THE SCHRÖDINGER-NEWTON SYSTEM

by

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This is to certify that I have examined this copy of a master's thesis by

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and have found that it is complete and satisfactory in all respects,
and that any and all revisions required by the final
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ABSTRACT

In this thesis, we study the Schrödinger-Newton system. A considerable amount is known about the stationary solutions of the Schrödinger-Newton system, [2], [9], [12], [13], [15], [20], [22], [23]. We give a summary of the work carried out in these papers.

We mainly study the initial boundary value problem for Dissipative Nonlinear Schrödinger-Newton Equation on a bounded domain in $\mathbb{R}^3$. We prove the existence of weak solution and we show that solutions go to zero as t tends to infinity. Also, we prove the existence and uniqueness of the strong solution.

ÖZETÇE

Bu tezde, Schrödinger-Newton Sistemini inceliyoruz. Sistemin durağan çözümlerini inceleyen pek çok çalışma yapılmıştır, [2], [9], [12], [13], [15], [20], [22], [23]. Bu makalelerde yapılan çalışmaların özetini veriyoruz.

Esas olarak Disipatif Nonlineer Schrödinger-Newton Sistemini $\mathbb{R}^3$te sınırlı bölgede inceliyoruz. Zayıf çözümün varlığını ve t sonsuza yakınsadığı zaman bu çözümün sıfıra yakınsadığını ispatlıyoruz. Son olarak sistemin kuvvetli çözümünün varlığını ve tekliğini ispatlıyoruz.

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NOTATION

(1) $\mathbb{R}^n := n-$ dimensional real Euclidean space, $\mathbb{R}^1 := \mathbb{R}$.

(2) Let U and V be open subsets of $\mathbb{R}^n$. We write

$$V \subset\subset U$$

if $V \subset \overline{V} \subset U$ and $\overline{V}$ is compact, and say V is compactly contained in U.

(3) $\partial\Omega =$ boundary of Ω.

(4) $\overline{\Omega} = \Omega \cup \partial\Omega =$ closure of Ω.

(5) $\Omega_T = \Omega \times (0, T]$

(6) $\Gamma_T = \overline{\Omega_T} - \Omega_T =$ parabolic boundary of U_T.

(7) $B^0(x, r) = \{y \in \mathbb{R}^n : |x - y| < r\} =$ open ball in $\mathbb{R}^n$ with center x and radius $r > 0$.

(8) $B(x, r) =$ closed ball with center x, radius $r > 0$.

(9) $\alpha(n) =$ volume of unit ball $B(0, 1)$ in $\mathbb{R}^n = \frac{\pi^{n/2}}{\Gamma(n/2)+1}$.

(10) $n\alpha(n) =$ surface of unit sphere $\partial B(0, 1)$ in $\mathbb{R}^n$.

(11) $\mathbb{C}^n = n-$dimensional complex space.

(12) $\mathbb{C} =$ complex plane.

(13) If $z \in \mathbb{C}$, we write $\Re(z)$ for the real part of z, and $\Im(z)$ for the imaginary part.

(14) Assume $u : U \to \mathbb{R},\ x \in U$.

$\frac{\partial u}{\partial x_i}(x) = \lim_{h\to 0} \frac{u(x+he_i)-u(x)}{h}$, provided this limit exists.

(15) We usually write u_{x_i} for $\frac{\partial u}{\partial x_i}$.

(16) Similarly

$$\frac{\partial^2 u}{\partial x_i \partial x_j} = u_{x_i x_j},$$
$$\frac{\partial^3 u}{\partial x_i \partial x_j \partial x_k} = u_{x_i x_j x_k}, \text{etc.}$$

(17) $\Delta u = \sum_{i=1}^{n} u_{x_i x_i}$.

(18) $C(\Omega) = \{u : \Omega \to \mathbb{R} : u \text{ continuous}\}$.

(19) $C(\overline{\Omega}) = \{u \in C(\Omega) : u \text{ is uniformly continuous on bounded subsets of } \ \Omega\}$.

(20) $C^k(\Omega) = \{u : \Omega \to \mathbb{R} : u \text{ is } k-\text{times continuously differentiable}\}$.

(21) $C_c(\Omega), C_c^k(\Omega)$, etc. denote these functions in $C(\Omega), C^k(\Omega)$, etc. with compact support.

(22) $L^p(\Omega) = \{u : \Omega \to \mathbb{R} \quad | \quad u \quad \text{is Lebesgue measurable}, \quad \|u\|_{L^p(\Omega)} < \infty\}$
where $\|u\|_{L^p(\Omega)} = \left(\int_\Omega |u|^p dx\right)^{1/p} \quad (1 \leqslant p < \infty)$.

(23) $L^\infty(\Omega) = \{u : \Omega \to \mathbb{R} \quad | \quad u \quad \text{is Lebesgue measurable}, \quad \|u\|_{L^\infty(\Omega)} < \infty\}$, where $\|u\|_{L^\infty(\Omega)} = esssup_\Omega |u|$.

(24) $\|Du\|_{L^p(\Omega)} = \||Du|\|_{L^p(\Omega)}$.

(25) $D^{1,2}(\mathbb{R}^3) = \{u \in L^6(\mathbb{R}^3) : |\nabla u| \in L^2(\mathbb{R}^3)\}$

$H_r^1(\mathbb{R}^3)$: the Sobolev space of radial functions u such that $u, \nabla u$ are in $L^2(\mathbb{R}^3)$.

(26) $D_r^{1,2}$ the subspace of radial functions corresponding to $D^{1,2}$.

(27) $(u, v) = \int_\Omega u(x)\overline{v(x)}dx$

(28) $\|\cdot\| := \|\cdot\|_{L^2(\Omega)}$.

(29) $\|\cdot\|_{L^p} := \|\cdot\|_{L^p(\Omega)}$.

(30) $\|\psi(t)\| := \|\psi(t, \cdot)\|$.

Chapter 1

INTRODUCTION

The Schrödinger-Newton System is the nonlinear system obtained by coupling of the linear Schrödinger equation of quantum mechanics with the Poisson equation from Newtonian mechanics. For a single particle of mass m the system consists of the following pair of partial differential equations:

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + mu\psi, \tag{1.1}$$

$$\Delta u = 4\pi Gm|\psi|^2, \tag{1.2}$$

where $2\pi\hbar$ is Planck's constant, ψ is the wave function, u is the potential, G is the gravitational constant and t is time.

The Schrödinger-Newton System was proposed by Penrose [17] for a theory of quantum state reduction. This problem is generally referred to as the measurement problem.

Under the following transformation

$$\psi(t,x) = \psi(x)e^{-i\frac{E}{\hbar}t},$$

$$u(t,x) = u(x)$$

the time-dependent Schrödinger-Newton System reduces to the time independent Schrödinger-Newton System:

$$\frac{-\hbar^2}{2m}\Delta\psi + mu\psi = E\psi, \tag{1.3}$$

$$\Delta u = 4\pi G m \mid \psi \mid^2 \tag{1.4}$$

where E is the energy eigenvalue.

The layout of the thesis is as follows. In the first introductory chapter we introduce the Schrödinger-Newton system and give a brief overview of the thesis.

In the second chapter, we give some preliminary facts that are used in the subsequent chapters.

Chapter 3 provides literature review on the time-independent Schrödinger-Newton system. We study stationary solutions in the case of spherical symmetry in Section 3.1. In Section 3.2, we give some results obtained on existence of stationary solutions and on the ground state energy of stationary solutions. In the remainder of this chapter we study stationary solutions of the Schrödinger-Newton system under the effect of the nonlinear term $|\psi|^{p-1}\psi$.

In Chapter 4, we study the initial boundary value problem for the Dissipative Nonlinear Schrödinger-Newton System on a bounded domain in $\mathbb{R}^3$. We prove the existence of a weak solution and we show that solution goes to zero as $t \to \infty$. In the Section 4.2 we prove the existence of the strong solution and in Section 4.3 we prove the uniqueness of the strong solution.

Chapter 2

PRELIMINARIES

2.1 Banach and Hilbert Spaces

Definition 2.1.1 *Suppose that* $(X, \|\cdot\|_X)$ *and* $(Y, \|\cdot\|_Y)$ *are normed spaces. We say that a function* $g : X \to Y$ *is Hölder continuous with exponent* $0 < \gamma \leqslant 1$ *if there exists a constant* C *such that*

$$\|g(x) - g(y)\|_Y \leqslant C\|x - y\|_X^\gamma, \quad x, y \in X$$

If $\gamma = 1$ *we say that* g *is a Lipschitz function, with Lipschitz constant* C.

Definition 2.1.2 *Let* A *be a subset of a vector space* E*. The subspace* H_A *is the subspace of* E *spanned by the subset* $A \subseteq E$ *(or the linear hull of* A*) and is denoted by* $Span(A)$.

2.2 Convergence Theorems

Definition 2.2.1 (Pointwise convergence) *A sequence* $\{f_n\}$ *of functions defined on a set* E *is said to converge pointwise on* E *to a function* f *if for every* x *in* E *we have*

$$f(x) = \lim f_n(x);$$

that is, if, given $x \in E$ *and* $\varepsilon > 0$*, there is an* N *such that for all* $n \geqslant N$*, we have* $|f(x) - f_n(x)| < \varepsilon$.

Definition 2.2.2 (Uniform convergence) *A sequence* $\{f_n\}$ *of functions defined on a set* E *is said to converge uniformly on* E *to a function* f *if given* $\varepsilon > 0$*, there* *is an N which depends only on ε such that for all $x \in E$ and all $n \geqslant N$, we have $|f(x) - f_n(x)| < \varepsilon$.*

Corollary $L^p(\Omega) \subset L^1_{loc}(\Omega)$ for $1 \leqslant p \leqslant \infty$ and any domain Ω.

In what follows X and Y are Banach spaces.

Definition 2.2.3 (Weak convergence) *We say a sequence $\{u_k\}_{k=1}^{\infty} \subset X$ converges weakly to $u \in X$, written*

$$u_k \rightharpoonup u$$

if

$$< u^*, u_k > \rightarrow < u^*, u >$$

for each bounded linear functional $u^* \in X$.

Definition 2.2.4 (Compact operator) *An operator $K : X \to Y$ is called compact if the image of any set W, which is bounded in X, has compact closure in Y:*

$\overline{K(W)}$ *is compact in Y for all bounded $W \subset X$.*

Definition 2.2.5 (Linear symmetric operator) *Let H be a Hilbert space. A linear operator $A \in \mathcal{L}(H, H)$ is symmetric if*

$$(u, Av) = (Au, v) \quad \text{for all} \quad u, v \in H.$$

Theorem 2.2.6 (Hilbert-Schmidt Theorem) *Let A be a linear, symmetric, compact operator acting on an infinite-dimensional Hilbert space H. Then all eigenvalues λ_j of A are real, and if they are ordered so that*

$$|\lambda_{n+1}| \leqslant |\lambda_n|$$

one has

$$\lim_{n\to\infty} \lambda_n = 0.$$

Furthermore, the eigenvalues ω_j can be chosen so that they from an orthonormal basis for $R(A)$, and the action of A on any $u \in H$ is given by

$$Au = \sum_{j=1}^{\infty} \lambda_j (u, \omega_j)\omega_j.$$

Definition 2.2.7 (Weak-* convergence) *A sequence $f_n \in X^*$ converges weakly-* to f, if $f_n(x) \to f(x)$ for every $x \in X$.*

Theorem 2.2.8 (Lebesgue Dominated Convergence Theorem) *Let g be integrable over E and let $\{f_n\}$ be a sequence of measurable functions such that $|f_n| \leqslant g$ on E and for almost all x in E we have $f(x) = \lim f_n(x)$. Then*

$$\int_E f = \lim \int_E f_n.$$

Theorem 2.2.9 (Alaoglu weak-* compactness) *Let X be a separable Banach space and let f_n be a bounded sequence in X^*. Then f_n has a weakly-* convergent subsequence.*

Corollary 2.2.10 (Reflexive weak compactness) *Let X be a reflexive Banach space and x_n a bounded sequence in X. Then x_n has a subsequence that converges weakly in X.*

Theorem 2.2.11 *([27]) Let B_0, B, B_1 be three Banach spaces where B_0, B_1 are reflexive. Suppose that B_0 is continuously imbedded into B, which is also continuously imbedded into B_1, and imbedding from B_0 into B_1 is compact. For any given p_0, p_1 with $1 < p_0, p_1 < \infty$, let*

$$W = \{v : v \in L^{p_0}(0, T : B_0), v_t \in L^{p_1}(0, T, B_1)\}$$

Then the imbedding from W into $L^{p_0}(0, T, B)$ is compact.

2.3 Elementary Inequalities

Theorem 2.3.1 (Young's inequality with ε.)

$$ab \leqslant \varepsilon a^p + C(\varepsilon) b^q \quad (a, b > 0, \quad \varepsilon > 0)$$

for $C(\varepsilon) = (\varepsilon p)^{\frac{-q}{p}} q^{-1}$.

Theorem 2.3.2 (Minkowski's inequality) *Assume* $1 \leqslant p \leqslant \infty$ *and* $u, v \in L^p(\Omega)$. *Then*

$$\|u+v\|_{L^p(\Omega)} \leqslant \|u\|_{L^p(\Omega)} + \|v\|_{L^q(\Omega)}$$

Theorem 2.3.3 (Generalized Hölder inequality.) *Let* $1 \leqslant p_1, \ldots, p_m \leqslant \infty$, *with* $\frac{1}{p_1} + \ldots + \frac{1}{p_m} = 1$, *and assume that* $u_k \in L^{p_k}(\Omega)$ *for* $k = 1, \ldots, m$. *Then*

$$\int_\Omega |u_1 \cdots u_m| dx \leqslant \prod_{k=1}^m \|u_i\|_{L^{p_k}(\Omega)}$$

Theorem 2.3.4 (Gronwall's inequality) *Let* K *be a nonnegative constant and let* f *and* g *be continuous nonnegative functions on some interval* $\alpha \leqslant t \leqslant \beta$ *satisfying the inequality*

$$f(t) \leqslant K + \int_\alpha^t f(s)g(s)ds$$

for $\alpha \leqslant t \leqslant \beta$. *Then*

$$f(t) \leqslant K \exp\left(\int_\alpha^t g(s)ds\right)$$

for $\alpha \leqslant t \leqslant \beta$.

Theorem 2.3.5 (An Interpolation Inequality) *Let* $1 \leqslant p < q < r$, *so that*

$$\frac{1}{q} = \frac{\theta}{p} + \frac{1-\theta}{r}$$

for some θ *satisfying* $0 < \theta < 1$. *If* $u \in L^p(\Omega) \cap L^r(\Omega)$, *then* $u \in L^q(\Omega)$ *and*

$$\|u\|_q \leqslant \|u\|_p^\theta \|u\|_r^{1-\theta}.$$

Theorem 2.3.6 (G. H. Hardy) *If*

$$a > 0,\ f(x) \geqslant 0,\ p > 1$$

and

$$\int_a^\infty f^p(x)dx \quad \textit{is convergent,}$$

then

$$\int_a^\infty \left(\frac{1}{x}\int_0^x f(t)dt\right)^p dx \leqslant \left(\frac{p}{p-1}\right) \int_a^\infty f^p(x)dx.$$

Theorem 2.3.7 (G. H. Hardy) *If*

$$a_n \geqslant 0, \quad p > 1$$

and

$$\sum_{k=1}^{n} a_k \quad \text{is convergent,}$$

then

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n} a_k\right)^p \leqslant \left(\frac{p}{p-1}\right)^p \sum_{n=1}^{\infty} a_n^p.$$

2.4 Convex functions.

Definition 2.4.1 *A function $f : \mathbb{R}^n \to \mathbb{R}$ is called convex provided*

$$f(\tau x + (1-\tau)y) \leqslant \tau f(x) + (1-\tau)f(y)$$

for all $x, y \in \mathbb{R}^n$ and each $0 \leqslant \tau \leqslant 1$.

Definition 2.4.2 *A function $f : \mathbb{R}^n \to \mathbb{R}$ is called strictly convex if and only if $\leqslant$ in the previous definition is replaced with $<$.*

Definition 2.4.3 (Gateaux derivative) *Let V, W be Banach spaces, $U \subset V$ open. The Gateaux derivative or directional derivative of a map $F : U \subset V \to W$ at the point $x \in U$ in direction $h \in V$ is defined by*

$$DF(x,h) = \lim_{t\to 0} \frac{1}{t}[F(x+th) - F(x)].$$

F is called Gateaux differentiable at $x \in U$ if $DF(x,h)$ exists for all $h \in V$.

Definition 2.4.4 (Gateaux-Levi derivative) *Let V, W be Banach spaces, $U \subset V$ open. A map $F : U \to W$ is called Gateaux-Levi differentiable at the point $x \in U$ if it is Gateaux differentiable at x and the map $h \in V \to DF(x,h)$ from V to W is linear* *and bounded. If F is Gateaux-Levi differentiable, we can set $DF(x)h = DF(x,h)$ and get*

$$F(x+h) = F(x) + DF(x)h + RF(x,h),$$

where

$$\lim_{t\to 0} \frac{RF(x,th)}{t} = 0 \text{ for each } h \in V.$$

Definition 2.4.5 (Frechet derivative) *Let V, W be Banach spaces, $U \subset V$ open. A map $F : U \to W$ is called Frechet differentiable at the point $x \in U$ if it can be approximated by a linear map in the form*

$$F(x+h) = F(x) + DF(x)h + RF(x,h),$$

where $DF(x)$, called the Frechet derivative of F at x,is a bounded (continuous) linear map from V to W, i.e.,$DF(x) \in L(V,W)$ and the remainder $RF(x,h)$ satisfies

$$\lim_{h\to 0} = \frac{RF(x,h)}{\|h\|} = 0.$$

If the map $DF : U \to L(V,W)$ is continuous in x, then F is called continuously differentiable or of class C^1. If F is Frechet differentiable at x then F is Gateaux-Levi differentiable at x and $DF(x)h = DF(x,h)$. If F is Gateaux-Levi differentiable at each point $x \in U$ and the map $DF : U \to L(V,W)$ is continuous, then F is Frechet differentiable at each x. In finite dimensions, i.e., $V = \mathbb{R}^n, W = \mathbb{R}^m$, this means $F : \mathbb{R}^n \to \mathbb{R}^m$ is Frechet differentiable if all partial derivatives of F exist and are continuous.

Notation H denotes a real Hilbert space, with inner product $(,)$.

Theorem 2.4.6 (Riesz Representation Theorem) *H^* can be canonically identified with H; more precisely, for each $u^* \in H^*$ there exists a unique element in H such that*

$$< u^*, v >= (u,v) \quad \text{for all} \quad v \in H.$$

The mapping $u^* \to u$ is a linear isomorphism of H^* onto H.

2.5 Sobolev Spaces

Let Ω be a nonempty open set in $\mathbb{R}^n$.

Definition 2.5.1 (test function) *A function f defined on Ω is called a test function if $f \in C^\infty(\Omega)$ and there is a compact set $K \subset \Omega$ such that the support of f lies in K. The set of all test functions on U is denoted by $C_c^\infty(\Omega)$.*

Definition 2.5.2 *Let $\Omega \subset \mathbb{R}^n$ be open and bounded, $k \in \{1, 2, \ldots\}$. We say $\partial\Omega$ is C^k if for each point $x_0 \in \partial\Omega$ there exist $r > 0$ and a C^k function $\gamma : \mathbb{R}^{n-1} \to \mathbb{R}$ such that - upon relabeling and reorienting the coordinate axes if necessary- we have*

$$\Omega \cap B(x_0, r) = \{x \in B(x_0, r) \mid x_n > \gamma(x_1, \ldots, x_{n-1})\}.$$

Likewise, $\partial\Omega$ is C^∞ if $\partial\Omega$ is C^k for $k = 1, 2, \ldots$.

Definition 2.5.3 (weak derivative) *Suppose $u, v \in L^1_{loc}(\Omega)$, and α is a multiindex. We say that v is the $\alpha^{th}-$weak partial derivative of u, written*

$$D^\alpha u = v,$$

provided

$$\int_\Omega u D^\alpha \phi dx = (-1)^\alpha \int_\Omega v \phi dx$$

for all test functions $\phi \in C_c^\infty(\Omega)$.

Proposition 2.5.4 *Weak limits are unique, and weakly convergent sequences are bounded.*

Definition 2.5.5 (Sobolev space) *Let $1 \leqslant p \leqslant \infty$ and let k be a nonnegative integer.*

The Sobolev space

$$W^{k,p}(\Omega)$$

consists of all locally summable functions $u : \Omega \to \mathbb{R}$ such that for each multiindex α with $|\alpha| \leqslant k$, $D^\alpha u$ exists in the weak sense and belongs to $L^p(\Omega)$. $W^{k,p}$ is a normed space equipped with the norm

$$\|u\|_{W^{k,p}(\Omega)} := \left(\sum_{|\alpha| \leqslant k} \int_\Omega |D^\alpha u|^p dx \right)^{1/p}, \quad \text{if} \quad 1 \leqslant p < \infty,$$

$$\|u\|_{W^{k,\infty}(\Omega)} := \sum_{|\alpha| \leqslant k} \text{ess sup}_\Omega |D^\alpha u|.$$

Remarks

(i) Among the spaces $W^{1,p}$, particular importance attaches to $W^{1,2}$, because it is a Hilbert-space, i.e., its norm comes from an inner product.

(ii) We usually write

$$H^k(\Omega) = W^{k,2}(\Omega) \quad (k = 0, 1, \ldots).$$

Definition 2.5.6

$$W_0^{k,p}(\Omega)$$

denotes the closure of $C_c^\infty(\Omega)$ in $W^{k,p}(\Omega)$.

Theorem 2.5.7 *For each $k = 1, \ldots$ and $1 \leqslant p \leqslant \infty$, the Sobolev space $W^{k,p}(\Omega)$ is a Banach space.*

Definition 2.5.8 *Let X and Y be Banach spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively. We say that X is continuously embedded into Y if there exists an injective linear map $i : X \to Y$ and a constant C such that*

$$\|i(x)\|_Y \leqslant C\|x\|_X$$

for all $x \in X$. We identify X with the image $i(X)$.

Definition 2.5.9 *We say that X is compactly embedded into Y if i is a compact map, that is, i maps bounded subsets of X into relatively compact subsets of Y.*

Definition 2.5.10 (Sobolev conjugate) *If $1 \leqslant p < n$, the Sobolev conjugate of p is*

$$p^* := \frac{np}{n-p}$$

Note that

$$\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}, \quad p^* > p.$$

Theorem 2.5.11 (Poincare - Friedrichs inequality) *If $\Omega \subset \mathbb{R}^n$ is a bounded domain, then there exists a constant C_Ω which depends only on $|\Omega|$ such that*

$$\int_\Omega u^2(x)dx \leqslant \lambda_1^{-1} \int_\Omega |Du(x)|^2 dx, \quad \forall u \in H_0^1(\Omega),$$

where λ_1 is the first eigenvalue of the operator $-\Delta$ under the homogeneous Dirichlet boundary condition.

Theorem 2.5.12 (Poincare inequality) *If $\Omega \subset \mathbb{R}^n$ is a bounded domain with C^1 boundary, then there exists a constant C_Ω depending only on $|\Omega|$ such that*

$$\int_\Omega u^2(x)dx \leqslant C_\Omega \left[\int (u(x)dx)^2 + \int_\Omega |Du(x)|^2 dx \right], \quad \forall u \in H^1(\Omega).$$

Theorem 2.5.13 (Sobolev inequality) *Assume $1 \leqslant p < n$. There exists a constant C, depending only on p and n, such that*

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leqslant C \|Du\|_{L^p(\mathbb{R}^n)},$$

for all $u \in C_c^1(\mathbb{R}^n)$, where p^* is the Sobolev conjugate of p.

Theorem 2.5.14 (O. A. Ladyzhenskaya) *For each $u \in H_0^1(\Omega)$, $\Omega \subset \mathbb{R}^2$ the following inequality holds true*

$$\|u(x)\|_{L^4(\Omega)} \leqslant 2^{1/4} \|u\|_{L^2(\Omega)}^{1/2} \|u_x\|_{L^2(\Omega)}^{1/2}.$$

Theorem 2.5.15 (O. A. Ladyzhenskaya) *For each $u \in H_0^1(\Omega)$, $\Omega \subset \mathbb{R}^3$ the following inequality holds true*

$$\|u\|_{L^4(\Omega)} \leqslant \sqrt{2}\|u\|_{L^2(\Omega)}^{1/4}\|\nabla u\|_{L^2(\Omega)}^{3/4}.$$

Theorem 2.5.16 (Gagliardo-Nirenberg inequality) *Let Ω be a bounded domain with smooth boundary. Let u be any function in $W^{m,r}(\Omega) \cap L^q(\Omega)$, $1 \leqslant r,\ q \leqslant \infty$. For any integer j, $0 \leqslant j \leqslant m$ and for any number $\hat{a}$ in the interval $\frac{j}{m} \leqslant \hat{a} \leqslant 1$, set*

$$\frac{1}{p} - \frac{j}{n} = \hat{a}\left(\frac{1}{r} - \frac{m}{n}\right) + (1-\hat{a})\frac{1}{q}.$$

If $m - j - \frac{n}{r}$ is a nonnegative integer, then

$$\|D^j u\|_{L^p(\Omega)} \leqslant C\|u\|_{W^{m,r}(\Omega)}^{\hat{a}}\|u\|_{L^q(\Omega)}^{1-\hat{a}}$$

where the constant C depends only on Ω, r, q, m, j and $\hat{a}$.

2.6 Negative Sobolev Spaces and Duality

Notation. We will write $<,>$ to denote the pairing between $H^{-1}(\Omega)$ and H_0^1.

Definition 2.6.1 *If $f \in H^{-1}(\Omega)$, we define the norm*

$$\| f \|_{H^{-1}(\Omega)} := \sup\{< f, u > | u \in H_0^1, \| u \|_{H_0^1(\Omega)} \leqslant 1\}.$$

Theorem 2.6.2 (Characterization of H^{-1})

(i) Assume $f \in H^{-1}(\Omega)$. Then there exist functions $f^0, f^1, \ldots, f^n$ in $L^2(\Omega)$ such that

$$(1) \qquad < f, v >= \int_\Omega f^0 v + \sum_{i=1}^n f_i v_{x_i} dx \quad (v \in H_0^1(\Omega)).$$

(ii) Furthermore,

$$\| f \|_{H^{-1}(\Omega)} = \inf \left\{ \left(\int_\Omega \sum_{i=0}^{n} |f^i|^2 dx \right)^{1/2} | f \quad satisfies \ (1) \ for \ f^0, \ldots, f^n \in L^2(\Omega) \right\}.$$

2.7 Imbedding Theorems

Definition 2.7.1 *We say that $\Omega \subset \mathbb{R}^m$ is a bounded domain of class C^k or a bounded C^k domain provided that at each point $x_0 \in \partial\Omega$ there is an $\varepsilon > 0$ and a C^k- diffeomorphism Φ of $B(x_0, \varepsilon)$ onto a subset $\widetilde{B}$ of $\mathbb{R}^m$ such that*

(i) $\Phi(x_0) = 0$,

(ii) $\Phi(B \cap \Omega) \subset \mathbb{R}^m_+$, *and*

(iii) $\Phi(B \cap \partial\Omega) \subset \partial\mathbb{R}^m_+$.

Theorem 2.7.2 (An Imbedding Theorem for L^p Spaces) *Suppose that*

$$vol(\Omega) = \int_\Omega 1 dx < \infty \qquad and \quad 1 \leqslant p \leqslant q \leqslant \infty.$$

If $u \in L^q(\Omega)$, then $u \in L^p(\Omega)$ and

$$\|u\|_p \leqslant (vol(\Omega))^{(1/p)-(1/q)} \|u\|_q.$$

Hence

$$L^q(\Omega) \subset L^p(\Omega).$$

If $u \in L^\infty(\Omega)$, then

$$\lim_{p \to \infty} \|u\|_p = \|u\|_\infty.$$

Finally, if $u \in L^p(\Omega)$ for $1 \leqslant p < \infty$ and if there exists a constant K such that for all such p

$$\|u\|_p \leqslant K,$$

then $u \in L^\infty(\Omega)$ *and*

$$\|u\|_\infty \leqslant K.$$

Theorem 2.7.3 *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded domain with sufficiently smooth boundary. Then there exists a constant* C_Ω *such that for each* $u \in H^2(\Omega) \cap H_0^1(\Omega)$ *the following inequality holds true*

$$\|u\|_{L^2(\Omega)} \leqslant C_\Omega \|\Delta u\|_{L^2(\Omega)}$$

Proposition 2.7.4 *If* $u \in H^1(a,b)$*, with* a, b *finite, then*

$$\max_{x \in [a,b]} |u(x)| \leqslant C\|u\|_{H^1(a,b)}$$

and in fact $u \in C^0([a,b])$*.*

Proposition 2.7.5 *If* $u \in C_0^1(\mathbb{R}^n)$*,* $n > 1$*, then*

$$\|u\|_{L^{n/(n-1)}} \leqslant \|Du\|_{L^1}.$$

Theorem 2.7.6 *If* $k < \frac{n}{2}$ *and* $u \in H^k(\mathbb{R}^n)$*, then*

$$u \in L^{\frac{2n}{n-2k}}(\mathbb{R}^n),$$

with

$$\|u\|_{L^{\frac{2n}{n-2k}}} \leqslant C(k,n)\|u\|_{H^k}. \tag{2.1}$$

Theorem 2.7.7 *If* Ω *is a domain in* $\mathbb{R}^n$*, then* $H_0^{1,p}(\Omega) \subset L^q(\Omega)$ *is a continuous embedding provided* $p < n$ *and* $p \leqslant q \leqslant \frac{np}{n-p}$*; in particular, the inequality*

$$\|u\|_q \leqslant C\|u\|_{1,p} \qquad C = C(n,p,q)$$

holds for all $u \in C_0^1(\Omega)$*, and by completion for all* $u \in H_0^{1,p}(\Omega)$

Theorem 2.7.8 *Let $\Omega \subset \mathbb{R}^n$ be a bounded C^k domain. If $u \in H^k(\Omega)$ and $k < \frac{n}{2}$, then*

$$u \in L^{\frac{2n}{n-2k}(\Omega)}(\Omega),$$

with

$$\|u\|_{L^{\frac{2n}{n-2k}}(\Omega)} \leqslant C\|u\|_{H^k(\Omega)}$$

If $k = \frac{n}{2}$ then $u \in L^p(\Omega)$ for each $1 \leqslant p < \infty$, and there exists a constant $C(p)$ such that

$$\|u\|_{L^p} \leqslant C(p)\|u\|_{H^1}$$

Proposition 2.7.9 *Let $p > n > 1$, and let Ω be a bounded subset of $\mathbb{R}^n$. Then if $u \in C^1_c(\Omega)$*

$$\|u\|_\infty \leqslant C\|Du\|_{L^p} \qquad C = C(n, p, \Omega)$$

Theorem 2.7.10 *Let $\Omega \subset \mathbb{R}^n$ be a bounded C^k domain. If $u \in H^k(\Omega)$ with $k > \frac{n}{2}$, then $u \in C^0(\overline{\Omega})$ and there exists a constant $C(n, k)$ such that*

$$\|u\|_\infty \leqslant C\|u\|_{H^k}$$

Corollary 2.7.11 *If $\Omega \subset \mathbb{R}^n$ is a bounded C^k domain and $k > \frac{n}{2} + j$ then $u \in H^k(\Omega)$ is an element of $C^j(\overline{\Omega})$, with*

$$\|u\|_{C^j} \leqslant C\|u\|_{H^k}$$

Theorem 2.7.12 (Sobolev Imbedding Theorem) *Let Ω be a bounded C^k domain in $\mathbb{R}^n$, and suppose that $u \in H^k(\Omega)$.*

(i) If $k < \frac{n}{2}$ then $u \in L^{\frac{2n}{n-2k}}(\Omega)$, and there exists a constant C independent of Ω such that

$$\|u\|_{L^{\frac{2n}{n-2k}}(\Omega)} \leqslant C\|u\|_{H^k(\Omega)}$$

(ii) If $k = \frac{n}{2}$ then $u \in L^p(\Omega)$ for every $1 \leqslant p < \infty$, and for each p there exists a constant $C = C(p, \Omega)$ such that

$$\|u\|_{L^p(\Omega)} \leqslant C\|u\|_{H^k(\Omega)}.$$

(iii) If $k > j + \frac{n}{2}$ *then* $u \in C^j(\overline{\Omega})$*, and there exists a constant* C *such that*

$$\|u(x)\|_{C^j(\overline{\Omega})} \leqslant C_j \|u\|_{H^k(\Omega)}$$

Theorem 2.7.13 (Rellich Compactness Theorem) *Let* Ω *be a bounded* C^1 *domain. Then* $H^1(\Omega)$ *is compactly embedded in* $L^2(\Omega)$*.*

Corollary 2.7.14 *Let* Ω *be bounded and* $\partial\Omega \quad C^{k+1}$*. Then* $H^{k+1}(\Omega)$ *is compactly embedded in* $H^k(\Omega)$*.*

Theorem 2.7.15 (Morrey's Imbedding Theorem for p>n) *If* $p > n$ *and* Ω *is a bounded domain in* $\mathbb{R}^n$*, then*

$$H_0^{1,p}(\Omega) \subset C^{\alpha}(\overline{\Omega})$$

is a continuous imbedding for $\alpha = 1 - (n/p)$*.*

Theorem 2.7.16 *If* Ω *is a bounded domain with* $C^{k+2}-$ *boundary, then the map*

$$\Delta : H^{k+2} \cap H_0^1(\Omega) \to H^k(\Omega)$$

is an isomorphism.

Theorem 2.7.17 *Let* $X \subset\subset H \subset Y$ *be Banach spaces, with* X *reflexive. Suppose that* $\{u_n\}$ *is a sequence that is uniformly bounded in* $L^2(0,T;X)$*, and* $\frac{du_n}{dt}$ *is uniformly bounded in* $L^p(0,T;Y)$*, for some* $p > 1$*. Then there is a subsequence that converges strongly in* $L^2(0,T;H)$*.*

2.8 Regularity

Definition 2.8.1 λ *is said to be an eigenvalue of the operator* $-\Delta$ *on a bounded, open set* Ω *in* $\mathbb{R}^n$*, subject to zero boundary condition, provided that there exists a function* u*, not identically zero, satisfying*

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

The function u is called a corresponding eigenfunction.

Theorem 2.8.2 *(i) Each eigenvalue of $-\Delta$ is real.*

(ii) Furthermore, the operator $-\Delta$ has infinitely many eigenvalues $\{\lambda_k\}_{k=1}^{\infty}$ where

$$0 < \lambda_1 \leqslant \lambda_2 \leqslant \lambda_3 \leqslant \dots,$$

and

$$\lambda_k \to \infty \quad \text{as} \quad k \to \infty.$$

(iii) Finally, there exists an orthonormal basis $\{u_k\}_{k=1}^{\infty}$ of $L^2(\Omega)$, where $u_k \in H_0^1(\Omega)$ is an eigenfunction corresponding to λ_k:

$$\begin{cases} -\Delta u_k = \lambda_k u_k & \text{in } \Omega \\ u_k = 0 & \text{on } \partial\Omega \end{cases}$$

for $k = 1, 2, \dots$.

Remark. The above theorem remains valid if we replace the operator $-\Delta$ by a symmetric elliptic operator L.

Corollary 2.8.3 *The eigenfunctions of the Laplace operator with Dirichlet boundary conditions*

$$\begin{cases} -\Delta u = \lambda u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega \end{cases}$$

are elements of $C^{\infty}(\Omega) \cap H_0^1(\Omega)$.

2.9 Calculus of Variations

Let $I[\cdot]$ be a functional defined on a subset M of a real normed space X.
Hereafter we assume that $I[\cdot]$ has the following explicit form:

$$I[u] := \int_\Omega L(Du(x), u(x), x)dx,$$

where

$$L = L(p, z, x) = L(p_1, \ldots, p_n, z, x_1, \ldots, x_n)$$

for $p \in \mathbb{R}^n, z \in \mathbb{R}$ and $x \in \Omega$.

Definition 2.9.1 (Uniform convexity) *L is called uniformly convex, if there exists a constant $\theta > 0$ such that*

$$\sum_{i,j=1}^{n} L_{p_i p_j}(p)\xi_i\xi_j \geqslant \theta|\xi|^2$$

for $p, \xi \in \mathbb{R}^n$.

Remark Uniform convexity implies strict convexity, and strict convexity implies convexity.

Definition 2.9.2 (Coercivity) *I is called (weakly) coercive if and only if*

$$I(u) \to \infty \quad as \quad \|u\| \to \infty$$

on M.

Definition 2.9.3 *We say that the function $I[\cdot]$ is (sequentially) weakly lower semicontinuous on $W^{1,q}(\Omega)$, provided*

$$I[u] \leqslant \liminf_{k\to\infty} I[u_k]$$

whenever

$$u_k \rightharpoonup u \quad weakly\ in \quad W^{1,q}(\Omega)$$

Theorem 2.9.4 *Suppose that $f : \mathcal{H} \to \mathbb{R}$, is coercive, strongly lower semicontinuous, convex function a Hilbert space $\mathcal{H}$. Then f is bounded from below and attains its minimum.*

Chapter 3

THE TIME-INDEPENDENT SCHRÖDINGER-NEWTON SYSTEM

3.1 Basic Definitions

Definition 3.1.1 (Stationary state) *A state is called a stationary state if it does not change with time. A stationary state of the system is also called a stationary solution.*

Definition 3.1.2 (Ground state) *The ground state is the state with the lowest energy a system can assume.*

3.2 Spherically-symmetric Stationary Solutions of the Schrödinger-Newton System

A number of studies of spherically-symmetric solutions of the Schrödinger-Newton system have been carried out, both numerical [15], [9] and analytical [24].

In [9] R. Harrison, I. Moroz and K. P. Tod tested numerically the linear stability of the spherically symmetric stationary solutions of the Schrödinger-Newton equations by linearizing the time-dependent Schrödinger-Newton equations about a spherically symmetric stationary solution, and then presented the fully nonlinear evolution of spherically symmetric data and found that the picture suggested by linear theory is supported: the ground state is stable but slight perturbations of the higher states decay.

In [15], I. M Moroz, R. Penrose and K. P. Tod computed numerically stationary solutions in the case of spherical symmetry.

In [24], the authors provide an analytical support for the results they obtained numerically in [15]. They define real functions S and V as follows

$$\begin{aligned} \psi &= \left[\left(\frac{\hbar^2}{8\pi G m^3}\right)\right]^{1/2} S, \\ E - U &= \frac{\hbar^2}{2m} V \end{aligned} \tag{3.1}$$

under this transformation (1.3) and (1.4) reduce to the pair of equations

$$\Delta S = -SV \tag{3.2}$$

$$\Delta V = -S^2 \tag{3.3}$$

In [24], P. Tod and I. M. Moroz investigate the case when S and V are functions of radius, r, only. The wavefunction is said to be normalized if

$$\int_0^\infty r^2 S^2 dr = \frac{2Gm^3}{\hbar^2} \tag{3.4}$$

and the energy eigenvalue E is given by

$$E = \frac{\hbar^2}{2m} V(\infty) \tag{3.5}$$

It is not initially assumed that S is normalizable, rather than P. Tod and I. M. Moroz search for solutions for which S and V are finite and smooth for all r.

In the case of spherical symmetry, (3.2) and (3.3) can be written as

$$\begin{aligned} \frac{1}{r}(rS)'' &= \frac{1}{r}(S + rS')' = \frac{1}{r}(2S' + rS'') = \\ &= \frac{1}{r^2}(2rS' + rS'') = \frac{1}{r^2}(r^2 S')' = -SV \end{aligned} \tag{3.6}$$

$$\frac{1}{r}(rV)'' = \frac{1}{r^2}(r^2 V')' = -S^2 \tag{3.7}$$

In [24], Tod and Moroz investigate the solutions $(S(r), V(r))$ which are finite and smooth at the origin and for which S' and V' vanish at $r = 0$. They prove the following results

Proposition 3.2.1 *Given initial data (S_0, V_0), there is a unique analytic solution of (3.6) and (3.7) on an interval $[0, r_0)$, where r_0 depends upon the initial data.*

Proposition 3.2.2 *Given initial data* (S_0, V_0)*, there is a unique* C^2 *solution of (3.6) and (3.7) on an interval* $[0, r_1)$ *where* r_1 *depends upon the initial data.*

Proof: We define an iteration by

$$S_{n+1} = S_0 - \int_0^r x(1 - \frac{x}{r}) S_n V_n dx \tag{3.8}$$

$$V_{n+1} = V_0 - \int_0^r x(1 - \frac{x}{r}) S_n^2 dx \tag{3.9}$$

We choose A so that $|S_0| < A$, $|V_0| < A$. First, we show by induction that all iterates are bounded by $2A$ at least for a range $[0, a)$ in r.

$$|V_{n+1}| \leqslant |V_0| + 4A^2 \left| \int_0^r x(1 - \frac{x}{r}) dx \right| < A + \frac{4A^2 r^2}{6}$$

If we take $a^2 = \frac{3}{2A}$ then we have

$$|V_{n+1}| < 2A$$

whenever $r \in [0, a)$. Next we define

$$e_{n+1} = V_{n+1} - V_n,$$

$$f_{n+1} = S_{n+1} - S_n$$

We want to prove that

$$|e_n| \leqslant \frac{(4A)^{n-1} r^{2n}}{(2n+1)!},$$

$$|f_n| \leqslant \frac{(4A)^{n-1} r^{2n}}{(2n+1)!}$$

We proceed as follows

$$
\begin{aligned}
|e_n| &= |V_n - V_{n-1}| \\
&= V_0 - \int_0^r x(1-\frac{x}{r})S_{n-1}^2 dx - V_0 + \int_0^r x(1-\frac{x}{r})S_{n-2}^2 dx \\
&= \int_0^r x(1-\frac{x}{r})(S_{n-1}-S_{n-2})(S_{n-1}+S_{n-2})dx \\
&< 4A\left|\int_0^r x(1-\frac{x}{r})(S_{n-1}-S_{n-2})dx\right| \\
&= 4A\left|\int_0^r x(1-\frac{x}{r})f_{n-1}dx\right| \\
&\leqslant 4A\left|\int_0^r x(1-\frac{x}{r})\frac{(4A)^{n-2}x^{2n-2}}{(2n-1)!}dx\right| \\
&= \frac{(4A)^{n-1}}{(2n-1)!}\left|\int_0^r (1-\frac{x}{r})x^{2n-1}dx\right| \\
&= \frac{(4A)^{n-1}r^{2n}}{(2n+1)!}.
\end{aligned} \tag{3.10}
$$

So we have

$$V = V_0 + \sum_{n=1}^{\infty} e_n, \quad \text{and} \quad S = S_0 + \sum_{n=1}^{\infty} f_n$$

which converge by the Weierstrass M-test.

To prove uniqueness, suppose that (S, V) and (T, W) are two solutions of (3.6) and (3.7) satisfying the same initial conditions (S_0, V_0), and assume that all four functions are bounded by M, on $[0, a)$. Define the differences:

$$e = |V - W| \quad \text{and} \quad f = |S - T|.$$

Then (3.6) and (3.7) imply that

$$
\begin{aligned}
e &= \left|V_0 - \int_0^r x(1-\frac{x}{r})S^2 dx - V_0 + \int_0^r x(1-\frac{x}{r})T^2 dx\right| \\
&= \left|\int_0^r x(1-\frac{x}{r})(S^2-T^2)dx\right| \\
&= 2M\left|\int_0^r x(1-\frac{x}{r})(S-T)dx\right| \\
&< 2M\left|\int_0^r x(1-\frac{x}{r})f dx\right|,
\end{aligned} \tag{3.11}
$$

and

$$\begin{aligned}
f &= \left|S_0 - \int_0^r x(1-\frac{x}{r})S(x)V(x)dx - S_0 + \int_0^r x(1-\frac{x}{r})T(x)W(x)dx\right| \\
&= \left|\int_0^r x(1-\frac{x}{r})(S(x)V(x) - T(x)W(x))dx\right| \\
&= \left|\int_0^r x(1-\frac{x}{r})(S(x)V(x) - W(x)S(x) + W(x)S(x) - T(x)W(x))dx\right| \\
&= \left|\int_0^r x(1-\frac{x}{r})[S(x)(V(x) - W(x)) + W(x)(S(x) - T(x))]dx\right| \\
&< M\left|\int_0^r x(1-\frac{x}{r})(e+f)dx\right|,
\end{aligned} \tag{3.12}$$

from which by induction it follows that

$$e < \frac{(2M)^{n+1}r^{2n}}{(2n+1)!}, \quad f < \frac{(2M)^{n+1}r^{2n}}{(2n+1)!}.$$

Therefore, taking the limit as $n \to \infty$, $e = f = 0$ and the solutions are unique.□

This proves the existence and uniqueness of solution of (3.6) and (3.7) satisfying the given initial conditions at $r = 0$. By using the same technique the following theorems can be proved.

Proposition 3.2.3 *Given data $S(a)$,$V(a)$,$S'(a)$,$V'(a)$ for $r = a > 0$,there is an analytic solution in a neighborhood of a which is unique as a C^2 solution.*

Proposition 3.2.4 *Given initial data (S_0, V_0) determining a solution (S, V) on an interval $[0, r_1)$. If $S(r_1)$ and $V(r_1)$ are finite, then the solution is defined, is unique and is therefore also analytic, on a larger range of r.*

Proposition 3.2.5 *Suppose a solution (S, V) of (3.6) and (3.7) with $S_0 = 1$ is bounded by some M for r in the interval $[0, a]$. Then S and V depend continuously on V_0 in this interval.*

Also they investigate how solutions behave with $S_0 = 1$, and V_0 increasing from negative values.

Proposition 3.2.6 *The solution to (3.6) and (3.7) cannot remain bounded for* $V_0 \leqslant 0$. *Necessarily both* $S \to \infty$ *and* $V \to -\infty$ *monotonically as* r *increases.*

Proposition 3.2.7 *If* $V_0 > 0$ *then* V *necessarily has a zero at a finite value of* r.

Proposition 3.2.8 *There exist numbers* α *and* β *with* $0 < \alpha < \beta < 1$ *such that for* V_0 *in the interval* $(0, \alpha)$, *all solutions have* $S \to \infty$ *with* $S > 0$ *and* $V \to -\infty$; *while for* V_0 *in the interval* $(\beta, 1]$, *all solutions have* $S \to -\infty$ *with* S *having a single zero, and* $V \to -\infty$.

Lemma 3.2.9 *If* $S \geqslant 0$ *and* $(rS)' > 0$ *for some* b *with* $V(b) < 0$, *then necessarily* $S \to +\infty$ *and* $V \to -\infty$, *and* $S > 0$ *for* $r > b$. *Similarly is* $S \leqslant 0$ *and* $(rS)' < 0$ *for some* b *with* $v(b) < 0$, *then* $S \to -\infty$ *and* $V \to -\infty$ *with* $S < 0$ *for* $r > b$.

Lemma 3.2.10 *If* $V_0 = S_0$ *then* $V = S\ \forall r$ *and* $V \to -\infty$ *as* r *increases.*

Lemma 3.2.11 *If* S *has a zero, at* $r = b$, *say, at which* $V \leqslant 0$, *then* $V \to -\infty$ *and* $S \to \pm\infty$ *monotonically, according as* $S'(b) > 0$ *or* $S'(b) < 0$ *respectively. In particular,* S *has no zero in* $r > b$.

Lemma 3.2.12 *Suppose* $S_0 = 1$ *and* $0 \leqslant V_0 \leqslant 1$. *If* $S = 1$ *for some* r, *then* $S \to -\infty$ *and* $V \to -\infty$. *If* $S = 0$ *for some* r, *then* $S \to -\infty$ *and* $V \to -\infty$. *If* $S' = 0$ *for some* $r > 0$, *then* $S \to +\infty$ *and* $V \to -\infty$.

Lemma 3.2.13 *Assume that* $S_0 = 1$ *and* $V_0 = \lambda$ *with* $0 \leqslant \lambda \leqslant 1$, *and consider the solutions* $S(r, \lambda)$ *and* $V(r, \lambda)$ *of (3.6) and (3.7). Suppose that* $S(r, \lambda) = 0$ *atr* $= b(\lambda)$, *at least for* λ *near* 1. *Then* $\frac{db}{d\lambda}$ *is negative. Suppose that* $S(r, \lambda) = 1$ *at* $r = c(\lambda)$, *at least for* λ *near zero, then* $\frac{dc}{d\lambda}$ *is positive.*

Theorem 3.2.14 *There is a unique value of* V_0 *in the interval* $[0, S_0]$ *for which the solution* S *remains in the interval* $[0, S_0]$. V *necessarily becomes negative. If at* $r = b$ *say,* $V(b) = -C^2$ *then for* $r > b$,S *lies in the interval*

$$0 \leqslant S \leqslant \frac{bS(b)}{r} \exp(-Cr) \tag{3.13}$$

Thus S *is normalizable and* V *tends to a negative constant as* $r \to \infty$.

Meoreover, P. Tod and I. M. Moroz prove that the higher bound states arise with $0 < S_0 < V_0$.

Lemma 3.2.15 *If* $V_0/S_0 < \pi N$ *then* S *has at least* N *zeros.*

Lemma 3.2.16 *Suppose the first zero of* S *(and therefore the second zero of* $X = rS$*) occurs at* $r = x_2$*, then* $_2$ *decreases with increasing* V_0*.*

Lemma 3.2.17 *The number* N *of zeros of* S *before* $V = 0$ *does not exceed*

$$N \leqslant \sqrt{6}\exp(2V_0^2/S_0^2).$$

Lemma 3.2.18 *With* $S_0 = 1$ *and* $V_0 > 1$*, if* $S(b) = 1$ *for some* $b > 0$ *then* $S \to +\infty$ *monotonically for* $r < b$*. If* $S(b) = -1$ *for some* $b > 0$ *then* $S \to -\infty$ *monotonically for* $r > b$*.*

Lemma 3.2.19 *If* S *and* S' *both vanish at some value* b *of* r*, then* S *is zero in a neighborhood of* b*.*

Theorem 3.2.20 *There exists at least one bound state with* n *zeros for each* $n \geqslant 0$*. The bound states are normalizable.*

Furthermore, P. Tod and I. M. Moroz prove that the singular solutions blow up at finite radius.

Suppose that $S \to +\infty$, $V \to -\infty$, specifically with S, S'(and consequently $(rS)'$) positive for $r > a$, say. Consider $Q = r(S + V)$, which satisfies

$$Q'' = -SQ.$$

Define a modified energy $\mathcal{F}$ by

$$\mathcal{F} = (Q')^2/S + Q^2. \tag{3.14}$$

Then $\mathcal{F}$ is positive for $r > a$ and

$$\begin{aligned} \mathcal{F}' &= \frac{2Q'Q''S - (Q')^2S'}{S^2} + 2QQ' \\ &= \frac{2Q'(-SQ)S - (Q')^2S'}{S^2} + 2QQ' \\ &= -S'(Q')^2/S^2 < 0 \end{aligned} \tag{3.15}$$

for $r > a$.

Thus

$$\mathcal{F}(r) < \mathcal{F}(a) \quad \text{and} \quad |Q| < \sqrt{\mathcal{F}(a)}.$$

From the definition of Q, it follows that

$$-\frac{\sqrt{\mathcal{F}(a)}}{r} < S + V < \frac{\sqrt{\mathcal{F}(a)}}{r} \tag{3.16}$$

for $r > a$, so that as $S \to +\infty$ and $V \to -\infty$, their sum remains small.

By adding $-S$ to each side of (3.16) we get

$$-\left(\sqrt{\mathcal{F}(a)} + X\right)/r < V < \left(\sqrt{\mathcal{F}(a)} - X\right)/r$$

for $r > a$, where $X = rS$. Since also $X > 0$ we find that

$$\begin{aligned} rX'' &= r(rS)'' = r(rS' + S)' = r^2 S'' + 2rS = (r^2 S')' \\ (r^2 S')' &= -rSV \qquad \text{by (3.6)} \\ &> r^2 S \frac{(X - \sqrt{\mathcal{F}(a)})}{r} \\ &> X(X - \sqrt{\mathcal{F}(a)}) \end{aligned}$$

Hence we have

$$X'' > (X^2 - X\sqrt{\mathcal{F}(a)})/r$$

Since $X \to \infty$, for some $b \geqslant a$, $X > 2\sqrt{\mathcal{F}(a)}$. Thus for $r > b$ we get $X'' > X^2/2r$. We also know that $X' > 0$ for $r > b$ so that $X'X'' > X'X^2/(2r)$, which yields

$$\begin{aligned} \left((X')^2\right)' &> \frac{(X^3)'}{6r} \\ (X')^2 &> \frac{X^3}{3r} + A \end{aligned} \tag{3.17}$$

for some constant A. Since $X'' > 0$ for $r > b$,we know that

$$X > X(b) + X'(b)(r - b)$$

for $r > b$. Thus for $r > c \geqslant b$, $\quad X^3 > 6|A|r$, for some c.

We can suppose that c is sufficiently large so that $X(c) > 6/c$, then by (3.17) $(X')^2 > X^3/(6r)$, which can be separated and integrated to yield

$$\begin{aligned} \frac{dX}{dr} &> \frac{X^{\frac{3}{2}}}{(6r)^{\frac{1}{2}}} \\ -2X^{-1/2} &> 2(r^{1/2} - B^{1/2})/\sqrt{6}, \end{aligned} \tag{3.18}$$

for $r > c$. Rearranging (3.18)we get

$$X > 6/(B^{1/2} - r^{1/2}),$$

so that X blows up at $r = B = r_0$, say, and substituting back for S gives

$$S > \frac{6(r_0^{1/2} + r^{1/2})^2}{r(r - r_0)^2} > 6/(r - r_0)^2, \tag{3.19}$$

and S blows up according to

$$S = \pm \frac{6}{(r - r_0)^2} + \ldots.$$

3.3 Stationary Solutions of the Schrödinger-Newton System

There are a number of papers devoted to the study of stationary solutions of Schrödinger-Newton system. (See [2],[23])

In [2], K. Benmlih studies the Schrödinger-Poisson system

$$i\partial_t \psi = -\frac{1}{2}\triangle\psi + (u + \widetilde{u})\psi \tag{3.20}$$

$$-\triangle u = |\psi|^2 - n^*. \tag{3.21}$$

where n^* and $\widetilde{u}$ are time independent real functions, with the initial condition

$$\psi(0, x) = \psi_0(x).$$

The main result of [2] can be summarized as follows:

Theorem 3.3.1 *Under the assumptions*

- $\widetilde{u}^+ \in L^1_{loc}(\mathbb{R}^3)$
- $\widetilde{u}^- \in L^1_{loc}(\mathbb{R}^3), \quad \widetilde{u}^- \geqslant 0 \quad$ *and*

  There exists $\quad q_0 \in [3/2, \infty]; \forall \lambda > 0 \quad \exists u_{1\lambda} \in L^{q_0}(\mathbb{R}^3)$

  $q_\lambda \in]3/2, \infty[\quad$ *and* $\quad u_{2\lambda} \in L^{q_\lambda}(\mathbb{R}^3) \quad$ *such that*

  $\widetilde{u}^- = u_{1\lambda} + u_{2\lambda} \quad$ *and* $\quad \lim_{\lambda \to 0} \|u_{1\lambda}\|_{L^{q_0}} = 0$
- $n^* \in L^1 \cap L^{\frac{6}{5}}(\mathbb{R}^3)$
- $\inf \left\{ \int_{\mathbb{R}^3} (|\nabla\varphi|^2 + (2\widetilde{u}(x) - \frac{1}{2\pi} \int_{\mathbb{R}^3} \frac{n^*(y)}{|x-y|} dy)\varphi^2) dx, \int_{\mathbb{R}^3} |\varphi|^2 = 1 \right\} < 0$

there exists $\omega_* > 0$ *such that for all* $0 < \omega < \omega_*$ *the equation*

$$-\frac{1}{2}\triangle u + (V(u) + \widetilde{V})u + \omega u = 0 \quad \text{in } \mathbb{R}^3. \tag{3.22}$$

has a nonnegative solution $u \not\equiv 0$ *which minimizes the functional* E*:*

$$E(\varphi) := \frac{1}{4} \int_{\mathbb{R}^3} |\nabla\varphi|^2 dx + \frac{1}{4} \int_{\mathbb{R}^3} |\nabla V(\varphi)|^2 dx + \frac{1}{2} \int_{\mathbb{R}^3} \widetilde{V}\varphi^2 dx + \frac{\omega}{2} \int_{\mathbb{R}^3} \varphi^2 dx \tag{3.23}$$

i.e.

$$E(u) = \min_{\varphi \in H^1(\mathbb{R}^3)} E(\varphi).$$

In other words, in [2] K. Benmlih proves the existence of standing wave solutions for a Schrödinger-Poisson system in $\mathbb{R}^3$.

Below we give a summary of the work carried out in this paper.

The author solves first explicitly the Poisson equation.

Lemma 3.3.2 *For all* $f \in L^{6/5}(\mathbb{R}^3)$*, the equation*

$$-\triangle W = f \quad \text{in } \mathbb{R}^3 \tag{3.24}$$

has a unique solution $W \in \mathcal{D}^{1,2}(\mathbb{R}^3)$ *given by*

$$W(f)(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{f(y)}{|x-y|} dy. \tag{3.25}$$

Proof: The existence of a unique solution of (3.24) follows from the existence and the uniqueness of the minimizer of

$$J(v) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla v|^2 dx - \int_{\mathbb{R}^3} fv dx$$

the functional corresponding to (3.24).

First we show that J is coercive: we want to show that $J(v_n) \to \infty$ as $\|v_n\|_{D^{1,2}} \to \infty$

$$\begin{aligned} J(v) &= \frac{1}{2}\int_{\mathbb{R}^3} |\nabla v|^2 dx - \int_{\mathbb{R}^3} fv dx \\ &\geqslant \text{(by Hölder's Inequality)} \geqslant \\ &\geqslant \frac{1}{2}\int_{\mathbb{R}^3} |\nabla v|^2 dx - \left(\int_{\mathbb{R}^3} |f|^{6/5} dx\right)^{5/6} \left(\int_{\mathbb{R}^3} |v|^6 dx\right)^{1/6} \\ &\geqslant \text{(by Sobolev Inequality)} \geqslant \\ &\geqslant \frac{1}{2}\int_{\mathbb{R}^3} |\nabla v|^2 dx - \left(\int_{\mathbb{R}^3} |f|^{6/5} dx\right)^{5/6} C \left(\int_{\mathbb{R}^3} |\nabla v|^2 dx\right)^{1/2}. \end{aligned}$$

Now we have an expression of the form $\frac{1}{2}x^2 + ax$ where a is a constant, hence as $x \to \infty$ $\frac{1}{2}x^2 + ax \to \infty$. Then we have

$$J(v_n) \to \infty \quad \text{as} \quad \|v_n\|_{D^{1,2}} \to \infty.$$

J is strictly convex:

$$\begin{aligned} J(v) &= \frac{1}{2}\int_{\mathbb{R}^3} (D_1 v^2 + D_2 v^2 + D_3 v^2) - \int_{\mathbb{R}^3} fv dx \\ &= \frac{1}{2}\int_{\mathbb{R}^3} |\nabla v|^2 dx - \int_{\mathbb{R}^3} fv dx \end{aligned}$$

$$\Rightarrow \sum_{i,j=1}^{3} L_{p_i p_j}(|\nabla v|^2, v, x)\xi_i \xi_j = 3(\xi_1^2 + \xi_2^2 + \xi_3^2).$$

J is lower semicontinuous: We have

$$\begin{aligned} \liminf J(v_n) &= \liminf \left(\frac{1}{2}\int_{\mathbb{R}^3} |\nabla v_n|^2 dx - \int_{\mathbb{R}^3} fv_n dx\right) \\ &\geqslant \liminf \frac{1}{2}\int_{\mathbb{R}^3} |\nabla v_n|^2 dx + \limsup \int_{\mathbb{R}^3} fv_n dx \\ &= \frac{1}{2}\int_{\mathbb{R}^3} |\nabla v|^2 dx - \int_{\mathbb{R}^3} fv dx \end{aligned}$$

since

$$\int fv_n \leqslant \left(\int f^{6/5}\right)^{5/6}\left(\int v_n^6\right)^{1/6} \leqslant \left(\int f^{6/5}\right)^{5/6}\left(\int |\nabla v_n|^2\right)^{1/2}$$

and

$$\begin{aligned} \left|\int fv_n - \int fv\right| &= \left|\int f(v_n - v)\right| \\ &\leqslant \left(\int f^{6/5}\right)^{5/6}\left(\int (v_n - v)^6\right)^{1/6} \\ &\leqslant \left(\int f^{6/5}\right)^{5/6}\left(\int |\nabla v_n - \nabla v|^2\right)^{1/2} \to 0. \end{aligned}$$

So we have shown that J is coercive, strictly convex, lower semicontinuous and since it is C^1 on $D^{1,2}(\mathbb{R}^3)$, it follows that J attains its minimum at $W \in D^{1,2}(\mathbb{R}^3)$,by Theorem 2.9.4, which is the unique solution of (3.24) .

□

In order to show that equation (3.22) has the variational structure we prove the following lemma.

Lemma 3.3.3 *Let $n^* \in L^{6/5}(\mathbb{R}^3)$. For $\varphi \in H^1(\mathbb{R}^3)$ we denote by*

$$V(\varphi) := W(|\varphi|^2 - n^*)$$

the unique solution of (3.24) when $f := |\varphi|^2 - n^*$. Define

$$I(\varphi) := \frac{1}{4}\int_{\mathbb{R}^3} |\nabla V(\varphi)|^2 dx.$$

Then I is C^1 on $H^1(\mathbb{R}^3)$ and its derivative is given by

$$\langle I'(\varphi), \psi\rangle = \int_{\mathbb{R}^3} V(\varphi)\varphi\psi dx \quad \forall\psi \in H^1)(\mathbb{R}^3) \tag{3.26}$$

Proof: For the sake of simplicity of calculations we take $n^* = 0$. Since $\varphi \in H^1(\mathbb{R}^3)$, by Sobolev inequality $|\varphi|^2 \in L^{6/5}(\mathbb{R}^3)$. In (3.24) letting $f = |\varphi|^2$ and multiplying it by $W(|\varphi|^2)$ we get

$$\begin{aligned} W(|\varphi|^2)(x) &= \frac{1}{4\pi}\int \frac{|\varphi|^2(y)}{|x-y|}dy \\ \Rightarrow \quad \|\nabla v(\varphi)\|^2 &= \frac{1}{4\pi}\int\int \frac{|\varphi|^2(x)|\varphi|^2(y)}{|x-y|}dxdy \\ \Rightarrow \quad I(\varphi) &= \frac{1}{16\pi}\int\int \frac{|\varphi|^2(x)|\varphi|^2(y)}{|x-y|}dxdy. \end{aligned} \tag{3.27}$$

Then for all $\varphi, \phi \in H^1(\mathbb{R}^3)$ we have

$$\lim_{t\to 0^+} \frac{I(\varphi+t\phi)-I(\varphi)}{t} =$$

$$= \lim_{t\to 0^+} \frac{1}{16\pi}\frac{1}{t}\int\int \frac{|\varphi+t\phi|^2(x)|\varphi+t\phi|^2(y)-|\varphi|^2(x)|\varphi|^2(y)}{|x-y|}dxdy$$

$$= \lim_{t\to 0^+} \frac{1}{16\pi}\frac{\int\int[2t|\varphi|^2(x) < \varphi,\phi > (y) + 2t < \varphi,\phi > (x)|\varphi|^2(y)]/|x-y|dxdy}{t}$$

$$= \int_{\mathbb{R}^3} V(\varphi)\varphi\phi dx$$

that is (3.26) holds for the Gâteaux differential of I. Hence I is C^1 on $H^1(\mathbb{R}^3)$ and I is Frechet differentiable.

□

To prove the main theorem one needs some estimates which are proven in the following lemma

Lemma 3.3.4 *(i) If $\theta \in L^r(\mathbb{R}^3)$ for some $r \geqslant 3/2$ then $\forall\delta > 0$, $\exists C_\delta > 0$ such that*

$$\int_{\mathbb{R}^3} \theta(x)|\varphi(x)|^2 \leqslant \delta\|\nabla\varphi\|^2 + C_\delta\|\varphi\|^2 \quad \forall\varphi \in H^1(\mathbb{R}^3). \tag{3.28}$$

(ii) For all $\varphi \in \mathcal{D}^{1,2}(\mathbb{R}^3)$ and $y \in \mathbb{R}^3$ one has

$$\int_{\mathbb{R}^3} \frac{|\varphi(x)|^2}{|x-y|^2}dx \leqslant 4\|\nabla\varphi\|^2. \tag{3.29}$$

(iii) For any $\delta > 0$ and all $y \in \mathbb{R}^3$

$$\int_{\mathbb{R}^3} \frac{|\varphi(x)|^2}{|x-y|}dx \leqslant \delta\|\nabla\varphi\|^2 + \frac{4}{\delta}\|\varphi\|^2 \quad \forall\varphi \in H^1(\mathbb{R}^3). \tag{3.30}$$

Proof of (i): First we show that (3.28) is valid for any $\theta \in L^\infty(\mathbb{R}^3) + L^{3/2}(\mathbb{R}^3)$.

Let $\theta = \theta_1 + \theta_2$ where $\theta_1 \in L^\infty(\mathbb{R}^3)$ and $\theta_2 \in L^{3/2}(\mathbb{R}^3)$. Then for each $\lambda > 0$ one has

$$\begin{aligned}
\int_{\mathbb{R}^3} \theta(x)|\varphi(x)|^2dx &\leqslant \|\theta_1\|_{L^\infty(\mathbb{R}^3)}\|\varphi\|^2 + \lambda\int_{[|\theta_2|\leqslant\lambda]}|\varphi|^2dx + \int_{[|\theta_2|>\lambda]}|\theta_2||\varphi|^2dx,\\
&\leqslant (\|\theta_1\|_{L^\infty(\mathbb{R}^3)}+\lambda)\|\varphi(x)\|^2 + \|\theta_2\|_{L^{3/2}([|\theta_2|>\lambda])}\|\varphi\|^2_{L^6(\mathbb{R}^3)},\\
&\leqslant (\|\theta_1\|_{L^\infty(\mathbb{R}^3)}+\lambda)\|\varphi(x)\|^2 + S_*\|\theta_2^\lambda\|_{L^{3/2}(\mathbb{R}^3)}\|\nabla\varphi\|^2,
\end{aligned}$$

where

$$S_* := \text{Sobolev constant},$$

and

$$\theta_2^\lambda := \theta_2 \mathbb{I}_{[|\theta_2|>\lambda]}.$$

Since

$$|\theta_2^\lambda| \leqslant \theta_2 \quad \text{for each} \quad \lambda > 0,$$

$$\theta_2^\lambda \to 0 \quad \text{pointwise a.e. whenever} \quad \lambda \to \infty$$

and

$$\theta_2 \in L^{3/2}(\mathbb{R}^3)$$

then by Lebesgue convergence theorem it follows that

$$\|\theta_2^\lambda\|_{L^{3/2}(\mathbb{R}^3)} \to 0.$$

Hence for any $\delta > 0, \quad \exists K_\delta > 0$ such that one has

$$S_* \|\theta_2^\lambda\|_{L^{3/2}(\mathbb{R}^3)} \leqslant \delta \quad \text{whenever} \quad \lambda \geqslant K_\delta.$$

Choosing

$$C_\delta := \|\theta_1\|_{L^\infty(\mathbb{R}^3)} + K_\delta$$

we conclude that (3.28) holds for all $\theta \in L^\infty(\mathbb{R}^3) + L^{3/2}(\mathbb{R}^3)$.

To complete the proof of part (i) it remains to show that

$$L^r(\mathbb{R}^3) \subset L^\infty(\mathbb{R}^3) + L^{3/2}(\mathbb{R}^3) \quad \text{for all} \quad r.$$

Claim:$L^r(\mathbb{R}^3) \subset L^\infty(\mathbb{R}^3) + L^{3/2}(\mathbb{R}^3)$ for all r.

Proof of Claim: Let $g \in L^r(\mathbb{R}^3)$ where $r \geqslant \frac{3}{2}$. We have

$$|g| = \underbrace{|g|\mathbb{I}_{[|g|\leqslant\lambda]}}_{\in L^\infty(\mathbb{R}^3)} + \underbrace{|g|\mathbb{I}_{[|g|>\lambda]}}_{L^{3/2}(\mathbb{R}^3)}$$

which completes the proof of Claim and the proof of part (i).□

Proof of (ii):(3.29) is a classical Hardy inequality, see Theorem 2.3.6 in Chapter 2.□

Proof of (iii): By using (3.29), for all $\delta > 0$ and any $y \in \mathbb{R}$ we have

$$\begin{aligned} \int_{\mathbb{R}^3} \frac{|\varphi(x)|^2}{|x-y|} dx &= \int_{|x-y|<\frac{\delta}{4}} \frac{|\varphi(x)|^2}{|x-y|^2}|x-y| dx + \int_{|x-y|\geqslant\frac{\delta}{4}} \frac{|\varphi(x)|^2}{|x-y|} dx \\ &\leqslant \frac{\delta}{4}\int_{\mathbb{R}^3} \frac{|\varphi(x)|^2}{|x-y|^2} dx + \frac{4}{\delta}\int_{\mathbb{R}^3} |\varphi(x)|^2 dx \\ &\leqslant \delta\|\nabla\varphi\|^2 + \frac{4}{\delta}\|\varphi\|^2. \end{aligned}$$

□

Another result that will be needed to prove the main theorem is

Lemma 3.3.5 *Let $\psi \in L^r(\mathbb{R}^3)$ for some $r > 3/2$. If $v_n \rightharpoonup 0$ weakly in $H^1(\mathbb{R}^3)$ then*

$$\int_{\mathbb{R}^3} \psi(x) v_n^2(x) dx \to 0 \quad as \quad n \to +\infty.$$

Proof: Let $A_\lambda := \{x \in \mathbb{R}^3 : |\psi(x)| > \lambda\}$ and K be a compact subset of A_λ to be determined later. Since $v_n \rightharpoonup v$ weakly in $H^1(\mathbb{R}^3) \Rightarrow \{v_n\}$ is bounded in $H^1(\mathbb{R}^3) \Rightarrow$ (by Sobolev inequality)$\Rightarrow \{v_n\}$ is bounded in $L^6(\mathbb{R}^3) \Rightarrow$(by Interpolation inequality)$\Rightarrow \{v_n\}$ is bounded in $L^{2r'}(\mathbb{R}^3)$ for $2 < 2r' < 6$, we have

$$\begin{aligned} \int_{\mathbb{R}^3} |\psi|(x) v_n^2(x) dx &= \int_{\mathbb{R}^3 - A_\lambda} |\psi| v_n^2 dx + \int_{A_\lambda - K} |\psi| v_n^2 dx + \int_K |\psi| v_n^2 dx \\ &\leqslant \lambda\|v_n\|^2 + \|\psi\|_{L^r(A_\lambda - K)}\|v_n\|_{L^{2r'}(\mathbb{R}^3)} + \|\psi\|_{L^r(K)}\|v_n\|^2_{L^{2r'}(K)} \\ &\leqslant \lambda C_0 + C_1\|\psi\|_{L^r(A_\lambda - K)} + \|\psi\|_{L^r(K)}\|v_n\|^2_{L^{2r'}(K)}, \end{aligned}$$

where $\frac{1}{r'} + \frac{1}{r} = 1$.

Next, for arbitrarily chosen but fixed $\delta > 0$, we choose λ with $\lambda C_0 \leqslant \frac{\delta}{3}$, and then we choose $K \subset A_\lambda$ so that

$$C_1\|\psi\|_{L^r(A_\lambda - K)} \leqslant \frac{\delta}{3}.$$

Since $v_n \rightharpoonup 0$ in $H^1(\mathbb{R}^3)$, then by Interpolation inequality it follows that $v_n \to v$ in $L^{2r'}(K)$ for $2 < 2r' < 6$. Hence there exists $N_\delta \in \mathbb{N}$ such that

$$\|\psi\|_{L^r(K)}\|v_n\|^2_{L^{2r'}(K)} \leqslant \frac{\delta}{3} \quad \text{whenever} \quad n \geqslant N_\delta.$$

This completes the proof of Lemma 3.3.5.□

To prove the main theorem we need to minimize the energy functional

$$E(\varphi) := \frac{1}{4}\int |\nabla\varphi|^2 dx + I(\varphi) + \frac{1}{2}\int \widetilde{V}\varphi^2 dx + \frac{\omega}{2}\int \varphi^2 dx$$

whose critical points correspond, on account of Lemma 3.3.3, to solutions of (3.22). We proceed in four steps with the aid of the previous lemmas:

Lemma 3.3.6 *For all $\omega > 0$ and $c \in \mathbb{R}$ the set $[E \leqslant c]$ is bounded in $L^2(\mathbb{R}^3)$.*

Lemma 3.3.7 *Let $\omega > 0$ and $c \in \mathbb{R}$. If the set $[E \leqslant c]$ is bounded in $L^2(\mathbb{R}^3)$ then it is also bounded in $H^1(\mathbb{R}^3)$.*

Lemma 3.3.8 *For any $\omega > 0$ the functional E is weakly lower semi continuous on $H^1(\mathbb{R}^3)$ and attains its minimum on $H^1(\mathbb{R}^3)$ at $u \geqslant 0$.*

Lemma 3.3.9 *There exists $\omega^* > 0$ such that if $0 < \omega < \omega^*$ then $E(u) < E(0)$ and thus $u \not\equiv 0$.*

This completes the proof of Theorem 3.3.1 and the summary of [2].

K. P. Tod in [23] investigates the system

$$-\Delta\psi + u\psi = E\psi \tag{3.31}$$

$$\Delta u = |\psi|^2 \tag{3.32}$$

subject to the normalization condition

$$\int |\psi|^2 dx = 1 \tag{3.33}$$

in $\mathbb{R}^3$.

(3.31) can be obtained from a variational problem

$$I = \frac{1}{2}\int |\nabla\psi|^2 dx + \frac{1}{2}\int u|\psi|^2 dx - \frac{1}{2}E\int |\psi|^2 dx$$

when u is determined by (3.32). Alternatively by solving (3.32) explicitly and substituting into (3.34), the system (3.31)-(3.32) can be obtained from

$$I = \frac{1}{2}\int |\nabla\psi|^2 dx + \frac{1}{8\pi}\int\int \frac{|\psi(x)|^2|\psi(y)|^2}{|x-y|}dxdy - \frac{1}{2}E\int |\psi|^2 dx. \tag{3.34}$$

The author shows that the energy eigenvalues are negative, and by using some elementary inequalities he proves the following proposition:

Proposition 3.3.10 *The functional I of (3.34) is bounded below.*

Proof: By Sobolev inequality, for $\psi \in W^{1,2}(\mathbb{R}^3)$ one has

$$\left(\int |\psi|^6\right)^{1/6} \leqslant K\left(\int |\nabla\psi|^2\right)^{1/2}, \tag{3.35}$$

where

$$K = \frac{2^{2/3}}{3^{1/2}\pi^{2/3}}. \tag{3.36}$$

Multiplying both sides of (3.32) by u, and then integrating over $\mathbb{R}^3$, one has

$$-\int |\nabla u|^2 dx = \int u|\psi|^2 dx.$$

Thus we have

$$\begin{aligned} \left(\int u^6 dx\right)^{1/6} &\leqslant K\left(\int |\nabla u|^2 dx\right)^{1/2} \\ &= K\left(\int -u|\psi|^2 dx\right)^{1/2} \\ &\leqslant \text{(by Hölder's inequality)} \\ &\leqslant K\left(\int u^6 dx\right)^{1/12}\left(\int |\psi|^{12/5} dx\right)^{5/12} \end{aligned}$$

$$\Rightarrow \left(\int u^6 dx\right)^{1/12} \leqslant K\left(\int |\psi|^{12/5} dx\right)^{5/12}.$$

Taking squares of both sides one obtains

$$\left(\int u^6 dx\right)^{1/6} \leqslant K^2 \left(\int |\psi|^{12/5} dx\right)^{5/6} \tag{3.37}$$

Then

$$\begin{aligned}
-\int u|\psi|^2 dx &\leqslant \left(\int u^6 dx\right)^{1/6} \left(\int |\psi|^{12/5} dx\right)^{5/6} \\
&\leqslant \text{(by} \quad (3.37)) \\
&\leqslant K^2 \left(\int |\psi|^{12/5} dx\right)^{5/3} \\
&\leqslant \text{(by Hölder's inequality)} \\
&\leqslant \left(\int |\psi|^2 dx\right)^{3/2} \left(\int |\psi|^6 dx\right)^{1/6} \\
&\leqslant \text{(by normalization condition (3.33) and by Sobolev inequality)} \\
&\leqslant K^3 \left(\int |\nabla\psi|^2 dx\right)^{1/2}.
\end{aligned}$$

By using the above estimates found, it follows that

$$\begin{aligned}
I &= \frac{1}{2}\int |\nabla\psi|^2 dx + \frac{1}{2}\int u|\psi|^2 dx \\
&\geqslant \frac{1}{2}\int |\nabla\psi|^2 dx - 2\frac{1}{4}K^3 \left(\int |\nabla\psi|^2 dx\right)^{1/2} \\
&= \frac{1}{2}\left(\int |\nabla\psi|^2 dx - \frac{1}{2}K^3\right)^2 - \frac{1}{8}K^6 \\
&\geqslant -\frac{1}{8}K^6 \\
&= \text{(by substituting (3.36))} = -\frac{2}{27\pi^4}
\end{aligned}$$

□

To find a bound for E, K. P. Tod defines the following tensor

$$T_{ij} = \psi_i\overline{\psi}_j + \psi_j\overline{\psi}_i + u_i u_j - \delta_{ij}(\psi_k\overline{\psi}_k + \frac{1}{2}u_k u_k + u|\psi|^2 - E|\psi|^2). \tag{3.38}$$

By straightforward calculations it follows that

$$T_{ij,j} = 0, \tag{3.39}$$

and hence

$$(T_{ij}x_i)_j = T_{ii}.$$

Then

$$0 = \int (T_{ij}x_i)_j dx = \int T_{ii} dx,$$

so we have

$$\begin{aligned} 0 = \int T_{ii} &= \int -|\nabla\psi|^2 - \frac{1}{2}|\nabla u|^2 - 3u|\psi|^2 + 3E|\psi|^2 dx \\ &= \int -|\nabla\psi|^2 - \frac{5}{2}u|\psi|^2 + 3E|\psi|^2 dx \end{aligned} \quad (3.40)$$

On the other hand, from (3.31) it follows that

$$\int |\nabla\psi|^2 + \int u|\psi|^2 = E\int |\psi|^2 dx = (\text{by normalization condition (3.33)}) = E. \quad (3.41)$$

From (3.40) and (t12), we attain

$$\int |\nabla\psi|^2 = -\frac{1}{3}E, \quad (3.42)$$

$$\int u|\psi|^2 = \frac{4}{3}E. \quad (3.43)$$

Using (3.32), (3.42) and (3.43), we infer that

$$I \geqslant \frac{1}{6}E \geqslant -\frac{1}{54\pi^4},$$

which implies that

$$E_0 \geqslant -\frac{4}{9\pi^4}. \quad (3.44)$$

By using appropriate scaling, from (3.44) it follows that

$$\varepsilon_0 \geqslant -1.44\frac{G^2m^5}{\hbar^2}$$

where ε_0 denotes the ground state energy of (1.3)-(1.4).

Furthermore, in [23], K. P. Tod shows that

$$\varepsilon_0 \leqslant -\frac{75}{512}\frac{G^2m^5}{\hbar^2} = -0.146\frac{G^2m^5}{\hbar^2}$$

by minimizing I over a special class of trial functions of the form

$$\psi(r) = \left(\frac{k^3}{\pi}\right)^{\frac{1}{2}} e^{-kr}.$$

3.4 Stationary Solutions of the Schrödinger-Newton System under the effect of a nonlinear term

In [20] D. Ruiz studies the stationary solutions of the following system

$$i\psi_t - \Delta\psi + \lambda u(x)\psi = |\psi|^{p-1}\psi \tag{3.45}$$

$$-\Delta u = |\psi|^2, \tag{3.46}$$

where $1 < p < 5$, $\lambda > 0$ and $\psi : \mathbb{R}^3 \times [0, T] \to \mathbb{C}$. That is, D. Ruiz studies the following stationary system

$$-\Delta\psi + \psi + \lambda u\psi = \psi^p \tag{3.47}$$

$$-\Delta u = |\psi|^2, \quad \lim_{|x|\to\infty} u(x) = 0 \tag{3.48}$$

where $\psi, u : \mathbb{R}^3 \to \mathbb{R}$ are positive functions.

The author in [20] investigates the existence of positive radial solutions of (3.47)-(3.48) depending on the parameter $\lambda > 0$. The approach is variational, the author looks for solutions of (3.47)-(3.48) as critical points of the associated energy functional $I : I_\lambda : H_r^1 \to \mathbb{R}$.

The main results obtained in this work can be summarized in following theorems:

Theorem 3.4.1 *If $p \leqslant 2$, (3.47)-(3.48) with $\lambda = 1$ does not admit any nontrivial solution. Moreover, if $2 < p < 5$, there exists a positive radial solution of (3.47)-(3.48) with $\lambda = 1$.*

In the following theorem a related Pohozaev identity is proved for the problem (3.47)-(3.48).

Theorem 3.4.2 *Let $(\psi, u) \in H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$ be a weak solution of the problem*

$$-a\Delta\psi + b\psi + cu\psi = d\psi^p \tag{3.49}$$

$$-\Delta u = \psi^2 \tag{3.50}$$

where a, b, c and d are real constants. Then there holds:

$$\int \frac{a}{2}|\nabla\psi|^2 + \frac{3b}{2}\psi^2 + \frac{5c}{4}u\psi^2 - \frac{3d}{p+1}\psi^{p+1} = 0.$$

In particular the solution of (3.47)-(3.48) *satisfy*

$$\int \frac{1}{2}|\nabla\psi|^2 + \frac{3}{2}\psi^2 + \frac{5\lambda}{4}u\psi^2 - \frac{3}{p+1}\psi^{p+1} = 0. \tag{3.51}$$

Here we generalize the previous result, and prove following Pohozaev identity:

Theorem 3.4.3 *Let $(\psi, u) \in H^1(\mathbb{R}^n) \times D^{1,2}(\mathbb{R}^n)$ be a weak solution of the problem*

$$-a\Delta\psi + b\psi + cu\psi = d\psi^p \tag{3.52}$$

$$-\Delta u = \psi^2 \tag{3.53}$$

where a, b, c and d are real constants. Then there holds:

$$\int a\frac{n-2}{2}|\nabla\psi|^2 + \frac{bn}{2}\psi^2 + \frac{c(n+2)}{4}u\psi^2 - \frac{dn}{p+1}\psi^{p+1} = 0$$

Proof: We multiply (3.52) by $\psi_{x_k}x_k$,

$$-a\Delta\psi\psi_{x_k}x_k + b\psi\psi_{x_k}x_k + cu\psi\psi_{x_k}x_k = d\psi^p\psi_{x_k}x_k$$

then sum k from 1 to n

$$\begin{aligned}
&-\sum_{k=1}^{n} a\Delta\psi\psi_{x_k}x_k + \sum_{k=1}^{n} b\psi\psi_{x_k}x_k + \sum_{k=1}^{n} cu\psi\psi_{x_k}x_k = \sum_{k=1}^{n} d\psi^p\psi_{x_k}x_k \\
&-\frac{a}{2}\sum_{k=1}^{n}(\psi_{x_k}^2)_{x_k}x_k + \frac{b}{2}\sum_{k=1}^{n}(\psi^2)_{x_k}x_k - a\sum_{i,k=1;i\neq k}^{n}\psi_{x_ix_i}\psi_{x_k}x_k + \\
&+\frac{cu}{2}\sum_{k=1}^{n}(\psi^2)_{x_k}x_k - \frac{d}{p+1}\sum_{k=1}^{n}[(\psi_{x_k}^{p+1})_{x_k} - \psi^{p+1}] = 0 \\
&-\frac{a}{2}\sum_{k=1}^{n}[(\psi_{x_k}^2x_k)_{x_k} - \psi_{x_k}^2] + \frac{b}{2}\sum_{k=1}^{n}[(\psi^2x_k)_{x_k} - \psi^2] \\
&-a\sum_{i,k=1;i\neq k}^{n}[(\psi_{x_i}\psi_{x_k})_{x_i}x_k - (\psi_{x_i}\psi_{x_kx_i})x_k] + \frac{cu}{2}\sum_{k=1}^{n}(\psi^2)_{x_k}x_k \\
&-\frac{d}{p+1}\sum_{k=1}^{n}[(\psi_{x_k}^{p+1})_{x_k} - \psi^{p+1}] = 0
\end{aligned} \tag{3.54}$$

Integrating over $\mathbb{R}^n$ we obtain

$$a\frac{n-2}{2}\int|\nabla\psi|^2 + \frac{nb}{2}\int\psi^2 - \frac{dn}{p+1}\int\psi^{p+1} - \frac{c}{2}\int\sum_{k=1}^{n}x_ku(\psi^2)_{x_k} = 0 \tag{3.55}$$

It is clear that

$$\sum_{k=1}^{n}x_ku(\psi^2)_{x_k} = \sum_{k=1}^{n}[(u\psi^2x_k)_{x_k} - u_{x_k}\psi^2x_k - u\psi^2]$$

Hence (3.55) implies

$$\begin{aligned}
&a\frac{n-2}{2}\int|\nabla\psi|^2 + \frac{nb}{2}\int\psi^2 - \frac{dn}{p+1}\int\psi^{p+1} \\
&\qquad - \frac{c}{2}\int\sum_{k=1}^{n}[(u\psi^2x_k)_{x_k} - u_{x_k}\psi^2x_k - u\psi^2] = 0
\end{aligned} \tag{3.56}$$

by using the equation (3.53), we obtain

$$a\frac{n-2}{2}\int|\nabla\psi|^2+\frac{nb}{2}\int\psi^2-\frac{dn}{p+1}\int\psi^{p+1}+\frac{cn}{2}\int u\psi^2+\frac{c}{2}\int\sum_{k=1}^{n}u_{x_k}\Delta ux_k=0 \quad (3.57)$$

It is clear that integrating by parts we obtain

$$\frac{c}{2}\int\sum_{k=1}^{n}u_{x_k}\Delta ux_k=\frac{c}{2}\int\sum_{k=1}^{n}u_{x_k}(\sum_{k=1}^{n}u_{x_ix_i})x_k=c\frac{n-2}{4}\int|\nabla u|^2$$

From (3.57) it follows that

$$a\frac{n-2}{2}\int|\nabla\psi|^2+\frac{nb}{2}\int\psi^2-\frac{dn}{p+1}\int\psi^{p+1}+c\frac{n+2}{4}\int u\psi^2=0.$$

□

The system (3.47)-(3.48) where $p=\frac{5}{3}$ is studied by O. Sánchez and J. Soler in [22]. The authors use a minimization procedure in an appropriate manifold to find a positive solution, and they also studied the evolution of the system in time.

Theorem 3.4.4 *Assume that $\lambda \leqslant 1/4$. Then, $u=0$ is the unique solution of the problem (3.47)-(3.48).*

Corollary 3.4.5 *Suppose that $p\in(1,2)$ and λ is small enough. Then there exist at least two different positive solutions of (3.47)-(3.48).*

Remark: The results obtained in the Theorem 3.4.4 and Corollary 3.4.5 are also true for the solutions of the following problem

$$-\Delta\psi+\psi+\lambda u\psi=|\psi|^{p-1}\psi, \quad (3.58)$$

$$-\Delta u=|\psi|^2, \quad (3.59)$$

for $p\in(1,2]$ and $(\psi,u)\in H^1(\mathbb{R}^3)\times D^{1,2}(\mathbb{R}^3)$. That is these results do not involve only radial functions.

In [13] M. K. Kwong proves the uniqueness of positive radial solutions in R^n of (3.47) with $\lambda = 0$ for $p \in (1,5)$ and $u_0 \in H^1(R^n)$. Using this result D. Ruiz in [20] concludes that for λ small enough there exists a radial solution u_λ of the problem (3.47)-(3.48) such that $u_\lambda \to u_0$ in H^1 as $\lambda \to 0$.

Chapter 4

DISSIPATIVE NONLINEAR SCHRÖDINGER-NEWTON SYSTEM

In this chapter we study the following problem

$$i\psi_t + \Delta\psi + ia\psi - bu\psi - k|\psi|^2\psi = 0, \quad x \in \Omega, \quad t > 0, \tag{4.1}$$

$$\Delta u = b|\psi|^2, \quad x \in \Omega, \quad t > 0, \tag{4.2}$$

$$\psi(0, x) = \psi_0(x), \quad x \in \Omega, \tag{4.3}$$

$$\psi\big|_{\partial\Omega} = u\big|_{\partial\Omega} = 0, \quad t > 0. \tag{4.4}$$

where Ω is a bounded domain in $\mathbb{R}^3$ with sufficiently smooth boundary $\partial\Omega$. We prove the existence of weak solution and that it tends to zero as t tends to infinity. We also prove the existence and uniqueness of the strong solution.

In 1989, the damped Schrödinger-Newton system with $k = 0$ was studied by G. Rünger in [21]. The author in this paper studies this system in one-dimensional case, and proves that the asymptotical behavior is described by a universal finite dimensional attractor.

In [21] G. Rünger instead of the system, equivalently, studies a weakly damped Schrödinger equation

$$i\psi_t + \psi_{xx} - d\psi + B(\psi) + i\gamma\psi = f, \quad \gamma > 0 \tag{4.5}$$

with

$$\psi(0) = \psi_0 \tag{4.6}$$

$$\psi(x + L, t) = \psi(x, t), \quad \forall x \in \mathbb{R}, \quad \forall t \in \mathbb{R} \tag{4.7}$$

where

- g is a real valued bounded continuous function,
- the external force

$$f \in L^{\infty}_{loc}(\mathbb{R}, \mathcal{H}), \quad f_t \in L^{\infty}_{loc}(\mathbb{R}, \mathcal{H}), \quad \mathcal{H} = L^2(0, L), \tag{4.8}$$

- the operator $B(\psi) = B(\psi, \psi)$ is defined by

$$B : \mathcal{V} \times \mathcal{V} \to C^2(0, L) \cap \mathcal{V}^2, \quad \mathcal{V} = \{\psi \in H^1(0, L) : \psi(0) = \psi(L)\},$$

$$\mathcal{V}^2 = \{\psi \in \mathcal{V} : \psi_x \in \mathcal{V}\},$$

$$B(\psi, \varphi) := \frac{1}{2}(x + \frac{1}{L}x^2)(\psi, \varphi) - \frac{x}{L}(x\psi, \varphi) - \int_0^x \int_0^y \psi\overline{\varphi} dz dy. \tag{4.9}$$

In [14] H. Lange and P. F. Zweifel have studied the Schrödinger-Newton system

$$i\partial_t \psi = -\frac{1}{2}\triangle\psi + u\psi, \tag{4.10}$$

$$-\triangle u = n - n^*, \tag{4.11}$$

$$n(x,t) = |\psi(x,t)|^2, \tag{4.12}$$

$$\psi(x, 0) = \phi(x); \tag{4.13}$$

under the various boundary conditions on the rectangle $Q = [-\frac{1}{2}L_x, \frac{1}{2}L_y] \times [0, \frac{1}{2}L_y] \subset \mathbb{R}^2$. In (4.11) n^* is a given time-independent dopant density which may be represented as

$$n^* = n_D^+ - n_A^-,$$

where n_D^+ is the density of donors and n_A^- the density of acceptors; ϕ is a given initial state function. They have proved the existence and uniqueness of the strong solution.

4.1 Existence of a Weak Solution in $\mathbb{R}^3$

We call a pair of functions $[\psi, u]$ a weak solution of $(4.1)-(4.4)$ if ψ and u satisfy the following conditions for all $T > 0$;

$(w1)$ $\psi \in L^\infty(0, T : H_0^1(\Omega))$, $\quad u \in L^\infty(0, T : H_0^1(\Omega))$;

$(w2)$ $\displaystyle\int_0^T [-i(\psi(t), \Psi_t(t)) - (\nabla\psi(t), \nabla\Psi(t)) + ia(\psi(t), \Psi(t)) - b(u(t)\psi(t), \Psi(t)) +$
$-k(|\psi(t)|^2\psi(t), \Psi(t))] = i(\psi(0), \Psi(0))$ for any complex function
$\Psi \in C^1(0, T : L^2(\Omega)) \cap C^0(0, T : H_0^1(\Omega))$ such that $\Psi(T) = 0$;

$(w3)$ $\displaystyle\int_0^T [(\nabla u(t), \nabla\Phi(t)) + b(\psi(t)\overline{\psi}(t), \Phi(t))]dt = 0$
for any real function $\Phi(t) \in C^0(0, T : H_0^1(\Omega))$

Theorem 4.1.1 (Existence of a Weak Solution) *: Suppose that $\psi_0 \in H_0^1(\Omega)$. Then there exists at least one solution $[\psi(t), u(t)]$ of the initial boundary value problem $(4.1)-(4.4)$. Furthermore, if $[\psi(t), u(t)]$ is a weak solution of the problem, then $\|\nabla\psi(t)\|$ and $\|\nabla u(t)\|$ tend to zero as $t \to \infty$.*

Proof of Theorem 4.1.1 We employ Galerkin's method to prove the existence of a weak solution. Let $v_j \quad (j = 1, 2, 3\ldots)$ be a system of eigenfunctions of $-\Delta$ in Ω with homogeneous Dirichlet boundary condition, that is

$$\begin{aligned} -\Delta v_j &= \lambda_j v_j, \quad x \in \Omega \\ v_j\big|_{\partial\Omega} &= 0, \quad j = 1, 2, 3\ldots\ . \end{aligned} \tag{4.14}$$

The system v_j composes a base of $L^2(\Omega)$ space. Each element belongs to
$H_0^1(\Omega) \cap H^n(\Omega) \quad n = 2, 3, \ldots$.
Let

$$\begin{aligned} \psi^m(t, x) &= \textstyle\sum_{j=1}^m \alpha_{jm}(t)v_j(t) \qquad (\alpha_{jm}(t) : \text{complex valued}), \quad \text{and} \\ u^m(t, x) &= \textstyle\sum_{j=1}^m \beta_{jm}(t)v_j(t) \qquad (\beta_{jm}(t) : \text{real-valued}) \end{aligned}$$

be solutions of the problem :

$$\begin{aligned} i\frac{d}{dt}(\psi^m(t), v_j) + (\Delta\psi^m(t), v_j) + ia(\psi^m(t), v_j) - b(u^m(t)\psi^m(t), v_j) \\ - k(|\psi^m(t)|^2\psi^m(t), v_j) = 0, \quad j = 1, 2, \ldots, m, \end{aligned} \tag{4.15}$$

$$(\Delta u^m(t), v_j) - b(\psi^m(t)\overline{\psi^m}(t), v_j) = 0, \quad j = 1, 2, \ldots, m, \tag{4.16}$$

$$\psi^m(0, x) = \psi_0^m(x)$$

and

$$\psi_0^m \to \psi_0 \quad \text{in the strong topology of} \quad H_0^1(\Omega) \tag{4.17}$$

Peano's theorem from the theory of ordinary differential equations ensure that this system has the solution $[\psi^m, u^m] \quad m = 1, 2, 3, \ldots$ locally in time which are uniquely determined by initial data, for each m.

From (4.15) we deduce that ψ^m satisfy the following four equalities:

$$i(\psi_t^m, \psi^m) + (\Delta\psi^m, \psi^m) - b(u^m\psi^m, \psi^m) + ia(\psi^m, \psi^m) - k(|\psi^m|^2\psi^m, \psi^m) = 0 \tag{4.18}$$

$$i(\psi_t^m, \psi_t^m) + (\Delta\psi^m, \psi_t^m) - b(u^m\psi^m, \psi_t^m) + ia(\psi^m, \psi_t^m) - k(|\psi^m|^2\psi^m, \psi_t^m) = 0 \tag{4.19}$$

$$-i(\psi^m, \psi_t^m) + (\psi^m, \Delta\psi^m) - b(\psi^m, u^m\psi^m) - ia(\psi^m, \psi^m) - k(\psi^m, |\psi^m|^2\psi^m) = 0 \tag{4.20}$$

$$-i(\psi_t^m, \psi_t^m) + (\psi_t^m, \Delta\psi^m) - b(\psi_t^m, u^m\psi^m) - ia(\psi_t^m, \psi^m) - k(\psi_t^m, |\psi^m|^2\psi^m) = 0 \tag{4.21}$$

From (4.18) and (4.20) it follows that

$$i\frac{d}{dt}\|\psi^m\|^2 + 2ia\|\psi^m\|^2 = 0.$$

$$\Rightarrow \|\psi^m\|^2 = e^{-2at}\|\psi^m(0)\|^2. \tag{4.22}$$

$$\Rightarrow \|\psi^m(t)\| \leqslant \|\psi^m(0)\| \leqslant c_1 < \infty. \tag{4.23}$$

Here c_1 is a positive constant independent of m. Throughout this work $c_j, j = 1, 2, 3, \ldots$ will denote various positive constants independent of m.

From (4.19) and (4.21), we obtain

$$-\frac{d}{dt}\|\nabla\psi^m\|^2 + ia[(\psi^m, \psi_t^m) - (\psi_t^m, \psi^m)] - b(u^m\psi^m, \psi_t^m)$$
$$- b(\psi_t^m, u^m\psi^m) - \frac{k}{2}\frac{d}{dt}\left(\int_\Omega |\psi^m|^4 dx\right) = 0$$

which is equivalent to

$$\frac{d}{dt}\|\nabla\psi^m\|^2 = -2a\Im(\psi^m, \psi_t^m) - b(u^m\psi^m, \psi_t^m) - b(\psi_t^m, u^m\psi^m) - \frac{k}{2}\frac{d}{dt}\left(\int_\Omega |\psi^m|^4 dx\right). \quad (4.24)$$

On the other hand u^m satisfy the following equality

$$(\Delta u^m(t), u_t^m(t)) - b(\psi^m\overline{\psi^m}, u_t^m(t)) = 0$$
$$\Rightarrow (\nabla u^m, \nabla u_t^m) + b(\psi^m\overline{\psi^m}, u_t^m(t)) = 0. \quad (4.25)$$

Also one has

$$\frac{d}{dt}(\psi^m\overline{\psi^m}, u^m) = (\psi_t^m\overline{\psi^m}, u^m) + (\psi^m\overline{\psi_t^m}, u^m) + (\psi^m\overline{\psi^m}, u_t^m). \quad (4.26)$$

From (4.24), (4.25) and (4.26), we obtain

$$\frac{d}{dt}\left(\|\nabla\psi^m\|^2 + \frac{1}{2}\|\nabla u^m\|^2 + b(\psi^m\overline{\psi}^m, u^m) + \frac{k}{2}\int_\Omega |\psi^m|^4 dx\right) + 2a\Im(\psi^m, \psi_t^m) = 0 \quad (4.27)$$

We have from (4.20)

$$(\psi^m, \psi_t^m) = i(\nabla\psi^m, \nabla\psi^m) - a(\psi^m, \psi^m) + ib(u^m\psi^m, \psi^m) + ik\int_\Omega |\psi^m|^4 dx$$
$$\Rightarrow \Im(\psi^m, \psi_t^m) = \|\nabla\psi^m\|^2 + b(u^m\psi^m, \psi^m) + k\int_\Omega |\psi^m|^4 dx.$$

Substituting the above result into (4.27) we obtain

$$\frac{d}{dt}(\|\nabla\psi^m\|^2 + \frac{1}{2}\|\nabla u^m\|^2 + b(u^m\psi^m, \psi^m) + \frac{k}{2}\int_\Omega |\psi^m|^4 dx) + 2a(\|\nabla\psi^m\|^2 + b(u^m\psi^m, \psi^m) + k\int_\Omega |\psi^m|^4 dx) = 0. \quad (4.28)$$

From (4.2) and (4.28) we get

$$\frac{d}{dt}(\|\nabla\psi^m\|^2 + \frac{1}{2}\|\nabla u^m\|^2 + b(u^m\psi^m, \psi^m) + \frac{k}{2}\int_\Omega |\psi^m|^4 dx) + a(\|\nabla\psi^m\|^2$$
$$+ \frac{1}{2}\|\nabla u^m\|^2 + b(u^m\psi^m, \psi^m) + \frac{k}{2}\int_\Omega |\psi^m|^4 dx) =$$
$$= -a(\|\nabla\psi^m\|^2 + \frac{1}{2}\|\nabla u^m\|^2 + 2b(u^m\psi^m, \psi^m) + \frac{3k}{2}\int_\Omega |\psi^m|^4 dx).$$

On the other hand we have

$$\begin{aligned}
|b(\psi^m\overline{\psi^m}, u^m)| \quad &\leqslant \quad (\text{Hölder Inequality}) \leqslant |b|\|\psi^m(t)\|_{L^4}\|u^m(t)\|_{L^4}\|\psi^m(t)\| \leqslant \\
&\leqslant (\text{Ladyzhenskaya Inequality in } \mathbb{R}^3) \leqslant \\
&\leqslant 2^{\frac{1}{2}}|b|\|\psi^m\|^{\frac{1}{4}}\|\nabla\psi^m\|^{\frac{3}{4}}\|u^m(t)\|_{L^4}\|\psi^m(t)\| \\
&\leqslant (\text{Sobolev Inequality}) \leqslant \\
&\leqslant 2^{\frac{1}{2}}|b|c\frac{1}{\varepsilon}\|\psi^m(t)\|^{\frac{5}{4}}\varepsilon^{\frac{1}{2}}\|\nabla\psi^m(t)\|^{\frac{3}{4}}\varepsilon^{\frac{1}{2}}\|\nabla u^m(t)\| \\
&\leqslant (\text{Young's Inequality}) \leqslant \\
&\leqslant \frac{2^4|b|^8c^8\|\psi^m(t,x)\|^{10}}{8\varepsilon^8} + \frac{3\varepsilon^{\frac{4}{3}}\|\nabla\psi^m(t)\|^2}{8} + \frac{\varepsilon\|\nabla u^m(t)\|^2}{2}. \qquad (4.29)
\end{aligned}$$

Then

$$\|\nabla\psi^m\|^2 + \frac{1}{2}\|\nabla u^m\|^2 + 2b(\psi^m\overline{\psi}^m, u^m) + \frac{3k}{2}\int_\Omega |\psi^m|^4 dx \geqslant$$
$$\geqslant \|\nabla\psi^m\|^2 + \frac{1}{2}\|\nabla u^m\|^2 + \frac{3k}{2}\int_\Omega |\psi^m|^4 dx - \frac{2^4|b|^8c^8\ |\psi^m(t,x)\|^{10}}{4\varepsilon^8}$$
$$- \frac{3\varepsilon^{\frac{4}{3}}\|\nabla\psi^m(t,x)\|^2}{4} + \varepsilon\|\nabla u^m(t,x)\|^2$$

Choose $\varepsilon = \frac{1}{2}$, and let

$$E(t) := \|\nabla\psi^m\|^2 + \frac{1}{2}\|\nabla u^m\|^2 + b(u^m\psi^m, \psi^m) + \frac{k}{2}\int_\Omega |\psi^m|^4 dx.$$

Then

$$\begin{aligned}\tfrac{d}{dt}E(t)+aE(t) &\leqslant -c_2\|\nabla\psi^m\|^2 - c_3\|\nabla u^m\|^2 - \frac{3k}{2}\int_\Omega |\psi^m|^4 dx + a\frac{2^4|b|^8c^8\|\psi^m(t,x)\|^{10}}{4\times 2^{-8}}\\ &\leqslant 2^{10}a|b|^8c^8\|\psi^m(t,x)\|^{10}\\ &= 2^{10}a|b|^8c^8e^{-20at}\|\psi^m(t)\|^{20}\\ &\leqslant c_4\end{aligned}$$

So we have

$$\begin{aligned}E'(t)+aE(t) &\leqslant h(t)e^{at},\\ \frac{d}{dt}(e^{at}E(t)) &\leqslant h(t)e^{at},\\ e^{at}E(t)-E(0) &\leqslant \int_0^t h(s)e^{as}ds,\\ E(t) &\leqslant E(0)e^{-at}+e^{-at}\int_0^t h(s)e^{as}ds.\end{aligned}$$

Since

$$\begin{aligned}\lim_{t\to\infty}\left[e^{-at}\int_0^t h(s)e^{as}ds\right] &= \lim \frac{h(t)e^{at}}{ae^{at}}\\ &= \frac{1}{a}\lim_{t\to\infty} h(t)\end{aligned}$$

and in our case $h(t)=2^{10}a|b|^8c^8e^{-19at}$ which tends to zero as $t\to\infty$. It follows that,

$$E(t)\to 0 \quad \text{as} \quad t\to\infty.$$

Also we have, for some $T_1>0$

$$E(t)\leqslant c_5 \quad \text{whenever} \quad t\geqslant T_1. \tag{4.30}$$

On the other hand we have

$$\begin{aligned}|E(t)| \geqslant \|\nabla\psi^m\|^2 + \frac{1}{2}\|\nabla u^m\|^2 +\\ + \frac{k}{2}\int_\Omega |\psi^m|^4 dx - \frac{2^4|b|^8c^8\|\psi^m(t)\|^{10}}{8\varepsilon^8} - \frac{3\varepsilon^{\frac{4}{3}}\|\nabla\psi^m(t)\|^2}{8} - \frac{\varepsilon\|\nabla u^m(t)\|^2}{2}\end{aligned} \tag{4.31}$$

Letting $\varepsilon = 1$ we have

$$|E(t)| \geqslant \frac{1}{2}\|\nabla\psi^m\|^2 + \frac{1}{4}\|\nabla u^m\|^2 + \frac{k}{2}\int_\Omega |\psi^m|^4 dx - \frac{2^4|b|^8c^8c_1^{10}}{8} \tag{4.32}$$

From (4.30) and (4.32) it follows that

$$\|\nabla\psi^m(t)\|, \|\nabla u^m(t)\| < c_6 \tag{4.33}$$

Next we obtain estimates for $\psi^m u^m$ and $\psi^m\overline{\psi^m}$

$$\begin{aligned}
\|\psi^m u^m\| &\leqslant \text{(by Hölder's Inequality)} \leqslant \|\psi^m(t)\|_{L^4}\|u^m(t)\|_{L^4} \\
&\leqslant \text{(by Ladyzhenskaya Inequality)} \leqslant \\
&\leqslant 2\|\psi^m(t)\|^{\frac{1}{4}}\|\nabla\psi^m(t)\|^{\frac{3}{4}}\|u^m(t)\|^{\frac{1}{4}}\|\nabla u^m(t)\|^{\frac{3}{4}} \\
&\leqslant \text{(by Young's Inequality)} \leqslant \\
&\leqslant 2\left(\frac{\|\psi^m(t)\|^2}{4} + \frac{\|\nabla\psi^m(t)\|^2}{\frac{4}{3}}\right)^{\frac{1}{2}}\left(\frac{\|u^m(t)\|^2}{4} + \frac{\|\nabla u^m(t)\|^2}{\frac{4}{3}}\right)^{\frac{1}{2}} \\
&\leqslant 2(\|\psi^m(t)\|^2 + \|\nabla\psi^m(t)\|^2)^{\frac{1}{2}}(\|u^m(t)\|^2 + \|\nabla u^m(t)\|^2)^{\frac{1}{2}} \\
&= 2\|\psi^m(t)\|_{H^1}\|u^m(t)\|_{H^1} \leqslant \\
&\leqslant c_7
\end{aligned} \tag{4.34}$$

From a similar computation it follows that

$$\|\psi^m\overline{\psi^m}\| \leqslant \|\psi^m(t)\|_{H^1} \leqslant c_8 \tag{4.35}$$

From $(4.23), (4.33), (4.34)$ and (4.35) it follows that we can extract subsequences $\{\psi^{m_j}\}$ and $\{u^{m_j}\}$ from $\{\psi^m\}$ and $\{u^m\}$, respectively, such that

$$\psi^{m_j} \to \psi \quad \text{in} \quad L^\infty(0, T : H_0^1(\Omega)) \quad \text{weakly star;} \tag{4.36}$$

$$u^{m_j} \to u \quad \text{in} \quad L^\infty(0, T : H_0^1(\Omega)) \quad \text{weakly star;} \tag{4.37}$$

$$\psi^{m_j}u^{m_j} \to \alpha \quad \text{in} \quad L^\infty(0, T : L^2(\Omega)) \quad \text{weakly star;} \tag{4.38}$$

$$\psi^{m_j}\overline{\psi^{m_j}} \to \beta \quad \text{in} \quad L^\infty(0, T : L^2(\Omega)) \quad \text{weakly star;} \tag{4.39}$$

$$|\psi^{m_j}|^2\psi^{m_j} \to \gamma \quad \text{in} \quad L^\infty(0, T : L^2(\Omega)) \quad \text{weakly star;} \tag{4.40}$$

Let us denote the projection from $L^2(\Omega)$ to the space spanned by $[v_1, v_2, \ldots, v_m]$ by P_m. Then from (4.15) and (4.16) we have

$$i\psi_t^m + P_m\Delta\psi^m + ia\psi^m - bP_m u^m\psi^m - kP_m|\psi^m|^2\psi^m = 0,$$
$$P_m\Delta u^m = bP_m|\psi^m|^2.$$

Then

$$\begin{aligned}\|\psi_t^m\|_{H^{-1}} &= \sup\frac{|(\psi_t^m, w)|}{\|w\|_{H_0^1}} = \sup\frac{|i(\psi_t^m, w)|}{\|w\|_{H_0^1}} \\ &\leqslant \|\nabla\psi^m\| + a\|\psi^m\| + b\|\psi^m u^m\| + k\|\psi^m\|_{L^4}^3 \\ &\leqslant c_9.\end{aligned}$$

So we have

$$\psi^m \in L^\infty(0, T : H_0^1(\Omega)) \quad \text{and} \quad \psi_t^m \in L^\infty(0, T : H^{-1}(\Omega)).$$

Then

$$\psi^m \in W_2 := \{v : v \in L^4(0, T : H_0^1(\Omega)), v_t \in L^4(0, T : H^{-1}(\Omega))\} \tag{4.41}$$

Since $H_0^1 \hookrightarrow L^4 \hookrightarrow H^{-1}$ by Theorem 2.2.11 it follows that the sequence $\{\psi^m\}$ is compact in $L^4(0, T : L^4(\Omega)) := L^4(Q)$. Thus it has a subsequence converging to ψ in $L^4(Q)$ and almost everywhere, we denote this subsequence by $\{\psi^m\}$.
So we have

$$\psi^m\overline{\psi}^m \to \psi\overline{\psi} \quad \text{almost everywhere}$$

and

$$\|\psi^m\overline{\psi}^m\|_{L^2(Q)} \leqslant C.$$

Thus

$$\psi^m\overline{\psi}^m \rightharpoonup \psi\overline{\psi} \quad \text{weakly in} \quad L^2(Q).$$

By similar reasoning we deduce that

$$\alpha = \psi u \quad \text{and} \quad \gamma = |\psi|^2\psi.$$

Let $c_j(t)$ be complex valued and $d_j(t)$ be real valued functions which are continuous and differentiable, with $c_j(T) = 0$. Multiplying (4.15) by $\overline{c_j(t)}$ and (4.16) by $d_j(t)$, and integrating from 0 to T we obtain

$$\int_0^T [-i(\psi^m(t), c_j'(t)v_j) + (\Delta\psi^m(t), c_j(t)v_j) + ia(\psi^m, c_j(t)v_j)$$
$$-b(u^m\psi^m, c_j(t)v_j) - k(|\psi^m|^2\psi^m, c_j(t)v_j)]dt = i(\psi^m(0), c_j'(0)v_j)$$
$$\int_0^T (\nabla u^m(t), d_j(t)\nabla v_j) + b(|\psi^m|^2, d_j(t)v_j)dt = 0.$$

Due to the previous results, as $m \to \infty$ we have

$$\int_0^T [-i(\psi(t), c_j'(t)v_j) + (\Delta\psi(t), c_j(t)v_j) + ia(\psi, c_j(t)v_j) - b(u\psi, c_j(t)v_j) - k(|\psi|^2\psi, c_j(t)v_j)]dt = i(\psi(0), c_j'(0)v_j)$$

$$\int_0^T (\nabla u(t), d_j(t)\nabla v_j) + b(|\psi|^2, d_j(t)v_j)dt = 0$$

The above equations by continuity are valid for all

$$\Psi \in C^1(0, T : L^2(\Omega)) \cap C^0(0, T : H_0^1(\Omega)) \quad \text{such that} \quad \Psi(T) = 0;$$
$$\Phi(t) \in C^0(0, T : H_0^1(\Omega)).$$

Moreover (4.23) and (4.30) imply that $\|\nabla u^m(t)\|$ and $\|\nabla\psi^m(t)\|$ are tending to zero with an exponential rate as $t \to \infty$. Hence $\|\nabla u(t)\|$ and $\|\nabla\psi(t)\|$ tend to zero as $t \to \infty$.

4.2 Existence of a Strong Solution

Let Ω be a bounded domain in $\mathbb{R}^3$ with sufficiently smooth boundary $\partial\Omega$.

We call a pair $[\psi, u]$ a strong solution of (4.1)−(4.4) in $[0, \infty)$ if the following conditions are satisfied for all $T > 0$

$$(s1) \qquad \psi \in L^\infty(0,T : H_0^1(\Omega) \cap H^3(\Omega)), \quad \psi_t \in L^\infty(0,T : H_0^1(\Omega))$$

$$u \in L^\infty(0,T : H_0^1(\Omega) \cap H^2(\Omega));$$

$$(s2) \qquad \int_0^T (i\psi_t - \Delta\psi + bu\psi + ia\psi + k|\psi|^2\psi, \Psi(t))dt = 0$$

for any complex function $\Psi \in C^0(0,T : L^2(\Omega))$;

$$(s3) \quad \int_0^T (\Delta u - b|\psi|^2, \Phi(t))dt = 0 \quad \text{for any real function} \quad \Phi \in C^0(0,T : L^2(\Omega)).$$

Theorem 4.2.1 (Existence of Strong Solutions) . *If $\psi_0 \in H_0^1(\Omega) \cap H^3(\Omega)$, then there exists a strong solution of the problem* (4.1) − (4.4).

Proof of Theorem4.2.1 As in the proof of Theorem 4.1.1, we employ Galerkin's method to prove the existence of a strong solution.

First we obtain estimates for the initial values of approximate solution. Let $[\psi^m, u^m]$ be solution of the equations (4.15) − (4.16) with initial data $\psi^m(0,x)$ satisfying

$$\psi^m(0,x) \to \psi_0 \quad \text{in the strong topology of} \quad H_0^1(\Omega) \cap H^3(\Omega). \tag{4.42}$$

From (4.15) and (4.16) with $t = 0$, we have

$$i\frac{d}{dt}(\psi^m(0), v_j) + (\Delta\psi^m(0), v_j) + ia(\psi^m(0), v_j) - b(u^m(0)\psi^m(0), v_j),$$
$$-k(|\psi^m(0)|^2\psi^m(0), v_j) = 0 \tag{4.43}$$

$$(\Delta u^m(0), v_j) - b(\psi^m(0)\overline{\psi^m}(0), v_j) = 0. \tag{4.44}$$

Multiplying (4.43) by $-\lambda_j\alpha'_{jm}(0)$ and summing in j from 1 to m we obtain from (4.14) and (4.15)

$$i(\psi_t^m(0), \Delta\psi_t^m(0)) + (\Delta\psi^m(0), \Delta\psi_t^m(0)) + ia(\psi^m(0), \Delta\psi_t^m(0))$$
$$- b(u^m(0)\psi^m(0), \Delta\psi_t^m(0)) - k(|\psi^m(0)|^2\psi^m(0), \Delta\psi_t^m(0)) = 0$$

The last equality is equivalent to

$$\begin{aligned} i\|\nabla\psi_t^m(0)\|^2 \quad &= (\nabla^3\psi^m(0), \nabla\psi_t^m(0)) + ia(\nabla\psi^m(0), \nabla\psi_t^m(0)) \\ &-b(\nabla u^m(0)\psi^m(0), \nabla\psi_t^m(0)) - b(u^m(0)\nabla\psi^m(0), \nabla\psi_t^m(0)). \\ &-2k(|\psi^m(0)|^2\nabla\psi^m(0), \nabla\psi_t^m(0)) - k(\nabla\overline{\psi^m}(0)(\psi^m(0))^2, \nabla\psi_t^m(0)) \end{aligned}$$

$$\begin{aligned} \Rightarrow \quad \|\nabla\psi_t^m(0)\|^2 \quad &\leqslant (\text{Hölder's Inequality}) \leqslant \\ &\leqslant (\|\nabla^3\psi^m(0)\| + |b||\nabla\|u^m(0)\|_{L^4}\|\psi^m(0)\|_{L^4} + \\ &+|b|\|u^m(0)\|_{L^4}\|\nabla\psi^m(0)\|_{L^4} + |a|\|\nabla\psi^m(0)\| \\ &+2|k|\|\psi^m(0)\|_{L^8}^2\|\nabla\psi^m(0)\|_{L^4} \\ &+|k|\|\nabla\overline{\psi^m}(0)\|_{L^4}\|\psi^m(0)\|_{L^8}^2)\|\nabla\psi_t^m(0)\|. \end{aligned}$$

By initial conditions the sum in the parenthesis is bounded, thus $\|\nabla\psi_t^m(0)\|$ is bounded.

Secondly we obtain a priori estimates. Suppose that $[\psi, u]$ is a smooth solution of $(4.1)-(4.4)$. We differentiate equation (4.1) and its adjoint with respect to t.

$$i\psi_{tt} + \Delta\psi_t + ia\psi_t - bu_t\psi - bu\psi_t - k(|\psi|^2\psi)_t = 0 \tag{4.45}$$

$$-i\overline{\psi}_{tt} + \Delta\overline{\psi}_t - ia\overline{\psi}_t - bu_t\overline{\psi} - bu\overline{\psi}_t - k(|\psi|^2\overline{\psi})_t = 0 \tag{4.46}$$

on multiplying (4.45) by $\overline{\psi}_{tt}$ and (4.46) by ψ_{tt}, respectively, we obtain

$$i(\psi_{tt}, \psi_{tt}) + (\Delta\psi_t, \psi_{tt}) + ia(\psi_t, \psi_{tt}) - b(u_t\psi, \psi_{tt}) - b(u\psi_t, \psi_{tt}) - k((|\psi|^2\psi)_t, \psi_{tt}) = 0$$

$$-i(\psi_{tt}, \psi_{tt}) + (\psi_{tt}, \Delta\psi_t) - ia(\psi_{tt}, \psi_t) - b(\psi_{tt}, u_t\psi) - b(\psi_{tt}, u\psi_t) - k(\psi_{tt}, (|\psi|^2\psi)_t) = \\ = 0$$

which are equivalent to

$$
\begin{aligned}
i(\psi_{tt},\psi_{tt}) &= (\nabla\psi_t,\nabla\psi_{tt}) - ia(\psi_t,\psi_{tt}) + b(u_t\psi,\psi_{tt}) + b(u\psi_t,\psi_{tt}) + k((|\psi|^2\psi)_t,\psi_{tt}) \\
-i(\psi_{tt},\psi_{tt}) &= (\nabla\psi_{tt},\nabla\psi_t) + ia(\psi_{tt},\psi_t) + b(\psi_{tt},u_t\psi) + b(\psi_{tt},u\psi_t) + k(\psi_{tt},(|\psi|^2\psi)_t)
\end{aligned}
$$

Adding both sides of the above equalities we get

$$
\begin{aligned}
\frac{d}{dt}\|\nabla\psi_t\|^2 = &-ia(\psi_t,\psi_{tt}) + b(u_t\psi,\psi_{tt}) + b(u\psi_t,\psi_{tt}) + k((|\psi|^2\psi)_t,\psi_{tt}) \\
&+ ia(\psi_{tt},\psi_t) + b(\psi_{tt},u_t\psi) + b(\psi_{tt},u\psi_t) + k(\psi_{tt},(|\psi|^2\psi)_t). \quad (4.47)
\end{aligned}
$$

Differentiating equation (4.2) in t and multiplying it by u_{tt}, we get

$$
(\Delta u_t,u_{tt}) = b(\overline{\psi}_t\psi,u_{tt}) + b(\overline{\psi}\psi_t,u_{tt})
$$

which is equivalent to

$$
\frac{1}{2}\frac{d}{dt}\|\nabla u_t\|^2 = -b(\overline{\psi}_t\psi,u_{tt}) - b(\overline{\psi}\psi_t,u_{tt}). \quad (4.48)
$$

One has

$$
\begin{aligned}
\frac{d}{dt}(\psi_t u,\psi_t) + \frac{d}{dt}(\psi_t u_t,\psi) + \frac{d}{dt}(\psi u_t,\psi_t) = (\psi_{tt}u,\psi_t) + \\
+ (\psi_t u,\psi_{tt}) + (\psi_{tt}u_t,\psi) + (\psi u_{tt},\psi) + \\
+ (\psi u_{tt},\psi_t) + (\psi u_t,\psi_{tt}) + 3(\psi_t u_t,\psi_t). \quad (4.49)
\end{aligned}
$$

Combining (4.47), (4.48) and (4.49) we obtain

$$
\begin{aligned}
&\frac{d}{dt}(2\|\nabla\psi_t\|^2 + \|\nabla u_t\|^2 + 2b(\psi_t u,\psi_t) + 2b(\psi_t u_t,\psi) + 2b(\psi u_t,\psi_t)) = \\
&= 6b(\psi_t u_t,\psi_t) - 4a\Im(\psi_t,\psi_{tt}) - 2k((|\psi|^2\psi)_t,\psi_{tt}) - 2k(\psi_{tt},(|\psi|^2\psi)_t).
\end{aligned}
$$

Integrating the above identity from 0 to t we get

$$
\begin{aligned}
2\|\nabla\psi_t(t)\|^2 + \|\nabla u_t(t)\|^2 + 2b(\psi_t(t)u(t),\psi_t(t)) +&\\
+ 2b(\psi_t(t)u_t(t),\psi(t)) + 2b(\psi(t)u_t(t),\psi_t(t)) = 2\|\nabla\psi_t(0)\|^2 + \|\nabla u_t(0)\|^2 +&\\
+ 2b(\psi_t(0)u(0),\psi_t(0)) + 2b(\psi_t(0)u_t(0),\psi(0)) + 2b(\psi(0)u_t(0),\psi_t(0)) +&\\
+ 6b\int_0^t(\psi_s(s)u_s(s),\psi_s(s))ds - 4a\Im\int_0^t(\psi_s(s),\psi_{ss}(s))ds&\\
- 4k\Re\int_0^t((|\psi(s)|^2\psi(s))_s,\psi_{ss}(s))ds& \quad (4.50)
\end{aligned}
$$

On the other hand we have

$$
\begin{aligned}
|b(\psi_t(t)u(t),\psi_t(t))| &\leqslant \text{(by Hölder Inequality)} \leqslant |b|\|\psi_t\|_{L^4}\|u\|_{L^4}\|\psi_t\|\\
&\leqslant \text{(by Ladyzhenskaya Inequality)} \leqslant\\
&\leqslant 2^{\frac{1}{2}}|b|\|\psi_t\|^{\frac{1}{4}}\|\nabla\psi_t\|^{\frac{3}{4}}\|u\|_{L^4}\|\psi_t\|\\
&\leqslant 2^{\frac{1}{2}}|b|\|\psi_t\|^{\frac{5}{4}}\|\nabla\psi_t\|^{\frac{3}{4}}\|u\|_{L^4}\\
&\leqslant \text{(by Sobolev Inequality)} \leqslant 2^{\frac{1}{2}}|b|\|\psi_t\|^{\frac{5}{4}}\|\nabla\psi_t\|^{\frac{3}{4}}\|u\|_1\\
&\leqslant \text{(by Young's Inequality)} \leqslant\\
&\leqslant \frac{3}{8}\|\nabla\psi_t\|^2 + \frac{5}{8}(2^{\frac{1}{2}}b)^{\frac{8}{5}}\|u(t)\|_1^{\frac{8}{5}}\|\psi_t\|^2;
\end{aligned}
$$

$$
\begin{aligned}
|b(\psi_t(t)u_t(t),\psi(t))| &\leqslant \text{(by Hölder Inequality)} \leqslant |b|\|\psi\|_{L^4}\|u_t\|_{L^4}\|\psi_t\|\\
&\leqslant \text{(by Sobolev Inequality)} \leqslant |b|\|\psi\|_1\|u_t\|_1\|\psi_t\|\\
&\leqslant \text{(by Young's Inequality)} \leqslant \frac{1}{2}\|\psi_t\|^2 + \frac{b^2}{2}\|u_t\|_1^2\|\psi\|_1^2;
\end{aligned}
$$

$$
\begin{aligned}
|b(\psi(t)u_t(t),\psi_t(t))| &\leqslant \text{(by Hölder Inequality)} \leqslant |b|\|\psi\|_{L^4}\|u_t\|_{L^4}\|\psi_t\|\\
&\leqslant \text{(by Sobolev Inequality)} \leqslant |b|\|\psi\|_1\|u_t\|_1\|\psi_t\|\\
&\leqslant \text{(by Young's Inequality)} \leqslant \frac{1}{2}\|\psi_t\|^2 + \frac{b^2}{2}\|u_t\|_1^2\|\psi\|_1^2;
\end{aligned}
$$

$$\begin{aligned}
\int_0^t (\psi_s(s)u_s(s),\psi_s(s))ds &\leqslant \int_0^t \|u_s(s)\|\|\psi_s(s)\|_{L^4}\|\psi_s(s)\|_{L^4}ds \\
&\leqslant \text{(by Sobolev Inequality)} \leqslant c\int_0^t \|\nabla u_s(s)\|\|\psi_s(s)\|^2_{L^4}ds \\
&\leqslant \text{(Ladyzhenskaya Inequality)} \leqslant \\
&\leqslant c\int_0^t \|\nabla u_s(s)\|\|\psi_s\|^{\frac{1}{2}}\|\nabla\psi_s(s)\|^{\frac{3}{2}}ds \\
&\leqslant \text{(Young's Inequality)} \leqslant \\
&\leqslant c\int_0^t \frac{\|\nabla u_s(s)\|^2}{2}ds + c\int_0^t \frac{\|\psi_s\|^2}{4}ds + c\int_0^t \frac{\|\nabla\psi_s(s)\|^2}{\frac{4}{3}}ds;
\end{aligned}$$

To find a bound for $\|\psi_t\|^2$ we differentiate (4.1) and its adjoint with respect to t and multiply the first one by $\overline{\psi}_t$ and the second one by ψ_t

$$i(\psi_t,\psi_{tt}) + (\Delta\psi_t,\psi_t) + ia(\psi_t,\psi_t) - b(u_t\psi,\psi_t) - b(u\psi_t,\psi_t) - k((|\psi|^2\psi)_t,\psi_t) = 0$$
$$-i(\psi_t,\psi_{tt}) + (\psi_t,\Delta\psi_t) - ia(\psi_t,\psi_t) - b(\psi_t,u_t\psi) - b(\psi_t,u\psi_t) - k(\psi_t,(|\psi|^2\psi)_t) = 0$$

Subtracting the second equation from the first we obtain

$$\begin{aligned}
&i\frac{d}{dt}\|\psi_t\|^2 + 2ai\|\psi_t\|^2 - 2ib\Im(u_t\psi,\psi_t) - 2ik\Im((|\psi|^2\psi)_t,\psi_t) = 0 \\
&\Rightarrow \frac{d}{dt}(e^{2at}\|\psi_t\|^2) = 2b\Im(u_t\psi,\psi_t) + 2k\Im((|\psi|^2\psi)_t,\psi_t)
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \frac{d}{dt}(e^{2at}\|\psi_t\|^2) &\leqslant 2|b||(u_t\psi,\psi_t)| + 2|k||((|\psi|^2\psi)_t,\psi_t)| \\
&\leqslant 2|b|\|\psi_t\|\|\psi(t)\|_{L^4}\|u_t(t)\|_{L^4} + 3|k|\int_\Omega |\psi|^2|\psi_t|^2dx \\
&\leqslant 2|b|\sqrt{a}\|\psi_t\|\frac{1}{\sqrt{a}}\|\psi(t)\|_{L^4}\|u_t(t)\|_{L^4} + 3|k|\|\psi(t)\|^2_{L^4}\|\psi_t(t)\|^2_{L^4} \\
&\leqslant \frac{a\|\psi_t\|^2}{2} + \frac{4b^2\|\psi(t)\|^2_{L^4}\|u_t(t)\|^2_{L^4}}{2a} + \frac{3^{\frac{4}{3}}2^{\frac{2}{3}}\|\psi(t)\|^{\frac{8}{3}}_{L^4}\|\nabla\psi_t\|^2}{\frac{4}{3}(2a)^{\frac{1}{3}}} + \\
&\qquad + \frac{2a\|\psi_t\|^2}{4}
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \frac{d}{dt}(e^{at}\|\psi_t\|^2) &\leqslant \frac{4b^2\|\psi(t)\|_{L^4}^2\|u_t(t)\|_{L^4}^2}{2a} + \frac{3^{\frac{4}{3}}2^{\frac{2}{3}}\|\psi(t)\|_{L^4}^{\frac{8}{3}}\|\nabla\psi_t\|^2}{\frac{4}{3}(2a)^{\frac{1}{3}}} \\
&\leqslant \frac{8b^2\|\psi(t)\|_{L^4}^2\|u_t(t)\|_4^{\frac{1}{2}}\|\nabla u_t(t)\|^{\frac{3}{2}}}{2a} + \frac{3^{\frac{4}{3}}2^{\frac{2}{3}}\|\psi(t)\|_{L^4}^{\frac{8}{3}}\|\nabla\psi_t\|^2}{\frac{4}{3}(2a)^{\frac{1}{3}}} \\
&\leqslant \frac{\|u_t\|^2}{8a} + \frac{(8b^2\|\psi(t)\|_{L^4}^2)^{\frac{4}{3}}\|\nabla u_t(t)\|^2}{\frac{8a}{3}} + c_{10}\|\nabla\psi_t(t)\|^2 \qquad (4.51)
\end{aligned}$$

Differentiating (4.2) in t we get

$$\Delta u_t = b\psi_t\overline{\psi} + b\psi\overline{\psi}_t \Rightarrow \|\Delta u_t\| \leqslant 2|b|\|\psi_t\|_{L^4}\|\psi\|_{L^4}.$$

$$\begin{aligned}
\Rightarrow \|\Delta u_t\|^2 &\leqslant 4b^2|\psi|_4^2\|\psi_t\|_{L^4}^2 \leqslant 8b^2|\psi|_4^2 2a^{\frac{1}{2}}\|\psi_t\|^{\frac{1}{2}}\frac{1}{2a^{\frac{1}{2}}}\|\nabla\psi_t\|^{\frac{3}{2}} \\
&\leqslant \frac{16a^2\|\psi_t\|^2}{4} + \frac{16b^{\frac{8}{3}}\|\psi\|_{L^4}^{\frac{8}{3}}\|\nabla\psi_t\|^2}{(16a^2)^{\frac{1}{3}}\frac{4}{3}} \\
\Rightarrow \|u_t\|_{H^2}^2 &\leqslant 4a^2\|\psi_t\|^2 + \frac{12b^{\frac{8}{3}}\|\psi\|_{L^4}^{\frac{8}{3}}\|\nabla\psi_t\|^2}{(16a^2)^{\frac{1}{3}}}. \qquad (4.52)
\end{aligned}$$

Combining (4.51) and (4.52) we obtain

$$\begin{aligned}
&\frac{d}{dt}(e^{at}\|\psi_t\|^2) \leqslant \frac{a}{2}\|\psi_t\|^2 + c_{11}\|\nabla u_t\|^2 + c_{10}\|\nabla\psi_t\|^2 \\
&\frac{d}{dt}(e^{\frac{a}{2}t}\|\psi_t\|^2) \leqslant c_{11}\|\nabla u_t\|^2 + c_{10}\|\nabla\psi_t\|^2 \\
&\|\psi_t\|^2 \leqslant e^{-\frac{a}{2}t}(c_{11}\int_0^t \|\nabla u_s\|^2 ds + c_{10}\int_0^t \|\nabla\psi_s\|^2 ds + \|\psi_t(0)\|^2). \qquad (4.53)
\end{aligned}$$

We also need to find a bound for $-4k\Re\int_0^t((|\psi(s)|^2\psi(s))_s, \psi_{ss}(s))ds$.

$$
\begin{aligned}
-4k\Re\int_0^t((|\psi(s)|^2\psi(s))_s,\psi_{ss}(s))ds &\leqslant 4|k|\left|\int_0^t((|\psi(s)|^2\psi(s))_s,\psi_{ss}(s))ds\right| \\
&= 4|k|\left|\int_0^t 2(\psi_s|\psi|^2,\psi_{ss})+(\psi^2\overline{\psi}_s,\psi_{ss})ds\right| \\
&\leqslant 4|k|\left(2\left|\int_0^t(\psi_s|\psi|^2,\psi_{ss})ds\right|+\left|\int_0^t(\psi^2\overline{\psi}_s,\psi_{ss})ds\right|\right) \\
&\leqslant 4|k|\left(2\int_0^t\int_\Omega|\psi|^2|\psi_s||\overline{\psi}_{ss}|ds+\int_0^t\int_\Omega|\psi|^2|\overline{\psi}_s||\psi_{ss}|ds\right) \\
&\leqslant 12|k|\int_0^t\int_\Omega|\psi|^2\frac{d}{ds}|\psi_s|^2ds \\
&= 12|k|\left|\int_0^t\frac{d}{ds}(|\psi|^2,|\psi_s|^2)-(\frac{d}{ds}|\psi|^2,|\psi_s|^2)ds\right| \\
&\leqslant 12|k|\left(|(|\psi(t)|^2,|\psi_t(t)|^2)-(|\psi(0)|^2,|\psi_t(0)|^2)|\right) \\
&+12|k|\left|\int_0^t(\psi\overline{\psi}_s,|\psi_s|^2)ds\right|+12|k|\left|\int_0^t(\overline{\psi}\psi_s,|\psi_s|^2)ds\right| \\
&\leqslant 12|k|\left[\|\psi(t)\|_{L^4}^2\|\psi_t(t)\|_{L^4}^2+\|\psi(0)\|_{L^4}^2\|\psi_t(0)\|_{L^4}^2\right] \\
&+24|k|\left|\int_0^t\|\psi\|_{L^4}|\psi_s|_4^3ds\right| \\
&\leqslant 12|k|\left[c_{12}\|\nabla\psi(t)\|\|\psi_t\|^{\frac{1}{2}}\|\nabla\psi_t\|^{\frac{3}{2}}+c_{13}\right] \\
&+c_{14}\int_0^t\|\nabla\psi_s\|\|\psi_s\|^{\frac{3}{4}}\|\nabla\psi_s\|^{\frac{9}{4}}ds \\
&\leqslant c_{15}\|\psi_t\|^2+6|k|\varepsilon\|\nabla\psi_t\|^2 \\
&+c_{16}\int_0^t\|\nabla\psi_s\|^2ds+c_{17}\quad\text{for}\quad 0\leqslant t\leqslant T. \quad (4.54)
\end{aligned}
$$

Now it remains to estimate the expression $-4a\Im\int_0^t(\psi_s(s),\psi_{ss}(s))ds$.

We have

$$
\begin{aligned}
(\psi_t,\psi_{tt}) &= -i(\nabla\psi,\nabla\psi_{tt})-a(\psi,\psi_{tt})-ib(u\psi,\psi_{tt})-ik(|\psi|^2\psi,\psi_{tt}) \\
&= -i(\frac{d}{dt}(\nabla\psi,\nabla\psi_t)-(\nabla\psi_t,\nabla\psi_t))-a(\frac{d}{dt}(\psi,\psi_t)-(\psi_t,\psi_t)) \\
&\quad -ib(\frac{d}{dt}(u\psi,\psi_t)-(u_t\psi,\psi_t)-(u\psi_t,\psi_t))-ik(\frac{d}{dt}(|\psi|^2\psi,\psi_t) \\
&\quad -((|\psi|^2\psi)_t,\psi_t)
\end{aligned}
$$

$$\begin{aligned}
\Rightarrow \quad \Im \textstyle\int_0^t (\psi_s, \psi_{ss}) ds \quad &= \Im[-i(\nabla\psi(t), \nabla\psi_t(t)) + i(\nabla\psi(0), \nabla\psi_t(0)) + i\int_0^t \|\nabla\psi_s\|^2 ds \\
&-a(\psi(t), \psi_t(t)) + a(\psi(0), \psi_t(0)) + a\int_0^t \|\psi_s\|^2 ds \\
&-ib(u(t)\psi(t), \psi_t(t)) + ib(u(0)\psi(0), \psi_t(0)) + \\
&+ib\int_0^t (u_s(s)\psi(s), \psi_s(s)) ds + ib\int_0^t (u(s)\psi_s(s), \psi_s(s)) ds \\
&-ik(|\psi|^2\psi(t), \psi_t(t)) + ik(|\psi|^2\psi(0), \psi_t(0)) + \\
&+ik\int_0^t ((|\psi(s)|^2\psi(s))_s, \psi_s(s)) ds]
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \quad 4a\Im \textstyle\int_0^t (\psi_s, \psi_{ss}) ds \quad &\leqslant 4a\|\nabla\psi(t)\|\|\nabla\psi_t(t)\| + 4a\|\nabla\psi(0)\|\|\nabla\psi_t(0)\| + \\
&+4a\int_0^t \|\nabla\psi_s\|^2 ds + 4a^2\|\psi(t)\|\|\psi_t(t)\| + 4a^2\|\psi(0)\|\|\psi_t(0)\| \\
&+4a|b|\int_0^t \|u_s\|_{L^4}\|\psi(s)\|_{L^4}\|\psi_s(s)\| ds + 4a|b|\int_0^t \|u\|\|\psi_s(s)\|_{L^4}^2 ds \\
&+4a|k|\|\psi(t)\|_{L^4}^3\|\psi_t(t)\|_{L^4} + 4a|k|\|\psi(0)\|_{L^4}^3\|\psi_t(0)\|_{L^4} \\
&+12a|k|\int_0^t \|\psi(s)\|_{L^4}^2\|\psi_s(s)\|_{L^4}^2 ds \\
&\leqslant \frac{4a\|\nabla\psi(t)\|^2}{2\varepsilon_1} + \frac{4a\varepsilon_1\|\nabla\psi_t(t)\|^2}{2} + c_{17}\int_0^t \|\nabla\psi_s(s)\|^2 ds \\
&\qquad +c_{18}\|\psi_t(t)\|^2 + \frac{\varepsilon_2\|\nabla\psi_t(t)\|^2}{2} + c_{19}. \qquad (4.55)
\end{aligned}$$

From $(4.50), (4.54), (4.55), (4.51), (4.51)$ and (4.51) it follows that

$$c_{20}\|\nabla\psi_t(t)\|^2 + \|\nabla\psi u_t\|^2 \leqslant c_{21}\int_0^t \|\nabla\psi_s(s)\|^2 ds + c_{22}\int_0^t \|\nabla\psi u_s(s)\|^2 ds + c_{23}.$$

From Gronwall's lemma it follows that

$$\|\nabla\psi_t(t)\|, \|\nabla\psi u_t\| \leqslant c_{24} \quad \text{for} \quad 0 \leqslant t \leqslant T \qquad (4.56)$$

where c_{24} depends on a, b, k, T and initial conditions.

Thirdly we show the convergence of the approximate solutions of (4.15) and (4.16) which satisfy the initial condition (4.42). The result we obtained in (4.56) is valid for a solution $[\psi^m, u^m]$ of $(4.15) - (4.16)$. So we have

$$\|\nabla \psi_t^m(t)\|, \|\nabla u_t^m(t)\| \leqslant c_{25}, \tag{4.57}$$

where c_{25} is independent of m. Therefore we can extract subsequences $\{\psi^{m_j}\}$ and $\{u^{m_j}\}$ from $\{\psi^m\}$ and $\{u^m\}$ such that

$$\psi^{m_j} \to \psi_t \quad \text{in} \quad L^\infty(0, T : H_0^1(\Omega)) \quad \text{weakly } * \tag{4.58}$$

$$u^{m_j} \to u_t \quad \text{in} \quad L^\infty(0, T : H_0^1(\Omega)) \quad \text{weakly } * \tag{4.59}$$

Finally we show that

$$\psi \in L^\infty(0, T : H_0^1(\Omega) \cap H^3(\Omega)) \quad \text{and} \quad u \in L^\infty(0, T : H_0^1(\Omega) \cap H^2(\Omega)).$$

We rewrite (4.1) and (4.2) as follows

$$\Delta\psi = -i\psi_t - ia\psi + bu\psi + k|\psi|^2\psi = 0,$$
$$\Delta u = b|\psi|^2.$$

By the above results it follows that

$$\Delta\psi \in L^\infty(0, T : L^2(\Omega)) \quad \text{and} \quad \Delta u \in L^\infty(0, T : L^2(\Omega))$$

and consequently

$$\psi \in L^\infty(0, T : H_0^1(\Omega) \cap H^2(\Omega)) \quad u \in L^\infty(0, T : H_0^1(\Omega) \cap H^2(\Omega)) \tag{4.60}$$

Moreover (4.60) yields $\psi u \in L^\infty(0, T : H_0^1(\Omega))$.
Therefore

$$\psi \in L^\infty(0, T : H_0^1(\Omega) \cap H^3(\Omega)).$$

4.3 Uniqueness of a Strong Solution

Theorem 4.3.1 (Uniqueness of Strong Solutions) : *A strong solution* $[\psi, u]$ *of the problem* (4.1) – (4.4) *is uniquely determined by the initial data.*

Proof of Theorem4.3.1: Assume that $[\psi^{(i)}, u^{(i)}] \quad (i = 1, 2)$ are two strong solutions of (4.1) – (4.4) with the same initial data. Let $\xi = \psi^{(1)} - \psi^{(2)}$ and $v = u^{(1)} - u^{(2)}$. Then the following equations hold:

$$i\xi_t + \Delta\xi + ia\xi - bu^{(1)}\psi^{(1)} + bu^{(2)}\psi^{(2)} - k|\psi^{(1)}|^2\psi^{(1)} + k|\psi^{(2)}|^2\psi^{(2)} = 0, \tag{4.61}$$

$$\Delta v = b|\psi^{(1)}|^2 - b|\psi^{(2)}|^2, \tag{4.62}$$

which are equivalent to

$$i\xi_t + \Delta\xi ia\xi - bv\psi^{(1)} - bu^{(2)}\xi - k|\psi^{(1)}|^2\overline{\xi} - k\overline{\psi^{(2)}}\psi^{(1)}\xi - k|\psi^{(2)}|^2\xi = 0, \tag{4.63}$$

$$\Delta v = -v\psi^{(1)} - u^{(2)}\xi. \tag{4.64}$$

We multiply (4.63) by $\overline{\xi}$

$$i(\xi_t, \xi) - \|\nabla\xi\|^2 + ia\|\xi\|^2 - b(v\psi^{(1)}, \xi) - b(u^{(2)}\xi, \xi) - \int_\Omega |\psi^{(1)}|^2\overline{\xi}^2 dx$$
$$- k\int_\Omega \overline{\psi^{(2)}}\psi^{(1)}|\xi|^2 dx - k\int_\Omega |\psi^{(2)}|^2|\xi|^2 dx = 0. \tag{4.65}$$

Taking the imaginary part of (4.65) we obtain

$$\frac{d}{dt}\|\xi\|^2 + 2a\|\xi\|^2 = 2b(v\psi^{(1)}, \xi) + 2k\Im\int_\Omega |\psi^{(1)}|^2\overline{\xi}^2 dx + 2k\Im\int_\Omega \overline{\psi^{(2)}}\psi^{(1)}|\xi|^2 dx$$

$$\begin{aligned}
\Rightarrow \frac{d}{dt}\|\xi\|^2 + 2a\|\xi\|^2 \;&\leqslant\; 2b\int_\Omega |v||\psi^{(1)}||\xi| dx + 2k\int_\Omega |\psi^{(1)}|^2|\xi|^2 dx + \\
&\qquad +2k\int_\Omega |\psi^{(2)}||\psi^{(1)}||\xi|^2 dx \\
&\leqslant\; 2b\|\psi^{(1)}\|_{L^\infty}\|v\|\|\xi\| + 2k(\|\psi^{(1)}\|^2_{L^\infty} + \|\psi^{(1)}\|_{L^\infty}\|\psi^{(2)}\|_{L^\infty})\|\xi\| \\
&\leqslant\; b\|\nabla v\|^2 + \frac{b}{\lambda_1}\|\psi^{(1)}\|^2_{L^\infty}\|\xi\|^2 + 2k(\|\psi^{(1)}\|^2_{L^\infty} + \\
&\qquad +\|\psi^{(1)}\|_{L^\infty}\|\psi^{(1)}\|_{L^\infty})\|\xi\|^2.
\end{aligned} \tag{4.66}$$

On the other hand we have

$$\begin{aligned}
&\Delta v = |\psi^{(1)}|^2 - |\psi^{(2)}|^2 = \xi\overline{\psi^{(1)}} + \psi^{(2)}\overline{\xi} \\
&\Rightarrow -\|\nabla v\|^2 = \int_\Omega \xi\overline{\psi^{(1)}} v dx + \int_\Omega \overline{\xi}\psi^{(2)} v dx
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \|\nabla v\|^2 &= (\|\psi^{(1)}\|_{L^\infty} + \|\psi^{(2)}\|_{L^\infty})\|\xi\|\|v\| \\
&\leqslant \lambda_1^{-\frac{1}{2}}(\|\psi^{(1)}\|_{L^\infty} + \|\psi^{(2)}\|_{L^\infty})\|\nabla v\|\|\xi\| \\
&\leqslant c_{26}\|\xi\|^2 + \frac{1}{2}\|\nabla v\|^2 \\
\Rightarrow \|\nabla v\|^2 &\leqslant c_{27}\|\xi\|^2. \qquad (4.67)
\end{aligned}$$

From (4.66) and (4.67) we get

$$\frac{d}{dt}\|\xi\|^2 \leqslant c_{28}\|\xi\|^2 \qquad (4.68)$$

where $c_{28} = c_{27} + (\frac{b}{\lambda_1} + 2k + 2k\|\psi^{(2)}\|_{L^\infty})\|\psi^{(1)}\|_{L^\infty}$.

$$(4.68) \Rightarrow \|\xi\|^2 = 0 \Rightarrow \xi \equiv 0 \qquad (4.69)$$

$$(4.69) \quad \text{and} \quad (4.67) \Rightarrow v \equiv 0.$$

Chapter 5

CONCLUSION

The Schrödinger-Newton equations as introduced by Penrose are closely related to the Schrödinger-Poisson equations which have been studied for much longer. They are also the non-relativistic limit of the Einstein equations with a complex Klein-Gordon field as source which is why their solutions are sometimes known as boson stars.

There is a remarkable amount of research done on time-independent Schrödinger-Newton system. The spherically-symmetric stationary solutions of Schrödinger-Newton system have been the subject of several independent numerical studies [9] and [15]. In [15] I. Moroz, R. Penrose and K. P. Tod with the aid of numerical techniques obtained the following results

- There is a discrete family of finite smooth solutions labeled by the positive integers; for the nth solution, $S(r)$ has $n-1$ zeros;
- For these solutions, $S(r)$ is normalizable;
- The energy eigenvalues are negative, increasing monotonically with n towards zero;
- These 'bound states' are the only solutions which are smooth and finite for all r;
- Singular solutions blow up at finite values of r with $V \to -\infty$ and $S \pm \infty$.

The solutions found numerically have been proven to exist by K. P. Tod and I. M. Moroz in [24].

The existence of stationary solutions for a Schrödinger-Poisson system in $\mathbb{R}^3$, under appropriate assumptions on the data is proved by K. Benmlih in [2].

D. Ruiz studied the Schrödinger-Newton system under the effect of a nonlinear local term in [20]. The author has proved results on existence and nonexistence of stationary solutions of the system in his paper.

We have shown the existence of weak solution of the initial boundary value problem for Dissipative Nonlinear Schrödinger-Newton Equation on a bounded domain in $\mathbb{R}^3$. We also have proved some results on the asymptotic behavior of solutions.

We have proved that the initial boundary value problem for Dissipative Nonlinear Schrödinger-Newton Equation has a unique strong solution on a bounded domain in $\mathbb{R}^3$.

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