DYNAMIC UPSELLING AND PRICING WITH PROMOTIONAL PRODUCTS

by

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This is to certify that I have examined this copy of a master's thesis by

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ABSTRACT

In this study, we consider a firm which offers two different products and a service which is related to these products to its customers. One of the product is called “regular product” which is assumed to have unlimited inventory, the other one is the “promotional product” which has a limited inventory which should be sold in a limited time. The firm also sells a “service” which has no inventory constraint. While regular product has only one sales-channel, promotional product and service can be sold in two different channels. In the first channel, promotional product (service) has its own demand and can be sold to customers at a fixed price which is announced at the beginning of the planning horizon. However, in the second channel, promotional product (service) is offered as an additional product to the customers who have just bought the regular product, possibly with a discount on the announced price. In our study, each regular product customer is offered a promotional product, a service or both in a bundle, at a dynamically adjusted price. Our aim is to decide which item to offer and at which price in order to maximize the total expected revenue over a finite horizon. This research will mostly focus on the stochastic and dynamic aspects of the problem using tools from Markov Decision Processes.

ÖZETÇE

Bu çalışmada müşterilerine iki farklı ürün ve bu ürünlerle ilişkili bir hizmet sunan bir firma göz önüne alınmıştır. “Düzenli ürün” olarak adlandırılan ürünün envanter kısıdının olmadığı kabul edilirken, diğer ürün “promosyon ürünü” diye adlandırdığımız ve belirli bir zaman diliminde satılması gereken sınırlı bir envantere sahip olan bir üründür. Firma ayrıca, envantere sahip olmayan bir hizmeti de müşterilerine sunar. Düzenli ürün sadece kendisine gelen talep karşılında satılırken, hizmet ve promosyon ürünü iki farklı kanaldan satılır: Birinci kanaldan, promosyon ürünü veya hizmeti doğrudan almaya gelen müşterilere daha önceden belirlenen fiyatlardan satış yapılır. Diğer kanaldan ise, promosyon ürün ve hizmet, daha önceden duyurulan fiyatları üzerinden, kalan zamana ve promosyon ürününün envanterine göre dinamik olarak hesaplanan bir indirim ile, düzenli ürünü almaya gelen müşterilere ek ürün olarak sunularak satış yapılır. Çalışmamızda her düzenli ürün müşterisine bir promosyon ürün, bir hizmet veya ikisi birden bir paket olarak dinamik olarak belirlenen bir fiyata satılabilir. Amacımız kazancımızı en iyileyecek şekilde, hangi ek ürünü hangi fiyata satacağımıza dinamik olarak karar vermektir. Bu problem Markov Karar Süreci olarak ele alınıp, zamana ve envantere göre en iyi fiyatın ve en iyi kararın yapısal özellikleri incelenmiştir.

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TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

Chapter 1

INTRODUCTION

Profit maximization is the common objective of almost all retailers. There are different ways of achieving profit maximization. One of the most useful strategies is price adjustments since price is one of the most effective variables that managers can use to increase or decrease the demand in the short run. The retailer can improve its revenues by using dynamic policies in which the prices of the products are adjusted over time, rather than adopting a fixed price throughout the selling season. With the ease of making price changes on the Internet, dynamic pricing strategies are now frequently used in many industries such as airlines, hotels, and electric utilities, where the capacity is fixed in the short-term and perishable (Elmaghraby and Keskinocak [8]). In recent years, dynamic pricing polices are adopted in retail and other industries, where the sellers have the ability to store inventory. Today, new technology allows retailers to collect demand and sales data, to adjust prices, to use decision support tools for analyzing demand data and for dynamic pricing. (Elmaghraby and Keskinocak [8]).

Another common strategy to improve profitability is to increase revenue generated per customer visit. For this purpose, retailers make promotions by selling an additional product related to the one the customer has bought or by selling two or more products (or services) together as a bundle and giving some discounts. Besides increasing revenue per customer visit, these strategies aim to develop and support long term relationship with customers. We refer to the practice of offering an additional product as upselling and creating offerings with several products as bundling herein. Motivated by these trends, we focus on a dynamic pricing problem in the context of bundling and upselling strategies.

More specifically, bundling is the practice of selling two or more different products (goods/services) in a package at a special price. There are two forms of bundling: price bundling and product bundling. Stremersch and Tellis [21] define price bundling as selling two or more separate products in a package, without any physical integration of the products. In this case, bundling itself does not create added value to customers. Therefore, a discount must be offered to the customers in order to encourage them to purchase the bundle. Price bundling is used in a wide range of industries especially in banks, hotels, airlines and restaurants. For example, some health clubs offers a package containing more than one activity with a discount or similarly some foreign language courses offer discounts to the customers who register to more than one level. Similarly, some airline companies offer a bundle which contains air travel together with car rentals and lodging (Guiltinan [11]). On the other hand, product bundling is defined as the integration and sale of two or more separate products or services at any price. A multimedia PC can be given as an example for product bundling. Many individual parts which have different functions are combined in a PC. This kind of bundling creates added value. Price bundling is easier to implement compared with product bundling. In our study we only focus on price bundling.

Upselling is offering an additional product to a customer who has just made a purchase. "Upselling can be considered as a special form of bundling, one in which the customer is given the option to purchase the bundle only after she has committed to buying a product" (Aydin and Ziya [2]). Upselling strategies are widely used by online retailers such as Amazon.com. Customers are offered generally an additional product related to the one they have bought, for example, a memory card or a carrying case can be offered with a discount to the customer who has just ordered a digital camera.

In our study we consider a firm that uses both upselling and bundling strategies. We will consider the case where the firm has two products and a service. The product definitions which are given in Aydin and Ziya [2] are used: One of the products is called as regular product which is assumed to be available whenever a demand comes, and the other one as promotional product which is offered with the regular product and also has its own demand. The promotional product has limited inventory. The firm also sells a service which has no inventory constraint and can be considered as a product with unlimited supply. For example, a mobile phone operator company can offer a cell phone and a service that can be used with this cell phone to customers who have purchased a service with the operator.

In this project, we assume as in Aydin and Ziya [2] that the firm has already chosen which product is the regular product and which one is the promotional product. A regular product customer who has just purchased a regular product is recommended an additional offering (promotional product or service). This is the upselling part of our study. We consider seven models corresponding to different upselling options, where the regular product customer is offered either one of the individual products or the bundle. These seven different scenarios constitute the bundling part of our work. Since upselling is used in all models, the models are named according to the bundling strategy used in the models.

Guiltinan [11] has classified bundling strategies into three main topics: Unbundling, pure bundling, mixed bundling. In the unbundling case, products are sold separately. Products are sold only in a bundled form in the pure bundling strategy. However, in mixed bundling, all products can be sold separately or in a bundle. There are two forms of mixed bundling strategy. One of them is called mixed-leader bundling and the other one is mixed-joint bundling. In mixed-leader bundling, one of the products is sold at its announced (regular) price and the additional product is offered with a discount. In the mixed-joint form, the customer is offered a single price for the bundle (Guiltinan [11]). As mentioned in Guiltinan [11], there are three main reasons that firms use a bundling strategy: To encourage the customers who are the target customers only for a product, to purchase the other product (upselling), encourage the new customers who do not purchase any of the two products, to purchase both products (acquisition of new customers), and encourage the customers who are in the target segment of both products, to continue to purchase the products (retention). In addition, obtaining more profit per customer visit and liquidation of excess inventories are also the main reasons for the use of a bundling strategy. According to the main objective of the firm, one of the three bundling strategies could be used.

In our problem, the regular product can be considered as the staple item which should be sold over an extended period of time, or as a fast-moving product. Therefore, we assume that the regular product is always available and focus on clearing the inventory of the perishable promotional product by a certain deadline.

The population of customers is characterized by a distribution of reservation prices, i.e, the maximum price that customers are willing to pay for the product. Customers only purchase the product if their reservation prices are greater than or equal to the price of the product.

The firm's main target is to maximize its expected revenues by selecting the appropriate dynamic controls. We characterize the optimal pricing strategy as a function of the inventory level of the promotional product and the remaining time in the planning horizon.

The remainder of the thesis is organized as follows. In Chapter 2, an overview of the previous studies about dynamic pricing and revenue management, bundling and upselling is provided. The problem description and the mathematical models for the problem are presented in Chapter 3. Basic structural results and optimal policies are stated in Chapter 4. In Chapter 5, results obtained under certain assumptions on reservation price distribution are provided. Then, in Chapter 6, the effect of different parameters, such as reservation price distribution parameters, arrival rates on the performances of the models, on the expected revenues and on the optimal policies are evaluated numerically. The limited inventory effect and bundle effect on the expected revenue and optimal polices are also analyzed in Chapter 6. Finally, Chapter 7 concludes the thesis, summarizes the main results, and gives an outline of possible extensions of the study.

Chapter 2

LITERATURE REVIEW

We discuss the relevant prior studies under three headings:

2.1 Dynamic Pricing and Revenue Management Literature

In this thesis, we will focus on the dynamic pricing of a perishable product and a service with no inventory constraints, over a finite time horizon in order to maximize the expected revenue.

There is a rich literature about pricing of perishable products in the context of revenue management. In this section we will review the most frequently cited studies related with the pricing of perishable inventories.

One of the first studies related to dynamic pricing of perishable goods is by Gallego and van Ryzin [9]. They consider the problem of dynamic pricing of a given stock of items over a finite time horizon. The demand, which is modeled as a homogeneous Poisson process with intensity $\lambda(p)$, is stochastic and price sensitive. In their model demand only depends on price and there is a negative correlation between price and the demand rate. Their aim is to maximize the expected revenues while the price is dynamically adjusted. They have shown structural properties of the model by using intensity control and demonstrated that for a family of exponential demand functions the solution can be found in closed form. By analyzing the deterministic version of the main problem they find an upper bound on the expected revenue.

Similar to Gallego and van Ryzin [9], Zhao and Zheng [25], consider dynamic pricing of a given inventory which must be sold by a certain deadline. Customers arrive according to a nonhomogeneous Poisson process. They study the dynamic pricing model in which the intensity of the customer arrival process and customer reservation price distribution may change over time. They derive a sufficient condition under which the optimal price decreases over time for a given inventory level. They also show that at a given time the optimal price decreases in the number of inventories left.

Monahan et al. [16] consider the problem of pricing a single product over multiple time periods after choosing the stocking level at the beginning of the season. They develop structural properties of the optimal pricing strategy over a finite horizon, and investigate the effects of the pricing policy on the optimal procurement policy of a newsvendor and on the optimal expected policy. In addition, an efficient algorithm for finding the optimal prices is given. Their numerical studies show the effects of the market parameters on the optimal solution.

Bitran and Mondschein [5] study dynamic pricing for seasonal products. In their model customers arrive according to a Poisson process. They first present a continuous time model in which the price is updated continuously, then use this model as a benchmark for a periodic pricing review model. They show that the price is decreasing in inventory level and increasing in time. Furthermore, demand uncertainty leads to higher prices, larger discounts and more unsold inventory.

Similarly, Chatwin [7] consider the dynamic pricing of perishable products. However, the prices are chosen from a finite set of available prices. It is shown that the maximum expected revenue function is nondecreasing and concave in the inventory level and in the remaining time. In addition, at a given time the optimal price is nonincreasing in the remaining inventory and nondecreasing in the time-to-go. These monotonicity results are shown to be satisfied when the price and corresponding demand intensities are functions of the remaining time; however, these results do not hold when the demand intensities are functions of the inventory level.

Gallego and van Ryzin [10] extended their single-product model to the multiple-product case. Similarly, Zhang and Cooper [24] and Maglaras and Meissner [14] analyze the dynamic pricing problem for multiple products in the revenue management context.

For extensive surveys of the areas of dynamic pricing and revenue management, see McGill and van Ryzin [15], and the book by Talluri and van Ryzin [22]. Moreover, the papers by Elmaghraby and Keskinocak [8] and Bitran and Caldentey [3] provide comprehensive overviews of the dynamic pricing and revenue management literatures.

2.2 Bundling Literature

There is a rich literature in marketing and economics about bundling. In this section, we will review the most significant studies about bundling and bundle pricing.

Most of the bundling papers are built on the early study of Stigler [20] in which he shows that bundling can increase the seller's profits when consumer valuations for two goods are negatively correlated.

In the economics literature, the cornerstone for many papers on product bundling was the study of Adams and Yellen [1], who consider a two-product monopoly bundling model. They assume that the consumer valuations are independent, additive and negatively correlated. Three bundling strategies (unbundling, pure bundling, mixed bundling) are compared in terms of seller profit. Using plausible cost structures and continuous distributions of reservation prices, they illustrate that pure bundling and mixed bundling can be more profitable than unbunling.

Similar to Adams and Yellen [1], Schmalensee [19] develops a two-product monopoly bundling model in which demands are independent. He keeps the reservation price additivity assumption while relaxing the assumption that the reservation prices of the individual products are negatively correlated. He compares three bundling strategies (mixed bundling, pure bundling and unbundling) under the assumption that reservation prices for two products follow a bivariate normal distribution.

Guiltinan [11] gives the basic concepts and economic principles of bundling strategies. He also divides the potential main objective of the firm into three categories (crossselling, acquisition of new customers, retention) and emphasizes the importance of determining the strategic objectives of the firms. Moreover, Guiltinan [11] indicates under which demand conditions which bundling form is more beneficial.

Different from the previous studies Hanson and Martin [12] focus on evaluating the optimal pricing of bundles. They investigate how a single firm determines optimal bundle price and provide a practical method for calculating optimal bundle prices. They use disjunctive programming in order to formulate the model as a mixed integer linear problem. An algorithm for finding optimal solutions is given along with computational results.

For a conceptual review about bundling, see Stremersch and Tellis [21]. The article provides a new synthesis of the field of bundling based on a critical review of the marketing, economics and law literatures. Stremersch and Tellis [21] clearly and consistently define bundling terms, give the difference between product bundling and price bundling and classify the bundle strategies. They also discuss the legality of bundling and give clear rules to identify the legality of each strategy. In addition, they explain under which conditions which strategy dominates the others. Prior research generally states that mixed bundling at least weakly dominates pure bundling since they assumes that pure bundling can never be optimal. However, Stremersch and Tellis [21] explain that mixed bundling is superior to pure bundling only in highly competitive environments or when consumer's reservation prices vary a fair amount. They also argue that pure bundling strategies tend to dominate mixed bundling strategies for new products.

Venkatesh and Kamakura [23] in their paper, decide which bundling strategy should be chosen for a given pair of complements or substitutes and which prices should be set for the products. They also examine the effect of complements and substitutes on the optimal bundling strategy and pricing decision. They define a parameter which represents the degree of contingency (i.e., degree of complementarity or substitutability) and observe the effect of this parameter on the optimal prices and on different strategies. They obtain the analytical results for the unbundling and pure bundling cases and these results are compared to the simulation results for the mixed bundling case since the closed form solution for mixed bundling strategy cannot be found analytically. They assume that the reservation prices are uniformly distributed.

Bulut, Gurler and Sen [6] consider the problem of selling two perishable products which have fixed initial inventories, over a finite planning horizon. They consider a retailer who provides three options to the customers: A customer either buys one of the individual products, buys the bundle or leaves without a purchase. The aim is to maximize the expected revenue while determining the optimal product and the bundle prices. They focus on the effect of different conditions such as various reservation price distributions, demand arrival rates and initial inventory on the performances of the three bundling strategies (mixed bundling, pure bundling, and unbundling). They also investigate the effect of the shape of the reservation price distribution on the optimal policies by considering both the bivariate normal and bivariate gamma densities. Their numerical results indicate that bundling is profitable when the reservation prices are positively correlated and the starting inventory levels are high. Moreover, they observe that the mixed bundling strategy dominates the other two especially when the starting inventory levels are not equal and the customer reservation prices are negatively correlated. In addition, in their numerical study, they show that when products are less substitutable and more complementary the bundling becomes more effective and the expected revenue increases.

In our study, we model different bundling strategies. We focus on the mixed bundling strategy and demonstrate the structural properties and optimal policies of the model. The prices of the individual products and the bundle is dynamically adjusted over the planning horizon. In our numerical study, we investigate the effect of complements and substitutes on the bundling strategies and pricing polices.

2.3 Upselling Literature

In this section, we continue with the survey of the upselling literature. We will present the works which are particularly related to our study. Recent work that is closely related to ours is by Netessine, Savin and Xiao [18] and Aydin and Ziya [2].

Netessine, Savin and Xiao [18] study the problem of dynamic pricing of product packages under cross selling in e-commerce settings. They focus on the problem of offering a bundle which includes the requested product and an additional product to the customer. The optimal package selection problem is modeled as a dynamic program which maximizes the profit while deciding which additional product to offer and how to price the bundle. In their study, they consider two basic inventory replenishment models: Emergency Replenishment Model, Lost-Sales Model. In Emergency Replenishment Model the company has an opportunity to procure additional product inventory at an extra cost in the case of a product stockout; however, in Lost-Sales Model it is not possible to procure an out-of-stock product when the product stocks out. They show that for Emergency Replenishment Model the problem can be decomposed in the initial inventory of all products. By using the decomposition property, they derive the structural properties of the dynamic pricing problem under any static packaging scheme for the emergency Replenishment Model. They use combinatorial optimization to find the best packaging complements. They also propose several packaging and pricing heuristics and compare the efficiency of these heuristics numerically for both models. Their numerical results show that dynamic cross selling is most beneficial for the case in which inventory is approximately equal to the expected demand.

In contrast, Aydin and Ziya [2] assume that the packaging complement has already been decided by the firm so do not consider the packaging problem. Therefore, their model could be considered as the two-product case of the problem considered by Netessine et al. [18]. Aydin and Ziya [2] study the dynamic pricing and discounting problem for a single promotional product under upselling settings. They evaluate the effects of inventory level, time, and the correlation between the reservation prices of the promotional and the regular products on the discounting decision. They also consider different scenarios based on the price-discount settings in which the price and the discount of the promotional product is dynamically adjusted or statically set. Under the always availability assumption of the regular product, when the reservation prices for promotional product and the regular product are negatively correlated, the firm always offers a discount, not considering the inventory level of the promotional product and the remaining time. However, if the reservation prices are positively correlated, the discounting decision depends on the inventory level and the remaining time. They also identify the conditions under which the customer purchase information which is the information that the customer has just bought a certain product with a certain price, is more beneficial. Their computational results indicate that in "static price-dynamic discount" case in which the price of the promotional product is set statically however, the given discount is determined dynamically in each state, the customer purchase information is more valuable.

Our model can be considered as an extension of the model presented by Aydin and Ziya [2]. Different from Aydin and Ziya [2], in our model, in addition to the promotional product, the firm can also offer a service to the customer as an option. Our model can be considered as the combination of the models which are studied by Netessine et al. [18] and Aydin and

Ziya [2]. We consider the packaging problem like Netessine et al. [18]; however, we assume like Aydin and Ziya [2] that the regular product is set in advance and our model finds only which item (promotional product or service) to add to the package. This simplification allows us to determine the structural properties of the value function.

Chapter 3

PROBLEM DESCRIPTION AND MATHEMATICAL MODEL

In this study, we consider a system in which a firm sells two products and a service to its customers. One of the products is called "Regular Product" which is assumed to have unlimited inventory. The other product, called "Promotional Product", has limited inventory to be sold over a finite time horizon. The promotional product can be considered as a seasonal or perishable product. Besides these products, the firm also sells a "service" which has no inventory. The service can be interpreted as a product with no capacity constraint. The regular product is sold only to customers who arrive to purchase the regular product directly. While regular product has only one sales-channel, the promotional product and the service can be sold in two different channels. In the first channel, the promotional product (service) has its own demand and can be sold to customers at a fixed price which is announced at the beginning of the planning horizon. However, in the second channel, promotional product (service) is offered as an additional product to the customers who have just bought the regular product, possibly with a discount on the announced price.

The main objective is to decide which product to upsell to the regular product customer if there is more than one choice and dynamically set the price of the add-on product in each period in order to maximize the revenue function over a finite time horizon. This problem is modeled as a Markov Decision Process and structural properties of optimal price and optimal decision with respect to time and the inventory level of the promotional product, are examined.

It is assumed that the planning horizon is divided into N periods, each of which is short enough such that at most one customer arrives. (This assumption is common in dynamic pricing models. See, for example, Bitran and Mondschein [5] and Aydin and Ziya [2].) The promotional product has a fixed initial inventory and there is no replenishment during the planning horizon.

In this study, no backlogging of demand is allowed, so the demands are considered as lost sales when the firm runs out of stock. We assume that the salvage value of unsold promotional products at the end of the planning horizon is zero and there is no holding cost.

In our model, n denotes the number of remaining periods in the planning horizon and I_1 is the inventory level of the promotional product. Both n and I_1 are system state parameters and all policies depend on these two parameters. We assume that there are three types of customers: Regular product customer, promotional product customer and service customer. The probability that a customer demands the regular product, the promotional product and the service in each period is denoted by λ_0, λ_1 and λ_2 respectively. In each period, no customer arrives with probability $(1 - \sum_{i=0}^{2} \lambda_i)$. All probabilities are assumed to be greater than zero. However, in some cases the promotional product and the service cannot have their own demands, i.e., $\lambda_1 = 0$ and $\lambda_2 = 0$. All results are valid in these cases.

The promotional product customers (service customers) are not offered any discount and purchase the product (service) at the fixed announced price. The announced price of the promotional product and service are represented by r_1 and r_2 respectively. In upselling, products (or services) are generally offered with a discount on the announced prices. p_1^{n,I_1} and p_2^{n,I_1} denotes the optimal discounted prices of the promotional product and of the service respectively, if there are n periods to go and inventory level of the promotional product is I_1. We will show in the following section that the price of the service only depends on the reservation price distribution of the service, independent from the remaining time and the inventory of the promotional product.

The firm has different upselling options, such as offering the promotional product or service individually or offering both in a bundle or making different combinations of individual and bundle scenarios. The optimal action is given by a^{n,I_1} which denotes the optimal choice (which choice should be offered as an add-on item to the regular product customer) when the inventory of the promotional product is I_1 and the remaining time is n. a^{n,I_1} takes values in the action set of the scenario which is denoted by A. The value of a^{n,I_1} is either 1, 2 or 12, depending on the scenario of the model, where 1, 2, and 12 denote offering the promotional product, the service

Table 3.1: Notation

n	number of remaining periods
I_1	inventory level of the promotional product
λ_0	probability of observing the regular product demand
λ_1	probability of observing the promotional product demand
λ_2	probability of observing the service demand
p_1^{n,I_1}	discounted price of the promotional product, in period n when the inventory level is I_1
p_2^{n,I_1}	discounted price of the service, in period n when the inventory level is I_1
a^{n,I_1}	optimal upsell item choice, in period n when the inventory level is I_1
r_1	announced price of the promotional product
r_2	announced price of the service
$F_1(.)$	reservation price distribution of promotional product
$F_2(.)$	reservation price distribution of service
$F_{12}(.)$	reservation price distribution of the bundle

and the bundle respectively.

The reservation price distribution for the promotional product, service and bundle are denoted by $F_1(.)$, $F_2(.)$ and $F_{12}(.)$ respectively. We assume that all reservation price distributions have a finite mean and are absolutely continuous with density $f_j(.),\ j \in \{1, 2, 12\}$.

In this study, our aim is to maximize the revenue function $V_n(I_1)$ which is the expected total revenue over a time horizon of N periods when the initial inventory level is I_1.

Under this setting, there are seven possible upselling scenarios, which can be grouped into three categories according to the number of choices (see Figure 3.1). In Figure 3.1, “S”, “P”, and “B” denote the service, the promotional product and the bundle respectively. The subsections are organized as follows: First, in Section 3.1, we consider the case when the firm has already chosen which item to upsell with the regular product. In this section, the only decision is to find the optimal price of the add-on product. Then, we model the case where the firm has two add-on options. Hence, in Section 3.2, the problem of which item to upsell is also considered. In both of these sections, we represent three scenarios. Finally, in section 3.3, the firm has three upsell options: Promotional product, service and bundle. The optimal decisions are given dynamically in each state.

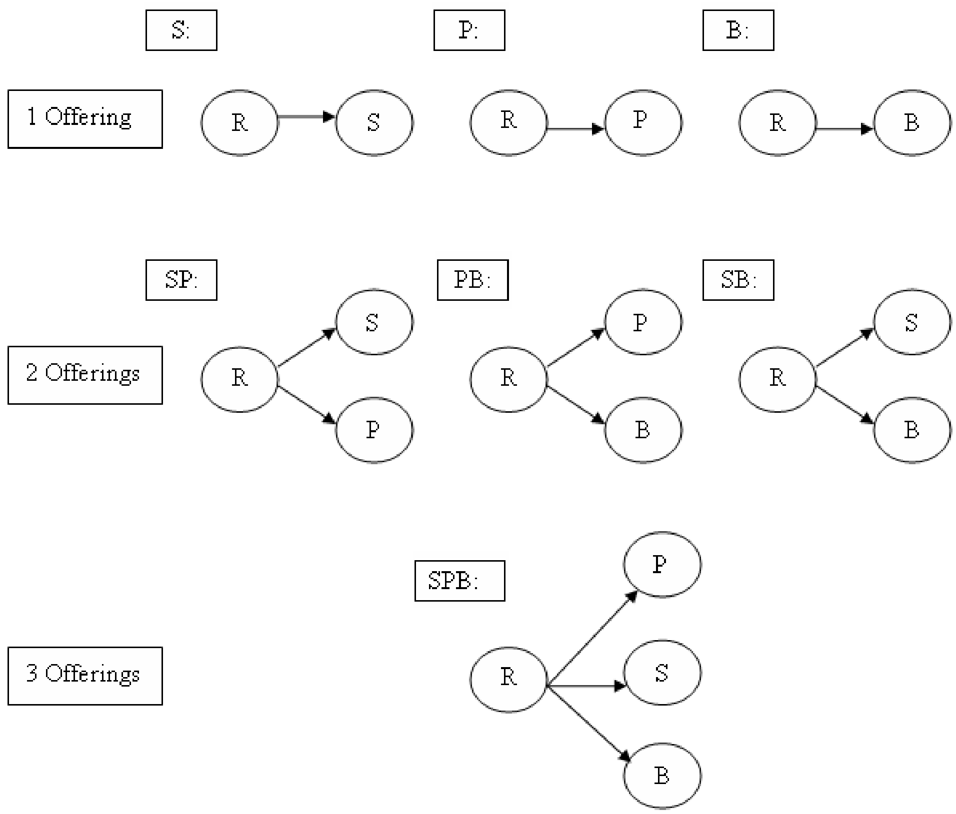

Figure 3.1: Combination of Offerings

3.1 Upselling: Single Add-On Choice

In this section, the firm has already decided which item(s) to offer to the regular product customer. Hence, the firm only focuses on the pricing decision. In the upselling part, the firm offers only a promotional product, only a service or only a package which contains both the promotional product and the service, to the regular product customers. The firm's main target is to set prices dynamically, with respect to the remaining time and the inventory level of the promotional product, for the add-on item to maximize the revenue.

3.1.1 Upselling of a Service

This section considers the model in which only the service is offered to the regular product customer for upselling. In this case, the promotional product can only be sold to the promotional product customer at its announced price. Hence, the firm needs to decide only on the price of the service. We will show that this price is constant for all n and I_1, so it is independent from the remaining time period and inventory level of the promotional product. The problem that we discuss here is a basic revenue management problem. There is no dynamic decision here since the optimal price of the service is independent from the system state parameters. There is a rich literature about this well-known revenue management problem such as Gallego and van Ryzin [9], Zhao and Zheng [25].

The revenue function can be given as:

$$\begin{aligned} V_n(I_1) = \lambda_0 \max_{p_2}\{V_{n-1}(I_1) + (1-F_2(p_2))p_2\} + \lambda_1(r_1 + V_{n-1}(I_1 - 1)) \\ +\lambda_2(r_2 + V_{n-1}(I_1)) + (1 - \sum_{i=0}^{2}\lambda_i)V_{n-1}(I_1) \qquad \forall\ I_1 \geq 1 \end{aligned} \tag{3.1}$$

with boundary conditions:

$$V_n(0) = V_{n-1}(0) + \lambda_0 \max_{p_2}\{(1-F_2(p_2))p_2\} + \lambda_2 r_2 \tag{3.2}$$

and

$$V_0(I_1) = 0 \qquad \forall\ I_1. \tag{3.3}$$

The first term of $V_n(I_1)$ is the expected revenue-to-go in the event that a regular product customer is offered a service as an add-on product: The customer purchases the service at price p_2 with probability $1 - F_2(p_2)$. The promotional product customer arrives to purchase only the promotional product with probability λ_1 and the service customer arrives to purchase only the service with probability λ_2 in each state; the second and the third terms of the revenue function denote the revenue-to-go for these cases respectively. Finally, the last term is the revenue-to-go if no customer arrives in that period. There are two terminal conditions: Equation (3.2) denotes the revenue-to-go if the firm runs out of stock. In that case the firm can offer the service to the regular product customer which does not depend on the inventory of the promotional product and also the service product customer can purchase the service at its announced price. Equation (3.3) indicates that the salvage value of the promotional product is zero.

The optimal p_2 value can be written in an explicit form. To do this, we define a function,

$$H_2(p_2) \quad = \quad V_{n-1}(I_1) + (1 - F_2(p_2))p_2$$

The optimal p_2^{n,I_1} value should maximize the function $H_2(p_2)$. Here, p_2^{n,I_1} denotes the optimal price of the service when inventory level of the promotional product is I_1 and the remaining time is n. Taking the first derivative with respect to p_2 and setting it to zero leads to:

$$1 - F_2(p_2) - p_2 F_2'(p_2) = 0 \tag{3.4}$$

Therefore, the first order necessary condition for service price to be optimal is given by:

$$p_2^{n,I_1} = \frac{1 - F_2(p_2^{n,I_1})}{F_2'(p_2^{n,I_1})}\ . \tag{3.5}$$

In Proposition 1 in Chapter 4, we will show that under a certain assumption this necessary condition is sufficient for p_2^{n,I_1} to be optimal. As seen in 3.4, the optimal price of the service does not depend on the state of the system and is fixed for all states, although it seems to be dynamically adjusted for each state. Hence, for notational simplicity, the optimal price of the service can be given as:

$$p_2^* = \frac{1 - F_2(p_2^*)}{F_2'(p_2^*)} \tag{3.6}$$

By using the optimal price definition, the revenue function can be written as the function of the optimal price. The modified value function can be given as:

$$V_n(I_1) = V_{n-1}(I_1) + \lambda_0 \frac{(1 - F_2(p_2^*))^2}{F_2'(p_2^*)} + \lambda_1(r_1 + V_{n-1}(I_1 - 1) - V_{n-1}(I_1)) + \lambda_2 r_2 \ . \tag{3.7}$$

3.1.2 Upselling of a Promotional Product with limited Inventory

In this section, the firm offers a promotional product, which has a limited inventory, to a regular product customer. The promotional product and the service are also sold to the customers who demand them directly. The aim is to determine the price of the promotional product which is offered as an add-on item in each state. As expected, the optimal price depends on the remaining time and the inventory level of the promotional product. A similar model is considered by Aydin and Ziya [2]. The revenue function is:

$$\begin{aligned} V_n(I_1) &= \lambda_0 \max_{p_1} \{F_1(p_1)V_{n-1}(I_1) + (1 - F_1(p_1))(p_1 + V_{n-1}(I_1 - 1))\} \\ &+ \lambda_1(r_1 + V_{n-1}(I_1 - 1)) + \lambda_2(r_2 + V_{n-1}(I_1)) \\ &+ (1 - \sum_{i=0}^{2} \lambda_i)V_{n-1}(I_1) \qquad \forall\ I_1 \geq 1 \end{aligned} \tag{3.8}$$

with boundary conditions:

$$V_n(0) = V_{n-1}(0) + \lambda_2 r_2 \qquad \forall n$$

and

$$V_0(I_1) = 0 \qquad \forall I_1 \ .$$

The first term of $V_n(I_1)$ is the revenue-to-go in the event that a regular product customer is offered a promotional product: The regular product customer does not purchase the offered promotional product at price p_1 with probability $F_1(p_1)$ and the inventory level of the promotional product does not change. On the other hand, the regular product customer buys the promotional product on an upsell offer with probability $1 - F_1(p_1)$, in that case both the remaining time and the inventory level decreases by one unit. The second term is the revenue-to-go if a promotional product customer arrives to purchase the promotional product and similarly the third term denotes the revenue-to-go if a service customer arrives to purchase only the service and the last term is the revenue-to-go if no customer arrives in that period. The first terminal condition denotes the revenue-to-go if there is no inventory of the promotional product in stock; in that case the firm only sells service to the service customer. The second boundary condition indicates that the salvage value of the promotional product is zero.

We can also write the optimal price of the promotional product as an explicit form. Let us define a function,

$$H_j(p_j) \quad = \quad F_j(p_j)V_{n-1}(I_1) + (1 - F_j(p_j))(p_j + V_{n-1}(I_1 - 1)) \tag{3.9}$$

Our aim is to find the optimal price p_j^{n,I_1} which maximizes the functions $H_j(p_j)$.

Taking the first derivative with respect to p_j and setting it equal to zero leads to:

$$F_j'(p_j)V_{n-1}(I_1) + 1 - F_j(p_j) - F_j'(p_j)(p_j + V_{n-1}(I_1 - 1)) = 0$$

By using the above equation the optimal price p_j^{n,I_1} can be found as:

$$p_j^{n,I_1} = V_{n-1}(I_1) - V_{n-1}(I_1 - 1) + \frac{1 - F_j(p_j^{n,I_1})}{F_j'(p_j^{n,I_1})} \ . \tag{3.10}$$

We refer to this price as the optimal price, since the first order necessary condition is also sufficient for the price to be optimal (See Proposition 1 in Chapter 4). Different from the optimal price of the service, the optimal price of the promotional product depends on the inventory level I_1 and the remaining time n. This is an expected result since the promotional product has limited inventory while the service has no inventory constraint. Substituting p_j^{n,I_1} into $H_j(p_j)$ leads to,

$$H_j(p_j^{n,I_1}) \quad = \quad V_{n-1}(I_1) + \frac{(1 - F_j(p_j^{n,I_1}))^2}{F_j'(p_j^{n,I_1})}$$

For $j = 1$, we have the optimal price as

$$p_1^{n,I_1} = V_{n-1}(I_1) - V_{n-1}(I_1 - 1) + \frac{1 - F_1(p_1^{n,I_1})}{F_1'(p_1^{n,I_1})} \ . \tag{3.11}$$

and

$$H_1(p_1^{n,I_1}) \quad = \quad V_{n-1}(I_1) + \frac{(1 - F_1(p_1^{n,I_1}))^2}{F_1'(p_1^{n,I_1})}$$

Substituting $H_1(p_1^{n,I_1})$ into the revenue function (3.8), gives a new form of the revenue function:

$$V_n(I_1) = V_{n-1}(I_1) + \lambda_0 \frac{(1 - F_1(p_1^{n,I_1}))^2}{F_1'(p_1^{n,I_1})} + \lambda_1(r_1 + V_{n-1}(I_1 - 1) - V_{n-1}(I_1)) + \lambda_2 r_2 \tag{3.12}$$

3.1.3 Upselling of a Bundle (Pure Bundling)

In this section, we consider a well-known case, pure bundling, in the bundling literature. In the pure bundling case, the promotional product and the service are offered to the regular product customer only in a package which contains both the promotional product and the service. Like in the previous cases, the promotional product and the service have their own demands. The revenue function is given by:

$$
\begin{aligned}
V_n(I_1) &= \lambda_0 \max_{p_{12}}\{F_{12}(p_{12})V_{n-1}(I_1) + (1 - F_{12}(p_{12}))(p_{12} + V_{n-1}(I_1 - 1))\} \\
&+ \lambda_1(r_1 + V_{n-1}(I_1 - 1)) + \lambda_2(r_2 + V_{n-1}(I_1)) \\
&+ (1 - \sum_{i=0}^{2} \lambda_i)V_{n-1}(I_1) \qquad \forall\ I_1 \geq 1
\end{aligned} \tag{3.13}
$$

with boundary conditions:

$$V_n(0) = V_{n-1}(0) + \lambda_2 r_2 \qquad \forall n,$$

and

$$V_0(I_1) = 0 \qquad \forall I_1\ .$$

The explanation of the revenue function terms are similar to the previous cases, the only difference is that the bundle is offered to the regular product customer at a price which maximizes the first term of the revenue function.

Similar to the optimal prices in previous sections, the optimal price of the bundle can be found explicitly by using (3.9) for $j = 12$. Then the optimal price of the bundle is given by (3.10) for $j = 12$:

$$p_{12}^{n,I_1} = V_{n-1}(I_1) - V_{n-1}(I_1 - 1) + \frac{1 - F_{12}(p_{12}^{n,I_1})}{F'_{12}(p_{12}^{n,I_1})}\ . \tag{3.14}$$

Similar to the optimal price of the promotional product, the optimal price of the bundle depends on the state parameters I_1 and n. Substituting p_{12}^{n,I_1} into (3.13), we can rewrite the revenue function in terms of the optimal price of the bundle:

$$V_n(I_1) = V_{n-1}(I_1) + \lambda_0 \frac{(1 - F_{12}(p_{12}^{n,I_1}))^2}{F'_{12}(p_{12}^{n,I_1})} + \lambda_1(r_1 + V_{n-1}(I_1 - 1) - V_{n-1}(I_1)) + \lambda_2 r_2\ . \tag{3.15}$$

3.2 Upselling: Two Add-On Choices

In this section, we model three different upselling scenarios each of which has two upselling options. In these scenarios, the firm determines which upselling option to offer to the regular product customer at which price dynamically, in order to maximize its profit. These models are pairwise combinations of three possible choices.

3.2.1 Offering a Promotional Product or a Service (Unbundling)

Unbundling is selling all products separately (not in a bundle). Whenever the demand comes for the regular product the firm offers either the promotional product or the service to the customer possibly with a discount. Our aim is to determine which item to offer (promotional product/service), at which price to the customer who just bought the regular product to maximize the revenue function. The revenue function can be written as follows:

$$
\begin{aligned}
V_n(I_1) &= \lambda_0 \max\{\max_{p_1}\{F_1(p_1)V_{n-1}(I_1) + (1-F_1(p_1))(p_1 + V_{n-1}(I_1-1))\}, \\
&\qquad \max_{p_2}\{V_{n-1}(I_1) + (1-F_2(p_2))p_2\}\} \qquad (3.16) \\
&+ \lambda_1(r_1 + V_{n-1}(I_1-1)) + \lambda_2(r_2 + V_{n-1}(I_1)) + (1 - \sum_{i=0}^{2}\lambda_i)V_{n-1}(I_1) \qquad \forall\ I_1 \geq 1
\end{aligned}
$$

with boundary conditions,

$$V_n(0) = V_{n-1}(0) + \lambda_0 \max_{p_2}\{(1-F_2(p_2))p_2\} + \lambda_2 r_2 \quad \forall n$$

and

$$V_0(I_1) = 0 \qquad \forall I_1 \ .$$

In the first term of the function, both the optimal prices for the service and the promotional product, and the optimal action for that state is chosen. When the inventory level of the promotional product is zero, the firm will offer the service to the regular product customer and will sell the service to the service customers. It is assumed that at the end of the planning horizon the unsold items have no value, therefore at time zero the value of the revenue function is zero.

In the unbundling strategy, we make an implicit assumption that the promotional product and service are either independent of or substitutes of each other because both products are offered individually to the customer who has just purchased the regular item. Offering the bundle is not an option in this strategy, implying that the products are not complements.

3.2.2 Offering a Promotional Product or a Bundle

In this case, the regular product customer is offered either the promotional product or a package that contains the promotional product and the service. The revenue function is:

$$
\begin{aligned}
V_n(I_1) &= \lambda_0 \max_{j \in \{1,12\}} \{\max_{p_j} \{F_j(p_j)V_{n-1}(I_1) + (1 - F_j(p_j))(p_j + V_{n-1}(I_1 - 1))\}\} \\
&+ \lambda_1(r_1 + V_{n-1}(I_1 - 1)) + \lambda_2(r_2 + V_{n-1}(I_1)) + (1 - \sum_{i=0}^{2} \lambda_i)V_{n-1}(I_1) \quad \forall\ I_1 \geq 1,
\end{aligned}
\tag{3.17}
$$

Boundary conditions are given as:

$$V_n(0) = V_{n-1}(0) + \lambda_2 r_2 \qquad \forall n,$$

and

$$V_0(I_1) = 0 \qquad \forall I_1\ .$$

The explanations of the terms of the revenue function are similar to that of cases given in the previous section.

3.2.3 Offering a Service or a Bundle

In this case, the firm offers to the regular product customer either the service or a bundle that includes the service and the promotional product together. The revenue function is given by,

$$\begin{aligned} V_n(I_1) &= \lambda_0 \max\{\max_{p_2}\{V_{n-1}(I_1) + (1 - F_2(p_2))p_2\}, \\ &\qquad \max_{p_{12}}\{F_{12}(p_{12})V_{n-1}(I_1) + (1 - F_{12}(p_{12}))(p_{12} + V_{n-1}(I_1 - 1))\}\} \qquad (3.18) \\ &+ \lambda_1(r_1 + V_{n-1}(I_1 - 1)) + \lambda_2(r_2 + V_{n-1}(I_1)) + (1 - \sum_{i=0}^{2}\lambda_i)V_{n-1}(I_1) \qquad \forall\ I_1 \geq 1 \end{aligned}$$

with boundary conditions:

$$V_n(0) = V_{n-1}(0) + \lambda_0 \max_{p_2}\{(1 - F_2(p_2))p_2\} + \lambda_2 r_2 \quad \forall n$$

and

$$V_0(I_1) = 0 \qquad \forall I_1\ .$$

The explanations of the terms of the revenue function are similar to that of cases given in Section 3.1.

3.3 Upselling: Three Add-On Choices (Mixed Bundling)

In this scenario, a regular product customer is offered one of the three upselling options: Only promotional product, only service or both in a bundle. This case is known as mixed bundling in the literature. There are two forms of mixed bundling strategy: Mixed leader and mixed-joint. In mixed-leader bundling, one of the products is sold at its announced price and the additional product is offered with a discount. In the mixed-joint form, the customer is offered a single price for the bundle (Guiltinan [11]). There are two different perspectives in our models. In the upselling part, similar to the mixed leader bundling, the regular product is sold at its announced price which does not be considered in our models, and the promotional product and/or service is offered with a discount. In the bundling part, similar to the mixed joint bundling, the promotional product and the service are sold at a single price which is determined dynamically.

This problem includes both packaging and pricing decisions like the problem considered by Netessine et al. [18]. In our model upselling is not offered to each customer but only to the customer who has purchased the predetermined product (regular product). On the other hand, Netessine et al. [18] consider an online retail company which sells a group of m products which are complementary and can therefore potentially be cross sold with each other. In their analysis they focus on dynamic pricing under static packaging in which the packaging complement for each product is fixed and does not change with the state of the product inventory or with time. Under the assumption of static packaging in the Emergency Replenishment Model, the m-dimensional dynamic pricing problem can be decomposed into m one-dimensional dynamic pricing problems which we consider in Section 3.1.

Like in the previous cases, choosing the best option and determining the optimum price decisions are given in each state dynamically.

The expected revenue can be written as:

$$\begin{aligned} V_n(I_1) &= \lambda_0 \max\{\max_{p_1}\{F_1(p_1)V_{n-1}(I_1) + (1-F_1(p_1))(p_1 + V_{n-1}(I_1-1))\}, \\ &\qquad \max_{p_2}\{V_{n-1}(I_1) + (1-F_2(p_2))p_2\}, \\ &\qquad \max_{p_{12}}\{F_{12}(p_{12})V_{n-1}(I_1) + (1-F_{12}(p_{12}))(p_{12} + V_{n-1}(I_1-1))\}\} \qquad (3.19) \\ &+ \lambda_1(r_1 + V_{n-1}(I_1-1)) + \lambda_2(r_2 + V_{n-1}(I_1)) + (1 - \sum_{i=0}^{2}\lambda_i)V_{n-1}(I_1) \quad \forall\, I_1 \geq 1, \end{aligned}$$

with boundary conditions:

$$V_n(0) = V_{n-1}(0) + \lambda_0 \max_{p_2}\{(1-F_2(p_2))p_2\} + \lambda_2 r_2 \quad \forall n,$$

and

$$V_0(I_1) = 0 \qquad \forall I_1 \ .$$

The first term of the revenue function is the revenue-to-go which depends on the best action and the corresponding optimal price decisions.

We can modify and rewrite the revenue function in two different forms which are more useful when proving the propositions. After some mathematical manipulations, the first form can be given as:

$$\begin{aligned} V_n(I_1) &= V_{n-1}(I_1) + \lambda_0 \max\{\max_{p_1}\{(1-F_1(p_1))(p_1 + V_{n-1}(I_1-1) - V_{n-1}(I_1))\} \\ &\qquad\qquad\qquad\quad \max_{p_2}\{(1-F_2(p_2))p_2\} \\ &\qquad\qquad\qquad\quad \max_{p_{12}}\{(1-F_{12}(p_{12}))(p_{12} + V_{n-1}(I_1-1) - V_{n-1}(I_1))\}\} \\ &+ \lambda_1(r_1 + V_{n-1}(I_1-1) - V_{n-1}(I_1)) + \lambda_2 r_2 \end{aligned} \tag{3.20}$$

The optimal prices p_1^{n,I_1}, p_2^{n,I_1} and p_{12}^{n,I_1} which are described in Section 3.1 are valid for all cases. Therefore, the revenue function as a function of the optimal prices can be given as:

$$\begin{aligned} V_n(I_1) &= V_{n-1}(I_1) + \lambda_0 \max_{j\in\{1,2,12\}}\{\frac{(1-F_j(p_j^{n,I_1}))^2}{F_j'(p_j^{n,I_1})}\} \\ &+ \lambda_1(r_1 + V_{n-1}(I_1-1) - V_{n-1}(I_1)) + \lambda_2 r_2 \ . \end{aligned} \tag{3.21}$$

Chapter 4

STRUCTURAL RESULTS

We provide all structural properties and analytical results for the mixed bundling model. Since each model is a special form of the mixed bundling model, all propositions (except for Proposition 5) and properties given below are satisfied for the remaining scenarios (for each section, a representative example of proofs will be given in Appendix A). Different from all other propositions, Proposition 5 exists only in cases in which offering service separately is an upselling option.

In the previous chapter, we presented certain necessary conditions for prices to be optimal:

$$\begin{aligned} p_j^{n,I_1} &= V_{n-1}(I_1) - V_{n-1}(I_1-1) + \frac{1-F_j(p_j^{n,I_1})}{F_j'(p_j^{n,I_1})}, \qquad j \in \{1,12\}, \\ p_2^{n,I_1} &= \frac{1-F_2(p_2^{n,I_1})}{F_2'(p_2^{n,I_1})} \end{aligned}$$

Now, we will give a condition under which the above necessary conditions for the optimal prices are also sufficient. This proposition is given by Bitran and Mondschein [4]. They study the dynamic pricing of a single perishable product over a limited time horizon. In Section 3.1.2, we consider the case in which the firm offers only a promotional product to the regular product customer in upselling part. Therefore, we consider the dynamic pricing of a promotional product which has a limited inventory that should be sold by a certain deadline. This problem is similar to the problem considered by Bitran and Mondschein [4]. Therefore, the necessary condition that we obtained is the same as that obtained by Bitran and Mondschein [4].

Proposition 1 *A necessary condition for the price p to be optimal at time n given an inventory* *level equal to I_1 corresponds to:*

$$p_1^{n,I_1} = V_{n-1}(I_1) - V_{n-1}(I_1 - 1) + \frac{1 - F_1(p_1^{n,I_1})}{F_1'(p_1^{n,I_1})}$$

If the function

$$\frac{(1 - F_1(p_1^{n,I_1}))^2}{F_1'(p_1^{n,I_1})}$$

is decreasing in p, then the first order condition has a unique solution and it corresponds to the optimal price.

Proof: See Bitran and Mondschein [4].

Similar conditions can also be given for optimal prices of the service and the bundle.

Bitran and Mondschein [4] also give a stronger condition for the price which satisfies the necessary condition to be optimal: If the hazard rate function $HR(p) = \frac{F'(p)}{(1-F(p))}$ is increasing, then the necessary condition is also sufficient for this price to be optimal. If the hazard rate function of the reservation price is an increasing function in p, then the term

$$\frac{(1 - F_j(p_j^{n,I_1}))^2}{F_j'(p_j^{n,I_1})} \quad j \in \{1, 2, 12\}$$

is decreasing in p, so that Proposition 1 holds whenever the hazard rate function of F_j, $j \in \{1, 2, 12\}$ is increasing in p.

The hazard function associated with the reservation price distribution, $HR(p) = \frac{F'(p)}{(1-F(p))}$, can also be written as

$$HR(p) = \frac{P(p < R < p + dp)}{P(R > p)}$$

where R denotes the reservation price of the customer. Then, the hazard function can be interpreted as the probability that the price p is the maximal price that the customer is willing to pay, given that she bought the product at price p.

The assumption of the increasing hazard rate can be interpreted as the probability that the price p is the maximal price that the customer is willing to pay, is increasing.

Throughout the study, we assume that the hazard rate functions associated with the reservation price distributions are increasing. This assumption will be further analyzed in Chapter 5.

4.1 Value Function Results

In this section, we will give the monotonicity properties of the revenue function. We will use the proof technique of Bitran and Mondschein [5].

Proposition 2 *The revenue function $V_n(I_1)$ is a non-decreasing function in inventory and time.*

$$\begin{aligned} V_n(I_1+1) &\geq V_n(I_1) \qquad \forall\ n,\ I_1, \\ V_n(I_1) &\geq V_{n-1}(I_1) \qquad \forall\ n,\ I_1\ . \end{aligned}$$

Proof:

i) For $n = 0$ the inequality is satisfied trivially. We assume that the inequality holds for n and we prove that it holds for $n+1$.

When the remaining time is $n+1$ and the inventory level of the promotional product is I_1+1, the revenue function is given by,

$$\begin{aligned} V_{n+1}(I_1+1) &= \lambda_0 \max\{\max_{p_1}\{F_1(p_1)V_n(I_1+1) + (1-F_1(p_1))(p_1+V_n(I_1))\}, \\ &\qquad \max_{p_2}\{V_n(I_1+1) + (1-F_2(p_2))p_2\}, \\ &\qquad \max_{p_{12}}\{F_{12}(p_{12})V_n(I_1+1) + (1-F_{12}(p_{12}))(p_{12}+V_n(I_1))\}\} \\ &+ \lambda_1(r_1+V_n(I_1)) + \lambda_2 r_2 + (1-\sum_{i=0}^{2}\lambda_i)V_n(I_1+1) \end{aligned}$$

using that the inequality is true for n, we obtain:

$$\begin{aligned}
V_{n+1}(I_1+1) \quad &\geq \quad \lambda_0 \max\{\max_{p_1}\{F_1(p_1)V_n(I_1) + (1-F_1(p_1))(p_1 + V_n(I_1-1))\}, \\
&\qquad \max_{p_2}\{V_n(I_1) + (1-F_2(p_2))p_2\}, \\
&\qquad \max_{p_{12}}\{F_{12}(p_{12})V_n(I_1) + (1-F_{12}(p_{12}))(p_{12} + V_n(I_1-1))\}\} \\
&+ \quad \lambda_1(r_1 + V_n(I_1-1)) + \lambda_2 r_2 + (1-\sum_{i=0}^{2}\lambda_i)V_n(I_1) \\
&= \quad V_{n+1}(I_1) \qquad \forall\ I_1,
\end{aligned}$$

which completes the proof.

ii) For $n = 1$ the inequality holds trivially, since $V_0(I_1) = 0$.

$$V_1(I_1) \geq V_0(I_1) \quad \forall\ I_1.$$

Assume that it holds for n, and prove it for $n+1$.

When the state parameters are $n+1$ and I_1, the revenue function is given as:

$$\begin{aligned}
V_{n+1}(I_1) \quad &= \quad \lambda_0 \max\{\max_{p_1}\{F_1(p_1)V_n(I_1) + (1-F_1(p_1))(p_1 + V_n(I_1-1))\}, \\
&\qquad \max_{p_2}\{V_n(I_1) + (1-F_2(p_2))p_2\}, \\
&\qquad \max_{p_{12}}\{F_{12}(p_{12})V_n(I_1) + (1-F_{12}(p_{12}))(p_{12} + V_n(I_1-1))\}\} \\
&+ \quad \lambda_1(r_1 + V_n(I_1-1)) + \lambda_2 r_2 + (1-\sum_{i=0}^{2}\lambda_i)V_n(I_1)
\end{aligned}$$

using that the proposition is true for n, we obtain:

$$\begin{aligned}
V_{n+1}(I_1) \quad &\geq \quad \lambda_0 \max\{\max_{p_1}\{F_1(p_1)V_{n-1}(I_1) + (1 - F_1(p_1))(p_1 + V_{n-1}(I_1 - 1))\}, \\
&\qquad \max_{p_2}\{V_{n-1}(I_1) + (1 - F_2(p_2))p_2\}, \\
&\qquad \max_{p_{12}}\{F_{12}(p_{12})V_{n-1}(I_1) + (1 - F_{12}(p_{12}))(p_{12} + V_{n-1}(I_1 - 1))\}\} \\
&+ \quad \lambda_1(r_1 + V_{n-1}(I_1 - 1)) + \lambda_2 r_2 + (1 - \sum_{i=0}^{2} \lambda_i)V_{n-1}(I_1) \\
&= \quad V_n(I_1),
\end{aligned}$$

which completes the proof.

These monotonicity results imply that for a fixed time period if there is one more unit of inventory revenue increases and for a fixed inventory level, opportunity cost of holding I_1 units of inventory for one more period is positive, i.e., $V_{n+1}(I_1) - V_n(I_1) \geq 0$

4.2 Optimal Pricing Policy

In this section, we will give our results of the optimal pricing policy and on the concavity of the revenue function.

Proposition 3

i*) The marginal value of one unit of promotional product is nondecreasing in n:*

$$V_n(I_1 + 1) - V_n(I_1) \leq V_{n+1}(I_1 + 1) - V_{n+1}(I_1) \quad \forall\ n,\ I_1\ .$$

ii*)The value function $V_n(I_1)$ is a concave function of n:*

$$V_{n+1}(I_1) - V_n(I_1) \geq V_{n+2}(I_1) - V_{n+1}(I_1) \qquad \forall\ n,\ I_1\ .$$

iii) *The function $V_n(I_1)$ is a concave function of the inventory:*

$$V_n(I_1) - V_n(I_1 + 1) \leq V_n(I_1 + 1) - V_n(I_1 + 2) \qquad \forall\ n,\ I_1\ .$$

The marginal value of one unit of additional promotional product is decreasing in the remaining number of periods, n, since the probability of selling the additional unit before the end of the planning horizon is decreasing in n. This intuition is proven in Proposition 3-*i*. Monotonicity property (proven in the Proposition (2)) ensures that the opportunity cost of holding I_1 units of inventory for one more period is positive. Proposition 3-*ii* implies that the opportunity costs are also non-increasing in the remaining time. Proposition 3-*iii* shows that at a fixed time, at high inventory levels of the promotional product, the marginal value of one unit of additional promotional product decreases, since in a limited time, the probability of selling the additional unit is decreasing in the inventory level of the promotional product.

Proof: We will prove the following three inequalities which correspond to the above claims, simultaneously:

$$V_{n+1}(I_1) - V_n(I_1) \leq V_{n+1}(I_1 + 1) - V_n(I_1 + 1) \quad \forall\ n, I_1, \tag{4.1}$$

$$V_{n+1}(I_1) - V_n(I_1) \geq V_{n+2}(I_1) - V_{n+1}(I_1) \qquad \forall\ n, I_1, \tag{4.2}$$

$$V_n(I_1) - V_n(I_1 + 1) \leq V_n(I_1 + 1) - V_n(I_1 + 2) \qquad \forall\ n, I_1, \tag{4.3}$$

Following Bitran and Mondschein [4], we will use an inductive argument on $k = n + I_1$. For $k = 0$ the inequalities are satisfied trivially since $n = 0$ and $I_1 = 0$. We assume that the inequalities are satisfied for $n + I_1 < k$ and we will prove that they hold for $n + I_1 = k$.

i) First, we will show that inequality (4.1) holds for $n + I_1 = k$.

$$V_{n+1}(I_1) - V_n(I_1) \leq V_{n+1}(I_1 + 1) - V_n(I_1 + 1) \quad \forall\ n,\ I_1$$

Let A be the optimal action and p_A be the optimal price for $V_{n+1}(I_1)$: $a^{n+1,I_1} = A$, $A \in \{1, 2, 12\}$, $p_A = p_A^{n+1,I_1}$.

Case 1: $A = j$, $j \in \{1, 12\}$,

Since the optimal action for $V_{n+1}(I_1)$ is not 2; as shown in Chapter 3.3, the revenue function can be written as:

$$\begin{aligned} V_{n+1}(I_1) - V_n(I_1) &= \lambda_0 \max_{p_j}\{(1 - F_j(p_j))(p_j + V_n(I_1 - 1) - V_n(I_1))\} \\ &+ \lambda_1(r_1 + V_n(I_1 - 1) - V_n(I_1)) + \lambda_2 r_2 \ . \end{aligned} \tag{4.4}$$

Since p_A is optimal for $V_{n+1}(I_1)$, equality (4.4) can be written as:

$$\begin{aligned} V_{n+1}(I_1) - V_n(I_1) &= \lambda_0(1 - F_j(p_A))(p_A + V_n(I_1 - 1) - V_n(I_1)) \\ &+ \lambda_1(r_1 + V_n(I_1 - 1) - V_n(I_1)) + \lambda_2 r_2 \ . \end{aligned} \tag{4.5}$$

Similarly for given j, the value function $V_{n+1}(I_1 + 1)$ can be written as:

$$\begin{aligned} V_{n+1}(I_1 + 1) - V_n(I_1 + 1) &\geq \lambda_0 \max_{p_j}\{(1 - F_j(p_j))(p_j + V_n(I_1) - V_n(I_1 + 1))\} \\ &+ \lambda_1(r_1 + V_n(I_1) - V_n(I_1 + 1)) + \lambda_2 r_2 \end{aligned}$$

If p_A is used for $V_{n+1}(I_1 + 1)$ instead of its optimal action, it can be written as:

$$\begin{aligned} V_{n+1}(I_1 + 1) - V_n(I_1 + 1) &\geq \lambda_0((1 - F_j(p_A))(p_A + V_n(I_1) - V_n(I_1 + 1))) \\ &+ \lambda_1(r_1 + V_n(I_1) - V_n(I_1 + 1)) + \lambda_2 r_2 \end{aligned} \tag{4.6}$$

We define:

$$\Delta = V_{n+1}(I_1) - V_n(I_1) - V_{n+1}(I_1 + 1) + V_n(I_1 + 1).$$

By using (4.5) and (4.6) we have:

$$\begin{aligned}
\Delta &\leq \lambda_0(1-F_j(p_A))(p_A+V_n(I_1-1)-V_n(I_1))+\lambda_1(r_1+V_n(I_1-1)-V_n(I_1))+\lambda_2 r_2 \\
&- \lambda_0(1-F_j(p_A))(p_A+V_n(I_1)-V_n(I_1+1))-\lambda_1(r_1+V_n(I_1)-V_n(I_1+1))-\lambda_2 r_2 \\
&= \lambda_0(1-F_j(p_A))(V_n(I_1-1)-V_n(I_1)-V_n(I_1)+V_n(I_1+1)) \\
&+ \lambda_1(V_n(I_1-1)-V_n(I_1)-V_n(I_1)+V_n(I_1+1)). \qquad (4.7)
\end{aligned}$$

We know that the inequality (4.3) holds for $I_1 - 1$, so that,

$$V_n(I_1)-V_n(I_1+1) \geq V_n(I_1-1)-V_n(I_1)\ . \qquad (4.8)$$

Using (4.8) in (4.7) leads to:

$$V_{n+1}(I_1)-V_n(I_1)-V_{n+1}(I_1+1)+V_n(I_1+1) \quad \leq \quad 0\ .$$

Case 2: $A = 2$,

Since the optimal action for $V_{n+1}(I_1)$ is 2, it can be written as:

$$V_{n+1}(I_1)-V_n(I_1)=\lambda_0 \max_{p_2}\{(1-F_2(p_2))p_2\}+\lambda_1(r_1+V_n(I_1-1)-V_n(I_1))+\lambda_2 r_2\ . \qquad (4.9)$$

The following inequality holds for $V_{n+1}(I_1+1)$:

$$\begin{aligned}
V_{n+1}(I_1+1)-V_n(I_1+1) &\geq \lambda_0 \max_{p_2}\{(1-F_2(p_2))p_2\} \\
&+ \lambda_1(r_1+V_n(I_1)-V_n(I_1+1))+\lambda_2 r_2\ . \qquad (4.10)
\end{aligned}$$

By using (4.9) and (4.10) we obtain:

$$\begin{aligned} V_{n+1}(I_1) - V_n(I_1) - V_{n+1}(I_1+1) + V_n(I_1+1) \quad &\leq \quad \lambda_0 \max_{p_2}\{(1-F_2(p_2))p_2\} \\ &+ \quad \lambda_1(r_1 + V_n(I_1-1) - V_n(I_1)) + \lambda_2 r_2 \\ &- \quad \lambda_0 \max_{p_2}\{(1-F_2(p_2))p_2\} \\ &- \quad \lambda_1(r_1 + V_n(I_1) - V_n(I_1+1)) - \lambda_2 r_2 \end{aligned} \tag{4.11}$$

Inequality (4.3) holds for $I_1 - 1$, so we have the following inequality:

$$V_n(I_1) - V_n(I_1+1) \geq V_n(I_1-1) - V_n(I_1), \tag{4.12}$$

which proves the statement along with (4.11):

$$V_{n+1}(I_1) - V_n(I_1) - V_{n+1}(I_1+1) + V_n(I_1+1) \quad \leq \quad 0\ .$$

ii) Now, we will show that inequality (4.2) holds for $n + I_1 = k$

$$V_{n+1}(I_1) - V_n(I_1) \geq V_{n+2}(I_1) - V_{n+1}(I_1) \qquad \forall\ I_1$$

Let B be the optimal action and p_B be the optimal price for $V_{n+2}(I_1)$: $a^{n+2,I_1} = B$, $B \in \{1,2,12\}$, $p_B = p_B^{n+2,I_1}$.

Case 1: $B = j$, $j \in \{1, 12\}$

Since the optimal action for $V_{n+2}(I_1)$ is not 2, it can be written as:

$$\begin{aligned} V_{n+2}(I_1) - V_{n+1}(I_1) \quad &= \quad \lambda_0 \max_{p_j}\{(1-F_j(p_j))(p_j + V_{n+1}(I_1-1) - V_{n+1}(I_1))\} \\ &+ \quad \lambda_1(r_1 + V_{n+1}(I_1-1) - V_{n+1}(I_1)) + \lambda_2 r_2 \end{aligned} \tag{4.13}$$

Since p_B is optimal for $V_{n+2}(I_1)$, equality (4.13) can be written as:

$$\begin{aligned} V_{n+2}(I_1) - V_{n+1}(I_1) &= \lambda_0(1 - F_j(p_B))(p_B + V_{n+1}(I_1 - 1) - V_{n+1}(I_1)) \\ &+ \lambda_1(r_1 + V_{n+1}(I_1 - 1) - V_{n+1}(I_1)) + \lambda_2 r_2 \end{aligned} \tag{4.14}$$

Similarly for given j, the value function $V_{n+1}(I_1)$ can be written as:

$$\begin{aligned} V_{n+1}(I_1) - V_n(I_1) &\geq \lambda_0 \max_{p_j}\{(1 - F_j(p_j))(p_j + V_n(I_1 - 1) - V_n(I_1))\} \\ &+ \lambda_1(r_1 + V_n(I_1 - 1) - V_n(I_1)) + \lambda_2 r_2 \ . \end{aligned}$$

If p_B is used for $V_{n+1}(I_1)$ instead of its optimal action, it can be written as:

$$\begin{aligned} V_{n+1}(I_1) - V_n(I_1) &\geq \lambda_0(1 - F_j(p_B))(p_B + V_n(I_1 - 1) - V_n(I_1)) \\ &+ \lambda_1(r_1 + V_n(I_1 - 1) - V_n(I_1)) + \lambda_2 r_2 \ . \end{aligned} \tag{4.15}$$

We define:

$$\Delta = V_{n+2}(I_1) - V_{n+1}(I_1) - V_{n+1}(I_1) + V_n(I_1)$$

By using (4.14) and (4.15) we obtain:

$$\begin{aligned} \Delta &\leq \lambda_0(1 - F_j(p_B))(p_B + V_{n+1}(I_1 - 1) - V_{n+1}(I_1)) \\ &+ \lambda_1(r_1 + V_{n+1}(I_1 - 1) - V_{n+1}(I_1)) + \lambda_2 r_2 \\ &- \lambda_0(1 - F_j(p_B))(p_B + V_n(I_1 - 1) - V_n(I_1)) \\ &- \lambda_1(r_1 + V_n(I_1 - 1) - V_n(I_1)) - \lambda_2 r_2 \\ &= \lambda_0(1 - F_j(p_B))(V_{n+1}(I_1 - 1) - V_{n+1}(I_1) - V_n(I_1 - 1) + V_n(I_1)) \\ &+ \lambda_1(V_{n+1}(I_1 - 1) - V_{n+1}(I_1) - V_n(I_1 - 1) + V_n(I_1)) \end{aligned} \tag{4.16}$$

Using inequality (4.1) for $I_1 - 1$ leads to:

$$V_n(I_1 - 1) - V_n(I_1) \geq V_{n+1}(I_1 - 1) - V_{n+1}(I_1) \ . \tag{4.17}$$

Using (4.17) in (4.16) leads to:

$$V_{n+2}(I_1) - V_{n+1}(I_1) - V_{n+1}(I_1) + V_n(I_1) \leq 0 \ .$$

Case 2: $B = 2$,

Since the optimal action for $V_{n+1}(I_1)$ is 2, it can be written as:

$$\begin{aligned} V_{n+2}(I_1) - V_{n+1}(I_1) &= \lambda_0 \max_{p_2}\{(1 - F_2(p_2))p_2\} \\ &+ \lambda_1(r_1 + V_{n+1}(I_1 - 1) - V_{n+1}(I_1)) + \lambda_2 r_2 \end{aligned} \tag{4.18}$$

The following inequality holds for $V_{n+1}(I_1)$:

$$\begin{aligned} V_{n+1}(I_1) - V_n(I_1) &\geq \lambda_0 \max_{p_2}\{(1 - F_2(p_2))p_2 \\ &+ \lambda_1(r_1 + V_n(I_1 - 1) - V_n(I_1)) + \lambda_2 r_2 \end{aligned} \tag{4.19}$$

By using (4.18) and (4.19) we have,

$$\begin{aligned} V_{n+2}(I_1) - V_{n+1}(I_1) - V_{n+1}(I_1) + V_n(I_1) &\leq \lambda_0 \max_{p_2}\{(1 - F_2(p_2))p_2 \\ &+ \lambda_1(r_1 + V_{n+1}(I_1 - 1) - V_{n+1}(I_1)) + \lambda_2 r_2 \\ &- \lambda_0 \max_{p_k}\{(1 - F_2(p_2))p_2 \\ &- \lambda_1(r_1 + V_n(I_1 - 1) - V_n(I_1)) - \lambda_2 r_2 \end{aligned} \tag{4.20}$$

Inequality (4.1) holds for $I_1 - 1$. Therefore:

$$V_n(I_1 - 1) - V_n(I_1) \geq V_{n+1}(I_1 - 1) - V_{n+1}(I_1) \ . \tag{4.21}$$

Inequalities (4.20) and (4.21) lead to:

$$V_{n+2}(I_1) - V_{n+1}(I_1) - V_{n+1}(I_1) + V_n(I_1) \leq 0$$

iii) Finally, we will show that inequality (4.3) holds for $n + I_1 = k$

$$V_n(I_1 + 2) - V_n(I_1 + 1) \leq V_n(I_1 + 1) - V_n(I_1) \qquad \forall\ n,\ I_1$$

Let C be the optimal action and p_C be optimal price for $V_n(I_1+2)$: $a^{n,I_1+2} = C$, $C \in \{1, 2, 12\}$, $p_C = p_C^{n,I_1+2}$.

Case 1: $C = j$, $j = \{1, 12\}$,

Since the optimal action for $V_n(I_1 + 2)$ is not 2, it can be written as:

$$\begin{aligned} V_n(I_1+2) &= V_{n-1}(I_1+2) + \lambda_0 \max_{p_j}\{(1 - F_j(p_j))(p_j + V_{n-1}(I_1+1) - V_{n-1}(I_1+2))\} \\ &+ \lambda_1(r_1 + V_{n-1}(I_1+1) - V_{n-1}(I_1+2)) + \lambda_2 r_2 \end{aligned}$$

Subtracting $V_{n-1}(I_1 + 1)$ from both sides yields,

$$\begin{aligned} V_n(I_1+2) - V_{n-1}(I_1+1) &= V_{n-1}(I_1+2) - V_{n-1}(I_1+1) \\ &+ \lambda_0 \max_{p_j}\{(1 - F_j(p_j))(p_j + V_{n-1}(I_1+1) - V_{n-1}(I_1+2))\} \\ &+ \lambda_1(r_1 + V_{n-1}(I_1+1) - V_{n-1}(I_1+2)) + \lambda_2 r_2 \end{aligned} \tag{4.22}$$

Since p_C is optimal for $V_n(I_1+2)$, equality (4.22) can be written as:

$$\begin{aligned} V_n(I_1+2) - V_{n-1}(I_1+1) &= V_{n-1}(I_1+2) - V_{n-1}(I_1+1) \\ &+ \lambda_0(1-F_j(p_C))(p_C + V_{n-1}(I_1+1) - V_{n-1}(I_1+2)) \\ &+ \lambda_1(r_1 + V_{n-1}(I_1+1) - V_{n-1}(I_1+2)) + \lambda_2 r_2 \end{aligned}$$

The above equality can be written as:

$$\begin{aligned} V_n(I_1+2) - V_{n-1}(I_1+1) &= (1-\lambda_0(1-F_j(p_C)) - \lambda_1)(V_{n-1}(I_1+2) - V_{n-1}(I_1+1)) \\ &+ \lambda_0(1-F_j(p_C))p_C + \lambda_1 r_1 + \lambda_2 r_2 \end{aligned} \tag{4.23}$$

Similarly for given j, the value function $V_{n+1}(I_1+1)$ can be written as:

$$\begin{aligned} V_{n+1}(I_1+1) &\geq V_n(I_1+1) + \lambda_0 \max_{p_j}\{(1-F_j(p_j))(p_j + V_n(I_1) - V_n(I_1+1))\} \\ &+ \lambda_1(r_1 + V_n(I_1) - V_n(I_1+1)) + \lambda_2 r_2\ . \end{aligned}$$

Subtracting $V_n(I_1)$ from both sides yields,

$$\begin{aligned} V_{n+1}(I_1+1) - V_n(I_1) &\geq V_n(I_1+1) - V_n(I_1) \\ &+ \lambda_0 \max_{p_j}\{(1-F_j(p_j))(p_j + V_n(I_1) - V_n(I_1+1))\} \\ &+ \lambda_1(r_1 + V_n(I_1) - V_n(I_1+1)) + \lambda_2 r_2\ . \end{aligned}$$

If p_C is used for $V_{n+1}(I_1+1)$ instead of its optimal action,it can be written as:

$$\begin{aligned} V_{n+1}(I_1+1) - V_n(I_1) &\geq V_n(I_1+1) - V_n(I_1) \\ &+ \lambda_0(1-F_j(p_C))(p_C + V_n(I_1) - V_n(I_1+1)) \\ &+ \lambda_1(r_1 + V_n(I_1) - V_n(I_1+1)) + \lambda_2 r_2\ . \end{aligned}$$

The above inequality can be written as:

$$\begin{aligned} V_{n+1}(I_1+1) - V_n(I_1) &\geq (1-\lambda_0(1-F_j(p_C))-\lambda_1)(V_n(I_1+1)-V_n(I_1)) \\ &+ \lambda_0(1-F_j(p_C))p_C + \lambda_1 r_1 + \lambda_2 r_2 \ . \end{aligned} \tag{4.24}$$

By using (4.1) and (4.3) for $(n-1)$ we have:

$$V_n(I_1+1) - V_n(I_1) \geq V_{n-1}(I_1+1) - V_{n-1}(I_1) \tag{4.25}$$

$$V_{n-1}(I_1+1) - V_{n-1}(I_1) \geq V_{n-1}(I_1+2) - V_{n-1}(I_1+1) \tag{4.26}$$

(4.25) and (4.26) together lead to:

$$V_n(I_1+1) - V_n(I_1) \geq V_{n-1}(I_1+2) - V_{n-1}(I_1+1) \tag{4.27}$$

By using inequality (4.27) in (4.24), we have:

$$\begin{aligned} V_{n+1}(I_1+1) - V_n(I_1) &\geq (1-\lambda_0(1-F_j(p_C))-\lambda_1)(V_{n-1}(I_1+2)-V_{n-1}(I_1+1)) \\ &+ \lambda_0(1-F_j(p_C))p_C + \lambda_1 r_1 + \lambda_2 r_2 \ . \end{aligned} \tag{4.28}$$

But the right hand side of this inequality is equal to $V_n(I_1+2) - V_{n-1}(I_1+1)$ by equality (4.23), so that it can be written as:

$$V_{n+1}(I_1+1) - V_n(I_1) \geq V_n(I_1+2) - V_{n-1}(I_1+1) \tag{4.29}$$

Additionally, (4.2) is satisfied for $(I_1+1, n-1)$, so we have:

$$V_n(I_1+1) - V_{n-1}(I_1+1) \geq V_{n+1}(I_1+1) - V_n(I_1+1)$$

or equivalently,

$$2V_n(I_1+1) \geq V_{n-1}(I_1+1) + V_{n+1}(I_1+1) \tag{4.30}$$

(4.29) and (4.30) together lead to the desired inequality:

$$V_n(I_1+1) - V_n(I_1) \geq V_n(I_1+2) - V_n(I_1+1)$$

Case 2: $C = 2$,

Since the optimal action for $V_n(I_1+2)$ is 2, it can be written as:

$$\begin{aligned} V_n(I_1+2) &= V_{n-1}(I_1+2) + \lambda_0 \max_{p_2}\{(1-F_2(p_2))p_2\} \\ &+ \lambda_1(r_1 + V_{n-1}(I_1+1) - V_{n-1}(I_1+2)) + \lambda_2 r_2 \end{aligned}$$

Subtracting $V_{n-1}(I_1+1)$ from both sides yields,

$$\begin{aligned} V_n(I_1+2) - V_{n-1}(I_1+1) &= V_{n-1}(I_1+2) - V_{n-1}(I_1+1) + \lambda_0 \max_{p_2}\{(1-F_2(p_2))p_2\} \\ &+ \lambda_1(r_1 + V_{n-1}(I_1+1) - V_{n-1}(I_1+2)) + \lambda_2 r_2 \end{aligned} \tag{4.31}$$

Since p_C is optimal for $V_n(I_1+2)$, this equality can be written as:

$$\begin{aligned} V_n(I_1+2) - V_{n-1}(I_1+1) &= V_{n-1}(I_1+2) - V_{n-1}(I_1+1) \\ &+ \lambda_0(1-F_2(p_C))p_C \\ &+ \lambda_1(r_1 + V_{n-1}(I_1+1) - V_{n-1}(I_1+2)) + \lambda_2 r_2 \\ &= (1-\lambda_1)(V_{n-1}(I_1+2) - V_{n-1}(I_1+1)) \\ &+ \lambda_0(1-F_2(p_C))p_C + \lambda_1 r_1 + \lambda_2 r_2 \ . \end{aligned} \tag{4.32}$$

Similarly, the following inequality holds for $V_{n+1}(I_1+1)$:

$$\begin{aligned} V_{n+1}(I_1+1) &\geq V_n(I_1+1) + \lambda_0 \max_{p_2}\{(1-F_2(p_2))p_2\} + \lambda_1(r_1 + V_n(I_1) - V_n(I_1+1)) \\ &+ \lambda_2 r_2 \ . \end{aligned}$$

Subtracting $V_n(I_1)$ from both sides yields,

$$\begin{aligned} V_{n+1}(I_1+1) - V_n(I_1) &\geq V_n(I_1+1) - V_n(I_1) + \lambda_0 \max_{p_2}\{(1-F_2(p_2))p_2\} \\ &+ \lambda_1(r_1 + V_n(I_1) - V_n(I_1+1)) + \lambda_2 r_2 \end{aligned} \tag{4.33}$$

If p_C is used for $V_{n+1}(I_1+1)$ instead of its optimal action, it can be written as:

$$\begin{aligned} V_{n+1}(I_1+1) - V_n(I_1) &\geq V_n(I_1+1) - V_n(I_1) \\ &+ \lambda_0(1-F_2(p_C))p_C \\ &+ \lambda_1(r_1 + V_n(I_1) - V_n(I_1+1)) + \lambda_2 r_2 \ . \end{aligned}$$

The above inequality can be written as:

$$\begin{aligned} V_{n+1}(I_1+1) - V_n(I_1) &\geq (1-\lambda_1)(V_n(I_1+1) - V_n(I_1)) \\ &+ \lambda_0(1-F_2(p_C))p_C + \lambda_1 r_1 + \lambda_2 r_2 \ . \end{aligned} \tag{4.34}$$

By using (4.1) and (4.3) for $n-1$ we have:

$$V_n(I_1+1) - V_n(I_1) \geq V_{n-1}(I_1+1) - V_{n-1}(I_1) \tag{4.35}$$

$$V_{n-1}(I_1+1) - V_{n-1}(I_1) \geq V_{n-1}(I_1+2) - V_{n-1}(I_1+1) \tag{4.36}$$

(4.35) and (4.36) together lead to:

$$V_n(I_1+1) - V_n(I_1) \geq V_{n-1}(I_1+2) - V_{n-1}(I_1+1) \tag{4.37}$$

By using inequality (4.37) in (4.34), we have:

$$\begin{aligned} V_{n+1}(I_1+1) - V_n(I_1) \quad &\geq \quad (1-\lambda_1)(V_{n-1}(I_1+2) - V_{n-1}(I_1+1)) \\ &+ \quad \lambda_0(1-F_2(p_C))p_C + \lambda_1 r_1 + \lambda_2 r_2 \ . \end{aligned} \tag{4.38}$$

The right hand side of the inequality is equal to $V_n(I_1+2) - V_{n-1}(I_1+1)$ by (4.32), so it can be written as:

$$V_{n+1}(I_1+1) - V_n(I_1) \geq V_n(I_1+2) - V_{n-1}(I_1+1) \tag{4.39}$$

Additionally, (4.2) is satisfied for $(I_1+1, n-1)$, so we have:

$$V_n(I_1+1) - V_{n-1}(I_1+1) \geq V_{n+1}(I_1+1) - V_n(I_1+1)$$

or equivalently,

$$2V_n(I_1+1) \geq V_{n-1}(I_1+1) + V_{n+1}(I_1+1) \tag{4.40}$$

(4.39) and (4.40) together lead to the desired inequality:

$$V_n(I_1+1) - V_n(I_1) \geq V_n(I_1+2) - V_n(I_1+1)$$

We have shown that all models satisfy the concavity properties with respect to time and inventory. By using these results we have also shown that optimal prices of the promotional product and the bundle are non-increasing in inventory, non-decreasing in time.

In the following proposition, we will give the optimal pricing policy: When the inventory level of the promotional product is low, the firm sets a higher price for the promotional product and the bundle in order to sell the products to those customers with high reservation prices. As the supply increases, the firm sets lower prices in order to promote the sales of the promotional product and to avoid the risk of having left over inventories on hand at the end of the planning horizon. Similarly at a fixed inventory level of the promotional product, if there is more time, the price of the promotional product and the bundle increases. This also means that if there is enough time, the firm may not offer any discount for the promotional product and the bundle. As we mentioned in the previous sections, the optimal price of the service is independent from the system state parameters and constant for all states.

Proposition 4

i*)For a given period of time, the optimal price is a non-increasing function of the inventory.*

$$p_1^{n+1,I_1} \geq p_1^{n+1,I_1+1} \qquad \forall n,$$
$$p_{12}^{n+1,I_1} \geq p_{12}^{n+1,I_1+1} \qquad \forall n.$$

ii*) For a given inventory, the optimal price is a non-decreasing function of time:*

$$p_1^{n+2,I_1} \geq p_1^{n+1,I_1} \qquad \forall I_1,$$
$$p_{12}^{n+2,I_1} \geq p_{12}^{n+1,I_1} \qquad \forall I_1.$$

Proof:

i)

The optimal prices for the promotional product at given states can be written as:

$$p_1^{n+1,I_1} = V_n(I_1) - V_n(I_1 - 1) + \frac{1}{HR_1(p_1^{n+1,I_1})}$$

$$p_1^{n+1,I_1+1} = V_n(I_1 + 1) - V_n(I_1) + \frac{1}{HR_1(p_1^{n+1,I_1+1})}$$

where $HR(p)$ is the hazard rate function associated with the corresponding reservation price distribution. The hazard rate function is defined as $HR_j(p) = \frac{F_j'(p)}{1-F_j(p)}$ and assumed to be increasing.

We prove this inequality by contradiction. Assume that $p_1^{n,I_1+1} \geq p_1^{n,I_1}$

$$\begin{aligned} p_1^{n+1,I_1+1} - p_1^{n+1,I_1} &= V_n(I_1+1) - V_n(I_1) - V_n(I_1) + V_n(I_1-1) \\ &+ \frac{1}{HR_1(p_1^{n+1,I_1+1})} - \frac{1}{HR_1(p_1^{n+1,I_1})} \end{aligned}$$

We know that inequality (4.3) is satisfied for $I_1 - 1$:

$$V_n(I_1+1) - V_n(I_1) - V_n(I_1) + V_n(I_1-1) \leq 0$$

and also that $\frac{1}{HR_1(p)}$ is decreasing in p, so

$$\frac{1}{HR_1(p_1^{n+1,I_1+1})} - \frac{1}{HR_1(p_1^{n+1,I_1})} \leq 0$$

Hence,

$$p_1^{n+1,I_1+1} - p_1^{n+1,I_1} \leq 0$$

This contradicts the assumption. Hence,

$$p_1^{n+1,I_1} \geq p_1^{n+1,I_1+1}$$

Similarly, it can be proven that:

$$p_{12}^{n+1,I_1} \geq p_{12}^{n+1,I_1+1} \qquad \forall n$$

ii) Similarly, we will use the contradiction method to prove this inequality. Assume that $p_1^{n+1,I_1} \geq p_1^{n+2,I_1}$ where

$$p_1^{n+2,I_1} = V_{n+1}(I_1) - V_{n+1}(I_1-1) + \frac{1}{HR_1(p_1^{n+2,I_1})},$$

$$p_1^{n+1,I_1} = V_n(I_1) - V_n(I_1-1) + \frac{1}{HR_1(p_1^{n+1,I_1})} \ .$$

Taking the difference leads to:

$$\begin{aligned} p_1^{n+1,I_1} - p_1^{n+2,I_1} &= V_n(I_1) - V_n(I_1-1) - V_{n+1}(I_1) + V_{n+1}(I_1-1) \\ &+ \frac{1}{HR_1(p_1^{n+1,I_1})} - \frac{1}{HR_1(p_1^{n+2,I_1})} \ . \end{aligned}$$

Inequality (4.1) holds for $I_1 - 1$, therefore,

$$V_n(I_1) - V_n(I_1-1) - V_{n+1}(I_1) + V_{n+1}(I_1-1) \leq 0,$$

and also $\frac{1}{HR_1(p)}$ is a decreasing function, so:

$$\frac{1}{HR_1(p_1^{n+1,I_1})} - \frac{1}{HR_1(p_1^{n+2,I_1})} \leq 0 \ .$$

Hence,

$$p_1^{n+1,I_1} - p_1^{n+2,I_1} \leq 0$$

This contradicts the assumption.

Hence,

$$p_1^{n+2,I_1} \geq p_1^{n+1,I_1}$$

Similarly, it can be shown that

$$p_{12}^{n+2,I_1} \geq p_{12}^{n+1,I_1} \qquad \forall I_1$$

4.3 Optimal Upselling Policy

In this section, we will give the optimal upselling policy of the mixed bundling model. In the following proposition, we will show that in a state with parameters n and I_1, if a promotional product or a bundle is offered to the customer, in all states with a higher inventory level than that state, and in all states which is closer to the end of the planning horizon, the customer will never be offered a service.

Proposition 5 ***i)*** *If for some I_1 the promotional product (or the bundle) is offered instead of the service, in all states with a higher inventory level than I_1, it is never optimal to offer the service.Equivalently:*
$\forall n$, *if* $\exists I_1^*$ *such that if* $a^{n,I_1^*} = 1$, *(or* 12*)* $a^{n,I} \neq 2$, $\forall I \geq I_1^*$
ii) *If for some n the promotional product (or the bundle) is offered instead of the service, in all states which is closer to the end of the planning horizon, it is never optimal to offer the service. Equivalently:*
$\forall I_1$, *if* $\exists n^*$ *such that if* $a^{n^*,I_1} = 1$ *(or* 12*)*, $a^{n,I} \neq 2$, $\forall n \leq n^*$

We will prove the proposition only for $a^{n,I_1^*} = 1$, since the proof for $a^{n,I_1^*} = 12$ is similar.

Proof: i) The revenue function of the mixed bundling case can be rewritten as:

$$\begin{aligned} V_n(I_1) &= V_{n-1}(I_1) + \lambda_0 \max_{j \in \{1,2,12\}} \{ \frac{(1 - F_j(p_j^{n,I_1}))^2}{F_j'(p_j^{n,I_1})} \} \\ &+ \lambda_1 (r_1 + V_{n-1}(I_1 - 1) - V_{n-1}(I_1)) + \lambda_2 r_2 \end{aligned}$$

where p_1^{n,I_1}, p_2^{n,I_1} and p_{12}^{n,I_1} are optimal prices and given explicitly as:

$$p_1^{n,I_1} = V_{n-1}(I_1) - V_{n-1}(I_1 - 1) + \frac{1 - F_1(p_1^{n,I_1})}{F_1'(p_1^{n,I_1})}$$

$$p_2 = \frac{1 - F_2(p_2)}{F_2'(p_2)}$$

$$p_{12}^{n,I_1} = V_{n-1}(I_1) - V_{n-1}(I_1 - 1) + \frac{1 - F_{12}(p_{12}^{n,I_1})}{F'_{12}(p_{12}^{n,I_1})} \ .$$

In the previous section, we have shown in Proposition 4 that

$$p_1^{n,I_1-1} \geq p_1^{n,I_1} \geq p_1^{n,I_1+1}$$

Since we assume that the hazard rate function is increasing in p, the function $\frac{(1-F_1(p))^2}{F_1'(p)}$ is a decreasing in p.
Therefore,

$$\frac{(1 - F_1(p_1^{n,I_1+1}))^2}{F_1'(p_1^{n,I_1+1})} \geq \frac{(1 - F_1(p_1^{n,I_1}))^2}{F_1'(p_1^{n,I_1})} \geq \frac{(1 - F_1(p_1^{n,I_1-1}))^2}{F_1'(p_1^{n,I_1-1})}$$

Since the price of the service is constant and independent from the inventory level of the promotional product (I_1) and the remaining time (n), if for some fixed time n the promotional product is offered instead of the service for an inventory level I_1, at the states in which the inventory levels are greater than I_1, the firm also offers promotional product. Therefore, if for some n and some I_1

$$\frac{(1 - F_1(p_1^{n,I_1}))^2}{F'(p_1^{n,I_1})} \geq \frac{(1 - F_2(p_2))^2}{F_2'(p_2)},$$

then,

$$\frac{(1 - F_1(p_1^{n,I}))^2}{F_1'(p_1^{n,I})} \geq \frac{(1 - F_1(p_1^{n,I_1}))^2}{F_1'(p_1^{n,I_1})} \geq \frac{(1 - F_2(p_2))^2}{F_2'(p_2)} \qquad \forall\ I \geq I_1$$

ii) We know by proposition 4 that

$$p_1^{n,I_1} \geq p_1^{n-1,I_1} \ .$$

Since we assume that the hazard rate function is increasing in p, the function $\frac{(1-F_1(p))^2}{F_1'(p)}$ is a decreasing in p.

Therefore,

$$\frac{(1-F_1(p_1^{n-1,I_1}))^2}{F_1'(p_1^{n-1,I_1})} \geq \frac{(1-F_1(p_1^{n,I_1}))^2}{F_1'(p_1^{n,I_1})}$$

Since the price of the service is constant and independent from the inventory level of the promotional product (I_1) and time (n), if for a fixed inventory I_1, the promotional product is offered instead of the service at a time n, then the firm offers the promotional product whenever the remaining time is less than or equal to n. If for any n any I_1

$$\frac{(1-F_1(p_1^{n,I_1}))^2}{F'(p_1^{n,I_1})} \geq \frac{(1-F_2(p_2))^2}{F_2'(p_2)},$$

then,

$$\frac{(1-F_1(p_1^{n',I}))^2}{F_1'(p_1^{n',I})} \geq \frac{(1-F_1(p_1^{n,I_1}))^2}{F_1'(p_1^{n,I_1})} \geq \frac{(1-F_2(p_2))^2}{F_2'(p_2)} \qquad \forall\ n' \leq n\ .$$

Chapter 5

RESULTS BASED ON SPECIFIC CUSTOMER PREFERENCE ASSUMPTIONS

In this chapter, we will make some assumptions on the reservation price distributions and give the additional results obtained under these assumptions. First, we will give the definitions of the hazard rate order, stochastic order and increasing concave order that we use throughout the chapter.

Definition-1 (Müller and Stoyan, [17]): Let Φ_1 and Φ_2 be two cumulative distribution functions with corresponding probability density functions ϕ_1 and ϕ_2. Φ_1 dominates Φ_2 in hazard (failure) rate ordering, denoted as $\Phi_1 \geq_{hr} \Phi_2$, if

$$\frac{\phi_1(x)}{1-\Phi_1(x)} \leq \frac{\phi_2(x)}{1-\Phi_2(x)} \quad \forall x \ .$$

Definition-2 (Müller and Stoyan, [17]): The random variable X is said to be smaller than the random variable Y with respect to stochastic order (written $X \geq_{hr} Y$), if

$$F_X(t) \geq F_Y(t) \quad \forall \ \ t$$

Definition-3 (Müller and Stoyan, [17]): The random variable X is less than the random variable Y in increasing concave order (written $X \geq_{icv} Y$), if

$$E[f(X)] \leq E[f(Y)]$$

for all increasing concave functions f such that the expectations exist.

These comparison methods are used to compare two random variables (or their distribution functions).

The hazard rate order is the strongest order and implies the others. The relation between the orders can be given as, the hazard rate order implies the stochastic order and the stochastic order implies the increasing concave order i.e., $X \leq_{hr} Y \rightarrow X \leq_{st} Y \rightarrow X \leq_{icv} Y$. In the following section, we will discuss the implication of these orders in the pricing context.

5.1 Reservation Price Distribution Assumptions

Up to this point, we did not specify any forms for the reservation price distributions. In this section, we will make some assumptions on the reservation price distributions and compare them to the assumptions that appear frequently in the pricing/revenue management literature.

Both optimal action decision (which option to choose) and optimal pricing decision (which price to offer) are mainly affected by the distribution of the reservation price. We make three assumptions on the reservation price distributions of the products. Before giving our assumptions, we will explain the work by Ziya et al. [26] in which they discuss three frequently used assumptions in pricing and revenue management literature. They explore the relationship between the following three assumptions: (1) decreasing marginal revenue with respect to demand, (2) decreasing marginal revenue with respect to price, and (3) increasing price elasticity of demand. These assumptions provide analytical tractability, and each ensures a well-behaved revenue function (revenue function is either unimodal or monotone under these assumptions). The first two assumptions are required for the revenue function to be concave in demand and price respectively. Moreover, the last assumption means $F(.)$ has an increasing generalized failure rate where $F(.)$ is the cumulative distribution function of the reservation price. Note that the generalized failure rate is denoted by $e(x)$, $e(x) = xHR(x)$ where $HR(x)$ is the hazard rate function for $F(.)$.

We make the following assumptions on the reservation price distributions and compare them with those of Ziya et al. [26].

We have made Assumption (i) in Chapter 4 and all structural result are provided under this assumption. Now, we will make additional assumptions which will be used only in this chapter.

i) The hazard rate function associated with the reservation price distribution is increasing in p where the hazard function $HR(p)$ is defined as $HR(p) = f(p)/(1 - F(p))$.

ii) $F_1 \leq_{hr} F_{12}$ and $F_2 \leq_{hr} F_{12}$

iii) The reservation price distribution of the promotional product stochastically dominates that of the service with respect to the hazard rate order: $F_2 \leq_{hr} F_1$

Increasing hazard rate assumption is stronger than the increasing failure rate assumption that means if the hazard rate function is increasing in p, the corresponding generalized failure rate function is also increasing in p. Therefore, Assumption (i) implies the increasing generalized failure rate assumption. Increasing hazard rate assumption holds for most common distributions such as the normal, exponential and Weibull (when shape parameter $k \geq 1$) (Lariviere and Porteus [13]). Assumption (ii) implies that the probability that an arbitrary price p is the maximal price that the customer is willing to pay for the promotional product (or service) is higher than that of for the bundle. Since the hazard rate order implies the stochastic order, Assumption (ii) also means that purchasing the bundle at a given price p is more probable than purchasing the promotional product (or service) at the same price p. This assumption would be applicable in most cases since in a bundle an additional product is offered. Finally, Assumption (iii) implies that at a given price, the purchase probability of the customers for the promotional product is higher than that of for the service.

Before giving the additional results on optimal prices and on the bundling under the reservation price assumptions given above, we will give a remark by using these assumptions. We have mentioned in Section 5.1 that some common distributions satisfy the increasing hazard rate property, such as the exponential, Weibull (when its shape parameter $k \geq 1$) and normal.

Under the assumption that reservation prices are normally distributed, we can give the following remark:

Remark: (i) The expected reservation price of the customers for the service is lower than that of the promotional product. (ii) Customers' reservation price variability for the service is greater than or equal to that of the promotional product, i.e. $Var(R_2) \geq Var(R_1)$.

This result seems consistent with the notion that a service is more difficult to evaluate for customers due to the presence of experience qualities. Experience qualities means that the customer must first purchase and use the service before evaluating it. Therefore, it is expected that the service has more variable reservation price than the promotional product.

Proof: We will prove both (i) and (ii) together.

We assume that F_1 stochastically dominates F_2 with respect to hazard rate order (Assumption (iii)). Since hazard rate order ($\leq_{hr}$) implies stochastic order ($\leq_{st}$) and stochastic order implies increasing concave order ($\leq_{icv}$) (also known as second order stochastic order ($\leq_{SSD}$)) , it can be written as:

$$F_2 \leq_{icv} F_1 \tag{5.1}$$

or equivalently,

$$F_2 \leq_{SSD} F_1 \tag{5.2}$$

In their book, Muller and Stoyan [17] state that for some important families of distributions mean-variance decision rules are consistent with second order stochastic dominance. They give an example that, in the case of normal distributed random variables X and Y, $X \leq_{SSD} Y$ if and only if $E[X] \leq E[Y]$ and $Var[X] \geq Var[Y]$

By using this information and inequality (5.2), we can write: $E(R_2) \leq E(R_1)$ and $Var(R_2) \geq Var(R_1)$ which is the end of the proof.

5.2 Results on Optimal Prices under Specific Reservation Price Distribution Assumptions

In this section, we will discuss the results on the optimal prices obtained under the reservation price distribution assumptions given in the previous section.

Proposition 6 *The optimal price of the promotional product is greater than or equal to the optimal price of the service in all states.*

Proof: Since F_1 dominates F_2 with respect to the hazard rate order, $F_2 \leq_{hr} F_1$, by the definition of the hazard rate order, it can be written as:

$$\frac{1 - F_2(p)}{F_2'(p)} \leq \frac{1 - F_1(p)}{F_1'(p)} \qquad \forall p \ . \tag{5.3}$$

We will use contradiction to prove the proposition. Let us assume that $p_1^{n,I_1} < p_2^{n,I_1}$ where

$$p_1^{n,I_1} = V_{n-1}(I_1) - V_{n-1}(I_1 - 1) + \frac{1 - F_1(p_1^{n,I_1})}{F_1'(p_1^{n,I_1})} \tag{5.4}$$

$$p_2^{n,I_1} = \frac{1 - F_2(p_2^{n,I_1})}{F_2'(p_2^{n,I_1})} \ . \tag{5.5}$$

We define:

$$\Delta = V_{n-1}(I_1) - V_{n-1}(I_1 - 1)$$

The difference of (5.4) and (5.5) leads to:

$$p_1^{n,I_1} - p_2^{n,I_1} = \Delta + \frac{1 - F_1(p_1^{n,I_1})}{F_1'(p_1^{n,I_1})} - \frac{1 - F_2(p_2^{n,I_1})}{F_2'(p_2^{n,I_1})}$$

Since the hazard rate function is increasing, the function $\frac{1-F_j(.)}{F_j'(.)}$ $j \in \{1, 2\}$ is decreasing. By using the assumption $p_1^{n,I_1} < p_2^{n,I_1}$, it can be written as:

$$\frac{1 - F_2(p_2^{n,I_1})}{F_2'(p_2^{n,I_1})} < \frac{1 - F_2(p_1^{n,I_1})}{F_2'(p_1^{n,I_1})},$$

and by equation 5.3:

$$\frac{1 - F_2(p_1^{n,I_1})}{F_2'(p_1^{n,I_1})} \leq \frac{1 - F_1(p_1^{n,I_1})}{F_1'(p_1^{n,I_1})} \ .$$

Hence,

$$\frac{1 - F_1(p_1^{n,I_1})}{F_1'(p_1^{n,I_1})} - \frac{1 - F_2(p_2^{n,I_1})}{F_2'(p_2^{n,I_1})} \geq 0 \ .$$

It is also known by Proposition 2 that, $\Delta \geq 0$. Therefore,

$$p_1^{n,I_1} - p_2^{n,I_1} \geq 0,$$

which contradicts to the assumption $p_1^{n,I_1} < p_2^{n,I_1}$, so if $F_2 \leq_{hr} F_1$, then the optimal prices are also ordered in the same direction:

$$p_2^{n,I_1} \leq p_1^{n,I_1} \ .$$

Proposition 7 *If the same prices are set for the promotional product and the service, it is optimal to offer the promotional product.*

Proof: We have shown in Chapter 3.3 that the revenue function as a function of the optimal prices can be given as:

$$\begin{aligned} V_n(I_1) &= V_{n-1}(I_1) + \lambda_0 \max_{j \in \{1,2,12\}} \{ \frac{(1 - F_j(p_j^{n,I_1}))^2}{F_j'(p_j^{n,I_1})} \} \\ &+ \lambda_1(r_1 + V_{n-1}(I_1 - 1) - V_{n-1}(I_1)) + \lambda_2 r_2 \ . \end{aligned} \tag{5.6}$$

We assume that F_1 dominates F_2 in hazard rate order. Hazard rate order implies the stochastic order:

$$F_2 \leq_{hr} F_1 \Rightarrow F_2 \leq_{st} F_1 \ .$$

By the definition of stochastic order (See Müller and Stoyan [17] for extensive definitions.), we have:

$$F_1(p) \leq F_2(p) \qquad \forall p \ . \tag{5.7}$$

By the definition of hazard rate order:

$$\frac{1-F_2(p)}{F_2'(p)} < \frac{1-F_1(p)}{F_1'(p)} \qquad \forall p \tag{5.8}$$

by 5.7 we know that:

$$1-F_2(p) \leq 1-F_1(p) \qquad \forall p \ . \tag{5.9}$$

(5.8) and (5.9) together lead to:

$$\frac{(1-F_2(p))^2}{F_2'(p)} < \frac{(1-F_1(p))^2}{F_1'(p)} \qquad \forall p \ . \tag{5.10}$$

If we consider the revenue function 5.6, this inequality implies, when the optimal prices are set for the promotional product and the service, offering promotional product always yields more profit.

Proposition 8 *The optimal price of the bundle is always greater than or equal to the optimal price of the promotional product and optimal price of the service.*

Proof: Let p_1^{n,I_1}, p_2^{n,I_1} and p_{12}^{n,I_1} denote the optimal prices for the promotional product, service and bundle for a given inventory I_1 and time n, respectively. We have shown in Chapter 3 that the optimal prices can be written in closed form as:

$$\begin{aligned} p_1^{n,I_1} &= V_{n-1}(I_1) - V_{n-1}(I_1-1) + \frac{1-F_1(p_1^{n,I_1})}{F_1'(p_1^{n,I_1})} \\ p_2^{n,I_1} &= \frac{1-F_2(p_2^{n,I_1})}{F_2'(p_2^{n,I_1})} \\ p_{12}^{n,I_1} &= V_{n-1}(I_1) - V_{n-1}(I_1-1) + \frac{1-F_{12}(p_{12}^{n,I_1})}{F_{12}'(p_{12}^{n,I_1})} \end{aligned}$$

Taking the difference of (5.11) and (5.11)leads to:

$$p_{12}^{n,I_1} - p_1^{n,I_1} = \frac{1 - F_{12}(p_{12}^{n,I_1})}{F'_{12}(p_{12}^{n,I_1})} - \frac{1 - F_1(p_1^{n,I_1})}{F'_1(p_1^{n,I_1})} \ . \tag{5.11}$$

Assume that $p_1^{n,I_1} \geq p_{12}^{n,I_1}$. Since the hazard rate function is increasing, $\frac{1}{Hr(p)}$ is decreasing in p. Therefore,

$$\frac{1 - F_{12}(p_{12}^{n,I_1})}{F'_{12}(p_{12}^{n,I_1})} \geq \frac{1 - F_{12}(p_1^{n,I_1})}{F'_{12}(p_1^{n,I_1})} \ . \tag{5.12}$$

Since $F_1 \leq_{hr} F_{12}$, using the definition of the hazard rate order, it can be written as:

$$\frac{1 - F_{12}(p_1^{n,I_1})}{F'_{12}(p_1^{n,I_1})} \geq \frac{1 - F_1(p_1^{n,I_1})}{F'_1(p_1^{n,I_1})} \ . \tag{5.13}$$

By using (5.12) and (5.13), we have:

$$\frac{1 - F_{12}(p_{12}^{n,I_1})}{F'_{12}(p_{12}^{n,I_1})} \geq \frac{1 - F_1(p_1^{n,I_1})}{F'_1(p_1^{n,I_1})} \ .$$

Using the last inequality in (5.11) leads to:

$$p_{12}^{n,I_1} - p_1^{n,I_1} \geq 0,$$

which contradicts the assumption $p_1^{n,I_1} \geq p_{12}^{n,I_1}$. Hence, $p_1^{n,I_1} \leq p_{12}^{n,I_1}$.
In Proposition 6 we have proven that:

$$p_2^{n,I_1} \leq p_1^{n,I_1} \qquad \forall \text{ n,}I_1$$

Therefore, it can be written as:

$$p_2^{n,I_1} \leq p_{12}^{n,I_1} \qquad \forall \text{ n,}I_1$$

5.3 Results on Bundling under Specific Reservation Price Distribution Assumptions

In this section, we will examine the relation between the models that we presented in Chapter 3. We will show under which assumptions which model dominates the other ones.

Assumption-1: The probability of not purchasing the promotional product at its optimal price is greater than or equal to that of bundle, i.e.,

$$F_1(p_1^{n,I_1}) \geq F_{12}(p_{12}^{n,I_1}), \tag{5.14}$$

where p_1^{n,I_1} and p_{12}^{n,I_1} are optimal prices of the promotional product and bundle respectively, when the inventory level of promotional product is I_1 and the remaining time is n.

Proposition 9 *Under Assumption-1, offering the bundle is always preferable to offering the promotional product.*

Proof: The revenue function in mixed bundling strategy can be given as:

$$\begin{aligned} V_n(I_1) \;&=\; V_{n-1}(I_1) + \lambda_0 \max\{\bar{F}_1(p_1^{n,I_1})(p_1^{n,I_1} + V_{n-1}(I_1-1) - V_{n-1}(I_1)), \\ &\qquad \bar{F}_2(p_2^{n,I_1})p_2^{n,I_1}, \\ &\qquad \bar{F}_{12}(p_{12}^{n,I_1})(p_{12}^{n,I_1} + V_{n-1}(I_1-1) - V_{n-1}(I_1))\} \\ &+ \; \lambda_1(r_1 + V_{n-1}(I_1-1) - V_{n-1}(I_1)) + \lambda_2 r_2. \end{aligned}$$

We define:

$$\Delta = V_{n-1}(I_1) - V_{n-1}(I_1 - 1).$$

Assumption-1 and Proposition 8 together lead to:

$$\bar{F}_{12}(p_{12}^{n,I_1})(p_{12}^{n,I_1} - \Delta) \geq \bar{F}_1(p_1^{n,I_1})(p_1^{n,I_1} - \Delta) \qquad \forall\ p_1^{n,I_1},\, p_{12}^{n,I_1}$$

Hence, we can conclude that under Assumption-1, in mixed bundling revenue function, the third term in the *max* operator is always greater than the first one. This implies that offering the bundle is always preferable to offering the promotional product, under this condition. This is an expected result, since at a given state, customer's purchase probability for promotional product is less than or equal to the that for the bundle (Assumption-1). Therefore, the firm increases the purchase probability by offering the bundle.

We can similarly show that under Assumption-1, Case SPB reduces to Case SB and Case PB reduces to Case B.

Assumption-2: The probability of not to purchase the service at its optimal price is greater than or equal to that of bundle, and the marginal price of the promotional product in a bundle is greater than or equal to the marginal value of one unit of the promotional product, i.e.,

i) $F_2(p_2^{n,I_1}) \geq F_{12}(p_{12}^{n,I_1})$

and

ii) $p_{12}^{n,I_1} - p_2^{n,I_1} \geq \Delta$ where $\Delta = V_{n-1}(I_1) - V_{n-1}(I_1 - 1)$.

Proposition 10 *Under Assumption-2, offering the bundle is more profitable than offering only service.*

Proof: We have shown that in mixed bundling strategy the revenue function can be written as:

$$\begin{aligned} V_n(I_1) &= V_{n-1}(I_1) + \lambda_0 \max_{j \in \{1,2,12\}} \{ \frac{(1 - F_j(p_j^{n,I_1}))^2}{F_j'(p_j^{n,I_1})} \} \\ &+ \lambda_1(r_1 + V_{n-1}(I_1 - 1) - V_{n-1}(I_1)) + \lambda_2 r_2 \end{aligned}$$

Since

$$p_{12}^{n,I_1} = \Delta + \frac{1 - F_{12}(p_{12}^{n,I_1})}{F_{12}'(p_{12}^{n,I_1})},$$

and

$$p_2^{n,I_1} = \frac{1 - F_2(p_2^{n,I_1})}{F_2'(p_2^{n,I_1})},$$

by (ii),

$$\frac{1 - F_{12}(p_{12}^{n,I_1})}{F'_{12}(p_{12}^{n,I_1})} \geq \frac{1 - F_2(p_2^{n,I_1})}{F'_2(p_2^{n,I_1})} \; .$$

Multiplying both sides by $(1 - F_{12}(p_{12}^{n,I_1}))$ leads to:

$$\frac{(1 - F_{12}(p_{12}^{n,I_1}))^2}{F'_{12}(p_{12}^{n,I_1})} \geq (1 - F_{12}(p_{12}^{n,I_1}))\frac{1 - F_2(p_2^{n,I_1})}{F'_2(p_2^{n,I_1})} \; .$$

By (i):

$$(1 - F_{12}(p_{12}^{n,I_1}))\frac{1 - F_2(p_2^{n,I_1})}{F'_2(p_2^{n,I_1})} \geq \frac{(1 - F_2(p_2^{n,I_1}))^2}{F'_2(p_2^{n,I_1})} \; .$$

Hence:

$$\frac{(1 - F_{12}(p_{12}^{n,I_1}))^2}{F'_{12}(p_{12}^{n,I_1})} \geq \frac{(1 - F_2(p_2^{n,I_1}))^2}{F'_2(p_2^{n,I_1})} \; .$$

The last inequality shows that offering the bundle is more profitable than offering only service under Assumption-2. It can be generalized that under Assumption-2 Case SPB reduces to Case PB and Case SB reduces to Case B.

If both Assumption-1 and Assumption-2 are satisfied then mixed bundling reduces to pure bundling.

Chapter 6

NUMERICAL ANALYSIS

In this chapter, we present the results of our numerical study. The purpose of our numerical study is to explain the impact of various factors such as reservation price distribution parameters, initial inventory levels and the customer arrival rate on pricing decisions in the presence of bundling. We particularly study the impact of these factors on expected revenues, optimal upselling decisions, bundle prices and individual product prices. We used a value iteration algorithm and wrote a MATLAB code to compute the expected revenues.

We assume that reservation price distributions of the promotional product, the service and the bundle have Weibull distributions. The Weibull distribution has two parameters: a shape parameter β and a scale parameter η. The shape parameter β has significant effects on the behavior of the distribution. For example, when $\beta = 1$ the pdf of the Weibull reduces to that of the exponential distribution or when $2.6 \leq \beta \leq 3.7$, the Weibull distribution appears similar to a normal distribution. The Weibull distribution has an increasing hazard rate, only when $\beta \geq 1$. In our study, the shape parameters are set to 3 for all distributions.

In prior studies, the reservation prices are generally assumed to have a normal distribution or a Weibull distribution. For example Aydin and Ziya [2] and Bitran and Mondschein [4] use a Weibull distribution in their numerical study; however, Bulut et al. [6] assume that the reservation prices are normally distributed. We use the Weibull distribution in our numerical study; however, we set the shape parameter to 3 and thus the pdf approximates the shape of a normal pdf.

Higher β values imply customers can purchase the product at a given price with a higher probability; however, the customers become more price sensitive.

Increasing the value of the scale parameter η while holding β constant stretches out the

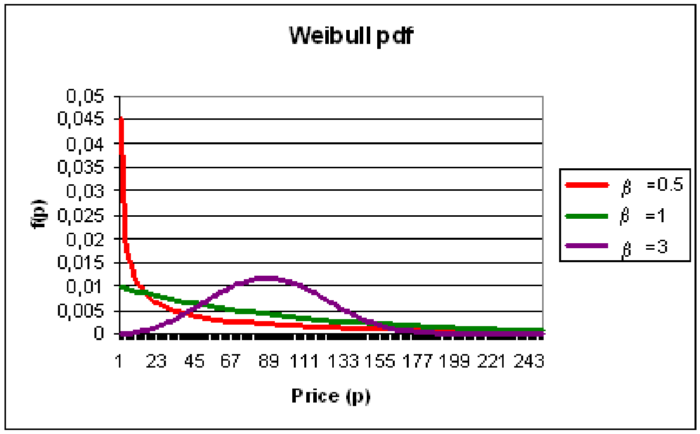

Figure 6.1: Weibull pdf with $0 \leq \beta \leq 1$, $\beta = 1$, $\beta \geq 1$

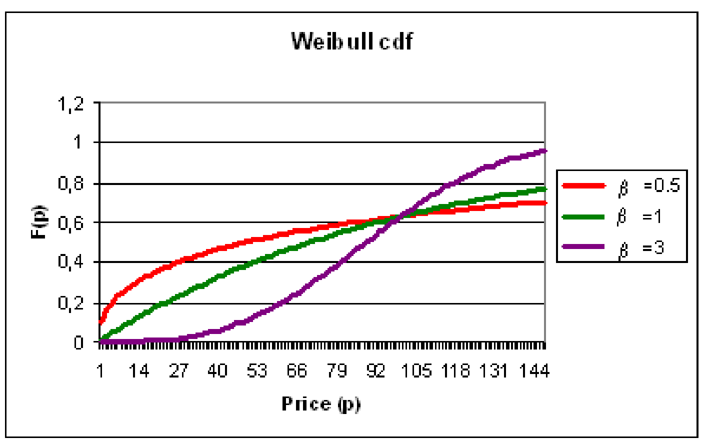

Figure 6.2: Effect of β on the cdf on a Weibull probability plot with a fixed value of η

pdf of the distribution. As indicated in Figure 6.3, if η is increased while β is kept the same, the distribution gets stretched out to the right and its peak decreases, while keeping its shape. Higher η values implies customers can purchase the product at a given price with a higher probability.

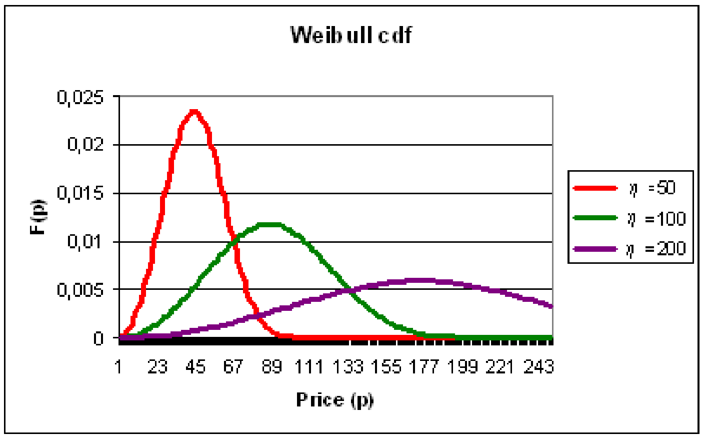

Figure 6.3: Effect of η on the Weibull pdf at a fixed value of β

The impact of the scale parameters of a Weibull distribution on expected revenue and pricing policies is explored in Section 6.1. In Section 6.2, we test the impact of adding a bundle option on the expected revenue and pricing decisions. Then, the effects of inventory limitations of the promotional product on expected revenues, optimal upselling decisions, bundle prices and individual product prices are observed in Section 6.3.

All analyses given below are performed with different demand arrival probabilities; however, the structure of the results did not change. Therefore, we restrict our analysis to the case with demand arrival probabilities, $\lambda_0 = 0.4$, $\lambda_1 = 0.3$, and $\lambda_2 = 0.2$.

6.1 The Impact of Complementarity

Complementarity implies that the reservation price for one product or service is increased if the other is purchased (Guiltinan [11]).

If the sale of one product favorably affects the sale of another product, these are called complementary products. When products are complements, a customer's reservation price for the bundle is superadditive (more than the sum of the reservation prices).

Alternatively, if the products are substitutable, the sale of one product adversely affects the sale of another product. In this case, a customer's reservation price for the bundle would be subadditive (less than the sum of the reservation prices).

The scale parameter of the Weibull distribution of the bundle gives the degree of complementarity or substitutability of the promotional product and the service by changing the probability of a bundle purchase at a given price. Let η_{12}^{L} and η_{12}^{H} denote the scale parameters for low correlation case and high correlation case respectively.

$$P(purchase\ the\ bundle\ at\ price\ p : \eta_{12}^{L}) \leq P(purchase\ the\ bundle\ at\ price\ p : \eta_{12}^{H})$$

For complementary promotional product and service, the probability of purchasing the bundle is higher than that for substitutable promotional product and service. Hence, by changing parameter η_{12} we can obtain different levels of complementarity/substitutabilility.

We study two different values for η_{12}, which are 130 and 170. All parameters are given in Table 6.1.

Table 6.1: Parameters -1-

Parameter	**Value Set**
λ_1	0.4
λ_2	0.3
λ_{12}	0.2
r_1	95
r_2	85
(β_1, η_1)	$(3, 95)$
(β_2, η_2)	$(3, 85)$
(β_{12}, η_{12})	$(3, 130)$, $(3, 170)$
time (N)	20

In this section, we test the performance of Case SBP for $\eta_{12} = 130$ and $\eta_{12} = 170$.

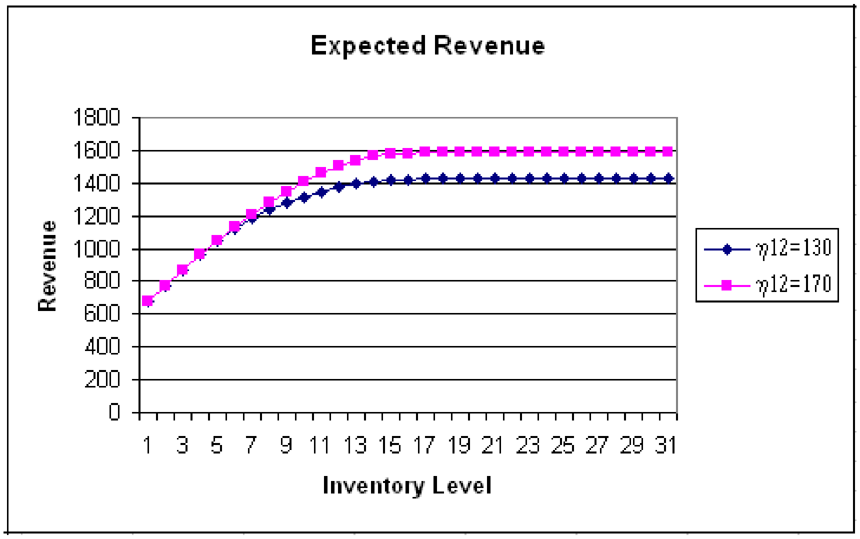

Figure 6.4: The Effect of Complementarity on the Expected Revenue

As seen in Figure 6.4, when complementarity increases between the promotional product and the service, the revenue also increases. The effect of the η_{12} is more significant for high inventory levels of the promotional product because if there is more inventory, more bundle will be sold. Figure 6.5 shows, when complementarity increases, the expected individual product sales decrease. This result is expected since as the promotional product and the service complement each other, the customers who are already willing to buy the promotional product (service) are more willing to buy the service (promotional product) as well, and thus the bundle becomes an option that is more attractive than offering only one of the individual products.

Despite the fact that the bundle price increases as the parameter η_{12} increases (See Figure 6.6), the expected sales amount of the bundle also increases as seen in Figure 6.5. Charging high bundle price will not dissuade the customers from buying it.

When the firm has limited promotional product, it sets higher bundle prices in order to sell the products to those customers with high reservation prices.

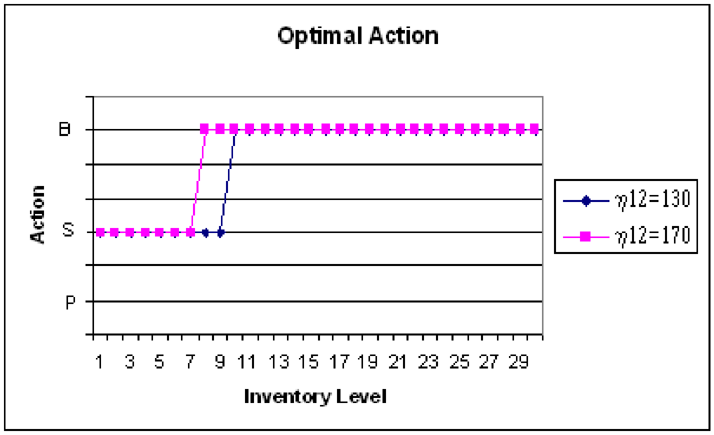

Figure 6.5: Optimal Action

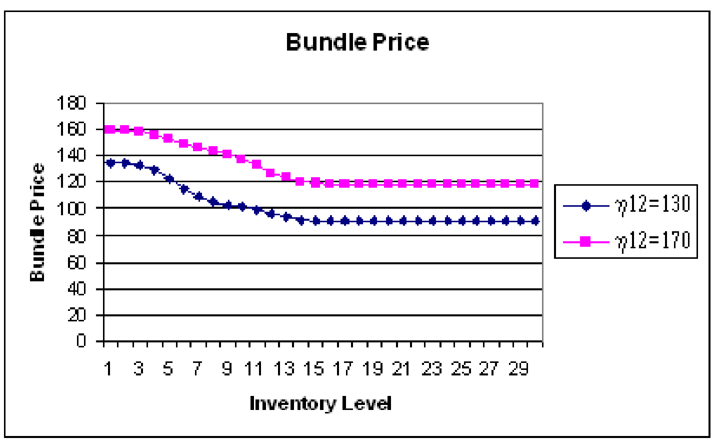

Figure 6.6: Optimal Bundle Price

As the supply increases, the firm sets lower prices in order to avoid the risk of having left over inventories on hand at the end of the planning horizon.

As seen in Figure 6.7, when substitutability increases, the price of the promotional product decreases. Since the products are substitutable, the purchase probability of the bundle decreases and the firm wants to sell the promotional product individually. Therefore, the firm decreases the individual price of the promotional product to encourage the customers to buy the promotional product.

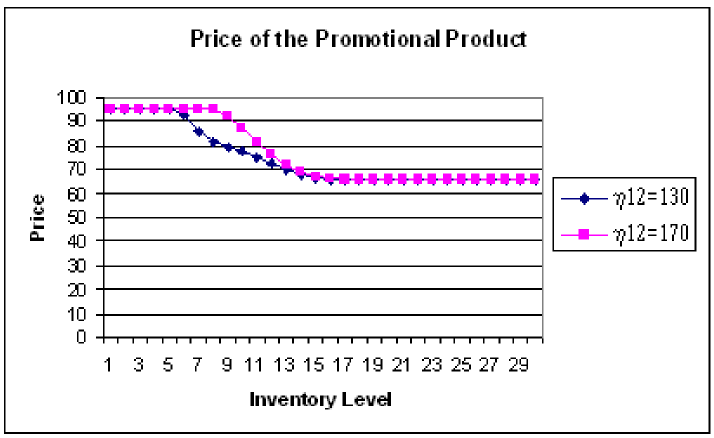

Figure 6.7: Optimal Price of the Promotional Product

6.2 Bundling Effect

In this section, we will examine the effects of considering the bundle as an upselling option on different cases. To observe the bundle effect, we consider three couples: (Case S, Case SB), (Case P, Case PB), (Case SP, Case SPB) and compare the effects of considering the bundle as an upselling option, on the revenue function and optimal offering decisions for these couples.

First, to see the bundle effect on the revenue function for three couples, we compute the percentage increase when the bundle is added to the upselling option list for each couple. The percentage increase for each couple is computed as:

$$\frac{V^{*B} - V^{*}}{V^{*}}$$

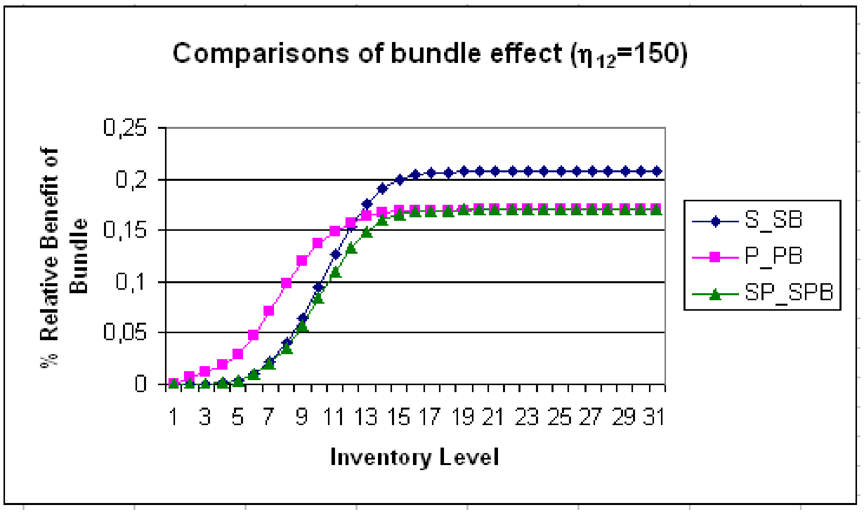

Figure 6.8: Comparison of the Relative Bundle Effect when $\eta_{12} = 150$, at a fixed time

Adding a bundle option always increases the revenue. However, as seen in Figure 6.8, the bundling is least effective when there is limited supply. When the starting inventory level is increased, bundling becomes more instrumental.

In high initial inventory level, adding a bundle option to only service case is more advantageous, because in Case S, the promotional product can only be sold to promotional product customers, directly. In this case adding the bundle option implies a new sales-channel for the promotional product. Since in Case P and Case SP the promotional product is already an upselling option, adding the bundle has more significant effect on Case S.

We also observe the bundle effect at $\eta_{12} = 130$ and $\eta_{12} = 170$. Similar effects are observed; however, when products are more complementary, i.e. $\eta_{12} = 170$, adding a bundle option is more beneficial for all couples because the bundle becomes a more attractive option than offering only one of the products individually. On the other hand, the marginal revenue of adding a bundle

Table 6.2: Bundle Sales Percentage

	$\eta_{12} = 130$	$\eta_{12} = 150$	$\eta_{12} = 170$
Case SB	0.8516	0.8716	0.89
Case PB	0.93	1	1
Case SPB	0.8516	0,8716	0,89

option decreases for substitute products, i.e. $\eta_{12} = 130$, because customers prefer individual products more than the bundle in this case.

The sales percentage of the bundle for Case SB, Case PB and Case SPB, at different η_{12} values are given in Table 6.2.

As seen in Table 6.2, the percentage of the products sold in bundled form is smaller when the products are substitutable ($\eta_{12} = 130$).

We observe an increase in the percentage of the products sold in bundled form as the supply increases for all cases. The reason is that, for high inventory levels the firm tries to sell more bundles to accomplish inventory depletion. We also observe that bundle sales percentage increases when the end of the planning horizon is closer.

6.3 Service vs Promotional Product

The main difference between the promotional product and the service is that the promotional product has a limited inventory while the service has an unlimited supply. We will observe the effects of having limited inventory on the expected revenue, bundle prices, optimal individual product prices and optimal actions. In this section, we study the case when the promotional product and the service are identically distributed, i.e. the scale and shape parameters of the Weibull distributions are the same. The initial inventory level is 30.

In this section, we compare Case S and Case P, and Case SB and Case PB to see the effect of limited inventory.

All parameters are given in Table (6.3).

We fix the time at $N = 20$ and observe the optimal prices and the expected revenues of Case

Table 6.3: Parameters -2-

Parameter	Value Set
λ_1	0.4
λ_2	0.3
λ_{12}	0.2
r_1	95
r_2	95
(β_1, η_1)	$(3, 95)$
(β_2, η_2)	$(3, 95)$
(β_{12}, η_{12})	$(3, 150)$
time (N)	20

S and Case P.

Figure 6.9 shows how the expected revenue changes with the inventory level for both cases. The optimal expected revenue is a non-decreasing function of the initial inventory of the promotional product for both cases.

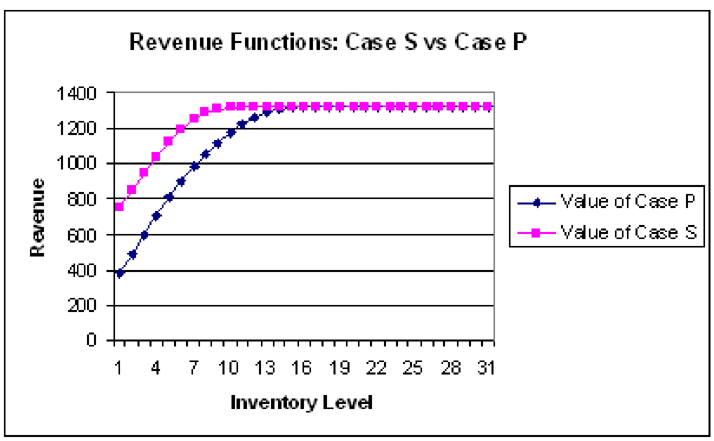

Figure 6.9: Revenue Function: Case S vs Case P

We also observe the percentage increase between the revenue function of Case S and Case P at two different time values, $n = 5$ and $n = 15$. As seen in Figure 6.10, the impact of limited inventory decreases as we come closer to the end of the planning horizon.

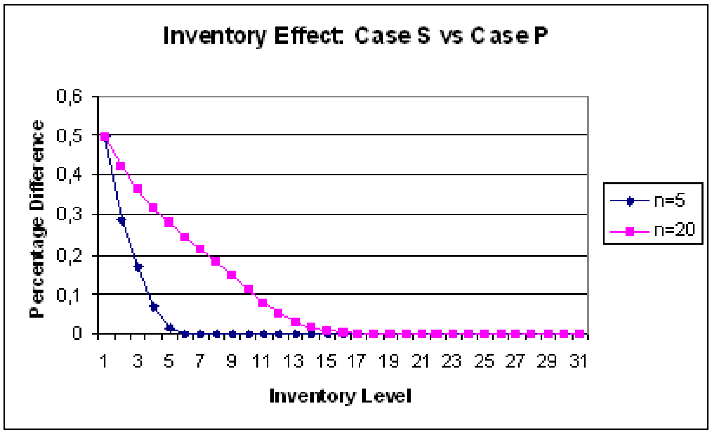

Figure 6.10: Percentage Difference between Revenue Function of Case S vs Case P at $n = 5$ and $n = 20$

As seen in Figure 6.11, the optimal price of the service is independent from the inventory level of the promotional product. However, the optimal price of the promotional product decreases in the initial inventory level.

The lower bound for the optimal price of the promotional product (for the optimal price of the limited inventory case) is the optimal price of the service (unlimited inventory case), as seen in Figure 6.11.

The firm increases the price of the promotional product because there is limited number of product and this limitation increases the price of the product. The firm sets an optimal price for the unlimited supply. This is the minimum value that the firm can determine. When the inventory level of the promotional product increases it behaves like the unlimited supply case, therefore, the price of the promotional product coincides with the service price. In our study, we fixed the upper bound of the price of the promotional product at the announced price of the promotional product, since it is assumed that the firm never offers a higher price than the announced price.

As the supply increases, the firm sets lower prices for the promotional product in order to avoid the risk of having left over inventories at the end of the planning horizon.

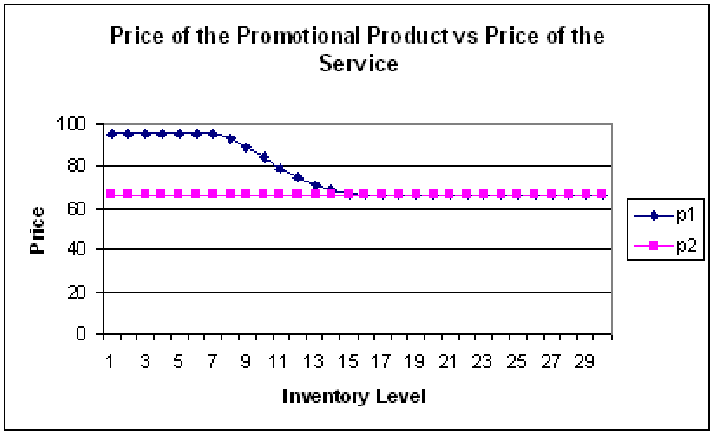

Figure 6.11: Price of the Promotional Product vs Price of the Service

Although the optimal price of the service is lower than the optimal price of the promotional product, Case S is always more profitable than Case P especially in low initial inventory levels, because there is no inventory limitations so the firm can sell more service.

When Case SB and Case PB are compared, the effect of limited inventory on expected revenues and optimal prices are the same.

In our analysis, we assume that the scale parameter of the Weibull distribution $\eta_{12} = 150$ for both cases.

When the firm has limited promotional product, it sets higher bundle prices in order to sell the products to those customers with high reservation prices.

In Case SB, a lower price is set for the bundle since there is only one channel to sell the promotional product as an upselling option. However, despite the high price, more bundle is sold in Case PB.

As seen in Figure (6.12) and Figure (6.13), despite having higher optimal price of the bundle higher revenues cannot be achieved in Case PB.

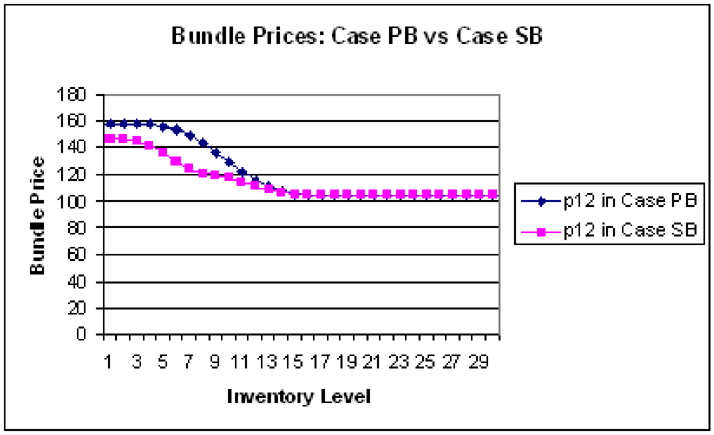

Figure 6.12: Bundle Prices: Case PB vs Case SB

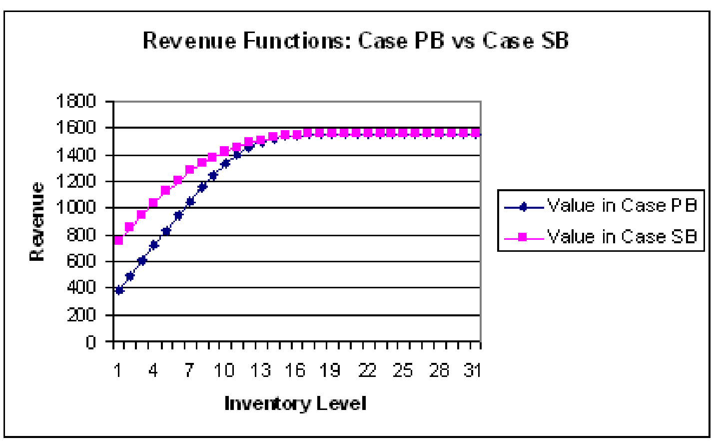

Figure 6.13: Expected Revenue: Case PB vs Case SB

Chapter 7

CONCLUSIONS

In our study we consider a firm that uses both upselling and bundling strategies. The firm sells two different types of products and a service. The staple item which is called the regular product has unlimited inventory so it is assumed to be always available when a customer demands it. The firm also has a perishable product, called the promotional product, which is available in limited quantities at the beginning of a fixed planning horizon and needs to be sold by a certain deadline. There are no replenishment opportunities for the promotional product. The firm also sells a service which has no inventory constraint and is always available like the regular product, whenever a demand comes.

The firm aims to maximize revenues by using a dynamic pricing strategy in the context of upselling and bundling. A customer who has just purchased a regular product is offered either one of the individual add-on products (promotional product or service) or a bundle which contains both the promotional product and the service. The price of the promotional product, the service and the bundle are dynamically adjusted over a finite time horizon in order to maximize the expected revenue. The customers can also purchase the promotional product /the service directly, at a fixed announced price. We modeled seven different scenarios each of which has different upselling options. The purchasing behavior of the customers is reflected by their reservation prices in the models.

Our first contribution is in modeling. The combined study of the possibility to upsell and to bundle while doing dynamic pricing has not been addressed in the literature. Each of these strategies has been studied separately or in combinations of two in the literature. Since the rapid evolution of information technologies and the corresponding growth of the internet provide the retailers to change the prices dynamically, applying the dynamic pricing strategies into the upselling and bundling applications has become more valuable. These practises are now frequently used by numerous companies.

The assumption that the service has no inventory constraint simplifies the problem and allows us to investigate structural properties of the models. A worthy but complex extension of our study could be considering multiple products with limited inventory in the upselling part. We ignored the inventory related issues such as replenishment opportunities or holding and backlogging cost. Another important research area for future studies may include replenishment opportunity for the promotional product. This extension will have a significant effect on the dynamic policies and it will be a complex task to implement. In addition, in a future research, the holding and backlogging cost should be considered. However, considering the costs seems not to change the structure of the problem significantly.

Our second contribution pertains to the structural properties of the models. We have shown that all models satisfy the monotonicity and concavity properties: The revenue function is a non-decreasing function of the initial inventory level of the promotional product and the remaining time. The revenue function is a concave function of the remaining time and the initial inventory level of the promotional product. Our first set of results are about the optimal pricing policy of the promotional product and the bundle. For a given period of time, the optimal price is a non-increasing function of the inventory and for a given inventory, the optimal price is a non-decreasing function of time. We have shown that the optimal price of the service is independent from the inventory level of the promotional product and the remaining time. We also demonstrated that if a promotional product or a bundle is offered to the customer, in all states with a higher inventory level than that state, and in all states which is closer to the end of the planning horizon, the customer will never be offered a service.

We then make some specific customer preference assumptions, which allows us to show an additional set of results. Assumptions on customer preferences allow us to compare results on optimal actions and optimal prices in some specific settings.

In our numerical analysis, we observe the effects of various factors on the pricing and upselling decisions. We analyze the effect of complementarity, having a bundle option, and product-service distinction on the optimal policies and on the expected revenue. The complementarity factor is modeled by a parameter of the Weibull distribution that we use. It is observed that the expected revenue increases with the increase of the complementarity between the promotional product and the service. The effect of complementarity is more significant for high inventory levels of the promotional product. The expected sales amount of the bundle increases with high complementarity factor, while the expected individual product sales decrease. In addition, the optimal price of the bundle and the promotional product increase when complementarity factor increases. When we change the initial inventory levels, we see that the effect of having a bundle option is more significant in higher starting inventory levels.

We have observed the effects of considering the bundle as an upselling option on different cases. For this purpose we compare the results of three couples: (Case S, Case SB), (Case P, Case PB), (Case SP, Case SPB). For all cases we observe that adding a bundle option always increases the revenue. When the starting inventory level is increased, bundling becomes more effective; especially, adding a bundle option to Case S has a significant effect on the revenue. We changed the complementarity factor and observe that adding a bundle option is more beneficial for all couples when products are more complementary.

As we mentioned before, the promotional product differs from the service with its limited inventory. This difference affects the optimal policies. To the best of our knowledge, the effect of the difference of limited supply and unlimited supply for the add-on products on the optimal policies and the expected revenue has not been examined in the literature. We compare Case S and Case P, and Case SB and Case PB to see the limited inventory effect. When the promotional product and the service are identically distributed, we observe that the impact of limited inventory decreases as we come closer to the end of the planning horizon. From the numerical study performed, we observe that although the optimal price of the service is lower than the optimal price of the promotional product, Case S (Case SB) is always more profitable than Case P (Case PB) especially in low initial inventory levels. When Case SB and Case PB are compared, it is observed that if the bundle is an option with a limited supply (Case PB), a higher price is set for the bundle. However, despite the high price, more bundle is sold in Case

PB.

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Appendix A

PROOFS FOR SECTION 3

We will give a representative example of proofs for Section 3.

Proposition 11

i*) Show that*

$$V_{n+1}(I_1+1) - V_{n+1}(I_1) \geq V_n(I_1+1) - V_n(I_1) \quad \forall\ n,\ I_1$$

ii*)The value function $V_n(I_1)$ is a concave function of n*

$$V_{n+1}(I_1) - V_n(I_1) \geq V_{n+2}(I_1) - V_{n+1}(I_1) \qquad \forall\ n,\ I_1$$

iii*)The function $V_n(I_1)$ is a concave function of the inventory.*

$$V_n(I_1+2) - V_n(I_1+1) \leq V_n(I_1+1) - V_n(I_1) \qquad \forall\ n,\ I_1$$

We will prove the following five inequalities which correspond to the above claims, simultaneously:

$$V_{n+1}(I_1) - V_n(I_1) \leq V_{n+1}(I_1+1) - V_n(I_1+1) \quad \forall\ n, I_1, \tag{A.1}$$

$$V_{n+1}(I_1) - V_n(I_1) \geq V_{n+2}(I_1) - V_{n+1}(I_1) \qquad \forall\ n, I_1, \tag{A.2}$$

$$V_n(I_1) - V_n(I_1+1) \leq V_n(I_1+1) - V_n(I_1+2) \qquad \forall\ n, I_1, \tag{A.3}$$

The proofs are done by induction in $k = n + I_1$. For $k = 0$ the inequalities are satisfied trivially since $n \geq 0$ and $I_1 \geq 0$. We assume that the inequalities $I1(n, I_1)$, $I2(n, I_1)$ and $I3(n, I_1)$ are satisfied for $n + I_1 \leq k$ and we will prove that they hold for $n + I_1 = k$.

A.1 Proofs for Section 3.1.2

i) First of all we will show that (A.1) holds for $n + I_1 = k$.

$$V_{n+1}(I_1) - V_n(I_1) \leq V_{n+1}(I_1 + 1) - V_n(I_1 + 1) \quad \forall\ n,\ I_1$$

Let $p_A = p_1^{n+1,I_1}$ be the optimal price for $V_{n+1}(I_1)$.

Since the optimal price for $V_{n+1}(I_1)$ is p_A, it can be written as:

$$\begin{aligned} V_{n+1}(I_1) - V_n(I_1) &= \lambda_0(1 - F_1(p_A))(p_A + V_n(I_1 - 1) - V_n(I_1))\} \\ &+ \lambda_1(r_1 + V_n(I_1 - 1) - V_n(I_1)) + \lambda_2 r_2 \ . \end{aligned} \tag{A.4}$$

If p_A is used for $V_{n+1}(I_1 + 1)$ instead of its optimal action, it can be written as:

$$\begin{aligned} V_{n+1}(I_1 + 1) - V_n(I_1 + 1) &\geq \lambda_0((1 - F_1(p_A))(p_A + V_n(I_1) - V_n(I_1 + 1))) \\ &+ \lambda_1(r_1 + V_n(I_1) - V_n(I_1 + 1)) + \lambda_2 r_2 \end{aligned} \tag{A.5}$$

We define:

$$\Delta = V_{n+1}(I_1) - V_n(I_1) - V_{n+1}(I_1 + 1) + V_n(I_1 + 1).$$

By using (A.4) and (A.5) we have:

$$\begin{aligned}
\Delta &\leq \lambda_0(1-F_1(p_A))(p_A+V_n(I_1-1)-V_n(I_1))+\lambda_1(r_1+V_n(I_1-1)-V_n(I_1))+\lambda_2 r_2 \\
&- \lambda_0(1-F_1(p_A))(p_A+V_n(I_1)-V_n(I_1+1))-\lambda_1(r_1+V_n(I_1)-V_n(I_1+1))-\lambda_2 r_2 \\
&= \lambda_0(1-F_1(p_A))(V_n(I_1-1)-V_n(I_1)-V_n(I_1)+V_n(I_1+1)) \\
&+ \lambda_1(V_n(I_1-1)-V_n(I_1)-V_n(I_1)+V_n(I_1+1)). \qquad \text{(A.6)}
\end{aligned}$$

We know that the inequality (A.3) holds for $I_1 - 1$, so that,

$$V_n(I_1)-V_n(I_1+1) \geq V_n(I_1-1)-V_n(I_1)\ . \qquad \text{(A.7)}$$

Using (A.7) in (A.6) leads to:

$$V_{n+1}(I_1)-V_n(I_1)-V_{n+1}(I_1+1)+V_n(I_1+1) \quad \leq \quad 0\ .$$

ii) Now, we will show that (A.2) holds for $n + I_1 = k$

$$V_{n+1}(I_1)-V_n(I_1) \geq V_{n+2}(I_1)-V_{n+1}(I_1) \qquad \forall\ I_1$$

Let $P_A = p_1^{n+1,I_1}$ be the optimal price for $V_{n+1}(I_1)$.

Since the optimal price for $V_{n+2}(I_1)$ is p_B, it can be written as:

$$\begin{aligned}
V_{n+2}(I_1)-V_{n+1}(I_1) &= \lambda_0(1-F_1(p_B))(p_B+V_{n+1}(I_1-1)-V_{n+1}(I_1)) \\
&+ \lambda_1(r_1+V_{n+1}(I_1-1)-V_{n+1}(I_1))+\lambda_2 r_2 \qquad \text{(A.8)}
\end{aligned}$$

If p_B is used for $V_{n+1}(I_1)$ instead of its optimal action, it can be written as:

$$\begin{aligned}
V_{n+1}(I_1)-V_n(I_1) &\geq \lambda_0(1-F_1(p_B))(p_B+V_n(I_1-1)-V_n(I_1)) \\
&+ \lambda_1(r_1+V_n(I_1-1)-V_n(I_1))+\lambda_2 r_2\ . \qquad \text{(A.9)}
\end{aligned}$$

We define:

$$\Delta = V_{n+2}(I_1) - V_{n+1}(I_1) - V_{n+1}(I_1) + V_n(I_1)$$

By using (A.8) and (A.9) we obtain:

$$\begin{aligned}
\Delta &\leq \lambda_0(1 - F_1(p_B))(p_B + V_{n+1}(I_1 - 1) - V_{n+1}(I_1)) \\
&+ \lambda_1(r_1 + V_{n+1}(I_1 - 1) - V_{n+1}(I_1)) + \lambda_2 r_2 \\
&- \lambda_0(1 - F_1(p_B))(p_B + V_n(I_1 - 1) - V_n(I_1)) \\
&- \lambda_1(r_1 + V_n(I_1 - 1) - V_n(I_1)) - \lambda_2 r_2 \\
&= \lambda_0(1 - F_1(p_B))(V_{n+1}(I_1 - 1) - V_{n+1}(I_1) - V_n(I_1 - 1) + V_n(I_1)) \\
&+ \lambda_1(V_{n+1}(I_1 - 1) - V_{n+1}(I_1) - V_n(I_1 - 1) + V_n(I_1)) \qquad \text{(A.10)}
\end{aligned}$$

Using inequality (A.1) for $I_1 - 1$ leads to:

$$V_n(I_1 - 1) - V_n(I_1) \geq V_{n+1}(I_1 - 1) - V_{n+1}(I_1) \ . \qquad \text{(A.11)}$$

Using (A.11) in (A.10) leads to:

$$V_{n+2}(I_1) - V_{n+1}(I_1) - V_{n+1}(I_1) + V_n(I_1) \leq 0 \ .$$

iii) Finally, we will show that (A.3) holds for $n + I_1 = k$

$$V_n(I_1 + 2) - V_n(I_1 + 1) \leq V_n(I_1 + 1) - V_n(I_1) \qquad \forall \text{ n,} I_1$$

Let p_C be the optimal price for $V_n(I_1 + 2)$.

Since the optimal price for $V_n(I_1+2)$ is p_C, it can be written as:

$$\begin{aligned} V_n(I_1+2) &= V_{n-1}(I_1+2)+\lambda_0(1-F_1(p_C))(p_C+V_{n-1}(I_1+1)-V_{n-1}(I_1+2)) \\ &+ \lambda_1(r_1+V_{n-1}(I_1+1)-V_{n-1}(I_1+2))+\lambda_2 r_2 \end{aligned}$$

Subtracting $V_n(I_1+1)$ from both sides yields,

$$\begin{aligned} V_n(I_1+2)-V_n(I_1+1) &= V_{n-1}(I_1+2)-V_n(I_1+1) \\ &+ \lambda_0(1-F_1(p_C))(p_C+V_{n-1}(I_1+1)-V_{n-1}(I_1+2)) \\ &+ \lambda_1(r_1+V_{n-1}(I_1+1)-V_{n-1}(I_1+2))+\lambda_2 r_2 \end{aligned} \tag{A.12}$$

If p_C is used for $V_n(I_1+1)$ instead of its optimal action,it can be written as:

$$\begin{aligned} V_n(I_1+1) &\geq V_{n-1}(I_1+1) \\ &+ \lambda_0(1-F_1(p_C))(p_C+V_{n-1}(I_1)-V_{n-1}(I_1+1)) \\ &+ \lambda_1(r_1+V_{n-1}(I_1)-V_{n-1}(I_1+1))+\lambda_2 r_2\ . \end{aligned}$$

Subtracting $V_n(I_1)$ from both sides yields,

$$\begin{aligned} V_n(I_1+1)-V_n(I_1) &\geq V_{n-1}(I_1+1)-V_n(I_1) \\ &+ \lambda_0(1-F_1(p_C))(p_C+V_{n-1}(I_1)-V_{n-1}(I_1+1)) \\ &+ \lambda_1(r_1+V_{n-1}(I_1)-V_{n-1}(I_1+1))+\lambda_2 r_2\ . \end{aligned} \tag{A.13}$$

We define:

$$\Delta = V_n(I_1+2)-V_n(I_1+1)-V_n(I_1+1)+V_n(I_1)\ .$$

By using (A.12) and (A.13) we obtain:

$$\begin{aligned}
\Delta &\leq V_{n-1}(I_1+2) - V_n(I_1+1) \\
&+ \lambda_0(1-F_1(p_C))(p_C + V_{n-1}(I_1+1) - V_{n-1}(I_1+2)) \\
&+ \lambda_1(r_1 + V_{n-1}(I_1+1) - V_{n-1}(I_1+2)) + \lambda_2 r_2 \\
&- V_{n-1}(I_1+1) + V_n(I_1) - \lambda_0(1-F_1(p_C))(p_C + V_{n-1}(I_1) - V_{n-1}(I_1+1)) \\
&- \lambda_1(r_1 + V_{n-1}(I_1) - V_{n-1}(I_1+1)) - \lambda_2 r_2 \\
&= V_{n-1}(I_1+2) - V_n(I_1+1) - V_{n-1}(I_1+1) + V_n(I_1) \\
&+ \lambda_0(1-F_1(p_C))(V_{n-1}(I_1+1) - V_{n-1}(I_1+2) - V_{n-1}(I_1) + V_{n-1}(I_1+1)) \\
&+ \lambda_1(V_{n-1}(I_1+1) - V_{n-1}(I_1+2) - V_{n-1}(I_1) + V_{n-1}(I_1+1)) \ .
\end{aligned}$$

Since $1 - F_1(p_C) \leq 1$, the above inequality can be written as:

$$\begin{aligned}
\Delta &\leq V_{n-1}(I_1+2) - V_n(I_1+1) - V_{n-1}(I_1+1) + V_n(I_1) \\
&+ (\lambda_0+\lambda_1)(V_{n-1}(I_1+1) - V_{n-1}(I_1+2) - V_{n-1}(I_1) + V_{n-1}(I_1+1)) \ .
\end{aligned}$$

We can modify above inequality:

$$\begin{aligned}
\Delta &= (1-\lambda_0-\lambda_1)(V_{n-1}(I_1+2) - V_n(I_1+1) - V_{n-1}(I_1+1) + V_n(I_1)) \\
&+ (\lambda_0+\lambda_1)(V_{n-1}(I_1+1) - V_{n-1}(I_1+2) - V_{n-1}(I_1) + V_{n-1}(I_1+1) \\
&\qquad + V_{n-1}(I_1+2) - V_n(I_1+1) - V_{n-1}(I_1+1) + V_n(I_1)) \\
&= (1-\lambda_0-\lambda_1)(V_{n-1}(I_1+2) - V_n(I_1+1) - V_{n-1}(I_1+1) + V_n(I_1)) \\
&+ (\lambda_0+\lambda_1)(V_{n-1}(I_1+1) - V_{n-1}(I_1) - V_n(I_1+1) + V_n(I_1))
\end{aligned} \tag{A.14}$$

Since inequality (A.1) holds for $n-1$,

$$V_{n-1}(I_1+1) - V_{n-1}(I_1) - V_n(I_1+1) + V_n(I_1) \leq 0 \tag{A.15}$$

and

$$V_{n-1}(I_1+2) - V_{n-1}(I_1+1) - V_n(I_1+2) + V_n(I_1+1) \leq 0$$

and by hypothesis

$$V_n(I_1+2) - V_n(I_1+1) - V_n(I_1+1) + V_n(I_1) \leq 0$$

Therefore,

$$V_{n-1}(I_1+2) - V_{n-1}(I_1+1) - V_n(I_1+1) + V_n(I_1) \leq 0 \tag{A.16}$$

By using the inequalities (A.15) and (A.16) in (A.14) yields:

$$\Delta \leq 0$$

which completes the proof.

A.2 Proofs for Section 3.2.1

i) First of all we will show that (A.1) holds for $n + I_1 = k$.

$$V_{n+1}(I_1) - V_n(I_1) \leq V_{n+1}(I_1+1) - V_n(I_1+1) \quad \forall\ n,\ I_1$$

Let A be the optimal action and p_A be the optimal price for $V_{n+1}(I_1)$: $a^{n+1,I_1} = A$, $A \in \{1,2\}$, $p_A = p_j^{n+1,I_1}$.

Case 1: $A = 1$,

Since the optimal action for $V_{n+1}(I_1)$ is 1 and optimal price for $V_{n+1}(I_1)$ is p_A, it can be written as:

$$\begin{aligned} V_{n+1}(I_1) - V_n(I_1) &= \lambda_0(1 - F_1(p_A))(p_A + V_n(I_1 - 1) - V_n(I_1)) \\ &+ \lambda_1(r_1 + V_n(I_1 - 1) - V_n(I_1)) + \lambda_2 r_2 \ . \end{aligned} \tag{A.17}$$

Similarly the value function $V_{n+1}(I_1 + 1)$ can be written as:

$$\begin{aligned} V_{n+1}(I_1 + 1) - V_n(I_1 + 1) &\geq \lambda_0 \max_{p_1}\{(1 - F_1(p_1))(p_j + V_n(I_1) - V_n(I_1 + 1))\} \\ &+ \lambda_1(r_1 + V_n(I_1) - V_n(I_1 + 1)) + \lambda_2 r_2 \end{aligned}$$

If p_A is used for $V_{n+1}(I_1 + 1)$ instead of its optimal action, it can be written as:

$$\begin{aligned} V_{n+1}(I_1 + 1) - V_n(I_1 + 1) &\geq \lambda_0((1 - F_1(p_A))(p_A + V_n(I_1) - V_n(I_1 + 1))) \\ &+ \lambda_1(r_1 + V_n(I_1) - V_n(I_1 + 1)) + \lambda_2 r_2 \end{aligned} \tag{A.18}$$

We define:

$$\Delta = V_{n+1}(I_1) - V_n(I_1) - V_{n+1}(I_1 + 1) + V_n(I_1 + 1).$$

By using (A.17) and (A.18) we have:

$$\begin{aligned} \Delta &\leq \lambda_0(1 - F_1(p_A))(p_A + V_n(I_1 - 1) - V_n(I_1)) + \lambda_1(r_1 + V_n(I_1 - 1) - V_n(I_1)) + \lambda_2 r_2 \\ &- \lambda_0(1 - F_1(p_A))(p_A + V_n(I_1) - V_n(I_1 + 1)) - \lambda_1(r_1 + V_n(I_1) - V_n(I_1 + 1)) - \lambda_2 r_2 \\ &= \lambda_0(1 - F_1(p_A))(V_n(I_1 - 1) - V_n(I_1) - V_n(I_1) + V_n(I_1 + 1)) \\ &+ \lambda_1(V_n(I_1 - 1) - V_n(I_1) - V_n(I_1) + V_n(I_1 + 1)). \end{aligned} \tag{A.19}$$

We know that the inequality (A.3) holds for $I_1 - 1$, so that,

$$V_n(I_1) - V_n(I_1 + 1) \geq V_n(I_1 - 1) - V_n(I_1) \ . \tag{A.20}$$

Using (A.20) in (A.19) leads to:

$$V_{n+1}(I_1) - V_n(I_1) - V_{n+1}(I_1+1) + V_n(I_1+1) \quad \leq \quad 0 \ .$$

Case 2: $A = 2$,

Since the optimal action for $V_{n+1}(I_1)$ is 2, it can be written as:

$$V_{n+1}(I_1) - V_n(I_1) = \lambda_0 \max_{p_2}\{(1 - F_2(p_2))p_2\} + \lambda_1(r_1 + V_n(I_1 - 1) - V_n(I_1)) + \lambda_2 r_2 \ . \tag{A.21}$$

The following inequality holds for $V_{n+1}(I_1 + 1)$:

$$\begin{aligned} V_{n+1}(I_1+1) - V_n(I_1+1) \quad &\geq \quad \lambda_0 \max_{p_2}\{(1 - F_2(p_2))p_2\} \\ &+ \quad \lambda_1(r_1 + V_n(I_1) - V_n(I_1+1)) + \lambda_2 r_2 \ . \end{aligned} \tag{A.22}$$

By using (A.21) and (A.22) we obtain:

$$\begin{aligned} V_{n+1}(I_1) - V_n(I_1) - V_{n+1}(I_1+1) + V_n(I_1+1) \quad &\leq \quad \lambda_0 \max_{p_2}\{(1 - F_2(p_2))p_2\} \\ &+ \quad \lambda_1(r_1 + V_n(I_1 - 1) - V_n(I_1)) + \lambda_2 r_2 \\ &- \quad \lambda_0 \max_{p_2}\{(1 - F_2(p_2))p_2\} \\ &- \quad \lambda_1(r_1 + V_n(I_1) - V_n(I_1+1)) - \lambda_2 r_2 \end{aligned} \tag{A.23}$$

Inequality (A.3) holds for $I_1 - 1$, so we have the following inequality:

$$V_n(I_1) - V_n(I_1+1) \geq V_n(I_1 - 1) - V_n(I_1),$$

which proves the statement along with (A.23):

$$V_{n+1}(I_1) - V_n(I_1) - V_{n+1}(I_1+1) + V_n(I_1+1) \quad \leq \quad 0 \ .$$

ii) Now, we will show that inequality (A.2) holds for $n + I_1 = k$

$$V_{n+1}(I_1) - V_n(I_1) \geq V_{n+2}(I_1) - V_{n+1}(I_1) \qquad \forall\ I_1$$

Let B be the optimal action and p_B be the optimal price for $V_{n+2}(I_1) : a^{n+2,I_1} = B$, $B \in \{1,2\}$, $p_B = p_j^{n+2,I_1}$.

Case 1: $B = 1$,

Since the optimal action for $V_{n+2}(I_1)$ is 1 and optimal price for $V_{n+2}(I_1)$ is p_B , it can be written as:

$$\begin{aligned} V_{n+2}(I_1) - V_{n+1}(I_1) &= \lambda_0(1 - F_1(p_B))(p_B + V_{n+1}(I_1 - 1) - V_{n+1}(I_1)) \\ &+ \lambda_1(r_1 + V_{n+1}(I_1 - 1) - V_{n+1}(I_1)) + \lambda_2 r_2 \end{aligned} \tag{A.24}$$

Similarly the value function $V_{n+1}(I_1)$ can be written as:

$$\begin{aligned} V_{n+1}(I_1) - V_n(I_1) &\geq \lambda_0 \max_{p_1}\{(1 - F_1(p_1))(p_j + V_n(I_1 - 1) - V_n(I_1))\} \\ &+ \lambda_1(r_1 + V_n(I_1 - 1) - V_n(I_1)) + \lambda_2 r_2\ . \end{aligned}$$

If p_B is used for $V_{n+1}(I_1)$ instead of its optimal action, it can be written as:

$$\begin{aligned} V_{n+1}(I_1) - V_n(I_1) &\geq \lambda_0(1 - F_1(p_B))(p_B + V_n(I_1 - 1) - V_n(I_1)) \\ &+ \lambda_1(r_1 + V_n(I_1 - 1) - V_n(I_1)) + \lambda_2 r_2\ . \end{aligned} \tag{A.25}$$

We define:

$$\Delta = V_{n+2}(I_1) - V_{n+1}(I_1) - V_{n+1}(I_1) + V_n(I_1)$$

By using (A.24) and (A.25) we obtain:

$$\begin{aligned}
\Delta &\leq \lambda_0(1-F_1(p_B))(p_B+V_{n+1}(I_1-1)-V_{n+1}(I_1)) \\
&+ \lambda_1(r_1+V_{n+1}(I_1-1)-V_{n+1}(I_1))+\lambda_2 r_2 \\
&- \lambda_0(1-F_1(p_B))(p_B+V_n(I_1-1)-V_n(I_1)) \\
&- \lambda_1(r_1+V_n(I_1-1)-V_n(I_1))-\lambda_2 r_2 \\
&= \lambda_0(1-F_1(p_B))(V_{n+1}(I_1-1)-V_{n+1}(I_1)-V_n(I_1-1)+V_n(I_1)) \\
&+ \lambda_1(V_{n+1}(I_1-1)-V_{n+1}(I_1)-V_n(I_1-1)+V_n(I_1))
\end{aligned} \tag{A.26}$$

Using inequality (A.1) for I_1-1 leads to:

$$V_n(I_1-1)-V_n(I_1) \geq V_{n+1}(I_1-1)-V_{n+1}(I_1)\ . \tag{A.27}$$

Using (A.27) in (A.26) leads to:

$$V_{n+2}(I_1)-V_{n+1}(I_1)-V_{n+1}(I_1)+V_n(I_1) \leq 0\ .$$

Case 2: $B=2$,

Since the optimal action for $V_{n+1}(I_1)$ is 2, it can be written as:

$$\begin{aligned}
V_{n+2}(I_1)-V_{n+1}(I_1) &= \lambda_0 \max_{p_2}\{(1-F_2(p_2))p_2\} \\
&+ \lambda_1(r_1+V_{n+1}(I_1-1)-V_{n+1}(I_1))+\lambda_2 r_2
\end{aligned} \tag{A.28}$$

The following inequality holds for $V_{n+1}(I_1)$:

$$\begin{aligned}
V_{n+1}(I_1)-V_n(I_1) &\geq \lambda_0 \max_{p_2}\{(1-F_2(p_2))p_2 \\
&+ \lambda_1(r_1+V_n(I_1-1)-V_n(I_1))+\lambda_2 r_2
\end{aligned} \tag{A.29}$$

By using (A.28) and (A.29) we have,

$$\begin{aligned}
V_{n+2}(I_1) - V_{n+1}(I_1) - V_{n+1}(I_1) + V_n(I_1) &\leq \lambda_0 \max_{p_2}\{(1 - F_2(p_2))p_2 \\
&+ \lambda_1(r_1 + V_{n+1}(I_1 - 1) - V_{n+1}(I_1)) + \lambda_2 r_2 \\
&- \lambda_0 \max_{p_k}\{(1 - F_2(p_2))p_2 \\
&- \lambda_1(r_1 + V_n(I_1 - 1) - V_n(I_1)) - \lambda_2 r_2
\end{aligned} \tag{A.30}$$

Inequality (A.1) holds for $I_1 - 1$. Therefore:

$$V_n(I_1 - 1) - V_n(I_1) \geq V_{n+1}(I_1 - 1) - V_{n+1}(I_1)\ . \tag{A.31}$$

Inequalities (A.30) and (A.31) lead to,

$$V_{n+2}(I_1) - V_{n+1}(I_1) - V_{n+1}(I_1) + V_n(I_1) \leq 0$$

iii) Finally, we will show that inequality (A.3) holds for $n + I_1 = k$

$$V_n(I_1 + 2) - V_n(I_1 + 1) \leq V_n(I_1 + 1) - V_n(I_1) \qquad \forall\ n,\ I_1$$

Let C be the optimal action and p_C be optimal price for $V_n(I_1 + 2)$: $a^{n,I_1+2} = C$, $C \in \{1, 2\}$, $p_C = p_j^{n,I_1+2}$.

Case 1: $C = 1$

Since the optimal action for $V_n(I_1 + 2)$ is 2 and optimal action for $V_n(I_1 + 2)$ is p_C, it can be written as:

$$\begin{aligned}
V_n(I_1 + 2) &= V_{n-1}(I_1 + 2) + \lambda_0(1 - F_1(p_C))(p_C + V_{n-1}(I_1 + 1) - V_{n-1}(I_1 + 2))\} \\
&+ \lambda_1(r_1 + V_{n-1}(I_1 + 1) - V_{n-1}(I_1 + 2)) + \lambda_2 r_2
\end{aligned}$$

Subtracting $V_n(I_1+1)$ from both sides yields,

$$\begin{aligned} V_n(I_1+2) - V_n(I_1+1) &= V_{n-1}(I_1+2) - V_n(I_1+1) \\ &+ \lambda_0(1-F_1(p_C))(p_C + V_{n-1}(I_1+1) - V_{n-1}(I_1+2))\} \\ &+ \lambda_1(r_1 + V_{n-1}(I_1+1) - V_{n-1}(I_1+2)) + \lambda_2 r_2 \end{aligned} \tag{A.32}$$

Similarly the value function $V_{n+1}(I_1)$ can be written as:

$$\begin{aligned} V_n(I_1+1) &\geq V_{n-1}(I_1+1) + \lambda_0 \max_{p_1}\{(1-F_1(p_1))(p_1 + V_{n-1}(I_1) - V_{n-1}(I_1+1))\} \\ &+ \lambda_1(r_1 + V_{n-1}(I_1) - V_{n-1}(I_1+1)) + \lambda_2 r_2 \ . \end{aligned}$$

Subtracting $V_n(I_1)$ from both sides yields,

$$\begin{aligned} V_n(I_1+1) - V_n(I_1) &\geq V_{n-1}(I_1+1) - V_n(I_1) \\ &+ \lambda_0 \max_{p_1}\{(1-F_1(p_1))(p_1 + V_{n-1}(I_1) - V_{n-1}(I_1+1))\} \\ &+ \lambda_1(r_1 + V_{n-1}(I_1) - V_{n-1}(I_1+1)) + \lambda_2 r_2 \ . \end{aligned}$$

If p_C is used for $V_n(I_1+1)$ instead of its optimal action,it can be written as:

$$\begin{aligned} V_n(I_1+1) - V_n(I_1) &\geq V_{n-1}(I_1+1) - V_n(I_1) \\ &+ \lambda_0(1-F_1(p_C))(p_C + V_{n-1}(I_1) - V_{n-1}(I_1+1)) \\ &+ \lambda_1(r_1 + V_{n-1}(I_1) - V_{n-1}(I_1+1)) + \lambda_2 r_2 \ . \end{aligned} \tag{A.33}$$

We define:

$$\Delta = V_n(I_1+2) - V_n(I_1+1) - V_n(I_1+1) + V_n(I_1) \ .$$

By using (A.32) and (A.33) we obtain:

$$
\begin{aligned}
\Delta &\leq V_{n-1}(I_1+2) - V_n(I_1+1) \\
&+ \lambda_0(1-F_1(p_C))(p_C + V_{n-1}(I_1+1) - V_{n-1}(I_1+2)) \\
&+ \lambda_1(r_1 + V_{n-1}(I_1+1) - V_{n-1}(I_1+2)) + \lambda_2 r_2 \\
&- V_{n-1}(I_1+1) + V_n(I_1) - \lambda_0(1-F_j(p_C))(p_C + V_{n-1}(I_1) - V_{n-1}(I_1+1)) \\
&- \lambda_1(r_1 + V_{n-1}(I_1) - V_{n-1}(I_1+1)) - \lambda_2 r_2 \\
&= V_{n-1}(I_1+2) - V_n(I_1+1) - V_{n-1}(I_1+1) + V_n(I_1) \\
&+ \lambda_0(1-F_1(p_C))(V_{n-1}(I_1+1) - V_{n-1}(I_1+2) - V_{n-1}(I_1) + V_{n-1}(I_1+1)) \\
&+ \lambda_1(V_{n-1}(I_1+1) - V_{n-1}(I_1+2) - V_{n-1}(I_1) + V_{n-1}(I_1+1)) \ .
\end{aligned}
$$

Since $1 - F_1(p_C) \leq 1$, the above inequality can be written as:

$$
\begin{aligned}
\Delta &\leq V_{n-1}(I_1+2) - V_n(I_1+1) - V_{n-1}(I_1+1) + V_n(I_1) \\
&+ (\lambda_0+\lambda_1)(V_{n-1}(I_1+1) - V_{n-1}(I_1+2) - V_{n-1}(I_1) + V_{n-1}(I_1+1)) \ .
\end{aligned}
$$

We can modify above inequality:

$$
\begin{aligned}
\Delta &= (1-\lambda_0-\lambda_1)(V_{n-1}(I_1+2) - V_n(I_1+1) - V_{n-1}(I_1+1) + V_n(I_1)) \\
&+ (\lambda_0+\lambda_1)(V_{n-1}(I_1+1) - V_{n-1}(I_1+2) - V_{n-1}(I_1) + V_{n-1}(I_1+1) \\
&\qquad + V_{n-1}(I_1+2) - V_n(I_1+1) - V_{n-1}(I_1+1) + V_n(I_1)) \\
&= (1-\lambda_0-\lambda_1)(V_{n-1}(I_1+2) - V_n(I_1+1) - V_{n-1}(I_1+1) + V_n(I_1)) \\
&+ (\lambda_0+\lambda_1)(V_{n-1}(I_1+1) - V_{n-1}(I_1) - V_n(I_1+1) + V_n(I_1))
\end{aligned} \tag{A.34}
$$

Since inequality (A.1) holds for $n-1$,

$$
V_{n-1}(I_1+1) - V_{n-1}(I_1) - V_n(I_1+1) + V_n(I_1) \leq 0 \tag{A.35}
$$

and

$$V_{n-1}(I_1+2) - V_{n-1}(I_1+1) - V_n(I_1+2) + V_n(I_1+1) \leq 0$$

and by hypothesis

$$V_n(I_1+2) - V_n(I_1+1) - V_n(I_1+1) + V_n(I_1) \leq 0$$

Therefore,

$$V_{n-1}(I_1+2) - V_{n-1}(I_1+1) - V_n(I_1+1) + V_n(I_1) \leq 0 \tag{A.36}$$

By using the inequalities (A.35) and (A.36) in (A.34) yields:

$$\Delta \leq 0$$

which completes the proof.

VITA

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