ALGEBRAIC (ÉTALE) FUNDAMENTAL GROUP

by

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A Thesis Submitted to the
Graduate School of Sciences & Engineering
in Partial Fulfillment of the Requirements for
the Degree of

Master of Science

in

Mathematics

Koç University

July, 2008

Koç University

Graduate School of Sciences and Engineering

This is to certify that I have examined this copy of a master's thesis by

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and have found that it is complete and satisfactory in all respects,
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To my family...

ABSTRACT

This thesis is devoted to a presentation of the étale algebraic fundamental group. In the first chapters, we give the prerequisites for the theory of schemes and the basic properties of schemes and morphisms to study the galois theory for schemes which classify the finite étale coverings of a connected scheme X in terms of the fundamental group, $\pi_1^{alg}(X, \cdot)$ of X, precisely in the same way as the finite coverings of a connected topological space. To emphasize the analogy, we remind the construction of the topological fundamental group from different aspects.

Also we state different characterizations of étaleness. In order to have a broader sense of the theory, we give an axiomatic characterization of the categories we are interested in: galois categories, which are equivalent to the categories of finite sets which a profinite group acts continuously.

Finally, we construct the algebraic étale fundamental group and illustrate the concept with some examples with a particular emphasis on the Riemann existence theorem.

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ÖZ

Bu tez çalışmasının esas konusu *é*tale yapı dönüşümleriyle tanımlanan cebirsel temel grupdur. İlk bölümlerde şema kavramına, şemaların temel özelliklerine ve aralarındaki yapı dönüşümlerinin özelliklerine yer verilmiştir. Bilindiği üzere, topolojik temel grup; $\pi_1^{top}(X,.)$, bağlı bir X uzayının örtü uzayları yardımıyla da tanımlanabilmektedir, şemalar için galois teorisi de bu gerçekten yola çıkarak cebirsel temel grubu; $\pi_1^{alg}(X,.)$ sonlu *é*tale örtü uzaylar sayesinde tanımlar. Bu analojiyi kurmak için, topolojik temel grup kavramına farklı açılardan bakılmıştır. Ayrıca, *é*tale yapı dönüşümlerinin karakteristik özellikleri işlenmiştir. Cebirsel temel grup kavramına geniş bir perspektif'den bakabilmek için galois kategori belitlerine ve örneklerine yer verilmiştir. Bu kategoriler sonlu grupların ters limiti şeklindeki bir grubun sürekli etki ettiği sonlu kümeler kategorisine denktir. Son olarak, cebirsel temel grup oluşturulup, Riemann varlık teoremi ışığında konu örneklerle aydınlatılmıştır.

ACKNOWLEDGMENTS

I would like to express my sincere gratitude to my thesis supervisor Sinan Ünver for his invaluable contributions and guidance. Trying to follow his way of understanding mathematics, I realized that a wonderful world behind those ideas lies. I thank him with all my heart for the encouragement, support and endless patience he showed me. I would like to thank to Burak Özbağcı and Alper Tunga Erdoğan for their participation in my thesis committee.

I consider myself fortunate to have had marvelous friends; Saliha Pehlivan, Şeyda Özdemir, Betül Özgür, Şirin Coşkun, Hatice Ekşi, Reyhane Kılcı and Zeynep Akçay. Without the encouragement and motivation they have given to me during the past two years, it would not be possible to complete this thesis.

I am thankful to all my officemates for a friendly atmosphere especially to Taylan Bilal and Barş Tümerkan whose presence and ideas were a great source of me.

Also, I would like to thank to the scientific and technological research council in Turkey (TÜBİTAK) for the support given to me during the past two years.

Finally and most importantly, I am grateful to my parents; Adalet and Mustafa and my angels; my sister Buşra and my brother Samet Gökhan for their endless support and confidence throughout my education.

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$\mathbb{R}$	:	real numbers
$\mathbb{C}$	:	complex numbers
$\mathbb{Z}_{>\mathbf{0}}$	:	positive integers
$\mathbf{S^{-1}A}$	:	localization by a multiplicative system
$\mathbf{A_p}$	:	localization by a prime ideal
$\mathbf{K}$	:	a field
$\mathbf{A_f}$	:	localization by an element
$\mathbf{A_k^n}$	:	affine n-space over k
$\mathbf{P_k^1}$	:	projective line over k
$\mathbf{K}[\mathbf{x_1}, ..., \mathbf{x_n}]$	:	polynomial ring
$\mathbf{Aut}$	:	group of automorphisms
$\mathbf{Aut}(\mathbf{K}/\mathbf{F})$	:	the group of automorphisms of a field K fixing the field F
$\sqrt{\mathbf{I}}$	:	radical of an ideal
$\mathbf{SpecA}$	:	spectrum of a ring
$\mathbf{ker}$	:	kernel
$\mathbf{im}$	:	image
$\underrightarrow{lim}\mathbf{A_i}$	:	the direct limit of the family of groups A_i
$\underleftarrow{lim}\mathbf{A_i}$	:	the inverse limit of the family of groups A_i
$(\mathbf{X}, \mathcal{O}_\mathbf{X})$	:	scheme

$\mathcal{O}_{\mathbf{X,p}}$	: the stalk of the structure sheaf at p
$\mathcal{O}_{\mathbf{X}}(\mathbf{U})$	: the ring of sections on an open set U in X
$\mathcal{F}_{\mathbf{Y}}$	: ideal sheaf of a closed subscheme
$\mathbf{Tr}(\mathbf{A})$	: the trace of the matrix A
$\mathbf{Hom}_{\mathbf{R}}(\mathbf{A},\mathbf{B})$	: the R-module homomorphisms from A to B
$\mathbf{Alg}_{\mathbf{K}}(\mathbf{A},\mathbf{B})$	: the set of ring homomorphisms that are identity on K
$\mathbf{M}\otimes_{\mathbf{R}}\mathbf{N}$	: the tensor product of modules M and N over R
$\mathbf{V}(\mathbf{I})$	: closed subset associated to an ideal I
$\mathbf{D}(\mathbf{f})$	: open set defined by f
$\mathbf{K}_{\mathbf{s}}$	: separable closure of a field
$\mathbf{X}\times_{\mathbf{S}}\mathbf{Y}$	: fiber product of schemes
$\mathbf{k}(\mathbf{p})$	: residue field at p
$\mathbf{\Omega}_{\mathbf{B/A}}$	: module of relative differential forms
$\mathbf{\Omega}_{\mathbf{X/Y}}$	: sheaf of relative differentials
$\pi_{\mathbf{1}}^{\mathbf{top}}(\mathbf{X},\mathbf{x})$	: topological fundamental group
$\pi_{\mathbf{1}}^{\mathbf{alg}}(\mathbf{X},\mathbf{x})$	: algebraic fundamental group
$\mathbf{Aut}(\mathcal{F})$	: Automorphism group of the functor
$_{\mathbf{k}}\mathbf{SAlg}$	: the category of free separable k-algebras
$\mathbf{FEt}_{\mathbf{X}}$	: the category of finite etale coverings of X
$\mathbf{Cov}(\mathbf{X})$	: the category of unramified coverings of X
$\mathbf{Cov}_{\mathbf{finite}}(\mathbf{X})$	: the category of finite unramified coverings of X

$\mathbf{Deck}(\widehat{\mathbf{X}}/\mathbf{X})$	:	the deck transformation of the covering space $\widehat{X}$
$\mathbf{G} * \mathbf{H}$	:	free product of G and H
$\mathbf{G} *_{\mathbf{K}} \mathbf{H}$	:	the free amalgamated product of G and H over K
$\mathbf{f}_*\mathcal{F}$	:	pushforward operation on presheaf $\mathcal{F}$

Chapter 1

PRELIMINARIES

We begin with some definitions from commutative algebra and at the end of this chapter we state the basic terms from category theory.

1.1 Tensor Products of Modules

Definition 1.1.1 Let R be a ring and let M and N be R-modules. Let F be the free R- module $R^{M\times N}$. The elements of F are formal linear combinations of elements of $M \times N$ with coefficients in R, such that

$$F = \{\sum r_i(m_i n_i); r_i \in R, m_i \in M, n_i \in N\}$$

Let D be the submodule of C generated by all elements of F of the following types;

$$(x + x^{'}, y) - (x, y) - (x^{'}, y) \text{ and } (x, y + y^{'}) - (x, y) - (x, y^{'})$$

$$(ax, y) - a \cdot (x, y) \text{ and } (x, ay) - a \cdot (x, y)$$

Let $T = F/D$ and let $x \otimes y$ denote the image of (x, y) in T. By the definition of the quotient, we obtain the relations

$$(x + x^{'}) \otimes y = (x \otimes y) + (x^{'} \otimes y)$$

$$x \otimes (y + y^{'}) = (x \otimes y) + (x \otimes y^{'})$$

$$(ax) \otimes y = x \otimes (ay) = a(x \otimes y)$$

The resulting quotient module T is denoted by $M \otimes_R N$ and is called the tensor product of M and N over R.

Definition 1.1.2 Let A and B are modules over a ring R, and C be an abelian group. We call a function $f : A \times B \to C$ a *middle linear map* if it satisfies the following properties;

$$\begin{aligned} f(a_1 + a_2, b) &= f(a_1, b) + f(a_2, b) \\ f(a, b_1 + b_2) &= f(a, b_1) + f(a, b_2) \\ f(ar, b) &= f(a, rb) \end{aligned}$$

for all $a, a_i \in A$, $b, b_i \in B$ and $r \in R$.

The map $i : A \times B \to A \otimes_R B$ given by $(a, b) \to a \otimes b$ is called canonical middle linear map.

Theorem 1.1.3 Let R be a ring and A and B be R-modules and let C be an abelian group. If $f : A \times B \to C$ is a middle linear map, then there exists a unique group homomorphism with homomorphisms $g : A \otimes_R B \to C$ such that $gi = f$, where $i : A \times B \to A \otimes_R B$.

Proof: Chpt. 4 Thm. 5.2 in [10] □

By this property $A \otimes_R B$ is uniquely determined up to isomorphism.

1.2 Localization

Let A be a ring and S be a multiplicative system, that is a subset of A containing 1_A and closed under multiplication. We construct a new ring $S^{-1}A$ which is the initial ring in which the elements of S become units. The localization of A at S, $S^{-1}A$ contains the fractions a/s with $a \in A$, $s \in S$, where we define an equivalance relation on $S^{-1}A$ by $a_1/s_1 \backsim a_2/s_2$ if and only if $s_3(a_1s_2 - a_2s_1) = 0$, for some $s_3 \in S$. The addition and multiplication in $S^{-1}A$ are

$$\frac{a_1}{s_1} + \frac{a_2}{s_2} = \frac{a_1 s_2 + a_2 s_1}{s_1 s_2} \text{ and } \frac{a_1}{s_1} \cdot \frac{a_2}{s_2} = \frac{a_1 a_2}{s_1 s_2}$$

Example 1.2.1 *i)* If A is an integral domain and if we take $S = A \setminus 0$, then we construct the most well-known case of localization; the field of fractions of A.

ii) Let P be a prime ideal of A and let $S = A \setminus P$ which is multiplicatively closed and contain 1_A. The ring $S^{-1}A$ is called the localization of A at P and $S^{-1}A$ is denoted by A_P.

*iii)*If $f \in A$, we may take $S = \{f^n \mid n = 0, 1...\}$ and denote $S^{-1}A$ as A_f.

1.3 Noetherian Rings

Proposition 1.3.1 Let A be a ring. The following are equivalent.

i) The set of ideals of A has ascending chain condition (ACC) that is any increasing chain

$$I_1 \subseteq I_2 \subseteq ... \subseteq I_n \subseteq ...$$

of ideals of A eventually stops in other words; there are no infinite increasing chains of ideals.

ii) Any non-empty collection of ideals of A have a maximal element

iii) Any ideal of A is finitely generated.

Proof: Chpt. 12 Thm. 1 in [1] □

Definition 1.3.2 If a ring satisfies the conditions of the proposition, then A is called Noetherian ring.

Lemma 1.3.3 *(Nakayama)* Let M be a finitely generated R- module and let I be an ideal of R which lies in all maximal ideals of R. If $IM = M$, then $M = 0$.

Proof: First of all, if $a \in I$ and the intersection of all maximal ideals of R contains I, then $(1 - a)$ is invertible. Suppose not, then $(1 - a)$ and a belong to a proper maximal ideal J but $(1 - a) + a = 1 \in J$ which is a contradiction. Since M

is a finitely generated R-module, let $S = \{w_1, ..., w_n\}$ be a set of minimal generators such that no proper subset of S can generate M. $IM = \{\sum_{i=1}^{n} a_i w_i \mid a_i \in I\}$. Let $|S| \neq 0$ and $IM = M$. Then $w_1 = \sum_{i=1}^{n} a_i w_i$. By moving the term with a_1, we have $(1 - a_1)w_1 = \sum_{i=2}^{n} a_i w_i$. Because $(1 - a_1)$ is invertible,

$$w_1 = \textstyle\sum_{i=2}^{n}(1 - a_1)^{-1} a_i w_i$$

which means M can be generated by $\{w_2, ..., w_n\}$ that contradicts the minimality of S. So $|S| = 0$, therefore $M = 0$. $\square$

1.4 Category Theory

Definition 1.4.1 A category $\mathbf{C}$ consists of

i) a class $Obj\mathbf{C}$ known as the objects of $\mathbf{C}$

ii) for every pair A, B of objects of $\mathbf{C}$, a set $Hom_{\mathbf{C}}(A, B)$ of morphisms from A to B.

iii) for every A, B, C of objects, a function $\circ : Hom_{\mathbf{C}}(A, B) \times Hom_{\mathbf{C}}(B, C) \to Hom_{\mathbf{C}}(A, C)$ called composition where $(f, g) \to g \circ f$. Composition of morphisms satisfy the following two axioms;

1) composition of morphisms is associative, such that $h(gf) = (hg)f$ for $f \in Hom_{\mathbf{C}}(A, B)$, $g \in Hom_{\mathbf{C}}(B, C)$ and $h \in Hom_{\mathbf{C}}(C, D)$

2) for each object A of $Obj\mathbf{C}$, there is an element $id_A \in Hom_{\mathbf{C}}(A, A)$ that is called identity morphism of A where $f id_A = f$, for all $f \in Hom_{\mathbf{C}}(A, B)$ and $id_A g = g$, for all $g \in Hom_{\mathbf{C}}(B, A)$.

Definition 1.4.2 Let $\mathbf{C}$ and $\mathbf{D}$ be categories. We say $\mathcal{F}$ is a *covariant functor* from $\mathbf{C}$ to $\mathbf{D}$ if;

i) for every object A in $\mathbf{C}$, $\mathcal{F}(A)$ is an object in $\mathbf{D}$

ii) for every $f \in Hom_{\mathbf{C}}(A, B)$, $\mathcal{F}(f) \in Hom_{\mathbf{C}}(\mathcal{F}A, \mathcal{F}B)$ such that

1) if $g \circ f$ is a composition of morphisms in $\mathbf{C}$ then $\mathcal{F}(f \circ g) = \mathcal{F}(f) \circ \mathcal{F}(g)$ in $\mathbf{D}$

2) $\mathcal{F}(id_A) = id_{\mathcal{F}(A)}$

Similarly a *contravariant functor* is defined by reversing the objects $\mathcal{F}(A)$ and $\mathcal{F}(B)$ in (ii) and reversing the order of composition in (1).

Definition 1.4.3 Let $\mathbf{C}$ and $\mathbf{D}$ be categories and $\mathcal{F}$, $\mathcal{G}$ be functors from $\mathbf{C}$ to $\mathbf{D}$. A morphism of functors (natural transformation) is a map $\eta : \mathcal{F} \to \mathcal{G}$ consists of a morphism $\eta_A \in Hom_{\mathbf{D}}(\mathcal{F}(A), \mathcal{G}(A))$ for every object A in $\mathbf{C}$, such that for every pair of objects A and B in $\mathbf{C}$ and for every $f \in Hom_{\mathbf{C}}(A, B)$, we have $\mathcal{G}(f)\eta_A = \eta_B \mathcal{F}(f)$.

If η_A is an isomorphism for each A, then η is called natural equivalence (isomorphism). Categories $\mathbf{C}$ and $\mathbf{D}$ are said to be equivalent (respectively antiequivalent) if there are two covariant (respectively contravariant) functors $\mathcal{F} : \mathbf{C} \to \mathbf{D}$ and $\mathcal{G} : \mathbf{D} \to \mathbf{C}$ such that the functors $\mathcal{F}\mathcal{G}$ and $\mathcal{G}\mathcal{F}$ are naturally equivalent to the identity functors $\mathcal{I}_{\mathbf{D}}$ and $\mathcal{I}_{\mathbf{C}}$ respectively.

A covariant (respectively contravariant) functor from a category $\mathbf{C}$ to **Set** where **Set** is the category of sets is said to be representable if it is naturally isomorphic to the Hom-functor $Hom_{\mathbf{C}}(A, \cdot)$ (respectively $Hom_{\mathbf{C}}(\cdot, A)$) for some object A of $\mathbf{C}$.

Chapter 2

SCHEMES

Unless otherwise stated the term ring will be considered as unitial commutative ring and all ring homomorphisms $\theta : A \to B$ are required to map 1_A to 1_B

2.1 Affine Schemes

2.1.1 Schemes as topological spaces

Definition 2.1.1 Let A be a ring. The spectrum of A is the set of all prime ideals of A, denoted by $SpecA$;

$$SpecA = \{P \subseteq A \mid P \textit{ is a prime ideal of } A\}$$

If S is any subset of A, $V(S) \subseteq SpecA$ consists of all the prime ideals in A that contains S;

$$V(S) = \{P \in SpecA \mid P \supseteq S\}$$

Lemma 2.1.2 The followings are the properties of the operation V

i) If S is a subset of A, then $V(S) = V((S))$ such that (S) is the ideal generated by the elements of S.

ii) Let $\{I_\alpha |\ \alpha \in \mathcal{L}\}$ be a set of ideals of A and $\sum_{i\alpha \in \mathcal{L}} I_\alpha$ be the smallest ideal of A containing all the ideals I_α. Then

$$V(\sum_{i_\alpha \in \mathcal{L}} I_\alpha) = \bigcap_{i_\alpha \in \mathcal{L}} V(I_\alpha)$$

iii) Let $I_1, ..., I_n$ be ideals of A. Then

$$V(\bigcap_{i=1}^{n} I_i) = \bigcup_{i=1}^{n} V(I_i)$$

iv) $V(0) = SpecA$ *and* $V(1) = \varnothing$

v) If $I_1 \subseteq I_2$ where I_1 and I_2 are ideals of A then $V(I_2) \subseteq V(I_1)$

Proof: Chpt. 2 Lemma 2.1 in [2] □

We can define a topology on $SpecA$ by specifying subsets of the form $V(I)$ where I is an ideal of A as closed subsets , since the following conditions hold;

i) $SpecA$ and the empty set $\emptyset$ are closed.

ii) Arbitrary intersections of $V(I_\alpha)$, $\alpha \in \mathcal{L}$ is closed.

iii) Finite unions of $V(I_\alpha)$, $\alpha \in \{1, ..., n\}$ is closed.

We call this topology as *Zariski topology* that is the standard topology in algebraic geometry. We use [-] to show the point $[P]$ in Spec which correspond to the prime ideal P.

Proposition 2.1.3 $V(P)$ is the closure of $[P] \in SpecA$ where P is a prime ideal of A.

Proof: Let $\mathcal{L} = \{V(I_\alpha) \mid P \in V(I_\alpha)\}$ be the set of closed sets containing P. By example

$$\bigcap_{I_\alpha \subseteq P} V(I_\alpha) = V(\sum_{I_\alpha \subseteq P} I_\alpha) \supseteq V(P)$$

Since $\sum_{\alpha \in \mathcal{L}} I_\alpha \subseteq P$. Also $P \in V(P)$ so $V(P)$ contains the intersection of closed sets containing P. □

Remark: If there is non-trivial containment of prime ideals $P_1 \nsubseteq P_2$ of A then $SpecA$ is not Hausdorff since $[P]$ is not closed; the closure of $[P_1]$; $V(P_1)$ contains the element $[P_2]$. That means the closed points are the only points corresponding to maximal ideals. If A is a domain, $SpecA$ is not Hausdorff unless A is a field since (0) is prime and $V((0)) = SpecA$.

Example 2.1.4 *i)* If $X = Spec\mathbb{Z}$, then closed sets are $\emptyset$, X and the sets of the form $\{P_1, ..., P_n\}$ where $P_i's$ are prime numbers since $V(P_i) = [P_i]$ for all $P_i \in \mathbb{Z}$ and finite union of closed sets is closed.

ii) If F is a field, then $F[x]$ is a unique factorization domain. In UFDs, every irreducible element is prime. Hence each non-zero prime ideal is of the form (f) where f is irreducible and monic polynomial.

$$SpecF[X] = \{(0)\} \cup \{[(f)] | f \in F[X] \text{ , } \textit{monic, irreducible polynomial}\}$$

Also note that $[(f)]$ where f is irreducible and monic polynomial is closed since (f) is maximal.

ii) Assume F is an algebraically closed field then an irreducible monic polynomial $f \in F[X]$ is of the form $f(x) = x - t$ for some $t \in F$. So, we have a $1-1$ correspondence between closed points of $Spec(F[X])$ and F.

Theorem 2.1.5 Let I be an ideal of A and define

$$\sqrt{I} = \{a \in A \mid a^n \in I \text{ for some } n \in Z_{>0}\}$$

called the radical of I. $\sqrt{I}$ is the intersection of all prime ideals P of A satisfying $I \subseteq P$.

Proof: See Chpt. 15 prop. 12 in [1] □

Definition 2.1.6 The ideal of a ring A which contains all nilpotent elements such that $a^n = 0$ for some $n \in \mathbb{Z}_{>0}$ is called nilradical of A, denoted by $\sqrt{0}$. By the above theorem, the nilradical of A is the intersection of all prime ideals in A.

Definition 2.1.7 For any subset S of $SpecA$, $\mathcal{I}(S)$ is the intersection of the prime ideals in S, i.e.

$$\mathcal{I}(S) = \bigcap_{[P] \in S} P$$

It follows from the definition, $\mathcal{I}(V(J)) = \bigcap_{[P] \in V(J)} P = \bigcap_{J \subseteq P} P = \sqrt{J}$ where J is an ideal of A.

Definition 2.1.8 A topological space T is noetherian if there are no infinite strictly decreasing chains of closed subspaces, i.e. whenever

$$F_0 \supsetneq F_1 \supsetneq F_2 \supsetneq ...$$

is a strictly decreasing chain of closed subspaces, then there is $m \in \mathbb{Z}_{>0}$ such that $F_k = F_m$ for all $k \geq m$

Proposition 2.1.9 If A is a noetherian ring, SpecA is a noetherian topological space.

Proof: Let A be a noetherian ring and $V(I_1) \supsetneq V(I_2) \supsetneq ...$ be a strictly decreasing chain of closed subsets of $SpecA$ then with the operation $\mathcal{I}$, we have $\mathcal{I}(V(I_1)) \supsetneq \mathcal{I}(V(I_2)) \supsetneq ...$ is a strictly increasing chain of ideals of A such that $\sqrt{I}_1 \subsetneq \sqrt{I}_2 \subsetneq ..$ Then since A is noetherian there exists $k \in \mathbb{Z}_{>0}$ such that for every $n > k$, $\sqrt{I}_n = \sqrt{I}_k$. So $V(I_n) = V(\sqrt{I}_n)) = V(\sqrt{I}_k) = V(I_k)$. Hence the decreasing chain of closed subsets of $SpecA$ is eventually constant. □

However the converse of the proposition is not true.

Example 2.1.10 Let us consider $A = K[x_1, x_2, ...] \setminus (x_1^2, x_2^2, ...)$ where k is a field. Now consider the ideal $M = (x_1, x_2, ...)$ of A. If we set a homomorphism $\varphi : A \to k$ such thata x_i goes to 0 for all i then $ker\varphi = M$. Because k is a field, M is a maximal ideal. Also $M \subseteq \sqrt{0}$ since $M^2 = 0$ in A, hence all prime ideals of A contain M. But M is a maximal ideal, so the only prime ideal containing M is itself. $SpecA$ has one element so noetherian but A is not noetherian, consider the infinite chain;

$$(x_1) \subset (x_1, x_2) \subset (x_1, x_2, x_3) \subset ...$$

Definition 2.1.11 For any element $f \in A$ and $X = SpecA$, $D(f)$ is the open complement of $V((f))$, which is called principal open set in $SpecA$;

$$D(f) = SpecA \setminus V((f)).$$

Proposition 2.1.12 Let $f \in A$ and let $D(f)$ be the corresponding principal open set in $X = SpecA$. Then,

i)$D(f) = X$ if and only if f is a unit and $D(f) = \emptyset$ if and only if f is nilpotent.

ii)$D(f) \cap D(g) = D(f.g)$

iii) $SpecA \setminus V(I) = \bigcup_{f \in I} D((f)$

Proof: See Chpt. 15 Prop. 56 in [1] □

The principal open sets form a basis for the Zariski topology on $SpecA$, by this proposition.

Definition 2.1.13 A topological space T is quasi-compact if every open cover of T has a finite subcover.

Proposition 2.1.14 $SpecA$ is quasi-compact for any ring A.

Proof: Every open set is a union of principal open sets, so it is sufficient to show if $X = SpecA$ is covered by principal open subsets $\{D(f_i)i \in I\}$ then $SpecA$ is a finite union of some $D(f_i)$. $X = \bigcup_{i\in I} D(f_i)$ is an open cover of X. Let J be the ideal generated by $\{f_i \quad | i \in I\}$, we have two cases; $J = A$ or there exists a maximal ideal M containing J. But in the latter case, $M \in SpecA$ would not be contained in any principal open set $D(f_i)$, since $f_i \in M$ for all $i \in I$. This contradicts with the assumption that $X = SpecA$ is covered by the $D(f_i)$. So $1 \in J$ and 1 can be written as a finite sum $1 = a_1 f_{i_1} + ... + a_n f_{i_n}$. Then $X = \bigcup_{i=1}^{n} D(f_i)$ since if $P \in \bigcup_{i=1}^{n} D(f_i)$ then P contains all $f_{i_1}, ...f_{i_n}$, hence it contains 1.

Definition 2.1.15 A closed subset is called irreducible if it cannot be written as a union of two proper closed subsets.

Proposition 2.1.16 If T is a noetherian topological space, then each closed subset of T is a finite union of irreducible closed subsets.

Proof: Let Σ be the set of all closed subsets of T that cannot be written as the union of finitely many irreducible closed sets. Assume that Σ is non-empty, then Σ has a minimal element V, since T is noetherian. Clearly, V is reducible and let's say $V = W_1 \bigcup W_2$ where W_1 and W_2 are proper closed subsets of V. Because V is minimal, both W_1 and W_2 are the union of finite numbers of closed subsets such that, $W_1 = \bigcup_{j=1}^{f(i)} W_{ij}, i = 1, 2$. It is easy to see that V can be written as a finite union of irreducible closed sets contradicting to $V \in \Sigma$. Hence Σ is empty. □

Lemma 2.1.17 Define $\phi_{A,S} : A \to S^{-1}A$ by $\phi_{A,S}(a) = a/1$. Then $\widetilde{\phi}_{A,S} : Spec(S^{-1}A) \to SpecA$ is a homeomorphism onto the subspace $\delta = \{[P] \mid P \cap S = \emptyset\}$ of $SpecA$.

Proof : See Chpt. 1 Lemma 1.3 in [12] □

Corollary 2.1.18 $SpecA_f$ and the open set $D(f) = SpecA \setminus V(f)$ are homeomorphic.

From the language of categories, the operation $\mathcal{F} : A \to SpecA$ defines a contravariant functor from the category of commutative rings to the category of topological spaces. Let $\varphi : A \to B$ a ring homomorphism then φ^{-1} defines a map between $SpecB$ and $SpecA$, since if $P \subset B$ is a prime ideal then $\varphi^{-1}(P)$ is a prime ideal of A.

So, sending $P \subset B$ to $\varphi^{-1}(P)$ defines a map;

$$\widetilde{\varphi} : SpecB \to SpecA$$

Proposition 2.1.19 The induced map $\widetilde{\varphi}$ is continuous.

Proof: Let $f \in A$, $X = SpecA$ and $Y = SpecB$. $\varphi(D_X(f))$ consists of $q \in Y$ such that $f \notin \widetilde{\varphi}(D_X(f))$ and $D_Y(\varphi(f))$ consists of $q \in Y$ such that $\varphi(f) \notin q$. We know $\varphi(f) \in q$ if and only if $f \in \varphi^{-1}(q)$, then $\widetilde{\varphi}^{-1}(D_X(f)) = D_Y(\varphi(f))$. Because $\{D_X(f) \mid f \in A\}$ form a basis for $SpecA$, for any open set $U \subseteq SpecA$, $\varphi^{-1}(U)$ is open. Hence $\widetilde{\varphi}$ is continuous. □

Proposition 2.1.20 If $\varphi : A \to A/I$ is the natural homomorphism then $\widetilde{\varphi} : SpecA/I \to V(I)$ is a homeomorphism.

Proof: Let q be a prime ideal of A/I. Then $I \subset \varphi^{-1}(q)$ and therefore $\widetilde{\varphi}(q) \subset V(I)$. This shows $\widetilde{\varphi}(SpecA/I) \subset V(I)$. Let p be a prime ideal of A such that $P \in V(I)$, and let $\varphi(p) = q$. Now $I \subset p$, and therefore $\varphi^{-1}(q) = I + p = p$. It follows from that q must be a proper prime ideal of A/I. Let $a, b \in A$ with $(I + a)(I + b) \in q$. Then $\varphi(ab) \in q$, and therefore $ab \in P$, which means $a \in P$ or $b \in P$, thus $I + a \in q$ or $I + b \in q$. Hence q is a prime ideal of A/I. We conclude that $\widetilde{\varphi}$ maps $SpecA/I$ onto $V(I)$.

If q_1 and q_2 are prime ideals of A/I and if $\widetilde{\varphi}(q_1) = \widetilde{\varphi}(q_2)$ then $\varphi^{-1}(q_1) = \varphi^{-1}(q_2)$ and therefore $q_1 = \varphi\varphi^{-1}(q_1) = \varphi\varphi^{-1}(q_2) = q_2$, so $\widetilde{\varphi}$ is injective. We proved that the induced map is continuous. Now the remaining part is that $\widetilde{\varphi}^{-1}$ is continuous. Let $V(J)$ is an arbitrary closed subset of $SpecA/I$ for J is an ideal of A/I.

$$\begin{aligned}
\widetilde{\varphi}(V(J)) &= \widetilde{\varphi}(\{q \in SpecA/I; J \subseteq q\}) \\
&= \widetilde{\varphi}(\{q \in SpecA/I; \varphi^{-1}(J) \subseteq \varphi^{-1}(q)\}) \\
&= \{P \in V(I); \varphi^{-1} \subseteq P\} \\
&= V(\varphi^{-1}(J) \cap V(I)
\end{aligned}$$

so $\widetilde{\varphi}$ is a closed map. We conclude that $SpecA/I$ and $V(I)$ are homeomorphic. □

2.1.2 Sheaf Theory

Definition 2.1.21 Suppose X is a topological space and Σ is the collection of open sets in X. A presheaf of abelian groups on X is a pair $(\mathcal{O}, \rho)$ consisting of

i) a family $\mathcal{O} = (\mathcal{O}(U))_{U \in \lambda}$ of abelian groups

ii) a family $\rho = (\rho_{UV})_{U,V \in \Sigma V \subset U}$ of group homomorphisms $\rho_{UV} : \mathcal{O}(U) \to \mathcal{O}(V)$ where V is open in U with the following properties;

a) $\mathcal{O}(\emptyset) = O$, where $\emptyset$ is the empty set

b) $\rho_{UU} = id_{\mathcal{O}(U)}$ for every $U \in \Sigma$

c) $\rho_{VW} \circ \rho_{UV} = \rho_{UW}$ for every $W \subseteq V \subseteq U$

We generally write $\mathcal{O}$ instead of $(\mathcal{O}, \rho)$. The homomorphism ρ_{UV} are called restriction homomorphisms and for $f \in \mathcal{O}$ we just write $f_{|V}$ instead of $\rho_{UV}(f)$. The elements f of $\mathcal{O}(U)$ are called the section of $\mathcal{O}$ over U. The elements of $\mathcal{O}(X)$ are called the global section of $\mathcal{O}$

If we look at the presheaf concept from the view of categories, we see that presheaf is the contravariant functor from the category $\mathbf{Top(X)}$ whose objects are the open subsets of X and where the only morphisms are the inclusion maps to the category **Ab** of abelian groups.

Definition 2.1.22 A presheaf $\mathcal{O}$ on a topogical space $\mathcal{X}$ is called a sheaf if for every open set $U \subseteq \mathcal{X}$ and every family of open subsets $U_i \subseteq U, i \in I$ such that $U = \bigcup_{i\in I} U_i$ the following conditions are satisfied;

*i)*If $f, g \in \mathcal{O}(U)$ are elements such that $f_{|U_i} = g_{|U_i}$ for every $i \in I$, then $f = g$

ii) If we have elements $f_i \in \mathcal{O}(U_i)$, $i \in I$ such that

$$f_{i|U_i \cap U_j} = f_{j|U_i \cap U_j} \quad \textit{for every} \quad i, j \in I,$$

then there exists an $f \in \mathcal{O}(U)$ such that $f_{|U_i} = f_i$ for every $i \in I$.

Observe that the uniqueness of f is assured by (i).

Example 2.1.23 i)(*Preheaf but not sheaf*)Let $X = \mathbb{R}$ and $U \subset X$ be an open subset. Suppose $\mathcal{O}(U)$ be the ring of constant functions on U, then $\mathcal{O}(U) \cong \mathbb{R}$ for all U. Let $\rho_{U,V}$ be the obvious restriction maps where $V \subset U$. Let $U = U_1 \cup U_2$ where $U_1 = (0, 2)$ and $U_2 = (3, 4)$ and let $f_1 : U_1 \to \mathbb{R}$ such that $f_1(x) = 0$ for all $x \in U_1$ and $f_2 : U_2 \to \mathbb{R}$ such that $f_2(x) = 1$ for all $x \in U_2$. $\mathcal{O}(U)$ is not a sheaf since sheaf axiom (ii) does not hold. For since $U_1 \cap U_2 = \emptyset$, we have $f_{1|U_1 \cap U_2} = f_{2|U_1 \cap U_2}$ but there is no a constant function $f \in \mathcal{O}(U)$ such that $f_{|U_1} = 0$ and $f_{|U_2} = 1$.

ii)(Sheaf) Suppose X an arbitrary topological space. For any open subset $U \subset X$, let $\mathcal{O}(U)$ be the sheaf of vector space of all continuous functions $f : U \to \mathbb{C}$. Let $\rho_{U,V} : \mathcal{O}(U) \to \mathcal{O}(V)$ be the usual restriction mapping for $V \subset U$. Both sheaf axioms are trivially satisfied, so $\mathcal{O}$ is a sheaf.

Definition 2.1.24 Let X be a topological space, $P \in X$, and $\mathcal{O}$ is a presheaf on X. Consider pairs (U, f) where U is an open neighborhood of P and $f \in \mathcal{O}$ is a section of $\mathcal{O}$ over U, we introduce an equivalence relation as follows; (U, f) and (V, g) are equivalent if there exists an open neighborhood W with $P \in W \subset U \cap V$ such that $f_{|W} = g_{|W}$. The set of all such pairs modulo this equivalence relation is called the *stalk* of $\mathcal{O}_P$ of $\mathcal{O}$ at P.

$$\mathcal{O}_P = (\coprod_{P \in U} (\mathcal{O}(U)))/ \sim$$

The elements of $\mathcal{O}_P$ are called germs of $\mathcal{O}$.

Definition 2.1.25 If $\mathcal{O}_1$ and $\mathcal{O}_2$ are presheaves on X, a morphism $\phi : \mathcal{O}_1 \to \mathcal{O}_2$ consists of a morphism of abelian groups $\phi(U) : \mathcal{O}_1(U) \to \mathcal{O}_2(U)$ for every U, and for $V \subset U$, the diagram

$$\begin{array}{ccc} \mathcal{O}_1(U) & \xrightarrow{\phi(U)} & \mathcal{O}_2(U) \\ {\scriptstyle \rho_{U,V}} \downarrow & & \downarrow {\scriptstyle \rho'_{U,V}} \\ \mathcal{O}_1(V) & \xrightarrow{\phi(V)} & \mathcal{O}_2(V) \end{array}$$

is commutative, which means $\phi(V) \circ \rho_{U,V} = \rho'_{U,V} \circ \phi(U)$

Definition 2.1.26 We define sheaf of rings $\mathcal{O}$ on $SpecA$ as the following way; for all $U \subset SpecA$, $\mathcal{O}(U)$ be the set of functions $s : U \to \coprod_{P \in U} A_P$ from U to the disjoint union of localizations A_P for $P \in U$ with the following properties;

i) $s(P) \in A_P$, for every $P \in U$

ii) for all $P \in U$ there is an open neighborhood $D(f)$ of P in U and an element $\frac{a}{f^n}$ in A_f defining s on $D(f)$; $s(Q) = \frac{a}{f^n} \in A_Q$ for every $Q \in D(f)$.

If s and t are elements in $\mathcal{O}(U)$ such that $s = \frac{a}{f_1^n}$ on $D(f_1)$ and $t = \frac{b}{f_2^m}$ on $D(f_2)$ then $st = \frac{abf_1^m f_2^n}{(f_1 f_2)^{m+n}}$ and $s + t = \frac{af_1^m f_2^{m+n} + bf_1^{m+n} f_2^n}{(f_1 f_2)^{m+n}}$ on $D(f_1 f_2)$ and id_A gives an identity for $\mathcal{O}(U)$. We conclude that $\mathcal{O}(U)$ is a commutative ring with identity.

Proposition 2.1.27 Let $X = SpecA$ and let $\mathcal{O}_X$ be its structure sheaf. If $D(f)$ is a principal open set in X for some $f \in A$, then $\mathcal{O}(D(f))$ is isomorphic to the localization A_f.

Proof : Chpt. 2 Prop. 2.2.b in [2] □

We deduce that the global sections of $\mathcal{O}$ are the elements of A, such that $\mathcal{O}(SpecA) \cong A$, since if we take $f = id_A$, then $D(f)$ is the whole space.

Proposition 2.1.28 Let $X = SpecA$ and let $\mathcal{O}_X$ be its structure sheaf. The stalk of $\mathcal{O}$ at the point $P \in X$; $\mathcal{O}_{X,P}$ is isomorphic to the localization A_P of A at P.

Proof : Let $\phi : \mathcal{O}_P \to A_P$ be a homomorphism sending the representative of s; (s, U) to $s(P)$. ϕ is well-defined since if $(s, U) \sim (s', V)$, then there exists W; $P \in W \subseteq U \cap V$ such that $s(P) = s'(P)$.

ϕ is injective. Let $\phi((s, U)) = \phi((s', V))$, then $s(P) = s'(P)$ in A_P. By definition(), $s = \frac{a}{f}$ on $D(f)$ and $s' = \frac{b}{g}$ on $D(g)$ where $f, g \notin P$. Since $\frac{a}{f}$ and $\frac{b}{g}$ have the same image $\frac{a}{f} = \frac{b}{g}$ in A_P, there exists $h \notin P$ such that $h(ag - bf) = 0$ in A. Let $g, f, h \notin Q$ then $\frac{a}{f} = \frac{b}{g}$ in A_Q. So s and s' agree on $D(fgh) = D(f) \cap D(g) \cap D(h)$ so $(s, U) \sim (s', V)$.

ϕ is onto. Let $\frac{a}{f}$ be an element in A_P with $a, f \in A$ and $f \notin P$. Let s be the function such that $s(Q) = \frac{a}{f}$ in A_Q. s gives us an element in $\mathcal{O}(D(f))$ since $\mathcal{O}(D(f)) = A_f$. So $(s, D(f))$ is the desired element such that $\phi((s, D(f))) = s(P) = \frac{a}{f}$. □

Note that the stalk $\mathcal{O}_{X,P}$ is a local ring which is a ring with a unique maximal ideal since A_P has a unique maximal ideal which is pA_p

Definition 2.1.29 A ringed space is a pair $(X, \mathcal{O}_X)$, where X is a topological space and $\mathcal{O}_X$ is a sheaf of rings on X. A morphism of ringed spaces $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ consists of a pair $(\tilde{\varphi}, \hat{\varphi})$ where $\tilde{\varphi} : X \to Y$ is a continuous map and $\hat{\varphi} : \mathcal{O}_Y \to \tilde{\varphi}_* \mathcal{O}_X$ is a homomorphism of sheaves of rings on Y where $\tilde{\varphi}_* \mathcal{O}_X(U) = \mathcal{O}_X(\tilde{\varphi}^{-1}(U))$ for $U \subseteq Y$. The last operation is the *pushforward* operation on sheaves that is if $\phi : X \to Y$ is a continuous map on topological spaces and $\mathcal{F}$ is a presheaf on X, we define the pushforward $\phi_* \mathcal{F}$ of $\mathcal{F}$ by ϕ to be the presheaf on Y given by

$$\phi_* \mathcal{F}(U) = \mathcal{F}(\phi^{-1}(U)) \textit{ for any open } U \subset Y$$

Example 2.1.30 Let T and S be topological spaces and let $\mathcal{O}_T$ and $\mathcal{O}_S$ be the sheaves such that for $U \subset T$, $\mathcal{O}_T(U) = \{f \mid f : U \to \mathbb{C}, f \text{ continuous } \}$. Each continuous map $\psi : T \to S$ of topological spaces induces to a morphism of ringed spaces $(\psi, \widetilde{\psi}) : (T, \mathcal{O}_T) \to (S, \mathcal{O}_S)$ such that $\widetilde{\psi}(g) = g \circ \psi$.

The morphism of sheaves $\hat{\varphi} : \mathcal{O}_Y \to \widetilde{\varphi}_* \mathcal{O}_X$ induces a homomorphism on stalks $\hat{\varphi}_p : \mathcal{O}_{Y,\widetilde{\varphi}} \to \widetilde{\varphi}_* \mathcal{O}_{X,p}$ such that

$$\hat{\varphi}_p \quad : \mathcal{O}_{Y,\widetilde{\varphi}} = \coprod_{\widetilde{\varphi}(p) \in U} (\mathcal{O}_Y(U))/\sim \quad \to \coprod_{p \in \widetilde{\varphi}^{-1}(U)} (\mathcal{O}_X(\tilde{\varphi})^{-1}(U))/\sim \quad = \quad \mathcal{O}_{X,p}$$

Definition 2.1.31 If A and B are local rings such that M_A, M_B are the unique maximal ideals, then the homomorphism $\varphi : A \to B$ is a local homomorphism if $\varphi^{-1}(M_B) = M_A$.

Definition 2.1.32 The ringed space $(X, \mathcal{O}_X)$ is a locally ringed space if for every $p \in X$, the stalk $\mathcal{O}_{X,p}$ is a local ring. A morphism of locally ringed spaces is a morphism $(\widetilde{\varphi}, \hat{\varphi})$ of ringed spaces such that the induced map on stalks $\hat{\varphi}_p : \mathcal{O}_{Y,\widetilde{\varphi}} \to \widetilde{\varphi}_* \mathcal{O}_{X,p}$ is a local homomorphism.

Example 2.1.33 $(SpecA, \mathcal{O}_{SpecA})$ is a locally ringed space since $\mathcal{O}_{SpecA,p} \cong A_p$ which is a local ring.

Theorem 2.1.34 If $\varphi : A \to B$ is a homomorphism of rings, then φ induces a natural morphism of locally ringed spaces

$$(\widetilde{\varphi}, \hat{\varphi}) : (SpecB, \mathcal{O}_{SpecB}) \to (SpecA, \mathcal{O}_{SpecA})$$

Conversely, every morphism of locally ringed spaces from $SpecB$ to $SpecA$ arises from a ring homomorphism from A to B.

Proof : Chpt. 15 Thm. 59 in [1] □

For morphism of locally ringed spaces, the extra condition that the induced map on stalks is a local homomorphism is necessary to force $(\widetilde{\varphi}, \hat{\varphi})$ to come from a ring homomorphism.

Example 2.1.35 Let $A = \mathbb{Z}_2$ and $B = \mathbb{Q}$. Let $(\widetilde{\varphi}, \hat{\varphi})$ be the morphism of ringed spaces between $(Spec\mathbb{Q}, \mathcal{O}_{Spec\mathbb{Q}})$ and $(Spec\mathbb{Z}_2, \mathcal{O}_{Spec\mathbb{Z}_2})$ such that $\widetilde{\varphi} : Spec\mathbb{Q} \to Spec\mathbb{Z}_2$ by $\widetilde{\varphi}((0)) = (2)$ and define $\hat{\varphi} : \mathcal{O}(SpecA) \to \mathcal{O}(SpecB)$ to be the inclusion map $\mathbb{Z}_2 \to \mathbb{Q}$ and for other $U \subseteq SpecA, \hat{\varphi}(U)$ is the zero map. Suppose that this morphism comes from a ring homomorphism φ, defined by $\hat{\varphi}$ on global sections, which is the inclusion map that maps $(0) \in Spec\mathbb{Q}$ to $(0) \in Spec\mathbb{Z}_2$ but φ^{-1} does not agree with $\tilde{\varphi}$ since $\tilde{\varphi}((0)) = (2)$. Observe that on stalks for $(0) \in Spec\mathbb{Q}$ and $\tilde{\varphi}((0)) = (2) \in Spec\mathbb{Z}_2$ the induced homomorphism $\hat{\varphi}_{(0)} : \mathcal{O}_{Spec\mathbb{Z}_2,(2)} \to \mathcal{O}_{Spec\mathbb{Q},(0)}$ is the injection $\mathbb{Z}_2 \to \mathbb{Q}$ which is not a local homomorphism.

Definition 2.1.36 An affine scheme is a locally ringed space $(X, \mathcal{O}_X)$ which is isomorphic to the spectrum of some ring.

Remark : The category of affine schemes **Aff** and the opposite category of commutative rings $\mathbf{Rings}^{op}$ are equivalent categories with the functors $\mathcal{F}$ and $\mathcal{G}$ such that $\mathcal{G}$ associates to every affine scheme to its ring of global sections and $\mathcal{F}$ maps every commutative ring to its spectrum.

2.2 Schemes

Definition 2.2.1 A scheme is a ringed space $(X, \mathcal{O}_X)$ which is locally isomorphic to an affine scheme that is for every point $p \in X$ there exists a neighborhood U of p in X and a homeomorphism ψ of U is an affine scheme $Y = SpecR$ such that $\psi_*(\mathcal{O}_{X_{|U}}) \cong \mathcal{O}_Y$ where $\psi_*(\mathcal{O}_{X_{|U}})$ is the sheaf given by $\psi_*(\mathcal{O}_{X_{|U}})(W) = \mathcal{O}_X(\psi^{-1}(W))$ for all open sets $W \subset Y$

Definition 2.2.2 A morphism of schemes $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces which is locally of the form $(\tilde{\varphi}, \hat{\varphi})$ for some homomorphism of commutative rings $\varphi : A \to B$, that is, for every $x \in X$ there are neighborhoods U of x and V of $f(x)$ such that f restricts to a map $f_{|U} : (U, \mathcal{O}_U) \to (V, \mathcal{O}_V)$, a homomorphism of commutative rings $\varphi : A \to B$ and isomorphism of ringed space $f : (U, \mathcal{O}_{X|U}) \to SpecB$, $h : (V, \mathcal{O}_{X|V}) \to SpecA$ such that the diagram

$$\begin{array}{ccc} (U, \mathcal{O}_{X|U}) & \xrightarrow{f_{|U}} & (V, \mathcal{O}_{X|V}) \\ \Big\downarrow g & & \Big\downarrow h \\ SpecB & \xrightarrow{(\widetilde{\varphi}, \widehat{\varphi})} & SpecA \end{array}$$

is commutative.

Example 2.2.3 *(Non affine scheme)* Let X be a topological space with points; p_1, q_1 and q_2 and let topologize X by setting $X_1 = \{p_1, q_1\}$ and $X_2 = \{p_1, q_2\}$ as open sets, then obviously $\emptyset, \mathcal{X}$ and $\{p_1\}$ are also open. Now define a sheaf of $\mathcal{O}$ of rings on X by setting

$$\mathcal{O}(X) = \mathcal{O}(X_1) = \mathcal{O}(X_2) = k[X]_{(x)} \text{ and } \mathcal{O}(\{p\}) = k(x)$$

where $k(x)$ is the field of rational functions and $k[X]_{(x)}$ is the localization of $k[X]$ in x. The restriction maps are $id : \mathcal{O}(X) \to \mathcal{O}(X_i)$ and $\mathcal{O}(X_i) \to \mathcal{O}(\{p\})$, the inclusion map. This ringed space is a scheme in which every point has an open neighborhood U such that $(U, \mathcal{O}_{x|U})$ is an affine scheme. But this is not an affine scheme since the corresponding points of the topological space gives us two maximal ideals, however $\mathcal{O}(X) = k[X]_{(x)}$ is a local ring with unique maximal ideal.

2.2.1 First Properties of Schemes

Definition 2.2.4 A scheme X is irreducible if its topological space is irreducible and it is connected if its topological space is connected.

Definition 2.2.5 A scheme X is reduced if for every open set $U \subseteq X$ the ring $\mathcal{O}_X(U)$ has no nilpotent elements.

Proposition 2.2.6 A scheme X is reduced if and only if the local ring $\mathcal{O}_{X,p}$ for all $p \in X$ have no nilpotent elements.

Proof : Assume that there is $p \in X$ and $g \in \mathcal{O}_{X,p}$ such that $g \neq 0$ and $g^n = 0$ for some $n \in N$. Then there is an open set $p \in U$ and $f \in \mathcal{O}_X(U)$ which corresponds g. But then $f^n = 0$ is a nilpotent element. Also assume that for some $U \subseteq X$, there is a nilpotent element f in $\mathcal{O}_X(U)$, such that $f^n = 0$. Then there is $p \in U$ for which the image $g \in \mathcal{O}_{X,p}$ of f is nonzero and $g^n = 0$. □

Proposition 2.2.7 An affine scheme $X = SpecR$ is reduced and irreducible if and only if R is a domain.

Proof : Assume R is a domain and $X = V(I_1) \cup V(I_2)$ then since (0) is a prime ideal, without loss of generality let $(0) \in V(I_1)$. Then all prime ideals of X is contained in $V(I_1)$, so $X = V(I_1)$ which means X is irreducible. Clearly $SpecR$ is reduced since for the distinguished open sets $D(f)$, $\mathcal{O}_X(D(f)) = R_f$ which has no nilpotent elements because R is a domain.

Now , suppose $\mathcal{O}_X(SpecR) = R$ is reduced and $SpecR$ is irreducible. Let $f \cdot g = 0 \in R$ then $D(f \cdot g) = D(f) \cap D(g) = D(0) = \emptyset$. By taking complements, $V(f) \cup V(g) = SpecR$, but since $SpecR$ is irreducible, say $V(f) = SpecR$ which means f lies in all prime ideals of R. By the definition of the nilradical, $f \in \sqrt{0}$. But R is reduced so $f = 0$, then R is a domain. □

Lemma 2.2.8 Let $X = SpecR$ be an affine scheme, and let $f \in R$. Then the distinguished open subset $D(f)$ is the affine scheme $SpecR_f$

Proof : First of all, $D(f) = SpecR_f = \{p \in X \mid f \notin p\}$, so the only remaining part is whether the structure sheaves on $D(f)$ and $SpecR_f$ are the same. Let $h \in R$, consider $D(f \cdot h) = (SpecR_f)_h$ then

$$\mathcal{O}_{D(f)}(D(f \cdot h)) = R_{f \cdot h} \text{ and } \mathcal{O}_{SpecR_f}((SpecR_f)_h) = (R_f)_h = R_{f \cdot h}$$

In other words, on every distinguished open subset, the rings of regular functions are same for $D(f)$ and $SpecR_f$. Since the distinguished open sets are principal basis, the sections over every open set must be same. □

Example 2.2.9 Finite disjoint union of affine schemes is affine. Let $\theta : X = SpecA \coprod SpecB \to Y = Spec(A \times B)$ be a morphism where P_A goes to $(P_A, 1)$ and P_B goes to $(1, P_B)$. θ is continuous, since $\theta^{-1}(Spec(A \times B)_{(a,b)}) = SpecA_{(a)} \coprod SpecB_{(b)}$ which is open. θ is open, $\theta(SpecA_a) = SpecA_a \times B = Spec(A \times B)_{(a,1)}$. So θ is a homeomorphism between topological spaces. To show the isomorphism between sheaves; for some $U = Spec(A \times B)(a, b)$

$$\widehat{\theta} : \mathcal{O}_Y(Spec(A \times B)_{(a,b)}) = A_a \times B_b \to \mathcal{O}_X(\theta^{-1}(Spec(A \times B)_{(a,b)})) = \mathcal{O}_X(SpecA_a \coprod SpecB_b)$$
$$= \mathcal{O}_X(SpecA_a) \times \mathcal{O}_X(SpecB_b) = A_a \times B_b$$

2.2.2 Closed and Open Subchemes

Definition 2.2.10 Let X be a scheme, then any open set $U \subset X$ is a scheme. Let $P \in U \subset X$ and let $Y = SpecR$ be an affine scheme containing P in X. There exists a distinguished open set in $Y \cap U$ containing P. By the lemma **2.2.8**, $(U, \mathcal{O}_{X|U})$ is covered by affine schemes, as required. We call $(U, \mathcal{O}_{X|U})$ as an open subscheme of X. An open immersion is a morphism $f : X \to Y$ which induces an isomorphism of X with an open subscheme of Y.

Consider an affine scheme $X = SpecR$, for any ideal I of R, we proved that $SpecR/I$ is canonically homeomorphic to $V(I)$. We may make the closed subset $V(I)$ into an affine scheme by identifying it with $Y = SpecR/I$. We define a closed subscheme Y of X to be a scheme of this form. In other words, a closed subscheme Y of an affine scheme X arises from an ideal. Now, we generalize the notion of a closed subscheme of an affine scheme to arbitrary schemes. Firstly, we need the concept of sheaf of modules.

Definition 2.2.11 An $\mathcal{O}_X$- module is a sheaf $\mathcal{F}$ such that for any open set $U \subset X$, $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$- module, with the restriction maps $\rho_{U,V}$; for $V \subset U$ $\mathcal{F}(V)$ is an $\mathcal{O}_X(U)$- module via the restriction map of sheaves $r_{U,V} : \mathcal{O}_X(U) \to \mathcal{O}_X(V)$. Then, $\rho_{U,V}(af) = r_{U,V}(a)\rho_{U,V}(f)$ for $a \in \mathcal{O}_X$ and $f \in \mathcal{F}(U)$.

Definition 2.2.12 The ideal sheaf of $Y = SpecR/I$ in $X = SpecR$, $\mathcal{F}_{Y|X}$ is the sheaf of ideals (submodules) of $\mathcal{O}_X$ such that for a distinguished open set $V = D(f)$ of $X, \mathcal{F}_{Y|X}(D(f)) = I \cdot R_f$.

Example 2.2.13 Let $R = k[X]_{(x)}$, the localization of the polynomial ring at the maximal ideal (x). The scheme $X = SpecR$ has only two points, $\{[(0)], [(X)]\}$. We may define a sheaf of ideals $\mathcal{F}$ by

$$\mathcal{F}(U) = \mathcal{O}_X(U) \text{ and } \mathcal{F}(X) = 0$$

Assume sheaf of ideals $\mathcal{F}$ come from an ideal of R, then we have $\mathcal{F}(U) = \mathcal{F}(D(x)) = I \cdot R_x = I \cdot k(x)$ and $I = \mathcal{F}(X)$ but there is a contradiction since $\mathcal{F}(X)$ is 0 while $\mathcal{F}(U)$ is $\mathcal{O}_X(U)$.

This example shows that not all sheaf of ideals come from an ideal. The sheaves of ideals that arise from an ideal are called *quasicoherent* sheaves of ideals.

Definition 2.2.14 A closed subscheme of a scheme X is a closed topological space Y together with a sheaf of rings $\mathcal{O}_Y$ that is a quotient sheaf of structure sheaf $\mathcal{O}_X$ by a *quasicoherent* sheaves of ideals $\mathcal{F} : \mathcal{O}_X/\mathcal{F}$

If U is an affine open subscheme of X such that $U = SpecA$ and $\mathcal{F}(U) = I$, then $Y \cap U = V(I)$.

Definition 2.2.15 A morphism $(\tilde{i}, \hat{i}) : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ is called a closed immersion if the followings are satisfied,

i) $\tilde{i} : Y \to X$ gives a homeomorphism of Y with a closed subset of X

ii) $\hat{i} : \mathcal{O}_X \to \tilde{i}_* \mathcal{O}_Y$ is surjective with kernel an ideal sheaf

2.2.3 Glueing Schemes

One can construct a scheme X by glueing the collection of schemes U_α, $\alpha \in I$ with open subschemes $U_{\alpha\beta} \subset U_\alpha$ such that $U_{\alpha\alpha} = U_\alpha$ and with a system of isomorphisms of schemes of schemes $\varphi_{\alpha\beta} : U_{\alpha\beta} \to U_{\beta\alpha}$ such that;

i) $\varphi_{\alpha\alpha} = id$ for $\alpha \in I$

ii) $\varphi_{\alpha\beta} \circ \varphi_{\beta\alpha} = id\alpha, \beta \in I$

iii) $\varphi_{\alpha\beta}(U_{\alpha\beta} \cap U_{\alpha\gamma}) = U_{\beta\alpha} \cap U_{\beta\gamma}$ and $\varphi_{\alpha\gamma} = \varphi_{\beta\gamma} \circ \varphi_{\alpha\beta}$ on $U_{\alpha\beta} \cap U_{\alpha\gamma}$

Let $Y = \bigcup_{\alpha \in I} U_\alpha$ and $X = Y/\sim$ where $x \sim y$ if $x \in U_{\alpha\beta}$, $y \in U_{\beta\alpha}$ and $\varphi_{\alpha\beta}(x) = y$ The conditions i, ii and iii satisfy the properties of the equivalence relation. For reflexivity, we can use condition i and for symmetry condition ii. As to transitivity, suppose $x, z \in U_{\alpha\beta} \cap U_{\alpha\gamma}$ and $y, z \in U_{\beta\alpha} \cap U_{\beta\gamma}$ such that $\varphi_{\alpha\beta}(x) = y$ and $\varphi_{\beta\gamma}(y) = z$, then $\varphi_{\alpha\gamma}(x) = z$.

Let $\pi : Y \to X$ be the quotient map and the quotient topology on X can be described as $U \subset X$ is open if $\pi^{-1}(U)$ is open in Y. π establishes a homeomorphism $\pi_\alpha : U_\alpha \to V_\alpha$ for $V_\alpha \subset X$, and $X = \bigcup_{\alpha \in I} V_\alpha$. Also $\pi_\alpha(U_{\alpha\beta}) = \pi_\alpha(U_\alpha) \cap \pi_\beta(U_\beta)$ and $\pi_\beta \circ \varphi_{\alpha\beta} = \pi_\alpha$ on $U_{\alpha\beta}$. The structure sheaf $\mathcal{O}_X$ is constructed by glueing the the structure sheaves $\mathcal{O}_{V_\alpha}$. For $W \subseteq V_\alpha$, we set $\mathcal{O}_X(W) = \mathcal{O}_{V_\alpha}(\pi_\alpha^{-1}(W))$. For a different $W \subset V_\beta$, we have $\mathcal{O}_{V_\beta}(\pi_\beta^{-1}(W))$ which is isomorphic to $\mathcal{O}_{V_\alpha}(\pi_\alpha^{-1}(W))$ since $W \subset V_\beta \cap V_\alpha$.

As a result, we say that X is obtained glueing the schemes U_α along the isomorphisms $\varphi_{\alpha\beta}$

2.2.4 Fiber Product of Schemes

Theorem 2.2.16 For any two schemes X and Y over a scheme S, then the fiber product $X \times_S Y$ exists. The fiber product $X \times_S Y$ is a scheme together with morphisms $p_1 : X \times_S Y \to X$ and $p_1 : X \times_S Y \to Y$ making the diagram;

$$\begin{array}{ccc} X \times_S Y & \xrightarrow{\rho_2} & Y \\ \downarrow{\scriptstyle \rho_1} & & \downarrow \\ X & \longrightarrow & S \end{array}$$

commute. Also it has the universal property.

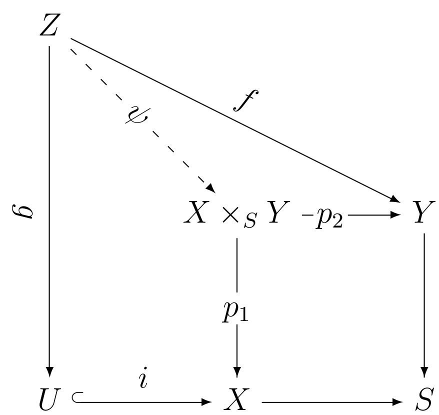
Proof : By using the tensor product of algebras, we can define the fiber product of affine schemes such that for S-algebras A and B, $Spec A \times_{Spec S} Spec B$ corresponds to $Spec(A \otimes_S B)$. This is because the diagram

$$\begin{array}{ccc} A \otimes_S B & \longleftarrow & A \\ \uparrow & & \uparrow \\ B & \longleftarrow & S \end{array}$$

has the opposite uniqueness property (universal property) to the one desired for the fiber products. In other words; for any scheme Z, to give a morphism of Z to

$Spec(A \otimes_S B)$ is the same as to give a homomorphism of $A \otimes_S B$ into the ring $\mathcal{O}(Z)$. This is the same as giving homomorphisms $A \to \mathcal{O}(Z)$ and $B \to \mathcal{O}(Z)$ inducing the same homomorphism on S. When we apply the contravariant functor $Spec$, we see that to give a morphism of Z into $Spec(A \otimes_S B)$ is the same as giving morphisms of Z into $SpecA$ and $SpecB$ that give rise to the same morphism from $Z \to S$. Thus $Spec(A \otimes_S B)$ is the desired product.

Then first of all, we show a fact that $p_1^{-1}(U)$ is a product $U \times_S Y$ for some open set U of X. For a given scheme Z with $g : Z \to U$ and $f : Z \to Y$. We obtain a map $g' = g \circ i : Z \to X$. By the universal property, there is a morphism $\psi : Z \to X \times_S Y$



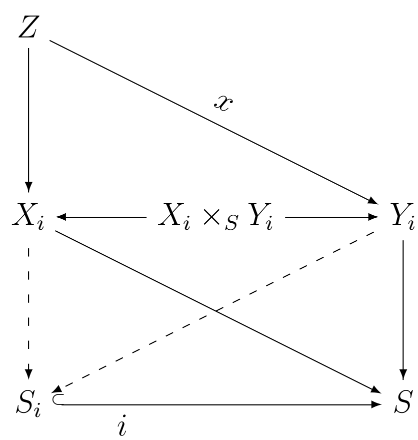
$g(Z) \subseteq U$ implies $\psi(Z) \subseteq p_1^{-1}(U)$, so $\psi : Z \to p_1^{-1}$ which is unique so $p_1^{-1} = U \times_S Y$.

Let's cover X by X_i and assume $X_i \times_S Y$ exists for each i. Let $X_{ij} = X_i \cap X_j$ and $U_{ij} = p_1^{-1}(X_{ij}) = X_{ij} \times_S Y$. Because the products are unique, we have isomorphisms $\varphi_{ij} : U_{ij} \to U_{ji}$ for each i, ȷ. Using the idea of gluing explained before, we can obtain a scheme via the isomorphisms φ_{ij}, since $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$ on $X_i \cap X_j \cap X_k$ and they are compatible with the projections. The claim is that the new scheme $X \times_S Y$ is the desired product. The projection morphisms p_1, p_2 arise from the projections $p_{1i} : X_i \times_S Y \to X_i$ and $p_{2i} : X_i \times_S Y \to Y$ where $p_{1i} = p_{1j}$ on $X_{ij} \times_S Y$.

For a scheme Z with morphisms $g : Z \to X$ and $f : Z \to Y$. Let $Z_i = g^{-1}(X_i)$, which gives us $\theta_i : Z_i \to X_i \times_S Y \hookrightarrow X \times_S Y$ where $\theta_{i|Z_i \bigcap Z_j} = \theta_{j|Z_i \bigcap Z_j}$. So we can glue θ_i to obtain a morphism $\theta : Z \to X \times_S Y$ which is compatible with f, g and projections.

Hence, we know for Y and S affine, $X \times_S Y$ exists. Let S be an affine scheme and for any scheme $Y = \bigcup_{i \in I} Y_i$ such that Y_i's are affine, $X \times_S Y_i$ exists. By the same gluing tool, we obtain $X \times_S Y$ for an affine scheme S.

The only remaining part is to show the existence for an arbitrary scheme S. Let S_i be an affine cover of S and let $\pi_1 : X \to S$ and $\pi_2 : Y \to S$ be the given morphism with $\pi_1^{-1}(S_i) = X_i$ and $\pi_2^{-1}(S_i) = Y_i$. We know $X_i \times_{S_i} Y_i$ exists. $X_i \times_{S_i} Y_i \cong X_i \times_S Y_i$ because $X_i \times_S Y_i$ satisfies the universal property of $X_i \times_{S_i} Y_i$.



□

2.3 Attributes of Morphisms

Definition 2.3.1 Let $f : Y \to X$ be a morphism of noetherian schemes. f is called affine if for each affine open subscheme of $U \subset X$, $f^{-1}(U)$ is an affine open subscheme of Y .

Definition 2.3.2 An affine morphism is called finite if theere exists a covering of Y by afine open subsets $V_i = SpecB_i$, such that $f^{-1}(V_i) = SpecA_i$ for each i, where A_i is a B_i-algebra which is finitely generated B_i-module.

Definition 2.3.3 A morphism $f : Y \to X$ is of finite type if X and Y admit finite affine covers $X = \bigcup_i U_i = \bigcup_i SpecA_i$, $Y = \bigcup_i V_i = \bigcup_i SpecB_i$ with for each i $f(V_i) \subset U_i$, B_i is a commutative A_i-algebra and B_i is isomorphic to quotient of a polynomial ring over A in finitely many variables:

$$B_i \cong A[X_1, ..., X_m]/I$$

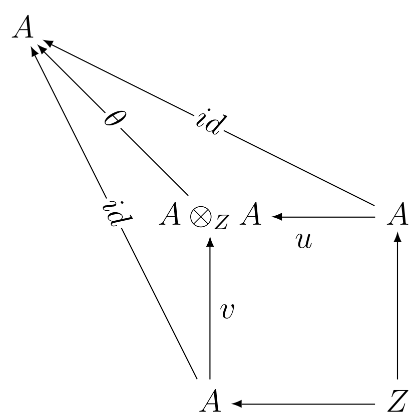
2.3.1 Separatedness

Except for the trivial cases, the topological space associated with a scheme is almost never Hausdorf. The property of being separated for schemes is analogous to that of being Hausdorf for topologies.

Definition 2.3.4 The morphism $\triangle = (id, id) : X \to (X \times_Y X)$ is called the diagonal morphism. We say that the morphism $f : X \to Y$ is separated if the diagonal morphism is a closed immersion. We know every scheme has a unique morphism to $Spec\mathbb{Z}$, we say that X is a separated scheme if $X \to Spec\mathbb{Z}$ is separated.

Proposition 2.3.5 Every affine scheme is separated.

Proof : Let $X = SpecA$. Since $X \times_{Spec\mathbb{Z}} X = Spec(A \otimes_{\mathbb{Z}} A)$, the morphism $\triangle : X \to (X \times_{Spec\mathbb{Z}} X)$ is associated with a homomorphism $\theta : A \otimes_{\mathbb{Z}} A \to A$



where $\theta \circ u = id$ and $\theta \circ u = id$ for homomorphisms $u, v : A \to A \otimes_{\mathbb{Z}} A$ such that $u(a) = a \otimes 1$, $v(a) = 1 \otimes a$. θ is surjective since $\theta(a \otimes b) = ab$. Let I be the ideal generated by elements of the form $a \otimes 1 - 1 \otimes a$. $SpecA \cong Spec(A \otimes_{\mathbb{Z}} A)/I$, so $\triangle$ is a closed immersion arising from the ideal I. □

Proposition 2.3.6 Let $f : X \to Y$ be a morphism of schemes. Then for any open subsets U and V of X, we have $U \cap V = \triangle(X) \cap (U \times_Y V)$

Proof : Chpt. 1 Prop. 1.40 in [12] □

Proposition 2.3.7 Let X be a separated scheme, U and V affine open subschemes. Then $U \cap V$ is also affine.

Proof : We know that $U \times_{Spec\mathbb{Z}} V$ is an affine scheme which is equal $Spec(A \otimes_{\mathbb{Z}} B)$ where $U = SpecA$ and $V = SpecB$. By the previous proposition $U \cap V \cong \triangle(X) \cap (U \times_{Spec\mathbb{Z}} V)$ which is a closed subscheme of an affine scheme $U \times_{Spec\mathbb{Z}} V)$ since X is separated. Hence $U \cap V = Spec(A \otimes B)/I$ for some ideal I. □

Example 2.3.8 (Non-separated scheme) Let $X = Speck[x_1, x_2]$ and $Y = Speck[y_1, y_2]$ and let $V_{12} = Speck[x_1, x_2] \setminus V(x_1) \cap V(x_2) = A^2_k \setminus \{0\}$ and as the same $V_{12} = A^2_k \setminus \{0\}$. We obtain a new scheme by gluing V_{12} and V_{21} by $f(x_i) = y_i$. We have an affine open cover U_1 and U_2 such that $U_1 = X$ and $U_2 = Y$. $U_1 \cap U_2 = V_{12}$ must be affine by the previous proposition. Let $f \in \mathcal{O}(A^2_k \setminus \{0\})$ be a rational function such that $f \notin k[x, y]$. Then $f \in \bigcap_{(a,b) \neq (0,0)} k[x, y]_{(x-a,y-b)}$ which means $f = \frac{g(x,y)}{h(x,y)}$ after factoring the polynomials, we find a polynomial $p(x, y)$ in the denominator which is relatively prime to the polynomials in the numerator since $k[x, y]_{(x-a,y-b)}$ is a UFD. If we choose a point satisfying $p(x)$ and the polynomials in the denominator, $f(x, y)$ is infinite at that point. So, $f \in k[x, y]$ and $\mathcal{O}(A^2_k) = \mathcal{O}(A^2_k \setminus \{0\}) = k[x, y]$. But $A^2_k \setminus \{0\}$ is not affine since $A^2_k \setminus \{0\} \neq Spec(A^2_k \setminus \{0\}) = Speck[x, y]$.

2.3.2 Flatness

Definition 2.3.9 A sequence of A- modules and A- homomorphisms

$$...M_{i-1} \to M_i \to M_{i+1} \to ...$$

is said to be exact at M_i if $Im(\psi_i) = Ker(\psi_{i+1})$. The sequence is exact if it is exact at each M_i.

Theorem 2.3.10 Assume M is an R-module if;

$$0 \to L \to K \to N \to 0$$

is exact then the associated sequence of abelian groups

$$L \otimes_R M \to K \otimes_R M \to N \otimes_R M \to 0$$

is exact where $f \otimes id$ denotes the tensor of two homomorphisms.

Proof : Chpt. 2 Prop. 2.18 in [4] □

Remark : The sequence

$$0 \to L \otimes_R M \to K \otimes_R M \to N \otimes_R M \to 0$$

is not in general exact since $\psi \otimes id$ does not have to be injective. For example; let $\psi : \mathbb{Z} \to \mathbb{Z}$ where $\psi(x) = 3x$ for all $x \in \mathbb{Z}$. When we tensor with $M = \mathbb{Z}/3\mathbb{Z}$, the map $\psi \otimes id : \mathbb{Z} \otimes Z/3\mathbb{Z} \to \mathbb{Z} \otimes \mathbb{Z}/3\mathbb{Z}$ is not injective since for any $z_1 \otimes z_2 \in \mathbb{Z}/3\mathbb{Z}$ we have $\psi \otimes id(z_1 \otimes z_2) = 3z_1 \otimes z_2 = z_1 \otimes 3z_2 = z_1 \otimes 0 = 0$

Definition 2.3.11 An R- module M is called flat if tensoring with M transforms all exact sequences into exact sequences. In other words, M is flat if $\psi : L \to K$ is injective then $\psi \otimes id : L \otimes_R M \to K \otimes_R M$ is injective for any R- modules L and K.

Example 2.3.12 Free modules are flat. Let R be a ring and $F = \bigoplus_{i \in I} R$ be a free R- module. Let $f : N_1 \to N_2$ be an injective map of R-modules N_1 and N_2. After tensoring $f \otimes id : N_1 \otimes (\bigoplus_{i \in I} R) \to N_2 \otimes (\bigoplus_{i \in I} R)$, we have $f \otimes id : \bigoplus_{i \in I} N_1 \to \bigoplus_{i \in I} N_2$ is just the natural map induced by the inclusion of f in each component, so injective.

Definition 2.3.13 A ring homomorphism $\psi : R \to S$ is flat if S is flat as an R-module. Let X and Y are schemes, a flat morphism $f : X \to Y$ is a morphism such that the induced map on every stalk is a flat homomorphism of rings, for every $p \in X$,

$$\widehat{f} : \mathcal{O}_{y,f(p)} \to \mathcal{O}_{X,p} \text{ is flat.}$$

Here are some facts related to the flatness.

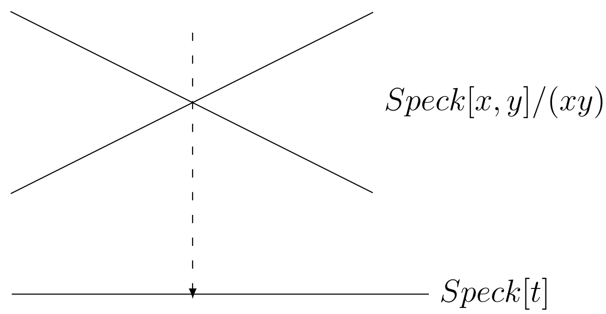
Proposition 2.3.14 Let R be a principal ideal domain and M be an R- module. Then for M to be flat it is necessary and sufficient that M be torsiom-free

Proof : Chpt. 1 2.4 prop. 3.ii in [11]

Proposition 2.3.15 If A is a noetherian ring and M is finitely generated A- module, M is flat over A if and only if M_p is free over A_p for every prime ideal p.

Proof : Chpt. 2 Thm. 2.9 in [3]

Example 2.3.16 Let $\phi : X = Speck[x,y]/(xy) \to Y = Speck[t]$ be the map of affine schemes pictured below.



The dual of the map is $k[t] \to k[x,y]$ where $t \to x+y$. To see the flatness of the map, we check the flatness of $(k[x,y]/(xy))_{(x,y)}$ as a $k[t]_t$-module. By the proposition, checking the torsion freeness of $(k[x,y]/(xy))_{(x,y)}$ as a $k[t]_t$-module via the map $\widetilde{\phi}$ is sufficient. Let $\frac{a(x,y)}{b(x,y)} \in (k[x,y]/(xy))_{(x,y)}$ such that $b(0,0) \neq 0$ and $\frac{c(t)}{d(t)} \in (k[t])_t$ where $d(t) \neq 0$. Then via the map $\widetilde{\phi}$,

$\frac{c(x+y)}{d(x+y)} \frac{a(x,y)}{b(x,y)} = 0$ which means $c(x+y)a(x,y) \in (x,y)$. After checking the coefficients of $c(x+y)$ and $a(x,y)$, we obtain $a(x,y) \in (xy)$. Hence the module is torsion-free.

2.3.3 Étale Morphisms

Definition 2.3.17 Let A be a ring, B be an A- algebra and let M be a B- module. An A- derivation of B into M is an additive map $d : B \to M$ satisfying the leibniz rule;

$$d(ab) = ad(b) + bd(a)$$

and $d(a) = 0$ for all $a \in A$.

Now we construct the module of Kähler differentials of B over A; $\Omega_{B/A}$ together with a n A- derivation $d : B \to \Omega_{B/A}$ with the following universal property; if d' is an A- derivation of B over A into a B-module M, then there is a unique B-module homomorphism $k : \Omega_{B/A} \to M$ such that $d^{'} = kd$.

Let A be a ring and B be an A- algebra, let $f : B \bigotimes_A B \to B$ and $g_1, g_2 : B \to B \bigotimes_A B$ defined by $f(b \otimes b^{'}) = bb^{'}, g_1(a) = a \otimes 1$ and $g_2(a) = 1 \otimes a$. Let $I = kerf$. By g_1, I/I^2 can be viewed as B- module. Let $\theta : B \otimes_A B \to (B \otimes_A B)/I^2$ is the natural homomorphism and let $g = g_2 - g_1$, then the B- module I/I^2 is the module of Kähler differentials of R over K with $d = \theta \circ g$ such that $da = 1 \otimes a - a \otimes 1 (modI^2)$. d is a derivation since for elements a, b of B, in I/I^2 we have,

$$\begin{aligned} 0 &= (a \otimes 1 - 1 \otimes a)(b \otimes 1 - 1 \otimes b) \\ &= ab \otimes 1 - a \otimes b - b \otimes a + 1 \otimes ab \\ &= a(b \otimes 1 - 1 \otimes b) + b(a \otimes 1 - 1 \otimes a) - (ab \otimes 1 - 1 \otimes ab) \end{aligned}$$

in other words, $d(ab) = ad(b) + bd(a)$

Now, we have to prove $d : B \to I/I^2$ satisfies the universal property. First, we show the existence of k. The map $d^{'}$ induces a B- module homomorphism $id \otimes_A d^{'}$: $B \otimes_A B \to M$ such that $a \otimes b \to ad^{'}(b)$. When we restrict this map to I, we obtain $\overline{k} : I \to M$ such that $\overline{k}(I^2) = 0$ since

$$\begin{aligned} \overline{k}(a \otimes 1 - 1 \otimes a)(b \otimes 1 - 1 \otimes b) &= \overline{k}(ab \otimes 1 - a \otimes b - b \otimes a + 1 \otimes ab) \\ &= -bd^{'}(a) - ad^{'}(b) + d^{'}(ab) \\ &= 0 \end{aligned}$$

because $d^{'}$ is a derivation. So $\overline{k}$ induces a homomorphism $k : I/I^2 \to M$ such that $k \circ d = d^{'}$.

$\Omega_{B/A} = I/I^2$ is generated by $\{d(b) \mid \quad b \in B\}$ as a B- module, since $\sum a_i \otimes b_i = \sum a_i b_i \otimes 1 + a_i(1 \otimes b_i - b_i \otimes 1) = \sum f(a_i \otimes b_i) + a_i g(b_i)$. Any element of $\Omega_{B/A} = I/I^2$ has the form $\sum a_i d(b_i)$ for $a_i, b(i) \in B$ since $\theta g(b_i) = d(b_i)$.

Finally, k is unique since $d : B \to I/I^2 = \Omega_{B/A}$ maps onto the B- module generators of I/I^2.

Example 2.3.18 1)Let $B = A[x_1 ... x_n]$ be a polynomial ring over A. The module of Kähler differentials $\Omega_{B/A}$ is generated by $\{dx_i\}$ as free B- module. Let $\sum h_i dx_i = 0$ for $h_i \in B$, and let $\frac{\partial}{\partial x_j} : B \to B$ denote the partial derivation, so there exists a map $f_j : \Omega_{B/A} \to B$ because of universal property such that $f_j(dx_i) = \frac{\partial x_i}{\partial x_j}$. $f_j(\sum h_i dx_i) = 0$, we find $h_j = 0$. By applying other f_j's, we obtain dx_i's are linearly independent over B.

2) Let F be a field and K a separable algebraic extension field of F. We know that $\Omega_{F/K}$ is generated by $\{db \mid b \in F\}$. Let $b \in F$, then there exist a polynomial $p(x) \in K[x]$ such that $p(b) = 0$. We have $d(p(b)) = p^{'}(b)d(b) = 0$, but since the extension is separable $p(x)$ does not have multiple roots, which means $p^{'}(b) \neq 0$, so $db = 0$. Then we get $\Omega_{F/K} = 0$.

Definition 2.3.19 An unramified morphism is a local homomorphism $f : R \to S$ of local rings satisfying the following conditions;

1)$f(M_R)S = M_S$

2)S/M_S is a finite and separable over R/M_R.

A morphism $\widetilde{\varphi} : Y \to X$ of schemes is an unramified morphism if it is of finite type and if the maps $\widehat{\varphi} : \mathcal{O}_{X,\widetilde{\varphi}(y)} \to \mathcal{O}_{Y,y}$ are unramified for all $y \in Y$.

Remark : The other definition of an unramified morphism $\varphi : Y \to X$ which is of finite type is that the diagonal morphism $\triangle : Y \to Y \times_X Y$ is an open immersion.

Proof : Chpt. 3 Prop. 3.5 in [3] □

Proposition 2.3.20 Let $f : Y \to X$ be a finite type. Then f is unramified if and only if the sheaf of differentials $\Omega_{Y/X}$ is zero.

Proof : First of all, the sheaf of differentials $\Omega_{Y/X}$ is compatible with the module of differentials defined before. If $U = SpecA$ is an affine open subset of X and $V = SpecB$ is an affine open subset of Y such that $f(V) \subseteq U$ then $V \times_U V = Spec(B \otimes_A B)$

and since $\triangle(Y)$ is locally closed; $\triangle(Y) \cap (V \times_U V)$ is a closed subscheme defined by the $Ker\widetilde{\triangle} = I$ where $\widetilde{\triangle} : B \otimes_A B \to B$. Let $\mathcal{F}$ be the sheaf of ideals of $\triangle(Y)$ in $V \times_U V$. We define the sheaf of relative differentials of Y over X to be the sheaf $\Omega_{Y/X} = \triangle^*(\mathcal{F}/\mathcal{F}^2)$. So the associated module is $I/I^2 = \Omega_{B/A}$ for $\mathcal{F}/\mathcal{F}^2$ which means $\Omega_{V/U} \cong (\Omega_{B/A})^\sim$ where $\sim$ is the functor from the category of B- modules to the category of $\mathcal{O}_Y$- modules. If we cover Y and X with affine schemes, we can obtain $\Omega_{Y/X}$ by gluing the corresponding sheaves $(\Omega_{B/A})^\sim$.

Assume f is unramified. Then we can reduce to the affine subsets such that $X = SpecA$ and $Y = SpecB$. B is finitely genarated A-algebra such that $A_{f^{-1}(p)} \to B_f$ is unramified which means $\widetilde{f}(M_A) = M_B$. Let $M_B \subset M_B^2 \subset ...$ but the chain must stop, so $M_B^n = M_B^{n+1}$ since M_B is the unique maximal ideal, $M_B = 0$ by Nakayama Lemma. Then it follows that B is finite separable field extension of A by the definition of the unramified morphism which was proved in example 2.3.18 Then $\Omega_{Y/X} \cong (\Omega_{B/A})^\sim = 0$.

For the converse, we know the diagonal $\triangle : Y \to Y \times_X Y$ is loclly closed, so as explained before $\triangle(Y) \cap V \times_U V$ is a closed subscheme which is defined I. Assume the sheaf $\Omega_{Y/X}$ is zero, so the associated module I/I^2 is zero. Then for all $p \in Y$, $I_p = I_p^2$, by Nakayama lemma, $I_p = 0$. So for some open subset W of $Y \times_X Y$ containing Y, $I = 0$. This means $(Y, \mathcal{O}_Y) \cong (W, \mathcal{O}_W)$ □

Definition 2.3.21 A morphism $\varphi : Y \to X$ of schemes is etale if it is flat and unramified. Equivalently the finite type of morphism $\varphi : Y \to X$ is etale if it is flat and $\Omega_{Y/X} = 0$.

Example 2.3.22 Let K be a field and $p(x)$ be an irreducible, monic and separable polynomial then the morphism $SpecK[x]/(p(x)) \to SpecK$ is etale. It is clearly flat since K is a field. And since $K[x]/(p(x))$ is a finite separable field extension of K, it is unramified.

Chapter 3

GALOIS CATEGORIES

In this chapter, we give an explanation about the axiomatic characterization of categories that are equivalent to the category of finite sets on which π acts continuously where π is a profinite group that is unique up to isomorphism. Also, we will discuss the relationship between the algebraic fundamental group and topological fundamental group.

3.0.4 Profinite Groups

Definition 3.0.23 A partially ordered set I is called directed if for any elements $i, j \in I$, there exists k satisfying $k \geq i, j$. A projective system is a family of objects $(A_i)_{i\in I}$ together with a family of morphisms $(f_{ij} : A_i \to A_j)_{i\geq j}$ such that $f_{ii} = id_{A_i}$ and $f_{ik} = f_{jk} \circ f_{ij}$ for all $i \leq j \leq k$. Given any such projective system, one has a projective limit;

$$\varprojlim A_i = \{(a_i)_{i\in I} \in \textstyle\prod_{i\in I} A_i : f_{ij}(a_i) = a_j \text{ for all } i, j \in I \text{ with } i \leq j\}$$

Definition 3.0.24 A topological group G is a group that is also a topological space such that the group operations ; the multiplication $G \times G \to G$; $(x, y) \to xy$ and the inversion $G \to G; x \to x^{-1}$ are continuous.

If $A_i's$ are finite groups induced with discrete topology then $\prod A_i$ is the product topology, while $\varprojlim A_i$ becomes a topological group with the subspace topology. By Tychonoff's theorem $\prod A_i$ is compact and $\varprojlim A_i$ is closed since every point of the complement of $\varprojlim A_i$ is interior. Let $(a_i) \notin \varprojlim A_i$, assume $f_{ij}(a_{i_0}) \neq (a_{j_0})$ then $(a_i)_{i\in I}$ has an open neighborhood $U = \{(b_i)_{i\in I} \in \prod A_i \mid b_{i_0} = a_{i_0} \text{ and } b_{j_0} = a_{j_0}\}$ where $\varprojlim A_i \cap U = \emptyset$, so $\varprojlim A_i$ is closed, also compact.

A topological group is called profinite, if it is isomorphic to the inverse limit of a system of finite groups.

Example 3.0.25 1) The collection of rings $(\mathbb{Z}/n\mathbb{Z})$ gives rise to a projective system by the relation of divisibility, $n/m \Leftrightarrow m \geq n$ where the transition maps are the canonical homomorphisms $\mathbb{Z}/m\mathbb{Z} \rightarrow \mathbb{Z}/n\mathbb{Z}$. Then;

$$\widehat{\mathbb{Z}} = \varprojlim \mathbb{Z}/n\mathbb{Z} = \{(a_n)_{n=1}^{\infty} \in \prod_{n=1}^{\infty}(\mathbb{Z}/n\mathbb{Z}), \quad \textit{for all} \quad n/m, a_m = a_n(modn)\}$$

2) Let p be a prime number, $(\mathbb{Z}/p^n\mathbb{Z})_{n>0}$ with the obvious transition maps $\mathbb{Z}/p^n\mathbb{Z} \rightarrow \mathbb{Z}/p^m\mathbb{Z}$ for $n > m$ is a projective system.

$$\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^n\mathbb{Z} = \{(a_n)_{n=1}^{\infty} \in \prod_{n=1}^{\infty}(\mathbb{Z}/p^n\mathbb{Z}), \quad \textit{for all} \quad n, a_{n+1} = a_n(modp^n)\}$$

3) Let L be a galois extension of K. The set of subfields E of L where E is finite galois extension of K form a directed set I since it is partially ordered by inclusion and the composite field of E_1E_2 is also a finite galois extension of K for $E_1, E_2 \in I$. So the galois groups $(Gal(E/K))_{E\in I}$ with the restriction homomorphisms; $Gal(E_2/K) \rightarrow Gal(E_1/K)$ by sending σ to $\sigma_{|E_1}$ for each $E_1 \subset E_2$ gives us a projective system. The inverse limit $\varprojlim Gal(E/K)$ is isomorphic to $Gal(L/K)$. Define the homomorphism $\varphi : Gal(L/K) \rightarrow \varprojlim Gal(E/K)$ by sending σ to $(\sigma_{|E})_{E\in I}$.

$(\sigma_{|E})_{E\in I}$ is in $\varprojlim Gal(E/K)$ since for $E^{'} \in I$, $\sigma_{|E_{|E'}} = \sigma_{|E'}$. φ is $1-1$; let $\sigma_{|E} = id_E$, for all $E \in I$, then $\sigma = id_L$ since $L = \bigcup_{E\in I} E$. L is a galois extension of K so, there is a set of polynomials $F = \{f_K^{\alpha} : f_K^{\alpha}$ is the minimal polynomial of $\alpha \in L\}$ such that L is the splitting field of F over K. The splitting field of $F^{'}$, which is a finite subset of F, over K belongs to I. So the union of the fields in I is the splitting field of F over K; that is L.

Let $(\sigma_{|E})_{E\in I}$ be in the projective limit, set $\eta : L \rightarrow L$ with $\eta(\alpha) = (\sigma_{|E_o}(\alpha))_{E_o\in I}$ Let $\alpha \in E_m$ then $\alpha \in E_oE_m$

$$\sigma_{|E_oE_m|E_m} = \sigma_{|E_m} \text{ and } \sigma_{|E_oE_m|E_o} = \sigma_{|E_o}$$

so $\varphi(\eta) = (\sigma_{|E})_{E \in I}$.

Definition 3.0.26 Let G be a profinite group. A G-set M is a set equipped with an action of G on M. Suppose M is induced by discrete topology and $G \times M$ has the product topology. We say the action is continuous if $G \times M \to M$ is continuous.

Corollary 3.0.27 Let π be a profinite group acting on a set E. The action is continuous if and only if for each $e \in E, Stab_\pi(e) = \{\sigma \in \pi : \sigma e = \sigma\}$ is open in π.

Proof Let $e \in E$, the inverse image of e under the map $G \times M \to M$ is the set

$$\{(\sigma, f) \in \pi \times E : \sigma f = e\} = \bigcup_{(\sigma,f)} f \times \sigma(Stab_\pi(e))$$

which is open , so action is continuous. For the converse, since each projection map is open, $Stab_\pi(e)$ is open. In particular, if E is finite, the kernel of the action $\pi^{'} = \{\sigma \in \pi : \sigma e = e, \text{ for all } e \in E\}$ is open since

$$\pi^{'} = \bigcap_{e \in E} Stab_\pi(e)$$

□

A morphism from a $\pi - set$ to A to a $\pi - set$ $A^{'}$; $f : A \to A^{'}$ satisfies $f(\sigma a) = \sigma f(a)$ for all $\sigma \in \pi$ and $a \in A$. So we can consider the category of finite $\pi - sets$ which is denoted by π**-sets**.

3.0.5 Separable Algebras

Definition 3.0.28 Let R be a ring and M a module over R, which is finitely generated with basis $v_1, v_{2,...}, v_n$. Let $\varphi : M \to M$ be an R-linear map. The trace of the matrix associated to the map φ with respect to the basis is the sum of the elements on the main diagonal

$$Tr(\varphi) = \textstyle\sum_{i=1}^{n} a_{ii} \ \textit{where} \ \varphi(v_i) = \sum_{i=1}^{n} a_{ij} v_j$$

With the map; $\psi_m : M \to M$ where $x \to mx$, we define $Tr_{M/R} : M \to R$ such that $Tr_{M/R}(m) = Tr_{M/R}(\psi_m)$. $\phi : M \to Hom_R(M, R)$ is defined by $(\phi(x))(y) = Tr(xy)$ for x, $y \in M$. M is free separable R- algebra if ϕ is an isomorphism.

Example 3.0.29 Let A be a ring, A^n is a free separable A-algebra for any $n \in \mathbb{Z}_{>0}$. Let $e_1 = (1, 0, ..., 0)$ $...e_n = (0, 0, ..., 1)$ be a basis for A^n. $\phi : A^n \to Hom_A(A^n, A)$ is injective. Let $(a_1, ...a_n) \in A^n$ and $\phi(x_1, ...x_n)(a_1...a_n) = Tr((a_1x_1, ..., a_nx_n)) = Tr(\psi_{(a_1x_1,...,a_nx_n)}) = 0$ for all $x = (x_1, ..., x_n) \in A^n$, then $\psi_{(a_1x_1,...,a_nx_n)}(e_i) = a_ix_i$. $Tr(\psi_{(a_1x_1,...,a_nx_n)}) = \sum_{i=1}^{n} a_ix_i = 0$ for all $x \in A^n$, so $a_i = 0$ for all $i \in \{1, ..., n\}$. Let $\vartheta \in Hom_A(A^n, A)$, we claim that $\vartheta = Tr(\psi_{(\vartheta(e_1),...,\vartheta(e_n))})$. Let $(a_1, ..., a_n) \in A^n$ then $\vartheta((a_1, ..., a_n)) = Tr(\psi_{(\vartheta(e_1)a_1,...,\vartheta(e_n)a_n)}) = \sum_{i=1}^{n} \vartheta(e_ia_i) = \vartheta(a_1, ..., a_n)$ so ϕ is onto.

Definition 3.0.30 A morphism $f : X \to Y$ is a finite etale morphism if there exists a covering of Y by affine open subsets $U_i = SpecA_i$ such that $f^{-1}(U_i)$ of X is affine; $f^{-1}(U_i) = SpecB_i$ where B_i is a free separable A_i- algebra.

Example 3.0.31 Let X be any scheme. The disjoint union $X \coprod X \coprod ... \coprod X$ of n copies for any $n \in \mathbb{Z}_{n>0}$ with the obvious morphism to X is a finite etale covering of X. Let $U_i = SpecB_i$ is an affine scheme, $f^{-1}(U_i) = SpecB_i \coprod SpecB_i \coprod ... \coprod SpecB_i \cong SpecB_i^n$, by example 3.0.29 B_i^n is a free separable algebra over B_i.

Lemma 3.0.32 Let A be a finite dimensional algebra over a field K. Then $A = \prod_{i=1}^{t} A_i$, where $t \in \mathbb{Z}_{>0}$ where A_i are local with nilpotent maximal ideals.

Proof : If A is a domain, then for all $a \in A/\{0\}$, $\varphi_a : A \to A$ where $x \to ax$ is injective. By dimension, it is onto which means there exists x such that $ax = 1$ so it is a field. We can deduce that every prime ideal is maximal. Since A is noetherian and has dimension 0, A is artinian satisfying the descending chain condition. Let $a_i \in P_i$ and and $a_i \notin P_{n+1}$, then $a = a_1...a_n \in \cap_{i=1}^{n} P_i$ and $a \notin \cap_{i=1}^{n+1} P_i$, so the chain $\cap_{i=1}^{n} P_i \supset \cap_{i=1}^{n+1} P_i...$ must stop, this means we have finitely many maximal ideals. Then $\cap_{i=1}^{t} M_i$ is the intersection of all prime ideals of A, so it is equal to the nilradical which is a nilpotent ideal in a noetherian ring;

$$\bigcap_{i=1}^{t} M_i^m \subseteq (\bigcap_{i=1}^{t} M_i) = 0$$

Since M_i^n are relatively comaximal, by the chinese remainder theorem $A \cong \prod_{i=1}^{t} A/M_i^n$. Let $A_i = A/M_i^n$ which has a unique nilpotent maximal ideal M_i/M_i^n. □

Theorem 3.0.33 Let $\overline{K}$ be the algebraic closure of the field K and A be a finite separable algebra over K. Then the followings are equivalent.

i) A is separable over K

ii) $A \otimes \overline{K}$ is separable over $\overline{K}$

iii) $A \otimes_K \overline{K}$ is isomorphic to a finite product of copies of $\overline{K}$

iv) A is isomorphic to a finite product of separable extensions of K

Proof

$(i) \Leftrightarrow (ii)$. Assume $w_1...w_n$ is a K- basis for A. Then $A \otimes_K \overline{K} \cong K^n \otimes_K \overline{K} \cong \overline{K}^n$ which shows $w_1 \otimes 1...w_n \otimes 1$ is a basis for $A \otimes_K \overline{K}$ over $\overline{K}$. Recall that $Tr_{A/K}(\alpha) = Tr_{A/K}(\psi_\alpha)$ such that $\psi_\alpha(x) = \alpha x$. $Tr_{A/K}(\alpha) = \sum a_{ii}$ where $\alpha w_i = \sum_{j=1}^{n} a_{ij} w_j$ and $(\alpha \otimes 1)(w_i \otimes 1) = \sum_{j=1}^{n} a_{ij}(w_j \otimes 1)$, so $Tr_{A/K}(\alpha) = Tr_{A\otimes_K \overline{K}/K}(\alpha \otimes 1)$. Then the natural inclusions for the horizontal arrows,

$$\begin{array}{ccc} A & \xrightarrow{\ i\ } & A \otimes_K \overline{K} \\ {\scriptstyle Tr_{A/K}} \downarrow & & \downarrow {\scriptstyle Tr_{A\otimes_K \overline{K}/K}} \\ K & \xrightarrow{\ i\ } & \overline{K} \end{array}$$

is commutative. A is separable over K so the map $\phi : A \to Hom_K(A, K)$ is invertible as a K-linear map. Observe that the determinant of the associated matrix for the linear map is unit if and only if the linear map is invertible. We claim that the associated matrix for $\phi : A \to Hom_K(A, K)$ is $(Tr(w_i w_j)_{1\le i,j\le n})$. Let $w_1^*...w_n^*$ be the dual basis for $Hom_A(A, K)$ where $w_i^*(w_j) = \delta_{ij}$. $\phi(w_i) = \psi(x) : x \to Tr(w_i x)$.

$$\phi(w_i) = \psi(x) = Tr(w_i w_1)w_1^* + ... + Tr(w_i w_n)w_n^*$$

since they are same in the basis;

$$\begin{aligned} \phi(w_j) = Tr(w_i w_j) &= Tr(w_i w_1)w_1^*(w_j) + ... + Tr(w_i w_n)w_n^*(w_j) \\ &= Tr(w_i w_j) \end{aligned}$$

By the commutative diagram; $Tr_{A/K}(w_iw_j) = Tr_{A\otimes_K \overline{K}/K}((w_i \otimes 1)(w_j \otimes 1))$ so the $det(Tr_{A\otimes_K \overline{K}/K}((w_i \otimes 1)(w_j \otimes 1))_{1\leq i,j\leq n})$ is also not zero, which means $\phi : A \otimes_K \overline{K} \to Hom_K(A \otimes_K \overline{K}, K)$ is also an isomorphism.

(ii) $\Rightarrow$ *(iii)* $A \otimes_K \overline{K}$ is a separable, finitely generated $\overline{K}$-algebra, so by lemma 3.0.32 $A \otimes_K \overline{K} = \prod_{i=1}^t A_i$ with nilpotent maximal ideals M_i. If $\phi : \prod_{i=1}^t A_i \to Hom_{\overline{K}}(\prod_{i=1}^t A_i, \overline{K})$ is an isomorphism, then $\phi_i : A_i \to Hom_{\overline{K}}(A_i, \overline{K})$ is an isomorphism; A_i is separable $\overline{K}$-algebra for each $i \in \{1, ..., t\}$. So for each $\varphi : A_i \to \overline{K}$, there exists an element $a \in A_i$ such that $\varphi(x) = Tr(ax)$ for all $x \in A_i$. If $x \in M_i$ where $x^m = 0$ for some $m \in \mathbb{Z}_{>0}$, $\psi_{ax}(y) = axy$ is a nilpotent map, $\psi^m = 0$. It is a fact that the trace of nilpotent maps is zero. All the eigenvalues of ψ_{ax} is in $\overline{K}$, there is a basis for $A \otimes_K \overline{K}$ with respect to which the matrix for ψ_{ax} is in the jordan canonical form, whose diagonals are the jordan blocks;

$$\begin{pmatrix} J_1 & 0 & \dots & 0 \\ 0 & J_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & J_n \end{pmatrix} \quad where \quad \begin{pmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{pmatrix}$$

Then the minimal polynomial for ψ_{ax} is divided by x^n, that implies the eigenvalues have to be zero. Then $Tr(\psi_{ax}) = 0$. This fact shows that $M_i = Ker\varphi$, for all $\varphi \in Hom_{\overline{K}}(A_i, \overline{K})$. For an injective map for instance, so $M_i = \{0\}$, each A_i is a field over $\overline{K}$. $\overline{K}$ is algebraically closed so $A_i = \overline{K}$.

(iii) $\Rightarrow$ *(iv)* By the lemma 3.0.32, we can write $A = \prod_{i=1}^t A_i$ where A_i has a unique nilpotent maximal ideal. Let $a \in A$, then $K[a] \cong K[x]/f_a$ for $f_a \neq 0$ and when we tensor $i : K[a] \hookrightarrow A$, we obtain;

$$i \otimes id : K[a] \otimes_K \overline{K} \cong \overline{K}[a] \hookrightarrow A \otimes_K \overline{K} \cong \overline{K}^n$$

Since $\overline{K}^n$ has no non-zero nilpotent element. Then f_a is separable.

Let $a \in A$ be a nilpotent element $a^n = 0$ for some $n \in \mathbb{Z}_{>0}$ then (f_a) contains x^n, but since f_a is separable $x \in (f_a)$ which means $a = 0$. Then all A_i's are field, otherwise set $a = (0, 0, .., a_i, , 0)$ where $a_i \in A_i$ is nilpotent.

These extensions are separable since if $a = (a_1...a_k) \in A$, then f_a is equal to the least common multiple of f_{a_i} for $i \in \{1, ..., k\}$. We know f_a is separable, so f_{a_i} is separable for all i.

$(iv) \Rightarrow (iii)$ The primitive element theorem asserts that if K/F is finite and separable then K/F is simple. Then $A = \prod_{i=1}^{t} A_i$, where $A_i = K[\alpha_i] \cong K[x]/f_{\alpha_i}$ that f_{α_i} is separable and irreducible. $A \otimes_K \overline{K} = (\prod_{i=1}^{t} A_i) \otimes_K \overline{K} = \prod_{i=1}^{t} A_i \otimes_K \overline{K}$ where $A_i \otimes_K \overline{K} = K[x]/f_{\alpha_i} \otimes_K \overline{K} \cong \overline{K}[x]/f_{\alpha_i}$. f_{α_i} splits into distinct linear factors $(x - \alpha_{ij})$ in $\overline{K}[x]$. By chinese remainder theorem;

$$\overline{K}[x]/f_{\alpha_i} \cong \prod_{j=1}^{m} \overline{K}[x]/x - \alpha_{ij} \cong \overline{K}^{degf_{\alpha_i}}$$

Hence $A \otimes_K \overline{K} \cong \overline{K}^n$ for some $n \in \mathbb{Z}_{>0}$.

$(iii) \Rightarrow (ii)$ By example 3.0.29, this is obvious. □

Proposition 3.0.34 Let Y be a noetherian scheme then finite $\acute{e}$tale morphism $f : X \to Y$ ie equivalent to finite and $\acute{e}$tale morphism.

Proof : First of all, by the definition of the finite $\acute{e}$tale morphism, it is also finite and locally free, so flat. So we reduce the problem to the assertion that B is separable over A if and only if $SpecB \to SpecA$ is unramified where B is an algebra over a ring A and B is finitely generated and free as an A-module

Step 1) $\widetilde{\varphi} : SpecB \to SpecA$ is unramified if and only if for all $q \in SpecB$, $\varphi(p) = q$ and $B_q/qB_q \cong B \otimes_A k(p)$ is a finite separable field extension of $A_p/pA_p = k(p)$. This is equivalent to say $Spec(B \otimes_A k(p)) \to Speck(p)$ is unramified for every $P \in SpecA$.

Step 2) B is separable over A if and only if $B \otimes_A k(p)$ is separable over $k(p)$ for every $p \in SpecA$. By the definition, B is separable over A if and only if the map $\phi : B \to Hom_A(B, A)$ is an isomorphism. Let $\{v_1, ..., v_n\}$ be A- basis for B, recall that ϕ is an isomorphism if and only if $det(Tr_{B/A}(v_i v_j))_{1 \leq i,j \leq n} = a$ is invertible in A. Note that $\{v_1 \otimes 1, ..., v_n \otimes 1\}$ is a $k(p)$- basis for $B \otimes_A k(p)$ and $B \otimes_A k(p) \to Hom_{k(p)}(B \otimes_A k(p), k(p))$ is an isomorphism if and only if $det(Tr_{B \otimes_A k(p)/k(p)}((v_i \otimes 1)(v_j \otimes 1))_{1 \leq i,j \leq n} = det(Tr_{B \otimes_A k(p)/k(p)}((v_i v_j \otimes 1))_{1 \leq i,j \leq n} = b$ is invertible in $k(p)$. There exists an f_p such that $\theta_1 : A \to A_p$ and $\theta_2 : A_p \to k(p)$ and $f_p = \theta_2\theta_1$. Then $k(p)$ is an A- module via f_p. Consider the diagram of natural maps

$$\begin{array}{ccc} End_A(B) & \xrightarrow{id_{k(p)}} & End_{k(p)}(B \otimes_A k(p)) \\ \Big\downarrow {\scriptstyle Tr_{B/A}} & & \Big\downarrow {\scriptstyle Tr_{B\otimes_A k(p)/k(p)}} \\ A & \xrightarrow[f_p]{} & k(p) \end{array}$$

Let $\psi \in End_A(B)$ and let $\mathcal{M}_\psi = (a_{ij})_{1\le i,j\le n}$ be the matrix of ψ with respect to $\{v_1, ..., v_n\}$, then $Tr_{B/A}(\psi) = \sum_{i=1}^n a_{ii}$. $id_{k(p)}(\psi)(v_i \otimes 1) = \psi(v_i) \otimes 1 = (\sum_{j=1}^n a_{ji}v_j) \otimes 1 = \sum_{j=1}^n (a_{ji}v_j) \otimes 1 = \sum_{j=1}^n v_j \otimes f_p(a_{ji}) = \sum_{j=1}^n f_p(a_{ji})(v_j \otimes 1)$ and $\mathcal{M}_{id_{k(p)}(\psi)} = (f_p(a_{ij}))_{1\le i,j\le}$, then $Tr_{B\otimes_A k(p)/k(p)}(id_{k(p)}(\psi)) = \sum_{i=1}^n f_p(a_{ii}) = f_p(\sum_{i=1}^n a_{ii}) = f_p(Tr_{B/A}(\psi))$. So the diagram is commutative. By this fact, $f_p(a) = f_p(det(Tr_{B/A}(v_iv_j))_{1\le i,j\le n}) = det(f_p((Tr_{B/A}(v_iv_j))_{1\le i,j\le n})) = det(Tr_{B\otimes_A k(p)/k(p)}(v_iv_j \otimes 1)_{1\le i,j\le n} = b$. Assume a is invertible in A then $a \notin p$ for every $p \in SpecA$. This is equivalent to $f_p(a) = b$ is invertible in $k(p)$.

Step 3)By combining Step 1 and Step 2, we change the assertion to the case that A is a field. So the remaining part is to show that $SpecB \to SpecA$ is unramified if and only if B is separable over A where A is a field. By lemma 3.0.32 $B = \prod_{i=1}^t B_i$ for some $t \in \mathbb{Z}_{>0}$ where B_i's are local rings with nilpotent maximal ideals. We prove in the lemma 3.1.5 that the localizations of B at all prime ideals of B are equal to $B_i's$. By the definition of unramified morphism and by the theorem 3.0.33, B_i is finite separable field extension of A which is equivalent to $B = \prod_{i=1}^t B_i$ is separable over A. □

3.1 Galois Categories and Examples

Definition 3.1.1 Let $\mathbf{C}$ be a category and $\mathcal{F}$, a covariant functor from $\mathbf{C}$ to the category of sets of finite sets. A category $\mathbf{C}$ satisfying the following conditions is called a galois category with fundamental functor; $\mathcal{F}$

G.1) $\mathcal{C}$ has a terminal object. A terminal object of a category $\mathcal{C}$ is an object Z which is unique up to ismorphism such that there exists exactly one morphism $X \to Z$ for every object X in $\mathcal{C}$. Also, the fiber product of any two objects over a third one exists in $\mathcal{C}$. The fiber product is defined as the same for schemes.

G.2) $\mathcal{C}$ has an initial object and finite sums exists in $\mathcal{C}$. The finite sum of the object $\bigoplus_{i=1}^{n} X_i$ with morphisms $\theta_i : X_i \to \bigoplus_{i=1}^{n} X_i$ for each $i = 1...n$ such that for any object Y with morphisms $\widetilde{\theta}_i : X_i \to Y$, there is a unique morphism $\rho : \bigoplus_{i=1}^{n} X_i \to Y$ where $\widetilde{\theta}_i = \rho\theta_i$. The quotient by a finite group automorphisms exists. The quotient X/G of X by $G \subseteq Aut(X)$ is an object in $\mathcal{C}$ with a morphism $\varphi : X \to X/G$ that satisfies $\varphi = \varphi\sigma$ for all $\sigma \in G$, such that for any morphism $\tilde{\varphi} : X \to Y$ in $\mathcal{C}$ satisfying $\tilde{\varphi} = \tilde{\varphi}\sigma$ for all $\sigma \in G$, there is a unique morphism $\rho : X/G \to Y$ such that $\tilde{\varphi} = \rho\varphi$.

G.3) Any morphism $\varphi : X \to Y$ in $\mathcal{C}$ factors as $\varphi'\varphi''$ where φ' is a monomorphism and φ'' is an epimorphism. $\varphi'' : X \to Y$ is an epimorphism if for any object Z and any morphisms $\psi, \eta : Y \to Z$ with $\psi\varphi'' = \eta\varphi''$, we have $\psi = \eta$. $\varphi' : X \to Y$ is n monomorphism if for any object Z and any morphisms $\psi, \eta : Z \to X$ with $\varphi'\psi = \varphi'\eta$, we have $\psi = \eta$. Any monomorphism $\varphi : X \to Y$ in $\mathcal{C}$ is an isomorphism of X with direct summand of Y.

F.1) $\mathcal{F}$ transforms terminal objects in terminal objects and $\mathcal{F}(X \times_S Y) = \mathcal{F}(X) \times_{\mathcal{F}(S)} \mathcal{F}(Y)$

F.2) $\mathcal{F}(\bigoplus_{i=1}^{n} X_i) = \coprod_{i=1}^{n} \mathcal{F}(X_i)$. $\mathcal{F}(f)$ is an epimorphism if f is an epimorphism. Also, $\mathcal{F}$ commutes with passage to the quotient by a finite group of automorphisms.

F.3) A morphism $\varphi : X \to Y$ is an isomorphism if $\mathcal{F}(\varphi) : \mathcal{F}(X) \to \mathcal{F}(Y)$ is an isomorphism .

Example 3.1.2 The category of sets is a galois category with the identity functor. The terminal object is single element set. For $f : X \to S$ and $g : Y \to S$ the fiber product is $X \times_S Y = \{(x, y) \in X \times_S Y f(x) = g(y)\}$. The empty set is the initial object and the disjoint union of sets is the finite sums of sets. The quotient is the set of orbits of X under G. Every morphism $f : X \to Y$ in sets can be written as $f : X \to_{\varphi_1} imf \to_{\varphi_2} Y$ where φ_2 is a monomorphism and φ_1 is an epimorphism.

Since the fundamental functor is identity, it clearly satisfies conditions (F.1), (F.2) and (F.3)

Definition 3.1.3 Let **C** be a galois category with the functor $\mathcal{F}$. An automorphism σ of functor $\mathcal{F}$; $\sigma \in Aut(\mathcal{F})$ is a collection of bijections $\sigma_X : \mathcal{F}(X) \to \mathcal{F}(X)$ for all $Obj(\mathbf{C})$ such that the diagram

$$\begin{array}{ccc} \mathcal{F}(X) & \xrightarrow{\mathcal{F}(f)} & \mathcal{F}(Y) \\ \downarrow \sigma_X & & \downarrow \sigma_Y \\ \mathcal{F}(X) & \xrightarrow{\mathcal{F}(f)} & \mathcal{F}(Y) \end{array}$$

is commutative for each morphism $f : X \to Y$ in **C**

Theorem 3.1.4 (Main Theorem) Let **C** a Galois category with the fundamental functor $\mathcal{F}$ and let **C** be an essentially small category which means **C** is equivalent to a category whose objects form a set Then **C** is equivalent to the category of finite sets on which $Aut(\mathcal{F})$ acts continuously; $Aut(\mathcal{F})$-**sets**

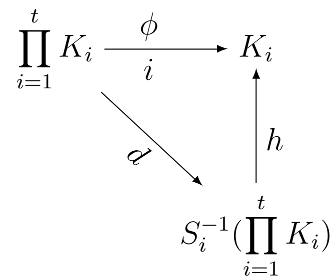
Proof : Chpt. 3 3.11-3.19 in [5] □

Lemma 3.1.5 Let $X = SpecK$ where K is a field. The finite, etale coverings $Y \to X$ of X are precisely given by $Y = \coprod_{i=1}^{t} SpecK_i$ where each K_i is a finite, separable field extension of K.

Proof : Assume $f : Y \to X = SpecK$ is a finite, etale morphism. Since the morphism is finite, $f^{-1}(SpecK) = SpecB$ where B is a K- algebra and finitely generated K- module. By lemma 3.0.32 $B \cong \prod_{i=1}^{t} K_i$ where each K_i has a unique, nilpotent maximal ideals M_i. The prime ideals of B are isomorphic to that of $\prod_{i=1}^{t} K_i$ and they are of the form ;

$$P_i = \{(b_1, ..., b_i, ..., b_t) \mid b_i \in M_i \text{ and } b_j \in K_j \text{ for } j \in 1, ..., t\}$$

Let $S_i = (\prod_{i=1}^t K_i)/P_i = \{(b_1, ..., b_t) \mid b_i \notin M_i\}$ and $\phi_i : \prod_{i=1}^t K_i \to K_i$ such that $s = (s_1...s_n) \in S_i$. $\phi_i(s) = s_i \notin M_i$ which means s_i is unit in K_i. Then there exists a unique ring homomorphism $h : S_i^{-1}(\prod_{i=1}^t K_i) \to K_i$ such that $\phi_i = h \circ f$.



clearly, h is surjective. To show h is injective; let $\frac{(b_1,...,b_n)}{(x_1,...,x_n)} \in S_i^{-1}(\prod_{i=1}^t K_i)$ such that $h(\frac{(b_1,...,b_n)}{(x_1,...,x_n)}) = b_i = 0$ then $(b_1, ..., b_n) = 0$ since there exists a non-zero $s = (0, ...1, 0..0)$ where 1 is in the ith coordinate such that $(b_1, ..., b_n)s = 0$ So each K_i is isomorphic to the localization of B. By the definition of the unramified morphism $K \to K_i$, K_i is a finite separable extension of K. By the fact in the proof of the theorem 3.1.8 $Spec \prod_{i=1}^t K_i \cong \bigsqcup_{i=1}^t SpecK_i$. Conversely, $Spec \prod_{i=1}^t K_i \to SpecK$ where K_i's are finite separable extensions of K is an etale morphism. It is flat since K is field and every localization of $\prod_{i=1}^t K_i$ is isomorphic to K_i, the map is unramified. □

Example 3.1.6 Consider the category FEt_{SpecK}, the category of finite, etale coverings of $SpecK$ which has objects of the form $SpecA$ where A is free separable K-algebra and the morphism are the morphisms between the schemes $h : SpecA_1 \to SpecA_2$ which is compatible with the morphisms $\theta_1 : SpecA_1 \to SpecK$ and $\theta : SpecA_2 \to SpecK$, such that; $\theta_1 = \theta_2 h$. Let x a geometric point of $SpecK$ which is a morphism $x : Spec\Omega \to SpecK$ for some algebraically closed field Ω. Define the functor

$$F_x : FEt_{SpecK} \to \{\text{Finite sets}\}$$

by $SpecA \to Hom_K(Spec\Omega, SpecA)$ Then FEt_{SpecK} is a galois category with the fundamental functor F_x. Since it satisfies the following conditions;

*G.1)*Let A, B and C are free separable K-algebras with the morphisms $C \to A$ and $C \to B$ then $A \otimes_C B$ exists in FEt_{SpecK}, since it is equal to $\prod_{j=1}^{n} A_{i_j} \otimes_{C_j} B_{l_j}$ where A_{i_j}, B_{l_j} and C_j are finite separable extensions of K. When we tensor with $\cdot \otimes_{C_j} \overline{C_j}$; $\prod_{j=1}^{n}(A_{i_j} \otimes_{C_j} B_{l_j}) \otimes_{C_j} \overline{C}_j \cong \prod_{j=1}^{n} A_{i_j} \otimes_{C_j} (B_{l_j} \otimes_{C_j} \overline{C}_j) \cong \prod_{j=1}^{n} A_{i_j} \otimes_{C_j} \overline{C}_j^n \cong \prod \overline{C}_j^{nm}$ for some $n, m \in \mathbb{Z}_{>0}$ and observe $\overline{C}_j = \overline{K}$, so $A \otimes_C B$ is a free separable K- algebra by proposition []. The terminal object is $SpecK \to SpecK$.

G.2) $\emptyset \to SpecK$ is the initial object and finite sums $\coprod_{i=1}^{n} SpecA_i \to SpecK$ exists in the category since $\coprod_{i=1}^{n} SpecA_i \cong Spec \prod_{i=1}^{n} A_i$ where each A_i is a separable algebra so $A_i = \prod_{j=1}^{t_i} K_{i_j}$ where K_{i_j}'s are finite separable extensions of K.

The quotient by a finite group automorphism exists. Let G be a finite group of $K-$ algebra automorphisms of A, extend G to $A \otimes_K \overline{K}$ which means for all $\sigma \in G$, $\sigma : A \to A$, there exists $\overline{\sigma} : A \otimes_K \overline{K} \to A \otimes_K \overline{K}$ since $\cdot \otimes_K \overline{K}$ is a flat functor. If $w_1, ..., w_n$ is a basis for A over K, then $w_1 \otimes 1, ..., w_n \otimes 1$ is a basis for $A \otimes_K \overline{K}$ over $\overline{K}$. Let G be the group of automorphisms fixing some basis elements, assume $w_1 ... w_r$ then $\overline{G}$ is the group of automorphisms fixing $w_1 \otimes 1 ... w_r \otimes 1$. It is obviously $(A \otimes \overline{K})^{\overline{G}} \cong A^G \otimes \overline{K}$. We know A is separable $\overline{K}$- algebra so $(A \otimes \overline{K})^{\overline{G}} \cong (\overline{K}^n)^{\overline{G}} \cong \overline{K}^r$ so $A^G \otimes \overline{K}$ is a separable $\overline{K}$- algebra. So the quotient $(SpecA)/G$ which is $SpecA^G$ exists in the category.

G.3) Let $\theta : A \to B$ be a K- algebra homomorphism, then it factors $\theta : A \to_{\varphi} im\varphi \to_{\psi} B$ where φ is an epimorphism and ψ is a monomorphism. If we tensor $\cdot \otimes_K \overline{K}$, then we have $A \otimes_K \overline{K} \to im\varphi \otimes_K \overline{K} \to B \otimes_K \overline{K}$ where $A \otimes_K \overline{K} \cong \overline{K}^n$ and $B \otimes_K \overline{K} \cong \overline{K}^m$ for some n, $m \in \mathbb{Z}_{n>0}$ and $\varphi \otimes id$ is an epimorphism and $\psi \otimes id$ is a monomorphism then $im\varphi \otimes_K \overline{K} \cong \overline{K}^t$ for some $t \in \mathbb{Z}_{n>0}$. Hence $im\varphi$ is a free separable K-algebra. Also since $\theta \otimes id$ behaves as a linear map between $\overline{K}^n$ and $\overline{K}^m$, $Ker(\theta \otimes id)$ is also a free separable $\overline{K}$- algebra, so $B/im\varphi$ is a free separable K-algebra. Thus we have verified $B = B/im\varphi \coprod im\varphi$

F.1) Since the terminal object $SpecK \to SpecK$, $\mathcal{F}_x(SpecK) = Hom(Spec\overline{K}, SpecK) \cong Alg_K(K, \overline{K})$ that contains only the inclusion map, which has single element set, So $\mathcal{F}_x$ transforms terminal object to the terminal object of set category. Any map from $K_1 \otimes_K K_2 \to \overline{K}$ gives us maps $K_1 \to \overline{K}$ and $K_2 \to \overline{K}$, and by the universal property, these maps lead to the map $K_1 \otimes_K K_2 \to \overline{K}$, so there is a correspondence between sets $Alg_K(K_1 \otimes_K K_2, \overline{K})$ and $Alg_K(K_1, \overline{K}) \times Alg_K(K_2, \overline{K})$ which means $\mathcal{F}_x$ commutes with the fiber product.

F.2) $\mathcal{F}_x$ commutes with the finite sums. Let $\coprod_{i=1}^{t} SpecA_i$ where A_i's are free separable K-algebras then $\coprod_{i=1}^{t} SpecA_i \cong Spec \prod_{i=1}^{t} A_i$ and also $A_i = \prod_{j=1}^{j=n_i} K_{i_j}$ where K_{i_j}'s finite, separable field extensions of K.

$$\mathcal{F}(\coprod_{i=1}^{t} SpecA_i) = Hom_{SpecK}(Spec\overline{K} Spec \prod_{i=1}^{t} A_i) \cong Alg_K(\prod_{i=1}^{t} A_i, \overline{K}) \cong$$
$$Alg_K(\prod_{i=1,j=1}^{i=t,j=n_i} K_{i_j}, \overline{K}) \cong \coprod_{i=1,j=1}^{i=t,j=n_i} Alg_K(K_{i_j}, \overline{K}) \cong \coprod_{i=1}^{t} Alg_K(\prod_{j=1}^{n_i} K_{i_j}, \overline{K}) \cong$$
$$\coprod_{i=1}^{t} Hom(Spec\overline{K}, SpecA_i) \cong \coprod_{i=1}^{t} \mathcal{F}_x(SpecA_i).$$

Also $Alg_K(A^G, \overline{K}) = (Alg_K(A, \overline{K}))^{\mathcal{F}_x(G)}$. Let $g : A^G \to \overline{K}$ then $g\sigma = g$, for all $\sigma \in G$ then $g \in Alg_K(A, \overline{K})^{\mathcal{F}_x(G)}$ since $\mathcal{F}_x(\sigma)g = g\sigma = g$, for all $\sigma \in G$ For the other side, let $g \in (Alg_K(A, \overline{K})^{\mathcal{F}_x(G)})$, then $\mathcal{F}_x(\sigma)g = g$ for all $\sigma \in G$ which means $\mathcal{F}_x(\sigma)g = g\sigma = g$, for all $\sigma \in G$ then $g \in Alg_K(A^G, \overline{K})$

F.3) Let $\tilde{f} : SpecA \to SpecB$ where A and B are free separable K-algebras, let $\mathcal{F}_x(f) : Alg_K(A, \overline{K}) \to Alg_K(B, \overline{K})$ which is an isomorphism. Then $\coprod Alg_K(A_i, \overline{K}) \to \coprod Alg_K(B_i, \overline{K})$ is an isomorphism where A_i and B_i are fields. Then there is a correspondence with these fields, so $f : Spec \prod A_i \to Spec \prod B_i$ is an isomorphism.

Theorem 3.1.7 Let X be a connected scheme. Then there exists a profinite group π, uniquely determined up to isomorphism, such that the category FEt_X of finite etale coverings of X is equivalent to the category of π-sets of finite set on which π acts continuously.

Proof : Chpt. 5 5.22-5.23 in [5] □

The profinite group π occurring in the above theorem is called the fundamental group of X. Now, we will prove the special case of the theorem above. For a certain profinite group π and for a field K, the category of free separable K-algebras; ${}_K SAlg$ is antiequivalent to the category of π-sets. Recall that we know, ${}_K SAlg$ is antiequivalent to FEt_{SpecK} for a field K.

Theorem 3.1.8 Let K be a field and $\pi = Gal(K^s/K)$. Then the categories ${}_K SAlg$ of free separable K-algebras and π-sets of finite sets with a continuous action are antiequivalent.

Proof : The idea is two find two contravariant functors $\mathcal{F} :_K SAlg \to \pi - sets$ and $\mathcal{G} : \pi - sets \to_K SAlg$ such that $\mathcal{FG}$ is naturally equivalent to the identity functor on π-sets and $\mathcal{GF}$ is naturally equivalent to the identity functor on ${}_K SAlg$. That is, to find a collection of isomorphisms $\theta_A : A \to \mathcal{GF}(A)$, making the diagram

$$\begin{array}{ccc} A & \xrightarrow{\ f\ } & B \\ {\scriptstyle \theta_A}\downarrow & & \downarrow{\scriptstyle \theta_B} \\ \mathcal{GF}(A) & \longrightarrow & \mathcal{GF}(B) \end{array}$$

commutative for any morphism $f : A \to B$ in ${}_K SAlg$ and similarly for the natural transformations between S and $\mathcal{FG}(S)$.

Step 1 : Set; $\mathcal{F}(B) = Alg_K(B, K_s)$ for each free separable K-algebra B. Here $Alg_K(B, K_s)$ is the set of field homomorphisms $B \to K_s$ that are identity on K. Define the π-action on $Alg_K(B, K_s)$ as;

$$\sigma \cdot g = \sigma \circ g \in Alg_K(B, K_s) \text{ for } g : B \to K_s \text{ and } \sigma \in \pi.$$

Since B is a separable K-algebra, $B = \prod_{i=1}^t K_i$ where K_i's are finite separable field extensions of K. So $K_i = K_s^{\pi_i}$ for some open subgroup π_i of π.

We can identify $Alg_K(\prod_{i=1}^t K_s^{\pi_i}, K_s)$ with the disjoint union of the sets; $Alg_K(K_s^{\pi_i}, K_s)$. Let $\varphi_i \in Alg_K(\prod_{i=1}^t K_s^{\pi_i}, K_s)$, then $\varphi_i : \prod_{i=1}^t K_s^{\pi_i} \to K_s$. Since $Ker\varphi_i$ is prime and every prime ideal of $\prod_{i=1}^t K_s^{\pi_i}$ is of the form $(1, .., 0, .., 1)$, we have $\overline{\varphi}_i : \prod_{i=1}^t K_s^{\pi_i}/Ker\varphi_i \cong K_s^{\pi_i} \to K_s$. This identification is onto since for any element in φ in $\coprod_{i=1}^t Alg_K(K_s^{\pi_i}, K_s)$; $\varphi : K_s^{\pi_i} \to K_s$, there exists $\psi : \prod_{i=1}^t K_s^{\pi_i} \to K_s$ such that $\psi = \varphi\rho_i$ where $\rho_i : \prod_{i=1}^t K_s^{\pi_i} \to K_s^{\pi_i}$. Then we have;

$$Alg_K(\prod K_s^{\pi_i}, K_s) \cong \coprod Alg_K(K_s^{\pi_i}, K_s)$$

$K_s^{\pi_i}$ correspond to $\pi_i = Aut_{K_s^{\pi_i}}(K_s)$, so there is a bijective map;

$$\pi/\pi_i \to Alg_K(K_s^{\pi_i}, K_s)$$

which assigns each $\sigma \in \pi/\pi_i$ to $\sigma_{|K_s^{\pi_i}}$. This map is injective if $\sigma_{|K_s^{\pi_i}} = id$, then $\sigma_{|K_s^{\pi_i}} \in Aut_{K_s^{\pi_i}}(K_s) = \pi_i$. To show surjectivity; each $\tau : K_s^{\pi_i} \to \tau[K_s^{\pi_i}] \subset K_s$ which $\tau_K = id_K$ can be extended to $\tau' : \overline{K} \to \overline{K}$ and $\tau'_{|K_s} \in \pi$ since K_s is normal over K. Then $\tau'_{|K_s^{\pi_i}} = \tau$. This identification is compatible with the π- action.

We know π_i is open, so π_i has a finite index. This means π/π_i is a finite set. The last thing we will show for this step is the continuity of the action;

$$\pi \times \coprod \pi/\pi_i \to \coprod \pi/\pi_i$$

$\coprod \pi/\pi_i$ is endowed with the discrete topology. Let $\sigma \in \pi/\pi_i$, the inverse image of σ under the action is equal to $\{(\tau, \pi/\pi_i) \mid \tau \in \pi_i\} = \pi_i \times \pi/\pi_i$ which is open.

Step 2 : Now we define $\mathcal{G}$. For a finite π- set S, $\mathcal{G}(S) = Mor_\pi(S, K_s)$. The set of morphisms of π-sets from S to K_s endowed with the operations $(+, \cdot)$ such that;

$$\begin{aligned}
f + g(s) = f(s) + g(s) \quad and \quad f + g(\sigma s) &= f(\sigma s) + g(\sigma s) \\
&= \sigma f(s) + \sigma g(s) \\
&= \sigma(f+g)(s) \\
fg(s) = f(s)g(s) \quad and \quad fg(\sigma s) &= f(\sigma s)g(\sigma s) \\
&= \sigma f(s)\sigma g(s) \\
&= \sigma(fg)(s) \\
kf(s) = k \cdot f(s) \quad and \quad kf(\sigma s) = k \cdot f(\sigma s) &= k \cdot \sigma f(s) \\
&= \sigma \cdot k(f(s)) \\
&= \sigma \cdot ((kf)(s)) \\
\forall s \in S \quad 1(S) = 1 \quad and \quad 1(\sigma s) = 1 = \sigma(1s)
\end{aligned}$$

So, $\mathcal{G}(S)$ is induced with K- algebra structure. We will verify $Mor_\pi(S, K_s)$ is a finite dimensional separable K-algebra. Firstly decompose S into its orbits under the action of π; $S = \coprod_{i=1}^n S_i$.

Let $\varphi \in Mor_\pi(\coprod_{i=1}^n S_i, K_s)$ gives us $\varphi_{|S_i} \in Mor_\pi(S_i, K_s)$ and $\psi \in \prod_{i=1}^n Mor_\pi(S_i, K_s)$ gives us $\coprod_{i=1}^n S_i \to K_s$, we can write;

$$G(S) = Mor_\pi(\coprod_{i=1}^n S_i, K_s) \cong \prod_{i=1}^n Mor_\pi(S_i, K_s)$$

As a finite π-set, S_i corresponds a subgroup, π_i the kernel of the action on S_i, that is open by the corollary 3.0.27. Hence we may identify S_i with π/π_i.

We claim that each element $a \in K_s^{\pi_i}$ defines a well-defined map of π-sets, $g_a : \pi/\pi_i \to K_s$ such that $g_a(\pi_i) = a$ Hence it satisfy;

$$g_a(\sigma\pi_i) = \sigma g_a(\pi_i) = \sigma(a)$$

and it is well-defined since for all $\sigma \in \pi_i$

$$g_a(\sigma(\pi_i)) = g_a(\pi_i) = a = \sigma(a) = \sigma g(\pi_i)$$

Hence we can write;

$$Mor_\pi(\pi/\pi_i, K_s) \cong K_s^{\pi_i}$$

which implies

$$\mathcal{G}(E) \cong \prod_{i=1}^n K_s^{\pi_i}$$

By theorem 3.0.33 $\mathcal{G}(E)$ is a finite dimensional separable K-algebra.

Step 3 : The functors $\mathcal{F}$ and $\mathcal{G}$ are contravariant. For A, $B \in_K SAlg$ and $f : A \to B$, a K- algebra morphism;

$$\mathcal{F}(f) : Alg_K(B, K_s) \to Alg_K(A, K_s)$$

where $\mathcal{F}(f)(g) = g \circ f$ for $g : B \to K_s$.

Let S, T be finite π-sets and $f : S \to T$ be a morphism of π-sets, then

$$\mathcal{G}(f) : Mor_\pi(T, K_s) \to Mor_\pi(S, K_s)$$

where $\mathcal{G}(f)(h) = h \circ f$ for $h : T \to K_s$.

Step 4 : Now we find natural transformations between $I_{_K SAlg} \to \mathcal{GF}$ and $I_{\pi-sets} \to \mathcal{FG}$. Let $B \in_K SAlg$, define

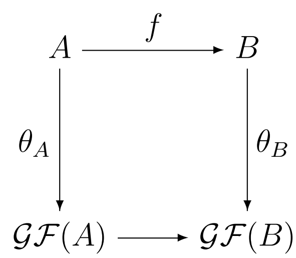
$$\theta_B : B \to \mathcal{GF}(B) = Mor_\pi(Alg_K(B, K_s), K_s)$$

which is defined by $\theta_B(b) : Alg_K(B, K_s) \to K_s$; $\theta_B(b)(h) = h(b)$, for $b \in B$ and $h \in Alg_K(B, K_s)$. $\theta_B(b)$ is a K-algebra homomorphism;

$$\theta_B(b + c)(h) = h(b + c) = h(b) + h(c) = \theta_B(b)(h) + \theta_B(c)(h)$$

$$\theta_B(k)(g) = g(k) = k \text{ since } g \in Alg_K(B, K_s).$$

Let $f : A \to B$ be a K-algebra morphism then the diagram



is commutative, it is because we have for $a \in A$ and $g \in Alg_K(B, K_s)$;

$$(\theta_B \circ f)(a)(g) = \theta_B(f(a))(g) = g(f(a))$$

$$(\mathcal{GF}(f))(\theta_A(a))(g) = \theta_A(a))(g \circ f) = g \circ f(a)$$

Now,we set a natural transformation between $I_{\pi-sets}$ and $\mathcal{FG}$. Let S be a finite π-set; define

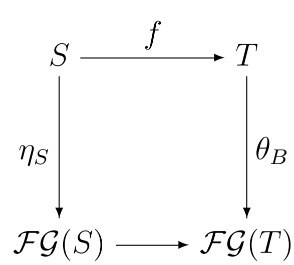
$$\eta_S : S \to \mathcal{FG}(S) = Alg_K(Mor_\pi(S, K_s), K_s)$$

where $\eta_S(s) : Mor_\pi(S, K_s) \to K_s$; $\eta_S(s)(g) = g(s)$.

$\eta_S(s)$ is a π-set morphism since;

$$\eta_S(s)(\sigma g) = \sigma g(s) = \sigma \eta_S(s)(g)$$

For any π-set morphism, $f : S \to T$, the diagram



is commutative, since we have

$$(\eta_T \circ f)(s)(g) = \eta_T(f(s))(g) = g(f(s))$$

$$(\mathcal{F}\mathcal{G}(f))(\eta_S(s))(g) = \eta_S(f(s))(g) = g(f(s))$$

Step 5 : Both θ_B and η_S are isomorphisms. For $B = \prod_{i=1}^{t} K_s^{\pi_i}$, we know from Step 1 and Step 2;

$$\begin{aligned} Mor_\pi(Alg_K(\prod_{i=1}^{t} K_s^{\pi_i}, K_s), K_s) &\cong Mor_\pi(\coprod_{i=1}^{t}(Alg_K(K_s^{\pi_i}, K_s)), K_s) \\ &\cong \prod_{i=1}^{t} Mor_\pi(Alg_K(K_s^{\pi_i}, K_s), K_s) \end{aligned}$$

We use the identifications in the previous steps;

$$\prod_{i=1}^{t} Mor_\pi(Alg_K(K_s^{\pi_i}, K_s), K_s) \cong \prod_{i=1}^{t} Mor_\pi(\pi/\pi_i, K_s) \cong \prod_{i=1}^{t} K_s^{\pi_i}$$

so θ_B is an isomorphism.

For $S = \coprod_{i=1}^{t} S_i \cong \coprod_{i=1}^{t} \pi/\pi_i$,

$$\begin{aligned} Alg_K(Mor_\pi(\coprod_{i=1}^{t} \pi/\pi_i, K_s), K_s) &\cong Alg_K(\prod_{i=1}^{t}(Mor_\pi(\pi/\pi_i, K_s)), K_s) \\ &\cong Alg_K(K_s^{\pi_i}, K_s) \\ &\cong \coprod_{i=1}^{t} Alg_K(K_s^{\pi_i}, K_s) \\ &\cong \coprod_{i=1}^{t} \pi/\pi_i \end{aligned}$$

so η_S is an isomorphism. □

Note that the theorem we proved above is the special case of the main theorem for Galois Categories.

Now we look at the classic topological fundamental group concept and state the relation with galois categories and algebraic fundamental group.

3.2 Topological Fundamental Group

Definition 3.2.1 Let X be a topological space. A path; α in X from p_0 to p_1 is a continuous map; $\alpha : [0,1] \to X$ such that $\alpha(0) = p_0$ and $\alpha(1) = p_1$.

Two paths, α_1 and α_2 on X with the same initial and end points such that $\alpha_1(0) = \alpha_2(0) = p_0$ and $\alpha_1(1) = \alpha_2(1) = p_1$ are homotopic if there exists a continuous map $h : [0,1] \times [0,1] \to X$ such that

$$h(0,s) = p_0 \text{ and } h(1,s) = p_1 \text{ for all } s \in [0,1]$$

$$h(t,0) = \alpha_1(t) \text{ and } h(t,1) = \alpha_2(t) \text{ for all } t \in [0,1]$$

Homotopy relation is an equivalence relation. Let α_1, α_2 and α_3 are paths on X.

i) $\alpha_1 \sim \alpha_1$

The homotopy can be defined as $h(t,s) = \alpha(t)$

ii) $\alpha_1 \sim \alpha_2$ implies $\alpha_2 \sim \alpha_1$

Let h_1 is the given homotopy which $h_1(t,0) = \alpha_1(t)$ and $h_1(t,1) = \alpha_2(t)$. Let h_2 be defined by $h_2(t,s) = h_1(1-t,s)$ which is continuous since $t \to 1-t$ is continuous. Hence $\alpha_2 \sim \alpha_1$ with the homotopy h_2.

iii) $\alpha_1 \sim \alpha_2$ and $\alpha_2 \sim \alpha_3$ implies $\alpha_1 \sim \alpha_3$

Let h_1 and h_2 be the given homotopies such that $h_1(t,0) = \alpha_1(t)$, $h_1(t,1) = \alpha_2(t)$ and $h_2(t,0) = \alpha_2(t)$ and $h_2(t,1) = \alpha_3(t)$. Now define;

$$h_3(t) = \begin{cases} h_1(t,2s), & 0 \leq s \leq \frac{1}{2}, \\ h_2(t,2s-1), & \frac{1}{2} \leq s \leq 1 \end{cases}$$

So; $h_3(t,0) = h_1(t,0) = \alpha_1(t)$ and $h_3(t,1) = h_2(t,1) = \alpha_3(t)$. h_3 is continuous since h_1 and h_2 are continuous and for $s = \frac{1}{2}$; $h_1(t,1) = h_2(t,0) = \alpha_2(t)$

Definition 3.2.2 Let α_1 be a path from p_0 to p_1 and α_2 be a path from p_1 to p_2. We define the product of two paths; $\alpha_1 * \alpha_2$ by

$$\alpha_1 * \alpha_2(t) = \begin{cases} \alpha_1(2t), & 0 \leq t \leq \frac{1}{2}, \\ \alpha_2(2t-1), & \frac{1}{2} \leq t \leq 1 \end{cases}$$

The inverse of α_1 is the path $\alpha_1^{-1}(t) = \alpha_1(1-t)$

Theorem 3.2.3 Let p_0, p_1 and $p_2 \in X$, Let α_1 and α_2 be paths from p_0 to p_1, and let β_1 and β_2 be paths from p_1 to p_2.

*1)*If $\alpha_1 \sim \alpha_2$ and $\beta_1 \sim \beta_2$, then $\alpha_1\beta_1 \sim \alpha_2\beta_2$.

*2)*If $\alpha_1 \sim \alpha_2$, then $\alpha_1^{-1} \sim \alpha_2^{-1}$

3) $\alpha_1\alpha_1^{-1} \sim e_a(t)$ which is the constant path at a; $e_a(t) = a$ for all $t \in [0,1]$.

Proof

1) Let h_1 and h_2 be homotopies from from α_1 to α_2 and β_1 to β_2 respectively. Then h_3 a homotopy from $\alpha_1\beta_1$ to $\alpha_2\beta_2$ is given by

$$h_3(t) = \begin{cases} h_1(2t, s), & 0 \leq t \leq \frac{1}{2}, \\ h_2(2t-1, s), & \frac{1}{2} \leq t \leq 1 \end{cases}$$

2) Let h_1 be the homotpy from α_1 to α_2. Then define h_2 a homotpy from α_1^{-1} to α_2^{-1};

$$h_2(t,s) = h_1(1-t, s)$$

3) We define the homotopy h_1 from e_a to $\alpha_1\alpha_1^{-1}$ such that

$$h_1(t,s) = \begin{cases} \alpha_1(2t), & 0 \leq t \leq \frac{s}{2}, \\ \alpha_1(s), & \frac{s}{2} \leq t \leq 1 - \frac{s}{2}, \\ \alpha_1(2-2t) & 1 - \frac{s}{2} \leq t \leq 1 \end{cases}$$

where $h_1(t,0) = e_a$ and

$$h_1(t,1) = \begin{cases} \alpha_1(2t), & 0 \leq t \leq \frac{1}{2}, \\ \alpha_1(2-2t), & \frac{1}{2} \leq t \leq 1 \end{cases}$$

□

Definition 3.2.4 Let X be a topological space, the set of equivalence classes of closed paths passing through p forms a group under the operations of product and inverse defined above. This group is called the fundamental group and denoted by $\pi_1(X, p)$.

Remark

1) If X is a path-connected topological space, for $p, q \in X$,

$$\pi_1(X, p) \cong \pi_1(X, q)$$

2) A space X is simply-connected if X is path-connected and $\pi_1(X, p)$ is trivial.

3) The fundamental group is topological invariant that is if $\varphi : X \to Y$ is a homeomorphism and if $p_1 \in Y$ and $\varphi(p_0) = p_1$ then

$$\pi_1(X, p_0) \cong \pi_1(Y, p_1)$$

The fundamental group of X classifies the covering spaces of X. In fact, one can define the fundamental group without using paths or homotopy.

Definition 3.2.5 A mapping of topological spaces $f : \tilde{X} \to X$ is said to be an unramified covering if each point $p \in X$ has an open neighborhood U such that $f^{-1}(U)$ is a disjoint union of open sets, each of which is homeomorphically onto U by f.

Theorem 3.2.6 (Path-lifting theorem) Let $f : \tilde{X} \to X$ be a covering map. Let α be a path with $\alpha(0) = p_0$. Let $\tilde{p_0} \in \tilde{X}$ with $f(\tilde{p_0}) = p_0$. Then there exists a unique path $\tilde{\alpha}$ in $\tilde{X}$ such that $\tilde{\alpha}(0) = \tilde{p_0}$ and $f \circ \tilde{\alpha} = \alpha$

Proof Chpt. 3 Cor. 2 in [6] □

It is a good time to ask the question when two paths in a covering space is equivalent. The answer is the Monodromy theorem.

Theorem 3.2.7 Let $f : \widetilde{X} \to X$ be a covering and let α and β are homotopic paths in X. Let $\widetilde{\alpha}$ and $\widetilde{\beta}$ be the lifts of α and β starting at the same point above $\alpha(0)$. Then $\widetilde{\alpha}$ is homotopic to $\widetilde{\beta}$ and in particular $\widetilde{\alpha}(1) = \widetilde{\beta}(1)$.

Proof Chpt. 5 Thm. 5.5 in [8] □

Let $f : X \to Y$ be a continuous map between path-connected spaces. Then for $p_0 \in X$, there is an induced homomorphism between fundamental groups. $f_* \pi_1(X, p_0) \to \pi_1(Y, f(p_0))$ such that;

$$f_*([\alpha]) = [f \circ \alpha]$$

where $[\alpha]$ is the set of all paths homotopic to α. Since we can define a homotopy $f \circ F$ from $f \circ \alpha$ to $f \circ \beta$ where F is a homotopy from α to β, this definition is well-defined.

Theorem 3.2.8 Let $f : \tilde{X} \to X$ be a covering map. Let $p \in X$ and $\tilde{p} \in \tilde{X}$ which $f(\tilde{p}) = p$. Then there exists a one-to-one correspondence between $f^{-1}(p)$ and $\pi_1(X,p)_{/} f_*\pi_1(\tilde{X},\tilde{p})$.

Proof : Define g : $\pi_1(X,p) \to f^{-1}(p)$ where $[\alpha]$ goes to $\tilde{\alpha}(1)$ which the end point of the unique lifting of α. By the monodromy theorem , this map is well-defined. Now we show that g is constant on cosets $f_*\pi_1(\tilde{X},\tilde{p})$. Let $[\alpha_1]$ and $[\alpha_2]$ lie in the same coset, which mean for some $[\beta] \in f_*\pi_1(\tilde{X},\tilde{p}), [\alpha_1] = [\beta] * [\alpha_2]$.

$$g([\alpha_1]) = g([\beta * \alpha_2]) = \widetilde{\beta * \alpha_2}(1) = \tilde{\alpha_2}(1) = g([\alpha_2])$$

Let $H = f_*\pi_1(\tilde{X},\tilde{p})$ then $\bar{g}(H[\alpha]) = g([\alpha])$ gives us the one-to-one, onto map. $\bar{g}$ is surjective since g is surjective. Let $\tilde{p} \in f^{-1}(p)$ and since X is path-connected we can construct a path starting from a point in $f^{-1}(p)$ and ending with $\tilde{p}$.

Let $\bar{g}(H[\alpha_1]) = \bar{g}(H[\alpha_2])$, which means $\tilde{\alpha}_1(1) = \tilde{\alpha}_2(1)$, so that $[\tilde{\alpha}_1\tilde{\alpha}_2^{-1}] \in \pi_1(\tilde{X},\tilde{p})$. $[\alpha_1]$ and $[\alpha_2]$ are in the same coset of H since for $h = f_*([\tilde{\alpha}_1\tilde{\alpha}_2^{-1}])$;

$$h[\alpha_2] = [f \circ (\tilde{\alpha}_1\tilde{\alpha}_2^{-1})][\alpha_2^{-1}] = [\alpha_1\alpha_2^{-1}][\alpha_2] = [\alpha_1].$$

□

Theorem 3.2.9 Let X be path-connected and locally simply connected topological space.

*1)*Let H be a subgroup of $\pi_1(X,p)$, then there exists a covering map $f : \tilde{X} \to X$ such that $f_*\pi_1(\tilde{X},\tilde{p}) = H$, where $\tilde{p} \in \tilde{X}$ with $f(\tilde{p}) = p$.

2) Let $f : \tilde{X}_1 \to X$ and $g : \tilde{X}_2 \to X$ be covering maps with $f(\tilde{p_1}) = g(\tilde{p_2})$ where $\tilde{p_i} \in \tilde{X}_i, i = 1,2$. If $f_*\pi_1(\tilde{X}_1,\tilde{p}_1) \subset g_*\pi_1(\tilde{X}_2,\tilde{p}_2)$ then there exists a unique covering map $h : \tilde{X}_1 \to \tilde{X}_2$ such that $h(\tilde{p}_1) = \tilde{p}_2$ and $gh = f$.

Proof :Chpt. 3 Thm. 4, Thm. 7 in [6] □

The fundamental group $\pi_1(X, p)$, always contains two trivial subgroups, id and itself. The covering space corresponding to $\pi_1(X, p)$ is the space itself with the identity mapping. We called the universal covering space $\hat{X}$ corresponding to the identity subgroup. It obviously has the universal property and $\pi_1(\hat{X}, \hat{p}) \cong id$ which makes $\hat{X}$ simply-connected.

Definition 3.2.10 Let $f : \tilde{X} \to X$ be a covering map. A covering (deck) transformation of the covering space $\tilde{X}$ is a homeomorphism $\varphi : \tilde{X} \to \tilde{X}$ such that $f \circ \varphi = f$. It is clear that the set of all covering transformations form a group under composition, it is denoted by $Deck(\tilde{X}, f)$

Theorem 3.2.11 Let X be a locally simply connected space. Let $f : \tilde{X} \to X$ be a covering map where the associated subgroup $f_*\pi_1(\tilde{X}, \tilde{p})$ is a normal subgroup of $\pi_1(X, p)$, then

$$Deck(\tilde{X}, f) \cong \pi_1(X, f(\tilde{p}))_{/} f_*\pi_1(\tilde{X}, \tilde{p})$$

In particular, if $\widehat{X}$ is the universal covering space $\pi_1(X, p) \cong Deck(\hat{X}, f)$.

Proof :Chpt. 3 Thm. 9 in [6] □

Actually, if the associated group is not normal, then the relation is;

$$Deck(\tilde{X}, f) \cong \ N(f_*\pi_1(\tilde{X}, \tilde{p}))/f_*\pi_1(\tilde{X}, \tilde{p})$$

where N denotes the normalizer. We are interested in covers corresponding subgroups are normal, which we call them as regular covers.

Example 3.2.12 $\mathbb{R}^1$ is the covering space of the circle $S^1 = \{(y_1, y_2) \in \mathbb{R}^2 : y_1^2 + y_2^2 = 1\}$ with the map $\phi : \mathbb{R}^1 \to S^1$ such that $\phi(r) = e^{2\pi i r}$. Note that $\pi_1(\mathbb{R}^1, 0)$ is trivial so, it is the universal covering of S^1. Consider $\rho \in Deck(\mathbb{R}^1, \phi)$ such that $\phi\rho = \phi$ that is, such that $e^{2\pi i \rho(r)} = e^{2\pi i r}$. Then for all $r \in \mathbb{R}^1$, $\rho(r) - r$ is an integer. Hence for a fixed k; $\rho(r) = r + k$, it is the translation by the integer k. Then, there is an isomorphism;

$$Deck(\mathbb{R}^1, \phi) \cong \pi_1(S^1, 1) \cong \mathbb{Z}$$

One of the main computational tools in calculating fundamental group is the Seifert- Van Kampen theorem. Before stating the theorem, we need some group theoretic preliminaries.

Definition 3.2.13 Let G and H be groups whose elements are arbitrary and their inverses. The elements of the free product of G and H, $G * H$ are equivalence classes of symbols

$$g_1 * h_1 * g_2 * h_2 * ... * g_n * h_n$$

where $g_i \in G$ and $h_i \in H$, and the equivalance relation is defined as

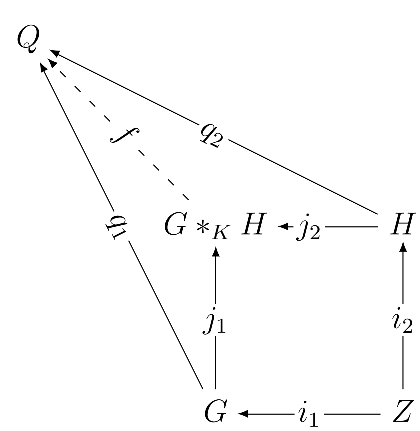
$$g_1 * h_1 ... * h_i * 1_G * h_{i+1} ... g_n * h_n \sim g_1 * h_1 * ... * h_i * h_{i+1} ... g_n * h_n$$

and similarly for $g_i * 1_H * g_{i+1}$. The product is defined by

$$(g_1 * ... * h_n) * (g_{n+1} * ... * h_m) = g_1 * ... * h_n * g_{n+1} * ... * h_m$$

Given any groups G, H and K with homomorphisms $i_1 : K \to G$, $i_2 : K \to H$, the free amalgamated product of G and H over K; $G *_K H$ is defined to be the quotient group of the free product $G * H$ by the subgroup containing all elements of the form $i_1(k) * i_2(k)^{-1}$

$G *_K H$ has the universal property, such that let Q be an arbitrary group with morphisms $q_1 : G \to Q$ and $q_2 : H \to Q$ satisfying $q_1 i_1 = q_2 i_2$. Then we have a unique morphism $f : G *_K H \to Q$ such that the diagram



is commutative.

The free product is the same as the free amalgamated product over the trivial subgroup where i_1 and i_2 are the obvious homomorphisms.

Theorem 3.2.14 (Seifert- Van Kampen Theorem) Let $X = U \cup V$ with U, V and $U \cap V$ all open, non-empty and path-connected. Let $x_o \in U \cap V$. Then the canonical maps of the fundamental groups of U, V and $U \cap V$ into that of X induce an isomorphism.

$$\theta : \pi_1(X, x_o) \to \pi_1(U, x_o) *_{\pi_1(U \cap V, x_o)} \pi_1(V, x_o)$$

Proof : Chpt. 4 Thm. 2.1 in [7] □

Example 3.2.15 *i)* Let X be a space which has shape like figure 8. Choose a and b from distinct circles of X. Let $U = X \setminus \{a\}$ and $V = X \setminus \{b\}$. U and V are both homeomorphic to a circle, which means their fundamental group is isomorphic to infinite cyclic groups. $U \cap V = X \setminus \{a, b\}$ is a simply-connected space, so $\pi_1(U \cap V, x_o) = id$. By the theorem, $\pi_1(X, x_o)$ is a free product of two infinite cyclic groups that is a free group on two generators.

*ii)*Let X be a sphere minus three points. Then this space is homeomorphic to the entire plane minus two points since there is a homeomorphism $f : S^n \setminus \{p\} \to \mathbb{R}^n$ called stereographic projection. We can find a subset Y of X such that Y is the union of two circles with a point in common like figure 8. Since Y is a deformation retract of X in other words; Y is homeomorphic to X, $\pi_1(X, x_o) \cong \pi_1(Y, x_o)$. We know from above, $\pi(Y, x_o)$ is free group on two generators. The generators can be the closed paths α and β based at x_o going once around a and b.

Now, we begin reviewing the fundamental group concept from the categorical point of view. Let X be a path-connected and locally simply connected topological space. Let $Cov(X)$ be the category of unramified covering spaces of X with morphisms of covering spaces. Let $f : \widetilde{X_1} \to X$ and $g : \widetilde{X_2} \to X$ be covering maps, then a morphism of covering maps $\alpha : \widetilde{X_1} \to \widetilde{X_2}$ is a continuous map such that $g\alpha = f$.

Consider the functor

$$\mathcal{F}_x : Cov(X) \to Sets \quad , \quad (f : \widetilde{X} \to X) \to f^{-1}(x)$$

The fiber has two actions;

1)The left action of $Deck(\widetilde{X}, f)$. If $\widetilde{X}$ is connected topological space, the stabilizers of the action is trivial.

Definition 3.2.16 Assume the covering $f : \widetilde{X} \to X$ is finite and connected. $\widetilde{X}$ is a galois cover if and only if the order of the group $Deck(\widetilde{X}, f)$ equals the degree of f which is the cardinal of the fiber.

2)The right action of $\pi_1(X, x)$

The uniqueness property of path-lifting theorem gives us a well-defined action, which is called monodromy action. Let $\alpha \in \pi_1(X, x)$ and $y \in f^{-1}(x)$

$$\alpha \cdot y = \widetilde{\alpha}(1)$$

where $\widetilde{\alpha}$ is the unique lift of α, such that $\widetilde{\alpha}(0) = y$

Let S be an orbit of the action of $\pi_1(X, x)$ and $s \in S$, consider $\widetilde{X}_S = \widehat{X}/Stab_s$ where $\widehat{X}$ is the universal covering. We can construct a covering by taking the direct sum of aver all orbits. If $\widetilde{X}$ is connected, then the action is transitive.

The group $Deck(\widetilde{X}, f)$ commutes with the monodromy action. Let $f' \in Deck(\widetilde{X}, f)$ and $\alpha \in \pi_1(X, x)$, $y \in f^{-1}(x)$. $f'(\alpha \cdot y) = f'(\widetilde{\alpha}(1))$ where $\widetilde{\alpha}(0) = y$. $f'(\widetilde{\alpha})$ is the lifting of α starting at $f'(\widetilde{\alpha}(0)) = f'(y)$, so

$$f'(\alpha \cdot y) = \alpha \cdot f'(y)$$

The monodromy action gives us;

$$\mathcal{F}_x : Cov(X) \to \pi_1(X, x) - sets$$

This action is compatible with the morphisms between two unramified coverings. Let $h : \widetilde{X}_1 \to \widetilde{X}_2$ and $\alpha \in \pi_1(X, x)$ then the diagram

$$\begin{array}{ccc} \mathcal{F}_x(\widetilde{X_1}) & \xrightarrow{\alpha} & \mathcal{F}_x(\widetilde{X_1}) \\ \mathcal{F}_x(h)\downarrow & & \downarrow\mathcal{F}_x(h) \\ \mathcal{F}_x(\widetilde{X_2}) & \xrightarrow{\alpha} & \mathcal{F}_x(\widetilde{X_2}) \end{array}$$

commutes. Since let $f : \widetilde{X_1} \to X$, $g : \widetilde{X_2} \to X$ and $\widetilde{x}_1, \widetilde{x}_2 \in f^{-1}(x)$ such that $h : \widetilde{x}_1 \to \widetilde{x}_2$.

$$\mathcal{F}_x(h)(\alpha \cdot \widetilde{x}_1) = h(\alpha \cdot \widetilde{x}_1) = h(\widetilde{\alpha}(1)) \text{ where } \widetilde{\alpha}(0) = \widetilde{x}_1$$

$$\alpha \cdot (\mathcal{F}_x(h)(\widetilde{x}_1)) = \alpha \cdot h(\widetilde{x}_1) = \alpha \cdot \widetilde{x}_2 = \widehat{\alpha}(1) \text{ where } \widehat{\alpha}(0) = \widetilde{x}_2$$

since h is a covering map for $\widetilde{X_2}$; $h(\widetilde{\alpha}(1) = \widehat{\alpha}(1)$.

So the action of α is a natural transformation for the functor $\mathcal{F}_x$ to itself. $Aut(\mathcal{F}_x)$ is the set of all natural transformation from $\mathcal{F}_x$ to itself. So we have the homomorphism;

$$\psi : \pi_1(X, x) \to Aut(\mathcal{F}_x)$$

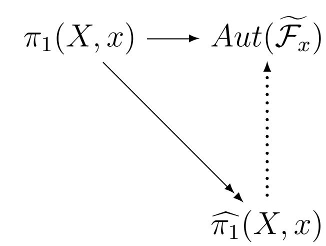
ψ is one-to-one. Let $\widehat{X}$ be the universal cover of X with $g : \widehat{X} \to X$ then by the theorem 3.2.8, $g^{-1}(x)$ has a one-to-one correspondence with $\pi_1(X, x)$.

ψ is onto. Let $\sigma \in Aut(\mathcal{F}_x)$ and $\alpha \in \pi_1(X, x)$ where $\widehat{\alpha}$ is the lift of α in $\widehat{X}$ such that $\widehat{\alpha}(0) = \widehat{x}$ and $\widehat{\alpha}(1) = \sigma\widehat{x}$ then $\psi(\alpha) = \sigma$. So we have;

$$\pi_1(X, x) \cong Aut(\mathcal{F}_x)$$

Remark: Let X be a connected topological space then the category of finite, unramified coverings of X, $Cov_{finite}(X)$ is a galois category (for the proof see Chpt. 3 3.9 in [5]). Let $\widetilde{\mathcal{F}}_x : Cov_{finite}(X) \to \{\text{finite sets}\}$ be the fundamental functor such that $f : Y \to X$ goes to $f^{-1}(x)$. We know $\pi_1(X, x) \cong Aut(\mathcal{F}_x)$ and by restriction we have $\pi_1(X, x) \to Aut(\widetilde{\mathcal{F}}_x)$. $Aut(\widetilde{\mathcal{F}}_x)$ is a profinite group since it consists of the automorphisms of the finite groups. Profinite completion $\hat{G}$ of an arbitrary group G

has such a universal property that let $\varphi : G \to \hat{G}$ is the natural homomorphism, for any profinite group H and any homomorphism $\psi : G \to H$, there exists a unique continuous group homomorphism $\phi : \hat{G} \to H$ such that $\phi\varphi = \psi$. So we have;



Considering the regular covers which correspond to the normal subgroups of $\pi_1(X, x)$ of finite index is sufficient to compute $Aut(\widetilde{\mathcal{F}_x})$. Therefore, we have

$$\varprojlim \pi_1(X, x)/N \cong Aut(\widetilde{\mathcal{F}_x})$$

3.3 Algebraic Fundamental Group

We construct the algebraic (*é*tale) fundamental group by imitating the characterization of the topological fundamental group as the group of deck (covering) transformations of a universal covering space. Also one can define it as the automorphism group of a fiber functor. So we consider a suitable analogue to the unramified covering and fiber functor. A finite etale morphism is the natural analogue of a finite unramified covering. Unfortunately, in algebraic case the fiber functor is not representable by the universal cover since there is usually no such object.

Example 3.3.1 *i)*Let F_q be a finite field and $\overline{F_q}$ be the algebraic closure. The fields $F_{q^k} = \{q \in F_q : a^{q^k} = a\}$ for $k \in \mathbb{Z}_{>0}$ are the only finite extensions of F_q in $\overline{F_q}$ Note that each F_{q^k} is a finite separable field extension of F_q, so $SpecF_{q^k} \to SpecF_q$ is a finite etale map. Among these coverings, there is no biggest one.

*ii)*A connected scheme X is said to be simply-connected if every etale covering of X is trivial that is a direct sum of copies of several copies of X. So $Spec\mathbb{C}$ is a simply-connected scheme and it is the universal cover $Spec\mathbb{R}$

However the fiber functor $\mathcal{F}_x : FEt_x \to Sets$ where $(f : Y \to X)$ goes to $f^{-1}(x)$, is pro-representable which means that there is a projective system $\widetilde{X} = (X_i)_{i \in I}$ of finite etale coverings of X indexed by a directed set I such that;

$$(\mathcal{F}_x(Y)) = \varinjlim_{i \in I} Hom_X(X_i, Y) = Hom(\widetilde{X}, Y)$$

So we can define the algebraic fundamental group;

$$\pi_1^{alg}(X, x) = Aut(\mathcal{F}_x) = Aut(Hom(\widetilde{X}, \cdot)) \cong Aut(\widetilde{X})^{opposite}$$

which is endowed with profinite group topology.

Example 3.3.2 *i)*Consider FEt_{SpecK} the category of finite etale coverings of $SpecK$ where K is a field. Let us define the fiber functor

$$\mathcal{F}_x(SpecR) = Hom_K(SpecK^{sep}, SpecR)$$

which is compatible with the definitions above: $SpecK^{sep} \to SpecK$ serves as the universal cover. Recall that $Gal(K^{sep}/K)$ is the inverse limit of $Gal(K^i/K)$ where K^i's are finite separable extensions of K.

$$\pi_1^{alg}(SpecK, x) = Aut(Hom_K(SpecK^{sep}, \cdot)) = Aut(SpecK^{sep})^{opp}$$

$$\pi_1^{alg}(SpecK, x) \cong Gal(K^{sep}/K)$$

*ii)*Let $\mathbb{A}_K^1 = SpecK[t]$ be the affine line over an algebraically closed field K of characteristic zero. The finite etale coverings of $\mathbb{A}_K^1 \setminus \{0\}$ are the maps;

$$SpecK[t] \setminus \{0\} \to SpecK[t] \setminus \{0\}$$

where t goes to t^n because $\frac{dx^n}{dx} = nx^{n-1}$ is nonsingular at all $x \neq 0$ since the characteristic is zero, so (by cor. 2.2 in [9]) φ is etale. Let $\mu_n(K)$ be the cyclic group of order n which is generated by the nth roots of unity in K, if $t \to t^n : \mathbb{A}_K^1\{0\} \to \mathbb{A}_K^1\{0\}$ then,

$$Aut(SpecK[t] \setminus \{0\} \to SpecK[t] \setminus \{0\} : t \to t^n) = \mu_n(K)$$

with $\zeta \in \mu_n(K)$ acting by $x \to \zeta x$.

We have an inverse system by the relation of divisibility; $n/m \Leftrightarrow m \geq n$ where the maps are;

$$Aut(\mathbb{A}^1_K \setminus \{0\} \to \mathbb{A}^1_K \setminus \{0\} : t \to t^m) \to Aut(\mathbb{A}^1_K \setminus \{0\} \to \mathbb{A}^1_K \setminus \{0\} : t \to t^n)$$

Note that $\mu_n(K)$ is non-canonically isomorphic to $\mathbb{Z}/n\mathbb{Z}$ and by the example..

$$\pi_1^{alg}(\mathbb{A}^1_K \setminus \{0\}, x) = \varprojlim_n \mathbb{Z}/n\mathbb{Z} \cong \widehat{\mathbb{Z}}.$$

Definition 3.3.3 Let X be a scheme finite type over $\mathbb{C}$. Let X^{top} be the topological space whose points are closed points of X and with base obtained as follows; consider an open set $U \subset X$, a finite number of regular functions $f_1, ...f_n$ on U and a number $\epsilon > 0$. Define $V(U, f_1, ...f_n; \quad \epsilon)$ as the set of points;

$$V(U, f_1, ...f_n; \quad \epsilon) = \{x \in U^{top} | \quad |f_i(x)| < \epsilon \quad for \quad i = 1, ..., n\}.$$

By taking the $V(U, f_1, ...f_n; \quad \epsilon)$ as a basis for the open sets, we make X^{top} into a topological space.

Theorem 3.3.4 *(Corollary of a Riemann Existence Theorem)* Let X be a scheme of finite type over $\mathbb{C}$. Then $\pi_1^{alg}(X, x)$ is isomorphic to $\widehat{\pi_1^{top}(X^{top}, x)}$, the profinite completion of the usual fundamental group of X^{top}.

Proof :App. E Thm. E.1 in [**?**] □

Example 3.3.5 Let $\mathbb{P}^1_{\mathbb{C}}$ be the projective line over $\mathbb{C}$. Define $X = \mathbb{P}^1_{\mathbb{C}} \setminus \{0, 1, \infty\}$, then the topological fundamental group is the free group on two generators. Namely;

$$\pi_1^{top}(X, x) = \langle x_0, x_1, x_\infty \mid x_0 \cdot x_1 \cdot x_\infty = 1 \rangle$$

where the x_i's correspond to the loops around 0, 1, and ∞ from the base point x. Then by the theorem $\pi_1^{alg}(X, x)$ is the profinite completion of the free group on two generators.

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