | Z_n = a\right]$ and finite state space $\mathbb{F} = \{a, b, c, ...\}$. The states of the environment are not observable and the Markov chain Z is hidden. This implies that information available to IMs is not perfect. The imperfect observations on the state of the environment are given by an observation process $Y = \{Y_n;\ n = 0, 1, 2, 3, \ldots\}$ with some finite state space $\mathbb{E} = \{i,\ j, ...\}$ where Y_n is the information available at time n. The environment evolves according to the unobserved process Z whose states depend on various economic and other factors; however, IMs can only see the observed process Y. Hence, they base their decisions on what they observe.

It is clear that the observed process Y is not necessarily a Markov chain and the state of the stochastic environment Z at any time depends on all of the past observations of Y. The relationship between the two processes Y and Z is made formal by enlarging the state spaces $\mathbb{E}$ and $\mathbb{F}$ so that $\mathbb{E}^{n+1} = \mathbb{E} \times \mathbb{E} \times \cdots \times \mathbb{E} = \{(i_0, i_2, \cdots, i_n) : i_m \in \mathbb{E}\}$ and $\mathbb{F}^{n+1} = \mathbb{F} \times \mathbb{F} \times \cdots \times \mathbb{F} = \{(a_0, a_1, a_2, \cdots, a_n) : a_m \in \mathbb{F}\}$. Then let $\bar{Y}_n = (Y_0, Y_1, Y_2, \cdots, Y_n)$ denote past observations of process Y until time n and $\bar{\imath}_n = (i_0, i_1, i_2, \cdots, i_n) \in \mathbb{E}^{n+1}$ denote the realizations of these observations. And let $\bar{Z}_n = (Z_0, Z_1, Z_2, \cdots, Z_n)$ denote the true state of the environment until time n and $\bar{a}_n = (a_0, a_1, a_2, \cdots, a_n) \in \mathbb{F}^{n+1}$ denote the realizations of this unobserved process Z. The information available to the IM at time n is $\bar{\imath}_n$ since he can only observe the process Y. We assume that the probabilistic evolution of Y depends purely on the state of Z such that

$$P\left[Y_n = i | Z_n = a\right] = E_n(a, i) \tag{4.1}$$

independent of all previous states of Z and Y in any period n. Borrowing the terminology that is commonly used in signal processing, the matrix E_n is often called the emission matrix. In signal processing, Z represents a process that emits signals such that if the nth signal is a, then it emits i with probability $E_n(a, i)$.

Simple probabilistic arguments give

$$\begin{aligned} O_n(\bar{\imath}_n, a) &= P\left[Z_n = a | \bar{Y}_n = \bar{\imath}_n\right] \\ &= \frac{\sum_{\bar{b}_{n-1} \in \mathbb{F}^n} P\left[Z_0 = b_0\right] E_0(b_0, i_0) Q_0(b_0, b_1) \cdots Q_{n-1}(b_{n-1}, a) E_n(a, i_n)}{\sum_{\bar{b}_n \in \mathbb{F}^{n+1}} P\left[Z_0 = b_0\right] E_0(b_0, i_0) Q_0(b_0, b_1) \cdots Q_{n-1}(b_{n-1}, b_n) E_n(b_n, i_n)} \end{aligned} \tag{4.2}$$

for $n \geq 1$, while

$$O_0(i_0, a) = P\left[Z_0 = a | Y_0 = i_0\right] = \frac{P\left[Z_0 = a\right] E_0(a, i_0)}{\sum_{b_0 \in \mathbb{F}} P\left[Z_0 = b_0\right] E_0(b_0, i_0)} \tag{4.3}$$

for $n = 0$. To simplify our notation, define

$$\begin{aligned} \Psi_n^k(a, j) &= P\left[Y_{n+k} = j | Z_n = a, \bar{Y}_n = \bar{\imath}_n\right] \\ &= P\left[Y_{n+k} = j | Z_n = a\right] \\ &= \sum_{b_{n+1}, \ldots, b_{n+k} \in \mathbb{F}} Q_n(a, b_{n+1}) \ldots Q_{n+l-1}(b_{n+k-1}, b_{n+k}) E_{n+l}(b_{n+k}, j). \end{aligned} \tag{4.4}$$

for $k \geq 1$. We use Ψ_n instead of Ψ_n^k when $k = 1$. The evolution of Y is now described probabilistically by

$$P_n^k(\bar{\imath}_n, j) = P\left[Y_{n+k} = j|\bar{Y}_n = \bar{\imath}_n\right] = \sum_{a\in\mathbb{F}} O_n(\bar{\imath}_n, a)\Psi_n^k\left(a, j\right). \tag{4.5}$$

And we use P_n instead of P_n^k when $k = 1$. Clearly, $P_n^0 = I$ so that

$$P_n^0(\bar{\imath}_n, j) = \begin{cases} 1 & i_n = j \\ 0 & i_n \neq j \end{cases}. \tag{4.6}$$

for all $\bar{\imath}_n$ and j. Furthermore,

$$\begin{aligned} P_n^{k+1}\left(\bar{\imath}_n, j\right) &= P\left[Y_{n+k+1} = j|\bar{Y}_n = \bar{\imath}_n\right] \\ &= \sum_{l\in\mathbb{E}} P\left[Y_{n+1} = l|\bar{Y}_n = \bar{\imath}_n\right] P\left[Y_{n+k+1} = j|\bar{Y}_n = \bar{\imath}_n, Y_{n+1} = l\right] \\ &= \sum_{l\in\mathbb{E}} P_n\left(\bar{\imath}_n, l\right) P_{n+1}^k(\left(\bar{\imath}_n, l\right), j). \end{aligned} \tag{4.7}$$

First equality follows from the definition of P_n^k in (4.5). After conducting simple probabilistic operations, we get second equality. The final equality follows from the definitions of P_n and P_{n+1}^k in (4.5).

Note that $\{P_n\}$ and $\{O_n\}$ matrices can easily be determined once the transition matrices $\{Q_n\}$, emission matrices $\{E_n\}$ and the initial distribution of the true state of the environment are known. It is also clear that the the next true state of the economy, as well as the next observation, both depend on all of the past observations. Moreover, we assume that $\{Q_n\}$and $\{E_n\}$ are time-dependent for single and multiple period analyses; however, they are time-homogenous for infinite period model so all time subscripts vanishes in infinite period analysis.

For a nonnegative function g, we note that

$$\begin{aligned} E\left[\sum_{k=0}^{m-1} \alpha^k g\left(Y_{n+k}\right) \middle| \bar{Y}_n = \bar{\imath}_n\right] &= \sum_{k=0}^{m-1} \alpha^k E\left[g\left(Y_{n+k}\right)\middle| \bar{Y}_n = \bar{\imath}_n\right] \\ &= \sum_{k=0}^{m-1} \alpha^k \sum_{j\in\mathbb{E}} P_n^k(\bar{\imath}_n, j) g\left(j\right) \\ &= \sum_{j\in\mathbb{E}} \left[\sum_{k=0}^{m-1} \alpha^n P_n^k(\bar{\imath}_n, j)\right] g\left(j\right) \\ &= \sum_{j\in\mathbb{E}} R_{n,m}^\alpha(\bar{\imath}_n, j) g\left(j\right) = R_{n,m}^\alpha g\left(\bar{\imath}_n\right) \end{aligned} \tag{4.8}$$

where we define $R^{\alpha}_{n,m}(\bar{\imath}_n, j) = \sum_{k=0}^{m-1} \alpha^k P^k_n(\bar{\imath}_n, j)$ for all $\bar{\imath}_n$ and j. If we consider $g(j)$ as the reward obtained when the observed state is j, then $R^{\alpha}_{n,m} g(\bar{\imath}_n)$ is the expected total discounted reward obtained during m transitions when all observations until time n is $\bar{\imath}_n$. Using a similar terminology as in Çınlar (1975), $R^{\alpha}_{n,m} g$ can be called $\alpha-$potential of g during first m transitions starting at time n. Similarly, $R^{\alpha}_{n,m}$ is called $\alpha-$potential matrix of Y during first m transitions starting at time n. Notice that Y is not a Markov chain; therefore, we cannot directly use results obtained by Çınlar (1975) regarding finite transition $\alpha-$potential matrices and functions.

For our analyses in the remaining sections, we need the relationship between total expected discounted reward from time n until the end of planning horizon and the total expected discounted reward from time $n+1$ until the end of the planning horizon. In other words, we need to find how $R^{\alpha}_{n,m} g$ is related to $R^{\alpha}_{n+1,m-1} g$. By (4.8), we know that

$$\begin{aligned}
R^{\alpha}_{n,m} g(\bar{\imath}_n) &= \sum_{j\in\mathbb{E}} R^{\alpha}_{n,m}(\bar{\imath}_n, k) g(k) \\
&= g(i_n) + \sum_{j\in\mathbb{E}} \left[\sum_{k=1}^{m-1} \alpha^k P^k_n(\bar{\imath}_n, j) \right] g(j) \\
&= g(i_n) + \sum_{j\in\mathbb{E}} \sum_{k=0}^{m-2} \alpha^{k+1} \left[\sum_{l\in\mathbb{E}} P_n(\bar{\imath}_n, l) P^k_{n+1}((\bar{\imath}_n, l), j) \right] g(j) \\
&= g(i_n) + \alpha \sum_{l\in\mathbb{E}} P_n(\bar{\imath}_n, l) \left[\sum_{j\in\mathbb{E}} \sum_{k=0}^{m-2} \alpha^k P^k_{n+1}((\bar{\imath}_n, l), j) g(j) \right] \\
&= g(i_n) + \alpha \sum_{j\in\mathbb{E}} P_n(\bar{\imath}_n, j) R^{\alpha}_{n+1,m-1} g(\bar{\imath}_n, j) \qquad (4.9)
\end{aligned}$$

for all $\bar{\imath}_n$. We get the second equality after we substituted $R^{\alpha}_{n,m}(\bar{\imath}_n, k)$. We then get the third equality by changing the index of inside summation in second line. Then using (4.7), we get the fourth equality. Finally, the last equality comes from the definition of $R^{\alpha}_{n+1,m-1} g$.

Let, as in Section 3.1.1, D_n and C_n denote the total demand and the random capacity of the supplier in period n, respectively. Here we assume that both demand and capacity of the producer are observed. However, their distributions depend on the unobserved state of environment. Hence, the demand process $D = \{D_n; n \geq 1\}$ and capacity process $C = \{C_n; n \geq 1\}$ are modulated by the Markov chain Z. Conditional cumulative distribution functions of demand and capacity are

$$M_a(z) = P[D_{n+1} \leq z \mid Z_n = a]$$

and

$$F_a(z) = P[C_{n+1} \leq z \mid Z_n = a]$$

for $a \in \mathbb{F}$ and $z \geq 0$, respectively. Moreover, we assume that they have probability density function m_a and f_a. Furthermore, we suppose that $P[C_{n+1} > z|Z_n = a] = 1 - F_a(z) > 0$ for all $z \geq 0$ which implies that the random capacity has no upper bound so that it is possible to receive all of ordered quantity.

As in Section 3.1.1, let U_n denote the proportion of produced amount which is received by the retailer in period n, where we again assume that either $U_n = 1$ or $U_n = 0$. As in the demand and capacity structure, we assume that availability of transporter at period n is observed, but the availability process $U = \{U_n;\, n \geq 1\}$ depends on unobserved environment so that

$$P[U_{n+1} = 1 \mid Z_n = a] = u_a$$

for some $0 \leq u_a \leq 1$ and all a. If $\sum_{a\in\mathbb{F}} u_a O_n(\bar{\imath}_n, a) = 0$, then $u_a = 0$ for all a; however, ordering is meaningless in such a case since nothing is received. In addition, if $\sum_{a\in\mathbb{F}} u_a O_n(\bar{\imath}_n, a) = 1$, then $u_a = 1$ for all a. But this is the case of total availability and is not the main concern in this study. Therefore, we assume that $0 < \sum_{a\in\mathbb{F}} u_a O_n(\bar{\imath}_n, a) < 1$ for all $\bar{\imath}_n$ and $n \geq 0$ which is not a restrictive assumption because of the reasons explained above.

Again, we let x_n denote the inventory level observed at time n. In each period, the information available to the retailer regarding the true state of the environment is the past observations of Y. Then information vector for the retailer at time n is $\bar{Y}_n = \bar{\imath}_n$. Given x_n and $\bar{\imath}_n$, the IM decides on the optimal order quantity, and this quantity brings the inventory level up to $y_n \geq x_n$. We assume that there is no delay in delivery so that an order is immediately delivered provided that the producer's capacity is sufficient and transporter is available. If the transporter is available, but the producer's capacity is not sufficient, then an amount equal to the capacity is delivered. However, if the transporter is not available, then nothing is received. Hence, amount received at time n is $U_{n+1} \min\{y_n - x_n, C_{n+1}\}$

and the supply is random. If demand in a period is not satisfied, then it is completely backlogged in that period and satisfied in the next period. Thus, the system equation for our problem is

$$X_{n+1} = X_n + U_{n+1} \min \{y(\bar{\imath}_n, X_n) - X_n, C_{n+1}\} - D_{n+1}$$

for $n \geq 0$. Note that $\{U_n\}$, $\{C_n\}$ and $\{D_n\}$ all depend on the unobserved process Z.

Obviously, all current cost parameters are observed by the IM. Therefore, we assume that all cost parameters depend on the observed environment Y. Given that the state of observed environment is i in period n, K_i is the fixed cost per order, c_i is the purchase cost per item, h_i is the holding cost per item per period, and p_i is the shortage cost per item per period. Both holding and shortage costs are incurred at the end of the period. Furthermore, we assume that $p_i > c_i$ and $c_i > 0$. Moreover, we assume that all cost parameters are finite so that they are bounded. Finally, we let α denote periodic discount factor and assume that $0 < \alpha < 1$.

4.2 Linear cost model with unobserved capacity

In this section, as in Section 3.1, we assume that $K_i = 0$ so that there is no fixed ordering cost. However, now, we assume that the random environment is only partially observable. Moreover, we assume that demand, capacity and availability processes are all modulated by the unobserved process Z while all cost parameters are modulated by the observed environmental process Y. In this section, we show via a numerical example that base-stock policy is not necessarily optimal. The counter example presented in this section shows that base-stock policy is not optimal even for the single-period problem.

We first give the formulation of the single-period problem. There is only one period to plan for so that $N = 1$; moreover, $v_1(\bar{\imath}_1, x_1) = 0$ for all $\bar{\imath}_1$ and x_1. Take $i_0 = i$ and $a_0 = a$. Assuming inventory level at the beginning of period is x, the single-period minimum cost function at the beginning of the first period satisfies

$$v_0(i, x) = \min\nolimits_{y \geq x} H_0(i, x, y) \qquad (4.10)$$

for all i and x, where y is the order-up-to level and

$$H_0(i,x,y) = \sum_{a\in\mathbb{F}} O_0(i,a)J_0(i,a,x,y) \tag{4.11}$$

$$\begin{aligned} J_0(i,a,x,y) &= u_a \int_0^{y-x} G_0(i,a,x+z)dF_a(z) + u_a G_0(i,a,y)\left[1-F_a(y-x)\right] \\ &\quad + (1-u_a)\, G_0(i,a,x) - c_i x \end{aligned} \tag{4.12}$$

$$G_0(i,a,y) = c_i y + L(i,a,y) \tag{4.13}$$

$$L(i,a,y) = h_i \int_0^y (y-z)dM_a(z) + p_i \int_y^\infty (z-y)dM_a(z). \tag{4.14}$$

Then first and second derivatives of $L(i,a,y)$ with respect to y are

$$L'(i,a,y) = (p_i+h_i)\, M_a(y) - p_i \tag{4.15}$$

$$L''(i,a,y) = (p_i+h_i)\, m_a(y). \tag{4.16}$$

Notice that (4.16) is always nonnegative so that $L(i,a,y)$ is a convex function in y.

Remark 2 *The optimal ordering policy for the single-period problem in (4.10) is not necessarily an environment-dependent base-stock policy.*

Example 2 *We assume that both observed and unobserved environments has two states so that $\mathbb{E} = \{1,2\}$ and $\mathbb{F} = \{1,2\}$, where state 1 represents a "good" environment and state 2 represents a "bad" environment. Moreover, we suppose that $P[Z_0 = 1] = 0.66$ and $P[Z_0 = 2] = 0.34$. Furthermore, the emission matrix for the first period is*

$$E_0 = \begin{bmatrix} 0.65 & 0.35 \\ 0.45 & 0.55 \end{bmatrix}.$$

Then, using (4.3),

$$O_0 = \begin{bmatrix} 0.7371 & 0.2629 \\ 0.5526 & 0.4474 \end{bmatrix}.$$

Demand distribution is assumed to be Poisson with mean $\lambda = [10,50]$ and truncated at $[22,75]$. In addition, capacity distribution is assumed to be geometric with probability of failure $q = [0.99, 0.8]$ so that $P[C=k] = pq^{k-1}$. We assume that λ and q change by the state of the unobserved environment Z so that demand and capacity distributions are modulated by real environment. In addition, we suppose that the availability process is also

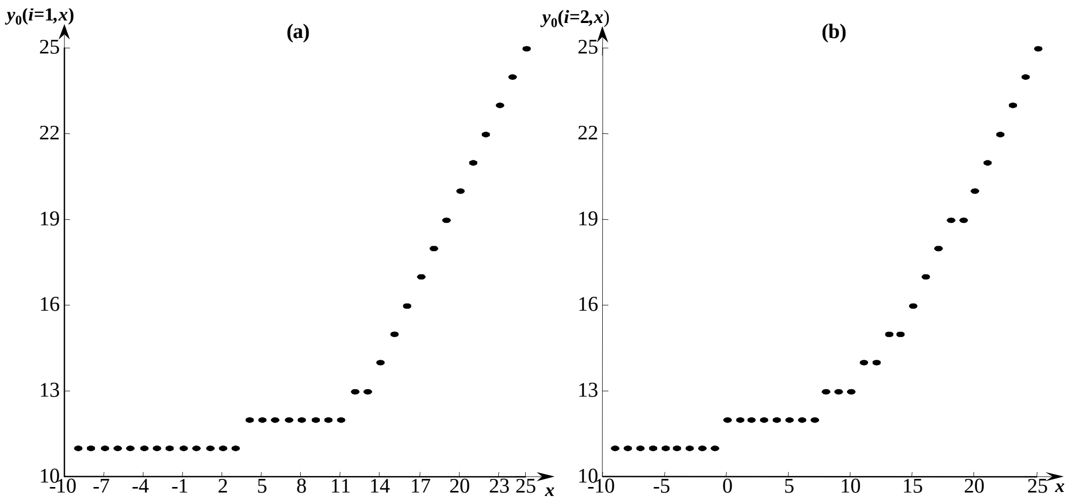

Figure 4.1: Optimal order-up-to levels

modulated by the real environment so that $u = [0.57, 0.43]$. *We have computed the optimal policy with* $c = [1, 2]\,, h = [3, 6]$ *and* $p = [9, 18]\,$. *We assume that cost parameters change with the state of the observed process* Y *so that they are known to the IM.*

We solved the single-period minimization problem in (4.10) with the given parameters and the results are shown in Figure 4.1. Figure 4.1 (a) shows the optimal order-up-to levels for each inventory level x *when the observed state of environment is good whereas Figure 4.1 (b) shows the optimal order-up-to levels for each inventory level* x *when the observed state of environment is bad. From Figure 4.1 (a) and (b), it is obvious that there is a critical inventory level under which we always order and over which we do not order. For example, this critical inventory level is* 13 *if the observed environmental state is good and it is* 19 *if the observed state is bad. Moreover, it is also obvious that the optimal order-up-to levels are constant up to a certain inventory level (i.e., optimal order-up-to level is* 15 *up to* $x = 3$ $(x = -1)$ *if the observed environmental state is good (bad). However, it is clear that the optimal order-up-to levels are not constant in between those two critical inventory levels and they change with varying level of inventory.*

Finally, we can generalize the same remark for multiple and infinite period problems. Therefore, base-stock policy is not necessarily optimal in multiple and infinite period settings for inventory problems with real environment modulated demand, capacity and availability processes and observed environment modulated cost parameters.

4.3 Linear cost model with observed capacity

In this section, as in Section 4.2, we assume that $K_i = 0$ and the random environment is only partially observable. The demand and availability processes are modulated by the unobserved environmental process Z. However, we assume that the capacity process C is modulated by observed environmental process Y. In addition, the random capacity has no upper bound so that $P[C_{n+1} \geq z | Y_n = i] = 1 - F_i(z) > 0$ for all $z \geq 0$. The remainder of this section proceeds as follows. In Section 4.3.1, we analyze the problem in single period and present our results. Next, in Section 4.3.2, we extend the planning horizon and analyze the same problem in multi-period setting.

4.3.1 Single-period model

Here we assume that there is only one period to plan for so that $N = 1$; moreover, $v_1(\bar{\imath}_1, x_1) = 0$ for all $\bar{\imath}_1$ and x_1. Take $i_0 = i$ and $a_0 = a$. Assuming inventory level at the beginning of period is x, the single-period problem is given by (4.10)-(4.14) where F_a is replaced by F_i in (4.12).

Let $y_0(i, x)$ denote the optimal order up to level which minimizes the expected discounted cost in (4.11), where F_a is replaced by F_i, when observed state is i and inventory level is x.In addition, let $v_0'(i, x)$ denote the first derivative of $v_0(i, x)$ with respect to x.

Theorem 8 *The optimal ordering policy for the single-period model is a state-dependent base-stock policy*

$$y_0(i, x) = \begin{cases} S_0^i & x \leq S_0^i \\ x & x > S_0^i \end{cases} \tag{4.17}$$

where S_0^i satisfies

$$\sum_{a \in \mathbb{F}} O_0(i, a) u_a \left(c_i + L'(i, a, S_0^i)\right) = 0$$

for all i. In addition, $H_0(i, x, y)$ is quasi-convex in y for all i and $y \geq x$. The optimal cost *incurred by this policy is*

$$v_0(i,x) = \begin{cases} H_0\left(i,x,S_0^i\right) & x \le S_0^i \\ \sum_{a\in\mathbb{F}} O_0(i,a)L(i,a,x) & x > S_0^i \end{cases} \tag{4.18}$$

for all i and x. Furthermore, $v_0(i,x)$ is convex in x, $\lim_{x\uparrow\infty} v_0'(i,x) = h_i$, and $\lim_{x\downarrow-\infty} v_0'(i,x) = -p_i$ for all i.

Proof. We need to find $y \ge x$ minimizing H_0. We denote first and second derivatives of H_0 with respect to y by H_0' and H_0'', respectively. Then,

$$H_0'(i,x,y) = [1-F_i(y-x)]\sum_{a\in\mathbb{F}} O_0(i,a)u_a\left(c_i + L'(i,a,y)\right) \tag{4.19}$$

$$H_0''(i,x,y) = [1-F_i(y-x)]\sum_{a\in\mathbb{F}} O_0(i,a)u_a\left(L''(i,a,y) - f_i(y-x)\left(c_i+L'(i,a,y)\right)\right) \tag{4.20}$$

for all $y \ge x$. Note that $1-F_i(y-x) > 0$ for all i and $y \ge x$; therefore, whether $H_0(i,x,y)$ is decreasing or increasing depends on the sign of the summation in (4.19). Let S_0^i be the smallest y satisfying

$$B_0(i,y) = \sum_{a\in\mathbb{F}} O_0(i,a)u_a\left(c_i + L'(i,a,y)\right) = 0. \tag{4.21}$$

Notice that $\sum_{a\in\mathbb{F}} O_0(i,a)u_a > 0$ for all i by our assumption. This implies that $O_0(i,a)u_a > 0$ at least for an a. Then, $c_i + L'(i,a,y)$ so that $B_0(i,y)$ is nondecreasing since L is a convex function. Therefore, $B_0(i,y) < 0$ for all $y < S_0^i$ and $B_0(i,y) \ge 0$ for $y \ge S_0^i$. In addition, $\lim_{y\uparrow\infty} B_0(i,y) = (h_i + c_i)\sum_{a\in\mathbb{F}} O_0(i,a)u_a > 0$ and $\lim_{y\downarrow-\infty} B_0(i,y) = -(p_i - c_i)\sum_{a\in\mathbb{F}} O_0(i,a)u_a < 0$ since $\sum_{a\in\mathbb{F}} O_0(i,a)u_a > 0$ by our assumption. Then, it follows that there exists finite S_0^i satisfying (4.21).

From (4.20), it is obvious that the first term is always nonnegative. Now, consider the two cases: $x \le S_0^i$ and $x > S_0^i$.

(i) $x \le S_0^i$: The second term in (4.20) is always nonnegative for $y \in \left[x, S_0^i\right)$ since $B_0(i,y) < 0$ for $y < S_0^i$; as a result, (4.20) is nonnegative. Therefore, $H_0(i,x,y)$ is convex decreasing for all y in $\left[x, S_0^i\right)$. In addition, (4.20) continues to be nonnegative for $y > S_0^i$ but very close to S_0^i; therefore, $H_0(i,x,y)$ is convex nondecreasing for y

close to S_0^i. However, (4.20) turns out to be negative for large values of $y > S_0^i$ since $\lim_{y\uparrow\infty}[1-F_i(y-x)] = 0$ and $\lim_{y\uparrow\infty}(c_i + L'(i,a,y)) = c_i + h_i > 0$. Hence, $H_0(i,x,y)$ is concave nondecreasing for sufficiently large values of $y \geq x$.

(ii) $x > S_0^i$: The second term in (4.20) is always nonpositive; however, (4.20) continues to be nonnegative for values of $y \geq x$ and close to S_0^i. Then, this implies that $H_0(i,x,y)$ is convex nondecreasing for values of y close to S_0^i. But for large values of y, (4.20) turns out to be negative since $\lim_{y\uparrow\infty}[1-F_i(y-x)] = 0$ and $\lim_{y\uparrow\infty}(c_i + L'(i,a,y)) = c_i + h_i > 0$. Hence, $H_0(i,x,y)$ is concave nondecreasing for sufficiently large values of $y \geq x$.

This analysis shows that $H_0(i,x,y)$ is quasi-convex since it satisfies all conditions in Lemma 28. Therefore, $y = S_0^i$ is a global minimum of $H_0(i,x,y)$ for $x \leq S_0^i$ and $y = x$ is the global minimum of $H_0(i,x,y)$ for $x > S_0^i$. This implies that S_0^i is the optimal order-up-to level when $x \leq S_0^i$ and that it is optimal not to order when $x > S_0^i$. As a result, (4.17) gives the optimal ordering policy. Because base-stock policy in (4.17) is optimal, the optimal cost is

$$v_0(i,x) = \begin{cases} H_0\left(i,x,S_0^i\right) & x \leq S_0^i \\ H_0(i,x,x) & x > S_0^i, \end{cases}$$

which leads to (4.18) since $H_0(i,x,x) = \sum_{a\in\mathbb{F}} O_0(i,a)L(i,a,x)$.

Now we prove that $v_0(i,x)$ is convex. First, we show that $v_0(i,x)$ is convex for $x < S_0^i$ and $x > S_0^i$, separately. Then we show that convexity is not violated at $x = S_0^i$.

(i) $x < S_0^i$: Using (4.18), the first and second derivatives of $v_0(i,x)$ are

$$\begin{aligned} v_0'(i,x) &= \sum_{a\in\mathbb{F}} O_0(i,a)\Bigg(u_a \int_0^{S_0^i - x} \left(c_i + L'(i,a,x+z)\right) dF_i(z) \\ &\quad + (1-u_a)\,L'(i,a,x) - c_i\Bigg) \qquad (4.22) \\ v_0''(i,x) &= \sum_{a\in\mathbb{F}} O_0(i,a)\Bigg(u_a \int_0^{S_0^i - x} L''(i,a,x+z)dF_i(z) + (1-u_a)\,L''(i,a,x) \\ &\quad - f_i\left(S_0^i - x\right) u_a \left(c_i + L'(i,a,S_0^i)\right)\Bigg). \qquad (4.23) \end{aligned}$$

In (4.23), the first and second sums are always nonnegative because L is convex. Moreover, the last sum in (4.23) is zero by (4.21). As a result, (4.23) is always nonnegative so that $v_0(i,x)$ is convex for all $x \leq S_0^i$.

(ii) $x > S_0^i$: Using (4.18), v_0 is convex because L is a convex function and sum of convex functions is also convex.

(iii) $x = S_0^i$: Finally we show that convexity of v_0 is not violated at $x = S_0^i$. For v_0 to be convex at $x = S_0^i$, the following condition must hold

$$\lim_{x \uparrow S_0^i} v_0'(i,x) \leq \lim_{x \downarrow S_0^i} v_0'(i,x) \tag{4.24}$$

and $v_0(i,x)$ must be continuous at $x = S_0^i$. Using (4.22),

$$\begin{aligned}
\lim_{x \uparrow S_0^i} v_0'(i,x) &= \lim_{x \uparrow S_0^i} \left\{ \sum_{a \in \mathbb{F}} O_0(i,a) \Big(u_a \int_0^{S_0^i - x} \big(c_i + L'(i,a,x+z)\big)\, dF_i(z) \right. \\
&\qquad \left. + (1-u_a)\, L'(i,a,x) - u_a c_i \Big) \right\} \\
&= \sum_{a \in \mathbb{F}} O_0(i,a) \Big(u_a F_i(0) \left(c_i + L'(i,a,S_0^i)\right) + L'(i,a,S_0^i) \\
&\qquad - u_a c_i + L'(i,a,S_0^i) \Big) \\
&= \sum_{a \in \mathbb{F}} O_0(i,a) L'(i,a,S_0^i).
\end{aligned}$$

The last equality follows from (4.21). Moreover, from (4.18),

$$\lim_{x \downarrow S_0^i} v_0'(i,x) = \sum_{a \in \mathbb{F}} O_0(i,a) L'(i,a,S_0^i).$$

Then it follows that $\lim_{x \uparrow S_0^i} v_0'(i,x) = \lim_{x \downarrow S_0^i} v_0'(i,x)$ so that (4.24) is satisfied. Finally, $v_0(i,x)$ is continuous at $x = S_0^i$ since

$$\lim_{x \uparrow S_0^i} v_0(i,x) = \lim_{x \downarrow S_0^i} v_0(i,x) = \sum_{a \in \mathbb{F}} O_0(i,a) L(i,a,S_0^i).$$

As a result, $v_0(i,x)$ is convex in x for all i. Furthermore, using (4.18), MCT and the fact that $\lim_{x \uparrow \infty} L'(i,a,x) = h_i$,

$$\lim_{x \uparrow \infty} v_0'(i,x) = \sum_{a \in \mathbb{F}} O_0(i,a) \lim_{x \uparrow \infty} L'(i,a,x) = h_i$$

for all i. Similarly, using (4.22), MCT and the fact that $\lim_{x\downarrow-\infty} L'(i,a,x) = -p_i$,

$$\lim_{x\downarrow-\infty} v_0'(i,x) = -p_i$$

for all i. This completes our proof. ■

Theorem 8 shows that a state-dependent base-stock policy is still optimal in single period for inventory problems with random capacity and random availability in a random environment if the unobserved environment modulates demand and availability process, but, not the capacity process. Capacity process and cost parameters are modulated by the observed environment. Moreover, base-stock level for single-period problem can be obtained by solving (4.21) for minimum y. From (4.21), it is obvious that the base-stock level must be decreasing when the availability probability of the transporter is increasing and it depends on demand distribution since L is dependent on M_a. However, the base-stock level for a single-period problem is independent of the capacity distribution and the current inventory level. In other words, the optimal base-stock level for a single-period inventory problem in a partially-observed random environment is exactly the same as when there is no random capacity if the capacity is modulated by the observed environment.

Furthermore, from (4.19), it is obvious that base-stock policy is not necessarily optimal if the capacity process is modulated by the unobserved environment since then $1 - F_a(y-x)$ would be inside the summation in (4.19) and H_0 would have many local minima and maxima. Moreover, in this case, global minimum of H_0 will depend on the current inventory level x.

4.3.2 Multi-period model

In the case with multiple periods, there are N periods to plan for and the dynamic programming equation involves the sum of single period costs in the current period plus the expected optimal discounted costs from the next period until the end of the planning horizon. We assume that $v_N(\bar{\imath}_N, x_N) = 0$ for all $\bar{\imath}_N$ and x_N. Moreover, we suppose that inventory level at the beginning of period $n+1$ is x and demand in period $n+1$ is D. Assuming $i_n = i$ and $a_n = a$, the minimum cost function at time n satisfies

$$v_n(\bar{\imath}_n, x) = \min\nolimits_{y\geq x} H_n(\bar{\imath}_n, x, y) \tag{4.25}$$

for all $\bar{\imath}_n$ and x, where y is the order-up-to level in period $n+1$ and

$$H_n(\bar{\imath}_n, x, y) = \sum_{a\in\mathbb{F}} O_n(\bar{\imath}_n, a) J_n(\bar{\imath}_n, a, x, y) \tag{4.26}$$

$$J_n(\bar{\imath}_n, a, x, y) = u_a \int_0^{y-x} G_n(\bar{\imath}_n, a, x+z) dF_i(z) + u_a G_n(\bar{\imath}_n, a, y)\left[1 - F_i(y-x)\right] + (1-u_a) G_n(\bar{\imath}_n, a, x) - c_i x \tag{4.27}$$

$$G_n(\bar{\imath}_n, a, y) = c_i y + L(i, a, y) + \alpha \sum_{j\in\mathbb{E}} \Psi_n(a, j) E_D^a\left[v_{n+1}\left((\bar{\imath}_n, j), y - D\right)\right] \tag{4.28}$$

with $L(i,a,y)$ as given in (4.14). Here, E_D^a denotes the notation that expectation is taken with respect to the random variable D with distribution M_a.

Let $y_n(\bar{\imath}_n, x)$ denote the optimal order-up-to level of the minimization problem in (4.25) given that all of the past observations until time n is $\bar{\imath}_n$, inventory level is x, and the observed environmental state is i at time n. In addition, let $v_n'(\bar{\imath}_n, x)$ denote the derivative of $v_n(\bar{\imath}_n, x)$ with respect to x. Finally, we assume that h and p are holding cost and shortage cost vectors, respectively.

Theorem 9 *The optimal ordering policy for N-period model is a state-dependent base-stock policy*

$$y_n(\bar{\imath}_n, x) = \begin{cases} S_n^{\bar{\imath}_n} & x \le S_n^{\bar{\imath}_n} \\ x & x > S_n^{\bar{\imath}_n} \end{cases} \tag{4.29}$$

where $S_n^{\bar{\imath}_n}$ satisfies

$$\sum_{a\in\mathbb{F}} O_n(\bar{\imath}_n, a) u_a \left(c_i + L'(i, a, S_n^{\bar{\imath}_n}) + \alpha \sum_{j\in\mathbb{E}} \Psi_n(a, j) E_D^a\left[v_{n+1}'\left((\bar{\imath}_n, j), S_n^{\bar{\imath}_n} - D\right)\right]\right) = 0$$

for all $\bar{\imath}_n$. In addition, $H_n(\bar{\imath}_n, x, y)$ is quasi-convex in y for all $\bar{\imath}_n$ and $y \le x$. The optimal cost incurred by this policy is

$$v_n(\bar{\imath}_n, x) = \begin{cases} H_n(\bar{\imath}_n, x, S_n^{\bar{\imath}_n}) & x \le S_n^{\bar{\imath}_n} \\ \sum_{a\in\mathbb{F}} O_n(\bar{\imath}_n, a)\Big(L(i,a,x) + \alpha \sum_{j\in\mathbb{E}} \Psi_n(a,j) E_D^a\left[v_{n+1}\left((\bar{\imath}_n, j), x - D\right)\right]\Big) & x > S_n^{\bar{\imath}_n} \end{cases} \tag{4.30}$$

for all $\bar{\imath}_n$ and x. Furthermore, $v_n(\bar{\imath}_n, x)$ is convex in x, $\lim_{x\uparrow\infty} v_n'(\bar{\imath}_n, x) = R_{n,N-n}^{\alpha} h(\bar{\imath}_n)$ and $\lim_{x\downarrow-\infty} v_n'(\bar{\imath}_n, x) = -R_{n,N-n}^{\alpha} p(\bar{\imath}_n)$ for all $\bar{\imath}_n$

Proof. The proof proceeds inductively. First of all, by Theorem 8, we know that $v_{N-1}(\bar{\imath}_{N-1}, x)$ is a convex function in x; moreover, base-stock policy is optimal for Nth period. Therefore, Theorem 9 is satisfied for time $N-1$. Notice that, here, we also assume that $v_N(\bar{\imath}_N, x_N) = 0$ for all $\bar{\imath}_N$ and x_N. Next, assume that the induction hypothesis is valid for times $n+1, n+2, ..., N-1$ so that $v_{n+1}(\bar{\imath}_{n+1}, x_{n+1})$ is convex in x_{n+1} for all $\bar{\imath}_{n+1}$, $\lim_{x\uparrow\infty} v'_{n+1}(\bar{\imath}_{n+1}, x) = R^{\alpha}_{n+1,N-n-1}h(\bar{\imath}_{n+1})$ and $\lim_{x\downarrow-\infty} v'_{n+1}(\bar{\imath}_{n+1}, x) = -R^{\alpha}_{n+1,N-n-1}p(\bar{\imath}_{n+1})$ for all $\bar{\imath}_{n+1} \in \mathbb{E}^{n+2}$. We now show that Theorem 9 also holds for time n.

The value function in period n is given by (4.25) and we need to analyze H_n. We let H'_n denote the derivative of H_n with respect to y. Then,

$$\begin{aligned} H'_n(\bar{\imath}_n, x, y) &= [1 - F_i(y-x)] \sum_{a\in\mathbb{F}} O_n(\bar{\imath}_n, a) u_a \bigg(c_i + L'(i, a, y) \\ &\quad + \alpha \sum_{j\in\mathbb{E}} \Psi_n(a, j) E^a_D \left[v'_{n+1}((\bar{\imath}_n, j), y - D) \right] \bigg) \end{aligned} \tag{4.31}$$

for all $y \geq x$. Note that $1 - F_i(y-x) > 0$ for all $y \geq x$. Then whether $H_n(\bar{\imath}_n, x, y)$ is increasing or decreasing depends on the sign of the summation in (4.31). Moreover, the summation is a nondecreasing function of y since L and v_{n+1} are convex. Then $H_n(\bar{\imath}_n, x, y)$ is nonincreasing if (4.31) is nonpositive, and it is increasing otherwise. Let $S^{\bar{\imath}_n}_n$ be the smallest y satisfying

$$\begin{aligned} B_n(\bar{\imath}_n, y) &= \sum_{a\in\mathbb{F}} O_n(\bar{\imath}_n, a) u_a \bigg(c_i + L'(i, a, y) \\ &\quad + \alpha \sum_{j\in\mathbb{E}} \Psi_n(a, j) E^a_D \left[v'_{n+1}((\bar{\imath}_n, j), y - D) \right] \bigg) = 0. \end{aligned} \tag{4.32}$$

Notice that, $B_n(\bar{\imath}_n, y) < 0$ so that (4.31) is negative for $x \leq y < S^{\bar{\imath}_n}_n$ and it is nonnegative for $x \leq y \geq S^{\bar{\imath}_n}_n$ since L and v_{n+1} are convex. Using Monotone Convergence Theorem (MCT) and the induction hypothesis

$$\lim_{y\uparrow\infty} B_n(\bar{\imath}_n, y) = \sum_{a\in\mathbb{F}} O_n(\bar{\imath}_n, a) u_a \bigg(c_i + h_i + \alpha \sum_{j\in\mathbb{E}} \Psi_n(a, j) R^{\alpha}_{n+1,N-n-1} h(\bar{\imath}_n, j) \bigg) > 0.$$

The inequality above follows from our assumptions that are $h_i > c_i > 0$ for all i and $\sum_{a\in\mathbb{F}} O_n(\bar{\imath}_n, a) u_a > 0$ for all $\bar{\imath}_n$. Similarly, using MCT, the induction hypothesis and the fact that $c_i < p_i$ for all i,

$$\lim_{y \downarrow -\infty} B_n\left(\bar{\imath}_n, y\right) = \sum_{a \in \mathbb{F}} O_n(\bar{\imath}_n, a) u_a \Bigg(c_i - p_i - \alpha \sum_{j \in \mathbb{E}} \Psi_n\left(a, j\right) R^{\alpha}_{n+1, N-n-1} p(\bar{\imath}_n, j) \Bigg) < 0.$$

Moreover, $B_n\left(\bar{\imath}_n, y\right)$ is continuous and nondecreasing in y since L and v_{n+1} are convex. Then, this implies that there exists a finite $S_n^{\bar{\imath}_n}$ satisfying (4.32). Now, we consider the two cases: $x \leq S_n^{\bar{\imath}_n}$ and $x > S_n^{\bar{\imath}_n}$.

(i) $x \leq S_n^{\bar{\imath}_n}$: Then (4.31) is negative so that $H_n\left(\bar{\imath}_n, x, y\right)$ is decreasing for $y \in [x, S_n^{\bar{\imath}_n})$. However, (4.31) is nonnegative so that $H_n\left(\bar{\imath}_n, x, y\right)$ is nondecreasing for $y \in [S_n^{\bar{\imath}_n}, +\infty)$. Obviously, $S_n^{\bar{\imath}_n}$ is the global minimum and $y_n\left(\bar{\imath}_n, x\right) = S_n^{\bar{\imath}_n}$ is the order-up-to level for $x \leq S_n^{\bar{\imath}_n}$

(ii) $x > S_n^{\bar{\imath}_n}$: Then (4.31) is nonnegative so that $H_n\left(\bar{\imath}_n, x, y\right)$ is nondecreasing for all $y \geq x$. As a result, no order should be given so that $y_n\left(\bar{\imath}_n, x\right) = x$.

This analysis shows that $H_n\left(i, x, y\right)$ is quasi-convex since it satisfies all conditions in Lemma 28. Hence, base-stock policy defined in (4.29) is the optimal ordering policy. Applying this optimal policy yields the following minimum cost function

$$v_n\left(\bar{\imath}_n, x\right) = \begin{cases} H_n\left(\bar{\imath}_n, x, S_n^{\bar{\imath}_n}\right) & x \leq S_n^{\bar{\imath}_n} \\ H_n\left(\bar{\imath}_n, x, x\right) & x > S_n^{\bar{\imath}_n} \end{cases}$$

which leads to (4.30) since

$$H_n\left(\bar{\imath}_n, x, x\right) = \sum_{a \in \mathbb{F}} O_n(\bar{\imath}_n, a) \Bigg(L(i, a, x) + \alpha \sum_{j \in \mathbb{E}} \Psi_n\left(a, j\right) E^a_D\left[v_{n+1}\left((\bar{\imath}_n, j), x - D\right)\right] \Bigg).$$

Now we prove that $v_n(\bar{\imath}_n, x)$ is convex. First, we show that $v_n(\bar{\imath}_n, x)$ is convex for $x < S_n^{\bar{\imath}_n}$ and $x > S_n^{\bar{\imath}_n}$, separately. Then we show that convexity is not violated at $x = S_n^{\bar{\imath}_n}$.

(i) $x < S_n^{\bar{\imath}_n}$: Using (4.30), the first and second derivatives of $v_n(\bar{\imath}_n, x)$ are

$$\begin{aligned} v_n'(\bar{\imath}_n, x) &= \sum_{a \in \mathbb{F}} O_n(\bar{\imath}_n, a) \Bigg(u_a \int_0^{S_n^{\bar{\imath}_n} - x} G_n'(\bar{\imath}_n, a, x+z) dF_i(z) \\ &\quad + (1-u_a) G_n'(\bar{\imath}_n, a, x) - c_i \Bigg) \qquad (4.33) \\ v_n''(\bar{\imath}_n, x) &= \sum_{a \in \mathbb{F}} O_n(\bar{\imath}_n, a) \Bigg(u_a \int_0^{S_n^{\bar{\imath}_n} - x} G_n''(\bar{\imath}_n, a, x+z) dF_i(z) \\ &\quad + (1-u_a) G_n''(\bar{\imath}_n, a, x) \\ &\quad - f_i\left(S_n^{\bar{\imath}_n} - x\right) u_a G_n'(\bar{\imath}_n, a, S_n^{\bar{\imath}_n}) \Bigg). \qquad (4.34) \end{aligned}$$

where

$$\begin{aligned} G_n'(\bar{\imath}_n, a, x) &= c_i + L'(i, a, x) + \alpha \sum_{j \in \mathbb{E}} \Psi_n(a, j) E_D^a \left[v_{n+1}'((\bar{\imath}_n, j), y - D) \right] \qquad (4.35) \\ G_n''(\bar{\imath}_n, a, x) &= L''(i, a, x) + \alpha \sum_{j \in \mathbb{E}} \Psi_n(a, j) E_D^a \left[v_{n+1}''((\bar{\imath}_n, j), y - D) \right]. \qquad (4.36) \end{aligned}$$

In (4.34), the first and second sums are always nonnegative because L and v_{n+1} are convex. Moreover, the last sum in (4.34) is zero by (4.32). As a result, (4.34) is always nonnegative so that $v_n(\bar{\imath}_n, x)$ is convex for all $x \leq S_n^{\bar{\imath}_n}$.

(ii) $x > S_n^{\bar{\imath}_n}$: Using (4.30), v_n is convex because L and v_{n+1} are convex, and sum of convex functions is also convex.

(iii) $x = S_n^{\bar{\imath}_n}$: Finally we show that convexity of v_n is not violated at $x = S_n^{\bar{\imath}_n}$. For v_n to be convex at $x = S_n^{\bar{\imath}_n}$, the following condition must hold

$$\lim_{x \uparrow S_n^{\bar{\imath}_n}} v_n'(\bar{\imath}_n, x) \leq \lim_{x \downarrow S_n^{\bar{\imath}_n}} v_n'(\bar{\imath}_n, x) \qquad (4.37)$$

and $v_n(\bar{\imath}_n, x)$ must be continuous at $x = S_n^{\bar{\imath}_n}$. Using (4.33),

$$\begin{aligned}
\lim_{x \uparrow S_n^{\bar{\imath}_n}} v_n'(\bar{\imath}_n, x) &= \lim_{x \uparrow S_n^{\bar{\imath}_n}} \Bigg\{ \sum_{a \in \mathbb{F}} O_n(\bar{\imath}_n, a) \Bigg(u_a \int_0^{S_n^{\bar{\imath}_n} - x} G_n'(\bar{\imath}_n, a, x+z) dF_i(z) \\
&\quad + (1-u_a) G_n'(\bar{\imath}_n, a, x) - c_i \Bigg) \Bigg\} \\
&= \sum_{a \in \mathbb{F}} O_n(\bar{\imath}_n, a) \left(u_a G_n'(\bar{\imath}_n, a, S_n^{\bar{\imath}_n}) F_i(0) \right. \\
&\quad \left. + (1-u_a) G_n'(\bar{\imath}_n, a, S_n^{\bar{\imath}_n}) - c_i \right) \\
&= \sum_{a \in \mathbb{F}} O_n(\bar{\imath}_n, a) \Bigg(L'(i, a, S_n^{\bar{\imath}_n}) \\
&\quad + \alpha \sum_{j \in \mathbb{E}} \Psi_n(a, j) E_D^a \left[v_{n+1}' \left((\bar{\imath}_n, j), S_n^{\bar{\imath}_n} - D \right) \right] \Bigg).
\end{aligned}$$

The last equality follows from (4.32). Moreover, from (4.30),

$$\lim_{x \downarrow S_n^{\bar{\imath}_n}} v_n'(\bar{\imath}_n, x) = \sum_{a \in \mathbb{F}} O_n(\bar{\imath}_n, a) \Bigg(L'(i, a, S_n^{\bar{\imath}_n}) + \alpha \sum_{j \in \mathbb{E}} \Psi_n(a, j) E_D^a \left[v_{n+1}' \left((\bar{\imath}_n, j), S_n^{\bar{\imath}_n} - D \right) \right] \Bigg).$$

Then it follows that $\lim_{x \uparrow S_n^{\bar{\imath}_n}} v_n'(\bar{\imath}_n, x) = \lim_{x \downarrow S_n^{\bar{\imath}_n}} v_n'(\bar{\imath}_n, x)$ so that (4.37) is satisfied. Finally, $v_n(\bar{\imath}_n, x)$ is continuous at $x = S_n^{\bar{\imath}_n}$ since

$$\begin{aligned}
\lim_{x \uparrow S_n^{\bar{\imath}_n}} v_n(\bar{\imath}_n, x) &= \lim_{x \downarrow S_n^{\bar{\imath}_n}} v_n(\bar{\imath}_n, x) \\
&= \sum_{a \in \mathbb{F}} O_n(\bar{\imath}_n, a) \Bigg(L(i, a, S_n^{\bar{\imath}_n}) \\
&\quad + \alpha \sum_{j \in \mathbb{E}} \Psi_n(a, j) E_D^a \left[v_{n+1} \left((\bar{\imath}_n, j), S_n^{\bar{\imath}_n} - D \right) \right] \Bigg).
\end{aligned}$$

As a result, $v_n(\bar{\imath}_n, x)$ is convex in x for all $\bar{\imath}_n$. Moreover, using induction hypothesis, (4.30) and MCT

$$\begin{aligned}
\lim_{x \uparrow \infty} v_n'(\bar{\imath}_n, x) &= h_i + \alpha \sum_{a \in \mathbb{F}} O_n(\bar{\imath}_n, a) \sum_{j \in \mathbb{E}} \Psi_n(a, j) R_{n+1, N-n-1}^{\alpha} h(\bar{\imath}_n, j) \\
&= h_i + \alpha \sum_{j \in \mathbb{E}} P_n(\bar{\imath}_n, j) R_{n+1, N-n-1}^{\alpha} h(\bar{\imath}_n, j) \\
&= R_{n, N-n}^{\alpha} h(\bar{\imath}_n)
\end{aligned}$$

The second equality follows from the definition of P_n in (4.5), and the last equality follows from (4.9). Similarly, using the induction hypothesis, (4.33), and MCT, we can also show that

$$\lim_{x \downarrow -\infty} v_n' (\bar{\imath}_n, x) = -R^{\alpha}_{n,N-n} p(\bar{\imath}_n).$$

This completes our proof. ■

By Theorem 9, we see that a state-dependent base-stock policy is still optimal in multiple periods. However, base-stock levels depend on all past observations regarding the true state of environment. In addition, from (4.32), it is obvious that the order-up-to level is independent of current inventory level x. Moreover, the cost function H_n is quasi-convex as in single-period model; therefore, $S_n^{\bar{\imath}_n}$ satisfying (4.32) is the global minimum. However, if the producer's capacity is modulated by the unobserved environmental process as in Section 4.3.1, the base-stock policy may not be optimal.

Using Theorem 9 and (4.8), we get that $\lim_{x \uparrow \infty} v_n' (\bar{\imath}_n, x) = E[\sum_{k=0}^{N-n-1} \alpha^k h_{Y_{n+k}} \Big| \bar{Y}_n = \bar{\imath}_n]$ and $\lim_{x \downarrow -\infty} v_n' (i, x) = -E[\sum_{k=0}^{N-n-1} \alpha^k p_{Y_{n+k}} \Big| \bar{Y}_n = \bar{\imath}_n]$. These results are intuitively understandable. If we assume that inventory level at time n is very large, inventory level in the remaining periods is also very large so that there is no stockouts. In this case, one unit increase on inventory level means that the retailer holds this extra unit until the end of the planning horizon. Thus, from time n until time $N-1$, the retailer incurs extra holding cost which depends on the state of environment at that time. Under discounting, expected present worth of increase in minimum cost given all past observations $\bar{\imath}_n$ is $E[\sum_{k=0}^{N-n-1} \alpha^k h_{Y_{n+k}} \Big| \bar{Y}_n = \bar{\imath}_n]$. Similarly, if we suppose that inventory level at time n is negative and very large in absolute value, this implies that that the retailer is always out of stock. In such a case, a unit increase in inventory level implies that the retailer's inventory level is one more in the remaining periods provided that everything is the same. As a result, in the remaining periods, the retailer stocks out one unit less compared to previous case. And this implies that the retailer pays p_{Y_n} less at each period until time $N-1$. Then, given $\bar{\imath}_n$, expected total decrease in the optimal cost is $E[\sum_{k=0}^{N-n-1} \alpha^k p_{Y_{n+k}} \Big| \bar{Y}_n = \bar{\imath}_n]$.

Unfortunately, there is a difficulty with the dynamic programming formulation above as the planning horizon gets longer since observations regarding the state of the environment increases. Therefore, the state space $\mathbb{E}^n$ gets larger. Then, this implies that the dimension of the state $(\bar{\imath}_n, x_n)$ increases. When the length of the planning horizon is infinite, this means that the dimension of the state space is infinite. As a result, analyzing infinite period problems with this formulation is very difficult. Therefore, we analyze the same problem by a different formulation in the following section. This new formulation enables us to analyze infinite-period problem as well as single-period and multi-period problems.

4.4 Sufficient statistics

The main problem about our formulation of inventory problem with imperfect information in Section 4.3 is that the state space is of increasing dimension. A new observation at period n causes an increase in the dimension of the information vector $\bar{Y}_n$. Then, the dimension of the state $(\bar{\imath}_n, x_n)$ increases accordingly. This situation clearly creates some problems as the number of periods increases. Moreover, it is computationally very hard to keep this much information. Hence, it is necessary to reduce the data. For this purpose, we need a process of smaller dimension than $\bar{Y}_n$. This process must summarize all information embedded in $\bar{Y}_n$. These are known as *sufficient statistics.*

Define $\pi_n^a = P\left[Z_n = a \left| \bar{Y}_n \right.\right]$ so that π_n^a is the probability that the true state of the environment at time n is a given all observations until that time. Let the vector $\pi_n = \left[\pi_n^a, \pi_n^b, ...\right]$ denote the conditional distribution of Z_n given $\bar{Y}_n$ where $\sum_{a \in \mathbb{F}} \pi_n^a = 1$ and $\pi_n^a \geq 0$ for all $a \in \mathbb{F}$. Additional information that we obtain at each period is the new state of the observed process Y. Therefore, information at time $n+1$ is information at time n plus Y_{n+1} so that $\bar{Y}_{n+1} = \left(\bar{Y}_n, Y_{n+1}\right)$. The conditional distribution of the true state of the environment at time $n+1$ is specified by

$$g_b\left(Y_{n+1}\right) = P\left[Z_{n+1} = b \left| \bar{Y}_{n+1} = \left(\bar{Y}_n, Y_{n+1}\right)\right.\right] \tag{4.38}$$

where

$$\begin{aligned} g_b\left(j\right) &= P\left[Z_{n+1} = b \left| \bar{Y}_n, Y_{n+1} = j \right.\right] \\ &= \frac{P\left[Z_{n+1} = b, Y_{n+1} = j \left| \bar{Y}_n \right.\right]}{P\left[Y_{n+1} = j \left| \bar{Y}_n \right.\right]} \\ &= \frac{\sum_{a \in \mathbb{F}} P\left[Z_n = a \left| \bar{Y}_n \right.\right] P\left[Z_{n+1} = b \left| Z_n = a \right.\right] P\left[Y_{n+1} = j \left| Z_{n+1} = b \right.\right]}{\sum_{a,b \in \mathbb{F}} P\left[Z_n = a \left| \bar{Y}_n \right.\right] P\left[Z_{n+1} = b \left| Z_n = a \right.\right] P\left[Y_{n+1} = j \left| Z_{n+1} = b \right.\right]} \\ &= \frac{\sum_{a \in \mathbb{F}} \pi_n^a Q_n(a,b) E_{n+1}(b,j)}{\sum_{a,c \in \mathbb{F}} \pi_n^a Q_n(a,c) E_{n+1}(c,j)} \end{aligned} \tag{4.39}$$

for $n \geq 0$. Note that π_0 is either known at the beginning or it can be determined from

$$\pi_0^b = \frac{P\left[Z_0 = b\right] E_0(b, Y_0)}{\sum_{c \in \mathbb{F}} P\left[Z_0 = c\right] E_0(c, Y_0)} \tag{4.40}$$

using the initial observation Y_0. This analysis shows that

$$\pi_{n+1}^b = g_b\left(Y_{n+1}\right) \tag{4.41}$$

after putting (4.38) and (4.39) together. The most important property of (4.41) is that calculation of conditional distribution of the true state of the environment after time $n+1$ requires only π_n, conditional probability of the true state of the environment after time n, and Y_{n+1} the new observation on the true state of the environment at time $n+1$. Therefore, π_n summarizes the information up to time n and represents a sufficient statistic for the complete past history of $\bar{Y}_n$. This result is also stated in Smallwood and Sondik (1973), Monahan (1982) and Bertsekas (2000 a). Moreover, it is stated in Monahan (1982) that $\{\pi_n; n \geq 0\}$ is a Markov chain. As a result, our problem can be modeled as a completely observable Markov decision chain, where π_n is the state of this Markov chain. The unobservable environmental process Z is defined on the finite state space $\mathbb{F}$ whereas the Markov chain π is defined on a continuous state space $\mathfrak{D}\left(\mathbb{F}\right)$ which is the set of all probability distributions on $\mathbb{F}$.

Expression in (4.41) is a transformation from π_n to π_{n+1} if the observations $\bar{Y}_n$ become $\bar{Y}_{n+1} = (\bar{Y}_n, Y_{n+1})$, and the transition function is

$$T_b\left(\pi_n \left| Y_{n+1}\right.\right) = \frac{\sum_{a \in \mathbb{F}} \pi_n^a Q_n(a,b) E_{n+1}(b, Y_{n+1})}{\sum_{a,c \in \mathbb{F}} \pi_n^a Q_n(a,c) E_{n+1}(c, Y_{n+1})} \tag{4.42}$$

for $n \geq 0$. Then we let $T = \{T_b; b \in \mathbb{F}\}$ denote the transition vector where $\sum_{b \in \mathbb{F}} T_b = 1$ and $T_b \geq 0$ for all $b \in \mathbb{F}$. Note that π_0 is again either known as an initial condition or it can be found by (4.40). In this case, we assume that $P\left[Z_0 = a\right]$ is externally specified so that it is initially known. In practice, $P\left[Z_0 = a\right]$ can be determined via preliminary analysis of the unobserved environment. Then, using $P\left[Z_0 = a\right]$, we can determine π_0 by (4.40) since we already know E_0.

Like (4.5), the evolution of Y is now described probabilistically by

$$P_n^k\left(\left(\pi_n, i\right), j\right) = P\left[Y_{n+k} = j \left| \bar{Y}_n, Y_n = i\right.\right] = \sum_{a \in \mathbb{F}} \pi_n^a \Psi_n^k\left(a, j\right) \tag{4.43}$$

for $k \geq 1$, where Ψ_n^k is given in (4.4). Moreover, we use P_n instead of P_n^k when $k = 1$. Clearly, $P_n^0 = I$ so that

$$P_n^0((\pi_n, i), j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \tag{4.44}$$

for all π_n, i and j. Note that $P_n^k((\pi_n, i), j)$ is independent of i for $k > 0$. In addition, through a similar analysis as in (4.7), we can write the following

$$P_n^{k+1}((\pi_n, i), j) = \sum_{l \in \mathbb{E}} P_n((\pi_n, i), l) P_{n+1}^k((T(\pi_n | l), l), j) \tag{4.45}$$

for all π_n, i and j.

Note that $\{P_n\}$ can easily be determined once the conditional distribution of true state of environment $\{\pi_n\}$, the transition matrices $\{Q_n\}$, emission matrices $\{E_n\}$ and the initial distribution of true state of the environment are known. Here, as in Section 4.1, we assume that $\{Q_n\}$, $\{E_n\}$ and $\{\Psi_n\}$ are time-dependent for single and multiple period analyses; however, they are time-homogenous for infinite period analysis so all time subscripts vanish.

For a nonnegative function g, we note

$$\begin{aligned} E\left[\sum_{k=0}^{m-1} \alpha^k g(Y_{n+k}) \middle| \bar{Y}_n, Y_n = i\right] &= \sum_{k=0}^{m-1} \alpha^k E\left[g(Y_{n+k}) \middle| \bar{Y}_n, Y_n = i\right] \\ &= \sum_{k=0}^{m-1} \alpha^k \sum_{j \in \mathbb{E}} P_n^k((\pi, i), j) g(j) \\ &= \sum_{j \in \mathbb{E}} \left[\sum_{k=0}^{m-1} \alpha^n P_n^k((\pi, i), j)\right] g(j) \\ &= \sum_{j \in \mathbb{E}} R_{n,m}^\alpha((\pi, i), j) g(j) = R_{n,m}^\alpha g(\pi, i) \end{aligned} \tag{4.46}$$

where we define $R_{n,m}^\alpha((\pi, i), j) = \sum_{k=0}^{m-1} \alpha^k P_n^k((\pi, i), j)$ for all π, i and j. If we consider $g(j)$ as the reward obtained when the observed state is j, then $R_{n,m}^\alpha g(\pi, i)$ is the expected total discounted reward obtained during m transitions given information vector π and observation i. Then, $R_{n,m}^\alpha g$ can be called $\alpha-$potential of g during first m transitions starting at time n. Similarly, $R_{n,m}^\alpha$ is called $\alpha-$potential matrix of Y during first m transitions starting at time n. Notice that we cannot directly use results obtained by Çınlar (1975) regarding finite and infinite transition $\alpha-$potential matrices and functions since Y is not a Markov chain. Therefore, potential theory of Markov chains must be modified so that it is also applicable for POMDPs.

As in Section 4.1, we need the relationship between $R^{\alpha}_{n,m}g$ and $R^{\alpha}_{n+1,m-1}g$ since we need it for our analyses in the following sections. By (4.46), we know that

$$\begin{aligned} R^{\alpha}_{n,m}g(\pi,i) &= \sum_{l\in\mathbb{E}} R^{\alpha}_{n,m}((\pi,i),l)g(l) \\ &= g(i)+\sum_{l\in\mathbb{E}}\left[\sum_{k=1}^{m-1}\alpha^k P^k_n((\pi,i),l)\right]g(l) \\ &= g(i)+\sum_{l\in\mathbb{E}}\sum_{k=0}^{m-2}\alpha^{k+1}\left[\sum_{j\in\mathbb{E}}P_n((\pi,i),j)P^k_{n+1}((T(\pi\,|j),j),l)\right]g(l) \\ &= g(i)+\alpha\sum_{j\in\mathbb{E}}P_n((\pi,i),j)\sum_{l\in\mathbb{E}}\sum_{k=0}^{m-2}\alpha^k P^k_{n+1}((T(\pi\,|j),j),l)\,g(l) \\ &= g(i)+\alpha\sum_{j\in\mathbb{E}}P_n((\pi,i),j)R^{\alpha}_{n+1,m-1}g(T(\pi\,|j),j) \end{aligned} \tag{4.47}$$

for all π and i. We get the second equality after we substituted $R^{\alpha}_{n,m}((\pi,i),l)$. Then, we get the third equality by changing the index of inside summation in second line. Using (4.45), we get the fourth equality. Finally, the last equality comes from the definition of $R^{\alpha}_{n+1,m-1}g$.

For infinite-period analysis, we assume that $\{Q_n\}$ and $\{E_n\}$ are time-homogenous so that $Q_n = Q$ and $E_n = E$. Therefore, we obtain simplifications that can be made in infinite-period analysis. Using (4.43), we can now write

$$P^k_n((\pi,i),j) = \sum_{a\in\mathbb{F}}\pi^a\Psi^k(a,j) \tag{4.48}$$

for all π, j, n and $k > 0$, where

$$\Psi^k(a,j) = \sum_{b\in\mathbb{F}}Q^k(a,b)E(b,j) \tag{4.49}$$

for all a and j. However, if $k = 0$, P^k_n is given by (4.44). Moreover, we define $R^{\alpha}_n g(\pi,i) = \lim_{m\uparrow\infty} R^{\alpha}_{n,m}g(\pi,i)$ for all π and i. It follows from (4.46) that

$$E\left[\sum_{k=0}^{\infty}\alpha^k g(Y_{n+k})\,\middle|\bar{Y}_n, Y_n = i\right] = \sum_{j\in\mathbb{E}}R^{\alpha}_n((\pi,i),j)g(j) = R^{\alpha}_n g(\pi,i) \tag{4.50}$$

for all π and i, where $R^{\alpha}_n((\pi,i),j) = \sum_{k=0}^{\infty}\alpha^k P^k_n((\pi,i),j)$ with P^k_n given in (4.44) and

(4.48). After substituting for P_n^k, we get

$$\begin{aligned} R_n^\alpha\left((\pi,i),j\right) &= I(i,j)+\sum_{k=1}^{\infty}\alpha^k\sum_{a\in\mathbb{F}}\pi^a\sum_{b\in\mathbb{F}}Q^k(a,b)E(b,j) \\ &= I(i,j)+\sum_{a\in\mathbb{F}}\pi^a\sum_{b\in\mathbb{F}}\left[\sum_{k=1}^{\infty}\alpha^kQ^k(a,b)\right]E(b,j) \\ &= I(i,j)+\sum_{a\in\mathbb{F}}\pi^a\sum_{b\in\mathbb{F}}\left[Q^\alpha(a,b)-I(a,b)\right]E(b,j) \qquad (4.51) \end{aligned}$$

for all π, i and j, where I is the identity matrix and $Q^\alpha(a,b)=\sum_{k=0}^{\infty}\alpha^kQ^k(a,b)$ for all a and b. It is obvious from (4.51) that R_n^α is independent of n. Therefore, we use R^α instead of R_n^α in the remaining sections. Note that $R_n^\alpha((\pi,i),j)$ is a matrix in $i,j\in\mathbb{E}$ for a given $\pi\in\mathfrak{D}(\mathbb{F})$. Then, by (4.51), $R^\alpha = I+\pi(Q^\alpha-I)E$ in matrix notation. Moreover, using (4.50) and (4.51),

$$R^\alpha g(\pi,i)=g(i)+\sum_{a\in\mathbb{F}}\pi^a\sum_{b\in\mathbb{F}}\left[Q^\alpha(a,b)-I(a,b)\right]\sum_{j\in\mathbb{E}}E(b,j)g(j) \qquad (4.52)$$

for all π and i. Note that $R^\alpha g(\pi,i)$ is a vector in $i\in\mathbb{E}$ for a given $\pi\in\mathfrak{D}(\mathbb{F})$ and we can write

$$R^\alpha g = g+\pi(Q^\alpha-I)Eg$$

in matrix notation. Notice from (4.52) that expected total reward obtained during infinite transitions is the sum of immediate reward in the current state plus the expected total reward after the first transition. Moreover, for a given π, expected reward after first transition is independent of the initial observation i. This is clearly intuitive since observing state j after first transition does not depend on the initial observation if we know the distribution of the true environmental state. Furthermore, by Corollary 1.8 in Çınlar (1975, p. 197), if the state space $\mathbb{F}$ is finite and $\alpha\in[0,1)$, then $Q^\alpha=(I-\alpha Q)^{-1}$. Since $\mathbb{F}$ is finite and $\alpha\in(0,1)$ in our model, we get from (4.51) that

$$R^\alpha g(\pi,i)=g(i)+\pi(I-\alpha Q)^{-1}Eg-\pi Eg \qquad (4.53)$$

for all π and i. If we use a similar terminology as in Çınlar (1975), then $R^\alpha g$ can be called $\alpha-$potential of g during infinite transitions. Similarly, R^α is called $\alpha-$potential matrix of Y during infinite transitions and its solution is (4.53). Note that $\pi(Q^\alpha-I)Eg$ is a constant and we can write

$$R^\alpha g(\pi,i)-R^\alpha g(\pi,j)=g(i)-g(j)$$

for any $\pi \in \mathfrak{D}(\mathbb{F})$.

Our notation and assumption regarding the demand, availability, capacity and costs are exactly the same as in Section 4.3. However, we now assume that $\sum_{a \in F} \pi^a u_a > 0$ for all π. This assumption requires that $u_a > 0$ for at least one a with $\pi^a > 0$. Moreover, this further implies that there is at least one environmental state a with $\pi^a > 0$ in which there is positive probability of receiving something such that $P[U_{n+1} = 0 \mid Z_n = a] < 1$. Consider cases where this requirement is not satisfied so that in all environmental states a with $\pi^a > 0$ receiving something is impossible so that $P[U_{n+1} = 0 \mid Z_n = a] = 1$ and $u_a = 0$. Clearly it is illogical to order in such situations. Therefore, this requirement is not restrictive. In our analysis of this section, we frequently refer to our assumption that $\sum_{a \in F} \pi^a u_a > 0$ for all π. This is equivalent to the condition that there is at least one environmental state a with $\pi^a > 0$ in which there is positive probability of receiving something. Note that if $u_a > 0$ for all a, then our assumption is trivially satisfied for all π since $\sum_{a \in F} \pi^a = 1$ and $\pi^a \geq 0$. Furthermore, $u_a \leq 1$ for all a since U_n is in $[0, 1]$. This fact together with our assumption imply that $0 < \sum_{a \in F} \pi^a u_a \leq 1$ for all π.

4.5 Linear cost model with sufficient statistics and observed capacity

As in Section 4.3, we assume that $K_i = 0$ for all i and the capacity is modulated by the observed environmental process Y. The remaining of this section proceeds as follows. In Section 4.5.1, we formulate and analyze the single-period problem and show that state-dependent base-stock policy is optimal. Next, in Section 4.5.2, we analyze multi-period problem and show the optimality of the state-dependent base-stock policy. Finally, in Section 4.5.3, we show that state-dependent base-stock policy is still optimal for infinite-period problem.

4.5.1 Single-period model

Here we assume that there is only one period to plan for so that $N = 1$; moreover, $v_1(\pi_1, i_1, x_1) = 0$ for all π_1, i_1 and x_1. Take $i_0 = i, a_0 = a$ and $\pi_0 = \pi$. Assuming inventory level at the beginning of period is x, the initial distribution of true state of environment is π and observed state of environment is i, single-period minimum cost function $v_0(\pi, i, x)$

satisfies

$$v_0(\pi, i, x) = \min_{y \geq x} H_0(\pi, i, x, y) \tag{4.54}$$

for all π, i and x, where y is the order-up-to level and

$$\begin{aligned} H_0(\pi, i, x, y) &= \sum_{a \in \mathbb{F}} \pi^a J_0(i, a, x, y) && (4.55) \\ J_0(i, a, x, y) &= u_a \int_0^{y-x} G_0(i, a, x+z) dF_i(z) + u_a G_0(i, a, y) [1 - F_i(y-x)] \\ &\quad + (1 - u_a) G_0(i, a, x) - c_i x && (4.56) \\ G_0(i, a, y) &= c_i y + L(i, a, y). && (4.57) \end{aligned}$$

where $L(i, a, y)$ is given in (4.14). From (4.57), it is obvious that G_0 is not a function of π. This also implies that J_0 doesn't depend on π. Remember that π is either initially known or can be determined by (4.40). In the latter case, it is obvious from (4.40) that $\pi^a = O_0(i, a)$. Therefore, our analysis in Section 4.3.1 is still applicable here. However, the subsequent analysis is conducted for the case where π is initially known.

The expected cost in a single period is the sum of expected purchase cost, and expected holding and shortage costs. Let $y_0(\pi, i, x)$ denote the optimal order-up-to level which minimizes the expected discounted cost in (4.55).In addition, let $v_0'(\pi, i, x)$ denote the derivative of $v_0(\pi, i, x)$ with respect to x.

Theorem 10 *The optimal ordering policy for the single-period model is a state-dependent base-stock policy*

$$y_0(\pi, i, x) = \begin{cases} S_0^{\pi,i} & x \leq S_0^{\pi,i} \\ x & x > S_0^{\pi,i}. \end{cases} \tag{4.58}$$

where $S_0^{\pi,i}$ *satisfies*

$$\sum_{a \in \mathbb{F}} \pi^a u_a \left(c_i + L'(i, a, S_0^{\pi,i}) \right) = 0$$

for all π *and* i. *In addition,* $H_0(\pi, i, x, y)$ *is quasi-convex in* y *for all* π, i *and* $y \geq x$. *The optimal cost incurred by this policy is*

$$v_0(\pi, i, x) = \begin{cases} H_0\left(\pi, i, x, S_0^{\pi,i}\right) & x \leq S_0^{\pi,i} \\ \sum_{a \in \mathbb{F}} \pi^a L(i, a, x) & x > S_0^{\pi,i} \end{cases} \tag{4.59}$$

for all π, i *and* x. *Furthermore,* $v_0(\pi, i, x)$ *is convex in* x, $\lim_{x \uparrow \infty} v_0'(\pi, i, x) = h_i$, *and* $\lim_{x \downarrow -\infty} v_0'(\pi, i, x) = -p_i$ *for all* π *and* i.

Proof. We need to find $y \geq x$ minimizing H_0. We denote first and second derivatives of H_0 with respect to y by H_0' and H_0'', respectively. Then,

$$H_0'(\pi, i, x, y) = [1 - F_i(y - x)] \sum_{a \in \mathbb{F}} \pi^a u_a \left(c_i + L'(i, a, y)\right) \tag{4.60}$$

$$H_0''(\pi, i, x, y) = [1 - F_i(y - x)] \sum_{a \in \mathbb{F}} \pi^a \left(u_a L''(i, a, y) - f_i(y - x)\left(c_i + L'(i, a, y)\right)\right) \tag{4.61}$$

for all π, i and $y \geq x$. Note that $1 - F_i(y - x) > 0$ for $y \geq x$; therefore, whether $H_0(\pi, i, x, y)$ is decreasing or increasing depends on the sign of the summation in (4.60). Let $S_0^{\pi,i}$ be the smallest y satisfying

$$B_0(\pi, i, y) = \sum_{a \in \mathbb{F}} \pi^a u_a \left(c_i + L'(i, a, y)\right) = 0. \tag{4.62}$$

Notice that, by our assumption, $\sum_{a \in F} \pi^a u_a > 0$ for all π. Then, $B_0(\pi, i, y)$ is nondecreasing since L is a convex function so that $c_i + L'(i, a, y)$ is nondecreasing. Therefore, $B_0(\pi, i, y) < 0$ for all $y < S_0^{\pi,i}$ and $B_0(\pi, i, y) \geq 0$ for $y \geq S_0^{\pi,i}$. In addition, $\lim_{y \uparrow \infty} B_0(\pi, i, y) = (h_i + c_i) \sum_{a \in \mathbb{F}} \pi^a u_a > 0$ and $\lim_{y \downarrow -\infty} B_0(\pi, i, y) = -(p_i - c_i) \sum_{a \in \mathbb{F}} \pi^a u_a < 0$. Then it follows that there exists finite $S_0^{\pi,i}$ satisfying (4.62).

From (4.61), it is obvious that the first term is always nonnegative. Now, consider the two cases: $x \leq S_0^{\pi,i}$ and $x > S_0^{\pi,i}$.

(i) $x \leq S_0^{\pi,i}$: The second term in (4.61) is always nonnegative for $y \in \left[x, S_0^{\pi,i}\right)$ since $B_0(\pi, i, y) < 0$ for $y < S_0^{\pi,i}$; as a result, (4.61) is nonnegative. Therefore, $H_0(\pi, i, x, y)$ is convex decreasing for all y in $\left[x, S_0^i\right)$. In addition, the second term in (4.61) continues to be nonnegative for y close to $S_0^{\pi,i}$; therefore, $H_0(\pi, i, x, y)$ is convex increasing for y close to $S_0^{\pi,i}$. However, the second term in (4.61) turns out to be negative for large values of $y > S_0^{\pi,i}$ since $\lim_{y \uparrow \infty}[1 - F_i(y - x)] = 0$ and $\lim_{y \uparrow \infty}(c_i + L'(i, a, y)) = c_i + h_i > 0$. Hence, $H_0(\pi, i, x, y)$ is concave increasing for sufficiently large values of $y \geq x$.

(ii) $x > S_0^{\pi,i}$: The second term in (4.61) is always nonpositive; however, (4.61) continues to be nonnegative for values of y close to $S_0^{\pi,i}$. Then, this implies that $H_0(\pi, i, x, y)$ is convex increasing for values of y close to $S_0^{\pi,i}$. But for large values of y, (4.61) turns out to be negative since $\lim_{y\uparrow\infty}[1-F_i(y-x)] = 0$ and $\lim_{y\uparrow\infty}(c_i + L'(i,a,y)) = c_i + h_i > 0$. Hence, $H_0(\pi, i, x, y)$ is concave nondecreasing for sufficiently large values of $y \geq x$.

This analysis shows that $H_0(\pi, i, x, y)$ is quasi-convex since it satisfies all conditions in Lemma 28. Therefore, $y = S_0^{\pi,i}$ is a global minimum of $H_0(\pi, i, x, y)$. This implies that $S_0^{\pi,i}$ is the optimal order-up-to level when $x \leq S_0^{\pi,i}$ and that it is optimal not to order when $x > S_0^{\pi,i}$. As a result, (4.58) gives the optimal ordering policy.

Because base-stock policy in (4.58) is optimal, the optimal cost is

$$v_0(\pi, i, x) = \begin{cases} H_0\left(\pi, i, x, S_0^{\pi,i}\right) & x \leq S_0^{\pi,i} \\ H_0(\pi, i, x, x) & x > S_0^{\pi,i} \end{cases}$$

which leads to (4.59) since $H_0(\pi, i, x, x) = \sum_{a\in\mathbb{F}} \pi^a L(i, a, x)$.

Now we prove that $v_0(\pi, i, x)$ is convex in x. First, we show that $v_0(\pi, i, x)$ is convex for $x < S_0^{\pi,i}$ and $x > S_0^{\pi,i}$ separately. Then we show that convexity is not violated at $x = S_0^{\pi,i}$.

(i) $x < S_0^{\pi,i}$: Using (4.59), the first and second derivatives of $v_0(\pi, i, x)$ are

$$\begin{aligned} v_0'(\pi, i, x) &= \sum_{a\in\mathbb{F}} \pi^a \Bigg(u_a \int_0^{S_0^{\pi,i}-x} \left(c_i + L'(i, a, x+z)\right) dF_i(z) \\ &\quad + (1-u_a)\, L'(i, a, x) - u_a c_i \Bigg) \end{aligned} \tag{4.63}$$

$$\begin{aligned} v_0''(\pi, i, x) &= \sum_{a\in\mathbb{F}} \pi^a \Bigg(u_a \int_0^{S_0^{\pi,i}-x} L''(i, a, x+z) dF_i(z) + (1-u_a)\, L''(i, a, x) \\ &\quad - f_i\left(S_0^{\pi,i} - x\right) u_a \left(c_i + L'(i, a, S_0^{\pi,i})\right) \Bigg). \end{aligned} \tag{4.64}$$

In (4.64), the first and the second summations are always nonnegative because L is convex. Moreover, the last expression in (4.64) is zero by (4.62). As a result, (4.64) is always nonnegative so that $v_0(\pi, i, x)$ is convex for all $x \leq S_0^{\pi,i}$.

(ii) $x > S_0^{\pi,i}$: Using (4.59), v_0 is convex because L is a convex function and the sum of convex functions is also convex.

(iii) $x = S_0^{\pi,i}$: Finally we show that convexity of v_0 is not violated at $x = S_0^{\pi,i}$. For v_0 to be convex at $x = S_0^{\pi,i}$, the following condition must hold

$$\lim_{x\uparrow S_0^{\pi,i}} v_0'(\pi, i, x) \leq \lim_{x\downarrow S_0^{\pi,i}} v_0'(\pi, i, x) \tag{4.65}$$

and $v_0(\pi,i,x)$ must be continuous at $x = S_0^{\pi,i}$. Using (4.63),

$$\begin{aligned}
\lim_{x\uparrow S_0^{\pi,i}} v_0'(\pi,i,x) &= \lim_{x\uparrow S_0^{\pi,i}} \left\{ \sum_{a\in\mathbb{F}} \pi^a \left(u_a \int_0^{S_0^{\pi,i}-x} \left(c_i + L'(i,a,x+z)\right) dF_i(z) \right.\right. \\
&\qquad \left.\left. + (1-u_a)\, L'(i,a,x) - u_a c_i \right) \right\} \\
&= \sum_{a\in\mathbb{F}} \pi^a \left(F_i(0) u_a \left(c_i + L'(i,a,S_0^{\pi,i}) \right) + L'(i,a,S_0^{\pi,i}) \right. \\
&\qquad \left. - u_a \left(c_i + L'(i,a,S_0^{\pi,i}) \right) \right) \\
&= \sum_{a\in\mathbb{F}} \pi^a L'(i,a,S_0^{\pi,i}).
\end{aligned}$$

The last equality follows from (4.62). Moreover, from (4.59),

$$\lim_{x\downarrow S_0^{\pi,i}} v_0'(\pi,i,x) = \sum_{a\in\mathbb{F}} \pi^a L'(i,a,S_0^{\pi,i}).$$

Then it follows that $\lim_{x\uparrow S_0^{\pi,i}} v_0'(\pi,i,x) = \lim_{x\downarrow S_0^{\pi,i}} v_0'(\pi,i,x)$ so that (4.65) is satisfied. Finally, $v_0(\pi,i,x)$ is continuous at $x = S_0^{\pi,i}$ since

$$\lim_{x\uparrow S_0^{\pi,i}} v_0(\pi,i,x) = \lim_{x\downarrow S_0^{\pi,i}} v_0(\pi,i,x) = \sum_{a\in\mathbb{F}} \pi^a L(i,a,S_0^{\pi,i}).$$

As a result, $v_0(\pi,i,x)$ is convex in x for all π and i. Furthermore, using (4.59), MCT and the fact that $\lim_{x\uparrow\infty} L'(i,a,x) = h_i$,

$$\lim_{x\uparrow\infty} v_0'(\pi,i,x) = \sum_{a\in\mathbb{F}} \pi^a \lim_{x\uparrow\infty} L'(i,a,x) = h_i$$

for all π and i. Similarly, using (4.63), MCT and the fact that $\lim_{x\downarrow-\infty} L'(i,a,x) = -p_i$,

$$\lim_{x\downarrow-\infty} v_0'(\pi,i,x) = -p_i$$

for all π and i. This completes our proof. ■

By Theorem 10, we get similar results as in Section 4.3.1. Namely, a state-dependent base-stock policy is optimal for the sufficient statistic formulation of the same problem in Section 4.3.1. The base-stock level can be obtained by solving (4.62) for minimum y. From (4.62), it is obvious that the base-stock level for a single-period problem is independent of capacity distribution and current inventory level. In other words, capacity has no effect on the optimal base-stock level for a single-period inventory problem in a partially-observed random environment.

Similar to Section 4.3.1, it is obvious from (4.60) that a base-stock policy is not necessarily optimal if the capacity process is modulated by the unobserved environment since then $1 - F_a\,(y - x)$ would be inside the summation in (4.60) and H_0 would have many local minima and maxima. Moreover, in this case, global minimum of H_0 will depend on the current inventory level x.

If we consider the case where the transporter is always available and the producer has infinite capacity so that $u_a = 1$ for all a and $F_i = 0$ for all i, then it is obvious from (4.56) that J_0 is convex in y since G_0 is convex. Then, it follows from (4.55) that H_0 is also convex in y. By following the same line of reasoning in the proof of Theorem 10, we can show that state-dependent base-stock policy is still optimal; however, notice from (4.62) that optimal base-stock level does not depend on availability probability. Moreover, the optimal cost function v_0 is still convex. Therefore, our results agree with those of Treharne and Sox (2002) who analyze inventory models in partially observed random demand environment.

4.5.2 Multi-period model

In the case with multiple periods, there are N periods to plan for and the dynamic programming equation involves the sum of single period costs in the current period plus the expected optimal discounted costs from the next period until the end of the planning horizon. We assume that $v_N\,(\pi_N, i_N, x_N) = 0$ for all π_N, i_N and x_N. Moreover, we suppose that inventory level is x, state of environment is i, distribution of true state of environment is π at time n. The minimum cost satisfies

$$v_n(\pi, i, x) = \min\nolimits_{y \geq x} H_n\,(\pi, i, x, y) \tag{4.66}$$

for all π, i and x, where y is the order-up-to level

$$H_n(\pi, i, x, y) = \sum_{a\in\mathbb{F}} \pi^a J_n(\pi, i, a, x, y) \tag{4.67}$$

$$J_n(\pi, i, a, x, y) = u_a \int_0^{y-x} G_n(\pi, i, a, x+z) dF_i(z) + u_a G_n(\pi, i, a, y)\left[1 - F_i(y-x)\right] + (1-u_a) G_n(\pi, i, a, x) - c_i x \tag{4.68}$$

$$G_n(\pi, i, a, y) = c_i y + L(i, a, y) + \alpha \sum_{j\in\mathbb{E}} \Psi_n(a, j) E_D^a \left[v_{n+1}\left(T(\pi\,|j), j, y-D\right)\right] \tag{4.69}$$

with $L(i,a,y)$ given in (4.14). Here, E_D^a again denotes the notation that expectation is taken with respect to the random variable D with distribution M_a.

Let $y_n(\pi, i, x)$ denote the optimal order-up-to level of the minimization problem in (4.66) given π, i and x. In addition, we assume that $R_{n,m}^{\alpha}((\pi, i), j) = \sum_{k=0}^{m-1} \alpha^k P_n^k((\pi, i), j)$ for all π, i and j and that, for a nonnegative function g, $R_{n,m}^{\alpha} g(\pi, i)$ is defined as (4.46). Finally, we let $v_n'(\pi, i, x)$ denote the derivative of $v_n(\pi, i, x)$ with respect to x and assume that h and p are holding cost and shortage cost vectors, respectively.

Theorem 11 *The optimal ordering policy for the N-period model is a state-dependent base-stock policy*

$$y_n(\pi, i, x) = \begin{cases} S_n^{\pi,i} & x \le S_n^{\pi,i} \\ x & x > S_n^{\pi,i} \end{cases} \tag{4.70}$$

where $S_n^{\pi,i}$ satisfies

$$\sum_{a\in\mathbb{F}} \pi^a u_a \left(c_i + L'(i, a, y) + \alpha \sum_{j\in\mathbb{E}} \Psi_n(a, j) E_D^a \left[v_{n+1}'\left(T(\pi\,|j), j, y-D\right)\right] \right) = 0$$

for all π and i. In addition, $H_n(\pi, i, x, y)$ is quasi-convex in y for all π, i and $y \ge x$. The optimal cost incurred by this policy is

$$v_n(\pi, i, x) = \begin{cases} H_n\left(\pi, i, x, S_n^{\pi,i}\right) & x \le S_n^{\pi,i} \\ \sum_{a\in\mathbb{F}} \pi^a \Big(L(i, a, x) + \alpha \sum_{j\in\mathbb{E}} \Psi_n(a, j) E_D^a\left[v_{n+1}\left(T(\pi\,|j), j, x-D\right)\right] \Big) & x > S_n^{\pi,i} \end{cases} \tag{4.71}$$

for all π, i and x. Furthermore, $v_n(\pi, i, x)$ is convex in x, $\lim_{x\uparrow\infty} v_n'(\pi, i, x) = R_{n,N-n}^{\alpha} h(\pi, i)$ and $\lim_{x\downarrow-\infty} v_n'(\pi, i, x) = -R_{n,N-n}^{\alpha} p(\pi, i)$ for all π and i.

Proof. The proof proceeds inductively. First of all, by Theorem 10, we know that Theorem 11 is satisfied for $n = N - 1$. Now assume that the induction hypothesis is valid for times $n+1, n+2, ..., N-1$ so that $v_{n+1}(\pi, i, x)$ is convex in x for all π and i; moreover, $\lim_{x\uparrow\infty} v'_{n+1}(\pi, j, x) = R^{\alpha}_{n+1,N-n-1}h(\pi, j)$ and $\lim_{x\downarrow-\infty} v'_{n+1}(\pi, j, x) = -R^{\alpha}_{n+1,N-n-1}p(\pi, j)$ for all π and j. We now show that Theorem 9 also holds for time n.

The value function in period n is given by (4.66) and we have to analyze H_n. We let $H'_n(\pi, i, x, y)$ denote derivative of $H_n(\pi, i, x, y)$ with respect to y. Then,

$$\begin{aligned} H'_n(\pi, i, x, y) \;\; = \;\; & [1 - F_i(y - x)] \sum_{a\in\mathbb{F}} \pi^a u_a \Bigg(c_i + L'(i, a, y) \\ & + \alpha \sum_{j\in\mathbb{E}} \Psi_n(a, j)\, E^a_D \left[v'_{n+1}(T(\pi\,|j), j, y - D) \right] \Bigg) \end{aligned} \tag{4.72}$$

for all π, i and $y \geq x$. Note that $1 - F_i(y - x) > 0$ for all $y \geq x$. Then, whether $H_n(\pi, i, x, y)$ is increasing or decreasing depends on the sign of the summation in (4.72). Moreover, the summation is a nondecreasing function of y since L and v_{n+1} are convex. Then, $H_n(\pi, i, x, y)$ is nonincreasing if (4.72) is nonpositive, and it is increasing otherwise. Let $S^{\pi,i}_n$ be the smallest y satisfying

$$\begin{aligned} B_n(\pi, i, y) \;\; = \;\; & \sum_{a\in\mathbb{F}} \pi^a u_a \Bigg(c_i + L'(i, a, y) \\ & + \alpha \sum_{j\in\mathbb{E}} \Psi_n(a, j)\, E^a_D \left[v'_{n+1}(T(\pi\,|j), j, y - D) \right] \Bigg) = 0. \end{aligned} \tag{4.73}$$

Note that, by our assumption, $\sum_{a\in F} \pi^a u_a > 0$ for all π. Then, this implies that $B_n(\pi, i, y)$ is nondecreasing since L and v_{n+1} are convex. Therefore, $B_n(\pi, i, y) < 0$ so that (4.72) is negative for $x \leq y < S^{\pi,i}_n$ and it is nonnegative for $x \leq y \geq S^{\pi,i}_n$ since L and v_{n+1} are convex. Using MCT and the induction hypothesis,

$$\lim_{y\uparrow\infty} B_n(\pi, i, y) = \sum_{a\in\mathbb{F}} \pi^a u_a \Bigg(c_i + h_i + \alpha \sum_{j\in\mathbb{E}} \Psi_n(a, j)\, R^{\alpha}_{n+1,N-n-1} h(T(\pi\,|j), j) \Bigg) > 0.$$

The inequality above follows from our assumptions that are $h_i > c_i > 0$ for all i and $\sum_{a\in F} \pi^a u_a > 0$ for all π. Similarly, using MCT and the induction hypothesis,

$$\lim_{y\downarrow-\infty} B_n(\pi, i, y) = \sum_{a\in\mathbb{F}} \pi^a u_a \Bigg(c_i - p_i - \alpha \sum_{j\in\mathbb{E}} \Psi_n(a, j)\, R^{\alpha}_{n+1,N-n-1} p(T(\pi\,|j), j) \Bigg) < 0.$$

This inequality comes from our assumptions that are $c_i < p_i$ for all i and $\sum_{a\in F} \pi^a u_a > 0$ for all π. Moreover, $B_n(\pi, i, y)$ is continuous and nondecreasing in y since L and v_{n+1} are convex. Then, it follows that there exists a finite $S_n^{\pi,i}$ satisfying (4.73). Now, we consider two cases: $x \leq S_n^{\pi,i}$ and $x > S_n^{\pi,i}$.

(i) $x \leq S_n^{\pi,i}$: Then (4.72) is negative so that $H_n(\pi, i, x, y)$ is decreasing for $y \in \left[x, S_n^{\pi,i}\right)$. However, (4.72) is nonnegative so that $H_n(\pi, i, x, y)$ is nondecreasing for $y \in \left[S_n^{\pi,i}, +\infty\right)$. Obviously, $S_n^{\pi,i}$ is the global minimum and $y_n(\pi, i, x) = S_n^{\pi,i}$ is the order-up-to level for $x \leq S_n^{\pi,i}$.

(ii) $x > S_n^{\pi,i}$: Then (4.72) is nonnegative so that $H_n(\pi, i, x, y)$ is nondecreasing for all $y \geq x$. As a result, no order should be given so that $y_n(\pi, i, x) = x$.

Furthermore, we can show by a similar discussion as in Section (4.5.1) that $H_n(\pi, i, x, y)$ is convex decreasing in y for all $y \leq S_n^{\pi,i}$, and it is convex increasing in y for all values of y greater than $S_n^{\pi,i}$ and close to $S_n^{\pi,i}$. We can also show that $H_n(\pi, i, x, y)$ is concave increasing in y for sufficiently large values of y greater than $S_n^{\pi,i}$. This analysis shows that $H_n(\pi, i, x, y)$ is quasi-convex since it satisfies all conditions in Lemma 28. Hence, base-stock policy defined in (4.70) is the optimal ordering policy. The minimum cost function corresponding to this optimal policy is

$$v_n(\pi, i, x) = \begin{cases} H_n\left(\pi, i, x, S_n^{\pi,i}\right) & x \leq S_n^{\pi,i} \\ H_n(\pi, i, x, x) & x > S_n^{\pi,i} \end{cases}$$

which leads to (4.71) since

$$H_n(\pi, i, x, x) = \sum_{a\in\mathbb{F}} \pi^a \left(L(i, a, x) + \alpha \sum_{j\in\mathbb{E}} \Psi_n(a, j)\, E_D^a\left[v_{n+1}\left(T(\pi\,|j), j, x - D\right)\right] \right).$$

We now prove that $v_n(\pi, i, x)$ is convex. First, we show that $v_n(\pi, i, x)$ is convex for $x < S_n^{\pi,i}$ and $x > S_n^{\pi,i}$, separately. Then we show that convexity is not violated at $x = S_n^{\pi,i}$.

(i) $x < S_n^{\pi,i}$: Using (4.71), the first and second derivatives of $v_n(i, x)$ are

$$\begin{aligned} v_n'(\pi, i, x) &= \sum_{a\in\mathbb{F}} \pi^a \Bigg(u_a \int_0^{S_n^{\pi,i}-x} G_n'(\pi, i, a, x+z)\, dF_i(z) \\ &\qquad + (1-u_a)\, G_n'(\pi, i, a, x) - c_i \Bigg) \end{aligned} \tag{4.74}$$

$$\begin{aligned} v_n''(\pi, i, x) &= \sum_{a\in\mathbb{F}} \pi^a \Bigg(u_a \int_0^{S_n^{\pi,i}-x} G_n''(\pi, i, a, x+z)\, dF_i(z) \\ &\qquad + (1-u_a)\, G_n''(\pi, i, a, x) - f_i\left(S_n^{\pi,i} - x\right) G_n'(\pi, i, a, S_n^{\pi,i}) \Bigg). \end{aligned} \tag{4.75}$$

where

$$G_n'(\pi,i,a,x) = c_i + L'(i,a,x) + \alpha \sum_{j\in\mathbb{E}} \Psi_n(a,j)\, E_D^a\left[v_{n+1}'\left(T(\pi\,|j),j,y-D\right)\right] \quad (4.76)$$

$$G_n''(\pi,i,a,x) = L''(i,a,x) + \alpha \sum_{j\in\mathbb{E}} \Psi_n(a,j)\, E_D^a\left[v_{n+1}''\left(T(\pi\,|j),j,y-D\right)\right]. \quad (4.77)$$

In (4.75), the first and second sums are always nonnegative because L and v_{n+1} are convex. Moreover, the last sum in (4.75) is zero by (4.73). As a result, (4.75) is always nonnegative so that $v_n(\pi,i,x)$ is convex for all $x \le S_n^{\pi,i}$.

(ii) $x > S_n^{\pi,i}$: Using (4.71), v_n is convex because L and v_{n+1} are convex, and sum of convex functions is also convex.

(iii) $x = S_n^{\pi,i}$: Finally we show that convexity of v_n is not violated at $x = S_n^{\pi,i}$. For v_n to be convex at $x = S_n^{\pi,i}$, the following condition must hold

$$\lim_{x\uparrow S_n^{\pi,i}} v_n'(\pi,i,x) \le \lim_{x\downarrow S_n^{\pi,i}} v_n'(\pi,i,x) \quad (4.78)$$

and $v_n(\pi,i,x)$ must be continuous at $x = S_n^{\pi,i}$. Using (4.74),

$$\begin{aligned}
\lim_{x\uparrow S_n^{\pi,i}} v_n'(\pi,i,x) &= \lim_{x\uparrow S_n^{\pi,i}} \Bigg\{ \sum_{a\in\mathbb{F}} \pi^a \Bigg(u_a \int_0^{S_n^{\pi,i}-x} G_n'(\pi,i,a,x+z)\,dF_i(z) \\
&\qquad + (1-u_a)\, G_n'(\pi,i,a,x) - c_i \Bigg) \Bigg\} \\
&= \sum_{a\in\mathbb{F}} \pi^a \left(u_a G_n'(\pi,i,a,S_n^{\pi,i}) F_i(0) + (1-u_a)\, G_n'(\pi,i,a,S_n^{\pi,i}) - c_i \right) \\
&= \sum_{a\in\mathbb{F}} \pi^a \Bigg(L'(i,a,S_n^{\pi,i}) \\
&\qquad + \alpha \sum_{j\in\mathbb{E}} \Psi_n(a,j)\, E_D^a\left[v_{n+1}'\left(T(\pi\,|j),j,S_n^{\pi,i}-D\right)\right] \Bigg).
\end{aligned}$$

The last equality follows from (4.73). Moreover, from (4.71),

$$\lim_{x\downarrow S_n^{\pi,i}} v_n'(\pi,i,x) = \sum_{a\in\mathbb{F}} \pi^a \Bigg(L'(i,a,S_n^{\pi,i}) + \alpha \sum_{j\in\mathbb{E}} \Psi_n(a,j)\, E_D^a\left[v_{n+1}'\left(T(\pi\,|j),j,S_n^{\pi,i}-D\right)\right] \Bigg).$$

Then it follows that $\lim_{x\uparrow S_n^{\pi,i}} v_n'(\pi,i,x) = \lim_{S_n^{\pi,i}} v_n'(\pi,i,x)$ so that (4.78) is satisfied.

Finally, $v_n(\pi, i, x)$ is continuous at $x = S_n^{\pi,i}$ since

$$\begin{aligned}\lim_{x \uparrow S_n^{\pi,i}} v_n(\pi, i, x) &= \lim_{x \downarrow S_n^{\pi,i}} v_n(\pi, i, x) \\ &= \sum_{a \in \mathbb{F}} \pi^a \Bigg(L(i, a, S_n^{\pi,i}) \\ &\qquad + \alpha \sum_{j \in \mathbb{E}} \Psi_n(a, j) \, E_D^a \left[v_{n+1}\left(T(\pi \,| j), j, S_n^{\pi,i} - D \right) \right] \Bigg).\end{aligned}$$

As a result, $v_n(\pi, i, x)$ is convex in x for all π and i. Moreover, using induction hypothesis, (4.71) and MCT

$$\begin{aligned}\lim_{x \uparrow \infty} v_n'(\pi, i, x) &= h_i + \alpha \sum_{a \in \mathbb{F}} \pi^a \sum_{j \in \mathbb{E}} \Psi_n(a, j) \, R^{\alpha}_{n+1, N-n-1} h(T(\pi \,| j), j) \\ &= h_i + \alpha \sum_{j \in \mathbb{E}} P_n((\pi_n, i), j) \, R^{\alpha}_{n+1, N-n-1} h(T(\pi \,| j), j) \\ &= R^{\alpha}_{n, N-n} h(\pi, i)\end{aligned}$$

The second equality follows from the definition of $P_n((\pi_n, i), j)$ in (4.43). And the last equality follows from (4.47). Similarly, using the induction hypothesis, (4.74), and MCT, we can also show that

$$\lim_{x \downarrow -\infty} v_n'(\pi, i, x) = -R^{\alpha}_{n, N-n} p(\pi, i).$$

This completes our proof. ■

Theorem 11 implies that a state-dependent base-stock policy is optimal in multiple periods for sufficient statistics formulation as it is optimal in Section 4.3.2. However, base-stock levels depend only on the information vector π and the observation i at any time. In addition, it is obvious from (4.73) that the order-up-to level is still independent of current inventory level x. The cost function H_n is quasi-convex as in single-period model; therefore, $S_n^{\pi,i}$ satisfying (4.73) is the global minimum. However, as in Section 4.5.1, if the producer's capacity is modulated by the unobserved environmental process, base-stock policy may not be optimal.

If we assume that the transporter is always available and the producer has infinite capacity so that $u_a = 1$ for all a and $F_i = 0$ for all i, we can show by induction that v_n is convex in x. In this case, it is clear from (4.68) that J_n is also convex since G_n is convex. This further implies that H_n in (4.67) is convex in x. Then, through a similar analysis as in the proof of Theorem 11, we can show that a state-dependent base-stock policy is still optimal for inventory models in a partially observed random demand environment. As in single-period, our results for multi-period problem still agree with results obtained by Treharne and Sox (2002) in finite-horizon.

4.5.3 Infinite-period model

In this section, we formulate and analyze the infinite-period problem. As stated earlier, we assume that $\{Q_n\}$ and $\{E_n\}$ are time-homogenous in infinite-period analysis so that $Q_n = Q$ and $E_n = E$ for all n. Then, this implies that $\Psi_n = \Psi$ is given by (4.49). Using Ψ, P_n can be determined from (4.48). Here, we show that the finite-horizon optimal cost function v_n in Section 4.5.2 converges to the infinite-horizon optimal cost function v. In addition, we show that a state-dependent base-stock policy is still optimal and the optimal discounted cost function of infinite-horizon problem is convex in inventory level x. By assuming that $k = N - n$ denotes the number of periods from time n until time N, we use the notation $v_{n,k}$ for the finite horizon optimal cost v_n in the remaining part of this section. Here, we show that, as k increases to infinity, the finite-horizon optimal cost function $v_{0,k}$ in (4.66) converges to the infinite-horizon optimal cost function v that satisfies

$$v(\pi, i, x) = \min_{y \geq x} H(\pi, i, x, y) \tag{4.79}$$

for all π, i and x, where y is the order-up-to level and

$$\begin{aligned} H(\pi, i, x, y) &= \sum_{a \in \mathbb{F}} \pi^a J(\pi, i, a, x, y) && (4.80) \\ J(\pi, i, a, x, y) &= u_a \int_0^{y-x} G(\pi, i, a, x+z) dF_i(z) + u_a G(\pi, i, a, y)\left[1 - F_i(y-x)\right] \\ &\quad + (1-u_a) G(\pi, i, a, x) - c_i x && (4.81) \\ G(\pi, i, a, y) &= c_i y + L(i, a, y) + \alpha \sum_{j \in \mathbb{E}} \Psi(a, j) E_D^a \left[v\left(T(\pi | j), j, y - D\right)\right] && (4.82) \end{aligned}$$

with $L(i, a, y)$ as given in (4.14). Again E_D^a denotes the notation that expectation is taken with respect to the random variable D with distribution M_a.

For any real valued function $f : \mathfrak{D}(\mathbb{F}) \times \mathbb{E} \times \mathbb{R} \to \mathbb{R}$ where $\mathfrak{D}(\mathbb{F})$ is the set of all probability distributions defined on state space $\mathbb{F}$, define the mapping $\mathcal{T}$ as

$$\mathcal{T} f(\pi, i, x) = \min_{y \geq x} H(\pi, i, x, y) \tag{4.83}$$

for all π, i and x, where H is given in (4.80) with J as given in (4.81) and

$$G(\pi, i, a, y) = c_i y + L(i, a, y) + \alpha \sum_{j \in \mathbb{E}} \Psi(a, j) E_D^a \left[f\left(T(\pi | j), j, y - D\right)\right]. \tag{4.84}$$

Using (4.54), $\mathcal{T} f$ can be interpreted as the optimal cost function for the one-period problem where the terminal cost function is $\sum_{j \in \mathbb{E}} \Psi(a, j) E_D^a \left[f\left(T(\pi | j), j, y - D\right)\right]$. Then, $\mathcal{T}^k$ denotes the composition of the mapping $\mathcal{T}$ with itself k times; that is, for all $k \geq 1$

$$\mathcal{T}^k f(\pi, i, x) = \mathcal{T}\mathcal{T}^{k-1} f(\pi, i, x) \tag{4.85}$$

with $\mathcal{T}^0 f = f$. Using (4.66), we can interpret $\mathcal{T}^k f$ as the optimal cost function for the $k-$period $\alpha-$discounted problem. Then, using (4.83) and (4.85),

$$\mathcal{T}^k f(\pi, i, x) = \min_{y \geq x} H_k(\pi, i, x, y) \tag{4.86}$$

where H_k is given in (4.80) with J_k as given in (4.81) with G replaced by

$$G_k(\pi, i, a, y) = c_i y + L(i, a, y) + \alpha \sum_{j \in \mathbb{E}} \Psi(a, j) E_D^a \left[\mathcal{T}^{k-1} f\left(T(\pi | j), j, y - D\right)\right]. \tag{4.87}$$

Let $f_0(\pi, i, x) = 0$ for all π, i and x. For our analysis in previous sections, we always assume that the terminal cost function is zero. Suppose that the initial cost function is $f_0(\pi, i, x)$ so that $\mathcal{T}^0 f(\pi, i, x) = f_0(\pi, i, x)$ for all π, i and x. Then, $k-$period optimal cost function is $v_{n,k}(\pi, i, x) = \mathcal{T}^k f_0(\pi, i, x)$ for all π, i, x and n.

Let $f_*(\pi, i, x)$ denote the optimal cost over infinite horizon and let

$$f_\infty(\pi, i, x) = \lim_{k \uparrow \infty} \mathcal{T}^k f_0(\pi, i, x) \tag{4.88}$$

for all π, i and x. Notice that f_∞ is well-defined provided we allow the possibility that f_∞ can take the value ∞. Our main aim in this section is to show that the finite-horizon optimal cost converges to the infinite-horizon optimal cost as the length of the planning horizon gets longer. In other words, we aim to show that $f_*(\pi, i, x) = f_\infty(\pi, i, x)$ for all π, i and x. As stated in Bertsekas (2000 b), it is analytically and computationally important to show that $f_*(\pi, i, x) = f_\infty(\pi, i, x)$ because if we know that $f_*(\pi, i, x) = \lim_{k \uparrow \infty} \mathcal{T}^k f_0(\pi, i, x)$, then we can infer the properties of $f_*(\pi, i, x)$ from the properties of $k-$period optimal cost functions $\mathcal{T}^k f_0(\pi, i, x)$.

Let $\mathcal{Z}_k$ denote the sets

$$\mathcal{Z}_k(\pi,i,x,\lambda)=\{y\geq x\,|H_k(\pi,i,x,y)\leq\lambda\} \tag{4.89}$$

for all π, i, x and $\lambda \in \mathbb{R}$. According to Proposition 1.7 in Bertsekas (2000 b, p. 148), if we show that the sets in (4.89) are compact for all π, i, x and λ, then $f_*(\pi,i,x) = f_\infty(\pi,i,x)$. By the following lemma, we accomplish this task.

Lemma 12 *Assume that* $\lim_{y\uparrow\infty} H_k(\pi,i,x,y) = \infty$ *for all* π, i, x *and* k. *The sets in (4.89) are compact subsets of the Euclidean space for all* π, i, x *and* λ.

Proof. We need to show that the sets in (4.89) are both bounded and closed in order to show that they are compact. Let us first show that the sets $\{\mathcal{Z}_k(\pi,i,x,\lambda)\}$ are bounded. Note that H_k is expected discounted cost when there are k periods until the end of planning horizon. Therefore, it is exactly the same as H_n in Section 4.5.2 where $n = N - k$. In multi-period analysis, we showed that $H_k(\pi,i,x,y)$ is nonincreasing for $y \in [x, S_k^{\pi,i}]$ and nondecreasing for $y \in [S_k^{\pi,i}, +\infty]$. Then, because we assume that $\lim_{y\uparrow\infty} H_k(\pi,i,x,y) = \infty$ for all π, i, x and k, the sets $\{\mathcal{Z}_k(\pi,i,x,\lambda)\}$ in (4.89) are bounded for all π, i, x and λ. Moreover, the sets $\{\mathcal{Z}_k(\pi,i,x,\lambda)\}$ are closed since $H_k(\pi,i,x,y)$ is continuous for $y \geq x$ and it is real valued. Thus, the sets in (4.89) are compact subsets of Euclidean space for all π, i, x and λ. ■

One of the cases when our assumption in Lemma 12 is satisfied is when $\lim_{y\uparrow\infty}[1 - F_i(y-x)] > 0$ for all i and x. Clearly, if $\lim_{y\uparrow\infty}[1 - F_i(y-x)] > 0$ for all i and x, then $\lim_{y\uparrow\infty} J_k(\pi,i,a,x,y) = \infty$ since $\lim_{y\uparrow\infty} G_k(\pi,i,a,y) = \infty$ and $\sum_{a\in F}\pi^a u_a > 0$ for all π, i, a and k. Clearly, this implies that $\lim_{y\uparrow\infty} H_k(\pi,i,x,y) = \infty$ for all π, i and x. Notice that this is not a restrictive requirement and it is only technically necessary. However, not all continuous distribution functions satisfy this requirement. As an example, assuming that the capacity distribution is exponential, cumulative distribution of capacity is $F_i(y-x) = 1 - e^{-\mu(y-x)}$ where $1/\mu$ is mean capacity. Then, it is obvious that $\lim_{y\uparrow\infty}[1 - F_i(y-x)] = 0$ for all i and x. Therefore, for probability distributions where $\lim_{y\uparrow\infty}[1 - F_i(y-x)] = 0$, we can use approximations such that $\lim_{y\uparrow\infty}[1 - F_i(y-x)] = \varepsilon > 0$ but it is very small like $\varepsilon = 10^{-10}$. In addition, we can also truncate the distribution F_i at a very large value and use this truncated distribution in place of F_i. Then, our assumption in Lemma 3 is satisfied so that the sets $\{\mathcal{Z}_k(\pi,i,x,\lambda)\}$ are compact.

The following proposition shows that f_∞ is a fixed point of the mapping $\mathcal{T}$; moreover, finite-horizon optimal cost function converges to the infinite-horizon optimal cost function.

Proposition 13 *The limit f_∞ is a fixed point of the mapping $\mathcal{T}$ so that*

$$f_\infty(\pi, i, x) = \mathcal{T} f_\infty(\pi, i, x) \tag{4.90}$$

for all π, i and x. Moreover,

$$f_\infty(\pi, i, x) = f_*(\pi, i, x) \tag{4.91}$$

for all π, i and x. Furthermore, there exists a stationary policy.

Proof. By Lemma 12, the sets in (4.89) are compact subsets of the Euclidean space for all π, i, x and λ. Then, using Proposition 1.7 in Bertsekas (2000 b, p. 148), f_∞ is a fixed point of $\mathcal{T}$ so that (4.90) is valid and there exists a stationary policy. In addition, notice that

$$f_0 \leq \mathcal{T} f_0 \leq ... \leq \mathcal{T}^k f_0 \leq ... \leq f_*$$

because expected cost per period is nonnegative. From this, we get $\lim_{k\uparrow\infty} \mathcal{T}^k f_0(\pi, i, x) \leq f_*(\pi, i, x)$ so that $f_\infty(\pi, i, x) \leq f_*(\pi, i, x)$. By (4.90), we know that f_∞ is a fixed point of $\mathcal{T}$. Then, by Proposition 1.2 in Bertsekas (2000 b, p. 140), we get that $f_*(\pi, i, x) \leq f_\infty(\pi, i, x)$. It follows that $f_\infty(\pi, i, x) = f_*(\pi, i, x)$. This completes our proof. ■

Notice that Proposition 13 implies also that f_∞, the optimal cost function that the finite-horizon cost function converges, satisfies the Bellman's equation since $f_\infty(i, x) = \mathcal{T} f_\infty(i, x)$ by (4.90). Hence,

$$f_\infty(\pi, i, x) = \min_{y \geq x} H(\pi, i, x, y) \tag{4.92}$$

for all π, i and x, where H is given in (4.80) with J as given in (4.81) and

$$G(\pi, i, a, y) = c_i y + L(i, a, y) + \alpha \sum_{j \in \mathbb{E}} \Psi(a, j) E_D^a \left[f_\infty(T(\pi | j), j, y - D) \right]. \tag{4.93}$$

As stated in Proposition 1.2 in Bertsekas (2000 b, p.140), f_∞ is not necessarily the unique optimal solution to Bellman's equation because single-period costs are not bounded under positivity assumption; however, f_∞ is the smallest fixed point of $\mathcal{T}$ since $f_\infty = f_*$.

Notice that, for a finite n, k goes to infinity as N goes to infinity. Then, our analysis shows that $\lim_{k\uparrow\infty} v_{0,k}(\pi, i, x) = v(\pi, i, x)$. Moreover, $v(\pi, i, x)$ satisfies (4.79) and there exists a stationary optimal policy $y(\pi, i, x)$ which minimizes the infinite-period total cost. However, notice that $H_k(\pi, i, x, y)$ is not bounded for $y \geq x$; therefore, v is not necessarily unique. Then, we take v as the minimal fixed point of (4.79). In other words, if $f = \mathcal{T}f$ is another solution, then $v \leq f$. Moreover, we also know that the optimal solution v is that fixed point of $\mathcal{T}$ which can be obtained as $v = \lim_{k\uparrow\infty} \mathcal{T}^k f_0$ with $f_0 = 0$.

Assuming π, i and x are current information vector, current observed state and current inventory level respectively, we let $y(\pi, i, x)$ denote the optimal order-up-to level of the minimization problem in (4.79). Let $v'(\pi, i, x)$ denote the partial derivative of $v(\pi, i, x)$ with respect to x. Finally, we again let h and p denote holding cost and shortage cost vectors respectively.

Theorem 14 *The optimal ordering policy for infinite-period model is a state-dependent base-stock policy*

$$y(\pi, i, x) = \begin{cases} S^{\pi,i} & x \leq S^{\pi,i} \\ x & x > S^{\pi,i} \end{cases} \tag{4.94}$$

where $S^{\pi,i}$ satisfies

$$\sum_{a\in\mathbb{F}} \pi^a u_a \bigg(c_i + L'(i, a, y) + \alpha \sum_{j\in\mathbb{E}} \Psi(a, j)\, E_D^a \left[v'\left(T(\pi\,|j), j, y - D\right)\right] \bigg) = 0$$

for all π and i. In addition, $H(\pi, i, x, y)$ is quasi-convex in y for all π, i and $y \geq x$. The optimal cost incurred by this policy is

$$v(\pi, i, x) = \begin{cases} H\left(\pi, i, x, S^{\pi,i}\right) & x \leq S^{\pi,i} \\ \sum_{a\in\mathbb{F}} \pi^a \Big(L(i, a, x) + \alpha \sum_{j\in\mathbb{E}} \Psi(a, j)\, E_D^a \left[v\left(T(\pi\,|j), j, x - D\right)\right] \Big) & x > S^{\pi,i} \end{cases} \tag{4.95}$$

for all π, i and x. Furthermore, $v(\pi, i, x)$ is convex in x $\lim_{x\uparrow\infty} v'(\pi, i, x) = R^\alpha h(\pi, i)$ and $\lim_{x\downarrow-\infty} v'(\pi, i, x) = -R^\alpha p(\pi, i)$ for all π and i.

Proof. As shown before, $v(\pi, i, x) = \lim_{k\uparrow\infty} v_{0,k}(\pi, i, x)$. Moreover, by Theorem 11, $v_{0,k}(\pi, i, x)$ is convex in x for all π and i. Then, $v(\pi, i, x)$ is convex because the limit of a convex function is also convex. In addition, by Theorem 11, $\lim_{x\uparrow\infty} v'_{0,k}(\pi, i, x) = R^\alpha h(\pi, i)$ and $\lim_{x\downarrow-\infty} v'_n(\pi, i, x) = -R^\alpha p(\pi, i)$ for all π and i, where R^α is given in (4.51). In addition, since $v_{0,k}$ is differentiable, it follows from Lemma 8-5 in Heyman and Sobel (1984) that

$v'(\pi,i,x) = \lim_{k\uparrow\infty} v'_{0,k}(\pi,i,x)$ for all π, i and x. Then, $\lim_{x\uparrow\infty} v'(\pi,i,x) = R^\alpha h(\pi,i)$ and $\lim_{x\downarrow-\infty} v'(\pi,i,x) = -R^\alpha p(\pi,i)$ for all π and i.

We denote the partial derivative of H with respect to y by H'. Then,

$$\begin{aligned} H'(\pi,i,x,y) &= [1-F_i(y-x)]\sum_{a\in\mathbb{F}}\pi^a u_a\Big(c_i + L'(i,a,y) \\ &\quad + \alpha\sum_{j\in\mathbb{E}}\Psi(a,j)\,E_D^a\left[v'(T(\pi\,|j\,),j,y-D)\right]\Big) \end{aligned} \tag{4.96}$$

for all $y \geq x$. Note that $1-F_i(y-x) > 0$ for all $y \geq x$. Then whether $H(\pi,i,x,y)$ is increasing or decreasing depends on the sign of the summation in (4.96). Moreover, the summation is a nondecreasing function of y since L and v are convex. Then, $H(\pi,i,x,y)$ is nonincreasing if (4.96) is nonpositive, and it is increasing otherwise. Let $S^{\pi,i}$ be the smallest y satisfying

$$\begin{aligned} B(\pi,i,y) &= \sum_{a\in\mathbb{F}}\pi^a u_a\Big(c_i + L'(i,a,y) \\ &\quad + \alpha\sum_{j\in\mathbb{E}}\Psi(a,j)\,E_D^a\left[v'(T(\pi\,|j\,),j,y-D)\right]\Big) = 0 \end{aligned} \tag{4.97}$$

Note that, by our assumption, $\sum_{a\in F}\pi^a u_a > 0$ for all π.. Then, this implies that $B(\pi,i,y)$ is nondecreasing since L and v are convex. Therefore, $B(\pi,i,y) < 0$ so that (4.96) is negative for $x \leq y < S^{\pi,i}$ and it is nonnegative for $x \leq y \geq S^{\pi,i}$ since L and v are convex. Using MCT and the fact that $\lim_{x\uparrow\infty} v'(\pi,j,x) = R^\alpha h(\pi,j)$

$$\lim_{y\uparrow\infty} B(\pi,i,y) = \sum_{a\in\mathbb{F}}\pi^a u_a\Big(c_i + h_i + \alpha\sum_{j\in\mathbb{E}}\Psi(a,j)\,R^\alpha h(T(\pi\,|j\,),j)\Big) > 0.$$

Similarly, using MCT and the fact that $\lim_{x\downarrow-\infty} v'(\pi,j,x) = -R^\alpha p(\pi,j)$

$$\lim_{y\downarrow-\infty} B(\pi,i,y) \leq \sum_{a\in\mathbb{F}}\pi^a u_a\Big(c_i - p_i - \alpha\sum_{j\in\mathbb{E}}\Psi(a,j)\,R^\alpha p(T(\pi\,|j\,),j)\Big) < 0.$$

Hence, there exists a finite $S^{\pi,i}$ satisfying (4.97). Moreover, by a similar discussion as in Section 4.5.1 and Section 4.5.2, we can show that $H(\pi,i,x,y)$ is decreasing for all $y < S^{\pi,i}$ and nondecreasing for all $y \geq S^{\pi,i}$. Therefore, it is quasi-convex. Then, it follows that a state-dependent base-stock policy in (4.94) is optimal. The optimal cost function through application of this base-stock policy is

$$v(\pi,i,x) = \begin{cases} H\left(\pi,i,x,S^{\pi,i}\right) & x \leq S^{\pi,i} \\ H(\pi,i,x,x) & x > S^{\pi,i} \end{cases}$$

which leads to (4.95) since

$$H(\pi, i, x, x) = \sum_{a \in \mathbb{F}} \pi^a \Bigg(L(i, a, x) + \alpha \sum_{j \in \mathbb{E}} \Psi(a, j) \, E_D^a \left[v\left(T\left(\pi \, |j\right), j, x - D \right) \right] \Bigg).$$

This completes our proof. ■

By Theorem 14, we show that a state-dependent base-stock policy is still optimal in infinite period as in single and multiple planning periods. At the beginning of each period, an order is given if an only if the inventory level is less than a particular value $S^{\pi,i}$ which depends on current information vector and observed state. However, unlike the multi-period model, $S^{\pi,i}$ is independent of the number of period in which we are planning for; therefore, they are the same in all periods with the same information vector and observed state. Thus, the optimal solution is stationary whereas it is not necessarily unique.

By Theorem 14, we also show that $\lim_{x \uparrow \infty} v'(\pi, i, x) = R^\alpha h(\pi, i)$ and $\lim_{x \downarrow -\infty} v'(\pi, i, x) = -R^\alpha p(\pi, i)$ for all π and i. Because state space $\mathbb{F}$ is finite and $\alpha \in (0, 1)$, by 4.53, $\lim_{x \uparrow \infty} v'(\pi, i, x) = h_i + \pi(I - \alpha Q)^{-1} Eh - \pi Eh$ and $\lim_{x \downarrow -\infty} v'(\pi, i, x) = -p_i - \pi(I - \alpha Q)^{-1} Ep + \pi Ep$ for all π and i, since h_i and p_i are nonnegative and bounded for all i.

If we assume that the transporter is always available and the producer has infinite capacity, then the formulation in (4.79)-(4.82) must be rearranged by assuming $u_a = 1$ for all a and $F_i = 0$ for all i. Note that Treharne and Sox (2002) analyze the finite-horizon problem only; therefore, inventory models of this type in infinite-period are not considered in the literature before. Treharne and Sox (2002) show that the finite-horizon optimal cost function is convex. Then, via a similar discussion as in this section, we can show that infinite-horizon optimal cost function v is also convex. It follows by (4.82) that G is convex in y so that both J and H are also convex. Through a similar discussion as in the proof of Theorem 14, we can show that a state-dependent base-stock policy is optimal in infinite planning period for inventory models in a partially observed random environment which modulates demand only.

4.6 Inventory models with unreliable suppliers

In this section, we assume that the capacity of the supplier is infinite so that all of order can be satisfied. However, the supplier is not always available. Let U_n in Section 4.1 denote the availability of the supplier at period n instead of availability of transporter. We assume that U_n is either 1 or 0. In other words, the supplier is either available or unavailable. As a result, the ordered quantity is either totally received or nothing is received. Obviously, this scenario covers the case where the supplier has infinite capacity; however, the transporter is randomly available. Moreover, we assume that the availability process $U = \{U_n; n \geq 1\}$ is modulated by the observed environment so that u_i is the probability that the supplier is available when the observed environment is i. Finally, without loss of generality, we assume that $u_i > 0$ for all i. This is a reasonable assumption since ordering is illogical when $u_i = 0$ so that nothing is received with certainty.

Özekici and Parlar (1999) analyze exactly the same problem in random environment with perfect information. They show that a state-dependent base-stock policy is optimal when there is no fixed cost of ordering and that a state-dependent (s, S) policy is optimal when there is fixed cost of ordering. In this section, we extend the results in Özekici and Parlar (1999) and assume that the random environment is not fully observed, but, it is only partially observed.

If we assume that the availability process of supplier is not observable so that U_n depends on the state of the unobserved environment, then our analysis in Section 4.5 obviously covers the case where there is no fixed cost of ordering so that $K_i = 0$ for all i. In other words, the inventory model with random supply due to randomly available suppliers in a partially-observed random environment is a special case of the inventory model in Section 4.5 where $F_i = 0$ for all i since the supplier always has enough capacity. The same analysis as in Section 4.5 is still applicable in this case. Moreover, similar results will be obtained (i.e., state-dependent base-stock policy is still optimal and optimal cost function is still convex) with one exception, that is the expected discounted cost function H in single, multiple and infinite planning periods is not only quasiconvex but also convex.

Moreover, if we assume that the availability process of supplier is observable, some modifications in dynamic programming formulations of Section 4.5 are necessary. For example, if we consider the multi-period problem only, the dynamic programming formulation for observed availability process case is similar to (4.66)-(4.69). However, u_a in (4.68) is replaced by u_i since availability process is observable and $F_i = 0$ for all i since the supplier always has sufficient capacity. Despite these modifications, basic results (i.e., a state-dependent base-stock policy is optimal and the optimal cost function is convex) of our analysis are still valid. But note that the expected discounted cost function H is not only quasi-convex but also convex in this case. Because the analysis in Section 4.5 with simple modifications can be applied, we do not analyze the linear cost model when the availability process of supplier is observed. In the remaining of this section, we analyze the case where there is a fixed cost of ordering so that $K_i > 0$ for all i. In Section 4.6.1, we analyze the fixed-cost model in single-period. Next, in Section 4.6.2, we study the problem in multi-period case. Finally, in Section 4.6.3, we analyze the infinite-period problem.

4.6.1 Single-period model

Here we assume that there is only one period to plan for so that $N = 1$; moreover, $v_1(\pi_1, i_1, x_1) = 0$ for all π_1, i_1 and x_1. Take $i_0 = i$ and $a_0 = a$ and $\pi_0 = \pi$. We assume that the vinformation vector, observed state of environment and inventory level are π, i and x at the beginning of the period respectively. Then, the single-period minimum cost function $v_0(\pi, i, x)$ satisfies

$$v_0(\pi, i, x) = \min_{y \geq x} \{K_i \delta(y - x) + J_0(\pi, i, x, y)\} \tag{4.98}$$

for all π, i and x, where $\delta(z)$ is the indicator function which is equal to 1 only if $z > 0$ and 0 otherwise, y is the order-up-to level and

$$\begin{aligned} J_0(\pi, i, x, y) &= u_i \sum_{a \in \mathbb{F}} \pi^a G_0(i, a, y) + (1 - u_i) \sum_{a \in \mathbb{F}} \pi^a G_0(i, a, x) - c_i x && (4.99) \\ G_0(i, a, y) &= c_i y + L(i, a, y) && (4.100) \end{aligned}$$

with $L(i, a, y)$ given in (4.14).

Expected cost in a single period is the sum of expected order cost, and expected holding and shortage costs. Let $y_0(\pi, i, x)$ denote the optimal order-up-to level for the minimization problem in (4.98).

Theorem 15 *The optimal ordering policy for single-period model is a state-dependent (s, S) policy*

$$y_0(\pi, i, x) = \begin{cases} S_0^{\pi,i} & x \leq s_0^{\pi,i} \\ x & x > s_0^{\pi,i} \end{cases} \tag{4.101}$$

where $S_0^{\pi,i}$ is the smallest value that satisfies

$$c_i + \sum_{a\in\mathbb{F}} \pi^a L'\left(i, a, S_0^{\pi,i}\right) = 0$$

and $s_0^{\pi,i} \leq S_0^{\pi,i}$ satisfies

$$u_i \sum_{a\in\mathbb{F}} \pi^a G_0\left(i, a, s_0^{\pi,i}\right) = K_i + u_i \sum_{a\in\mathbb{F}} \pi^a G_0\left(i, a, S_0^{\pi,i}\right)$$

for all π and i. In addition, $J_0(\pi, i, x, y)$ is K_i-convex in y for all π, i and x. The optimal cost incurred by this policy is

$$v_0(\pi, i, x) = \begin{cases} u_i \sum_{a\in\mathbb{F}} \pi^a G_0\left(i, a, s_0^{\pi,i}\right) & \\ \quad + (1-u_i) \sum_{a\in\mathbb{F}} \pi^a G_0(i, a, x) - c_i x & x \leq s_0^{\pi,i} \\ \sum_{a\in\mathbb{F}} \pi^a G_0(i, a, x) - c_i x & x > s_0^{\pi,i} \end{cases} \tag{4.102}$$

for all π and i. Moreover, $v_0(\pi, i, x)$ is continuous and K_i-convex in x for all π and i.

Proof. We denote first and second derivatives of J_0 with respect to y by

$$J_0'(\pi, i, x, y) = u_i \sum_{a\in\mathbb{F}} \pi^a \left(c_i + L'(i, a, y)\right) \tag{4.103}$$

$$J_0''(\pi, i, x, y) = u_i \sum_{a\in\mathbb{F}} \pi^a L''(i, a, y) \tag{4.104}$$

for all π, i, x and y. Notice that $u_i > 0$ for all i. In addition, $L'' \geq 0$ since L is convex. As a result, $J_0'' \geq 0$ so that $J_0(\pi, i, x, y)$ is convex in y. Moreover, by part (a) of Lemma 29, we know that every convex function is also K_i-convex for $K_i \geq 0$. Therefore, J_0 is K_i-convex in y for all π, i and x. In addition, J_0 is continuous in y for all π, i and x since every convex function is also continuous.

By (4.99), $\lim_{|y|\uparrow\infty} J_0(\pi, i, x, y) = +\infty$. Furthermore, J_0 is continuous. Then, by part (d) of Lemma 29, there exist scalars $s_0^{\pi,i}$ and $S_0^{\pi,i}$ with $s_0^{\pi,i} \leq S_0^{\pi,i}$ satisfying four conditions in part (d) of the same lemma. Clearly, by (i) of part (d) of Lemma 29, $S_0^{\pi,i}$ minimizes J_0. Then, the first order optimality condition must be satisfied at $y = S_0^{\pi,i}$ so that

$$c_i + \sum_{a\in\mathbb{F}} \pi^a L'\left(i, a, S_0^{\pi,i}\right) = 0. \tag{4.105}$$

In addition, by (ii) of part (d) of Lemma 29, $s_0^{\pi,i}$ satisfies

$$J_0\left(\pi, i, x, s_0^{\pi,i}\right) = K_i + J_0\left(\pi, i, x, S_0^{\pi,i}\right)$$

or

$$u_i \sum_{a\in\mathbb{F}} \pi^a G_0\left(i,a,s_0^{\pi,i}\right) = K_i + u_i \sum_{a\in\mathbb{F}} \pi^a G_0\left(i,a,S_0^{\pi,i}\right). \tag{4.106}$$

Together with (iii) and (iv) of part (d) of Lemma 29, we can claim that a state-dependent (s,S) policy

$$y_0(\pi,i,x) = \begin{cases} S_0^{\pi,i} & x \le s_0^{\pi,i} \\ x & x > s_0^{\pi,i} \end{cases}$$

is the optimal policy. From (4.105) and (4.106), it is clear that both $s_0^{\pi,i}$ and $S_0^{\pi,i}$ depend on the current information vector and current state of environment; however, they are independent of the current inventory level. Then, the optimal cost incurred by this policy is

$$v_0(\pi,i,x) = \begin{cases} K_i + u_i \sum_{a\in\mathbb{F}} \pi^a G_0(i,a,S_0^{\pi,i}) + (1-u_i)\sum_{a\in\mathbb{F}} \pi^a G_0(i,a,x) - c_i x & x \le s_0^{\pi,i} \\ \sum_{a\in\mathbb{F}} \pi^a G_0(i,a,x) - c_i x & x > s_0^{\pi,i} \end{cases}$$

which leads to (4.102) by using (4.106).

In addition, the right and left derivatives of v_0 at $x = s_0^{\pi,i}$ are

$$\lim_{x\uparrow s_0^{\pi,i}} v_0'(\pi,i,x) = \sum_{a\in\mathbb{F}} \pi^a L'\left(i,a,s_0^{\pi,i}\right) - u_i \sum_{a\in\mathbb{F}} \pi^a \left(c_i + L'\left(i,a,s_0^{\pi,i}\right)\right)$$

and

$$\lim_{x\downarrow s_0^{\pi,i}} v_0'(\pi,i,x) = \sum_{a\in\mathbb{F}} \pi^a L'\left(i,a,s_0^{\pi,i}\right)$$

for all π and i. Notice that $\lim_{x\uparrow s_0^{\pi,i}} v_0'(\pi,i,x) \ge \lim_{x\downarrow s_0^{\pi,i}} v_0'(\pi,i,x)$ because, by (4.105), $\sum_{a\in\mathbb{F}} \pi^a (c_i + L'(i,a,x)) < 0$ for all π, i and $x < S_0^{\pi,i}$, and $u_i > 0$ for all i. Then it follows that the left derivative of v_0 at $s_0^{\pi,i}$ is greater than the right derivative at the same point. This clearly implies that v_0 is not convex at $x = s_0^{\pi,i}$.

Finally, we now show that $v_0(\pi,i,x)$ is continuous and K_i−convex in x for all π and i. We must verify that for all $z \ge 0, b > 0$, and x, we have

$$K_i + v_0(\pi,i,x+z) \ge v_0(\pi,i,x) + \frac{z}{b}\left[v_0(\pi,i,x) - v_0(\pi,i,x-b)\right]. \tag{4.107}$$

We distinguish three cases:

Case 1: $x > s_0^{\pi,i}$. If $x - b > s_0^{\pi,i}$, then in this region of values of z, b and x, v_0, by (4.102), is the sum of a convex function and a linear function; therefore, it is convex. Then, by part

(a) of Lemma 29, v_0 is K_i-convex and (4.107) holds if $x-b>s_0^{\pi,i}$. If $x-b\leq s_0^{\pi,i}$, then in view of (4.102) we can write the condition (4.107) after some simplification as

$$\begin{aligned} K_i+\sum_{a\in\mathbb{F}}\pi^a G_0\left(i,a,x+z\right) \quad &\geq \quad \sum_{a\in\mathbb{F}}\pi^a G_0\left(i,a,x\right) \\ &\qquad +u_i\left(\frac{z}{b}\right)\sum_{a\in\mathbb{F}}\pi^a\left[G_0\left(i,a,x\right)-G_0\left(i,a,s_0^{\pi,i}\right)\right] \\ &\qquad +\left(1-u_i\right)\left(\frac{z}{b}\right)\sum_{a\in\mathbb{F}}\pi^a \\ &\qquad \times\left[G_0\left(i,a,x\right)-G_0\left(i,a,x-b\right)\right]. \qquad (4.108)\end{aligned}$$

Now, if x is such that $\sum_{a\in\mathbb{F}}\pi^a\left[G_0\left(i,a,x\right)-G_0\left(i,a,s_0^{\pi,i}\right)\right]\leq 0$, then we have

$$\begin{aligned} K_i+u_i\sum_{a\in\mathbb{F}}\pi^a G_0\left(i,a,x+z\right) \quad &\geq \quad K_i+u_i\sum_{a\in\mathbb{F}}\pi^a G_0\left(i,a,S_0^{\pi,i}\right) \\ &= \quad u_i\sum_{a\in\mathbb{F}}\pi^a G_0\left(i,a,s_0^{\pi,i}\right) \\ &\geq \quad u_i\sum_{a\in\mathbb{F}}\pi^a G_0\left(i,a,x\right) \\ &\geq \quad u_i\sum_{a\in\mathbb{F}}\pi^a G_0\left(i,a,x\right)+u_i\left(\frac{z}{b}\right)\sum_{a\in\mathbb{F}}\pi^a \\ &\qquad \times\left[G_0\left(i,a,x\right)-G_0\left(i,a,s_0^{\pi,i}\right)\right]. \qquad (4.109)\end{aligned}$$

The first inequality follows from (i) of part (d) of Lemma 29 and the fact that $S_0^{\pi,i}$ is optimal x of the minimization problem in (4.98). The next equality follows from (4.106). The third and the last inequalities follow from the fact that $\sum_{a\in\mathbb{F}}\pi_a^a\left[G_0\left(i,a,x\right)-G_0\left(i,a,s_0^{\pi,i}\right)\right]\leq 0$. Moreover, because $G_0\left(i,a,x\right)$ is convex and the sum of convex functions is also convex, $\left(1-u_i\right)\sum_{a\in\mathbb{F}}\pi^a G_0\left(i,a,x\right)$ is convex so that

$$\begin{aligned} \left(1-u_i\right)\sum_{a\in\mathbb{F}}\pi^a G_0\left(i,a,x+z\right) \quad &\geq \quad \left(1-u_i\right)\sum_{a\in\mathbb{F}}\pi^a G_0(i,a,x) \\ &\qquad +\left(1-u_i\right)\left(\frac{z}{b}\right)\sum_{a\in\mathbb{F}}\pi^a\Big[G_0\left(i,a,x\right) \\ &\qquad -G_0\left(i,a,x-b\right)\Big] \qquad (4.110)\end{aligned}$$

By summing (4.109) and (4.110), we get (4.108). If $\sum_{a\in\mathbb{F}}\pi^a\left[G_0\left(i,a,x\right)-G_0\left(i,a,s_0^{\pi,i}\right)\right]\leq 0$, (4.107) and (4.108) hold. If x is such that $\sum_{a\in\mathbb{F}}\pi^a\left[G_0\left(i,a,x\right)-G_0\left(i,a,s_0^{\pi,i}\right)\right]>0$, then we have

$$\begin{aligned} K_i + u_i \sum_{a \in \mathbb{F}} \pi^a G_0\left(i, a, x+z\right) \quad &\geq \quad u_i \sum_{a \in \mathbb{F}} \pi^a G_0\left(i, a, x\right) \\ &+ u_i \left(\frac{z}{x - s_0^{\pi,i}} \right) \sum_{a \in \mathbb{F}} \pi^a \left[G_0\left(i, a, x\right) - G_0\left(i, a, s_0^{\pi,i}\right) \right] \end{aligned}$$

because $u_i \sum_{a \in \mathbb{F}} \pi^a G_0\left(i, a, x\right)$ is convex, it is also K_i-convex. Note that $0 < x - s_0^{\pi,i} \leq b$ and $\sum_{a \in \mathbb{F}} \pi^a \left[G_0\left(i, a, x\right) - G_0\left(i, a, s_0^{\pi,i}\right) \right] > 0$. Then, it follows that

$$\begin{aligned} K_i + u_i \sum_{a \in \mathbb{F}} \pi^a G_0\left(i, a, x+z\right) \quad &\geq \quad u_i \sum_{a \in \mathbb{F}} \pi^a G_0\left(i, a, x\right) \\ &+ u_i \left(\frac{z}{b} \right) \sum_{a \in \mathbb{F}} \pi^a \\ &\times \left[G_0\left(i, a, x\right) - G_0\left(i, a, s_0^{\pi,i}\right) \right]. \qquad (4.111) \end{aligned}$$

By summing (4.110) and (4.111), we get (4.108). If $\sum_{a \in \mathbb{F}} \pi^a \left[G_0\left(i, a, x\right) - G_0\left(i, a, s_0^{\pi,i}\right) \right] > 0$, (4.107) and (4.108) also hold.

Case 2: $x \leq x + z \leq s_0^{\pi,i}$. In this region, by (4.102), the function v_0 is sum of a convex function and a linear function; therefore, v_0 is convex. Then, by part (a) of Lemma 29, it is also K_i-convex.

Case 3: $x < s_0^{\pi,i} < x + z$. For this case, in view of (4.102), we can write condition (4.107) as

$$\begin{aligned} K_i + \sum_{a \in \mathbb{F}} \pi^a G_0\left(i, a, x+z\right) \quad &\geq \quad u_i \sum_{a \in \mathbb{F}} \pi^a G_0\left(i, a, s_0^{\pi,i}\right) + \left(1 - u_i\right) \sum_{a \in \mathbb{F}} \pi^a G_0\left(i, a, x\right) \\ &+ \left(1 - u_i\right) \left(\frac{z}{b} \right) \sum_{a \in \mathbb{F}} \pi^a \\ &\times \left[G_0\left(i, a, x\right) - G_0\left(i, a, x - b\right) \right]. \qquad (4.112) \end{aligned}$$

Notice that $u_i \sum_{a \in \mathbb{F}} \pi^a G_0\left(i, a, x\right)$ is convex and, by part (a) of Lemma 29, it is also K_i-convex. Then, we have

$$K_i + u_i \sum_{a \in \mathbb{F}} \pi^a G_0\left(i, a, x+z\right) \geq u_i \sum_{a \in \mathbb{F}} \pi^a G_0\left(i, a, s_0^{\pi,i}\right) \qquad (4.113)$$

by (iv) of part (d) of Lemma 29. Moreover, convexity of $\left(1 - u_i\right) \sum_{a \in \mathbb{F}} \pi^a G_0\left(i, a, x\right)$ implies (4.110). Then, by summing (4.110) and (4.113), we get (4.112). As a result, (4.107) holds for case 3. Finally, for all three cases, v_0 is K_i-convex in x.

We only need to check $v_0(\pi, i, x)$ at $x = s_0^{\pi,i}$ for continuity. Note that $v_0(\pi, i, x)$ is continuous at $s_0^{\pi,i}$ because

$$\lim_{x \uparrow s_0^{\pi,i}} v_0(\pi, i, x) = \lim_{x \downarrow s_0^{\pi,i}} v_0(\pi, i, x) = \sum_{a \in \mathbb{F}} \pi^a G_0(i, a, s_0^{\pi,i}) - c_i s_0^{\pi,i}.$$

This completes our proof. ■

By Theorem 15, we show that v_0 is K_i-convex so that (4.107) is valid. Note that (4.107) is still valid if we use K_i/u_i instead of K_i in the left hand side of this equation since $K_i/u_i > K_i$ for $u_i \in (0, 1)$. This clearly implies that $v_0(\pi, i, x)$ is also $(K_i/u_i)-$convex in x for all π and i. In addition, we show by Theorem 15 that a state-dependent (s, S) policy continues to be optimal in single-period for inventory problems with unreliable suppliers in a partially observed random environment. However, s and S now depend not only on the current state of observed environment but also on the current information vector.

Notice also that the analysis in this section is still valid when the availability of supplier is modulated by the unobserved process. Then, u_a denotes the probability of the supplier being available when state of unobserved environment is a. In this case several modifications in J_0 and in the proof of Theorem 15 are necessary. For example, u_a and $(1 - u_a)$ will be inside the summation in (4.99). Then, the proof must be rearranged for new J_0 accordingly. However, the basic results in Theorem 15 are still valid. Moreover, our analysis is also valid when the cost parameters and supplier availability probabilities are constant so that they are independent of observation process Y and unobserved process Z.

4.6.2 Multi-period model

Suppose that there are N periods to plan for and the dynamic programming equation involves the sum of expected single period costs in the current period plus the expected optimal discounted costs from the next period until the end of the planning horizon. We assume that $v_N(\pi_N, i_N, x_N) = 0$ for all π_N, i_N and x_N. The distribution π of the true state of environment, the state i of the observed environment and inventory level x at any time are given; moreover, demand in period $n + 1$ is D. Then, the minimum cost satisfies

$$v_n(\pi, i, x) = \min_{y \geq x} \{K_i \delta(y - x) + J_n(\pi, i, x, y)\} \tag{4.114}$$

for all π, i and x, where y is the order-up-to level and

$$J_n(\pi, i, x, y) = u_i \sum_{a \in \mathbb{F}} \pi^a G_n(\pi, i, a, y) + (1 - u_i) \sum_{a \in \mathbb{F}} \pi^a G_n(\pi, i, a, x) - c_i x \quad (4.115)$$

$$G_n(\pi, i, a, y) = c_i y + L(i, a, y) + \alpha \sum_{j \in \mathbb{E}} \Psi_n(a, j) E_D^a [v_{n+1}(T(\pi | j), j, y - D)] \quad (4.116)$$

with $L(i, a, y)$ given in (4.14). Here, E_D^a again denotes the notation that expectation is taken with respect to the random variable D with distribution M_a.

In this section, we show that a state-dependent (s, S) policy is still optimal; however, this requires an additional assumption on fixed-ordering costs $\{K_i\}$ and availability probabilities $\{u_i\}$. In particular, we assume that

$$\left(\frac{K_i}{u_i}\right) \geq \alpha \sum_{j \in \mathbb{E}} \Psi_n(a, j) \left(\frac{K_j}{u_j}\right) \quad (4.117)$$

for all i, a and n, where Ψ_n is given by (4.4). One of the special cases satisfying the condition in (4.117) trivially is when the fixed ordering costs and availability probabilities are constant. In this case, $K_i = K$ and $u_i = u$ for all i. This is possible only when the outside environment has no effect on the fixed ordering costs and the availability of supplier, and this may be true in stationary environments. A less restrictive case is when K_i/u_i is equal to some constant $\bar{K}$ for all i. In this case, individual K_i and u_i values may be different but their ratios must be constant for all i. In addition, this case also covers the case where fixed ordering costs and availability probabilities are constant. Note that the condition in (4.117) is quite restrictive since it must be satisfied for all a and n. However, the condition in (4.117) may not be too restrictive when we have a time-homogenous model so that $Q_n = Q$, $E_n = E$ and $\Psi_n = \Psi$ as given in (4.49). As shown by the following example, if we assume that $\{Q_n\}$ and $\{E_n\}$ are time-homogenous, we can find many values of K_i and u_i which satisfy the condition in (4.117) .

Example 3 *Suppose that both observed and unobserved environments have two states so that $\mathbb{E} = \{1, 2\}$ and $\mathbb{F} = \{1, 2\}$, where state 1 represents a "good" environment and state 2 represents a "bad" environment. In addition, we suppose that $\{Q_n\}$ and $\{E_n\}$ are time-homogenous and*

$$Q = \begin{bmatrix} 0.65 & 0.35 \\ 0.43 & 0.57 \end{bmatrix}, \qquad E = \begin{bmatrix} 0.47 & 0.53 \\ 0.4 & 0.6 \end{bmatrix}$$

so that

$$\Psi = \begin{bmatrix} 0.45 & 0.55 \\ 0.43 & 0.57 \end{bmatrix}.$$

Moreover, we assume that fixed ordering cost vector is $K = [34, 10]$, *the availability probability vector is* $u = [0.6, 0.25]$ *and the discount factor is* $\alpha = 0.8$. *Using these values, we can easily compute* $(K/u) = [50.67, 40]$ *and* $\alpha\Psi\,(K/u) = [37.94, 37.73]$. *Then, it now follows that condition (4.117) is satisfied.*

Let $y_n(\pi, i, x)$ denote the optimal order-up-to level for the minimization problem in (4.114).

Theorem 16 *Suppose that the assumption in (4.117) is valid. Then, the optimal ordering policy for* N*-period model is a state-dependent* (s, S) *policy*

$$y_n(\pi, i, x) = \begin{cases} S_n^{\pi,i} & x \le s_n^{\pi,i} \\ x & x > s_n^{\pi,i} \end{cases} \tag{4.118}$$

where $S_n^{\pi,i}$ *is the smallest value that satisfies*

$$\sum_{a\in\mathbb{F}} \pi^a G_n\left(\pi, i, a, S_n^{\pi,i}\right) \le \sum_{a\in\mathbb{F}} \pi^a G_n(\pi, i, a, y)$$

and $s_n^{\pi,i} \le S_n^{\pi,i}$ *satisfies*

$$u_i \sum_{a\in\mathbb{F}} \pi^a G_n\left(\pi, i, a, s_n^{\pi,i}\right) = K_i + u_i \sum_{a\in\mathbb{F}} \pi^a G_n\left(\pi, i, a, S_n^{\pi,i}\right)$$

for all π, i *and* y. *In addition,* $J_n(\pi, i, x, y)$ *is continuous and* K_i-*convex in* y *for all* π, i *and* x. *The optimal cost incurred by this policy is*

$$v_n(\pi, i, x) = \begin{cases} u_i \sum_{a\in\mathbb{F}} \pi^a G_n\left(\pi, i, a, s_n^{\pi,i}\right) & \\ \quad + (1 - u_i) \sum_{a\in\mathbb{F}} \pi^a G_n(\pi, i, a, x) - c_i x & x \le s_n^{\pi,i} \\ \sum_{a\in\mathbb{F}} \pi^a G_n(\pi, i, a, x) - c_i x & x > s_n^{\pi,i} \end{cases} \tag{4.119}$$

for all π, i *and* x. *Moreover,* $v_n(\pi, i, x)$ *is continuous and* $(K_i/u_i)-$*convex in* x *for all* π *and* i.

Proof. The proof proceeds by induction. We know by Theorem 15 that Theorem 16 is valid for $n = N - 1$. Now assume that Theorem 16 is valid for $n + 1, n + 2, ..., N - 1$.

Therefore, $v_{n+1}(\pi, j, x)$ is $(K_j/u_j)-$convex and continuous in x for all π and j. Next, we show that Theorem 16 holds also for time n.

First of all, notice that, by part (c) of Lemma 29, $E_D^a\left[v_{n+1}\left(T\left(\pi\left|j\right.\right), j, y-D\right)\right]$ is also $(K_j/u_j)-$convex in y and that, by part (b) of Lemma 29,

$$\alpha\sum_{j\in\mathbb{E}}\Psi_n(a,j)\,E_D^a\left[v_{n+1}\left(T\left(\pi\left|j\right.\right),j,y-D\right)\right]$$

is $\left[\alpha\sum_{j\in\mathbb{E}}\Psi_n(a,j)\,(K_j/u_j)\right]-$convex in y for all a and n. By our assumption (4.117), it is also $(K_i/u_i)-$convex. In addition, because L is convex in y and $c_i y$ is linear in y, it follows from (4.116) that G_n is $(K_i/u_i)-$convex in y. Together with part (b) of Lemma 29, this also implies that J_n in (4.115) is K_i-convex in y for all π, i and x. Moreover, J_n in (4.115) is continuous in y for all π, i and x because it is sum of continuous functions.

Notice that $\lim_{|y|\uparrow\infty} J_n(\pi, i, x, y) = +\infty$; moreover, J_n is continuous and K_i-convex in y for all π, i and x. Then, there exist scalars $s_n^{\pi,i}$ and $S_n^{\pi,i}$ with $s_n^{\pi,i} \le S_n^{\pi,i}$ satisfying four conditions in part (d) of Lemma 29 where $S_n^{\pi,i}$ is the smallest minimizer of J_n. Using (4.115), this implies that $S_n^{\pi,i}$ satisfies $\sum_{a\in\mathbb{F}}\pi^a G_n\left(\pi, i, a, S_n^{\pi,i}\right) \le \sum_{a\in\mathbb{F}}\pi^a G_n(\pi, i, a, y)$ for all π, i and y. Moreover, using the fact that $J_n(\pi, i, x, y)$ is K_i-convex in y, $s_n^{\pi,i} \le S_n^{\pi,i}$ can be computed by solving

$$u_i\sum_{a\in\mathbb{F}}\pi^a G_n(\pi,i,a,s_n^{\pi,i}) = K_i + u_i\sum_{a\in\mathbb{F}}\pi^a G_n(\pi,i,a,S_n^{\pi,i}) \tag{4.120}$$

for all π and i. Together with (iii) and (iv) of part (d) of Lemma 29, we can claim that a state-dependent (s, S) policy defined as

$$y_n(\pi,i,x) = \begin{cases} S_n^{\pi,i} & x \le s_n^{\pi,i} \\ x & x > s_n^{\pi,i} \end{cases}$$

is optimal. The optimal cost incurred by this policy is

$$v_n(\pi,i,x) = \begin{cases} J_n\left(\pi,i,x,s_n^{\pi,i}\right) & x \le s_n^{\pi,i} \\ J_n(\pi,i,x,x) & x > s_n^{\pi,i}. \end{cases}$$

which leads to (4.119) by using (4.115) and (4.120).

We now show that v_n is $(K_i/u_i)-$convex in x. For this to be true, we must verify that for all $z \ge 0, b > 0$, and x, we have

$$\frac{K_i}{u_i} + v_n(\pi,i,x+z) \ge v_n(\pi,i,x) + \frac{z}{b}\left[v_n(\pi,i,x) - v_n(\pi,i,x-b)\right]. \tag{4.121}$$

First of all, notice that G_n in (4.116) is $(K_i/u_i)-$convex in y. Then, $u_i \sum_{a\in\mathbb{F}} \pi^a G_n(\pi,i,a,x)$ is K_i-convex so that it satisfies

$$\begin{aligned} K_i + u_i \sum_{a\in\mathbb{F}} \pi^a G_n(\pi,i,a,x+z) \quad \geq \quad & u_i \sum_{a\in\mathbb{F}} \pi^a G_n(\pi,i,a,x) \\ & +u_i \left(\frac{z}{x-s_n^{\pi,i}}\right) \sum_{a\in\mathbb{F}} \pi^a \\ & \times \Big[G_n(\pi,i,a,x) - G_n(\pi,i,a,s_n^{\pi,i})\Big] \quad (4.122) \end{aligned}$$

for all $z \geq 0$ and $x > s_n^{\pi,i}$, and $(1-u_i)\sum_{a\in\mathbb{F}} \pi^a G_n(\pi,i,a,x)$ is $\left(K_i\left(\frac{1}{u_i}-1\right)\right)-$convex so that it satisfies

$$\begin{aligned} K_i\left(\frac{1}{u_i}-1\right) + (1-u_i)\sum_{a\in\mathbb{F}} \pi^a G_n(\pi,i,a,x+z) \quad \geq \quad & (1-u_i)\Bigg(\sum_{a\in\mathbb{F}} \pi^a G_n(\pi,i,a,x) \\ & +\left(\frac{z}{b}\right)\sum_{a\in\mathbb{F}} \pi^a \left[G_n(\pi,i,a,x)\right. \\ & \left. -G_n(\pi,i,a,x-b)\right]\Bigg) \quad (4.123) \end{aligned}$$

for all $z \geq 0, b > 0$, and x.

Again, we distinguish three cases:

Case 1: $x > s_n^{\pi,i}$. If $x - b > s_n^{\pi,i}$, then in this region of values of z, b and x, v_n, by (4.119), is sum of a $(K_i/u_i)-$convex function and a linear function; therefore, it is $(K_i/u_i)-$convex. Then, by part (b) of Lemma 29, v_n is $(K_i/u_i)-$convex and (4.121) holds if $x-b > s_n^{\pi,i}$. If $x - b \leq s_n^{\pi,i}$, then in view of (4.119) we can write (4.121) after some simplification as

$$\begin{aligned} \frac{K_i}{u_i} + \sum_{a\in\mathbb{F}} \pi^a G_n(\pi,i,a,x+z) \quad \geq \quad & \sum_{a\in\mathbb{F}} \pi^a G_n(\pi,i,a,x) \\ & +u_i\left(\frac{z}{b}\right)\sum_{a\in\mathbb{F}} \pi^a \Big[G_n(\pi,i,a,x) - G_n(\pi,i,a,s_n^{\pi,i})\Big] \\ & +(1-u_i)\left(\frac{z}{b}\right)\sum_{a\in\mathbb{F}} \pi^a \\ & \times \left[G_n(\pi,i,a,x) - G_n(\pi,i,a,x-b)\right]. \quad (4.124) \end{aligned}$$

Notice that $x > s_n^{\pi,i}$ so that (4.122) is applicable here; moreover, $x - b \leq s_n^{\pi,i}$ so that $x - s_n^{\pi,i} \leq b$. Then, if x is such that $\sum_{a\in\mathbb{F}} \pi^a \left[G_n(\pi,i,a,x) - G_n\left(\pi,i,a,s_n^{\pi,i}\right)\right] > 0$, we have together with (4.122) that

$$\begin{aligned} K_i + u_i \sum_{a\in\mathbb{F}} \pi^a G_n(\pi,i,a,x+z) \quad &\geq \quad u_i \sum_{a\in\mathbb{F}} \pi^a G_n(\pi,i,a,x) \\ &\quad + u_i\left(\frac{z}{b}\right)\sum_{a\in\mathbb{F}} \pi^a \\ &\quad \times\left[G_n(\pi,i,a,x) - G_n(\pi,i,a,s_n^{\pi,i})\right] \end{aligned} \tag{4.125}$$

Then, by summing (4.125) and (4.123), we get (4.124). So if $x - b \leq s_n^{\pi,i}$ and x is such that $\sum_{a\in\mathbb{F}} \pi^a \left[G_n(\pi,i,a,x) - G_n\left(\pi,i,a,s_n^{\pi,i}\right)\right] > 0$, (4.121) and (4.124) hold.

However, if x is such that $\sum_{a\in\mathbb{F}} \pi^a \left[G_n(\pi,i,a,x) - G_n\left(\pi,i,a,s_n^{\pi,i}\right)\right] \leq 0$, then we have

$$\begin{aligned} K_i + u_i \sum_{a\in\mathbb{F}} \pi^a G_n(\pi,i,a,x+z) \quad &\geq \quad K_i + u_i \sum_{a\in\mathbb{F}} \pi^a G_n(\pi,i,a,S_n^{\pi,i}) \\ &= \quad u_i \sum_{a\in\mathbb{F}} \pi^a G_n(\pi,i,a,s_n^{\pi,i}) \\ &\geq \quad u_i \sum_{a\in\mathbb{F}} \pi^a G_n(\pi,i,a,x) \\ &\quad + u_i\left(\frac{z}{b}\right)\sum_{a\in\mathbb{F}} \pi^a \\ &\quad \times\left[G_n(\pi,i,a,x) - G_n(\pi,i,a,s_n^{\pi,i})\right]. \end{aligned} \tag{4.126}$$

The first inequality follows from (i) of part (d) of Lemma 29 and the fact that $S_n^{\pi,i}$ is optimal x to the minimization problem in (4.114). The second equality follows from (4.120). The third and the last inequalities follow from the fact that $\sum_{a\in\mathbb{F}} \pi^a \left[G_n(\pi,i,a,x) - G_n\left(\pi,i,a,s_n^{\pi,i}\right)\right] \leq 0$. Then, if we sum (4.123) and (4.126), we get (4.124). So if $x - b \leq s_n^{\pi,i}$ and x is such that $\sum_{a\in\mathbb{F}} \pi^a \left[G_n(\pi,i,a,x) - G_n\left(\pi,i,a,s_n^{\pi,i}\right)\right] \leq 0$, (4.121) and (4.124) also hold.

Case 2: $x \leq x + z \leq s_n^{\pi,i}$. In this region, by (4.119), the function v_n is sum of a (K_i/u_i) −convex function and a linear function; therefore, v_n is (K_i/u_i) −convex.

Case 3: $x < s_n^{\pi,i} < x + z$. For this case, in view of (4.119), we can write (4.121) as

$$\begin{aligned} K_i + \sum_{a\in\mathbb{F}} \pi^a G_n(\pi,i,a,x+z) \quad &\geq \quad u_i \sum_{a\in\mathbb{F}} \pi^a G_n(\pi,i,a,s_n^{\pi,i}) + (1-u_i)\sum_{a\in\mathbb{F}} \pi^a G_n(\pi,i,a,x) \\ &\quad + (1-u_i)\left(\frac{z}{b}\right)\sum_{a\in\mathbb{F}} \pi^a \\ &\quad \times\left[G_n(\pi,i,a,x) - G_n(\pi,i,a,x-b)\right]. \end{aligned} \tag{4.127}$$

Notice that $u_i \sum_{a\in\mathbb{F}} \pi^a G_n(\pi_n, i, a, x)$ is K_i-convex since $\sum_{a\in\mathbb{F}} \pi^a G_n(\pi_n, i, a, x)$ is $(K_i/u_i)-$convex. Then, we have

$$K_i + u_i \sum_{a\in\mathbb{F}} \pi^a G_n(\pi, i, a, x+z) \geq \sum_{a\in\mathbb{F}} \pi^a u_i G_n(\pi, i, a, s_n^{\pi,i}) \tag{4.128}$$

by (iv) of part (d) of Lemma 29 and (4.120) since $s_n^{\pi,i} < x+z$. Moreover, we know that $(1-u_i)\sum_{a\in\mathbb{F}} \pi^a G_n(\pi, i, a, x)$ satisfies (4.123). Then, by summing (4.123) and (4.128), we get (4.127). As a result, (4.121) holds for case 3. Finally, for all three cases, v_n is $(K_i/u_i)-$convex in x for all π and i.

We need to check $v_n(\pi, i, x)$ at $x = s_n^{\pi,i}$ for continuity. Note that $v_n(\pi, i, x)$ is continuous at $s_n^{\pi,i}$ because

$$\lim_{x\uparrow s_n^{\pi,i}} v_n(\pi, i, x) = \lim_{x\downarrow s_n^{\pi,i}} v_n(\pi, i, x) = \sum_{a\in\mathbb{F}} \pi^a G_n(\pi, i, a, s_n^{\pi,i}) - c_i s_n^{\pi,i}.$$

This completes our proof. ■

By Theorem 16, we get similar results as Özekici and Parlar (1999) who analyze exactly the same problem as ours but the environment is fully-observed. We show that an (s, S) type inventory control policy is still optimal in multi-period setting for inventory problems with unreliable suppliers in a partially-observed random environment. However, when the environment is partially-observed, both s and S depend not only on the observed state but also on the information vector at the same time. This implies that, for inventory problems with unreliable suppliers, observing the random environment partially does not affect the type of optimal control policy but it affects the parameters of that policy.

In Theorem 16, we assume that $\{K_i/u_i\}$ satisfies (4.117) for all i, a and n. This assumption clearly implies that

$$\left(\frac{K_i}{u_i}\right) \geq \alpha \sum_{j\in\mathbb{E}} P_n\left((\pi, i), j\right)\left(\frac{K_j}{u_j}\right) \tag{4.129}$$

for all π, i and n. Notice that $(1/u_i)$ is the expected number of orders until a successful one so that there is a replenishment. Then, (K_i/u_i) is the actual expected cost incurred per successful order if the observed state is i. By (4.129), this implies that the actual fixed cost of ordering in any environment is greater than or equal to the actual expected discounted fixed cost of ordering that will be incurred if the order is given after one more period. If we consider the fixed costs only, this is an important reason for the IM not to give an order at the beginning of a period, but to wait one more period to pay less in expectation. This is the fundamental motivation behind (s, S) inventory control policies.

Note that we always assume that cost parameters and demand distribution only depend on the observed environmental state; but, they may also be dependent on time. For example, Sethi and Cheng (1997) assume that all costs depend on both time and environmental state so that $K_{n,i}$ denotes the fixed ordering cost at time n when environmental state is i. However, they assume that environment is fully observed, and the supplier is always available with infinite capacity. By allowing partial observation and randomly available supplier, we can extend Sethi and Cheng (1997). If we assume that costs, availability of supplier and demand distribution depend on both time and the observed environment, then formulation of the extended model in multiple periods will be the same as in (4.114)-(4.116). The only difference would be that the cost parameters and availability probability of supplier now depend on the time as well. Notice also that L should be replaced by

$$L_n(i, a, y) = h_{n,i} \int_0^y (y - z) dM_{n+1,a}(z) + p_{n,i} \int_y^\infty (z - y) dM_{n+1,a}(z)$$

for all i, a and n. However, Theorem 16 and its proof are still valid in this case if we assume that

$$\left(\frac{K_{n,i}}{u_{n,i}}\right) \geq \alpha \sum_{j \in \mathbb{E}} \Psi_n(a, j) \left(\frac{K_{n+1,j}}{u_{n+1,j}}\right)$$

for all i, a and n. This is clearly true when fixed ordering costs and availability probability are constants. Another case is when fixed ordering costs are nonincreasing in time and the availability of the supplier increases as time increases. Fixed ordering costs and the supplier availability may be nonincreasing and nondecreasing respectively due to the learning curve effect. In such a situation, $K_{n,i} \geq K_{n+1,j}$ and $u_{n,i} \leq u_{n+1,j}$ for all i, j and n. Then, this implies that $(K_{n+1,j}/u_{n+1,j}) \leq (K_{n,i}/u_{n,i})$ for all i, j and n so that the condition in (4.117) is satisfied.

4.6.3 Infinite-period model

In this section, we formulate and analyze the infinite-period problem. As we stated earlier, $\{Q_n\}$ and $\{E_n\}$ matrices are assumed to be time-homogenous in infinite-period analysis so that $Q_n = Q$, $E_n = E$ and $\Psi_n = \Psi$ as given in (4.49). Then, we show that the finite-horizon solution in Section 4.6.2 converges to the infinite-horizon solution. In other words, we show that a state-dependent (s, S) policy is still optimal and optimal discounted cost function is K_i-convex. By assuming that $k = N - n$ denotes the number of periods from time n until time N, we use the notation $v_{n,k}$ for the finite horizon optimal cost v_n in the remaining part of this section. Here, we show that, as k increases to infinity, the finite-horizon optimal cost function $v_{0,k}$ in (4.114) converges to the infinite-horizon optimal cost function v that satisfies

$$v(\pi, i, x) = \min_{y \geq x} \{K_i \delta (y - x) + J (\pi, i, x, y)\} \tag{4.130}$$

for all π, i and x, where y is order-up-to level and

$$J (\pi, i, x, y) = u_i \sum_{a \in \mathbb{F}} \pi^a G(\pi, i, a, y) + (1 - u_i) \sum_{a \in \mathbb{F}} \pi^a G(\pi, i, a, x) - c_i x \tag{4.131}$$

$$G(\pi, i, a, y) = c_i y + L (i, a, y) + \alpha \sum_{j \in \mathbb{E}} \Psi (a, j) E_D^a [v (T (\pi |j) , j, y - D)] \tag{4.132}$$

with $L (i, a, y)$ given in (4.14). Again E_D^a denotes the notation that expectation is taken with respect to the random variable D with distribution M_a.

For any real valued function $f : \mathfrak{D} (\mathbb{F}) \times \mathbb{E} \times \mathbb{R} \to \mathbb{R}$ where $\mathfrak{D} (\mathbb{F})$ is the set of all probability distributions defined on state space $\mathbb{F}$, we define the mapping $\mathcal{T}$ as

$$\mathcal{T} f(\pi, i, x) = \min_{y \geq x} \{K_i \delta (y - x) + J (\pi, i, x, y)\} \tag{4.133}$$

where $J (\pi, i, x, y)$ is given in (4.131) with

$$G(\pi, i, a, y) = c_i y + L (i, a, y) + \alpha \sum_{j \in \mathbb{E}} \Psi (a, j) E_D^a [f (T (\pi |j) , j, y - D)] . \tag{4.134}$$

Using relations in (4.98), $\mathcal{T} f$ can be interpreted as the optimal cost function for the one-step problem where $\sum_{j \in \mathbb{E}} \Psi (a, j) E_D^a [f (T (\pi |j) , j, y - D)]$ is the terminal cost function. Then, $\mathcal{T}^k$ denotes the composition of the mapping with itself k times; that is, for all $k \geq 1$

$$\mathcal{T}^k f (\pi, i, x) = \mathcal{T} \mathcal{T}^{k-1} f (\pi, i, x) \tag{4.135}$$

with $\mathcal{T}^0 f = f$. Using (4.114), we can interpret $\mathcal{T}^k f$ as the optimal cost function for the $k-$period $\alpha-$discounted problem given information vector π. Then, using (4.133) and (4.135),

$$\mathcal{T}^k f(\pi, i, x) = \min_{y \geq x} \{K_i \delta (y - x) + J (\pi, i, x, y)\} \tag{4.136}$$

where J is given in (4.131) with G replaced by

$$G_k(\pi, i, a, y) = c_i y + L(i, a, y) + \alpha \sum_{j \in \mathbb{E}} \Psi(a, j) E_D^a \left[\mathcal{T}^{k-1} f(T(\pi | j), j, y - D) \right]. \quad (4.137)$$

Let $f_0(\pi, i, x) = 0$ for all π and x. For our analysis in previous sections, we always assume that the terminal cost function is zero. Suppose that the initial cost function is $f_0(\pi, i, x)$ so that $\mathcal{T}^0 f(\pi, i, x) = f_0(\pi, i, x)$ for all π, i and x. As a result, $k-$period optimal cost function is $v_{n,k}(\pi, i, x) = \mathcal{T}^k f_0(\pi, i, x)$ for all π, i, x and n.

Let $f_*(\pi, i, x)$ denote the optimal cost over infinite horizon and let

$$f_\infty(\pi, i, x) = \lim_{k \uparrow \infty} \mathcal{T}^k f_0(\pi, i, x) \quad (4.138)$$

for all π, i and x. Notice that f_∞ is well-defined provided we allow the possibility that f_∞ can take the value ∞. Our main aim in this section is to show that the finite-horizon optimal cost converges to the infinite-horizon optimal cost as the number of periods increases. In other words, we aim to show that $f_*(\pi, i, x) = f_\infty(\pi, i, x)$ for all π, i and x. As stated in Bertsekas (2000 b), it is analytically and computationally important to show that $f_*(\pi, i, x) = f_\infty(\pi, i, x)$ because if we know that $f_*(\pi, i, x) = \lim_{k \uparrow \infty} \mathcal{T}^k f_0(\pi, i, x)$, then we can infer the properties of $f_*(\pi, i, x)$ from the properties of $k-$period optimal cost functions $\mathcal{T}^k f_0(\pi, i, x)$.

Let $\mathcal{Z}_k$ denote the sets

$$\mathcal{Z}_k(\pi, i, x, \lambda) = \{ y \geq x \,| K_i \delta(y - x) + J_k(\pi, i, x, y) \leq \lambda \} \quad (4.139)$$

for all π, i, x and $\lambda \in \mathbb{R}$. According to Proposition 1.7 in Bertsekas (2000 b, p. 148), if we show that the sets in (4.139) are compact for all π, i, x and λ, then $f_*(\pi, i, x) = f_\infty(\pi, i, x)$. Hence, by the following lemma, we accomplish this task.

Lemma 17 *The sets in (4.139) are compact subsets of the Euclidean space for all π, i, x and λ.*

Proof. We need to show that the sets in (4.139) are both bounded and closed in order to show that they are compact. Note that J_k is expected discounted cost when there are k periods until the end of planning horizon. Therefore, it is exactly the same as J_n in Section 4.5.2 where $n = N - k$. In multi-period analysis, we show that J_k is continuous and K_i-convex;

moreover, $\lim_{y\uparrow\infty} J_k(\pi, i, x, y) = \infty$ for all π, i, x and k. Hence, the sets $\{\mathcal{Z}_k(\pi, i, x, \lambda)\}$ in (4.139) are bounded for all π, i, x and λ. Moreover, the sets $\{\mathcal{Z}_k(\pi, i, x, \lambda)\}$ are closed since $J_k(\pi, i, x, y)$ is continuous for $y \geq x$ and it is real valued. Thus, the sets in (4.89) are compact subsets of Euclidean space for all π, i, x and λ. ■

The following proposition shows that f_∞ is a fixed point of the mapping $\mathcal{T}$; moreover, finite-horizon optimal cost function converges to the infinite-horizon optimal cost function. In addition, this proposition shows that there is a stationary optimal policy.

Proposition 18 *The limit f_∞ is a fixed point of the mapping $\mathcal{T}$ so that*

$$f_\infty(\pi, i, x) = \mathcal{T} f_\infty(\pi, i, x) \tag{4.140}$$

for all π, i and x. Moreover,

$$f_\infty(\pi, i, x) = f_*(\pi, i, x) \tag{4.141}$$

for all π, i and x. Furthermore, there exists a stationary optimal policy.

Proof. By Lemma 17, the sets in (4.139) are compact subsets of the Euclidean space for all π, i, x and λ. Then, using Proposition 1.7 in Bertsekas (2000 b, p. 148), f_∞ is a fixed point of $\mathcal{T}$ so that (4.140) is valid, and there exists a stationary optimal policy. In addition, notice that

$$f_0 \leq \mathcal{T} f_0 \leq ... \leq \mathcal{T}^k f_0 \leq ... \leq f_*$$

because expected cost per period is nonnegative. From this, we get $\lim_{k\uparrow\infty} \mathcal{T}^k f_0(\pi, i, x) \leq f_*(\pi, i, x)$ so that $f_\infty(\pi, i, x) \leq f_*(\pi, i, x)$. By (4.140), we know that $f_\infty(\pi, i, x)$ is a fixed point of $\mathcal{T}$. Then, by Proposition 1.2 in Bertsekas (2000 b, p. 140), we get that $f_*(\pi, i, x) \leq f_\infty(\pi, i, x)$. It follows that $f_\infty(\pi, i, x) = f_*(\pi, i, x)$. This completes our proof. ■

Notice that Proposition 18 implies also that f_∞ satisfies the Bellman's equation since $f_\infty(\pi, i, x) = \mathcal{T} f_\infty(\pi, i, x)$ by (4.140). Hence,

$$f_\infty(\pi, i, x) = \min_{y\geq x}\{K_i\delta(y-x) + J(\pi, i, x, y)\} \tag{4.142}$$

for all π and x, where $J(\pi, i, x, y)$ is given in (4.131) with G replaced by

$$G(\pi, i, a, y) = c_i y + L(i, a, y) + \alpha \sum_{j\in\mathbb{E}} \Psi(a, j) E_D^a [f_\infty(T(\pi|j), j, y - D)]. \tag{4.143}$$

As stated in Proposition 1.2 in Bertsekas (2000 b, p. 140), f_∞ is not necessarily the unique optimal solution to the Bellman's equation because single-period costs are not bounded under positivity assumption; however, f_∞ is the smallest fixed point of $\mathcal{T}$ since $f_\infty = f_*$.

Notice that, for a finite n, k goes to infinity as N goes to infinity. Then, above analysis shows that $\lim_{k\uparrow\infty} v_{0,k}(\pi, i, x) = v(\pi, i, x)$. Moreover, $v\,(\pi, i, x)$ satisfies (4.130) and there exists a stationary optimal policy $y\,(\pi, i, x)$ which minimizes the infinite-period total cost. However, notice that $J\,(\pi, i, x, y)$ is not bounded for $y \geq x$; therefore, $v\,(\pi, i, x)$ is not necessarily unique. Then, we take $v\,(\pi, i, x)$ as the minimal fixed point of (4.130). In other words, if $f = \mathcal{T}f$, then $v \leq f$. Moreover, we also know that the optimal solution v is that fixed point of $\mathcal{T}$ which can be obtained as $v = \lim_{k\uparrow\infty} \mathcal{T}^k f_0$ with $f_0 = 0$.

Here we will also show that state-dependent (s, S) policy is optimal for infinite-horizon problem. However, this requires a similar assumption as in (4.117). We assume that

$$\frac{K_i}{u_i} \geq \alpha \sum_{j\in\mathbb{E}} \Psi\,(a, j)\,\frac{K_j}{u_j} \tag{4.144}$$

for all i and a, where Ψ is given by (4.49). As in multi-period case, this assumption is also satisfied when the fixed ordering costs and availability probabilities are constant.

Let $y\,(\pi, i, x)$ denote the optimal order-up-to level for the minimization problem in (4.130).

Theorem 19 *Suppose that assumption (4.144) is valid. Then, the optimal ordering policy for infinite-period model is a state-dependent (s, S) policy*

$$y\,(\pi, i, x) = \begin{cases} S^{\pi,i} & x \leq s^{\pi,i} \\ x & x > s^{\pi,i} \end{cases} \tag{4.145}$$

where $S^{\pi,i}$ is the smallest value that satisfies

$$\sum_{a\in\mathbb{F}} \pi^a G\left(\pi, i, a, S^{\pi,i}\right) \leq \sum_{a\in\mathbb{F}} \pi^a G\,(\pi, i, a, y)$$

and $s^{\pi,i} \leq S^{\pi,i}$ satisfies

$$u_i \sum_{a\in\mathbb{F}} \pi^a G\left(\pi, i, a, s^{\pi,i}\right) = K_i + u_i \sum_{a\in\mathbb{F}} \pi^a G\left(\pi, i, a, S^{\pi,i}\right)$$

for all π, i and y. In addition, $J(\pi, i, x, y)$ is continuous and K_i-convex in y for all π, i and x. The optimal cost incurred by this policy is

$$v(\pi, i, x) = \begin{cases} u_i \sum_{a\in\mathbb{F}} \pi^a G\left(\pi, i, a, s^{\pi,i}\right) & \\ \quad + (1-u_i)\sum_{a\in\mathbb{F}} \pi^a G(\pi, i, a, x) - c_i x & x \le s^{\pi,i} \\ \sum_{a\in\mathbb{F}} \pi^a G(\pi, i, a, x) - c_i x & x > s^{\pi,i} \end{cases} \tag{4.146}$$

for all π, i and x. Moreover, $v(\pi, i, x)$ is continuous and $(K_i/u_i)-$convex in x for all π and i.

Proof. As shown before, $v(\pi, i, x) = \lim_{k\uparrow\infty} v_{0,k}(\pi, i, x)$ for all π, i and x. The limit of a continuous function is also continuous. Hence, v is continuous since $v_{0,k}$ is continuous for all k. Moreover, by Theorem 16, $v_{0,k}(\pi, i, x)$ is $(K_i/u_i)-$convex in x for all π and i. Therefore, for all $z \ge 0, b > 0$, and x, we have

$$\frac{K_i}{u_i} + v_{0,k}(\pi, i, x+z) \ge v_{0,k}(\pi, i, x) + \frac{z}{b}\left[v_{0,k}(\pi, i, x) - v_{0,k}(\pi, i, x-b)\right].$$

for all π, i and x. After taking limit of both sides as k goes to infinity in above inequality, we get

$$\frac{K_i}{u_i} + v(\pi, i, x+z) \ge v(\pi, i, x) + \frac{z}{b}\left[v(\pi, i, x) - v(\pi, i, x-b)\right] \tag{4.147}$$

since $v(\pi, i, x) = \lim_{k\uparrow\infty} v_{0,k}(\pi, i, x)$ for all π, i and x. Then, (4.147) implies that $v(\pi, i, x)$ is $(K_i/u_i)-$convex in x for all π and i.

Moreover, by part (c) of Lemma 29, $E_D^a\left[v\left(T\left(\pi\,|j\right), j, y-D\right)\right]$ is $(K_j/u_j)-$convex in y and that, by part (b) of Lemma 29,

$$\alpha \sum_{j\in\mathbb{E}} \Psi(a, j)\, E_D^a\left[v\left(T\left(\pi\,|j\right), j, y-D\right)\right]$$

is $\left[\alpha \sum_{j\in\mathbb{E}} \Psi(a, j)(K_j/u_j)\right]-$convex in y for all a. By our assumption (4.144), it is also $(K_i/u_i)-$convex. Now, because L is convex in y and $c_i y$ is linear, thus convex, in y, it follows from (4.132) that G is $(K_i/u_i)-$convex in y. Together with part (b) of Lemma 29, this also implies that J in (4.131) is K_i-convex in y for all π, i and x. Moreover, J is continuous in y since v and L are continuous, and sum of continuous functions is also continuous.

Note that J is continuous and K_i-convex in y for all π, i and x; moreover, $\lim_{|y|\uparrow\infty} J(\pi, i, x, y) = \infty$. Then, there exist scalars $s^{\pi,i}$ and $S^{\pi,i}$ with $s^{\pi,i} \le S^{\pi,i}$ satisfying four conditions in part (d) of Lemma 29. $S^{\pi,i}$ is the smallest minimizer of J. Using (4.131), $S^{\pi,i}$ satisfies $\sum_{a\in\mathbb{F}} \pi^a G\left(\pi, i, a, S^{\pi,i}\right) \leq \sum_{a\in\mathbb{F}} \pi^a G\left(\pi, i, a, y\right)$ for all π, i and y. Moreover, using the fact that J is K_i-convex in y, $s^{\pi,i} \leq S^{\pi,i}$ can be computed by solving

$$u_i \sum_{a\in\mathbb{F}} \pi^a G(\pi, i, a, s^{\pi,i}) = K_i + u_i \sum_{a\in\mathbb{F}} \pi^a G(\pi, i, a, S^{\pi,i}) \tag{4.148}$$

for all π, i and x. Together with (iii) and (iv) of part (d) of Lemma 29, we can claim that state-dependent (s, S) policy defined

$$y\left(\pi, i, x\right) = \begin{cases} S^{\pi,i} & x \leq s^{\pi,i} \\ x & x > s^{\pi,i} \end{cases}$$

is optimal. The optimal cost incurred by this policy is

$$v\left(\pi, i, x\right) = \begin{cases} J\left(\pi, i, x, s^{\pi,i}\right) & x \leq s^{\pi,i} \\ J\left(\pi, i, x, x\right) & x > s^{\pi,i}. \end{cases}$$

which leads to (4.146) by using (4.131) and (4.148).

Clearly, $v\left(\pi, i, x\right)$ is continuous in x for all $x \leq s^{\pi,i}$ and $x > s^{\pi,i}$ separately since it is the sum of continuous functions. Moreover, $v\left(\pi, i, x\right)$ is continuous at $x = s^{\pi,i}$ because

$$\lim_{x\uparrow s^{\pi,i}} v\left(\pi, i, x\right) = \lim_{x\downarrow s^{\pi,i}} v\left(\pi, i, x\right) = \sum_{a\in\mathbb{F}} \pi^a G(\pi, i, a, s^{\pi,i}) - c_i s^{\pi,i}.$$

Therefore, $v\left(\pi, i, x\right)$ is continuous and $\left(K_i/u_i\right)-$convex function in x for all π and i. This completes our proof. ■

By Theorem 19, we get similar results as in single-period and multi-period models. This clearly implies that results obtained by Özekici and Parlar (1999) are valid in infinite period so that state-dependent (s, S) policy is optimal. But, as in single and multiple planning periods, s and S depend on both current observed state and information vector. However, they are independent of number of the current period that we are planning for; therefore, they are the same in all periods with the same information vector and observed state. Moreover, we show that multi-period optimal cost function converges to the infinite-period optimal cost as the length of planning period gets longer.

By Theorem 19, we show that state-dependent (s, S) policy is still optimal in infinite-horizon for inventory problems with random availability and fixed ordering cost in a partially observed random environment. Our results agree with those obtained by Sethi and Cheng (1997). They show that state and time dependent (s, S) policy is optimal in infinite-horizon for inventory problems with available supplier having infinite capacity in a random environment where all costs depend on both environmental state and time. However, in our model, optimal policy is independent of time since we assume that costs, availability probability and demand distribution are only environment-dependent (not time dependent). Our model can be extended by considering time-dependence as well. But, this extension is not so trivial as in the multi-period case. In this case, extra attention must be paid since multi-period cost function may not converge due to time-dependence.

4.7 Inventory models with finite capacity and random yield

In this section, we consider a discrete time, single product, single location, periodic-review inventory model with finite production capacity and random proportional yield where the demand process, supply process and all cost parameters are modulated by a partially-observed environment. As in Section 3.1, we assume that the inventory system is composed of a retailer, a producer and a transporter. However, we assume that the producer has a finite or fixed capacity; moreover, the transporter can deliver a random proportion of produced quantity. And we assume as in Section 3.1 that there is a random environment in which demand, supply and all cost parameters depend. Markov modulated demand, supply and costs are analyzed by many researchers, like Özekici and Parlar (1999), Erdem and Özekici (2002), and Gallego and Hu (2004), among many others. We analyze exactly the same inventory problem in Gallego and Hu (2004); however, our main contribution is that the Markov modulated environment is not fully observed, it is only partially observed.

The remainder of this section is organized as follow. In Section 4.7.1, we introduce our notation and the basic model, and state our assumptions. In Section 4.7.2, we analyze the single-period problem. Section 4.7.3 focuses on the multi-period problem. In Section 4.7.4, we study the infinite-period problem.

4.7.1 Models and preliminaries

We consider a single product inventory system which is inspected periodically over a planning horizon of length N. As before, we let $Z = \{Z_n;\ n = 0, 1, 2, 3, \ldots\}$ and $Y = \{Y_n;\ n = 0, 1, 2, 3, \ldots\}$ denote the state of real and observed environmental processes respectively. Our assumptions and notations regarding Z and Y processes are exactly the same as in Section 4.1.

As stated in Section 4.1, the state of the real environment Z at any time depends on all of the past observations of Y. Therefore, we need $\bar{Y}_n$ to make inferences regarding the true state of the real environment Z at time n. However, as the number of period gets longer, dimension of $\bar{Y}_n$ increases without bounds. Therefore, we use sufficient statistics which are measures that summarize all information embedded in $\bar{Y}_n$. From Section 4.4, we know that the distribution π_n of the true state of the environment at time n given all observations until that time is a sufficient statistic for $\bar{Y}_n$. In this section, our assumptions and notations regarding the information vector π_n and its transition vector $T(\pi_n\,|j\,)$ are exactly the same as in Section 4.4. Therefore, interested readers are referred to this section for a detailed description and analysis.

We let D_n denote the total demand in period n and M_a denote the conditional cumulative distribution function of demand when unobserved environment is a. However, unlike in previous sections, we assume in this section that M_a is increasing (not only nodecreasing). Furthermore, we assume that there is a maximum finite inventory capacity A.

Moreover, we let x_n and y_n denote the inventory level and order-up-to level at time n respectively. At each time n, the inventory level x_n is checked and order $y_n - x_n$, if any, is placed from an outside supplier which is delivered instantaneously. During the period, demand is realized and unsatisfied demand is backlogged. We assume that some proportion of order quantity is lost. We define $U_n \in [0, 1]$ to be the proportion of the produced amount which is received by the retailer in period n. Clearly supply or amount received by the retailer at time n is random and equal to $U_{n+1} \min\{A, y_n - x_n\}$. As for demand process, we assume that realizations of yield process $U = \{U_n; n \geq 1\}$ is also observable. However, its distribution depends on the real environmental process Z. Then, the conditional cumulative distribution function of the proportional yield is

$$F_a(u) = P[U_{n+1} \leq u \mid Z_n = a]$$

and F_a is assumed to be differentiable so that it has probability density function f_a. We define $\mu_a = E\left[U_{n+1} \middle| Z_n = a\right]$ to be the mean value of proportional yield when real environmental state is a and we define $\bar{\mu}_\pi = \sum_{a\in\mathbb{F}} \pi^a \mu_a$ for any $\pi \in \mathfrak{D}(\mathbb{F})$. Here, we assume that $\bar{\mu}_\pi > 0$ for all π. This assumption requires that $\mu_a > 0$ for at least an a with $\pi^a > 0$. Moreover, this further implies that there is at least one environmental state a with $\pi^a > 0$ in which there is positive probability of receiving something such that $P[U_{n+1} = 0 \mid Z_n = a] < 1$. Consider cases where this requirement is not satisfied so that in all environmental states a with $\pi^a > 0$ receiving something is impossible so that $P[U_{n+1} = 0 \mid Z_n = a] = 1$ and $\mu_a = 0$. Clearly it is illogical to order in such situations. Therefore, this requirement is not restrictive. In our analysis of this section, we frequently refer to our assumption that $\bar{\mu}_\pi > 0$ for a given π. This is equivalent to the condition that there is at least one environmental state a with $\pi^a > 0$ in which there is positive probability of receiving something. Note that if $\mu_a > 0$ for all a, then our assumption is trivially satisfied for all π since $\sum_{a\in\mathbb{F}} \pi^a = 1$ and $\bar{\mu}_\pi = \sum_{a\in\mathbb{F}} \pi^a \mu_a > 0$. Furthermore, $\mu_a \leq 1$ for all a since U_n is in $[0, 1]$. This fact together with our assumption imply that $0 < \bar{\mu}_\pi \leq 1$ for all π.

In addition, we define

$$\beta_a(\pi) = \frac{\pi^a \mu_a}{\sum_{a\in\mathbb{F}} \pi^a \mu_a} \tag{4.149}$$

for all a. Moreover, using (4.149), we denote

$$M_\pi(z) = \sum_{a\in\mathbb{F}} \beta_a(\pi) M_a(z) \tag{4.150}$$

to be a mixture of the cumulative distribution functions $\{M_a\}$ since $\sum_{a\in\mathbb{F}} \beta_a(\pi) = 1$ for all π. Finally, our assumptions and notations regarding the cost parameters and the discount factor are exactly the same as in Section 4.1.

4.7.2 Single-period model

Here we assume that there is only one period to plan for so that $N = 1$; moreover, $v_1(\pi_1, i_1, x_1) = 0$ for all π_1, i_1 and x_1. Assuming inventory level at the beginning of period is x, the initial distribution of true state of environment is π, and observed state of environment is i, single-period minimum cost function $v_0(\pi, i, x)$ satisfies

$$v_0(\pi, i, x) = \min_{x \leq y \leq x+A} H_0(\pi, i, x, y) \tag{4.151}$$

for all π, i and x, where y is the order-up-to level and

$$H_0(\pi, i, x, y) = \sum_{a \in \mathbb{F}} \pi^a J_0(i, a, x, y) \tag{4.152}$$

$$J_0(i, a, x, y) = \int_0^1 G_0(i, a, x + u(y - x))\, dF_a(u) - c_i x \tag{4.153}$$

$$G_0(i, a, y) = c_i y + L(i, a, y) \tag{4.154}$$

where $L(i, a, y)$ is given in (4.14). Notice that L is a strictly convex function in y since we assume that demand distribution M_a is increasing. In addition, from (4.153), it is obvious that J_0 is not a function of π. Remember that π is either initially known or can be determined by (4.40).

The following lemma, which is similar to the one in Gallego and Hu (2004), is useful in demonstrating the strict convexity of cost functions. The proof can be conducted as in Gallego and Hu (2004) by using the fact that function ϕ is strictly convex in this case. If e and y are column vectors, then $e'y$ is the inner product of these vectors.

Lemma 20 *Let $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ be strictly convex, then for any constant vector $e \in \mathbb{R}^n$ and any scalar d, $\psi(y) = \phi(e'y - d) : \mathbb{R}^n \longrightarrow \mathbb{R}$ is also strictly convex.*

Moreover, we use the following lemma to prove the convexity of optimal cost function. Therefore, we state and prove it here.

Lemma 21 *Assume that g is a function of x and u. Then,*

$$\left[\sum_{a \in \mathbb{F}} \pi^a \int_0^1 u L''(i, a, g(x, u))\, dF_a(u)\right]^2 \leq \left[\sum_{a \in \mathbb{F}} \pi^a \int_0^1 L''(i, a, g(x, u))\, dF_a(u)\right] \times \left[\sum_{a \in \mathbb{F}} \pi^a \int_0^1 u^2 \times L''(i, a, g(x, u))\, dF_a(u)\right] \tag{4.155}$$

for all π, i and x.

Proof. First of all, notice that $dF_a(u) = f_a(u)\,du$ for all a since we assume that F_a is differentiable. It is clear that

$$\sum_{a\in\mathbb{F}} \pi^a \int_0^1 u^k L''(i,a,g(x,u))\, f_a(u)du = \int_0^1 u^k \left(\sum_{a\in\mathbb{F}} \pi^a L''(i,a,g(x,u))\, f_a(u) \right) du.$$

for all π, i, x and integer $k \geq 0$. Then, using this fact for $k \in \{0,1,2\}$, condition (4.155) becomes

$$\begin{aligned}\left[\int_0^1 u \Bigl(\sum_{a\in\mathbb{F}} \pi^a L''(i,a,g(x,u))\, f_a(u) \Bigr) du \right]^2 \leq & \left[\int_0^1 \Bigl(\sum_{a\in\mathbb{F}} \pi^a L''(i,a,g(x,u))\, f_a(u) \Bigr) du \right] \\ & \times \left[\int_0^1 u^2 \Bigl(\sum_{a\in\mathbb{F}} \pi^a \right. \\ & \left. \times L''(i,a,g(x,u))\, f_a(u) \Bigr) du \right]. \quad (4.156)\end{aligned}$$

Moreover, for any two integrable nonnegative functions h and t, Cauchy-Schwarz inequality implies that

$$\left[\int_a^b h(u)\, t(u)\, du\right]^2 \leq \int_a^b h^2(u)\, du \int_a^b t^2(u)\, du. \quad (4.157)$$

Then, after substituting $h(u) = \left[\sum_{a\in\mathbb{F}} \pi^a L''(i,a,g(x,u))\, f_a(u)\right]^{1/2}$ and $t(u) = uh(u)$ in (4.157), we get (4.156). As a result, condition (4.155) is satisfied. ■

Expected cost in a single period is the sum of expected purchase cost, and expected holding and shortage costs. Let $y_0(\pi,i,x)$ denote the optimal order-up-to level for the minimization problem in (4.151).

Theorem 22 *The optimal ordering policy for the single-period model is a state-dependent modified inflated base-stock policy*

$$y_0(\pi,i,x) = \begin{cases} x + A & x < s_0^{\pi,i} \\ y_0^{\pi,i}(x) & s_0^{\pi,i} \leq x < S_0^{\pi,i} \\ x & x \geq S_0^{\pi,i} \end{cases} \quad (4.158)$$

where $y_0^{\pi,i}(x)$, $S_0^{\pi,i}$ *and* $s_0^{\pi,i}$ *are unique* y *values which satisfy*

$$\begin{aligned} \sum_{a\in\mathbb{F}} \pi^a \int_0^1 u\left(c_i + L'(i,a,x+u(y-x))\right) dF_a(u) &= 0 \\ \sum_{a\in\mathbb{F}} \pi^a \mu_a \left(c_i + L'(i,a,y)\right) &= 0 \\ \sum_{a\in\mathbb{F}} \pi^a \int_0^1 u\left(c_i + L'(i,a,y+uA)\right) dF_a(u) &= 0 \end{aligned}$$

for all π and i respectively. Moreover, $y_0^{\pi,i}(x)$ is nonincreasing in x with $\lim_{x\downarrow-\infty} y_0^{\pi,i}(x) = \infty$ for all π and i. In addition, $x \leq y_0^{\pi,i}(x) \leq x+A$ for all $x \in \left[s_0^{\pi,i}, S_0^{\pi,i}\right)$ with $y_0^{\pi,i}\left(s_0^{\pi,i}\right) = s_0^{\pi,i} + A$ and $y_0^{\pi,i}\left(S_0^{\pi,i}\right) = S_0^{\pi,i}$. The cost function $H_0(\pi, i, x, y)$ is strictly convex in (x, y) for all π and i, and the optimal cost is

$$v_0(\pi, i, x) = \begin{cases} \sum_{a\in\mathbb{F}} \pi^a \int_0^1 (c_i uA + L(i, a, x + uA))\, dF_a(u) & x < s_0^{\pi,i} \\ H_0\left(\pi, i, x, y_0^{\pi,i}(x)\right) & s_0^{\pi,i} \leq x < S_0^{\pi,i} \\ \sum_{a\in\mathbb{F}} \pi^a L(i, a, x) & x \geq S_0^{\pi,i} \end{cases} \tag{4.159}$$

for all π, i and x. Furthermore, $v_0(\pi, i, x)$ is convex in x, $\lim_{x\uparrow\infty} v_0'(\pi, i, x) = h_i$ and $\lim_{x\downarrow-\infty} v_0'(\pi, i, x) = -p_i$ for all π and i.

Proof. We need to find $x \leq y \leq x + A$ minimizing H_0. Note that $G_0(i, a, y)$ is strictly convex in y for all i and a since L is strictly convex. Then, it follows by Lemma 20 that $G_0(i, a, x + u(y - x))$ is also strictly convex in (x, y) for all i, a and $u \in [0, 1]$. This clearly implies that J_0 is strictly convex in (x, y) because strict convexity is preserved by expectation. Finally, H_0 is strictly convex in (x, y) since sum of strictly convex functions is also strictly convex. In addition, we denote partial derivative of H_0 with respect to y by H_0'. Then,

$$H_0'(\pi, i, x, y) = \sum_{a\in\mathbb{F}} \pi^a \int_0^1 u\left(c_i + L'(i, a, x + u(y - x))\right) dF_a(u) \tag{4.160}$$

for all π, i, x and y.

We define $y_0^{\pi,i}(x)$ to be the y value minimizing $H_0(\pi, i, x, y)$ without the constraint $x \leq y \leq x + A$. By the first order optimality condition, $y_0^{\pi,i}(x)$ satisfies

$$\sum_{a\in\mathbb{F}} \pi^a \int_0^1 u\left(c_i + L'\left(i, a, x + u(y_0^{\pi,i}(x) - x)\right)\right) dF_a(u) = 0 \tag{4.161}$$

for all π, i and x. Note that H_0' is increasing, $\lim_{y\downarrow-\infty} H_0'(\pi, i, x, y) = (c_i - p_i)\,\bar{\mu}_\pi < 0$ and $\lim_{y\uparrow\infty} H_0'(\pi, i, x, y) = (c_i + h_i)\,\bar{\mu}_\pi > 0$. This together with the fact that H_0 is strictly convex imply that $y_0^{\pi,i}(x)$ satisfying (4.161) is finite and unique. We now show that $\lim_{x\downarrow-\infty} y_0^{\pi,i}(x) = \infty$. Suppose that $\lim_{x\downarrow-\infty} y_0^{\pi,i}(x) < \infty$. Then, left-hand side of (4.161) goes to $(c_i - p_i)\,\bar{\mu}_\pi$, which is strictly less than 0 since $\bar{\mu}_\pi > 0$ by our assumption, as x goes to $-\infty$. This is a contradiction and $\lim_{x\downarrow-\infty} y_0^{\pi,i}(x) = \infty$ for all π and i.

Next, we show that $y_0^{\pi,i}(x)$ is nonincreasing in x. Suppose that there exist $x_1 < x_2$ such that $H_0'\left(\pi, i, x_1, y_0^{\pi,i}(x_1)\right) = H_0'\left(\pi, i, x_2, y_0^{\pi,i}(x_2)\right) = 0$ with $y_0^{\pi,i}(x_1) < y_0^{\pi,i}(x_2)$. This clearly implies that $x_1 + u(y_0^{\pi,i}(x_1) - x_1) < x_2 + u(y_0^{\pi,i}(x_2) - x_2)$ for all $u \in [0,1]$. This further implies that $L'\left(i, a, x_2 + u(y_0^{\pi,i}(x_2) - x_2)\right) > L'\left(i, a, x_1 + u(y_0^{\pi,i}(x_1) - x_1)\right)$ for all i and a since L' is increasing. Then, using (4.160) and our assumption that $\bar{\mu}_\pi > 0$, it follows that

$$H_0'\left(\pi, i, x_2, y_0^{\pi,i}(x_2)\right) > H_0'\left(\pi, i, x_1, y_0^{\pi,i}(x_1)\right) = 0$$

for all π and i. However, notice that this is a clear contradiction to the definition of $y_0^{\pi,i}(x_2)$. In other words, condition in (4.161) is not satisfied for $\left(x_2, y_0^{\pi,i}(x_2)\right)$ pair. Therefore, $y_0^{\pi,i}(x)$ is nonincreasing in x. Furthermore, after differentiating (4.161) with respect to x and making some simplifications, we get

$$\frac{\partial y_0^{\pi,i}(x)}{\partial x} = \frac{\sum_{a\in\mathbb{F}} \pi^a \int_0^1 \left(u^2 - u\right) L''\left(i, a, x + u(y_0^{\pi,i}(x) - x)\right) dF_a(u)}{\sum_{a\in\mathbb{F}} \pi^a \int_0^1 u^2 L''\left(i, a, x + u(y_0^{\pi,i}(x) - x)\right) dF_a(u)} \leq 0 \tag{4.162}$$

for all π and i. Note that the denominator in (4.162) is always positive since $L'' > 0$ and, by our assumption, $\bar{\mu}_\pi > 0$. In addition, the numerator of the same equation is nonpositive since $u^2 - u \leq 0$ for $u \in [0,1]$ and $L'' > 0$. As a result, inequality in (4.162) is valid. Note that (4.162) is, in general, not a constant; moreover, it implies also that $y_0^{\pi,i}(x)$ is nonincreasing in x. Then, it follows that $y_0^{\pi,i}(x)$ has the structure in Figure 4.2.

Notice that $y_0^{\pi,i}(x)$ satisfying (4.161) is not necessarily a feasible solution for the problem in (4.151) since it may also be less than x or greater than $x + A$. However, as shown in Figure 4.2, a feasible order-up-to level is bounded above by the line $y = x + A$ and below by line $y = x$. Suppose that $S_0^{\pi,i}$ is the smallest inventory level at which it is optimal not to order so that $y_0^{\pi,i}\left(S_0^{\pi,i}\right) - S_0^{\pi,i} = 0$. Then, for $x = S_0^{\pi,i}$, (4.161) becomes

$$\sum_{a\in\mathbb{F}} \pi^a \mu_a \left(c_i + L'\left(i, a, S_0^{\pi,i}\right)\right) = 0 \tag{4.163}$$

for all π and i. In addition, (4.163) implies the following

$$\sum_{a\in\mathbb{F}} \pi^a \mu_a L'\left(i, a, S_0^{\pi,i}\right) = -c_i \bar{\mu}_\pi < 0. \tag{4.164}$$

Note that, by our assumption, $\mu_a > 0$ for at least one a with $\pi^a > 0$; moreover, L' is increasing in y, $\lim_{y\uparrow\infty} L'(i, a, y) = c_i$ and $\lim_{y\downarrow-\infty} L'(i, a, y) = -p_i$ for all i and a. Hence,

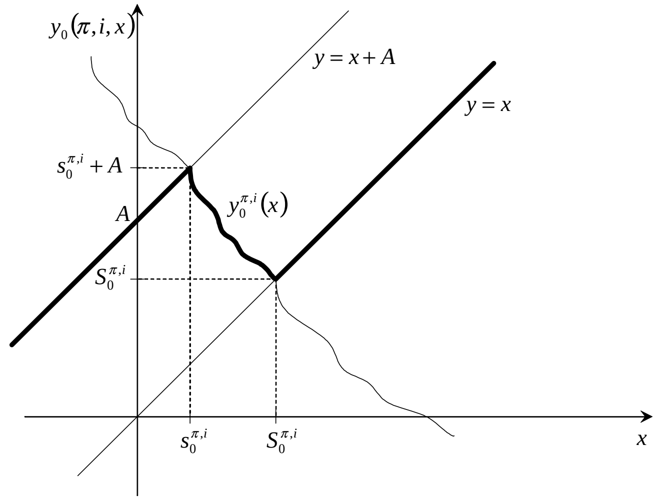

Figure 4.2: State-dependent modified inflated base-stock policy

(4.164) further implies that $S_0^{\pi,i}$ is finite and unique. Moreover, uniqueness of $S_0^{\pi,i}$ is also obvious from Figure 4.2 since line $y=x$ can cross the curve $y_0^{\pi,i}(x)$, which is nonincreasing, at most at one point. Next, let us assume that $x=s_0^{\pi,i}$ is the largest inventory level at which it is optimal to order as much as the finite capacity A, as a result, $y_0^{\pi,i}\left(s_0^{\pi,i}\right)=s_0^{\pi,i}+A$. Using (4.161), $s_0^{\pi,i}$ satisfies

$$\sum_{a\in\mathbb{F}}\pi^a\int_0^1 u\left(c_i+L'\left(i,a,s_0^{\pi,i}+uA\right)\right)dF_a(u)=0 \tag{4.165}$$

for all π and i. Then, from (4.165), it follows that

$$\sum_{a\in\mathbb{F}}\pi^a\int_0^1 uL'\left(i,a,s_0^{\pi,i}+uA\right)dF_a(u)=-c_i\bar{\mu}_\pi<0. \tag{4.166}$$

Notice that $\bar{\mu}_\pi>0$ by our assumption; moreover, L' is increasing in y, $\lim_{y\uparrow\infty}L'(i,a,y)=c_i$ and $\lim_{y\downarrow-\infty}L'(i,a,y)=-p_i$ for all i and a. Thus, $s_0^{\pi,i}$ is also finite and unique as $S_0^{\pi,i}$. Similarly, uniqueness of $s_0^{\pi,i}$ can easily be verified from Figure 4.2 since line $y=x+A$ can intersect the curve $y_0^{\pi,i}(x)$ at most at one point. Notice that $s_0^{\pi,i}\leq S_0^{\pi,i}$ since $y_0^{\pi,i}(x)$ is nonincreasing in x. Now consider the three cases: $x<s_0^{\pi,i}$, $s_0^{\pi,i}\leq x<S_0^{\pi,i}$ and $x\geq S_0^{\pi,i}$.

(i) $x<s_0^{\pi,i}$: Note that $y_0^{\pi,i}(x)$ is nonincreasing in x; therefore, $y_0^{\pi,i}(x)\geq y_0^{\pi,i}(s_0^{\pi,i})=s_0^{\pi,i}+A$ for all $x<s_0^{\pi,i}$. It follows that $y_0^{\pi,i}(x)>x+A$ for all $x<s_0^{\pi,i}$. Moreover, we

know also that H_0 is strictly convex in y and $y_0^{\pi,i}(x)$ is the unique minimizer of this strictly convex function. Then, this clearly implies that H_0 is decreasing for values of y in $[x, y_0^{\pi,i}(x))$. If the production capacity was infinite, ordering up to $y_0^{\pi,i}(x)$ would be optimal. However, the order quantity is limited by A and ordering up to $y_0^{\pi,i}(x)$ requires an order of $y_0^{\pi,i}(x) - x > A$. In this case, ordering as much as the fixed capacity A is optimal since $H_0(\pi, i, x, y)$ is decreasing for values of y in $[x, x + A)$ and $x + A$ is the minimum that can be attained. Therefore, optimal order-up-to level for all $x < s_0^{\pi,i}$ is $y_0(\pi, i, x) = x + A$.

(ii) $s_0^{\pi,i} \leq x < S_0^{\pi,i}$: Note that $y_0^{\pi,i}(x)$ is nonincreasing in x; therefore, $s_0^{\pi,i} + A \geq y_0^{\pi,i}(x) \geq S_0^{\pi,i}$ for values of x in $[s_0^{\pi,i}, S_0^{\pi,i})$. Then it follows that $x \leq y_0^{\pi,i}(x) \leq x + A$ for $s_0^{\pi,i} \leq x < S_0^{\pi,i}$. This further implies that the global minimum $y_0^{\pi,i}(x)$ is attainable. Therefore, optimal order-up-to level is $y_0(\pi, i, x) = y_0^{\pi,i}(x)$ for values of x in $[s_0^{\pi,i}, S_0^{\pi,i})$.

(iii) $x \geq S_0^{\pi,i}$: Note that $y_0^{\pi,i}(x) \leq y_0^{\pi,i}(S_0^{\pi,i}) = S_0^{\pi,i}$ since $y_0^{\pi,i}(x)$ is nonincreasing in x. This clearly implies that $y_0^{\pi,i}(x) \leq x$. Ordering a negative amount is not possible; therefore, it is not possible to attain the global minimum $y_0^{\pi,i}(x)$ if $x \geq S_0^{\pi,i}$. However, strict convexity of H_0 implies that it is increasing for all $y \in [x, \infty)$. As a result, optimal policy is not to order if inventory level is greater than or equal to $S_0^{\pi,i}$ so that $y_0(\pi, i, x) = x$.

In conclusion, the optimal ordering policy is given by $y_0(\pi, i, x)$ in (4.158) as shown in Figure 4.2. Notice that $y_0(\pi, i, x) = y_0^{\pi,i}(x)$ for all x in $[s_0^{\pi,i}, S_0^{\pi,i})$ and $y_0^{\pi,i}(x)$ is nonincreasing in x. Moreover, $y_0^{\pi,i}(S_0^{\pi,i}) = S_0^{\pi,i}$ by definition. Then, it follows that $y_0(\pi, i, x) \geq S_0^{\pi,i}$ for all x in $[s_0^{\pi,i}, S_0^{\pi,i})$.

The minimum cost function corresponding to the optimal policy in (4.158) is

$$v_0(\pi, i, x) = \begin{cases} H_0(\pi, i, x, x + A) & x < s_0^{\pi,i} \\ H_0\left(\pi, i, x, y_0^{\pi,i}(x)\right) & s_0^{\pi,i} \leq x < S_0^{\pi,i} \\ H_0(\pi, i, x, x) & x \geq S_0^{\pi,i} \end{cases}$$

which leads to (4.159) since

$$H_0(\pi, i, x, x + A) = \sum_{a \in \mathbb{F}} \pi^a \int_0^1 (c_i u A + L(i, a, x + uA))\, dF_a(u)$$

and

$$H_0\left(\pi,i,x,x\right)=\sum_{a\in\mathbb{F}}\pi^a L\left(i,a,x\right)$$

for all π, i and x.

Now, we prove that $v_0\left(\pi,i,x\right)$ is convex in x. First, we show that $v_0\left(\pi,i,x\right)$ is convex for $x<s_0^{\pi,i}, s_0^{\pi,i}\le x<S_0^{\pi,i}$ and $x\ge S_0^{\pi,i}$ separately. Then, we show that convexity is not violated at $x=s_0^{\pi,i}$ and $x=S_0^{\pi,i}$.

(i) $x<s_0^{\pi,i}$: Using (4.159), first and second derivatives of $v_0(\pi,i,x)$ are

$$v_0'(\pi,i,x) = \sum_{a\in\mathbb{F}}\pi^a\int_0^1 L'(i,a,x+uA)dF_a(u) \tag{4.167}$$

$$v_0''\left(\pi,i,x\right) = \sum_{a\in\mathbb{F}}\pi^a\int_0^1 L''(i,a,x+uA)dF_a(u) \tag{4.168}$$

for all π, i and x. Notice that $v_0''\left(\pi,i,x\right)$ in (4.168) is always positive since L is a strictly convex function. This implies that, for $x<s_0^{\pi,i}$, $v_0\left(\pi,i,x\right)$ is convex in x for all π and i.

(ii) $s_0^{\pi,i}\le x<S_0^{\pi,i}$: Using (4.152) and (4.159), the first derivative of $v_0(\pi,i,x)$ is

$$\begin{aligned} v_0'(\pi,i,x) &= \sum_{a\in\mathbb{F}}\pi^a\int_0^1\left(1+u\Big(\frac{\partial y_0^{\pi,i}\left(x\right)}{\partial x}-1\Big)\right) \\ &\quad\times\left(c_i+L'\left(i,a,x+u\left(y_0^{\pi,i}\left(x\right)-x\right)\right)\right)dF_a(u)-c_i \\ &= \Big(\frac{\partial y_0^{\pi,i}\left(x\right)}{\partial x}-1\Big)\sum_{a\in\mathbb{F}}\pi^a\int_0^1 u\left(c_i+L'(i,a,x+u\left(y_0^{\pi,i}\left(x\right)-x\right)\right)dF_a(u) \\ &\quad+\sum_{a\in\mathbb{F}}\pi^a\int_0^1 L'(i,a,x+u\left(y_0^{\pi,i}\left(x\right)-x\right)dF_a(u) \\ &= \sum_{a\in\mathbb{F}}\pi^a\int_0^1 L'(i,a,x+u\left(y_0^{\pi,i}\left(x\right)-x\right)dF_a(u) \end{aligned} \tag{4.169}$$

for all π, i and x. We get the second equality in (4.169) after we make some mathematical simplifications. Then, using (4.161), we get the third equality in the same equation. Moreover, after differentiating $v_0'(\pi,i,x)$ in (4.169) one more time, we get

$$\begin{aligned} v_0''(\pi,i,x) &= \sum_{a\in\mathbb{F}}\pi^a\int_0^1 L''(i,a,x+u\left(y_0^{\pi,i}(x)-x\right)dF_a(u) \\ &\quad +\Big(\frac{\partial y_0^{\pi,i}(x)}{\partial x}-1\Big)\sum_{a\in\mathbb{F}}\pi^a\int_0^1 uL''(i,a,x+u\left(y_0^{\pi,i}(x)-x\right)dF_a(u) \\ &= \Bigg\{\sum_{a\in\mathbb{F}}\pi^a\int_0^1 L''(i,a,x+u\left(y_0^{\pi,i}(x)-x\right)dF_a(u) \\ &\quad \times\sum_{a\in\mathbb{F}}\pi^a\int_0^1 u^2L''(i,a,x+u\left(y_0^{\pi,i}(x)-x\right)dF_a(u) \\ &\quad -\Big(\sum_{a\in\mathbb{F}}\pi^a\int_0^1 uL''(i,a,x+u\left(y_0^{\pi,i}(x)-x\right)dF_a(u)\Big)^2\Bigg\} \\ &\quad \Big/\Big(\sum_{a\in\mathbb{F}}\pi^a\int_0^1 u^2L''(i,a,x+u\left(y_0^{\pi,i}(x)-x\right)dF_a(u)\Big) \qquad (4.170) \\ &\geq 0 \end{aligned}$$

for all π, i and x. Note that the second equality in (4.170) is gathered by substituting (4.162) in the first equality of the same equation. Notice also that the denominator of (4.170) is always positive since $L''>0$ and, by our assumption, $\bar{\mu}_\pi>0$. In addition, by Lemma 21, the numerator of (4.170) is also nonnegative. Therefore, for $s_0^{\pi,i}\leq x<S_0^{\pi,i}$, $v_0(\pi,i,x)$ is convex in x for all π and i.

(iii) $x\geq S_0^{\pi,i}$: Using (4.159), first and second derivatives of $v_0(\pi,i,x)$ are

$$v_0'(\pi,i,x) = \sum_{a\in\mathbb{F}}\pi^aL'(i,a,x) \qquad (4.171)$$

$$v_0''(\pi,i,x) = \sum_{a\in\mathbb{F}}\pi^aL''(i,a,x) \qquad (4.172)$$

for all π, i and x. It is clear that $v_0''(\pi,i,x)$ is always positive since L is strictly convex. Then, it follows that, for $x\geq S_0^{\pi,i}$, $v_0(\pi,i,x)$ is convex in x for all π and i.

(iv) $x=s_0^{\pi,i}$: Here, we show that convexity of $v_0(\pi,i,x)$ is not violated at $x=s_0^{\pi,i}$. For v_0 to be convex at $x=s_0^{\pi,i}$, the following conditions must hold:

$$\lim_{x\uparrow s_0^{\pi,i}} v_0'(\pi,i,x)\leq\lim_{x\downarrow s_0^{\pi,i}} v_0'(\pi,i,x) \qquad (4.173)$$

and v_0 must be continuous at $x = s_0^{\pi,i}$. Firstly, by (4.167) and (4.169),

$$\lim_{x\uparrow s_0^{\pi,i}} v_0'(\pi,i,x) = \lim_{x\downarrow s_0^{\pi,i}} v_0'(\pi,i,x) = \sum_{a\in\mathbb{F}} \pi^a \int_0^1 L'(i,a,s_0^{\pi,i}+uA)dF_a(u)$$

for all π and i so that condition (4.173) is satisfied. Secondly, by (4.159),

$$\lim_{x\uparrow s_0^{\pi,i}} v_0(\pi,i,x) = \lim_{x\downarrow s_0^{\pi,i}} v_0(\pi,i,x) = \sum_{a\in\mathbb{F}} \pi^a \int_0^1 \left(c_i uA + L\left(i,a,s_0^{\pi,i}+uA\right)\right) dF_a(u)$$

for all π and i; as a result, $v_0(\pi,i,x)$ is continuous at $x = s_0^{\pi,i}$. Then, it follows that $v_0(\pi,i,x)$ is convex at $x = s_0^{\pi,i}$.

(v) $x = S_0^{\pi,i}$: We now show that $v_0(\pi,i,x)$ is also convex at $x = S_0^{\pi,i}$. Similarly, for v_0 to be convex at $x = S_0^{\pi,i}$ condition (4.173) must also be satisfied at $x = S_0^{\pi,i}$; moreover, v_0 must be continuous at at $x = S_0^{\pi,i}$. Note by (4.169) and (4.171) that

$$\lim_{x\uparrow S_0^{\pi,i}} v_0'(\pi,i,x) = \lim_{x\downarrow S_0^{\pi,i}} v_0'(\pi,i,x) = \sum_{a\in\mathbb{F}} \pi^a L'(i,a,S_0^{\pi,i})$$

for all π and i so that condition (4.173) is satisfied at $x = S_0^{\pi,i}$. Moreover, by (4.159),

$$\lim_{x\uparrow S_0^{\pi,i}} v_0(\pi,i,x) = \lim_{x\downarrow S_0^{\pi,i}} v_0(\pi,i,x) = \sum_{a\in\mathbb{F}} \pi^a L(i,a,S_0^{\pi,i})$$

for all π and i; therefore, v_0 is continuous at $x = S_0^{\pi,i}$. Hence, $v_0(\pi,i,x)$ is convex at $x = S_0^{\pi,i}$ for all π and i.

Finally, using (4.15), (4.167) and MCT,

$$\lim_{x\downarrow -\infty} v_0'(\pi,i,x) = \sum_{a\in\mathbb{F}} \pi^a \lim_{x\downarrow -\infty} L'(i,a,x) = -p_i$$

for all π and i. Similarly, using (4.15), (4.171) and MCT,

$$\lim_{x\uparrow\infty} v_0'(\pi,i,x) = h_i$$

for all π and i. This completes our proof. ■

Given π, i and x, $y_0(\pi,i,x) - x$ is the optimal amount of inventory to order and $y_0(\pi,i,x)$ is the optimal order-up-to level. Federgruen and Zipkin (1986 b) studied finite capacity problems in stationary environment for the first time. Their results are similar to ours except that optimal order-up-to level in our policy depends on inventory level, information vector and state of the observed environment as well. There, the optimal policy (modified base-stock) orders a positive amount if the initial inventory level is less than a critical level S and no order is given otherwise. Moreover, there is another critical level $s < S$ over which optimal order-up-to level is constant and equal to S, and below which it is optimal to order as much as the finite capacity A. Since they do not assume proportional yield, the optimal order-up-to level is independent of initial inventory level.

Later, Henig and Gerchak (1990) study inventory problems with random proportional yield in stationary environment and our results are similar to theirs in some respect. They show that it is optimal to order only if the inventory level is below a critical level S and it is optimal not to order otherwise. In addition, their optimal order-up-to level decreases with current inventory level and always stays above S. Since they did not assume finite capacity, their ordering quantity is arbitrary. On the other hand, we assume finite capacity and partially observed random environment. Hence, the policy has a second critical level and all critical levels depend on inventory level, information vector and state of observed environment.

Recently, Gallego and Hu (2004) show that the optimal policy is a combination of both modified and inflated base-stock policies for problems with finite capacity and random yield in a fully observed random environment. They prefer to use the term "modified inflated" base-stock for this type of policies. Specifically, they show that there is an inventory level S (above) below which it is optimal (not) to order. They also show that there is another inventory level $s < S$ below which order quantity is constant and equal to the finite capacity. Furthermore, when the inventory level is between s and S, order-up-to level decreases with inventory level and is above S. Finally, they show that both s and S depend on the environmental state. By Theorem 22, we get similar results as Gallego and Hu (2004). However, all critical levels and order-up-to level depend also on current information vector π since we assume partially observed random environment.

Moreover, by Theorem 22, we show that $S_0^{\pi,i}$ satisfies (4.163). Then, using (4.15), (4.149), (4.150) and (4.163), it follows that $S_0^{\pi,i}$ satisfies

$$M_\pi \left(S_0^{\pi,i}\right) = \sum_{a\in\mathbb{F}} \beta_a(\pi) M_a \left(S_0^{\pi,i}\right) = \frac{p_i - c_i}{p_i + h_i} \tag{4.174}$$

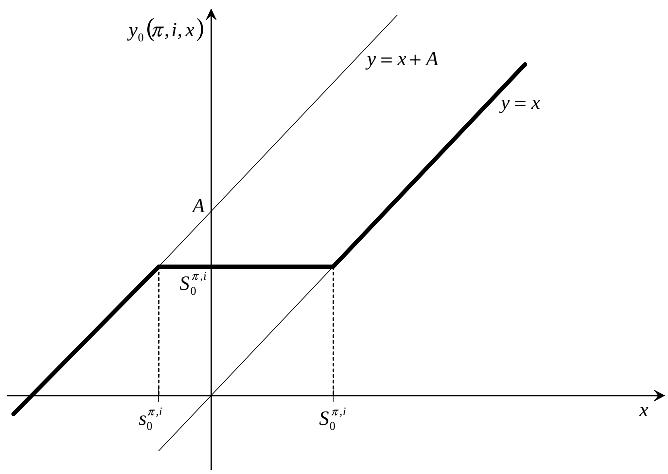

Figure 4.3: State-dependent modified base-stock policy

for all π and i. From (4.149) and (4.174), it is obvious that $S_0^{\pi,i}$ depends on the mean of proportional yield U. In addition, we show by (4.162) that the optimal order-up-to level is decreasing in x for all π, i and $x \in [s_0^{\pi,i}, S_0^{\pi,i})$. This implies that the optimal policy is not a modified base-stock policy but a modified inflated base-stock policy. However, if $U = 1$ with probability 1 in all environments, then (4.162) becomes 0 so that optimal order-up-to level is independent of inventory level when x is in $[s_0^{\pi,i}, S_0^{\pi,i})$. As a result, optimal ordering policy is modified base-stock and has the form shown in Figure 4.3. On the other hand, if we assume that there is no capacity limitation so that $A = \infty$, then the line $y = x + A$ in Figure 4.2, which is bounding $y_0^{\pi,i}(x)$ from above, will cross x and y axes at minus and plus infinity respectively. As a result, optimal policy has the form as shown in Figure 4.4. In this type of policy, it will always be optimal to order if the inventory level is below $S_0^{\pi,i}$ and the optimal order-up-to level is $y_0^{\pi,i}(x)$. Optimal policy structure for inventory models with random proportional yield is, in general, in this form and called by Zipkin (2000) as inflated base stock policy.

4.7.3 Multi-period model

In the case with multiple periods, there are N periods to plan for and the dynamic programming equation involves the sum of single period costs in the current period plus the

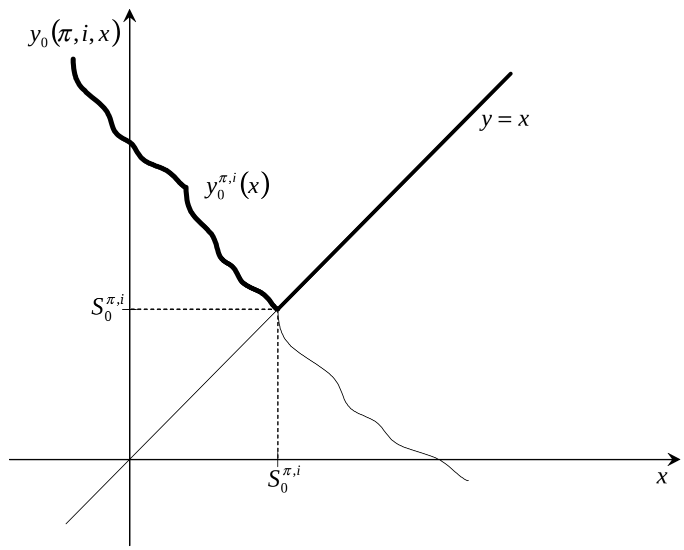

Figure 4.4: State-dependent inflated base-stock policy

expected optimal discounted costs from the next period until the end of the planning horizon. We assume that $v_N(\pi_N, i_N, x_N) = 0$ for all π_N, i_N and x_N. Moreover, we suppose that inventory level is x, state of the environment is i, distribution of true state of the environment is π at time n. The minimum cost satisfies

$$v_n(\pi, i, x) = \min_{x \le y \le x+A} H_n(\pi, i, x, y) \tag{4.175}$$

for all π, i and x, where y is the order-up-to level and

$$\begin{aligned} H_n(\pi, i, x, y) &= \sum_{a \in \mathbb{F}} \pi^a J_n(\pi, i, a, x, y) && (4.176) \\ J_n(\pi, i, a, x, y) &= \int_0^1 G_n(\pi, i, a, x + u(y - x))\, dF_a(u) - c_i x && (4.177) \\ G_n(\pi, i, a, y) &= c_i y + L(i, a, y) \\ &\quad + \alpha \sum_{j \in \mathbb{E}} \Psi_n(a, j)\, E_D^a \left[v_{n+1}(T(\pi | j), j, y - D) \right] && (4.178) \end{aligned}$$

with $L(i, a, y)$ given in (4.14). Here, E_D^a again denotes the notation that expectation is taken with respect to the random variable D with distribution M_a. We use G_n' and G_n'' to denote first and second derivatives of G_n with respect to y respectively. Then,

$$G'_n(\pi, i, a, y) = c_i + L'(i, a, y) + \alpha \sum_{j \in \mathbb{E}} \Psi_n(a, j) E^a_D \left[v'_{n+1}(T(\pi | j), j, y - D) \right] \quad (4.179)$$

$$G''_n(\pi, i, a, y) = L''(i, a, y) + \alpha \sum_{j \in \mathbb{E}} \Psi_n(a, j) E^a_D \left[v''_{n+1}(T(\pi | j), j, y - D) \right] \quad (4.180)$$

for all π, i, a and y.

As Lemma 21, we use the following lemma to prove the convexity of optimal cost function in multi-period. However, we skip the proof since it follows exactly the same line of reasoning as the proof of Lemma 21.

Lemma 23 *Assume that g is a function of x and u. Then,*

$$\left[\sum_{a \in \mathbb{F}} \pi^a \int_0^1 u G''_n(\pi, i, a, g(x, u)) \, dF_a(u) \right]^2 \leq \left[\sum_{a \in \mathbb{F}} \pi^a \int_0^1 G''_n(\pi, i, a, g(x, u)) \, dF_a(u) \right] \times \left[\sum_{a \in \mathbb{F}} \pi^a \int_0^1 u^2 G''_n(\pi, i, a, g(x, u)) \, dF_a(u) \right]$$

for all π, i and x.

Let $y_n(\pi, i, x)$ denote the optimal order-up-to level of the minimization problem in (4.175) given π, i and x. Finally, we let $v'_n(\pi, i, x)$ denote the derivative of $v_n(\pi, i, x)$ with respect to x, and assume that h and p are holding cost and shortage cost vectors respectively. In addition, we recall that $R^\alpha_{n,m}((\pi, i), j) = \sum_{k=0}^{m-1} \alpha^k P^k_n((\pi, i), j)$ for all π, i and j and that, for a nonnegative function g, $R^\alpha_{n,m} g(\pi, i)$ is defined as in (4.46).

Theorem 24 *The optimal ordering policy for N-period model is a state-dependent modified inflated base-stock policy*

$$y_n(\pi, i, x) = \begin{cases} x + A & x < s^{\pi,i}_n \\ y^{\pi,i}_n(x) & s^{\pi,i}_n \leq x < S^{\pi,i}_n \\ x & x \geq S^{\pi,i}_n \end{cases} \quad (4.181)$$

where $y^{\pi,i}_n(x), S^{\pi,i}_n$ and $s^{\pi,i}_n$ are unique y values which satisfy

$$\sum_{a \in \mathbb{F}} \pi^a \int_0^1 u G'_n(\pi, i, a, x + u(y - x)) \, dF_a(u) = 0$$

$$\sum_{a \in \mathbb{F}} \pi^a \mu_a G'_n(\pi, i, a, y) = 0$$

$$\sum_{a \in \mathbb{F}} \pi^a \int_0^1 u G'_n(\pi, i, a, y + uA) \, dF_a(u) = 0$$

for all π and i respectively. Moreover, $y_n^{\pi,i}(x)$ is nonincreasing in x with $\lim_{x\downarrow-\infty} y_n^{\pi,i}(x) = \infty$ for all π and i. In addition, $x \leq y_n^{\pi,i}(x) \leq x+A$ for all $x \in \left[s_n^{\pi,i}, S_n^{\pi,i}\right)$ with $y_n^{\pi,i}\left(s_n^{\pi,i}\right) = s_n^{\pi,i} + A$ and $y_n^{\pi,i}\left(S_n^{\pi,i}\right) = S_n^{\pi,i}$. The cost function $H_n(\pi,i,x,y)$ is strictly convex in (x,y) for all π and i, and the optimal cost is

$$v_n(\pi,i,x) = \begin{cases} \sum_{a\in\mathbb{F}} \pi^a \int_0^1 G_n(\pi,i,a,x+uA)\,dF_a(u) - c_i x & x < s_n^{\pi,i} \\ H_n\left(\pi,i,x,y_n^{\pi,i}(x)\right) & s_n^{\pi,i} \leq x < S_n^{\pi,i} \\ \sum_{a\in\mathbb{F}} \pi^a G_n(\pi,i,a,x) - c_i x & x \geq S_n^{\pi,i}. \end{cases} \tag{4.182}$$

for all π, i and x. Furthermore, $v_n(\pi,i,x)$ is convex in x, $\lim_{x\uparrow\infty} v_n'(\pi,i,x) = R_{n,N-n}^{\alpha} h(\pi,i)$ and $\lim_{x\downarrow-\infty} v_n'(\pi,i,x) = -R_{n,N-n}^{\alpha} p(\pi,i)$ for all π and i.

Proof. The proof proceeds inductively. By Theorem 22, we know that Theorem 24 is satisfied for $n = N-1$. Assume that the induction hypothesis is valid for times $n+1, n+2, ..., N-1$ so that $v_{n+1}(\pi,i,x)$ is convex in x for all π and i; moreover, $\lim_{x\uparrow\infty} v_{n+1}'(\pi,j,x) = R_{n+1,N-n-1}^{\alpha} h(\pi,j)$ and $\lim_{x\downarrow-\infty} v_{n+1}'(\pi,j,x) = -R_{n+1,N-n-1}^{\alpha} p(\pi,j)$ for all π and j. Then, using (4.179), MCT and the induction hypothesis, we get

$$\begin{aligned} \lim_{y\uparrow\infty} G_n'(\pi,i,a,y) &= c_i + h_i + \alpha \sum_{j\in\mathbb{E}} \Psi_n(a,j)\, R_{n+1,N-n-1}^{\alpha} h(T(\pi|\,j),j) \qquad (4.183) \\ &> 0 \\ \lim_{y\downarrow-\infty} G_n'(\pi,i,a,y) &= c_i - p_i - \alpha \sum_{j\in\mathbb{E}} \Psi_n(a,j)\, R_{n+1,N-n-1}^{\alpha} p(T(\pi|\,j),j) \qquad (4.184) \\ &< 0 \end{aligned}$$

for all i and a. We now show that Theorem 24 also holds for time n.

Note that $G_n(\pi,i,a,y)$ is strictly convex in y for all π, i and a since L is strictly convex and v_{n+1} is convex. Then it follows from Lemma 20 that $G_n(\pi,i,a,x+u(y-x))$ is strictly convex in (x,y) for all π, i, a and $u \in [0,1]$. As a result, J_n is also strictly convex in (x,y) because expectation preserves the strict convexity. Finally, it follows that H_n is strictly convex since sum of strictly convex functions is also strictly convex. In addition, we use H_n' to denote partial derivative of H_n with respect to y. Hence,

$$H_n'(\pi,i,x,y) = \sum_{a\in\mathbb{F}} \pi^a \int_0^1 u G_n'(\pi,i,a,x+u(y-x))\,dF_a(u) \tag{4.185}$$

for all π, i, x and y.

Suppose that $y_n^{\pi,i}(x)$ is the y value minimizing H_n without the constraint $x \leq y \leq x+A$. Because H_n is strictly convex in y, using (4.185), $y_n^{\pi,i}(x)$ satisfies

$$\sum_{a\in\mathbb{F}} \pi^a \int_0^1 uG_n'\left(\pi, i, a, x + u(y_n^{\pi,i}(x) - x)\right) dF_a(u) = 0 \tag{4.186}$$

for a given π, i and x. Using (4.185), induction hypothesis and (4.184), we get

$$\lim_{y\downarrow-\infty} H_n'(\pi, i, x, y) = (c_i - p_i)\,\bar{\mu}_\pi - \alpha \sum_{j\in\mathbb{E}} \sum_{a\in\mathbb{F}} \pi^a \mu_a \Psi_n(a, j)\, R^\alpha_{n+1,N-n-1} p(T(\pi\,|j), j) < 0$$

for all π, i and x. Similarly, using (4.185), induction hypothesis and (4.183), we get

$$\lim_{y\uparrow\infty} H_n'(\pi, i, x, y) = (c_i + h_i)\,\bar{\mu}_\pi + \alpha \sum_{j\in\mathbb{E}} \sum_{a\in\mathbb{F}} \pi^a \mu_a \Psi_n(a, j)\, R^\alpha_{n+1,N-n-1} h(T(\pi\,|j), j) > 0$$

for all π, i and x. Then, it follows that there exists a finite $y_n^{\pi,i}(x)$ satisfying (4.186). Furthermore, strict convexity of H_n implies that $y_n^{\pi,i}(x)$ is unique. We now show that $\lim_{x\downarrow-\infty} y_n^{\pi,i}(x) = \infty$. Suppose that $\lim_{x\downarrow-\infty} y_n^{\pi,i}(x) < \infty$. Then, using (4.185) and MCT,

$$\begin{aligned}
\lim_{x\downarrow-\infty} H_n'\left(\pi, i, x, y_n^{\pi,i}(x)\right) &= \sum_{a\in\mathbb{F}} \pi^a \int_0^1 u \lim_{x\downarrow-\infty} G_n'\left(\pi, i, a, x + u(y_n^{\pi,i}(x) - x)\right) dF_a(u) \\
&= (c_i - p_i)\,\bar{\mu}_\pi - \alpha \sum_{j\in\mathbb{E}} \sum_{a\in\mathbb{F}} \pi^a \mu_a \Psi_n(a, j)\, R^\alpha p(T(\pi|\, j), j) < 0
\end{aligned}$$

for all π and i. The first equality is gathered by MCT. If we assume that $\lim_{x\downarrow-\infty} y_n^{\pi,i}(x) < \infty$, then $\lim_{x\downarrow-\infty}\left[x + u(y_n^{\pi,i}(x) - x)\right] = -\infty$ for all π, i and $u \in [0,1]$. Then, using (4.184), we get the second equality. However, we assume that $p_i > c_i$ for all i. This implies that $\lim_{x\downarrow-\infty} H_n'\left(\pi, i, x, y_n^{\pi,i}(x)\right) < 0$. But, this is a clear contradiction and $\lim_{x\downarrow-\infty} y_n^{\pi,i}(x) = \infty$ for all π and i.

Next, we show that $y_n^{\pi,i}(x)$ is nonincreasing in x. Suppose that there exist $x_1 < x_2$ such that $H_n'\left(\pi, i, x_1, y_n^{\pi,i}(x_1)\right) = H_n'\left(\pi, i, x_2, y_n^{\pi,i}(x_2)\right) = 0$ with $y_n^{\pi,i}(x_1) < y_n^{\pi,i}(x_2)$. This clearly implies that $x_1 + u(y_n^{\pi,i}(x_1) - x_1) < x_2 + u(y_n^{\pi,i}(x_2) - x_2)$ for all $u \in [0,1]$. This further implies that $G_n'\left(\pi, i, a, x_2 + u(y_n^{\pi,i}(x_2) - x_2)\right) > G_n'\left(\pi, i, a, x_1 + u(y_n^{\pi,i}(x_1) - x_1)\right)$ for all π, i and a since L' is increasing and v_{n+1}' is nondecreasing. Then, using (4.185) and our assumption that $\bar{\mu}_\pi > 0$, it follows that

$$H_n'\left(\pi, i, x_2, y_n^{\pi,i}(x_2)\right) > H_n'\left(\pi, i, x_1, y_n^{\pi,i}(x_1)\right) = 0$$

for all π and i. However, notice that this is a clear contradiction to the definition of $y_n^{\pi,i}(x_2)$. Therefore, $y_n^{\pi,i}(x)$ is nonincreasing in x. Furthermore, after we differentiate (4.186) with respect to x, we get

$$\frac{\partial y_n^{\pi,i}(x)}{\partial x} = \frac{\sum_{a\in\mathbb{F}} \pi^a \int_0^1 \left(u^2-u\right) G_n''\left(\pi,i,a,x+u(y_n^{\pi,i}(x)-x)\right) dF_a(u)}{\sum_{a\in\mathbb{F}} \pi^a \int_0^1 u^2 G_n''\left(\pi,i,a,x+u(y_n^{\pi,i}(x)-x)\right) dF_a(u)} \leq 0. \tag{4.187}$$

Notice that denominator in (4.187) is always positive since $G_n'' > 0$ and, by our assumption, $\bar{\mu}_\pi > 0$. On the other hand, numerator of the same equation is nonpositive since $u^2 - u \leq 0$ for $u \in [0,1]$ and $G'' > 0$. As a result, inequality in (4.162) is valid and $y_n^{\pi,i}(x)$ is nonincreasing in x. Therefore, $y_n^{\pi,i}(x)$ has a similar structure as $y_0^{\pi,i}(x)$ in Figure 4.2.

Obviously $y_n^{\pi,i}(x)$ satisfying (4.186) may not be a feasible solution for the problem in (4.175). But, a feasible order-up-to level must satisfy $x \leq y \leq x + A$. If $S_n^{\pi,i}$ is the smallest inventory level at which it is optimal not to order, then, for $x = S_n^{\pi,i}$, (4.186) becomes

$$\sum_{a\in\mathbb{F}} \pi^a \mu_a G_n'\left(\pi,i,a,S_n^{\pi,i}\right) = 0. \tag{4.188}$$

Note that, by our assumption, $\mu_a > 0$ for at least one a with $\pi^a > 0$; moreover, G_n' is increasing in y, and $\lim_{y\uparrow\infty} G_n'(\pi,i,a,y) > 0$ and $\lim_{y\downarrow-\infty} G_n'(\pi,i,a,y) < 0$ by (4.183) and (4.184). Hence, $S_n^{\pi,i}$ satisfying (4.188) is finite and unique. We next assume that $s_n^{\pi,i}$ is the largest inventory level at which ordering as much as finite capacity A is optimal so that $y_n^{\pi,i}\left(s_n^{\pi,i}\right) = s_n^{\pi,i} + A$. Then, $s_n^{\pi,i}$ satisfies

$$\sum_{a\in\mathbb{F}} \pi^a \int_0^1 u G_n'\left(\pi,i,a,s_n^{\pi,i}+uA\right) dF_a(u) = 0 \tag{4.189}$$

for all π, i and a. Using exactly the same line of reasoning that we did for $S_n^{\pi,i}$, we can show that $s_n^{\pi,i}$ is also finite and unique. Note that $y_n^{\pi,i}(x)$ is nonincreasing in x. This clearly implies that $s_n^{\pi,i} \leq S_n^{\pi,i}$ for all π and i. Now, as in single-period model, we consider the three cases: $x < s_n^{\pi,i}$, $s_n^{\pi,i} \leq x < S_n^{\pi,i}$ and $x \geq S_n^{\pi,i}$.

(i) $x < s_n^{\pi,i}$: Note that $y_n^{\pi,i}(x) > y_n^{\pi,i}(s_n^{\pi,i}) = s_n^{\pi,i} + A$ for all $x < s_n^{\pi,i}$ since $y_n^{\pi,i}(x)$ is nonincreasing in x. As a result, $y_n^{\pi,i}(x) > x + A$ for all $x < s_n^{\pi,i}$. Since the production capacity is limited by A, ordering up to $y_n^{\pi,i}(x)$ is not possible when $x < s_n^{\pi,i}$. In addition, H_n is a strictly convex function whose unique minimizer is $y_n^{\pi,i}(x)$; therefore, it is decreasing for values of y in $[x, y_n^{\pi,i}(x))$. Hence, ordering as much as the fixed capacity A is optimal for $x < s_n^{\pi,i}$. Thus, optimal order-up-to level for all $x < s_n^{\pi,i}$ is $y_n(\pi, i, x) = x + A$.

(ii) $s_n^{\pi,i} \leq x < S_n^{\pi,i}$: Note that $y_n^{\pi,i}(x)$ is nonincreasing in x; therefore, $s_n^{\pi,i} + A \geq y_n^{\pi,i}(x) > S_n^{\pi,i}$ for values of x in $[s_n^{\pi,i}, S_n^{\pi,i})$. This implies that $x < y_n^{\pi,i}(x) \leq x + A$ because x is in $[s_n^{\pi,i}, S_n^{\pi,i})$. Then it follows that the global minimum $y_n^{\pi,i}(x)$ is attainable. As a result, optimal order-up-to level is $y_n(\pi, i, x) = y_n^{\pi,i}(x)$.

(iii) $x \geq S_n^{\pi,i}$: Note that $y_n^{\pi,i}(x) \leq S_n^{\pi,i}$ since $y_n^{\pi,i}(x)$ is nonincreasing in x. This clearly implies that $y_n^{\pi,i}(x) \leq x$ because x is greater than or equal to $S_n^{\pi,i}$. As a result, global minimum $y_n^{\pi,i}(x)$ is not attainable since ordering a negative amount is not possible. However, H_n is increasing for all $y \in [x, \infty)$ since it is strictly convex and $y_n^{\pi,i}(x) \leq x$. Therefore, it is optimal not to order so that $y_n(\pi, i, x) = x$ if the inventory level is greater than or equal to $S_n^{\pi,i}$.

As a result, $y_n(\pi, i, x)$ in (4.181) is the optimal ordering policy and has a similar form as $y_0(\pi, i, x)$ in Figure 4.2. Notice that $y_n(\pi, i, x) = y_n^{\pi,i}(x)$ for all x in $[s_n^{\pi,i}, S_n^{\pi,i})$ and $y_n^{\pi,i}(x)$ is nonincreasing in x; moreover, $y_n^{\pi,i}(S_n^{\pi,i}) = S_n^{\pi,i}$ by definition. Then, it follows that $y_n(\pi, i, x) \geq S_n^{\pi,i}$ for all x in $[s_n^{\pi,i}, S_n^{\pi,i})$.

If we apply the optimal policy in (4.181), then the optimal cost becomes

$$v_n(\pi, i, x) = \begin{cases} H_n(\pi, i, x, x + A) & x < s_n^{\pi,i} \\ H_n\left(\pi, i, x, y_n^{\pi,i}(x)\right) & s_n^{\pi,i} \leq x < S_n^{\pi,i} \\ H_n(\pi, i, x, x) & x \geq S_n^{\pi,i} \end{cases}$$

which leads to (4.182) since

$$H_n(\pi, i, x, x + A) = \sum_{a \in \mathbb{F}} \pi^a \int_0^1 G_n(\pi, i, a, x + uA)\, dF_a(u) - c_i x$$

and

$$H_n(\pi, i, x, x) = \sum_{a \in \mathbb{F}} \pi^a G_n(\pi, i, a, x) - c_i x$$

for all π, i and x where G_n is given in (4.178).

Now, we prove that $v_n(\pi, i, x)$ is convex in x. First, we show that $v_n(\pi, i, x)$ is convex for $x < s_n^{\pi,i}, s_n^{\pi,i} \leq x < S_n^{\pi,i}$ and $x \geq S_n^{\pi,i}$ separately. Then, we show that convexity is not violated at $x = s_n^{\pi,i}$ and $x = S_n^{\pi,i}$.

(i) $x < s_n^{\pi,i}$: Using (4.182), the first and second derivatives of $v_n(\pi, i, x)$ are

$$v_n'(\pi, i, x) = \sum_{a \in \mathbb{F}} \pi^a \int_0^1 G_n'(\pi, i, a, x + uA) dF_a(u) - c_i \quad (4.190)$$

$$v_n''(\pi, i, x) = \sum_{a \in \mathbb{F}} \pi^a \int_0^1 G_n''(\pi, i, a, x + uA) dF_a(u) \quad (4.191)$$

for all π, i and x. Notice that $v_n''(\pi, i, x)$ in (4.191) is always positive since G is a strictly convex function. This implies that, for $x < s_n^{\pi,i}$, $v_n(\pi, i, x)$ is convex in x for all π and i.

(ii) $s_n^{\pi,i} \leq x < S_n^{\pi,i}$: Using (4.176) and (4.182), the first derivative of $v_n(\pi, i, x)$ is

$$\begin{aligned} v_n'(\pi, i, x) &= \Big(\frac{\partial y_n^{\pi,i}(x)}{\partial x} - 1\Big) \sum_{a \in \mathbb{F}} \pi^a \int_0^1 u G_n' \left(\pi, i, a, x + u \left(y_n^{\pi,i}(x) - x\right)\right) dF_a(u) \\ &\quad + \sum_{a \in \mathbb{F}} \pi^a \int_0^1 G_n' \left(\pi, i, a, x + u \left(y_n^{\pi,i}(x) - x\right)\right) dF_a(u) - c_i \\ &= \sum_{a \in \mathbb{F}} \pi^a \int_0^1 G_n' \left(\pi, i, a, x + u \left(y_n^{\pi,i}(x) - x\right)\right) dF_a(u) - c_i \end{aligned} \quad (4.192)$$

for all π, i and x. After we make some simplifications, we get the second equality in (4.192). Then, using (4.186), we get the third equality in the same equation. Moreover, using (4.192), we get

$$
\begin{aligned}
v_n''(\pi,i,x) &= \sum_{a\in\mathbb{F}}\pi^a\int_0^1 G_n''\left(\pi,i,a,x+u\left(y_n^{\pi,i}(x)-x\right)\right)dF_a(u) \\
&\quad +\Big(\frac{\partial y_n^{\pi,i}(x)}{\partial x}-1\Big)\sum_{a\in\mathbb{F}}\pi^a\int_0^1 uG_n''\left(\pi,i,a,x+u\left(y_n^{\pi,i}(x)-x\right)\right)dF_a(u) \\
&= \Bigg\{\sum_{a\in\mathbb{F}}\pi^a\int_0^1 G_n''\left(\pi,i,a,x+u\left(y_n^{\pi,i}(x)-x\right)\right)dF_a(u) \\
&\quad \times\sum_{a\in\mathbb{F}}\pi^a\int_0^1 u^2G_n''\left(\pi,i,a,x+u\left(y_n^{\pi,i}(x)-x\right)\right)dF_a(u) \\
&\quad -\Big(\sum_{a\in\mathbb{F}}\pi^a\int_0^1 uG_n''\left(\pi,i,a,x+u\left(y_n^{\pi,i}(x)-x\right)\right)dF_a(u)\Big)^2\Bigg\} \\
&\quad \Big/\Big(\sum_{a\in\mathbb{F}}\pi^a\int_0^1 u^2G_n''\left(\pi,i,a,x+u\left(y_n^{\pi,i}(x)-x\right)\right)dF_a(u)\Big) \qquad (4.193)\\
&\geq 0
\end{aligned}
$$

for all π,i and x. Note that the second equality in (4.193) is gathered by substituting (4.187) in the first equality of the same equation. Clearly the denominator of (4.193) is always positive since $G_n''>0$ and, by our assumption, $\bar{\mu}_\pi>0$. In addition, by Lemma 23, the numerator of (4.193) is nonnegative. Therefore, for $s_n^{\pi,i}\leq x<S_n^{\pi,i}$, $v_n(\pi,i,x)$ is convex in x for all π and i.

(iii) $x\geq S_n^{\pi,i}$: Using (4.182), the first and second derivatives of $v_n(\pi,i,x)$ are

$$v_n'(\pi,i,x) = \sum_{a\in\mathbb{F}}\pi^aG_n'(\pi,i,a,x)-c_i \qquad (4.194)$$

$$v_n''(\pi,i,x) = \sum_{a\in\mathbb{F}}\pi^aG_n''(\pi,i,a,x) \qquad (4.195)$$

for all π,i and x. Notice from (4.195), it is clear that $v_n''(\pi,i,x)$ is always positive since G_n is a strictly convex function. Then, it follows that, for $x\geq S_n^{\pi,i}$, $v_n(\pi,i,x)$ is convex in x for all π and i.

(iv) $x=s_n^{\pi,i}$: Here, we show that convexity of $v_n(\pi,i,x)$ is not violated at $x=s_n^{\pi,i}$. For v_n to be convex at $x=s_n^{\pi,i}$, the following conditions must hold:

$$\lim_{x\uparrow s_n^{\pi,i}} v_n'(\pi,i,x)\leq\lim_{x\downarrow s_n^{\pi,i}} v_n'(\pi,i,x) \qquad (4.196)$$

and v_n must be continuous at $x = s_n^{\pi,i}$. Firstly, by (4.190) and (4.192),

$$\lim_{x\uparrow s_n^{\pi,i}} v_n'(\pi,i,x) = \lim_{x\downarrow s_n^{\pi,i}} v_n'(\pi,i,x) = \sum_{a\in\mathbb{F}} \pi^a \int_0^1 G_n'(\pi,i,a,s_n^{\pi,i}+uA)dF_a(u) - c_i$$

for all π and i so that condition (4.173) is satisfied. Secondly, by (4.182),

$$\lim_{x\uparrow s_n^{\pi,i}} v_n(\pi,i,x) = \lim_{x\downarrow s_n^{\pi,i}} v_n(\pi,i,x) = \sum_{a\in\mathbb{F}} \pi^a \int_0^1 G_n(\pi,i,a,s_n^{\pi,i}+uA)dF_a(u) - c_i s_n^{\pi,i}$$

for all π and i; as a result, $v_n(\pi,i,x)$ is continuous at $x = s_n^{\pi,i}$. Then, it follows that $v_n(\pi,i,x)$ is convex at $x = s_n^{\pi,i}$.

(v) $x = S_n^{\pi,i}$: We now show that $v_n(\pi,i,x)$ is also convex at $x = S_n^{\pi,i}$. Similarly, for v_n to be convex at $x = S_n^{\pi,i}$, condition (4.173) must also be satisfied at $x = S_n^{\pi,i}$; moreover, v_n must be continuous at at $x = S_n^{\pi,i}$. Note by (4.192) and (4.194) that

$$\lim_{x\uparrow S_n^{\pi,i}} v_n'(\pi,i,x) = \lim_{x\downarrow S_n^{\pi,i}} v_n'(\pi,i,x) = \sum_{a\in\mathbb{F}} \pi^a G_n'(\pi,i,a,S_n^{\pi,i}) - c_i$$

for all π and i so that condition (4.173) is satisfied at $x = S_n^{\pi,i}$. Moreover, by (4.182),

$$\lim_{x\uparrow S_n^{\pi,i}} v_n(\pi,i,x) = \lim_{x\downarrow S_n^{\pi,i}} v_n(\pi,i,x) = \sum_{a\in\mathbb{F}} \pi^a G_n(\pi,i,a,S_n^{\pi,i}) - c_i S_n^{\pi,i}$$

for all π and i; therefore, v_n is continuous at $x = S_n^{\pi,i}$. Hence, $v_n(\pi,i,x)$ is convex at $x = S_n^{\pi,i}$ for all π and i.

Finally, using (4.179), (4.190), MCT and the induction hypothesis, we get

$$\begin{aligned}
\lim_{x\downarrow -\infty} v_n'(\pi,i,x) &= \sum_{a\in\mathbb{F}} \pi^a \lim_{x\downarrow -\infty} G_n'(\pi,i,a,x) - c_i \\
&= -p_i - \alpha \sum_{j\in\mathbb{E}} \left[\sum_{a\in\mathbb{F}} \pi^a \Psi_n(a,j)\right] R_{n+1,N-n-1}^{\alpha} p(\pi,j) \\
&= -p_i - \alpha \sum_{j\in\mathbb{E}} P_n((\pi,i),j)\, R_{n+1,N-n-1}^{\alpha} p(\pi,j) \\
&= -R_{n,N-n}^{\alpha} p(\pi,i).
\end{aligned}$$

for all π and i. We get the second equality by (4.184). Then, we get the third equality by (4.43). The last equality is gathered from (4.47). Similarly, using (4.194), MCT and induction hypothesis, we get

$$\lim_{x\uparrow\infty} v_n'(\pi,i,x) = R_{n,N-n}^{\alpha} h(\pi,i)$$

for all π and i. This completes our proof. ■

By Theorem 24, we show that the optimal policy has the form in (4.181). This policy satisfies all necessary properties of an inflated base-stock policy for x in $[s_n^{\pi,i}, S_n^{\pi,i})$. Therefore, it is a state-dependent modified inflated base-stock policy. This implies that results obtained by Gallego and Hu (2004) are still valid when the random environment is partially observed. However, the critical values of the modified inflated base-stock policy such as $y_n^{\pi,i}(x), s_n^{\pi,i}$ and $S_n^{\pi,i}$ do not only depend on observed state at time n, but they also depend on the information vector π.

Moreover, it is obvious from (4.187) that optimal order-up-to level for $s_n^{\pi,i} \leq x < S_n^{\pi,i}$ depends on the inventory level x. This clearly implies that modified base-stock policy is not optimal for inventory problems with finite capacity and random yield. But, when $U = 1$ with probability 1 which is the finite capacity only case, we get by (4.187) that $\partial y_n^{\pi,i}(x)/\partial x = 0$. As a result, for $s_n^{\pi,i} \leq x < S_n^{\pi,i}$, optimal order-up-to level is independent of inventory level; moreover, it is equal to $S_n^{\pi,i}$. In this case, optimal order-up-to level in (4.181) turns out to be

$$y_n(\pi, i, x) = \begin{cases} x + A & x < s_n^{\pi,i} \\ S_n^{\pi,i} & s_n^{\pi,i} \leq x < S_n^{\pi,i} \\ x & x \geq S_n^{\pi,i} \end{cases}$$

for all π and i. Therefore, the state-dependent modified base-stock policy in Figure 4.3 is optimal in multi-period if only finite capacity in a partially observed random environment is considered.

Furthermore, if we assume that there is no capacity limitation ($A = \infty$) but only random proportional yield, then, using (4.189), $s_n^{\pi,i} = -\infty$ for all π and i. Note that if we assume that $s_n^{\pi,i} > -\infty$, then, using (4.185), we get

$$\begin{aligned} \lim_{A\uparrow\infty} H_n'\left(\pi, i, s_n^{\pi,i}, s_n^{\pi,i} + A\right) &= \sum_{a\in\mathbb{F}} \pi^a \int_0^1 u \lim_{A\uparrow\infty} G_n'\left(\pi, i, a, s_n^{\pi,i} + uA\right) dF_a(u) \\ &= (c_i + h_i)\,\bar{\mu}_\pi + \alpha \sum_{j\in\mathbb{E}} \sum_{a\in\mathbb{F}} \pi^a \mu_a \Psi_n(a, j)\, R^\alpha h(T(\pi| j), j) > 0 \end{aligned}$$

for all π and i. First equality follows from the MCT. Then, using (4.183) and making some simplifications, we get second equality. Notice that right hand side of second equality is always positive since all terms on the right are positive. However, this is a clear contradiction to (4.189). Hence, $s_n^{\pi,i} = -\infty$ for all π and i when $A = \infty$. Then, optimal order-up-to level in (4.181) becomes

$$y_n(\pi, i, x) = \begin{cases} y_n^{\pi,i}(x) & x < S_n^{\pi,i} \\ x & x \geq S_n^{\pi,i} \end{cases}$$

for all π and i. In other words, optimal policy structure in multi-period is state-dependent inflated base-stock, which is shown in Figure 4.4, if we assume that there is no capacity limitation.

4.7.4 Infinite-period model

In this section, we consider the infinite-period problem. As stated earlier, we assume that $\{Q_n\}$ and $\{E_n\}$ are time-homogenous in infinite-period analysis so that $Q_n = Q$ and $E_n = E$ for all n. Then, this implies that $\Psi_n = \Psi$ as given by (4.49). Using Ψ, P_n can be computed by (4.48). Here, we show that the finite-horizon optimal cost function v_n in Section 4.5.2 converges to the infinite-horizon optimal cost function v. In addition, we show that a state-dependent modified inflated base-stock policy is optimal when minimizing the expected discounted costs over an infinite-horizon. By assuming that $k = N - n$ denotes the number of periods from time n until time N, we use the notation $v_{n,k}$ for the finite horizon optimal cost v_n in the remaining part of this section. We show that, as k increases to infinity, the finite-horizon optimal cost function $v_{0,k}$ in (4.175) converges to the infinite-horizon optimal cost function v that satisfies

$$v(\pi, i, x) = \min_{x \leq y \leq x+A} H(\pi, i, x, y) \tag{4.197}$$

for all π, i and x, where y is order-up-to level and

$$\begin{aligned} H(\pi, i, x, y) &= \sum_{a \in \mathbb{F}} \pi^a J(\pi, i, a, x, y) && (4.198) \\ J(\pi, i, a, x, y) &= \int_0^1 G(\pi, i, a, x + u(y - x))\, dF_a(u) - c_i x && (4.199) \\ G(\pi, i, a, y) &= c_i y + L(i, a, y) + \alpha \sum_{j \in \mathbb{E}} \Psi(a, j)\, E_D^a [v(T(\pi | j), j, y - D)] && (4.200) \end{aligned}$$

with $L(i, a, y)$ as given in (4.14). Again E_D^a denotes the notation that expectation is taken with respect to the random variable D with distribution M_a. As in multi-period analysis, we use G' and G'' to denote first and second derivatives of G with respect to y, respectively. As a result,

$$G'(\pi, i, a, y) = c_i + L'(i, a, y) + \alpha \sum_{j \in \mathbb{E}} \Psi(a, j) E_D^a \left[v'(T(\pi | j), j, y - D) \right] \quad (4.201)$$

$$G''(\pi, i, a, y) = L''(i, a, y) + \alpha \sum_{j \in \mathbb{E}} \Psi(a, j) E_D^a \left[v''(T(\pi | j), j, y - D) \right] \quad (4.202)$$

for all π, i, a and y.

For any real valued function $f : \mathfrak{D}(\mathbb{F}) \times \mathbb{E} \times \mathbb{R} \rightarrow \mathbb{R}$ where $\mathfrak{D}(\mathbb{F})$ is the set of all probability distributions defined on state space $\mathbb{F}$, we define the mapping $\mathcal{T}$ as

$$\mathcal{T} f(\pi, i, x) = \min_{x \leq y \leq x+A} H(\pi, i, x, y) \quad (4.203)$$

for all π, i and x, where H is given in (4.198) with J as given in (4.199) and

$$G(\pi, i, a, y) = c_i y + L(i, a, y) + \alpha \sum_{j \in \mathbb{E}} \Psi(a, j) E_D^a \left[f(T(\pi | j), j, y - D) \right]. \quad (4.204)$$

Using (4.151), $\mathcal{T} f$ can be interpreted as the optimal cost function for the one-period problem where the terminal cost function is $\alpha \sum_{j \in \mathbb{E}} P_n((\pi_n, i), j) E_D^a \left[f(T(\pi | j), j, y - D) \right]$. Then, $\mathcal{T}^k$ denotes the composition of the mapping $\mathcal{T}$ with itself k times; that is, for all $k \geq 1$

$$\mathcal{T}^k f(\pi, i, x) = \mathcal{T} \mathcal{T}^{k-1} f(\pi, i, x) \quad (4.205)$$

with $\mathcal{T}^0 f = f$. Using (4.175), we can interpret $\mathcal{T}^k f$ as the optimal cost function for the $k-$period $\alpha-$discounted problem. Then, using (4.203) and (4.205),

$$\mathcal{T}^k f(\pi, i, x) = \min_{x \leq y \leq x+A} H_k(\pi, i, x, y) \quad (4.206)$$

where H_k given in (4.198) with J as given in (4.199) where G is replaced by

$$G_k(\pi, i, a, y) = c_i y + L(i, a, y) + \alpha \sum_{j \in \mathbb{E}} \Psi(a, j) E_D^a \left[\mathcal{T}^{k-1} f(T(\pi | j), j, y - D) \right]. \quad (4.207)$$

Let $f_0(\pi, i, x) = 0$ for all π, i and x. In our single-period and multi-period analyses, we always assume that the terminal cost function is zero. Suppose that the initial cost function is $f_0(\pi, i, x)$ so that $\mathcal{T}^0 f(\pi, i, x) = f_0(\pi, i, x)$ for all π, i and x. Then, $k-$period optimal cost function is $v_{n,k}(\pi, i, x) = \mathcal{T}^k f_0(\pi, i, x)$ for all π, i, x and n.

Let $f_*(\pi, i, x)$ denote the optimal cost over infinite horizon and let

$$f_\infty(\pi, i, x) = \lim_{k\uparrow\infty} \mathcal{T}^k f_0(\pi, i, x) \tag{4.208}$$

for all π, i and x. Notice that f_∞ is well-defined provided we allow the possibility that f_∞ can take the value ∞. If we can show that the finite-horizon optimal cost converges to the infinite-horizon optimal cost as the length of the planning horizon gets longer, then we can infer the properties of $f_*(\pi, i, x)$ from the properties of $k-$period optimal cost functions $\mathcal{T}^k f_0(\pi, i, x)$.

Let $\mathcal{Z}_k$ denote the sets

$$\mathcal{Z}_k(\pi, i, x, \lambda) = \{x \leq y \leq x + A \,| H_k(\pi, i, x, y) \leq \lambda\} \tag{4.209}$$

for all π, i, x and $\lambda \in \mathbb{R}$. According to Proposition 1.7 in Bertsekas (2000 b, p. 148), if we show that the sets in (4.209) are compact for all π, i, x and λ, then $f_*(\pi, i, x) = f_\infty(\pi, i, x)$ for all π, i and x. By the following lemma, we accomplish this task.

Lemma 25 *The sets in (4.209) are compact subsets of the Euclidean space for all π, i, x and λ.*

Proof. Notice that the sets $\{\mathcal{Z}_k(\pi, i, x, \lambda)\}$ are bounded since y must be in $[x, x + A]$, which is a bounded interval. Moreover, the sets $\{\mathcal{Z}_k(\pi, i, x, \lambda)\}$ are closed since H_n is strictly convex, so continuous, in y and it is real valued. Thus, the sets in (4.209) are compact subsets of Euclidean space for all π, i, x and λ. ■

The following proposition shows that f_∞ is a fixed point of the mapping $\mathcal{T}$; moreover, finite-horizon optimal cost function converges to the infinite-horizon optimal cost function.

Proposition 26 *The limit f_∞ is a fixed point of the mapping $\mathcal{T}$ so that*

$$f_\infty(\pi, i, x) = \mathcal{T} f_\infty(\pi, i, x) \tag{4.210}$$

for all π, i and x. Moreover,

$$f_\infty(\pi, i, x) = f_*(\pi, i, x) \tag{4.211}$$

for all π, i and x. Furthermore, there exists a stationary optimal policy.

Proof. By Lemma 25, the sets in (4.209) are compact subsets of the Euclidean space for all π, i, x and λ. Then, using Proposition 1.7 in Bertsekas (2000 b, p. 148), f_∞ is a fixed point of $\mathcal{T}$ so that (4.210) is valid and there exists a stationary optimal policy. In addition, notice that

$$f_0 \leq \mathcal{T} f_0 \leq ... \leq \mathcal{T}^k f_0 \leq ... \leq f_*$$

because expected cost per period is nonnegative. From this, we get $\lim_{k\uparrow\infty} \mathcal{T}^k f_0(\pi, i, x) \leq f_*(\pi, i, x)$ so that $f_\infty(\pi, i, x) \leq f_*(\pi, i, x)$. By (4.210), we know that f_∞ is a fixed point of $\mathcal{T}$. Then, by Proposition 1.2 in Bertsekas (2000 b, p. 140), we get that $f_*(\pi, i, x) \leq f_\infty(\pi, i, x)$. It follows that $f_\infty(\pi, i, x) = f_*(\pi, i, x)$. This completes our proof. ■

Notice that Proposition 26 implies also that f_∞, the optimal cost function that the finite-horizon cost function converges, satisfies the Bellman's equation since $f_\infty(i, x) = \mathcal{T} f_\infty(i, x)$ by (4.210). Hence,

$$f_\infty(\pi, i, x) = \min_{x \leq y \leq x+A} H(\pi, i, x, y) \tag{4.212}$$

for all π, i and x, where H is given in (4.198) with J as given in (4.199) and

$$G(\pi, i, a, y) = c_i y + L(i, a, y) + \alpha \sum_{j \in \mathbb{E}} \Psi(a, j) E_D^a \left[f_\infty(T(\pi \,|j), j, y - D) \right]. \tag{4.213}$$

As stated in Proposition 1.2 in Bertsekas (2000 b, p.140), f_∞ is not necessarily the unique optimal solution to Bellman's equation because single-period costs are not bounded under positivity assumption; however, f_∞ is the smallest fixed point of $\mathcal{T}$ since $f_\infty = f_*$.

Notice that, for a finite n, k goes to infinity as N goes to infinity. Then, above analysis shows that $\lim_{k\uparrow\infty} v_{0,k}(\pi, i, x) = v(\pi, i, x)$. Moreover, $v(\pi, i, x)$ satisfies (4.197) and there exists a stationary optimal policy $y(\pi, i, x)$ for the problem in (4.197). However, notice that single-period cost and $H_n(\pi, i, x, y)$ are not bounded; therefore, v is not necessarily unique. Then, we take v as the minimal fixed point of (4.197). In other words, if $f = \mathcal{T} f$ is another solution, then $v \leq f$. Moreover, we also know that the optimal solution v is that fixed point of $\mathcal{T}$ which can be obtained as $v = \lim_{k\uparrow\infty} \mathcal{T}^k f_0$ with $f_0 = 0$.

Assuming π, i and x are current information vector, current observed state and current inventory level respectively, we let $y(\pi, i, x)$ denote the optimal order-up-to level to the minimization problem in (4.197) and let $v'(\pi, i, x)$ denote the derivative of $v(\pi, i, x)$ with respect to x. Finally, we again let h and p denote holding cost and shortage cost vectors, respectively. Moreover, we recall that $R^{\alpha}((\pi,i),j)=\sum_{k=0}^{\infty}\alpha^{k}P_{0}^{k}((\pi,i),j)$ for all π,i and j and that, for a nonnegative function g, $R^{\alpha}g(\pi,i)$ is defined as in (4.52).

Theorem 27 *The optimal ordering policy for infinite-period model is a state-dependent modified inflated base-stock policy*

$$y(\pi,i,x)=\begin{cases} x+A & x<s^{\pi,i}\\ y^{\pi,i}(x) & s^{\pi,i}\leq x<S^{\pi,i}\\ x & x\geq S^{\pi,i}\end{cases} \tag{4.214}$$

where $y^{\pi,i}(x), S^{\pi,i}$ *and* $s^{\pi,i}$ *are unique* y *values which satisfy*

$$\begin{aligned} \sum_{a\in\mathbb{F}}\pi^{a}\int_{0}^{1}uG'(\pi,i,a,x+u(y-x))\,dF_{a}(u) &= 0\\ \sum_{a\in\mathbb{F}}\pi^{a}\mu_{a}G'(\pi,i,a,y) &= 0\\ \sum_{a\in\mathbb{F}}\pi^{a}\int_{0}^{1}uG'(\pi,i,a,y+uA)\,dF_{a}(u) &= 0\end{aligned}$$

for all π *and* i *respectively. Moreover,* $y^{\pi,i}(x)$ *is nonincreasing in* x *with* $\lim_{x\downarrow-\infty}y^{\pi,i}(x)=\infty$ *for all* π *and* i*. In addition,* $x\leq y^{\pi,i}(x)\leq x+A$ *for all* $x\in\left[s^{\pi,i},S^{\pi,i}\right)$ *with* $y^{\pi,i}\left(s^{\pi,i}\right)=s^{\pi,i}+A$ *and* $y^{\pi,i}\left(S^{\pi,i}\right)=S^{\pi,i}$*. The cost function* $H(\pi,i,x,y)$ *is strictly convex in* (x,y) *for all* π *and* i*, and the optimal cost is*

$$v(\pi,i,x)=\begin{cases}\sum_{a\in\mathbb{F}}\pi^{a}\int_{0}^{1}G(\pi,i,a,x+uA)\,dF_{a}(u)-c_{i}x & x<s^{\pi,i}\\ H\left(\pi,i,x,y^{\pi,i}(x)\right) & s^{\pi,i}\leq x<S^{\pi,i}\\ \sum_{a\in\mathbb{F}}\pi^{a}G(\pi,i,a,x)-c_{i}x & x\geq S^{\pi,i}.\end{cases} \tag{4.215}$$

for all π,i *and* x*. Furthermore,* $v(\pi,i,x)$ *is convex in* x*,* $\lim_{x\uparrow\infty}v'(\pi,i,x)=R^{\alpha}h(\pi,i)$ *and* $\lim_{x\downarrow-\infty}v'(\pi,i,x)=-R^{\alpha}p(\pi,i)$ *for all* π *and* i*.*

Proof. We know that $v(\pi,i,x)=\lim_{k\uparrow\infty}v_{0,k}(\pi,i,x)$. In addition, by Theorem 24, $v_{0,k}(\pi,i,x)$ is convex in x for all π and i. Then, $v(\pi,i,x)$ is also convex because the limit of a convex function is also convex. Also we know by Theorem 24 that $\lim_{x\uparrow\infty}v_{0,k}'(\pi,i,x)=R_{0,k}^{\alpha}h(\pi,i)$ and $\lim_{x\downarrow-\infty}v_{n}'(\pi,i,x)=-R_{0,k}^{\alpha}p(\pi,i)$ for all π and i. Moreover, since $v_{0,k}$ is differentiable, it follows from Lemma 8-5 in Heyman and Sobel (1984) that $v'(\pi,i,x)=$

$\lim_{k\uparrow\infty} v'_{0,k}(\pi,i,x)$ for all π, i and x. As a result, $\lim_{x\uparrow\infty} v'(\pi,i,x) = R^\alpha h(\pi,i)$ and $\lim_{x\downarrow-\infty} v'(\pi,i,x) = -R^\alpha p(\pi,i)$ for all π and i. Then, using (4.15) and MCT, we get

$$\lim_{y\uparrow\infty} G'(\pi,i,a,y) = c_i + h_i + \alpha \sum_{j\in\mathbb{E}} \Psi(a,j)\, R^\alpha h(T(\pi|\, j), j) > 0 \tag{4.216}$$

and

$$\lim_{y\downarrow-\infty} G'(\pi,i,a,y) = c_i - p_i - \alpha \sum_{j\in\mathbb{E}} \Psi(a,j)\, R^\alpha p(T(\pi|\, j), j) < 0 \tag{4.217}$$

for all i and a.

Notice that G in (4.200) is strictly convex since L is a strictly convex function and v is convex. Moreover, Lemma 20 implies that $G(\pi,i,a,x+u(y-x))$ is strictly convex in (x,y) for all π, i, a and $u \in [0,1]$. In addition, J is also strictly convex in (x,y) because expectation preserves strict convexity. As a result, H is strictly convex in (x,y) for all π and i since sum of strictly convex functions is also strictly convex. Furthermore, we let H' to denote the partial derivative of H with respect to y. Then, using (4.198),

$$H'(\pi,i,x,y) = \sum_{a\in\mathbb{F}} \pi^a \int_0^1 uG'(\pi,i,a,x+u(y-x))\, dF_a(u) \tag{4.218}$$

for all π, i, x and y.

Let us suppose that $y^{\pi,i}(x)$ satisfies

$$\sum_{a\in\mathbb{F}} \pi^a \int_0^1 uG'\left(\pi,i,a,x+u(y^{\pi,i}(x)-x)\right) dF_a(u) = 0 \tag{4.219}$$

for all π, i and x. Note that $\lim_{|y|\uparrow\infty} H_n(\pi,i,x,y) = \infty$ for all π, i and x. Moreover, using (4.201), (4.218), MCT, and the induction hypotheses that $\lim_{x\uparrow\infty} v'(\pi,i,x) = R^\alpha h(\pi,i)$ and $\lim_{x\downarrow-\infty} v'(\pi,i,x) = -R^\alpha p(\pi,i)$,

$$\lim_{y\downarrow-\infty} H'(\pi,i,x,y) = (c_i - p_i)\,\bar{\mu}_\pi - \alpha \sum_{j\in\mathbb{E}}\sum_{a\in\mathbb{F}} \pi^a \mu_a \Psi(a,j)\, R^\alpha p(T(\pi\,|j), j) < 0$$

and

$$\lim_{y\uparrow\infty} H'(\pi,i,x,y) = (c_i + h_i)\,\bar{\mu}_\pi + \alpha \sum_{j\in\mathbb{E}}\sum_{a\in\mathbb{F}} \pi^a \mu_a \Psi(a,j)\, R^\alpha h(T(\pi\,|j), j) > 0$$

for all π, i and x, where R^α is given in (4.53). These together with the fact that H is strictly convex imply that there exists a finite and unique $y^{\pi,i}(x)$ satisfying (4.219). In addition, we now show that $\lim_{x\downarrow-\infty} y^{\pi,i}(x) = \infty$. Suppose that $\lim_{x\downarrow-\infty} y^{\pi,i}(x) < \infty$. Then, using (4.218) and MCT,

$$\begin{aligned}\lim_{x\downarrow-\infty} H'\left(\pi,i,x,y^{\pi,i}(x)\right) &= \sum_{a\in\mathbb{F}} \pi^a \int_0^1 u \lim_{x\downarrow-\infty} G'\left(\pi,i,a,x+u(y^{\pi,i}(x)-x)\right) dF_a(u) \\ &= (c_i - p_i)\,\bar{\mu}_\pi - \alpha \sum_{j\in\mathbb{E}}\sum_{a\in\mathbb{F}} \pi^a \mu_a \Psi(a,j)\, R^\alpha p(T(\pi|\, j), j) < 0\end{aligned}$$

for all π and i. The first equality is gathered by MCT. If we assume that $\lim_{x\downarrow-\infty} y^{\pi,i}(x) < \infty$, then $\lim_{x\downarrow-\infty}\left[x+u(y^{\pi,i}(x)-x)\right] = -\infty$ for all π, i and $u\in[0,1]$. Then, using (4.217), we get the second equality. However, we assume that $p_i > c_i$ for all i and $\bar{\mu}_\pi > 0$ for all π. This implies that $\lim_{x\downarrow-\infty} H'\left(\pi,i,x,y^{\pi,i}(x)\right) < 0$. But, this is a clear contradiction and $\lim_{x\downarrow-\infty} y^{\pi,i}(x) = \infty$ for all π and i.

Next, we show that $y^{\pi,i}(x)$ is nonincreasing in x. Suppose that there exist $x_1 < x_2$ such that $H'\left(\pi,i,x_1,y^{\pi,i}(x_1)\right) = H'\left(\pi,i,x_2,y^{\pi,i}(x_2)\right) = 0$ with $y^{\pi,i}(x_1) < y^{\pi,i}(x_2)$. This clearly implies that $x_1+u(y^{\pi,i}(x_1)-x_1) < x_2+u(y^{\pi,i}(x_2)-x_2)$ for all $u\in[0,1]$. This further implies that $G'\left(\pi,i,a,x_2+u(y^{\pi,i}(x_2)-x_2)\right) > G'\left(\pi,i,a,x_1+u(y^{\pi,i}(x_1)-x_1)\right)$ for all π, i and a since L' is increasing and v' is nondecreasing. Moreover, we assume that there is at least one a with $\pi^a > 0$ such that there is positive probability of receiving something. Then, using (4.218), it follows that

$$H'\left(\pi,i,x_2,y_n^{\pi,i}(x_2)\right) > H'\left(\pi,i,x_1,y_n^{\pi,i}(x_1)\right) = 0$$

for all π and i. However, notice that this is a clear contradiction to the definition of $y^{\pi,i}(x_2)$. Therefore, $y^{\pi,i}(x)$ is nonincreasing in x.Moreover, if we differentiate (4.219) with respect to x, we get

$$\frac{\partial y^{\pi,i}(x)}{\partial x} = \frac{\sum_{a\in\mathbb{F}} \pi^a \int_0^1 \left(u^2-u\right) G''\left(\pi,i,a,x+u(y^{\pi,i}(x)-x)\right) dF_a(u)}{\sum_{a\in\mathbb{F}} \pi^a \int_0^1 u^2 G''\left(\pi,i,a,x+u(y^{\pi,i}(x)-x)\right) dF_a(u)} \le 0 \qquad (4.220)$$

for all π, i and x. Note that denominator of (4.220) is always positive since G is strictly convex and there exists, by our assumption, at least one a with $\pi^a > 0$ such that there is positive probability of receiving something. However, the numerator in the same equation is nonpositive since $u^2 - u \le 0$ and and $G'' > 0$. This is another way of showing that $\partial y^{\pi,i}(x)/\partial x \le 0$ so that $y^{\pi,i}(x)$ is nonincreasing in x.Therefore, $y^{\pi,i}(x)$ has a similar structure as $y_0^{\pi,i}(x)$ in Figure 4.2.

However, $y^{\pi,i}(x)$ may not be a feasible solution for our problem in (4.197). In other words, $y^{\pi,i}(x)$ is not necessarily in $[x, x+A]$; moreover, it may also be less than x. If $S^{\pi,i}$ is the smallest inventory level at which it is optimal not to order, then, for $x = S^{\pi,i}$, (4.219) turns out to be

$$\sum_{a\in\mathbb{F}} \pi^a \mu_a G'\left(\pi, i, a, S^{\pi,i}\right) = 0 \tag{4.221}$$

for all π and i. Note that, by our assumption, $\mu_a > 0$ for at least one a with $\pi^a > 0$; moreover, G' is increasing in y, and $\lim_{y\uparrow\infty} G'(\pi, i, a, y) > 0$ and $\lim_{y\downarrow-\infty} G'(\pi, i, a, y) < 0$ by (4.216) and (4.217). Hence, $S^{\pi,i}$ satisfying (4.221) is finite and unique. Moreover, we assume that $x = s^{\pi,i}$ is the inventory level at which ordering as much as finite capacity A is optimal so that $y^{\pi,i}\left(s^{\pi,i}\right) = s^{\pi,i} + A$. Then, using (4.219), $s^{\pi,i}$ satisfies

$$\sum_{a\in\mathbb{F}} \pi^a \int_0^1 uG'\left(\pi, i, a, s^{\pi,i} + uA\right) dF_a(u) = 0$$

for all π and i. Via a similar discussion as we did for $S^{\pi,i}$, we can show that $s^{\pi,i}$ is also finite and unique. Note that $y^{\pi,i}(x)$ is nonincreasing in x. This clearly implies that $s^{\pi,i} \leq S^{\pi,i}$ for all π and i. Now, if we consider three cases as in single and multiple-period analyses, we see that optimal order-up-to level is $x + A$, $y^{\pi,i}(x)$ and x when $x < s^{\pi,i}$, $s^{\pi,i} \leq x < S^{\pi,i}$ and $x \geq S^{\pi,i}$ respectively. Therefore, $y(\pi, i, x)$ in (4.214) is the optimal ordering policy. Clearly, for $s^{\pi,i} \leq x < S^{\pi,i}$, $y(\pi, i, x) = y^{\pi,i}(x) \geq S^{\pi,i}$ since $y^{\pi,i}(x)$ is nonincreasing in x. If this optimal policy is applied, the corresponding optimal cost is given by (4.215). This completes our proof. ■

By Theorem 27, we show that the optimal policy for finite capacity and random yield inventory model in a partially observed random environment is still modified inflated base-stock policy, which depends on current information vector and observed state. At the beginning of each period, an order is given if and only if the inventory level is less than a particular value which depends on information vector and observed state. However, order size is not unlimited because of finite capacity. There is another value, which depends also on current information vector and observed state, below which it is always optimal to order as much as finite capacity. In addition, the order-up-to level depends on inventory level as well as current information vector and observed state when inventory level is in between these two particular values. But, all of these critical values, i.e., $y^{\pi,i}(x)$, $s^{\pi,i}$ and $S^{\pi,i}$, are independent of the number of period in which we are planning for; therefore, they are the same in all periods with the same information vector and observed state. Hence, the optimal solution is stationary whereas it is not necessarily unique.

In addition, by Theorem 27, we also show that $\lim_{x\uparrow\infty} v'(\pi,i,x) = R^{\alpha}h(\pi,i)$ and $\lim_{x\downarrow-\infty} v'(\pi,i,x) = -R^{\alpha}p(\pi,i)$ for all π and i. Because the state space $\mathbb{F}$ is finite and $\alpha \in (0,1)$, by (4.53), $\lim_{x\uparrow\infty} v'(\pi,i,x) = h_i + \pi(I-\alpha Q)^{-1}Eh - \pi Eh$ and $\lim_{x\downarrow-\infty} v'(\pi,i,x) = -p_i - \pi(I-\alpha Q)^{-1}Ep + \pi Ep$ for all π and i, since h_i and p_i are nonnegative and bounded for all i.

As in single and multiple period models, the order-up-to level is not independent of inventory level for $s^{\pi,i} \leq x < S^{\pi,i}$. But, for finite capacity only case, (i.e., $U = 1$ with probability 1), it is clear from (4.220) that $\partial y^{\pi,i}(x)/\partial x = 0$. As a result, for $s^{\pi,i} \leq x < S^{\pi,i}$, optimal order-up-to level is independent of inventory level; moreover, it is equal to $S^{\pi,i}$. Hence, state-dependent modified base-stock policy is optimal. On the other hand, if we assume that finite capacity is infinite, then via a similar discussion as in multi-period, we can show that state-dependent inflated base-stock policy is still optimal in infinite-period.

4.8 Summary of Results

In this part of the thesis, our main aim is to characterize the optimal policy structures for inventory problems with different random supply scenarios in a partially observed random environment. As in the first part of the thesis, the environment follows a Markov chain which affects demand, supply and costs. However, we assume in this part that the state of this Markov chain is not fully observed. In this case, there is another process, not necessarily a Markov chain, which gives partial information about the real environment. The results regarding different random supply scenarios show that policy structures that are optimal in a fully observed random environment are not necessarily optimal when the random environment is partially observed. Moreover, policies optimal when the capacity process is observable are no more optimal when the capacity is modulated by the real environment.

In Section 4.2, we assume that the supply is random because the supply part of the system is composed of a producer having random capacity and a transporter being randomly available. Here, we also assume that both demand and supply depend on the unobserved environment. In addition, we use current inventory level and all observations until current time as the state of our system. Through a counter example, we show in the first model that state-dependent base-stock policy is not necessarily optimal in single-period when the environment is partially observed. We can generalize the same result for multi-period and infinite-period problems as well since single-period problem is a special case of both. Next in Section 4.3, we analyze a similar model as in Section 4.2; however, we assume that the capacity of the producer is observable. Here, we study single and multiple period problems and show that a state-dependent base-stock policy is optimal in both cases. Future research on inventory models with random capacity and random availability in a partially observed random environment may aim to characterize the optimal policy structure. However, it is obvious from the structure of expected discounted cost function and figures in Section 4.2 that the optimal policy structure is not fairly simple.

In Section 4.5, we change our formulation and use conditional distribution of the true environmental state as a sufficient statistic. With this new formulation, the state of our system becomes current inventory level and the current distribution of real environmental state. Moreover, this formulation of the problem in Section 4.3 enables us to analyze the infinite-period problem as well as single and multiple period problems. As a result of our analysis, we show that state-dependent base-stock policy is optimal in single, multiple and infinite planning periods. Moreover, same results are still applicable if we assume that the transporter is always available and the producer has infinite capacity. These results are similar to those obtained by other researchers in the literature. For example, Treharne and Sox (2002) use exactly the same formulation as our model in Section 4.5 by assuming that the producer has infinite capacity and the transporter is always available. They show that a state-dependent base-stock policy is optimal.

Certainly, we can extend the model in Section 4.5 by considering the fixed ordering costs. Normally, we can expect that a state-dependent (s, S) policy is optimal in this case. However, we know by our analysis in Section 3.2 that a state-dependent (s, S) policy is not optimal for random capacity models even in a fully observed random environment. This model is clearly a special case of inventory models with fully observed random capacity and partially observed availability. Therefore, state-dependent (s, S) policy is not optimal for inventory models in Section 3.2 with fixed ordering costs. In this regard, future research may aim to characterize the optimal policy structure for these inventory models. However, by Shaoxiang and Lambrecht (1996) and Gallego and Wolf (2000), we know that optimal policy structure is not simple even when there is no random environment and full observation. Hence, optimal policy structure for inventory models with fixed ordering cost, fully observed random capacity and partially observed availability won't be simple.

In Section 4.6, we assume that the supplier is randomly available and has infinite capacity. In this section, we further assume that there is fixed cost of ordering. Moreover, we use sufficient statistic formulation as in Section 4.5 so that state of our system is current inventory level and current distribution of real environmental state. Finally, as in Özekici and Parlar (1999), we show that state-dependent (s, S) policy is optimal in single, multiple and infinite planning periods if the supplier availability process is observable. During our analysis, we also observed that state-dependent (s, S) policy is optimal when the availability process is unobservable; however, this requires a more restrictive assumption on fixed ordering costs and availability probabilities. Our model in Section 4.6 can be extended by assuming random proportional yield. In other words, we can assume that the supplier can deliver any amount between 0 and the order quantity. We know by Henig and Gerchak (1990) that inflated base-stock or nonorder-up-to type of policies are optimal in stationary environment when we consider random proportional yield only. Therefore, the well-known (s, S) policy structure may not be optimal when we incorporate random proportional yield into the model in Section 4.6. In this regard, future research may aim to characterize the optimal policy structure for the inventory models with fixed ordering cost and random proportional yield in a partially observed random environment.

Finally in Section 4.7, we again assume that the supplier is composed of a producer having finite capacity and a transporter with random transportation yield. Assuming that the environment is partially observed and using the sufficient statistic formulation, we show that state-dependent modified inflated base-stock policy is optimal. Similar results are obtained by Gallego and Hu (2004) that analyze exactly the same inventory model as ours by assuming that the environment is fully observed. Then, it follows that partial observation has no effect on the optimal policy structure, but on optimal policy parameters. For example, parameters of optimal policy in our model depend also on the distribution of real environment as well as the state of observed environment. By considering random capacity, we can extend our model. However, we know from Section 4.2 that state-dependent base-stock policy is not optimal for inventory models with random capacity and random availability in a partially observed random environment. Moreover, inventory models with random capacity and random availability is a special case of inventory models with random capacity and random yield. Therefore, we can expect that state-dependent base-stock policy is not optimal for inventory models with random capacity and random proportional yield in a partially observed random environment. In this regard, future research on these inventory models must aim to characterize the optimal policy structure.

Chapter 5

CONCLUSIONS AND FUTURE RESEARCH

In this thesis, our main aim was to discuss inventory models with different supply scenarios in a random environment. These topics are studied in the literature; however, the setting that we consider in this thesis is the most general until now. Firstly, we assume that the supply can be random due not only to random proportional yield but also to random capacity. Considering this fact, we aim to find out the circumstances under which well-characterized inventory policies (i.e., base-stock, (s, S) and inflated base-stock) are optimal. Secondly, we consider the fact that it is not always possible to observe the random environment fully. Then we aim to find the structure of optimal inventory policies for different random supply scenarios when the environment is only partially observed.

The underlying assumption of random environment is due to the fact that inventory systems cannot be isolated from the fluctuating environment. There can be significant interruptions due to exogenous factors that affect not only the demand but also the supply and the costs. But, in the great majority of existing inventory literature, the source of uncertainty for the random component is specified by a single distribution function. Moreover, in the literature, parameters of this distribution are assumed to be constant. Hence, the environment is assumed to be stationary. Clearly, these inventory models cannot utilize the information gathered from the environment. Therefore, they are not valid in most real life situations. In order to create a more realistic model, we introduced a random environment that follows a time-homogenous Markov chain and affects all costs, distributions of demand and supply.

Moreover, we also consider the fact that observations regarding the real environment are not always perfect. Parameters of demand and supply distributions, and distributions by themselves are defined based on the observed data. These observations do not match reality in most of the time. Under such circumstances, assuming full observation of the random environment is misleading. For this purpose, we introduced a random process other than the Markov chain which represents the random environment in this thesis. Moreover, we assumed that this new process is observable and it gives partial information about the real environment. Under this major setting, we divide this thesis into two main parts based on the observability of the random environment.

In the first part of the thesis, we assume that the environment is fully observed and the supply is random due to random availability and random capacity. Through the first model, we show that environment-dependent base-stock policy is still optimal when there is a supplier having random capacity and a randomly available transporter. Then, by the second model, we show that environment-dependent (s, S) policy is no more optimal when there is random capacity only and fixed ordering cost; moreover, the optimal policy does not have a simple structure.

In the second part of the thesis, we assume that the environment is only partially observed and supply is random. Main results of this part and the direction of future research are summarized in Section 4.8. However, if we must restate important points of our analysis, we should point out that inventory policies that are optimal in fully observed environment are not necessarily optimal when the environment is partially observed. Moreover, the optimal policy structures for random capacity inventory models in fully observed environment are no more optimal when the environment is partially observed. In other words, optimal policy structures are not well-defined and hard to characterize if the capacity process is unobservable.

Although inventory system considered in this thesis is the most general in the literature, further extensions of our models are still possible. For example, we assume in all models that the unsatisfied demand is backordered and satisfied in the next period. Therefore, a possible extension would be to consider the case where the sales are lost. The future research may aim to question whether the policies being optimal in backordering case are still optimal in lost sales case or not. If not, researchers may aim to characterize the optimal policy structure for lost sales case.

In addition, we consider the retailer side only in our all models. In other words, we assume that the retailer is the strongest party in the chain. However, we know that this is not true in some real life situations and the supplier has some incentive on the ordering quantity. As a result, our models can be extended by considering the supplier side as well. In this regard, the future research may aim to characterize types of contracts which coordinate the chain. An important question in this case would be whether contracts coordinating the chain in full observation case still coordinate the chain in partial observation case or not.

Furthermore, we analyze the single vendor case in our all models. However, by working with multiple suppliers, the retailer may want to eliminate additional stockout risk derived from the producer with random capacity and/or the transporter with random availability. In this case, the ordered quantity would be diversified among many suppliers. Therefore, we can extend our models by considering multiple suppliers case. An important research question in this case would be whether policy structures being optimal in single supplier case are still optimal when there are multiple suppliers or not. In this regard, future research may analyze the effect of multiple suppliers on the total order quantity. Through multiple suppliers analysis, the effect of individual supplier characteristics on orders from each supplier may also be determined.

Appendix A

PRELIMINARY CONCEPTS

In this appendix, we review some important concepts for our analyses in this thesis. Among many others, the most important concepts are convexity, quasi-convexity, $K-$convexity and potential theory. We make use of these concepts frequently. This appendix is organized as follows. In the next section, we give a brief introduction of quasi-convex functions. In Section A.2, we give the definition of $K-$convexity and present important properties of $K-$convex functions. Finally, we present some results on the potential theory of Markov chains in Section A.3.

A.1 Quasi-convex functions

In general, the expected total discounted cost function is convex for inventory problems where the base-stock policy is optimal. This is due to the structure of the convex functions. Namely, a convex function has a minimum (not necessarily unique) to the left of which the function is nonincreasing and to the right of which the function is nondecreasing. However, there are other functions which posses this property although they are not convex. Quasi-convex functions are examples of these types of functions. They are more general than convex functions. In other words, every convex function is also quasi-convex; however, the converse is not true.

In this study, expected total discounted cost functions for our different models are mostly not convex. However, they will in general be quasi-convex. Therefore, we note the following definition of quasi-convexity since we make use of quasi-convex functions in our proofs.

Definition 1 *We say that a real-valued function f is quasi-convex if, for two points x_1 and x_2,*

$$f(\lambda x_1 + (1-\lambda)x_2) \leq \max(f(x_1), f(x_2))$$

for $0 \leq \lambda \leq 1$.

From this definition taken from Bazaraa et al. (1993), a function f is quasi-convex if, whenever $f(x_2) \geq f(x_1)$, $f(x_2)$ is greater than or equal to f at all convex combinations of x_1 and x_2. Therefore, if f increases from its value at a point along any direction, it must remain nondecreasing along that direction (Bazaraa et al., 1993). In addition, by the following lemma, we show that a function decreasing on the left of a particular point and nondecreasing on the right of the same point is quasi-convex.

Lemma 28 *Suppose that f is a real-valued continuous function, and there exists an $\bar{x}$ such that $f(x)$ is decreasing in x for all $x \leq \bar{x}$ and $f(x)$ is nondecreasing in x for all $x > \bar{x}$. Then, $f(x)$ is quasi-convex.*

Proof. Suppose that f is a function satisfying the properties in the lemma; but, it is not quasi-convex. Then, since f is not quasi-convex, there exists either $x < \bar{x}$ or $x \geq \bar{x}$ such that

$$f(\lambda x + (1-\lambda)\bar{x}) > \max(f(x), f(\bar{x})) \tag{A.1}$$

for some $0 \leq \lambda \leq 1$. Consider the two cases: $x < \bar{x}$ and $x \geq \bar{x}$.

(i) $x < \bar{x}$: It follows from (A.1) that $f(\lambda x + (1-\lambda)\bar{x}) > f(x)$ where $\lambda x + (1-\lambda)\bar{x} \geq x$. However, this is a clear contradiction to the fact that $f(x)$ is decreasing in x for all $x < \bar{x}$.

(ii) $x \geq \bar{x}$: Again it follows from (A.1) that $f(\lambda x + (1-\lambda)\bar{x}) > f(x)$ where $\lambda x + (1-\lambda)\bar{x} \leq x$. But this is also a clear contradiction to the fact that $f(x)$ is nondecreasing in x for all $x \geq \bar{x}$.

Therefore, f is quasi-convex and this completes our proof. ■

A.2 $K-$convex functions

The concept of $K-$convexity is very important in the analysis of inventory problems with fixed ordering costs. In general, the cost function for an inventory problem without fixed ordering cost has a nice structure such that the cost function has a global minimum (not necessarily unique). However, the structure of the cost function becomes very complex when there is fixed cost of ordering. Despite the complex structure of the cost function, optimal policy for inventory problems with fixed ordering cost is well-characterized in general. Mostly (s, S) policy is optimal for these types of problems. Moreover, if the cost function for inventory problems with fixed ordering cost belongs to a certain class, then (s, S) policy is certainly optimal. This class of functions are called $K-$convex functions and introduced by Scarf (1960) for the first time. Scarf (1960) shows that although there are some fluctuations in $K-$convex functions, these fluctuations are not very much to cause a deviation from (s, S) policy.

First of all, let us note the definition of $K-$convexity taken from Bertsekas (2000 a, p. 158).

Definition 2 *We say that a real-valued function f is $K-$convex, where $K \geq 0$, if, for all $z \geq 0, b > 0$, and y,*

$$K + f(y + z) \geq f(y) + \frac{z}{b}\left[f(y) - f(y - b)\right].$$

Some properties of $K-$convex functions taken from Bertsekas (2000 a, p. 159) are provided in the following lemma and interested readers are referred to this reference for details. Notice that part (d) of the following lemma implies the optimality of (s, S) policy for $K-$convex cost functions.

Lemma 29

a. *A real valued convex function f is also $0-$convex and, hence, also $K-$convex for all $K \geq 0$,*

b. *If f_1 and f_2 are $K-$convex and $L-$convex $(K \geq 0, L \geq 0)$, respectively, then $\alpha f_1 + \beta f_2$ is $(\alpha K + \beta L)$-convex for all $\alpha > 0$ and $\beta > 0$,*

c. *If f is $K-$convex and W is a random variable, then $E_W\left[f(y - W)\right]$ is also $K-$convex, provided that $E_W\left[f(y - W)\right] < \infty$ for all y,*

d. *If f is a continuous $K-$convex function and $f(y) \to \infty$ as $|y| \to \infty$, then there exist scalars s and S with $s \leq S$ such that* *i.* $f(S) \le f(y)$ *for all* y,

ii. $K + f(S) = f(s) < f(y)$ *for all* $y < s$,

iii. $f(y)$ *is a decreasing function on* $(-\infty, s)$,

iv. $f(y) \le f(z) + K$ *for all* y, z *with* $s \le y \le z$.

A.3 Potential functions

Let $Z = \{Z_n; n \ge 0\}$ be a Markov chain with state space $\mathbb{E}$ and transition matrix P, and let g be a function defined on $\mathbb{E}$ and taking real values. Suppose that at each time n, we are given a reward whose amount depends on the state Z is in at that time: if Z is in state j, then the reward is $g(j)$. Furthermore, assume that all future rewards are being discounted in such a way that one unit of reward at time n has the present worth of α^n. Here, α is the periodic discount factor taking values in $[0, 1]$. Markov chain Z visits states $Z_0, Z_1, Z_2, \ldots$, and the rewards received at times $0, 1, 2, \ldots$, have the respective present worths of $g(Z_0), \alpha g(Z_1), \alpha^2 g(Z_2), \ldots$. Our aim is to find expected total discounted reward during first m transitions given the initial state. Then, the expected total discounted return during the first m transitions given initial state i is

$$\begin{aligned} E\left[\left.\sum_{n=0}^{m-1} \alpha^n g(Z_n) \right| Z_0 = i\right] &= \sum_{n=0}^{m-1} \alpha^n E\left[g(Z_n) \,|\, Z_0 = i\right] \\ &= \sum_{n=0}^{m-1} \alpha^n \sum_{j \in \mathbb{E}} P^n(i,j)\, g(j) \\ &= \sum_{j \in \mathbb{E}} \left[\sum_{n=0}^{m-1} \alpha^n P^n(i,j)\right] g(j) \\ &= \sum_{j \in \mathbb{E}} R_m^\alpha(i,j) g(j) = R_m^\alpha g(i) \end{aligned} \tag{A.2}$$

where $R_m^\alpha(i,j) = \sum_{n=0}^{m-1} \alpha^n P^n(i,j)$ for all i and j. Notice that R_m^α is a matrix, g is the reward vector and $R_m^\alpha g$ is the multiplication of matrix R_m^α and vector g. Therefore, $R_m^\alpha g$ is a vector and $R_m^\alpha g(i)$ is the ith entry of this vector. However, given the initial state, we may also aim to find expected total discounted reward over infinite transitions. Clearly, it is the limit of $R_m^\alpha g$ where m goes to ∞. Therefore, given that the initial state is i, expected total discounted return over infinite transitions is

$$R^{\alpha}g(i) = E\left[\sum_{n=0}^{\infty}\alpha^{n}g\left(Z_{n}\right)\middle| Z_{0}=i\right] = \sum_{j\in\mathbb{E}}R^{\alpha}\left(i,j\right)g\left(j\right) \tag{A.3}$$

if the initial state is i, where R_{∞}^{α} is denoted by R^{α}. Çınlar (1975) calls R^{α} and $R^{\alpha}g$ as $\alpha-$potential matrix of Z and $\alpha-$potential of g, respectively. A detailed analysis of potential functions can be found in Çınlar (1975, Chap. 7). As shown in Çınlar (1975, p. 200),

$$R_{m}^{\alpha}g(i) = R^{\alpha}g(i) - \alpha^{m}P^{m}R^{\alpha}g(i) = \left(I-\alpha^{m}P^{m}\right)R^{\alpha}g(i) \tag{A.4}$$

for all i. This relation is derived using the fact that for someone who looks into the future after time m, that future looks the same probabilistically as the future after time 0 would look to someone considering it at time 0 provided that they observe the same state. The relation between $R_{m}^{\alpha}g$ and $R_{m-1}^{\alpha}g$ can be obtained as follows:

$$\begin{aligned}
R_{m}^{\alpha}g(i) &= g(i) + E\left[\sum_{n=1}^{m-1}\alpha^{n}g\left(Z_{n}\right)\middle| Z_{0}=i\right] \\
&= g(i) + \alpha E\left[\sum_{k=0}^{m-2}\alpha^{k}g\left(Z_{k+1}\right)\middle| Z_{0}=i\right] \\
&= g(i) + \alpha\sum_{j\in\mathbb{E}}P(i,j)E\left[\sum_{k=0}^{m-2}\alpha^{k}g\left(Z_{k}\right)\middle| Z_{0}=j\right] \\
&= g(i) + \alpha\sum_{j\in\mathbb{E}}P(i,j)R_{m-1}^{\alpha}g(j).
\end{aligned} \tag{A.5}$$

First equality above follows from (A.2). If we make change of index, then the second equality is obtained. Via some probabilistic operations, we get the third equality. Finally, the last equality again follows from (A.2).

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