Stabilization of Linear and Nonlinear Schrödinger Equations

by

Türker Özsarı

A Thesis Submitted to the
Graduate School of Sciences & Engineering
in Partial Fulfillment of the Requirements for
the Degree of

Master of Science

in

Mathematics

Koç University

Fall, 2006

Koç University

Graduate School of Sciences & Engineering

This is to certify that I have examined this copy of a master's thesis by

Türker Özsarı

and have found that it is complete and satisfactory in all respects,
and that any and all revisions required by the final
examining committee have been made.

Committee Members:

Prof. Tekin Dereli

Prof. Varga K. Kalantarov (Thesis Advisor)

Prof. Ali Mostafazadeh

Date:

Dedicated to Prof. Ali Ülger

ABSTRACT

In this thesis, the main aim is to study stabilization problems in the contex of linear and nonlinear Schrödinger equations. These include both boundary and internal stabilization problems. We also get a new result on a stabilization problem for Schrödinger equation with inhomogeneous boundary condition. In this result, we prove the decay of energy of solutions of the weakly damped Schrödinger equation with inhomogeneous Dirichlet boundary condition. We prove that if we impose a decaying condition on the boundary condition in a reasonable sense then we get stabilization of the energy. In addition, we observe that decay rate of the boundary function controls the decay rate of energy of the solutions.

Keywords: Stabilization, exponential stabilization, boundary stabilization, internal stabilization, linear Schrödinger equation, nonlinear Schrödinger equation, inhomogeneous Dirichlet boundary condition, weakly damped Schrödinger equation, localized damping

ÖZET

Tezin esas amacı doğrusal ve doğrusal olmayan Schrödinger denklemlerinin çözümlerinin kararlılaştırılmasının incelenmesidir. Bu çalışma hem sınır hem de iç kararlılaştırma problemlerini içermektedir. Ayrıca bu çalışmada homojen olmayan sınır koşulu altındaki Schrödinger denkleminin kararlılaştırılması ile ilgili yeni bir sonuç elde edilmektedir. Bu sonuçta homojen olmayan sınır koşulu altındaki Schrödinger denklemi için başlangıç sınır değer probleminin çözümünün zamana göre davranışı incelenmektedir. İspat edilmiştir ki, eğer zaman sonsuza yaklaştıkça sınır değer fonksiyonu uygun manada sıfıra yaklaşıyorsa, incelenen problemin çözümünün enerjisi de zaman sonsuza yaklaştıkça sıfıra gitmek zorundadır. Buna ek olarak sınır değer fonksiyonunun sıfıra yaklaşım hızının çözümün enerjisinin sıfıra yaklaşım hızını kontrol ettiği de tespit edilmiştir.

ACKNOWLEDGMENTS

It is a pleasure to thank the people who made this thesis possible.

It is difficult to overstate my gratitude to my advisor, Prof. Varga K. Kalantarov. With his enthusiasm, his inspiration, and his great efforts to explain things clearly and simply, he helped to make mathematics more enjoyable for me. Throughout my years in Koç University, he provided encouragement, sound advice, excellent teaching, excellent company, and lots of excellent ideas. I would have been lost without him.

I would like to thank some people from whom I have learnt mathematics, my childhood teacher Hasan Akman, my high school teacher Aysun Şendağ, my undergraduate teachers Yaman Arkun, Attila Aşkar, Mine Çağlar, Selda Küçükçifçi, Burak Özbağcı, my graduate teachers Albert Erkip, Alkan Kabakçıoğlu, Müge Kanuni, John Pym. I especially thank to Prof. Ali Mostafazadeh for his excellent teaching of the fundamental notions of abstract mathematics that I will always use as my main tools while doing mathematics.

I wish to thank many best friends, Zeynep Akçay, Ahmet Batal, Ahmet Bölükbaşı, Ümit Demirbaş, Cem Doğan, Cengiz Yönezer, Oytun Kasaboğlu, Fatih Kuz, Mehmet Kuzu, Rıfat Memiş, Murat Öktem, Hakkı Yazıcı for helping me get through the most difficult times, and for all the emotional support, fellowship, amusement, and compassion they provided.

I wish to thank my entire family, particularly to my parents Çetin Özsarı and Necla Özsarı, my brothers Berker Özsarı and Kağan Özsarı and to my grandmother Sacide Kale for providing constant support for me to be a mathematician.

I am also grateful to Prof. Irena Lasiecka and Prof. Roberto Triggiani for their worthwhile mathematical discussions with me.

Lastly, I wish to thank Prof. Ali Ülger who provided lots of inspiration and a model

for me on which road I need to follow to be a mathematician in my further life. To him I dedicate this thesis.

TABLE OF CONTENTS

NOTATION

a.e.	almost everywhere
div	divergence
X^*	the topological dual of the vector space X
α	$= (\alpha_1, ..., \alpha_n)$ where $\alpha_i \in \mathbb{N}$ and we define $\vert\alpha\vert = \sum_{i=1}^n \alpha_i$
D^α	$= \frac{\partial^{\vert\alpha\vert}}{\partial^{\alpha_1} x_1 ... \partial^{\alpha_n} x_n}$
Ω	open subset of $\mathbb{R}^n$
$\bar{\Omega}$	closure of Ω in $\mathbb{R}^n$
$\partial\Omega$	boundary of Ω, $= \bar{\Omega} - \Omega$
$B(x_0, r)$	ball of radius r with center x_0
$C(\Omega)$	continuous functions on Ω
$C(\bar{\Omega})$	uniformly continuous functions on Ω
$C^k(\Omega)$	k times continuously differentiable functions on Ω
$C^k(\bar{\Omega})$	$= \{u \in C^k(\Omega) \vert D^\alpha u$ is uniformly continuous on Ω for $\vert\alpha\vert \leq k\}$
$\mathcal{D}(\Omega)$	the space of C^∞ functions with compact support in Ω
$\triangle$	$= \sum_{i=1}^n \frac{\partial^2}{\partial {x_i}^2}$
∇u	$= (\partial_{x_1} u, ..., \partial_{x_n} u)$
u_t	$= \partial_t u = \frac{\partial u}{\partial t} = u' = \frac{du}{dt}$
$L^p(\Omega)$	the space of measurable functions u on Ω such that $\vert u\vert^p$ is integrable, $p \in [1, \infty]$
$\vert\vert u\vert\vert_{L^p(\Omega)}$	$= (\int_\Omega \vert u\vert^p)^{1/p}$, where $u \in L^p(\Omega)$
p'	the conjugate exponent of p, that is $p' = \frac{p}{p-1}$ for $p \in [1, \infty]$
$L^\infty(\Omega)$	the space of measurable functions u on Ω such that there is a constant C with $\vert u(x)\vert \leq C$ for a.e. $x \in \Omega$
$\vert\vert u\vert\vert_{L^\infty(\Omega)}$	$= \inf\{C > 0 \vert \vert u(x)\vert \leq C$ a.e.$\}$ for $u \in L^\infty(\Omega)$

$\|\| \cdot \|\|$	$= \|\| \cdot \|\|_{L^2(\Omega)}$
$(\cdot,\cdot)$	$= (\cdot,\cdot)_{L^2(\Omega)}$
$W^{m,p}(\Omega)$	$= \{u \in L^p(\Omega), D^\alpha u \in L^p(\Omega)$ for all $\alpha \in \mathbb{N}^m$ such that $\|\alpha\| \leq m\}$
$\|\|u\|\|_{W^{m,p}(\Omega)}$	$= (\sum_{\|\alpha\|\leq m} \|\|D^\alpha u\|\|^p_{L^p(\Omega)})^{\frac{1}{p}}$ for $u \in W^{m,p}(\Omega)$
$W^{m,p}_0(\Omega)$	the closure of $\mathcal{D}(\Omega)$ with respect to the norm $\|\| \cdot \|\|_{W^{m,p}(\Omega)}$
$H^m(\Omega)$	$= W^{m,2}(\Omega)$
$H^m_0(\Omega)$	$= W^{m,2}_0(\Omega)$
$C^k([a,b];X)$	Banach space of m times continuously differentiable functions $u : [a,b] \to X$ with sup norm
$L^p([a,b];X)$	Banach space of measurable functions $u : [a,b] \to X$ such that $\|\|u\|\|_{L^p([a,b];X)} < \infty$
$\|\|u\|\|_{L^p([a,b];X)}$	$= (\int_\Omega \|\|u\|\|^p_X)^{1/p}$, where $p < \infty$
$\|\|u\|\|_{L^\infty([a,b];X)}$	$= ess\sup_{t\in[a,b]} \|\|u(t)\|\|_X$

Chapter 1

INTRODUCTION

The Schrödinger equation, which was first proposed by the famous physicist Erwin Schrödinger in 1925, is the model which describes the space and time dependence of quantum mechanical systems. Therefore, this equation helps to predict the future behavior of a dynamical system. Indeed, Schrödinger equation is interpreted as the analogous of Newton's second law and conservation of energy in classical mechanics. In quantum mechanics, we represent a system by means of a complex Hilbert space where each state of this system corresponds to a unit vector in this Hilbert space. This vector includes the information of the probabilities of the outcomes of all measurements on the system. Since the state of a system may change over time, this vector depends on time. Hence, the Schrödinger equation gives the information about the rate of change of the state of the system. This equation can be either linear or nonlinear and may include various kinds of damping terms according to the physical situation it applies.

In this work, we make a mathematical study of some linear and nonlinear Schrödinger equations in the context of stabilization. We study decay rates of solutions for different scenarios which are important from physical point of view. In addition to these, we get a new result on the weakly damped Schrödinger equation with inhomogeneous Dirichlet boundary condition.

In Chapter 2, we give a brief reminder for some common mathematical tools that we will use in the subsequent chapters. These include some calculus facts, various inequalities, Sobolev spaces, Banach space valued functions, functional analysis and a brief semigroup theory.

In Chapter 3, we study the asymptotic behavior of solutions of linear Schrödinger equation on a bounded domain with sufficiently smooth boundary. We consider both the boundary stabilization problem and the internal stabilization problem. In the boundary stabilization problem, in Section 3.1 we prove existence of solutions and in Section 3.2, we prove exponential decay of solutions in H^1-sense by applying a dissipative boundary control which is indeed the trace of velocity on the boundary. In the internal stabilization problem, in Section 3.3, we prove exponential decay of solutions in L^2-sense by adding a localized linear damping in the main equation which is supported only on a small neighborhood of the boundary.

In Chapter 4, we study the long time behaviour of solutions of linearly damped nonlinear Schrödinger equation with homogeneous and inhomogeneous Dirichlet boundary condition on a bounded domain with smooth boundary. In Section 4.1, we prove exponential decay of solutions in H^1-sense when we use zero boundary condition. In Section 4.2 and 4.3, we prove a new stabilization result on Schrödinger equation with inhomogeneous Dirichlet Boundary value. In this result, the equation can be both linear and nonlinear (with positive signed nonlinearity). We prove that the solutions of linearly damped Schrödinger equation decays to zero in H^1-sense by applying a decaying Dirichlet control on the boundary. Our result shows that the decay rate of the solution is at least in the rate of slowest of exponential rate and the decay rate of the boundary condition.

Our result is not only for the sake of mathematical analysis, but it has also physical implications. A nonlinear Schrödinger equation with inhomogeneous boundary condition has a physical meaning. For instance, in ionospheric modification experiments of one space dimension, one directs a radio frequency wave at the ionosphere. At the reflection point of the wave, a sufficient level of electron plasma waves is excited to make the nonlinear behavior important. This may be described by the nonlinear Schrödinger equation with the cubic nonlinear term and a Dirichlet type of boundary condition, [12]. The possible damping term in the nonlinear Schrödinger equation has also various physical implications. For instance, one can describe the high frequency electrostatic plasma oscillations under the presence of a damping, [24]. There are two well known plasma heating problems which are approximated by the nonlinear Schrödinger equation. The first problem is the Langmuir turbulence when the plasma is assumed in equilibrium with the ponderomotive pressure from the high-frequency fields. The second problem is a nonlinear stage of the mode-converted wave in the lower hybrid heating of large tokamaks. When such a wave heats and transfers energy to particles of the plasma, a dissipation term appears in the nonlinear Schrödinger equation, which results in the damped equation, [25].

We finish this work in Section 4.4 by briefly listing some open problems based on the analysis we do in the previous sections. These problems might be of interest for further research. They ask about the stabilization result in the cases of negative nonlinearity, localized damping, less smooth boundary condition or nonlinear damping.

Chapter 2

PRELIMINARIES

This section is a very brief reminder of some mathematical tools for reading the main chapters more comfortable. We only include the tools which we will need in our analysis in the main sections. Most results are given without proof since the proofs can be found in many sources. We may only give the proofs of results which have particular interest in our analysis.

2.1 Calculus

We refer to [7] for this section which includes basic calculus facts.
Suppose Ω is an open and bounded subset of $\mathbb{R}^n$ where $n \geq 1$.

Definition 2.1.1. *We say $\partial\Omega$ is C^k if for each point $x^0 \in \partial\Omega$ there exist $r > 0$ and a C^k function $\gamma : \mathbb{R}^{n-1} \to \mathbb{R}$ such that-upon relabeling and reorienting the coordinate axes if necessary-we have*

$$\Omega \cap B(x^0, r) = \{x \in B(x^0, r) | x_n > \gamma(x_1, ..., x_{n-1})\}.$$

Likewise, $\partial\Omega$ is C^∞ if $\partial\Omega$ is C^k for $k = 1, 2, ...$

Definition 2.1.2. *(i) If $\partial\Omega$ is C^1, then along $\partial\Omega$ is defined the outward pointing unit normal vector field $\nu = (\nu_1, ..., \nu_n)$. The unit normal at any point $x_0 \in \partial\Omega$ is $\nu(x_0)$.*
(ii)For $u \in C^1(\bar{\Omega})$,

$$\frac{\partial u}{\partial \nu} := \nabla u \cdot \nu$$

is called the (outward) normal derivative of u.

Theorem 2.1.3. *(Divergence Theorem) Suppose* $\Gamma := \partial\Omega$ *is* C^1 *and* $F \in C^1$ *vector field on* Ω*. Then, the following identity holds.*

$$\int_\Omega div(F)dx = \int_{\partial\Omega} F \cdot \nu d\Gamma.$$

In the following Remark, we give different variations of Divergence Theorem that we will use in the main section s without pointing out to the particular one we use.

Remarks 2.1.4. *(i) In the above theorem, choosing* $F = (0, ..., \overbrace{u}^{F_j}, ..., 0)$*, where* $u \in C^1(\bar{\Omega})$*, we have*

$$\int_\Omega u_{x_j} dx = \int_{\partial\Omega} u\nu_j d\Gamma.$$

(ii) Taking $u := fg$ *in (i) where* $f, g \in C^1(\bar{\Omega})$*, we get the integration by parts formula*

$$\int_\Omega u_{x_j} v dx = -\int_\Omega u v_{x_j} dx + \int_{\partial\Omega} u v \nu_j d\Gamma.$$

(iii) Taking $F := u\nabla v$ *in the Divergence Theorem where* $u, v \in C^2(\bar{\Omega})$*, we have*

$$\int_\Omega u \triangle v dx + \int_\Omega \nabla u \cdot \nabla v dx = \int_{\partial\Omega} u \frac{\partial v}{\partial \nu} d\Gamma.$$

(iv) Taking F:=uG, where $u \in C^1(\bar{\Omega})$ *and* G *is a* C^1 *vector field on* Ω*, we have*

$$\int_\Omega u div(G) dx = -\int_\Omega \nabla u \cdot G dx + \int_\Gamma u G \cdot \nu d\Gamma.$$

2.2 *Inequalities*

We refer to [7], [17] for this section. We will use the following inequalities throughout the main chapters.

Young's Inequality. *Let* $1 < p, q < \infty$ *and* $\frac{1}{p} + \frac{1}{q} = 1$*. Then,*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \tag{2.1}$$

where $a, b > 0$*.*

Young's Inequality with $\epsilon > 0$**.** *Let* $1 < p, q < \infty$ *and* $\frac{1}{p} + \frac{1}{q} = 1$*. Then,*

$$ab \leq \epsilon a^p + (\epsilon p)^{-q/p} \frac{b^q}{q} \tag{2.2}$$

where $a, b > 0$*.*

In the case, $p = q = 2$, Young's inequalities are also called Cauchy's inequalities.

Hölder's Inequality. *Let* $1 \leq p, q \leq \infty$ *and* $\frac{1}{p} + \frac{1}{q} = 1$*. Then, if* $u \in L^p(\Omega), v \in L^q(\Omega)$*, we have*

$$\int_\Omega |uv| dx \leq ||u||_{L^p(\Omega)} ||v||_{L^q(\Omega)}. \tag{2.3}$$

Gronwall's Inequality. *If for* $t \in [t_0, t_1]$*,* $\phi(t) \geq 0$ *and* $\psi(t) \geq 0$ *are continuous such that the inequality*

$$\phi(t) \leq K + L \int_{t_0}^t \phi(s)\psi(s) ds$$

holds on $[t_0, t_1]$ *with* K *and* L *positive constants, then*

$$\phi(t) \leq K e^{L \int_{t_0}^t \psi(s) ds} \tag{2.4}$$

on $[t_0, t_1]$*.*

Lemma 2.2.1. *Let* $A(t)$ *be a nonnegative continuous function of* t *satisfying the inequality*

$$A(t) \leq C_1 + C_2 A(t)^\gamma$$

in some interval containing 0*, where* C_1 *and* C_2 *are positive constants and* $\gamma \geq 1$*. If* $A(0) < C_1$ *and*

$$C_1 C_2^{(\gamma-1)^{-1}} < (1 - \gamma^{-1}) \gamma^{-(\gamma-1)^{-1}},$$

then in the same interval

$$A(t) < \frac{C_1}{1-\gamma^{-1}}.$$

Lemma 2.2.2. *Let $a, b \geq 0$ and $0 \leq \lambda \leq 1$. Then we have the inequality*

$$(a+b)^\lambda \leq a^\lambda + b^\lambda. \tag{2.5}$$

Whenever, $\lambda \geq 1$, then we have the inequality

$$a^\lambda + b^\lambda \leq (a+b)^\lambda. \tag{2.6}$$

Indeed, for any $\lambda \geq 0$, there exist appropriate constants c_1 and c_2 depending only on λ such that the following inequalities hold.

$$(a+b)^\lambda \leq c_1(a^\lambda + b^\lambda) \tag{2.7}$$

and

$$a^\lambda + b^\lambda \leq c_2(a+b)^\lambda. \tag{2.8}$$

2.3 Sobolev Spaces

We refer to [3], [7] for this section which includes main definitions and facts on Sobolev spaces. These spaces are the fundamental spaces for our analysis.
Let $\Omega \subset \mathbb{R}^n$ be a domain with smooth boundary $\Gamma = \partial\Omega, m \in \mathbb{N}, p \in [1, \infty]$, $p' = \frac{p}{p-1}$.

Definition 2.3.1.

$$W^{m,p}(\Omega) := \{u \in L^p(\Omega), D^\alpha u \in L^p(\Omega) \textit{ for all } \alpha \in \mathbb{N}^m \textit{ such that } |\alpha| \leq m\}.$$

Remark 2.3.2. *$W^{m,p}(\Omega)$ is a Banach space equipped with the norm*

$$||u||_{W^{m,p}(\Omega)} := (\sum_{|\alpha|\leq m} ||D^\alpha u||^p_{L^p(\Omega)})^{\frac{1}{p}}.$$

Definition 2.3.3.

$$W_0^{m,p}(\Omega) := \overline{\mathcal{D}}(\Omega)^{||\cdot||_{W^{m,p}(\Omega)}}.$$

Remark 2.3.4. *When $p = 2$, we say $W^{m,2}(\Omega) =: H^m(\Omega)$ and $W_0^{m,2}(\Omega) =: H_0^m(\Omega)$ where H^m is then a Hilbert space with inner product*

$$(u, v)_{H^m(\Omega)} = \sum_{|\alpha|\leq m} \int_\Omega D^\alpha u D^\beta v dx.$$

Definition 2.3.5. *We define $W^{-m,p'}(\Omega)$ as the topological dual of $W_0^{m,p}(\Omega)$.*

Remark 2.3.6. *Similarly, we say $H^{-m}(\Omega) := W^{-m,2}(\Omega) = (H_0^m(\Omega))^*$*

Theorem 2.3.7. *(Trace Theorem) Assume Ω is bounded and $\partial\Omega$ is C^1. Then there exists a bounded linear operator*

$$T : W^{1,p}(\Omega) \to L^p(\partial\Omega)$$

such that

$$(i)\ Tu = u|_{\partial\Omega} \textit{ if } u \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$$

and

$$(ii)\ ||Tu||_{L^p(\partial\Omega)} \leq C||u||_{W^{1,p}(\Omega)} \textit{ for each } u \in W^{1,p}(\Omega), \textit{ with the constant } C = C(p, \Omega).$$

Theorem 2.3.8. *Suppose Ω is a bounded domain in $\mathbb{R}^n$ $(n \geq 1)$ with a boundary $\Gamma = \partial\Omega$ of class C^3. Then there exists a positive constant $C > 0$ such that for any function $u \in H^1(\Omega)$ which vanishes on a part of the boundary Γ_1 that has a non-empty interior in Γ, we have the inequality*

$$||u||_{L^2(\Omega)} \leq C||\nabla u||_{L^2(\Omega)}. \tag{2.9}$$

Proof: Suppose the above inequality does not hold. Then for all $m > 0$ there is a function $u_m \in H^1(\Omega)$ which vanishes on Γ_1 satisfies

$$||u_m||_{L^2(\Omega)} > m||\nabla u_m||_{L^2(\Omega)}.$$

Without loss of generality we can choose u_m (just take $\frac{u_m}{||u_m||_{L^2(\Omega)}}$ instead of u_m) so that

$$||u_m||_{L^2(\Omega)} = 1. \tag{2.10}$$

So, we have

$$||\nabla u_m||_{L^2(\Omega)} \to 0 \text{ as } m \to \infty. \tag{2.11}$$

Since,

$$||u_m||_{H^1(\Omega)} = (||u_m||^2_{L^2(\Omega)} + ||\nabla u_m||^2_{L^2(\Omega)})^{1/2},$$

(2.10) and (2.11) imply that (u_m) is bounded in $H^1(\Omega)$. Since, $H^1(\Omega)$ is compactly embedded in $L^2(\Omega)$, there exists a subsequence (u_{mj}) of (u_m) which converges to some u_0 in $L^2(\Omega)$.

So, using (2.10) we have that

$$||u_0||_{L^2(\Omega)} = 1. \tag{2.12}$$

Note that

$$\int_\Omega u_0 \phi_{x_i} dx = \lim_{j\to\infty} \int_\Omega u_{mj} \phi_{x_i} dx = -\lim_{j\to\infty} \int_\Omega u_{mj_{x_i}} \phi dx = 0$$

for all $\phi \in C_c^{(\infty)}(\Omega)$. Hence, $u_0 \in H^1(\Omega)$ with $\nabla u_0 = \mathbf{0}$.

Hence $||\nabla u_0||_{L^2(\Omega)} = 0$ which means u_0 is constant, but we know u_0 is zero on some part of the boundary, hence u_0 is zero everywhere. This contradicts the result given in (2.12). Hence, our assumption is wrong, so (2.9) must hold.■

Theorem 2.3.9. *(Gagliardo-Nirenberg Inequality) Let Ω be a bounded domain with $\partial\Omega$ in C^m, and let u be any function in $W^{m,r}(\Omega) \cap L^q(\Omega)$, $1 \leq r, q \leq \infty$. For any integer j, $0 \leq j < m$, and for any number θ in the interval $j/m \leq \theta \leq 1$, set*

$$\frac{1}{p} = \frac{j}{n} + \theta(\frac{1}{r} - \frac{m}{n}) + (1-\theta)\frac{1}{q}.$$

If $m - j - \frac{n}{r}$ is not a nonnegative integer, then

$$||D^j u||_{L^p(\Omega)} \leq C||u||^{\theta}_{W^{m,r}(\Omega)}||u||^{1-\theta}_{L^q(\Omega)}. \tag{2.13}$$

If $m - j - \frac{n}{r}$ is a nonnegative integer, then (2.13) holds for $\theta = \frac{j}{m}$. The constant C depends only on $\Omega, r, q, m, j, \theta$.

2.4 Banach Space Valued Functions

We refer to [21] for this section.

Consider a Banach space $\mathbb{X}$ and let $p \in [1, \infty), -\infty < a < b < \infty$. Then we make the following definitions and remarks.

Definition 2.4.1. *$L^p((a,b);\mathbb{X})$ is said to be the space of L^p functions from (a,b) to $\mathbb{X}$ with the norm*

$$||u||_{L^p((a,b);\mathbb{X})} := (\int_a^b ||u(t)||^p_{\mathbb{X}} dt)^{\frac{1}{p}}.$$

Remark 2.4.2. *$L^p((a,b);\mathbb{X})$ equipped with its norm becomes a Banach space.*

In the case, $p = \infty$, we make the following definition.

Definition 2.4.3. *$L^\infty((a,b);\mathbb{X})$ is said to be the space of measurable functions from (a,b) to $\mathbb{X}$ which are essentially bounded, i.e.,*

$$||u||_{L^\infty((a,b);\mathbb{X})} := \sup_{t\in(a,b)} ess||u(t)||_{\mathbb{X}} < \infty.$$

Remark 2.4.4. *$L^\infty((a,b);\mathbb{X})$ is also a Banach space.*

Remark 2.4.5. *We also define continuous (and continuously differentiable) functions valued in Banach spaces as $C^k([a,b];\mathbb{X})$ with the norm*

$$||u||_{C^k([a,b];\mathbb{X})} := \sum_{i=0}^{k} \max_{t\in[a,b]} ||\frac{d^i u}{dt^i}(t)||_{\mathbb{X}},$$

which is again a Banach space.

2.5 Functional Analysis

We refer to [2], [4], [5], [6], [18], [19], [22], [23] for this section. We will need the following various definitions and results in the main chapters.

Theorem 2.5.1. *(The Banach Fixed Point Theorem) Let (E,d) be a complete metric space and let $f : E \to E$ be a mapping such that there exists $k \in [0, 1)$ satisfying $d(f(x), f(y)) \leq kd(x, y)$ for all $(x, y) \in E \times E$. Then there exists a unique point $x_0 \in E$ such that $f(x_0) = x_0$.*

Definition 2.5.2. *Let $\mathbb{X}$ be a Banach space. We say a sequence $\{x_n\}_{k=1}^{\infty} \subset \mathbb{X}$ converges weakly to $x \in \mathbb{X}$, written*

$$x_n \rightharpoonup x,$$

if

$$\phi(x_n) \to \phi(x)$$

for each linear bounded functional $\phi \in \mathbb{X}^*$.

Definition 2.5.3. *Let $\mathbb{X}$ be a Banach space. We say ϕ is the weak-* limit of a sequence $\phi_n \in \mathbb{X}^*$ if*

$$\phi_n(x) \to \phi(x)$$

for all $x \in \mathbb{X}$, and we write

$$\phi_n \overset{*}{\rightharpoonup} \phi.$$

Remark 2.5.4. *If $\mathbb{X}$ is reflexive weak and weak-* convergence are equivalent. For example, L^p spaces for $1 < p < \infty$ satisfy this. However, in the case $p = \infty$, we know from analysis that $L^1(\Omega)^* = L^\infty(\Omega)$ but $L^\infty(\Omega)^* \neq L^1(\Omega)$, in fact $L^\infty(\Omega)^*$ is larger than $L^1(\Omega)$. Hence, weak and weak* convergence does not coincide on L^∞ space.*

Theorem 2.5.5. *(Banach Alaoglu Theorem) Let $\mathbb{X}$ be a Banach space. The closed unit ball in $\mathbb{X}^*$ is weak-* compact.*

Remark 2.5.6. *Every strongly convergent sequence is also weakly convergent.*

Definition 2.5.7. *An operator A on a real Hilbert space H is called dissipative if $(Ax, x)_H \leq 0$ for all $x \in D(A)$.*

Definition 2.5.8. *An unbounded linear operator A on a Hilbert space H is called m-dissipative if A is dissipative and $\lambda I - A$ is onto for all $\lambda > 0$.*

Theorem 2.5.9. *Let A be an unbounded linear operator on a Hilbert space H. Then the operator A is m-dissipative if and only if $I - A$ is onto.*

Definition 2.5.10. *An operator A on a real Hilbert space H is called monotone if $(x_2 - x_1, Ax_2 - Ax_1)_H \geq 0$ for all $x_1, x_2 \in D(A)$.*

Definition 2.5.11. *Let X and Y be two sets and let $A : X \to Y$ be a mapping. Then, the graph of A is $G(A) := \{(x, y) \in X \times Y : y = Ax\}$.*

Definition 2.5.12. *An operator A on a real Hilbert space H is called maximal monotone if A is monotone and for any monotone operator B, $G(B) \subset G(A)$.*

Definition 2.5.13. *An operator A on a Hilbert space H is called positive if $(Ax, x)_H \geq 0$ for all $x \in D(A)$.*

Theorem 2.5.14. *(Minty's Theorem) A monotone map A is maximal if and only if the map $I + A$ is surjective.*

Lemma 2.5.15. *Let*

$$X := \{u \in L^2([0, \infty); H_0^1(\Omega)) : u_t \in L^2([0, \infty); H^{-1}(\Omega))\},$$

then bounded subsets in X are relatively compact in $L^2([0, \infty); L^2(\Omega))$.

Lemma 2.5.16. *Let Q be a bounded subset in $\mathbb{R}^n_X \times \mathbb{R}_t$, g_μ and g are functions in $L^q(Q) \leq C$ where $q \in (1, \infty)$, such that $||g_\mu||_{L^q(Q)} \leq C$ and $g_\mu \to g$ a.e. in Q. Then, $g_\mu \to g$ weakly in $L^q(Q)$.*

2.6 Semigroup Theory

We refer to [1], [6] for this section.

Let $\mathbb{X}$ be a Banach space and $u^0 \in \mathbb{X}$. We consider the general problem

$$\begin{cases} u'(t) = Au(t), \quad \forall t \in [0, \infty), \\ \quad u(0) = u^0 \end{cases}$$

where $A : D(A) \subset \mathbb{X} \to \mathbb{X}$ is a (possibly) unbounded operator and $D(A)$ is a linear subspace of $\mathbb{X}$. We are looking for solutions $u : [0, \infty) \to \mathbb{X}$, Then, we have the following definitions and results.

Definition 2.6.1. *A family $\{S(t)\}_{t\geq 0}$ of bounded linear operators from $\mathbb{X}$ into $\mathbb{X}$ is called a strongly continuous semigroup if the conditions*
(i) $S(0)u^0 = u^0, u^0 \in \mathbb{X}$,
(ii) $S(t+s)u^0 = S(t)S(s)u^0 = S(s)S(t)u^0, u^0 \in \mathbb{X}, s, t \geq 0$,
(iii) $\lim_{t\searrow 0} ||S(t)u^0 - u^0|| = 0, u^0 \in \mathbb{X}$
hold.
We say $\{S(t)\}_{t\geq 0}$ is a contraction semigroup if in addition $||S(t)|| \leq 1$ for each $t \geq 0$.

Remark 2.6.2. *If $\{S(t)\}_{t\geq 0}$ is a strongly continuous semigroup on $\mathbb{X}$, then for all $u^0 \in \mathbb{X}$, the mapping $t \to S(t)u^0$ is continuous.*

Definition 2.6.3. *Let $\{S(t)\}_{t\geq 0}$ be a strongly continuous semigroup on $\mathbb{X}$. The infinitesimal generator of the semigroup $\{S(t)\}_{t\geq 0}$ is the unbounded operator A, defined by*

$$A\phi := \lim_{t\searrow 0} \frac{S(t)\phi - \phi}{t}$$

with domain $D(A)$ which is given by

$$D(A) := \{\phi | \lim_{t\searrow 0} \frac{S(t)\phi - \phi}{t} \text{ exists in } \mathbb{X}\}.$$

Lemma 2.6.4. *If A is the infinitesimal generator of a strongly continuous semigroup $\{S(t)\}_{t\geq 0}$ on $\mathbb{X}$, then $D(A)$ is dense in $\mathbb{X}$ and A is closed.*

Theorem 2.6.5. *(Philips' Theorem) An unbounded linear operator $A : D(A) \subset H \to H$ where H is a Hilbert space is the infinitesimal generator of a semigroup of contractions on H if and only if A is m-dissipative in H.*

Theorem 2.6.6. *If a linear operator $A : D(A) \subset H \to H$ is the infinitesimal generator of a strongly continuous semigroup $(S(t))_{t\geq 0}$, then for each $u^0 \in D(A)$, $u(t) = S(t)u^0$ is the unique solution of the problem*

$$\begin{cases} u'(t) = Au(t), \quad \forall t \in [0, \infty), \\ \quad u(0) = u^0 \end{cases}$$

which is from the class $C([0, \infty); D(A)) \cap C^1([0, \infty); H)$.

Chapter 3

STABILIZATION OF THE LINEAR SCHRÖDINGER EQUATION

In this chapter, we review the results (studied in [9]) on the boundary and internal stabilization of the linear Schrödinger equation in a bounded domain with boundary of class C^3. In addition to these results, we do the proof of the existence and uniqueness of the solution and give more complete proofs for the stabilization results given in [9]. Now, suppose Ω is a bounded domain in $\mathbb{R}^n$ $(n \geq 1)$ with a boundary $\Gamma = \partial\Omega$ of class C^3. Consider the following initial value problem with zero boundary value.

$$\begin{cases} iu_t + \triangle u = 0, & \text{in } \Omega \times (0, \infty), \\ u = 0, & \text{on } \Gamma \times (0, \infty), \\ u(0) = u^0, & \text{in } \Omega. \end{cases} \tag{3.1}$$

Some Properties:

1. For the above problem H^1 and L^2 norms of the solutions are conserved.

Multiplying the above equation with $\bar{u}_t$, taking the real parts and using integration by parts one gets

$$0 = \mathrm{Re} \int_\Omega (i|u_t|^2 + \bar{u}_t \triangle u) dx = \mathrm{Re} \int_\Omega \bar{u}_t \triangle u dx$$
$$= -\mathrm{Re} \int_\Omega \nabla \bar{u}_t \cdot \nabla u dx + \int_\Gamma \bar{u}_t \frac{\partial u}{\partial \nu} d\Gamma$$
$$= -\mathrm{Re} \int_\Omega \nabla \bar{u}_t \cdot \nabla u dx = -\frac{1}{2}\frac{d}{dt} \int_\Omega |\nabla u|^2 dx = -\frac{1}{2}\frac{d}{dt} ||\nabla u(t)||^2_{L^2(\Omega)}.$$

So, we get

$$\frac{d}{dt} ||\nabla u(t)||_{L^2(\Omega)} = 0,$$

that is $||\nabla u(t)||_{L^2(\Omega)}$ is constant, which implies

$$||\nabla u(t)||_{L^2(\Omega)} = ||\nabla u^0||_{L^2(\Omega)}. \tag{3.2}$$

Also multiplying the above equation with $\bar{u}$, taking the imaginary parts and using integration by parts one gets

$$0 = \mathrm{Im}\int_\Omega (iu_t\bar{u} + \bar{u}\triangle u)dx = \mathrm{Re}\int_\Omega u_t\bar{u}dx - \mathrm{Im}\int_\Omega \nabla\bar{u}\cdot\nabla u dx$$

$$+\mathrm{Im}\int_\Omega \bar{u}\frac{\partial u}{\partial\nu}d\Gamma dx = \mathrm{Re}\int_\Omega u_t\bar{u}dx = \frac{1}{2}\frac{d}{dt}\int_\Omega |u|^2dx = \frac{1}{2}\frac{d}{dt}||u(t)||^2_{L^2(\Omega)}.$$

Hence,

$$\frac{d}{dt}||u(t)||^2_{L^2(\Omega)} = 0,$$

that is $||u(t)||^2_{L^2(\Omega)}$ is constant. Therefore, we have

$$||u(t)||_{L^2(\Omega)} = ||u^0||_{L^2(\Omega)}. \tag{3.3}$$

Hence, with the zero Dirichlet boundary condition, the system is not dissipative.

2. Solutions of the Schrödinger equation are also solutions of the plate equation

$$u_{tt} + \triangle^2 u = 0,$$

because one can write

$$u_{tt} + \triangle^2 u = -(i\partial_t - \triangle)(i\partial_t + \triangle)u = 0.$$

From this observation, one can try to solve the stabilization problem for Schrödinger equation by using the results obtained on plate equation. However, our approach will be directly to study the Schrödinger equation without using the properties of plate equation.

Stabilization Problem: To introduce a damping term in a system which ensures (desirably exponential) decay of solutions in a physically appropriate norm as time becomes large.

Now, we define,

$$\Gamma_0 := \{x \in \Gamma; m(x)\cdot\nu(x) > 0\}, \Gamma_1 := \{x \in \Gamma; m(x)\cdot\nu(x) \le 0\} = \Gamma\backslash\Gamma_0$$

where $\Gamma = \Gamma_0 \cup \Gamma_1$, $m(x) := x - x_0$ ($x_0 \in \mathbb{R}^n$) and ν is the unit outward normal.

Consider now, the following initial value problem with a non-zero boundary condition supported on Γ_0.

$$\begin{cases} iu_t + \triangle u = 0, & \text{in } \Omega \times (0,\infty), \\ \frac{\partial u}{\partial \nu} = -(m(x)\cdot \nu(x))u_t, & \text{on } \Gamma_0 \times (0,\infty), \\ u = 0, & \text{on } \Gamma_1 \times (0,\infty), \\ u(x,0) = u^0(x), & \text{in } \Omega \end{cases} \tag{3.4}$$

where the space of initial data is

$$\mathbb{X} = \{u \in H^1(\Omega) | u = 0 \text{ on } \Gamma_1\}.$$

Multiplying the above equation with $\bar{u_t}$, integrating by parts and taking the real parts we get

$$0 = \mathrm{Re}\int_\Omega [i|u_t|^2 + \triangle u \bar{u_t}]dx = -\mathrm{Re}\int_\Omega \nabla u \cdot \nabla \bar{u_t} dx + \mathrm{Re}\int_\Gamma \frac{\partial u}{\partial \nu}\bar{u_t} d\Gamma$$

which implies

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u|^2 dx = -\int_{\Gamma_0}(m\cdot v)|u_t|^2 d\Gamma, \tag{3.5}$$

where $\frac{1}{2}\int_\Omega |\nabla u|^2 dx =: E(t)$ is the energy function.

Consider the inner product $(u,v)_{\mathbb{X}} := \mathrm{Re}\int_\Omega \nabla u \cdot \nabla \bar{v} dx$ on the initial value set $\mathbb{X}$. Then the norm $||\cdot||_{\mathbb{X}}$ induced by this inner product is equivalent to the classical H^1 norm $||\cdot||_{H^1(\Omega)}$ on $\mathbb{X}$ since the Poincare inequality (see Theorem 2.3.8)

$$||u||_{L^2(\Omega)} \leq C||\nabla u||_{L^2(\Omega)}$$

is valid for functions in $\mathbb{X}$ (in the case Γ_1 has nonempty interior).

The existence (and uniqueness) of the solution of the system (3.4) can be proved by reducing the problem into a semigroup problem and using the results of the linear semigroup theory. The proof of existence is not present in [9], but we give a proof in section 3.1 for the sake of completeness and we will show that there exists a strongly continuous semigroup of contractions $\{S(t)\}_{t\geq 0}$ in $\mathbb{X}$ such that $u(t) = S(t)u^0$ is a solution of (3.4).

The boundary conditions in this problem indeed play a role of a damping and ensures the exponential decay (in H^1-sense) of the energy which we prove in section 3.2. One can also conider the linear Schrödinger equation by using a positive damping not on the boundary but supported in a neighborhood of the boundary. This problem is called the internal stabilization problem which is defined formally as follows.
Let $\omega \subset \Omega$ be a neighborhood of $\overline{\Gamma_0}$ in Ω and $a \in L^\infty(\Omega)$ such that

$$\begin{cases} a \geq 0, & \text{a.e. in } \Omega, \\ \exists a_0 > 0, & a \geq a_0 \text{ a.e. in } \omega \end{cases} \tag{3.6}$$

and we consider the following damped Schrödinger equation.

$$\begin{cases} iu_t + \triangle u + ia(x)u = 0, & \text{in } \Omega \times (0, \infty), \\ u = 0, & \text{on } \Gamma \times (0, \infty), \\ u(0) = u^0, & \text{in } \Omega. \end{cases} \tag{3.7}$$

As noted in [9], for any $u^0 \in L^2(\Omega)$, there exists a unique solution of the system (3.7) from the class

$$C([0, \infty); L^2(\Omega)) \cap C^1([0, \infty); (H^2(\Omega) \cap H_0^1(\Omega))'). \tag{3.8}$$

Here we consider the $L^2(\Omega)-$norm of the solution $F(t)$,

$$F(t) := \frac{1}{2}\int_\Omega |u(t)|^2 dx = \frac{1}{2}||u(t)||^2_{L^2(\Omega)}, \tag{3.9}$$

which is shown to be decaying exponentially in section 3.3.

3.1 Existence and Uniqueness: Boundary Stabilization

Note that, the equation (3.4) is equivalent to the ordinary differential equation

$$\begin{cases} u'(t) = Au(t), \quad \forall t \in [0, \infty), \\ u(0) = u^0 \end{cases}$$

where $A = i\triangle$ with domain $D(A) \subset \mathbb{X}$ defined as follows.

$$D(A) := \{u \in \mathbb{X} : \triangle u \in \mathbb{X}, \frac{\partial u}{\partial \nu} = -i(m \cdot \nu)\triangle u \text{ on } \Gamma_0\}.$$

Now, the operator $A = i\triangle$ is dissipative, because

$$(Au, u)_{\mathbb{X}} = \mathrm{Re}(\int_\Omega \nabla(i\triangle u) \cdot \nabla \bar{u} dx) = \mathrm{Re}(-\int_\Omega \triangle \bar{u}(i\triangle u)dx + i\int_\Gamma \triangle u \frac{\partial \bar{u}}{\partial \nu} d\Gamma)$$

$$= \mathrm{Re}(i\int_{\Gamma_0} \triangle u \frac{\partial \bar{u}}{\partial \nu} d\Gamma + i\int_{\Gamma_1} \triangle u \frac{\partial \bar{u}}{\partial \nu} d\Gamma) = \mathrm{Re}(i\int_{\Gamma_0} \triangle u(i(m \cdot \nu)\triangle \bar{u})d\Gamma) =$$

$$-\mathrm{Re}(\int_{\Gamma_0} (m \cdot \nu)|\triangle u|^2 d\Gamma) \leq 0 \text{ for all } u \in D(A).$$

We also have $(I - A) : D(A) \subset \mathbb{X} \to \mathbb{X}$ is surjective. To see this, let $f \in \mathbb{X}$ and consider the equation $u - Au = f$. Now, define the operator $L : \mathbb{X} \to D(A) \subset \mathbb{X}$ as $L\tilde{u} = u$ if and only if

$$\begin{cases} -\triangle u = i\tilde{u}, & \text{in } \Omega, \\ u = 0, & \text{on } \Gamma_1, \\ \frac{\partial u}{\partial \nu} = -(m \cdot \nu)\tilde{u}, & \text{on } \Gamma_0. \end{cases}$$

Then, $u - Au = f \Rightarrow u - i\triangle u = f \Rightarrow L\tilde{u} - \tilde{u} = f \Rightarrow \tilde{u} - L\tilde{u} = -f$. Hence, to prove $(I - A)$ is onto, it is enough to prove that $I - L : \mathbb{X} \to \mathbb{X}$ is onto, but if $-L$ is monotone, this is equivalent to prove that $-L$ is a maximal monotone operator by Minty's Theorem (See Theorem 2.5.14).

First, observe that

$$(\tilde{u}, -L\tilde{u})_{\mathbb{X}} = \mathrm{Re}(\int_\Omega \nabla\tilde{u} \cdot \nabla(\overline{-L\tilde{u}})dx) = \mathrm{Re}(\int_\Omega \triangle(\overline{L\tilde{u}})\tilde{u}dx - \int_\Gamma \frac{\partial(\overline{L\tilde{u}})}{\partial \nu}\tilde{u}d\Gamma) =$$

$$\mathrm{Re}(i\int_\Omega \tilde{u}\bar{\tilde{u}}dx + \int_{\Gamma_0} (m \cdot \nu)\tilde{u}\bar{\tilde{u}}d\Gamma) = \int_{\Gamma_0} (m \cdot \nu)|\tilde{u}|^2 d\Gamma > 0.$$

Hence,

$$(\tilde{u_2} - \tilde{u_1}, -L(\tilde{u_2} - \tilde{u_1}))_{\mathbb{X}} = (\tilde{u_2} - \tilde{u_1}, -L\tilde{u_2} - ((-L\tilde{u_1}))_{\mathbb{X}} > 0.$$

We conclude that $-L$ is a monotone operator.

Also, since $(\tilde{u}, -L\tilde{u})_{\mathbb{X}} = (-L\tilde{u}, \tilde{u})_{\mathbb{X}}$, we have $(-L\tilde{u}, \tilde{u})_{\mathbb{X}} > 0$, that is $-L$ is a positive operator on the Hilbert space $\mathbb{X}$. This implies $-L$ is maximal monotone, since any positive linear operator on a Hilbert space is maximal monotone (See Examples 1.5-b in [5]). Now thanks to Minty's Theorem to say that $I - L$ is onto, hence $I - A$ is onto. Thus, A is an m-accretive operator.

Then, by Philip's Theorem (See Theorem 2.6.5), we can say that A is the infinitesimal generator of a semigroup of contractions $(S(t))_{t\geq 0}$ on $\mathbb{X}$. Therefore $(S(t))_{t\geq 0}$ is a strongly continuous semigroup of contractions and for all $u^0 \in D(A)$, $u(t) = S(t)u^0$ is the unique solution to our problem from the class $C([0,\infty); D(A)) \cap C^1([0,\infty); \mathbb{X})$ (See Theorem 2.6.6).

3.2 Boundary Stabilization

In this part, the boundary stabilization problem is considered for system 3.4 which includes dissipative boundary conditions. The problem is solved using the multiplier techniques. In the end, the exponential decay of the energy of the solution is proved. We introduce the function

$$E_\epsilon(t) := E(t) + \epsilon\rho(t) \tag{3.10}$$

where

$$\rho(t) := \mathrm{Im}\int_\Omega u(x,t)m(x)\cdot\nabla\bar{u}(x,t)dx$$

for $t \geq 0$. From Poincare inequality, we get

$$|\rho(t)| \leq ||u(t)||_{L^2(\Omega)}||m\cdot\nabla u(t)||_{L^2(\Omega)} \leq C||m||_{L^\infty(\Omega)}||u(t)||^2_{\mathbb{X}} = C_1E(t) \tag{3.11}$$

for each $t \geq 0$ where $R := ||m||_{L^\infty(\Omega)}$ and $C_1 := 2RC$.
Differentiating $\rho(t)$ gives

$$\rho'(t) = \mathrm{Im}\int_\Omega u_t m\cdot\nabla\bar{u}dx + \mathrm{Im}\int_\Omega um\cdot\nabla\bar{u}_t dx.$$

Using the divergence theorem on the vector function $u\bar{u}_t m$ we have

$$\mathrm{Im}\int_\Omega \mathrm{div}(u\bar{u}_t m)dx = \mathrm{Im}\int_\Gamma u\bar{u}_t(m\cdot\nu)d\Gamma.$$

Since,

$$\mathrm{div}(u\bar{u}_t m) = \Sigma_{i=1}^n(u\bar{u}_t m)_{x_i} = \Sigma_{i=1}^n u_{x_i}\bar{u}_t m + \Sigma_{i=1}^n u(\bar{u}_t)_{x_i}m + \Sigma_{i=1}^n u\bar{u}_t m_{x_i}$$

$$= (\nabla u\cdot m)\bar{u}_t + (\nabla\bar{u}_t\cdot m)u + nu\bar{u}_t,$$

we have the equality

$$\mathrm{Im}\int_\Omega um\cdot\nabla\bar{u}_t dx = \mathrm{Im}\int_\Gamma(m\cdot\nu)u\bar{u}_t d\Gamma - \mathrm{Im}\int_\Omega m\cdot\nabla u\bar{u}_t dx - n\mathrm{Im}\int_\Omega u\bar{u}_t dx$$

$$= \mathrm{Im}\int_{\Gamma_0}(m\cdot\nu)u\bar{u}_t d\Gamma + \mathrm{Im}\int_\Omega m\cdot\nabla\bar{u}u_t dx - n\mathrm{Im}\int_\Omega u\bar{u}_t dx.$$

Using (3.4) we get

$$\mathrm{Im}\int_\Omega u\bar{u}_t dx = -\mathrm{Re}\int_\Omega \triangle u\bar{u}dx = \mathrm{Re}\int_\Omega \nabla u\cdot\nabla\bar{u}dx - \mathrm{Re}\int_\Gamma \frac{\partial u}{\partial\nu}\bar{u}d\Gamma$$

$$= \int_\Omega |\nabla u|^2 dx + \mathrm{Re}\int_{\Gamma_0}(m\cdot\nu)u_t\bar{u}d\Gamma$$

and

$$\mathrm{Im}\int_\Omega m\cdot\nabla u\bar{u}_t dx = -\mathrm{Re}\int_\Omega m\cdot\nabla\bar{u}\triangle u dx.$$

We also have

$$\mathrm{Im}\int_{\Gamma_0}(m\cdot\nu)u\bar{u}_t d\Gamma = -\mathrm{Re}\int_{\Gamma_0}(m\cdot\nu)iu\bar{u}_t d\Gamma.$$

Combining the above equalities we get

$$\rho'(t) = 2\mathrm{Re}\int_\Omega m\cdot\nabla\bar{u}\triangle u dx - n\int_\Omega |\nabla u|^2 dx - \mathrm{Re}\int_{\Gamma_0}(m\cdot\nu)(i+n)u_t\bar{u}d\Gamma. \quad (3.12)$$

Now, to get an estimate on the term $2\mathrm{Re}\int_\Omega m\cdot\nabla\bar{u}\triangle u dx$, we use the following Lemma (a technical proof is given in [8]).

Lemma 3.2.1. *If $n \leq 3$, $\varphi, \psi \in V$ are real valued functions such that $\triangle\varphi \in L^2(\Omega)$ and*

$$\frac{\partial\varphi}{\partial\nu} = -(m\cdot\nu)\psi \text{ on } \Gamma_0,$$

then

$$2\int_\Omega \triangle\varphi m\cdot\nabla\varphi \leq (n-2)\int_\Omega |\nabla\varphi|^2 dx + 2\int_\Gamma \frac{\partial\varphi}{\partial\nu}m\cdot\nabla\varphi d\Gamma - \int_\Gamma (m\cdot\nu)|\nabla\varphi|^2 d\Gamma.$$

Since, u is a complex valued function, there exist real valued functions u_1 and u_2 such that $u = u_1 + iu_2$.

Hence,

$$\mathrm{Re}(m\cdot\nabla\bar{u}\triangle u) = m\cdot\nabla u_1\triangle u_1 + m\cdot\nabla u_2\triangle u_2.$$

Therefore, using Lemma 3.2.1, we get

$$2\mathrm{Re}\int_\Omega m\cdot\nabla\bar{u}\triangle u dx = 2\int_\Omega m\cdot\nabla(u_1\triangle u_1 + m\cdot\nabla u_2\triangle u_2)dx$$

$$\leq (n-2)\int_{\Omega}(|\nabla u_1|^2+|\nabla u_2|^2)dx+2\int_{\Gamma}(\frac{\partial u_1}{\partial \nu}m\cdot\nabla u_1+\frac{\partial u_2}{\partial \nu}m\cdot\nabla u_2)d\Gamma$$

$$-\int_{\Gamma}(m\cdot\nu)(|\nabla u_1|^2+(|\nabla u_2|^2)d\Gamma$$

$$=(n-2)\int_{\Omega}|\nabla u|^2dx+2\text{Re}\int_{\Gamma}\frac{\partial u}{\partial \nu}m\cdot\nabla\bar{u}d\Gamma-\int_{\Gamma}(m\cdot\nu)|\nabla u|^2d\Gamma. \tag{3.13}$$

Since, $u=0$ on $\Gamma_1\times(0,\infty)$, we have $\nabla u=\frac{\partial u}{\partial \nu}\nu$ on $\Gamma_1\times(0,\infty)$. Using this and inserting (3.13) into (3.12) we get the inequality

$$\rho'(t)\leq -2\int_{\Omega}|\nabla u|^2dx-\text{Re}\int_{\Gamma_0}(m\cdot\nu)[2u_t(m\cdot\nabla\bar{u})+|\nabla u|^2+(i+n)u\bar{u}_t]d\Gamma$$

$$+\text{Re}\int_{\Gamma_1}(m\cdot\nu)|\frac{\partial u}{\partial \nu}|^2d\Gamma-\int_{\Gamma_1}(m\cdot\nu)|\nabla u|^2d\Gamma=$$

$$-2\int_{\Omega}|\nabla u|^2dx-\text{Re}\int_{\Gamma_0}(m\cdot\nu)[2u_t(m\cdot\nabla\bar{u})+|\nabla u|^2+(i+n)u\bar{u}_t]d\Gamma \tag{3.14}$$

(3.5), (3.10) and (3.14) imply that

$$E'_\epsilon(t)\leq -4\epsilon E(t)-\text{Re}\int_{\Gamma_0}(m\cdot\nu)[|u_t|^2+2\epsilon u_t(m\cdot\nabla\bar{u})+\epsilon|\nabla u|^2+\epsilon(i+n)u\bar{u}_t]d\Gamma. \tag{3.15}$$

Note that, $\forall\varphi\in V$, the Poincare inequality (2.9) and the trace inequality in $H^1(\Omega)$ imply

$$\int_{\Gamma_0}(m\cdot\nu)|\varphi|^2d\Gamma\leq\beta\int_{\Omega}|\nabla\varphi|^2dx,$$

$\forall\varphi\in V$, and for some $\beta>0$.

From this inequality and Young's inequality we get

$$|\int_{\Gamma}(m\cdot\nu)(n+i)u\bar{u}_td\Gamma|\leq\frac{\beta}{2}(n^2+1)\int_{\Gamma_0}(m\cdot\nu)|u_t|^2d\Gamma+\frac{1}{2\beta}\int_{\Gamma_0}(m\cdot\nu)|u|^2dx$$

$$\leq\frac{\beta}{2}(n^2+1)\int_{\Gamma_0}(m\cdot\nu)|u_t|^2d\Gamma+\frac{1}{2}\int_{\Omega}|\nabla u|^2dx. \tag{3.16}$$

Again using Young's inequality on $\Gamma_0\times(0,\infty)$ we also have that

$$|2u_tm\cdot\nabla\bar{u}|\leq R^2|u_t|^2+|\nabla u|^2. \tag{3.17}$$

Inserting (3.16) and (3.17) into (3.15) we get

$$E'_\epsilon(t)\leq -3\epsilon E(t)-\text{Re}\int_{\Gamma_0}(m\cdot\nu)[1-\epsilon(R^2+\frac{(n^2+1)\beta}{2})]|u_t|^2d\Gamma.$$

Since $m \cdot \nu > 0$ on Γ_0, for $0 < \epsilon \le \epsilon_0 := \frac{2}{2R^2+(n^2+1)\beta}$, we have

$$E'_\epsilon(t) \le -3\epsilon E(t). \tag{3.18}$$

Using (3.10) and (3.11),

$$E_\epsilon(t) = E(t) + \epsilon\rho(t) \le E(t) + \epsilon C_1 E(t) = (1+\epsilon C_1)E(t)$$

which implies

$$\frac{-3\epsilon}{1+\epsilon C_1}E_\epsilon(t) \ge -3\epsilon E(t). \tag{3.19}$$

From (3.18) and (3.19), we have

$$E'_\epsilon(t) \le -C_2\epsilon E_\epsilon(t)$$

where $C_2 = \frac{3}{1+\epsilon C_1}$. This implies

$$E_\epsilon(t) \le E_\epsilon(0)e^{-C_2\epsilon t}.$$

Replacing $E_\epsilon(t)$ with $E(t)+\epsilon\rho(t)$ and $E_\epsilon(0)$ with $E(0)+\epsilon\rho(0)$ and using (3.11), for all sufficiently small ϵ we get

$$(1-\epsilon C_1)E(t) \le (1+\epsilon C_1)E(0)e^{-C_2\epsilon t} \Rightarrow E(t) \le \frac{1+\epsilon C_1}{1-\epsilon C_1}E(0)e^{-C_2\epsilon t}$$

which shows that the energy $E(t)$ decreases exponentially, provided that Γ_1 has non-empty interior in Γ and $n \le 3$.

Hence, we conclude with the following theorem.

Theorem 3.2.2. *Suppose $\Omega \subset \mathbb{R}^n$ $(1 \le n \le 3)$ is a bounded domain with boundary Γ of class C^3, $x_0 \in \mathbb{R}^n$ is a fixed point such that int$(\Gamma_1) \neq \emptyset$. Then there exist positive constants C, γ such that*

$$E(t) \le CE(0)e^{-\gamma t}, \forall t \in \mathbb{R}_+.$$

3.3 Internal Stabilization

In this section we solve the internal stabilization problem.

Multiplying the system (3.7) by $\bar{u}$, integrating by parts and taking the imaginary parts we get

$$0 = \mathrm{Im}\int_\Omega (i\bar{u}u_t + \bar{u}\triangle u + iau\bar{u})dx =$$
$$\mathrm{Re}\int_\Omega \bar{u}u_t dx - \mathrm{Im}\int_\Omega \nabla\bar{u}\cdot\nabla u dx + \mathrm{Im}\int_\Gamma \bar{u}\frac{\partial u}{\partial\nu}d\Gamma + \mathrm{Im}\int_\Omega ia|u|^2dx =$$
$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u|^2dx + \int_\Omega a|u|^2dx.$$

Hence, the function F defined by identity (3.9) satisfies

$$F'(t) = -\int_\Omega a|u|^2dx, \tag{3.20}$$

or we can compute

$$\int_{t_1}^{t_2} F'(t)dt = F(t_2) - F(t_1) = -\int_{t_1}^{t_2}\int_\Omega a|u|^2dxdt \tag{3.21}$$

where $t_2 > t_1 \geq 0$. Thus, one can conclude that the $L^2(\Omega)-$norm is non-increasing, but (3.21) or (3.20) may not imply any decay at a first glance, because if u is zero on the neighborhood of the boundary where damping is supported we may not conclude stabilization, but we will prove that this is really not the case. Indeed, our aim is to prove a stronger result which is to show that $L^2(\Omega)-$norm of the solution has an exponential decay.

In order to prove the exponential decay of the solutions of the Schrödinger equation, it is enough to prove the inequality

$$F(T) \leq C_0\int_0^T\int_\Omega a|u|^2dxdt, \tag{3.22}$$

for a time $T > 0$ and a positive constant C_0, because if (3.22) holds, then using (3.21) we have

$$F(T) - F(0) \leq -\int_0^T\int_\Omega a|u|^2dxdt \leq \frac{-1}{C_0}F(T)$$

which implies

$$F(T) \leq \frac{C_0}{1+C_0} F(0). \tag{3.23}$$

The fact that (3.23) implies the exponential decay of the $L^2(\Omega)-$norm of the solution is noted but not proved in [9]. Now, we show this for the completeness. Note that (3.23) implies

$$||u(t)||^2 = ||S(T)u^0||^2_{L^2(\Omega)} \leq \frac{C_0}{1+C_0}||u^0||^2_{L^2(\Omega)},$$

that is

$$\frac{||S(T)u^0||^2_{L^2(\Omega)}}{||u^0||^2_{L^2(\Omega)}} \leq \frac{C_0}{1+C_0}.$$

Now, using this we observe that

$$||S(T)||^2 = \sup_{||u^0||_{L^2(\Omega)} \neq 0} \frac{||S(T)u^0||^2_{L^2(\Omega)}}{||u^0||^2_{L^2(\Omega)}} \leq \frac{C_0}{1+C_0}.$$

Since, $\frac{C_0}{1+C_0} < 1$ and $T > 0$, there exists a $\gamma > 0$ such that $\frac{C_0}{1+C_0} = e^{-\gamma T}$. Taking logarithm of both sides, we get $\gamma = \frac{1}{T}\ln(1+\frac{1}{C_0})$. Hence, we now have

$$||S(T)||^2 \leq e^{-\gamma T}.$$

For any large t, there exist $k \in \mathbb{N}$ and $0 \leq \tilde{t} < T$ such that $t = kT + \tilde{t}$. From this equality, we observe that $k > \frac{t}{T} - 1$. Now, from the semigroup property we have

$$S(t) = S(kT+\tilde{t}) = S(\tilde{t})S(T)^k,$$

which implies

$$||S(t)||^2 = ||S(kT+\tilde{t})||^2 = ||S(\tilde{t})S(T)^k||^2 \leq ||S(\tilde{t})||^2||S(T)||^{2k}$$

$$\leq ||S(0)||^2||S(T)||^{2k} = ||S(T)||^{2k} \leq e^{-\gamma Tk} \leq e^{-\gamma T(\frac{t}{T}-1)} = e^{\gamma T}e^{-\gamma t} = (1+\frac{1}{C_0})e^{-\gamma t}.$$

Hence,

$$||S(t)u^0||^2 = ||u(t)||^2 \leq (1+\frac{1}{C_0})e^{-\gamma t}||u^0||^2,$$

that is

$$F(t) \leq (1+\frac{1}{C_0})F(0)e^{-\gamma t}.$$

However, as we noted in the earlier lines, for this estimate to be true, we still need to prove (3.22). To prove this, we first consider the problem

$$\begin{cases} i\varphi_t + \triangle\varphi = 0, & \text{in } \Omega \times (0,\infty), \\ \varphi = 0, & \text{on } \Gamma \times (0,\infty), \\ \varphi(0) = u^0, & \text{in } \Omega. \end{cases} \tag{3.24}$$

Then, we define the function $z := u - \varphi$ where u is the solution of the system (3.7) and φ is the solution of the system (3.24). Then z satisfies the following problem.

$$\begin{cases} iz_t + \triangle z = -ia(x)u, & \text{in } \Omega \times (0,\infty), \\ z = 0, & \text{on } \Gamma \times (0,\infty), \\ z(0) = 0, & \text{in } \Omega. \end{cases} \tag{3.25}$$

First note that from (3.23), we have

$$F(T) \leq F(0) = \frac{1}{2}||u^0||^2_{L^2(\Omega)}. \tag{3.26}$$

On $||u^0||^2_{L^2(\Omega)}$, we have the estimate given in the following proposition given in [10].

Proposition 3.3.1. *Let $\omega \subset \Omega$ be a neighborhood of $\overline{\Gamma_0}$ in Ω. Then, for $T > 0$, there is a constant $C_1 = C_1(T) > 0$ such that*

$$||u^0||^2_{L^2(\Omega)} \leq C_1 \int_0^T \int_\omega |\varphi|^2 dxdt \tag{3.27}$$

where φ is the solution of the system (3.24).

Now, using (3.26), (3.27) and (3.6) we have

$$\begin{aligned} F(T) \leq \frac{1}{2}||u^0||^2_{L^2(\Omega)} \leq \frac{C_1}{2}\int_0^T \int_\omega |\varphi|^2 dxdt \leq \frac{C_1}{2a_0}\int_0^T \int_\Omega a|\varphi|^2 dxdt \\ \leq \frac{C_1}{a_0}\int_0^T \int_\Omega a[|u|^2 + |z|^2] dxdt \leq \\ \frac{C_1}{a_0}\int_0^T \int_\Omega a|u|^2 dxdt + \frac{C_1||a||_{L^\infty(\Omega)}}{a_0}\int_0^T ||z(t)||^2_{L^2(\Omega)} dt. \end{aligned} \tag{3.28}$$

Now, we will prove a Schrödinger estimate (3.29), which is stated in [9] without proof. Multiplying (3.25) by $\bar{z}$, integrating by parts and taking the imaginary parts, we have

$$\mathrm{Im}\int_\Omega [iz_t\bar{z} + \bar{z}\triangle z]dx = -\mathrm{Im}\int_\Omega iau\bar{z}dx \Rightarrow$$

$$\mathrm{Re}\int_\Omega z_t\bar{z}dx - \mathrm{Im}\int_\Omega \nabla\bar{z}\cdot\nabla z dx + \mathrm{Im}\int_\Gamma \bar{z}\frac{\partial z}{\partial\nu}d\Gamma = -\mathrm{Re}\int_\Omega au\bar{z}dx \Rightarrow$$

$$\mathrm{Re}\int_\Omega z_t\bar{z}dx = -\mathrm{Re}\int_\Omega au\bar{z}dx.$$

Since, $(|z|^2)_t = (z\bar{z})_t = z_t\bar{z} + z\bar{z}_t = 2\mathrm{Re}(z_t\bar{z})$, that is $\mathrm{Re}(z_t\bar{z}) = \frac{1}{2}(|z|^2)_t$, we have

$$\mathrm{Re}\int_\Omega z_t\bar{z}dx = \frac{1}{2}\frac{d}{dt}\int_\Omega |z|^2dx = -\mathrm{Re}\int_\Omega au\bar{z}dx \Rightarrow$$

$$\frac{d}{dt}||z(t)||^2_{L^2(\Omega)} = -2\mathrm{Re}\int_\Omega au\bar{z}dx.$$

Integrating from 0 to t, we have

$$\int_0^t \frac{d}{ds}||z(s)||^2_{L^2(\Omega)}ds = ||z(t)||^2_{L^2(\Omega)} - ||z(0)||^2_{L^2(\Omega)} = ||z(t)||^2_{L^2(\Omega)}$$

$$= -2\mathrm{Re}\int_0^t\int_\Omega au\bar{z}dxds \le 2\big(\int_0^T\int_\Omega |au|^2dxds\big)^{1/2}\big(\int_0^T\int_\Omega |z|^2dxds\big)^{1/2}.$$

Hence, we have

$$||z(t)||^2_{L^2(\Omega)} \le 2||au||_{L^2(\Omega\times(0,T))}\big(\int_0^T ||z(s)||^2_{L^2(\Omega)}ds\big)^{1/2}$$

$$\le 2||au||_{L^2(\Omega\times(0,T))}\sqrt{T}||z||_{L^\infty(0,T;L^2(\Omega))}.$$

This is true for each t, hence we have

$$||z||^2_{L^\infty(0,T;L^2(\Omega))} \le 2||au||_{L^2(\Omega\times(0,T))}\sqrt{T}||z||_{L^\infty(0,T;L^2(\Omega))}$$

which implies

$$||z||_{L^\infty(0,T;L^2(\Omega))} \le 2\sqrt{T}||au||_{L^2(\Omega\times(0,T))} \le 2\sqrt{T}||a||_{L^\infty(\Omega)}\int_0^T\int_\Omega a|u|^2dxdt. \tag{3.29}$$

Inserting (3.29) into (3.28) we have

$$F(T) \le \frac{C_1}{a_0}\int_0^T\int_\Omega a|u|^2dxdt + \frac{2T\sqrt{T}C_1||a||^2_{L^\infty(\Omega)}}{a_0}\int_0^T\int_\Omega a|u|^2dxdt$$

$$= (\frac{C_1}{a_0} + \frac{2T\sqrt{T}C_1||a||^2_{L^\infty(\Omega)}}{a_0}) \int_0^T \int_\Omega a|u|^2 dxdt = C(T) \int_0^T \int_\Omega a|u|^2 dxdt$$

where

$$C(T) = (\frac{C_1}{a_0} + \frac{2T\sqrt{T}C_1||a||^2_{L^\infty(\Omega)}}{a_0}).$$

Hence, we have just proved (3.22), which gives us the exponential decay of the $L^2(\Omega)$-norm of the solution.

Hence, we conclude with the following theorem.

Theorem 3.3.2. *Suppose $\Omega \subset \mathbb{R}^n$ ($n \geq 1$) is a bounded domain with boundary Γ of class C^3, $x_0 \in \mathbb{R}^n$, $\omega \subset \Omega$ a neighbourhood of $\overline{\Gamma}_0$ and $a \in L^\infty(\Omega)$ satisfying (3.6). Then there exist positive constants C, γ such that*

$$F(t) \leq CF(0)e^{-\gamma t}, \forall t \in \mathbb{R}_+.$$

Chapter 4

STABILIZATION OF THE NON-LINEAR SCHRÖDINGER EQUATION

We consider the following Schrödinger equation,

$$iu_t - \triangle u + f(|u|^2)u + iau = 0 \text{ in } \Omega \times (0, \infty), \tag{4.1}$$

with initial data and (inhomogeneous) boundary condition

$$\begin{cases} u = Q & \text{on } \Gamma \times (0, \infty), \\ u(0) = u^0 & \text{in } \Omega \end{cases} \tag{4.2}$$

where a is a strictly positive constant and $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ is a bounded domain with sufficiently smooth boundary Γ and $f(|u|^2) = g|u|^p$ where $p > 0$, $g \geq 0$, $u^0 \in H^1(\Omega)$, $Q \in C^3(\Gamma \times [0, \infty))$ and $u^0(x) \equiv Q(x, 0)$ on $\partial\Omega$ in the sense of traces.

The existence of solution of the nonlinear equation (4.1)-(4.2) (without the damping term iau) have already been considered by Strauss and Bu in [11] with $g > 0$ in which case a solution exists for all $p \in (0, \infty)$ and by Bu, Tsutaya, and Zhang in [12] with the same nonlinear term, but with $g < 0$ in which case the solution exists at least for $p \in (0, \frac{2}{n}]$. There are also some earlier results due to Bu [14] and Carroll and Bu [15] considering only the corresponding $1-$dimensional equations. In addition, the corresponding nonlinear damped equation with homogenous boundary condition (i.e., $Q \equiv 0$) was considered by Tsutsumi in his paper [13]. We will first review this homogeneous case in the next section before solving the inhomogeneous boundary value problem.

However, the stabilization problem of these strictly inhomogeneous Dirichlet boundary value problems with a damping term has not been addressed as far as we know, so we will consider the stabilization problem of the damped linear and nonlinear Schrödinger equations with inhomogeneous Dirichlet boundary condition. There are a number of techniques to prove stabilization for PDEs, we will use a direct method of multipliers. See for example Zuazua and Machtyngier [9]. Our main objective is to get H^1-stabilization of solutions under the assumption that the boundary function decays in the sense of a reasonable norm.

4.1 Existence and Stabilization: Homogenous Boundary Condition

In this section we consider the damped nonlinear Schrödinger equation with zero boundary condition, i.e.,

$$\begin{cases} iu_t - \triangle u + f(|u|^2)u + iau = 0 & \text{in } \Omega \times (0,\infty), \\ u = 0 & \text{on } \Gamma \times (0,\infty), \\ u(0) = u^0 & \text{in } \Omega, \end{cases} \tag{4.3}$$

where $a > 0, \Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$. In addition, we assume the nonlinearity f is a function from the class $C^2(\mathbb{R}_+)$ with $|f(s)| \le Cs^{p/2}$ $(p > 0)$ for all $s \in \mathbb{R}_+$. Then, we have the following existence and stabilization result.

Theorem 4.1.1. *Consider $p \in (0,\infty)$ if $n = 1,2$ and $p \in (0, \frac{4}{n-2})$ if $n \ge 3$, then*
(i) If $p \in (0, \frac{4}{n})$, then for any $u^0 \in H_0^1(\Omega)$, there is a solution $u \in L^\infty([0,\infty); H_0^1(\Omega))$ to problem (4.3) with exponential decay rate

$$||u(t)||_{H_0^1(\Omega)} \le Ce^{-bt}$$

where b can be any arbitrary positive number $< a$.
(ii) If $p \ge \frac{4}{n}$, then there is a constant $M > 0$ such that if $u^0 \in H_0^1(\Omega)$ with $||u^0||_{H_0^1(\Omega)} \le C$, then there is a solution $u \in L^\infty([0,\infty); H_0^1(\Omega))$ to problem (4.3) with exponential decay rate

$$||u(t)||_{H_0^1(\Omega)} \le Ce^{-at}.$$

In the remaining part of this section, we will prove both the existence and the stabilization results. Now, let $w_j \in H_0^1(\Omega) \cap H^\infty(\Omega)$ be a complete orthonormal basis in $L^2(\Omega)$ satisfying

$$\begin{cases} -\triangle w_j = \lambda_j w_j & \text{in } \Omega, \\ w_j = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.4}$$

Let's define $u_m(t) = \sum_{j=1}^m g_{jm}(t)w_j$ where $g_{jm}(t)$ are found by the solution of the following system of ordinary differential equations.

$$\begin{cases} i(u_m'(t), w_j) + (-\triangle u_m(t), w_j) + (f(|u_m(t)|^2)u_m(t), w_j) & \\ \qquad + ia(u_m(t), w_j) = 0 & j \in (1, ..., m), \\ u_m(0) = u_{0m} = \sum_{j=1}^m g_{jm}(0)w_j & \text{on } \partial\Omega \end{cases} \tag{4.5}$$

where $u_{0m} \to u^0$ strongly in $H^1(\Omega)$.

By the theory of ODE's there is a unique solution u_m of the problem (4.5) on some interval $[0, t_m)$. Our aim is to prove convergence of the approximate solutions u_m to a solution u of the problem (4.3). To achieve this, we will first prove some uniform estimates on u_m independent of time and m. For the time being assume that u_m is valid for the whole time interval, the remaining part of this section will readily formalize this.

We multiply (4.5) by $\bar{g}_{jm}(t)$, then

$$i(u'_m(t), g_{jm}(t)w_j) + (-\triangle u_m(t), g_{jm}(t)w_j)$$

$$+(f(|u_m(t)|^2)u_m(t), g_{jm}(t)w_j) + ia(u_m(t), g_{jm}(t)w_j) = 0.$$

We sum in j,

$$\begin{aligned} &i(u'_m(t), \sum_{j=1}^{m} g_{jm}(t)w_j) + (-\triangle u_m(t), \sum_{j=1}^{m} g_{jm}(t)w_j) \\ &+(f(|u_m(t)|^2)u_m(t), \sum_{j=1}^{m} g_{jm}(t)w_j) + ia(u_m(t), \sum_{j=1}^{m} g_{jm}(t)w_j) \\ &= i(u'_m(t), u_m(t)) + (-\triangle u_m(t), u_m(t)) + (f(|u_m(t)|^2)u_m(t), u_m(t)) \\ &+ia(u_m(t), u_m(t)) = 0. \end{aligned} \tag{4.6}$$

Taking the imaginary part and using the divergence theorem, we have

$$\frac{d}{dt}||u_m(t)||^2 = -2a||u_m(t)||^2.$$

Multiplying this with e^{2at} and integrating in time, we get

$$||u_m(t)||^2 = e^{-2at}||u_{0m}||^2 \tag{4.7}$$

However taking the real part of (4.6) and again using the divergence theorem we have

$$i(u'_m(t), u_m(t)) - i(u_m(t), u'_m(t)) + 2||\nabla u||^2 + 2\int_\Omega f(|u_m(t)|^2)|u_m(t)|^2 dx = 0. \tag{4.8}$$

Multiplying (4.5) by $\bar{g}'_{jm}(t)$, we have

$$i(u'_m(t), g'_{jm}(t)w_j) + (-\triangle u_m(t), g'_{jm}(t)w_j)$$

$$+(f(|u_m(t)|^2)u_m(t), g'_{jm}(t)w_j) + ia(u_m(t), g'_{jm}(t)w_j) = 0.$$

We sum in j, hence we have

$$i(u'_m(t), \sum_{j=1}^{m} g'_{jm}(t)w_j) + (-\triangle u_m(t), \sum_{j=1}^{m} g'_{jm}(t)w_j)$$

$$+(f(|u_m(t)|^2)u_m(t), \sum_{j=1}^{m} g'_{jm}(t)w_j) + ia(u_m(t), \sum_{j=1}^{m} g'_{jm}(t)w_j)$$

$$= i(u'_m(t), u'_m(t)) + (-\triangle u_m(t), u'_m(t))$$

$$+(f(|u_m(t)|^2)u_m(t), u'_m(t)) + ia(u_m(t), u'_m(t)) = 0. \tag{4.9}$$

Taking the real part of (4.9) and using the divergence theorem, we have

$$\frac{d}{dt}(||\nabla u_m||^2 + \int_\Omega F(|u_m|^2)dx) = a(i(u'_m(t), u_m(t)) - i(u_m(t), u'_m(t)))$$

where F is the antiderivative of f, i.e.,

$$F(s) := \int_0^s f(\tau)d\tau.$$

Now, using (4.8), we have

$$\frac{d}{dt}(||\nabla u_m||^2 + \int_\Omega F(|u_m|^2)dx) = -2a(||\nabla u||^2 + \int_\Omega f(|u_m(t)|^2)|u_m(t)|^2dx).$$

Integrating this in time, we have,

$$\big(||\nabla u_m||^2_{L^2(\Omega)} + \int_\Omega F(|u_m|^2)dx\big)e^{2at} = ||\nabla u_{m0}||^2_{L^2(\Omega)} + \int_\Omega F(|u_{m0}|^2)dx$$

$$+2a\int_0^t e^{2as}\int_\Omega (F(|u_m|^2) - f(|u_m|^2)|u_m|^2)dxds. \tag{4.10}$$

Now, taking $j = 0, m = 1, r = q = 2$ in the Gagliardo-Nirenberg inequality (2.3.9) and using Poincare inequality since $u_m \in H_0^1(\Omega)$, in particular we have

$$||u_m||_{L^{p+2}(\Omega)} \leq C||\nabla u_m||^\theta ||u_m||^{1-\theta}.$$

where

$$\frac{1}{p+2} = \frac{1}{2} - \frac{\theta}{n}.$$

Now, since $F(s) = \int_0^s f(s)ds$, we get

$$|F(s)| = |\int_0^s f(s)ds| \leq \int_0^s |f(s)|ds \leq C\int_0^s s^{p/2}ds = Cs^{\frac{p+2}{2}}$$

which implies

$$|F(s^2)| \leq Cs^{p+2}.$$

Using this and (4.7) we have

$$\begin{aligned}
&|\int_\Omega F(|u_m|^2)dx|, |\int_\Omega (F(|u_m|^2) - f(|u_m|^2)|u_m|^2)dx| \\
&\leq M||u||_{p+2}^{p+2} \leq M||\nabla u_m||^{\theta(p+2)}||u_m||^{(1-\theta)(p+2)} \\
&\leq M||\nabla u_m||^{\theta(p+2)}e^{-a(1-\theta)(p+2)t}||u_{0m}||^{(1-\theta)(p+2)} \qquad (4.11)
\end{aligned}$$

Now, consider the case $p \in (0, \frac{4}{n})$. In this case, we have

$$0 < \theta = \frac{np}{2(p+2)} = \frac{n}{2(1+\frac{2}{p})} < \frac{n}{2(1+\frac{n}{2})} = \frac{n}{n+2} < 1$$

and

$$0 < \theta(p+2) = \frac{np}{2} < 2.$$

Hence, we can use Young's inequality and right hand side of (4.11) is dominated by

$$\delta||\nabla u_m||^2 + C_\delta e^{-\mu at}$$

for some $\delta \in (0,1)$ and $\mu = \frac{2((2-n)p+4)}{4-np}$. Hence, using (4.10) we have

$$\begin{aligned}
(1-\delta)||\nabla u_m||^2 e^{2at} \leq C_\delta e^{(2-\mu)at} + C_0 + 2a\delta\int_0^t ||\nabla u_m(s)||^2 e^{2as}ds \\
+2aC_\delta\int_0^t e^{(2-\mu)as}ds
\end{aligned}$$

where $C_0 = C_0(||u^0||_{H^1(\Omega)})$ is a constant with

$$C_0 \geq ||\nabla u_{m0}||_{L^2(\Omega)}^2 + |\int_\Omega F(|u_{m0}|^2)dx|$$

for all m, such a constant exists since we took the sequence u_{m0} strongly converging to u^0 in H^1-sense. Note that $2-\mu<0$, hence we have

$$||\nabla u_m||^2 e^{2at} \leq C + \frac{2a\delta}{1-\delta}\int_0^t ||\nabla u_m(s)||^2 e^{2as} ds.$$

Now, using Gronwall's inequality, we have

$$||\nabla u_m||^2 e^{2at} \leq Ce^{\frac{2a\delta}{1-\delta}t}$$

which implies

$$||\nabla u_m|| \leq Ce^{\frac{(2\delta-1)a}{1-\delta}t}.$$

Hence, taking δ so small, we have the desired stabilization result.

Now, we consider the case $p=\frac{4}{n}$. Then, we have

$$\theta(p+2)=2$$

and

$$(1-\theta)(p+2)=p.$$

Suppose that we have the fact that $||u_{m0}|| \leq (\frac{M}{2})^{1/p}$. Then, (4.11) gives

$$|\int_\Omega F(|u_m|^2)dx| \leq \frac{1}{2}||\nabla u||^2$$

and

$$|\int_\Omega (F(|u_m|^2) - f(|u_m|^2)|u_m|^2)dx| \leq \frac{1}{2}||\nabla u||^2 e^{-apt}.$$

Combining these with (4.10) we have

$$||\nabla u_m||^2 e^{2at} \leq 2C_0 + 4Ma\int_0^t e^{-aps}||\nabla u_m(s)||^2 e^{2as} ds.$$

Then, by Gronwall's inequality,

$$||\nabla u_m||^2 \leq 2C_0 e^{-2at} e^{4Ma\int_0^t e^{-aps}ds} \leq Ce^{-2at}$$

which is the desired stabilization result.

Now, we consider the case $p > \frac{4}{n}$. Then using (4.10) we have

$$||\nabla u_m(t)||^2(1 - C_2||\nabla u_m||^{\theta(p+2)-2})e^{2at}$$
$$\leq C_0 + 2aC_2 \int_0^t e^{2as}||\nabla u_m||^{\theta(p+2)}e^{-(1-\theta)(p+2)as}ds \tag{4.12}$$

where $C_2 := M||u_{m0}||^{(1-\theta)(p+2)}$ and C_0 as before. Now, assume that $||u^0||_{H^1(\Omega)}$ chosen so that

$$C_0 < \frac{np-4}{2np2^qC_2^q}\min\{n,2\}^{-q}$$

where $q = \frac{4}{np-4} = \frac{1}{\theta(p+2)-2}$. Now, we claim that for each t, we have the stabilization result

$$||\nabla u_m(t)||^2 < \frac{2np}{np-4}C_0e^{-2at}.$$

To prove this claim, let's define

$$I := \{t > 0 : ||\nabla u_m(t)||^2 < \frac{2np}{np-4}C_0e^{-2at}\}.$$

Then, $I \neq \emptyset$, because otherwise

$$||\nabla u_m(t)||^2 \geq \frac{2np}{np-4}C_0e^{-2at}$$

for all t and since $p > \frac{4}{n}$, we have $\frac{np}{np-4} > 1$ and hence $||\nabla u_m(t)||^2 \geq 2C_0$ which is a contradiction since we already know $||\nabla u_m(t)||^2 \leq C_0$. Hence, $I \neq \emptyset$, so now by the continuity of norm, we know that I is a nonempty open interval. Now, let

$$t_{max} := \sup_{t\in I}\{t\} \leq \infty.$$

Now, if $t_{max} = \infty$ we are done. If not, taking the $\frac{\theta(p+2)-2}{2}$th powers, for $[0, t_{max}]$, we have

$$||\nabla u_m(t)||^{\theta(p+2)-2} < (\frac{2np}{np-4}C_0e^{-2at})^{\theta(p+2)-2}$$
$$< (\frac{2np}{np-4}\frac{np-4}{2np2^qC_2^q}\min\{n,2\}^{-q})^{\frac{1}{q}} = \frac{1}{2C_2}. \tag{4.13}$$

Now, using this and (4.12), we have

$$||\nabla u_m(t)||^2 < 2C_0 + 4aC_2\int_0^t e^{2as}||\nabla u_m||^{\theta(p+2)}e^{-(1-\theta)(p+2)as}ds.$$

Now, we define

$$A(T) := \sup_{t\in[0,T]} ||\nabla u_m(t)||^2 e^{2at}$$

where $T \in [0, t_{max}]$.

Then, using (4.13), we have

$$A(T) \le 2C_0 + 4aC_2 A(T)^{\frac{\theta(p+2)}{2}} \int_0^T e^{-aps} ds \le 2C_0 + \frac{4}{p} C_2 A(T)^{\frac{\theta(p+2)}{2}}.$$

Since we know that, $\theta(p+2) > 1$, by Strauss's inequality we have

$$A(T) \le \frac{\frac{\theta(p+2)}{2}}{\frac{\theta(p+2)-2}{2}} C_0 = \frac{2np}{np-4} C_0.$$

This implies, $t_{max} \in I$, hence I is both open and closed, therefore $I = \mathbb{R}$, that is t_{max} must be equal to ∞. Hence, we have the desired stabilization result for the case $p > 4/n$, too.

By this stabilization result, we see that

$$||u_m||_{L^\infty([0,\infty);H_0^1(\Omega))} \le C \tag{4.14}$$

and

$$|||u_m|^p u_m||_{L^\infty([0,\infty);L^{(p+2)/(p+1)}(\Omega))} \le C. \tag{4.15}$$

Also, by (4.5), taking any $w = \sum_{j=1}^\infty a_j w_j \in H_0^1(\Omega)$ with $||w||_{H_0^1(\Omega)} = 1$, we have

$$i(u_m'(t), w) + (-\triangle u_m(t), w) + (f(|u_m(t)|^2)u_m(t), w) + ia(u_m(t), w) = 0$$

which implies

$$i(u_m'(t), w) + (\nabla u_m(t), \nabla w) + (f(|u_m(t)|^2)u_m(t), w) + ia(u_m(t), w) = 0.$$

This gives,

$$||u_m'(t)||_{H^{-1}(\Omega)} = \sup_{||w||_{H_0^1(\Omega)}=1} |i(u_m'(t), w)| \le ||\nabla u_m(t)|| + C||u_m(t)||_{L^{p+2}}^{p+2} \le C$$

by (4.14) and (4.15). Hence, we have

$$||u_m'||_{L^\infty([0,\infty);H^{-1}(\Omega))} \le C. \tag{4.16}$$

Hence, by (4.14), (4.15) and (4.16), we can assume that

$$u_m \to u \text{ weakly* in } L^\infty([0,\infty); H_0^1(\Omega)),$$

$$|u_m|^p u_m \to g \text{ weakly* in } L^\infty([0,\infty); L^{(p+2)/(p+1)}(\Omega)),$$

$$u_m' \to u \text{ weakly* in } L^\infty([0,\infty); H^{-1}(\Omega))$$

where g is indeed equal to $|u|^p u$ by Lemma 2.5.15 and Lemma 2.5.16. Hence, this concludes the existence part.

4.2 Existence: Inhomogeneous Boundary Condition

In this section, we will prove the existence of the solutions of the damped Schrodinger equation with inhomogeneous boundary condition, the proof is similar to the proof for the undamped problem given in [11]. The equation that we are concerned is formally given as follows.

$$\begin{cases} iu_t - \triangle u + f(|u|^2)u + iau = 0, & \text{in } \Omega \times (0,\infty), \\ u = Q, & \text{on } \Gamma \times (0,\infty), \\ u(0) = u^0, & \text{in } \Omega. \end{cases} \tag{4.17}$$

Here, $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ is a bounded domain with sufficiently smooth boundary Γ, $a > 0$ is a positive constant, and $f(s) = gs^{\frac{p}{2}}$ where $g \geq 0, p > 0$. Note that when $g = 0$, we have the corresponding damped linear equation and when $g > 0$ we have the corresponding nonlinear damped equation with positive sign. One can also consider the stabilization problem when the nonlinearity has negative sign, i.e., $g < 0$ problem, but we omit such a discussion.

Multiplying the equation (4.17) with $\bar{u}$, taking the imaginary parts and integrating by parts, we have

$$\text{Im}\int_\Omega [iu_t\bar{u} - \bar{u}\triangle u + f(|u|^2)u\bar{u} + iau\bar{u}]dx$$

$$= \text{Re}\int_\Omega u_t\bar{u}dx + \overbrace{\text{Im}\int_\Omega \nabla u\cdot\nabla\bar{u}dx}^{0} - \text{Im}\int_\Gamma \bar{u}\frac{\partial u}{\partial\nu}d\Gamma + \overbrace{\text{Im}\int_\Omega f(|u|^2)|u|^2dx}^{0}$$
$$+ \int_\Omega a|u|^2dx = 0.$$

Since we have

$$\text{Re}\int_\Omega u_t\bar{u}dx = \frac{1}{2}\frac{d}{dt}\int_\Omega |u|^2dx,$$

we get the L^2-identity

$$\frac{d}{dt}\int_\Omega |u|^2dx = 2\text{Im}\int_\Gamma \bar{u}\frac{\partial u}{\partial\nu}d\Gamma - 2a\int_\Omega |u|^2dx$$
$$= 2\text{Im}\int_\Gamma \bar{Q}(\nabla u\cdot\nu)d\Gamma - 2a\int_\Omega |u|^2dx,$$

that is

$$\frac{d}{dt}||u(t)||^2_{L^2(\Omega)} = -2a||u(t)||^2_{L^2(\Omega)} + 2\text{Im}\int_\Gamma \bar{Q}(\nabla u\cdot\nu)d\Gamma. \tag{4.18}$$

Multiplying the above identity with e^{2at} and integrating over time, we get the identity

$$||u||^2_{L^2(\Omega)} = e^{-2at}||u^0||^2_{L^2(\Omega)} + 2e^{-2at}\text{Im}\int_0^t e^{2as}\int_\Gamma \bar{Q}(\nabla u\cdot\nu)d\Gamma ds. \tag{4.19}$$

After multiplying the equation (4.17) with $\bar{u}$, taking the real parts and integrating over Ω, we obtain

$$\text{Re}\int_\Omega [iu_t\bar{u} - \bar{u}\triangle u + g|u|^p u\bar{u} + iau\bar{u}]dx = 0.$$

Integrating by parts we get,

$$\text{Re}\int_\Omega iu_t\bar{u}dx + \int_\Omega |\nabla u|^2 dx - \text{Re}\int_\Gamma \bar{u}\frac{\partial u}{\partial\nu}d\Gamma + \int_\Omega g|u|^{p+2}dx + \overbrace{\text{Re}\int_\Omega ia|u|^2dx}^{0} = 0.$$

Hence, we have

$$\text{Re}\int_\Omega iu_t\bar{u}dx = -\text{Re}\int_\Omega i\bar{u}_t u dx = -\int_\Omega(|\nabla u|^2 + g|u|^{p+2})dx + \text{Re}\int_\Gamma \bar{Q}(\nabla u\cdot\nu)d\Gamma. \tag{4.20}$$

Multiplying the main equation (4.17) with $\bar{u}_t$, taking the real parts and integrating by parts, we have

$$\text{Re}\int_\Omega [iu_t\bar{u}_t - \bar{u}_t\triangle u + g|u|^p u\bar{u}_t + iau\bar{u}_t]dx = 0$$

$$\Rightarrow \overbrace{\text{Re}\int_\Omega i|u_t|^2dx}^{0} + \text{Re}\int_\Omega \nabla u\cdot\nabla\bar{u}_t dx - \text{Re}\int_\Gamma \bar{u}_t\frac{\partial u}{\partial\nu}d\Gamma + \text{Re}\int_\Omega g|u|^p u\bar{u}_t dx$$
$$+a\text{Re}\int_\Omega iu\bar{u}_t dx = 0.$$

In the above, we can use the equalities

$$\text{Re}[|u|^p u\bar{u}_t] = \frac{1}{p+2}\frac{d}{dt}|u|^{p+2} \text{ and } \text{Re}[\nabla u\cdot\nabla\bar{u}_t] = \frac{1}{2}\frac{d}{dt}|\nabla u|^2.$$

Hence, using (4.20) we get the (energy) identity

$$\frac{d}{dt}\int_\Omega(|\nabla u|^2 + \frac{2g}{p+2}|u|^{p+2})dx = 2\text{Re}\int_\Gamma \bar{u}_t\frac{\partial u}{\partial\nu}d\Gamma - 2a\text{Re}\int_\Omega iu\bar{u}_t dx$$

$$= 2\mathrm{Re}\int_\Gamma \bar{Q}_t(\nabla u\cdot\nu)d\Gamma - 2a\big(\int_\Omega(|\nabla u|^2+g|u|^{p+2})dx - \mathrm{Re}\int_\Gamma \bar{Q}(\nabla u\cdot\nu)d\Gamma\big),$$

that is

$$\frac{d}{dt}(||\nabla u||^2_{L^2(\Omega)}+\int_\Omega F(|u|^2)dx)+2a\big(||\nabla u||^2_{L^2(\Omega)}+\int_\Omega F(|u|^2)dx)=$$

$$2\mathrm{Re}\int_\Gamma(\bar{Q}_t+a\bar{Q})(\nabla u\cdot\nu)d\Gamma+2a\int_\Omega(F(|u|^2)-g|u|^{p+2})dx, \tag{4.21}$$

where

$$F(|u|^2)=\frac{2g}{p+2}|u|^{p+2}.$$

Multiplying (4.21) with e^{2at} and integrating over time, we get the identity

$$\big(||\nabla u||^2_{L^2(\Omega)}+\int_\Omega F(|u|^2)dx\big)e^{2at}=||\nabla u^0||^2_{L^2(\Omega)}+\int_\Omega F(|u^0|^2)dx$$

$$+2\mathrm{Re}\int_0^t e^{2as}\int_\Gamma(\bar{Q}_t+a\bar{Q})(\nabla u\cdot\nu)d\Gamma ds+2a\int_0^t e^{2as}\int_\Omega(F(|u|^2)-g|u|^{p+2})dxds. \tag{4.22}$$

Since, Γ is of class C^∞, there exists a smooth vector field $q:\mathbb{R}^n\to\mathbb{R}^n$ such that $q|_\Gamma=\nu$, which is the outward unit normal vector field.

Now, we have the equality

$$u_t\bar{u}_{x_j}-\bar{u}_tu_{x_j}=(u\bar{u}_{x_j})_t-(u\bar{u}_t)_{x_j}.$$

Multiplying this by q_j, we obtain

$$q_j(u_t\bar{u}_{x_j}-\bar{u}_tu_{x_j})=q_j(u\bar{u}_{x_j})_t-q_j(u\bar{u}_t)_{x_j}=(q_ju\bar{u}_{x_j})_t-(q_ju\bar{u}_t)_{x_j}+(q_j)_{x_j}u\bar{u}_t.$$

Integrating this equality on Ω, we get

$$\int_\Omega q_j(u_t\bar{u}_{x_j}-\bar{u}_tu_{x_j})=\frac{d}{dt}\int_\Omega q_ju\bar{u}_{x_j}dx-\int_\Gamma\overbrace{\nu_jq_j}^{\nu_j^2}u\bar{u}_td\Gamma+\int_\Omega(q_j)_{x_j}u\bar{u}_tdx. \tag{4.23}$$

On the other hand, using the main equation, we can also write

$$q_j(u_t\bar{u}_{x_j}-\bar{u}_tu_{x_j})=q_j\{2i\mathrm{Im}(u_t\bar{u}_{x_j})\}=q_j\{2i\mathrm{Im}\big[(-i\triangle u+ig|u|^pu-au)\bar{u}_{x_j}\big]\}$$

$$= q_j\{-2i\text{Re}(\triangle u\bar{u}_{x_j}) + 2ig\text{Re}(|u|^p u\bar{u}_{x_j}) - 2ia\text{Im}(u\bar{u}_{x_j})\}$$

$$= q_j\{-2i\text{Re}\sum_{m=1}^{n}[(u_{x_m}\bar{u}_{x_j})_{x_m} - u_{x_m}\bar{u}_{x_j x_m}] + 2ig\text{Re}(|u|^p u\bar{u}_{x_j}) - 2ia\text{Im}(u\bar{u}_{x_j})\}$$

$$= -2i\text{Re}\sum_{m=1}^{n}[(q_j u_{x_m}\bar{u}_{x_j})_{x_m} - (q_j)_{x_m} u_{x_m}\bar{u}_{x_j}] + \sum_{m=1}^{n}[i(q_j|u_{x_m}|^2)_{x_j} - i(q_j)_{x_j}|u_{x_m}|^2] +$$

$$i\frac{2g}{p+2}[(q_j|u|^{p+2})_{x_j} - (q_j)_{x_j}|u|^{p+2}] - 2ia\text{Im}(u\bar{u}_{x_j}).$$

Integrating this equality over Ω, we obtain

$$\int_\Omega q_j(u_t\bar{u}_{x_j} - \bar{u}_t u_{x_j})dx = -2i\text{Re}\sum_{m=1}^{n}\int_\Gamma q_j u_{x_m}\bar{u}_{x_j}\nu_m d\Gamma + 2i\text{Re}\sum_{m=1}^{n}\int_\Omega (q_j)_{x_m} u_{x_m}\bar{u}_{x_j}dx$$
$$+i\sum_{m=1}^{n}\int_\Gamma q_j|u_{x_m}|^2\nu_j d\Gamma - i\sum_{m=1}^{n}\int_\Omega (q_j)_{x_j}|u_{x_m}|^2 dx + i\frac{2g}{p+2}\int_\Gamma q_j|u|^{p+2}\nu_j d\Gamma$$
$$-i\frac{2g}{p+2}\int_\Omega (q_j)_{x_j}|u|^{p+2}dx - 2ia\text{Im}\int_\Omega u\bar{u}_{x_j}dx. \qquad (4.24)$$

Using (4.23) and (4.24) together and adding the terms corresponding to $j = 1, ..., n$, we have

$$\frac{d}{dt}\int_\Omega u(q\cdot\nabla\bar{u})dx - \int_\Gamma Q\bar{Q}_t d\Gamma + \int_\Omega u\bar{u}_t(divq)dx$$

$$= -2i\int_\Gamma |\nu\cdot\nabla u|^2 d\Gamma + 2i\sum_{m,j=1}^{n}\int_\Omega (q_j)_{x_m} u_{x_m}\bar{u}_{x_j}dx + i\int_\Gamma |\nabla u|^2 d\Gamma$$
$$-i\int_\Omega (divq)|\nabla u|^2 dx + i\frac{2g}{p+2}\int_\Gamma |u|^{p+2}d\Gamma - i\frac{2g}{p+2}\int_\Omega (divq)|u|^{p+2}dx$$
$$-2ia\text{Im}\sum_{j=1}^{n}\int_\Omega u\bar{u}_{x_j}dx. \qquad (4.25)$$

Now, we multiply (4.17) by $(divq)\bar{u}$ and integrate on Ω, which gives

$$0 = \int_\Omega (iu_t - \triangle u + g|u|^p u + iau)\bar{u}(divq)dx =$$

$$i\int_\Omega (divq)u_t\bar{u}dx + \int_\Omega (divq)|\nabla u|^2 dx + \int_\Omega \bar{u}(\nabla(divq)\cdot\nabla u)dx$$

$$-\int_\Gamma (divq)\bar{u}\frac{\partial u}{\partial \nu}d\Gamma + \int_\Omega g(divq)|u|^{p+2}dx + ia\int_\Omega (divq)|u|^2 dx.$$

Multiplying the above identity with $-i$, taking the complex conjugate, we have

$$\int_\Omega u\bar{u}_t(divq)dx = i\int_\Gamma (\nabla\bar{u}\cdot\nu)Q(divq)d\Gamma - i\int_\Omega (divq)|\nabla u|^2 dx$$
$$-i\int_\Omega u(\nabla(divq)\cdot\nabla\bar{u})dx - i\int_\Omega g(divq)|u|^{p+2}dx - a\int_\Omega (divq)|u|^2 dx. \tag{4.26}$$

Using this in (4.25) we get the following identity

$$\frac{d}{dt}\int_\Omega u(q\cdot\nabla\bar{u})dx - \int_\Gamma Q\bar{Q}_t d\Gamma + i\int_\Gamma (\nabla\bar{u}\cdot\nu)Q(divq)d\Gamma$$

$$-i\int_\Omega u(\nabla(divq)\cdot\nabla\bar{u})dx - i\int_\Omega g(divq)|u|^{p+2}dx - a\int_\Omega (divq)|u|^2 dx$$

$$= -2i\int_\Gamma |\nu\cdot\nabla u|^2 d\Gamma + 2i\sum_{m,j=1}^n \int_\Omega (q_j)_{x_m} u_{x_m}\bar{u}_{x_j}dx + i\int_\Gamma |\nabla u|^2 d\Gamma$$
$$+i\frac{2g}{p+2}\int_\Gamma |u|^{p+2}d\Gamma - i\frac{2g}{p+2}\int_\Omega (divq)|u|^{p+2}dx - 2ia\mathrm{Im}\sum_{j=1}^n\int_\Omega u\bar{u}_{x_j}dx. \tag{4.27}$$

Now, let M_Q be a constant such that $|Q| < M_Q$. Then, let's define the following truncated version f_k of the nonlinear term f as

$$f_k(|u|^2) = \begin{cases} g|u|^p, & |u| < k \\ gk^p, & |u| \geq k \end{cases}$$

where we take $k > M_Q$. Now, consider the following truncated system

$$\begin{cases} iu_t^{(k)} - \triangle u^{(k)} + f_k(|u^{(k)}|^2)u^{(k)} + iau^{(k)} = 0, & \text{in } \Omega\times(0,\infty), \\ u^{(k)} = Q, & \text{on } \Gamma\times(0,\infty), \\ u^{(k)}(0) = u^0, & \text{in } \Omega. \end{cases} \tag{4.28}$$

It is easy to see that for any k the mapping $u \longrightarrow f_k(|u|^2)u$ is globally Lipschitz. Also since $k > M_Q$, we have $f_k(|Q|^2) = f(|Q|^2)$.
Now, we choose some $\tilde{Q} \in C^3(\bar{\Omega} \times [0, \infty))$ such that

$$\begin{cases} \tilde{Q}|_\Gamma = Q, \\ \triangle \tilde{Q}|_\Gamma = f(|Q|^2)Q + iQ_t + iaQ. \end{cases}$$

We remark that $v^{(k)} := u^{(k)} - \tilde{Q}$ converts the problem (4.28) into the following problem that has homogeneous boundary condition.

$$\begin{cases} iv_t^{(k)} - \triangle v^{(k)} + iav^{(k)} = \tilde{f}(v^{(k)}, \tilde{Q}), & \text{in } \Omega \times (0, \infty), \\ v^{(k)} = 0, & \text{on } \Gamma \times (0, \infty), \\ v^{(k)}(0) = v^0 \equiv u^0 - \tilde{Q}(0), & \text{in } \Omega, \end{cases} \tag{4.29}$$

where

$$\tilde{f}_k(v^{(k)}, \tilde{Q}) = -f_k(|v^{(k)} + \tilde{Q}|^2)(v^{(k)} + \tilde{Q}) - i\tilde{Q}_t + \triangle \tilde{Q} - ia\tilde{Q}.$$

Now, let $U_0(t) = e^{-i\triangle t}$ be the evolution operator for the free Schrodinger equation, i.e., a group of unitary operators on $H_0^1(\Omega)$. Then, we can write the system (4.29) alternatively as an integral equation as follows.

$$v^{(k)}(t) = U_0(t)v^{(k)}(0) + \int_0^t U_0(t-\tau)\tilde{f}_k(\tau)d\tau =: \mathcal{N}v^{(k)}(t)$$

Now, looking at the $H_0^1(\Omega)$ norm of the above identity on each $[0, T]$, we get the inequality

$$\begin{aligned} ||\mathcal{N}v^{(k)}(t)||_{H_0^1(\Omega)} &\leq ||v^{(k)}(0)||_{H_0^1(\Omega)} + \int_0^t ||\tilde{f}_k(v^{(k)}(s), \tilde{Q}(s))||_{H_0^1(\Omega)} ds \\ &\leq ||v^{(k)}(0)||_{H_0^1(\Omega)} + C_k \int_0^t ||v^{(k)}(s)||_{H_0^1(\Omega)} ds + \tilde{C}_{k,T}. \end{aligned} \tag{4.30}$$

We also have

$$||\mathcal{N}v^{(k)}(t) - \mathcal{N}w^{(k)}(t)||_{H_0^1(\Omega)} \leq ||v^{(k)}(0) - w^{(k)}(0)||_{H_0^1(\Omega)}$$

$$+ \int_0^t ||f_k(|v^{(k)} + \tilde{Q}|^2)(v^{(k)} + \tilde{Q}) - f_k(|w^{(k)} + \tilde{Q}|^2)(w^{(k)} + \tilde{Q})||_{H_0^1(\Omega)} ds$$

$$\leq ||v^{(k)}(0) - w^{(k)}(0)||_{H_0^1(\Omega)} + C_k \int_0^t ||v^{(k)}(s) - w^{(k)}(s)||_{H_0^1(\Omega)} ds. \tag{4.31}$$

We define the space

$$B := \{v \in C([0, T_0]; H_0^1(\Omega)) : ||v||_{C([0,T_0];H_0^1(\Omega))} \leq c^*, v(0) = u^0 - \tilde{Q}(0)\}$$

where $T_0 := \frac{1}{2}C_k$ and $c^* := 2(\tilde{C}_{k,T} + ||u^0||_{H^1(\Omega)} + ||\tilde{Q}(0)||_{H^1(\Omega)})$. Then, $\mathcal{N}$ is a contraction on B by (4.30) and (4.31). Hence, there is a unique solution $v^{(k)}$ in B to the problem (4.29) in $[0, T_0]$ which implies there is a unique solution $u^{(k)} = v^{(k)} + \tilde{Q}$ to the problem (4.28) in $[0, T_0]$.

Now, if $u^{(k)} \in C([0, T]; H^1(\Omega))$ is a solution, we want to extend it to the whole positive time interval, therefore, we need a uniform bound on $||u^{(k)}(t)||_{H^1(\Omega)}$.

First note that replacing u by $u^{(k)}$ the identities (4.22) and (4.27) take the forms

$$\begin{aligned}\big(||\nabla u^{(k)}||^2_{L^2(\Omega)} + \int_\Omega F_k(|u^{(k)}|^2)dx\big)e^{2at} &= ||\nabla u^0||^2_{L^2(\Omega)} + \int_\Omega F_k(|u^0|^2)dx \\ &+2\text{Re}\int_0^t e^{2as}\int_\Gamma (\bar{Q}_t + a\bar{Q})(\nabla u^{(k)} \cdot \nu)d\Gamma ds \\ &+2a\int_0^t e^{2as}\int_\Omega (F_k(|u^{(k)}|^2) - f_k(|u^{(k)}|^2)|u^{(k)}|^2)dxds.\end{aligned} \tag{4.32}$$

and

$$\frac{d}{dt}\int_\Omega u^{(k)}(q \cdot \nabla \bar{u}^{(k)})dx - \int_\Gamma Q\bar{Q}_t d\Gamma + i\int_\Gamma (\nabla \bar{u}^{(k)} \cdot \nu)Q(divq)d\Gamma$$

$$-i\int_\Omega u^{(k)}(\nabla(divq) \cdot \nabla \bar{u}^{(k)})dx - i\int_\Omega (divq)f_k(|u^{(k)}|^2)|u^{(k)}|^2dx - a\int_\Omega (divq)|u^{(k)}|^2dx$$

$$\begin{aligned}&= -2i\int_\Gamma |\nu \cdot \nabla u|^2 d\Gamma + 2i\sum_{m,j=1}^n \int_\Omega (q_j)_{x_m} u_{x_m}\bar{u}_{x_j}dx + i\int_\Gamma |\nabla u|^2 d\Gamma \\ &+i\frac{2g}{p+2}\int_\Gamma |u|^{p+2}d\Gamma - i\int_\Omega (divq)F_k(|u^{(k)}|^2)dx - 2ia\text{Im}\sum_{j=1}^n \int_\Omega u\bar{u}_{x_j}dx.\end{aligned} \tag{4.33}$$

Since $v^{(k)}$ is constant on the boundary Γ, we have that $\nabla v^{(k)} = \frac{\partial v^{(k)}}{\partial \nu}\nu$, i.e., $\nabla v^{(k)}$ is in the direction of the outward unit normal. Hence, the tangential component of $\nabla v^{(k)}$ on the boundary is zero. Thus, using the definition of $v^{(k)}$, we have

$$\nabla u^{(k)} \cdot A = \overbrace{\nabla v^{(k)} \cdot A}^{0} + \nabla \tilde{Q} \cdot A = \nabla \tilde{Q} \cdot A$$

where A is the unit tangential vector, so the dot product with A gives the tangential components. Hence, we can write

$$|\nabla u^{(k)}|^2 = |\nabla u^{(k)} \cdot \nu|^2 + |\nabla u^{(k)} \cdot A|^2 = |\nabla u^{(k)} \cdot \nu|^2 + |\nabla \tilde{Q} \cdot A|^2. \tag{4.34}$$

Substituting (4.34) into (4.33), integrating over time, and taking the absolute values, we get the estimate

$$\begin{aligned}\int_0^t \int_\Gamma |\nu \cdot \nabla u^{(k)}|^2 d\Gamma d\tau \leq \int_0^t \int_\Gamma |\nabla \tilde{Q} \cdot A|^2 d\Gamma d\tau + \Big| \int_\Omega u^{(k)}(q \cdot \nabla \bar{u}^{(k)}) dx \Big| \\ + \Big| \int_\Omega u^0 (q \cdot \nabla \bar{u}^0) dx \Big| + \int_0^t \int_\Gamma |Q \bar{Q}_t| d\Gamma d\tau + c \int_0^t \int_\Gamma |(\nu \cdot \nabla \bar{u^{(k)}}) Q| d\Gamma d\tau \\ + c \int_0^t \int_\Omega |u^{(k)}||\nabla u^{(k)}| dx d\tau + c \int_0^t \int_\Omega |\nabla u^{(k)}|^2 dx d\tau \\ + c \int_0^t \int_\Omega |u^{(k)}|^2 dx d\tau + c \int_0^t \int_\Gamma |Q|^{p+2} d\Gamma d\tau + c \int_0^t \int_\Omega F_k(|u^{(k)}|^2) dx d\tau.\end{aligned} \tag{4.35}$$

Hence, we have

$$\begin{aligned}\int_0^t \int_\Gamma |\nu \cdot \nabla u^{(k)}|^2 d\Gamma d\tau \leq c + c(||u^{(k)}||^2_{L^2(\Omega)} + ||\nabla u^{(k)}||^2_{L^2(\Omega)}) + \\ c \int_0^t (||u^{(k)}||^2_{L^2(\Omega)} + ||\nabla u^{(k)}||^2_{L^2(\Omega)} + F_k(|u^{(k)}|^2)) d\tau\end{aligned} \tag{4.36}$$

where c is a constant which does not depend on time and which may have different values at each place it is used.

Now, let's define

$$J^2 := \int_0^t \int_\Gamma |\nabla u^{(k)} \cdot \nu|^2 d\Gamma d\tau.$$

Then, we can write (4.36) in compact form $J^2 \le A^2$, where A^2 is the sum of all the terms in the right hand side of (4.36). Taking square root of both sides, we get

$$J \le A. \tag{4.37}$$

Now, let's define

$$G(t) := \int_\Omega (|u^{(k)}|^2 + |\nabla u^{(k)}|^2 + F_k(|u^{(k)}|^2))dx.$$

Then, we can write

$$\begin{aligned} A^2 &= c + c(||u^{(k)}||^2_{L^2(\Omega)} + ||\nabla u^{(k)}||^2_{L^2(\Omega)}) \\ &+c\int_0^t (||u^{(k)}||^2_{L^2(\Omega)} + ||\nabla u^{(k)}||^2_{L^2(\Omega)} + F_k(|u^{(k)}|^2))d\tau \\ &\le c + c(||u^{(k)}||^2_{L^2(\Omega)} + ||\nabla u^{(k)}||^2_{L^2(\Omega)} + \int_\Omega F_k(|u^{(k)}|^2)dx) \\ &+c\int_0^t (||u^{(k)}||^2_{L^2(\Omega)} + ||\nabla u^{(k)}||^2_{L^2(\Omega)} + F_k(|u^{(k)}|^2))d\tau \end{aligned}$$

that is,

$$A^2 \le c + cG + c\int_0^t G d\tau. \tag{4.38}$$

Now, from the mass identity (4.19), using Hölder's Inequality we have

$$\begin{aligned} ||u^{(k)}||^2_{L^2(\Omega)} &= e^{-2at}||u^0||^2_{L^2(\Omega)} + 2e^{-2at}\mathrm{Im}\int_0^t e^{2a\tau}\int_\Gamma \bar{Q}(\nabla u^{(k)}\cdot \nu)d\Gamma d\tau \\ &\le c + ce^{-2at}\big(\int_0^t e^{4a\tau}||Q||^2_{L^2(\Gamma)}d\tau\big)^{\frac{1}{2}}\big(\int_0^t\int_\Gamma |\nabla u^{(k)}\cdot\nu|^2 d\Gamma d\tau\big)^{\frac{1}{2}}, \end{aligned}$$

that is,

$$||u^{(k)}||^2_{L^2(\Omega)} \le c + cJ. \tag{4.39}$$

Now, using the energy identity (4.32) and Hölder's Inequality, we have

$$\begin{aligned} ||\nabla u^{(k)}||^2_{L^2(\Omega)} + \int_\Omega F_k(|u^{(k)}|^2)dx &= ||\nabla u^0||^2_{L^2(\Omega)}e^{-2at} + e^{-2at}\int_\Omega F_k(|u^0|^2)dx \\ &+2e^{-2at}\mathrm{Re}\int_0^t e^{2a\tau}\int_\Gamma (\bar{Q}_t + a\bar{Q})(\nabla u^{(k)}\cdot\nu)d\Gamma d\tau \\ &+2ae^{-2at}\int_0^t e^{2a\tau}\int_\Omega (F_k(|u^{(k)}|^2) - f_k(|u^{(k)}|^2)|u^{(k)}|^2)dxd\tau \end{aligned}$$

$$\leq c + ce^{-2at}\Big(\int_0^t e^{4a\tau}(||Q||^2_{L^2(\Gamma)} + ||Q_t||^2_{L^2(\Gamma)})d\tau\Big)^{\frac{1}{2}}\Big(\int_0^t\int_\Gamma |\nabla u^{(k)}\cdot\nu|^2 d\Gamma d\tau\Big)^{\frac{1}{2}}$$
$$+ce^{-2at}\int_0^t e^{2a\tau}\int_\Omega F_k(|u^{(k)}|^2)dxd\tau \leq c + cJ + c\int_0^t Gd\tau,$$

that is

$$||\nabla u^{(k)}||^2_{L^2(\Omega)} + \int_\Omega F_k(|u^{(k)}|^2)dx \leq c + cJ + c\int_0^t Gd\tau. \tag{4.40}$$

Summing (4.39) and (4.40), we obtain

$$G = ||u^{(k)}||^2_{L^2(\Omega)} + ||\nabla u^{(k)}||^2_{L^2(\Omega)} + \int_\Omega F_k(|u^{(k)}|^2)dx \leq c + cJ + c\int_0^t Gd\tau. \tag{4.41}$$

We observe that, (4.37), (4.38), the inequality (2.5) and Young's Inequality imply that

$$J \leq A \leq c + c\sqrt{G} + c\sqrt{\int_0^t Gd\tau} \leq c + c\sqrt{G} + c\int_0^t Gd\tau \tag{4.42}$$

Inserting (4.42) into (4.41) and using Young's Inequality on $c\sqrt{G}$, we obtain

$$G \leq c + c\sqrt{G} + c\int_0^t Gd\tau \Rightarrow G \leq c + c\int_0^t Gd\tau. \tag{4.43}$$

Now using Gronwall's Inequality in (4.43), we have $G \leq e^{ct}$ and hence $J \leq e^{ct}$. Now, (4.40) and positivity of F_k implies $||u^{(k)}(t)||_{H^1(\Omega)}$ is bounded for bounded T. Hence, now we can extend $u^{(k)}$ to $[0,\infty)$ so that $u^{(k)}$ will be in $C([0,T];H^1(\Omega))$ for each T and for some constant C_T, we will have

$$\sup_{t\in[0,T]} ||u^{(k)}(t)||_{H^1(\Omega)} \leq C_T.$$

Since, C_T is independent of k, for $T = 1$ we have a weakly* convergent subsequence $u_1^{(k)}$ of $u^{(k)}$ in $L^\infty([0,1];H^1(\Omega))$, and similarly a weakly* convergent subsequence $u_2^{(k)}$ of $u_1^{(k)}$ in $L^\infty([0,2];H^1(\Omega))$, etc. Hence, choosing the diagonal sequence $u_k^{(k)}$, we have a function $u \in L^\infty_{loc}([0,\infty];H^1(\Omega))$ such that $u_k^{(k)}$ converges weakly* to u in $L^\infty([0,T];H^1(\Omega))$ for each T, call this again $u^{(k)}$. Now, boundedness of $\int_\Omega F_k(|u^{(k)}|^2)dx$ implies boundedness of $f_k(|u^{(k)}|^2)u^{(k)}$ in $L^\infty([0,T];L^1+L^2)$. Since,

$u_t^{(k)} = -i\triangle u^{(k)} + if_k(|u^{(k)}|^2)u^{(k)}$, $u_t^{(k)} \in L^\infty([0,T]; L^1 + H^{-1})$. Using relative compactness lemma and Cantor's diagonalization again, there is a subsequence again called $u^{(k)}$ without loss of generality, that converges to u almost everywhere on $\Omega \times [0,\infty)$. Hence, we also have $f_k(|u^{(k)}|^2)u^{(k)}$ converges a.e. to $f(|u|^2)u$. Again boundedness of $\int_\Omega F_k(|u^{(k)}|^2)dx$ and Egoroff's Theorem imply $f_k(|u^{(k)}|^2)u^{(k)}$ converges to $f(|u|^2)u$ uniformly in $L^1(\Omega')$ for any bounded $\Omega' \subset \Omega \times [0,\infty)$. Therefore, we conclude that u is a solution and from the class $L^\infty_{loc}((0,\infty]; H^1(\Omega) \cap L^{p+2}(\Omega))$.

Hence, we conclude with the following theorem

Theorem 4.2.1. *There exists a solution u of the system given in (4.17) from the class $L^\infty_{loc}([0,\infty); H^1(\Omega) \cap L^{p+2}(\Omega))$ for all $p \in (0,\infty)$.*

4.3 Stabilization: Inhomogeneous Boundary Condition

In this section, we will prove the H^1-stabilization of the damped nonlinear equation with inhomogeneous boundary condition for which we have proved the existence result in Section 4.2.

Let's first define the following multiplier function

$$\rho(t) := \mathrm{Im} \int_\Omega u(x,t)h(x) \cdot \nabla \bar{u}(x,t)dx.$$

where $h(x)$ is a sufficiently smooth real vector field and $H(x)$ be the $n \times n$ matrix with entries $H_{ij} = \frac{\partial h_i}{\partial x_j}$.

Differentiating $\rho(t)$ gives

$$\rho'(t) = \mathrm{Im} \int_\Omega u_t h \cdot \nabla \bar{u} dx + \mathrm{Im} \int_\Omega uh \cdot \nabla \bar{u}_t dx.$$

Using the divergence theorem on the vector function $u\bar{u}_t h$ we have

$$\mathrm{Im} \int_\Omega \mathrm{div}(u\bar{u}_t h)dx = \mathrm{Im} \int_\Gamma u\bar{u}_t(h \cdot \nu)d\Gamma.$$

Since,

$$\mathrm{div}(u\bar{u}_t h) = \Sigma_{i=1}^n (u\bar{u}_t h)_{x_i} = \Sigma_{i=1}^n u_{x_i}\bar{u}_t h + \Sigma_{i=1}^n u(\bar{u}_t)_{x_i} h + \Sigma_{i=1}^n u\bar{u}_t h_{x_i}$$

$$= (\nabla u \cdot h)\bar{u}_t + (\nabla \bar{u}_t \cdot h)u + div(h)u\bar{u}_t,$$

we have the equality

$$\mathrm{Im} \int_\Omega uh \cdot \nabla \bar{u}_t dx = \mathrm{Im} \int_\Gamma (h \cdot \nu)u\bar{u}_t d\Gamma - \mathrm{Im} \int_\Omega h \cdot \nabla u\bar{u}_t dx - \mathrm{Im} \int_\Omega div(h)u\bar{u}_t dx.$$

Now using (4.17), we get

$$-\mathrm{Im} \int_\Omega div(h)u\bar{u}_t dx = \mathrm{Im} \int_\Omega div(h)\bar{u}u_t dx = -\mathrm{Re} \int_\Omega div(h)\bar{u}(iu_t)dx$$

$$= -\mathrm{Re} \int_\Omega div(h)\bar{u}(\triangle u - f(|u|^2)u - iau)dx$$

$$= -\mathrm{Re} \int_\Omega div(h)\bar{u}\triangle u dx + \int_\Omega div(h)f(|u|^2)|u|^2 dx$$

$$= \text{Re}\int_\Omega \nabla(div(h)\bar{u})\cdot\nabla u dx + \int_\Omega div(h)f(|u|^2)|u|^2dx - \text{Re}\int_\Gamma (div(h)\bar{u})(\nabla u\cdot\nu)d\Gamma$$

$$= \int_\Omega div(h)|\nabla u|^2dx + \text{Re}\int_\Omega \bar{u}\nabla(div(h))\cdot\nabla u dx$$

$$+ \int_\Omega div(h)f(|u|^2)|u|^2dx - \text{Re}\int_\Gamma (div(h)\bar{u})(\nabla u\cdot\nu)d\Gamma$$

We also have

$$-\text{Im}\int_\Omega h\cdot\nabla u\bar{u}_tdx = \text{Im}\int_\Omega h\cdot\nabla\bar{u}u_tdx = -\text{Re}\int_\Omega h\cdot\nabla\bar{u}iu_tdx$$

$$-\text{Re}\int_\Omega h\cdot\nabla\bar{u}(\triangle u - f(|u|^2)u - iau)dx$$

$$= -\text{Re}\int_\Omega h\cdot\nabla\bar{u}\triangle udx + \text{Re}\int_\Omega (h\cdot\nabla\bar{u})f(|u|^2)udx + \text{Re}\int_\Omega iau(h\cdot\nabla\bar{u})dx$$

Hence, combining all above we have

$$\rho'(t) = -2\text{Re}\int_\Omega h\cdot\nabla\bar{u}\triangle udx + 2\text{Re}\int_\Omega iau(h\cdot\nabla\bar{u})dx + 2\text{Re}\int_\Omega (h\cdot\nabla\bar{u})f(|u|^2)udx$$

$$+ \int_\Omega div(h)|\nabla u|^2dx + \text{Re}\int_\Omega \bar{u}\nabla(div(h))\cdot\nabla udx + \int_\Omega div(h)f(|u|^2)|u|^2dx$$

$$-\text{Re}\int_\Gamma (div(h)\bar{u})(\nabla u\cdot\nu)d\Gamma + \text{Im}\int_\Gamma (h\cdot\nu)u\bar{u}_td\Gamma.$$

Now, note that we have

$$-2\text{Re}\int_\Omega h\cdot\nabla\bar{u}\triangle udx = 2\text{Re}\int_\Omega \nabla u\cdot\nabla(h\cdot\nabla\bar{u})dx - 2\text{Re}\int_\Gamma (\nabla\bar{u}\cdot h)(\nabla u\cdot v)d\Gamma$$

$$= 2\text{Re}\int_\Omega (H\nabla u)\cdot\nabla\bar{u}dx + \int_\Omega h\cdot\nabla(|\nabla u|^2)dx - 2\text{Re}\int_\Gamma (\nabla\bar{u}\cdot h)(\nabla u\cdot v)d\Gamma$$

$$= 2\text{Re}\int_\Omega (H\nabla u)\cdot\nabla\bar{u}dx - \int_\Omega div(h)|\nabla u|^2dx + \int_\Gamma (h\cdot\nu)|\nabla u|^2d\Gamma$$

$$-2\text{Re}\int_\Gamma (\nabla\bar{u}\cdot h)(\nabla u\cdot v)d\Gamma.$$

Note that we also have,

$$2\text{Re}\int_\Omega (h\cdot\nabla\bar{u})f(|u|^2)udx = \int_\Gamma F(|Q|^2)h\cdot\nu d\Gamma - \int_\Omega div(h)F(|u|^2)dx$$

Hence, rewriting $\rho'(t)$, we have

$$\rho'(t) = 2\text{Re}\int_\Omega (H\nabla u)\cdot\nabla\bar{u}dx + \int_\Gamma (h\cdot\nu)|\nabla u|^2d\Gamma - 2\text{Re}\int_\Gamma (\nabla\bar{u}\cdot h)(\nabla u\cdot v)d\Gamma$$

$$+2\mathrm{Re}\int_\Omega iau(h\cdot\nabla\bar{u})dx+\mathrm{Re}\int_\Omega \bar{u}\nabla(div(h))\cdot\nabla udx-\mathrm{Re}\int_\Gamma(div(h)\bar{u})(\nabla u\cdot\nu)d\Gamma$$

$$+\mathrm{Im}\int_\Gamma(h\cdot\nu)u\bar{u}_td\Gamma+\int_\Gamma F(|Q|^2)h\cdot\nu d\Gamma-\int_\Omega div(h)(F(|u|^2)-f(|u|^2)|u|^2)dx.$$

Let's define

$$||Q(t)||_b^2:=||Q(t)||^2_{H^1(\Gamma)}+||Q_t(t)||^2_{L^2(\Gamma)}+||Q(t)||^{p+2}_{L^{p+2}(\Gamma)}.$$

Now, let's define

$$G(t):=||u||^2_{H^1(\Omega)}+\int_\Omega F(|u|^2)dx.$$

Then by (4.18) and (4.21), we have

$$G'(t)=-2aG(t)+2a\int_\Omega(F(|u|^2)-f(|u|^2)|u|^2)dx$$

$$+2\mathrm{Re}\int_\Gamma((a-i)\bar{Q}+\bar{Q}_t)(\nabla u\cdot\nu)d\Gamma\le -2aG(t)+2\mathrm{Re}\int_\Gamma((a-i)\bar{Q}+\bar{Q}_t)(\nabla u\cdot\nu)d\Gamma,$$

since

$$F(|u|^2)-f(|u|^2)|u|^2<0.$$

Hence, we have

$$G'(t)\le -2aG(t)+2C_0(||Q(t)||_{L^2(\Gamma)}+||Q_t(t)||_{L^2(\Gamma)})||\nabla u\cdot\nu||_{L^2(\Gamma)}$$

$$\le -2aG(t)+\frac{8C_0^2}{\epsilon}(||Q(t)||^2_{L^2(\Gamma)}+||Q_t(t)||^2_{L^2(\Gamma)})+\frac{\epsilon}{4}||\nabla u\cdot\nu||^2_{L^2(\Gamma)}$$

$$\le -2aG(t)+\frac{8C_0^2}{\epsilon}||Q(t)||_b^2+\frac{\epsilon}{4}||\nabla u\cdot\nu||^2_{L^2(\Gamma)}$$

where $\epsilon>0$ is a constant to be chosen later.

Note that we have

$$|\rho(t)|\le ||u(t)||_{L^2(\Omega)}||h\cdot\nabla u(t)||_{L^2(\Omega)}\le ||u(t)||_{L^2(\Omega)}||h||_{L^\infty(\Omega)}||\nabla u(t)||_{L^2(\Omega)}$$

$$\le\frac{1}{2}(||u(t)||^2_{L^2(\Omega)}+||h||^2_{L^\infty(\Omega)}||\nabla u(t)||^2_{L^2(\Omega)})$$

$$\le\frac{1}{2}(1+||h||^2_{L^\infty(\Omega)})(||u(t)||^2_{L^2(\Omega)}+||\nabla u(t)||^2_{L^2(\Omega)}).$$

Hence, we have

$$|\rho(t)|\le C_1G(t) \tag{4.44}$$

where $C_1 := \frac{1}{2}(1 + ||h||^2_{L^\infty(\Omega)})$.

Then, choosing h so that $h|_\Gamma = \nu$ we have the following estimate on $\rho'(t)$,

$$\rho'(t) \leq C_2||u(t)||^2_{H^1(\Omega)}$$

$$+||\nabla u||^2_{L^2(\Gamma)} - 2||\nabla u \cdot \nu||^2_{L^2(\Gamma)} + C_3||Q(t)||_{L^2(\Gamma)}||\nabla u \cdot \nu||_{L^2(\Gamma)}$$

$$+||Q(t)||_{L^2(\Gamma)}||Q_t(t)||_{L^2(\Gamma)} + \int_\Gamma F(|Q|^2)d\Gamma + C_4\int_\Omega |F(|u|^2) - f(|u|^2)|u|^2|dx$$

Now, suppose that $\tilde{Q}(x,t)$ be such that $\tilde{Q}|_\Gamma = Q$. We remark that $v := u - \tilde{Q}$ converts the problem (4.17) into the following problem that has homogeneous boundary condition.

$$\begin{cases} iv_t - \triangle v + iav = \tilde{f}(v, \tilde{Q}), & \text{in } \Omega \times (0,\infty), \\ v = 0, & \text{on } \Gamma \times (0,\infty), \\ v(0) = u^0 - \tilde{Q}(0), & \text{in } \Omega, \end{cases} \tag{4.45}$$

where

$$\tilde{f}(v, \tilde{Q}) = -f(|v + \tilde{Q}|^2)(v + \tilde{Q}) - i\tilde{Q}_t + \triangle\tilde{Q} - ia\tilde{Q}.$$

Since v is constant on the boundary Γ, we have that $\nabla v = \frac{\partial v}{\partial \nu}\nu$, i.e., ∇v is in the direction of the outward unit normal. Hence, the tangential component of ∇v on the boundary is zero. Thus, using the definition of v, we have

$$\nabla u \cdot A = \overbrace{\nabla v \cdot A}^{0} + \nabla\tilde{Q} \cdot A = \nabla\tilde{Q} \cdot A$$

where A is the unit tangential vector, so the dot product with A gives the tangential components. Hence, we can write

$$|\nabla u|^2 = |\nabla u \cdot \nu|^2 + |\nabla u \cdot A|^2 = |\nabla u \cdot \nu|^2 + |\nabla\tilde{Q} \cdot A|^2 = |\nabla u \cdot \nu|^2 + |\nabla_A Q|^2. \tag{4.46}$$

on the boundary.

Using (4.46) into the estimate above we derived for $\rho'(t)$, we obtain,

$$\rho'(t) \leq C_2||u(t)||^2_{H^1(\Omega)}$$

$$-||\nabla u \cdot \nu||^2_{L^2(\Gamma)} + ||\nabla_A Q(t)||^2_{L^2(\Gamma)} + C_3||Q(t)||_{L^2(\Gamma)}||\nabla u \cdot \nu||_{L^2(\Gamma)}$$

$$+||Q(t)||_{L^2(\Gamma)}||Q_t(t)||_{L^2(\Gamma)} + \int_\Gamma F(|Q|^2)d\Gamma + C_4 \int_\Omega |F(|u|^2) - f(|u|^2)|u|^2|dx.$$

Hence, we have

$$G'(t) + \epsilon\rho'(t) \leq (C_5\epsilon - 2a)G(t) - \frac{\epsilon}{4}||\nabla u \cdot \nu||^2_{L^2(\Gamma)}$$

$$+(\epsilon + \frac{C_3^2\epsilon}{2} + 8C_0^2)||Q(t)||_b^2.$$

Hence, choosing a fixed $\epsilon \in (0, \frac{2a}{C_5})$, we have

$$G'_\epsilon(t) < -C_6G(t) + C_7||Q(t)||_b^2.$$

Since, we have $|\rho'(t)| \leq C_1G(t)$

$$G_\epsilon(t) = G(t) + \epsilon\rho(t) \leq G(t) + \epsilon C_1G(t) = (1 + \epsilon C_1)G(t)$$

which implies

$$\frac{-C_6}{1 + \epsilon C_1}G_\epsilon(t) \geq -C_6G(t). \tag{4.47}$$

Therefore,

$$G'_\epsilon(t) \leq -C_8G_\epsilon(t) + C_7||Q(t)||_b^2$$

where $C_8 = \frac{-C_6}{1+\epsilon C_1}$. This implies

$$G_\epsilon(t) \leq G_\epsilon(0)e^{-C_8t} + C_7e^{-C_8t}\int_0^t e^{C_8s}||Q(s)||_b^2ds.$$

Replacing $G_\epsilon(t)$ with $G(t) + \epsilon\rho(t)$ and $G_\epsilon(0)$ with $G(0) + \epsilon\rho(0)$ and using (4.44), for ϵ chosen sufficiently small we get

$$(1 - \epsilon C_1)G(t) \leq (1 + \epsilon C_1)G(0)e^{-C_8t} + C_5e^{-C_8t}\int_0^t e^{C_8t}||Q(s)||_b^2ds$$

$$\Rightarrow G(t) \leq \frac{1 + \epsilon C_1}{1 - \epsilon C_1}G(0)e^{-C_8t} + \frac{C_7}{1 - \epsilon C_1}e^{-C_8t}\int_0^t e^{C_8s}||Q(s)||_b^2ds$$

$$\Rightarrow G(t) \leq G(0)e^{-C_8t} + C_7e^{-C_8t}\int_0^t e^{C_8s}||Q(s)||_b^2ds).$$

Hence, we conclude with the following theorem.

Theorem 4.3.1. *Let u be a solution of the system given in (4.17) and assume that* $\lim_{t\to\infty}||Q(t)||_b = 0$*, then* $\lim_{t\to\infty}||u(t)||_{(H^1(\Omega)\cap L^{p+2}(\Omega))} = 0$.

Note that this theorem implies

$$\lim_{t\to\infty} ||\nabla u(t)||_{L^2(\Omega)} = 0$$

which implies the decay of the energy of the solutions.

Remark 4.3.2. ***(A remark on the rate of decay)*** *Note, that in the above system, the decay rate of b-norm of Q plays a fundamental role in the decay rate of the energy, because for example if Q decays exponentially, we get exponential stabilization, or if Q decays polynomially, we get at least a polynomial stabilization. However, if the decay rate of Q is faster than exponential such as super exponential, this does not necessarily make energy to decay faster than exponential, because of the exponential term $G(0)e^{-C_8 t}$.*

4.4 Open Problems

In this section we briefly list some open problems based on the analysis we do in the previous sections of this study. These problems might be of interest for further research.

Stabilization under Negative Nonlinearity

In Section 4.3, we prove that the energy of solutions of the weakly damped nonlinear Schrödinger equation with inhomogeneous boundary condition decays to zero as time goes to infinity under the assumption that the boundary condition decays to zero in a reasonable sense. However, one of the assumptions we make is the sign of the nonlinearity in the equation. We assume that the nonlinearity has positive sign, that is the equation is attractive. However, there are also physical situations which yield the same equation with negative signed nonlinearity, that is the equation is repulsive. Therefore it is also interesting to consider the stabilization problem for the same equation with negative signed nonlinearity. The mathematical motivations to consider this problem can be the decay of solutions of the weakly damped nonlinear Schrödinger equation with homogeneous boundary condition and the existence result for the Schrödinger equation with inhomogeneous boundary condition with negative signed nonlinearity which is done in [12] for $p \leq \frac{2}{n}$.

Stabilization with Localized Damping

In Section 3.3, we prove that the solutions of the linear Schrödinger equation decay to zero in L^2-sense in the case the equation contains a linear damping term which is supported only a small neighborhood of the boundary. Therefore, a natural question is also to ask for a similar stabilization result for the nonlinear equation with some localized damping, even in the case of homogeneous boundary condition.

Stabilization with Less Smooth Boundary Condition

Note that in Section 4.3, another assumption that we make is the smoothness of the boundary condition. We assume that the boundary condition Q is from the class of functions $C^3(\partial\Omega \times (0,\infty))$. Hence, a new question we can ask is the stabilization problem with a boundary condition at a lower regularity, for example one can consider the case where $Q \in H^1$.

Stabilization with Nonlinear Damping

In Section 4.3, we consider the equation with a weak linear damping. However, there are also some physical situations where the equation is driven by a nonlinear damping instead of a linear damping. Hence, another question that we might consider for further analysis can be the stabilization result for the nonlinear Schrödinger equation with nonlinear damping.

BIBLIOGRAPHY

[1] Arendt, W., *Semigroups Generated by Elliptic Operators*, ISEM, 1999/2000.

[2] Aubin, J. P., Frankowska H., *Set Valued Analysis*, Systems & Control: Foundations & Applications, 2. Birkhuser Boston, Inc., Boston, MA, 1990.

[3] Friedman, A., *Partial Differential Equations*, Holt, Rinehart and Winston, Inc., New York-Montreal, Que.-London, 1969.

[4] Kurdila, A., Zabarankin M., *Convex Functional Analysis*, Systems & Control: Foundations & Applications. Birkhuser Verlag, Basel, 2005.

[5] Phelps, R. R., *Lectures on Maximal Monotone Operators.*

[6] Raymond, J. P., *Control and Homegenization Notes.*

[7] Evans, L. C., *Partial Differential Equations*, Graduate Studies in Mathematics, 19. American Mathematical Society, Providence, RI, 1998.

[8] Komornik, V., Zuazua E., *A Direct Method For Boundary Stabilization of the Wave Equation*, J. Math. Pures Appl. (9) 69 (1990), no. 1, 33–54.

[9] Machtyngier E., Zuazua E., *Stabilization of the Schrödinger Equation*, Portugaliae Mathematica 51 (1994), no. 2, 243–256.

[10] Machtyngier E., *Exact Controllability For Schrödinger Equation*, SIAM J. Control Optim. 32 (1994), no. 1, 24–34.

[11] Strauss, W., Bu, C. *An Inhomogeneous Boundary Value Problem for Nonlinear Schrödinger Equations*, J. Differential Equations 173 (2001), no. 1, 79–91.

[12] Bu, C., Tsutaya, K., Zhang, C. *Nonlinear Schrödinger Equation with Inhomogeneous Dirichlet Boundary Data*, J. Math. Phys. 46 (2005), no. 8, 083504, 6 pp.

[13] Tsutsumi, M. *On Global Solutions to the Initial Boundary Value Problem for the Damped Nonlinear Schrödinger Equations*, J. Math. Anal. Appl. 145 (1990), no. 2, 328–341.

[14] Bu, C. *Nonlinear Schrödinger Equation on the Semi-infinite Line*, Chinese J. Contemp. Math. 21 (2000), no. 3, 209–222.

[15] Carroll, R., Bu, C. *Solution of the Forced Nonlinear Schrödinger Equation (NLS) using PDE techniques*, Appl. Anal. 41 (1991), no. 1-4, 33–51.

[16] Revuz, D., Yor, M. *Continuous martingales and Brownian motion*, Fundamental Principles of Mathematical Sciences, 293. Springer-Verlag, Berlin, 1999.

[17] Strauss, W. *Decay and Asymptotics for* $\Box u = F(u)$, Journal of Funct. Anal. 2 (1968), 409-457.

[18] Lions, J. L. "Quelques methodes de resolution des problemes aux limites non lineaires." Dunod, Paris, 1969.

[19] Cazenave, T., Haraux A. *"An Introduction to Semilinear Evolution Equations."* Oxford University Press, New York, 1998.

[20] Cazenave, T. *"Semilinear Schrödinger Equations."* Courant Lecture Notes 10, 2003.

[21] Zheng, S. *"Nonlinear parabolic equations and hyperbolic-parabolic coupled systems."* Pitman Monographs and Surveys in Pure and Applied Mathematics, 76. Longman Scientific & Technical, Harlow; copublished in the United States with John Wiley & Sons, Inc., New York, 1995.

[22] Hunter, J., Nachtergaele, B. *"Applied analysis."* World Scientific Publishing Co., Inc., River Edge, NJ, 2001.

[23] Reddy, B. D. *"Introductory functional analysis: with applications to boundary value problems and finite elements."* Texts in Applied Mathematics, 27. Springer-Verlag, New York, 1998.

[24] Nicholson, D. R., Goldman, M. V. *"Damped nonlinear Schrödinger equation."* Phys. Fluids 19 (1976), no. 10, 1621–1625.

[25] Pereira, N. R. *"Soliton in the damped nonlinear Schrödinger equation."* Phys. Fluids 20 (1977), no. 10, part 1, 1735–1743.

VITA

Türker Özsarı was born in Aksaray, Turkey, on 24 June 1981, the son of Necla Özsarı and Çetin Özsarı. After completing his work at Süleyman Demirel Anatolian High School in June 1999, he went on to the Koç University where he studied mathematics and computer engineering and he received his Bachelor of Science degrees in both fields in June 2004. From September 2004 to June 2006, he worked as a teaching assistant in Koç University, Turkey and had completed his masters degree in this university in the field of mathematics under the supervision of Prof. Varga K. Kalantarov. Then, he started to his PhD studies in the Mathematics Department of University of Virginia under the supervision of Prof. Irena Lasiecka. He is still a PhD student and teaching assistant in the University of Virginia as of January 2007.