FRACTIONAL BROWNIAN MOTION IN FINANCE FROM ARBITRAGE POINT OF VIEW

by

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This is to certify that I have examined this copy of a master's thesis by

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and have found that it is complete and satisfactory in all respects,
and that any and all revisions required by the final
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To the memory of my mother.

ABSTRACT

Fractional Brownian motion is a centered Gaussian process with stationary increments that is stochastically self-similar. It is suggested as a model in various disciplines, one of which is finance. Arbitrage is a trading strategy where positive earning is guaranteed with no risk. It is not expected in fair markets. Despite the fact that fractional Brownian motion allows for arbitrage, it has found a place in finance by capturing the long-range dependence observed in stock prices.

We review the results recently obtained for arbitrage strategies when the stock price process is based on fractional Brownian motion. These are fractional Bachelier and fractional Black-Scholes models for the stock price or its logarithm. The suggested modifications in the model or in the trading to avoid arbitrage opportunities are analyzed. Existing stock price models which approximate a fractional Brownian motion in the limit are also studied.

We construct two agent based stock price models as integrals with respect to a Poisson random measure. These processes are analyzed as the trading occurs more frequently and in smaller quantities. Fractional Brownian motion is obtained in the limit in the sense of finite dimensional distributions. We show that our simplified scaling is equivalent to time scaling used frequently for such limits.

ÖZET

Kesirli Brown hareketi, durağan artışlara sahip ortalanmış, özbenzer bir Gauss sürecidir. Pek çok disiplinde olduğu gibi finans alanında da model olarak kullanılması önerilmiştir. Arbitraj, risk almadan pozitif kazancın garanti edildiği bir alım satım stratejisidir. Adil piyasalarda beklenilen bir olgu değildir. Kesirli Brown hareketi, arbitraja izin vermesine rağmen, hisse senedi fiyatlarındaki uzun sureli bağımlılığı modellemesi sayesinde finans alanında kendine bir yer edinmiştir.

Kesirli Brown hareketi temelli hisse senedi fiyatı modelleri için yakın zamanda bulunmuş arbitraj stratejileri ile ilgili sonuçları gözden geçirmekteyiz. Bunlar, hisse senedi fiyatı veya logaritmasi icin oluşturulmuş kesirli Bachelier ve kesirli Black-Scholes modelleridir. Arbitrajı önlemek için alım satımda veya modellerde önerilen değişiklikler incelenmiştir. Limitte kesirli Brown hareketine yaklaşan mevcut hisse senedi modelleri de çalışılmıştır.

Poisson rastsal ölcüme göre integrallerden oluşan, çok sayıda acentaya dayanan iki hisse senedi modeli kurmaktayız. Bu süreçler, alım satım sıklığının artması ve miktarın küçülmesi durumlarında incelenmektedir. Sonlu boyutlu dağılımlar anlamında, limitte kesirli Brown hareketi elde edilir. Benzer limitlerde sıkça kullanılan zaman ölçeklemesinin kullandığımız basitleştirilmiş ölçeklemeyle denk olduğunu göstermekteyiz.

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a.e.	:	Almost everywhere
a.s.	:	Almost sure(ly)
$\mathbf{aS}((\mathcal{F}_t))$	:	The set of almost simple predictable integrands
B	:	Brownian motion
B^H	:	Fractional Brownian motion
EMM	:	Equivalent Martingale Measure
$\mathbb{E}X$	:	Expected value of a random variable X
$\mathbb{E}_{\mathbb{P}}[X\|(\mathcal{F}_t)]$	:	Expectation of the random variable X given the filtration $(\mathcal{F}_t)$
$\mathcal{F}_t$	:	The $\sigma-$ algebra generated by $\{X_s; 0 \leq s \leq t\}$
$(\mathcal{F}_t)$	:	A filtration
$G(\theta)$	:	Gains process of a portfolio
H	:	The Hurst parameter
K	:	Strike price of an option
μ	:	A measure
$M = (M_t)_{t\in\mathbb{T}}$	:	A martingale
N	:	Poisson random measure

$\mathbb{N}$	:	The set of natural numbers
φ_X	:	The characteristic function of X
$\mathbb{R}$	:	The set of real numbers
S	:	Price of an asset
$\mathbf{S}((\mathcal{F}_t))$	:	The set of simple predictable integrands
$\mathbb{T}$	:	Time set which refers to either $\{0, 1, \ldots, T\}$ or $[0, T]$
$\theta_t = (\theta_t^i)_{0 \leq i \leq d}$	:	Portfolio of an investor
(θ_t)	:	Trading strategy
$\Theta^{\mathbf{S}}((\mathcal{F}_t))$	:	The set of simple predictable trading strategies
$\Theta^{\mathbf{S}}_{sf}((\mathcal{F}_t))$	:	The set of simple predictable self-financing trading strategies
$\Theta^{\mathbf{S}}_{adm}((\mathcal{F}_t))$	:	The set of simple predictable admissible trading strategies
$\Theta^{\mathbf{aS}}((\mathcal{F}_t))$	:	The set of almost simple predictable trading strategies
$\Theta^{\mathbf{aS}}_{sf}((\mathcal{F}_t))$	:	The set of almost simple predictable self-financing trading strategies
$\Theta^{\mathbf{aS}}_{adm}((\mathcal{F}_t))$	:	The set of almost simple predictable admissible trading strategies
$(\Omega, \mathcal{F}, \mathbb{P})$	:	A probability space
$V_t(\theta)$	:	Value process of a portfolio θ at time t
X	:	Price of a riskless asset
Y	:	Price of a risky asset
$\mathbb{Z}$	:	The set of integers

Chapter 1

INTRODUCTION

Finding a stochastic process that fits best as a stock price process has occupied many financial mathematicians for decades. Brownian motion and exponential Brownian motion are the most common models. On one hand, they are preferred since they model perfectly the fluctuations of price and do not allow arbitrage opportunities. On the other hand, they are not the most suitable models since their increments are independent. This situation does not reflect the reality that stock price increments at different times are dependent with each other. For this reason, a search for a better model has continued.

Fractional Brownian motion (fBm) has recently been considered as a model for stock prices [2, 8, 27, 37, 40]. Although it is known to allow for arbitrage and explicit arbitrage strategies have been constructed, fBm models the long-range dependence structure in the price process properly [5, 30].

Arbitrage is a trading strategy, in which positive earning is guaranteed without taking any risk. The initial cost of the trading is zero, but the final result is a positive gain. It occurs as a result of mispricing. It is an "unfair earning". Therefore, it is not a desired event in economics. One of the first assumptions of economic theory is that there is no arbitrage opportunity in the market. Therefore, in modeling the stock prices, processes that lead to arbitrage are not preferred.

Brownian motion and geometric Brownian motion have widely been used as stock price models [33, 42]. It is shown that since these processes are semimartingales they do not allow arbitrage [14]. They are suitable to be used in modeling stock prices from this perspective. On the other hand, they have independent increments in contradiction to the observation that price at different times depend on each other in real markets [5, 30]. Although fBm is not a semimartingale and cannot satisfy the NFLVR property [13], it is suggested since its increments are dependent.

In addition to stochastic analysis, there are several studies that model stock prices using deterministic approach in which the price is determined by the total demand [15, 45, 20, 4, 6]. In some of these studies, the agents are considered to be in two categories as chartists and fundamentalist, according to their trading behavior [18, 21, 23].

In this thesis, we first review and compare the results recently obtained on the interplay of fBm and arbitrage. We start by giving necessary definitions and results from Probability and Finance in Chapter 2. In Chapter 3, we study the stock price models that are related with fBm. In [2], there are N types of agents in a stock price model based on a semi-Markov process and the limit of the price process is found to be an integral with respect to fBm under certain scaling. Another stock price model is constructed with a Poisson shot-noise process in [27]. In this model, the price is assumed to be determined by the shocks that fall in the market, such as a political decision or some rumor concerning a merger. The limit is again found to be fBm. Although these studies suggest fBm as a limiting model, there are two well known stochastic models based on fBm. These are fractional Bachelier model

$$X_t = 1, \qquad Y_t = Y_0 + \nu t + \sigma B_t^H, \quad t \in [0,T]$$

and fractional Black- Scholes model

$$X_t = e^{rt}, \qquad Y_t = Y_0 e^{(r+\nu)t + \sigma B_t^H}, \, t \in [0,T]$$

where X and Y denote the bond and stock price processes, respectively, B_t^H is an fBm, ν is a real constant and σ is a positive constant on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F})_{t\in\mathbb{R}_+}, \mathbb{P})$. Fractional Brownian motion's increments are positively correlated at different times for the Hurst index $H \in (\frac{1}{2}, 1)$. This case is suitable for reflecting the positive correlation on prices. But fBm is not a semimartingale, therefore, an equivalent martingale measure cannot be constructed for it. This leads to the opportunity of arbitrage. Explicit arbitrage strategies have been constructed in fBm models [1, 8, 37, 40]. We analyze and compare these strategies in Chapter 3. Modifications to avoid arbitrage opportunities in the model or trading are suggested in [8, 37]. We will also review these suggestions.

In Chapter 4, we construct two agent based stochastic models for stock price processes. In these models, every buy and sell order for a stock causes an increase or a decrease in the logarithm of its price. These effects are aggregated and determine the price by the following relation

$$X(t) = \int_{-\infty}^{\infty} \int_{0}^{\infty} \int_{-\infty}^{\infty} (g(t-s,u,q) - g(-s,u,q))N(ds,du,dq) \qquad t \geq 0.$$

where $X(t)$ denotes the logarithm of the price, N is a Poisson random measure and g is the effect function, also called a pulse, depending on the arrival time, amount of the order and duration which are random. We analyze the scaling of these processes as the trading occurs more and more frequently and in smaller and smaller quantities so that the limit is a fBm.

The models which result from aggregating effects and approximate fBm have been well studied [9, 10, 27, 25]. They are used for modeling stock prices [27], internet traffic [10] and telecommunication processes [25]. In these studies, processes that are obtained by aggregating the effects of some type of pulses are analyzed. Arrival of these pulses constitutes a Poisson process in all of them. The pulses which are generated according to a compound Poisson process are studied in [10] and in [25]. Also the kind of pulses that are generated continuously by a constant random rate are studied in [25]. The effects do not vanish in time and there are no negative effects. Every pulse that is born after time 0 is taken into consideration. Hence, the mean value of the process increases to infinity as t tends to infinity. Therefore, to calculate the limit of the process, first the mean is subtracted. On the other hand, pulses whose effect vanish in a random amount of time are studied in [9]. Two kinds of pulses are considered. In this case, the expected effect is simply zero and the mean need not be subtracted. In [27], on the other hand, Poisson shot-noise processes in which the pulses have effects that last forever are studied. In this case, the pulses come with a jump and decrease from that time onwards exponentially. In this case, all of the pulses that were born before time t are considered. From this value, the total value of pulses that are alive at time 0 are subtracted. When a pulse is born, its effect never dies. The pulses can take positive and negative values coming from a symmetric probability distribution, therefore the mean value stays at 0.

We study two kinds of pulses that are effective on the logarithm of the stock price. One is the case studied in [9], and the other is studied partly in [25]. In both situations, the pulse changes the price continuously. It does not cause a jump in the price. In the first case, the effect can be of many different shapes satisfying certain conditions. One example to this is the case when the effect increases in the beginning and then decreases linearly until it takes back its effect totally in a finite amount of time. This kind of pulses help us model the following situation. If a buy order comes to the market, it causes the price increase for some amount and then when its effect vanishes after a period the price falls back to its previous value. Even if there are only buy orders for the stock, this will not cause a continuous increase in the price. The effects vanish and the price before the order arrival is reached again. In the second case, the effect of the pulse increases linearly and takes its maximum value at the instant it dies. If a buy order comes to the market, it causes the price increase some amount and then when its effect vanish in some period the price will stay at this higher value and will not fall back to its previous value. So, in this case, if there are only buy orders for the stock, the price will continuously increase since these effects will never vanish. Therefore, we impose that the buy and sell orders come from a symmetric distribution to have the mean stay at 0. However, the positive and negative effects need not be symmetric for the mean of the price stay unchanged in the first case. This is due to the vanishing effects of the pulses.

A type similar to ours is studied in [25]. Our model differs from [25] since only positive effects are considered in [25] and then the mean needs to be subtracted to analyze the asymptotic behavior of the process. We let the pulses take both positive and negative values coming from a symmetric distribution and hence we do not need to subtract the mean to approximate an fBm which is always zero. Also in [27], effects having positive and negative values from a symmetric distribution are considered. Our model differs from it in the sense that our pulses have finite lifetime, whereas the lifetime is infinite in [27].

Our study also contributes to the unification of the various scalings for an fBm limit by reviewing their equivalence. Namely, all such scalings boil down to increasing the arrival rate of Poisson events as well as decreasing the quantity. Chapter 5 includes these conclusions and summarizes our findings on arbitrage strategies and fBm.

Chapter 2

PRELIMINARIES

In this chapter, we give some facts from probability theory and finance that are necessary for the development of the other chapters. Then, we state some definitions from probability in Section 2.1 ,and from finance in Section 2.2. Finally, Section 2.3 covers some fundamental theorems about asset pricing and two examples.

2.1 Probability Preliminaries

Recall that a **probability space** $(\Omega, \mathcal{F}, \mathbb{P})$ is a special measure space [12] where the measure has total mass 1. The subsets of Ω which belong to the σ-algebra $\mathcal{F}$ are called **events**. A probability space is **complete** if all subsets of the events with probability 0 are in the σ-algebra. A real-valued **random variable** $X : \Omega \to \mathbb{R}$ is a mapping such that $\{\omega \in \Omega : X(\omega) \leq x\}$ is an event for each $x \in \mathbb{R}$ and a **stochastic process** $X = \{X_t : t \in \mathbb{R}\}$ is a collection of random variables.

We begin with some definitions from [12]. Expectation notion is equivalent to integration in measure theory. The integral of a measurable function X with respect to a probability measure $\mathbb{P}$ corresponds to the expected value of X with respect to that probability measure. The precise definition is given below.

Definition 2.1.1 *Let X be an $\mathbb{R}$ valued random variable on $(\Omega, \mathcal{F}, \mathbb{P})$. We call its integral with respect to the measure $\mathbb{P}$, **expected value** of X and denote by any of the following:*

$$\mathbb{E}X = \int_{\Omega} \mathbb{P}(d\omega)X(\omega) = \int_{\Omega} X d\mathbb{P}.$$

If we have more that one probability measure on a measurable space $(\Omega, \mathcal{F})$ then we denote the expectation with respect to the probability measure $\mathbb{P}$ as $\mathbb{E}_{\mathbb{P}}$.

The **characteristic function** φ_X of a random variable X is defined [24] on $\mathbb{R}$ as

$$\varphi_X = \mathbb{E}e^{iuX}.$$

The characteristic function of a random variable is its Fourier transform and it always exists. As in ordinary functions, there is a one to one correspondence with the space of random variables and their characteristic functions. Therefore, a characteristic function is sufficient to characterize the probability distribution of a random variable.

We continue with the definitions of random measures and Poisson random measure.

Definition 2.1.2 *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(\Psi, \mathcal{G})$ be a measurable space. A mapping $N : \Omega \times \mathcal{G} \to \mathbb{R}_+$ is called a **random measure** on $(\Psi, \mathcal{G})$ if $\omega \to N(\omega, A)$ is a random variable for each $A \in \mathcal{G}$ and $A \to N(\omega, A)$ is a measure on $(\Psi, \mathcal{G})$ for each $\omega \in \Omega$.*

Definition 2.1.3 *Let μ be a measure on $(\Psi, \mathcal{G})$. A **Poisson random measure (Prm)** with mean μ on $(\Psi, \mathcal{G})$ is a random measure on $(\Psi, \mathcal{G})$ with the properties*

a)for every $A \in \mathcal{G}$, the random variable $N(A)$ has the Poisson distribution with mean $\mu(A)$,

b) whenever $A_1, \dots, A_n$ are in $\mathcal{G}$ and disjoint, the random variables $N(A_1), \dots, N(A_n)$ are independent, for every $n \geq 2$.

Let $\mathcal{G}_+$ denote all $\mathcal{G}$-measurable functions on $(\Psi, \mathcal{G})$ taking values in $\mathbb{R}_+$. For each $f \in \mathcal{G}_+$, the Prm N defines a positive random variable $N f$ with the relation

$$Nf(\omega) = \int_\Psi N(\omega, dx) f(x), \ \omega \in \Omega.$$

The expected value of this random variable is given by

$$\mathbb{E}\, N\, f = \mu\, f, \tag{2.1}$$

its variance by

$$\mathrm{Var}\, N\, f = \mu\,(f^2) - (\mu\, f)^2, \tag{2.2}$$

and its characteristic function by

$$\mathbb{E}\, e^{iNf} = e^{\mu(1-e^{-if})}. \tag{2.3}$$

We define convergence in probability below which is the same as convergence in measure. This mode of convergence is central to much of modern stochastics. Stochastic calculus and stochastic differential equations employ convergence in probability as their basic mode of limit taking.

Definition 2.1.4 *The sequence (X_n) is said to converge to X in probability if, for every $\varepsilon > 0$,*

$$\lim_{n\to\infty} \mathbb{P}\{|X_n - X| > \varepsilon\} = 0.$$

We now give some definitions from [11] that are the building blocks of stochastic calculus.

Definition 2.1.5 *A random vector $X = (X_1, \ldots, X_n)$ taking values in $\mathbb{R}^n$ is said to be **Gaussian** provided that every possible linear combination of $X_1, \ldots, X_n$ is Gaussian. Let $X = \{X_t : t \in \mathbb{R}_+\}$ be a stochastic process with state space $\mathbb{R}$. It is said to be a **Gaussian process** if the vector $(X_{t_1}, \ldots, X_{t_n})$ is Gaussian for every choice of the integer $n \geq 1$ and times $t_1, \ldots, t_n$.*

Definition 2.1.6 *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. For each $t \in \mathbb{R}$, let $\mathcal{F}_t$ be a sub-σ-algebra of $\mathcal{F}$. If*

i) $\mathcal{F}_s \subset \mathcal{F}_t$ for $s \leq t$,

ii) $\mathcal{F}_t = \cap_{\varepsilon>0}\mathcal{F}_{t+\varepsilon}$ for every t and

iii) $\mathcal{F}_0$ contains all the $\mathbb{P}$-null sets of $\mathcal{F}$

*then the family $(\mathcal{F}_t) \equiv \{\mathcal{F}_t : t \in \mathbb{R}\}$ is called a **filtration**. We refer to a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ satisfying these conditions as a filtered, complete probability space satisfying the usual hypothesis. Let $\{X_t : t \in T\}$ be a stochastic process. Let $(\mathcal{F}_t)$ be a filtration such that $\mathcal{F}_t$ is the smallest σ-algebra that contains all the sets $\{X_t \leq r\}$ for $s \leq t$ and $r \in \mathbb{R}$. Then $(\mathcal{F}_t)$ is said to be the **filtration generated by the process** X.*

Sometimes we need a different index set than $\mathbb{R}$, and denote it by $\mathbb{T}$. Then, the notation $(\mathcal{F}_t)_{t\in\mathbb{T}}$ is used.

Definition 2.1.7 *Let $(\mathcal{F}_t)$ be a filtration. Let X be a stochastic process. Then X is said to be* ***adapted*** *to $(\mathcal{F}_t)$ if X_t is measurable with respect to $\mathcal{F}_t$ for each t. A random time $T : \Omega \to \mathbb{R}$ is said to be a* ***stopping time*** *of $(\mathcal{F}_t)$ if the process $t \to 1_{[T,\infty)}(t)$ is adapted to $(\mathcal{F}_t)$.*

Definition 2.1.8 *Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ be a filtered probability space. The real-valued process $M = (M_t)_{t\in\mathbb{R}}$ is said to be a* ***martingale*** *with respect to the filtration $(\mathcal{F}_t)$ provided that*

a)M is adapted to $(\mathcal{F}_t)$,

b)M_t has a finite mean for each t,

c)for every $s \leq 0$ and $t > s$,

$$\mathbb{E}[M_t - M_s|\mathcal{F}_s] = 0.$$

The process $M = (M_t)_{t\in\mathbb{R}}$ is called a ***supermartingale*** *if c) is replaced by $\mathbb{E}[M_t - M_s|\mathcal{F}_s] \leq 0$, and a* ***submartingale*** *if c) is replaced by $\mathbb{E}[M_t - M_s|\mathcal{F}_s] \geq 0$.*

Definition 2.1.9 *A stochastic process X on $\mathbb{R}^n$ is* ***stochastically continuous*** *or* ***continuous in probability*** *if, for every $t \geq 0$ and $\varepsilon > 0$,*

$$\lim_{s\to t} \mathbb{P}\{|X_s - X_t| > \varepsilon\} = 0.$$

The p^{th} variation of a function X can defined as

$$\lim_{n\to\infty} \sum_{k=0}^{2^n-1} \left| X\left(\frac{k+1}{2^n}t\right) - X\left(\frac{k}{2^n}t\right)\right|^p .$$

If X is a random variable, we consider the behavior of its sample paths, that is for $\omega \in \Omega$, we evaluate a.s.

$$\lim_{n\to\infty} \sum_{k=0}^{2^n-1} \left| X\left(\omega, \frac{k+1}{2^n}t\right) - X\left(\omega, \frac{k}{2^n}t\right)\right|^p$$

The cases $p = 1$ and $p = 2$ are called the **variation** and the **quadratic variation** of X, respectively. If the variation of X is finite, X is said to have finite variation.

Definition 2.1.10 *Let $(\mathcal{F}_t)$ be a filtration and let $M = (M_t)$ be a continuous process adapted to $(\mathcal{F}_t)$. Then, M is said to be a* ***local martingale*** *if there exists a sequence of stopping times T_n with $\lim_{n\to\infty} T_n = \infty$ such that $\{M_{t\wedge T_n} : t \in \mathbb{R}_+\}$ is a bounded martingale for each n. We shall call a continuous process X a* ***Stieltjes process*** *if, for almost every ω, the path $t \to X_t(\omega)$ has finite variation over finite intervals. A continuous process X is called a* ***semimartingale*** *if it can be decomposed as*

$$X = M + V$$

where M is a local martingale and V is Stieltjes.

We can now describe stochastic integration. Let (F_t) be a filtration on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let X be a continuous semimartingale, and F be a continuous process both adapted to $(\mathcal{F}_t)$. Let σ be a subdivision of $\mathbb{R}_+$, that is, σ is a sequence of times $0 = t_0 < t_1 < t_2 < \dots$ with $\lim_{n\to\infty} t_n = \infty$. We let $||\sigma||$ denote the mesh of σ, that is, $||\sigma|| = \sup_{i\in\mathbb{Z}} |t_{i+1} - t_i|$. We define

$$F_\sigma(\omega, t) = F(\omega, t_i) \qquad \text{for} \qquad t_i < t \leq t_{i+1}.$$

Thus, for each ω, the function $t \to F_\sigma(\omega, t)$ is a left-continuous step function. For such a function, we define the integral

$$Y_\sigma(\omega, t) = \int_{(0,t]} F_\sigma(\omega, s) dX(\omega, s) \tag{2.4}$$

in the fashion

$$Y_\sigma(\omega, t) = F(\omega, t_i)\big[X(\omega, t) - X(\omega, t_i)\big] + \sum_{j=0}^{i-1} F(\omega, t_j)\big[X(\omega, t_{j+1}) - X(\omega, t_j)\big]$$

for $t_i < t \leq t_{i+1}$. In this manner, for each subdivision σ, the integral (2.4) yields a continuous process $\{Y_\sigma(t) : t \geq 0\}$, which is an integral in all possible senses of the term. As $||\sigma|| \to 0$, it is clear that the paths $s \to F_\sigma(\omega, s)$ approach the path $s \to F(\omega, s)$. Hence, if the process Y_σ converges to some process Y in a reasonable sense, then we call Y the integral of F with respect to X. This is accomplished by the next theorem.

Theorem 2.1.11 *There exists a unique process Y such that*

$$\lim_{||\sigma||\to 0} \mathbb{P}\{\sup_{0\le s\le t} |Y_\sigma(s) - Y(s)| > \varepsilon\} = 0 \tag{2.5}$$

for every $\varepsilon > 0$ and $t < \infty$.

Definition 2.1.12 *We define the **stochastic integral** of F with respect to X to be the unique process Y described in Theorem 2.1.11 , that is, we set*

$$\int_0^t F dX = Y(t), \qquad t \in \mathbb{R}_+.$$

The following is the simplest version of Itô's Formula. The general ones can be found in [11].

Theorem 2.1.13 *(Itô's Formula) Let X be an $\mathbb{R}$-valued continuous semimartingale adapted to the filtration $(\mathcal{F}_t)$. For $f \in C^2$, $f(X)$ is a semimartingale and*

$$df(X) = f'(X)dX + \frac{1}{2}f''(X)dX\,dX$$

where $dX\,dX$ is a notation for dQ where Q is the quadratic variation process for X.

2.2 Definitions in Finance

An investment instrument that can be bought and sold is frequently called an ***asset***. Examples of an asset is cash, real estate and other properties that are owned by an individual or corporation which have economic value and could be converted to cash. A ***security*** is an asset which is not a real good in the sense of having intrinsic value but instead is traded in the financial markets only as pieces of paper, or as entries in a computer database. ***Risk*** is defined as the uncertainty contained in the value of an asset, in mathematical terms, an asset is a non-risky asset if its value is deterministic, and it is a risky asset, if its value follows a stochastic process [31].

We assume that we are given a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which we will define the financial market. Let $\mathbb{T}$ be a time set. We will either have $\mathbb{T} =$

$\{0, 1, \ldots, T\}$ or $\mathbb{T} = [0, T]$, where the points of $\mathbb{T}$ are the trading dates and T is the ***terminal date of the economic activity*** being modeled [16].

A ***derivative*** or ***contingent claim*** is a security whose value or ***pay off*** depends on the value of some underlying security. The underlying security on which a derivative is based on could be a security in a financial market, such as a stock, bond or a derivative itself. There are many types of contingent claims in financial markets. We will only focus on ***European contingent claims*** with the following definition [46].

Definition 2.2.1 *A European contingent claim H is an $\mathcal{F}_T$ measurable random variable.*

A European contingent claim is defined as the pay off or the value of a security which is itself referred as the contingent claim in general. An example of a European contingent claim is an ***option*** which is the right, but not the obligation, to buy (or sell) an asset under specified terms [31]. The specifications include a description of what can be bought or sold, the period of time for which the option is valid and the exercise price, or ***strike price***, which is the price at which the asset can be purchased upon exercise of the option, denoted by K below. An option which gives the right to purchase something is called a ***call*** option, whereas an option that gives the right to sell something is called a ***put*** option. Usually an option itself has a price referred as the ***option premium***, denoted by C below. A European call option with strike price $K \in (0, \infty)$ and expiration date T for a stock Y is represented by $H = (Y_T - K)^+$.

We fix a natural number d, the dimension of the market model which is the number of different risky assets that are tradable. We assume that $(S_t)_{t\in\mathbb{T}} = (X_t, Y_t^1 \ldots, Y_t^d)_{t\in\mathbb{T}}$ represents the time evolution of securities price process [16]. The process X_t denotes the price process of the riskless asset which is bond (or bank account) while $Y_t^1, Y_t^2, \ldots Y_t^d$, denote the price processes of the risky assets (for example stocks) at time t. We assume that the information structure available to the investors is given by the filtration $(\mathcal{F}_t)$ on $(\Omega, \mathcal{F}, \mathbb{P})$ where $\mathcal{F}_t = \sigma(S_u : u \leq t)$ and without loss of generality we will assume $\mathcal{F} = \mathcal{F}_T$. We call the tuple $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{T}, (\mathcal{F}_t), S)$ the ***securities***

market model [16]. In some cases we will need trading to be available only in bond and the stocks. In this case, we call the model the ***primary market model***.

We denote the interest rate with $r > 0$, and the discount factor with β. The relation between these is given by $\beta = (1+r)^{-1}$ for discrete time trading and by $\beta = e^{-rt}$ for continuous time trading. We denote the discounted process βX with $\bar{X}$.

We require at least one of the processes to be strictly positive throughout, known as the ***numéraire***. Numéraire is any process Z with $Z_t > 0\ \forall t \in \mathbb{T}$. It is the unit in which prices are measured. It may be a currency, but in real models, the numéraire is usually one of the goods such as a bond whose price is then set at 1 [16]. Without loss of generality, we will in general assume $X_0 = 1$ so that the initial value of X yields the units relative to which all other quantities are expressed. The following two definitions can be found in [16].

Definition 2.2.2 *In a d-dimensional securities market or primary market, an investor's* ***portfolio*** *at time* $t \in \mathbb{T}$ *is given by the* $\mathbb{R}^{d+1}$*-valued random variable* $\theta_t = (\theta_t^i)_{0 \le i \le d}$ *which shows the amounts held in the assets during the time interval* $(t-1, t]$ *for* $t \in \mathbb{T} = \{0, 1, \ldots, T\}, t \ge 1$ *or during the time interval* $(s, t]$ *for* $s, t \in \mathbb{T} = [0, T], 0 \le s \le t \le T$. *The stochastic process* $\theta = \{\theta_t : t \in \mathbb{T}\}$ *is called the* ***trading strategy***.

In the above definition, θ^0 represents the amount held in the bond while $\theta^1, \ldots, \theta^d$ represent the amount held in the risky assets. The following are some definitions concerning the trading strategies [8].

Definition 2.2.3 *Let* $a = \tau_1 \le \ldots \le \tau_n = b$ *and* $\{\tau_j\}_{j=1}^n$ *be* $(\mathcal{F}_t)$*-stopping times. Let* g_0 *be an* $\mathbb{R}$*-valued,* $\mathcal{F}_a$*-measurable random variable and* $\{g_j\}_{j=1}^n$ *are* $\mathbb{R}$*-valued,* $\mathcal{F}_{\tau_j}$*-measurable random variables for* $n \ge 2$.

i) The set of ***simple predictable integrands*** *is given by*
$\mathbf{S}((\mathcal{F}_t)) := \left\{ g_0 1_{\{a\}} + \sum_{j=1}^{n-1} g_j 1_{(\tau_j, \tau_{j+1}]} \right\}$. *The set of* ***simple predictable trading strategies*** *is given by* $\Theta^{\mathbf{S}}((\mathcal{F}_t)) := \{\theta = (\theta^0, \theta^1) : \theta^0, \theta^1 \in \mathbf{S}((\mathcal{F}_t))\}$.

ii) The set of ***almost simple predictable integrands*** *is given by*

$\mathbf{aS}((\mathcal{F}_t)) := \left\{g_0 1_{\{a\}} + \sum_{j=1}^{\infty} g_j 1_{(\tau_j,\tau_{j+1}]} : \mathbb{P}\{\exists\, j \ \text{ such that } \tau_j = b\} = 1\right\}$ *The set of **almost simple predictable trading strategies** is given by*
$\Theta^{\mathbf{aS}}((\mathcal{F}_t)) := \{\theta = (\theta^0, \theta^1) : \theta^0, \theta^1 \in \mathbf{aS}((\mathcal{F}_t))\}$.

Definition 2.2.4 *The **value process** or the investor's **wealth** of a trading strategy $\theta \in \Theta^{\mathbf{S}}$ at time $t \in \mathbb{T} = \{0, 1, \ldots, T\}$ is denoted by $V_t(\theta)$ which is given by*

$$\begin{aligned} V_0(\theta) &= \theta_1 \cdot S_0, \\ V_t(\theta) &= \theta_t \cdot S_t = \theta_t^0 X_t + \sum_{i=1}^{d} \theta_t^i Y_t^i \qquad (t \in \mathbb{T}, t \geq 1). \end{aligned}$$

*where $(\cdot)$ denotes the usual dot product in $\mathbb{R}^{d+1}$. The **gains process** associated with θ is defined as*

$$G_0 = 0,\ G_t(\theta) = \theta_1 \cdot \Delta S_1 + \theta_2 \cdot \Delta S_2 + \ldots + \theta_t \cdot \Delta S_t$$

where $\Delta Z_t = Z_t - Z_{t-1}$ for any function Z on $\mathbb{T}$.

We now define the replication strategy for a European contingent claim H below [46].

Definition 2.2.5 *A **replicating strategy** in the primary market for a European contingent claim H is a trading strategy $\theta \in \Theta^{\mathbf{S}}$ such that $V_T(\theta) = H$ a.s. If there exists such a replication strategy, the contingent claim is said to be **attainable**.*

We define an important notion about trading strategies in the following definition [16].

Definition 2.2.6 *A trading strategy $\theta \in \Theta^{\mathbf{S}}$ (resp. $\theta \in \Theta^{\mathbf{aS}}$) in the securities market is **self-financing** if any changes in the value $V_t(\theta)$ result entirely from net gains (or losses) realized on the investments. The set of simple simple self-financing trading strategies are denoted by $\Theta_{sf}^{\mathbf{S}}$ (resp. $\Theta_{sf}^{\mathbf{aS}}$).*

For the discrete time set $\mathbb{T} = \{0, 1, \ldots T\}$ we have the following explanation. The value of the portfolio after trading has occurred at time t and before stock prices at time $t+1$ are known is given by $\theta_{t+1} \cdot S_t$. If the total value of the portfolio has been used for these adjustments, in other words, there are no withdrawals and no new funds are invested, then this means that for all $t = 1, 2, \ldots, T-1$

$$\theta_{t+1} \cdot S_t = \theta_t \cdot S_t. \tag{2.6}$$

Using this equality for the calculation of the change in the value, we have

$$\begin{aligned} \Delta V_t(\theta) &= \theta_t \cdot S_t - \theta_{t-1} \cdot S_{t-1} \\ &= \theta_t \cdot S_t - \theta_t \cdot S_{t-1} = \theta_t \cdot \Delta S_t \end{aligned} \tag{2.7}$$

We therefore see that θ is self-financing if and only if

$$V_t = V_0(\theta) + G_t(\theta) \qquad \text{for all} \qquad t \in \mathbb{T}. \tag{2.8}$$

where G is the gains process in Definition 2.2.4.

The continuous-time analogue of the self-financing condition is [46]

$$V_t(\theta) = V_0(\theta) + \int_0^t \theta_s^0 dX_s + \int_0^t \theta_s^1 dY_s^1 + \cdots + \int_0^t \theta_s^d dY_s^d \tag{2.9}$$

or

$$dV_t(\theta) = \theta_s^0 dX_s + \theta_s^1 dY_s^1 + \cdots + \theta_s^d dY_s^d.$$

Thus, for both discrete and continuous cases, changes in the value $V_t(\theta)$ of the portfolio result only from changes in the values of the assets so that there is no external infusion of capital and no spending of wealth. We denote the set of all self-financing strategies by Θ.

Definition 2.2.7 *We call a trading strategy $\theta \in \Theta_{sf}^{\mathbf{S}}$ (resp. $\theta \in \Theta_{sf}^{\mathbf{aS}}$)* ***admissible*** *if*

$$V_t(\theta) \geq 0 \qquad \textit{for all} \qquad t \in [0, T].$$

We denote the set they define with $\Theta_{adm}^{\mathbf{S}}$ (resp. $\Theta_{adm}^{\mathbf{aS}}$). Furthermore, for $c \geq 0$, we call $\theta \in \Theta_{sf}^{\mathbf{aS}}$ ***c-admissible*** *if*

$$\inf_{t \in [0,T]} (V_t(\theta) - V_0(\theta)) \geq -c \qquad a.s.$$

We now give the definition of an equivalent martingale measure which plays a fundamental role in the analysis of arbitrage properties of financial markets [46].

Definition 2.2.8 *Two probability measures, $\mathbb{P}$ and $\mathbb{Q}$, defined on $\Omega, \mathcal{F}$ are equivalent provided for each $A \in \mathcal{F}$, $\mathbb{Q}(A) = 0$ if and only if $\mathbb{P}(A) = 0$. An **equivalent martingale measure (EMM)** (resp. equivalent local martingale measure) is a probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ such that $\mathbb{Q}$ is equivalent to $\mathbb{P}$ and the discounted price process $(\bar{Y})_{t\in\mathbb{T}}$ is a martingale (resp. local martingale) under $\mathbb{Q}$.*

We now give the following definitions of arbitrage and related terms in continuous markets from [8] with obvious simplifications.

Definition 2.2.9 *Let the time set be either $\mathbb{T} = [0, T]$ or $\mathbb{T} = \{0, 1, \ldots, T\}$ and ξ be a $[0, \infty]$- valued random variable with $\mathbb{P}\{\xi > 0\} > 0$.*
*i) A sequence of trading strategies $\{\theta(n)\}_{n=1}^{\infty}$ is a **FLVR** (free lunch with vanishing risk) if*

$$\lim_{n\to\infty} (V_T(\theta(n)) - V_0(\theta(n))) = \xi \quad \textit{in probability}$$
$$\lim_{n\to\infty} \left|(V_T(\theta(n)) - V_0(\theta(n)))^{-}\right| = 0.$$

*The sequence $\{\theta(n)\}_{n=1}^{\infty}$ is also called ξ-**FLVR** when ξ is to be indicated.*
*ii) A trading strategy θ is an **arbitrage** if $V_T(\theta) - V_0(\theta) = \xi$ a.s.. The strategy θ is also called a $\xi-$**arbitrage** when ξ is to be indicated.*
*iii) A trading strategy θ is a **strong arbitrage** if there exists a constant $c > 0$ such that $V_T(\theta) - V_0(\theta) \geq c$ a.s..*

The definition of arbitrage in the securities market is usually given as a trading strategy $\theta \in \Theta_{adm}^{\mathbf{as}}$ such that

$$V_0(\theta) = 0 \quad \text{and} \quad \mathbb{E}[V_T(\theta)] > 0 \tag{2.10}$$

which is equivalent to what we have described in Definition 2.2.9. These definitions state that arbitrage is a trading strategy in which the initial value is zero, the value always stays positive and it has a nonzero expectation. The price of an asset in a securities market is said to be ***arbitrage free*** if no arbitrage strategy can be constructed by using this asset. The securities market is called arbitrage free, if all the assets have arbitrage free prices.

We now define Brownian motion [35]. Next, we state a result frequently referred as Girsanov Theorem or Girsanov Transformation below [16]. This theorem will be used in deriving the Black-Scholes option pricing formula below.

Definition 2.2.10 *An adapted stochastic process $B = (B_t)_{0\leq t<\infty}$ taking values in $\mathbb{R}^n$ is called an* ***n-dimensional Brownian motion (Bm)*** *if*
(i) for $0 \leq s < t < \infty$, $B_t - B_s$ is independent of $\mathcal{F}_s$;
(ii) for $0 < s < t$, $B_t - B_s$ is a Gaussian with mean zero and variance matrix $(t-s)C$, for a given, non random matrix C.

A Bm has continuous sample paths. When $C = I$, the identity matrix, and $B_0 = x$ for some $x \in \mathbb{R}^n$, we call it a standard Brownian motion.

Theorem 2.2.11 *Suppose $(\theta_t), t \in [0, T]$, is an adapted measurable process such that $\int_0^T \theta_s^2 ds < \infty$ a.s. and also so that the process $\Lambda_t = \exp(-\int_0^t \theta_s dB_s - \frac{1}{2}\theta_s^2 ds)$ is an $(\mathcal{F}_t, \mathbb{P})$ martingale where B denotes one dimensional Brownian motion. Define a new measure $\mathbb{Q}_\theta$ on $\mathcal{F}_T$ by putting*

$$\frac{d\mathbb{Q}_\theta}{d\mathbb{P}}|_{\mathcal{F}_T} = \Lambda_T.$$

Then the process $W_t := B_t + \int_0^t \theta_s ds$ is a standard Brownian motion on $(\mathcal{F}_t, \mathbb{Q}_\theta)$.

2.3 Pricing Stock Options

In this section, we first give an example of single period Binomial market model. Next, we state the fundamental theorems of asset pricing. We then consider the problem of pricing an option that would not allow arbitrage opportunities. We will demonstrate the connection between the fair price, in other words the arbitrage-free price of a European call option and a replicating portfolio with an example. Finally, we derive the Black-Scholes option pricing formula.

Let $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{T}, (\mathcal{F}_t), S)$ be a two dimensional market composed of a bond X which is the numèraire, a stock Y and a European contingent claim H. We take $\mathbb{T} = \{0, 1\}$. We will find a replicating strategy $\theta = (\theta^0, \theta^1)$ in the primary market for the given European contingent claim H according to Definition 2.2.5. In other words, we seek $\theta^0, \theta^1 \in \mathbb{R}$ such that

$$V_1 = \theta^0 X_1 + \theta^1 Y_1 = H. \tag{2.11}$$

The price process Y of the stock is as follows. At time 0, Y_0 is known and Y_1 is random with the probabilities

$$\mathbb{P}\{Y_1 = uY_0\} = p \text{ and } \mathbb{P}\{Y_1 = dY_0\} = 1 - p$$

for some $p \in (0, 1)$ and $0 < d < r < u$ where r is the interest rate. This is called a Binomial model. Denoting by H^u and H^d the value of H when $Y_1 = uY_0$, the "up" price, and $Y_1 = dY_0$, the down price, respectively. We take $X_0 = 1$ since it is the numéraire. By Equation (2.11) we must have

$$\theta^0(1 + r) + \theta^1 uY_0 = H^u \tag{2.12}$$

$$\theta^0(1 + r) + \theta^1 dY_0 = H^d. \tag{2.13}$$

Solving for θ^0, θ^1 yields

$$\theta^0 = \frac{1}{1+r}\left(\frac{uH^d - dH^u}{u - d}\right), \tag{2.14}$$

$$\theta^1 = \frac{H^u - H^d}{(u - d)Y_0}. \tag{2.15}$$

The initial value of this strategy which is the wealth needed to finance the strategy is

$$\begin{aligned} V_0(\theta) &= \theta^0 X_0 + \theta^1 Y_0 \\ &= \frac{1}{(1+r)(u-d)}\left((1 + r - d)H^u + (u - (1 + r))H^d\right) \\ &= \frac{1}{1+r}\left(qH^u + (1 - q)H^d\right) \\ &= \mathbb{E}_{\mathbb{Q}}[\bar{H}], \end{aligned} \tag{2.16}$$

where $\mathbb{Q}$ is the equivalent martingale measure for H with

$$\mathbb{Q}\{Y_T = uY_0\} = q = \frac{1 + r - d}{u - d}. \tag{2.17}$$

We will now state a theorem that gives the arbitrage-free price of a European contingent claim. For this purpose let $\theta = (\tilde{\theta}^0, \tilde{\theta}^1, \tilde{\theta}^2)$ be a portfolio in the securities market according to Definition 2.2.2 where the $\mathcal{F}_0$-measurable random variables $\tilde{\theta}^0$, $\tilde{\theta}^1$, and $\tilde{\theta}^2$ represent the number of units of the bond, shares of stock and units of the contingent claim held over $(0, 1]$ respectively. The initial and time 1 values of θ are respectively $V_0(\theta) = \tilde{\theta}^0 X_0 + \tilde{\theta}^1 Y_0 + \tilde{\theta}^2 C$ and $V_1(\theta) = \tilde{\theta}^0 X_1 + \tilde{\theta}^1 Y_1 + \tilde{\theta}^2 H$ by Definition 2.2.4 where C is the price of the contingent claim. We now state the theorem.

Theorem 2.3.1 *$V_0 = \mathbb{E}_{\mathbb{Q}}[\beta H]$ is the unique arbitrage free price for the contingent claim H where $\mathbb{Q}$ is defined by (2.17).*

Proof: Let $\theta = (\theta^0, \theta^1)$ denote the replicating strategy in the primary market for the contingent claim H given in Equations (2.14) and (2.15).

We will first show that there is an arbitrage opportunity in the securities market if the price C of the contingent claim is different than V_0.

Suppose that $C > V_0$. Then an investor could sell one contingent claim ($\theta^2 = -1$) for C, invest V_0 in the replicating strategy $\theta = (\theta^0, \theta^1)$ and invest the remainder $C - V_0$, in bond. Thus, the trading strategy in the securities market would be $\bar{\theta} = (\theta^0 + C - V_0, \theta^1, -1)$. By Equation (2.16) we obtain the value process for this portfolio as follows

$$V_0(\bar{\theta}) = \theta^0 + C - V_0 + \theta^1 Y_0 - C = 0.$$

The value at time 1 is

$$V_1(\bar{\theta}) = (\theta^0 + C - V_0) X_1 + \theta^1 Y_1 - H. \tag{2.18}$$

But the strategy θ was chosen so that

$$\theta^0 X_1 + \theta^1 Y_1 = H, \tag{2.19}$$

and so it follows that the value at time 1 of the portfolio is

$$(C - V_0) X_1 > 0 \tag{2.20}$$

showing that we have an arbitrage opportunity. For the case $C < V_0$, if the investor similarly uses the strategy $(-\theta^0 + V_0 - C, -\theta^1, 1)$ which is exactly -1 times $\bar{\theta}$ an arbitrage with a total amount $(V_0 - C)X_1 > 0$ occurs. In either case, we would have arbitrage opportunities if $V_0 \neq C$.

Conversely, if $C = V_0$, then we will show that there is no arbitrage opportunity in the securities market by contradiction. Assume that $\theta = (\theta^0, \theta^1, \theta^2)$ is a trading strategy in the securities market with initial value $V_0(\theta) = 0$ and non-negative value $V_1(\theta) = \theta^0 X_1 + \theta^1 Y_1 + \theta^2 H$ at time T. So we have

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}} V_1(\theta) &= (1+r)\theta^0 + \theta^1 \mathbb{E}_{\mathbb{Q}} Y_1 + \theta^2 \mathbb{E}_Q H \\
&= (1+r)\theta^0 + (1+r)\theta^1 Y_0 + (1+R)\theta^2 C \\
&= (1+r)[\theta^0 + \theta^1 Y_0 + \theta^2 C] \\
&= (1+r)V_0(\theta) \\
&= 0,
\end{aligned}$$

where we have used the fact that $\theta^0, \theta^1, \theta^2$ are constants, the martingale property of the discounted stock price process under the measure $\mathbb{Q}$ and the assumption that $C = V_0 = \mathbb{E}_{\mathbb{Q}}[\bar{H}]$. Thus, there cannot be an arbitrage opportunity. ■

We now give two versions of the Fundamental Theorems of Asset Pricing related with the no-arbitrage and completeness concepts of financial markets. Roughly speaking, no-arbitrage principle is equivalent to the existence of a martingale measure for the price process of the assets and completeness of the market is equivalent to the uniqueness of this martingale measure.

We first state the first fundamental theorem of asset pricing for both discrete and continuous time market models [46].

Theorem 2.3.2 *There does not exist any arbitrage opportunities in the discrete time market model if and only if there exists an equivalent martingale measure* $\mathbb{Q}$ *for the price processes.*

We will now give the general version of the previous theorem when the price process is a semimartingale [13].

Theorem 2.3.3 *Let S be a bounded real valued semimartingale. There is an equivalent martingale measure for S if and only if S satisfies NFLVR.*

The previous theorem relates the existence of local martingales with the no FLVR condition given in Definition 2.2.9. Now, we turn our attention to another important property of financial markets: completeness. We define completeness as follows [28].

Definition 2.3.4 *A financial market is* ***complete*** *if every contingent claim is attainable.*

We now state the second fundamental theorem of asset pricing again for discrete and continuous time market models which relates the completeness concept of the market with the uniqueness of martingale measures. We start with the discrete time version.

Theorem 2.3.5 *An arbitrage free discrete financial market is complete if and only if it admits a unique equivalent martingale measure.*

The continuous time version of this theorem is as follows.

Theorem 2.3.6 *Suppose there is an equivalent martingale measure $\mathbb{Q}$ for the price processes. Then, the following two conditions are equivalent.*

(i) $\mathbb{Q}$ is the unique equivalent local martingale measure.

(ii) Every $\mathcal{F}_T$ measurable random variable X satisfying $\mathbb{E}_{\mathbb{Q}}|X| < \infty$ is attainable.

We now give and analyze a numerical example of a single-term binomial model [16, 46]. Let $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{T}, (\mathcal{F}_t), S)$ be a two dimensional discrete time market model. For simplicity, we set the interest rate r as zero. The time set is $\mathbb{T} = \{0, 1\}$. There is a bank account $X_0 = 1$ that is used as a numéraire, and a stock Y and a European call option H. The price process Y of the stock is as follows. $Y_0 = 10$ and Y_1 is random with the probabilities

$$\mathbb{P}\{Y_1 = 20\} = p \text{ and } \mathbb{P}\{Y_1 = 7.5\} = 1 - p$$

for some $p \in (0, 1)$. Therefore, we find $u = 20/10 = 2$ and $d = 7.5/10 = 0.75$.

Let H be a given by $H = (Y_1 - K)^+$ where $K = 15$ is the strike price at time 1. By the probability distribution of Y, we have

$$\mathbb{P}\{H = 5\} = p \text{ and } \mathbb{P}\{H = 0\} = 1 - p.$$

We look for an equivalent martingale measure $\mathbb{Q}$ on $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{T}, (\mathcal{F}_t), S)$ which will turn the process Y into a martingale and allow us to find the unique arbitrage free price for H by Theorem 2.3.1. If Y is a martingale under the measure $\mathbb{Q}$, then we would have

$$\mathbb{E}_{\mathbb{Q}}[Y_1] = Y_0. \tag{2.21}$$

We take $\mathcal{F}_0 = \{\emptyset, \Omega\}$ the trivial σ-algebra and $\mathcal{F}_1 = \sigma(Y_1)$. Putting the given data into the Equation (2.21), we get

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}} Y_1 &= (20)\mathbb{Q}\{Y_1 = 20\} + (7.5)\mathbb{Q}\{Y_1 = 7.5\} \\ &= (20)\mathbb{Q} \circ Y_1^{-1}\{20\} + (7.5)\mathbb{Q} \circ Y_T^{-1}\{7.5\} \\ &= 20q + 7.5(1 - q) = 10.\end{aligned}$$

Solving for q, we obtain the distribution of the equivalent martingale measure as $\mathbb{Q}\{Y_1 = 20\} = 0.2$ and $\mathbb{Q}\{Y_1 = 7.5\} = 0.8$.

We now use this EMM to price the contingent claim. The price C of the option must be equal to the expected value H with respect to the measure $\mathbb{Q}$ by Theorem 2.17. Therefore, the arbitrage-free price of the option is

$$C = \mathbb{E}_{\mathbb{Q}}[H] = 5(0.2) + 0(0.8) = 1.$$

We will now calculate the replicating portfolio for this option to analyze the arbitrage opportunities. Let $\theta = (\theta^0, \theta^1)$ be the replicating portfolio for the European option, where θ^0 is the amount of cash held and θ^1 the number of shares of stock held during the time $(0, T]$. By Definition 2.2.4 the value of this portfolio is

$$V_t = \theta^0 \cdot 1 + \theta^1 \cdot Y_t \qquad t = 0, 1.$$

By Definition 2.2.6 this is a self-financing portfolio since θ^0 and θ^1 are fixed and $\Delta V = \theta^0 + \theta^1 \cdot Y_1 - (\theta^0 + \theta^1 \cdot Y_0) = \Delta Y \cdot \theta^1$. We have the following equalities for the gain and value processes.

$$G_1 = \theta^1 \cdot \Delta Y \qquad V_1 = V_0 + G_1. \tag{2.22}$$

By the choice of the probability measure $\mathbb{Q}$, we have

$$V_0 = \mathbb{E}_{\mathbb{Q}} V_0 = \mathbb{E}_{\mathbb{Q}}[V_1 - G_1] = \mathbb{E}_{\mathbb{Q}} V_1 - \mathbb{E}_{\mathbb{Q}} G_1 = \mathbb{E}_{\mathbb{Q}} V_1,$$

where the last equality comes from the fact that $\mathbb{E}_{\mathbb{Q}} G_1 = \mathbb{E}_{\mathbb{Q}}[\theta^1 \cdot \Delta Y] = \theta^1 \mathbb{E}_{\mathbb{Q}}[Y_1 - Y_0] = 0$ by Equations (2.21) and (2.22). We observe that V is also a martingale which prevents any arbitrage opportunities, since for any portfolio with $V_0 = 0$ we will have $\mathbb{E} V_1 = 0$.

By Theorem 2.3.6, H is attainable and hence we can find $\theta = (\theta^0, \theta^1)$ consisting of a bank account and the stock that replicates the option by Definition 2.2.5. For example, using Equations (2.14) and (2.15) with $u = 2$, $d = 0.75$ we obtain $\theta^0 = -3$ and $\theta^1 = 0.4$.

We now describe the trading strategy consisting of bank account, stock and option as in the setting of Theorem 2.3.1. Let $\tilde{\theta} = (\tilde{\theta}^0, \tilde{\theta}^1, \tilde{\theta}^2)$ be the trading strategy by Definition 2.2.2, $\tilde{\theta}^0, \tilde{\theta}^1, \tilde{\theta}^2$ denoting the amount of cash, number of shares of stocks and number of units of options held, respectively.

Our portfolio is $\tilde{\theta}_0 = (3, -0.4, 1)$ which is constructed by buying the replicating portfolio and selling one unit of option which can be explained as

borrowing 3 dollars,

buying 0.4 stock shares with a cost of 4 dollars,

selling the option with a gain of 1 dollar.

Here, selling the option means promising to sell one share of the stock at 15 dollars at time 1 which is a written contract containing this information, instead of the formal Definition 2.2.1. Time 0 value of this portfolio is $V_0(\tilde{\theta}) = 3 + (-0.4)10 + 1 = 0$ by Definition 2.2.4 for the securities market model.

At time 1, either if $Y_1 = 20$, then the holder will exercise the option and the value of this portfolio will be $V_1(\tilde{\theta}) = -3 + (0.4)20 - 5 = 0$ explained by

repaying 3 dollars,

selling the shares with a gain of 8 dollars,

the cost of 5 dollars of the exercised option.

Or else if $Y_1 = 7.5$, the holder will not exercise the option and the value will be $V_1(\tilde{\theta}) = -3 + (0.4)7.5 + 0 = 0$ meaning

repaying 3 dollars,

selling the shares with a gain of 3 dollars,

zero cost of the option since it is not exercised.

In both cases, the $V_1(\tilde{\theta}) = 0$, so $\mathbb{E}[V_1(\tilde{\theta})] = 0p + 0(1-p) = 0$, hence no arbitrage is possible.

We now analyze the arbitrage opportunities if $C = \mathbb{E}_{\mathbb{Q}}[H] = 1$ is not the case. Suppose without loss of generality $C = 1.5 > 1$, then we construct our portfolio as in the proof of Theorem 2.3.1. As we have constructed the trading strategy $\tilde{\theta}$ above, we buy the replicating portfolio $\theta = (\theta_t^0, \theta_t^1)$ and sell one unit of option and moreover invest the excess amount 1.5-1=0.5 in the account. This gives the portfolio $\vartheta = (-3 + 0.5, 0.4, 1)$ and the value $V_0(\vartheta) = 2.5 - (0.4)10 + 1.5 = 0$.

At time 1, either if $Y_1 = 20$ then the holder of the option will exercise it and the value will be $V_1(\vartheta) = -2.5 + (0.4)20 - 5 = 0.5$ or else if $Y_1 = 7.5$, then the option will not be exercised and the value will be $V_0(\vartheta) = -2.5 + (0.4)7.5 + 0 = 0.5$. We have $V_0(\vartheta) = 0$, $V_1(\vartheta) = 0.5 \geq 0$ and $\mathbb{E}[V_1(\vartheta)] = 0.5 > 0$. Hence by Definition 2.10 an arbitrage occurs.

We next cover Black-Scholes option pricing model as given in [46]. Let (Ω, $\mathcal{F}$, $\mathbb{P}$, $[0, T]$, $(\mathcal{F}_t)$, S) be a two dimensional securities market model consisting of a bond with price process X and a stock with price process Y in the primary market given by

$$X_t = e^{rt},\ t \in [0,T] \tag{2.23}$$

$$Y_t = Y_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma B_t\right),\ t \in [0,T] \tag{2.24}$$

where $r \geq 0$, $\mu \in \mathbb{R}$, $\sigma > 0$ and $Y_0 > 0$ is a positive constant. We assume that $(\mathcal{F}_t)$ is the standard filtration generated by the Brownian motion B and without loss of generality we take $\mathcal{F} = \mathcal{F}_T$. Since there is continuous trading, we have continuous discount. The discounted price process of the stock is given by

$$\bar{Y}_t = Y_0 \exp\left(\left(\mu - r - \frac{1}{2}\sigma^2\right) t + \sigma B_t\right) \tag{2.25}$$

and in particular, P-a.s.,

$$d\bar{Y}_t = (\mu - r)\bar{Y}_t dt + \sigma \bar{Y}_t dB_t,\ t \in [0,T]. \tag{2.26}$$

Our aim is to find an EMM $\mathbb{Q}$ to determine the arbitrage free price for a given European contingent claim. We want the discounted price process $\bar{Y}$ to satisfy a differential equation like (2.26) but without the drift term $(\mu - r)\bar{Y}_t dt$. We look for an EMM $\mathbb{Q}$ for $\mathbb{P}$ that will serve for this purpose. To change the drift of a Bm, we use Girsanov transformation 2.2.11. If we let

$$\theta = \frac{\mu - r}{\sigma}, \tag{2.27}$$

$$\Lambda_t = \exp\left(-\theta B_t - \frac{1}{2}\theta^2 t\right),\ t \in [0,T], \tag{2.28}$$

then, $(\Lambda_t)_{t\in[0,T]}$ is a positive martingale under $\mathbb{P}$. On $(\Omega, \mathcal{F})$, our new probability measure $\mathbb{Q}$ satisfies

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \Lambda_T \ \text{ that is }\ \mathbb{Q}\{A\} = \mathbb{E}_{\mathbb{P}}[1_A \Lambda_T], \qquad A \in \mathcal{F}.$$

We can check if $\bar{Y}$ is a martingale under $\mathbb{Q}$. Let

$$\tilde{B}_t = B_t + \theta t, \qquad t \in [0,T]. \tag{2.29}$$

By Girsanov Theorem (Theorem 2.2.11), $\{\tilde{B}_t,\, t \in [0,T]\}$ is a standard Brownian motion hence a martingale under $\mathbb{Q}$ and the same filtration $(\mathcal{F}_t)$. Thus, for each $t \in [0,T]$, we have

$$\begin{aligned} \bar{Y}_t &= Y_0 \exp\left(\left(\mu - r - \frac{1}{2}\sigma^2\right)t + \sigma\left(\tilde{B}_t - \theta t\right)\right) \\ &= Y_0 \exp\left(-\frac{1}{2}\sigma^2 t + \sigma\tilde{B}_t\right), \end{aligned} \tag{2.30}$$

by Equations (2.25) and (2.27) and so $\mathbb{Q}$-a.s.,

$$d\bar{Y}_t = \sigma\bar{Y}_t d\tilde{B}_t \tag{2.31}$$

where $\tilde{B}$ is a standard Brownian motion under $\mathbb{Q}$. Thus, $\mathbb{Q}$ is an EMM.

We now state a theorem which will be used to calculate the arbitrage free price of a European call option below [46].

Theorem 2.3.7 *The unique arbitrage free price process for the $\mathbb{Q}$-integrable European contingent claim H is a continuous modification of*

$$C_t = \{e^{rt}\mathbb{E}_{\mathbb{Q}}[\bar{H}|\mathcal{F}_t],\, t \in [0,T]\}. \tag{2.32}$$

We sketch the proof given in [46]. By Theorem 4.4.2 in [46], there is a replicating strategy $\theta = (\theta^0, \theta^1)$ for H with value process $(V_t(\theta))_{t\in[0,T]}$ equal to $\{e^{rt}\mathbb{E}_{\mathbb{Q}}[\bar{H}|\mathcal{F}_t],\, t \in [0,T]\}$. Supposing $\mathbb{Q}\{C_t \neq V_t(\theta) \text{ for some } t \in [0,T]\} > 0$, an arbitrage strategy is explicitly defined. Hence, pricing the contingent claim different than (2.32) results in an arbitrage. Therefore, the unique arbitrage free price is (2.32).

Now, we can calculate the arbitrage free price of a European call option in other words, derive the Black-Scholes option pricing formula using the tools given above.

We know that $H = (Y_T - K)^+$ represents the value of a European call option with strike price $K \in (0,\infty)$ and expiration time T. By Theorem 2.3.7 above, the arbitrage free price process for the call option H is given by $\{C_t, t \in [0,T]\}$. This is a continuous process such that for each $t \in [0,T]$, if $\bar{C}_t = C_t e^{-rt}$ as before, then we have $\mathbb{Q}$-a.s.,

$$\bar{C}_t = \mathbb{E}_{\mathbb{Q}}[\bar{H}|F_t] = \mathbb{E}_{\mathbb{Q}}[(\bar{Y}_T - \bar{K})^+|\mathcal{F}_t], \tag{2.33}$$

where $\bar{K} = e^{-rT}K$. By writing $\bar{Y}_T$ according to (2.30) and multiplying and dividing the right hand side with $\bar{Y}_t$ obtain $\mathbb{Q}$-a.s., for each $t \in [0,T]$,

$$\begin{aligned} \bar{Y}_T &= \bar{Y}_t \frac{Y_0 \exp\left(\sigma\tilde{B}_T - \frac{1}{2}\sigma^2 T\right)}{Y_0 \exp\left(\sigma\tilde{B}_t - \frac{1}{2}\sigma^2 t\right)} \\ &= \bar{Y}_t \exp\left(\sigma\left(\tilde{B}_T - \tilde{B}_t\right) - \frac{1}{2}\sigma^2(T-t)\right). \end{aligned}$$

If we substitute this into (2.33) we get for $t \in [0,T], \mathbb{Q}$-a.s.,

$$\bar{C}_t = \mathbb{E}_{\mathbb{Q}}\left[\left(\bar{Y}_t \exp\left(\sigma\ \left(\tilde{B}_T - \tilde{B}_t\right) - \frac{1}{2}\sigma^2(T-t)\right) - \bar{K}\right)^+ \middle|\mathcal{F}_t\right]. \tag{2.34}$$

By the fact $\bar{Y}_t \in \mathcal{F}_t$ and $\tilde{B}_T - \tilde{B}_t$ is a normal random variable with mean zero and variance $T-t$ that is independent of $\mathcal{F}_t$ under $\mathbb{Q}$, it follows that for $t \in [0,T), \mathbb{Q}$-a.s., the price of the contingent claim can be calculated as follows,

$$C_0 = Y_0\Phi\left(\frac{\log(\frac{Y_0}{\bar{K}})}{\sigma\sqrt{T}} + \frac{1}{2}\sigma\sqrt{T}\right) - \bar{K}\Phi\left(\frac{\log(\frac{Y_0}{\bar{K}})}{\sigma\sqrt{T}} - \frac{1}{2}\sigma\sqrt{T}\right)$$

which is the Black Scholes option pricing formula.

Chapter 3

FRACTIONAL BROWNIAN MOTION AND ARBITRAGE

In this chapter, we study the use of fBm in finance as a stock price process. We will consider two models that approximate fBm in the limit. These models aim to provide an explanation for why the stock price process may follow a fBm. Then, we study more common stock price models involving fBm directly, namely fractional Bachelier and fractional Black-Scholes models. After introducing fBm as a suitable model, we cover the arbitrage strategies constructed in [1, 8, 37, 40]. Then, we review the remedies to prevent arbitrage opportunities, either by making minor changes in the process or putting some restrictions on the trading rules.

3.1 Properties of fBm

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A fractional Brownian motion is a centered Gaussian process with stationary increments that is stochastically self-similar. It has continuous sample paths and zero quadratic variation. We give three equivalent definitions of fBm [29].

Definition 3.1.1 *A centered Gaussian process* $\{B_t^H : 0 \leq t \leq \infty\}$ *with* $B_0^H = 0$ *is a fBm, if*

i. $\left(B_{t_2}^H - B_{t_1}^H, B_{s_2}^H - B_{s_1}^H\right) \stackrel{D}{=} (B_{t_2+h}^H - B_{t_1+h}^H, B_{s_2+h}^H - B_{s_1+h}^H)$ *for all* t_1, t_2, s_1, s_2 *and* $h \geq 0$.

ii. There is an $H \in (0,1)$ *such that*

$$B_{t+\tau}^H - B_t^H \stackrel{D}{=} h^{-H}(B_{t+h\tau}^H - B_t^H)$$

for all $t, \tau, h \geq 0$.

Definition 3.1.2 *A centered Gaussian process* $\{B_t^H : 0 \le t < \infty\}$ *with* $B_0^H = 0$ *is a fBm, if*

$$Var(B_t^H - B_s^H) = |t-s|^{2H} \cdot Var(B_1^H)$$

for all $s, t \ge 0$.

Definition 3.1.3 *A centered Gaussian process* $\{B_t^H : 0 \le t < \infty\}$ *with* $B_0^H = 0$ *is a fBm, if*

$$Cov(B_t^H, B_s^H) = \frac{1}{2} Var(B_1^H)(t^{2H} + s^{2H} - |t-s|^{2H})$$

for all $t, s \ge 0$.

We go over the proof of the following theorem given in [37], while providing further details.

Theorem 3.1.4 *The fBm* B^H *with self-similarity parameter* $H \in (0, 1/2) \cup (1/2, 1)$ *is a semimartingale only if* $H = \frac{1}{2}$.

Proof: We fix the parameter H of fBm and consider for $p > 0$ fixed

$$Z_{n,p} := \sum_{j=1}^{2^n} \left| B^H(\frac{j}{2^n}) - B^H(\frac{j-1}{2^n}) \right|^p (2^n)^{pH-1}.$$

We see that the following holds

$$\begin{aligned} \mathbb{E} Z_{n,p} &= 2^{npH} 2^{-n} \sum_{j=1}^{2^n} \mathbb{E} \left| B^H(\frac{j}{2^n}) - B^H(\frac{j-1}{2^n}) \right|^p \\ &\overset{D}{=} 2^{npH} \, \mathbb{E} \left| B^H(\frac{1}{2^n}) \right|^p \\ &= \mathbb{E} \left| 2^{nH} B^H(\frac{1}{2^n}) \right|^p \\ &\overset{D}{=} \mathbb{E} \left| B^H(2^n \frac{1}{2^n}) \right|^p = \mathbb{E} \left| B_1^H \right|^p \end{aligned} \tag{3.1}$$

where the equalities in distribution come from stationarity and self-similarity, respectively. If we now consider

$$\tilde{Z}_{n,p} := 2^{-n} \sum_{j=1}^{2^n} |B_j^H - B_{j-1}^H|^p,$$

for each n, this has the same law as $Z_{n,p}$, that is,

$$\begin{aligned} Z_{n,p} &= 2^{-n}\sum_{j=1}^{2^n} 2^{npH}\left|B^H(\frac{j}{2^n}) - B^H(\frac{j-1}{2^n})\right|^p \\ &= 2^{-n}\sum_{j=1}^{2^n}\left|2^{nH}B^H(\frac{j}{2^n}) - 2^{nH}B^H(\frac{j-1}{2^n})\right|^p \\ &\overset{D}{=} 2^{-n}\sum_{j=1}^{2^n}\left|B^H(2^n\frac{j}{2^n}) - B^H(2^n\frac{j-1}{2^n})\right|^p \\ &\overset{D}{=} 2^{-n}\sum_{j=1}^{2^n}|B^H_j - B^H_{j-1}|^p = \tilde{Z}_{n,p} \end{aligned} \tag{3.2}$$

where again the two equalities in distribution come from self-similarity and stationarity, respectively. Noticing that the sequence $(B^H_k - B^H_{k-1})_{k\in\mathbb{Z}}$ is stationary and ergodic, the ergodic theorem tells us that

$$\tilde{Z}_{n,p} \longrightarrow \mathbb{E}|B^H_1 - B^H_0|^p =: c_p$$

as $n \to \infty$ a.s. and in L_1 due to Equations (3.1) and (3.2). Hence,

$$Z_{n,p} \longrightarrow c_p$$

as $n \to \infty$ in probability. Therefore,

$$V_{n,p} := \sum_{j=1}^{2^n}\left|B^H(\frac{j}{2^n}) - B^H(\frac{j-1}{2^n})\right|^p \longrightarrow \begin{cases} 0 & if \quad pH > 1 \\ +\infty & if \quad pH < 1 \end{cases} . \tag{3.3}$$

$n \to \infty$. Now, we consider the cases $H < 1/2$ and $H > 1/2$ separately.

i) $H < \frac{1}{2}$

We can choose $p > 2$ such that $pH < 1$, so the pth variation of X on $[0, 1]$ must be ∞. Then the quadratic variation of X is also ∞, otherwise the pth variation would be zero. But the quadratic variation of a continuous semimartingale is finite ([36], Proposition 1.18). Hence, for $H < 1/2$, X cannot be a semimartingale.

ii) $H > \frac{1}{2}$

We can choose $p \in (\frac{1}{H}, 2)$ then $pH > 1$ so, $V_{n,p} \to 0$ in probability by (3.3), therefore there exists a subsequence $v_{n,p}$ of $V_{n,p}$ such that $v_{n,p} \to 0$ a.s. Since $p < 2$, the quadratic variation is also zero. On the other hand, for $p \in (1, \frac{1}{H})$, we have $pH < 1$, it follows that $V_{n,p} \to \infty$ in probability. Hence there exists a subsequence $v_{n,p}$ of $V_{n,p}$ such that $v_{n,p} \to 0$ a.s. and by scaling, variation on any interval is infinite. So, X has infinite total variation and finite quadratic variation. A continuous semimartingale $X = M + A$, where M is a local martingale and A is a Stieltjes process has a finite quadratic variation and the quadratic variation of X and M are equal ([36], Proposition 1.18). If X were a semimartingale, we would get that the quadratic variation of M is zero. But, then $M = 0$ ([26], Theorem 7.8). This will imply that $X = A$, a Stieltjes process, which contradicts to the fact that X has infinite variation. Hence, X cannot be a semimartingale. ■

We now define the stochastic integral with respect to fBm B^H with $H \in (1/2, 1)$ [40]. We suppose that $f : \mathbb{R} \to \mathbb{R}$ belongs to C^1. If we apply Taylor's formula with remainder term in the integral form [44] to

$$F(x) = F(0) + \int_0^y f(y)dy$$

around the point y, we get

$$F(x) = F(y) + f(y)(x-y) + \int_y^x f'(u)(x-u)du. \tag{3.4}$$

For each sequence $T^n := \left\{t^{(n)}(m), m \geq 1\right\}, n \geq 1$, of times $t^{(n)}(m)$ with $0 = t^{(n)}(1) \leq t^{(n)}(2) \leq \ldots$, we have the following equality from 3.4

$$\begin{aligned} F\left(B_t^H\right) - F\left(B_0^H\right) &= \sum_m \left[F\left(B^H_{t\wedge t^{(n)}(m+1)}\right) - F\left(B^H_{t\wedge t^{(n)}(m)}\right)\right] \\ &= \sum_m f\left(B^H_{t\wedge t^{(n)}(m)}\right)\left(B^H_{t\wedge t^{(n)}(m+1)} - B^H_{t\wedge t^{(n)}(m)}\right) + R_t^{(n)}, \end{aligned} \tag{3.5}$$

where

$$R_t^{(n)} = \sum_m \int_{B_{t\wedge t^{(n)}(m)}}^{B_{t\wedge t^{(n)}(m+1)}} f'(u)\left(B_{t\wedge t^{(n)}(m+1)} - u\right)du.$$

Since $f \in C^1$, we have $\mathbb{P}\left\{\sup_{0\leq u\leq t}\left|f'\left(B_u^H\right)\right| < \infty\right\} = 1$. Also, since quadratic variation of B_u^H is zero for $H \in \left(\frac{1}{2}, 1\right)$ ([29],Theorem 2.1), we obtain

$$\left|R_t^{(n)}\right| \leq \frac{1}{2}\sup_{0\leq u\leq t}\left|f'\left(B_u^H\right)\right| \cdot \sum_m \left|B_{t\wedge t^{(n)}(m+1)} - B_{t\wedge t^{(n)}(m)}\right|^2 \to 0$$

where the limit is in probability.

The left hand side of (3.5), that is, $F\left(B_t^H\right) - F\left(B_0^H\right)$ is independent of n and $R_t^{(n)} \to 0$ in probability. So,

$$\mathbb{P} - \lim_n \sum_m f\left(B_{t\wedge t^{(n)}(m)}\right)\left(B_{t\wedge t^{(n)}(m+1)} - B_{t\wedge t^{(n)}(m)}\right)$$

denoted by

$$\int_0^t f\left(B_u^H\right) dB_u^H \tag{3.6}$$

exists and is called the ***stochastic integral with respect to fractional Brownian motion*** $B^H = (B_u^H)_{u\le t}, H \in \left(\frac{1}{2}, 1\right), f \in C'$.

3.2 Stock price Models with fBm

We study the use of fBm in modeling stock prices in this section. We will first give a model from [27] that approximates fBm in the limit. Then, we will give another model from [2] which approximates an integral with respect to fBm in the limit. Finally, we will introduce two types of stock price models involving fBm following [33, 42].

3.2.1 fBm as a Limit of a Stock Price Process

In [27], Poisson shot-noise processes are used to model the stock prices. The model converges weakly to fractional Brownian motion under a regular variation condition. Whereas fBm allows for arbitrage, the shot-noise process itself can be constructed not to allow arbitrage. Also, an economic reason for the long-range dependence in asset returns is provided. A possible economic explanation of logarithmic stock price processes to follow fBm is given.

The suggested model is given by,

$$Y(t) = e^{B(t)+S(t)}, \;\; t \ge 0$$

where B is a Brownian motion and S is a shot-noise model defined by

$$S(t) = \sum_{i=1}^{N(t)} X_i(t - T_i) + \sum_{i=-1}^{-\infty} [X_i(t - T_i) - X_i(-T_i)], \;\; t \ge 0. \tag{3.7}$$

Here, $X_i = (X_i(t))_{t \in \mathbb{R}_+}$, $i \in \mathbb{Z} \backslash \{0\}$, are independent and identically distributed stochastic processes on $\mathbb{R}$, such that $X_i(t) = 0$ for $t < 0$. X_i is also independent of the two-sided homogeneous Poisson process N with rate $\alpha > 0$, and points T_i, $i \in \mathbb{Z} \backslash \{0\}$.

This shot-noise model S demonstrates the information provided from various sources. The arrival times of information constitutes a Poisson process N. The arrival of information acts like a shock to the market and may change the price as well as having some influence on future price movements. Some effects may vanish but certain information has a long lasting effect on the price. Long memory is introduced into the model by this way.

The case when $X_i(u) = g(u)Y_i$, $u \geq 0$, where Y_i are i.i.d. innovations with $\mathbb{E}Y_1 = 0$ and $EY_1^2 \in (0, \infty)$ is considered only. The function $g : \mathbb{R}_+ \to \mathbb{R}$ is a continuously differentiable function with $g'(u) = O(u^{-1/2-\varepsilon})$, $u \to \infty$, for some $\varepsilon > 0$.

With these choices, S is shown to be a semimartingale with its natural filtration. A probability measure $\mathbb{Q} \sim \mathbb{P}$ is found explicitly such that S is a local $\mathbb{Q}$-martingale. Therefore, by Thm 2.3.3, there is NFLVR.

For the limiting result, it is assumed that g is a normalized regularly varying at ∞ with index $\delta \in (-1/2, 1/2)$. For $t > 0$, the scaled process is

$$S_n(t) = \frac{S(xt)}{\delta(t)},$$

where $n \in [0, \infty)$ and $\sigma^2(t) = \mathrm{Var}[S(t)]$ is introduced. The main result is the fact that S_n tends to B^H as $n \to \infty$, where the convergence holds in $D[0, \infty)$ equipped with the metric of uniform convergence on compacta.

In [2], the effect of investor inertia on stock price fluctuations is studied. Investor inertia refers to the tendency of investors to remain inactive in trading. Long uninterrupted periods of inactivity may be viewed as a form of investor inertia. In this model, incoming buy orders increase the price while sell orders decrease it. The fact that infrequent trading gives rise to long-range dependence in stock prices and arbitrage opportunities for other more sophisticated traders is demonstrated in this study. A functional central limit theorem for stationary semi-Markov processes is also proven.

There assumed to be N agents in the market, namely $A := \{a_1, a_2, \ldots, a_N\}$.

For each agent a, a stationary semi-Markov process $x^a = (x_t^a)_{t\geq 0}$ is defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with finite state space E. The process x^a denotes the agent's propensity for trading. It can be positive, negative or zero indicating that the agent is buying, selling or in inactive state, respectively. It has heavy-tailed sojourn times in inert states and relatively thin-tailed sojourn times in other states.

The size of trading in the stock market is defined by the random variable Ψ, where $\Psi_t > 0$ denotes the size of a typical trading at time t. It is assumed that the agents are homogeneous, x^a and Ψ are independent.

The process x^a is determined in terms of the random variables $\xi_n : \Omega \to E$ and $T_n : \Omega \to \mathbb{R}_+$ by the relation

$$x_t = \sum_{n\geq 0} \xi_n 1_{[T_n, T_{n+1})}(t),$$

in other words, the agent's mood in the random time interval $[T_n, T_{n+1})$ is given by the random variable ξ_n. The process x is assumed to be homogeneous under $\mathbb{P}$, that is,

$$\mathbb{P}\{\xi_{n+1} = j, T_{n+1} - T_n \leq t | \xi_n = i\} = Q(i, j, t)$$

does not depend on $n \in N$.

With these assumptions, the holdings of the agent $a \in A$ and the market imbalance at time $t \geq 0$ are given by

$$\int_0^t \Psi_s x_s^a ds \qquad \text{and} \qquad I_t^N := \sum_{a\in A} \int_0^t \Psi_s x_s^a ds$$

respectively. On the other hand, the pricing rule is

$$dS_t^N = \sum_{a\in A} \Psi_t x_t^a dt \qquad \text{so} \qquad S_t^N = S_0 + I_t^N$$

for the evolution of the logarithmic stock price process S^N.

A scaling parameter $\varepsilon > 0$ is introduced and the rescaled process $x_{t/\varepsilon}^a$ is considered which denotes a speeded-up semi-Markov process when ε is small. The aggregate order at time t is given by

$$Y_t^{\varepsilon,N} = \sum_{a\in A} \Psi_t x_{t/\varepsilon}^a.$$

Finally, the price process $X_t^{\varepsilon,N} = (X_t^{\varepsilon,N})_{0\leq t\leq T}$ is defined by

$$X_t^{\varepsilon,N} := \int_0^t \sum_{a\in A} \Psi_s(x^a_{s/\varepsilon} - \mu)ds$$

where $\mu := \mathbb{E}x_t$.

Under further technical assumptions, the following result is obtained

$$\lim_{\varepsilon\to 0}\lim_{N\to\infty}\left(\frac{X_t^{\varepsilon,N}}{\varepsilon^{1-H}\sqrt{NL(\varepsilon^{-1})}}\right)_{0\leq t\leq T} = \left(c\int_0^t \Psi_s dB_s^H\right)_{0\leq t\leq T},$$

and in the special case when $\Psi \equiv 1$, fBm is obtained in the limit, that is

$$\lim_{\varepsilon\to 0}\lim_{N\to\infty}\left(\frac{X_t^{\varepsilon,N}}{\varepsilon^{1-H}\sqrt{NL(\varepsilon^{-1})}}\right)_{0\leq t\leq T} = \left(cB_t^H\right)_{0\leq t\leq T}.$$

where L is a function that is slowly varying at infinity, the limits are in probability and H is restricted to be greater than 1/2.

3.2.2 Stock Price Models Involving fBm

In financial mathematics, there are two "classical" models which form a basis for description of dynamics of the stock prices [42]. The first one, the *Bachelier model*, is described in one dimensional primary markets [33] as

$$X_t = 1, \qquad Y_t = Y_0 + \nu t + \sigma B_t, \;\; t\in[0,T],$$

where X and Y denote the bond and stock price processes, respectively, B is a Brownian motion, $\nu \in \mathbb{R}$ and $\sigma, T \in \mathbb{R}_+$ on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F})_{t\in[0,T]}, \mathbb{P})$. The constant σ is called the ***volatility***. The ***fractional Bachelier model*** is the version of the Bachelier model where Brownian motion B_t is replaced by fBm.

$$X_t = 1, \qquad Y_t = Y_0 + \nu t + \sigma B_t^H, \;\; t\in[0,T]. \tag{3.8}$$

The second one, generally referred as the *Black-Scholes model*, (also as Samuelson-Black-Merton-Scholes model), can be described as follows

$$X_t = e^{rt}, \qquad Y_t = Y_0 e^{(r+\nu)t+\sigma B_t}, \;\; t\in[0,T] \tag{3.9}$$

where $r \in \mathbb{R}_+$ is the interest rate. Similarly, the ***fractional Black-Scholes model*** is the version of the Black-Scholes model where B_t is replaced with fBm. We will use the following model and refer as the fractional Black-Scholes model

$$X_t = e^{rt}, \qquad Y_t = Y_0 e^{(r+\nu)t+\sigma B_t^H},\ t \in [0,T]. \tag{3.10}$$

3.3 Arbitrage with fBm models

In this section, we review construction of arbitrage strategies for the models introduced in Section 3.2.2. Since it is known that fBm is not a semimartingale, we cannot conclude that there does not exist any arbitrage opportunities in financial market models from Theorem 2.3.3. Recently, research efforts have gone to find out arbitrage strategies in the fractional models [1, 8, 37, 40]. We will review these strategies and the suggested solutions to exclude arbitrage.

In [37], only fBm-modulated Bachelier model is examined. In [40], fractional versions of both Bachelier and Black-Scholes models are analyzed. In [8],four cases are examined which are the combinations of the Bachelier and Black-Scholes models with the cases $H \in (0, \frac{1}{2})$ and $H \in (\frac{1}{2}, 1)$. Finally, in [1] a different case is analyzed where the volatility is a stochastic process rather than a constant. The fractional Bachelier model with $H \in (0, 1]$ is studied.

We will analyze arbitrage strategies constructed in fBm models and their modifications to avoid such opportunities since in fair markets arbitrage is not expected. These studies imply that fBm is not a perfect candidate for modeling stock prices due to arbitrage but can be made suitable with some changes.

3.3.1 Arbitrage with Fractional Bachelier Model

Arbitrage strategies with the fractional Bachelier model have been found by [8],[37] and [40]. In [8], model (3.8) is considered in its general case while only the case $\nu = 0$

is given in [37] and only the case $H \in \left(\frac{1}{2}, 1\right)$ is given in [40]. We will now study these strategies in detail.

In [8], an arbitrage strategy for a one dimensional fractional Bachelier model is described. Money in the money market evolves according to $(X_t)_{t\in[0,T]}$ and the stock price follows $(Y_t)_{t\in[0,T]}$ in the securities market $(\Omega, \mathcal{F}, \mathbb{P}, [0,T], (\mathcal{F}_t), S)$. We let X be the numéraire as usual. We will denote the filtration generated by a price process Z with $(\mathcal{F}_t)^Z$ and $\bar{Z}$ will again denote the discounted price process. We state the results of [8] below.

Theorem 3.3.1 *Suppose $\nu \in C^1[0,T]$ and $\sigma > 0$,*

$$\bar{Y}_t = \nu(t) + \sigma B_t^H, \qquad t \in [0,T],$$

Then, in both cases i)$H \in (\frac{1}{2}, 1)$ and ii)$(0, \frac{1}{2})$, for every constant $c > 0$ and all $n \in \mathbb{N}$, there exists $\theta^1(n) \in \boldsymbol{S}((\mathcal{F}_t)^{\bar{Y}})$ such that

$$a) \qquad \mathbb{P}\left\{\int_0^T \theta^1(n) d\bar{Y} = c\right\} > 1 - \frac{1}{n} \qquad \textit{and}$$

$$b) \qquad \inf_{t\in[0,T]} \int_0^t \theta^1(n) d\bar{Y} \geq -\frac{1}{n}.$$

In particular, the strategies $\theta(n) = (\theta^0(n), \theta^1(n)) \in \Theta^{\mathbf{S}}_{adm}((\mathcal{F}_t)^{\bar{Y}}), n \in \mathbb{N}$, where $\theta^0(n)$ is given by

$$\theta_t^0(n) = \left(\theta^1(n) \cdot \bar{Y}\right)_t - \theta_t^1(n)\bar{Y}_t,\ t \in [0,T], n \in \mathbb{N}$$

form a c-FLVR. In the second case, $\theta^1(n)$ can be chosen such that

$$c) \qquad \left|\theta^1(n)\right| \leq \frac{1}{n}.$$

We go over the proof by sketching the facts shown in [8] as well as providing extra details in some places for clarification.

Proof: It is enough to prove the theorem for $T = 1$ since B^H is self-similar.

(i) $H \in \left(\frac{1}{2}, 1\right)$:

Since $(\bar{Y}_t)_{t\in[0,1]}$ is a Gaussian process, it satisfies $\mathbb{P}\{\bar{Y}_0 = \bar{Y}_1\} = 0$. The function ν is in $C^1[0,T]$, so it is Lipschitz continuous. Almost all paths of $(B_t^H)_{t\in[0,1]}$ are Hölder continuous of order α for every $\alpha \in (0,H)$ [43]. We get

$$\begin{aligned}
&\max_{t\in[0,1]}\sum_{j=0}^{n-1}\left(\bar{Y}_{\frac{j+1}{n}\wedge t}-\bar{Y}_{\frac{j}{n}\wedge t}\right)^2 \\
&= \max_{t\in[0,1]}\sum_{j=0}^{n-1}\left[\left(\nu(\frac{j+1}{n}\wedge t)+\sigma B^H(\frac{j+1}{n}\wedge t)\right)-\left(\nu(\frac{j}{n}\wedge t)+\sigma B^H(\frac{j}{n}\wedge t)\right)\right]^2 \\
&= \max_{t\in[0,1]}\sum_{j=0}^{n-1}\left[\left(\nu(\frac{j+1}{n}\wedge t)-\nu(\frac{j}{n}\wedge t)\right)+\left(\sigma B^H(\frac{j+1}{n}\wedge t)-\sigma B^H(\frac{j}{n}\wedge t)\right)\right]^2 \\
&= \max_{t\in[0,1]}\sum_{j=0}^{n-1}\left[\left(\nu(\frac{j+1}{n}\wedge t)-\nu(\frac{j}{n}\wedge t)\right)^2+\left(\sigma B_t^H(\frac{j+1}{n}\wedge t)-\sigma B_t^H(\frac{j}{n}\wedge t)\right)^2\right. \\
&\quad \left.+\left(\nu(\frac{j+1}{n}\wedge t)-\nu(\frac{j}{n}\wedge t)\right)\left(\sigma B_t^H(\frac{j+1}{n}\wedge t)-\sigma B_t^H(\frac{j}{n}\wedge t)\right)\right] \\
&\leq \max_{t\in[0,1]}\left[\sum_{j=0}^{n-1}C_1^2\left(\frac{1}{n}\right)^2+\sum_{j=0}^{n-1}C_2^2\left(\frac{1}{n}\right)^{2\alpha}+\sum_{j=0}^{n-1}2C_1\cdot C_2\frac{1}{n}\left(\frac{1}{n}\right)^{\alpha}\right] \\
&= \max_{t\in[0,1]}\left[C_1^2\cdot n\left(\frac{1}{n^2}\right)+C_2^2\cdot n\left(\frac{1}{n^{2\alpha}}\right)+2C_1\cdot C_2\cdot n\left(\frac{1}{n^{\alpha+1}}\right)\right] \\
&= \max_{t\in[0,1]}\left[C_1^2\left(\frac{1}{n}\right)+C_2^2\left(\frac{1}{n^{2\alpha-1}}\right)+2C_1\cdot C_2\left(\frac{1}{n^{\alpha}}\right)\right] \\
&\longrightarrow 0 \qquad (3.11)
\end{aligned}$$

as $n\to\infty$ a.s., since $\alpha > \frac{1}{2}$. This implies convergence in probability also. So, $(\bar{Y}_t)_{t\in[0,1]}$ satisfies the hypothesis of Lemma 3.3 in [8]. Therefore, for all $n \in \mathbb{N}$, there exists $\beta(n) \in \mathbf{S}((\mathcal{F}_t)^{\bar{Y}})$ such that

$$\text{(a) } \mathbb{P}\left\{(\beta(n)\cdot Y)_1 < c\right\} < \frac{1}{n} \qquad \text{and} \qquad \text{(b) } \inf_{t\in[0,1]}(\beta(n)\cdot\bar{Y})_t \geq -\frac{1}{n}. \qquad (3.12)$$

For every $n \in \mathbb{N}$,

$$\xi_n := \inf\left\{t : \left(\beta(n)\cdot\bar{Y}\right)_t = c\right\} \qquad \text{(we set } \inf\emptyset = 1\text{)}$$

is an $(\mathcal{F}_t)^Z$ stopping time. We set $\theta^1(n) := \beta(n)1_{[0,\xi_n]} \in \mathbf{S}((\mathcal{F}_t)^{\bar{Y}})$. Hence, $\theta^1(n)$ satisfies the statements (a) and (b) of the theorem.

(ii) $H \in (0,\frac{1}{2})$, $\bar{Y}_t = \nu(t) + \sigma B_t^H$, $t \in [0,1]$:

Let $L > 0$. It follows from Lemma 3.5 in [8] a) with $p = 1$ and $q = H$ that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \left| B^H_{\frac{j+1}{n}} - B^H_{\frac{j}{n}} \right| = 0 \qquad \text{in} \qquad L^1.$$

Hence,

$$\sum_{j=0}^{n-1} 2 \left| \left(\nu\left(\frac{j+1}{n}\right) - \nu\left(\frac{j}{n}\right)\right) \left(\sigma B^H_{\frac{j+1}{n}} - \sigma B^H_{\frac{j}{n}}\right) \right| \leq 2\|\nu'\|_\infty \frac{1}{n}\sigma \sum_{j=0}^{n-1} \left| B^H_{\frac{j+1}{n}} - B^H_{\frac{j}{n}} \right|$$

where $||(a_1, \ldots, a_n)||_\infty = \max\{|a_1|, \ldots, |a_n|\}$ for $(a_1, \ldots, a_n) \in \mathbb{R}^n$. The right hand side of the above term goes to 0 in L^1, so does the left hand side. This implies convergence in probability. In particular, there exists $n_1 \in \mathbb{N}$, such that for all $n \geq n_1$,

$$\mathbb{P}\left\{ \sum_{j=0}^{n-1} 2 \left| (\nu(\frac{j+1}{n}) - \nu(\frac{j}{n})) \left(\sigma B^H_{\frac{j+1}{n}} - \sigma B^H_{\frac{j}{n}}\right) \right| > L \right\} < \frac{1}{2L}. \tag{3.13}$$

On the other hand, there exists $n_2 \in \mathbb{N}$, such that for all $n \geq n_2$,

$$\mathbb{P}\left\{ \sum_{j=0}^{n-1} \left(\sigma B^H_{\frac{j+1}{n}} - \sigma B^H_{\frac{j}{n}}\right)^2 < 2L \right\} < \frac{1}{2L} \tag{3.14}$$

with $p = 1$ and $q = H$ in Lemma 3.5 [8] b). Hence, for all $n \geq \max(n_1, n_2)$,

$$\begin{aligned}
&\mathbb{P}\left\{ \sum_{j=0}^{n-1} \left(\bar{Y}_{\frac{j+1}{n}} - \bar{Y}_{\frac{j}{n}}\right)^2 < L \right\} \\
\leq \;& \mathbb{P}\left\{ \sum_{j=0}^{n-1} \left(\sigma B^H_{\frac{j+1}{n}} - \sigma B^H_{\frac{j}{n}}\right)^2 + 2(\nu(\frac{j+1}{n}) - \nu(\frac{j}{n})) \left(\sigma B^H_{\frac{j+1}{n}} - \sigma B^H_{\frac{j}{n}}\right) < L \right\} \\
\leq \;& \mathbb{P}\left\{ \sum_{j=0}^{n-1} \left(\sigma B^H_{\frac{j+1}{n}} - \sigma B^H_{\frac{j}{n}}\right)^2 < 2L \right\} \\
&\quad + \mathbb{P}\left\{ \sum_{j=0}^{n-1} 2 \left| (\nu t(\frac{j+1}{n}) - \nu(\frac{j}{n})) \left(\sigma B^H_{\frac{j+1}{n}} - \sigma B^H_{\frac{j}{n}}\right) \right| > L \right\} \\
< \;& \frac{1}{L}.
\end{aligned}$$

The last inequality follows from (3.13) and (3.14). The second one follows from the argument $C \subset (A \cup B)$ where

$$A := \left\{ \omega : \sum_{j=0}^{n-1} \left(\sigma B^H_{\frac{j+1}{n}}(\omega) - \sigma B^H_{\frac{j}{n}}(\omega)\right)^2 \geq 2L \right\}$$

$$B := \left\{\omega : \sum_{j=0}^{n-1} 2\left|(\nu(\frac{j+1}{n}) - \nu(\frac{j}{n}))\left(\sigma B^H_{\frac{j+1}{n}}(\omega) - \sigma B^H_{\frac{j}{n}}(\omega)\right)\right| \leq L\right\}$$

and

$$C := \left\{\omega : \sum_{j=0}^{n-1}\left(\sigma B^H_{\frac{j+1}{n}} - \sigma B^H_{\frac{j}{n}}\right)^2 + 2\left(\nu\left(\frac{j+1}{n}\right) - \nu\left(\frac{j}{n}\right)\right)\left(\sigma B^H_{\frac{j+1}{n}} - \sigma B^H_{\frac{j}{n}}\right) \geq L\right\}$$

so, $P(C) \leq P(A \cup B) \leq P(A) + P(B)$. This shows that $\bar{Y}_{t\in[0,1]}$ satisfies the conditions in Lemma 3.4 in [8] which implies that $\theta^1(n)$ can be constructed as in (i). The result follows. ■

Theorem 3.3.2 *In both cases of Theorem 3.3.1 there exists for every constant $c > 0$, a $\frac{1}{c}$-admissible c-arbitrage $\theta \in \Theta^{\mathbf{aS}}_{adm}((\mathcal{F}_t)^{\bar{Y}})$. In the second case, θ can be chosen such that $|\theta^1| \leq \frac{1}{c}$.*

Proof: Since B^H is self-similar, it is enough to prove the theorem for $T = 1$. We split $(0, 1]$ into the subintervals

$$I_n := (a_n = 1 - 2^{1-n}, b_n = 1 - 2^{-n}], n \in \mathbb{N}$$

$\bar{Y}_n$ denotes the restriction of $\bar{Y}$ to I_n and $(\mathcal{F}_t^{\bar{Y}_n})_{t\in I_n}$ the filtration generated by $\bar{Y}_n$. $\mathcal{F}_t^{\bar{Y}_n} \subset \mathcal{F}_t^{\bar{Y}}$ for all $n \in \mathbb{N}$ and $t \in I_n$ since $\bar{Y}_n$ is the restriction of $\bar{Y}$.

Since B^H has stationary increments, from Theorem 3.3.1 there exists for all $n \in \mathbb{N}$, a $\beta(n) \in \mathbf{S}((\mathcal{F}_t)^{\bar{Y}_n})$ such that

(a) $\mathbb{P}\left\{\int_0^{b_n} \beta(n) d\bar{Y}_n = c + \frac{1}{c}\right\} > 1 - \frac{1}{n}$ so

$\mathbb{P}\left\{\int_0^{b_n} \beta(n) d\bar{Y}_n < c + \frac{1}{c}\right\} + \mathbb{P}\left\{\int_0^{b_n} \beta(n) d\bar{Y}_n > c + \frac{1}{c}\right\} < \frac{1}{n}$

$\mathbb{P}\left\{\int_0^{b_n} \beta(n) d\bar{Y}_n < c + \frac{1}{c}\right\} < \frac{1}{n}$

(b) $\inf_{t\in I_n} \int_0^t \beta(n) d\bar{Y}_n \geq -\frac{2^{-n}}{c}.$

For $\beta := \sum_{n=1}^{\infty} \beta(n) 1_{I_n}$, the random variable $\xi := \inf \left\{ t \in [0,1] : \int_0^t \beta d\bar{Y} = c \right\}$ is an $(\mathcal{F}_t)^{\bar{Y}}$-stopping time. We see from (b) that

$$\inf \int_0^1 \beta d\bar{Y} \geq \sum_{n=1}^{\infty} -\frac{1}{2^n c} = -\frac{1}{c}$$

We also obtain from (a) by letting n go to infinity that

$$\mathbb{P}\left\{ \int_0^1 \beta d\bar{Y} < c + \frac{1}{c} \right\} = 0$$

So, a.s. $\int_0^t \beta d\bar{Y} = c$ for some $t < 1$. Hence $\mathbb{P}\{\xi < 1\} = 1$. Therefore,
$\theta^1 := \beta 1_{[0,1]}$ belongs to $\mathbf{aS}((\mathcal{F}_t)^{\bar{Y}})$ and (θ^0, θ^1) is a $\frac{1}{c}$-admissible c-arbitrage in $\Theta_{adm}^{\mathbf{aS}}$ where

$$\theta_t^0 := \int_0^t \theta^1 d\bar{Y} - \theta_t^1 \bar{Y}_t, t \in [0,T].$$

In the second case $|\theta^1| \leq \frac{1}{c}$ can be obtained by choosing $|\beta(n)| \leq \frac{1}{c}$. ■

Another arbitrage strategy for the fractional Bachelier model is given in [37] where a discrete time strategy for the case $\nu = 0$ in the model (3.8) is described. We analyze the result of [37] next.

The arbitrage strategy in [37] is constructed by splitting $[-1,0)$ into the union of intervals $\left(-\frac{1}{2^{n-1}}, -\frac{1}{2^n}\right)$. Trading is restricted to some special time intervals which are called promising since they promise a positive expected value of the price. The price of the asset stays in a price interval during trading. If it gets too high or too low, the asset is immediately sold. In this way, bounded gains are obtained on each interval. It is first shown that there exist time intervals which look promising. Then, it is concluded that they are infinitely many by stationarity. For each $n \in \mathbb{Z}$, the process

$$Y_n(t) := \left\{ B^H\left(\frac{t}{2^n} - \frac{1}{2^{n-1}}\right) - B^H\left(\frac{-1}{2^{n-1}}\right) \right\} 2^{nH}, \qquad (0 \leq t \leq 1)$$

is defined. The sequence $(Y_n)_{n \in \mathbb{Z}}$ of $C([0,1])$-valued random variables is stationary and ergodic by the scaling properties of B^H and is adapted to $\mathcal{G}_n := \sigma(-2^{-n})$. There exist $\alpha < 0 < \beta$ and $\varepsilon > 0$ such that

$$\mathbb{P}\{\mathbb{E}[Y_n(\tau_n)|\mathcal{G}_{n-1}] \geq \varepsilon\} \geq \varepsilon \text{ with } \tau_n := \inf\{t : Y_n(t) \notin [\alpha, \beta]\} \wedge 1. \tag{3.15}$$

By the ergodic theorem, it is concluded that

$$\mathbb{P}\{\mathbb{E}[Y_n(\tau_n)|\mathcal{G}_{n-1}] \geq \varepsilon \text{ for infinitely many } n \geq 0\} = 1. \tag{3.16}$$

A period is called *promising* if $\mathbb{E}[Y(\tau_n)|\mathcal{G}_{n-1}] \geq \varepsilon$. Trading is allowed only in promising periods. If the price process Y_n leaves $[\alpha, \beta]$ during a promising period, the shares are immediately sold and no trade is done until the next promising period. Thus, the gain made during a promising period is bounded and has positive mean. The accumulated gain at the end of period n is denoted by η_n. We have $\alpha \leq \eta_n - \eta_{n-1} \leq \beta$, since the gains are bounded, and $\mathbb{E}[\eta_n - \eta_{n-1}|\mathcal{G}_{n-1}] \geq \varepsilon$, since the period is promising. We can write the following Taylor expansion

$$\mathbb{E}\left[e^{-\lambda\eta_n} - e^{-\lambda\eta_{n-1}}|\mathcal{G}_n\right] = (-\lambda)(\eta_n - \eta_{n-1})e^{-\lambda\eta_{n-1}} + (-\lambda)^2\frac{(\eta_n - \eta_{n-1})^2}{2!}e^{-\lambda\xi}$$

Since the terms other than λ are positive, by choosing $\lambda > 0$ small enough, we can have this equation less than or equal to zero. Then, we have for all n

$$\mathbb{E}[e^{-\lambda\eta_n}|\mathcal{G}_{n-1}] \leq e^{-\lambda\eta_{n-1}}.$$

Thus $e^{-\lambda\eta_n}$ is a non-negative supermartingale. It is decreasing and bounded by zero, so convergent. It is clear that it converges to zero. If we stop η at the first time ν that $\nu < \alpha$, then we have

$$\mathbb{P}\{\nu < \infty\} \leq \exp(\lambda\alpha) = 1 - \theta < 1$$

and $\eta_n \to \infty$, on the event $\{\nu = +\infty\}$.

By the self-similarity and stationarity properties of fBm, we observe that

$$Y_n(t) = \left\{B^H\left(\frac{t}{2^n} - \frac{1}{2^{n-1}}\right) - B^H\left(-\frac{1}{2^{n-1}}\right)\right\}2^{nH} \stackrel{D}{=} B^H(t-2) - B^H(-2) \stackrel{D}{=} B^H(t)$$

We describe the arbitrage strategy as follows. We invest only in promising periods. We start with investing an amount of $|\alpha|$ in the stock. We stop if the accumulated gain η rises to 1 or falls below α. If it falls below α, we loose $|\alpha| + |\alpha| = 2|\alpha|$. Then we invest $\frac{|\alpha|}{2}$ in the next promising period and stop if the accumulated gain η rises to 1 or falls below $2|\alpha| + \frac{|\alpha|}{2} = \frac{5|\alpha|}{2}$. If the latter happens we loose $\frac{|\alpha|}{2} + \frac{5|\alpha|}{2} = 3|\alpha|$. Similarly, in the next promising period, we invest $\frac{|\alpha|}{4}$ and stop if the accumulated gain η rises to 1 or falls below $3|\alpha| + \frac{|\alpha|}{4} = \frac{13|\alpha|}{4}$. If the latter happens we loose $\frac{|\alpha|}{4} + \frac{13|\alpha|}{4} = \frac{14|\alpha|}{4}$. If we continue in this way, halving the amount we invest, and stopping on the described stopping times, we will eventually be successful and make a net gain of at least 1. In the worst case, our wealth can fall to $2|\alpha| \sum_{n=0}^{\infty} 2^n = 4|\alpha|$.

We have examined the discrete time trading strategy in [37] where it is constructed for both the cases $H \in \left(0, \frac{1}{2}\right)$ and $H \in \left(\frac{1}{2}, 1\right)$.

Finally, we will analyze the arbitrage strategy in [40]. We are given a one dimensional securities market model $(\Omega, \mathcal{F}, \mathbb{P}, [0,T], (\mathcal{F}_t), S)$ where $S = (X, Y)$ is the price process of the two assets, X being the numéraire. In [40], only the case $H \in \left(\frac{1}{2}, 1\right)$ is considered for the model (3.8). For simplicity, it is assumed that $r = 0$ and $X_t(0) = 1$ and

$$Y_t = Y_0 + \mu t + B_t^H. \tag{3.17}$$

It is shown that for $B^H, \frac{1}{2} < H < 1$, the corresponding market has an arbitrage property. For this purpose, we consider the following portfolio $\theta = (\theta^0, \theta^1)$

$$\begin{aligned} \theta_t^0 &= -\left(B_t^H\right)^2 - 2B_t^H. \\ \theta_t^1 &= 2B_t^H \end{aligned} \tag{3.18}$$

By choosing $Y_0 = 1$ and $\mu = 1$, its value is

$$\begin{aligned} V_t(\theta) &= \theta_t^0 X_t + \theta_t^1 Y_t \\ &= \left(-(B_t^H)^2 - 2B_t^H\right) + 2B_t^H \left(1 + t + B_t^H\right) \\ &= -(B_t^H)^2 - 2B_t^H + 2B_t^H + 2tB_t^H + 2(B_t^H)^2 \\ &= (B_t^H)^2 \end{aligned}$$

By Itô's formula (Theorem 2.1.13), we obtain

$$dV_t(\theta) = 2B_t^H dB_t^H = \theta_t^1 dY_t.$$

The portfolio $\theta = (\theta^0, \theta^1)$ is self-financing due to Definition 2.2.6. Since $V_0(\theta) = 0$ and $V_t(\theta) = (B_t^H)^2 > 0$ for $t > 0$, it follows that arbitrage occurs in the securities market for all $t > 0$. We see that there is continuous time trading in this strategy different from [8] and [37]. Therefore, the tools of stochastic calculus are used.

3.3.2 Arbitrage with Fractional Black-Scholes Model

Arbitrage strategies with fractional Black-Scholes model have been described in [1, 8, 40]. In [8], the model (3.10) is considered for the general case $H \in (0, 1/2) \cup (1/2, 1)$ while [40] considers only the case $H \in \left(\frac{1}{2}, 1\right)$ and [1] the general case when the volatility σ is a stochastic process. We review these strategies in this section.

In [8], the same assumptions in Bachelier model of the previous section are made and the construction of the arbitrage strategies is similar to the construction already done in there. The following results are the main results.

Theorem 3.3.3 *Suppose $\nu \in C^1[0, T]$ and $\sigma > 0$,*

$$\bar{Y}_t = \exp\left(\nu(t) + \sigma B_t^H\right), \qquad t \in [0, T].$$

Then in both cases i) $H \in (\frac{1}{2}, 1)$ and ii) $H \in (0, \frac{1}{2})$, for every constant $c > 0$ and all $n \in \mathbb{N}$, there exists $\theta^1(n) \in \boldsymbol{S}((\mathcal{F}_t)^{\bar{Y}})$ that satisfy the inequalities a) and b) in Theorem 3.3.1. In particular, the strategies $\theta(n) = (\theta^0(n), \theta^1(n)) \in \Theta^{\mathbf{S}}_{adm}((\mathcal{F}_t)^{\bar{Y}}), n \in \mathbb{N}$, where $\theta^0(n)$ is given by

$$\theta_t^0(n) = \int_0^t \theta^1(n) d\bar{Y} - \theta_t^1(n)\bar{Y}_t,\ t \in [0, T], n \in \mathbb{N}$$

form a c-FLVR as before. In the second case, c) in Theorem 3.3.1 is again satisfied. The result of Theorem 3.3.2 holds for both cases.

Proof: It is enough to prove the theorem for $T = 1$ since B^H is self-similar.

(i)$(\bar{Y}_t)_{t\in[0,1]}$ satisfies the conditions of Lemma 3.3 in [8]. We also have

$$\begin{aligned}
&\max_{t\in[0,1]}\sum_{j=0}^{n-1}\left(\bar{Y}_{\frac{j+1}{n}\wedge t}-\bar{Y}_{\frac{j}{n}\wedge t}\right)^2\\
&= \quad \max_{t\in[0,1]}\sum_{j=0}^{n-1}\left(e^{\nu(\frac{j+1}{n}\wedge t)+\sigma B^H(\frac{j+1}{n}\wedge t)}-e^{\nu(\frac{j}{n}\wedge t)+\sigma B^H(\frac{j}{n}\wedge t)}\right)^2\\
&\leq \quad \max_{t\in[0,1]}\sum_{j=0}^{n-1}c^2\left[\left(\nu(\frac{j+1}{n}\wedge t)+\sigma B^H(\frac{j+1}{n}\wedge t)\right)-\left(\nu(\frac{j}{n}\wedge t)+\sigma B^H(\frac{j}{n}\wedge t)\right)\right]^2\\
&\longrightarrow \quad 0 \qquad \text{as} \qquad n\to\infty \qquad \text{a.s.}
\end{aligned}$$

as in (3.11).

Hence the second condition in Lemma 3.3 in [8] is also satisfied. The proof follows as in case (i) of Theorem 3.3.1.

(ii) Since $(\bar{Y}_t)_{t\in[0,1]}$ is positive and continuous, $\min_{t\in[0,1]}\bar{Y}_t>0$. Let $\tilde{y}$ be the $\frac{1}{2L}$th percentile of $\min_{t\in[0,1]}\bar{Y}_t$. Then for all $y<\tilde{y}$ we have

$$\mathbb{P}\left\{\min_{t\in[0,1]}\bar{Y}_t\leq y\right\}<\frac{1}{2L}.$$

We choose $\varepsilon<\min(\frac{1}{\sqrt{2}},\tilde{y})$ then we have

$$\mathbb{P}\left\{\min_{t\in[0,1]}\bar{Y}_t\leq \varepsilon\right\}<\frac{1}{2L}.$$

and from what we have shown in the proof of (ii) of Theorem 3.3.1, there exists $n\in N$ such that

$$\mathbb{P}\left\{\sum_{j=0}^{n-1}\left(\ln\bar{Y}_{\frac{j+1}{n}}-\ln\bar{Y}_{\frac{j}{n}}\right)^2<\frac{1}{\varepsilon^2}L\right\}<\frac{1}{2L}.$$

By Intermediate Value Theorem, $e^x-e^y\geq\min_{t\in[0,1]}e^t(x-y)$.

For $x=\nu(\frac{j+1}{n})+\sigma B^H_{\frac{j+1}{n}}$ and $y=\nu(\frac{j}{n})+\sigma B^H_{\frac{j}{n}}$ we get,

$$\left|\bar{Y}_{\frac{j+1}{n}}-\bar{Y}_{\frac{j}{n}}\right|\geq\min_{t\in[0,1]}\bar{Y}_t\left|\ln\tilde{Y}_{\frac{j+1}{n}}-\ln\bar{Y}_{\frac{j}{n}}\right|.$$

Denoting

$$\begin{aligned} A &:= \left\{\omega : \min_{t\in[0,1]} \bar{Y}_t(\omega) \le \varepsilon\right\} \\ B &:= \left\{\omega : \sum_{j=0}^{n-1} \left(\ln \bar{Y}_{\frac{j+1}{n}}(\omega) - \ln \bar{Y}_{\frac{j}{n}}(\omega)\right)^2 < \frac{1}{\varepsilon^2} L\right\} \\ C &:= \left\{\omega : \sum_{j=0}^{n-1} \left(\bar{Y}_{\frac{j+1}{n}}(\omega) - \bar{Y}_{\frac{j}{n}}(\omega)\right)^2 < L\right\} \end{aligned}$$

we again have $C \subset A \cup B$ in other words, $P(C) \le P(A \cup B) \le P(A) + P(B)$ so we obtain

$$\begin{aligned} &\mathbb{P}\left\{\sum_{j=0}^{n-1} \left(\bar{Y}_{\frac{j+1}{n}} - \bar{Y}_{\frac{j}{n}}\right)^2 < L\right\} \\ \le\ &\mathbb{P}\left\{\min_{t\in[0,1]} \bar{Y}_t < \varepsilon\right\} + \mathbb{P}\left\{\sum_{j=0}^{n-1} \left(\ln \bar{Y}_{\frac{j+1}{n}} - \ln \bar{Y}_{\frac{j}{n}}\right)^2 < \frac{1}{\varepsilon^2} L\right\} < \frac{1}{L}. \end{aligned}$$

Thus, Lemma 3.4 in [8] applies and $\theta^1(n)$ can be constructed as in case (ii) of Theorem 3.3.1. Again, $|\theta^1(n)| \le \frac{1}{n}$. This completes the proof of the first part. The second part follows by choosing $\beta(n)$'s such that $|\beta(n)| \le \frac{1}{c}$. Then, $|\theta^1| \le \frac{1}{c}$ too, and the theorem is proved. ■

We now analyze the strategy given in [40] where an arbitrage strategy is constructed in model (3.10) for the case $H \in \left(\frac{1}{2}, 1\right)$ in the market model $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{T}, (\mathcal{F}_t), S)$ with $\mathbb{T} = [0,1]$. For simplicity, we let $\nu = 0, Y_0 = \delta = 1$. In particular the model is given by

$$\begin{aligned} X_t &= e^{rt}, \\ Y_t &= e^{rt + B_t^H}. \end{aligned} \tag{3.19}$$

In view of Itô's formula (Theorem 2.1.13), we have

$$dY_t = Y_t \left(rdt + dB_t^H\right) \tag{3.20}$$

and we also have

$$dX_t = rX_t dt.$$

Now, the portfolio $\theta = (\theta^0, \theta^1)$ with

$$\begin{aligned} \theta_t^0 &= 1 - e^{2B_t^H}, \\ \theta_t^1 &= 2\left(e^{2B_t^H} - 1\right) \end{aligned} \tag{3.21}$$

is considered. For this portfolio, we have

$$\begin{aligned} V_t(\theta) &= (1 - e^{2B_t^H})e^{rt} + 2(e^{B_t^H} - 1)e^{rt+B_t^H} \\ &= e^{rt} - e^{2B_t^H+rt} + 2e^{2B_t^H+rt} - 2e^{rt+B_t^H} \\ &= e^{rt} + e^{2B_t^H+rt} - 2e^{rt+B_t^H} \\ &= e^{rt}(1 - 2e^{B_t^H} + e^{2B_t^H}) \\ &= e^{rt}(e^{B_t^H} - 1)^2 \end{aligned}$$

and

$$\begin{aligned} dV_t(\theta) &= re^{rt}(e^{B_t^H} - 1)^2 dt + 2e^{rt+B_t^H}(e^{B_t^H} - 1)dB_t^H \\ &= \theta_t^0 dX_t + \theta_t^1 Y_t \end{aligned}$$

Hence, the portfolio is self-financing according to Equation (2.9). Since for this portfolio we also have $V_0(\theta) = 0$ and $V_t(\theta) > 0$ for $t > 0$, this market model leaves space for arbitrage for each $t > 0$.

Finally, we will analyze the arbitrage strategy introduced in [1] where the following market model is assumed

$$\begin{aligned} X_t &= e^{rt} \qquad ,0 \le t \le T \\ dY_t &= Y_t(\nu dt + \sigma_t Z_t), \qquad 0 \le t \le T. \end{aligned} \tag{3.22}$$

Here, σ is a stochastic volatility process which is not a constant, and Z denotes a process with zero quadratic variation. The cases $Z_t = B_t^H$ or $Z_t = B_{A_t}^H$ where A is a continuous non-decreasing process are considered in the two cases where σ is a semimartingale or a C^1 function of the modulating fBm. These cases for σ are large enough to include many volatility models in the literature. The following is the main result giving a strategy that leads to arbitrage.

Theorem 3.3.4 *Consider a market modeled by (3.22). Let Z be an a.s. continuous adapted process with zero quadratic variation. Suppose (σ_t) is an adapted process such that $(\int_0^t \sigma_s dZ_s)_{t\in[0,T]}$ exists, has zero quadratic variation, and is continuous. Then, for each $c > 0$, the following portfolio is an arbitrage strategy*

$$\begin{aligned} \theta_t^0 &= cY_0\left(1 - \exp\left(2(\nu - r)t + 2\int_0^t \sigma_s dZ_s\right)\right) \qquad (3.23) \\ \theta_t^1 &= 2c\left(\exp\left((\nu - r)t + \int_0^t \sigma_s dZ_s\right) - 1\right). \end{aligned}$$

Proof: We will sketch the proof and omit the calculations which can be found in [1]. First, it is shown that $V_t(\theta) > 0, \forall t$ in the following

$$\begin{aligned} V_t(\theta) &= \theta_t^0 \exp(rt) + \theta_t^1 Y_t \qquad (3.24) \\ &= cY_0\left(1 - \exp\left(2(\nu - r)t + 2\int_0^t \sigma_s dZ_s\right)\right)\exp(rt) \\ &\quad +2c\left(\exp\left((\nu - r)t + \int_0^t \sigma_s dZ_s\right) - 1\right)Y_0 \exp\left(\nu t + \int_0^t \sigma_s dZ_s\right) \\ &= cY_0 \exp(rt)\left(\exp\left(\nu t - rt + \int_0^t \sigma_s dZ_s\right) - 1\right)^2 \\ &> 0. \end{aligned}$$

This was the first step to show that arbitrage exists. Next, it is shown that the portfolio is self-financing. By (3.24) we see that $V_t(\theta)$ is a smooth enough function of t and $\int_0^t \sigma_s dZ_s$, both of which have zero quadratic variation and are continuous. Therefore using the modified Itô formula (Theorem 2.1.13), we have

$$\begin{aligned} dV_t(\theta) &= crY_0 \exp(rt)\left(\left(\exp\left((\nu - r)t + \int_0^t \sigma_s dZ_s\right) - 1\right)^2 \right. \qquad (3.25) \\ &\quad +2c(\nu - r)Y_0 \exp(rt)\left(\exp\left((\nu - r)t + \int_0^t \sigma_s dZ_s\right) - 1\right) \\ &\quad \left. \times \exp\left((\nu - r)t + \int_0^t \sigma_s dZ_s\right)\right)dt \\ &\quad +2cY_0 \exp(rt)\left(\exp\left((\nu - r)t + \int_0^t \sigma_s dZ_s\right) - 1\right) \\ &\quad \times \exp\left((\nu - r)t + \int_0^t \sigma_s dZ_s\right)\sigma_t dZ_t. \end{aligned}$$

On the other hand, we have

$$\begin{aligned}\theta_t^0 dX_t + \theta_t^1 dY_t &= \theta_t^0 r\exp(rt)dt + \theta_t^1 Y_t(\nu dt dZ_t) \\ &= (\theta_t^0 r\exp(rt) + \theta_t^1 Y_t\nu)dt + \theta_t^1 Y_t \sigma_t dZ_t\end{aligned} \tag{3.26}$$

$$\begin{aligned}= \ & \Bigg\{ cY_0 \exp(rt) \left(\left(1 - \exp\left(2(\nu - r)t + 2\int_0^t \sigma_s dZ_s \right) \right) \right. \\ & + \left. \left(\exp\left(2(\nu-r)t + 2\int_0^t \sigma_s dZ_s \right) \right) - 2\exp\left((\nu - r)t + \int_0^t \sigma_s dZ_s \right) \right) \\ & + 2cY_0(\nu - r)\exp(rt) \left(\exp\left((\nu - r)t + \int_0^t \sigma_s dZ_s \right) - 1 \right) \\ & \quad \times \exp\left((\nu - r)t + \int_0^t \sigma_s dZ_s \right) \Bigg\} dt \\ & + 2\,cY_0 \exp(rt) \left(\exp\left((\nu - r)t + \int_0^t \sigma_s dZ_s \right) \right) \exp\left((\nu - r)t + \int_0^t \sigma_s dZ_s \right) \sigma_t dZ_t.\end{aligned}$$

Since the right-hand side of (3.26) is equal to the right-hand side of (3.25) this strategy is self-financing according to Equation (2.9). It is positive at all times. Hence this is an arbitrage strategy by Definition 2.10. ■

It is shown in [1] that there is a wide class of processes that satisfy the assumptions of Theorem 3.3.4. Two separate types of stochastic volatility models are considered; those in which volatility is a semimartingale, and those in which it is a function of the integrator B^H. We will not give the arbitrage strategies constructed for these models here. They can be found in [1]. The following corollary gives an arbitrage strategy in a model involving Brownian motion (Bm) and fBm.

Corollary 3.3.5 *Suppose the dynamics of the stock price are modeled by the following stochastic differential equations:*

$$\begin{aligned}dY_t &= Y_t(\nu dt + \phi(W_t)dB_t^H) \\ dW_t &= \mu_W(t, W_t)dt + \sigma_W(t, W_t)dB_t,\end{aligned}$$

where B^H is a fBm with $H \in (\frac{1}{2}, 1]$, and B is a Bm. Assume that $\int_0^T \mu_W(s, W_s)ds < \infty$ and $\int_0^T (\sigma_W(s, W_s))^2 ds < \infty$, and that $\phi \in C^2$. Then for each $c > 0$, the following *portfolio is an arbitrage strategy:*

$$\begin{aligned}\theta_t^0 &= cY_0\left(1-\exp\left(2(\nu-r)t+2\int_0^t\phi(W_s)dZ_s\right)\right)\\ \theta_t^1 &= 2c\left(\exp\left((\nu-r)t+\int_0^t\phi(W_s)dZ_s\right)-1\right).\end{aligned}$$

3.3.3 Exclusion of Arbitrage in fBm Models

For the models in [8] and [37] described in Sections 3.3.1 and 3.3.2, the corresponding authors suggest methods to exclude arbitrage. We will now briefly review these methods.

The arbitrage strategy constructed in Theorem 3.3.2 and 3.3.3 is a discrete time strategy but it needs trading in very small time intervals. A suggestion to prevent arbitrage in these models is to introduce some small unit of time $h > 0$ that must lie between two transactions. If trading is not allowed before h units of time has passed after the last trading, then no arbitrage strategy can be constructed. The following definition is given for imposing this restriction to the trading rules.

Definition 3.3.6 *Let* $(\mathcal{F}_t)$ *be a filtration and* $h > 0$. *We define*

$$\begin{aligned}\mathbf{S}^h((\mathcal{F}_t)) &:= \left\{g_0 1_{\{0\}} + \sum_{j=1}^{n-1} g_j 1_{(\tau_j,\tau_{j+1}]} \in \mathbf{S}((\mathcal{F}_t)) : \forall j, \tau_{j+1} \geq \tau_j + h\right\} \qquad \text{and}\\ \Theta_{sf}^h((\mathcal{F}_t)) &:= \left\{\theta = (\theta^0,\theta^1) \in \Theta_{sf}^{\mathbf{S}} : \theta^0,\theta^1 \in \mathbf{S}^h((\mathcal{F}_t))\right\}.\end{aligned}$$

The following theorem states a sufficient condition to exclude arbitrage opportunities from the models defined in theorems 3.3.1 and 3.3.3.

Theorem 3.3.7 *Let* B^H *be a fBm with* $H \in (0,\frac{1}{2}) \cup (\frac{1}{2},1)$. *Let* $T \in (0,\infty), \sigma > 0$ *and* $\nu : [0,T] \to \mathbb{R}$ *be a measurable function such that* $\sup_{t\in[0,T]} |\nu(t)| < \infty$. *Consider the case*

$$\bar{Y}_t = \nu(t) + \sigma B_t^H,\ t \in [0,T]$$

If

$$\theta^1 = g_0 1_{\{0\}}(t) + \sum_{j=1}^{n-1} g_j 1_{(\tau_j, \tau_{j+1}]}(t) \in \bigcup_{h>0} \mathbf{S}^h((\mathcal{F}_t)^{\bar{Y}})$$

and there exists $j \in \{1, \ldots, n-1\}$ *with* $\mathbb{P}\{g_j \neq 0\} > 0$, *then,*

$$\mathbb{P}\left\{\int_0^T \theta^1 d\bar{Y} \leq -c\right\} > 0 \qquad \textit{for all} \qquad c \geq 0.$$

Another method to exclude arbitrage from the model (3.8) is suggested in [37]. In this model arbitrage arises because of the behavior of the kernel $\varphi(t) = t^{H-\frac{1}{2}} I_{\{t>0\}}$ on small time scales. To get rid of this problem without sacrificing the long-range dependence of fBm, the following process is defined

$$\tilde{B}_t = \int_{-\infty}^{t} \varphi(t-s) dB(s) - \int_{-\infty}^{0} \varphi(-s) dB(s)$$

where $\varphi \in C^2(\mathbb{R}), \varphi(0) = 1, \varphi'(0) = 0$ and $\lim_{t\to\infty} \varphi''(t) t^{\frac{5}{2}-H}$ existing in $(0, \infty)$. Then, we have that $\bar{B}_t$ is a Gaussian process with the same long-range dependence as fBm and yet

$$\begin{aligned}
\tilde{B}_t &= \varphi(t-t)B_t - \varphi(t+\infty)B_{-\infty} + \int_{-\infty}^{t} \varphi'(t-s)B_s ds - < \varphi, B > \\
&\quad -\varphi(0)B_0 + \varphi(\infty)B_\infty - \int_{-\infty}^{0} B_s ds + < \varphi, B > \\
&= B_t + \int_{-\infty}^{t} \varphi'(t-s)B_s ds - \int_{-\infty}^{0} \varphi'(-s)B_s ds \\
&= B_t + \int_{-\infty}^{0} \left[\varphi'(t-s) - \varphi'(-s)\right] B_s ds + \int_{0}^{t} \left[\varphi'(t-s) - \varphi'(s-s)\right] B_s ds \\
&= B_t + \int_{-\infty}^{0} \left(\int_{0}^{t} \varphi''(v-s) dv\right) B_s ds + \int_{0}^{t} \left(\int_{s}^{t} \varphi''(v-s) dv\right) B_s ds \\
&= B_t + \int_{0}^{t} \left(\int_{-\infty}^{0} \varphi''(v-s) B_s ds\right) dv + \int_{0}^{t} \left(\int_{0}^{v} \varphi''(v-s) B_s ds\right) dv \\
&= B_t + \int_{0}^{t} \left(\int_{-\infty}^{0} \varphi''(s-v) B_v dv\right) ds + \int_{0}^{t} \left(\int_{0}^{s} \varphi''(s-v) B_v dv\right) ds \\
&= B_t + \int_{0}^{t} \left(\int_{-\infty}^{s} \varphi''(s-v) B_v dv\right) ds
\end{aligned}$$

showing that $\tilde{B}$ is a semimartingale, hence does not allow arbitrage opportunities. For example, if

$$\varphi(t) = (\varepsilon + t^2)^{\frac{(2H-1)}{4}}$$

is taken as the kernel, then no arbitrage will occur.

Chapter 4

AGENT BASED STOCK PRICE MODELS

Agent based modeling is widely used to find a model that best fits stock price processes. In some studies, agents are divided into two groups, mostly named as chartists and fundamentalists. In these studies, the two agent groups have different demand functions for the stock [18, 23, 23]. The price is generally determined via the total excess demand [15, 45, 20, 4, 6].

In this chapter, we construct two different agent based models. Agents can belong to different groups such as fundamentalists and chartists and possibly others. In both models, under the assumption of positive correlation between total net demand and the price change, we assume that each buy order increases the price whereas each sell order decreases it. Each order given to the market has an effect proportional to its quantity. The difference of the two models is the kind of effect function. We show that the price process is long-range dependent and prove that the limit of the price process is fBm as the number of order arrivals increases and the quantity of the orders decreases. Although the limiting process is fBm, our constructions are semimartingales which do not allow arbitrage.

4.1 Vanishing Effects

Agent based models that involve heterogenous agents divide them into two separate groups in general as chartists and fundamentalists according to their trading behavior. We generalize this situation by assuming that there are I types of agents in the market as in [2].

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $\mathcal{B}_{\mathbb{R}}$ denote the Borel σ-algebra on $\mathbb{R}$. For $i = 1, \ldots, I$, let N^i be Poisson random measures on $(\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}, \mathcal{B}_{\mathbb{R}} \otimes \mathcal{B}_{\mathbb{R}_+} \otimes \mathcal{B}_{\mathbb{R}})$

with mean measures

$$\mu^i(ds, du, dq) = \lambda^i u^{-\delta^i - 1} ds du F^i(dq)$$

where $\lambda^i > 0$ is the arrival rate of orders of agent i, $1 < \delta^i < 3$ and F^i is the distribution of a random variable Q^i with finite second moment. Each atom (S^i_j, U^i_j, Q^i_j) represents an order from an agent that belongs to the i^{th} type; the arrival time S^i_j of the order, the duration of its effect U^i_j and its quantity Q^i_j, $j = 1, 2, \ldots$ .Each agent has its own effect to the price process. Let $g^i(t - s, u, q) : \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be the effect function. The effect of an order starting at s and ending at $s + u$ depends on the quantity of the order and equals

$$g^i(t - s, u, q)$$

at time t.

Let $Z(t)$ denote the logarithm of the price of the stock at time t. We denote with $Z^i(t)$ the total effect given to the price by the agents of type i. The logarithm of the price of the stock at time t is the sum of those effects. We think $Z^i(t) - Z^i(0)$ as the sum of all effects due to all active agents of type i between times 0 and t. We set $Z^i(0) = 0$. Hence, the price at time t is

$$Y(t) = e^{Z(t)} \qquad \text{with} \qquad Z(t) = \sum_{i=1}^{n} Z^i(t)$$

where $Z^i(t)$ is given by

$$Z^i(t) = \int_{-\infty}^{\infty} \int_0^{\infty} \int_{-\infty}^{\infty} (g^i(t - s, u, q) - g^i(-s, u, q)) N^i(ds, du, dq) \qquad t \geq 0.$$

Adding up the effects corresponds to the integration of the difference in the effect amplitude at times 0 and t with respect to the Poisson random measure.

We turn our attention to finding the limit of the price process as a result of frequent trading in smaller and smaller amounts. We choose one type of agent and analyze the limiting process for this particular type. The others would have similar limits in case of heavy trading. The limit of the logarithm of the price process $Z(t)$ will be the sum of the limits of these processes. For simplicity, we denote the Prm of our particular agent group by N, the effect function by g and the mean measure by μ with the terms λ and δ. We assume that the effect function g has the form

$$g(t-s,u,q) = q\,u\,f\left(\frac{t-s}{u}\right).$$

Here, f is a deterministic effect function which is scaled linearly by q as in the micropulses of [9]. What we have in mind is a function f with compact support. In particular, its effect starts at a point in time and finishes after a period.

We introduce a scaling factor $n \in \mathbb{Z}_+$, that will eventually tend to ∞ and consider only the effects in which the quantity is rescaled to q/n and s, u remain unchanged. Thus, the effect at time t of a rescaled effect with coordinates s, q and u equals

$$\frac{q}{n}\,u\,f\left(\frac{t-s}{u}\right).$$

For each $n \in \mathbb{Z}_+$, we consider a Prm N_n on $(\mathbb{R}\times\mathbb{R}_+\times\mathbb{R}, \mathcal{B}_\mathbb{R}\otimes\mathcal{B}_{\mathbb{R}_+}\otimes\mathcal{B}_\mathbb{R})$ with mean measure

$$\mu_n(ds,du,dq) = n^2\lambda u^{-\delta-1}dsduF(dq). \tag{4.1}$$

The scaled price process is given by

$$Z_n(t) = \int_{-\infty}^{\infty}\int_0^{\infty}\int_{-\infty}^{\infty}\frac{q}{n}\,u\left[f\left(\frac{t-s}{u}\right) - f\left(\frac{-s}{u}\right)\right]N_n(ds,du,dq). \tag{4.2}$$

We first prove the existence of the process $\{Z_n(t), t \geq 0\}$. If

$$\int_{-\infty}^{\infty}\int_0^{\infty}\int_{-\infty}^{\infty}\frac{q}{n}\,u\left|f\left(\frac{t-s}{u}\right) - f\left(\frac{-s}{u}\right)\right|\mu_n(ds,du,dq) < \infty,$$

then Z_n is well defined since then $\mathbb{E}|Z_n(t)| < \infty$. However, if $\mathbb{E}|Z_n(t)|$ is not finite, but

$$\int_{-\infty}^{\infty}\int_0^{\infty}\int_{-\infty}^{\infty}\frac{q^2}{n^2}\,u^2\left[f\left(\frac{t-s}{u}\right) - f\left(\frac{-s}{u}\right)\right]^2\mu_n(ds,du,dq) < \infty$$

then $Z_n(t)$ is still well-defined as shown in the next proposition.

Proposition 4.1.1 *Suppose that* $\int_{-\infty}^{\infty}\int_a^{\infty}\int_{-\infty}^{\infty}\frac{q}{n}\,u\left[f\left(\frac{t-s}{u}\right) - f\left(\frac{-s}{u}\right)\right]\mu_n(ds,du,dq) = 0$ *for all* $a > 0$ *and* $\int_{-\infty}^{\infty}\int_0^{\infty}\int_{-\infty}^{\infty}\frac{q^2}{n^2}\,u^2\left[f\left(\frac{t-s}{u}\right) - f\left(\frac{-s}{u}\right)\right]^2\mu_n(ds,du,dq) < \infty$ *for every* *n. Then, the process* $\{Z_n(t), t \geq 0\}$ *is well-defined for* $1 < \delta < 3$ *and its characteristic function at* $\xi \in \mathbb{R}$ *is given by*

$$\exp\left\{\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{-\infty}^{\infty}\left(e^{i\xi\frac{q}{n}u\left[f\left(\frac{t-s}{u}\right)-f\left(\frac{-s}{u}\right)\right]}-1 \right.\right.$$
$$\left.\left. -i\xi\frac{q}{n}\left[f\left(\frac{t-s}{u}\right)-f\left(\frac{-s}{u}\right)\right]\right)\mu_n(ds,du,dq)\right\} \tag{4.3}$$

Proof: Let $A_k = (2^{-k}, 2^{-k+1}]$, $k = 1, 2, \ldots$, $A_0 = (1, \infty)$ be a partition of $\mathbb{R}^+ \equiv (0, \infty)$. Clearly,

$$\int_{-\infty}^{\infty}\int_{A_k}\int_{-\infty}^{\infty}\frac{q}{n}u\left[f\left(\frac{t-s}{u}\right)-f\left(\frac{-s}{u}\right)\right]\mu_n(ds,du,dq)=0,$$

for $k = 1, 2, \ldots$, due to the form of the effect function and compactness of A_k, and also for $k = 0$ due to the hypothesis of the Proposition. Hence, random variables

$$\int_{-\infty}^{\infty}\int_{A_k}\int_{-\infty}^{\infty}\frac{q}{n}u\left[f(\frac{t-s}{u})-f(\frac{-s}{u})\right]N_n(ds,du,dq)\ \ k=0,1,\ldots, \tag{4.4}$$

are well-defined and have zero expectations. Due to the hypothesis, the sum of their variances is finite, which is found from formula (2.2) as

$$\mathrm{Var}\sum_{k=0}^{\infty}\int_{-\infty}^{\infty}\int_{A_k}\int_{-\infty}^{\infty}\frac{q}{n}u\left[f\left(\frac{t-s}{u}\right)-f\left(\frac{-s}{u}\right)\right]N_n(ds,du,dq)=$$
$$\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{-\infty}^{\infty}\frac{q^2}{n^2}u^2\left[f\left(\frac{t-s}{u}\right)-f\left(\frac{-s}{u}\right)\right]^2\mu_n(ds,du,dq).$$

Therefore, the series

$$\sum_{k=0}^{\infty}\int_{-\infty}^{\infty}\int_{A_k}\int_{-\infty}^{\infty}\frac{q}{n}u\left[f\left(\frac{t-s}{u}\right)-f\left(\frac{-s}{u}\right)\right]N_n(ds,du,dq) \tag{4.5}$$

is a.s. convergent by [12], Theorem 7.5. We denote the limit by

$$Z_n(t)=\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{-\infty}^{\infty}\frac{q}{n}\,u\left[f\left(\frac{t-s}{u}\right)-f\left(\frac{-s}{u}\right)\right]N_n(ds,du,dq)$$

which is $Z_n(t)$ of (4.2). Its characteristic function at $\xi \in \mathbb{R}$ equals

$$\mathbb{E}\exp\left\{i\xi\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{-\infty}^{\infty}\frac{q}{n}u\left[f\left(\frac{t-s}{u}\right)-f\left(\frac{-s}{u}\right)\right]N_n(ds,du,dq)\right\}. \tag{4.6}$$

To compute the above characteristic function, we proceed by finding the characteristic functions of the random variables (4.4) which are equal to

$$\exp\left\{\int_{-\infty}^{\infty}\int_{A_k}\int_{-\infty}^{\infty}\left[e^{i\xi\frac{q}{n}u\left[f\left(\frac{t-s}{u}\right)-f\left(\frac{-s}{u}\right)\right]}-1\right]\mu_n(ds,du,dq)\right\}$$

by formula 2.3. The logarithm of this is also equal to

$$\int_{-\infty}^{\infty}\int_{A_k}\int_{-\infty}^{\infty}\left[e^{i\xi\frac{q}{n}u\left[f\left(\frac{t-s}{u}\right)-f\left(\frac{-s}{u}\right)\right]}-1-i\xi\frac{q}{n}u\left[f\left(\frac{t-s}{u}\right)-f\left(\frac{-s}{u}\right)\right]\right]\mu_n(ds,du,dq) \tag{4.7}$$

since (4.4) has zero mean. Due to the inequality $|e^{ix}-1-ix|<\frac{1}{2}x^2$ for $x\in\mathbb{R}$ and $\int_{-\infty}^{\infty}\int_0^{\infty}\int_{-\infty}^{\infty}\frac{1}{2}\left|-\xi\frac{q}{n}u\left[f\left(\frac{t-s}{u}\right)-f\left(\frac{-s}{u}\right)\right]\right|^2\mu_n(ds,du,dq)<\infty$ by assumption, the dominated convergence theorem applies. Hence, the series

$$\sum_{k=0}^{\infty}\left\{\int_{-\infty}^{\infty}\int_{A_k}\int_{-\infty}^{\infty}\left[e^{i\xi\frac{q}{n}u\left[f\left(\frac{t-s}{u}\right)-f\left(\frac{-s}{u}\right)\right]}-1\right.\right.$$
$$\left.\left.-\,i\xi\frac{q}{n}u\left[f\left(\frac{t-s}{u}\right)-f\left(\frac{-s}{u}\right)\right]\right]\mu_n(ds,du,dq)\right\}$$

converges, which is clearly (4.3). The a.s convergence of the series (4.5) to Z_n is shown above. This implies convergence in distribution. Hence, (4.3) is the characteristic function of Z_n. Therefore, (4.6) is well defined and is given by (4.3). ■

We now state and prove convergence to fBm.

Theorem 4.1.2 *Suppose that*

$$\int_{-\infty}^{\infty}\int_a^{\infty}\int_{-\infty}^{\infty}q\left[f\left(\frac{t-s}{u}\right)-f\left(\frac{-s}{u}\right)\right]n\lambda u^{-\delta}ds\,du\,F(dq)=0 \tag{4.8}$$

and

$$\int_0^{\infty}\int_{-\infty}^{\infty}\left[f\left(\frac{1-s}{u}\right)-f\left(\frac{-s}{u}\right)\right]^2u^{1-\delta}dsdu<\infty \tag{4.9}$$

and $\mathbb{E}Q^2<\infty$. *Then, the finite dimensional distributions of* $\{Z_n(t),t\geq 0\}$, *for* $1<\delta<3$ *converge to those of a fBm, with variance*

$$\lambda\mathbb{E}Q^2\int_0^{\infty}\int_{-\infty}^{\infty}\left[f\left(\frac{1-s}{u}\right)-f\left(\frac{-s}{u}\right)\right]^2u^{1-\delta}dsdu$$

as $n\to\infty$.

Proof: To prove the convergence in the sense of finite dimensional distributions of $\{Z_n(t), t \geq 0\}$, we consider the characteristic function of $(Z_n(t_1), \ldots, Z_n(t_m))$, that is, $\mathbb{E}\exp(i\sum_{k=1}^m \xi_k Z_n(t_k),\ t_k \geq 0)$, $\xi_k \in \mathbb{R},\ m \in \mathbb{N}$ which equals

$$\mathbb{E}\exp\left\{ i\sum_{k=1}^{m}\xi_k \int_{-\infty}^{\infty}\int_0^{\infty}\int_{-\infty}^{\infty}\frac{q}{n}u\left[f\left(\frac{t-s}{u}\right)-f\left(\frac{-s}{u}\right)\right]N_n(ds,du,dq)\right\}$$

$$= \exp\left\{\int_{-\infty}^{\infty}\int_0^{\infty}\int_{-\infty}^{\infty}\left[e^{i\sum_{k=1}^m \xi_k \frac{q}{n}u\left[f\left(\frac{t-s}{u}\right)-f\left(\frac{-s}{u}\right)\right]}-1\right.\right.$$

$$\left.\left.-i\sum_{k=1}^{m}\xi_k\frac{q}{n}u\left[f\left(\frac{t-s}{u}\right)-f\left(\frac{-s}{u}\right)\right]\right]n^2\lambda\, u^{-\delta-1}ds\,du\,F(dq)\right\} \quad (4.10)$$

by the same derivation given in the proof of Proposition 4.1.1 since the assumptions (4.8) and (4.9) with $\mathbb{E}Q^2 < \infty$ imply those of the proposition. As $n \to \infty$, we will show that the above characteristic function converges to

$$\exp\left\{-\int_{-\infty}^{\infty}\int_0^{\infty}\int_{-\infty}^{\infty}\frac{1}{2}\sum_{k=1}^{m}\sum_{j=1}^{m}\xi_j\xi_k q^2\right.$$

$$\left.\left[f\left(\frac{t_j-s}{u}\right)-f\left(\frac{-s}{u}\right)\right]\left[f\left(\frac{t_k-s}{u}\right)-f\left(\frac{-s}{u}\right)\right]\lambda u^{1-\delta}dsduF(dq)\right\} \quad (4.11)$$

which is the characteristic function of $\sum_{k=1}^m \xi_k Z(t_k)$, where $Z = (Z(t_1), \ldots, Z(t_m))$ is a Gaussian vector with zero mean and covariance matrix

$$\lambda\mathbb{E}Q^2\int_0^{\infty}\int_{-\infty}^{\infty}\left[f\left(\frac{t_j-s}{u}\right)-f\left(\frac{-s}{u}\right)\right]\left[f\left(\frac{t_k-s}{u}\right)-f\left(\frac{-s}{u}\right)\right]u^{1-\delta}dsdu. \quad (4.12)$$

Due to the inequality $|e^{ix}-1-ix| < \frac{1}{2}x^2$ for $x \in \mathbb{R}$, the integrand in (4.11) is an upper bound to

$$\left|e^{i\sum_{k=1}^m \xi_k \frac{q}{n}u\left[f\left(\frac{t-s}{u}\right)-f\left(\frac{-s}{u}\right)\right]}-1-i\sum_{k=1}^{m}\xi_k\frac{q}{n}u\left[f\left(\frac{t-s}{u}\right)-f\left(\frac{-s}{u}\right)\right]\right|$$

which is the absolute value of the integrand in (4.10). Therefore, the dominated convergence theorem allows us to take the limit inside the integral in (4.10). That is, we must find

$$\lim_{n\to\infty}\left(e^{i\sum_{k=1}^m \xi_k \frac{q}{n}u\left[f\left(\frac{t-s}{u}\right)-f\left(\frac{-s}{u}\right)\right]}-1-i\sum_{k=1}^{m}\xi_k\frac{q}{n}u\left[f\left(\frac{t-s}{u}\right)-f\left(\frac{-s}{u}\right)\right]\right)n^2. \quad (4.13)$$

We claim that the limit is

$$-\frac{1}{2}\sum_{k=1}^{m}\sum_{j=1}^{m}\xi_j\xi_k q^2\left[f\left(\frac{t_j-s}{u}\right)-f\left(\frac{-s}{u}\right)\right]\left[f\left(\frac{t_k-s}{u}\right)-f\left(\frac{-s}{u}\right)\right] \tag{4.14}$$

For fixed n, the absolute value of the difference between the expression in (4.13) and (4.14) equals

$$n^2|R_2(n)|$$

where $R_2(\varepsilon)$ is the remainder term after 3 terms of the Taylor expansion of the function $h(\varepsilon) = e^{i\sum_{k=1}^{m}\varepsilon q u\left[f\left(\frac{t_k-s}{u}\right)-f\left(\frac{-s}{u}\right)\right]}$ on the disk $D = \{\varepsilon \in \mathbb{C} : |\varepsilon| < 2\}$ around 0 expanded for ε [19]. For all $\varepsilon \in D$ we have the following bound for $R_2(\varepsilon)$

$$|R_2(\varepsilon)| < \frac{rM}{r-r_0}\left(\frac{r_0}{r}\right)^3$$

where $r_0 = |\varepsilon - 0| = |\varepsilon|$, $r \in (|\varepsilon|, 2)$ and $M = \max_{|\varepsilon|=r}\left|e^{i\sum_{k=1}^{m}\xi_k\varepsilon q u\left[f\left(\frac{t_j-s}{u}\right)-f\left(\frac{-s}{u}\right)\right]}\right| = 1$. Hence, with $\varepsilon = \frac{1}{n}$

$$|R_2(n)| < \frac{r}{r-1/n}\left(\frac{1/n}{r}\right)^3.$$

Taking the limit as $n \to \infty$, we obtain $|R_2(n)|n^2 \to 0$. This shows that (4.10) converges to (4.11) as $n \to \infty$.

When (4.12) is evaluated, the variance is found to be

$$\mathrm{Var}(Z_n(t)) = \lambda \mathbb{E}Q^2 \int_0^\infty \int_{-\infty}^\infty \left[f\left(\frac{t-s}{u}\right)-f\left(\frac{-s}{u}\right)\right]^2 u^{1-\delta} ds du.$$

and the covariance of $Z_n(t)$ is

$$\mathrm{Cov}(Z_n(t_1), Z_n(t_2)) = \frac{1}{2}(|t_1|^{2H} + |t_2|^{2H} - |t_1-t_2|^{2H})\mathrm{Var}(Z_n(1))$$

with $H = (3-\delta)/2$ as defined in Definition 3.1.3. ■

The following proposition is essentially Proposition 3.1 of [9] which gives a sufficient condition for the effect function f to satisfy Equation (4.9). Its proof can be found in [9].

Proposition 4.1.3 *Suppose $f : [0,1] \to \mathbb{R}$ is Hölder continuous in $[0,1]$ with an exponent $\alpha > 0$, that is,*

$$|f(x) - f(y)| \le M|x-y|^\alpha$$

for some $M > 0$ and any $x, y \in [0,1]$ and $f(0) = f(1) = 0$. Then the statement in (4.9) which is

$$\int_0^\infty \int_{-\infty}^\infty \left[f\left(\frac{1-s}{u}\right) - f\left(\frac{-s}{u}\right)\right]^2 u^{1-\delta} ds du < \infty$$

is true for $3 - 2\alpha < \delta < 3$.

Remark. When f has compact support and is continuous as in Proposition 4.1.3, we can show that (4.8) also holds. Since then

$$\int_a^\infty \int_{-\infty}^\infty \left[f\left(\frac{t-s}{u}\right) - f\left(\frac{t-s}{u}\right)\right] u^{-\delta} ds\, du = 0.$$

We now give a simple example of such an effect function [9]. The effect function given by $g(t,s,u) = (\frac{u}{2} - |t - s - \frac{u}{2}|)$ on $t \in [s, s+u]$ can be written in the form $u\, f(\frac{t-s}{u}) = u\left(\frac{1}{2} - \left|\frac{t-s}{u} - \frac{1}{2}\right|\right)$. The function f is Hölder continuous with order 1 on $[0,1]$ and satisfy $f(0) = f(1) = 0$ and hence the assumptions of Proposition 4.1.3 hold. It has the shape of an isosceles triangle. In this case, the effect starts at time s as the buy or sell order is first given. It increases linearly as the amount traded increases, reaches its maximum value at time $s + \frac{u}{2}$ as the total order amount is reached and starts decreasing from that point on until it vanishes at time $s + u$ and brings back the price level back to the original, locally.

4.2 Non-vanishing Effects

In this part, we study the other type of pulses. We have I type of agents as in the previous section and analyze only one type. This time, we assume that an order given to the market results in a change in the price which does not vanish in time. A buy order increases the price to some value over a time interval and then the pulse dies. It leaves the price in a level upper than the level it found at arrival. This model is a modification of the continuous flow rate model studied in [25].

With each market order, an arrival time S, a duration U and an effect rate R of a pulse is associated as in the previous model. The orders arrive according to a Poisson process on $\mathbb{R}$ with intensity $\lambda > 0$. The lifetime of an effect is represented by the random variable $U > 0$ with distribution function $F_U(u) = P(U \leq u)$ and expected value $\nu = \mathbb{E}U < \infty$. We have $P(U < u) \sim \frac{L_U(u)u^{-\delta}}{\delta}$ as $u \to \infty$, where $1 < \delta < 2$. The arrival of an order changes the price with a constant effect rate. This rate valid during a pulse is given by a random variable R which has a symmetric distribution around 0 with $F_R(r) = P(R \leq r)$, $\mathbb{E}R = 0$ and $\mathbb{E}R^2 < \infty$. The maximum effect caused by an order is $UR = Q$ where Q denotes the quantity traded as before. The effect linearly increases from 0 to this value and it dies at that instant. This causes a right angled triangle shaped pulse with its hypotenuse is on the left side. To simplify the representation we will set $L_U = 1$. The modifications needed to be done in case of general slowly varying functions L_U are trivial [25].

Let $Z(t)$ denote the aggregated effect of orders in the time interval $[0, t]$ which gives the logarithm of the stock price at time t. Let $N(ds, du, dr)$ denote a Prm on $\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}$ with mean measure

$$\mu(ds, du, dr) = \lambda ds F_U(du) F_R(dr).$$

A pulse is active during the time interval $[s, s+u]$ and its effect increases at rate r throughout its lifetime.

Let

$$Z(t) = \int_{-\infty}^{\infty} \int_0^{\infty} \int_{-\infty}^{\infty} ((t-s)^+ \wedge u - (-s)^+ \wedge u) r N(ds, du, dr). \tag{4.15}$$

The kernel

$$K_t(s, u) = ((t-s)^+ \wedge u - (-s)^+ \wedge u) \tag{4.16}$$

can be written as $u\left[f\left(\frac{t-s}{u}\right) - f\left(\frac{-s}{u}\right)\right]$ in the notation of the previous subsection where $f\left(\frac{t-s}{u}\right) = \left(\frac{t-s}{u}\right)^+ \wedge 1$. In this case, f satisfies the Hölder continuity condition of Proposition 4.1.3 with $\alpha = 1$ in $[0, 1)$ but it is not compactly supported, its support is $[0, \infty)$. This causes the mean to increase and not to stay at zero. Here, $f(0) = 0$

also holds but the condition $f(1) = 0$ is not satisfied. The kernel $K_t(s,u)$ equals

$$0 \le K_t(s,u) = \begin{cases} 0 & \text{if } s+u \le 0 \text{ or } s \ge t \\ s+u & \text{if } s \le 0 \le s+u \le t \\ t & \text{if } s \le 0, t \le s+u \\ u & \text{if } 0 \le s, s+u \le t \\ t-s & \text{if } 0 \le s \le t \le s+u \end{cases}$$

as given in [25]. Hence, $K_t(s,u) \le t$ is a function of the starting time s and the duration u of an effect that measures the length of the time interval contained in $[0,t]$ during which the pulse is active. We can express $Z(t)$ as

$$Z(t) = \int_{-\infty}^{\infty} \int_{0}^{\infty} \int_{-\infty}^{\infty} K_t(s,u) r N(ds, du, dr) \tag{4.17}$$

The relation

$$(t-s)^+ \wedge u - (-s)^+ \wedge u = \int_{-s}^{t-s} 1_{\{0<y<u\}} dy$$

yields

$$K_t(s,u) = \int_0^t 1_{\{s<y<s+u\}} dy. \tag{4.18}$$

By applying relation (4.18) to (4.17), the accumulated effects of orders can be represented as

$$Z(t) = \int_0^t \bar{Z}(y) dy, \tag{4.19}$$

where

$$\bar{Z}(y) = \int_{-\infty}^{\infty} \int_{0}^{\infty} \int_{-\infty}^{\infty} 1_{\{s<y<s+u\}} q N(ds, du, dr). \tag{4.20}$$

The integrand $(\bar{Z}(y))$, $-\infty < y < \infty$, is a well-defined random instantaneous price change rate process and $Z(t)$ is the cumulative price process. In the following, we state a well known result about the stationarity of these processes.

Lemma 4.2.1 *The instantaneous price change rate process* $\{\bar{Z}(y), -\infty < y < \infty\}$ *is stationary and the cumulative price process* $\{Z(t), t \ge 0\}$ *has stationary increments.*

Proof: We make a change of variable with $s' = s + \tau$, and $ds' = ds + \tau$ as follows

$$\bar{Z}(y+\tau) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{0}^{\infty} 1_{\{s'<y+\tau<s'+u\}} q N(ds' + \tau, du, dr)$$

Note that

$$\bar{N}(ds, du, dr) = N(ds + \tau, du, dr)$$

is also a Prm with the same distribution as N since it is just a translation of N in time by τ units and N is time homogeneous. Therefore,

$$\bar{Z}(y + \tau) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{0}^{\infty} 1_{\{s' < y+\tau < s'+u\}} q \bar{N}(ds' + \tau, du, dr)$$

has the same distribution as $\bar{Z}$ itself and hence has the same distribution as $\bar{Z}$. Therefore, price change rate process is stationary, in other words, the price process $\{Z(t), t \geq 0\}$ has stationary increments. ■

We are interested in the limit process that arises when the speed of time increases in proportion to the intensity of order arrivals. This limit process corresponds to more frequent trading. To balance the increasing trading intensity λ_n, time is speeded up by a factor n and the size is normalized by a factor $\lambda_n^{1/2} n^{(3-\delta)/2}$. The scaled price process has the form

$$\begin{aligned} Z_n(t) &= \frac{Z(nt)}{\lambda_n^{1/2} n^{(3-\delta)/2}} \\ &= \frac{1}{\lambda_n^{1/2} n^{(3-\delta)/2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{0}^{\infty} K_{nt}(s, u) N(ds, du, dr) \end{aligned} \tag{4.21}$$

By using the scaling property

$$K_{nt}(ns, nr) = n K_t(s, r) \tag{4.22}$$

we can write the scaled process in the form

$$\frac{Z(nt)}{\lambda_n^{1/2} n^{(3-\delta)/2}} = (n^{\delta-1}/\lambda_n)^{1/2} \int_{-\infty}^{\infty} \int_{0}^{\infty} \int_{-\infty}^{\infty} K_t(s, u) q N(nds, adu, dr) \tag{4.23}$$

When the trading intensity λ_n increases to infinity, time is speeded up by a factor n, which tends to infinity, and the size is divided by $\lambda_n^{1/2} n^{(3-\delta)/2}$ which also tends to infinity, it is possible to obtain different limiting processes of (4.21) as proven in [25]. The fact that changes these limiting processes is the relative speed at which λ_n and n increase. It is the asymptotic behavior of the ratio $\lambda_n / n^{\delta-1}$ which determines the limit process. For a choice of sequences λ_n and n, the random variable $\sharp(\lambda_n, n)$ denotes the number of effects still active at time n. It measures the amount of very long pulses that are alive and how much they contribute to the total price. The expected value of the random variable $\sharp(\lambda_n, n)$ is

$$\mathbb{E}\ \sharp(\lambda_n, n) \sim \frac{\lambda_n}{n^{\delta-1}}. \tag{4.24}$$

The limit is considered in the cases where this value tends to a finite, positive constant, to infinity, or to zero as λ_n and n go to infinity. We will study the case of fast connection rate in which $\lambda_n/n^{\delta-1} \to \infty$. The cases when the limit is zero or a finite nonzero number in the processes similar to ours are studied in [25]. In our case, a large number of very long pulses contribute in the asymptotic limit of aggregating the price. We will obtain that the limit is fBm which is stated below.

We will find that the scaling done in this section is the same with the scaling done in Section 4.1. This will let us conclude the result by the same tools used in that section.

Theorem 4.2.2 *Let* $1 < \delta < 2$, *and assume*

$$\lambda_n \to \infty \quad \textit{and} \quad \frac{\lambda_n}{n^{\delta-1}} \to \infty. \tag{4.25}$$

as $n \to \infty$. *Then, the process*

$$\begin{aligned} \frac{Z(nt)}{\lambda_n^{1/2} n^{(3-\delta)/2}} &= \frac{1}{\lambda_n^{1/2} n^{(3-\delta)/2}} \int_0^{nt} \bar{Z}(y) dy \\ &= \frac{1}{\lambda_n^{1/2} n^{(3-\delta)/2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_0^{\infty} K_{nt}(s,u) N(ds, du, dr) \end{aligned}$$

tends to a fBm

$$\mathbb{E}(R^2)^{1/2} \sigma B_H(t)$$

with index

$$H = (3-\delta)/2 \in (1/2, 1),$$

where

$$\sigma^2 = \int_0^{\infty} \int_{-\infty}^{\infty} K_1(s,u)^2 ds u^{-(\delta+1)} du = \frac{1}{(2-\delta)(3-\delta)}. \tag{4.26}$$

as $n \to \infty$.

Proof: It is shown in [25] that $\mathbb{E}\, Z_n(t) = \lambda\, \nu\, t\, \mathbb{E} R$. So, $\mathbb{E}\, Z_n(t) = 0$ since $\mathbb{E}\, R = 0$. It is also known that $\lambda\, \mathbb{E} R^2 \int_0^\infty \int_{-\infty}^\infty K_t^2(s,u) ds u^{-\delta-1} du < \infty$ [25]. Recall that the kernel can be written as $u\left[f\left(\frac{t-s}{u}\right) - f\left(\frac{-s}{u}\right)\right]$ in the notation of the previous subsection where $f\left(\frac{t-s}{u}\right) = \left(\frac{t-s}{u}\right)^+ \wedge 1$. Since $\mathbb{E} Z_n(t) = 0$, it can be shown as in Proposition 4.1.1 that the characteristic function of $Z_n(t)$ is (4.6).

On the other hand, without loss of generality, letting $\lambda_n = \lambda\, n^{\varepsilon+\delta-1}$ with $\varepsilon > 0$, we have the mean measure

$$\begin{aligned}
\mathbb{E}\ N(nds, ndu, dr) &= \lambda_n\, (nds)\, F_U(n\, du) F_R(dr) \\
&\sim \lambda\, n^{\varepsilon+\delta-1}\, n\, n^{-\delta} ds\, u^{-\delta-1} du\, F_R(dr) \\
&= \lambda n^{\varepsilon} ds\, u^{-\delta-1} du\, F_R(dr).
\end{aligned}$$

Calling the measure $\lambda n^{\varepsilon} ds\, u^{-\delta-1} du\, F_R(dr)$ as μ_n, we have obtained a scaled measure similar to the one in 4.2.

Therefore, we can apply Theorem 4.1.2 to obtain the limit as a fBm given by $\mathbb{E} R^2 \sigma B^H(t)$ where $H = (3-\delta)/2$ and σ is given by (4.26). ∎

As it is explained in the previous subsection, both of the models we have constructed are semimartingales. According to Theorem 2.3.3, there is NFLVR if there is an equivalent martingale measure for Z. A martingale measure can be found as in [27] for our processes which show that they do not allow FLVR opportunities.

Chapter 5

CONCLUSION

In this thesis, we have studied the usage of fBm in finance. We have first analyzed two models that obtain fBm in the limit. One of them is [1] which uses semi-Markov processes to model the stock prices with the assumption that agents' demands determine the price. The other one is [27], which uses Poisson shot-noise processes, with the assumption that the important events concerning politics or business life determine price. In [1], an integral with respect to fBm is obtained, where fBm is a special case of the limit. In [27], the limit is fBm.

We have studied the arbitrage strategies constructed for fBm models. We have reviewed the strategies for fractional Bachelier model [8, 37, 40], then for fractional Black-Scholes model [8, 40], and finally for a more general model with a process of zero quadratic variation containing fBm in the special case [1]. We have analyzed the ways suggested to turn the market free of arbitrage [8, 37].acknowlegments

Since fBm is not a semimartingale [37], it is shown that it allows arbitrage [13]. Despite this fact, fBm is preferred for modeling the long range dependence structure of price processes. It can be used with some modifications [37]. Another way for getting advantage of the long range dependence of fBm is to construct models that do not allow arbitrage which may approximate fBm in the limit.

We have constructed two stock price models, in which we used ideas from [9] and [25]. In the first model pulses with vanishing effects are used whereas in the second one the effects do not vanish. The limiting processes are studied in both models. In the first model, the limit is found as the frequency of trading is increased and the quantity of the orders is decreased following [9]. The limit is found to be fBm. In the second case, the limit is found as time is speeded up, intensity of the order arrivals is increased while the size of the process itself is divided by some scaling factor [25]. The limit is again fBm. The equivalence of these two ways of scalings has been shown for the first time in this thesis. Indeed, the scaling by increasing the intensity and decreasing the quantity is more intuitive and easier than the other one.

In [25], the limit is considered in the cases where $\lambda_n/n^{\delta-1}$ tends to a finite, positive constant, to infinity, or to zero as λ_n and n tend to infinity. We have studied the case of fast connection rate in which $\lambda_n/n^{\delta-1} \to \infty$ only. In the other cases called the slow and the intermediate connection rates, the limit is a Lévy process and another stochastic processes. Moreover, our stock price models can be investigated for such limits. The Lévy process limit is interesting on its own as it is also used as a model for stock prices. Any equivalence of the scalings in the other cases to simpler scalings remains as future work.

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