LINEARIZATION OF STOCHASTIC DIFFERENTIAL EQUATIONS DRIVEN BY LÉVY PROCESSES

by

İsmail İyigünler

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This is to certify that I have examined this copy of a master's thesis by

İsmail İyigünler

and have found that it is complete and satisfactory in all respects,
and that any and all revisions required by the final
examining committee have been made.

Committee Members:

Assoc. Prof. Mine Çağlar

Prof. Süleyman Özekici

Prof. Ali Mostafazadeh

Date:

To my parents

ABSTRACT

Stochastic differential equations with jumps are important in physics, finance and engineering as they represent systems with sudden random effects. Analytical solutions of stochastic differential equations not only allow us to study the underlying stochastic processes, but also provide the means to test the numerical schemes. Therefore, analytical methods for the integration of nonlinear stochastic differential equations are of paramount importance.

We consider linearizing transformations of the one-dimensional nonlinear stochastic differential equations driven by Wiener and compound Poisson processes, namely finite activity Lévy processes. We present linearizability criteria and derive the required transformations. We introduce a stochastic integrating factor method to solve the linearized equations and provide closed-form solutions.

We apply our method to a number of stochastic differential equations including Cox-Ingersoll-Ross short-term interest rate model, log-mean reverting asset pricing model and geometric Ornstein-Uhlenbeck equation all with additional jump terms. We use their analytical solutions to evaluate the accuracy of the numerical approximations obtained from Euler and Maghsoodi discretization schemes. The means of the solutions are estimated through Monte Carlo method.

ÖZETÇE

Sıçramalı stokastik diferansiyel denklemler, ani rassal değişimlerin görüldüğü sistemleri temsil etmeleri nedeniyle fizik, finans ve mühendislik alanlarında önemli bir yer tutar. Bu denklemlerin analitik çözümleri ise sadece temeldeki stokastik süreçlerin incelenmesini değil, aynı zamanda sayısal yöntemlerin sınanmasını da sağlamaktadır. Bu yüzden doğrusal olmayan stokastik diferansiyel denklemler için analitik çözüm yöntemleri son derece önemlidir.

Bu çalışmada, Wiener ve bileşik Poisson süreçleriyle yani sonlu etkinliğe sahip Lévy süreçleriyle sürülmüş, tek boyutlu doğrusal olmayan stokastik diferansiyel denklemleri ele almaktayız. Doğrusallaştırma ölçütleri ortaya çıkarılıp, denklemleri doğrusallaştırmak için gerekli dönüşümler bulunmuştur. Adi diferansiyel denklemlerde bilinen integrasyon çarpan yöntemi stokastik diferansiyel denklemlere uyarlanarak, doğrusal denklemlerin çözümleri elde edilmektedir.

Doğrusallaştırma yöntemimiz, sıçrama terimi içeren Cox-Ingersoll-Ross modeli, log-ortalamaya çekilen fiyatlama modeli ve geometrik Ornstein-Uhlenbeck denklemi gibi çeşitli stokastik diferansiyel denklemleri çözmek için uygulanmıştır. Bulduğumuz analitik çözümler, sözü geçen denklemlerin Euler ve Maghsoodi sayısal yöntemleriyle yaklaştırımlarıyla karşılaştırılmıştır. Çözümlerin beklenen değeri ise Monte Carlo yöntemi ile kestirilmiştir.

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Chapter 1

INTRODUCTION

Interest in the study of stochastic phenomena has increased dramatically in recent years. Intensified research activity in this area has been stimulated by the need to take random effects into account in complicated physical systems which are usually described by differential equations. One way to incorporate randomness is to add a stochastic term to the deterministic differential equation at hand, which is then called a stochastic differential equation. The theory of stochastic differential equations has recently enjoyed significant reputation as a result of its impact on physics, finance and engineering [7, 15, 26, 27, 8, 32]. Furthermore, stochastic differential equations with jump terms, driven by Lévy processes in general, appear to be more realistic in cases where sudden events play prominent role [17, 19, 20, 25, 28, 8, 32].

Analytical solutions of stochastic differential equations not only allow us to study the underlying stochastic processes, but also provide the means to test the numerical schemes [14, 23]. Therefore, analytical methods for the integration of nonlinear stochastic differential equations are of paramount importance.

Lévy processes are basically stochastic processes with stationary and independent increments. They are analogues of the random walks in continuous time. Moreover, they form a subclass of semimartingales and Markov processes which include very important special cases such as Brownian motion, Poisson process, subordinators and stable processes. Although much of the basic theory was established earlier, a great deal of new theoretical developments as well as novel applications in diverse areas have emerged in recent years. [1, 2, 30].

We derive linearizing transformations of one dimensional stochastic differential equations driven by Wiener and compound Poisson processes which form a finite activity Lévy process. We provide the conditions for linearization and resulting exact solutions via stochastic integrating factors. We propose an analytical method of integration which is based on a linearizing a nonlinear stochastic differential equation and solving the linear stochastic differential equation via stochastic integrating factor method which exactly originates from the theory of ordinary differential equations.

Let $(\Omega, \mathcal{F}, P)$ be a filtered probability space and $W = \{W_t, t \geq 0\}$ be a standard Wiener process [19] and $N = \{N_t, t \geq 0\}$ be a homogeneous Poisson process [32]. We consider a stochastic process $X = \{X_t, t \geq 0\}$ adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the nonlinear stochastic differential equation with jump terms of the form

$$dX_t = f(X_{t^-}, t)dt + g(X_{t^-}, t)dW_t + \sum_{j=1}^{m} r_j(X_{t^-}, t)dN_t^j, \qquad X_0 = x_0 \tag{1.1}$$

where f, g and r are $\mathbb{R}$ valued continuously differentiable functions, dW_t is the infinitesimal increment of the Wiener process [19, 27] and independently $dN_t^1, dN_t^2, \ldots, dN_t^m$ are the infinitesimal increments of the independent Poisson processes with intensities $\lambda_1, \lambda_2, \ldots, \lambda_m$ respectively [17, 32]. We present necessary and sufficient conditions for the linearization of the stochastic differential equation given in (1.1) via an invertible transformation. As applications, it is shown that Cox-Ingersoll-Ross short-term interest rate model [7], log-mean reverting asset pricing model [9, 31, 34] and Geometric Ornstein-Uhlenbeck [12, 11, 27, 33] model with additional jump terms and several more examples [17, 24] are linearizable under specified conditions on the functions f, g and r. Exact solutions to these linearizable equations obtained. We then compare our analytical solutions with the numerical approximations found by Euler and Maghsoodi schemes to demonstrate the agreement. The means are found by Monte Carlo approach to estimate the expected value of X.

This thesis is organized as follows. We first give the necessary definitions and results from the probability theory in Chapter 2. We present the preliminary descriptions from measure theory, probability spaces and random processes. In Chapter 3, we briefly review the theory of Lévy processes, its special cases and their simulations.

The main results related to stochastic integration and Itô's formula are given in Chapter 4. In Chapter 5, we present results about the linearization of nonlinear stochastic differential equations and give the necessary and sufficient conditions. The analytical solution of linear stochastic differential equations via stochastic integrating factors are also found in this chapter. A number of examples to demonstrate our method of integration are given. In Chapter 6, the numerical results are given for several examples of linearizable stochastic differential equations. Finally, the conclusions are stated in Chapter 7.

Chapter 2

PRELIMINARIES

We present necessary definitions and results from the probability theory. Preliminary descriptions from measure theory, probability spaces and random processes are given in this chapter.

2.1 Measure Theory

We begin with some elementary definitions. First, we define a σ-algebra [21] .

Definition 2.1.1 *A non-empty collection $\mathcal{E}$ of subsets of E is a $\sigma - algebra$ if*

- *$\mathcal{E}$ contains the empty set: $\emptyset \in \mathcal{E}$ (and contains E),*
- *$\mathcal{E}$ is closed under countable unions of disjoint subsets: $A_n \in \mathcal{E}$, $n = 1, 2, \ldots \Rightarrow \bigcup_n A_n \in \mathcal{E}$,*
- *$\mathcal{E}$ is closed under complements: $A \in \mathcal{E} \Rightarrow A^c \in \mathcal{E}$.*

The $\sigma-$algebra generated by the open sets in E (that is, the smallest $\sigma-$algebra that contains all open sets in E) is called the Borel $\sigma-$algebra on E and is denoted by $\mathcal{B}(E)$. For example, the $\sigma-$algebra generated by the collection of all intervals in $\mathbb{R}$ is the Borel $\sigma-$algebra on $\mathbb{R}$.

The following definitions can be found in [6].

Definition 2.1.2 *Let $\mathcal{E}$ be a σ-algebra on a set E. The pair $(E, \mathcal{E})$ is called a measurable space. The elements of $\mathcal{E}$ is called a measurable set.*

Definition 2.1.3 *Let* $(E, \mathcal{E})$ *&* $(F, \mathcal{F})$ *be two measurable spaces. A function* $f : E \to F$ *is called measurable if for any* $A \in \mathcal{F}$

$$f^{-1}(A) = \{x \in E,\ f(x) \in A\}$$

is a measurable subset of $E : f^{-1}(A) \in \mathcal{E}$.

A function from E into $\mathbb{R}$ is said to be $\mathcal{E} - measurable$ if it is measurable relative to $\mathcal{E}$ and $\mathcal{B}(\mathbb{R})$.

Definition 2.1.4 *A measure on a measurable space* $(E, \mathcal{E})$ *is a function* $\mu : \mathcal{E} \to \mathbb{R}_+$ *such that*

- $\mu(\emptyset) = 0$,
- $\mu(\bigcup_n A_n) = \sum_n \mu(A_n)$ *for any sequence of disjoint sets:* $A_n \in \mathcal{E}$, $n = 1, 2, \ldots$

The triplet $(E, \mathcal{E}, \mu)$ is called a *measure space*.

We then consider following examples from [6].

Example 2.1.1 *Dirac Measures*

Let $(E, \mathcal{E})$ be a measurable space. The Dirac measure δ_x on $(E, \mathcal{E})$ associated to a fixed point $x \in E$ is

$$\delta_x(A) = \begin{cases} 1,\ x \in A, \\ 0,\ x \notin A. \end{cases},$$

for $A \in \mathcal{E}$.

Example 2.1.2 *Counting Measures*

Let $(E, \mathcal{E})$ *be a measurable space. The counting measure* v *on* $(E, \mathcal{E})$ *for* $D \subset E$ *is*

$$v(A) = \sum_{x \in D} \delta_x(A),$$

for $A \in \mathcal{E}$. *Intuitively,* $v(A)$ *is the number of elements in* $A \cap D$ *for a countable subset* D*, which is possibly infinite.*

Example 2.1.3 *Discrete Measures*

Let $(E, \mathcal{E})$ *be a measurable space and let* m *be a measure with* $m(x) \in \mathbb{N}$ *for* $x \in E$. *The discrete measure* μ *on* $(E, \mathcal{E})$ *is*

$$\mu(A) = \sum_{x \in D} m(x) \delta_x(A),$$

for $A \in \mathcal{E}$. *Intuitively,* μ *can be conceived as the weight of the set* A *where* m *can be considered as the mass of* x.

Example 2.1.4 *Lebesgue Measure*

The Lebesgue measure λ *on* $\left(\mathbb{R}^d, \mathcal{B}\left(\mathbb{R}^d\right)\right)$ *is denoted by*

$$\lambda(A) = \int_A dx,$$

which represents the (d-dimensional) volume of a set $A \in \mathcal{B}\left(\mathbb{R}^d\right)$.

Let $(E, \mathcal{E}, \mu)$ be a measure space and f be a measurable function in $\mathcal{E}$. The *integral* of f with respect to the measure μ is denoted by

$$\mu f = \mu(f) = \int_E \mu(dx) f(x) = \int_E f d\mu.$$

Definition 2.1.5 *A function on* E *to* $\mathbb{R} \cup \{-\infty, \infty\}$ *is called a simple function if it has the form*

$$f = \sum_1^n a_i 1_{A_i},$$

for some $n \in \mathbb{N}$, $a_1, \ldots, a_n \in \mathbb{R}$ *and* $A_1, \ldots, A_n$ *are measurable sets belonging to* $\mathcal{E}$, *where* 1_{A_i} *is the indicator function of* A_i *given by*

$$1_{A_i}(x) = \begin{cases} 1, & x \in A_i, \\ 0, & x \notin A_i. \end{cases}$$

Therefore, there exists $m \in \mathbb{N}$ and distinct real numbers $b_1, \ldots, b_m$ and a measurable partition $\{B_1, \ldots, B_m\}$ of E such that

$$f = \sum_1^m b_i 1_{B_i}.$$

This representation is called the *canonical form* of the simple function f.

Definition 2.1.6 *Let f be simple and positive. If it has the canonical form $f = \sum_1^n a_i 1_{A_i}$, then we define the integral as follows ,*

$$\mu f = \sum_1^n a_i \mu(A_i).$$

A positive function on E is $\mathcal{E}$-measurable if and only if it is the pointwise limit of an increasing sequence of positive simple functions which is denoted simply $f \in \mathcal{E}_+$. Therefore, let $f \in \mathcal{E}_+$ and let (f_n) denote a sequence that converges pointwise to f where each f_n is simple and positive. The integral μf_n is defined for each n by the preceding step and we define

$$\mu f = \lim f_n.$$

On the other hand, let $f \in \mathcal{E}$. Then, $f^+ = f \vee 0$ and $f^- = -(f \wedge 0)$ where $\vee$ denotes the maximum and $\wedge$ denotes the minimum and their integrals $\mu(f^+)$ and $\mu(f^-)$ are defined by the last step. Noting that $f = f^+ - f^-$ we define

$$\mu f = \mu\left(f^+\right) - \mu\left(f^-\right),$$

provided that at least one term on the right hand side is finite.

Specifically, let E be a Lebesgue measurable subset of $\mathbb{R}^d$, let $\mathcal{E} = \mathcal{B}(E)$ and μ be the Lebesgue measure on $(E, \mathcal{E})$. The integral

$$\mu f = Leb_E f = \int_E Leb(dx) f(x) =: \int_E f(x) dx$$

is called the *Lebesgue integral* of f on E.

If the Riemann integral of f exists, then it is equal to its Lebesgue integral. However, Lebesgue integral exists for a larger class of functions than the Riemann integral does. Intuitively, the Lebesgue integral is computed by partitioning the range of f whereas the well-known Riemann integral is computed by partitioning the domain of the function f [13].

2.2 Probability Spaces and Stochastic Processes

A measure space $(\Omega, \mathcal{F}, \mathbb{P})$ is called a *probability space* if $\mathbb{P}$ is a measure on $(\Omega, \mathcal{F})$ with total mass 1 [6, 8] that is, $\mathbb{P}(\Omega) = 1$. The set Ω is interpreted as the collection of all possible outcomes and is called the *sample space*. Each element $\omega \in \Omega$ is called an *outcome*. The σ-algebra $\mathcal{F}$ is called the *history* and each measurable set $A \in \mathcal{F}$, called an *event*, is a set of outcomes to which a probability can be assigned. The measure $\mathbb{P}$ is called a *probability measure*.

The following definitions from probability theory can be found in [6].

Definition 2.2.1 *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(E, \mathcal{E})$ be a measurable space. A measurable function*

$$X : \Omega \to E$$

is called a random variable. In other words, the function X is a random variable if

$$X^{-1}(A) = \{\omega \in \Omega : X(\omega) \in A\}$$

is an event for $A \in E$.

Definition 2.2.2 *Let $(E, \mathcal{E})$ be a measurable space and X be a random variable taking values in it. The distribution of X is a probability measure μ_X on $(E, \mathcal{E})$ denoted by*

$$\mu_X(A) = \mathbb{P}\left(X^{-1}(A)\right) = \mathbb{P}(X \in A),$$

where $\mathbb{P}(X \in A)$ is the probability that X is in $A \in \mathcal{E}$.

Definition 2.2.3 *Let $(E, \mathcal{E})$ be a measurable space and X_t be a random variable taking values in $(E, \mathcal{E})$ for $t \in \mathbb{T}$. The family of random variables $X = \{X_t : t \in \mathbb{T}\}$ is called a stochastic process with state space $(E, \mathcal{E})$ and parameter set $\mathbb{T}$.*

For each fixed $\omega \in \Omega$ the function

$$X(\omega) : t \to X_t(\omega)$$

is called the *sample path* of the process.

Definition 2.2.4 *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A $filtration$ on $\mathbb{T}$ is an increasing sequence of σ-algebras $(\mathcal{F}_t)_{t\in T}$, that is,*

$$\mathcal{F}_s \subset \mathcal{F}_t \text{ whenever } s < t$$

and each $\mathcal{F}_t$ is a sub-σ-algebra of $\mathcal{F}$.

Let X be a random variable taking values in a measurable space $(E, \mathcal{E})$. A σ-algebra $\sigma(X) = \{X^{-1}(B) : B \in \mathcal{B}\}$ is called the *$\sigma - algebra$ generated by a random variable* X.

The filtration $\mathcal{F}_t = \sigma\{X_s : s \leq t\}$ for a stochastic process $X = (X_t)_{t\in\mathbb{T}}$ with a state space $(E, \mathcal{E})$ is called the *$filtration$ generated by* X and the process X is said to be *adapted* to the filtration $(\mathcal{F}_t)$.

Definition 2.2.5 *Let X be a stochastic process defined on a probability space adapted to a filtration $(\mathcal{F}_t)$ satisfying*

$$E(|X_t|) < \infty.$$

The process X is a martingale if for $0 \leq s < t < \infty$

$$E(X_t|\mathcal{F}_s) = X_s.$$

Definition 2.2.6 *A stopping time is a random variable $T : \Omega \to [0, \infty)$ for which the event $(T \leq t) \in \mathcal{F}_t$ for each $t \geq 0$.*

Definition 2.2.7 *A stochastic process M is a local martingale for which there exists a sequence of stopping times $\tau_1 \leq \ldots \leq \tau_n \to \infty$ almost surely such that each of the processes $\{M_{t\wedge\tau_n},\ t \geq 0\}$ is a bounded martingale.*

Definition 2.2.8 *A stochastic process $C = \{C_t,\ t \geq 0\}$ is of $finite$ $variation$ if the paths $\{C_t(\omega),\ t \geq 0\}$ are of finite variation for almost all $\omega \in \Omega$.*

Definition 2.2.9 *A stochastic process X is a semimartingale if for each $t \geq 0$*

$$X_t = X_0 + M_t + C_t$$

where $M = \{M_t,\ t \geq 0\}$ is a local martingale and $C = \{C_t,\ t \geq 0\}$ is an adapted finite variation process.

An important and one of the most intensively studied semimartingale is Brownian motion $B = \{B_t,\ t \geq 0\}$. One dimensional Brownian motion starts from zero, has stationary and independent increments, its marginal probability distribution is Gaussian, that is, $B_t \sim N(0,t)$ and finally its sample paths are almost surely continuous and nowhere differentiable.

2.3 Random Measures

Let $(E, \mathcal{E})$ be a measurable space. A function

$$M : \Omega \times \mathcal{E} \rightarrow \mathbb{R}_+$$

is called a *random measure* [6] if the map

$$(\omega, .) \rightarrow M(\omega, A)$$

is a random variable for $A \in \mathcal{E}$ and

$$(., A) \rightarrow M(\omega, A)$$

is a measure on $(E, \mathcal{E})$ for $\omega \in \Omega$. Furthermore,

$$Mf = Mf(\omega) = \int_E M(\omega, dx) f(x)$$

is a random variable for every f in the set of all positive measurable functions on $(E, \mathcal{E})$ and

$$\mu(A) = \mathbb{E}M(A) = \int_\Omega \mathbb{P}(d\omega) M(\omega, A)$$

is a measure μ on $(E, \mathcal{E})$, called the *mean* measure of M, therefore

$$\mathbb{E}Mf = \mu f.$$

Definition 2.3.1 *Let $(E, \mathcal{E})$ be a measurable space and v be a measure on it. A random measure N on $(E, \mathcal{E})$ is called a Poisson random measure [6] with mean (intensity) v provided that*

- *$N(A)$ has the Poisson distribution (a Poisson random variable) with intensity measure $v(A)$ for $A \in \mathcal{E}$,*

$$\mathbb{P}\left(N\left(A\right)=k\right)=e^{-v(A)}\frac{\left(v\left(A\right)\right)^{k}}{k!},\ \ k\in\mathbb{N}.$$

- *A sequence of Poisson random variables $N(A_1),\ N(A_2),\ldots,\ N(A_n)$ are independent for disjoint measurable sets $A_1,\ A_2,\ldots,\ A_n \in \mathcal{E}$ for $n \geq 2$.*

A Poisson random measure N on a measurable space $(E, \mathcal{E})$ can be described as a *counting measure* associated with a random configuration of points in E such that

$$N\left(\omega, A\right)=\sum_{n\geq 1}\delta_{X_n(\omega)}\left(A\right),$$

for $A \in \mathcal{E}$ and $X_1, X_2, \ldots$ finite, independent and identically distributed random variables in $(E, \mathcal{E})$. Therefore, $N\left(\omega, A\right)$ is the number of random points in A.

Chapter 3

LÉVY PROCESSES AND THEIR SIMULATION

Lévy processes are right-continuous stochastic processes that start at 0 and have stationary and independent increments named in honor of Paul Lévy. In this chapter, we characterize their main properties by giving necessary definitions and examples. We also present algorithms and descriptions to simulate the Lévy processes.

3.1 Characterization of Lévy Processes

A cadlag, right continuous with left-hand limits, stochastic process $X = \{X_t,\ t \geq 0\}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with values in $\mathbb{R}^d$ such that $X_0 = 0$ is called a Lévy process if it possesses the following properties [2, 30]:

- **Independent increments**: for every increasing sequence of times $t_0, t_1, \ldots, t_n$ the random vectors X_{t_0}, $X_{t_1} - X_{t_0}$,..., $X_{t_n} - X_{t_{n-1}}$ are independent.

- **Stationary increments**: distribution of $(X_{t+h} - X_t)$ does not depend on t.

- **Stochastic continuity**: sample paths of X are stochastically continuous; $\lim_{h\to 0} \mathbb{P}\left(|X_{t+h} - X_t| \geq \epsilon\right) = 0$ for $\forall \epsilon > 0$, $t \geq 0$.

The following definition can be found in [8].

Definition 3.1.1 *A probability distribution μ_X on $\mathbb{R}^d$ is said to be $infinitely$ $divisible$ if for any integer $n \geq 2$, there exists n i.i.d. random variables X_1, $X_2, \ldots,$ X_n such that $X_1 + X_2 + \ldots + X_n$ has distribution μ_X.*

Let $X = \{X_t,\ t \geq 0\}$ be a Lévy process. Then, for every t, X_t has an infinitely divisible distribution. Conversely, if μ_X is an infinitely divisible distribution then there exists a Lévy process $X = \{X_t,\ t \geq 0\}$ such that the distribution of X_t is given by μ_X [8].

The characteristic function of a random variable X_t is (the Fourier transform of its distribution μ_X) given by

$$\phi_t(u) = E\left(e^{iu \cdot X_t}\right) = \int_{\mathbb{R}^d} e^{iu \cdot x} \mu_X(dx)$$

for $u \in \mathbb{R}^d$. The distribution of a Lévy process is characterized by its characteristic function which can be written as in the following theorem.

A measure v [2, 30] on $\mathcal{B}\left(\mathbb{R}^d\right)$ is called a Lévy measure if

$$v(\{0\}) = 0 \text{ and } \int_{\mathbb{R}^d} \left(1 \wedge |x|^2\right) v(dx) < \infty.$$

Theorem 3.1.1 (Lévy-Khintchine Representation) *If $X = \{X_t,\ t \geq 0\}$ is a Lévy process, then the characteristic function of X_t, $\phi_t(u) = E\left(e^{iu \cdot X_t}\right)$ satisfies*

$$\phi_t(u) = e^{t\psi(u)}$$

$$\psi(u) = ib.u - \frac{1}{2}u.Au + \int_{\mathbb{R}^d} \left(e^{iu \cdot x} - 1 - iu.x 1_{|x| \leq 1}\right) v(dx),$$

where b is a vector on $\mathbb{R}^d$ called the drift term, A is the covariance matrix of a Brownian motion B_t on $\mathbb{R}^d$ called the Gaussian coefficient and v is the Lévy measure. The triplet (A, b, v) is called the Lévy triplet or the characteristic triplet of X. The function $\psi(u)$ is called the characteristic exponent of the Lévy process X.

Lévy process can be expressed as a sum of two independent parts: a continuous part and a discontinuous jump part. The latter part cannot be expressed as the sum of jumps, since the sum of all jumps up to some time may be divergent . However, one can overcome this problem by a compensated sum of independent jumps: summation of random quantities with simultaneously subtracted means. The celebrated Lévy-Itô decomposition is given below. Its proof can be found in [30].

Theorem 3.1.2 (Lévy-Itô decomposition) *If $X = \{X_t,\ t \geq 0\}$ is a Lévy Process, then there exists $b \in \mathbb{R}^d$, a Brownian motion B_t with covariance matrix A and an* *independent Poisson random measure N on $\mathbb{R}_+ \times \mathbb{R}^d$ such that for each $t \geq 0$*

$$X_t = bt + B_t + \int_{|x|<1} x\left(N\left(t, dx\right) - tv\left(dx\right)\right) + \int_{|x|\geq 1} xN(t, dx). \tag{3.1}$$

An important implication of the Lévy-Itô decomposition is that every Lévy process is a sum of a Brownian motion with drift and a possibly infinite sum of independent compound Poisson processes [2]. Therefore, every Lévy process can be approximated by the sum of Brownian motion with drift and a compound Poisson process.

A Lévy process X has infinite activity if the total mass of the Lévy measure on the real line is infinite. Therefore, this expression characterizes a high rate of arrival of jumps of different sizes without a diffusion component [16]. Furthermore, the set of jump times of every trajectory of the Lévy process that has infinite activity is countably infinite in $\mathbb{R}_+$. The countability follows directly from the fact that the paths are cadlag [8]. The following proposition states these sample path properties [8].

Proposition 3.1.3 *Let $X = \{X_t,\ t \geq 0\}$ be a Lévy process with characteristic triplet (A, b, v).*

- *If $v(\mathbb{R}) < \infty$ then almost all paths of X have a finite number of jumps on every interval (The Lévy process X has finite activity).*

- *If $v(\mathbb{R}) = \infty$ then almost all paths of X have an infinite number of jumps on every interval (The Lévy process X has infinite activity).*

A Lévy process with triplet (A, b, v) is said to be of finite variation if its trajectories are functions of finite variation with probability 1. Therefore, we must have $A = 0$ for X_t to be of finite variation since the trajectories of Brownian motion are almost surely of infinite variation. Consequently, we have the following proposition [8].

Proposition 3.1.4 *Let $X = \{X_t,\ t \geq 0\}$ be a Lévy process with characteristic triplet (A, b, v).*

- *If* $A = 0$ *and* $\int_{|x|\leq 1} |x|\, v\,(dx) < \infty$ *then almost all paths of* X *have finite variation.*

- *If* $A \neq 0$ *and* $\int_{|x|\leq 1} |x|\, v\,(dx) = \infty$ *then almost all paths of* X *have infinite variation.*

We examine two special Lévy processes next.

3.2 Wiener Process and Compound Poisson Process

As a special case, a Brownian motion taking values in $\mathbb{R}^d$ happens to be a Lévy process with a characteristic triplet $(a, 0, 0)$ where the process is denoted by $B = \{B(t), t \geq 0\}$. It has mean zero and covariance $E\left(B^i(s)\, B^j(t)\right) = a^{ij}(s \wedge t)$ where $B^i(s)$ is the ith component of the vector $B(s)$. Wiener process is further special case of a Brownian motion with $a \equiv I$, the identity matrix.

A sample path of one dimensional Brownian motion can be simulated on $[0, T]$ with the following algorithm [8]:

Simulate n *independent standard normal variables* $N_1, \ldots, N_n$,

Set $\Delta X_i = aN_i\sqrt{t_i - t_{i-1}} + b\,(t_i - t_{i-1})$ *where* $t_0 = 0$.

The discretized trajectory is given by $X(t_i) = \sum_{k=1} \Delta X_i$.

Note that, this is not an exact simulation as it is a finite discretization of a continuous process.

A Poisson process N taking values in $\mathbb{N}$ is a Lévy process with a characteristic triplet $(0, 0, \lambda\delta_1)$ where $\lambda > 0$ is the intensity of the Poisson process and δ_1 is the Dirac measure concentrated at 1. The paths of N are piecewise constant on each finite interval, with jumps of size 1 at random times $\tau_n = \inf\{t \geq 0, N(t) = n\}$.

Let $\{Y_n, n \in \mathbb{N}\}$ be a sequence of independent and identically distributed random variables with distribution μ_Y and N be a Poisson process with intensity $\lambda > 0$. Compound Poisson process is a Lévy process $X_t = \sum_{i=1}^{N_t} Y_i$ with the characteristic exponent

$$\psi(u) = \int_{\mathbb{R}^d} \left(e^{iu\cdot x} - 1\right) \lambda\mu_Y(dx).$$

A sample path of this process can be simulated exactly on $[0,T]$ with one of the following algorithms [8]. The first one is

Initialize $k := 0$

Repeat while $\sum_{i=1}^{k} T_i < T$

Set $k = k+1$

Simulate $T_k \sim exp(\lambda)$

Simulate Y_k *from the distribution* $\mu = \frac{\nu}{\lambda}$

The trajectory is given by

$$X(t) = \gamma b + \sum_{i=1}^{N(t)} Y_{\ i} \text{ where } N(t) = \sup\left\{k : \sum_{i=1}^{k} T_i \leq t\right\}.$$

Another algorithm is

Simulate $N \sim Poisson\,(\lambda T)$

Simulate $U_i \sim Uniform(0,T)$, $i = 1, \ldots, N$

Simulate Y_i *with* $\mu = \frac{\nu}{\lambda}$

The trajectory is given by

$$X(t) = bt + \sum_{i=1}^{N} 1_{U_i<t} Y_i.$$

3.3 Infinite Activity Lévy Processes

Simulation of a finite activity Lévy process can be done by generating Brownian and Poissonian-type components independently of each other. The main problem is to simulate a Poissonian-type component of a Lévy process having an infinite Lévy measure. In that case, by the Lévy-Itô decomposition (3.1), sample paths of $\{X_t,\ t \geq 0\}$ have infinitely many jumps in each finite interval. Exact simulation of such process is obviously impossible.

There are three main simulation methods for Lévy processes;

- Discretization [8]:

Discretization (also known as random walk approximation) procedure is discretizing the process $X(t)$ into $X(jh) : j = 0, 1, \ldots$ Therefore, if we can simulate $X(h)$, then there is an approximate simulation for the process $X(t)$. A complication of these discretization methods is that, location and the magnitude of the large jumps cannot be determined exactly. Especially in a heavy tailed case, large jumps are crucial because they determine many functionals of a Lévy process. Another disadvantage may be that a simulation of $X(h)$ may be computationally heavy.

- Series representations for Lévy processes [29]:

Series representation [29] provides uniform along sample paths approximation of Lévy processes and often easy to simulate. Usually largest jumps of a Lévy process are included in the first few terms of the series. A disadvantage of this method is that some series may converge very slowly. Therefore, huge number of terms may be needed to reach a desired accuracy of the approximation.

- Poissonian & Gaussian approximations [29, 3]:

If small jumps on the right hand side of (3.1) are removed or substituted by their mean value then the subsequent process is a compound Poisson process with a drift. This is a Poisson approximation of a Lévy process. As the magnitude of the removed jumps tends to zero it converges uniformly on each finite interval, because of the growth of the Lévy measure is not too fast. Large jumps are exactly simulated. However, when small jumps have high intensity, removing them (as in the series representation) brings a substantial error. Then, the small jump part of a Lévy process can be approximated by a Brownian motion with small variance instead of removing from the right hand side of (3.1). Therefore, small jumps are truncated and substituted with a properly scaled Brownian motion. This Gaussian approximation complements the series representation method because it is practical to use even the series converges slowly.

Chapter 4

STOCHASTIC CALCULUS

In this chapter, we first summarize the theory of stochastic integration. Then the quadratic variation of a stochastic process is discussed. Finally, we state the celebrated Itô's formula for Lévy processes in particular for Wiener and Poisson processes.

4.1 Stochastic Integration

We introduce stochastic integration with respect to Wiener processes, Poisson random measures and Lévy processes. Therefore, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $X_1, \ldots, X_n$ be random variables.

4.1.1 Stochastic Integration with respect to a Wiener Process

Let $X = \{X_t,\ t \geq 0\}$ be a simple predictable process defined by

$$X_t = \sum_{i=0}^{n} X_i 1_{[T_i, T_{i+1})}(t)$$

where $\{T_0 = 0 < T_1 < \ldots < T_{n-1} < T_n = T\}$ is a time grid and let $W = \{W_t,\ t \geq 0\}$ be a Wiener process. The stochastic integral of X with respect to the Wiener process [19, 26] is defined as

$$\int_0^T X_t dW_t = \sum_{i=0}^{n} X_i \left(W_{T_{i+1}} - W_{T_i} \right).$$

For many applications, it is important to consider a wider class of integrands, instead of just simple predictable processes. Suppose $Y = \{Y_t,\ t \geq 0\}$ is a stochastic process adapted to a filtration generated by a Wiener process satisfying

$$E\left(\int_0^T |Y_t|^2 \, dt \right) < \infty$$

on $[0, T]$. Then, we can find a sequence $Y^n = \{Y_t^n,\ t \geq 0\}$ of simple processes such that

$$E\left(\int_0^T (Y_t - Y_t^n)^2 \, dt\right) \to 0$$

as $n \to \infty$ [6, 26]. Hence, we can compute the stochastic integral as

$$\int_0^T Y_t dW_t = \lim_{n\to\infty} \int_0^T Y_t^n dW_t$$

in probability [19, 26].

4.1.2 Stochastic Integration with respect to Poisson Random Measures

Let $X : \Omega \times [0, T] \times \mathbb{R} \to \mathbb{R}$ be a simple and predictable function given by

$$X(t, y) = \sum_{i=1}^{n} \sum_{j=1}^{m} X_{ij} 1_{[T_i, T_{i+1})}(t) \, 1_{A_j}(y)$$

where A_j are disjoint subsets and N is a Poisson random measure with intensity measure v on $[0, T] \times \mathbb{R}$. The stochastic integral with respect to a Poisson random measure [1, 8] is defined as

$$\int_0^T \int_{\mathbb{R}} X(t, y) \, N(dt, dy) = \sum_{i=1}^{n} \sum_{j=1}^{m} X_{ij} N([T_i, T_{i+1}) \times A_j).$$

As above, to consider a wider class of integrands, we can define a stochastic process $Y = \{Y_t,\ t \geq 0\}$ satisfying

$$E\left(\int_0^T \int_{\mathbb{R}} |Y(t, y)|^2 \, v(dt, dy)\right) < \infty$$

as the limit of a sequence of simple processes $Y^n = \{Y_t^n,\ t \geq 0\}$ [6, 26]. Therefore, we can compute the stochastic integral of Y with respect to N by

$$\int_0^T \int_{\mathbb{R}} Y(t, y) \, N(dt, dy) = \lim_{n\to\infty} \int_0^T \int_{\mathbb{R}} Y^n(t, y) \, N(dt, dy).$$

which converges almost surely.

4.1.3 Stochastic Integration with respect to a Lévy Process

Let $\phi = \{\phi_t,\ t \geq 0\}$ and $\gamma = \{\gamma_t,\ t \geq 0\}$ simple predictable processes and $\psi : \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$, $\varphi : \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$ be simple predictable functions as described above. The stochastic integral with respect to a Lévy process with a Lévy measure v on $[0,T] \times \mathbb{R}$ is defined [8] as follows

$$\begin{aligned} & \int_0^T \gamma_t dt + \int_0^T \phi_t dW_t + \int_0^T \int_{|z|\geq 1} \psi(t,z)\, N(dt,dz) + \int_0^T \int_{|z|<1} \varphi(t,y)\, \tilde{N}(dt,dz) \\ = \;& \sum_{i=0}^{n} \gamma_i (T_{i+1} - T_i) + \sum_{i=0}^{n} \phi_i (W_{T_i+1} - W_{T_i}) \\ & + \sum_{i=1}^{n} \sum_{j=1}^{m} \psi_{ij} \left[N_{T_{i+1}}(A_j) - N_{T_i}(A_j) \right] \\ & + \sum_{i=1}^{n} \sum_{j=1}^{m} \varphi_{ij} \left[N([T_i, T_{i+1}] \times A_j) - v([T_i, T_{i+1}] \times A_j) \right]. \end{aligned}$$

We can extend this definition to the general integrands as above sections by defining the integrand processes on $\mathbb{L}^2(0,T)$ as the limit of a sequence of simple processes [1, 6]. Therefore, we can compute stochastic integral by combining the results defined in 4.1.1 and 4.1.2.

4.2 Quadratic Variation

Consider a semimartingale X with $X_0 = 0$. It is impossible to define the integral

$$\int_0^t X_t dX_t \tag{4.1}$$

path by path as a Riemann-Stieltjes integral [26, 27] since X_t defined in Definition 2.11 may not be of bounded variation. However, one can define this integral as a mean square limit of

$$\sum_{t_i \in \tau} X_{t_i} \left(X_{t_{i+1}} - X_{t_i} \right) = \frac{1}{2} X_t^2 - \frac{1}{2} \sum_{t_i \in \tau} \left(X_{t_{i+1}} - X_{t_i} \right)^2 \tag{4.2}$$

on a time grid $\tau = \{t_0 = 0 < t_1 < \ldots < t_{n-1} < t_n = t\}$ as $n \longrightarrow \infty$. If (4.1) was a classical integral, the result of (4.1) would be $\frac{X_t^2}{2}$. However, the relation (4.2) in the limit produces an extra term as $\frac{X_t^2}{2} - \frac{Q_t}{2}$. The extra term Q_t is called *quadratic variation process* of the stochastic process X and is defined by

$$[X,X]_t := Q_t = \lim_{n\to\infty} \sum_{i=0}^{n} (X_{t_i+1} - X_{t_i})^2$$

in probability.

As an example, let W be a Wiener process. The quadratic variation of W can be found as

$$[W,W]_t = t.$$

On the other hand, let M be a semimartingale of the form

$$M_t = \int_0^t \gamma_s ds + \int_0^t \sigma_s dW_s.$$

The quadratic variation of M can now be found as

$$[M,M]_t = \int_0^t \sigma_s^2 ds.$$

We state the quadratic variation of a Poisson process as another example. Let N be a Poisson process, its quadratic variation process is given as

$$[N,N]_t = N_t.$$

More generally, let N be a Poisson random measure with intensity measure v on $[0,T]\times\mathbb{R}$ and $\phi : \Omega\times[0,T]\times\mathbb{R}\to\mathbb{R}$ be a simple predictable function. The quadratic variation of the integral process

$$X_t = \int_0^t \int_{\mathbb{R}} \phi(t,y)\, N(dt,dy)$$

is found as

$$[X,X]_t = \int_0^t \int_{\mathbb{R}} |\phi(t,y)|^2\, N(dt,dy).$$

If X is a Lévy process with characteristic triplet (σ, v, γ), its quadratic variation is found as

$$[X,X]_t = \sigma^2 t + \int_0^t \int_{\mathbb{R}} y^2 N(ds,dy).$$

More generally, let Y be a *Lévy stochastic integral* of the form,

$$dY_t = \gamma_t dt + \sigma_t dW_t + \int_{|z|<1} \psi(t,y)\tilde{N}(dt,dy) + \int_{|z|\geq 1} \phi(t,y)N(dt,dy).$$

Then, the quadratic variation process of a Lévy stochastic integral is found as

$$[Y,Y]_t = \int_0^t \sigma_s^2 ds + \int_0^t \int_{|z|<1}^2 \psi^2(t,y)N(ds,dy) + \int_0^t \int_{|z|\geq 1}^2 \phi^2(t,y)N(ds,dy).$$

Given two semimartingales X and Y, the *quadratic covariation process* $[X,Y]$ is the semimartingale defined by

$$[X,Y]_t = X_tY_t - X_0Y_0 - \int_0^t X_{s^-}dY_s - \int_0^t Y_{s^-}dX_s.$$

If X and Y are semimartingales, then we can define the Itô's product rule [1] as follows,

$$X_tY_t = X_0Y_0 + \int_0^t X_{s^-}dY_s + \int_0^t Y_{s^-}dX_s + [X,Y]_t\,.$$

We can interpret this result to the differential form which will be more useful for many applications

$$d\left(X_tY_t\right) = X_{s^-}dY_s + Y_{s^-}dX_s + d\left[X,Y\right]_t. \tag{4.3}$$

We can now investigate the quadratic covariation processes of Brownian, Poisson and Lévy stochastic integrals [8]. Quadratic covariation of correlated Brownian integrals of the form

$$X_t^i = \int_0^t \sigma_s^i dW_s^i,\ i = 1,2$$

where W^1 and W^2 are correlated Wiener processes with correlation ρ is given by

$$\left[B^1, B^2\right]_t = \int_0^t \rho\sigma_s^1\sigma_s^2 ds.$$

Quadratic covariation of integrals with respect to a Poisson random measure N on $[0,T]\times\mathbb{R}$ of the form

$$X_t^i = \int_0^t \int_{\mathbb{R}} \psi^i\left(s,y\right)\tilde{N}(ds,dy),\ i = 1,2$$

is found as

$$\left[X^1, X^2\right]_t = \int_0^t \int_{\mathbb{R}} \psi^1(s,y)\, \psi^2(s,y)\, N(ds,dy).$$

Finally, quadratic covariation process of Lévy stochastic integrals given by

$$\begin{aligned} Y_t^i &= \int_0^t \gamma_s^i ds + \int_0^t \sigma_s^i dW_s + \int_0^t \int_{|z|<1} \psi^i(s,y)\, \tilde{N}(ds,dy) \\ &\quad + \int_0^t \int_{|z|\geq R} \phi^i(s,y) N(ds,dy) \end{aligned}$$

for $i = 1, 2$ is as follows

$$\begin{aligned} \left[Y^1, Y^2\right] &= \int_0^t \sigma_s^1 \sigma_s^2 ds + \int_0^t \int_{|z|<1} \psi^1(s,y)\, \psi^2(s,y)\, N(ds,dy) \\ &\quad + \int_0^t \int_{|z|\geq 1} \phi^1(s,y)\phi^2(s,y) N(ds,dy). \end{aligned}$$

4.3 Itô's Formula

Itô's formula is the touchstone of stochastic calculus and is used to find the differential of a function of a given semimartingale. It is the stochastic calculus analogue of the classical chain rule $(f(g(x)))' = f'(g(x))\, g'(x)$ in ordinary calculus and is obtained by using the well-known Taylor series expansion and retaining the second order term related to the stochastic component change.

The following formulas are taken and blended from [1, 28].

Theorem 4.3.1 *Let a Brownian Motion X be defined by*

$$dX_t = \gamma_t dt + \sigma_t dW_t$$

$f : \mathbb{R}^2 \to \mathbb{R}$ is a C^2 function and Y is given by

$$Y_t = f(X_t, t)$$

then we have

$$dY_t = \partial_t f(X_t, t)\, dt + \partial_x f(X_t, t)\left[\gamma_t dt + \sigma_t dW_t\right] + \frac{1}{2}\sigma_t^2 \partial_{xx} f(X_t, t)\, dt.$$

4.3.1 Itô's Formula for a Poisson Integral

Theorem 4.3.2 *If X is an integral with respect to a Poisson random measure*

$$dX_t = \int_{|z|\geq 1} \phi(t,z)N(dt,dz)$$

$f : \mathbb{R}^2 \to \mathbb{R}$ is a C^2 function and Y is given by

$$Y_t = f\left(X_t, t\right),$$

then

$$dY_t = \int_{|z|\geq 1} \left[f\left(X_{t^-} + \phi(t,z), t\right) - f\left(X_{t^-}, t\right)\right] N(dt,dz).$$

4.3.2 Itô's Formula for a Lévy Process

Theorem 4.3.3 *Let a Lévy process X be defined by*

$$dX_t = \gamma_t dt + \sigma_t dW_t + \int_{|z|<1} \psi(t,z)\tilde{N}(dt,dz) + \int_{|z|\geq 1} \phi(t,z)N(dt,dz)$$

$f : \mathbb{R}^2 \to \mathbb{R}$ is a C^2 function and Y is given by

$$Y_t = f\left(X_t, t\right)$$

then

$$\begin{aligned} dY_t &= \partial_t f\left(X_t, t\right) dt + \partial_x f\left(X_t, t\right)\left[\gamma_t dt + \sigma_t dW_t\right] + \frac{1}{2}\sigma_t^2 \partial_{xx} f\left(X_t, t\right) dt \qquad (4.4) \\ &+ \int_{|z|<1} \left[f\left(X_{t^-} + \psi(t,z), t\right) - f\left(X_{t^-}, t\right) - \psi(t,z)\partial_x f\left(X_{t^-}, t\right)\right] v\left(dz\right) dt \\ &+ \int_{|z|<1} \left[f\left(X_{t^-} + \psi(t,z), t\right) - f\left(X_{t^-}, t\right)\right] \tilde{N}(dt,dz) \\ &+ \int_{|z|\geq 1} \left[f\left(X_{t^-} + \phi(t,z), t\right) - f\left(X_{t^-}, t\right)\right] N(dt,dz). \end{aligned}$$

In general, we can write Itô's formula in the following formal expression.

Theorem 4.3.4 *Let X be a semimartingale, $f : \mathbb{R}^2 \to \mathbb{R}$ is a C^2 function and Y is given by*

$$Y_t = f\left(X_t, t\right)$$

then

$$
\begin{aligned}
Y_t &= \int_0^t \partial_s f\left(X_s, s\right) ds + \int_0^t \partial_x f\left(X_s, s\right) dX_s + \frac{1}{2}\int_0^t \partial_{xx} f\left(X_s, s\right) d\left[X, X\right]_s \\
&\quad + \sum_{0 \le s \le t} \left[f\left(X_s, s\right) - f\left(X_{s^-}, s\right) - \Delta X_s \partial_x f\left(X_{s^-}, s\right)\right].
\end{aligned}
$$

Chapter 5

CONDITIONS FOR LINEARIZATION AND A METHOD OF SOLUTION

Let $W = \{W_t, t \geq 0\}$ be a Wiener process and $N^j = \{N_t^j, t \geq 0\}$ be Poisson processes with arrival rates $\lambda_1, \lambda_2, \ldots, \lambda_m$ on a probability space $(\Omega, \mathcal{F}, P)$. Let $V_i^j = \{V_i^j, i = 1, 2, \ldots\}$ be independent and identically distributed random variables to form the compound Poisson process

$$C_t^j = \sum_{i=1}^{N_t^j} V_i^j \tag{5.1}$$

$j = 1, 2, \ldots, m$.

We consider a real-valued stochastic process X, starting at time $t = 0$ adapted to the filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the nonlinear stochastic differential equation with jump terms of the form

$$dX_t = f(X_{t^-}, t)dt + g(X_{t^-}, t)dW_t + \sum_{j=1}^{m} r_j(X_{t^-}, t)dC_t^j, \qquad X_0 = x_0 \tag{5.2}$$

where dW_t is the infinitesimal increment of the Wiener process and independently dC_t^j is the infinitesimal increment of the compound Poisson processes for $j = 1, 2, \ldots, m$ [17, 27]. Since a finite activity Lévy process can be decomposed into a Wiener and a compound Poisson process as a special case of Lévy-Itô decomposition (3.1), one can obviously say that (5.2) driven by Lévy components that makes (5.2) more general than (3.1).

5.1 Linearization

We seek a transformation $h : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ of X_t to

$$Y_t = h(X_t, t) \tag{5.3}$$

which will transform the nonlinear stochastic differential equation given in (5.2) into a linear equation of the form

$$\begin{aligned} dY_t &= (a_1(t)Y_{t^-} + a_2(t))\,dt + (b_1(t)Y_{t^-} + b_2(t))\,dW_t \\ &+ \sum_{j=1}^{m} \left(c_1^j \left(t, V^j_{N^j_t} \right) Y_{t^-} + c_2^j \left(t, V^j_{N^j_t} \right) \right) dN^j_t. \end{aligned} \tag{5.4}$$

since

$$dC^j_t = V^j_{N^j_t} dN^j_t$$

in (5.2). Note that, the magnitude of the jump at time t can be found as $X_{t^+} - X_{t^-}$ from the cadlag property, that is, $X_{t^-} = \lim_{s\uparrow t} X_s$.

Itô's formula (4.4) for $h(X_t, t)$ leads to

$$\begin{aligned} dY_t &= \left[\partial_t h(X_{t^-}, t) + f(X_{t^-}, t)\partial_x h(X_{t^-}, t) + \frac{1}{2} g^2(X_{t^-}, t)\partial_{xx} h(X_{t^-}, t) \right] dt \\ &+ g(X_{t^-}, t)\partial_x h(X_{t^-}, t) dW_t + \sum_{j=1}^{m} \left[h(X_{t^-} + r_j\left(X_{t^-}, t\right) V^j_{N^j_t}, t) - h(X_{t^-}, t) \right] dN^j_t. \end{aligned} \tag{5.5}$$

Using equations (5.2) and (5.4) we obtain

$$\partial_t h(X_{t^-}, t) + f(X_t, t)\partial_x h(X_{t^-}, t) + \frac{1}{2} g^2(X_t, t)\partial_{xx} h(X_{t^-}, t) = a_1(t)h(X_t, t) + a_2(t) \tag{5.6}$$

$$g(X_t, t)\partial_x h(X_{t^-}, t) = b_1(t)h(X_t, t) + b_2(t) \tag{5.7}$$

and

$$\sum_{j=1}^{m} \left[h(X_t + r_j(X_t, V^j_{N^j_t}, t), t) - h(X_t, t) \right] dN^j_t = \sum_{j=1}^{m} \left(c_1^j \left(t, V^j_{N^j_t} \right) Y_t + c_2^j \left(t, V^j_{N^j_t} \right) \right) dN^j_t \tag{5.8}$$

Ordinary differential equation (5.7) has two distinct solutions for *i*) $b_1(t) = 0$ and *ii*) $b_1(t) \neq 0$. We now consider each case separately.

Case 1 $b_1(t) = 0$ and $b_2(t) \neq 0$

In this case, (5.7) is satisfied if

$$g(x,t)\partial_x h(x,t) = b_2(t),$$

for all $x \in \mathbb{R}$ and $t \in \mathbb{R}_+$. Then the transformation h can be found as

$$h(x,t) = \int^x \frac{b_2(t)}{g(\tilde{x},t)} d\tilde{x} \tag{5.9}$$

where we have chosen the arbitrary function of integration to be zero and assumed $g(\tilde{x},t) \neq 0$. Substituting (5.9) into (5.6) yields

$$\int^x \partial_t \left(\frac{b_2\,(t)}{g(\tilde{x},t)} \right) d\tilde{x} + f(x,t)\frac{b_2\,(t)}{g(x,t)} - \frac{1}{2} b_2\,(t)\,\partial_x g(x,t) = a_1(t) \int^x \frac{b_2(t)}{g(\tilde{x},t)} d\tilde{x} + a_2(t). \tag{5.10}$$

Arranging the terms in (5.10) leads to

$$\int^x \partial_t \left(\frac{b_2\,(t)}{g(\tilde{x},t)} \right) d\tilde{x} + b_2\,(t) \left(\frac{f(x,t)}{g(x,t)} - \frac{1}{2}\partial_x g(x,t) \right) = a_1(t) \int^x \frac{b_2(t)}{g(\tilde{x},t)} d\tilde{x} + a_2(t),$$

and

$$\int^x \partial_t \left(\frac{b_2\,(t)}{g(\tilde{x},t)} \right) d\tilde{x} - a_1\,(t) \int^x \frac{b_2(t)}{g(\tilde{x},t)} d\tilde{x} + b_2\,(t) \left(\frac{f(x,t)}{g(x,t)} - \frac{1}{2}\partial_x g(x,t) \right) = a_2(t). \tag{5.11}$$

Differentiating (5.11) with respect to x gives

$$\frac{b_2'\,(t)}{g(x,t)} + b_2\,(t) \left(L(x,t) - \frac{a_1(t)}{g(x,t)} \right) = 0 \tag{5.12}$$

where

$$L(x,t) = \partial_t \left(\frac{1}{g\,(x,t)} \right) + \partial_x \left(\frac{f\,(x,t)}{g\,(x,t)} - \frac{1}{2}\partial_x g\,(x,t) \right).$$

Multiplying both sides of (5.12) by $g(x,t)$ we find

$$b_2'\,(t) + b_2\,(t)\,(g(x,t)L - a_1(t)) = 0,$$

That is, we get

$$g(x,t)L(x,t) = a_1(t) - \frac{b_2'(t)}{b_2(t)} \tag{5.13}$$

which does not depend on x. Therefore, differentiating (5.13) with respect to x leads to,

$$\partial_x [g(x,t)L(x,t)] = 0. \tag{5.14}$$

Then, we can arbitrarily choose $b_2(t)$ and set

$$a_1(t) = g(x,t)L(x,t) + \frac{b_2'(t)}{b_2(t)}.$$

and find $a_2(t)$ from (5.11) for $f \in C^2$ and $g \in C^3$.

Now we consider (5.8), which is satisfied if

$$h(X_t + r_i(X_t,t)V_{N_t^j}^j, t) = \left(c_1^i\left(t, V_{N_t^j}^j\right) + 1\right) h(X_t,t) + c_2^i\left(t, V_{N_t^j}^j\right)$$

$i = 1, 2, \ldots, m$, for all $x \in \mathbb{R}$, $z \in \mathbb{R}$, $t \in \mathbb{R}_+$. Equivalently, we must have

$$\int^{x + r_i(x,t)z} \frac{b_2(t)}{g(\tilde{x},t)} d\tilde{x} = \left(c_1^i(t,z) + 1\right) \int^x \frac{b_2(t)}{g(\tilde{x},t)} d\tilde{x} + c_2^i(t,z) \tag{5.15}$$

and differentiating it with respect to x, we obtain

$$\frac{b_2(t)}{g(x + r_i(x,t)z, t)} (\partial_x r_i(x,t)z + 1) = \frac{b_2(t)}{g(x,t)} (c_1^i(t,z) + 1).$$

We rewrite the above equation as

$$A_i(x,t) := (\partial_x r_i(x,t)z + 1) \frac{g(x,t)}{g(x + r_i(x,t)z, t)} = c_1^i(t,z) + 1, \tag{5.16}$$

Now, c_1 can be found from (5.16) and c_2 can be found from (5.15). Differentiating (5.16) with respect to x yields

$$\partial_x A_i(x,t) = 0 \tag{5.17}$$

$i = 1, 2, \ldots, m$, for $r_j \in C^2$.

Therefore, (5.14) and (5.17) are the linearization conditions.

Case 2 $b_1(t) \neq 0$ and $b_2(t) = 0$

The solution of (5.7) in this case is $h(x,t) = K(t)e^{\int^x \frac{b_1(t)}{g(\tilde{x},t)}d\tilde{x}}$. We simply choose $K(t) = 1$ and seek for b_1. Thus, we get

$$h(x,t) = e^{\int^x \frac{b_1(t)}{g(\tilde{x},t)}d\tilde{x}}. \tag{5.18}$$

Substitution of (5.18) into (5.6) yields

$$\partial_t h(x,t) + f(x,t)\partial_x h(x,t) + \frac{1}{2}g^2(x,t)\partial_{xx}h(x,t) = a_1(t)h(x,t) + a_2(t),$$

which is equivalent to

$$\left[\int^x \partial_t \left(\frac{b_1(t)}{g(\tilde{x},t)}\right) d\tilde{x} + b_1(t)\left(\frac{f(x,t)}{g(x,t)} - \frac{1}{2}\partial_x g(x,t)\right) + \frac{b_1^2(t)}{2} - a_1(t)\right] e^{\int^x \frac{b_1(t)}{g(\tilde{x},t)}d\tilde{x}} = a_2(t). \tag{5.19}$$

Differentiating both sides of (5.19) with respect to x leads to

$$\begin{aligned} b_1'(t) + b_1(t)\, g(x,t)L(x,t) + b_1(t) \textstyle\int^x \partial_t \left(\frac{b_1(t)}{g(\tilde{x},t)}\right) d\tilde{x} \\ +b_1^2(t)\left(\frac{f(x,t)}{g(x,t)} - \frac{1}{2}\partial_x g(x,t)\right) + \frac{b_1^3(t)}{2} - b_1(t)\, a_1(t) = 0 \end{aligned} \tag{5.20}$$

and

$$\begin{aligned} b_1'(t) + b_1(t)\left[g(x,t)L(x,t) + \textstyle\int^x \partial_t \left(\frac{b_1(t)}{g(\tilde{x},t)}\right) d\tilde{x} - a_1(t)\right] \\ +b_1^2(t)\left[\frac{f(x,t)}{g(x,t)} - \frac{1}{2}\partial_x g(x,t)\right] + \frac{b_1^3(t)}{2} = 0 \end{aligned}$$

by simplification. We aim to find $b_1(t)$ first. Differentiating with respect to x and simplifying $b_1(t)$ as it is nonzero, we get

$$\left[\partial_x\left[g(x,t)L(x,t)\right] + \frac{b_1'(t)}{g(x,t)}\right] + b_1(t)\left[\partial_x\left(\frac{f(x,t)}{g(x,t)} - \frac{1}{2}\partial_x g(x,t)\right) + \partial_t\left(\frac{1}{g(x,t)}\right)\right] = 0.$$

Therefore, we have

$$b_1'(t) + b_1(t)\, g(x,t)L(x,t) = -g(x,t)\partial_x\left[g(x,t)L(x,t)\right]. \tag{5.21}$$

Differentiating (5.21) with respect to x, we get $b_1(t)$

$$b_1(t) = -\frac{\partial_x\left(g(x,t)\partial_x\left[g(x,t)L(x,t)\right]\right)}{\partial_x\left[g(x,t)L(x,t)\right]}. \tag{5.22}$$

Differentiating (5.22) with respect to x yields

$$\partial_x M(x,t) = 0, \tag{5.23}$$

where we have introduced

$$M\left(x,t\right) := \frac{\partial_x \left(g(x,t)\partial_x \left[g(x,t)L(x,t)\right]\right)}{\partial_x \left[g(x,t)L(x,t)\right]}.$$

for $f \in C^2$ and $g \in C^3$, as a term in the linearization criterion (5.23).

The transformation given in Equation (5.18) is

$$h(x,t) = e^{b_1(t)\int^x \frac{1}{g(\tilde{x},t)}d\tilde{x}}.$$

Differentiating (5.18) with respect to x yields

$$\partial_x h(x,t) = \frac{b_1(t)}{g(x,t)} e^{b_1(t)\int^x \frac{1}{g(\tilde{x},t)}d\tilde{x}}. \tag{5.24}$$

Substitution of (5.24) into (5.8) leads to

$$e^{\int^{x+r_i(x,t)z} \frac{b_1(t)}{g(\tilde{x},t)}d\tilde{x}} = \left(c_1^i(t,z)+1\right) e^{\int^x \frac{b_1(t)}{g(\tilde{x},t)}d\tilde{x}} + c_2^i(t,z). \tag{5.25}$$

Differentiating with respect to x, in order to eliminate $c_2^i(t,z)$, we find

$$\frac{b_1(t)}{g(x+r_i(x,t)z,t)}\left(\partial_x r_i(x,t)z+1\right) e^{\int^{x+r_i(x,t)z} \frac{b_1(t)}{g(\tilde{x},t)}d\tilde{x}} = \frac{b_1(t)}{g(x,t)} e^{\int^x \frac{b_1(t)}{g(\tilde{x},t)}d\tilde{x}} \left(c_1^i(t,z)+1\right), \tag{5.26}$$

and cancelling $b_1(t)$ we get

$$A_i(x,t)e^{\left(\int^{x+r_i(x,t)z} \frac{b_1(t)}{g(\tilde{x},t)}d\tilde{x} - \int^x \frac{b_1(t)}{g(\tilde{x},t)}d\tilde{x}\right)} = \left(c_1^i\left(t,z\right)+1\right) \tag{5.27}$$

$i = 1,2,\ldots,m$, in terms of A_i of (5.16). Differentiating Equation (5.27) with respect to x yields

$$\partial_x A_i(x,t) + A_i(x,t)b_1\left(t\right)\left(\frac{A_i(x,t)-1}{g(x,t)}\right) = 0. \tag{5.28}$$

Therefore, equations (5.23) and (5.28) are the linearization conditions in this case for $r_j \in C^2$. Now, one can obtain $a_1(t)$ from (5.20), $a_2(t)$ from (5.19), $b_1(t)$ from(5.22), $c_1(t)$ from (5.27) and $c_2(t)$ from (5.25).

We now state our findings as a theorem.

Theorem 5.1.1 *A nonlinear stochastic differential equation (5.2) is linearizable via the transformation*

$$h(x,t) = \int^{x} \frac{d\tilde{x}}{g(\tilde{x},t)}$$

if (5.14) and (5.17) are satisfied or via

$$h(x,t) = e^{\int^{x} \frac{b_1(t)}{g(\tilde{x},t)} d\tilde{x}}$$

if (5.23) and (5.28) are satisfied where $b_1(t)$ *is given in* (5.22).

Note that, the two sets of conditions in Theorem 5.1.1 are mutually exclusive. If (5.14) is satisfied, then (5.23) is not possible as it originates from (5.21) where b_1 is nonzero and (5.14) cannot hold. Clearly, this argument holds both ways.

5.2 Stochastic Integrating Factors

A first order linear ordinary differential equation can be solved using the integrating factor method [22]. We extend this well-known integrating factor method for solving linear ODEs to the linear SDEs driven by compound Poisson processes and develop stochastic integrating factors. We define a stochastic integrating factor as follows.

Definition 5.2.1 *The function* $\mu_t = \mu_t(W_t, C_t^1, C_t^2, \ldots, C_t^m) =: \mu_t(W_t, C_t)$ *with property*

$$d(\mu_t Y_t) = D_1(t)dt + D_2(t)dW_t + \sum_{j=1}^{m} D_3^j(t) V_{N_t^j}^j dN_t^j$$

is called a stochastic integrating factor for the one-dimensional linear jump-diffusion SDE .

We apply the stochastic product rule for semimartingales [1] as

$$d(\mu_t Y_t) = \mu_{t^-} dY_{t^-} + Y_{t^-} d\mu_t + d\left[\mu, Y\right]_t \tag{5.29}$$

for

$$\begin{aligned} dY_t \quad &= \quad \left(a_1(t)Y_{t^-} + a_2(t)\right)dt + \left(b_1(t)Y_{t^-} + b_2(t)\right)dW_t \\ &\quad + \sum_{j=1}^{m} \left(c_1^j\left(t, V_{N_t^j}^j\right)Y_{t^-} + c_2^j\left(t, V_{N_t^j}^j\right)\right)dN_t^j. \end{aligned} \tag{5.30}$$

Then, Itô's formula for μ reads,

$$d\mu_t = \left(\partial_t \mu_{t-} + \frac{1}{2}\partial_{xx}\mu_{t-}\right) dt + \partial_x \mu_{t-} dW_t + \sum_{j=1}^{m} (\mu_t - \mu_{t-})\, dN_t^j.$$

Therefore, from (4.3) we can write the terms which will form the differential product $d(\mu_t Y_t)$ as

$$\begin{aligned} \mu_{t-} dY_t &= \mu_{t-}\,(a_1(t)Y_{t-} + a_2(t))\, dt + \mu_{t-}\,(b_1(t)Y_{t-} + b_2(t))\, dW_t \\ &\quad + \mu_{t-} \sum_{j=1}^{m} \left(c_1^j(t, V_{N_t^j}^j)Y_{t-} + c_2^j(t, V_{N_t^j}^j)\right) dN_t^j, \end{aligned} \tag{5.31}$$

and

$$Y_{t-} d\mu_t = Y_{t-}\left(\partial_t \mu_{t-} + \frac{1}{2}\partial_{xx}\mu_{t-}\right) dt + Y_{t-}\partial_x \mu_{t-} dW_t + Y_{t-}\sum_{j=1}^{m} (\mu_t - \mu_{t-})\, dN_t^j \tag{5.32}$$

and the quadratic variation

$$\begin{aligned} d\,[\mu, Y]_t &= \partial_x \mu_{t-}\,[b_1\,(t)\,Y_{t-} + b_2\,(t)]\, dt \\ &\quad + \sum_{j=1}^{m} [\mu_t - \mu_{t-}]\left[c_1^j\left(t, V_{N_t^j}^j\right) Y_{t-} + c_2^j\left(t, V_{N_t^j}^j\right)\right] dN_t^j. \end{aligned} \tag{5.33}$$

Using the results in Equations (5.31), (5.32) and (5.33), Equation (5.29) transforms into,

$$\begin{aligned} d(\mu_t Y_t) &= \underbrace{\begin{bmatrix} \mu_{t-}\,[a_1(t)Y_{t-} + a_2(t)]\, dt + \mu_{t-}\,[b_1(t)Y_{t-} + b_2(t)]\, dW_t \\ +\mu_{t-} \sum_{j=1}^{m} \left[c_1^j(t, V_{N_t^j}^j)Y_{t-} + c_2^j(t, V_{N_t^j}^j)\right] dN_t^j \end{bmatrix}}_{\mu_{t-} dY_t} \\ &\quad + \underbrace{\left[Y_{t-}\left(\partial_t \mu_{t-} + \frac{1}{2}\partial_{xx}\mu_{t-}\right) dt + Y_{t-}\partial_x\mu_{t-} dW_t + Y_{t-}\sum_{j=1}^{m}(\mu_t - \mu_{t-})\, dN_t^j\right]}_{Y_{t-} d\mu_t} \\ &\quad + \underbrace{\begin{bmatrix} \partial_x \mu_{t-}\,[b_1\,(t)\,Y_{t-} + b_2\,(t)]\, dt \\ + \sum_{j=1}^{m} [\mu_t - \mu_{t-}]\left[c_1^j\left(t, V_{N_t^j}^j\right) Y_{t-} + c_2^j\left(t, V_{N_t^j}^j\right)\right] dN_t^j \end{bmatrix}}_{d[\mu, Y]_t}. \end{aligned} \tag{5.34}$$

The right hand side of Equation (5.34) should not involve the variable Y_{t-} to comply with the definition of the integrating factor. Hence, arranging the terms in

Equation (5.34) yields,

$$\begin{aligned} d(\mu_t Y_t) &= Y_{t^-}\left(\partial_t \mu_{t^-} + \frac{1}{2}\partial_{xx}\mu_{t^-} + b_1(t)\partial_x \mu_{t^-} + a_1(t)\mu_{t^-}\right)dt \\ &\quad + Y_{t^-}\left(\partial_x \mu_{t^-} + b_1(t)\mu_{t^-}\right)dW_t \\ &\quad + Y_{t^-}\left(\sum_{j=1}^{m}\left(c_1^j\left(t, V^j_{N_t^j}\right)+1\right)\left(\mu_t - \mu_{t^-}\right)dN_t^j + \mu_{t^-}\sum_{j=1}^{m} c_1^j\left(t, V^j_{N_t^j}\right)dN_t^j\right) \\ &\quad + \left[\mu_{t^-} a_2(t) + \partial_x \mu_{t^-} b_2(t)\right]dt + \mu_{t^-} b_2(t)\,dW_t + \sum_{j=1}^{m}\mu_t c_2^j\left(t, V^j_{N_t^j}\right)dN_t^j. \end{aligned} \tag{5.35}$$

which leads to

$$\partial_t \mu_{t^-} + \frac{1}{2}\partial_{xx}\mu_{t^-} + b_1(t)\partial_x \mu_{t^-} + a_1(t)\mu_{t^-} = 0 \tag{5.36}$$

$$\partial_x \mu_{t^-} + b_1(t)\mu_{t^-} = 0 \tag{5.37}$$

and

$$\sum_{j=1}^{m}\left(c_1^j\left(t, V^j_{N_t^j}\right)+1\right)\left(\mu_t - \mu_{t^-}\right)dN_t^j + \mu_{t^-}\sum_{j=1}^{m} c_1^j\left(t, V^j_{N_t^j}\right)dN_t^j = 0. \tag{5.38}$$

Let us seek a solution to the system of PDEs (5.36) and (5.37) together with (5.38). First, consider Equation (5.38), which simplifies to

$$\sum_{j=1}^{m}\left(c_1^j\left(t, V^j_{N_t^j}\right)+1\right)\mu_t dN_t^j = \sum_{j=1}^{m}\mu_{t^-} dN_t^j. \tag{5.39}$$

To find a solution to (5.39), let us set

$$\mu_t \equiv \mu_t(W_t, C_t) = M_1(t, W_t)\prod_{j=1}^{m} M_2^j\left(t, C_t^j\right) \tag{5.40}$$

for a continuous function M_1 and let $M_2^j\left(t, C_t^j\right)$ be given by

$$M_2^j\left(t, C_t^j\right) = \prod_{i=1}^{N_t^j}\frac{1}{c_1^j\left(t, V_i^j\right)+1} \tag{5.41}$$

in view of

$$C_t^j = \sum_{i=1}^{N_t^j} V_i^j$$

$j = 1, \ldots, m$, for the compound Poisson processes C_t^j given in (5.1). By definition, note that

$$M_2^j \left(t, C_{t-}^j\right) = \prod_{i=1}^{N_{t-}^j} \frac{1}{c_1^j \left(t, V_i^j\right) + 1} \tag{5.42}$$

that is, $M_2^j \left(t, C_{t-}^j\right)$ involves all jumps of the Poisson process N_t^j except for the one at time t, if any. We will show that (5.41) satisfies (5.39) in almost sure sense next. Since the Poisson processes N^j, $j = 1, \ldots, m$ are independent, at any any instant, at most one jump occurs from only one of $N^1, \ldots, N^m$ with probability 1. That is, for each $t \in \mathbb{R}$, $dN_t^j = 0$ for $j = 1, \ldots, m$, or for $j \neq j^0$ and some $j^0 \in \{1, \ldots, m\}$ almost surely in (5.39). For each $t \in \mathbb{R}$ at which there is a jump, (5.39) reduces to

$$\left(c_1^{j^0} \left(t, V_{N_t^{j^0}}^{j^0}\right) + 1\right) \mu_t dN_t^{j^0} = \mu_{t-} dN_t^{j^0}. \tag{5.43}$$

Now, for fixed $t \in \mathbb{R}$, let us specify μ_{t-} in relation to μ_t defined by (5.40). We have

$$\mu_{t-} \equiv M_1 \left(t, W_t\right) \prod_{j=1}^{m} M_2^j \left(t, C_{t-}^j\right) \tag{5.44}$$

as $M_1 \left(t^-, W_{t-}\right) = M_1 \left(t, W_t\right)$ by continuity of M_1 and W_t. On the other hand, since there is a jump from only $N_t^{j^0}$ at time t, we have

$$\mu_t \equiv M_1 \left(t, W_t\right) M_2^{j^0} \left(t, C_t^{j^0}\right) \prod_{j=1, j \neq j^0}^{m} M_2^j \left(t, C_{t-}^j\right). \tag{5.45}$$

From (5.44) and (5.45), we observe that

$$\mu_{t-} = \mu_t \frac{M_2^{j^0} \left(t, C_{t-}^{j^0}\right)}{M_2^{j^0} \left(t, C_t^{j^0}\right)} = \mu_t \frac{1}{c_1^{j^0} \left(t, V_{N_t^{j^0}}^{j^0}\right) + 1} \tag{5.46}$$

also in view of (5.41) and (5.42). Now, (5.46) satisfies (5.43). Hence, (5.40) satisfies (5.39) almost surely and the integrating factor μ_t takes the form of

$$\mu_t \equiv \mu_t(W_t, C_t^j) = M_1(t, W_t) \prod_{j=1}^{m} \prod_{i=1}^{N_t^j} \frac{1}{c_1^j \left(t, V_i^j\right) + 1}. \tag{5.47}$$

Now, we will find M_1 using (5.37) and (5.38). Substitution of (5.47) into (5.37) leads to

$$\partial_x M_1(t, W_t) \prod_{j=1}^{m} \prod_{i=1}^{N_{t-}^j} \frac{1}{c_1^j\left(t, V_i^j\right) + 1} + b_1(t) M_1(t, W_t) \prod_{j=1}^{m} \prod_{i=1}^{N_{t-}^j} \frac{1}{c_1^j\left(t, V_i^j\right) + 1} = 0,$$

which is

$$\partial_x M_1(t, W_t) + b_1(t) M_1(t, W_t) = 0. \tag{5.48}$$

Now, by Itô's formula, we have

$$dM_1(t, W_t) = \partial_t M_1(t, W_t)dt + \partial_x M_1(t, W_t)dW_t + \frac{1}{2}\partial_{xx} M_1(t, W_t)dt.$$

In view of (5.48), this simplifies to

$$dM_1(t, W_t) = \partial_t M_1(t, W_t)dt - b_1\left(t\right) M_1(t, W_t)dW_t + \frac{1}{2}b_1^2\left(t\right) M_1(t, W_t)dt. \tag{5.49}$$

To solve (5.49), we try $M_1(t, W_t)$ which satisfies

$$\partial_t M_1(t, W_t) = M_1(t, W_t)q'\left(t\right) - \frac{1}{2}b_1^2\left(t\right) M_1(t, W_t) \tag{5.50}$$

for a Stieltjes function q and we have taken q' for the sake of brevity in the sequel. Hence, (5.49) reduces to

$$dM_1(t, W_t) = -b_1\left(t\right) M_1(t, W_t)dW_t + M_1(t, W_t)q'\left(t\right)dt \tag{5.51}$$

and solution to (5.51) is

$$M_1(t, W_t) = e^{-\int_0^t b_1(s)dW_s - \frac{1}{2}\int_0^t b_1^2(s)ds + q(t)} \tag{5.52}$$

where the integral $\int_0^t b_1(s)dW_s$ is a Riemann-Stieltjes integral since $b_1\left(t\right)$ is a deterministic function [26]. Note that the solution (5.52) indeed satisfies (5.50). Thus, the stochastic integrating factor takes the form

$$\mu_t = e^{-\int_0^t b_1(s)dW_s - \frac{1}{2}\int_0^t b_1^2(s)ds + q(t)} \prod_{j=1}^{m} \prod_{i=1}^{N_t^j} \frac{1}{c_1^j\left(t, V_i^j\right) + 1}. \tag{5.53}$$

We now use μ_t of (5.53) in (5.36) to obtain

$$-\frac{1}{2}b_1^2(t)+q'(t)-\sum_{j=1}^{m}\sum_{i=1}^{N_{t-}^j}\frac{c_1^{j\prime}\left(t,V_i^j\right)}{c_1^j\left(t,V_i^j\right)+1}+\frac{1}{2}b_1^2(t)-b_1^2(t)+a_1(t)=0.$$

Then, we get

$$q(t)=\int_0^t\left(\sum_{j=1}^{m}\sum_{i=1}^{N_{s-}^j}\frac{c_1^{j\prime}\left(s,V_i^j\right)}{c_1^j\left(s,V_i^j\right)+1}+b_1^2(s)-a_1(s)\right)ds.$$

Therefore the integrating factor from (5.47) becomes

$$\begin{aligned}\mu_t &= \exp\left[-\int_0^t b_1(s)\,dW_s+\int_0^t\left(\sum_{j=1}^{m}\sum_{i=1}^{N_{s-}^j}\frac{c_1^{j\prime}\left(s,V_i^j\right)}{c_1^j\left(s,V_i^j\right)+1}\right.\right.\\ &\qquad\left.\left.+\frac{1}{2}b_1^2(s)-a_1(s)\right)ds\right]\prod_{j=1}^{m}\prod_{i=1}^{N_t^j}\frac{1}{c_1^j\left(t,V_i^j\right)+1}. \qquad (5.54)\end{aligned}$$

Equation (5.35) now reads

$$d(\mu_tY_t)=\left(\mu_{t-}a_2(t)+\partial_x\mu_{t-}b_2(t)\right)dt+\mu_{t-}b_2(t)\,dW_t+\sum_{j=1}^{m}\mu_tc_2^j\left(t,V_{N_t^j}^j\right)dN_t^j. \qquad (5.55)$$

Integration of (5.55) yields

$$\begin{aligned}Y_t &= \mu_t^{-1}\left(\int_0^t(\mu_{s-}a_2(s)+\partial_x\mu_{s-}b_2(s))ds+\int_0^t\mu_{s-}b_2(s)\,dW_s\right.\\ &\qquad\left.+\sum_{j=1}^{m}\int_0^t\mu_sc_2^j\left(s,V_{N_s^j}^j\right)dN_s^j\right). \qquad (5.56)\end{aligned}$$

Substituting (5.54) the Equation (5.56) leads us to the solution of the linear stochastic differential equation (5.30) ,

$$\begin{aligned}Y_t &= \left[\begin{array}{c}\int_0^t e^{H(s)}\left(\prod_{j=1}^m\prod_{i=1}^{N_{s-}^j}\frac{1}{c_1^j\left(s,V_i^j\right)+1}\right)(a_2(s)-b_1(s)\,b_2(s))\,ds\\ +\int_0^t e^{H(s)}\left(\prod_{j=1}^m\prod_{i=1}^{N_{s-}^j}\frac{1}{c_1^j\left(s,V_i^j\right)+1}\right)b_2(s)\,dW_s\\ +\sum_{j=1}^{m}\int_0^t e^{H(s)}\left(\prod_{j=1}^m\prod_{i=1}^{N_s^j}\frac{1}{c_1^j\left(s,V_i^j\right)+1}\right)c_2^j\left(s,V_{N_s^j}^j\right)dN_s^j\end{array}\right]\\ &\quad\cdot e^{-H(t)}\prod_{j=1}^{m}\prod_{i=1}^{N_t^j}\left(c_1^j\left(t,V_i^j\right)+1\right) \qquad (5.57)\end{aligned}$$

where

$$H(t) = -\int_0^t b_1(s)\,dW_s + \int_0^t \left(\sum_{j=1}^{m} \sum_{i=1}^{N_{s-}^j} \frac{c_1^{j\prime}\left(s, V_i^j\right)}{c_1^j\left(s, V_i^j\right) + 1} + \frac{1}{2} b_1^2(s) - a_1(s) \right) ds.$$

5.3 Analytical Solutions of Specific Examples

We now consider some linearizable SDEs driven by compound Poisson processes. Calculations are given explicitly for the first equation, other examples are partially discussed and the solutions are given in Table 5.1.

Example 1 is from [24] originally and it reads

$$dX_t = \frac{1}{3} X_{t^-}^{\frac{1}{3}} dt + X_{t^-}^{\frac{2}{3}} dW_t + \sum_{j=1}^{m} \nu_j X_{t^-} dC_t^j, \qquad X_0 = x_0 \tag{5.58}$$

when the jump terms driven by C_t^j are added. Here, we have

$$f(x) = \frac{1}{3} x^{\frac{1}{3}},\ g(x) = x^{\frac{2}{3}},\ r_j(x) = \nu_j x.$$

These functions satisfy the linearization criteria (5.14) and (5.17). Indeed,

$$\partial_x\left[g(x,t)L\right] = \partial_x \left[x^{\frac{2}{3}} \left(\partial_t \left(x^{-\frac{2}{3}} \right) + \partial_x \left(\frac{1}{3} x^{-\frac{1}{3}} - \frac{1}{2} \partial_x x^{\frac{2}{3}} \right) \right) \right] = 0$$

and

$$\partial_x A_i = \partial_x \left[\left(\nu_i V_{N_t^i}^i + 1 \right) \frac{x^{\frac{2}{3}}}{\left(x + \nu_i x V_{N_t^i}^i \right)^{\frac{2}{3}}} \right] = 0,\ i = 1, \ldots, m.$$

Hence, the transformation

$$h(x,t) = \int_{x_0}^{x} \frac{d\tilde{x}}{g(\tilde{x},t)} = 3x^{\frac{1}{3}} - 3x_0^{\frac{1}{3}} \tag{5.59}$$

linearizes Equation (5.58) as

$$dY_t = dW_t + \sum_{j=1}^{m} \left[\left(\left(1 + \nu_j V_{N_t^j}^j \right)^{\frac{1}{3}} - 1 \right) Y_{t^-} + 3x_0^{\frac{1}{3}} \left(\left(1 + \nu_j V_{N_t^j}^j \right)^{\frac{1}{3}} - 1 \right) \right] dN_t^j \tag{5.60}$$

by Itô's formula (5.5). We have $a_1(t) = 0$, $a_2(t) = 0$, $b_1(t) = 0$, $b_2(t) = 1$, $c_1(t,z) = (1 + \nu_j z)^{\frac{1}{3}} - 1$ and $c_2(t,z) = 3x_0^{\frac{1}{3}} \left((1 + \nu_j z)^{\frac{1}{3}} - 1 \right)$.

By (5.57), integrating (5.60) yields

$$Y_t = \mu_t^{-1}\left(\int_0^t \mu_{s-} dW_s + 3x_0^{\frac{1}{3}} \sum_{j=1}^m \int_0^t \mu_s \left(\left(1+\nu_j V_{N_t^j}^j\right)^{\frac{1}{3}} - 1\right) dN_s^j\right),$$

where

$$\mu_t = \prod_{j=1}^m \prod_{i=1}^{N_t^j} \left(1+\nu_j V_i^j\right)^{-\frac{1}{3}}.$$

Hence, the solution of (5.58) is given by

$$X_t = \left(x_0^{\frac{1}{3}} + \mu_t^{-1}\left(\frac{1}{3}\int_0^t \mu_{s-} dW_s + x_0^{\frac{1}{3}} \sum_{j=1}^m \int_0^t \mu_{s-} dN_s^j\right)\right)^3$$

in view of the transformation (5.59).

Example 2 is taken from [17], again with extra jump terms given by

$$dX_t = \left(\alpha(t) X_{t^-}^{\frac{3}{4}} + \frac{3}{8}\beta^2 X_{t^-}^{\frac{1}{2}}\right) dt + \beta X_{t^-}^{\frac{3}{4}} dW_t + \sum_{j=1}^m \gamma_j X_{t^-} dC_t^j, \qquad X_0 = x_0 \quad (5.61)$$

where

$$f(x,t) = \alpha(t) x^{\frac{3}{4}} + \frac{3}{8}\beta^2 x^{\frac{1}{2}},\ g(x) = \beta x^{\frac{3}{4}},\ r_j(x) = \gamma_j x.$$

and $\alpha(t)$, β, γ_j, $j = 1, \ldots, m$, are positive real valued. As indicated in Table 5.1, together with all examples of this section (5.61) satisfies criteria (5.14) and (5.17). The transformation

$$Y_t = \frac{4}{\beta}\left(X_t^{\frac{1}{4}} - x_0^{\frac{1}{4}}\right) \quad (5.62)$$

linearizes Equation (5.61) into

$$dY_t = \frac{\alpha(t)}{\beta} dt + dW_t + \sum_{j=1}^m \left(\left(1+\gamma_j V_{N_t^j}^j\right)^{\frac{1}{4}} - 1\right) Y_{t^-} dN_t^j$$

which corresponds to (5.4) with $a_1(t) = 0$, $a_2(t) = \frac{\alpha(t)}{\beta}$, $b_1(t) = 0$, $b_2(t) = 1$, $c_1(t,z) = \left(1+\gamma_j z\right)^{\frac{1}{4}} - 1$ and $c_2(t,z) = \frac{4}{\beta} x_0^{\frac{1}{4}} \left(\left(1+\gamma_j z\right)^{\frac{1}{4}} - 1\right)$. The solution is

$$Y_t = \mu_t^{-1}\left(\int_0^t \mu_{s-} \frac{\alpha(s)}{\beta} ds + \int_0^t \mu_{s-} dW_s + \frac{4}{\beta} x_0^{\frac{1}{4}} \sum_{j=1}^m \int_0^t \mu_s \left(\left(1+\gamma_j V_{N_t^j}^j\right)^{\frac{1}{4}} - 1\right)\right) dN_t^j$$

where the integrating factor is

$$\mu_t = \prod_{j=1}^{m} \prod_{i=1}^{N_t^j} \left(1 + c_j V_i^j\right)^{-\frac{1}{4}}.$$

Transformation (5.62) leads to the solution given in Table 5.1, that is,

$$X_t = \left(x_0^{\frac{1}{4}} + \frac{b}{4}\mu_t^{-1} \left(\int_0^t \mu_{s-} \frac{a(s)}{b} ds + \int_0^t \mu_{s-} dW_s + \frac{4}{b} x_0^{\frac{1}{4}} \sum_{j=1}^{m} \int_0^t \mu_{s-} dN_t^j \right) \right)^4.$$

Example 3 is taken from [7] but with an additional jump term and it reads

$$dX_t = \alpha(\beta - X_{t^-}) dt + \sigma X_{t^-}^{\frac{1}{2}} dW_t + \sum_{j=1}^{m} \gamma_j X_{t^-} dC_t^j, \qquad X_0 = x_0. \tag{5.63}$$

where α, β, σ and γ_j, $j = 1, \ldots, m$ are positive real valued parameters.

In (6.2.3), we have

$$f(x) = \alpha(\beta - x), \; g(x) = \sigma x^{\frac{1}{2}}, \; r_j(x) = \gamma_j x$$

These functions satisfy the linearization criteria (5.14) and (5.17) when

$$\sigma = 2\sqrt{\alpha\beta}.$$

Indeed,

$$\partial_x [g(x,t)L] = \partial_x \left[\sigma x^{\frac{1}{2}} \left(\partial_x \left[\left(\frac{\alpha\beta}{\sigma} - \frac{1}{4}\sigma \right) x^{-\frac{1}{2}} - \frac{\alpha}{\sigma} x^{\frac{1}{2}} \right] \right) \right] = 0$$

is satisfied if $\sigma = 2\sqrt{\alpha\beta}$. Therefore, using the transformation

$$h(x,t) = \int_{x_0}^{x} \frac{d\tilde{x}}{\sigma \tilde{x}^{\frac{1}{2}}}$$

we get

$$Y_t = \frac{1}{\sqrt{\alpha\beta}} \left(X_t^{\frac{1}{2}} - x_0^{\frac{1}{2}} \right). \tag{5.64}$$

We see that Equation (5.63) transforms into

$$\begin{aligned} dY_t \;\; = \;\; & -\frac{\alpha}{2} Y_{t^-} dt + dW_t + \sum_{j=1}^{m} \left[\left(\left(1 + \gamma_j V_{N_t^j}^j \right)^{\frac{1}{2}} - 1 \right) Y_{t^-} \right. \\ & \left. + \frac{1}{\sqrt{\alpha\beta}} x_0^{\frac{1}{2}} \left(\left(1 + \gamma_j V_{N_t^j}^j \right)^{\frac{1}{2}} - 1 \right) \right] dN_t^j \end{aligned}$$

where we have $a_1(t) = -\frac{\alpha}{2}$, $a_2(t) = 0$, $b_1(t) = 0$, $b_2(t) = 1$, $c_1(t,z) = \left(1+\gamma_j z\right)^{\frac{1}{2}} - 1$ and $c_2(t,z) = \frac{1}{\sqrt{\alpha\beta}} x_0^{\frac{1}{2}} \left(\left(1+\gamma_j z\right)^{\frac{1}{2}} - 1\right)$. Integration yields

$$Y_t = \mu_t^{-1}\left(\int_0^t \mu_{s^-} dW_s + \frac{1}{\sqrt{\alpha\beta}} x_0^{\frac{1}{2}} \sum_{j=1}^m \int_0^t \mu_{s^-} dN_s^j\right) \frac{1}{\sqrt{\alpha\beta}}\left(X_t^{\frac{1}{2}} - x_0^{\frac{1}{2}}\right)$$

where

$$\mu_t = e^{-\frac{\alpha}{2}t} \prod_{j=1}^m \prod_{i=1}^{N_t^j} \left(1+\gamma_j V_i^j\right)^{-\frac{1}{2}}.$$

Transformation (5.64) leads to the solution given in Table 5.1 which is

$$X_t = \left(x_0^{\frac{1}{2}} + \mu_t^{-1}\left(\sqrt{\alpha\beta}\int_0^t \mu_{s^-} dW_s + x_0^{\frac{1}{2}} \sum_{j=1}^m \int_0^t \mu_{s^-} dN_s^j\right)\right)^2.$$

Example 4 is a log-mean-reverting model [9, 31, 34] with a jump term. We have

$$dX_t = \eta X_{t^-}\left(\theta(t) - \ln X_{t^-}\right) dt + \rho X_{t^-} dW_t + \sum_{j=1}^m \zeta_j X_{t^-} dC_t^j, \qquad X_0 = x_0. \tag{5.65}$$

where η, $\theta(t)$, ρ, ζ_j, $j = 1, \ldots, m$, are positive real valued.

Here, we have

$$f(x,t) = \eta x\left(\theta(t) - \ln x\right),\; g(x) = \rho x,\; r_j(x) = \zeta_j x.$$

The transformation given by

$$Y_t = \frac{1}{\rho}\ln\frac{X_t}{x_0}, \tag{5.66}$$

linearizes the Equation (5.65) into

$$dY_t = \left(-\eta Y_{t^-} + \left(\frac{\rho}{2} + \frac{\mu\theta(t)}{\rho}\right)\right) dt + dW_t + \sum_{j=1}^m \frac{1}{\rho}\ln\left(1+\zeta_j V_{N_t^j}^j\right) dN_t^j$$

with $a_1(t) = -\eta$, $a_2(t) = \frac{\rho}{2} + \frac{\mu\theta(t)}{\rho}$, $b_1(t) = 0$, $b_2(t) = 1$, $c_1(t,z) = \frac{1}{\rho}\ln\left(1+\zeta_j z\right)$ and $c_2(t,z) = 0$. Integration gives

$$Y_t = \mu_t^{-1}\left(\int_0^t \mu_{s^-}\left(\frac{\rho}{2} + \frac{\mu\theta(s)}{\rho}\right) dt + \int_0^t \mu_{s^-} dW_s + \sum_{j=1}^m \int_0^t \mu_{s^-}\frac{1}{\rho}\ln\left(1+\zeta_j V_{N_s^j}^j\right) dN_s^j\right)$$

where

$$\mu_t = e^{\eta t}.$$

Finally, the use of (5.66) leads to the solution given in Table 5.1 which is

$$\begin{aligned} X_t &= x_0 \exp\left[\rho\mu_t^{-1}\left(\int_0^t \mu_{s-}\left(\frac{\rho}{2}+\frac{\mu\theta(s)}{\rho}\right)dt + \int_0^t \mu_{s-}dW_s \right.\right. \\ &\left.\left. + \sum_{j=1}^m \int_0^t \mu_{s-}\frac{1}{\rho}\ln\left(1+\zeta_j V^j_{N_s^j}\right)dN_s^j\right)\right]. \end{aligned}$$

Example 5 is from [11, 12, 27, 33], a geometric Ornstein-Uhlenbeck equation with an additional jump term. We have

$$dX_t = \xi(t) X_{t-}(\eta(t) - X_{t-})\,dt + \delta X_{t-}dW_t + \sum_{j=1}^m \lambda_j X_{t-}dC_t^j, \qquad X_0 = x_0. \quad (5.67)$$

where $\xi(t), \eta(t), \delta, \lambda_j, j = 1, \ldots, m,$ are positive real valued.

Here, the functions in (5.2) correspond to

$$f(x) = \xi(t)x(\eta(t) - x),\ g(x) = \delta x,\ r_j(x) = \lambda_j x.$$

The transformation given by

$$Y_t = \left(\frac{X_t}{x_0}\right)^{-1} \quad (5.68)$$

linearizes Equation (5.67) into

$$dY_t = \left[\left(\delta^2 - \xi(t)\eta(t)\right)Y_{t-} + \xi(t)x_0\right]dt + (-\delta)Y_{t-}dW_t + \sum_{j=1}^m \left(\frac{-\lambda_j V^j_{N_t^j}}{1+\lambda_j V^j_{N_t^j}}\right)Y_{t-}dN_t^j.$$

with $a_1(t) = \delta^2 - \xi(t)\eta(t)$, $a_2(t) = \xi(t)x_0$, $b_1(t) = -\delta$, $b_2(t) = 0$, $c_1(t,z) = \frac{-\lambda_j z}{1+\lambda_j z}$ and $c_2(t,z) = 0$. Integration gives

$$Y_t = \mu_t^{-1}\left(\mu_0 + x_0\int_0^t \mu_{s-}\xi(s)\,ds\right)$$

where

$$\mu_t = \exp\left(-\delta W_t - \frac{1}{2}\delta^2 t + \int_0^t \xi(s)\eta(s)\,ds\right)\prod_{j=1}^m\prod_{i=1}^{N_t^j}\left(1+\lambda_j V_i^j\right).$$

As before, (5.68) leads to the solution given in Table 5.1 which is

$$X_t = x_0\mu_t\left(\mu_0 + x_0\int_0^t \mu_{s-}\xi(s)\,ds\right)^{-1}.$$

Table 5.1: Linearizable Equations and Solutions

Equation Criteria (5.14,5.17)	$dX_t = \left(\alpha(t) X_{t^-}^{\frac{3}{4}} + \frac{3}{8}\beta^2 X_{t^-}^{\frac{1}{2}}\right) dt + \beta X_{t^-}^{\frac{3}{4}} dW_t + \sum_{j=1}^{m} \gamma_j X_{t^-} dC_t^j$ Solution $X_t = \left(x_0^{\frac{1}{4}} + \frac{\beta}{4}\mu_t^{-1}\left(\int_0^t \mu_{s^-}\frac{\alpha(s)}{\beta}ds + \int_0^t \mu_{s^-} dW_s\right)\right)^4$ $\mu_t = \prod_{j=1}^m \prod_{i=1}^{N_t^j}\left(1+\gamma_j V_i^j\right)^{-\frac{1}{4}}$
Equation	$dX_t = \alpha\left(\beta - X_{t^-}\right)dt + \sigma X_{t^-}^{\frac{1}{2}} dW_t + \sum_{j=1}^m \gamma_j X_{t^-} dC_t^j$
Criteria (5.14,5.17)	Solution $X_t = \left(x_0^{\frac{1}{2}} + \sqrt{\alpha\beta}\mu_t^{-1}\int_0^t \mu_{s^-} dW_s\right)^2$ $\mu_t = e^{-\frac{\alpha}{2}t}\prod_{j=1}^m \prod_{i=1}^{N_t^j}\left(1+\gamma_j V_i^j\right)^{-\frac{1}{2}}$
Equation	$dX_t = \eta X_{t^-}\left(\theta(t) - \ln X_{t^-}\right)dt + \rho X_{t^-} dW_t + \sum_{j=1}^m \zeta_j X_{t^-} dC_t^j$
Criteria (5.14,5.17)	Solution $X_t = x_0 \exp\left(\rho\mu_t^{-1}\left(\begin{array}{c}\int_0^t \mu_{s^-}\left(\frac{\rho}{2}+\frac{\mu\theta(s)}{\rho}\right)dt + \int_0^t\mu_{s^-}dW_s \\ + \sum_{j=1}^m \int_0^t \mu_{s^-}\frac{1}{\rho}\ln\left(1+\zeta_j V_{N_s^j}^j\right)dN_s^j\end{array}\right)\right)$ $\mu_t = e^{\eta t}$
Equation	$dX_t = \xi(t) X_{t^-}\left(\eta(t) - X_{t^-}\right)dt + \delta X_{t^-}dW_t + \sum_{j=1}^m \lambda_j(t) X_{t^-} dC_t^j$
Criteria (5.23,5.28)	Solution $X_t = x_0\mu_t\left(\mu_0 + x_0\int_0^t \mu_{s^-}\xi(s)ds\right)^{-1}$ $\mu_t = \exp\left(-\delta W_t - \frac{1}{2}\delta^2 t + \int_0^t \xi(s)\eta(s)ds\right)\prod_{j=1}^m \prod_{i=1}^{N_t^j}\left(1+\lambda_j V_i^j\right)$

Chapter 6

NUMERICAL EXPERIMENTS

As exact solutions are rarely known, numerical methods for stochastic differential equations are of supreme importance [4, 5, 24]. We have not specified a numerical approach to stochastic differential equations so far. In this chapter, we will discuss two important pathwise approximation methods: Euler-Maruyama and Milstein-Maghsoodi. We shall then consider our examples introduced in Chapter 5 to simulate their sample paths using these methods and compare with the numerical evaluation of their analytical solutions.

6.1 Numerical Schemes for Stochastic Differential Equations

The general form of a nonlinear stochastic differential equation driven by Wiener and compound Poisson processes reads

$$dX_t = f(X_{t^-}, t)dt + g(X_{t^-}, t)dW_t + \sum_{j=1}^{m} r_j(X_{t^-}, t)dC_t^j, \qquad X_0 = x_0 \tag{6.1}$$

as introduced in Chapter 5. We can simply rewrite (6.1) in the integral form as

$$X_t = X_0 + \int_0^t f(X_{s^-}, s)ds + \int_0^t g(X_{s^-}, s)dW_s + \sum_{j=1}^{m} \int_0^t r_j(X_{s^-}, s)V_{N_s^j}^j dN_s^j$$

where the first integral is a Riemann-Stieltjes integral, the second and the third integrals are stochastic integrals with respect to a Wiener process and a Poisson random measure, respectively [1, 17].

Here we will use equidistant stepsize $\Delta t = t_{n+1} - t_n$ for the partition $t_0 = 0 < t_1 < \ldots < t_k = T$ and the increments ΔW_n and ΔN_n of Wiener and the Poisson processes to discretize (6.1) and obtain an approximated solution Y_k at $t_k = T$. Moreover, the approximation Y with stepsize δ converges strongly to X with order γ at time T if there exists a positive constant C such that

$$\varepsilon(\delta) = \sqrt{E\left(|X_T - Y(T)|^2\right)} \leq C\delta^{\gamma}$$

for each $\delta > 0$ [5].

6.1.1 Euler-Maruyama Scheme

The simplest discretization procedure for stochastic differential equations, generalized Euler-Maruyama scheme with jumps [10, 24] is given by

$$Y_{n+1} = Y_n + f(Y_n, t_n)\Delta t + g(Y_n, t_n)\Delta W_{n+1} + r(Y_n, t_n) V_{N_s} \Delta N_{n+1}. \tag{6.2}$$

The sequence $\{Y_n, n = 0, 1, \ldots, N\}$ of values of the Euler-Maruyama approximation at times $\{t_n, n = 0, 1, \ldots, N\}$ can be computed by generating the random increments ΔW and ΔN. This scheme has a strong order of convergence $\gamma = 0.5$ [5, 24].

6.1.2 Milstein-Maghsoodi Scheme

Maghsoodi [24] generalizes Milstein's second order diffusion scheme for stochastic differential equations with jumps to improve the order of accuracy and derive a higher order scheme in the mean square sense by obtaining

$$\begin{aligned} Y_{n+1} &= Y_n + \left(f(Y_n, t_n) - \frac{1}{2} g(Y_n, t_n)\partial_x g(Y_n, t_n)\right)\Delta t \\ &\quad + g(Y_n, t_n)\Delta W_{n+1} + \frac{1}{2} g(Y_n, t_n)\partial_x g(Y_n, t_n)\Delta W_{n+1}^2 \\ &\quad + \frac{1}{2}\left(3r(Y_n, t_n)V_{N_n} - r(Y_n + r(Y_n, t_n)V_{N_n}, t_n)V_{N_n}\right)\Delta N_{n+1} \\ &\quad + \left(g(Y_n + r(Y_n, t_n)V_{N_n}, t_n) - g(Y_n, t_n)\right)\Delta W_{n+1}\Delta N_{n+1} \\ &\quad + \frac{1}{2}\left(r(Y_n, t_n)V_{N_n} - r(Y_n + r(Y_n, t_n)V_{N_n}, t_n)V_{N_n}\right)\Delta N_{n+1}^2 \\ &\quad + \left(g(Y_n, t_n)\partial_x r(Y_n, t_n)V_{N_n} - g(Y_n + r(Y_n, t_n)V_{N_n}, t_n) + g(Y_n, t_n)\right)\Delta Z_{n+1} \end{aligned} \tag{6.3}$$

where

$$\Delta Z_{n+1} := \int_{t_n}^{t_{n+1}} (W_s - W_{t_n})\, dN_s.$$

It has been proved that this scheme has strong order of convergence $\gamma = 1$ [5, 24].

We apply (6.2) and (6.3) with r_j, $j = 1, \ldots, m$, next.

6.2 Analytical versus Numerical Solutions

We will now illustrate and examine our examples in detail. As we have found the explicit solutions we can compare the Euler-Maruyama and Milstein-Maghsoodi approximations with our exact solutions and compute the errors. Approximations have been simulated for $N = 10000$ trajectories of the stochastic differential equations given in Section 6.3. Computations have been done in MATLAB. The CPU times range from 60 to 90 minutes for the completion of 10000 trials.

6.2.1 Introductory Example 1

We first consider our first example which is from [24] given by

$$dX_t = \frac{1}{3} X_{t^-}^{\frac{1}{3}} dt + X_{t^-}^{\frac{2}{3}} dW_t + \sum_{j=1}^{m} \nu_j X_{t^-} dC_t^j \tag{6.4}$$

for $t \in [0, T]$ with the initial value $X_0 = x_0$. From Table 5.1 we know that (6.4) has an analytical solution

$$X_t = \left(x_0^{\frac{1}{3}} + \frac{1}{3} \prod_{j=1}^{m} \prod_{i=1}^{N_t^j} \left(1 + \nu_j V_i^j\right)^{\frac{1}{3}} \left(\begin{array}{c} \int_0^t \prod_{j=1}^m \prod_{i=1}^{N_{s^-}^j} \left(1 + \nu_j V_i^j\right)^{-\frac{1}{3}} dW_s \\ + \int_0^t \prod_{j=1}^m \prod_{i=1}^{N_{s^-}^j} \left(1 + \nu_j V_i^j\right)^{-\frac{1}{3}} dN_s \end{array} \right) \right)^3$$

for $t \in [0, T]$.

We have simulated $N = 10000$ trajectories for all numerical approximations with $x_0 = 0.6$, $T = 1$, $v = 1$, jump intensity rate $\lambda = 3$ and stepsizes $\Delta t = 2^{-3}$, 2^{-6}, 2^{-9}, 2^{-12}. Notice that, the analytical solution almost coincides with the Euler and Maghsoodi approximations as given in Fig. 6.2.

It can be seen that the mean trajectories in Figure 6.3 are in each other's 95% confidence interval, which is computed for every time point for 10000 simulations by calculating means and their standard errors. These confidence intervals show the reliability of our estimates.

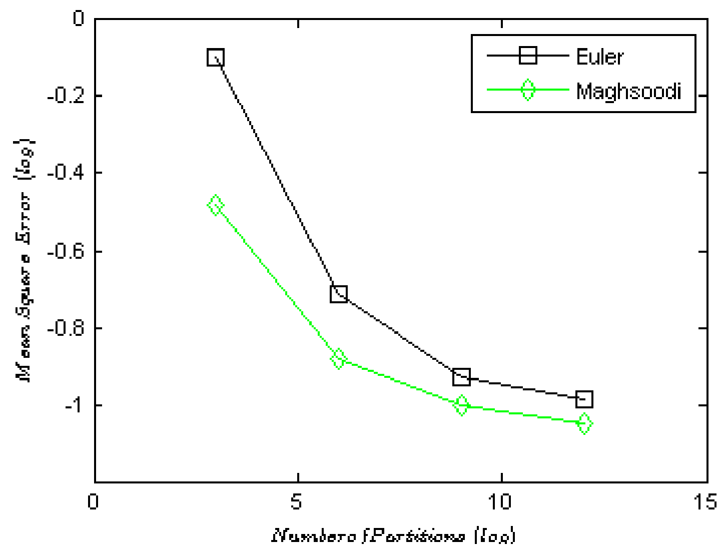

Figure 6.1: Mean square errors for Euler and Maghsoodi approximations for the first example.

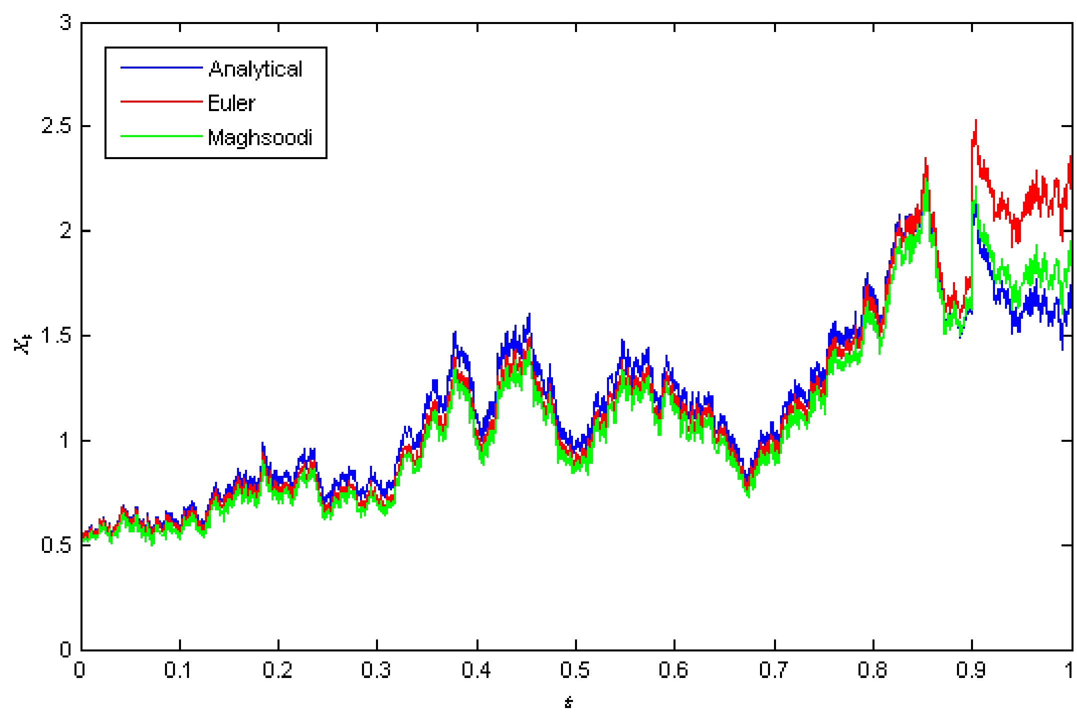

Figure 6.2: Simulation of the exact solution and the numerical approximations, $\Delta t = 2^{-12}$. Jumps are at $t = 0.0862$, 0.2068, and 0.8992 with sizes 0.0924, 00.0078 and 0.4231 respectively.

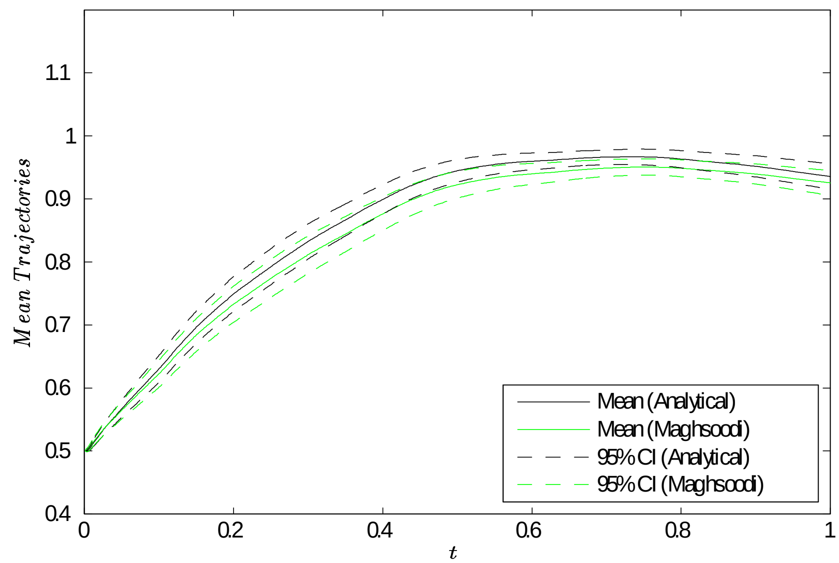

Figure 6.3: Mean trajectories estimated from 10000 independent replications for the first example.

Comparing the approximation results in Fig. 6.1, it can be noticed that by decreasing the stepsize, the estimate of the mean square error decreases too. Moreover, it can be seen that the Maghsoodi approximation produces more accurate results as expected.

6.2.2 Introductory Example 2

We now consider our next example taken from [17]

$$dX_t = \left(\alpha(t) X_{t^-}^{\frac{3}{4}} + \frac{3}{8}\beta^2 X_{t^-}^{\frac{1}{2}}\right) dt + \beta X_{t^-}^{\frac{3}{4}} dW_t + \sum_{j=1}^{m} \gamma_j X_{t^-} dC_t^j, \qquad X_0 = x_0. \quad (6.5)$$

with additional jump terms for $t \in [0, T]$ with the initial value $X_0 = x_0$. From Table 5.1 we know that (6.5) has an explicit solution

$$X_t = \left(x_0^{\frac{1}{4}} + \frac{\beta}{4} \prod_{j=1}^{m} \prod_{i=1}^{N_t^j} \left(1 + \gamma_j V_i^j\right)^{\frac{1}{4}} \left(\begin{array}{c} \int_0^t \prod_{j=1}^m \prod_{i=1}^{N_{s^-}^j} \left(1 + \gamma_j V_i^j\right)^{-\frac{1}{4}} \frac{\alpha(s)}{\beta} ds \\ + \int_0^t \prod_{j=1}^m \prod_{i=1}^{N_{s^-}^j} \left(1 + \gamma_j V_i^j\right)^{-\frac{1}{4}} dW_s \end{array} \right) \right)^4$$

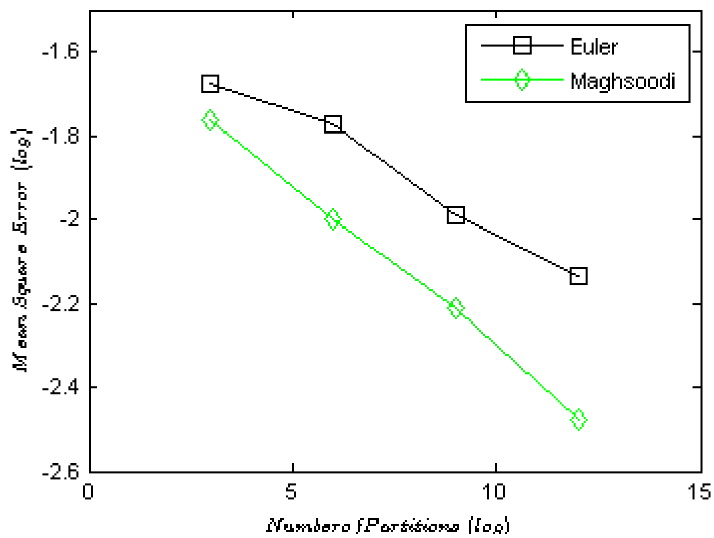

Figure 6.4: Mean square errors for Euler and Maghsoodi approximations the second example.

for $t \in [0, T]$. As we found the explicit solution, we can compare the Euler-Maruyama and Milstein-Maghsoodi approximations with our exact solution and compute the errors.

We have simulated $N = 10000$ trajectories with $x_0 = 0.5$, $T = 1$, $\alpha(t) = 2$, $\beta = 0.5$, $\gamma = 1$, jump intensity rate $\lambda = 3$ and stepsizes $\Delta t = 2^{-3}$, 2^{-6}, 2^{-9}, 2^{-12}. The analytical solution almost coincides with the Euler and Maghsoodi approximations in Fig. 6.5.

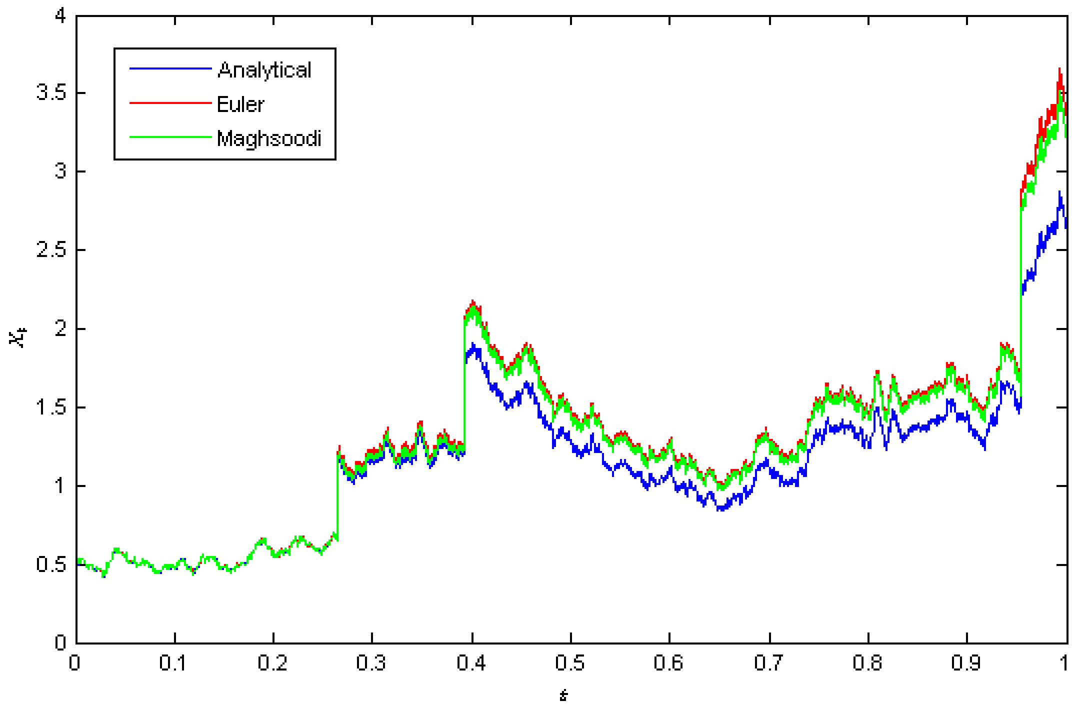

Figure 6.5: Simulation of the exact solution and the numerical approximations, $\Delta t = 2^{-12}$. Jumps are at $t = 0.2651$, 0.3918, and 0.9548 with sizes 0.8645, 0.5881, and 0.6704 respectively.

It can be seen that the estimated analytical mean trajectory in Figure 6.6 nearly coincides with the mean trajectory estimated from the numerical approximation. Furthermore, comparing the approximation results in Figure 6.4, the estimate of the mean square error decreases by decreasing the stepsize and Maghsoodi produces more accurate results as expected.

6.2.3 Cox-Ingersoll-Ross Model

We now consider our second example from [7] with additional jump terms given by

$$dX_t = \alpha\left(\beta - X_{t^-}\right)dt + \sigma X_{t^-}^{\frac{1}{2}}\,dW_t + \sum_{j=1}^{m} \gamma_j X_{t^-} dC_t^j \tag{6.6}$$

for $t \in [0, T]$ with the initial value $X_0 = x_0$. This equation is known as the Cox-Ingersoll-Ross interest rate model, before the jump terms are added. Here, X_t repre-

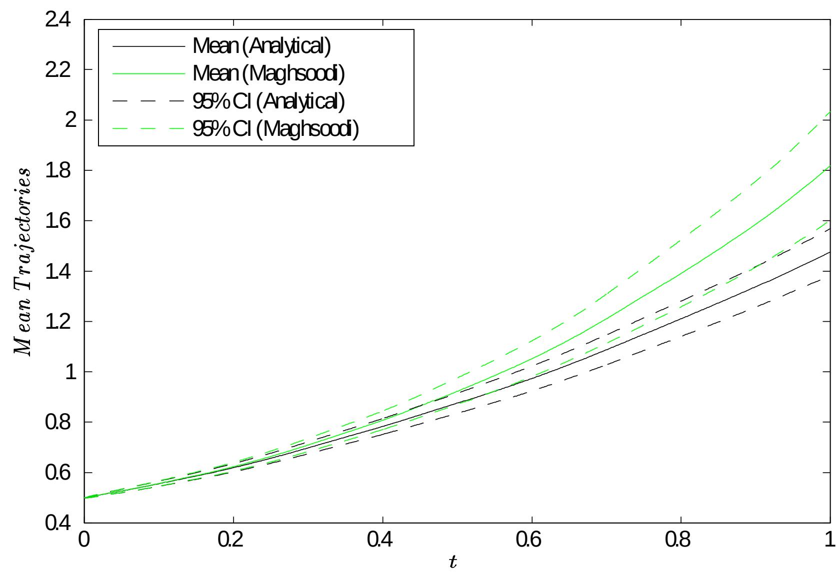

Figure 6.6: Mean trajectories estimated from 10000 independent replications for the second example.

sents the mean-reverting short-term interest rate. In this model, β is the long-term average value of interest rate with jumps, α is the intensity (strength) of mean reversion, σ is the interest rate volatility where X_t is the instantaneous interest rate at period t, maturing at period T.

Mean reversion is a tendency for a stochastic process to remain near, or tend to return over time to a long-run average value. If the interest rate (or the spot price of a commodity) is below the mean, the mean reversion component will be positive, resulting in an upward influence on the spot price. Alternatively, if the spot price is above this level, the mean reversion component will be negative, thus causing a downward influence on the interest rate. Over time, this results in a price path that drifts towards the mean, at a period (or speed) determined by the mean reversion rate $\alpha > 0$. Moreover, short-term interest rates are the interest rates on loan contracts-or debt instruments such as Treasury bills, bank certificates of deposit or commercial paper-having maturities of less than one year, which are modeled by X.

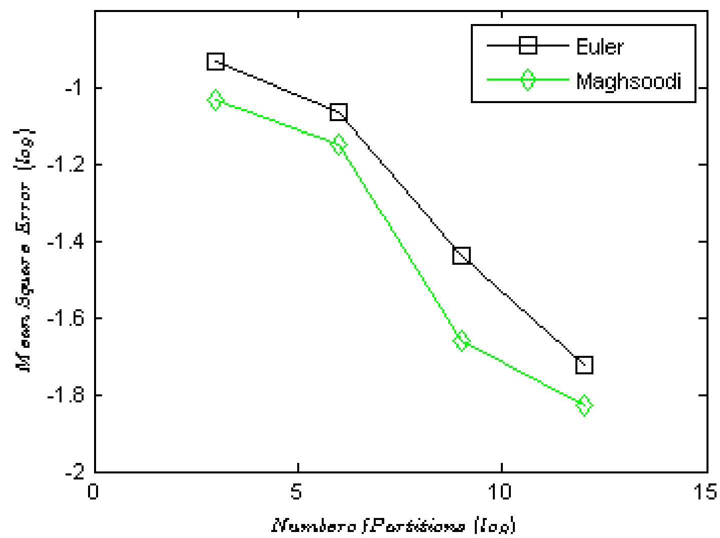

Figure 6.7: Mean square errors for Euler and Maghsoodi approximations for the CIR model.

From Table 5.1 we know that (6.6) has an explicit solution

$$X_t = \left(x_0^{\frac{1}{2}} + \sqrt{\alpha\beta} e^{\frac{\alpha}{2}t} \prod_{j=1}^{m} \prod_{i=1}^{N_t^j} \left(1 + \gamma_j V_i^j\right)^{\frac{1}{2}} \int_0^t e^{-\frac{\alpha}{2}t} \prod_{j=1}^{m} \prod_{i=1}^{N_{s-}^j} \left(1 + \gamma_j V_i^j\right)^{-\frac{1}{2}} dW_s \right)^2$$

for $t \in [0, T]$. As we have the explicit solution, we can compare the Euler-Maruyama and Milstein-Maghsoodi approximations with our exact solution and compute the errors.

We have simulated $N = 10000$ trajectories with $x_0 = 0.5$, $T = 100$, $\alpha = 0.4$, $\beta = 0.6$, $\gamma = 1$, jump intensity rate $\lambda = 0.03$ and stepsizes $\Delta t = 2^{-3}$, 2^{-6}, 2^{-9}, 2^{-12}. Note that, the Euler and Maghsoodi approximations are close to the analytical solution as shown in Fig. 6.8. The process is expected to stabilize around its mean value $\beta = 0.6$ in the long run.

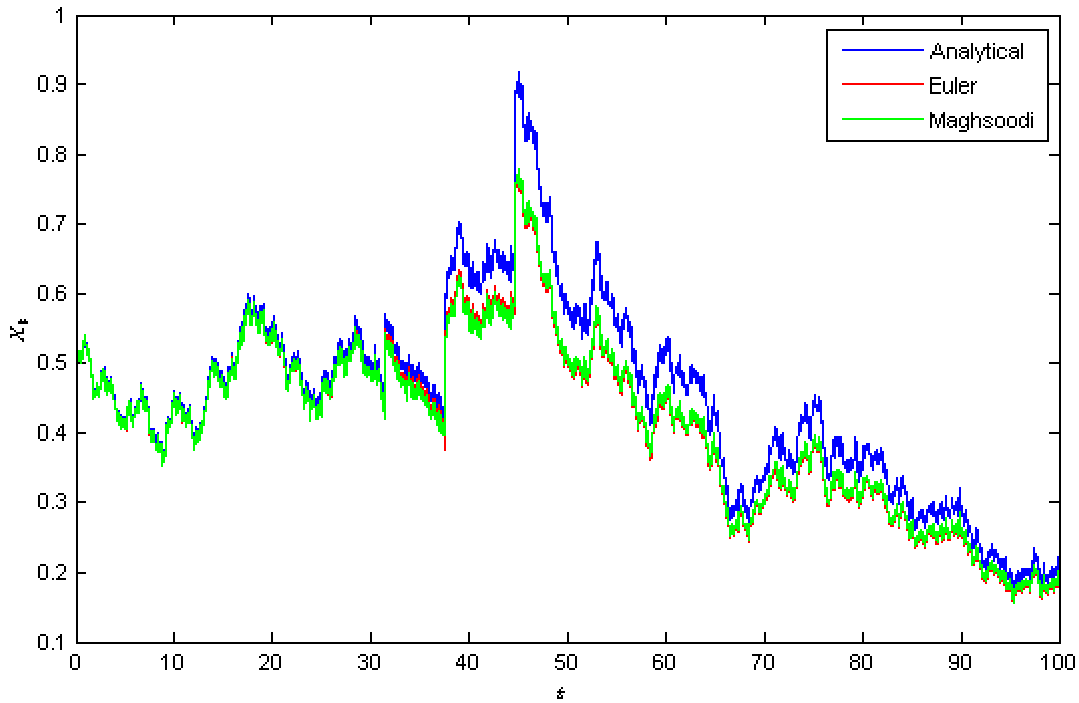

Figure 6.8: Simulation of the exact solution and the numerical approximations, $\Delta t = 2^{-12}$. Jumps are at $t = 31.32$, 37.47, and 44.77 with sizes 0.2662, 0.3513, and 0.2779 respectively.

It can be seen that the estimated analytical mean trajectory in Fig. 6.9 nearly coincides with the mean trajectory estimated from the numerical approximation.

As shown in Figure 6.7, as the stepsize decreases, the estimate of the mean square error decreases. Moreover, it can be seen that the Maghsoodi approximation produces more accurate results as expected.

6.2.4 Log-Mean-Reverting Model

We now consider our second example from [9, 31, 34] with jump terms

$$dX_t = \eta X_{t^-} \left(\theta\left(t\right) - \ln X_{t^-}\right) dt + \rho X_{t^-} dW_t + \sum_{j=1}^{m} \zeta_j X_{t^-} dC_t^j, \qquad X_0 = x_0. \qquad (6.7)$$

for $t \in [0, T]$ with the initial value $X_0 = x_0$. This log mean-reverting equation with jumps is commonly used in modeling assets subject to supply and demand such as

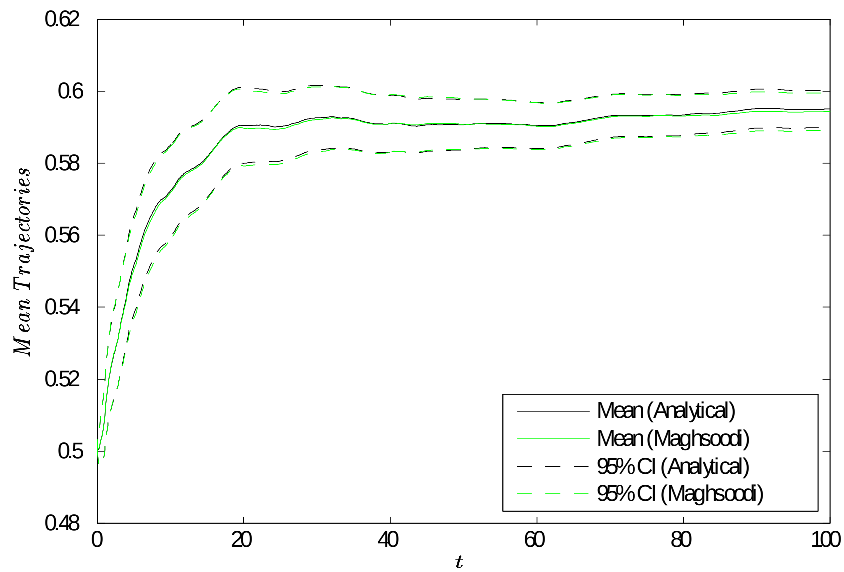

Figure 6.9: Mean trajectories estimated from 10000 independent replications for the CIR model.

commodities. Due to advantage of easiness of simulation, modeling and parameters estimation, this model is widely preferred. Therefore, X_t now corresponds to the spot price of the commodity. In this model, θ is the long-run mean of the logarithm of the price with jumps, η is the mean reversion speed (intensity) of the price and ρ is the price volatility.

From Table 5.1, we know that (6.7) has an explicit solution

$$X_t = x_0 \exp\left(\rho e^{-\eta t}\left(\int_0^t e^{\eta s}\left(\frac{\rho}{2}+\frac{\mu\theta(s)}{\rho}\right)dt + \int_0^t e^{\eta s}dW_s + \sum_{j=1}^{m}\int_0^t e^{\eta s}\frac{1}{\rho}\ln\left(1+\zeta_j V_i^j\right)dN_s^j\right)\right)$$

for $t \in [0, T]$. As we have found the explicit solution, we can compare the Euler-Maruyama and Milstein-Maghsoodi approximations with our closed-form solution and compute the errors.

We have simulated $N = 10000$ trajectories with $x_0 = 0.5$, $T = 100$, $\eta = 0.4$, $\theta(t) = 0.6$, $\rho = 0.5$, $\zeta = 1$, jump intensity rate $\lambda = 0.03$ and stepsizes $\Delta t = 2^{-3}$, 2^{-6}, 2^{-9}, 2^{-12}. Note that, the analytical solution almost coincides with the Euler and

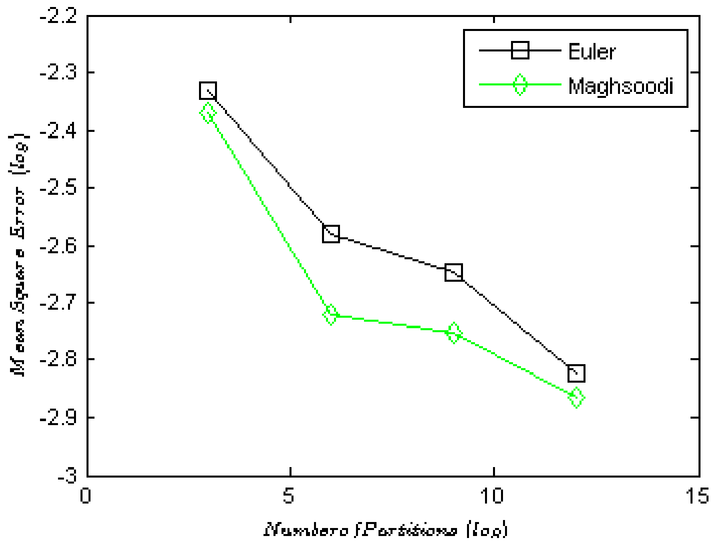

Figure 6.10: Mean square errors for Euler and Maghsoodi approximations for the log mean-reverting model.

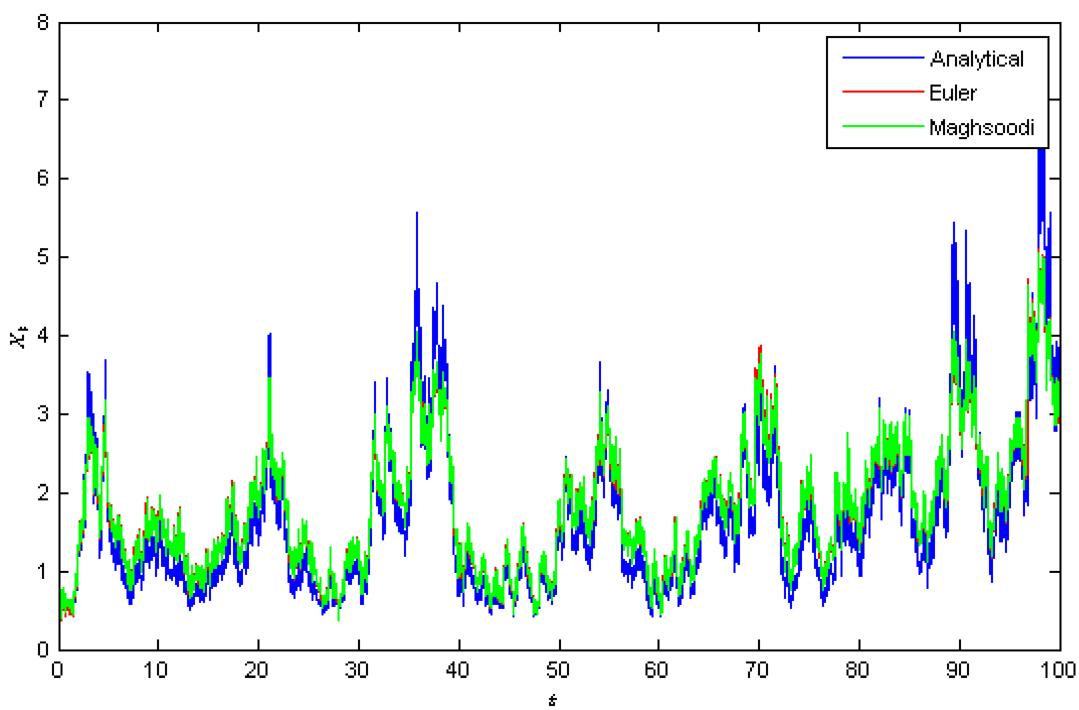

Figure 6.11: Simulation of the exact solution and the numerical approximations, $\Delta t = 2^{-12}$. Jumps are at $t = 5.21$, 69.66, and 96.79 with sizes 0.0915, 0.6802, and 0.9426 respectively.

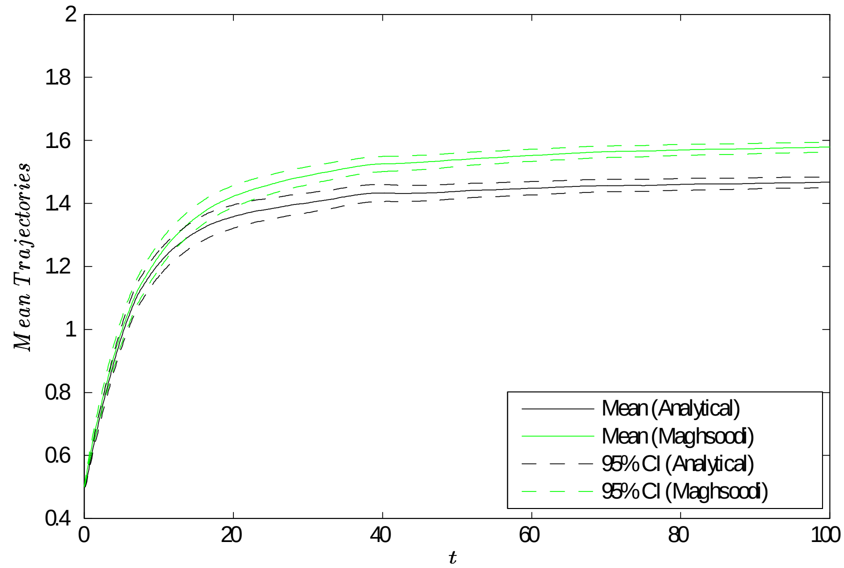

Figure 6.12: Mean trajectories estimated from 10000 independent replications for the log mean-reverting model.

Maghsoodi approximations in Fig. 6.11. The estimated analytical mean trajectory given in Figure 6.12 is close to the mean trajectory estimated from the numerical approximations.

Fig. 6.10 demonstrates again the higher accuracy achieved with Maghsoodi.

6.2.5 Geometric O-U Model

We now consider our next example taken from [12, 11, 27, 33] given by

$$dX_t = \xi(t) X_{t^-} (\eta(t) - X_{t^-}) dt + \delta X_{t^-} dW_t + \sum_{j=1}^{m} \lambda_j(t) X_{t^-} dC_t^j \tag{6.8}$$

for $t \in [0, T]$ with the initial value $X_0 = x_0$. Equation (6.8) is also known as Geometric Ornstein-Uhlenbeck or Dixit & Pindyck Model, now including additional jump terms. This model is based on a mean-reverting commodity price or interest rate X_t. In this equation, the mean reversion component is governed by the difference between the current price and the mean η as well as by the mean reversion rate ξ where δ is the

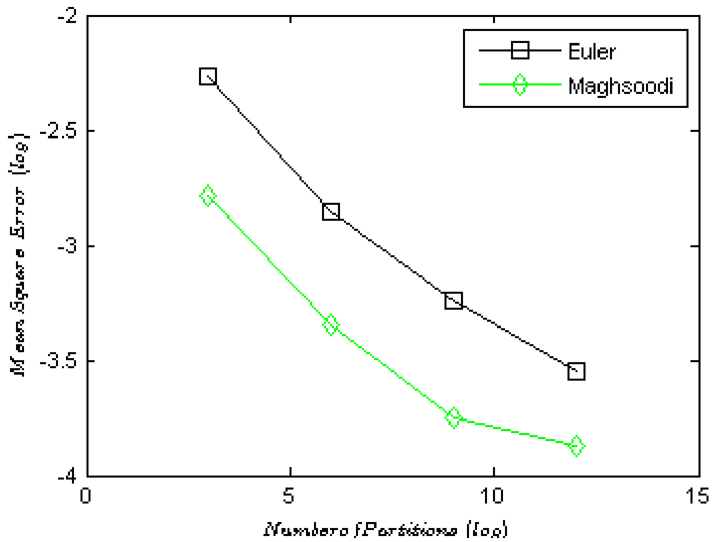

Figure 6.13: Mean square errors for Euler and Maghsoodi approximations for the geometric O-U model.

volatility of the spot price. Note that, spot price X_t is always positive.

From Table 5.1, we know that (6.8) has an explicit solution

$$X_t = x_0 \left(\begin{array}{c} e^{\delta W_t + \frac{1}{2}\delta^2 t - \int_0^t \xi(s)\eta(s)ds} \prod_{j=1}^{m} \prod_{i=1}^{N_t^j} \left(1+\lambda_j V_i^j\right)^{-1} \\ +x_0 \int_0^t \frac{\prod_{j=1}^{m}\prod_{i=1}^{N_{s-}^j}\left(1+\lambda_j V_i^j\right)}{\prod_{j=1}^{m}\prod_{i=1}^{N_t^j}\left(1+\lambda_j V_i^j\right)} e^{-\delta(W_s - W_t) - \frac{1}{2}\delta^2(s-t) + \int_0^{s-t} \xi(\zeta)\eta(\zeta)d\zeta} \xi\left(s\right) ds \end{array} \right)^{-1}$$

for $t \in [0, T]$. As we have the explicit solution, we again compare the numerical solutions of (6.8) with our analytical solution and compute the errors.

We have simulated $N = 10000$ trajectories with $x_0 = 3$, $T = 100$, $\xi(t) = 0.3$, $\eta(t) = 1.9$, $\delta = 0.2$, $\lambda = 1$, jump intensity rate $\Lambda = 0.03$ and stepsizes $\Delta t = 2^{-3}$, 2^{-6}, 2^{-9}, 2^{-12}.

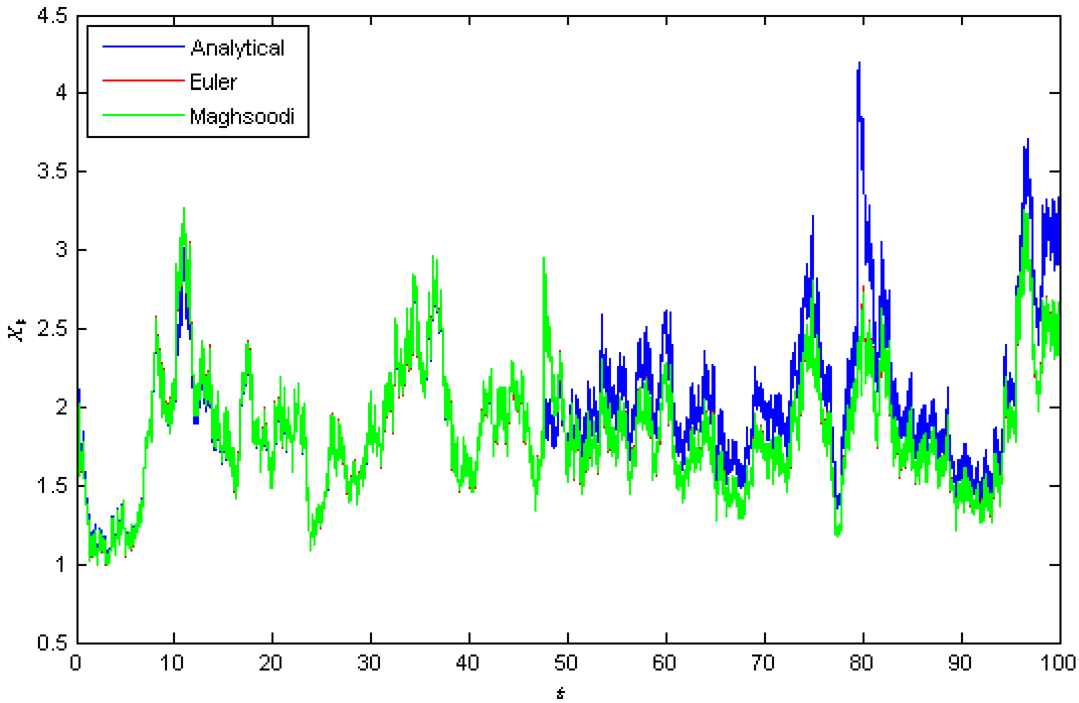

Figure 6.14: Simulation of the exact solution and the numerical approximations, $\Delta t = 2^{-12}$. Jumps are at $t = 10.05$, 47.54, 79.5400, 79.91, and 95.68 with sizes 0.1549, 0.6437, 0.0965, 0.2673, and 0.3547 respectively.

It can be seen that the estimated analytical mean trajectory in Fig. 6.15 nearly coincides with the mean trajectory estimated from the numerical approximation. In this case, the means are close, the standard errors are very small, which implies that most trajectories must agree as good as the ones shown in Fig. 6.14.

As we see in Fig. 6.13, Maghsoodi approximation achieves more accurate results in the mean square sense.

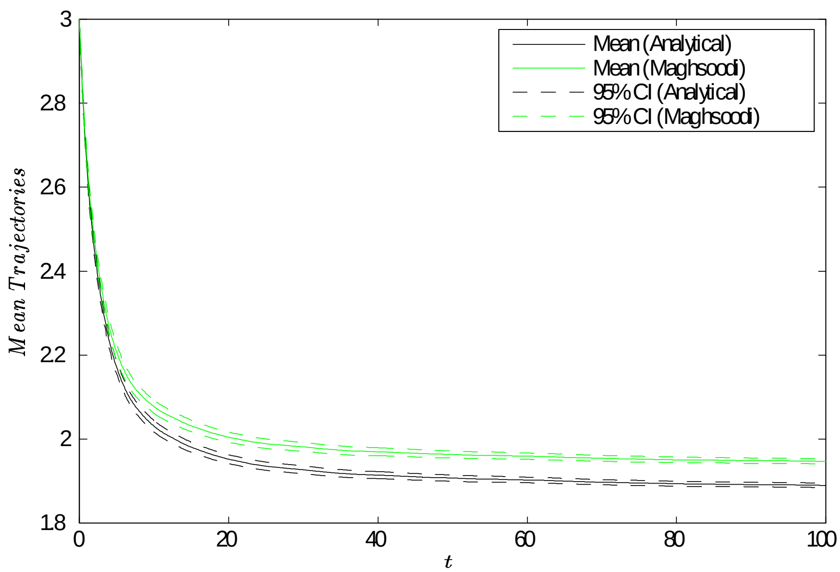

Figure 6.15: Mean trajectories estimated from 10000 independent replications for the geometric O-U model.

Chapter 7

CONCLUSION

In this thesis, we have contributed to integration of nonlinear stochastic differential equations driven by compound Poisson processes, namely finite activity Lévy processes by a novel method of integration based on linearization.

We have shown that a nonlinear stochastic differential equation of the form (5.2) which is driven by a finite activity Lévy process consisting of a Wiener process and a compound Poisson process is linearizable to (5.4) via the transformations given in Theorem 5.1 under certain conditions. We have then introduced the stochastic integrating factors (5.54) to solve the linear stochastic differential equation (5.57).

We have studied several examples which appeared in previous work. We have applied our method to Cox-Ingersoll-Ross interest rate model [7], a log-mean reverting model [9, 31, 34], Geometric Ornstein-Uhlenbeck [12, 11, 27, 33] models and two equations borrowed from [24] and [17]. The first three are important models in applications. Although the original equations are diffusion models, we have accomplished to generalize them to jump-diffusion cases by adding jump terms. We have found the analytical solutions explicitly when the linearizability conditions are satisfied.

Moreover, we have compared our closed-form analytical solutions with the numerical discretizations of Euler and Maghsoodi approximations. We have illustrated their sample paths and showed their agreement. Mean square errors of numerical approximations have also been computed and demonstrated for several stepsizes. Monte Carlo approach has been used to estimate the expected value of X by computing the means of analytical solutions and numerical discretizations.

This integration method can also be investigated for general Lévy driven nonlinear stochastic differential equations including infinite activity. However, since there will be additional terms in Itô's formula in the infinite activity case, we will have integro-differential conditions for linearization. Solution to the linearized equations by integrating factors is also a challenging problem since we have a new Itô's formula as the stochastic chain rule. Moreover, any numerical methods with general Lévy driven stochastic differential equations also remains as our future work.

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