WEAK COLORINGS OF STEINER TRIPLE SYSTEMS

by

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This is to certify that I have examined this copy of a master's thesis by

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and have found that it is complete and satisfactory in all respects,
and that any and all revisions required by the final
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To my parents

ABSTRACT

A Steiner triple system (STS) is an ordered pair $(\mathcal{S}, T)$ where $\mathcal{S}$ denotes a set of points and T denotes a set of three element subsets of $\mathcal{S}$ called triples with the property that any pair of elements of $\mathcal{S}$ is a subset of exactly one triple. Let C be a set of colors. A (weak) k-coloring of a $STS(v)$ is a map $\Phi : \mathcal{S} \to C$ such that $|\{\Phi(x), \Phi(y), \Phi(z)\}| \geq 2$ for every triple $\{x, y, z\} \in T$. A Steiner triple system is k-chromatic if it admits a k-coloring but not a $(k-1)$-coloring. In this case we say that the STS has chromatic number k and we write $Chr(\mathcal{S}) = k$. This thesis is a survey on colorings of Steiner triple systems.

ÖZETÇE

Bir Steiner üçlü sistemi ($S\ddot{U}S$), $(\mathcal{S}, T)$ şeklinde bir sıralı ikili ile ifade edilebilir, öyleki, bu ifadede $\mathcal{S}$ sistemin elemanlar kümesini, T bu elemanlar kümesinin 3 elemanlı bazı altkümelerinden oluşan bir kümeyi temsil eder. T kümesi üzerindeki koşul, $\mathcal{S}$'ye ait her eleman çiftinin birlikte T'nin elemanlarından yalnız ve ancak birinde yer almasıdır. C kümesini renkler kümesi olarak kabul ettiğimizde, köşe renklendirmesi, $\mathcal{S}$ kümesinden C kümesine ($\Phi : \mathcal{S} \rightarrow C$) tanımlanmış bir fonksiyondur, öyle ki, her $\{x,\, y,\, z\} \in T$ için, $|\Phi(x),\, \Phi(y),\, \Phi(z)| \geq 2$ olsun. Bir Steiner üçlü sistemi, hiç bir üçlü tek renkle boyanmayacak şekilde k renkle renklendirilip daha az renkle renklendirilemediğinde sisteme, k-kromatik diyoruz. Bu tez, bu konuda daha önce yapılmış çalışmalar üstüne bir inceleme niteliğindedir.

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NOMENCLATURE

STS	Steiner triple system
$S^{\star}(2^n - 1)$	Totally associative Steiner triple system of order $2^n - 1$
F_n	Field of n elements
$\mathbb{A}G(n, k)$	Affine geometry of dimension n over the field F_k
$\mathbb{P}G(n, k)$	Projective geometry of dimension n over the field F_k
$Chr(\mathcal{S})$	Chromatic number of a Steiner triple system $\mathcal{S}$
K_n	Complete graph with n vertices
F_3^n	Vector space over F_3 of dimension n

Chapter 1

PRELIMINARIES AND DEFINITIONS

1.1 Basic definitions

Definition 1.1.1 *A* ***binary operation*** $*$ *on a set* G *is a function* $* : G \times G \to G$*. For any* $a, b \in G$ *we shall write* $a * b$ *for* $*(a, b)$*.*

Definition 1.1.2 *If* $*$ *is a binary operation on a set* G *we say elements* a *and* b *of* G ***commutes*** *if* $a * b = b * a$*. We say* $*$ *(or* G*) is* ***commutative*** *if for all* $a, b \in G$ $a * b = b * a$*.*

Definition 1.1.3 *For a map* $f : A \to B$*, for a subset* A' *of* A*, the* ***induced map*** $f_{A'} : A' \to B$ *is defined as follows: for any* $a \in A'$*,* $f_{A'}(a) = f(a)$*.*

Definition 1.1.4 *A* ***group*** *is an ordered pair* $(G, *)$ *where* G *is a non-empty set and* $*$ *is a binary operation on* G *satisfying the following axioms:*

(i) For all a*,* b*,* $c \in G$*,* $(a * b) * c = a * (b * c)$*, that is,* $*$ *is associative.*

(ii) There exists an element $e \in G$*, called an* ***identity*** *of* G *such that for all* $g \in G$ *we have* $e * g = g * e = g$*.*

(iii) For each $g \in G$ *there is an element* g^{-1} *of* G *called an* ***inverse*** *of* g *such that* $g^{-1} * g = g * g^{-1} = e$*.*

The group $(G, *)$ *is called* ***abelian*** *if for all* $a, b \in G$ $a * b = b * a$*.*

Definition 1.1.5 *A* ***subgroup*** H *of a group* G *is a nonempty subset which is closed under the operation of the group* G*, and satisfies the condition that* $x \in H$ *implies* $x^{-1} \in H$*.*

Definition 1.1.6 *A* ***field*** *is a set* $\boldsymbol{F}$ *together with two binary operations* $+$ *and* $\cdot$ *on* F *such that* $(F, +)$ *is an abelian group (call its identity* 0*) and* $(F - \{0\}, \cdot)$ *is also an abelian group, and the following distributive law holds:*

$$a \cdot (b + c) = a \cdot b + a \cdot c \text{ for all } a, b, c \in F.$$

Definition 1.1.7 *A* ***vector space*** V *is a set along with a binary operation, namely,* ***addition*** *on* V *and a* ***scalar multiplication*** *(a function that assigns an element* $av \in V$ *to each pair* (a, v) *for* $a \in F$ *and* $v \in V$*) on* V *such that the following properties hold:*

(i) $x + y = y + x$ *for all* $x, y \in V$ *(commutativity of vector addition).*

(ii) $x + (y + z) = (x + y) + z$ *for all* $x, y, z \in V$ *(associativity of vector addition).*

(iii) *There exists an element* $0 \in V$ *such that* $x + 0 = 0 + x$ *(additive identity).*

(iv) *For any* $x \in V$ *there exists* $-x \in V$ *such that* $x + (-x) = (-x) + x = 0$ *(existence of additive inverse).*

(v) $r(sx) = (rs)x$ *for all* $r, s \in F$ *and all* $x \in V$ *(associativity of scalar multiplication).*

(vi) $(r + s)x = rx + sx$ *for all* $r, s \in F$ *and all* $x \in V$ *(distributivity of scalar sums).*

(vii) $r(x + y) = rx + ry$ *for all* $r \in F$ *and all* $x, y \in V$ *(distributivity of vector sums).*

(viii) *There exists* $1 \in F$ *such that* $1x = x1 = x$ *for all* $x \in V$ *(existence of multiplicative identity).*

Definition 1.1.8 *A* ***latin square*** *of* ***size*** n *is an* $n \times n$ *array* $L = (l_{i,j})$ *such that each entry* $l_{i,j}$ *contains a single symbol from an* n*-set* $S = \{a_1, ..., a_n\}$ *of symbols such that each symbol occurs in each row and column exactly once.*

Definition 1.1.9 *A* ***partial latin square*** *of* ***size*** n *is an* $n \times n$ *array* $L = (l_{i,j})$ *such that each entry* $l_{i,j}$ *contains either a single symbol from an* n*-set* S *of symbols or empty such that each symbol occurs in each row and column at most once.*

Definition 1.1.10 *A* ***quasigroup*** $(Q, \circ)$ *is a pair where* Q *is a set of size* n *and* $\circ$ *is a binary operation on* Q *such that for every pair* a, b *of* Q*, the equations* $a \circ x = b$ *and* $y \circ a = b$ *have unique solutions. The* ***order*** *of the quasigroup is the number of the elements of the set* Q*.*

Note that a quasigroup is just a latin square with a headline and a sideline.

Let $Q = \{a_1, ..., a_n\}$*. We can use a latin square of size* n *to obtain a quasigroup of order* n *by defining* $a_i \circ a_j = a_k$ *where the entry* $l_{i,j}$ *of the latin square is* a_k*, for all* $i, j, k \in \{1, 2, ..., n\}$*.*

A quasigroup $(Q, \circ)$ *is* ***idempotent*** *if for all* $a_i \in Q$*,* $a_i \circ a_i = a_i$*.*

A quasigroup $(Q, \circ)$ *of order* $2n$ *is* ***half idempotent*** *if for all* a_i *and* $a_{i+n} \in Q$ *with* $i \leq n$*,* $a_i \circ a_i = a_i$ *and* $a_{i+n} \circ a_{i+n} = a_i$*.*

Definition 1.1.11 *For a finite set* S *of* v *objects a* ***block design*** *based on* S *is a collection of* k *element subsets of* S *which are called* ***blocks****, where each* t *element subset of* V *occurs in* λ *blocks together. We denote this design as a* $t - (v, k, \lambda)$ *design.*

When $t = 2$*, a block design is called a* ***balanced incomplete block design*** *(****BIBD****). We denote a BIBD by* $(v, k, \lambda) - BIBD$*.*

When the size of the blocks in a block design is 3*, we call the blocks* ***triples****, and the design a* ***triple system****. We denote a triple system as* $TS(v, \lambda)$

When $\lambda = 1$ *in a block design we call the design a* ***Steiner system****,* $S(t, k, v)$*. A* ***Steiner triple system*** *(****STS****) is a* $S(2, 3, v)$*. We denote the STS as* $(\mathcal{S}, T)$ *where* $\mathcal{S}$ *is the set of points (or vertices) and* T *is the set of triples. The* ***order****, that is, the number of elements of the Steiner triple system is denoted by* $|\mathcal{S}|$*. For the ease of notation, we sometimes denote the STS* $(\mathcal{S}, T)$ *as* $\mathcal{S}$*.*

Definition 1.1.12 *A* ***partial triple system*** $PTS(v, \lambda)$ *is a set of* v *elements* V *and a collection of triples* B*, so that each unordered pair of elements from* V *occurs in at most* λ *triples of* B*.*

Definition 1.1.13 *An* ***incomplete triple system****,* ITS *of order* v *is a* $PTS(v, \lambda)$*, with a set* V *of* v *elements and a collection* B *of triples, such that the following*

additional property is satisfied: there is a set $W \subseteq V$ of size w such that,
(i) if $x, y \in W$, then no triple of $\mathcal{B}$ contains $\{x, y\}$,
(ii) if $x \in V \setminus W$ and $y \in V$ then exactly λ triples of $\mathcal{B}$ contains $\{x, y\}$.

Definition 1.1.14 *For a STS $(\mathcal{S}, T)$, a subset of $\mathcal{S}$ is said to be* ***independent*** *if no three element subset of it is a triple.*

Definition 1.1.15 *If for a $PTS(v, \lambda)$, defined on a set of points V with the triple set T, there is a TS(w, λ) defined on a set of points V' with the triple set T', such that $V \subseteq V'$, and $T \subseteq T'$, then we call the enclosing an* ***embedding****.*

Definition 1.1.16 *For a pair of Steiner triple systems, namely $(\mathcal{S}_1, T_1)$ and $(\mathcal{S}_2, T_2)$, an* ***isomorphism*** *is a bijection $\Phi : \mathcal{S}_1 \to \mathcal{S}_2$ for which the induced mapping $\overline{\Phi} : T_1 \to T_2$ is also a bijection. Two Steiner triple systems are* ***isomorphic*** *if there is such a bijection.*

Definition 1.1.17 *An* ***automorphism*** *of a STS $(\mathcal{S}, T)$, is a bijection $f : \mathcal{S} \to \mathcal{S}$ such that $\{f(x), f(y), f(z)\} \in T$ for every triple $\{x, y, z\} \in T$. We denote the automorphism group of the system as $Aut(\mathcal{S})$.*

Definition 1.1.18 *The* ***totally associative*** Steiner triple system *of order $2^n - 1$, which is unique up to isomorphism, denoted by $\mathcal{S}^\star(2^n - 1)$ is a STS with the property that, when the elements of the system are considered as elements of a commutative quasigroup which is defined by:*
(i) $aa = a$,
(ii) $ab = c$, $a \neq b$ if $\{a, b, c\}$ is a triple in $\mathcal{S}^\star(2^n - 1)$,
then, we have for any points a, b, c from the system not forming a triple, $a(bc) = b(ac) = c(ab)$. We denote the set of points of the totally associative STS as $V(\mathcal{S}^\star(2^n - 1))$. It is known that any 3 elements not forming a triple in the system generates a $STS(7)$ in $\mathcal{S}^\star(2^n - 1)$. $\mathcal{S}^\star(2^n - 1)$ is isomorphic to $\mathbb{P}G(n - 1, 2)$.

Definition 1.1.19 *A* ***vertex coloring*** *of a Steiner triple system is an onto map $f : V \to C$ where C is the set of colors. If the order of C, $|C| = m$, we call the* *coloring an m-coloring. For each $c \in C$, $f^{-1}(c) = \{x \in V : f(x) = c\}$ is a* ***color class****.*

Definition 1.1.20 *A triple is* ***monochromatic*** *if all of its vertices are colored by the same color.*

There are many ways of coloring Steiner triple systems. But among these, we will study the properties of the **weak colorings**. Those are the colorings with the condition that no triple in the STS is **monochromatic**. When this condition is satisfied, we say the coloring is a **proper coloring**.

Definition 1.1.21 *A* ***hypergraph*** *H is an ordered pair (V, E) where V is a set of* ***vertices*** *and E is a set of* ***edges****, where edges are subsets of V. A STS is an example of a hypergraph where each triple corresponds to an edge. A hypergraph is k-****uniform*** *if its edges all have size k. The* ***chromatic number*** *$Chr(H)$ of a hypergraph H is the minimum number of colors needed to label the vertices so that no edge is monochromatic.*

Notice that a 3-uniform hypergraph is a STS.

Definition 1.1.22 *A* ***partition sequence*** *of an r-chromatic $STS(v)$ is a sequence of integers $t_1 \geq t_2 \geq \ldots \geq t_r$ that corresponds to the sizes of color classes in some r-coloring of the r-chromatic $STS(v)$.*

Definition 1.1.23 *A weakly m-chromatic $STS(v)$ is* ***uniquely colorable*** *if every weak m-coloring of the STS produces the same partition of the element set into color classes.*

Definition 1.1.24 *A STS has a* ***bicoloring*** *with m-color classes if the points are partitioned into m subsets and the three points in every triple are contained in exactly two of the color classes. Bicolorings are weak colorings.*

Definition 1.1.25 *The* ***chromatic number*** *of a STS is the least integer k such that the system admits a proper vertex coloring with k colors. We then say that the STS is k-****chromatic*** *and we denote this as $Chr(\mathcal{S}) = k$.*

Definition 1.1.26 *The **spectrum** is the set of integers m for which there exists a weakly m-chromatic $STS(v)$. It is denoted as $C(v)$. So in other words $C(v) = \{k :$ there exist a k-chromatic $STS(v)\}$.*

Definition 1.1.27 *A k-coloring of a STS is a **biased coloring** if at least $\frac{v+3}{2}$ of the elements are colored by $(k-2)$ of the colors.*

Definition 1.1.28 *A **graph** G is a triple consisting of a **vertex set** $V(G)$, an **edge set** $E(G)$ and a relation that associates with each edge two vertices (not necessarily distinct) called its **endpoints**. When the vertices u and v are endpoints of an edge, they are **adjacent**. We call an edge a **loop** if its endpoints are the same vertex. A loopless graph is called a **simple graph**.*

Definition 1.1.29 *For a graph G, a **subgraph** G' of G is a graph such that $V(G') \subseteq V(G)$ and $E(G') \subseteq E(G)$. If $V(G) = V(G')$, the subgraph is called a **spanning subgraph**.*

Definition 1.1.30 *If vertex v is an endpoint of an edge e, then v and e are **incident**. The **degree** of a vertex is the number of edges incident to it.*

Definition 1.1.31 *A **complete graph** is a simple graph whose vertices are pairwise adjacent. If the number of vertices is n, the complete graph is denoted as K_n.*

Definition 1.1.32 *A **walk** is a list $v_0, e_1, v_1, ..., e_k, v_k$ of vertices and edges such that for $1 \leq i \leq k$, the edge e_i has endpoints v_{i-1} and v_i. A **trail** is a walk with no repeated edge. A circuit is a closed trail, that means $v_0 = v_k$. For a graph G if a circuit is also a spanning subgraph, we call it a **Hamiltonian circuit**.*

Definition 1.1.33 *A 1**-factor** of a graph G is a spanning subgraph in which all the vertices have degree 1.*

Definition 1.1.34 *A 1**-factorization** of a graph G is a set of 1-factors which partition the edge set of the graph.*

Definition 1.1.35 *For a graph G with odd number of vertices n, a* ***near** 1**-factor*** *is a subgraph of G, where exactly one vertex has degree 0, all the rest have degree 1.*

Definition 1.1.36 *For a graph with odd number of vertices, a* ***near** 1**-factorization*** *is a collection of near 1-factors whose union is the graph itself.*

Definition 1.1.37 *For any subgroup N of a group G and for any $g \in G$ let $gN = \{gn|n \in N\}$ and $Ng = \{ng|n \in N\}$ called respectively a* ***left coset*** *and* ***right coset*** *of N in G.*

Definition 1.1.38 *The* ***affine geometry of dimension n over the field*** *F_3 is the set of all cosets of the vector space F_3^n, which is the vector space of dimension n over the field F_3, the field of 3 elements. We denote an affine geometry as $\mathbb{A}G(n,3)$.*

A **k-*flat*** *of $\mathbb{A}G(n,3)$ for $k \in \{0,1,2,...,n\}$ is a coset of a subspace of dimension k. 1-flats are called lines, 2-flats are called planes, and $(n-1)$-flats are called hyperplanes. The points of $\mathbb{A}G(n,3)$ which are vectors of F_3^n are 0-flats.*

$\mathbb{A}G(n,3)$ can be thought as a $STS(3^n)$ where the points of $\mathbb{A}G(n,3)$ are the points of the STS, the lines of it are triples of the STS.

Definition 1.1.39 *The* ***projective geometry****, $\mathbb{P}G(n,3)$ is defined as the space of equivalence classes $(\mathbb{A}G(n+1,3)\setminus \{\overrightarrow{0}\})/\sim$ where $x \sim y$ if for some element c of the field F, $x = cy$. For all $k \geq 0$, the images of a $(k+1)$-flat in $\mathbb{A}G(n,3)$ are defined to be* **k-*flats*** *in $\mathbb{P}G(n,3)$.*

Definition 1.1.40 *A subset of an affine geometry is a* ***cap*** *if no three of its points are collinear, that is, if no three of its points lie in the same 1-flat. A cap of cardinality k is called a k-cap.*

Moreover, the caps of the $\mathbb{A}G(n,3)$ can be thought as the color classes of the STS, since no three points are collinear, they will not be on the same line, and on the same triple.

Definition 1.1.41 *A subset S of a vector space V is called a set of **linearly independent vectors** if an equation $a_1v_1 + a_2v_2 + ... + a_nv_n = 0$ with $a_1, a_2, ..., a_n \in F$ and $v_1, v_2, ..., v_n \in S$ implies $a_1 = a_2 = ... = a_n = 0$.*

Definition 1.1.42 *A **basis** of a vector space V is a list of vector in V that are linearly independent and spans (that is, any vector in V can be written as a linear combination of the vectors from this list) V.*

Definition 1.1.43 ***Complexity theory** is the theory of classifying problems based on how difficult they are to solve. A problem is assigned to the **P-problem** (polynomial-time) class if the number of steps needed to solve it, is bounded by some power of the size of the problem. A **non-deterministic Turing machine** is a a model of behavior composed of a finite number of states, transitions between those states, and actions which is associated with an external storage or memory medium and for each pair of state and input symbol there may be several possible next states. An **NP** (Non-deterministic Polynomial time) problem is a decision problem (a problem whose answer is either yes or no) solvable in polynomial time on a non-deterministic Turing machine. The class of P-problems is a subset of the class of NP-problems, but there also exist problems which are not NP.*

Definition 1.1.44 *An **NP-complete problem** is a problem which is both **NP** and **NP-hard** (any NP-problem can be translated into this problem). In other words, a problem is NP-complete if it is NP and an algorithm for solving it can be translated into one for solving any other NP-problem.*

Definition 1.1.45 *A **group divisible design** (GDD) is a triple (X, G, B), where X is a point set, which satisfies the following properties:*
(i) G is a partition of X into subsets called groups.
(ii) B is a set of subsets of X (called blocks) such that a group and a block contain at most one element in common.
(iii) Every pair of points from distinct groups occurs in a unique block.

1.2 Bose Construction and Skolem Construction

1.2.1 Bose Construction

Let $(Q, \circ)$ be an idempotent commutative quasigroup of order $2u + 1$. Let $Q = \{1, 2, 3, ..., 2u + 1\}$ and $\mathcal{S} = Q \times \{1, 2, 3\}$. Define a collection of triples T of $\mathcal{S}$ as follows:

(i) For every $x \in Q$ $\{(x, 1), (x, 2), (x, 3)\} \in T$.

(ii) For every x, $y \in Q$, $x \neq y$, then $\{(x, 1), (y, 1), (x \circ y, 2)\}$, $\{(x, 2), (y, 2), (x \circ y, 3)\}$,$\{(x, 3), (y, 3), (x \circ y, 1)\} \in T$.

It is easy to see that $(\mathcal{S}, T)$ is a $STS(6u + 3)$, by counting the number of triples in T and checking that any pair of elements is seen in exactly one triple. Note that the number of triples should be $\frac{v(v-1)}{6}$ in a STS.

1.2.2 Skolem Construction

Let $(Q, \circ)$ be an half idempotent commutative quasigroup of order $2u$ where $Q = \{1, 2, 3, ..., 2u\}$ and set $\mathcal{S} = Q \times \{1, 2, 3\} \cup \{\infty\}$. Define a collection of triples T of $\mathcal{S}$ as follows:

(i) for every $x \in Q$, $x \leq u$, $x \in Q$, $\{(x, 1), (x, 2), (x, 3)\} \in T$,

(ii) for each $x > u$, the three triples $\{\infty, (x, 1), (x - u, 2)\}$, $\{\infty, (x, 2), (x - u, 3)\}$, $\{\infty, (x, 3), (x - u, 1)\} \in T$,

(iii) for x, $y \in Q$, $x \neq y$, then $\{(x, 1), (y, 1), (x \circ y, 2)\}$, $\{(x, 2), (y, 2), (x \circ y, 3)\}$, $\{(x, 3), (y, 3), (xoy, 1)\} \in T$.

Similar to the Bose Construction, it is easy to see that $(\mathcal{S}, T)$ is an $STS(6u + 1)$.

Notice that in both Bose and Skolem Constructions subquasigroups will always produce subsystems in the resulting STS [5].

1.3 Some preliminary theorems

1.3.1 Preliminary results for studies of Rosa

We will see in the following chapters, for any admissable $v \geq 7$, a $STS(v)$ is at least 3-chromatic. We summarize here some theorems about the existence of 2-chromatic PTSs and 3-chromatic TSs. We will also state some of the theorems due to A. Rosa. [30].

The following result gives us an upper bound for the number of triples of the 2-chromatic partial triple system.

Proposition 1.3.1 [9] *Let $(V, \mathcal{B})$ be a weakly 2-chromatic partial $TS(v, \lambda)$. Then, $|\mathcal{B}| \leq \lambda v^2/8$.*

Proof: For the two colors α and β, suppose p of the elements are colored by α and the remaining $v-p$ of them are colored by β. Since none of the triples are monochromatic, in each triple, two of the elements will be colored by one color, and the other element will be colored by the other color. So in each triple there are two unordered pairs of elements which are colored by different colors. There are $v(v-p)$ pairs of elements which are not monochromatic. Each of them can be seen in λ triples together. But since there are two such pairs for each triple, to count the triples, we divide the product $\lambda p(v-p)$ by 2. This is a partial triple system. So $|\mathcal{B}| \leq \lambda p(v-p)/2 \leq \lambda v^2/8$. The second inequality holds since $p(v-p)/2$ takes a maximum value when $p = v/2$.■

Consider a triple system with a 2-coloring. To decrease the number of the monochromatic triples, we should have color classes of nearly equal size. The following result belongs to Phelps, and is about the number of the monochromatic triples of a $TS(v, \lambda)$ with a 2-coloring.

Lemma 1.3.2 [9] *When a $TS(v, \lambda)$ has a coloring with color classes of size $\lfloor \frac{v}{2} \rfloor$ and $\lceil \frac{v}{2} \rceil$, the number of monochromatic triples equals $\lambda \lfloor \frac{v}{2} \rfloor (\lceil \frac{v}{2} \rceil - 2)/6$ and does not depend on the particular triple system.*

This shows that there exists a 2-chromatic partial triple system with b triples for any b satisfying $1 \leq b \leq \lfloor \lambda v^2/8 \rfloor$. Take any $TS(v, \lambda)$. Color any $\lfloor \frac{v}{2} \rfloor$ of its elements by one color and the rest by a second color. The number of monochromatic triples is assured to be $\lambda \lfloor \frac{v}{2} \rfloor (\lceil \frac{v}{2} \rceil - 2)/6$ by Lemma 1.3.2 independent from how we color the vertices. When we delete these monochromatic triples, we will have $\lfloor \lambda v^2/8 \rfloor$ triples. To obtain a partial triple system with b triples, delete any $\lfloor \lambda v^2/8 \rfloor - b$ triples.

By Proposition 1.3.1 we can deduce, the chromatic number of a $TS(v, \lambda)$ for $v \geq 5$ is at least 3.

Lemma 1.3.3 [30] *If there are two STSs named $\mathcal{S}_1$, $\mathcal{S}_2$ having orders n_1 and n_2, respectively, then there exists a Steiner triple system $(\mathcal{S}, B)$ of order $n_1 n_2$, moreover $\mathcal{S}$ includes $\mathcal{S}_1$ and $\mathcal{S}_2$ as subsystems.*

Lemma 1.3.4 [30] *If there are three STSs named $\mathcal{S}_1$, $\mathcal{S}_2$, $\mathcal{S}_3$, where $\mathcal{S}_3$ is a subsystem of $\mathcal{S}_2$, then there exists a STS of order $n_3 + n_1(n_2 - n_3)$ containing the all three STSs as subsystems.*

Theorem 1.3.5 [30] *If $\mathcal{S}_1$ and $\mathcal{S}_2$ are two STSs of orders $6k+1$ and n, respectively, where k is an integer, $k \geq 0$ and $n \geq 6k+1$, then there exists a STS of order $2n + 6k + 1$ containing $\mathcal{S}_1$ and $\mathcal{S}_2$ as disjoint subsystems.*

Theorem 1.3.6 [30] *If $\mathcal{S}_1$ and $\mathcal{S}_2$ are two STSs of orders $6k+3$ and n, respectively, where k is an integer, $k \geq 0$, $n \equiv 3 \,(mod\ 6)$, $n \geq 6k+3$, then there exists a STS of order $2n + 6k + 3$ containing $\mathcal{S}_1$ and $\mathcal{S}_2$ as disjoint subsystems.*

Corollary 1.3.7 [30] *Let $\mathcal{S}$ be a STS of order $n \equiv 3 \,(mod\ 6)$. For every admissible s, $2n + 1 \leq s \leq 3n$, there exists a STS of order s containing $\mathcal{S}$ as a subsystem.*

Proof: We apply the two previous theorems for every $k \in \{0, 1, ..., t\}$, where $n = 6t + 3$, and obtain Steiner triple systems $\mathcal{S}_i$'s, where $0 \leq i \leq t$, so that in each $\mathcal{S}_i$, we have copies of $\mathcal{S}$ and a STS of order $6k + 1$. ∎

1.3.2 Preliminary results for coloring finite geometries

Let C be an independent set of the vector space $V(S^{\star}(63))$.

Lemma 1.3.8 [16] *If C does not contain a basis, then there is a hyperplane K such that $|C \cap K| \leq 2$.*

Let $C_4 = \{0, 1, 2, 3, 4, 5, 0123, 0124, 0125, 012345, 034, 035, 045, 134, 135, 145, 234, 235, 245, 345\}$. It has the property that, it meets every hyperplane of $S^{\star}(63)$ in at least 4 points.

Theorem 1.3.9 [16] *If the independent set C satisfies $|C \cap K| \geq 4$ for every hyperplane K, then $C = f(C_4)$ for some $f \in Aut(S^{\star}(63))$.*

This shows the size of an independent set that meets every hyperplane of $S^{\star}(2^6 - 1)$ in at least 4 points is 20.

Corollary 1.3.10 [16] *$V(S^{\star}(63))$ has no independent subset C which meets every hyperplane of $S^{\star}(2^6 - 1)$ in 5 or more points.*

Lemma 1.3.8 and Theorem 1.3.9 together imply the following corollary, which will be used in the proof of Theorem 4.1.4.

Corollary 1.3.11 [16] *If C is independent and satisfies $|C| \leq 19$, then there is a hyperplane K such that $|C \cap K| \leq 3$.*

1.3.3 Preliminary results for the existence of k-chromatic STS for large k

Lemma 1.3.12 [9] *A commutative quasigroup of order n exists for all $n \geq 0$. A commutative idempotent quasigroup of order n exists if and only if $n = 0$ or n is odd.*

Proof: Index the rows, columns and the symbols of the latin square of size n by the elements of the additive group $\mathbb{Z}_n$. Consider the quasigroup corresponding to this latin square obtained by putting $l_{i,j} = l_{j,i} = i + j \pmod{n}$. This is commutative by construction.

For the idempotent commutative quasigroup, the case $n = 0$ is trivial. Take an odd number n. Consider a latin square just as above. It is commutative by construction. Every symbol is seen just once on the main diagonal. By renaming the symbols in the latin square, we can obtain an idempotent latin square. For the converse, consider we have an idempotent commutative latin square whose order is more than 0. Every symbol is seen n times on the latin square, once on the main diagonal and by symmetry, even number of times outside the main diagonal. So, n must be odd. ■

Erdös and Hajnal used methods of probability to show the following theorem [13].

Theorem 1.3.13 *For any integer $k \geq 2$ there exists a PTS such that chromatic number of the system $C \geq k$.*

Treash proved that, any partial Steiner triple system can be extended to a Steiner triple system.

Theorem 1.3.14 [31] *A partial triple system has an embedding.*

For $\lambda = 1$, this means a $PSTS$ can be completed to a STS.

The following theorem of Lindner also gives us an idea about the comparative sizes of the $PSTS$ and STS.

Theorem 1.3.15 [21] *A $PSTS(v)$ can be embedded in a $STS(w)$ if $w \geq 12v+7$ and $w \equiv 1, 3\,(mod\ 6)$.*

Andersen, Hilton, and Mendelsohn [1] proved the following theorem which improves the previous result of Lindner:

Theorem 1.3.16 *A partial Steiner triple system of order v can be embedded in a $STS(v)$ whenever $w \geq 4v + 1$ and $w \equiv 1, 3\,(mod\ 6)$.*

Finally, very recently Bryant proved the following [7]:

Theorem 1.3.17 *Any partial Steiner triple system of order u can be embedded in a Steiner triple system of order v if $v \equiv 1,\ 3\ (mod\ 6)$ and $v \geq 3u - 2$.*

Chapter 2

INTRODUCTION

Combinatorics as an area of mathematics can be considered as the science of counting discrete objects. Design theory, which is a branch of combinatorics, is the science of counting, choosing, arranging and classifying discrete objects. A *combinatorial design* is a way of selecting subsets from a finite set that meets certain requirements. For instance, these subsets can be selected in such a way that the intersections of them have certain properties. Therefore, *combinatorial design theory* deals with the construction of the necessary and sufficient conditions for the existence of combinatorial designs. Like many other problems in combinatorics, coloring problems are counting problems in general. Various methods can be used in studies for colorings of designs, including some probabilistic and computational methods [10]. There are many studies investigating the coloring properties of Steiner triple systems. This thesis is a survey on weak colorings of Steiner triple systems, the focus is particularly on dealing with the constructive methods that are used for the coloring problems.

Steiner triple systems were defined for the first time by W. S. B. Woolhouse in 1844. In 1847, T. P. Kirkman proved that for all $v \equiv 1$ or $3 \pmod 6$, there exists a Steiner triple system of order v [24]. The weak chromatic number of a hypergraph, will be one of our main concerns since STSs are special cases of uniform hypergraphs. Erdös, Hajnal [13], and Lovaśz [25] considered the weak vertex colorings of hypergraphs for the first time. Chromatic number of hypergraphs has been defined by Berge [3].

Here we first gave some required definitions for our study and state some of the main theorems about coloring of Steiner triple systems. In the following chapters, we will discuss construction methods for k-chromatic Steiner triple systems for small k and the existence of k-chromatic Steiner triple systems for large k. We will discuss the constructive methods used in the coloring problems of Steiner triple systems.

Blaniuk and Mendelsohn have made a survey on the colorings of Steiner systems in [4] where the studies about vertex colorings and edge colorings of Steiner triple systems were explained. Some notes on open problems and comments on the possible future studies also took place in it. A latter survey about colorings of block designs belong to Colbourn and Rosa [10].

2.1 Overview

In the weak colorings of Steiner triple systems, the condition that no triple is monochromatic is required. There are many methods to color Steiner triple systems, while we will only deal with the weak colorings.

A. Rosa worked on the weak chromatic number of STSs. He showed that there exists no nontrivial 2-chromatic STS [30]. Mathon, Phelps, and Rosa showed all $STS(v)$'s with $v \leq 15$ are 3-chromatic in their detailed study [26] on small Steiner triple systems and their properties. Rosa proved there exists a weakly 3-chromatic STS of all orders except of order 3 elements [29]. He also gave some constructions for weakly 4-chromatic STSs. For instance, he showed the totally associative Steiner triple system of order 31 is 4-chromatic [29]. Also, he constructed a 4-chromatic $STS(49)$ by using a product of two Steiner triple systems of order 7 [29].

Rödl, Brandes, and Phelps used probabilistic approach to find upper and lower bounds for the number u_m which corresponds to the smallest order for which there exists a partial $STS(u_m)$ [5]. In particular, they showed $C_1 m^2 \log m < u_m < C_2 m^2 \log m$ where C_1 and C_2 are constants. By using the embedding theorem for partial STSs, and the existence of m-chromatic partial STSs, they proved that for any $m \geq 3$, there exists an n_m such that for all $v \equiv 1$ or $3 \pmod 6$, $v \geq n_m$ there exists an m-chromatic $STS(v)$. In addition, they showed that for any $v \geq 25$ ($v \equiv 1$ or $3 \pmod 6$) there exists a weakly 4-chromatic STS except 39, 43, 45. The examples of 4-chromatic Steiner triple systems of orders 25, 27, 33, 37 were found by Rödl, Phelps and Brandes using computer algorithms. Haddad showed there exist 4-chromatic $STS(21)$ and $STS(39)$

[17]. The 4-chromatic STS of order 39 is again found by using computer algorithms. The existence of 4-chromatic $STS(21)$ implies the existence of 4-chromatic $STS(43)$ and 4-chromatic $STS(45)$ by the following Theorem:

Theorem 2.1.1 [9] *If a k-chromatic $STS(v)$ exists with $v \equiv 3\,(mod\ 6)$ and $k \geq 4$ in which size of the three of the color classes bounded by $v/3$, then there exists a k-chromatic $STS(u)$ for every admissible $u \geq 2v + 1$.*

Therefore, there are 4-chromatic STS for all admissible values of $v \geq 21$ and the only value in doubt is 19. Since there are $11,084,874,829$ known nonisomorphic $STS(19)$ [20], proving if there exists a 4-chromatic $STS(19)$ is not a trivial problem and requires a constructive solution.

A 5-chromatic $STS(v)$ for every admissible $v \geq 127$ is constructed by Fugère, Haddad, and Wehlau [16]. They also gave a specific example of a 5-chromatic $STS(v)$. Before that, for $k \geq 5$, there had been no specific examples of k-chromatic STS, the existence of such Steiner triple systems were shown by nonconstructive methods [5]. After four years, the first example of a 6-chromatic STS was given [6] by using some computer algorithms. They proved that there exists a 6-chromatic $STS(u)$ for every admissible $u \geq 487$ by recursive methods.

It is known that as k increases, it is hard to give specific examples of k-chromatic STSs. Colbourn worked on the complexity of the problem of finding the chromatic number of a given STS.

In the previous chapter, we gave the required definitions for our study, statements of some preliminary theorems, and proofs to some of them. In the third chapter, we will focus on the earlier studies on the problem which mostly deals with k-chromatic Steiner triple systems for small k. In the fourth chapter, we will deal with the former studies on the k-chromatic Steiner triple systems for small k, which are mostly using methods in finite geometry. In the fifth chapter, we will summarize the studies on the k-chromatic Steiner triple for large k, which are mostly existence theorems. In the fifth chapter, we will also include some related results for weak colorings of Steiner triple systems related to complexity of this problem and some specific weak colorings of Steiner triple systems. We will conclude our study by summarizing the main results that are included in it.

Chapter 3

EARLY STUDIES ON α-CHROMATIC STEINER TRIPLE SYSTEMS FOR SMALL α

Early contributions of the weak colorings of STSs made by A. Rosa. We will first discuss the early studies of Rosa about weak colorings of Steiner triple systems. Some very important contributions about weak colorings of Steiner triple systems were done by the studies of Brandes, Phelps, and Rödl. We will also discuss some of the main theorems they proved.

3.1 Studies of Rosa

Mathon, Phelps and Rosa studied on the structure of Steiner triple systems of small orders. In their detailed study [26], they summarized all known structural properties of the Steiner triple systems up to date which includes the list of triples and the color classes. The following two examples of STSs, one of order 13, and the other of order 15 are taken from this nice work. Actually, all Steiner triple systems whose order less than or equal to 15 are included in the article. There is a unique $STS(7)$ up to isomorphism, there are exactly two nonisomorphic $STS(13)$, 80 nonisomorphic $STS(15)$. After the order 19, the number of nonisomorphic Steiner triple systems are comparably very large, that is why, it is not easy to classify them. Number of nonisomorphic Steiner triple systems of small orders or lower bounds on these orders are given.

In addition, some selected Steiner triple systems whose order is less than 27 are also included in the article [26].

Example 3.1.1 $STS(7)$

Triples: $\{1,2,3\}$, $\{1,4,5\}$, $\{1,6,7\}$, $\{2,4,6\}$, $\{2,5,7\}$, $\{3,4,7\}$, $\{3,5,6\}$.
Color classes: $\{1,2\}$, $\{3,4\}$, $\{5,6,7\}$.

Example 3.1.2 $STS(15)$

Triples: $\{1,2,3\}$, $\{1,4,5\}$, $\{1,6,7\}$, $\{1,8,9\}$, $\{1,10,11\}$, $\{1,12,13\}$, $\{1,14,15\}$, $\{2,4,6\}$, $\{2,5,7\}$, $\{2,8,10\}$, $\{2,9,11\}$, $\{2,12,14\}$, $\{2,13,15\}$, $\{3,4,7\}$, $\{3,5,6\}$, $\{3,8,11\}$, $\{3,9,10\}$, $\{3,12,15\}$, $\{3,13,14\}$, $\{4,8,12\}$, $\{4,9,13\}$, $\{4,10,14\}$, $\{4,11,15\}$, $\{5,8,14\}$, $\{5,9,12\}$, $\{5,10,15\}$, $\{5,11,13\}$, $\{6,8,15\}$, $\{6,9,14\}$, $\{6,10,13\}$, $\{6,11,12\}$, $\{7,8,13\}$, $\{7,9,15\}$, $\{7,10,12\}$, $\{7,11,14\}$.
Color classes: $\{1,2,4,7,9\}$, $\{3,5,8,12,13\}$, $\{6,10,11,14,15\}$.

Example 3.1.3 $STS(13)$

Triples: $\{1,2,3\}$, $\{1,4,5\}$, $\{1,6,7\}$, $\{1,8,9\}$, $\{1,10,11\}$, $\{1,12,13\}$, $\{2,4,6\}$, $\{2,5,7\}$, $\{2,8,10\}$, $\{2,9,12\}$, $\{2,11,13\}$, $\{3,4,8\}$, $\{3,5,12\}$, $\{3,6,13\}$, $\{3,7,11\}$, $\{3,9,10\}$, $\{4,7,9\}$, $\{4,10,13\}$, $\{4,11,12\}$, $\{5,6,10\}$, $\{5,8,11\}$, $\{5,9,13\}$, $\{6,8,12\}$, $\{6,9,11\}$, $\{7,8,13\}$, $\{7,10,12\}$.
Color classes: $\{1,2,4,7,8\}$, $\{3,5,6,9\}$, $\{10,11,12,13\}$.

Theorem 3.1.4 [29] *$Chr(S(v)) \geq 3$ whenever $v \geq 7$.*

Proof: Suppose for some v we have $STS(v)$ which can be colored by two colors, say p of them by red, $v-p$, where $p \leq v-p$ of them by blue. None of the triples is monochromatic. Corresponding to each triple, there are 2 pairs of elements which are multicolored. So, if we denote the number of triples by $|T|$, $2|T| = p(v-p)$. But $|T| = \frac{v(v-1)}{6}$ [24]. This equality holds if and only if when $p = 1, v = 3$. that is a STS is 2-chromatic if and only if $v \doteq 3$ and it is at least 3-chromatic when $v \geq 7$. ■

The existence of Steiner triple systems of large chromatic numbers can be deduced by earlier studies. Treash proved that any partial Steiner triple system can be embedded in a Steiner triple system [31]. Erdös and Hajnal [13] proved the existence of a partial Steiner triple system which is k-chromatic for any positive integer k. The extension of this $PSTS$ to a STS will include an isomorphic copy of the $PSTS$ in the STS. So the STS is at least k-chromatic.

Theorem 3.1.5 *If $v \equiv 1$ or 3 (mod 6), then there exists a 3-chromatic $STS(v)$ [29].*

Proof: For the case $v \equiv 3$ (mod 6) the Bose Construction is used. In the Bose Construction, a commutative quasigroup of order $2n+1$, $(Q, \circ)$ where $Q = \{1, 2, ..., 2n+1\}$ is used. Let $\mathcal{S} = Q \times \{1, 2, 3\}$. We color $(i, 1)$s by blue, $(i, 2)$s by green and $(i, 3)$s by red for $1 \leq i \leq 2n + 1$.

For the case $v \equiv 1 \,(\text{mod}\, 6)$, there exists a 3-chromatic $STS(v)$ constructed as follows by Rosa: $\mathcal{S} = \{a_1, ..., a_{2k}, b_1, ..., b_{2k}, c_1, ..., c_{2k}, \infty\}$. Let G be the complete graph on the vertices $1, 2, ..., 2k$ and let $L = \{L_1, ..., L_{2k-1}\}$ be a decomposition of G into $2k - 1$ 1-factors. Where we denote the vertex $2k$ by ∞ in this decomposition. $L_i = \{(k+i-1, \infty), (i+j, i-j-1) \,(\text{mod}\, 2k-1), 0 \leq j \leq k-2\}$. $L' = \{L'_i : 1 \leq i \leq 2k\}$ is a set of subgraphs of K_{2k} which forms a decomposition of K_{2k}.

$$L'_i = \begin{cases} L_i, & \text{if i is odd;} \\ L_i - (i-1, i), & \text{if i is even and}\, i < 2k; \\ (2j-2, 2j), & 1 \leq j \leq k-1 \text{ if } i = 2k. \end{cases}$$

For x, y in $1, 2, ..., 2k$ and $x \neq y$ define $f(x, y)$ by $f(x, y) = i$ if and only if $(x, y) \in L'_i$. Then, the blocks of the $STS(v)$ are given by $(a_x, a_y, b_{f(x,y)})$, $(b_x, b_y, c_{f(x,y)})$, $(c_x, c_y, a_{f(x,y)})$ for $1 \leq x < y \leq 2k$, (a_{2i}, b_{2i}, c_{2i}), for $1 \leq i \leq k$; and (a_i, b_{i+1}, ∞), (b_i, c_{i+1}, ∞), (c_i, a_{i+1}, ∞) for $i = 1, 3, ..., 2k-1$. By coloring a_i's by blue, b_i's by yellow, c_i's by red and ∞ by any of those colors, the three coloring can be verified. ■

Example 3.1.6 *$STS(19)$ can be constructed and colored by the above construction where $k = 3$.*

Let $S = \{a_1, ..., a_6, b_1, ..., b_6, c_1, ..., c_6, \infty\}$. G is a complete graph of 6 vertices. $L = \{L_1, L_2, L_3, L_4, L_5\}$ is a set of 1-factors which partites G.
$L_1 = \{(3, 6), (1, 5), (2, 4)\}$, $L_2 = \{(6, 4), (2, 1), (5, 3), \}$, $L_3 = \{(5, 6), (3, 2), (4, 1)\}$,

$L_4 = \{(6,1),(4,3),(5,2)\}$, $L_5 = \{(6,2),(5,4),(1,3)\}$.

$L' = \{L'_1, L'_2, L'_3, L'_4, L'_5\}$, $L'_1 = L_1$, $L'_3 = L_3$, $L'_5 = L_5$,

$L'_2 = \{(4,6),(3,5)\}$, $L'_4 = \{(6,1),(5,2)\}$, $L'_6 = \{(3,4),(1,2)\}$.

$f(1,2) = 6$, $f(1,3) = 5$, $f(2,3) = 3$, $f(1,4) = 3$, $f(2,4) = 1$, $f(1,5) = 1$, $f(2,5) = 4$, $f(1,6) = 4$, $f(2,6) = 5$, $f(3,3) = 6$, $f(4,5) = 5$, $f(3,5) = 2$, $f(4,6) = 2$, $f(3,6) = 1$, $f(5,6) = 3$.

Type 1 triples: $\{a_1,a_2,b_6\}$, $\{a_1,a_3,b_5\}$, $\{a_2,a_3,b_3\}$, $\{b_1,b_2,c_6\}$, $\{b_1,b_3,c_5\}$, $\{b_2,b_3,c_3\}$, $\{c_1,c_2,a_6\}$, $\{c_1,c_3,a_5\}$, $\{c_2,c_3,a_3\}$, $\{a_1,a_4,b_3\}$, $\{a_1,a_6,b_4\}$, $\{a_2,a_5,b_4\}$, $\{b_1,b_4,c_3\}$, $\{b_1,b_6,c_4\}$, $\{b_2,b_5,c_4\}$, $\{c_1,c_4,a_3\}$, $\{c_1,c_6,a_4\}$, $\{c_2,c_5,a_4\}$, $\{a_2,a_4,b_1\}$, $\{a_2,a_6,b_5\}$, $\{c_1,c_5,a_1\}$, $\{c_2,c_4,a_1\}$, $\{c_2,c_6,a_5\}$, $\{b_1,b_5,c_1\}$, $\{b_2,b_4,c_1\}$, $\{b_2,b_6,c_5\}$, $\{a_3,a_4,b_6\}$, $\{a_3,a_5,b_2\}$, $\{a_3,a_6,b_1\}$, $\{b_3,b_4,c_6\}$, $\{b_3,b_5,c_2\}$, $\{b_3,b_6,c_1\}$, $\{c_3,c_6,a_1\}$, $\{a_4,a_5,b_5\}$, $\{a_4,a_6,b_2\}$, $\{a_5,a_6,b_3\}$, $\{b_4,b_5,c_5\}$, $\{b_4,b_6,c_2\}$, $\{b_5,b_6,c_3\}$, $\{c_3,c_4,a_6\}$, $\{c_3,c_5,a_2\}$, $\{c_4,c_5,a_5\}$, $\{c_4,c_6,a_2\}$, $\{c_5,c_6,a_3\}$, $\{a_1,a_5,b_1\}$.

Type 2 triples: $\{a_2,b_2,c_2\}$, $\{a_4,b_4,c_4\}$, $\{a_6,b_6,c_6\}$,

Type 3 triples: $\{a_1,b_2,\infty\}$, $\{b_1,c_2,\infty\}$, $\{c_1,a_2,\infty\}$, $\{a_3,b_4,\infty\}$, $\{b_3,c_4,\infty\}$, $\{c_3,a_4,\infty\}$, $\{a_5,b_6,\infty\}$, $\{b_5,c_6,\infty\}$, $\{c_5,a_6,\infty\}$.

Example 3.1.7 *A STS of order 21 colored with 3 colors.*

This example illustrates a $STS(21)$ which is constructed by the Bose Construction. The triples are as follows:

Type 1 triples: {1 ,8, 15}, {2 ,9, 16}, {3 ,10, 17}, {4 ,11, 18}, {5 ,12, 19}, {6 ,13, 20}, {7 ,14, 21}.

Type 2 triples: {1 ,2, 12}, {1 ,3, 9}, {1 ,4, 13}, {1 ,5, 10}, {1 ,6, 14}, {1 ,7, 11}, {2 ,3, 13}, {2 ,4, 10}, {2 ,5, 14}, {2 ,6, 11}, {2 ,7, 8}, {3 ,4, 14}, {3 ,5, 11}, {3 ,6, 8}, {3 ,7, 12}, {4 ,5, 8}, {4 ,6, 12}, {4 ,7, 9}, {5 ,6, 9}, {5 ,7, 13}, {6 ,7, 10}, {8 ,9, 19}, {8 ,10, 16}, {8 ,11, 20}, {8 ,12, 17}, {8 ,13, 21}, {8 ,14, 18}, {9 ,10, 20}, {9 ,11, 17}, {9 ,12, 21}, {9 ,13, 18}, {9 ,14, 15}, {10 ,11, 21}, {10 ,12, 18}, {10 ,13, 15}, {10 ,14, 19}, {11 ,12, 15}, {11 ,13, 19}, {11 ,14, 16}, {12 ,13, 16}, {12 ,14, 20}, {13 ,14, 17}, {15 ,16, 5}, {15 ,17, 2}, {15 ,18, 6},

$\{15,19,3\}$, $\{15,20,7\}$, $\{15,21,4\}$, $\{16,17,6\}$, $\{16,18,3\}$, $\{16,19,7\}$, $\{16,20,4\}$, $\{16,21,1\}$, $\{17,18,7\}$, $\{17,19,4\}$, $\{17,20,1\}$, $\{17,21,5\}$, $\{18,19,1\}$, $\{18,20,5\}$, $\{18,21,2\}$, $\{19,20,2\}$, $\{19,21,6\}$, $\{20,21,3\}$.

In this construction, the color classes are as follows:

$\{1, 2, 3, 4, 5, 6, 7\}$, $\{8, 9, 10, 11, 12, 13, 14\}$, $\{15, 16, 17, 18, 19, 20, 21\}$.

All triples above are colored by 3 colors.

The **totally associative** Steiner triple system of order $2^n - 1$, which is unique up to isomorphism, denoted by $\mathcal{S}^\star(2^n - 1)$. Denote the elements of the system by: $a_1, a_2, ..., a_n$, $a_1a_2, ..., a_1a_2......a_n$. Notice that the elements are formed according to the subsets of the set $\{1, 2, ..., n\}$. We can represent the elements of the $S^*(2^n - 1)$ by using the vectors in F_2^n except the zero vector. The basis elements of F_2^n can be represented as $e_1, ..., e_n$ where $e_i = 000...1...00$ where i is the i^{th} entry for $1 \leq i \leq n$. The triples are two dimensional subspaces of F_2^n. For instance, let a_1a_3 and a_1a_5 are two elements of $S^*(2^n - 1)$ and the corresponding vectors are $e_1 + e_3$ and $e_1 + e_5$ respectively, name $(e_1 + e_3) + (e_1 + e_5) = e_3 + e_5$ as a_3a_5. Then the triple which includes a_1a_3 and a_1a_5 is $\{a_1a_3, a_1a_5, a_3a_5\}$. $Chr(S^\star(15)) = 3$. We will prove now, $S^\star(31) = 4$.

Lemma 3.1.8 *$Chr(S^\star(31))$ is 4.*

Proof: By contradiction, suppose $S^\star(31) = 3$. We suppose there are five independent elements a_1, a_2, a_3, a_4, a_5 colored by the same color. (If all of the independent sets have size less than 5, then the order 31 can not be obtained.) Color these points by blue. Then no point a_ia_j where $i, j \in \{1, 2, 3, 4, 5\}$ can be colored by blue. If the elements a_1a_2, a_1a_3, a_1a_4 are colored by green, then $a_2a_3 = a_1a_2a_1a_3$, $a_2a_4 = a_1a_4a_1a_2$, $a_3a_4 = a_1a_3a_1a_4$ can not be green, and can not be blue, so they should be red. But, this can not be the case since they form a triple in $S^\star(31)$ ($a_2a_3a_2a_4 = a_3a_4$). We can suppose a_1a_2, a_1a_3, a_2a_4, a_3a_5, a_4a_5 are colored green, a_1a_5, a_2a_3, a_2a_5, a_3a_4, a_1a_4 are colored red by taking in consideration symmetry. $a_1a_5a_2a_3 = a_1a_5a_2a_3 = a_1a_2a_3a_5$ must be blue, since it can not be red or green. Similarly $a_ia_ja_ka_l$ must be blue where

$i, j, k, l \in \{1, 2, 3, 4, 5\}$ and are distinct. Then $a_1a_2a_3a_4a_5$ must not blue, $a_ia_ja_ka_l$ can not be blue, which can not be true where $i, j, k, l \in \{1, 2, 3, 4, 5\}$ and are distinct. If $a_1a_2a_3a_4a_5$ is green, then $a_2a_4a_5$, $a_3a_4a_5$ must be red, but this can not hold since $\{a_2a_3,\ a_2a_4a_5,\ a_3a_4a_5\}$ is a triple. If $a_1a_2a_3a_4a_5$ is red, then elements $a_2a_3a_5$ and $a_2a_3a_4$ must be green but this can not be the case, since $\{a_4a_5, a_2a_3a_4, a_2a_3a_5\}$ is a triple. So we cannot color $S^\star(31)$ by 3 colors. On the other hand, we can show $Chr(S^\star(31)) \leq 4$. Take a 3 coloring of $S^\star(15)$ which is generated by a_1, a_2, a_3, a_4. Color x and xa_5 with the color of x in the coloring of $S^\star(15)$. Color a_5 with the fourth color. This gives a 4-coloring of $S^\star(31)$. ■

From the proof of lemma 3.1.8 we can conclude that then

$Chr(S^\star(2^n - 1)) \leq Chr(S^\star(2^{n+1} - 1)) \leq 1 + Chr(S^\star(2^n - 1)$.

Lemma 3.1.9 *There exists a* 4 *chromatic* $STS(49)$ [30].

Proof: Rosa constructed a $STS(49)$ by the product rule from two Steiner triple systems of order 7. Then he showed the chromatic number of the system is more than 3.

We will name the $STS(49)$ which is obtained by the product rule as $S'(49)$ and name the $STS(7)$ as $S(7)$, and the elements of $S'(49)$ will be denoted as $c_{i,j}$ where i, $j \in \{1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7\}$. Three elements $c_{m,r}$, $c_{n,s}$, $c_{p,t}$ form a triple in $S'(49)$ if one of the following cases holds:

(*i*) $m = n = p$, $\{r,\ s,\ t\}$ *is a triple in* $S(7)$,

(*ii*) $r = s = t$, $\{m,\ n,\ p\}$ *is a triple in* $S(7)$,

(*iii*) $\{r,\ s,\ t\}$ *is a triple in* $S(7)$ and $\{m,\ n,\ p\}$ *is a triple in* $S(7)$.

Every row or column of the 7×7 square $C = c_{i,j}$, i, $j \in \{1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7\}$ form a $STS(7)$. We will name some subsets of C. Any set of 7 elements no two of them are on the same row or column will be named as a diagonal. If no three element subset of a diagonal is a triple in $S'(49)$ then we call it an M-diagonal. If it contains at least one triple, we call it a P-diagonal, if a P-diagonal is isomorphic to $S(7)$ we call it an S-diagonal.

Suppose we have the three colors, red, blue, and green. Since $Chr(STS(7)) = 3$, in each row and column, there must be at least one element from each of the three color classes.

In an M-diagonal, no triple of the $S'(49)$ is included and there is an M-diagonal whose all elements are colored by the same color, say blue. Call this diagonal D. The set of elements, which can not be colored blue since they form together with the elements of D a triple in $S'(49)$, will be denoted as $\alpha(D)$. This set contains 21 elements. Suppose we have an element $c_{x,y} \in \alpha(D)$ if namely c_{i_1,j_1}, c_{i_2,j_2}, c_{i_3,j_3}, c_{i_4,j_4} are in D, while $\{c_{i_1,j_1}, c_{i_2,j_2}, c_{x,y}\}$ and $\{c_{i_3,j_3} c_{i_4,j_4}, c_{x,y}\}$ form triples in $S'(49)$, then the remaining three elements of D would form a triple in $S'(49)$ since $\{i_5, i_6, x\}$ and $\{j_5, j_6, y\}$ must be triples in $S(7)$. However, this can not be the case since D is an M-diagonal. Except the elements in D, we have 21 elements in $\alpha(D)$. Name these elements by $\beta(D)$. In each row and column, because of the properties of $S(7)$ there are exactly 3 elements of $\alpha(D)$ and 3 elements of $\beta(D)$.

We will now show that for some column or row, the three elements of the row which belong to $\beta(D)$ is a triple in $S'(49)$. If we denote the elements of D by c_{i,j_i}, $i = 1, 2, 3, 4, 5, 6, 7$ and the elements of the k^{th} row belonging to $\alpha(D)$ by $c_{k,x}, c_{k,y}, c_{k,z}$ then there will be the following triples: $\{c_{s_1,t_1}, c_{s_2,t_2}, c_{k,x}\}$, $\{c_{s_3,t_3}, c_{s_4,t_4}, c_{k,y}\}$, $\{c_{s_5,t_5}, c_{s_6,t_6}, c_{k,z}\}$ in $S'(49)$. The first two elements in each of the three triples are from D, the elements s_1, ..., s_7 and t_1, ..., t_7 form a permutation of the elements 1, ..., 7 with $s_7 = k$ and $t_7 = j_k$. Then, $\{t_1, t_2, x\}$, $\{t_3, t_4, y\}$, $\{t_5, t_6, z\}$ are triples in $S(7)$. Because of the structure of $S(7)$, x is equal to t_3 or t_4 or y is equal to t_1 or t_2. Suppose $x = t_3$. Then t is equal to t_1 or t_2. With the previous assumptions, y is equal to either t_5 or t_6. Assuming $y = t_5$ implies $z = t_1$. So we have the triples $\{t_1, t_2, t_3\}$, $\{t_3, t_4, t_5\}$, $\{t_5, t_6, t_1\}$. For the elements t_1, t_2, t_3, t_4, t_5, t_6, t_7, the $STS(7)$ with the above three triples must have the following four triples: $\{t_1, t_4, t_7\}$, $\{t_2, t_5, t_7\}$, $\{t_3, t_6, t_7\}$, $\{t_2, t_4, t_6\}$. As a result, c_{k,t_2}, c_{k,t_4}, c_{k,t_6} are the elements of the k^{th} row belonging to $\beta(D)$ and they form a triple in $S'(49)$.

We will now prove that for an M-diagonal D, for each element $c_{x,y} \in \beta(D)$, there is an S-diagonal, namely $D^\star$ which contains the element $c_{x,y}$ and 6 elements from $\alpha(D)$, in addition, D and $D^\star$ are disjoint. To prove this claim, we will first prove the following statement. Let $c_{m,n}$ and $c_{k,l}$ be two elements of an M-diagonal D, then one of the c_m, l and c_k, n elements belong to $\alpha(D)$ and the other belongs to $\beta(D)$. Suppose $c_{m,l}$ and $c_{k,n}$ are both in $\alpha(D)$, name the remaining elements of the diagonal D as $C = \{c_{a_1,b_1},$ $c_{a_2,b_2}, \dots , c_{a_5,b_5}\}$ where $\{m, k, a_1, ..., a_5\} = \{l, n, b_1, ..., b_5\} = \{1, 2, ..., 7\}$. $c_{m,l}$ is known to be in $\alpha(D)$, therefore, two elements of C namely c_{a_1,b_1}, c_{a_2,b_2} form a triple in $S'(49)$ with $c_{m,l}$. Similarly, $c_{k,n}$ is known to be in $\alpha(D)$, therefore, two elements of C namely c_{a_3,b_3}, c_{a_4,b_4} form a triple in $S'(49)$ with $c_{k,n}$. But then, in $S(7)$ we would have the triples $\{a_1, a_2, m\}$ and $\{a_3, a_4, k\}$ but this can not be the case since in a $S(7)$, no pair of triples is disjoint. Assume now, $\{c_{a_1,b_1}, c_{a_2,b_2}, c_{m,l}\}$ and $\{c_{a_1,b_1}, c_{a_3,b_3}, c_{k,n}\}$ are the two triples in $S'(49)$. Again because of the structure of $S(7)$, this would imply having $\{c_{a_1,b_1}, c_{a_4,b_4}, c_{a_5,b_5}\}$ as a triple in $S'(49)$. This is impossible since D is an M-diagonal. If we suppose $c_{m,l}$ and $c_{k,n}$ both belong to $\beta(D)$, then there is a pair of elements of D such that, since $\alpha(D)$ and $\beta(D)$ have the same number of elements, the corresponding "cross-elements" belong to $\alpha(D)$. So, our claim "one of the elements $c_{m,l}$ and $c_{k,n}$ belong to $\alpha(D)$ and the other belongs to $\beta(D)$" holds.

Consider D to be an M-diagonal and let $c_{kl} \in \beta(D)$. Let $c_{a,l}$ and $c_{k,b}$ be elements of D. By what we have just proved, since $c_{k,l}$ belongs to $\beta(D)$, $c_{a,b}$ belongs to $\alpha(D)$. Furthermore, there must be two elements in D, namely $c_{x,y}$ and $c_{u,v}$ such that $\{c_{x,y},$ $c_{u,v}, c_{a,b}\}$ forms a triple in $S'(49)$. Similarly, for the following four elements $c_{x,y}$, c_{al}, $c_{k,b}$, $c_{u,v}$ there is a corresponding set of 6 elements which belong to $\alpha(D)$ (there are 6 possible pairs and for each pair we have an element in $\alpha(D)$). These 6 elements together with the element $c_{k,l}$ forms a set of 7 elements which we will call $D^\star$. We will now show, this set is in fact a S-diagonal. The first indices are a, k, x, u. The other elements of the $S(7)$ will be w, m, n. When one of the triples is known to be $\{x, u, a\}$, the others are expected to be as follows: $\{a, k, w\}$, $\{x, k, m\}$, $\{k, u, n\}$, $\{a, m, n\}$, $\{x, w, n\}$, $\{w, u, m\}$. For the second indices b, l, y, v, we apply a similar procedure; we take z, p, q as the extra elements to complete these to an element set of a $STS(7)$ for the extension of the triple $\{y, v, b\}$ to a $S(7)$. The other 6 triples are: $\{y, l, p\}$, $\{l, b, z\}$, $\{l, v, q\}$, $\{b, p, q\}$, $\{y, z, q\}$, $\{v, z, p\}$. So, $D^\star = \{c_{k,l}, c_{a,b}, c_{w,z}, c_{x,q}, c_{u,p}, c_{m,v}, c_{n,y}\}$ is an S-diagonal. We have shown that for an M-diagonal D, for each element $c_{x,y}$ in $\beta(D)$, there is an S-diagonal, namely $D^\star$ which contains the element $c_{x,y}$ and 6 elements from $\alpha(D)$, in addition, D and $D^\star$ are disjoint.

Among the blue vertices, there must be an M-diagonal D. We proved that for each element $c_{k,l} \in \beta(D)$, there is an S-diagonal namely $D^\star$ disjoint from D, containing $c_{k,l}$ with other 6 elements of $\alpha(D)$. $D^\star$ is isomorphic to $S(7)$, so one of its elements at least should be blue. If not $Chr(S'(49))$ would be more than 3. Since elements of $\alpha(D)$ can not be blue, $c_{k,l}$ is blue. But $c_{k,l}$ is an arbitrary element in $\beta(D)$, so all the elements of $\beta(D)$ are blue. But some three element subsets of $\beta(D)$ are triples in $S'(49)$, so, $S'(49)$ can not be 3-chromatic.

The next step is to show $Chr(S'(49)) \leq 4$ which will result $Chr(S'(49)) = 4$.

Let the triples in $S(7)$ be defined as $\{i, i+1, i+3\}$ (mod 7), $i = 1, 2, 3, 4, 5, 6, 7$. Take an M-diagonal $D = \{c_{1,7}, c_{2,6}, c_{3,5}, c_{4,4}, c_{7,1}, c_{6,2}, c_{5,3}\}$. For $i \in \{1, 2, ..., 7\}$, $D_i = \{c_{1,(7+i)}, c_{2,(6+i)}, c_{3,(5+i)}, c_{4,(4+i)}, c_{7,(1+i)}, c_{6,(2+i)}, c_{5,(3+i)}\}$ (mod 7). Then color the elements of D, D_1 by blue, the ones of D_2, D_3 by red, the ones of D_4, D_5 by green, and the ones of D_6 by brown. With this coloring, no triple will be monochromatic. So $Chr(S'(49)) = 4$. ■

Rosa concluded the following:

Corollary 3.1.10 *Suppose there are two STSs named $\mathcal{S}_1$, $\mathcal{S}_2$ of orders n_1 and n_2 respectively, both containing a subsystem of order 7. Then, $Chr(\mathcal{S}) \geq 4$, where $\mathcal{S}$ is the Steiner triple system obtained by the product of $\mathcal{S}_1$ and $\mathcal{S}_2$.*

Theorem 3.1.11 *For any $v \equiv 1\,,3\,(mod 6)$, $v \geq 123$, there exists a $STS(v)$ $\mathcal{S}$, where $Chr(\mathcal{S}) \geq 4$.*

Proof: Rosa used recursive methods to prove this theorem.

We know that there is a STS of order 31 (namely $\mathcal{S}^\star(31)$) and a STS of order 49 (namely $\mathcal{S}'(49)$) whose chromatic number is 4. In Theorem 1.3.5:

(i) By letting $n = 31$ and $k = 1, 7, 13, 19, 25, 31$ respectively, we can conclude there are STSs of order $63, 69, 75, 81, 87,$ and 93 containing a subsystem $S^\star(31)$ having chromatic number $k \geq 4$.

(ii) By letting $n = 49$ and $k = 1, 7, 13, 19, 25,$ respectively, we can conclude there are STSs of order $99, 105, 111, 117,$ and 123 containing a subsystem $S'(49)$, with chromatic number $k \geq 4$.

Recall Lemmas 3.1.8 and 3.1.9. In Lemma 1.3.4:

(i) If we let $n_1 = 3$, $n_2 = 31$, and $n_3 = 7$, we obtain a Steiner triple system of order 79.

(ii) If we let $n_1 = 3$, $n_2 = 31$, and $n_3 = 1$, we obtain a Steiner triple system of order 91.

Both of the above systems include $S^\star(31)$ as a subsystem, so have chromatic number $k \geq 4$.

We obtained $STS(63)$ and $STS(93)$ previously. Applying Corollary 1.3.7:

(i) For any admissible order k such that $127 \leq k \leq 189$ we obtain a $\mathcal{S}(k)$,

(ii) For any admissible order t such that $187 \leq t \leq 279$ we obtain a $\mathcal{S}(k)$.

By Theorem 1.3.5 for any $n \geq 279$:

(i) Let $|S_1| = 7$, and $|S_2| = 6k - 3$ in Theorem 1.3.5 to obtain a $STS(n)$, where $n = 12k + 1$.

(ii) Let $|S_1| = 1$, and $|S_2| = 6k + 1$ in Theorem 1.3.5 to obtain $STS(n)$, where $n = 12k + 3$.

(iii) Let $|S_1| = 1$, and $|S_2| = 6k + 3$ in Theorem 1.3.5 to obtain $STS(n)$, where $n = 12k + 7$.

(iv) Let $|S_1| = 7$, and $|S_2| = 6k + 1$ in Theorem 1.3.5 to obtain $STS(n)$ where $n = 12k + 9$.

Since all of those constructed STSs include isomorphic copies of STSs of smaller orders which are at least 4-chromatic, they are also at least 4-chromatic. ∎

Corollary 3.1.12 *For every $v \equiv 1$ or $3\,(mod\,6)$ there are at least two nonisomorphic STSs.*

For any admissible v, we have a 3-chromatic STS and a STS which can be colored by at least k colors, where $k \geq 4$. But two STSs with different chromatic numbers are nonisomorphic, this existence proves the existence of two nonisomorphic STSs. It is not easy to show that a STS is not m-chromatic for some $m \geq 4$.

3.2 Studies of Phelps, Brandes, and Rödl

Phelps, Brandes, and Rödl made important contributions to the weak coloring problems of the STSs. In their study [5], several results on the chromatic number of Steiner triple systems are established. They proved that for any $k \geq 3$ there exists an n_k such that for all admissible $v \geq n_k$, there exists a k-chromatic Steiner triple systems of order v. In addition they proved that for all $v \geq 49$ there exists a 4-chromatic Steiner triple system of order v.

Lemma 3.2.1 [5] *There exists a 4-chromatic $STS(v)$ for $v = 25, 27, 33,$ and 37.*

Proof: For the above orders, the points of the Steiner triple systems are considered to be $1, 2, ..., v$ and STSs are cyclic, that is, there is a map $\Phi : \mathcal{S} \rightarrow \mathcal{S}$ such that $\Phi(i) = i{+}1\,(\text{mod}\,v)$ for all i in $\mathcal{S}$ which satisfies the condition that $\{i, j, k\} \in T$ implies $\{\Phi(i), \Phi(j), \Phi(k)\} \in T$. Computational methods are used to show these STSs are not 3-chromatic.

v = 25

Base triples: $\{1, 2, 4\}, \{1, 5, 24\}, \{1, 6, 12\}, \{1, 8, 18\}$.

Color classes: {1, 2, 3, 6, 7, 8, 11} {5, 9, 10, 13, 15, 14, 19}, {12, 16, 17, 18, 21, 22}, {4, 20, 23, 24, 25}.

v = 27

Base triples: $\{1, 2, 4\}, \{1, 5, 12\}, \{1, 6, 18\}, \{1, 7, 15\}$.

Color classes: {1, 2, 3, 6, 7, 8, 11, 12, 22, 25}, {10, 13, 15, 14, 19, 18, 20, 23, 24}, {4, 5, 9, 17, 26}, {16, 21, 27}.

v = 33

Base triples:$\{1,2,4\},\{1,5,15\},\{1,6,14\},\{1,7,19\}$.

Color classes: $\{1, 2, 3, 6, 7, 8, 12, 13, 30\}$, $\{15, 14, 16, 19, 20, 21, 24, 25, 26, 31\}$, $\{4,5,11,27,28,29,32,33\}$, $\{9,10,17,18,2,23\}$.

v = 37

Base triples: $\{1,2,4\},\{1,5,15\},\{1,6,14\},\{1,7,22\}$.

Color classes: $\{1,2,3,6,7,8,11,12,13,18,32\}$, $\{15,14,16,19,20,21,24,25,26,37\}$, $\{4,9,23,29,30,31,34,35,36\}$, $\{5,10,17,22,27,28,33\}$. ■

Theorem 3.2.2 [5] *If there exists a k-chromatic $STS(v)$, then there exists a k-chromatic $STS(2v+1)$.*

Proof: Let $(\mathcal{S}, B)$ be a k-chromatic $STS(v)$, where $\mathcal{S} = \{a_1, a_2, ..., a_v\}$. Consider two disjoint sets T_1 and T_2, with $T_1 \cup T_2 = \mathcal{T}$, $\mathcal{T} \cap \mathcal{S} = \emptyset$ where $|\mathcal{T}| = v+1$ which is an even number. Let $v+1 = 2n$, $|T_1| = n = |T_2|$.

Case (a): $n \equiv 0 \pmod 2$

Take a 1-factorization $(\mathcal{T}, F)$ of K_{2n}, whose vertices represent elements of $\mathcal{T}$. $F = \{F_1, ..., F_{2n-1}\}$. Let the first $n-1$ of them be in such a way that they are unions of two isomorphic copies of 1-factorizations of K_n, namely (T_i, F^i) for $i = 1, 2$, one for T_1, one for T_2. So, if $F^i = \{F_1^i, \ldots, F_{n-1}^i\}$, for $j = 1,\ 2,\ \ldots n-1$, let $F_j = F_j^1 \cup F_j^2$.

Case (b): $n \equiv 1 \pmod 2$

Take a 1-factorization $(\mathcal{T}, F)$ of K_{2n}, whose vertices represent elements of $\mathcal{T}$. $F = \{F_1, ..., F_{2n-1}\}$. Let the first n of them be in such a way that they are unions of two isomorphic copies of near 1-factorizations of K_n, namely (T_i, F^i) for $i = 1, 2$, one for T_1, one for T_2, plus an edge which is selected as the edge between the vertex which is out of the near 1-factorization of T_1 namely x_j and the vertex which is out of the near 1- factorization of T_2, namely $\alpha(x_j)$. That is $F_j = F_j^1 \cup F_j^2 \cup \{x_j, \alpha(x_j)\}$, $j = 1, 2, ..., n$ where α is a bijection between vertices of two copies of K_n and $F^i = \{F_1^i, \ldots, F_n^i\}$. The existence of such a 1-factorization of K_{2n} where n is odd, is known [2].

For both of the cases, let $S^* = \mathcal{S} \cup \mathcal{T}$, $B^* = B \cup D$ where $D = \{\{a_i, x, y\} : x, y \in F_i,\ i = 1, 2, ..., 2n-1\}$. We will first show (S^*, B^*) is a STS. To do this we will check if any pair of elements of S^* is seen in exactly one triple. If both of these elements, namely x and y are from $\mathcal{S}$, then they are seen exactly once in B, so only once in B^*. If both of them are from $\mathcal{T}$, the edge $\{x, y\}$ will be in exactly one 1-factor, call it F_k. Then the only triple in B^* containing the pair x, y would be $\{x, y, a_k\}$. If $x \in \mathcal{S}$, $y \in \mathcal{T}$, then $x = a_i$ for some $i \in \{1, 2, ..., 2n-1 = v\}$, and $y \in \mathcal{T}$ then find the vertex adjacent to y in the 1-factor F_i, say z, then $\{x, y, z\}$ would be a triple in B^*. The more interesting problem is to show the new system is still k-chromatic. Color the elements of T_1 black, elements of T_2 white, and let the elements of $\mathcal{S}$ have the same color as in the coloring C. There are no monochromatic triples, this is obviously true for triples of B. If $\{a_i, x, y\}$ is a triple of D with $i \in \{1, 2, ., n\}$, then a_i is colored by one of the $k-2$ colors other than black or white while $x, y (\in T)$ can be only black or white. If $i \in \{n+1, n+2, .., 2n-1\}$ then one of x, y is black and the other is white.■

Theorem 3.2.3 *If $v \equiv 1$ or 9 $(mod\, 12)$, and there exist a k-chromatic $STS(v)$, then there exists a k-chromatic $STS(2v+7)$.*

Proof: Let (S, B) be a k-chromatic $STS(v)$ where $\mathcal{S} = \{a_1, a_2, ..., a_v\}$. Let $v + 7 = 2m$. Since $v \equiv 1$ or 9 $(mod\, 12)$, m is even. Let $\mathcal{T} = X \cup Y$ where $X \cap Y = \emptyset$, $|X| = m = |Y|$. Let (X, F), $F = \{F_1, ..., F_{m-1}\}$ a 1-factorization of K_m whose two 1-factors give an hamiltonian circuit when their union is taken. Call them F_{m-2} and F_{m-1}. Let $F_{m-2} \cup F_{m-1} = (x_1, x_2, ..., x_m, x_1)$, where $x_i \in X$, $i = 1, 2, ..., m$. The existence of such a 1-factorization is known [22]. We define the new triples as follows:

$C = \{\{y_i, x_{i+3}, x_{i+4}\}, \{y_i, y_{i+1}, x_{i+2}\} \mid i = 1, 2, ..., m\}$,

$D = \{\{a_i, x_p, x_q\}, \{a_i, y_p, y_q\} \mid \{x_p, x_q\} \in F_i, i = 1, 2, ..., m-3\}$,

$E = \{\{a_{m-2+k}, x_j, y_{j+k}\} \mid j = 1, 2, ..., m; k = 0, 1, ..., m-5\}$,

$S^* = S \cup T$, $B^* = B \cup C \cup D \cup E$ (the subscripts of x's and y's reduced modulo m when necessary).

(S^*, B^*) is a $STS(2v+7)$. The triples which include pairs of elements from $\mathcal{S}$ namely a_i, a_j pairs is only seen in B. The pairs of the form x_i, x_j are seen in 1-factors of X. An edge $\{x_i, x_j\}$ with the condition that $|j - i| = 1$ if and only if it is in F_{m-1} or in F_{m-2}. So for $|j-i|=1$, $\{x_i, x_j\}$ is in C, otherwise it is in D and there is a unique triple including this pair for the second case since there is only one 1-factor including every edge. For the first case, it is obvious. For the pairs $\{y_i, y_j\}$ where $|j-i|>1$ the situation is same as above. For $|j-i|=1$, $\{y_i, y_j\}$ pairs are in C. For the pairs $\{x_i, y_j\}$ where $1 \leq |j-i| \leq 4$, it is easy to see the pairs are in C. For $0=|j-i|$ and $|j-i|>4$, the pairs are in E. The remaining pairs are $\{a_i, y_j\}$ and $\{x_i, a_j\}$ pairs. Consider the 1-factor F_i. It spans the complete graph of order m. So, for any vertex x_j there is an edge in F_i which contains this vertex. So there is a triple including the pair $\{a_i, x_j\}$ for $i=1,2,...,m-3$. The same argument also works for $\{a_i, y_j\}$.

To show that (S^*, B^*) is k-chromatic, color the elements of X black, those of Y white and those of S as in the coloring of (S, B). There are no monochromatic triples in B^*. This is obvious for triples of B, C and E as the latter two contain only triples with at least one black and at least one white element. On the other hand, no element a_i with $i \in \{1,2,...,m-3\}$ is colored black or white; thus no triple of D can be monochromatic. ■

Theorem 3.2.4 *If $v \equiv 3$ or 7 $(mod\, 12)$, $k \geq 5$, and there exists a biased k-chromatic $STS(v)$, then there exists a k-chromatic $STS(2v+7)$.*

Proof: Let $(\mathcal{S}, B)$ be a k-chromatic $STS(v)$ with $\mathcal{S} = \{a_1, a_2, ..., a_v\}$ and C be a biased k-coloring of the system. Let $v+7=2m$. Let $X=\{x_1, x_2, ..., x_m\}$ and $Y=\{y_1, y_2, ..., y_m\}$ be two disjoint sets where $X \cup Y = \mathcal{T}$ and $\mathcal{T} \cap \mathcal{S} = \emptyset$. Let (X, F) be a near 1-factorization, where $F=\{F_1, F_2, ..., F_m\}$ containing two near 1-factors, namely F_{m-1} and F_m whose union is a hamiltonian path $F_{m-1} \cup F_m = (x_1, x_2, ..., x_m)$. Such a near 1-factorization is known to exist [2]. Suppose also the edge $\{x_1, x_m\} \in F_{m-2}$.

Let $C = \{\{y_i, x_{i+3}, x_{i+4}\}, \{y_i, y_{i+1}, x_{i+2}\} \,|\, i=1, 2, ..., m\}$,

$D = \{\{a_i, x_p, x_q\}, \{a_i, y_p, y_q\} \,|, \{x_p, x_q\} \in F_i, i=1, 2, ..., m-3\}$,

$D' = \{\{a_i, x_{j(i)}, y_{j(i)}\} \,|\, i=1, 2, ..., m-2, x_{j(i)}$ is an isolated vertex of $F_i\}$,

$D'' = \{\{a_{m-2}, x_p, x_q\}, \{a_{m-2}, y_p, y_q\} \,|\, \{x_p, x_q\} \in F_{m-2} \backslash \{x_1, x_m\} \cup \{\{a_{m-2}, x_1, y_1\}, \{a_{m-2}, y_m, x_m\}\}$,

$E = \{\{a_{m-2+k},\, x_j,\, y_{j+k}\} \mid j = 1, 2, ..., m;\ k = 1, 2, ..., m-5\}$.

(The subscripts are considered to be in modulo m.) Let $\mathcal{S}^* = \mathcal{S} \cup \mathcal{T}$, $B^* = B \cup C \cup D \cup D' \cup E$. $(\mathcal{S}^*, B^*)$ is a $STS(2v+7)$. $(\mathcal{S}^*, B^*)$ is also k-chromatic. If elements of X are black, elements of Y are white and elements of $\mathcal{S}$ are as in the coloring of $(\mathcal{S}, B)$, then there are no monochromatic triples in B^*. Triples of B, C, D' and E are not also monochromatic. Since $m - 2 = \frac{v+3}{2}$ and the coloring of $(\mathcal{S}, B)$ is biased, neither of the elements $a_1, a_2, ..., a_{m-2}$ is black or white; so no triple of D or D' can be monochromatic. ■

Lemma 3.2.5 *There exists a 4-chromatic $STS(v)$ for every admissible $v \geq 49$.*

Proof: There are three cases:

(**i**) $\mathbf{v \equiv 1\ (\bmod 6)}$ Recall that in the Skolem Construction, we use an half idempotent commutative quasigroup. For any even number $2t$, there exist a half idempotent commutative quasigroup of order $2t$ which contains a half idempotent commutative quasigroup of order 8. Here notice that with the given constraint, our set is of order $6t+1$ where $2t \geq 18$. Therefore, such a $STS(6t+1)$ will include a $STS(25)$. But we have a 4-chromatic $STS(25)$. Then we can replace the 3-chromatic subSTS with the 4-chromatic STS, but we will make the three of the color classes of the new $STS(25)$ in a way that they are subsets of the old $STS(25)$ with color classes of sizes 8, 8, and 9.

(**ii**) $\mathbf{v \equiv 3\ (\bmod 6)}$ We follow a procedure similar to above, this time, our idempotent commutative quasigroups are known to exist for every odd number say $2t+1$ where $v = 3(2t+1)$. There exists a commutative idempotent quasigroup of order $2t+1 \geq 19$, which contains a commutative idempotent subquasigroup of order 9. We will use the 4-chromatic $STS(27)$, and replace the 3-chromatic subsystem by the 4-chromatic $STS(27)$ again with the condition that the color classes are subsets of the old $STS(27)$.

(**iii**) $\mathbf{v = 49}$ **or** $\mathbf{51}$ For $v = 49$ Rosa gave a solution (see Lemma 3.1.9). For $v = 51$ recall that we have a 4-chromatic $STS(25)$, recall Theorem 3.2.2; for k-chromatic $STS(v)$, we have a k-chromatic $STS(2v+1)$. ■

The following theorem follows from the previously established Lemmas 3.1.9, 3.1.8 and 3.2.1.

Theorem 3.2.6 *There exists a* 4*-chromatic* $STS(v)$ *for all* $v \geq 25$*,* $v \equiv 1$ *or* 3 $(mod\ 6)$ *except possibly* 39*,* 43*, and* 45*.*

By using Bose Construction, we gave an example of a 3-chromatic $STS(21)$. Haddad constructed a 4-chromatic $STS(21)$ [17]. Forbes, Grannell and Griggs [15] prove that every $STS(21)$ is 5-colorable. They then showed every $STS(21)$ is 4-colorable [14]. The chromatic number of any $STS(21)$ is either 3 or 4.

Haddad used the following construction to show the existence of a 4-chromatic $STS(21)$. A similar construction was used in [5], for the proof of Theorem 5.1.5. Let $(\mathcal{S}, T)$ be an $STS(v)$ with $\mathcal{S} = \{0, 1, ..., v-1\}$. Define the $STS(3v)$ $(\mathcal{S}', T')$ by $\mathcal{S}' = \{(x, i), x \in \mathcal{S}, i = 1, 2, 3\}$ and for every triple $\{x, y, z\} \in T$, the following triples are included in T': $\{(x, 1), (y, 1), (z, 1)\}$, $\{(x, 1), (x, 2), (x, 3)\}$, $\{(y, 1), (y, 2), (y, 3)\}$, $\{(z, 1),(z, 2), (z, 3)\}$, $\{(x, 2), (z, 1), (y, 2)\}$, $\{(x, 2), (z, 2), (y, 1)\}$, $\{(x, 2), (z, 3), (y, 3)\}$, $\{(x, 1), (y, 3), (z, 2)\}$, $\{(x, 1), (z, 3), (y, 2)\}$, $\{(x, 3), (z, 1), (y, 3)\}$, $\{(x, 3), (y, 1), (z, 3)\}$, $\{(x, 3), (y, 2), (z, 2)\}$. Note that the triples $\{(x, 1), (x, 2), (x, 3)\}$, $\{(y, 1), (y, 2), (y, 3)\}$, $\{(z, 1), (z, 2), (z, 3)\}$, are included only once. Suppose that $(\mathcal{S}, T)$ is k-chromatic, let C be a set of colors whose order is k, $\Phi : \mathcal{S} \to C$ be a k-coloring of it. $(\mathcal{S}', T')$ includes an isomorphic copy of $(\mathcal{S}, T)$, so is at least k-chromatic. $\Phi' : \mathcal{S}' \to C \cup \{\infty\}$, where ∞ is a color which is not in C. For any $x \in \mathcal{S}$, $\Phi'(x, 1) = \Phi'(x, 2) = \Phi(x)$ and $\Phi(x, 3) = \infty$. This is a proper $k + 1$ coloring, so, chromatic number of the system is either k or $k + 1$.

The following results belong to Haddad. He studied on the $STS(21)$ which can be obtained from a cyclic $STS(7)$ and on the $STS(39)$ which can be obtained from a cyclic $STS(13)$ by the above construction. By using some computational methods, he proved for this $STS(21)$ and this $STS(39)$ that these two can not be colored by 3 colors, but can be colored by 4-colors properly. The only values in doubt were $19, 21, 39, 43$ and 45 [5]. The existence of 4-chromatic $STS(21)$ implies the existence of 4-chromatic $STS(43)$ and 4-chromatic $STS(45)$ (See Theorem 4.1.5) [5]. It can be deduced that [17]:

Theorem 3.2.7 *There exists a 4-chromatic $STS(v)$ for every admissible $v \geq 21$. In particular $n_4 \leq 21$.*

So the only order for which we are not sure about the existence of a 4-chromatic STS is 19.

About the existence of k-chromatic Steiner triple systems for large k, some results proven by Brandes, Phelps and Rödl will be discussed in the fifth chapter.

Chapter 4

FINITE GEOMETRIES AND STEINER TRIPLE SYSTEMS

In this chapter, we will deal with the solutions found to the problem of finding specific examples of k-chromatic Steiner triple systems for $k > 4$. Up to date, only 5-chromatic and 6-chromatic STSs could be found. The first example of a 5-chromatic STS took place in the study of Fugere, Haddad, and Wehlau [16]. Bruen, Haddad, and Wehlau gave the first specific example of a 6-chromatic STS [6].

4.1 5-chromatic STS

Previously, Rosa worked on the coloring properties of the totally associative Steiner triple systems which we denote as $S^\star(2^n-1)$. A $S^\star(2^n-1)$ can be constructed by considering F_2^n, the **vector space** of dimension n **over the field** of order 2. Here, the **points** are one dimensional subspaces of F_2^n or the points of the projective $n-1$-space, and the triples are two dimensional subspaces or the **lines** of the projective $n-1$-space, $\mathbb{P}G(n-1, 2)$. Each such subspace is represented by the non-zero vector contained in it. For the sake of simplicity, if $\{e_1, \ldots, e_n\}$ is a basis of F_2^n and x is an element of $S^\star(2^n-1)$, if $x = e_1 + e_2 + e_3$ in the given basis then we write it as 123. The following example is a $S^\star(15)$. Recall that for a vector space of dimension n over the field F, the projective space $\mathbb{P}G(n, F)$ is the geometry whose points, lines, planes, . . . are the vector subspaces of the vector space of dimensions $1, 2, 3, \ldots$.

Example 4.1.1 *Consider F_2^4. The basis elements of the system can be denoted as follows: $e_1 = \langle 0,0,0,1\rangle$, $e_2 = \langle 0,0,1,0\rangle$, $e_3 = \langle 0,1,0,0\rangle$, and $e_4 = \langle 1,0,0,0\rangle$. We denote the points of the STS as:* 1, 2, 12, 3, 13, 23, 123, 4, 14, 24, 124, 34, 134, 234,

1234. *The triples are as follows:* $\{1, 2, 12\}$, $\{1, 3, 13\}$, $\{1, 4, 14\}$, $\{1, 234, 1234\}$, $\{1, 23, 123\}$, $\{1, 24, 124\}$, $\{1, 34, 134\}$, $\{2, 3, 23\}$, $\{2, 4, 24\}$, $\{2, 14, 124\}$, $\{2, 13, 123\}$, $\{2, 34, 234\}$, $\{12, 3, 123\}$, $\{12, 13, 23\}$, $\{12, 4, 124\}$, $\{12, 24, 14\}$, $\{12, 34, 1234\}$, $\{3, 4, 34\}$, $\{3, 14, 134\}$, $\{3, 24, 234\}$, $\{3, 124, 1234\}$, $\{23, 4, 234\}$, $\{2, 134, 1234\}$, $\{123, 4, 1234\}$, $\{13, 24, 1234\}$, $\{13, 4, 134\}$, $\{13, 14, 34\}$, $\{13, 234, 124\}$, $\{23, 14, 1234\}$, $\{23, 24, 34\}$, $\{23, 124, 134\}$, $\{123, 134, 24\}$, $\{123, 234, 14\}$, $\{123, 124, 34\}$, $\{12, 134, 234\}$.

We will call elements of $S^\star(2^n-1)$ words and we say for instance 123 is a word of length 3. Note that these words are commutative. If a vector in F_2^n is written as a sum of k basis elements, then the corresponding word will be of length k. A three element subset of the set $S^\star(2^n-1)$ is a triple if and only if the sum of the corresponding vectors in F_2^n is $\vec{0}$, the zero vector.

Rosa showed that $Chr(S^\star(31)) = 4$ and gave a constraint on the chromatic number of such systems: $Chr(S^\star(2^n-1)) \leq Chr(S^\star(2^{n+1}-1)) \leq Chr(S^\star(2^n-1)) + 1$.

J. Pelikán [27] showed also that $Chr(S^\star(31)) = 4$.

Recall that a subset of $V(S^\star(2^n-1))$ is called **independent** if it does not contain any lines or in other words, it does not contain any triples. Moreover, it is called linearly independent if it is a linearly independent subset of F_2^n. The study of Fugere, Haddad, and Wehlau [16] shows $Chr(S^\star(63)) = 5$. This is done by studying **independent sets** and the size of the sets of intersection of the independent sets sets with hyperplanes.

Furthermore, it is shown that if a k-chromatic $STS(v)$ exists for some $v \equiv 3 \pmod 6$ with $k \geq 5$, then a k-chromatic $STS(w)$ exists for every admissible $w \geq 2v+1$. From these results, it can be deduced that for every admissible $v \geq 127$, there is a 5-chromatic $STS(v)$.

For $2 \leq k \leq n-1$, a subset $K \subseteq V(S^\star(2^n-1))$ is a k-flat if $K \cup \{\vec{0}\}$ is a $(k+1)$-dimensional subspace of F_2^n. The set of triples T restricted to a k-flat K induces the structure of a Steiner triple system isomorphic to $S^\star(2^k-1)$ on K.

For any $X \subseteq V(S^\star(2^n-1))$ we denote the smallest k-flat containing X as $\langle X \rangle$. A plane is a 1-flat while a hyperplane is an $(n-2)$-flat, that is, an $n-1$-dimensional subspace of F_2^n.

A subset of $V(S^\star(2^n-1))$, namely $K(v)$ is an hyperplane if and only if there is a vector v in $V(S^\star(2^n-1))$ such that for all u in $K(v)$ the scalar product results in $\overrightarrow{0}$, that is $uv = \overrightarrow{0}$.

4.1.1 Main results

The following is proved by Pelikán [27]:

Proposition 4.1.2 *In any* 3*-coloring of* $S^\star(15)$ *all color classes have the same size and every plane* (1*-flat*) *contains at most* 3 *points of the same color.*

This means any color class consists of 4 linearly independent vectors and the linear combinations of these vectors.

Proposition 4.1.3 $\mathcal{S}^\star(15)$ *is uniquely colorable up to an isomorphism.*

Proof: Let $\langle 0,1,2,3 \rangle$ be a copy of $\mathcal{S}^\star(15)$ with $\phi : V(\mathcal{S}^\star(15)) \to \{1,2,3\}$ a 3-coloring. Set $S_i = \Phi^{-1}(i)$, $i = 1,2,3$. By Proposition 4.1.2, we can deduce no matter how we 3-color $\mathcal{S}^\star(15)$, any color class will consist of 4 linearly independent vectors and their sum. We may apply an element of the automorphism group of $\mathcal{S}^\star(15)$ to ensure that $\mathcal{S}_1 = \{0,1,2,3,0123\}$. Since there are 6 words of length two, one color class, say $\mathcal{S}_2$, must contain at least three of them. Call those three words of length two a, b and c. Since a, b and c are independent, they generate a plane P. Now P contains only words of even length and by Proposition 4.1.2, $P \cap \mathcal{S}_2 = \{a, b, c\}$. This shows that for $i = 2, 3$, $\mathcal{S}_i$ contains exactly three words of length two and two words of length three. Since $\mathcal{S}_2$ contains three words of length two, at least two of them must have a common letter. Put $\{i, j, k, s\} = \{0, 1, 2, 3\}$ and let ij, $ik \in \mathcal{S}_2$. Then kj is not in $\mathcal{S}_2$ and moreover $is \in \mathcal{S}_2$ would imply $\{jk, js, ks\} \subseteq \mathcal{S}_3$, a contradiction. Thus $\mathcal{S}_2$ must contain a subset of the form $\{ij, ik, js\}$. We want to show that given three words of length two in $\mathcal{S}_2$, the remaining two words of length three in $\mathcal{S}_2$ are uniquely determined. So let $\{ij, ik, js\} \subseteq \mathcal{S}_2$ and suppose that $\mathcal{S}_2$ contains the word jks. Then this would imply $ij + ik + js + jks = j \in \mathcal{S}_2$, a contradiction. By a similar argument we have iks is not in $\mathcal{S}_2$ and we are left with $\{ijk, ijs\} \subseteq \mathcal{S}_2$. Hence $\mathcal{S}_2 = \{ij, ik, js, ijk, ijs\}$. Since the group of permutations of $\{0, 1, 2, 3\}$ is a subgroup of the automorphism group of $\mathcal{S}^\star(15)$, the result is proved. ■

With the following color classes, we have a proper 3-coloring of $\mathcal{S}^\star(15)$: $C_1 = \{0, 1, 2, 3, 0123\}$, $C_2 = \{01, 12, 23, 012, 123\}$, $C_3 = \{02, 03, 13, 013, 023\}$.

Theorem 4.1.4 *$S^\star(63)$ is a 5-chromatic $STS(63)$.*

Proof: Rosa previously showed $Chr(S^\star(63))$ is either 4 or 5. We will show that it is not possible to color $S^\star(63)$ by 4 colors properly. By contradiction, suppose that $S^\star(63)$ is 4-colorable with color classes C_1, C_2, C_3 and C_4. Then one of these color classes, say C_4, has size at most 15. By Corollary 1.3.11, there is a hyperplane K such that $|C_4 \cap K| \leq 3$. The hyperplane K is generated by 5 independent points which we can denote by $0, 1, 2, 3, 4$. There is a 3-flat, $H \subset K$ such that $C_4 \cap H = \emptyset$. Without loss of generality $H = \langle 0, 1, 2, 3 \rangle$. The 4-coloring of $S^\star(63)$ induces a 3-coloring of H since $C_4 \cap H = \emptyset$. By Proposition 4.1.3, we may assume that $C_1 \cap H = \{0, 1, 2, 3, 0123\}$, $C_2 \cap H = \{01, 12, 23, 012, 123\}$, and $C_3 \cap H = \{02, 03, 13, 013, 023\}$. A computer program was used to check exhaustively that such a partition of the vertices of H together with the condition that $|C_4 \cap K| \leq 3$ cannot be extended to a 4-coloring of $S^\star(63)$ and thus we have a proof of the theorem. ■

The following Theorem which resembles Theorem 2.1.1, is proved with a similar method.

Theorem 4.1.5 *Let $k \geq 5$ and $v \equiv 3 \ (mod\ 6)$. If there exists a k-chromatic $STS(v)$, then there exists a k-chromatic $STS(w)$ for every admissible $w \geq 2v + 1$.*

Proof: Let $k \geq 5$ and $\mathcal{S}$ be a k-chromatic $STS(v)$ where $v = 6n+3$. Let $S_1, S_2, ..., S_k$ be the color classes and let $s_1 \geq s_2 \geq ... \geq s_k$ be the corresponding partition sequence where $|S_i| = s_i$ for $1 \leq i \leq k$. Then, $2n+1 > s_1 > s_2 > s_3$. We can find three sets of points of size $2n+1$, namely A_1, A_2, and A_3 where $S_1 \subseteq A_1$, $S_2 \subseteq A_2$, and $S_3 \subseteq A_3$. Let $w \geq 2v+1$ be an admissible integer. We have two cases:

Case 1: $\mathbf{w \equiv 1 (mod\ 6)}$. Then $w = 6u+1$ where $u \geq 2n+1$. There exists a half-idempotent commutative quasigroup $(Q, \circ)$ of order $2u$ containing an idempotent commutative quasigroup of order $2n+1$ [23]. By Skolem Construction with $(Q, \circ)$, we obtain a $STS(6u+1)$ containing a subsystem of order $6n+3$. Take this subsystem of order $6n+3$ out and replace it with a copy of $\mathcal{S}$ in such a way that $A_i \subseteq Q \times i$ for $i = 1, 2, 3$. This gives a k-chromatic $STS(w)$. Since it includes a k-chromatic subsystem, it is at least k-chromatic. It can also be colored properly by k-colors. Let $C = \{c_1, ..., c_k\}$ be the k different colors used to color $\mathcal{S}$, where $\mathcal{S}_i$ is the color class of c_i for $i = 1, ..., k$. Define Φ be the coloring, $\Phi : (Q \times \{1, 2, 3\}) \cup \{\infty\} \to C$ by $\Phi(x) := c_i$ if $x \in \mathcal{S}_i$, $\Phi(x) := c_j$ if $x \in Q \times \{j\} \backslash \mathcal{S}$, $j = 1, 2, 3$ and $\Phi(\infty)$ may be chosen to be any color. Φ is a k-coloring of our $STS(w)$.

Case 2: $\mathbf{w \equiv 3}$ (mod 6). Then $w = 6u+3 \geq 2(6n+3)+1$ implies that $u \geq 2n+1$. By Cruses Theorem [12], there exists an idempotent commutative quasigroup $(Q, \circ)$ of odd order $2u+1$ containing an idempotent commutative quasigroup of order $2n+1$. Bose construction applied to the quasigroup $(Q, \circ)$ produces a 3-chromatic $STS(6u+3)$ with a (3-chromatic) subsystem of order $6n+3$. Take this subsystem of order $6n+3$ and replace it with a copy of $\mathcal{S}$ in the same fashion as for the case 1. Again it is easy to check that the resulting $STS(6u+3)$ is k-chromatic. ■

$S^\star(63)$ was the first specific example of a 5-chromatic STS and it was shown in the study [16]. A combination of Theorem 4.1.4 and Theorem 4.1.5 results in the following corollary.

Corollary 4.1.6 *There exists a 5-chromatic $STS(v)$ for every admissible $v \geq 127$.*

Haddad proved the following [17]:

Theorem 4.1.7 $\mathbb{A}G(4,3)$ *is a 5-chromatic $STS(81)$.*

Proof: Haddad proved first the largest k for which there exists a k-cap for $\mathbb{A}G(4,3)$ is 20 [17]. So, we can not color the system with 4 colors properly. The following 5-coloring given by Haddad proves $\mathbb{A}G(4,3)$ is 5-chromatic.

C_1={0000, 0001, 0010, 0102, 1000, 1001, 1010, 1102, 1120, 1122, 2011, 2012, 2021, 2100, 2101, 2122},

C_2={0002, 0020, 0022, 0211, 0212, 0221, 1101, 1211, 1212, 1221, 2000, 2001, 2010, 2202, 2220, 2222},

C_3={0111, 0112, 0121, 0200; 0201, 0210, 1111, 1112, 1121, 1200, 1201, 1210, 2102, 2120, 2211, 2212, 2221},

C_4={0011, 0012, 0021, 0100, 0101, 0110, 1011, 1012, 1021, 1100, 2002, 2020, 2022, 2111, 2112, 2121},

C_5={0120, 0122, 0202, 0220, 0222, 1002, 1020, 1022, 1110, 1202, 1220, 1222, 2110, 2200, 2201, 2210}. ■

4.2 6-chromatic STS

Bruen, Haddad, and Wehlau gave the first specific example of a 6-chromatic STS [6]. To do this, they showed that, the size of a cap in $\mathbb{A}G(5,3)$ is bounded above by 48. They also found three disjoint 45-caps in $\mathbb{A}G(5,3)$. They combined these results to prove the corresponding $STS(243)$ of $\mathbb{A}G(5,3)$ is 6-chromatic. They first showed the structure does not admit a 5-coloring. For a $STS(243)$ to be 5-colorable, it should have a color class with at least 49 elements. If all color classes have less than 49 elements, then the order of the STS, $v \leq 240$, which is not the case. But $\mathbb{A}G(5,3)$ has caps of size at most 48. Therefore, $\mathbb{A}G(5,3)$ is not 5-colorable.

The 6-coloring of the $STS(243)$ is as follows. C_1, C_2, C_3 are the three disjoint 45 caps of $\mathbb{A}G(5,3)$, C_4, C_5, C_6 are obtained by a 3-coloring of the partial STS, which is the restriction of the $\mathbb{A}G(5,3)$ to the remaining $108 = 243 - 3(45)$ points by a computer program.

C_1= {02201, 02101, 12202, 01211, 21111, 00120, 10221, 00112, 21100, 20210, 11211, 00002, 02020, 10020, 21000, 00010, 11110, 21210, 20120, 11121,00212, 00201, 22220, 02220, 20001, 22001, 21221, 21101, 10112, 22222, 22212, 00110, 02021, 22121, 10111, 21220, 01210, 02102, 20100, 01102, 01110, 22021, 02200, 11221, 22101};

C_2= {00200, 11112, 10102, 01120, 00222, 20121, 00100, 11101, 02222, 11212, 22012, 20022, 22200, 12220, 22211, 02221, 01202, 10212, 22022, 21122, 22122, 21201, 22210, 02120, 10011, 01201, 00111, 20111, 02011, 21211, 02211, 01101, 00101, 00001, 20201, 21121, 21021, 10100, 00020, 22112, 02012, 21212, 21102, 11202, 02122};

C_3 = {02000, 20010, 22202, 10121, 10002, 12211, 22000, 11122, 21110, 02111, 02202, 21202, 20102, 02212, 11200, 02110, 22100, 01220, 00122, 11220, 01122, 22110, 01222, 21012, 22201, 21120, 00022, 21200, 00102, 11100, 10200, 22010, 00210, 02210, 10120, 00011, 01111, 00121, 00221, 22221, 02002, 20112, 21112, 20222, 21222};

C_4 = {11021, 21002, 01022, 10022, 21022, 12120, 02121, 12212, 11020, 20020, 22120, 00000, 10000, 12002, 01000, 12000, 12100, 01200, 11120, 01112, 11010, 20220, 01002, 10122, 20021, 12022, 02010, 11201, 10201, 10202, 12101, 20110, 00202, 22002, 20212, 12222};

C_5 = {20000, 11000, 02100, 12110, 10210, 11210, 10001, 01001, 21001, 12001, 10101, 12201, 01011, 21011, 12011, 22011, 00211, 10211, 01121, 20221, 12221, 20002, 11002, 11102, 12102, 22102, 20202, 00012, 10012, 01012, 12012, 12112, 01212, 11022, 02022, 20122};

C_6 = {01100, 20200, 12200, 10010, 01010, 21010, 12010, 10110, 12210, 01020, 21020, 12020, 22020, 00220, 10220, 11001, 02001, 20101, 20011, 11011, 11111, 12111, 22111, 20211, 00021, 10021, 01021, 12021, 12121, 01221, 20012, 11012, 02112, 12122, 10222, 11222}.

So, $\mathbb{A}G(5,3)$ is a 6-chromatic $STS(243)$. ■

The combination of Theorem 4.1.5 and the existence of the 6-chromatic $STS(243)$ gives us the following corollary:

Corollary 4.2.1 *There exists a* 6*-chromatic* $STS(u)$ *for every admissible* $u \geq 487$.

Chapter 5

RELATED RESULTS

5.1 Existence of k-chromatic Steiner triple systems for large k

Erdös and Hajnal showed by probabilistic methods, the existence of a k-chromatic partial triple system (see Theorem 1.3.13) for any $k \geq 2$. Treash showed any partial Steiner triple system can be embedded into a Steiner triple system (see Theorem 1.3.14). Since this extension includes an isomorphic copy of the k-chromatic $PSTS$, chromatic number of the STS is at least k. Rosa concluded, for any positive integer k, there exists a Steiner triple system whose chromatic number is at least k. There is no general way to embed $PSTS$s to STSs while preserving the chromatic number. For instance, a $PSTS$ can be 3-chromatic, but a nontrivial STS is at least 3 chromatic [5]. So, due to the above results, it could not be concluded that, for any k there is a STS, S, for which $Chr(S) = k$. However, result is obtained by Brandes, Phelps and Rödl [5].

Theorem 5.1.1 [5] *Let u_m be the number of the smallest order for which there exist a m-chromatic partial STS. Then $C_1 m^2 \log m < u_m < C_2 m^2 \log m$, where C_1 and C_2 are constants.*

Lindner gave an embedding of a $PSTS$ to a STS [23], but this did not necessarily preserve the chromatic number. However, Brandes, Phelps, and Rödl had a similar approach with Lindner and showed there exists k-chromatic $STS(v)$'s for large admissible v.

The following theorem which is proved by Brandes, Rödl, and Phelps [5] shows that there are infinitely many STSs of every chromatic number c, $c \geq 3$. Before giving the statement and the proof of the theorem, we will state the required propositions and lemmas in the proof.

Lemma 5.1.2 *Let u_k be the smallest order of a weakly m-chromatic partial STS. Then, $c_1k^2 \log k > u_k > c_2k^2$ where c_1 and c_2 are constants.*

Lemma 5.1.3 [12], *A partial idempotent commutative quasigroup of order n can be embedded in an idempotent commutative quasigroup of order t for every odd $t \geq 2n+1$ and in a commutative quasigroup of order t for every even $t \geq 2n$.*

Lemma 5.1.4 [23] *A partial idempotent commutative quasigroup of order n can be embedded in an half idempotent commutative quasigroup of order $2t$ for every $t \geq n$.*

Theorem 5.1.5 [5] *For all $k \geq 3$ there exists a n_k such that for every $v \equiv 1$ or $3 \,(mod\ 6), v \geq n_k$, there exists a k-chromatic $STS(v)$.*

Proof: We know the existence of the k-chromatic partial Steiner system of order u_k by Lemma 5.1.2. From this $PSTS$ we can obtain a partial idempotent commutative quasigroup and by Lemmas 5.1.3 and 5.1.4, we can embed it into an idempotent commutative quasigroup of order $2t+1$ and half idempotent commutative quasigroup of order $2t$ for every $t \geq u_k$. Then by the Bose and Skolem Constructions, we can obtain STSs of order $6t+3$ and $6t+1$ which are 3-chromatic by construction. For any triple $\{x,y,z\}$ in the $PSTS$, we will have a $STS(9)$ as a subsystem of the new $STS(6t+1)$ or $STS(6t+3)$, with the following triples: $\{(x,1),(y,1),(z,2)\}$, $\{(x,1),(z,1),(y,2)\}$, $\{(y,1),(z,1),(x,1)\}$, $\{(x,2),(y,2),(z,3)\}$, $\{(x,2),(z,2),(y,3)\}$, $\{(y,2),(z,2),(x,3)\}$, $\{(x,3),(y,3),(z,1)\}$, $\{(x,3),(z,3),(y,1)\}$, $\{(y,3),(z,3),(x,1)\}$, $\{(x,1),(x,2),(x,3)\}$, $\{(y,1),(y,2),(y,3)\}$, $\{(z,1),(z,2),(z,3)\}$. Then this $STS(9)$ is replaced with the $STS(9)$ which includes the triples $\{(x,1),(y,1),(z,1)\}$, $\{(x,1),(x,2), (x,3)\}$, $\{(y,1),(y,2),(y,3)\}$, $\{(z,1), (z,2),(z,3)\}$. (The remaining triples are: $\{(x,2), (z,1), (y,2)\}$, $\{(x,2),(z,2),(y,1)\}$, $\{(x,2),(z,3),(y,3)\}$, $\{(x,1),(y,3),(z,2)\}$, $\{(x,1),(z,3), (y,2)\}$, $\{(x,3),(z,1),(y,3)\}$, $\{(x,3),(y,1),(z,3)\}$, $\{(x,3),(y,2),(z,2)\}$.) When we apply this procedure to our STS for all of the triples of the original $PSTS$, our resulting STS will include an isomorphic copy of our $PSTS$. So the resulting STS is at least k-chromatic. We can also show that during this procedure of replacing STSs of order 9 with the new STSs of the above kind for all triples of the $PSTS$, at some point we have a STS which is k-chromatic since this replacing procedure increases the least required number of colors for a proper coloring at most once.

Consider one step of exchanging a subsystem of order 9 with a $STS(9)$ as described above. If there are no monochromatic triples after the replacement, this means, our coloring is still proper. Say the $STS(v)$ is i-chromatic before the exchange. Notice that, only the triples consisting of the elements of the $STS(9)$ are changed, the rest of the triples remained unchanged, so a monochromatic triple may occur only in the new $STS(9)$. Before the replacement, the $STS(9)$ was colored with 3-colors, since every $STS(9)$ is 3-chromatic. None of the 3 color classes can have more than 4 elements. If one of the color classes has for instance 5 elements, there should be $\frac{5.4}{2} = 10$ triples including pairs of elements of this color class (no three of them can be in a triple since this is a proper coloring). There are 4 remaining elements, namely, a, b, c, d. For any of the 10 triples we had already, we may have only one of a, b, c, d. Without loss of generality, consider a. There must one triple including both a, b, one including both a, c and including both a, d. If we have a triple $\{a, b, c\}$, we must have one triple $\{a, d, x\}$ and one $\{b, d, y\}$. These two triples are distinct, otherwise $x = b$ and $y = a$, but we had a and b together in one triple before. However this means we should have more than 12 triples, which can not be the case. For 3 colors, the exchange may result in at most 3 monochromatic triples. We can choose one element from each monochromatic triple (there are at most 3) so that the resulting set contains no triple of the subsystem of order 9 (and, hence, can contain no triple of the system either). Assigning $(i+1)^{st}$ color to this set obviously gives a proper $(i+1)$-coloring. ■

The motivation behind the vertex colorings of STSs was to find non isomorphic STSs. Isomorphisms preserve chromatic number. The above theorem also assures that for any large v there are a large number of nonisomorphic $STS(v)$s.

5.2 Equitable and balanced colorings

There are some types of weak colorings with additional properties. **Equitable colorings** and **balanced colorings** are two of those. Since the proofs of the theorems in this section are based on GDDs, and beyond the scope of our survey, we will only define these terms and summarize the main results. We will only give the proof of Theorem 5.2.1 since the proof is done by using a counting argument.

The equitable colorings are the colorings when the size of the color classes differ by at most 1. Each of the 80 $STS(15)$ have an equitable weak 3-coloring [26]. Each 3-coloring of $S^\star(15)$ is equitable [27]. The problem of finding if every $STS(v)$ admits an equitable weak coloring was a major problem.

The following theorem, which is about the sizes of color classes of STSs, belongs to Haddad and Rödl.

Theorem 5.2.1 *Let $S = (V, \mathcal{B})$ be a weakly 3-chromatic $STS(v)$, and V_i, $i =$ 1, 2, 3, be the color classes of weak 3-colorings of $\mathcal{S}$. Then*

$$v = v_1 + v_2 + v_3 \geq \frac{1}{2}[(v_1 - v_2)^2 + (v_1 - v_3)^2 + (v_2 - v_3)^2]$$

Thus, for sufficiently large v, the color classes must have approximately the same size.

Proof: For $i, j \in \{1, 2, 3\}$, let x_{ij} be the number of triples that intersect V_i in 2 elements, and V_j in one element. Let x be the number of triples having one element from each color class. The number of triples which is known to be $\frac{v(v-1)}{6}$ is equal to $x + \sum x_{ij}$. So we have,

$$\begin{aligned}
\tfrac{v(v-1)}{6} &= x + \textstyle\sum x_{ij},\\
x_{12} + x_{13} &= \tbinom{v_1}{2},\\
x_{21} + x_{23} &= \tbinom{v_2}{2},\\
x_{31} + x_{32} &= \tbinom{v_3}{2},\\
2x_{12} + 2x_{21} + x &= v_1 v_2,\\
2x_{23} + 2x_{32} + x &= v_2 v_3,
\end{aligned}$$

$$2x_{13} + 2x_{31} + x = v_1v_3.$$

The following equalities follow from the above.

$x_{13} + x_{23} - x_{21} - x_{12} = \binom{v_1}{2} + \binom{v_2}{2} - v_1v_2 + x \geq \frac{(v_1-v_2)^2-(v_1+v_2)}{2}$,

and similarly,

$x_{32} + x_{12} - x_{31} + -x_{13} \geq \frac{(v_1-v_3)^2-(v_1+v_3)}{2}$,

$x_{31} + x_{21} - x_{23} + -x_{32} \geq \frac{(v_2-v_3)^2-(v_2+v_3)}{2}$.

The inequality follows from adding the previous three inequalities and by rearranging the terms. ■

For larger numbers, the following theorem suggests not to expect to find an equitable m-coloring [18].

Theorem 5.2.2 *For every $0 < \varepsilon < 1$, $m \geq 6$, and $t \geq m$, there exists a weakly m-chromatic $STS(v)$ such that for every t-coloring of the STS, there are $m-3$ color classes whose union contains at most εv elements.*

Corollary 5.2.3 *For every $m \geq 6$ there exists a weakly m-chromatic $STS(v)$ that does not admit an equitable m-coloring.*

Colbourn, Haddad, and Linek had shown that, when the order v is large enough with respect to the number r of colors, and $v \equiv 1$ or $3 \pmod 6$ an equitably r-colored r-chromatic Steiner triple system of order v exists.

Lemma 5.2.4 *Suppose there exists an r-chromatic $STS(w)$ with $w \equiv 3\ (mod 6)$ and partition sequence $t_1 \geq t_2 \geq \ldots \geq t_r$. If v satisfies:*
(i) $v \equiv 3\,(mod\ 6)$,
(ii) $v \geq max\{2w, rt_1\}$,
(iii) $\lceil \frac{v}{r} \rceil - t_r \leq \frac{1}{3}(v - w)$.
Then there exists an equitably r-colored, r-chromatic $STS(v)$.

Lemma 5.2.5 *Suppose there exists an r-chromatic $STS(w)$ with $w \equiv 3 \ (mod 6)$ with partition sequence $t_1 \geq t_2 \geq \ ... \geq t_r$. If v satisfies:*
(i) $v \equiv 1 \, (mod 6)$,
(ii) $v > max\{2w, rt_1\}$,
(iii) $\lceil \frac{v}{r} \rceil - t_r \leq \frac{1}{3}(v - w - 1)$.
Then, there exists an equitably r-colored, r-chromatic $STS(v)$.

Combination of Lemma 5.2.4 and Lemma 5.2.5 implies the following theorem.

Theorem 5.2.6 *If there exists an equitably r-colored, r-chromatic $STS(w)$ with $w \equiv 1 \, or \, 3 \ (mod \ 6)$ and $r \geq 4$, then there exists an equitably r-colored, r-chromatic $STS(v)$ for each admissible $v \geq 2w$.*

Lemma 5.2.7 *Any r-chromatic $STS(w)$ with $r \geq 4$ can be embedded in an r-chromatic $STS(3w)$. In addition, if the r-chromatic $STS(w)$ is equitably r-colored, then it can be embedded in a equitably r-colored, r-chromatic $STS(3w)$.*

Theorem 5.2.8 *If there exists an equitably r-colored, r-chromatic $STS(w)$ with $w \equiv 1 \ (mod \ 6)$, and $r \geq 4$, then there exists an equitably r-colored, r-chromatic $STS(w)$ for each admissible $v \geq 6w + 1$.*

Lemma 5.2.9 *Suppose there exists an equitably r-chromatic $STS(w)$ with $w \equiv 1 \ (mod \ 6)$ with partition sequence $t_1 \geq t_2 \geq \ ... \geq t_r$. If v satisfies*
(i) $v \equiv 1 \ (mod 6)$,
(ii) $v > max(2w, \, rt_1)$,
(iii) $\lceil \frac{v}{r} \rceil - t_r \leq \frac{1}{3}(v - w)$
then there exists an equitably r-colored, r-chromatic $STS(v)$.

If every r-coloring of a triple system is equitable, we call the system r-**balanced**. An r- balanced system is necessarily r-chromatic.

Theorem 5.2.10 *With the possible exceptions of* $v \in \{19, 21, 37, 49, 55, 57, 67, 69, 85, 109, 139\}$ *for all* $w \equiv 1 \ or \ 3 \, (mod \ 6)$ *and* $v \geq 15$, *there exists a* 3*-balanced Steiner triple system of order v.*

Some examples of 3-chromatic $STS(v)$ which do not admit equitable 3-colorings are given recently in [14]. In this study, also further examples of systems with unique and balanced colorings are presented.

5.3 Some related results

In this section we will summarize the results from complexity theory about weak colorings of STS. We will also note some questions arised about chromatic number of STSs.

C. J. Colbourn, M. J. Colbourn, Phelps and Rödl worked on the problem of finding the difficulty of deciding if the chromatic number of a given STS is k or not. They proved the following theorem:

Theorem 5.3.1 [11]

(*i*) *Deciding if a* $PSTS$ *is* t*-colorable is* NP*-complete for any fixed* $t \geq 3$.

(*ii*) *Deciding if a block design is* t*-colorable is* NP*-complete for any fixed* $t \geq 9$.

Phelps and Rödl proved the following [28]:

Theorem 5.3.2 *Deciding if a simple* k*-uniform hypergraph is* t*-colorable is NP-complete for* $t \geq 3$.

Corollary 5.3.3 [28] *Deciding* 14*-colorability of a* STS *is* NP*-complete.*

A uniquely colorable $STS(33)$ was found, then the following question arised: for what orders do such systems exist? In [5] it is asked that if there is a uniquely colorable m-chromatic $STS(v)$ for all m or not? [5] also introduced the question "is $C(v)$ is an interval?" More generally, "what is the chromatic spectrum of $STS(v)$?"

Colbourn, Dinitz, and Rosa answered in [8] partially one of the open problems they noted in the previous study. They showed the spectrum $C(v)$ for bicolorings needs not to be an interval. For instance, $C(31) = \{3, 5\}$, is not an interval. They added that 31 may be the only possible order for a STS whose $C(v)$ is not an interval.

They noted if a $STS(v)$ is m-bicolorable, then $m \leq [\log_2(v+1)]$ and for all $v = 2n-1$, there exists a $STS(v)$ for which the bound is attained. There are some recent studies on the chromatic spectrum of STSs. In [19] it is shown that the chromatic spectrum of $STS(25)$ is $\{3, 4\}$.

Chapter 6

CONCLUSION

In our study, we tried to demonstrate the findings about the weak colorings of Steiner triple systems. The most interesting part of the problem is the fact that the solutions come from various branches of mathematics and even computer science.

In the first chapter, we gave the required definitions for our study, statements of the preliminary theorems that we need in the following chapters and proofs of some of them.

In the second chapter, we summarized early studies on the topic on the k-chromatic STSs for small k. Rosa made the first contributions to the problem [29, 30]. He showed the existence of 3-chromatic STS for any admissible order. He also gave examples of 4-chromatic $STS(31)$ and $STS(49)$. The study of Brandes, Phelps and Rödl [5] includes results about the existence of k-chromatic STS for any k by using probabilistic methods, in addition, some 4-chromatic STSs are also included in the study.

In the third chapter, we focused on the studies in which finite geometry was used. Fugère, Haddad, and Wehlau gave the first specific example of a 5-chromatic STS [16]. Bruen, Haddad and Wehlau gave the first specific example of a 6-chromatic STS [6]. Haddad gave examples of 4-chromatic $STS(21)$ and $STS(39)$ by using computational methods, and he gave an example of a 5-chromatic $STS(81)$ by using finite geometry techniques [17].

In the 4^{th} chapter, we summarized the arguments for the existence of k-chromatic STSs for large k.

In the 5^{th} chapter, we summarized the recent studies about colorings of STS, which are mostly on specific types of weak colorings such as equitable and balanced colorings. We only defined the terms and stated the main theorems for those colorings. We also included some related results in this chapter. For instance the statements of the theorems about NP-completeness of solving weak coloring problems are included.

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