IMPACTS OF ADVANCE DEMAND INFORMATION AND ADMISSION CONTROL IN PRODUCTION-INVENTORY SYSTEMS

by

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This is to certify that I have examined this copy of a master's thesis by

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and have found that it is complete and satisfactory in all respects,
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To my family

ABSTRACT

In this thesis, we concentrate on two aspects of production-inventory systems: advance demand information and admission control. We firstly consider a continuous review, infinite horizon replenishment and admission control model. We propose static and dynamic admission control methods to determine the acceptable demand rate for the supplier who produces a single product for a single class of customers. The profit function of the supplier is investigated under different parameter settings and the benefits of the dynamic admission control policies are found to be significant. Therefore, we continue our study with dynamic admission control in a periodic-review, infinite horizon model with multiple customer classes and advance orders. We investigate the role of inventory and advance demand information to manage multi-class demand through dynamic admission control policies. The main aim of the latter model is capturing the tradeoffs in a situation where different customers provide different levels of advance demand information. We analyze the model using event-based stochastic dynamic programming and investigate the structure of the optimal replenishment and admission control policies. The numerical results of this model underline the benefits of advance demand information and admission control.

ÖZETÇE

Bu tez üretim-envanter sistemlerinin iki özelliği üzerinde odaklanmaktadır: ileri talep bilgisi ve kabul kontrolü. Öncelikle sonsuz zamanlı, sürekli kontrollü bir envanter ve kabul kontrol modeli ele alınmıştır. Bu modelde tek tip müşteriye tek tip ürünle hizmet eden bir tedarikçi için en iyi kabul edilebilir talep oranı bulunmak üzere statik ve dinamik kabul kontrol yöntemleri önerilmiştir. Kar fonksiyonu farklı sistem parametreleriyle sayısal olarak analiz edildiğinde dinamik kabul kontrol yöntemlerinin statik kabul kontrol yöntemlerine üstünlük sağladığı görülmüştür. Bu nedenle, tezin geri kalan kısmında dinamik kabul kontrol yöntemleri kullanılarak devam edilmiştir. İkinci olarak ele alınan periyodik kontrollü çoklu müşterili modelde bir kısım müşterilerin tedarikçi ile yaptıkları anlaşma sebebiyle her zaman belirli bir süre önce sipariş verdikleri bir sistem ele alınmıştır. Bu model yoluyla çoklu müşterili sistemler için envanter ve ileri talep bilgisinin rolü incelenmiştir. Bu modeli oluştururken asıl amaç farklı seviyelerde ileri talep bilgisi sağlayan çok sayıda müşteriye sahip sistemler hakkında bilgi sahibi olmaktır. Model olay-bazlı stokastik dinamik programlama kullanılarak analiz edilmiştir; en iyi envanter ve kabul kontrol politikalarının yapısı araştırılmıştır. Bu model için yapılan sayısal analizler ileri talep bilgisi ve kabul kontrolünün yararlarını vurgulamaktadır.

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NOMENCLATURE

ADI	Advance Demand Information
DP	Dynamic Programming
FCFS	First Come First Served
H	Length of the Information Horizon
MDP	Markov Decision Process
MMFE	Martingale Model of Forecast Evolutions
MRP	Materials Requirement Planning
OEM	Original Equipment Manufacturer
RFID	Radio Frequency Identification
SDP	Stochastic Dynamic Programming
X	Order Acceptance Threshold
$X_{\mathbf{d}}$	Order Acceptance Threshold depending on the Demand Vector
Z	Target Level
$Z_{\mathbf{d}}$	Target Level depending on the Demand Vector

Chapter 1

INTRODUCTION

It is well known that capacity limitations and demand and processing time uncertainties have a negative effect on the performance of a supply chain. Inventory has to be carried through the supply chain in order to satisfy these deficiencies with an acceptable service level. However, carrying inventory is a costly activity. Thus, the suppliers aim to decrease their inventory levels while keeping service levels sufficiently high. In this context, increasing popularity of technologies such as Internet and Radio Frequency Identification (RFID) is worth mentioning. These technologies enable better control and sharing of information so that they can be used to decrease uncertainties in the supply chain.

Capacity limitations are common in today's demanding market. The market consists of several segments of customers with different rewards, lost sales costs and demand leadtimes. On the other hand, the supplier has a fixed capacity that may not be sufficient to serve the whole market demand available and the capacity investments are so costly that they can not be considered in the short term. Therefore, the supplier aims to control the portion of demand to be satisfied in order to better match demand and supply. For instance, it is plausible that, she admits or rejects to satisfy demand according to the inventory level and other system parameters in order to maximize her benefits. Additionally, she may choose to provide only a fraction of the demand of a specific customer class and keep inventory for the other classes. This is called the "admission control" problem of the supplier. One of the issues that we try to shed some light on is determining the customer portfolio of a supplier with fixed capacity through different admission control methods. The motivation behind our study is the growing consensus on having a portfolio of different customers, since this may lead to higher revenues and better capacity utilizations.

Demand and processing time uncertainties are the other factors that reduce the performance of a production-inventory system. The performance losses caused by the uncertainties can be compensated by keeping inventory or sharing demand information in advance. As a result of the recent advances in information technologies, demand information can easily be shared in advance between the members of a supply chain. Therefore, it is possible to decrease the role of inventory against uncertainty by the help of advance demand information (ADI). The literature on ADI advocates its use and outlines the benefits if it is smartly integrated. Additionally, improvements in technology enable keeping better track of inventory and production information in the system. Collecting better information about demand, inventory level etc. allows the supplier to take uncertainty in control through dynamic control policies. In this thesis, we investigate the interactions between ADI and admission control in a multi-customer setting where dynamic control is performed.

We concentrate on two specific production-inventory systems with admission control. In the first model, we focus on the method of admission control. We perform analyses of both static and dynamic admission control policies; these analyses reveal the dominance of dynamic admission control policies. Thus, we use dynamic admission control in the second model to investigate impacts of admission control and ADI at the same time. We establish the structure of the optimal policies. In general, we investigate the role of inventory and ADI to manage multi-class demand through dynamic admission control policies. We aim to determine the optimal portfolio of customers among the ones providing different levels of advance demand information. To our knowledge, there are only a few and specific studies which concentrate on dynamic admission control and ADI at the same time.

The rest of this thesis is organized as follows. In Chapter 2, we provide the necessary background on inventory management with advance demand information and admission control.

In Chapter 3, we consider a supplier's admission control problem. We propose static and dynamic admission control methods and discuss the advantages and disadvantages of the methods. According to the numerical results, we find out that the benefits of dynamic admission control surely dominates the static admission control methods.

In Chapter 4, we apply dynamic admission control to a system with multiple customers and advance orders. We propose event-based operators to analyze the model. We prove optimality of threshold type replenishment and admission control policies using stochastic dynamic programming.

Finally, in the last chapter, we summarize the performed study and its results.

Chapter 2

LITERATURE SURVEY

Inventory systems with ADI have been frequently studied in the literature and they are still of interest today. Research has been done in order to incorporate advance demand information in different types of production/inventory systems. In order to expand the use of ADI, its benefits and its interactions with the system parameters are investigated.

This literature review is structured as follows. Firstly, the literature on ADI in capacitated production/inventory systems is presented. It is followed by the papers about uncapacitated systems with ADI. Later, the related literature about demand management focusing on stock rationing is discussed. In the subsequent part, the papers combining the ADI and stock rationing issues are mentioned. The review is concluded with the contribution of the model in this thesis.

Buzacott and Shanthikumar [2] is one of the earliest papers considering advance demand information in a production/inventory system. In this study, safety time is compared with safety stocks in a single-stage capacitated production/inventory system with advance demand information coming from a MRP system. Despite the conventional discrete time MRP systems, they propose release rules for continuous time systems. They provide insights into the performance of the MRP system by comparing the use of safety time with safety stocks in systems with different levels of advance demand information. According to their results, the accuracy of the ADI is effective on the choice of safety stocks against safety time; if the ADI about the future demand is poor, safety stocks will be preferable. However, with accurate ADI, safety time will bring more benefits to the system. Karaesmen et al. [23] study a single-stage, single-customer capacitated production/inventory system modeled as a discrete time make-to-stock queue with advance orders. Their work resembles Buzacott and Shanthikumar [2] in several ways except its discrete time nature. However, their main interest is in identifying optimal production policies. By stochastic dynamic programming, they find out that advance orders can be optimally incorporated in the system by a basestock mechanism; however, an easily implemented heuristic policy with a release leadtime parameter also performs well. They compare these two policies and emphasize the benefits of advance demand information when optimally integrated into such a system.

Karaesmen et al. [24] provide an extensive analysis for both make-to-order and make-to-stock systems with advance demand information. The intuition that the advance demand information increases the performance of a system is highlighted in this study. Additionally, the factors that decrease costs in a system with advance demand information are investigated. The authors primarily consider the single stage systems where the inputs are only controlled at the beginning of the stage. For the single stage make-to-order systems, they find the optimal release leadtime L^*. Thus, if the order is released exactly L^* units of time in advance of the due-date, the sum of average holding and backordering costs is minimized. The effects of the production leadtime, i.e. the time between the release of an order to the production stage and its delivery to the finished goods stock, are significant in make-to-order systems. Increased production leadtime variability causes high optimal costs. Moreover, high production leadtimes require more ADI in make-to-order systems with generally distributed processing times. For capacitated $M/M/1$ make-to-order systems, the authors find out that sharing ADI earlier exponentially decreases the expected number of backorders; on the other hand, increasing the release leadtime increases the expected number of inventory. As further research, a system operating in make-to-stock mode is considered under base-stock policy with ADI. The basestock levels and the optimal average costs depending on the demand leadtime, i.e. the horizon of visibility, are determined according to this policy. They state that demand leadtime decreases the basestock levels and the optimal average costs in comparison with the no information case. Additionally, the make-to-stock system with general processing times under geometric tail approximation is considered in order to examine the effects of increased production leadtime variability. Highly variable production leadtimes increase the optimal basestock levels and optimal average costs and decrease the benefits of ADI. Besides, in highly loaded systems, the demand leadtimes should be sufficiently long to make use of ADI. Karaesmen et al. handle the multi-stage systems where the analysis is even difficult without advance demand information. They examine the system with two stages and state that if the unit holding cost of the upstream stage is greater than the downstream stage, the system behaves as a single stage system. Otherwise, the analysis of multi-stage systems poses several challenges. In another study, Karaesmen et al. [25] reconsider make-to-order and make-to-stock systems with advance demand information. In addition to their previous study, this time, they emphasize the value of advance demand information in such systems. The early deliveries and timely deliveries cases are separately examined. They consider the timely delivery systems under the basestock policy with a fixed release time parameter. On the other hand, for the early delivery systems, they apply an order-based basestock policy where the backorder cost is incurred after the horizon of visibility ends. The relative value of advance demand information is found to be high in the systems where early deliveries are allowed; although it is moderate in timely delivery systems. Additionally, for the horizon dependent costs of advance demand information, if the customer has a more dominant role in price setting, the supplier may not prefer giving discounts in return for advance demand information. Thus, inefficiencies in the supply chain that could be removed by sharing information remain unchanged.

DeCroix and Mookerjee [8] study the incorporation of advance demand information in a different setting. The aim of this paper is to determine the value of advance demand information in a periodic-review inventory system where the advance demand information is available at a price. They analyze the problem with both perfect and imperfect ADI over finite and infinite horizons. The optimal information purchasing and replenishment policies are addressed. DeCroix and Mookerjee state that the value of information increases as the inventory level decreases and the length of remaining horizons increases. Another study that aims to measure the benefits of advance demand information is the work of Gavirneni et al. [13]. They consider a two-echelon capacitated inventory model under three assumptions. They assume the scenarios where no information is shared, only demand distribution and the optimal production policy of the retailer is shared, and lastly, full information, i.e. information of the periodic inventory levels, is shared. In all cases, the order-up-to $((s, S))$ policies are optimal. They compare these three scenarios in order to distinguish the system conditions where the benefits of advance demand information are maximized. They show that it is always better to have full information than restricted information. However, when the variance of the demand and the $S - s$ are of moderate values, the benefits of ADI become significant. Additionally, low capacity levels cause a decrease in the benefits of ADI.

In some of the systems, the forecasts are considered as advance demand information.

Güllü [15] provides one of the first of these studies. He evaluates the benefits of information about future demands provided from forecast evolutions in a dynamic production/inventory system with zero production leadtime. He compares the inventory model with Martingale model of forecast evolutions (MMFE) to a standard inventory model under capacity constraints; then, he extends his work for the uncapacitated versions of the systems. The results, similar with those for advance orders, reveal the fact that information about future demands decreases the expected costs and inventory levels. In one of his later studies [16], Güllü works on a two-echelon model, consisting of a depot and several retailers, with the aim of analyzing the benefits of the MMFE incorporated in the system. He finds the approximate system-wide order-up-to level and compares the system costs and inventory levels to a standard system. The results show that forecasts improve the system costs and performance for the two-echelon systems, too. Another study with advance forecast information is Toktay and Wein [33]. Their aim is to determine the replenishment policy structure under the MMFE in discrete time with limited capacity. They state that the optimal policy is designated as the state-dependent forecast-corrected basestock policy. They benchmark the optimal policy with a replenishment policy in which the advance forecast information and state-dependency is not considered. Additionally, an easily implemented approximation is derived for the optimal basestock level to gain insights on the benefits of the forecasts and the forecast updates. One of the results of their work parallels the previous literature on advance demand information; information is mostly beneficial at moderate levels of system load. They observe that the optimal planning horizon length increases by the system load, demand variability and desired service level as also highlighted in Karaesmen et al. [23]. Their main result reveals the increasing use of capacity instead of safety stocks when systems with longer horizon of visibilities are handled. As in forecasts, the supplier is only able to reach imperfect information about future demand in some cases. Tan et al. [31] carry out one of the noteworthy studies about the systems with imperfect advance demand information. They define imperfect ADI as early uncertain indication of prospective future demands. They provide a generalized probability structure to model the imperfect ADI. By using this model and the dynamic programming, they address the structure of the optimal inventory policies for the discrete-time finite-horizon model with backordering. They show that the optimal policy is of order-up-to type depending on the available ADI. Since the order-up-to levels are shown to be non-decreasing in time with same ADI, they develop an upper-bound on the order-up-to level of any period. In order to make better use of imperfect ADI, they propose a segmentation framework for the ADI sources; by this way, they observe a decrease in the system costs. They assess the value of imperfect ADI in the single-period problem and generalize it for the N-period problem. They state that their work can be extended by using updating schemes like Bayesian updates, time-series models and forecast evolution methods as in Güllü [15].

In order to underline the use of ADI, Hu [20] compares it with outsourcing against demand and capacity uncertainty. Hu considers a system with advance demand information where production and outsourcing decisions are made at each period. The optimal policy has two parameters; the production threshold and the outsourcing threshold. The system costs decrease in expected capacity; however, the variability of capacity increases the costs. Hu proposes the practical uses of the outsourcing and the advance demand information underlining the decreasing benefits of the advance demand information with the increases in variability of the capacity, system load and service level.

Özer and Wei [29] scrutinize the capacitated production/inventory system with advance demand information and fixed costs. Özer and Wei suggest optimal replenishment policies for the periodic-review problems over finite and infinite horizons. They conclude that for both problems, with zero fixed cost a state dependent modified basestock policy is optimal. For the fixed cost case, below a state dependent threshold, the supplier should produce at full capacity; otherwise, she should produce none over finite and infinite horizons. Additionally, Özer and Wei emphasize the role of advance demand information as a substitute for both capacity and inventory in order to provide incentives to share advance demand information.

Above the literature on capacitated systems with ADI is mentioned; however, there are important results corresponding uncapacitated systems with advance demand information. Hariharan and Zipkin [19] investigate the effects of advance demand information by extending basic inventory models to allow advance ordering in the systems with single class of customers and backorders. They investigate the single stage system under deterministic and stochastic leadtimes while the multi-stage extension is done for only deterministic leadtimes. They conclude that the optimal policies for conventional systems can be simply and effectively used for the cases with advance demand information, especially under some specifications, the conventional policies are optimal. Their results show that the horizon of visibility improves the system performance in the same way as a reduction in supply leadtimes and the reduction in costs is more significant as future uncertainty is resolved earlier. Gallego and Özer [11] also consider an uncapacitated production/inventory system with advance demand information. Their work is similar to the Hariharan and Zipkin [19] in most aspects; however, Gallego and Özer consider a system with periodic reviews and set up costs. They state that a portfolio of customers with different demand leadtimes can lead to higher revenues and better utilizations. With a given portfolio of customers, they check the optimality of the conventional policies on the single stage systems with and without set up costs over finite and infinite horizons. According to their results, when the horizon of visibility is shorter than the planning horizon, the systems reduce to the classical inventory systems. However, when the horizon of visibility is longer than the planning horizon, a state-dependent basestock policy is optimal for the system with zero set up costs and a state dependent (s, S) policy is optimal for the fixed set up costs. The results parallel the ones in Özer and Wei [29] that is the capacitated version of this paper. Gallego and Özer [12] extend their work for the periodic-review multi-echelon inventory systems. The system considered is a centralized system with a single-item where the number of customers arriving each period, the order size and the demand leadtimes are random. Gallego and Özer decompose the system into single stages and solve each dynamic program starting from the last stage until the first stage. After this stage-based analysis, they conclude that for the finite and infinite horizons, the state-dependent, echelon basestock policies are optimal. Additionally, they investigate the case where demand and cost parameters are stationary and propose the optimal policy as a myopic policy where the information beyond the leadtimes does not affect the optimal basestock levels and costs. By their numerical analysis, it is shown that the value of information displays the same behavior as in the single-echelon case. ADI is more valuable in the systems with high penalty costs. Moreover, when the ADI reaches a certain level, the system begins to operate in make-to-order mode.

Up to this point, the papers about the effects of ADI on the optimal replenishment policies are presented. However, there are other relevant issues which have received less attention. The firms usually have capacity constraints and different customers who provide different levels of advance demand information. Thus, in this thesis, we consider joint replenishment and admission control in multi-customer systems to be more realistic. In the remaining part of this chapter, the literature about segmentation of customers and demand management with fixed capacity will be discussed.

Chen [6] provides an important study since it combines the effects of market segmentation with advance demand information. Chen considers a system with heterogeneous customers exhibiting different degrees of aversion to waiting. The pricing and replenishment policies are investigated to optimally incorporate the market segmentation and ADI in the system. The optimal replenishment policy is designated as a basestock policy. Heuristic algorithms are proposed to compute the optimal price schedule. In the numerical analysis, the benefits of information and segmentation are evaluated. It is always better to have both ADI and segmentation. However, value of segmentation becomes significant when the number of more patient customers, difference between customers (in terms of aversion), total leadtime (for all stages) and the number of stages increase. On the other hand, the value of information decreases in the number of stages and increases in the total leadtime and unit backorder cost.

When the capacity is limited, management of demand is an important tool to improve performance in supply chains. Tang [32] provides an extensive review for risk management in supply chains and classifies the methods of demand management. In this review, advance booking and advance-purchase discounts are mentioned in order to shift demand across time. Additionally, product rollovers, substitutions and bundling are stated as methods of demand management. However, different methods of demand management exist that do not belong to these classifications. The work of Carr and Lovejoy [5] is an example of these methods. The regular newsvendor problem aims choosing the best capacity that responds to a known demand distribution. Carr and Lovejoy address the "inverse newsvendor" problem where a demand distribution is chosen among prioritized customer classes that best fits the known, fixed capacity. They assume continuous opportunity sets consisting of several normally distributed customer clusters; the opportunity sets are completely or partially known. For the case with completely known opportunity sets, they firstly work on the single customer scenario and propose the optimal demand distribution in the newsvendor setting. They use the single class solution to approximate the multi-customer setting. Additionally, bounds are developed on the optimal mean demand when the opportunity set is only partially known.

Li [28] investigates the factors that engender the supplier to operate in a make-to-stock manner rather than a make-to-order manner. The role of inventory is examined in a production system where the supplier faces delivery time competition. In oligopoly racing markets, duplicate orders are allowed; so the customer buys the product from the supplier who provides earliest service. The oligopoly racing market, the monopoly market and the demand sharing market are compared in terms of inventory policies and the incentive for make-to-stock regime is shown to be decreasing in the mentioned order. The results show that the optimal inventory level decreases in the number of competitors if there is a demand sharing market. However, the increase in the number of competitors results in an increase in the optimal inventory level when the firm competes in an oligopoly market. Li's study provides insights in the way of managing demand since it enables the supplier to determine the basestock level that maximizes her profit. In fact, with the optimal basestock level determined, the supplier chooses to respond only a fraction of customers. Duenyas [9] reports another study about the time concern of customers and its use in managing demand. He analyzes a make-to-order queue with multi customers. He considers the due date setting and order sequencing problem that arises when the customer orders depend on the due date quoted by the supplier. Duenyas emphasizes the deficiencies of the FCFS model and introduces a heuristic approach that outperforms classical due date setting rules. By choosing optimal policies in consideration of the due date and price preferences of the customers, the supplier indirectly chooses the fraction of demand that she wants to serve. There are many papers in the literature that address the stock rationing policies as a way of managing demand with fixed capacity. Ha [17] studies a make-to-stock queue with backorders and two priority classes. For the exponential interarrival and service times scenario, he proposes the production and stock-rationing policies by modeling the system as a queuing control problem. The production policy is of basestock type and the stock rationing policy has the rationing level parameter that decreases in the number of backorders of the low-priority class. De Vericourt et al. [7] also consider a capacitated make- to-stock production system where they compare the performance of three different stock rationing policies. A multi-customer single item model with backordering is examined and is extended with a fill-rate constraint. The optimal parameters are found under standard FCFS policy, a strict priority policy, i.e. a policy with priority that does not reserve inventory, and a multilevel rationing policy. The performance of the policies improves with the degree of bias offered to more costly customers. However, with identical backorder costs, all policies are identical. De Vericourt et al. highlight the benefits of stock rationing for the systems with significantly different backorder costs.

Frank et al. [10] consider the optimal replenishment and stock rationing policies for a system with priority demand classes. In their periodic-review model, there are two customers. The first class has constant deterministic demand that should be fully satisfied. On the other hand, the secondary class has stochastic demand that can be accepted or rejected in order to ration inventory for the first class. The demand of the second class customers is lost when rejected. The authors present the optimal replenishment and inventory rationing policies for the given system; however, since the policies are complicated, implementation is difficult in practice. Thus, they propose a simpler heuristic algorithm including both policies. By the numerical analysis, they test their heuristic against the optimal policy and observe that the heuristic performs well. In particular, they underline the use of inventory rationing in the systems with large fixed costs and low-volume stochastic demand. Carr and Duenyas [4] consider the admission control and sequencing problem arising in a multi-customer production system. They investigate a system where there are both contractual and spontaneous customers arriving according to distinct Poisson distributions. They assume that the demand of contractual customers is satisfied from the inventory; however, make-to-order discipline is applied to the spontaneous customers. The processing times have identical exponential distributions and preemptions are allowed. The main aim in their study is determining the amount of demand that will be fixed by contracts to maximize the profit and finding out the structure of the optimal admission and inventory control policies. The optimal policies are characterized by switching curves, i.e. each area formed by intersection of curves represents a different combination of the optimal decisions. Since the optimal policies have complex structure, they propose a simpler policy that seems to perform well in numerical study. They extend their model with Erlang arrival and service distributions.

Recently, the effects of ADI in the systems where replenishment and stock rationing are jointly controlled have been an attractive issue for the researchers. Gayon et al. [14]

investigate a continuous-time production/inventory system with joint production control and inventory allocation. They study the problem with several customer classes who provide different levels of advance demand information. In the study, they evaluate the effects of the information that is imperfect because of random demand leadtimes and cancellation of customer orders. The optimal production control policy is characterized as a state dependent basestock policy with basestock level non-decreasing in the number of orders. Additionally, the structure of the inventory allocation policy is addressed as a state-dependent multi-level rationing policy, with a non-decreasing rationing level in the number of announced orders. Gayon et al. emphasize the value of advance demand information through both suppliers' and customers' perspectives. According to their numerical results, suppliers tend to benefit from advance demand information by reducing the inventory levels, thus the service levels and the utilization of customers. For the customers to share the benefits of advance demand information, high lost sales costs should be imposed. Additionally, Gayon et al. observe that the value of advance demand information is considerably high in moderate levels of expected demand leadtimes and lost sales costs while it is insignificant when the system load is high.

Iravani et al. [21] also study a similar production-inventory system including ADI and stock rationing. They consider a two-class, single item system where only one of the classes give advance order information. They define the optimal replenishment and admission control policies. In addition to this, the benefits of advance demand information in a multi-class setting are evaluated. They model the problem with Poisson arrivals and exponential processing times. The first class of customers, who gives advance demand information, orders a random quantity periodically although the secondary customers order an individual item randomly. The orders of the first class are fully satisfied; otherwise, a penalty cost is incurred. However, the orders of the second class of customers are lost without any cost when rejected. By stochastic dynamic programming, they characterize the optimal replenishment and stock rationing policies as threshold type where the thresholds depend on the future demand quantity of the first class as well as the current time. By their numerical analysis, they investigate the effects of some system parameters on the value of information. When the share of Original Equipment Manufacturers (OEM) customers who give advance demand information is significantly larger than the secondary customers, an increase in capacity increases the value of information. However, when the share of the secondary customers is significantly larger than the OEM market; the value of information decreases in the capacity. Additionally, they emphasize the fact that as the ratio between the reward of the secondary customers and the penalty cost of not satisfying OEM orders increases, the value of the information decreases. These numerical results imply that the importance of the OEM market increases the value of information.

In this chapter, firstly, a review of the literature about the impacts of ADI is presented. Demand management literature focusing on the stock rationing is given in the subsequent part. In this thesis, a joint study of these issues is carried out. In the model considered in Chapter 3, a static admission control model is proposed and it is compared with a dynamic admission control model. There are only a few studies in the literature that consider the static admission control of customers in this framework. In addition, the model in Chapter 4 investigates the joint problem of replenishment and admission control in the presence of ADI. Another aspect of this model is its multi-customer setting. This model is one of the limited studies in the literature that combine admission control with ADI. Similar studies in the literature model the system in continuous time setting. The dominance of our model on the existing models is because of the constant information horizon, random OEM customer arrivals, batch arrivals and batch admissions assumptions. We also enable outsourcing of items for both classes. Our contribution with this model can be summarized as addressing the optimal replenishment and admission control policies in a discrete time multi-customer capacitated system with ADI.

Chapter 3

DETERMINING THE ACCEPTABLE DEMAND RATE

3.1 Introduction

Capacity is a common constraint that a supplier faces while serving customers. Since capacity investments are costly, it is difficult to adapt capacity according to the conditions in the supply chain. Thus, adapting other system components is usually preferable for most of the suppliers. For a supplier with limited capacity, demand management is a way of getting better utilizations with the existing capacity limitations.

There are different ways of managing demand when the capacity of the supplier is limited. Shifting demand across time, product substitutions, holding inventory and optimizing the stock-rationing policies are some of the useful demand management methods. Each method has its own area of use. For long term customers with contractual agreements, the firm may determine the amount of supply at the beginning while contracting. However, in some cases the supplier may face unplanned demands and has to admit or reject the customer at the time of the demand arrival. In this type of settings, the supplier is able to make such decisions at the time of the customer arrivals. Our main interest in this chapter is to optimally perform an admission control for a single server, single customer make-to-stock queue. Firstly, we investigate the optimal acceptable demand rate of customers that will be contracted to be admitted. We determine the acceptable demand rate that gives the best expected profit. Secondly, we look at the problem from a dynamic perspective and investigate the structure of the optimal admission control policy. The aim of this chapter is to analyze the admission control problem of a supplier and to compare the methods used for the analysis.

The remainder of this chapter is organized as follows. In Section 3.2, the system is modeled and in Sections 3.3 and 3.4, the analyses with static and dynamic perspectives are done respectively. The structure of the optimal policies are determined in Section 3.4.1. In

Section 3.5, numerical examples for both methods and the comparison of the methods are presented. Finally, in Section 3.6, we summarize the conclusions driven by the structural and numerical analysis model.

3.2 Problem Definition

We consider a firm who supplies a single product with limited capacity. Since supplier's capacity is limited, the supplier does not prefer satisfying the demand of the whole market. In the market, there are customers with no distinction. Whenever the demand of the customers is satisfied, the supplier gains a reward of R per item. Unit demand arrivals occur according to a Poisson distribution with rate λ. The service time of the supplier is exponential with rate μ and she produces a single item at a time with a cost of c per item. In this environment, the supplier satisfies the demand of the customers from the inventory. If there is no inventory, backordering is allowed with a cost of b per unit time per item. The supplier incurs an inventory holding cost of h per unit time per item. The production system is modeled as a continuous-time $M/M/1$ queue. More precisely, since items can be produced in advance and held in inventory, the resulting model is a make-to-stock queue.

3.3 Analysis of the Model in the Static Framework

When the model is considered with a long term customer working through contracts, the admission control problem may be investigated in the static framework. In the beginning, before the contract, the supplier should decide on the fraction of the customers that will be satisfied. At that moment, the supplier only knows the potential rates of the customer arrivals and the capacity. In addition, the reward and costs are fully known. Thus, the supplier chooses the fraction of demand that maximizes her expected profit per unit time which depends on the customer arrival parameter λ. With the profit function $\Pi(\lambda)$, the supplier maximizes the expected revenue gained from sales after subtracting the expected inventory and backorder costs over unit time. The expected profit $\Pi(\lambda)$ of the supplier is given as follows where $E[I]$ is the expected inventory and $E[B]$ is the expected backorder:

$$\Pi(\lambda) = \lambda(R - c) - hE[I] - bE[B] \tag{3.1}$$

The optimal amount of demand to be satisfied by the supplier is the fraction of customers that maximizes the expected profit $\Pi(\lambda)$ given in Equation 3.1. The problem is simply:

$$\max_{\lambda} \Pi(\lambda) = \Pi(\lambda^*)$$

In the literature, $M/M/1$ make-to-stock queues are studied in detail and there are standard formulas for the expected inventory and backorder in the $M/M/1$ queues. For the target inventory level of Z and $\rho = \lambda/\mu$, the expected inventory and backorder are given as follows [1]:

$$\begin{aligned} E[I] &= Z - \frac{\rho}{1-\rho}(1-\rho^Z) \\ E[B] &= \frac{\rho^{Z+1}}{1-\rho} \end{aligned}$$

Since our system is also modeled as an $M/M/1$ make-to-stock queue, standard formulas for the expected inventory and the expected backorder can be used while writing the profit function of the supplier given in Equation 3.1. The expected profit of the supplier becomes:

$$\Pi(\lambda) = \lambda(R-c) - h(Z - \frac{\rho}{1-\rho}(1-\rho^Z)) - b(\frac{\rho^{Z+1}}{1-\rho}) \tag{3.2}$$

In order to find the optimal customer fraction λ^* that maximizes the profit function $\Pi(\lambda)$ given in Equation 3.1, $\Pi(\lambda)$ should be concave in λ. The second order derivative should be non-positive to satisfy concavity in λ. The first order derivative and second order derivatives are as follows:

$$\frac{d\Pi(\lambda)}{d\lambda} = (R-c) - h\left(-\left(\frac{\lambda\left(1-\rho^Z\right)}{(1-\rho)^2\mu^2}\right) + \frac{Z\lambda\rho^{-1+Z}}{(1-\rho)\mu^2} - \frac{1-\rho^Z}{(1-\rho)\mu}\right) - \frac{b\,(1+Z)\,\rho^Z}{(1-\rho)\mu} - \frac{b\,\rho^{1+Z}}{(1-\rho)^2\mu}$$

$$\begin{aligned} \frac{d^2\Pi(\lambda)}{d\lambda^2} = & -h\left(\frac{-2\lambda\left(1-\rho^Z\right)}{(1-\rho)^3\mu^3} + \frac{(-1+Z)\,Z\lambda\rho^{-2+Z}}{(1-\rho)\mu^3} + \frac{2Z\lambda\rho^{-1+Z}}{(1-\rho)^2\mu^3} - \frac{2\left(1-\rho^Z\right)}{(1-\rho)^2\mu^2}\right) \\ & -h\left(\frac{2Z\rho^{-1+Z}}{(1-\rho)\mu^2}\right) - \frac{bZ\,(1+Z)\,\rho^{-1+Z}}{(1-\rho)\mu^2} - \frac{2b\,(1+Z)\,\rho^Z}{(1-\rho)^2\mu^2} - \frac{2b\rho^{1+Z}}{(1-\rho)^3\mu^2} \end{aligned}$$

We check the inequality $\frac{d^2\Pi(\lambda)}{d\lambda^2} \leq 0$ for concavity. However, it is impossible to prove or disprove this inequality for $\Pi(\lambda)$. Therefore, the structure of the profit function cannot be easily addressed. In particular, it seems difficult to find an analytical expression for the optimal acceptable demand rate λ^*.

Remark 1 *Since the profit function is complex, we may use approximations in order to gain insights about the system. In this context, we investigate a similar system without processing time uncertainty and capacity limitations where normal approximations are used. This approximated system can be modeled as an $M/D/\infty$ queue. The analysis of this model is given in Appendix A.1.*

Remark 2 *In the literature there are papers about the structural properties of queuing systems. The results of these papers may shed some light on the structure of the complex profit function considered here at least under certain assumptions. Harel and Zipkin [18] investigate $M/M/c$ queues and prove convexity of the expected cost function in the arrival and service rates. These results are for convexity of the expected cost functions; so they are also valid for concavity of the expected profit functions. Although, in general, our problem is more complex since it is a make-to-stock queue; Harel and Zipkin's results hold for the special case of our problem where make-to-order discipline is applied, i.e. $Z = 0$.*

3.3.1 Analysis of the Model in the Static Framework with Approximations

In the previous section, we considered the admission control problem where the supplier aims to choose the demand rate that maximizes her expected profit at once. However, because of the complexity of the profit function given in Equation 3.2; we were unable to reach an explicit solution for the problem. In this section, we are going to simplify the profit function of the supplier by employing some approximations. By the help of these approximations, we may gain insights about optimal acceptable demand rate and the behavior of the supplier's profit function.

In Section 3.3, the profit function of the supplier was defined as follows:

$$\Pi(\lambda) = \lambda(R - c) - hE[I] - bE[B].$$

$E[I]$ and $E[B]$ were replaced with the formulas where a basestock policy with a target inventory level of Z is used. Recall that, the optimal basestock level for this system is:

$$Z^* = \left\lfloor \frac{\log(\frac{h}{b+h})}{\log \rho} \right\rfloor \tag{3.3}$$

where $\lfloor x \rfloor$ denotes the largest integer that is less than or equal to x.

The exact optimal basestock level for the system is the integer value given by Equation 3.3. However, since it is not practical to work with integer round-offs, we use the continuous approximation given in Equation 3.4. Hence:

$$Z^* \approx \frac{\log(\frac{h}{b+h})}{\log \rho}. \tag{3.4}$$

The profit function of the supplier can be written as follows by using the approximations (See [22]):

$$\begin{aligned} \Pi(\lambda) &\approx \lambda(R-c) - hZ^* \\ \Pi_{app}(\lambda) &= \lambda(R-c) - hZ^* \\ \Pi_{app}(\lambda) &= \lambda(R-c) - h\frac{\log(\frac{h}{b+h})}{\log \rho}. \end{aligned} \tag{3.5}$$

The static admission control problem then reduces to:

$$\max_{\lambda} \Pi_{app}(\lambda) = \Pi_{app}(\lambda^*).$$

In order to maximize the approximate profit function $\Pi_{app}(\lambda)$ given in Equation 3.5, we should check the concavity of $\Pi_{app}(\lambda)$ in λ. The first and second order derivatives are found for $\Pi_{app}(\lambda)$:

$$\begin{aligned} \frac{d\Pi_{app}(\lambda)}{d\lambda} &= (R-c) + \frac{h\,\log(\frac{h}{b+h})}{\lambda \log \rho^2} \\ \frac{d^2\Pi_{app}(\lambda)}{d\lambda^2} &= \frac{-2\,h\,\log(\frac{h}{b+h})}{\lambda^2 \log \rho^3} - \frac{h\,\log(\frac{h}{b+h})}{\lambda^2 \log \rho^2} \end{aligned}$$

If the inequality $\frac{d^2\Pi_{app}(\lambda)}{d\lambda^2} \leq 0$ is satisfied, the approximate profit function $\Pi_{app}(\lambda)$ is shown to be concave. However, for this profit function $\frac{d^2\Pi_{app}(\lambda)}{d\lambda^2} \leq 0$ has some restrictions on the value of λ:

$$\begin{aligned}\frac{d^2\Pi_{app}(\lambda)}{d\lambda^2} &= \frac{-2\,h\,\log(\frac{h}{b+h})}{\lambda^2\log\rho^3} - \frac{h\,\log(\frac{h}{b+h})}{\lambda^2\log\rho^2}\\ \frac{d^2\Pi_{app}(\lambda)}{d\lambda^2} &= -\frac{(2+\log(\frac{\lambda}{\mu}))\,h\,\log(\frac{h}{b+h})}{\lambda^2\log\rho^3}\end{aligned}$$

Since $\log\rho \leq 0$, $\log(\frac{h}{b+h}) \leq 0$ and $h \geq 0$; $2+\log\rho \geq 0$ should be verified in order to show concavity. Thus,

$$\begin{aligned}\log\rho &\geq -2\\ \rho &\geq e^{-2}\\ \rho &\geq 0.136.\end{aligned}$$

The approximate profit function $\Pi_{app}(\lambda)$ is concave for the values of $0.136 \leq \rho \leq 1$. Since concavity for at least some specific values of λ is satisfied, we check for $\frac{d\Pi_{app}(\lambda)}{d\lambda} = 0$ to find the optimal acceptable demand rate λ^*. However, this approximation does not yield an explicit solution for λ^*.

We further analyze the system with an additional approximation in order to obtain an explicit solution.

Another well-known approximation is (See [3]):

$$\log\rho \approx -(\mu-\lambda) \tag{3.6}$$

This approximation is known to be accurate as $\rho = \frac{\lambda}{\mu}$ approaches 1. The optimal basestock level is given as follows by the help of Equation 3.6:

$$Z^* \approx \frac{\log(\frac{h}{b+h})}{-(\mu-\lambda)}$$

After using the two approximations mentioned, the profit function becomes:

$$\begin{aligned}\Pi(\lambda) &\approx \lambda(R-c) - hZ^*\\ \Pi_{app2}(\lambda) &= \lambda(R-c) - h\frac{\log(\frac{h}{b+h})}{-(\mu-\lambda)}\end{aligned} \tag{3.7}$$

The profit function given by Equation 3.7 is simpler than the Equations 3.2 and 3.5. An explicit solution for the optimal acceptable demand rate may be found if $\Pi_{app2}(\lambda) \leq 0$. The first and second order derivatives of the approximated profit function given in Equation 3.7 are:

$$\frac{d\Pi_{app2}(\lambda)}{d\lambda} = (R-c) + \frac{h \log(\frac{h}{b+h})}{(\lambda-\mu)^2}$$
$$\frac{d^2\Pi_{app2}(\lambda)}{d\lambda^2} = \frac{-2\,h \log(\frac{h}{b+h})}{(\lambda-\mu)^3}.$$

The second order derivative check ensures concavity of $\Pi_{app2}(\lambda)$; since $\frac{d^2\Pi_{app2}(\lambda)}{d\lambda^2} \leq 0$ is satisfied without restrictions. Moreover, an optimal explicit solution for λ is found by solving $\frac{d\Pi_{app2}(\lambda)}{d\lambda} = 0$:

$$\lambda^* = \mu - \sqrt{-\frac{h \log(\frac{h}{b+h})}{(R-c)}}. \tag{3.8}$$

By the approximations used, the static admission control problem is solved and the acceptable demand rate that maximizes the expected profit of the supplier is found. The supplier should contract with the long term customer to satisfy the demand arriving according to a Poisson distribution with parameter λ^*. Two cases may occur:

1. If $\lambda^* < \lambda$, the supplier randomly eliminates certain proportion of customers. She serves the customers arriving according to a Poisson distribution with parameter λ^* by randomly eliminating the others.

2. Else if $\lambda^* \geq \lambda$, the supplier satisfies the demand of all customers. She serves the customers arriving according to a Poisson distribution with parameter λ.

The optimal amount of the customers admitted (λ^*) is smaller than the service rate μ of the supplier which reveals the fact that the supplier takes precautions against backordering. The capacity left for safety depends on the cost of inventory holding, backordering and production as well as the reward gained per supply.

3.3.2 The Performance of the Approximations

While analyzing the model in the static framework, we use two approximations (3.4, 3.6) on the average profit function given in Equation 3.1. It is not possible to prove concavity of the exact profit function; however, we are able to find the optimal acceptable demand rate numerically for each specific parameter setting.

In this section, we compare the approximations with the optimal results obtained through a simulation of the exact profit function. We numerically find λ^*_{exact} that maximizes the exact profit function. We set the problem parameters and then increase λ with 0.01 step sizes up to the value of μ. We choose λ^*_{exact} that has the maximum profit. We compare λ^*_{exact} with λ^* that is found by the Equation 3.8. The approximated profit is computed by plugging λ^* in the exact profit function given in 3.2. In these numerical analysis, we always assume that $c = 0$.

In the numerical tests, it is observed that the optimal exact parameter of customers that should be accepted is always lower than the approximated number of customers. Thus, the supplier keeps higher basestock levels and expects lower profits in the approximated case.

Table 3.1: Comparison of Exact Analysis with Approximations

				Exact Values			Approximated Values		
R	b	μ	h	λ^*_{exact}	Basestock	Exact Profit	λ^*	Basestock	App.Profit
100	5	0.4	1	0.31	7	23.974	0.3253	9	23.8191
100	10	0.4	1	0.3	8	21.6963	0.3354	14	19.8796
100	15	0.4	1	0.29	8	20.4166	0.3399	18	16.8106
100	20	0.4	1	0.29	9	19.5725	0.3427	20	14.5509
100	25	0.4	1	0.29	10	18.8862	0.3446	22	12.5936
100	30	0.4	1	0.28	9	18.4144	0.346	24	10.8803
100	35	0.4	1	0.28	10	17.9605	0.3472	26	9.3372
100	40	0.4	1	0.28	10	17.631	0.3481	27	8.0619
100	45	0.4	1	0.28	10	17.3014	0.3489	29	6.7551
100	50	0.4	1	0.27	10	16.9971	0.3496	30	5.6859

In Table 3.1, there is a comparison of the exact analysis with the approximations for specific parameter settings. With the approximations, the supplier tends to accept more customers. We observe that $\lambda^* \geq \lambda^*_{exact}$ in all of the examples. The supplier keeps more basestock since she aims to accept more customers when approximations are used. Thus, her optimal average profit is lower with the approximations. However, when the problem with parameters $b = 10$, $h = 1$, $\mu = 0.4$, $c = 0$ is considered, the difference between the exact and approximated profits is not very high (Figure 3.1). The percentage deviation from the exact profit for this problem is -12 on average.

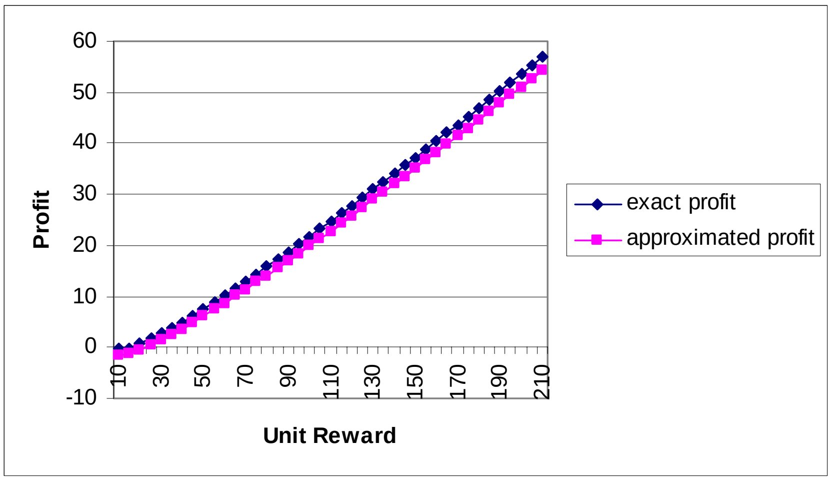

Figure 3.1: Approximated vs exact profit over R.

The difference is increasing in the unit backorder cost. At extremely high values of unit backordering cost, the approximation performs worse (Table 3.1). We observe that the performance of the approximation is sufficient for the systems that do not incur extreme unit holding and unit backordering costs (Table 3.2). Using approximations enables us to gain insights into the behaviors of the supplier while she is determining the optimal acceptable demand rate.

Table 3.2: Comparison of Exact Analysis with Approximations with respect to h

				Exact Values			Approximated Values		
R	b	μ	h	λ^*_{exact}	Basestock	Exact Profit	λ^*	Basestock	App.Profit
150	10	0.4	1	0.32	10	37.2755	0.3473	16	35.1298
150	10	0.4	2	0.3	6	32.5928	0.3137	7	32.3645
150	10	0.4	3	0.29	4	29.9402	0.2832	4	29.8343
150	10	0.4	4	0.28	3	28.1287	0.2541	2	27.2421
150	10	0.4	5	0.27	2	26.6901	0.2258	1	24.3761

3.4 Analysis of the Model in the Dynamic Framework

In certain cases, suppliers may determine the amount of demand to be satisfied by making spontaneous admission control decisions. In this section, we are going to focus on this type of supplier's admission control problem.

We assume that the supplier admits or rejects the customer demand whenever a customer order arrives. Unit demand arrivals occur and a reward of R is gained if the demand is satisfied. Since backordering is allowed, $x < 0$ is possible. Additionally, the supplier does the replenishment control. She incurs a cost of c per item produced. By using stochastic dynamic programming, we are going to address the structure of the optimal replenishment and admission control policies. In Section 3.2, the system is modeled as a continuous time $M/M/1$ queue. However, it is difficult to work with continuous-time systems. To this end, we first convert the continuous-time problem to an equivalent discrete-time problem using uniformization ([27]). After uniformization, we only consider the embedded discrete-time Markov decision chain and look at the chain at the transition instants that occur according to a Poisson process with rate $\lambda + \mu$. We rescale the time by taking $\lambda + \mu + \beta = 1$ without loss of generality. Thus, unit holding and backorder costs should be adjusted to this time scale.

The supplier aims to find the optimal replenishment and admission control policies that minimize the average cost per unit time including the cost of inventory holding and backordering minus the revenue gained. However, we first focus on the finite-horizon total discounted cost problem and then, extend it for the infinite horizon since the results for the infinite horizon discounted cost problem are used for the problem with the average cost criterion under certain conditions.

Let $V_n(x)$ be the optimal discounted cost for the discrete-time equivalent of the given system with a starting inventory level of x and n transitions to go. The value function $V_n(x)$ is defined as follows:

$$V_{n+1}(x) = c'(x) + [\lambda \min\{V_n(x-1) - R, V_n(x)\} + \mu \min\{V_n(x+1) + c, V_n(x)\}] \qquad (3.9)$$

where β is the discount rate and

$$c'(x) = \begin{cases} hx & \text{if } x > 0 \\ -bx & \text{if } x \leq 0 \end{cases}$$

The critical step in proving structural properties in the dynamic setting is to prove the convexity of the value function. Koole [27] proposes event-based operators to easily prove convexity. Following [27], the events and transitions in the system can be represented by event-based operators. The value function is a concatenation of the event-based operators. In the model with dynamic admission control, there are four events: production control, admission control, discounting and the inventory and backordering costs, uniformization of production and admission events.

$$T_{rep}f(x) = \min\{f(x+1) + c, f(x)\}$$

T_{rep} represents the replenishment decision of the supplier The supplier chooses the action that minimizes her costs. If she decides to produce, her inventory level increases by one. Otherwise, her state does not change. The next lemma establishes the convexity problems for T_{rep} operator.

Lemma 3.4.1. For every convex function $f(x)$, $T_{rep}f(x)$ preserves convexity in x.

Proof. Let $f(x)$ be convex in x, i.e. $\Delta f(x+1) \geq \Delta f(x)$ where $\Delta f(x+1) = f(x+1) - f(x)$. We are going to show convexity of $T_{rep}f(x)$:

$$\begin{aligned} \Delta T_{rep}f(x+1) &\geq \Delta T_{rep}f(x) \\ T_{rep}f(x+1) - T_{rep}f(x) &\geq T_{rep}f(x) - T_{rep}f(x-1) \\ \min\{f(x+2)+c, f(x+1)\} &- \min\{f(x+1)+c, f(x)\} \geq \\ \min\{f(x+1)+c, f(x)\} &- \min\{f(x)+c, f(x-1)\} \end{aligned}$$

All possible cases resulting through minimizations should be considered separately to verify convexity. The possible cases are listed in Table 3.3. Some cases are eliminated because they conflict with the convexity property of $f(x)$.

Table 3.3: Possible Situations: Replenishment Control Operator

	$T_{rep}f(x+1)$	$T_{rep}f(x)$	$T_{rep}f(x-1)$
1	$f(x+2)+c$	$f(x+1)+c$	$f(x)+c$
2	$f(x+1)$	$f(x+1)+c$	$f(x)+c$
3	$f(x+1)$	$f(x)$	$f(x)+c$
4	$f(x+1)$	$f(x)$	$f(x-1)$

Case 1. We have to verify the inequality:

$$f(x+2) - f(x+1) \geq f(x+1) - f(x)$$

Since $f(x)$ is assumed to be convex in x, for this case, it is directly verified that T_{rep} preserves convexity in x.

Case 2. We have to verify the inequality:

$$f(x+1) - f(x+1) - c \geq f(x+1) + c - f(x) - c$$

$$f(x) \geq f(x+1) + c$$

By the results of the minimizations, we have:

$$\begin{aligned} f(x+1) &\leq f(x+2) + c \\ f(x+1) + c &\leq f(x) \\ f(x) + c &\leq f(x-1) \end{aligned}$$

These results establish that $f(x) \geq f(x+1) + c$. So, for this case, it is verified that T_{rep} preserves convexity in x.

Case 3. We have to verify the inequality:

$$f(x+1) - f(x) \geq f(x) - f(x) - c$$

$$f(x+1) + c \geq f(x)$$

By the results of the minimizations, we have:

$$\begin{aligned} f(x+1) &\leq f(x+2)+c \\ f(x) &\leq f(x+1)+c \\ f(x)+c &\leq f(x-1) \end{aligned}$$

These results show that $f(x+1)+c \geq f(x)$. So, for this case, it is verified that T_{rep} preserves convexity in x.

Case 4. We have to verify the inequality:

$$f(x+1)-f(x) \geq f(x)-f(x-1)$$

Since $f(x)$ is assumed to be convex in x, for this case, it is directly verified that T_{rep} preserves convexity in x.

Thus, $T_{rep}f(x)$ is proven to be convex in x. □

Next, we focus on the admission control operator:

$$T_{acc}f(x) = \min\{f(x-1)-R, f(x)\}$$

T_{acc} represents the order acceptance decision of the supplier. She chooses the action that minimizes her costs. If she decides to accept the order of the customer, she will give one item to the customer from her inventory. So, her inventory level will decrease by one and she will get a reward of R for the demand satisfied. If she chooses to reject the order, her state will not change. The next lemma establishes the convexity problems for T_{acc} operator.

Lemma 3.4.2. For every convex function $f(x)$, $T_{acc}f(x)$ preserves convexity in x.

Proof. Let $f(x)$ be convex in x, i.e. $\Delta f(x+1) \geq \Delta f(x)$ where $\Delta f(x+1) = f(x+1)-f(x)$. We are going to show convexity of $T_{acc}f(x)$:

$$\begin{aligned} \Delta T_{acc}f(x+1) &\geq \Delta T_{acc}f(x) \\ T_{acc}f(x+1)-T_{acc}f(x) &\geq T_{acc}f(x)-T_{acc}f(x-1) \end{aligned}$$

$$\min\{f(x)-R, f(x+1)\}-\min\{f(x-1)-R, f(x)\} \geq \min\{f(x-1)-R, f(x)\}-\min\{f(x-2)-R, f(x-1)\}$$

Once again, all possible cases resulting through minimizations should be considered separately to verify convexity. The possible cases are listed in Table 3.4. As in Lemma 3.4.1, some cases are eliminated because they conflict with the convexity property of $f(x)$.

Table 3.4: Possible Situations: Unit Admission Control Operator

	$T_{acc}f(x+1)$	$T_{acc}f(x)$	$T_{acc}f(x-1)$
1	$f(x)-R$	$f(x-1)-R$	$f(x-2)-R$
2	$f(x)-R$	$f(x-1)-R$	$f(x-1)$
3	$f(x)-R$	$f(x)$	$f(x-1)$
4	$f(x+1)$	$f(x)$	$f(x-1)$

Case 1. We have to verify the inequality:

$$\begin{aligned} f(x)-R-f(x-1)+R &\geq f(x-1)-R-f(x-2)+R \\ f(x)-f(x-1) &\geq f(x-1)-f(x-2) \end{aligned}$$

Since $f(x)$ is assumed to be convex in x, for this case, it is directly verified that T_{acc} preserves convexity in x.

Case 2. We have to verify the inequality:

$$\begin{aligned} f(x)-R-f(x-1)+R &\geq f(x-1)-R-f(x-1) \\ f(x) &\geq f(x-1)-R \end{aligned}$$

By the results of the minimizations, we have:

$$\begin{aligned} f(x)-R &\leq f(x+1) \\ f(x-1)-R &\leq f(x) \\ f(x-1) &\leq f(x-2)-R \end{aligned}$$

These results yield that $f(x) \geq f(x-1)-R$. So, for this case, it is verified that T_{acc} preserves convexity in x.

Case 3. We have to verify the inequality:

$$\begin{aligned} f(x) - R - f(x) &\geq f(x) - f(x-1) \\ f(x-1) - R &\geq f(x) \end{aligned}$$

By the results of the minimizations, we have:

$$\begin{aligned} f(x) - R &\leq f(x+1) \\ f(x) &\leq f(x-1) - R \\ f(x-1) &\leq f(x-2) - R \end{aligned}$$

These results establish that $f(x-1) - R \geq f(x)$. So, for this case, it is verified that T_{acc} preserves convexity in x.

Case 4. We have to verify the inequality:

$$f(x+1) - f(x) \geq f(x) - f(x-1)$$

Since $f(x)$ is assumed to be convex in x, for this case, it is directly verified that T_{acc} preserves convexity in x.

Thus, $T_{acc}f(x)$ is proven to be convex in x. □

Next, we investigate the cost operator:

$$T_{disc}f(x) = c'(x) + f(x)$$

T_{disc} represents the direct holding and backordering costs depending on the state of the supplier. The supplier incurs backordering cost of b per period per item if she cannot satisfy the demand from inventory, i.e. $x \leq 0$. If there are products in the inventory, the supplier incurs the inventory holding cost of h per period per item.

Lemma 3.4.3. For every convex function $f(x)$, $T_{disc}f(x)$ preserves convexity in x.

Proof. The proof follows since $c'(x)$ and $f(x)$ are convex functions [27]. □

Finally, we focus on the operator for the uniformization of events:

$$T_{uni}(f_1, f_2)(x) = \lambda f_1(x) + \mu f_2(x)$$

T_{uni} does the uniformization of events in a continuous time Markovian model; T_{uni} represents the random events that may happen at an instant. The time is discretized whenever the events occur.

Lemma 3.4.4. For convex functions $f_j(x)$, $\forall j$, $T_{uni}(f_1(x), f_2(x))$ preserves convexity in x.

Proof. The proof follows since $f_j(x)$ is convex $\forall j$ [27]. □

The value function, given in Equation 3.9, can be rewritten by using the above operators:

$$V_{n+1}(x) = T_{disc}(T_{uni}(T_{acc}V_n, T_{rep}V_n))(x)$$

The discounted problem allows us to use induction in showing the structure of the value function. To start induction, we assume that $V_0(x) = 0$, $\forall x$, and verify convexity for $n = 0$. Then, we assume that convexity is verified for n and check for $n + 1$. Following is the main result of induction and the event-based dynamic analysis:

Proposition 3.4.5. The value function $V_{n+1}(x)$ is convex in x since it can be represented as a linear combination of operators that are proven to preserve convexity of $V_n(x)$ in x.

As we stated earlier, our aim is to determine the structure of the optimal policies when the infinite horizon problem is considered. However, by Proposition 3.4.5, we address the structure of the optimal policies with the aim of minimizing the total discounted costs for a finite number of transitions n. The results of the finite horizon problem can be related with the infinite horizon problems with the objective of minimizing the discounted and the long-run average costs through value iteration if certain standard conditions hold [30]. Since these conditions hold, $V_n(x)$ converges to the optimal total discounted cost $V(x)$ of the infinite horizon problem, i.e. $V(x) = \lim_{n\to\infty} V_n(x)$. This shows that the infinite horizon discounted cost problem possesses the same structural properties. In addition, the structure of the optimal policies with the aim of minimizing the average cost per unit time is similar to the structure of the policies with the finite horizon discounted problem as n goes to infinity and β approaches 0. We, therefore, conclude that Proposition 3.4.5 extends to infinite-horizon models.

3.4.1 Structure of the Optimal Policies

The value function of the dynamic program written for the infinite-horizon problem is proven to be convex in x. Since we have shown the structural properties of the value function, we are able to address the structure of the optimal policy of replenishment and admission control:

Corollary 3.4.6. The optimal replenishment policy is a threshold type policy since the value function is convex in x. The optimal replenishment policy is a basestock policy with an optimal basestock level of Z^*. If the supplier has less than Z^* items in her inventory, she always produces. If her inventory level is greater than or equal to Z^*, she never produces.

Next, we focus on the structure of the optimal admission control policy:

Corollary 3.4.7. The optimal admission control policy is a threshold type policy since the value function is convex in x. The supplier has a threshold type admission control policy with an order acceptance threshold of X^*. If the supplier has more than X^* items in her inventory, she admits the customer's order. However, if she has less inventory, she rejects the order.

3.4.2 Optimal Acceptable Demand Rates

In this chapter, our aim is to determine the optimal number of customers to be admitted by a capacitated supplier. When the problem is considered in the dynamic framework, the optimal acceptable demand rate may be found through a dynamic program. This DP should use as input the admission decisions from the main dynamic model that is represented by the value function 3.9. At each admission, the supplier is assumed to collect a reward of 1. Thus, the supplier is able to count the number of customers admitted per unit time by counting the total revenue gained. As in the main DP, we assume without loss of generality that $\lambda + \mu + \beta = 1$.

The supplier has two decisions: production and order acceptance. Let $a(x)$ be the decision about order acceptance made by the supplier at state x.

$$a(x) = \begin{cases} 1 & \text{if the supplier decides to admit the customer at } x \\ 0 & \text{if the supplier decides to reject the customer at } x \end{cases}$$

Proposition 3.4.8. The below value function gives the expected total β-discounted number of customers accepted over n transitions:

$$\hat{V}_{n+1}(x) = \lambda[a(x)\,(\hat{V}_n(x-1)+1) + (1-a(x))\,\hat{V}_n(x)] \tag{3.10}$$

As before, letting $n \to \infty$ and $\beta \to 0$, we can obtain the average revenue per unit time which yields the optimal arrival rate λ_d^*.

Remark 3 λ_d^* *can also be obtained by numerically solving the Markov Chain corresponding to the optimal admission policy.*

3.5 Numerical Analysis

In this section, we will present the results obtained by generating sample runs with different system parameters. The objective of this numerical study is to observe the effects of system parameters when static and dynamic admission control policies are separately applied in a system. Additionally, we are going to compare these admission control methods and underline the advantages and disadvantages of the methods. The average profit criterion is used to analyze the admission control methods. The value iteration algorithm [30] is coded in C in order to report the average profit per unit time for the dynamic control problem. In the numerical examples considered, we assume that fifty items can be stored and backordered at maximum and the unit production cost $c = 0$.

3.5.1 Results for the Static Admission Control

In the numerical analysis of the static method, we use the explicit solution of λ^* given in Equation 3.8. However, we work with the exact profit function by placing λ^* in the $\Pi(\lambda)$ given by Equation 3.2.

Figure 3.2 displays the classical relationship of unit reward and unit backordering cost with the average profit on a specific example with parameters $\mu = 0.4, h = 1$. An increase in the unit backordering cost shifts the average profit curve downwards. On the other hand, an increase in the unit reward stimulates the increase of the average profit.

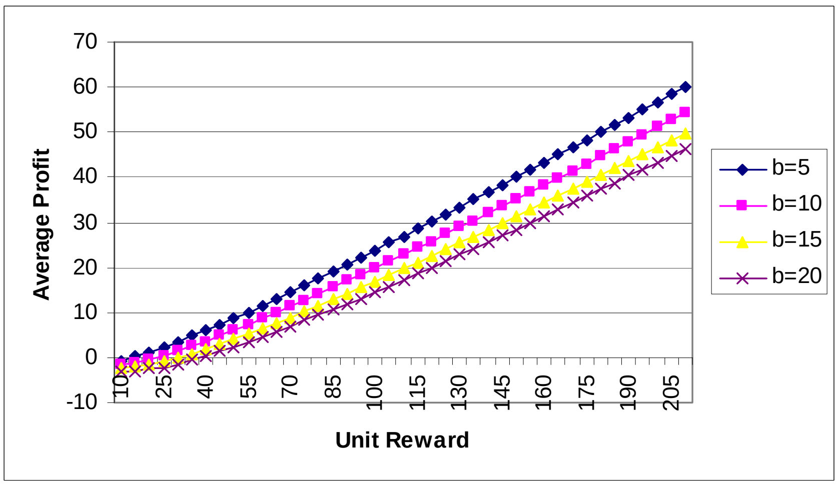

Figure 3.2: R vs average profit.

In Figure 3.3, the problem with $b = 10, h = 1$ is investigated. The figure verifies that capacity is the constraint on the number of customers to be accepted. The supplier with higher capacity accepts more customers and gains more. The unit reward has the same effect; the optimal customer arrival rate increases in the unit reward.

In the static method, the supplier determines her basestock level according to the unit reward and the customer arrival rate. In Figure 3.4, the basestock level of the supplier is shown for the problem with parameters $\mu = 0.4$, $h = 1$. The supplier increases her inventory level when the unit reward and the customer arrival rate increase.

3.5.2 Results for the Dynamic Admission Control

In the numerical analysis of dynamic control, the average profit is reported by using the value iteration algorithm [30]. Without loss of generality, it is assumed that $\lambda + \mu = 1$. Since the supplier has demand that she cannot fully satisfy, $\lambda > \mu$ by the problem definition.

Figure 3.3: R vs μ.

In Figure 3.5, the problem with parameters $\lambda + \mu = 1, \lambda = 0.6, b = 10, R = 50$ is considered. The increase in the unit holding cost decreases the safety stocks held and the average profit.

In the dynamic control model, the supplier usually accepts relatively high amounts of customers. Especially when her capacity is low, the supplier usually works at full capacity. Therefore, the increase in unit reward affects the supplier whose capacity is relatively higher. This effect is obvious in Figure 3.6 for the problem with $b = 10, h = 1$.

The supplier who faces higher demand with respect to her capacity keeps a higher basestock level to satisfy more customers. The problem with $b = 5, h = 1$ is considered in Figure 3.7. If the supplier gains more from unit customer demand, she holds more inventory to guarantee receiving the reward.

3.5.3 Comparison of Static and Dynamic Admission Control Policies

In this subsection, the aim is to compare admission control methods used to analyze the continuous-review model considered in this chapter. The problem is studied with different parameters.

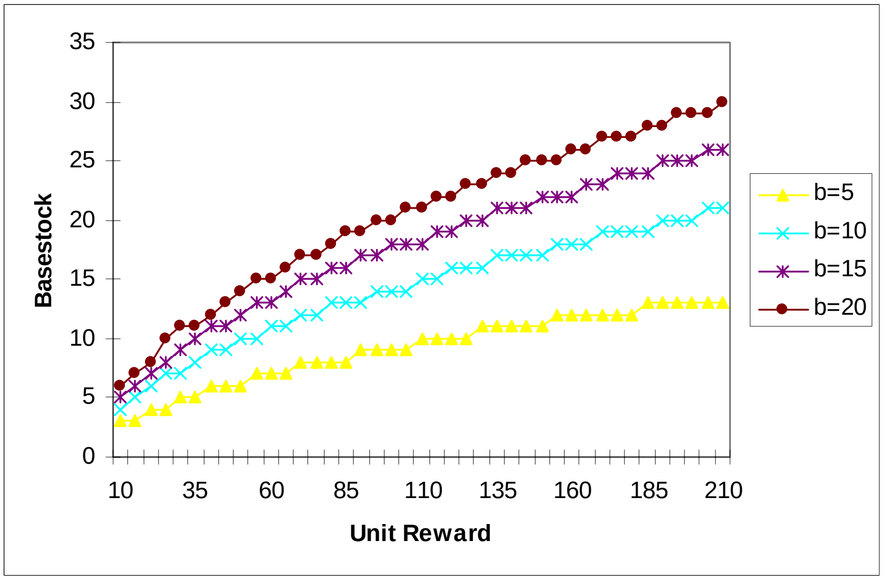

Figure 3.4: R vs basestock.

Below in Figures 3.8 and 3.9, the impacts of capacity on the static and dynamic models are displayed. In the figure, the problem with parameters $R = 250, b = 10, h = 1$ is considered. $\lambda + \mu = 1$ is still valid for the dynamic control case. The graphics display the importance of capacity as a constraint on the optimal accepted demand rate. When the capacity increases, the number of accepted customers significantly increases. The optimal basestock level also shows a non-decreasing behavior which causes non-decreasing inventory holding costs. This is because of the increasing number of customers admitted. Since more customers are admitted, more inventory should be held. Despite holding costs, the average profit increases as the capacity parameter increases. Therefore, we conclude that the revenue gained from the optimally satisfied demand is so significant that it compensates the increasing holding costs. It is worth mentioning that determining the amount of demand to be satisfied optimally is very effective in the system performance.

In Figures 3.10 and 3.11, the effects of the unit holding cost h is investigated for both dynamic and static methods. The impact of the unit holding cost is as expected on base-

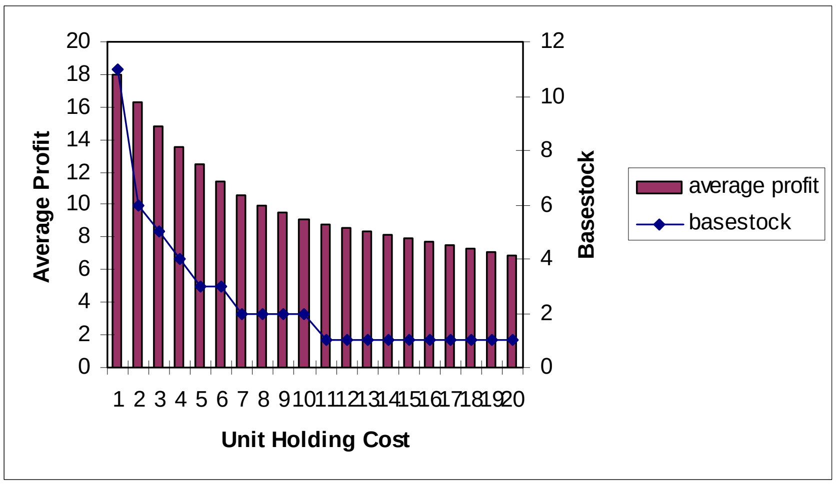

Figure 3.5: h vs average profit and basestock Level.

Figure 3.6: R vs λ^*.

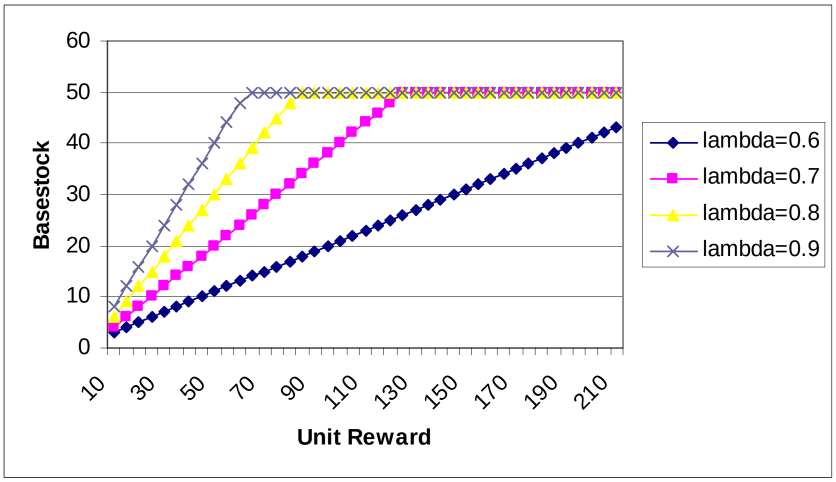

Figure 3.7: R vs basestock and λ^*.

stock levels and average profit. Since holding inventory is more costly, the supplier carries less inventory. The average profit of the supplier decreases because of the increase in the expected inventory costs. However, λ^* also decreases and fewer customers are accepted since inventory levels decrease. The supplier can guarantee less demand fulfillment with less safety stock. The dynamic and static admission control methods display same behavior with respect to the unit holding cost.

Table 3.5 aims to summarize the advantages and disadvantages of the static and dynamic methods. For some representative cases, related results are listed. The basestock levels in two methods do not show a specific behavior; the static basestock level is sometimes less than the dynamic basestock level and sometimes it is greater. However, at each case, dynamic λ^* is greater than or equal to the static λ^*. So, the dynamic average profit is also greater than or equal to the static average profit. In the right-most columns, the percentage differences in the average profits of the two methods are listed. The percentage difference between the approximated profit and the dynamic profit is computed by $\Delta_1 = 100(\Pi_d^*(\lambda) - \Pi_s^*(\lambda))/\Pi_s^*(\lambda)$. Additionally, the percentage difference between the exact profit (found by simulation) and the dynamic profit is computed by $\Delta_2 = 100(\Pi_d^*(\lambda) - \Pi_{exact}^*(\lambda))/\Pi_{exact}^*(\lambda)$.

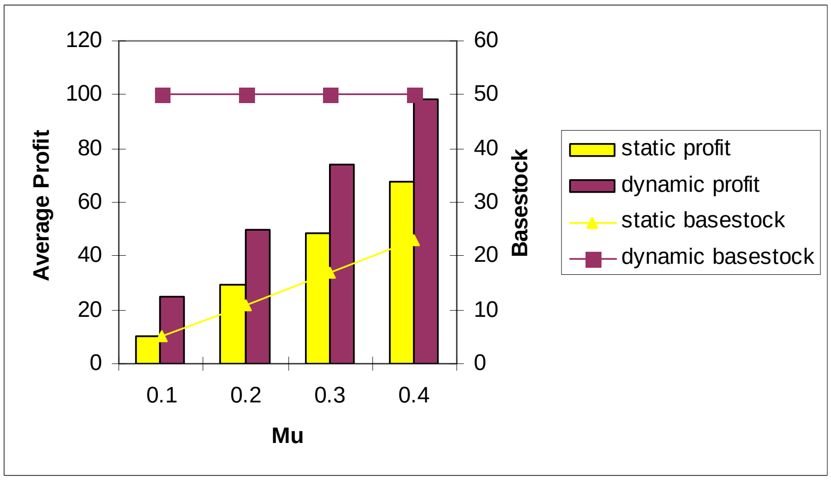

Figure 3.8: Static profit vs dynamic profit over μ and basestock.

We can conclude that the static analysis brings strict precautions against backordering, thus the accepted number of customers stays low. In the dynamic framework, the supplier is able to adjust order acceptance to inventory levels so that she may work at full capacity with inconsiderable backordering. In this problem, dynamic control brings high benefits to the system. However, for some systems, dynamic control may not be suitable or may be difficult to apply. Therefore, static analysis is still valuable for the cases where dynamic control is not possible.

Remark 4 *In the numerical analysis, it is observed that the order acceptance threshold is always less than or equal to zero, i.e.* $X^* \leq 0$*. There is no reason to keep inventory rather than satisfying demand since there is only one customer class. Therefore the supplier always accepts the customer if there is inventory on hand.*

3.6 Concluding Remarks

In this chapter, we consider a production-inventory system with a single customer and backorders. We model the system as a make-to-stock queue with a single server. We compare the dynamic admission control of customers with a static one where the admitted

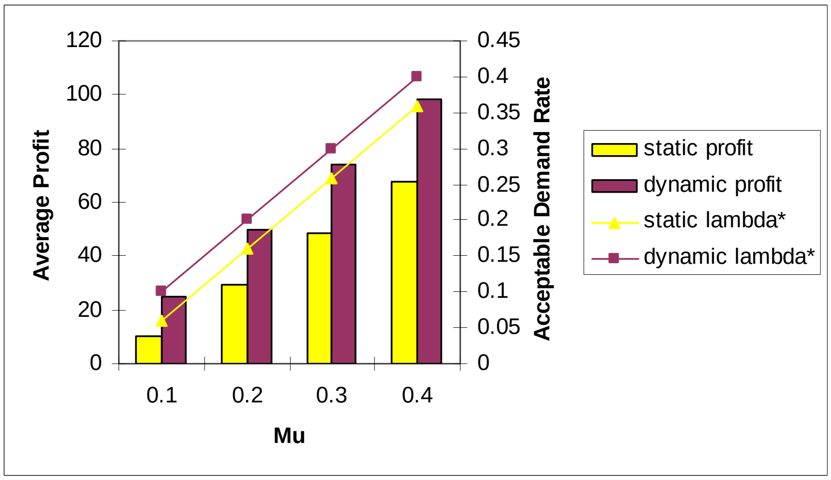

Figure 3.9: Static profit vs dynamic profit over μ and λ^*.

amount is determined at the beginning. Using dynamic programming, the production and admission control policies are found out to be of threshold type. An approximate explicit solution is proposed for the amount to be admitted at the beginning of the corresponding planning period. However, the numerical results display the fact that dynamic control still brings significant benefits to the system in means of better service level and higher profits. Thus, we are going to focus on dynamic admission control of customers in the remaining part of this thesis .

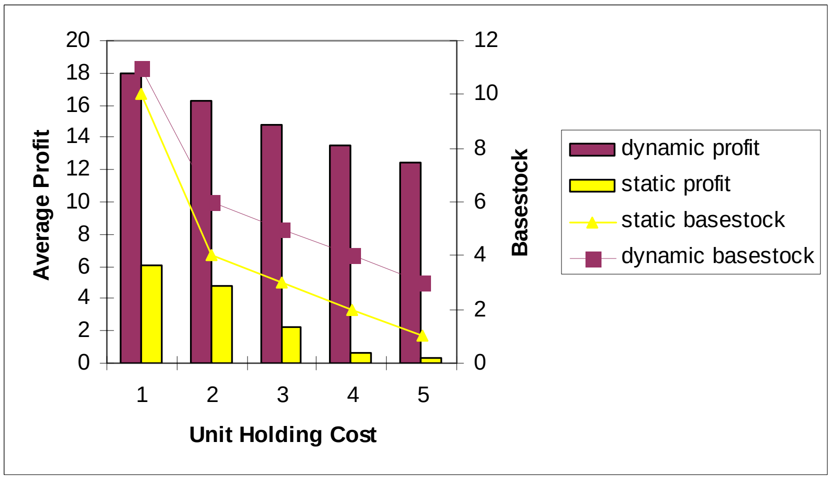

Figure 3.10: Static profit vs dynamic profit over h and basestock.

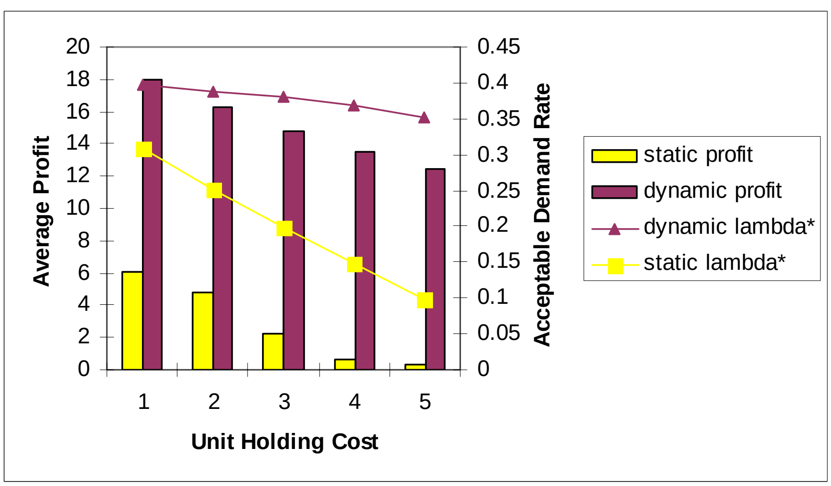

Figure 3.11: Static profit vs dynamic profit over h and λ^*.

Table 3.5: Comparison of Static and Dynamic Analysis

		Static App.		**Static Exact**		**Dynamic**				**Differences**	
R	**b**	λ_s^*	Z_s^*	λ_{exact}^*	Z_{exact}^*	λ_d^*	Z_d^*	X^*	$\Pi_d^*(\lambda)$	% Δ_1	% Δ_2
50	5	0.29	6	0.27	4	0.4	11	0	18.02	103.58	100.39
55	5	0.30	7	0.29	5	0.4	12	0	20.01	98.29	92.08
55	10	0.31	10	0.26	5	0.4	12	0	20.01	170.03	127.73
60	5	0.30	7	0.29	5	0.4	13	0	22.01	89.33	85.43
60	10	0.32	11	0.27	6	0.4	13	0	22.01	155.85	117.55
60	15	0.32	13	0.26	6	0.4	13	0	22.01	240.43	138.79
65	5	0.31	7	0.29	5	0.4	14	0	24.00	82.58	80.24
65	10	0.32	11	0.28	6	0.4	14	0	24.00	139.41	108.49
65	15	0.33	14	0.26	6	0.4	14	0	24.00	214.91	128.27
65	20	0.33	16	0.26	7	0.4	14	0	24.00	316.65	143.82
70	5	0.31	8	0.3	6	0.4	15	0	26.00	79.80	75.74
70	10	0.32	12	0.28	6	0.4	15	0	26.00	130.78	101.36
70	15	0.33	15	0.28	7	0.4	15	0	26.00	196.08	119.27
70	20	0.33	17	0.27	7	0.4	15	0	26.00	278.30	132.33
70	25	0.33	19	0.26	7	0.4	15	0	26.00	400.37	143.24
75	5	0.31	8	0.3	6	0.4	16	0	28.00	74.56	71.83
75	10	0.33	12	0.29	7	0.4	16	0	28.00	119.90	95.37
75	15	0.33	15	0.28	7	0.4	16	0	28.00	175.06	111.20
75	20	0.33	17	0.27	7	0.4	16	0	28.00	242.08	123.26
75	25	0.34	19	0.27	8	0.4	16	0	28.00	332.58	133.35
75	30	0.34	21	0.27	8	0.4	16	0	28.00	465.71	142.40

Chapter 4

OPTIMAL PRODUCTION AND ADMISSION CONTROL IN A MAKE-TO-STOCK SYSTEM WITH ADI

4.1 Introduction

Suppliers carry inventory because of several reasons like demand uncertainty and production leadtime. Carrying inventory is a costly activity that does not add value to the product. Thus, decreasing inventory and its costs is crucial for most of the suppliers. Since demand uncertainty is a reason to carry inventory, sharing demand information in advance between the members of a supply chain acts as a substitute for safety stocks. Demand information can easily be shared in advance with the use of the latest information technologies in the supply chain. In order to benefit from advance demand information (ADI) in decreasing inventory costs, ADI should be smartly integrated into the supply chain.

The role of ADI in multi-class systems has recently become an area of interest. The tradeoffs between carrying inventory and sharing ADI are important when operating in a multi-customer setting. In particular, the supplier may have to determine the right portfolio of customers through an admission control policy in order to maximize her benefits. In the multi-customer setting, it is possible to optimally manage demand by the help of both ADI and admission control.

In this chapter, we consider a supplier facing demand and processing time uncertainty with limited capacity and multi customers. We aim at minimizing the total of inventory, outsourcing and production costs minus the revenue gained from the accepted orders. We focus on one type of advance demand information -advance order information- that represents the fixed orders by contractual agreements. Our main modeling interest is in capturing the tradeoffs in a situation where different customers provide different levels of advance demand information. In this setting, we investigate the role of inventory and ADI to manage multi-class demand through dynamic admission control policies.

The remainder of this chapter is structured as follows. In Section 4.2, the problem is defined and the model is built. Section 4.3 analyzes the model with Stochastic Dynamic Programming. The structure of the optimal policies is described in Section 4.4. In Section 4.5, numerical examples analyzing the impacts of ADI and admission control are presented. Extensions done for the model are summarized in Section 4.6. Lastly, in Section 4.7, the observations and conclusions are given for the ADI model.

4.2 Problem Definition and Model Formulation

We consider a supplier who produces a single item at a time for two different customer classes. The first customer class corresponds to contractual customers. According to the contract, the customer orders H periods in advance of the due date and the supplier has to satisfy all demands of this class at the due-date. The second class corresponds to non-contractual customers who desire to purchase the item immediately but the supplier can reject such demands if necessary. The decision of whether to accept or reject the customer order depends on the inventory level and future demand information at that time.

The processing times of the product have a geometric distribution, that is, the production will be completed during the same period with probability p if started at the beginning of the period. At each period, unit demand arrivals occur randomly. First and second class demand inter-arrival times have independent geometric distributions with parameters q_1 and q_2 respectively. A reward R_1 is gained by the supplier, if an order of the first customer class is received. Similarly, R_2 is gained whenever an order of the second class customer is fulfilled.

The supplier has the capacity to produce one unit at a time with a cost of c per item. If there is inventory, the orders are fulfilled by the inventory. However, if there is no inventory, the orders can be fulfilled by outsourcing the product with a cost of F per item. Since the same product is sold to both customers, the outsourcing costs are assumed not to differ according to customers. Additionally, backordering is not allowed. An inventory cost (h) is incurred per period per item for the stocked products.

We model the production-inventory system as a discrete time make-to-stock queue. In this setting, the supplier has two decisions in a period: replenishment and order acceptance. The supplier considers these decisions with the aim of minimizing the inventory, outsourcing and variable production costs minus the sales revenue discounted over an infinite horizon. To enhance our understanding of the supplier's problem, it is crucial to investigate the structure of the optimal policy of replenishment and admission control. We assume that the events in a period occur in a sequence: production decision, production completion, arrival of the first customer, arrival of the second customer and order acceptance decision.

Remark 1 *In this problem we assume that a reward is gained by the supplier when a first class customer orders. However, the reward may be received when the demand is satisfied H periods after receiving the order. If the reward was gained at the time of the demand fulfillment, nothing would change structurally. The only thing that would change is that the reward would be discounted for H periods by the discount factor α.*

4.3 Analysis of the Model

Stochastic Dynamic Programming (SDP) is used to address the structural properties for the supplier's replenishment and order acceptance problems. The supplier aims to find the optimal replenishment and admission control policies that minimize the average cost per unit time including the inventory holding, outsourcing and variable production costs minus the revenue gained. However, we first focus on the finite-horizon total discounted cost problem and then, extend it for the infinite horizon since the results for the infinite horizon discounted cost problem are used for the problem with the average cost criterion under certain conditions. The value function of the stochastic dynamic program that minimizes the discounted cost with n periods to go until the end of the horizon is $V_n(\,)$. We assume that $V_0(x, d) = 0$, $\forall\, x, \mathbf{d}$. The state space consists of two components: the inventory level, x ($x \geq 0$), and the demand vector $\mathbf{d} = [d_1\ d_2\ \ldots d_H]$ for the first class customers with d_i as the first class demand to be fulfilled i periods later. At every period, the first element (d_1) of this vector is fulfilled by the supplier. $\mathbf{d}^+$ is the demand vector after $\mathbf{d}$ is shifted left or in other words, after the demand of that period (d_1) is fulfilled. e_H is the vector with one in the H^{th} entry and zeroes elsewhere, used to display the arrival of first class customer order that will be satisfied H periods later. Additionally, α $(0 < \alpha < 1)$ is the discount factor.

Below, the value function is given, divided into smaller parts for practical purposes:

$$
\begin{aligned}
V_n^{(1)}(x,\mathbf{d}) &= V_n((x-d_1)^+, d^+) - F(x-d_1)^- \\
V_n^{(2)}(x,\mathbf{d}) &= \min\{V_n((x-d_1-1)^+, d^+) - R_2 - F\,(x-d_1-1)^-, \\
&\quad\; V_n((x-d_1)^+, d^+) - F\,(x-d_1)^-\} \\
V_n^{(3)}(x,\mathbf{d}) &= V_n((x-d_1)^+, d^+ + e_H) - R_1 - F(x-d_1)^- \\
V_n^{(4)}(x,\mathbf{d}) &= \min\{V_n((x-d_1-1)^+, d^+ + e_H) - R_1 - R_2 - F(x-d_1-1)^-, \\
&\quad\; V_n((x-d_1)^+, d^+ + e_H) - R_1 - F\,(x-d_1)^-\} \\
V_n^{(5)}(x,\mathbf{d}) &= q_1q_2V_n^{(4)}(x,\mathbf{d}) + q_1(1-q_2)V_n^{(3)}(x,\mathbf{d}) + (1-q_1)q_2V_n^{(2)}(x,\mathbf{d}) \\
&\quad\; +(1-q_1)(1-q_2)V_n^{(1)}(x,\mathbf{d}) \\
V_n^{(6)}(x,\mathbf{d}) &= \min\{V_n^{(5)}(x+1,\mathbf{d}) + c, V_n^{(5)}(x,\mathbf{d})\} \\
V_n^{(7)}\;(x,\mathbf{d}) &= p\,V_n^{(6)}(x,\mathbf{d}) + (1-p)V_n^{(5)}(x,\mathbf{d}) \\
V_{n+1}(x,\mathbf{d}) &= hx + \alpha V_n^{(7)}(x,\mathbf{d})
\end{aligned}
$$

If the small parts are put together, the value function becomes:

$$
\begin{aligned}
V_{n+1}(x,\mathbf{d}) \;=\;& hx+\alpha p\,\min\{(q_1q_2\min\{V_n((x-d_1)^+,d^++e_H)-R_1-R_2\\
&-F(x-d_1)^-,V_n((x-d_1+1)^+,d^++e_H)-R_1-F\,(x-d_1+1)^-\}\\
&+q_1(1-q_2)V_n((x-d_1+1)^+,d^++e_H)-R_1-F\,(x-d_1+1)^-\\
&+(1-q_1)q_2\min\{V_n((x-d_1)^+,d^+)-R_2-F\,(x-d_1)^-,\\
&V_n((x-d_1+1)^+,d^+)-F\,(x-d_1+1)^-\}+(1-q_1)(1-q_2)\\
&V_n((x-d_1+1)^+,d^+)-F\,(x-d_1+1)^-)+c,\\
&q_1q_2\min\{V_n((x-d_1-1)^+,d^++e_H)-R_1-R_2-F\,(x-d_1-1)^-,\\
&V_n((x-d_1)^+,d^++e_H)-R_1-F\,(x-d_1)^-\}+q_1(1-q_2)\\
&V_n((x-d_1)^+,d^++e_H)-R_1-F(x-d_1)^-+(1-q_1)q_2\\
&\min\{V_n((x-d_1-1)^+,d^+)-R_2-F\,(x-d_1-1)^-,V_n((x-d_1)^+,d^+)\\
&-F\,(x-d_1)^-\}+(1-q_1)(1-q_2)V_n((x-d_1)^+,d^+)-F(x-d_1)^-\}\\
&+(1-p)q_1q_2\min\{V_n((x-d_1-1)^+,d^++e_H)-R_1-R_2\\
&-F\,(x-d_1-1)^-,V_n((x-d_1)^+,d^++e_H)-R_1-F\,(x-d_1)^-\}+q_1\\
&(1-q_2)V_n((x-d_1)^+,d^++e_H)-R_1-F(x-d_1)^-+(1-q_1)q_2\\
&\min\{V_n((x-d_1-1)^+,d^+)-R_2-F\,(x-d_1-1)^-,\\
&V_n((x-d_1)^+,d^+)-F\,(x-d_1)^-\}+(1-q_1)(1-q_2)\\
&V_n((x-d_1)^+,d^+)-F(x-d_1)^-
\end{aligned}
$$

In order to analyze the model with SDP, we should handle the value function and investigate its characteristics. However, it is hard to work with the given complex value function. Therefore, we are going to analyze the value function in pieces by the help of the Event-Based Stochastic Dynamic Programming framework as in Chapter 3 [26].

In our model, we seek to prove convexity of the value function in x, $x \geq 0$. Two properties should be preserved by the event-based operators in this context:

$$
\begin{aligned}
Conv(x) &: \Delta f(x+1,\mathbf{d}) \geq \Delta f(x,\mathbf{d})\\
LBD(F) &: \Delta f(x+1,\mathbf{d}) \geq -F
\end{aligned}
$$

where $\Delta f(x+1,\mathbf{d}) = f(x+1,\mathbf{d}) - f(x,\mathbf{d})$. $Conv(x)$ represents convexity property. Since our model does not allow backordering, x is always non-negative. Therefore, another property

$LBD(F)$ is used in the analysis. In particular, $LBD(F)$ reveals that outsourcing is not done if there is no demand to be satisfied.

In our model, nine event-based operators are used. The event-based operators used to define the value function are:

$$T_{costs}f(x, \mathbf{d}) = hx + \alpha f(x, \mathbf{d})$$

T_{costs} represents the direct holding costs and the discounting. If there are products in the inventory, the supplier will incur the inventory holding cost of h per period per item. The costs and the revenue are discounted over time.

Lemma 4.3.1. If $f(x, \mathbf{d})$ is convex in x, $\forall\, \mathbf{d}$, $T_{costs}f(x, \mathbf{d})$ preserves convexity in x, $\forall\, \mathbf{d}$.

Proof. The proof follows since hx and $f(x, \mathbf{d})$ are convex. □

Next, we focus on the operator denoting the event of production completion:

$$T_{unifpro}(f_1, f_2)\,(x, \mathbf{d}) = pf_1(x, \mathbf{d}) + (1 - p)f_2(x, \mathbf{d})$$

$T_{unifpro}$ is the convex combination of f_j. This operator does the uniformization of events; $T_{unifpro}$ represents the random production completions that may happen in a period. $T_{unifpro}$ shows the expectation of production completion.

Lemma 4.3.2. If $f_j(x, \mathbf{d})$ is convex in x, $\forall\, j, \mathbf{d}$, $T_{unifpro}(f_1, f_2)(x, \mathbf{d})$ preserves convexity in x, $\forall\, \mathbf{d}$.

Proof. The proof follows from the closedness of convexity under convex combinations. □

Next, we focus on the operators representing the events of first and second class customer arrival:

$$T_{unifar1}(f_1, f_2)\,(x, \mathbf{d}) = q_1 f_1(x, \mathbf{d}) + (1 - q_1)f_2(x, \mathbf{d})$$

$$T_{unifar2}(f_1, f_2)\,(x, \mathbf{d}) = q_2 f_1(x, \mathbf{d}) + (1 - q_2)f_2(x, \mathbf{d})$$

These operators arise as applications of Lemma 4.3.2. $T_{unifar1}$ and $T_{unifar2}$ represent arrivals of first and second class customers respectively.

Next, we define the operator used for the production decision of the supplier:

$$T_{AC}f\,(x,\mathbf{d}) = \min\{f\,(x+1,\mathbf{d})\ +c\,,f\,(x,\mathbf{d})\}$$

T_{AC} is the operator that represents production decision. The supplier chooses the action that minimizes her costs. If she decides to produce, she will incur a cost of c and her inventory level will increase by one. If she decides not to produce, her position does not change.

Lemma 4.3.3. If $f(x,\mathbf{d})$ is convex in x, $\forall\,\mathbf{d}$, $T_{AC}f(x,\mathbf{d})$ preserves convexity in x, $\forall\,\mathbf{d}$.

Proof. The proof is in the Appendix A.2. □

The fulfillment of first class orders are denoted by T_{ofl}:

$$T_{ofl}f\,(x,\mathbf{d}) = f\,(x-d_1,d^+)$$

Order fulfillment of the first class customers is displayed by the operator T_{ofl}. The supplier satisfies the demand of the first class (d_1) from the inventory. So, her inventory level decreases by the demanded amount. Additionally, the demand vector is shifted leftwards, since the demand of the current period is no more collected.

Lemma 4.3.4. If $f(x,\mathbf{d})$ is convex in x, $\forall\,\mathbf{d}$, and satisfies $LBD(F)$, $T_{ofl}f(x,\mathbf{d})$ preserves convexity and $LBD(F)$ property in x, $\forall\,\mathbf{d}$.

Proof. The proof is in the Appendix A.3. □

The first class order arrivals are represented by T_{coa}:

$$T_{coa}f\,(x,\mathbf{d}) = f\,(x,d+e_H) - R_1$$

The contractual order arrivals, the orders of the first class customers, are represented by T_{coa}. As soon as the first class order is received, the supplier gets a reward of R_1. In addition, the amount of demand that will be satisfied H periods later is added as the H^{th} entry of the demand vector.

Lemma 4.3.5. If $f(x,\mathbf{d})$ is convex in x, $\forall\,\mathbf{d}$, $T_{coa}f(x,\mathbf{d})$ preserves convexity in x, $\forall\,\mathbf{d}$.

Proof. The proof is in the Appendix A.4. □

Next, we focus on the operator for admission control:

$$T_{aca}f\left(x,\mathbf{d}\right)=\min\{f\left(x-1,\mathbf{d}\right)-R_2,f\left(x,\mathbf{d}\right)\}$$

T_{aca} denotes the admission control decision for the second class customers. The supplier decides to admit or reject the second class customers in order to minimize her costs. If she decides to admit the orders, her inventory level decreases by one and she gains a reward of R_2. If she decides to reject the second class orders, her state does not change.

Lemma 4.3.6. If $f(x,\mathbf{d})$ is convex in x, $\forall\,\mathbf{d}$, and satisfies $LBD(F)$, $T_{aca}f(x,\mathbf{d})$ preserves convexity and $LBD(F)$ property in x, $\forall\,\mathbf{d}$.

Proof. The proof is in the Appendix A.5. □

Finally, the operator for outsourcing is investigated:

$$T_{out}f\left(x,\mathbf{d}\right)=f\left(x^{+},\mathbf{d}\right)-Fx^{-}$$

T_{out} is the operator showing that the inventory level x is always non-negative. If there is no inventory, the supplier outsources the deficient amount of demand with a cost of F per item.

Lemma 4.3.7. If $f(x,\mathbf{d})$ is convex in x, $\forall\,\mathbf{d}$, and satisfies $LBD(F)$, $T_{out}f(x,\mathbf{d})$ preserves convexity and $LBD(F)$ property in x, $\forall\,\mathbf{d}$.

Proof. The proof is in the Appendix A.6. □

To sum up, the properties that are preserved by the operators for $x\geq 0$ may be listed as follows (See Appendix A for proofs):

- T_{costs}: $Conv(x)\rightarrow Conv(x)$; $LBD(F)\rightarrow LBD(F)$.
- $T_{unifpro}$: $Conv(x)\rightarrow Conv(x)$; $LBD(F)\rightarrow LBD(F)$.
- T_{AC}: $Conv(x)\rightarrow Conv(x)$; $LBD(F)\rightarrow LBD(F)$.

- T_{ofl}: $Conv(x), LBD(F) \rightarrow Conv(x)$; $LBD(F) \rightarrow LBD(F)$.
- T_{coa}: $Conv(x) \rightarrow Conv(x)$; $LBD(F) \rightarrow LBD(F)$.
- T_{aca}: $Conv(x), LBD(F) \rightarrow Conv(x)$; $LBD(F) \rightarrow LBD(F)$.
- T_{out}: $Conv(x), LBD(F) \rightarrow Conv(x)$; $LBD(F) \rightarrow LBD(F)$.

Now, we are ready to combine the individual operators to obtain the value function (4.3):

$$\begin{aligned} V_{n+1}(x,\mathbf{d}) \quad = \quad & T_{costs}(T_{ofl}(T_{unifpro}(T_{AC}(T_{unifar1}(T_{coa}(T_{unifar2}(T_{aca}(T_{out}V_n), T_{out}V_n)), \\ & T_{unifar2}(T_{aca}(T_{out}\,V_n), T_{out}\,V_n))), T_{unifar1}(T_{coa}(T_{unifar2}(T_{aca}(T_{out}\,V_n), \\ & T_{out}V_n)), T_{unifar2}(T_{aca}(T_{out}\,V_n), T_{out}\,V_n))))) \,(x,\mathbf{d}) \end{aligned}$$

Since we showed that all operators used to form the value function preserve convexity in x, $\forall\,\mathbf{d}$; we can address the characteristics of the value function by induction. To start the induction, we specify $V_0(x,\mathbf{d}) = 0$, $\forall\,x$, $\mathbf{d}$; so, convexity is verified for $n = 0$. We, then, assume that convexity is verified for $V_n(x,\mathbf{d})$ and check for $V_{n+1}(x,\mathbf{d})$.

Proposition 4.3.8. The value function $V_{n+1}(x,\mathbf{d})$ is convex in x, $\forall\,\mathbf{d}$, since it can be represented by the operators that are proven to preserve convexity of $V_n(x,\mathbf{d})$ in x, $\forall\,\mathbf{d}$.

As we stated earlier, our aim is to determine the structure of the optimal policies when the infinite horizon problem is considered. However, by Proposition 4.3.8, we address the structure of the optimal policies with the aim of minimizing the total discounted costs for a finite number of transitions n. Since the results of the finite horizon problem can be related with the infinite horizon problems, we conclude that Proposition 4.3.8 extends to the infinite-horizon models as in Chapter 3.

4.4 Structure of the Optimal Policies

We have proven convexity of the value function $\forall\,\mathbf{d}$. The next corollary explains the policy implications of this property. We first consider the replenishment policy that minimizes the average cost per unit time:

Corollary 4.4.1. A state dependent basestock policy is optimal for replenishment with a basestock level depending on the future demand vector $\mathbf{d}$.

For the replenishment policy, there is a threshold inventory level $S_{\mathbf{d}}$ for every demand vector $\mathbf{d}$. Below the inventory level $S_{\mathbf{d}}$, the decision maker always decides to produce and above the inventory level $S_{\mathbf{d}}$, the decision maker always decides not to produce. The threshold inventory level $S_{\mathbf{d}}$ depends on the demand vector $\mathbf{d}$, so the replenishment policy is said to be state dependent. This is a modified version of the basestock policy that is called "the state dependent basestock policy" in the literature.

Next, we define the structure of the optimal admission control policy that minimizes the average cost per unit time:

Corollary 4.4.2. A state dependent threshold type policy is optimal for admission control with an order acceptance threshold depending on the future demand vector $\mathbf{d}$.

For admission control of the second class customers, there is an order acceptance threshold $X_{\mathbf{d}}$ that depends on the demand vector $\mathbf{d}$. Above that order acceptance threshold, the second class customer should be admitted. The second class customer should be rejected below that order acceptance threshold for the same demand vector $\mathbf{d}$. Since the order acceptance threshold level depends on the demand vector $\mathbf{d}$, the admission control policy is a state dependent threshold type policy.

To sum up, in Section 4.3, we consider a production-inventory system with two customer classes one of which provides ADI. We model the system as a discrete time make-to-stock queue and analyze it using Event-Based SDP. Since all the operators written for the events and transitions in the system are proven to preserve convexity in x, $\forall\, \mathbf{d}$, the value function is also convex in x, $\forall\, \mathbf{d}$. Therefore, the optimal replenishment and admission control policies are threshold type policies.

4.5 Numerical Analysis

In this section, our aim is to display the impacts of ADI on the system performance in terms of costs and threshold levels. The improvements in profit and basestock level support the structural results of the paper. Moreover, we observe the behavior of the proposed policies with different parameter settings. We use the value iteration algorithm [30] to compute the optimal average profit. Since basestock levels are state dependent, without loss of generality, we display the basestock levels at state $\mathbf{d} = [0\,0\,...\,0]$ where there are no orders in the information horizon. Similarly, the threshold levels depend on the inventory level, decisions and events before the order acceptance decision. The order acceptance thresholds have the same characteristics, so we representatively use the threshold level of order acceptance when production decision is made, first customer arrived and $\mathbf{d} = [0\,0\,...\,0]$.

4.5.1 Impact of the horizon of visibility

The ADI has positive effects on the basestock and customer acceptance threshold levels. In Figures 4.1 and 4.2, we investigate the problem with parameters $h = 1, R_1 = 25, R_2 = 50$, $F = 300, p = 0.5, q_1 = 0.45, q_2 = 0.4$. In Figure 4.1, we observe that the basestock level and the customer acceptance threshold are non-increasing in the horizon of visibility. As the horizon of visibility increases, it is obvious that the demand uncertainty decreases and the supplier does not need to carry high safety stocks. This is the intuition behind the decrease of the basestock levels. Decreasing demand uncertainty facilitates a better control of inventory and customer admission. Therefore, the acceptance threshold for second class of customers decreases, meaning that the supplier can serve more customers of second class. The revenue gained from the second class of customers increases if ADI is provided earlier. The increase in the revenue is due to the advantage of having multiple classes of customers. Reduction in inventory costs and the increase in the revenue stimulate an increase in the optimal average profit as the horizon of visibility increases.

4.5.2 Impacts of the system load and capacity

The ratio between the first class customer arrival rate and the production rate (q_1/p) defined as system load affects the average profit of the system (See Figure 4.3). If we increase system load with a constant capacity, the optimal average profit reduces in the given problem with parameters $h = 1$, $R_1 = 25$, $R_2 = 50$, $F = 300$, $p = 0.5$, $q_2 = 0.4$. This shows that the capacity is not enough to fully satisfy the demand from the inventory; the supplier has to satisfy the demand by outsourcing. The outsourcing cost can be the reason of this reduction in the average profit. However, the average profit can also increase in some conditions when the system load increases. The reason for this increase is the utilization of idle capacity

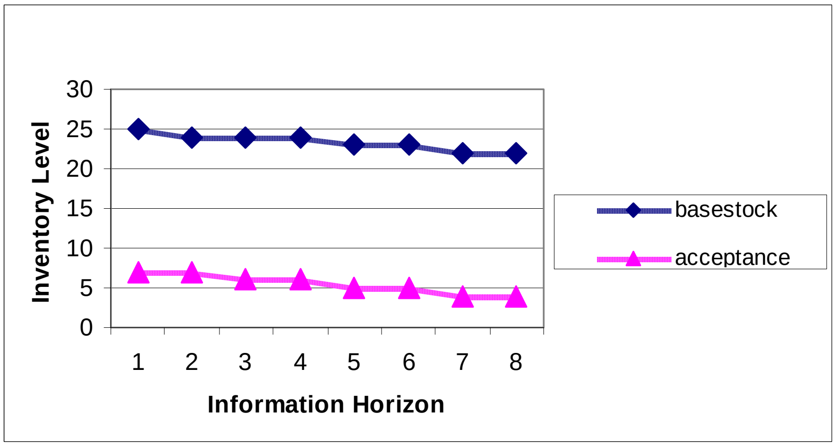

Figure 4.1: The effect of the information horizon on the basestock and the order acceptance threshold.

when the system load increases.

The parameter p represents the production rate. For system load (q_1/p) held constant, p can be viewed as a measure of variability. High production rates designate high capacities; low production rates designate low capacities. In Figure 4.4, the problem with parameters $h = 1$, $R_1 = 25$, $R_2 = 50$, $F = 300$, $q_1 = 0.4$, $q_1/p = 0.9$, $q_2 = 0.4$ is investigated. In this figure as the capacity increases, the average profit significantly increases for constant system load. This increase is caused by the increased number of satisfied customers. With a better replenishment and admission control, high capacity enables obtaining higher profits when ADI is provided.

4.5.3 Impacts of advance orders

The integration of ADI into our production-inventory system is optimally done by a state dependent basestock policy for replenishment. Within this policy, the supplier makes a decision according to the current demand vector and the inventory level. In Figure 4.5 optimal basestock and order acceptance threshold levels are given for a problem with parameters

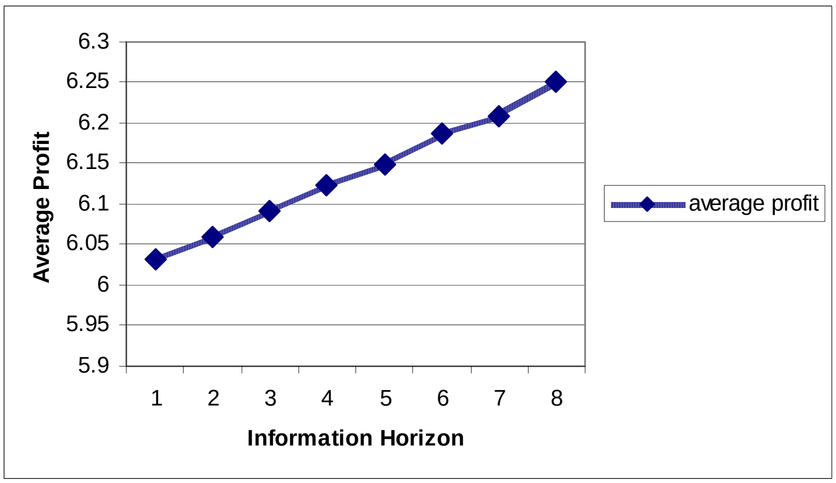

Figure 4.2: The effect of the information horizon on the average profit.

$H = 5$; $R_1 = 25$; $R_2 = 50$; $F = 300$; $p = 0.5$; $q_1 = 0.45$; $q_2 = 0.8$. The basestock and the order acceptance threshold levels depend on the current demand vector and the inventory level. Additionally, they behave similarly with respect to the number of orders (See Figure 4.5). Table 4.1 reveals the fact that the basestock and the order acceptance threshold levels increase with unit step sizes when the number of orders in the information horizon increases. This property is observed in the numerical examples with $0 \leq H \leq 8$.

Remark 2 *In this subsection, optimality of order-basestock policies is numerically shown in a limited number of examples. However, it would be an interesting extension if this property is tested structurally. Wijngaard and Karaesmen [34] show optimality of order-basestock policies for a single class $M/D/1$ make-to-stock queue with ADI. Their paper may be a guide to show this property for our model although our model is more complex.*

4.6 Extensions: Batch Arrivals and Batch Processing

So far, we consider the model with unit production, unit admission control and unit demand arrivals. However, it is possible to relax these assumptions where different amounts of production, admission and demand arrivals may occur.

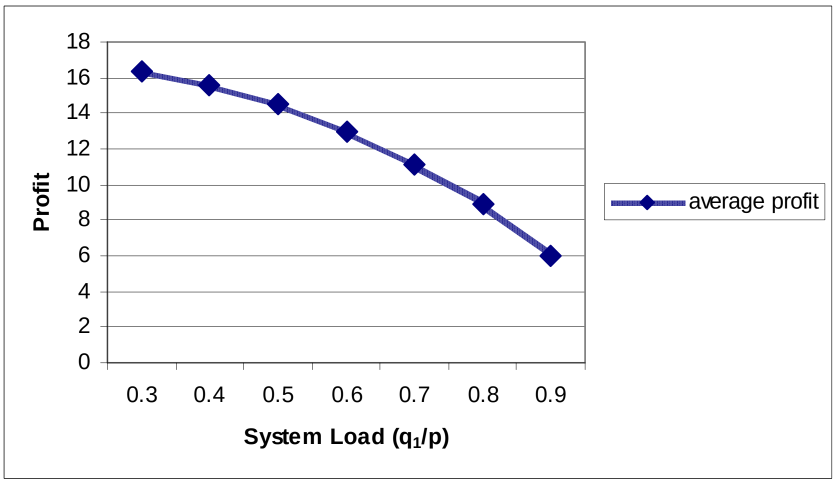

Figure 4.3: The effect of the system load on the optimal average profit.

In this section, we consider a production-inventory system with two customer classes and advance orders. The first customer class corresponds to contractual customers. According to the contract, the customer orders H periods in advance of the due date. The first class customers may order any amount m, $0 \leq m \leq M$ and the supplier has to satisfy all demands of this class at the due-date. The second class corresponds to non-contractual customers who desire to purchase the item immediately but the supplier can reject some or all of the second class demand if necessary. The second class customers order any amount $\bar{t}$, $0 \leq \bar{t} \leq T$ and the supplier admits any amount t, $0 \leq t \leq \bar{t}$ that minimizes her costs.

Let us assume that there is no processing time uncertainty. The production will be surely completed if started. The supplier has a capacity of producing K items with a cost of c per item and she chooses to produce any amount k, $0 \leq k \leq K$ in a period. At each period, demand arrivals of two customer classes occur randomly. First class orders m, $0 \leq m \leq M$ items with probability q_m, $\sum_{m=0}^{M} q_m = 1$. The second class orders $\bar{t}$, $0 \leq \bar{t} \leq T$ items with probability $r_{\bar{t}}$, $\sum_{\bar{t}=0}^{\bar{t}=T} r_{\bar{t}} = 1$. A reward R_1 is gained by the supplier for each item ordered by the first customer class. Similarly, R_2 is gained for each second class order fulfilled.

If there is inventory, the orders are fulfilled by the inventory. However, if there is no

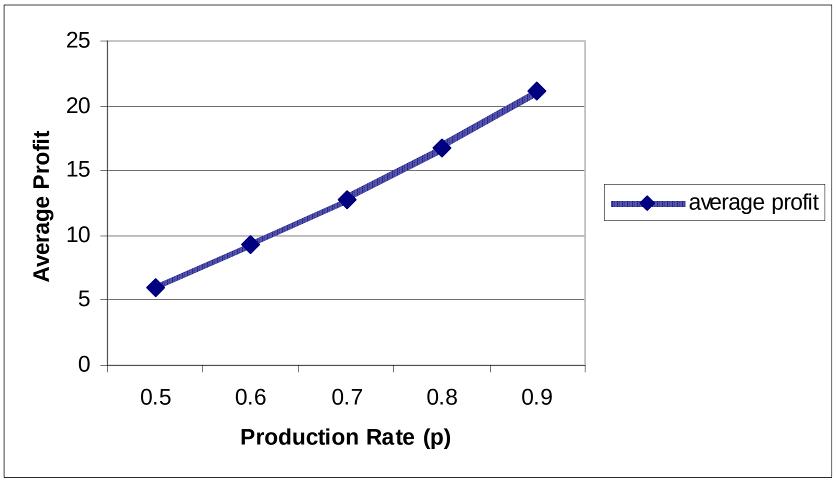

Figure 4.4: The effect of the production rate on the average profit for a given system load.

inventory, the orders can be fulfilled by outsourcing the product with a cost of F per item. Since the same product is sold to both customers, the outsourcing costs do not differ according to customers. Additionally, backordering is not allowed. An inventory cost (h) is incurred per period per item for the stocked products.

We model the production-inventory system as a discrete time make-to-stock queue. In this setting, the supplier has two decisions in a period: replenishment and admission control. The supplier considers these decisions with the aim of minimizing the inventory, outsourcing and variable production costs minus the sales revenue discounted over an infinite horizon. She investigates the structure of the optimal policy of replenishment and admission control. We assume that the events in a period occur in a sequence: production decision, production completion, arrival of the first customer, arrival of the second customer and order acceptance decision.

4.6.1 Analysis of the Extended Model

Stochastic Dynamic Programming (SDP) is used to address the structural properties for the supplier's replenishment and order acceptance problems. The supplier again aims to

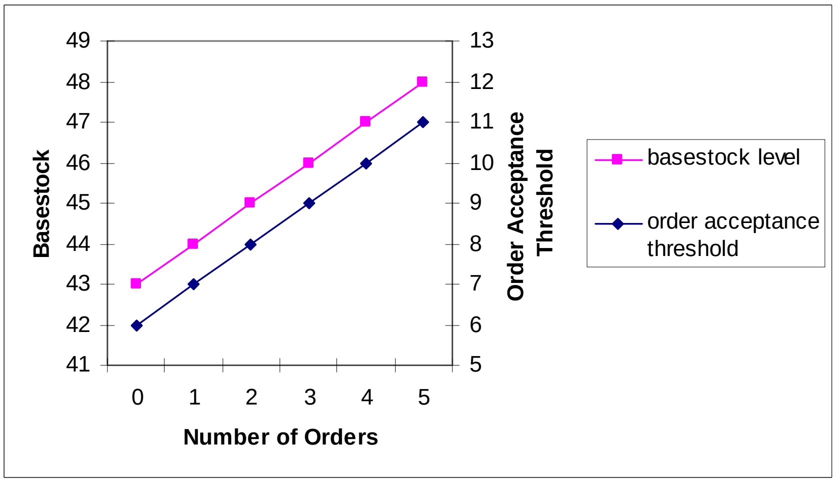

Figure 4.5: The effect of the number of orders on the basestock and the order acceptance thresholds.

find the optimal replenishment and admission control policies that minimize the average cost per unit time. However, as before we will start with the finite-horizon discounted cost problem to reach the average cost per unit time. The value function of the stochastic dynamic program that minimizes the discounted cost with n periods to go until the end of the horizon is $V_n(\,)$. The state space consists of two components: the inventory level, x $(x \geq 0)$, and the demand vector $\mathbf{d} = [d_1\ d_2\ \ldots d_H]$ for the first class customers with d_i as the first class demand to be fulfilled i periods later. At every period, the first element (d_1) of this vector is fulfilled by the supplier. $\mathbf{d}^+$ is the demand vector after $\mathbf{d}$ is shifted left or in other words, after the demand of that period (d_1) is fulfilled. $\mathbf{m}_H$ is the vector with m in the H^{th} entry and zeroes elsewhere, used to display the arrival of m first class orders that will be satisfied H periods later. Additionally, α $(0 < \alpha < 1)$ is the discount factor.

Table 4.1: Impacts of the Number of Orders in the Information Horizon

Demand Vector	Number of Orders	Basestock	Order Acceptance Threshold
$\mathbf{d} = [00000]$	0	43	6
$\mathbf{d} = [00001]$	1	44	7
$\mathbf{d} = [00010]$	1	44	7
$\mathbf{d} = [00011]$	2	45	8
$\mathbf{d} = [00100]$	1	44	7
$\mathbf{d} = [00101]$	2	45	8

Below, the value function is given, divided into smaller parts for practical purposes:

$$\begin{aligned}
V_n^{(1)}(x,\mathbf{d}) &= V_n((x-d_1)^+,\mathbf{d}^+) - F\,(x-d_1)^- \\
V_n^{(2)}(x,\mathbf{d}) &= \min_{0\le t\le \bar{t}}\{V_n^{(1)}((x-t),\mathbf{d}) - t\,R_2\} \\
V_n^{(3)}(x,\mathbf{d}) &= \sum_{\bar{t}=0}^{\bar{t}=T} r_{\bar{t}} V_n^{(2)}(x,\mathbf{d}) \\
V_n^{(4)}(x,\mathbf{d}) &= \sum_{m=0}^{M} q_m\,(V_n^{(3)}(x,\mathbf{d}+\mathbf{m}_H) - m\,R_1) \\
V_n^{(5)}\;(x,\mathbf{d}) &= \min_{0\le k\le K}\{V_n^{(4)}(x+k,\mathbf{d}) + k\,c\} \\
V_{n+1}(x,\mathbf{d}) &= hx + \alpha V_n^{(5)}(x,\mathbf{d})
\end{aligned}$$

If the small parts are put together, the value function becomes:

$$\begin{aligned}
V_{n+1}(x,\mathbf{d}) \;=\;& hx + \alpha \min_{0\le k\le K}\{(\sum_{m=0}^{M} q_m[(\sum_{\bar{t}=0}^{\bar{t}=T} r_{\bar{t}} \min_{0\le t\le \bar{t}}\{V_n((x+k-d_1-t),\mathbf{d}^+ + m) - t\,R_2 \\
& -F\,(x+k-d_1-t)^-\}) - m\,R_1]) + k\,c\}
\end{aligned}$$

In order to analyze the model with SDP, we should handle the value function and investigate its characteristics. For this purpose, once again, we are going to analyze the value function in pieces by the help of the Event-Based SDP. In this extended model, below event-based operators are used. Firstly, we focus on the operator representing the batch production:

$$T_{pro}f\,(x,\mathbf{d}) = \min_{0\le k\le K}\{f(x+k,\mathbf{d}) + k\,c\}$$

T_{pro} represents supplier's decision of production amount. The supplier has a capacity of producing K items. She chooses any production amount k $(0 \leq k \leq K)$ that will minimize her costs. Additionally, she incurs a cost of c per item produced.

Lemma 4.6.1. If $f(x, \mathbf{d})$ is convex in x, $\forall\, \mathbf{d}$, $T_{pro} f(x, \mathbf{d})$ preserves convexity in x, $\forall\, \mathbf{d}$.

Proof. The proof is in the Appendix A.7. □

Next, the operator for batch admission control is given:

$$T_{adm} f\,(x, \mathbf{d}) = \min_{0 \leq t \leq \bar{t}} \{ f((x - t), \mathbf{d}) - t\, R_2 \}$$

T_{adm} represents supplier's decision of admission amount. The second class customer orders $\bar{t}$ items at that period and the supplier can choose to satisfy t $(0 \leq t \leq \bar{t})$ items of the demand in order to minimize her costs.

Lemma 4.6.2. If $f(x, \mathbf{d})$ is convex in x, $\forall\, \mathbf{d}$, and satisfies $LBD(F)$ property, $T_{adm} f(x, \mathbf{d})$ preserves convexity and $LBD(F)$ in x, $\forall\, \mathbf{d}$.

Proof. The proof is in the Appendix A.8. □

Next, we focus on the operator of first class customer arrivals:

$$T_{far} f_m\,(x, \mathbf{d}) = \sum_{m=0}^{M} q_m f_m\,(x, \mathbf{d})$$

T_{far} is the convex combination of f_m. T_{far} represents the random amount of the first class customer arrivals that may happen in a period.

Lemma 4.6.3. If $f_m(x, \mathbf{d})$ is convex in x, $\forall\, m, \mathbf{d}$, $T_{far} f_m(x, \mathbf{d})$ preserves convexity in x, $\forall\, \mathbf{d}$.

Proof. The proof follows from the closedness of convexity under convex combinations. □

The below operator, T_{sar}, is an application of Lemma 4.6.3 and it represents the random amount of the second class customer arrivals that may happen in a period:

$$T_{sar} f_{\bar{t}}\,(x, \mathbf{d}) = \sum_{\bar{t}=0}^{\bar{t}=T} r_{\bar{t}} f_{\bar{t}}\,(x, \mathbf{d})$$

Lastly, we define the operator used for contractual order arrivals:

$$T_{fco} f\,(x, \mathbf{d}) = f\,(x, \mathbf{d} + \mathbf{m}_H) - m\,R_1$$

The contractual order arrivals, the orders of the first class customers, are represented by T_{fco}. The customers order any amount m $(0 \leq m \leq M)$. As soon as the first class order is received, the supplier gets a reward of R_1 per item. $\mathbf{m}_H$ is the vector with m in the H^{th} entry and zeroes elsewhere. By adding $\mathbf{m}_H$ to the demand vector, the amount of demand that will be satisfied H periods later is collected in the H^{th} entry of the demand vector.

Lemma 4.6.4. If $f(x, \mathbf{d})$ is convex in x, $\forall\, \mathbf{d}$, $T_{fco} f(x, \mathbf{d})$ preserves convexity in x, $\forall\, \mathbf{d}$.

Proof. The proof is in the Appendix A.9. □

To sum up, the properties that are preserved by the operators for $x \geq 0$ may be listed as follows:

- T_{pro}: $Conv(x) \rightarrow Conv(x)$; $LBD(F) \rightarrow LBD(F)$.
- T_{adm}: $Conv(x), LBD(F) \rightarrow Conv(x)$; $LBD(F) \rightarrow LBD(F)$.
- T_{far}: $Conv(x) \rightarrow Conv(x)$; $LBD(F) \rightarrow LBD(F)$.
- T_{sar}: $Conv(x) \rightarrow Conv(x)$; $LBD(F) \rightarrow LBD(F)$.
- T_{fco}: $Conv(x) \rightarrow Conv(x)$; $LBD(F) \rightarrow UBD(F)$.

The value function can be rewritten as follows by using the event-based operators:

$$V_{n+1}(x, \mathbf{d}) \quad = \quad T_{costs}(T_{pro}(T_{far}(T_{fco}(T_{sar}(T_{adm}(T_{ofl}(T_{out}\,V_n)))))))\,(x, \mathbf{d})$$

Since we showed that all operators used to form the value function preserve convexity in x, $\forall\, \mathbf{d}$; we can address the characteristics of the value function by induction. To start the induction, we specify $V_0(x, \mathbf{d}) = 0$, $\forall\, x$, $\mathbf{d}$; so, convexity is verified for $n = 0$. We, then, assume that convexity is verified for $V_n(x, \mathbf{d})$ and check for $V_{n+1}(x, \mathbf{d})$.

Proposition 4.6.5. The value function $V_{n+1}(x, \mathbf{d})$ is convex in x, $\forall\, \mathbf{d}$, since it can be represented by the operators that are proven to preserve convexity of $V_n(x, \mathbf{d})$ in x, $\forall\, \mathbf{d}$.

As in the basic ADI model, our aim is to determine the structure of the optimal policies when the infinite horizon problem is considered. By Proposition 4.6.5, we address the structure of the optimal policies with the aim of minimizing the total discounted costs for a finite number of transitions n. Since necessary conditions hold ([30]) as in section 4.3, we conclude that Proposition 4.6.5, written for the α-discounted case, extends to the infinite-horizon models.

4.6.2 Structure of the Optimal Policies

We have proven the convexity of the operators and conclude that the finite horizon value function is convex in x, $\forall\, \mathbf{d}$. Additionally, we relate the finite horizon model with the infinite horizon model with the average cost criterion. Thus, we are able to address the structure of the optimal replenishment policy that minimizes the average cost per unit time:

Corollary 4.6.6. A state dependent basestock policy is optimal for replenishment with a basestock level depending on the future demand vector $\mathbf{d}$.

For the replenishment policy, there is a threshold inventory level $S_{\mathbf{d}}$ for every demand vector $\mathbf{d}$. Below the inventory level $S_{\mathbf{d}}$, the decision maker chooses to produce the amount k to reach the optimal basestock level. Above the inventory level $S_{\mathbf{d}}$, the decision maker always decides not to produce. The threshold inventory level $S_{\mathbf{d}}$ depends on the demand vector $\mathbf{d}$.

Next, we define the structure of the optimal admission control policy that minimizes the average cost per unit time:

Corollary 4.6.7. A state dependent threshold type policy is optimal for admission control with an order acceptance threshold depending on the future demand vector $\mathbf{d}$.

For admission control of second class customers, there is an order acceptance threshold $X_{\mathbf{d}}$ that depends on the demand vector $\mathbf{d}$. Above that order acceptance threshold, the second class customer order is accepted until the inventory level reaches that threshold. The second class customer should be rejected below that order acceptance threshold for the same demand vector **d**. Since the order acceptance threshold level depends on the demand vector **d**, the admission control policy is a state dependent threshold type policy.

4.7 Concluding Remarks

In this chapter, we examine a production-inventory system with multiple classes of customers and advance orders. We consider the supplier's production planning and order acceptance problems. By using SDP, we propose optimal replenishment and admission control policies that highlight the advantages of ADI.

We provide insights into implementing ADI in a multi-class production-inventory system by the proposed replenishment policy. It is shown that the replenishment in such a system with ADI is of state dependent basestock type. Additionally, by the proposed admission control policy, flexibility is attained for systems with different customers providing different levels of ADI. The admission control policy is also of state dependent threshold type in such a system.

The behaviors of the proposed policies are examined under different parameter settings. Numerical results display the impacts of ADI on the optimal policies and on the optimal profit. Our results show that sharing ADI yields significant improvements in the system performance as costs and threshold levels. Thus, contractual agreements with the supply chain partners are encouraged in multi-class production-inventory systems. As the first class customer orders earlier, the average profit increases. Therefore, the downstream partner should be persuaded to share the demand information as early as possible. As the first class customer orders earlier, the supplier is able to extract higher benefits from the second class customers. This underlines the importance of having the right portfolio of customers in case of ADI.

The model is extended by relaxing the unit production, order and admission amount assumptions. The optimal replenishment and admission control policies are state dependent threshold type policies in the extended model.

Remark 3 *In the model considered in this chapter, we assume that the supplier has information about the future demands of a class of customers. The demand of the first class customers determines the state of the system and affects the supplier and her decisions. However, the supplier cannot control the demand of the first class, it randomly fluctuates.*

The future demand of the first class customers is represented by the demand vector $\boldsymbol{d}$. *Since the first class demand has the properties of a random environment, the results of the ADI model hold for any random environment represented by* $\boldsymbol{d}$. *Additionally, for the ADI model, the demand vector may permit cancellation of orders or any other uncontrollable random event that affects the demand vector. Therefore, the model and its results are also valid for more general systems.*

Chapter 5

CONCLUSIONS

In this thesis, infinite-horizon inventory and admission control problems with capacitated supply are analyzed. In the models considered, we focus on the systems with limited capacity that have to utilize through different admission control policies. The main aim is to gain insights into the effects of admission control in different systems and to compare the two different admission control models applied in business. One of them is the static admission control method where the admitted number of orders is determined for once and remains fixed over time. The other method is the dynamic admission control where the decision maker admits or rejects the customer at arrival instants depending on the current state of the system.

Firstly, we investigate a model with a single supplier that aims to determine the optimal acceptable demand rate. Choosing the optimal acceptable demand rate can be done in two ways: static and dynamic admission control. We propose static admission control policies for the suppliers determining the amount that will be satisfied through contracts for long term customers. Additionally, we apply dynamic admission control on the same system for the spot customers whom the supplier may accept or reject depending on her inventory level. For the model with Poisson inter-arrival times and exponential servers, we propose an explicit formula for the optimal acceptable demand rate. In the dynamic case, we find that threshold type policies are optimal for replenishment and admission control.

In this problem our main aim is to compare the admission control methods and discover the specific systems that are suitable for a specific admission control method. Through numerical analysis, we realize the dominance of dynamic admission control. However, choosing the admission control policy that will be applied is still a fundamental question for most of the suppliers. The dynamic admission control method seems to be more advantageous however it may have many drawbacks since the supplier may lose reputation on the customers. Customers once rejected may be less likely to employ demand next time. These type of suppliers who have more loyal relations with their customers may still apply static admission control.

Secondly, we consider a periodic-review production-inventory system with ADI. We examine a supplier with two customer classes, one of which orders certain number of periods in advance. We model this system as a discrete time make-to-stock with geometric processing and inter-arrival times. In this study, we aim to gain insights into the way of implementing ADI in a multi-customer setting. Thus, we analyze supplier's replenishment and admission control problem with stochastic dynamic programming. Dynamic admission control is applied in this model since there are spot customers who desire to purchase the item immediately. In the analysis, event-based operators are proposed for each event and transition occurring in the system. These event-based operators constitute a framework for the analysis of the models including similar events. We address the structure of the optimal replenishment and admission control policies using these event-based operators. The optimal replenishment policy is proven to be of state dependent basestock type. The supplier aims to keep her inventory at the basestock level; she produces if and only if her inventory is below the basestock level. Similarly, the optimal admission control policy is a state dependent threshold type policy. There is an order acceptance threshold below which second class customers are never served.

The effects of ADI and admission control in the multi-customer setting are examined in the numerical analysis. As in the single class systems, ADI has positive effects on the inventory levels. Additionally, as the first class customers order earlier, more profit can be extracted from the second class customers. These results underline the importance of information sharing in the multiple class systems.

The assumption of unit arrivals and processing is relaxed for the ADI model. For the batch arrivals and processing case, some additional event-based operators are used to address the structure of the optimal policies. State dependent threshold type policies are found to be optimal for this model too.

A natural extension of the ADI model considered in this thesis is to allow backordering. The results of the model with equal unit backorder costs for both customer classes are trivial. However, extending the model with distinct backordering costs is not a trivial issue. When the unit backorder costs are distinct, backorders should be collected separately and it becomes difficult to manage the state space. Another challenge would be to increase the number of customer classes and allow each of them to give different levels of ADI.

The continuous-review model with both static and dynamic admission control considers to optimize the number of customers to be accepted from a single class. However, it seems interesting to construct the model with multi-customer classes. Analyzing the multi-class model with and without priorities may lead to gain insights about the optimal customer portfolio of the supplier. Another attractive issue might be to add ADI into this model.

Both of the models considered in this thesis ignore the fact that admission control may affect the customer arrival rates in the long term. Therefore, another challenging extension would be to investigate the models with arrival rates depending on the admission control decisions of the supplier.

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Appendix A

STRUCTURAL RESULTS

A.1 *Analysis of the Acceptable Demand Rate Model with Normal Approximations*

In this model, the aim is to find the acceptable demand rate that maximizes the expected profit in an uncapacitated production-inventory system with a fixed production leadtime L. Let us define Y as the stationary inventory level (a negative value indicates backorders) and $N = Z - Y$ which denotes the stationary number of outstanding orders with respect to the basestock level Z. N can be approximated by a normal distribution with parameters $(\mu = \lambda L, \sigma = \sqrt{\lambda L})$ (See [35] for details). The expected profit with normal approximations that includes the revenue and the costs of inventory, backorder and production is:

$$\Pi_n = \lambda(R - c) - (h + b) f_z(z^*) \sqrt{\lambda L}$$

We apply second order derivative test to analyze the structure of the profit function. The first and second order derivatives are:

$$\begin{aligned} \frac{d\Pi_n(\lambda)}{d\lambda} &= (R - c) - \frac{(b + h)\ f_z(z^*)\, L}{2\sqrt{L\,\lambda}} \\ \frac{d^2\Pi_n(\lambda)}{d\lambda^2} &= \frac{(b + h)\ f_z(z^*)\, L^2}{4\,(L\,\lambda)^{\frac{3}{2}}} \end{aligned}$$

$f_z(z^*)$ is the probability density function and $f_z(z^*) \geq 0$. Additionally, $h \geq 0, b \geq 0$. Therefore, $\frac{d^2\Pi_n(\lambda)}{d\lambda^2} \geq 0$ and the profit function is convex. This approximation does not yield a concave profit function that can be maximized at an inner point. However, if the profit function is nondecreasing, i.e. $\frac{d\Pi_n(\lambda)}{d\lambda} \geq 0$, the supplier always decides to accept all of the customers in order to maximize her profit. Otherwise, she rejects all customers. To sum up, the supplier either accepts or rejects all customers for this model.

A.2 Unit Production Decision Operator

In this section, we investigate the structure of the unit production decision operator:

$$T_{AC}f\,(x,\mathbf{d}) = \min\{f\,(x+1,\mathbf{d})\ + c\,, f\,(x,\mathbf{d})\}$$

For $f(x,\mathbf{d})$ convex in x, $\forall\mathbf{d}$, T_{AC} preserves convexity in x, $\forall\mathbf{d}$:

Proof. Let $f(x,\mathbf{d})$ be convex in x, $\forall\mathbf{d}$, i.e. $\Delta f(x+1,\mathbf{d}) \geq \Delta f(x,\mathbf{d})$. We are going to show convexity of $T_{AC}f(x,\mathbf{d})$:

$$\begin{aligned} \Delta T_{AC}f(x+1,\mathbf{d}) &\geq \Delta T_{AC}f(x,\mathbf{d}) \\ T_{AC}f(x+1,\mathbf{d}) - T_{AC}f(x,\mathbf{d}) &\geq T_{AC}f(x,\mathbf{d}) - T_{AC}f(x-1,\mathbf{d}) \end{aligned}$$

$$\begin{aligned} \min\{f(x+2,\mathbf{d})+c, f(x+1,\mathbf{d})\} &- \min\{f(x+1,\mathbf{d})+c, f(x,\mathbf{d})\} \geq \\ \min\{f(x+1,\mathbf{d})+c, f(x,\mathbf{d})\} &- \min\{f(x,\mathbf{d})+c, f(x-1,\mathbf{d})\} \end{aligned}$$

All possible cases resulting through minimizations should be considered separately to verify convexity. The possible cases are listed in Table A.1. However, some cases are eliminated because they conflict with the convexity property of $f(x,\mathbf{d})$. The cases that are given numbers should be shown to verify convexity.

Table A.1: Possible Situations: Unit Production Decision Operator

	$T_{AC}f(x+1,\mathbf{d})$	$T_{AC}f(x,\mathbf{d})$	$T_{AC}f(x-1,\mathbf{d})$
1	$f(x+2,\mathbf{d})+c$	$f(x+1,\mathbf{d})+c$	$f(x,\mathbf{d})+c$
2	$f(x+1,\mathbf{d})$	$f(x+1,\mathbf{d})+c$	$f(x,\mathbf{d})+c$
3	$f(x+1,\mathbf{d})$	$f(x,\mathbf{d})$	$f(x,\mathbf{d})+c$
4	$f(x+1,\mathbf{d})$	$f(x,\mathbf{d})$	$f(x-1,\mathbf{d})$

Case 1. We have to verify the inequality:

$$f(x+2,\mathbf{d})+c-f(x+1,\mathbf{d})-c \geq f(x+1,\mathbf{d})+c-f(x,\mathbf{d})-c$$

Since $f(x,\mathbf{d})$ is assumed to be convex in x, $\forall\,\mathbf{d}$, for this case, it is directly verified that T_{AC} preserves convexity in x, $\forall\,\mathbf{d}$.

Case 2. We have to verify the inequality:

$$f(x+1,\mathbf{d}) - f(x+1,\mathbf{d}) - c \geq f(x+1,\mathbf{d}) + c - f(x,\mathbf{d}) - c$$

$$f(x,\mathbf{d}) \geq f(x+1,\mathbf{d}) + c$$

By the results of the minimizations, we have:

$$\begin{aligned} f(x+1,\mathbf{d}) &\leq f(x+2,\mathbf{d}) + c \\ f(x+1,\mathbf{d}) + c &\leq f(x,\mathbf{d}) \\ f(x,\mathbf{d}) + c &\leq f(x-1,\mathbf{d}) \end{aligned}$$

These results display that $f(x,\mathbf{d}) \geq f(x+1,\mathbf{d}) + c$, $\forall\,\mathbf{d}$. So, for this case, it is verified that T_{AC} preserves convexity in x, $\forall\,\mathbf{d}$.

Case 3. We have to verify the inequality:

$$f(x+1,\mathbf{d}) - f(x,\mathbf{d}) \geq f(x,\mathbf{d}) - f(x,\mathbf{d}) - c$$

$$f(x+1,\mathbf{d}) + c \geq f(x,\mathbf{d})$$

By the results of the minimizations, we have:

$$\begin{aligned} f(x+1,\mathbf{d}) &\leq f(x+2,\mathbf{d}) + c \\ f(x,\mathbf{d}) &\leq f(x+1,\mathbf{d}) + c \\ f(x,\mathbf{d}) + c &\leq f(x-1,\mathbf{d}) \end{aligned}$$

These results display that $f(x+1,\mathbf{d}) + c \geq f(x,\mathbf{d})$, $\forall\,\mathbf{d}$. So, for this case, it is verified that T_{AC} preserves convexity in x, $\forall\,\mathbf{d}$.

Case 4. We have to verify the inequality:

$$f(x+1,\mathbf{d}) - f(x,\mathbf{d}) \geq f(x,\mathbf{d}) - f(x-1,\mathbf{d})$$

Since $f(x,\mathbf{d})$ is assumed to be convex in x, $\forall\,\mathbf{d}$, for this case, it is directly verified that T_{AC} preserves convexity in x, $\forall\,\mathbf{d}$.

Thus, $T_{AC}f(x, \mathbf{d})$ is proven to be convex in x, $\forall\, \mathbf{d}$. □

For $f(x, \mathbf{d})$ satisfying $LBD(F)$, $\forall\, \mathbf{d}$, T_{AC} preserves $LBD(F)$ property, $\forall\, \mathbf{d}$:

Proof. Since $f(x, \mathbf{d})$ satisfying $LBD(F)$, below equation is valid:

$$f(x+1, \mathbf{d}) - f(x, \mathbf{d}) \geq -F$$

We aim to show:

$$\begin{aligned} \Delta T_{AC}f(x+1, \mathbf{d}) &\geq -F \\ \min\{f(x+1, \mathbf{d}) + c, f(x, \mathbf{d})\} - \min\{f(x, \mathbf{d}) + c, f(x-1, \mathbf{d})\} &\geq -F \end{aligned}$$

Through the results of the minimizations, three cases may occur:

Case 1. We have to verify the inequality:

$$f(x+1, \mathbf{d}) - f(x, \mathbf{d}) \geq -F$$

Since $f(x, \mathbf{d})$ satisfies $LBD(F)$, $\forall\, \mathbf{d}$, T_{AC} preserves $LBD(F)$ property, $\forall\, \mathbf{d}$.

Case 2. We have to verify the inequality:

$$f(x, \mathbf{d}) - f(x, \mathbf{d}) - c \geq -F$$

Since outsourcing is more costly than producing, i.e. $c \leq F$, $LBD(F)$ property is preserved by the operator T_{AC}.

Case 3. We have to verify the inequality:

$$f(x, \mathbf{d}) - f(x-1, \mathbf{d}) \geq -F$$

Since $f(x, \mathbf{d})$ satisfies $LBD(F)$, $\forall\, \mathbf{d}$, T_{AC} preserves $LBD(F)$ property, $\forall\, \mathbf{d}$. □

A.3 Order Fulfillment Operator

In this section, we investigate the structure of the order fulfillment operator:

$$T_{ofl}f\,(x, \mathbf{d}) = f\,(x - d_1, d^+)$$

For $f(x, \mathbf{d})$ convex in x, $\forall\mathbf{d}$, T_{ofl} preserves convexity in x, $\forall\mathbf{d}$:

Proof. Let $f(x, \mathbf{d})$ be convex in x, $\forall\,\mathbf{d}$, i.e. $\Delta f\,(x+1, \mathbf{d}) \geq \Delta f(x, \mathbf{d})$. We are going to show convexity of $T_{ofl}f(x, \mathbf{d})$:

$$\begin{aligned} \Delta T_{ofl}f\,(x+1, \mathbf{d}) &\geq \Delta T_{ofl}f(x, \mathbf{d}) \\ T_{ofl}f\,(x+1, \mathbf{d}) - T_{ofl}f(x, \mathbf{d}) &\geq T_{ofl}f(x, \mathbf{d}) - T_{ofl}f\,(x-1, \mathbf{d}) \end{aligned}$$

Since we define x as $x \geq 0$, we may check the boundary values of x as well as the non-extreme cases. For the non-extreme cases ($x - 1 - d_1 \geq 0$), we have:

$$f\,(x+1-d_1, d^+) - f\,(x-d_1, d^+) \geq f\,(x-d_1, d^+) - f\,(x-1-d_1, d^+)$$

Since $f(x, \mathbf{d})$ is convex in x, $\forall\,\mathbf{d}$, $T_{ofl}f\,(x, \mathbf{d})$ is proven to be convex in x, $\forall\,\mathbf{d}$ for the non-extreme cases. However, three extreme cases may occur that should be checked:

Case 1. We have to verify convexity at $x - d_1 = 0$:

$$\begin{aligned} f\,((x+1-d_1)^+, d^+) - (x+1-d_1)^-\,F &- f\,((x-d_1)^+, d^+) + (x-d_1)^-\,F \;\geq \\ f\,((x-d_1)^+, d^+) - (x-d_1)^-\,F &- f\,((x-1-d_1)^+, d^+) + (x-1-d_1)^-\,F \\ f\,(1, d^+) - f\,(0, d^+) &\geq f\,(0, d^+) - f\,(0, d^+) - F \\ f\,(1, d^+) - f\,(0, d^+) &\geq -F \end{aligned}$$

So, $T_{ofl}f\,(x, \mathbf{d})$ preserves convexity in x, $\forall\,\mathbf{d}$ for the functions satisfying the inequality $f\,(x+1, d^+) - f\,(x, d^+) \geq -F$.

Case 2. We have to verify convexity at $x + 1 - d_1 = 0$:

$$\begin{aligned} f\,((x+1-d_1)^+, d^+) - (x+1-d_1)^-\,F &- f\,((x-d_1)^+, d^+) + (x-d_1)^-\,F \;\geq \\ f\,((x-d_1)^+, d^+) - (x-d_1)^-\,F &- f\,((x-1-d_1)^+, d^+) + (x-1-d_1)^-\,F \\ f\,(0, d^+) - f\,(0, d^+) - F &\geq f\,(0, d^+) + F - f\,(0, d^+) - 2F \\ -F &\geq -F \end{aligned}$$

$T_{ofl}f\,(x, \mathbf{d})$ is proven to be convex in x, $\forall\,\mathbf{d}$ for this case.

Case 3. We have to verify convexity at $x+1-d_1<0$:

$$\begin{aligned}
f\left((x+1-d_1)^+,d^+\right)-(x+1-d_1)^-\,F &- f\left((x-d_1)^+,d^+\right)+(x-d_1)^-\,F \;\geq\\
f\left((x-d_1)^+,d^+\right)-(x-d_1)^-\,F &- f\left((x-1-d_1)^+,d^+\right)+(x-1-d_1)^-\,F\\
f\left(0,d^+\right)-(x+1-d_1)F &- f\left(0,d^+\right)+(x-d_1)F \;\geq\\
f\left(0,d^+\right)-(x-d_1)F &- f\left(0,d^+\right)+(x-1-d_1)F\\
F &\geq F
\end{aligned}$$

$T_{ofl}f\left(x,\mathbf{d}\right)$ is proven to be convex in x, $\forall\,\mathbf{d}$ for this case.

In conclusion, $T_{ofl}f\left(x,\mathbf{d}\right)$ is proven to be convex in x, $\forall\,\mathbf{d}$ for all convex functions $f\left(x,\mathbf{d}\right)$ satisfying the inequality $f\left(x+1,d^+\right)-f\left(x,d^+\right)\geq -F$. □

For $f(x,\mathbf{d})$ satisfying $LBD(F)$, $\forall\,\mathbf{d}$, T_{ofl} preserves $LBD(F)$ property, $\forall\,\mathbf{d}$:

Proof.

$$\begin{aligned}
\Delta T_{ofl}f\left(x+1,\mathbf{d}\right) &\geq -F\\
T_{ofl}f\left(x+1,\mathbf{d}\right)-T_{ofl}f(x,\mathbf{d}) &\geq -F\\
f\left(x+1-d_1,d^+\right)-f\left(x-d_1,d^+\right) &\geq -F
\end{aligned}$$

Since $f(x,\mathbf{d})$ satisfies $LBD(F)$, $\forall\,\mathbf{d}$, T_{ofl} preserves $LBD(F)$ property, $\forall\,\mathbf{d}$ for the non-extreme cases. However, for the extreme cases $(x+1-d_1\leq 0)$, we have to verify the inequality:

$$\begin{aligned}
f\left(0,d^+\right)-(x+1-d_1)F-f\left(0,d^+\right)+(x-d_1)F &\geq -F\\
-F &\geq -F
\end{aligned}$$

T_{ofl} preserves $LBD(F)$ property, $\forall\,\mathbf{d}$. □

A.4 Unit Contractual Order Arrivals Operator

In this section, we investigate the structure of the unit contractual order arrivals operator:

$$T_{coa}f\left(x,\mathbf{d}\right)=f\left(x,d+e_H\right)-R_1$$

For $f(x,\mathbf{d})$ convex in x, $\forall\mathbf{d}$, T_{coa} preserves convexity in x, $\forall\mathbf{d}$:

Proof. Let $f(x,\mathbf{d})$ be convex in x, $\forall\,\mathbf{d}$, i.e. $\Delta f\left(x+1,\mathbf{d}\right)\geq\Delta f(x,\mathbf{d})$. We are going to show convexity of $T_{coa}f(x,\mathbf{d})$:

$$\begin{aligned}
\Delta T_{coa} f\,(x+1,\mathbf{d}) &\geq \Delta T_{coa} f(x,\mathbf{d}) \\
T_{coa} f\,(x+1,\mathbf{d}) - T_{coa} f(x,\mathbf{d}) &\geq T_{coa} f(x,\mathbf{d}) - T_{coa} f\,(x-1,\mathbf{d})
\end{aligned}$$

$$f\,(x+1,\mathbf{d}+\mathbf{e}_H) - R_1 - f\,(x,d+\mathbf{e}_H) + R_1 \geq f\,(x,\mathbf{d}+\mathbf{e}_H) - R_1 - f\,(x-1,\mathbf{d}+\mathbf{e}_H) + R_1$$

$$f\,(x+1,\mathbf{d}+\mathbf{e}_H) - f\,(x,\mathbf{d}+\mathbf{e}_H) \geq f\,(x,\mathbf{d}+\mathbf{e}_H) - f\,(x-1,\mathbf{d}+\mathbf{e}_H)$$

Since $f(x,\mathbf{d})$ is convex in x, $\forall\,\mathbf{d}$, $T_{coa} f\,(x,\mathbf{d})$ is proven to be convex in x, $\forall\,\mathbf{d}$. □

For $f(x,\mathbf{d})$ satisfying $LBD(F)$, $\forall\mathbf{d}$, T_{coa} preserves $LBD(F)$ property, $\forall\mathbf{d}$:

Proof.

$$\begin{aligned}
\Delta T_{coa} f\,(x+1,\mathbf{d}) &\geq -F \\
T_{coa} f\,(x+1,\mathbf{d}) - T_{coa} f(x,\mathbf{d}) &\geq -F \\
f\,(x+1,\mathbf{d}+\mathbf{e}_H) - R_1 - f\,(x,d+\mathbf{e}_H) + R_1 &\geq -F
\end{aligned}$$

Since $f(x,\mathbf{d})$ satisfies $LBD(F)$, $\forall\mathbf{d}$, T_{coa} preserves $LBD(F)$ property, $\forall\mathbf{d}$. □

A.5 Unit Admission Control Operator

In this section, we investigate the structure of the unit admission control operator:

$$T_{aca} f\,(x,\mathbf{d}) = \min\{f\,(x-1,\mathbf{d}) - R_2, f\,(x,\mathbf{d})\}$$

For $f(x,\mathbf{d})$ convex in x, $\forall\mathbf{d}$, T_{aca} preserves convexity in x, $\forall\mathbf{d}$:

Proof. Let $f(x,\mathbf{d})$ be convex in x, $\forall\,\mathbf{d}$, i.e. $\Delta f\,(x+1,\mathbf{d}) \geq \Delta f(x,\mathbf{d})$. We are going to show convexity of $T_{aca} f(x,\mathbf{d})$:

$$\begin{aligned}
\Delta T_{aca} f\,(x+1,\mathbf{d}) &\geq \Delta T_{aca} f(x,\mathbf{d}) \\
T_{aca} f\,(x+1,\mathbf{d}) - T_{aca} f(x,\mathbf{d}) &\geq T_{aca} f(x,\mathbf{d}) - T_{aca} f\,(x-1,\mathbf{d})
\end{aligned}$$

$$\begin{aligned}
\min\{f(x,\mathbf{d}) - R_2, f\,(x+1,\mathbf{d})\} &- \min\{f\,(x-1,\mathbf{d}) - R_2, f(x,\mathbf{d})\} \geq \\
\min\{f\,(x-1,\mathbf{d}) - R_2, f(x,\mathbf{d})\} &- \min\{f\,(x-2,\mathbf{d}) - R_2, f\,(x-1,\mathbf{d})\}
\end{aligned}$$

All possible cases resulting through minimizations should be considered separately to verify convexity. The possible cases are listed in Table A.2.

Table A.2: Possible Situations: Unit Admission Control Operator

	$T_{aca}f\left(x+1,\mathbf{d}\right)$	$T_{aca}f(x,\mathbf{d})$	$T_{aca}f\left(x-1,\mathbf{d}\right)$
1	$f(x,\mathbf{d})-R_2$	$f\left(x-1,\mathbf{d}\right)-R_2$	$f\left(x-2,\mathbf{d}\right)-R_2$
2	$f(x,\mathbf{d})-R_2$	$f\left(x-1,\mathbf{d}\right)-R_2$	$f\left(x-1,\mathbf{d}\right)$
3	$f(x,\mathbf{d})-R_2$	$f(x,\mathbf{d})$	$f\left(x-1,\mathbf{d}\right)$
4	$f\left(x+1,\mathbf{d}\right)$	$f(x,\mathbf{d})$	$f\left(x-1,\mathbf{d}\right)$

Case 1. We have to verify the inequality:

$$\begin{aligned} f(x,\mathbf{d})-R_2-f\left(x-1,\mathbf{d}\right)+R_2 &\geq f\left(x-1,\mathbf{d}\right)-R_2-f\left(x-2,\mathbf{d}\right)+R_2 \\ f(x,\mathbf{d})-f\left(x-1,\mathbf{d}\right) &\geq f\left(x-1,\mathbf{d}\right)-f\left(x-2,\mathbf{d}\right) \end{aligned}$$

Since $f(x,\mathbf{d})$ is assumed to be convex in x, $\forall\,\mathbf{d}$, for this case, it is directly verified that T_{aca} preserves convexity in x, $\forall\,\mathbf{d}$. However, we assume that $x\geq 0$. So, we have to verify convexity at boundary values as well as the non-extreme cases. Above inequality verifies convexity at non-extreme values of x. Three extreme cases may occur for this case:

1. We have to verify convexity at $x-1=0$:

$$\begin{aligned} f(x^+,\mathbf{d})-x^-\,F &- f\left((x-1)^+,\mathbf{d}\right)+(x-1)^-\,F \;\geq \\ f\left((x-1)^+,\mathbf{d}\right)-(x-1)^-\,F &- f\left((x-2)^+,\mathbf{d}\right)+(x-2)^-\,F \\ f\left(1,\mathbf{d}\right)-f\left(0,\mathbf{d}\right) &\geq f\left(0,\mathbf{d}\right)-f\left(0,\mathbf{d}\right)-F \\ f\left(1,\mathbf{d}\right)-f\left(0,\mathbf{d}\right) &\geq -F \end{aligned}$$

   So, $T_{aca}f\left(x,\mathbf{d}\right)$ preserves convexity in x, $\forall\,\mathbf{d}$, for the functions satisfying the inequality $f\left(x+1,d\right)-f\left(x,d\right)\geq -F$.

2. We have to verify convexity at $x=0$:

$$\begin{aligned} f(x^+,\mathbf{d})-x^-\,F &- f\left((x-1)^+,\mathbf{d}\right)+(x-1)^-\,F \;\geq \\ f\left((x-1)^+,\mathbf{d}\right)-(x-1)^-\,F &- f\left((x-2)^+,\mathbf{d}\right)+(x-2)^-\,F \\ f\left(0,\mathbf{d}\right)-f\left(0,\mathbf{d}\right)-F &\geq f\left(0,\mathbf{d}\right)+F-f\left(0,\mathbf{d}\right)-2F \\ -F &\geq -F \end{aligned}$$

$T_{aca}f\,(x, \mathbf{d})$ is proven to be convex in x, $\forall\,\mathbf{d}$, for this case.

3. We have to verify convexity at $x < 0$:

$$\begin{aligned}
f(x^{+}, \mathbf{d}) - x^{-}\,F \quad & - \quad f\,((x-1)^{+}, \mathbf{d}) + (x-1)^{-}\,F \;\geq \\
f\,((x-1)^{+}, \mathbf{d}) - (x-1)^{-}\,F \quad & - \quad f\,((x-2)^{+}, \mathbf{d}) + (x-2)^{-}\,F \\
f\,(0, \mathbf{d}) - xF \quad & - \quad f\,(0, \mathbf{d}) + (x-1)F \;\geq \\
f\,(0, \mathbf{d}) - (x-1)F \quad & - \quad f\,(0, \mathbf{d}) + (x-2)F \\
F \quad & \geq \quad F
\end{aligned}$$

For Case 1, $T_{aca}f\,(x, \mathbf{d})$ is proven to be convex in x, $\forall\,\mathbf{d}$, for the functions satisfying the inequality $f\,(x+1, \mathbf{d}) - f\,(x, \mathbf{d}) \geq -F$.

Case 2. We have to verify the inequality:

$$\begin{aligned}
f(x, \mathbf{d}) - R_2 - f\,(x-1, \mathbf{d}) + R_2 \;&\geq\; f\,(x-1, \mathbf{d}) - R_2 - f\,(x-1, \mathbf{d}) \\
f(x, \mathbf{d}) \;&\geq\; f\,(x-1, \mathbf{d}) - R_2
\end{aligned}$$

By the results of the minimizations, we have:

$$\begin{aligned}
f(x, \mathbf{d}) - R_2 \;&\leq\; f\,(x+1, \mathbf{d}) \\
f\,(x-1, \mathbf{d}) - R_2 \;&\leq\; f(x, \mathbf{d}) \\
f\,(x-1, \mathbf{d}) \;&\leq\; f\,(x-2, \mathbf{d}) - R_2
\end{aligned}$$

These results display that $f(x, \mathbf{d}) \geq f\,(x-1, \mathbf{d}) - R_2$. So, for this case, it is verified that T_{aca} preserves convexity in x, $\forall\,\mathbf{d}$. However, we assume that $x \geq 0$. So, we have to verify convexity at boundary values as well as the non-extreme cases. Above inequality verifies convexity at non-extreme values of $x(x-2 \geq 0)$. Two extreme cases may occur for this case:

1. We have to verify convexity at $x = 0$:

$$\begin{aligned} f(x^{+},\mathbf{d}) - x^{-}\,F &\geq f\left((x-1)^{+},\mathbf{d}\right) - R_2 - (x-1)^{-}\,F \\ f\left(0,\mathbf{d}\right) &\geq f\left(0,\mathbf{d}\right) - R_2 + F \\ R_2 &\geq F \end{aligned}$$

By the results of the minimizations, we have:

$$\begin{aligned} f(0,\mathbf{d}) - R_2 &\leq f\left(1,\mathbf{d}\right) \\ f\left(0,\mathbf{d}\right) - R_2 + F &\leq f(0,\mathbf{d}) \\ f\left(0,\mathbf{d}\right) + F &\leq f\left(0,\mathbf{d}\right) - R_2 + 2F \\ R_2 &= F \end{aligned}$$

Since $R_2 \geq F$ is satisfied, $T_{aca}f\left(x,\mathbf{d}\right)$ preserves convexity in x, $\forall\,\mathbf{d}$, for this case.

2. We have to verify convexity at $x-1<0$:

$$\begin{aligned} f(x^{+},\mathbf{d}) - x^{-}\,F &\geq f\left((x-1)^{+},\mathbf{d}\right) - R_2 - (x-1)^{-}\,F \\ f(0,\mathbf{d}) - x\,F &\geq f\left(0,\mathbf{d}\right) - R_2 - (x-1)\,F \\ R_2 &\geq F \end{aligned}$$

By the results of the minimizations, we have:

$$\begin{aligned} f(x,\mathbf{d}) - R_2 + F &\leq f\left(x+1,\mathbf{d}\right) \\ f\left(x-1,\mathbf{d}\right) - R_2 + F &\leq f(x,\mathbf{d}) \\ f\left(x-1,\mathbf{d}\right) &\leq f\left(x-2,\mathbf{d}\right) - R_2 + F \\ R_2 &= F \end{aligned}$$

Since $R_2 \geq F$ is satisfied, $T_{aca}f\left(x,\mathbf{d}\right)$ is proven to be convex in x, $\forall\,\mathbf{d}$, for this case.

For Case 2, $T_{aca}f\left(x,\mathbf{d}\right)$ is proven to be convex in x, $\forall\,\mathbf{d}$.

Case 3. We have to verify the inequality:

$$\begin{aligned} f(x,\mathbf{d}) - R_2 - f(x,\mathbf{d}) &\geq f(x,\mathbf{d}) - f(x-1,\mathbf{d}) \\ f(x-1,\mathbf{d}) - R_2 &\geq f(x,\mathbf{d}) \end{aligned}$$

By the results of the minimizations, we have:

$$\begin{aligned} f(x,\mathbf{d}) - R_2 &\leq f(x+1,\mathbf{d}) \\ f(x,\mathbf{d}) &\leq f(x-1,\mathbf{d}) - R_2 \\ f(x-1,\mathbf{d}) &\leq f(x-2,\mathbf{d}) - R_2 \end{aligned}$$

These results display that $f(x-1,\mathbf{d}) - R_2 \geq f(x,\mathbf{d})$. So, for this case, it is verified that T_{aca} preserves convexity in x, $\forall\,\mathbf{d}$ for the non-extreme cases.

1. We have to verify convexity at $x = 0$:

$$\begin{aligned} f((x-1)^{+},\mathbf{d}) - R_2 - (x-1)^{-}F &\geq f(x^{+},\mathbf{d}) - x^{-}F \\ f(0,\mathbf{d}) - R_2 + F &\geq f(0,\mathbf{d}) \\ F &\geq R_2 \end{aligned}$$

By the results of the minimizations, we have:

$$\begin{aligned} f(0,\mathbf{d}) - R_2 &\leq f(1,\mathbf{d}) \\ f(0,\mathbf{d}) &\leq f(0,\mathbf{d}) - R_2 + F \\ f(0,\mathbf{d}) + F &\leq f(0,\mathbf{d}) - R_2 + 2F \\ R_2 &\leq F \end{aligned}$$

Since $F \geq R_2$ is satisfied, $T_{aca}f(x,\mathbf{d})$ is proven to be convex in x, $\forall\,\mathbf{d}$, for this case.

2. We have to verify convexity at $x < 0$:

$$\begin{aligned} f\left((x-1)^{+}, \mathbf{d}\right) - R_2 - (x-1)^{-} F &\geq f(x^{+}, \mathbf{d}) - x^{-} F \\ f\left(0, \mathbf{d}\right) - R_2 - (x-1) F &\geq f(0, \mathbf{d}) - x F \\ f\left(0, \mathbf{d}\right) - R_2 - (x-1) F &\geq f(0, \mathbf{d}) - x F \\ F &\geq R_2 \end{aligned}$$

By the results of the minimizations, we have:

$$\begin{aligned} f(x^{+}, \mathbf{d}) - R_2 - x^{-} F &\leq f\left((x+1)^{+}, \mathbf{d}\right) - (x+1)^{-} F \\ f(x^{+}, \mathbf{d}) - x^{-} F &\leq f\left((x-1)^{+}, \mathbf{d}\right) - R_2 - (x-1)^{-} F \\ f\left((x-1)^{+}, \mathbf{d}\right) - (x-1)^{-} F &\leq f\left((x-2)^{+}, \mathbf{d}\right) - R_2 - (x-2)^{-} F \\ R_2 &\leq F \end{aligned}$$

Since $R_2 \leq F$ is satisfied, $T_{aca} f\left(x, \mathbf{d}\right)$ is proven to be convex in x, $\forall\, \mathbf{d}$, for this case.

For Case 3, $T_{aca} f\left(x, \mathbf{d}\right)$ is proven to be convex in x, $\forall\, \mathbf{d}$.

Case 4. We have to verify the inequality:

$$f\left(x+1, \mathbf{d}\right) - f(x, \mathbf{d}) \geq f(x, \mathbf{d}) - f\left(x-1, \mathbf{d}\right)$$

Since $f(x, \mathbf{d})$ is assumed to be convex in x, $\forall\, \mathbf{d}$, for this case, it is directly verified that T_{aca} preserves convexity in x, $\forall\, \mathbf{d}$. However, these are valid for the non-extreme cases only. We have to verify convexity for the extreme cases:

1. We have to verify convexity at $x = 0$:

$$\begin{aligned} & f((x+1)^{+}, \mathbf{d}) - (x+1)^{-} F && - \; f\left(x^{+}, \mathbf{d}\right) + x^{-} F \;\geq \\ & f\left(x^{+}, \mathbf{d}\right) - x^{-} F && - \; f\left((x-1)^{+}, \mathbf{d}\right) + (x-1)^{-} F \\ & f\left(1, \mathbf{d}\right) - f\left(0, \mathbf{d}\right) && \geq \; f\left(0, \mathbf{d}\right) - f\left(0, \mathbf{d}\right) - F \\ & f\left(1, \mathbf{d}\right) - f\left(0, \mathbf{d}\right) && \geq \; -F \end{aligned}$$

So, $T_{aca}f\,(x,\mathbf{d})$ preserves convexity in x, $\forall\,\mathbf{d}$, for the functions satisfying the inequality $f\,(x+1,d)-f\,(x,d)\geq -F$.

2. We have to verify convexity at $x+1=0$:

$$\begin{aligned}
f((x+1)^{+},\mathbf{d})-(x+1)^{-}\,F &\;-\; f\,(x^{+},\mathbf{d})+x^{-}\,F \;\geq \\
f\,(x^{+},\mathbf{d})-x^{-}\,F &\;-\; f\,((x-1)^{+},\mathbf{d})+(x-1)^{-}\,F \\
f\,(0,\mathbf{d})-f\,(0,\mathbf{d})-F &\;\geq\; f\,(0,\mathbf{d})+F-f\,(0,\mathbf{d})-2F \\
-F &\;\geq\; -F
\end{aligned}$$

$T_{aca}f\,(x,\mathbf{d})$ is proven to be convex in x, $\forall\,\mathbf{d}$, for this case.

3. We have to verify convexity at $x+1<0$:

$$\begin{aligned}
f((x+1)^{+},\mathbf{d})-(x+1)^{-}\,F &\;-\; f\,(x^{+},\mathbf{d})+x^{-}\,F \;\geq \\
f\,(x^{+},\mathbf{d})-x^{-}\,F &\;-\; f\,((x-1)^{+},\mathbf{d})+(x-1)^{-}\,F \\
f\,(0,\mathbf{d})-(x+1)F &\;-\; f\,(0,\mathbf{d})+xF \;\geq \\
f\,(0,\mathbf{d})-x\,F &\;-\; f\,(0,\mathbf{d})+(x-1)F \\
F &\;\geq\; F
\end{aligned}$$

For Case 4, $T_{aca}f\,(x,\mathbf{d})$ is proven to be convex in x, $\forall\,\mathbf{d}$, for the functions satisfying the inequality $f\,(x+1,\mathbf{d})-f\,(x,d)\geq -F$.

In conclusion, $T_{aca}f(x,\mathbf{d})$ is proven to be convex in x, $\forall\,\mathbf{d}$, for the functions satisfying the inequality $f\,(x+1,d)-f\,(x,d)\geq -F$. □

For $f(x,\mathbf{d})$ satisfying $LBD(F)$, $\forall\,\mathbf{d}$, T_{aca} preserves $LBD(F)$ property, $\forall\,\mathbf{d}$:

Proof. We have to verify the inequality:

$$\begin{aligned}
\Delta T_{aca}f\,(x+1,\mathbf{d}) &\;\geq\; -F \\
T_{aca}f\,(x+1,\mathbf{d})-T_{aca}f(x,\mathbf{d}) &\;\geq\; -F \\
\min\{f(x,\mathbf{d})-R_2,f\,(x+1,\mathbf{d})\}-\min\{f\,(x-1,\mathbf{d})-R_2,f(x,\mathbf{d})\} &\;\geq\; -F
\end{aligned}$$

According to the results of the minimizations, three cases may occur:

Case 1. We have to verify the inequality:

$$\begin{aligned} f(x,\mathbf{d}) - R_2 - f(x-1,\mathbf{d}) + R_2 &\geq -F \\ f(x,\mathbf{d}) - f(x-1,\mathbf{d}) &\geq -F \end{aligned}$$

Since $f(x,\mathbf{d})$ satisfies $LBD(F)$, $\forall\,\mathbf{d}$, T_{aca} preserves $LBD(F)$ property, $\forall\,\mathbf{d}$ for this case.

Case 2. We have to verify the inequality:

$$\begin{aligned} f(x,\mathbf{d}) - R_2 - f(x,\mathbf{d}) &\geq -F \\ -R_2 &\geq -F \end{aligned}$$

According to the results of the minimizations at the extreme values, we know that:

$$\begin{aligned} f(0,\mathbf{d}) - R_2 - Fx &\leq f(0,\mathbf{d}) - F(x+1) \\ -R_2 &\leq -F \\ f(0,\mathbf{d}) - Fx &\leq f(0,\mathbf{d}) - R_2 - F(x-1) \\ -F &\leq -R_2 \end{aligned}$$

Thus, $R_2 = F$ is shown for this case and T_{aca} preserves $LBD(F)$ property, $\forall\,\mathbf{d}$ for this case.

Case 3. We have to verify the inequality:

$$f(x+1,\mathbf{d}) - f(x,\mathbf{d}) \geq -F$$

Since $f(x,\mathbf{d})$ satisfies $LBD(F)$, $\forall\,\mathbf{d}$, T_{aca} preserves $LBD(F)$ property, $\forall\,\mathbf{d}$ for this case. □

A.6 Outsourcing Operator

In this section, we investigate the structure of the outsourcing operator:

$$T_{out}f(x,\mathbf{d}) = f(x^+,\mathbf{d}) - x^- F$$

For $f(x,\mathbf{d})$ convex in x, $\forall\mathbf{d}$, T_{out} preserves convexity in x, $\forall\mathbf{d}$:

Proof. Let $f(x,\mathbf{d})$ be convex in x, $\forall\,\mathbf{d}$, i.e. $\Delta f(x+1,\mathbf{d}) \geq \Delta f(x,\mathbf{d})$. We are going to show convexity of $T_{out}f(x,\mathbf{d})$:

$$\begin{aligned}
\Delta T_{out} f\left(x+1, \mathbf{d}\right) &\geq \Delta T_{out} f(x, \mathbf{d}) \\
T_{out} f\left(x+1, \mathbf{d}\right) - T_{out} f(x, \mathbf{d}) &\geq T_{out} f(x, \mathbf{d}) - T_{out} f\left(x-1, \mathbf{d}\right) \\
f\left((x+1)^{+}, \mathbf{d}\right) - (x+1)^{-} F &- f\left(x^{+}, \mathbf{d}\right) + x^{-} F \quad \geq \\
f\left(x^{+}, \mathbf{d}\right) - x^{-} F &- f\left((x-1)^{+}, \mathbf{d}\right) + (x-1)^{-} F
\end{aligned}$$

Since we define x as $x \geq 0$, we may check the boundary values of x as well as the non-extreme cases. For the non-extreme cases ($x - 1 \geq 0$), we have:

$$f\left(x+1, d\right) - f\left(x, d\right) \geq f\left(x, \mathbf{d}\right) - f\left(x-1, \mathbf{d}\right)$$

Since $f(x, \mathbf{d})$ is convex in x, $\forall\, \mathbf{d}$, $T_{out} f\left(x, \mathbf{d}\right)$ is proven to be convex in x, $\forall\, \mathbf{d}$ for the non-extreme cases.However, three extreme cases occur that should be checked:

Case 1. We have to verify the convexity where $x = 0$:

$$\begin{aligned}
f\left((x+1)^{+}, \mathbf{d}\right) - (x+1)^{-} F &- f\left(x^{+}, \mathbf{d}\right) + x^{-} F \quad \geq \\
f\left(x^{+}, \mathbf{d}\right) - x^{-} F &- f\left((x-1)^{+}, \mathbf{d}\right) + (x-1)^{-} F \\
f\left(1, \mathbf{d}\right) - f\left(0, \mathbf{d}\right) &\geq f\left(0, \mathbf{d}\right) - f\left(0, d\right) - F \\
f\left(1, d\right) - f\left(0, \mathbf{d}\right) &\geq -F
\end{aligned}$$

So, $T_{out} f\left(x, \mathbf{d}\right)$ preserves convexity in x, $\forall\, \mathbf{d}$ for the functions satisfying the inequality $f\left(x+1, \mathbf{d}\right) - f\left(x, \mathbf{d}\right) \geq -F$.

Case 2. We have to verify the convexity where $x + 1 = 0$:

$$\begin{aligned}
f\left((x+1)^{+}, \mathbf{d}\right) - (x+1)^{-} F &- f\left(x^{+}, \mathbf{d}\right) + x^{-} F \quad \geq \\
f\left(x^{+}, \mathbf{d}\right) - x^{-} F &- f\left((x-1)^{+}, \mathbf{d}\right) + (x-1)^{-} F \\
f\left(0, d^{+}\right) - f\left(0, d^{+}\right) - F &\geq f\left(0, d^{+}\right) + F - f\left(0, d^{+}\right) - 2F \\
-F &\geq -F
\end{aligned}$$

$T_{out} f\left(x, \mathbf{d}\right)$ is proven to be convex in x, $\forall\, \mathbf{d}$ for this case.

Case 3. We have to verify the convexity where $x+1<0$:

$$\begin{aligned}
f\left((x+1)^{+},\mathbf{d}\right)-(x+1)^{-}F &- f\left(x^{+},\mathbf{d}\right)+x^{-}F \geq \\
f\left(x^{+},\mathbf{d}\right)-x^{-}F &- f\left((x-1)^{+},\mathbf{d}\right)+(x-1)^{-}F \\
f\left(0,d^{+}\right)-(x+1)F &- f\left(0,d^{+}\right)+x\,F \geq \\
f\left(0,d^{+}\right)-x\,F &- f\left(0,d^{+}\right)+(x-1)\,F \\
F &\geq F
\end{aligned}$$

$T_{out}f\left(x,\mathbf{d}\right)$ is proven to be convex in x, $\forall\,\mathbf{d}$ for this case.

In conclusion, $T_{out}f\left(x,\mathbf{d}\right)$ is proven to be convex in x, $\forall\,\mathbf{d}$ for all convex functions $f\left(x,\mathbf{d}\right)$ satisfying the inequality $f\left(x+1,\mathbf{d}\right)-f\left(x,\mathbf{d}\right)\geq -F$. □

For $f(x,\mathbf{d})$ satisfying $LBD(F)$, $\forall\,\mathbf{d}$, T_{out} preserves $LBD(F)$ property, $\forall\,\mathbf{d}$:

Proof. We have to verify the inequality.

$$\begin{aligned}
\Delta T_{out}f\left(x+1,\mathbf{d}\right) &\geq -F \\
T_{out}f\left(x+1,\mathbf{d}\right)-T_{out}f(x,\mathbf{d}) &\geq -F \\
f\left((x+1)^{+},\mathbf{d}\right)-(x+1)^{-}F-f\left(x^{+},\mathbf{d}\right)+x^{-}F &\geq -F
\end{aligned}$$

Three cases may occur:

Case 1. For $x\geq 0$:

$$f\left(x+1,\mathbf{d}\right)-f\left(x,\mathbf{d}\right)\geq -F$$

Since $f(x,\mathbf{d})$ satisfies $LBD(F)$, $\forall\,\mathbf{d}$, T_{out} preserves $LBD(F)$ property, $\forall\,\mathbf{d}$ for this case.

Case 2. For $x+1=0$:

$$\begin{aligned}
f\left(0,\mathbf{d}\right)-f\left(0,\mathbf{d}\right)-F &\geq -F \\
-F &\geq -F
\end{aligned}$$

Case 3. For $x+1\leq 0$:

$$\begin{aligned}
f\left(0,\mathbf{d}\right)-(x+1)^{-}F-f\left(0,\mathbf{d}\right)+x^{-}F &\geq -F \\
-F &\geq -F
\end{aligned}$$

Thus, for $f(x,\mathbf{d})$ satisfying $LBD(F)$, $\forall\,\mathbf{d}$, T_{out} is proven to preserve $LBD(F)$ property,

$\forall\, \mathbf{d}$. □

A.7 Production Amount Operator

In this section, we investigate the structure of the operator for production decision of any amount k:

$$T_{pro}f\,(x, \mathbf{d}) = \min_{0 \le k \le K}\{f(x + k, \mathbf{d}) + k\,c\}$$

For $f(x, \mathbf{d})$ convex in x, $\forall \mathbf{d}$, T_{pro} preserves convexity in x, $\forall \mathbf{d}$:

Proof. Let $f(x, \mathbf{d})$ be convex in x, $\forall\, \mathbf{d}$, i.e. $\Delta f\,(x+1, \mathbf{d}) \ge \Delta f(x, \mathbf{d})$. We are going to show convexity of $T_{pro}f(x, \mathbf{d})$:

$$\begin{aligned}
\Delta T_{pro}f\,(x+1, \mathbf{d}) &\ge \Delta T_{pro}f(x, \mathbf{d}) \\
T_{pro}f\,(x+1, \mathbf{d}) - T_{pro}f(x, \mathbf{d}) &\ge T_{pro}f(x, \mathbf{d}) - T_{pro}f\,(x-1, \mathbf{d}) \\
\min_{0 \le k \le K}\{f(x+1+k, \mathbf{d}) + k\,c\} &- \min_{0 \le k \le K}\{f(x+k, \mathbf{d}) + k\,c\} \ge \\
\min_{0 \le k \le K}\{f(x+k, \mathbf{d}) + k\,c\} &- \min_{0 \le k \le K}\{f(x-1+k, \mathbf{d}) + k\,c\}
\end{aligned}$$

$$\begin{aligned}
& \min\{f(x+1, \mathbf{d}), ..., f(x-1+K, \mathbf{d}) + (K-2)c, f(x+K, \mathbf{d}) + (K-1)c, f(x+1+K, \mathbf{d}) + Kc\} \\
- \quad & \min\{f(x, \mathbf{d}), f(x+1, \mathbf{d}) + c, ..., f(x-1+K, \mathbf{d}) + (K-1)c, f(x+K, \mathbf{d}) + Kc\} \ge \\
& \min\{f(x, \mathbf{d}), f(x+1, \mathbf{d}) + c, ..., f(x-1+K, \mathbf{d}) + (K-1)c, f(x+d) + Kc\} \\
- \quad & \min\{f(x-1, \mathbf{d}), f(x, \mathbf{d}) + c, f(x+1, \mathbf{d}) + 2c..., f(x-1+K, \mathbf{d}) + Kc\}
\end{aligned}$$

There are similar terms in the minimizations. If we define:

$$a = \min\{f(x+1, \mathbf{d}), f(x+2, \mathbf{d}) + c, f(x+3, \mathbf{d}) + 2c, ..., f(x-1+K, \mathbf{d}) + (K-2)c\}$$

we can simplify the convexity inequality as follows:

$$\begin{aligned}
& \min\{a, f(x+K, \mathbf{d}) + (K-1)c, f(x+1+K, \mathbf{d}) + Kc\} \\
- \quad & \min\{f(x, \mathbf{d}), a + c, f(x+K, \mathbf{d}) + Kc\} \ge \\
& \min\{f(x, \mathbf{d}), a + c, f(x+K, \mathbf{d}) + Kc\} \\
- \quad & \min\{f(x-1, \mathbf{d}), f(x, \mathbf{d}) + c, a + 2c\}
\end{aligned}$$

All possible result combinations for the minimizations should be examined to verify convexity. In Table A.3 all possible results are listed. Since we assume that $f(x, \mathbf{d})$ is convex in x, $\forall\,\mathbf{d}$, only nine of the possible results are valid.

Table A.3: Possible Situations: Production Amount

$T_{pro}f(x+1,\mathbf{d})$		$T_{pro}f(x,\mathbf{d})$		$T_{pro}(x-1,\mathbf{d})$		
Min	P.Amt.	Min	P.Amt.	Min	P.Amt.	No
a	0	$f(x,\mathbf{d})$	0	$f(x-1,\mathbf{d})$	0	1
a	0	$f(x,\mathbf{d})$	0	$f(x,\mathbf{d})+c$	1	2
a	0	$f(x,\mathbf{d})$	0	$a+2c$	2	3
a	0	$a+c$	1	$f(x,\mathbf{d})+c$	1	4
a	0	$a+c$	1	$a+2c$	2	5
a	$(K-2)$	$f(x+K,\mathbf{d})+Kc$	K	$a+2c$	K	6
$f(x+K,\mathbf{d})+(K-1)c$	$K-1$	$a+c$	$K-1$	$a+2c$	K	7
$f(x+K,\mathbf{d})+(K-1)c$	$K-1$	$f(x+K,\mathbf{d})+Kc$	K	$a+2c$	K	8
$f(x+1+K,\mathbf{d})+Kc$	K	$f(x+K,\mathbf{d})+Kc$	K	$a+2c$	K	9

Case 1. We have to verify the inequality:

$$f(x+1,\mathbf{d}) - f(x,\mathbf{d}) \geq f(x,\mathbf{d}) - f(x-1,\mathbf{d})$$

Since $f(x, \mathbf{d})$ is convex in x, $\forall\,\mathbf{d}$, $T_{pro}f\,(x, \mathbf{d})$ is proven to be convex in x, $\forall\,\mathbf{d}$ for this case where no production is done.

Case 2. We have to verify the inequality:

$$\begin{aligned} f(x+1,\mathbf{d}) - f(x,\mathbf{d}) &\geq f(x,\mathbf{d}) - f(x,\mathbf{d}) - c \\ f(x+1,\mathbf{d}) + c &\geq f(x,\mathbf{d}) \end{aligned}$$

By the results of the minimizations, we have:

$$\begin{aligned} a &= f(x+1,\mathbf{d}) \\ f(x+K,\mathbf{d})+(K-1)c &\geq f(x+1,\mathbf{d}) \\ f(x+1+K,\mathbf{d})+Kc &\geq f(x+1,\mathbf{d}) \\ f(x+1,\mathbf{d})+c &\geq f(x,\mathbf{d}) \\ f(x+K,\mathbf{d})+Kc &\geq f(x,\mathbf{d}) \\ f(x+1,\mathbf{d})+2c &\geq f(x,\mathbf{d})+c \\ f(x-1,\mathbf{d}) &\geq f(x,\mathbf{d})+c \end{aligned} \tag{A.1}$$

Since $f(x+1,\mathbf{d})+c \geq f(x,\mathbf{d})$ is verified by Equation A.1, convexity is verified for this case.

Case 3. We have to verify the inequality:

$$f(x+1,\mathbf{d})-f(x,\mathbf{d}) \quad \geq \quad f(x,\mathbf{d})-f(x+1,\mathbf{d})-2c$$

By the results of the minimizations, we have:

$$\begin{aligned} a &= f(x+1,\mathbf{d}) \qquad \text{(A.2)} \\ f(x+K,\mathbf{d})+(K-1)c &\geq f(x+1,\mathbf{d}) \\ f(x+1+K,\mathbf{d})+Kc &\geq f(x+1,\mathbf{d}) \\ f(x+1,\mathbf{d})+c &\geq f(x,\mathbf{d}) \\ f(x+K,\mathbf{d})+Kc &\geq f(x,\mathbf{d}) \\ f(x,\mathbf{d})+c &\geq f(x+1,\mathbf{d})+2c \\ f(x-1,\mathbf{d}) &\geq f(x+1,\mathbf{d})+2c \end{aligned}$$

Since $f(x+1,\mathbf{d})+c = f(x,\mathbf{d})$ by the Equations A.3 and A.3, the convexity inequality is verified for this case.

Case 4. We have to verify the inequality:

$$\begin{array}{lcl} f(x+1,\mathbf{d}) - f(x+1,\mathbf{d}) - c & \geq & f(x+1,\mathbf{d}) + c - f(x,\mathbf{d}) - c \\ f(x,\mathbf{d}) & \geq & f(x+1,\mathbf{d}) + c \end{array}$$

By the results of the minimizations, we have:

$$\begin{array}{rcl} a & = & f(x+1,\mathbf{d}) \\ f(x+K,\mathbf{d}) + (K-1)c & \geq & f(x+1,\mathbf{d}) \\ f(x+1+K,\mathbf{d}) + Kc & \geq & f(x+1,\mathbf{d}) \\ f(x,\mathbf{d}) & \geq & f(x+1,\mathbf{d}) + c \\ f(x+K,\mathbf{d}) + Kc & \geq & f(x+1,\mathbf{d}) + c \\ f(x+1,\mathbf{d}) + 2c & \geq & f(x,\mathbf{d}) + c \\ f(x-1,\mathbf{d}) & \geq & f(x,\mathbf{d}) + c \end{array} \tag{A.3}$$

Since $f(x+1,\mathbf{d}) + c \leq f(x,\mathbf{d})$ by Equation A.4, the convexity inequality is verified for this case.

Case 5. We have to verify the inequality:

$$\begin{array}{lcl} f(x+1,\mathbf{d}) - f(x+1,\mathbf{d}) - c & \geq & f(x+1,\mathbf{d}) + c - f(x+1,\mathbf{d}) - 2c \\ -c & \geq & -c \end{array}$$

Here, convexity is verified for $a = f(x+1,\mathbf{d})$, however it is true for all values of a.

Case 6. We have to verify the inequality:

$$f(x+K-1,\mathbf{d}) + (K-2)c - f(x+K,\mathbf{d}) - Kc \;\; \geq \;\; f(x+K,\mathbf{d}) + Kc - f(x+K-1,\mathbf{d}) - Kc$$

By the results of the minimizations, we have:

$$\begin{aligned}
a &= f(x+K-1,\mathbf{d})+(K-2)c \\
f(x+K,\mathbf{d})+(K-1)c &\geq f(x+K-1,\mathbf{d})+(K-2)c \qquad \text{(A.4)} \\
f(x+1+K,\mathbf{d})+Kc &\geq f(x+K-1,\mathbf{d})+(K-2)c \\
f(x+K-1,\mathbf{d})+(K-1)c &\geq f(x+K,\mathbf{d})+Kc \qquad \text{(A.5)} \\
f(x,\mathbf{d}) &\geq f(x+K,\mathbf{d})+Kc \\
f(x-1,\mathbf{d}) &\geq f(x+K-1,\mathbf{d})+Kc \\
f(x,\mathbf{d})+c &\geq f(x+K-1,\mathbf{d})+Kc
\end{aligned}$$

By Equations A.4 and A.5, the convexity inequality is verified for this case.

Case 7. We have to verify the inequality:

$$\begin{aligned}
f(x+K,\mathbf{d})+(K-1)c \quad &- \quad f(x+K-1,\mathbf{d})-(K-1)c \geq \\
f(x+K-1,\mathbf{d})+(K-1)c \quad &- \quad f(x+K-1,\mathbf{d})-Kc \\
f(x+K,\mathbf{d})+c \quad &\geq \quad f(x+K-1,\mathbf{d})
\end{aligned}$$

By the results of the minimizations, we have:

$$\begin{aligned}
a &= f(x+K-1,\mathbf{d})+(K-2)c \\
f(x+K-1,\mathbf{d})+(K-2)c &\geq f(x+K,\mathbf{d})+(K-1)c \\
f(x+1+K,\mathbf{d})+Kc &\geq f(x+K,\mathbf{d})+(K-1)c \\
f(x+K,\mathbf{d})+Kc &\geq f(x+K-1,\mathbf{d})+(K-1)c \qquad \text{(A.6)} \\
f(x,\mathbf{d}) &\geq f(x+K-1,\mathbf{d})+(K-1)c \\
f(x-1,\mathbf{d}) &\geq f(x+K-1,\mathbf{d})+Kc \\
f(x,\mathbf{d})+c &\geq f(x+K-1,\mathbf{d})+Kc
\end{aligned}$$

By Equation A.6, the convexity inequality is verified for this case.

Case 8. We have to verify the inequality:

$$\begin{array}{lcl} f(x+K,\mathbf{d})+(K-1)c & - & f(x+K,\mathbf{d})-Kc \geq \\ f(x+K,\mathbf{d})+Kc & - & f(x+K-1,\mathbf{d})-Kc \\ f(x+K-1,\mathbf{d})+c & \geq & f(x+K,\mathbf{d})+c \end{array}$$

By the results of the minimizations, we have:

$$\begin{array}{rcl} a & = & f(x+K-1,\mathbf{d})+(K-2)c \\ f(x+K-1,\mathbf{d})+(K-2)c & \geq & f(x+K,\mathbf{d})+(K-1)c \\ f(x+1+K,\mathbf{d})+Kc & \geq & f(x+K,\mathbf{d})+(K-1)c \\ f(x+K-1,\mathbf{d})+(K-1)c & \geq & f(x+K,\mathbf{d})+Kc \\ f(x,\mathbf{d}) & \geq & f(x+K,\mathbf{d})+Kc \\ f(x-1,\mathbf{d}) & \geq & f(x+K-1,\mathbf{d})+Kc \\ f(x,\mathbf{d})+c & \geq & f(x+K-1,\mathbf{d})+Kc \end{array} \tag{A.7}$$

By Equation A.7, the convexity inequality is verified for this case.

Case 9. We have to verify the inequality:

$$\begin{array}{lcl} f(x+1+K,\mathbf{d})+Kc & - & f(x+K,\mathbf{d})-Kc \geq \\ f(x+K,\mathbf{d})+Kc & - & f(x+K-1,\mathbf{d})-Kc \\ f(x+1+K,\mathbf{d})-f(x+K,\mathbf{d}) & \geq & f(x+K,\mathbf{d})-f(x+K-1,\mathbf{d}) \end{array}$$

If $f(x,\mathbf{d})$ is convex in x, $\forall\,\mathbf{d}$, convexity is verified for this case.

In conclusion, if $f(x,\mathbf{d})$ is convex in x, $\forall\,\mathbf{d}$, $T_{pro}f\,(x,\mathbf{d})$ is proven to be convex in x, $\forall\,\mathbf{d}$. □

For $f(x,\mathbf{d})$ satisfying $LBD(F)$, $\forall\,\mathbf{d}$, T_{AC} preserves $LBD(F)$ property, $\forall\,\mathbf{d}$:

Proof. We have to verify the inequality:

$$\begin{array}{lcl} \Delta T_{pro}f\,(x+1,\mathbf{d}) & \geq & -F \\ T_{pro}f\,(x+1,\mathbf{d})-T_{pro}f(x,\mathbf{d}) & \geq & -F \\ \min\limits_{0\leq k\leq K}\{f(x+1+k,\mathbf{d})+k\,c\}-\min\limits_{0\leq k\leq K}\{f(x+k,\mathbf{d})+k\,c\} & \geq & -F \end{array}$$

Three cases may occur according to the results of the minimizations:

Case 1. $(0, 0)$ production case:

$$f(x+1,\mathbf{d}) - f(x,\mathbf{d}) \quad \geq \quad -F$$

This is valid since $f(x,\mathbf{d})$ satisfies $LBD(F)$, $\forall\,\mathbf{d}$.

Case 2. $(k, k+1)$ production case:

$$\begin{aligned} f(x+1+k,\mathbf{d}) + kc - f(x+k+1,\mathbf{d}) - (k+1)c \quad &\geq \quad -F \\ -c \quad &\geq \quad -F \end{aligned}$$

This is valid since production is always less costly than outsourcing, $c \leq F$.

Case 3. (K, K) production case:

$$\begin{aligned} f(x+1+K,\mathbf{d}) + Kc - f(x+K,\mathbf{d}) - Kc \quad &\geq \quad -F \\ f(x+1+K,\mathbf{d}) - f(x+K,\mathbf{d}) \quad &\geq \quad -F \end{aligned}$$

This is valid since $f(x,\mathbf{d})$ satisfies $LBD(F)$, $\forall\,\mathbf{d}$. □

A.8 Admission Amount Operator

In this section, we investigate the structure of the operator for admission control with any amount t:

$$T_{adm} f\,(x,\mathbf{d}) = \min_{0 \leq t \leq \bar{t}} \{f((x-t),\mathbf{d}) - t\,R_2\}$$

For $f(x,\mathbf{d})$ convex in x, $\forall\mathbf{d}$, T_{adm} preserves convexity in x, $\forall\mathbf{d}$:

Proof.

$$T_{adm} f\,(x,\mathbf{d}) = \min_{0 \leq t \leq \bar{t}} \{f(x-t,\mathbf{d}) - t\ R\}$$

The convexity inequality is:

$$\begin{aligned} &\min_{0 \leq t \leq \bar{t}} \{f(x+1-t,\mathbf{d}) - t\ R\} - \min_{0 \leq t \leq \bar{t}} \{f(x-t,\mathbf{d}) - t\ R\} \geq \\ &\min_{0 \leq t \leq \bar{t}} \{f(x-t,\mathbf{d}) - t\ R\} - \min_{0 \leq t \leq \bar{t}} \{f(x-1-t,\mathbf{d}) - t\ R\} \end{aligned}$$

It can be expressed as follows:

$$
\begin{aligned}
& \min\{f(x+1,\mathbf{d}), f(x,\mathbf{d})-R, f(x-1,\mathbf{d})-2R, f(x-2,\mathbf{d})-3R, ..., f(x+1-\bar{t},\mathbf{d})-\bar{t}\ R\} \\
- \quad & \min\{f(x,\mathbf{d}), f(x-1,\mathbf{d})-R, ..., f(x+1-\bar{t},\mathbf{d})-(\bar{t}-1)\ R, f(x-\bar{t},\mathbf{d})-\bar{t}\ R\} \geq \\
& \min\{f(x,\mathbf{d}), f(x-1,\mathbf{d})-R, ..., f(x+1-\bar{t},\mathbf{d})-(\bar{t}-1)\ R, f(x-\bar{t},\mathbf{d})-\bar{t}\ R\} \\
- \quad & \min\{f(x-1,\mathbf{d}), f(x-2,\mathbf{d})-R, ..., f(x+1-\bar{t},\mathbf{d})-(\bar{t}-2)\ R, ..., f(x-1-\bar{t},\mathbf{d})-\bar{t}\ R\}
\end{aligned}
$$

In order to simplify the minimizations, we can define a variable:

$$a = \min\{f(x-1,\mathbf{d}), f(x-2,\mathbf{d})-R, ..., f(x+1-\bar{t},\mathbf{d})-(\bar{t}-2)\ R\}$$

So, convexity inequality is simplified as follows:

$$
\begin{aligned}
& \min\{f(x+1,\mathbf{d}), f(x,\mathbf{d})-R, a-2R\} - \min\{f(x,\mathbf{d}), a-R, f(x-\bar{t},\mathbf{d})-\bar{t}\ R\} \geq \\
& \min\{f(x,\mathbf{d}), a-R, f(x-\bar{t},\mathbf{d})-\bar{t}\ R\} - \min\{a, f(x-\bar{t},\mathbf{d})-(\bar{t}-1)\ R, f(x-1-\bar{t},\mathbf{d})-\bar{t}\ R\}
\end{aligned}
$$

All possible result combinations of the minimizations should be examined to verify convexity. There are nine possible results as shown in A.4, some are eliminated since they conflict with the convex nature of $f(x,\mathbf{d})$:

Table A.4: Possible Situations: Admission Amount

$T_{adm}f(x+1,\mathbf{d})$		$T_{adm}f(x,\mathbf{d})$		$T_{adm}f(x-1,\mathbf{d})$		
Min	A.Amt.	Min	A.Amt.	Min	A.Amt.	*No*
$f(x+1,\mathbf{d})$	0	$f(x,\mathbf{d})$	0	a	0	1
$f(x,\mathbf{d})-R$	1	$f(x,\mathbf{d})$	0	a	0	2
$f(x,\mathbf{d})-R$	1	$a-R$	1	a	0	3
$a-2R$	2	$f(x,\mathbf{d})$	0	a	0	4
$a-2R$	2	$a-R$	1	a	0	5
$a-2R$	$\bar{t}$	$a-R$	$\bar{t}-1$	$f(x-\bar{t},\mathbf{d})-(\bar{t}-1)R$	$\bar{t}-1$	6
$a-2R$	$\bar{t}$	$f(x-\bar{t},\mathbf{d})-\bar{t}R$	$\bar{t}$	a	$\bar{t}-2$	7
$a-2R$	$\bar{t}$	$f(x-\bar{t},\mathbf{d})-\bar{t}R$	$\bar{t}$	$f(x-\bar{t},\mathbf{d})-(\bar{t}-1)R$	$\bar{t}-1$	8
$a-2R$	$\bar{t}$	$f(x-\bar{t},\mathbf{d})-\bar{t}R$	$\bar{t}$	$f(x-1-\bar{t},\mathbf{d})-\bar{t}R$	$\bar{t}$	9

Case 1. We have to verify the inequality:

$$f(x+1,\mathbf{d}) - f(x,\mathbf{d}) \quad \geq \quad f(x,\mathbf{d}) - f(x-1,\mathbf{d})$$

Since we assume that $f(x,\mathbf{d})$ is convex in x,$\forall\mathbf{d}$, convexity is directly verified in this case.

Case 2. We have to verify the inequality:

$$\begin{aligned} f(x,\mathbf{d}) - R - f(x,\mathbf{d}) &\geq f(x,\mathbf{d}) - f(x-1,\mathbf{d}) \\ f(x-1,\mathbf{d}) - R &\geq f(x,\mathbf{d}) \end{aligned}$$

By the results of the minimizations:

$$\begin{aligned} a &= f(x-1,\mathbf{d}) \\ f(x+1,\mathbf{d}) &\geq f(x,\mathbf{d}) - R \\ f(x-1,\mathbf{d}) - 2R &\geq f(x,\mathbf{d}) - R \\ f(x-1,\mathbf{d}) - R &\geq f(x,\mathbf{d}) \qquad \text{(A.8)} \\ f(x-\bar{t},\mathbf{d}) - \bar{t}R &\geq f(x,\mathbf{d}) \\ f(x-\bar{t},\mathbf{d}) - (\bar{t}-1)R &\geq f(x-1,\mathbf{d}) \\ f(x-1-\bar{t},\mathbf{d}) - \bar{t}R &\geq f(x-1,\mathbf{d}) \end{aligned}$$

(A.9)

By the Equation A.8, convexity is verified for this case.

Case 3. We have to verify the inequality:

$$\begin{aligned} f(x,\mathbf{d}) - R - f(x-1,\mathbf{d}) + R &\geq f(x-1,\mathbf{d}) - R - f(x-1,\mathbf{d}) \\ f(x,\mathbf{d}) &\geq f(x-1,\mathbf{d}) - R \end{aligned}$$

By the results of the minimizations:

$$\begin{aligned} a &= f(x-1,\mathbf{d}) \\ f(x+1,\mathbf{d}) &\geq f(x,\mathbf{d}) - R \\ f(x-1,\mathbf{d}) - 2R &\geq f(x,\mathbf{d}) - R \\ f(x,\mathbf{d}) &\geq f(x-1,\mathbf{d}) - R \qquad \text{(A.10)} \\ f(x-\bar{t},\mathbf{d}) - \bar{t}R &\geq f(x-1,\mathbf{d}) - R \\ f(x-\bar{t},\mathbf{d}) - (\bar{t}-1)R &\geq f(x-1,\mathbf{d}) \\ f(x-1-\bar{t},\mathbf{d}) - \bar{t}R &\geq f(x-1,\mathbf{d}) \end{aligned}$$

By the Equation A.10, convexity is verified for this case.

Case 4. We have to verify the inequality:

$$\begin{aligned} f(x-1,\mathbf{d}) - 2R - f(x,\mathbf{d}) &\geq f(x,\mathbf{d}) - f(x-1,\mathbf{d}) \\ f(x-1,\mathbf{d}) - R - f(x,\mathbf{d}) &\geq f(x,\mathbf{d}) - f(x-1,\mathbf{d}) + R \end{aligned}$$

By the results of the minimizations:

$$\begin{aligned} a &= f(x-1,\mathbf{d}) \\ f(x+1,\mathbf{d}) &\geq f(x-1,\mathbf{d}) - 2R \\ f(x,\mathbf{d}) - R &\geq f(x-1,\mathbf{d}) - 2R \qquad \text{(A.11)} \\ f(x-1,\mathbf{d}) - R &\geq f(x,\mathbf{d}) \qquad \text{(A.12)} \\ f(x-\bar{t},\mathbf{d}) - \bar{t}R &\geq f(x,\mathbf{d}) \\ f(x-\bar{t},\mathbf{d}) - (\bar{t}-1)R &\geq f(x-1,\mathbf{d}) \\ f(x-1-\bar{t},\mathbf{d}) - \bar{t}R &\geq f(x-1,\mathbf{d}) \end{aligned}$$

By the Equations A.11 and A.12, convexity is verified for this case.

Case 5. We have to verify the inequality:

$$\begin{aligned} f(x-1,\mathbf{d}) - 2R - f(x-1,\mathbf{d}) + R &\geq f(x-1,\mathbf{d}) - R - f(x-1,\mathbf{d}) \\ -R &\geq -R \end{aligned}$$

Here, convexity is verified for $a = f(x-1,\mathbf{d})$, however it can be verified for all values of a.

Case 6. We have to verify the inequality:

$$\begin{aligned} f(x+1-\bar{t},\mathbf{d}) - \bar{t}R &- f(x-\bar{t}+1,\mathbf{d}) + (\bar{t}-1)R \geq \\ f(x-\bar{t}+1,\mathbf{d}) - (\bar{t}-1)R &- f(x-\bar{t},\mathbf{d}) + (\bar{t}-1)R \\ f(x-\bar{t},\mathbf{d}) - \bar{t}R &\geq f(x-\bar{t}+1,\mathbf{d}) - (\bar{t}-1)R \end{aligned}$$

By the results of the minimizations:

$$\begin{aligned} a &= f(x+1-\bar{t},\mathbf{d}) - (\bar{t}-2)R \\ f(x+1,\mathbf{d}) &\geq f(x+1-\bar{t},\mathbf{d}) - \bar{t}R \\ f(x,\mathbf{d}) - R &\geq f(x+1-\bar{t},\mathbf{d}) - \bar{t}R \\ f(x,\mathbf{d}) &\geq f(x+1-\bar{t},\mathbf{d}) - (\bar{t}-1)R \\ f(x-\bar{t},\mathbf{d}) - \bar{t}R &\geq f(x+1-\bar{t},\mathbf{d}) - (\bar{t}-1)R \qquad \text{(A.13)} \\ f(x-1-\bar{t},\mathbf{d}) - \bar{t}R &\geq f(x-\bar{t},\mathbf{d}) - (\bar{t}-1)R \\ f(x+1-\bar{t},\mathbf{d}) - (\bar{t}-2)R &\geq f(x-\bar{t},\mathbf{d}) - (\bar{t}-1)R \end{aligned}$$

For $x-\bar{t} \geq 0$, the above computations are sufficient for convexity. However, for $x-\bar{t}+1 \leq 0$:

$$\begin{aligned} &f(0,\mathbf{d}) - \bar{t}R - F(x+1-\bar{t})^- - f(0,\mathbf{d}) + (\bar{t}-1)R + F(x-\bar{t}+1)^- \geq \\ &f(0,\mathbf{d}) - (\bar{t}-1)R - F(x-\bar{t}+1)^- - f(0,\mathbf{d}) + (\bar{t}-1)R + F(x-\bar{t})^- \end{aligned}$$

$$-R \geq -F$$

This inequality is related with the decision at x. Since the decision maker preferred to accept $\bar{t}-1$ customers rather than accepting $\bar{t}$ customers at x, the below inequality is formed:

$$\begin{aligned} f(0,\mathbf{d}) - (\bar{t}-1)R - F(x-\bar{t}+1)^- &\leq f(0,\mathbf{d}) - \bar{t}R - F(x-\bar{t})^- \\ R &\leq F \\ -R &\geq -F \end{aligned}$$

Thus, convexity is verified for this case.

Case 7. We have to verify the inequality:

$$\begin{aligned} f(x+1-\bar{t},\mathbf{d})-\bar{t}R &- f(x-\bar{t},\mathbf{d})+\bar{t}R \geq \\ f(x-\bar{t},\mathbf{d})-\bar{t}R &- f(x+1-\bar{t},\mathbf{d})+(\bar{t}-2)R \end{aligned}$$

By the results of the minimizations:

$$\begin{aligned} a &= f(x+1-\bar{t},\mathbf{d})-(\bar{t}-2)R \\ f(x+1,\mathbf{d}) &\geq f(x+1-\bar{t},\mathbf{d})-\bar{t}R \\ f(x,\mathbf{d})-R &\geq f(x+1-\bar{t},\mathbf{d})-\bar{t}R \\ f(x,\mathbf{d}) &\geq f(x-\bar{t},\mathbf{d})-\bar{t}R \\ f(x+1-\bar{t},\mathbf{d})-(\bar{t}-1)R &\geq f(x-\bar{t},\mathbf{d})-\bar{t}R \qquad \text{(A.14)} \\ f(x-1-\bar{t},\mathbf{d})-\bar{t}R &\geq f(x+1-\bar{t},\mathbf{d})-(\bar{t}-2)R \\ f(x-\bar{t},\mathbf{d})-(\bar{t}-1)R &\geq f(x+1-\bar{t},\mathbf{d})-(\bar{t}-2)R \qquad \text{(A.15)} \end{aligned}$$

By the Equations A.14 and A.15, convexity is verified for this case.

$$\begin{aligned} f(0,\mathbf{d})-\bar{t}R-F(x+1-\bar{t})^- &- f(0,\mathbf{d})+\bar{t}R+F(x-\bar{t})^- \geq \\ f(0,\mathbf{d})-\bar{t}R-F(x-\bar{t})^- &- f(0,\mathbf{d})+(\bar{t}-2)R+F(x+1-\bar{t})^- \end{aligned}$$

$$R \geq F$$

According to the decisions at x and $x-1$, it is obvious that $F=R$ for this case. Therefore, convexity is verified for this case.

Case 8. We have to verify the inequality:

$$\begin{aligned} f(x+1-\bar{t},\mathbf{d})-\bar{t}R &- f(x-\bar{t},\mathbf{d})+\bar{t}R \geq \\ f(x-\bar{t},\mathbf{d})-\bar{t}R &- f(x-\bar{t},\mathbf{d})+(\bar{t}-1)R \\ f(x+1-\bar{t},\mathbf{d})-(\bar{t}-1)R &\geq f(x-\bar{t},\mathbf{d})-\bar{t}R \end{aligned}$$

By the results of the minimizations:

$$\begin{aligned}
a &= f(x+1-\bar{t},\mathbf{d}) - (\bar{t}-2)R \\
f(x+1,\mathbf{d}) &\geq f(x+1-\bar{t},\mathbf{d}) - \bar{t}R \\
f(x,\mathbf{d}) - R &\geq f(x+1-\bar{t},\mathbf{d}) - \bar{t}R \\
f(x,\mathbf{d}) &\geq f(x-\bar{t},\mathbf{d}) - \bar{t}R \\
f(x+1-\bar{t},\mathbf{d}) - (\bar{t}-1)R &\geq f(x-\bar{t},\mathbf{d}) - \bar{t}R \qquad \text{(A.16)} \\
f(x-1-\bar{t},\mathbf{d}) - \bar{t}R &\geq f(x-\bar{t},\mathbf{d}) - (\bar{t}-1)R \\
f(x+1-\bar{t},\mathbf{d}) - (\bar{t}-2)R &\geq f(x-\bar{t},\mathbf{d}) - (\bar{t}-1)R
\end{aligned}$$

By the Equation A.16, convexity is verified for this case. The extreme cases are trivial here.

Case 9. admission amount $= \bar{t} - \bar{t} - \bar{t}$

$$\begin{aligned}
f(x+1-\bar{t},\mathbf{d}) - \bar{t}R - f(x-\bar{t},\mathbf{d}) + \bar{t}R &\geq f(x-\bar{t},\mathbf{d}) - \bar{t}R - f(x-1-\bar{t},\mathbf{d}) + \bar{t}R \\
f(x+1-\bar{t},\mathbf{d}) - f(x-\bar{t},\mathbf{d}) &\geq f(x-\bar{t},\mathbf{d}) - f(x-1-\bar{t},\mathbf{d})
\end{aligned}$$

Since we assume that $f(x,\mathbf{d})$ is convex in x, $\forall\mathbf{d}$, convexity is directly verified for this case for the non-extreme cases. However, for $x-\bar{t}=0$, we have:

$$\begin{aligned}
f(1,\mathbf{d}) - \bar{t}R - f(0,\mathbf{d}) + \bar{t}R &\geq f(0,\mathbf{d}) - \bar{t}R - f(0,\mathbf{d}) + \bar{t}R - F \\
f(1,\mathbf{d}) - f(0,\mathbf{d}) &\geq -F
\end{aligned}$$

So, $f(1,\mathbf{d}) - f(0,\mathbf{d}) \geq -F$ for proving convexity in x. For this case, $f(x+1,\mathbf{d}) - f(x,\mathbf{d}) \geq -F$ should be verified for convexity.

In conclusion, $T_{adm}f(x,\mathbf{d}) = \min_{0\leq t\leq\bar{t}}\{f(x-t,\mathbf{d}) - t\,R_2\}$ preserves convexity in x, $\forall\mathbf{d}$ for every $f(x,\mathbf{d})$ convex in x and satisfying $LBD(F)$, $\forall\mathbf{d}$. □

For $f(x,\mathbf{d})$ satisfying $LBD(F)$, $\forall\mathbf{d}$, T_{adm} preserves $LBD(F)$ property, $\forall\mathbf{d}$:

Proof. We have to verify the inequality:

$$\min_{0\leq t\leq\bar{t}}\{f(x+1-t,\mathbf{d}) - t\ R\} - \min_{0\leq t\leq\bar{t}}\{f(x-t,\mathbf{d}) - t\ R\} \geq -F$$

There are three possible cases according to the results of the minimizations:

Case 1. $(0,0)$ admission amount:

$$f(x+1,\mathbf{d}) - f(x,\mathbf{d}) \geq -F$$

Since $f(x,\mathbf{d})$ satisfies $LBD(F)$, $\forall\mathbf{d}$, T_{adm} preserves $LBD(F)$ property, $\forall\mathbf{d}$.

Case 2. $(t,t-1)$ admission amount:

$$\begin{aligned} f((x+1-t)^+,\mathbf{d}) &- tR - F(x+1-t)^- \\ -f((x-t+1)^+,\mathbf{d}) &+ (t-1)R + F(x-t+1)^- \geq -F \\ -R &\geq -F \end{aligned}$$

This result is related with the decision at x. Although it is possible to accept t customers at x, the decision maker chooses to accept $t-1$ customers. Thus, below inequality is formed:

$$f((x-t+1)^+,\mathbf{d}) - (t-1)R - F(x-t+1) \leq f((x-t)^+,\mathbf{d}) - tR - F(x-t)$$

For $x-t+1 \leq 0$:

$$\begin{aligned} f(0,\mathbf{d}) - (t-1)R - F(x-t+1) &\leq f(0,\mathbf{d}) - tR - F(x-t) \\ -F &\leq -R \end{aligned}$$

Thus, $LBD(F)$ property, $\forall\mathbf{d}$, is verified for this case.

Case 3. $(\bar{t},\bar{t})$ admission amount:

$$\begin{aligned} f(x+1-\bar{t},\mathbf{d}) - \bar{t}R - f(x-\bar{t},\mathbf{d}) + \bar{t}R &\geq -F \\ f(x+1-\bar{t},\mathbf{d}) - f(x-\bar{t},\mathbf{d}) &\geq -F \end{aligned}$$

Since $f(x,\mathbf{d})$ satisfies $LBD(F)$, $\forall\mathbf{d}$, T_{adm} preserves $LBD(F)$ property, $\forall\mathbf{d}$. □

A.9 First Class Order Arrival Operator

In this section, we investigate the structure of the operator for the first class order arrivals of any amount m:

$$T_{fco}f\,(x,\mathbf{d}) = f\,(x,\mathbf{d}+\mathbf{m}_H) - m\,R_1$$

For $f(x,\mathbf{d})$ convex in x, $\forall\mathbf{d}$, T_{fco} preserves convexity in x, $\forall\mathbf{d}$:

Proof. Let $f(x,\mathbf{d})$ be convex in x, $\forall\,\mathbf{d}$, i.e. $\Delta f\,(x+1,\mathbf{d}) \geq \Delta f(x,\mathbf{d})$. We are going to show convexity of $T_{fco}f(x,\mathbf{d})$:

$$\begin{aligned}
\Delta T_{fco}f\,(x+1,\mathbf{d}) &\geq \Delta T_{fco}f(x,\mathbf{d})\\
T_{fco}f\,(x+1,\mathbf{d}) - T_{fco}f(x,\mathbf{d}) &\geq T_{fco}f(x,\mathbf{d}) - T_{fco}f\,(x-1,\mathbf{d})
\end{aligned}$$

$$\begin{aligned}
f\,(x+1,\mathbf{d}+\mathbf{m}_H) - mR_1 &- f\,(x,d+\mathbf{m}_H) + mR_1 \geq\\
f\,(x,\mathbf{d}+\mathbf{m}_H) - mR_1 &- f\,(x-1,\mathbf{d}+\mathbf{m}_H) + mR_1
\end{aligned}$$

$$f\,(x+1,\mathbf{d}+\mathbf{m}_H) - f\,(x,\mathbf{d}+\mathbf{m}_H) \geq f\,(x,\mathbf{d}+\mathbf{m}_H) - f\,(x-1,\mathbf{d}+\mathbf{m}_H)$$

Since $f(x,\mathbf{d})$ is convex in x, $\forall\,\mathbf{d}$, $T_{fco}f\,(x,\mathbf{d})$ is proven to be convex in x, $\forall\,\mathbf{d}$. □

For $f(x,\mathbf{d})$ satisfying $LBD(F)$, $\forall\,\mathbf{d}$, T_{fco} preserves $LBD(F)$ property, $\forall\,\mathbf{d}$:

Proof. We have to verify the inequality:

$$\begin{aligned}
\Delta T_{fco}f\,(x+1,\mathbf{d}) &\geq -F\\
T_{fco}f\,(x+1,\mathbf{d}) - T_{fco}f(x,\mathbf{d}) &\geq -F\\
f\,(x+1,\mathbf{d}+\mathbf{m}_H) - mR_1 - f\,(x,d+\mathbf{m}_H) + mR_1 &\geq -F\\
f\,(x+1,\mathbf{d}+\mathbf{m}_H) - f\,(x,d+\mathbf{m}_H) &\geq -F
\end{aligned}$$

Since $f(x,\mathbf{d})$ satisfies $LBD(F)$, $\forall\,\mathbf{d}$, T_{fco} preserves $LBD(F)$ property, $\forall\,\mathbf{d}$. □

VITA

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