Dynamic Inventory Allocation Problems with Stochastic Demand

by

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To my fiancée Çağlar

ABSTRACT

This thesis presents two dynamic inventory allocation problems with stochastic demand. The objective in both of these problems is to maximize the seller's total expected revenue over a finite horizon. In the first problem, the seller has a discrete-time, single-product inventory control problem in which he maximizes the total expected revenue by selecting a dynamic rule that controls the allocation of capacity to requests from different demand classes. This problem is a well-studied revenue management problem in the literature; however, our work differs from the earlier studies. Besides showing the basic structural properties of the problem, we investigate how the varying system parameters affect the optimal policy, and highlight the effects of the random problem parameters, such as the probability distributions of demand, on the optimal policy. The second problem is the seller's pricing problem of perishable products with random quality. To the best of our knowledge, our work is the first attempt of solving the pricing problem of perishable products with random quality. In this problem, the seller dynamically chooses the price to maximize the total expected revenue by adjusting the current distribution of the product quality according to the number of the remaining inventory at some point in time during the selling season. We present the numerical results aiming to illustrate the behavior of the model and to assess the impact of the varying problem parameters on the optimal revenue and the optimal prices. We compare the static pricing strategy with the dynamic pricing strategy under the same settings, and prove that updating the cumulative distribution function of the product quality significantly increases the seller's total expected profit.

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ÖZETÇE

Bu çalışmada iki farklı dinamik envanter denetimi problemi rastsal talep modellemesiyle göz önüne alınmıştır. Her iki problemin de amacı tedarikçinin kazancını en iyilemektir. İlk problem, sonlu zamanda mevsimsel ve tek tip ürünlü envanter denetimi problemidir. Çalışmada öne sürülen modelde tedarikçi çok çeşitli müşteri gruplarından gelen talebi karşılayıp karşılamama kararını verir. Bir defada tek bir müşteri tipinden toplu halde talep gelebilir ve tedarikçi, müşteri tipine, kalan zamana ve envantere göre bu talebin tamamını ya da sadece bir kısmını karşılama hakkına sahiptir. Bu problem bilinen bir kazanç yönetimi problemidir, fakat bu çalışmada önceki çalışmalardan farklı olarak sistemdeki parametrelerin değişiminin en iyi kurala etkisi incelenmiştir. İkinci problem ise bir dinamik fiyatlandırma problemidir. Bu problemde, tedarikçi mevsimsel ve tek tip ürün satar, fakat sattığı ürünler kalite bakımından farklılık gösterir. Problemdeki kilit nokta tedarikçinin mevsimin başından sonuna kadar ürünlerinin kalitesini tam olarak gözlemleyememesi, ya da anlayamamasıdır. Ürünlerin kalitesinin tedarikçi tarafından tam olarak bilinmemesi, tedarikçinin mevsim başında varsayılan kalite dağılımını, geçmiş satışları ve zaman içerisinde müşterilerin ürünlere biçtikleri değeri gözlemleyerek yenilemesini gerektirir. Kalite dağılımı yenilendikçe ürünlerin fiyatı da yenilenir, daha gerçeğe uygun bir hal alır. Böylece mevsim boyunca elde edilecek kazanç en iyilenmeye çalışılır. Bu problem literatürde var olan mevsimsel ürünlerin dinamik fiyatlandırılması problemlerinden farklıdır çünkü farklı kaliteki tek tip ürünler göz önüne alınmıştır. Bu açıdan, bu çalışma bildiğimiz kadarıyla literatürde bir ilk olma özelliğini taşır.

Chapter 1

INTRODUCTION

Many industries and organizations use inventory management and pricing strategies to maximize the total expected profit when the selling season is finite and the products are perishable. Inventory management is concerned with the allocation of capacity to multiple customer classes, and the pricing decisions are made to determine the sale price. The practice of applying pricing strategies together with allocation of the capacity to different customer classes is known as revenue management. Among the pricing strategies, dynamic pricing strategy, which adjusts the sale price dynamically as a function of time and capacity, is commonly used to control inventory level so as to maximize the total expected revenue. Dynamic pricing decisions are made to determine the optimal price according to remaining inventory level, demand size, and remaining time.

We consider two different problems throughout this thesis. In the first problem, the seller applies a capacity allocation rule that controls which and how many of the randomly arriving requests should be accepted. On the other hand, the second problem uses dynamic pricing policy for the products with random quality and determines how to adjust the price to enhance the expected total revenue during a finite selling season by updating the distribution of the product quality of the remaining inventory for the rest of the selling season.

The two main contributions of this thesis are the following. We first study the structural properties of a discrete-time single product inventory control problem first introduced in [19] and called as the "omnibus model". This model is fairly general and is in fact a unifying formulation which compromises between static and dynamic models. We then focus on a dynamic optimization problem for pricing of perishable products with random quality. The problem of the seller who must sell the available inventory within a desired period of time is a fairly common situation, in practice, with seasonal products and well-known problem in the pricing literature. However, in this work, we study the optimal pricing of perishable products with random qualities. To the best of our knowledge, this is the first research work associating dynamic pricing of perishable products with random quality.

The main purpose of the first part of this work is to develop a dynamic programming (DP) model for the partial admission problem aiming to optimize the expected revenue of a seller who is making acceptance-rejection decisions. Partial admission is the *partial* fulfillment of demand for accepted requests. The key modelling assumption of the problem is that even though customer requests can be for multiple units of the product (*batch* orders) coming from multiple demand classes, time is divided into decision periods such that at most one request is received in any given period. The demand intensity of a batch of any customer class at a point in time is taken as a request probability, and these probabilities are allowed to vary with time. The core problem consists of whether or not to accept requests for products coming from different customer classes. The accept/reject decisions are made by comparing the expected cost of rejecting the current request with the expected revenue of admitting the current request. By rejecting the current request, the seller waits for future demands that may come from more profitable demand classes. On the other hand, he takes the risk of having to reject a more profitable future demand due to lack of capacity by accepting the current request. There are established results on the structure of the optimal policy, but little is known on the effects of varying parameters such as probability distributions of demands, revenues, etc. In this thesis, the effects of varying system parameters are examined for this model, and besides proving some basic and even more complex structural properties, we provide guidelines for predicting the response of the optimal policy to changes in random system parameters, such as arrival rates of batch orders of any customer class at any decision period. In practice, arrival probabilities of demands may not be known with certainty, and may be estimated from the sales information. Due to randomness in some system parameters, the outcome of a decision in any decision period can be only predictable to some extent. Therefore, in this problem, it is the expected total revenue which we aim to maximize.

The second part of this work aims to determine the optimal pricing policy when the seller has perishable products which cannot be stored for use thereafter and which have random quality. In our problem, there is a homogeneity in customers' willingness to pay for a product with the same quality. We assume that the seller does not perceive the exact product quality of available inventory, so sets a single price for a pool of products with different quality. The reservation prices of customers reflect the value that customers assign to the product quality. Thus, customers only buy the product if their valuations of the product quality are higher than or equal to the product's price. The seller does not perceive the exact quality of each product on hand, thus begins with a projected and subjective cumulative distribution for the product quality of his inventories. He infers about the product quality as he monitors the sales information. The goal of the seller is to determine the pricing policy during the planning horizon that maximizes the total expected profit. We first present a model where the price is updated at a point in time during the selling season according to the updated distribution of the quality of products which are left for the remaining part of the season, and call this model as "information update scenario". We obtain the optimal pricing strategy and optimal total expected profit as a function of the initial inventory level, and initial distribution of the product quality by numerical examples. Later, we simplify the basic model to incorporate static pricing where price does not change during the selling season. We call this model as "No information update scenario". Static pricing strategy simplifies the implementation of the model since there is no update of the product quality distribution. However, it is shown by numerical results that the loss experienced by the seller when implementing static pricing policy instead of dynamic pricing policy is not negligible. We study these models with information update and with no information update under two different demand settings. First, we investigate the optimal pricing policies for the models when the seller observes random demand during the selling season, then study the problem with no demand restriction for both of the scenarios. Next, we focus on the special case of the problem with a single product for both scenarios and show some analytical results that are used as benchmark statistics in the numerical results. Finally, in order to incorporate the simplest version of the seller's problem, we investigate the optimal pricing of perishable products with constant quality (no randomness).

The thesis is structured as follows. Chapter 2 provides literature surveys related to our two problems: the inventory control problem, and the dynamic pricing problem. Chapter 3 presents the model of discrete-time single product inventory control problem with its optimal policy and structural properties. Next, in Chapter 4 we study the dynamic pricing of perishable products with random quality, and give numerical results. Finally, Chapter 5 summarizes the performed study, and mention the future research perspectives.

Chapter 2

LITERATURE REVIEW

There is a huge literature on inventory control as well as dynamic pricing and revenue management. In this chapter, we provide details and references on advances in areas related to different aspects of this thesis.

2.1 Inventory and Admission Control Decisions

The purpose of this section is to review the literature on revenue management focusing the subject of integrated pricing and inventory management and admission control decisions. Excellent survey and classification of research on coordination of pricing and inventory decisions, can be found in Chan, Shen, Levi and Swann [8], and a complete review on coordinated pricing and production/procurement decisions is presented by Yano, Gilbert [34] which provide the summary of the research papers on this area by focusing on different aspects of the problem. In addition, a number of excellent introductions to the subject of revenue management, especially, to the airline revenue management problem exist in the literature, particularly in the paper by Talluri and Van Ryzin [31]. An excellent review of the literature, along with examples of applications of revenue management in areas other than the airline industry, can be found in McGill and van Ryzin [23].

Revenue management studies the theory and the applications of making efficient use of a given fixed resource that perishes after a given time. To do so, it uses some basic controls such as booking or sales limits at various price levels. In the early literature on this subject, many models are static. In static models, fare classes are assumed to book sequentially in order of increasing fare level. These models do not explicitly consider the passenger arrival process over time, requiring instead only the total demand for each class [19]. In that category, the papers by Belobaba [3], Brumelle et al. [7], Curry [10], Wollmer [33] and Brumelle and McGill [6] offer an analysis; Robinson [25] removes the last assumption.

Robinson [25] develops the most general of the static models. He uses continuous variables to model passenger demand and makes no assumptions concerning the order in which fare classes arrive. Brumelle and McGill [6], Wollmer [33], and Curry [10] consider a static problem similar to Robinson, with the added restriction that fare classes arrive in order of increasing fare level. Curry [10] gives a mathematical programming formulation for a multiple-flight-leg problem in a manner similar to that of Robinson and models demand as a continuous quantity. Brumelle and McGill [6] prove the optimality of a booking limit policy for the case in which lower fare classes book first. They formulate a model capable of handling both continuous and discrete demand.

Li and Oum [21] presented a brief note on the single leg multi-fare seat allocation problem. The note consists of three models which were independently proposed by Curry [10], Wollmer [33], and Brumelle and McGill [6] and compares their optimality conditions. It is shown that these three models give analytically equivalent optimality conditions and Wollmer's model is just a discrete version of Curry's model. The equivalence of optimality conditions stated in the Curry's and Brumelle's models is given in this note, thus it is claimed that none of them may have any computational advantage over the others.

Additionally, dynamic programming models, similar to the methodology of this thesis, have been applied to the related problems in airline management such as overbooking and inventory management. A comprehensive approach was presented later in Lee and Hersh [20], and then in Subramanian, Stidham, and Lautenbacher [29] and Lautenbacher and Stidham [19] (all in discrete time with multiple classes, the latter comments on extensions to Poisson arrivals and splittable batches).

First, we consider the dynamic model by Lee and Hersh [20]. The aim of the paper was to determine whether a booking request for seats in a certain booking class occurring at some point in time during the booking period should be accepted or denied and to develop a discrete-time dynamic programming model for finding an optimal booking policy, which can be reduced to a set of critical values. The Lee and Hersh model does not require any assumption about the arrival pattern for the various booking classes and multiple seat bookings are also incorporated into the model. Concavity and monotonicity of the total expected revenue is demonstrated. In this thesis, we present similar approach for modelling the inventory problem when there are multiple fare classes for a pool of identical products. Several of the papers we consider extend this model. The indispensable extension of Lee and Hersh model is the "omnibus" model of Lautenbacher and Stidham [19] which combine the static and dynamic models in a unifying formulation.

Lautenbacher and Stidham [19] studied the single leg yield management problem without cancellations, overbooking, or discounting . At the moment the request arrives, the decision to accept or reject involves three factors: 1) The number of seats previously allocated 2) The time remaining in the reservations horizon 3) The fare class of the request. They develop a discrete-time, finite horizon markov decision process model and solve by backward induction on the number of periods remaining before departure. They show that the maximal undiscounted expected revenue function is concave and non-increasing in x by following an earlier result from Stidham [28]. Lautenbacher, and Stidham, [19] convert Robinson's demand assumption into discrete demand and highlight his main result which says that optimal value function is concave in x. Lautenbacher, and Stidham, [19] solve for the optimal booking limits by means of a backward recursion, so that the optimal booking limit for the last period during which a class arrives is determined first. In addition, Subramanian, Stidham, and Lautenbacher [29] appeared subsequent to Lee and Hersh [20] supplementing their model with cancellations, overbooking and discounting.

Brumelle, and Walczak [26] also considered a problem where when a customer requests a discount fare, the airline must decide whether to sell the seat at the requested discount or to hold the seat in hope that a customer will arrive later who will pay more. They model this situation for a single-leg flight with multiple fare classes and customers who arrive according to a semi-Markov process (possibly nonhomogeneous). They also provide counterexamples to show that this structural property of the optimal policy need not hold for more general arrival processes if the requests can be for more than one seat and must be accepted or rejected as a whole. They also include a counterexample to a claim by Lee and Hersh [20] that a critical time property holds when accept/reject requests are for more than one seat.

On the technical sides, we use the paper from, Koole [17], [18] and Karaesmen, Örmeci and Cil [2] not only provide a general framework to prove the structural properties of the value functions and optimal policies but also proposes to define distinct event operators and examine the structural properties of the operators rather than the whole value function. We have used these papers to show that if the event operators preserve some structural properties then the value function which is the combination of the operators will also satisfy the same properties.

In this thesis, we have removed cancellations, overbooking and discounting assumptions of the discussion of Subramanian, Stidham, and Lautenbacher [29]. The model presented in this work is essentially that of the "omnibus" model in Stidham, and Lautenbacher [19], but in addition to the work of Stidham, and Lautenbacher, we demonstrate the structural properties of the model and perform sensitivity analysis.

2.2 Dynamic Pricing Decisions

The purpose of this section is to review the literature on revenue management focusing the subject of dynamic pricing decisions. Many industries have the opportunity to increase their revenues through the dynamic pricing of their perishable products. The seller can improve its revenues by dynamically adjusting the price of the product rather than adopting a fixed price throughout the selling season. In this section we review the most frequently referred works studied dynamically pricing of perishable inventories. In fact, most of the papers in the literature study the question of how a seller should dynamically adjust the price of a perishable product as the time at which the product will perish approaches and the inventory of the product diminishes, and various scenarios of dynamic pricing of perishable inventories have been mentioned in the literature so far; however, to our knowledge, dynamically pricing the products with random quality has not been studied yet.

First of all, we refer the reader to the literature review [14] for an extensive review of the works discussing pricing problems. In their paper [14], Gallego and van Ryzin study the problem of dynamic pricing of a given number of inventories over a finite time horizon. Their demand model is a homogeneous Poisson process, and demand is assumed to be price dependent. Together with showing some structural properties, they aim maximizing the expected revenue through dynamic pricing. In [15], Gallego and van Ryzin extend their single-product model to the multiple-product case.

Even though implementing a policy in which the price is always changing is optimal, it is not practical. In this regard, the paper of Gallego and van Ryzin [14] presents their "at-most-one-price-change heuristic" by investigating the performance of dynamic pricing policies by comparing them with the deterministic versions. When the set of prices is a continuous interval, they show that a fixed price policy is asymptotically optimal. When the set of prices is finite, the price change at most once is shown to be asymptotically optimal. Following the Gallego and Van Ryzin [14], Feng and Gallego [12] show the optimal timing of a single price change from a given initial price to either a given lower or higher second price. They show that it is optimal to decrease the price as soon as the time-to-go falls below a time threshold that depends on the number of unsold items. In [13], Feng and Gallego extend their work and study the problem of deciding the optimal timing of price changes within a given set of allowable, time dependent price paths.

In some papers, the retailers are usually constrained to choosing between a limited and certainly finite, set of allowable prices. For example, Chatwin [9] assumes that the set of allowable prices is finite and in his paper a continuous-time dynamic programming model in which the state is the number of inventory at any given time and the seller's decision is to choose the price at which to sell the product is employed. He shows that the maximum expected revenue function is nondecreasing and concave in the inventory level and in the remaining time. Moreover, he verifies that at a given time the optimal price is non-increasing in the remaining inventory and non-decreasing in the remaining time till the end of the season.

After choosing the stocking level at the beginning of the selling season, Monahan [24] investigates pricing of a single product over multiple time periods. In his paper, structural properties of the optimal pricing policy over a finite horizon, and the effects of the pricing policy on the optimal procurement policy of a news-vendor problem are studied. He gives an efficient algorithm to find optimal prices, and investigates the effects of the market parameters on the optimal policy using numerical examples.

Bitran and Mondschein [5] extend the search for optimal policies and they call it as "periodic pricing review policies". They focus on the sale of seasonal products for which the retailers make discount during the season. They study a continuous time model in which the price is updated continuously, and present this model as a benchmark for a periodic pricing review model. The authors use empirical analysis to establish the structure of the optimal policy and revenue but no theoretical results are presented. They show that the price is decreasing in inventory level and increasing in time, and claim that the demand uncertainty leads to higher prices, larger discounts and more unsold inventory. This work is extended in Bitran et al. [4] where they focus on the retail chains with several stores which coordinate prices, and develop heuristic solutions.

Zhao and Zheng [36] consider dynamic pricing of inventories that will perish at the end of the selling season. In their problem, the demand model is assumed to be a nonhomogeneous Poisson process, the rate of an arrival of the customer, and the distribution of reservation price for the product is allowed to change over time. They conclude that the optimal price decreases over time for a given inventory level, and it decreases as the remaining inventory level increases at a given period. In our work, we will show that their results may not be true under the assumption that each inventory is with a random quality.

Finally, for detailed descriptions of yield management and references to works discussing solution techniques, the reader is suggested to see Weatherford and Bodily [32]. The term yield management has typically been used to mention the airlines' practices to improve revenues through the seat control. When these practices began to be used by the other industries and applied to related problems, Weatherford and Bodily [32] suggested the more general and more descriptive term instead of the term yield management, and they called it as perishable-asset revenue management (PARM). Zhang and Cooper [35] and Maglaras and Meissner [22] analyze the dynamic pricing problem for multiple products in revenue management context. For excellent surveys of dynamic pricing in the context of revenue management, see McGill and van Ryzin [23], and the book by Talluri and van Ryzin [30].

In this thesis, we present the dynamic pricing of perishable products in the presence of quality component. Hence, our results differ from the earlier results in the related literature. To our knowledge, this is the first work that considers the pricing problem of perishable products with quality components.

Chapter 3

STRUCTURAL PROPERTIES OF A DISCRETE-TIME SINGLE PRODUCT INVENTORY CONTROL PROBLEM WITH REVENUE MANAGEMENT APPLICATIONS

3.1 Introduction

In this chapter, we consider a finite horizon, single-product inventory control problem in which the decision-maker accepts or rejects customer requests coming from multiple demand classes. Customer requests can be for multiple units of the product (*batch* orders) and we allow *partial* fulfillment of demand for accepted requests. The optimal decisions must incorporate the following core tradeoff; by accepting the current request, the decision-maker foregoes the opportunity of using the inventory for more profitable future requests. On the other hand, if the current request is rejected, the decision-maker faces the risk of having to accept a request from a less profitable demand class or even being left with unsold inventory at the end of the finite horizon (due to lack of demand). The optimal decision depends on factors such as the available inventory, relative profitability of demand classes, projected volume and mix of future demand (distribution of future demand), and time to go till the end of the time horizon. If the seller is assumed to be profit-maximizer, an optimal strategy is the strategy providing the highest net revenues. Since all parameters of the problem are time dependent, the problem is allowed to model in a dynamic aspect. Clearly, this is a typical revenue management problem, which has garnered great interest both from practitioners as well as researchers (we refer the reader to [30] for a comprehensive survey of revenue management literature).

3.2 Problem Description and Formulation

Our problem is a discrete-time single product inventory control problem, and we study this problem in the presence of partial admission policy. We have multiple demand classes and the values of these classes are not the same to the seller. The valuations of the customer classes are made according to the prices that they offer for the product, so there may be more valuable classes such that satisfying the requests coming from these classes is more important than satisfying the other classes' requests.

In this problem, the decision maker is the seller, and when the inventory level is low, he decides to reserve a particular amount of the inventory for more profitable demand classes that he anticipates to observe in the future. The reservation of the capacity is made by rejecting the demand from less valuable classes, and there are threshold values for each demand classes below which it is optimal to reject requests coming from these classes and reserve the available capacity for more profitable future demands. Hence, the main problem of the seller is to determine whether a request from a certain demand class arriving at some point in time during the selling season should be accepted or denied. Moreover, as we have mentioned in the previous section, customer requests can be for multiple units of the product (*batch* orders), but the decision of admitting or rejecting a customer class does not depend on the size of the request, since we allow partial admission in this model. So, the seller also needs to determine the number of orders that will be satisfied from an arriving batch. The decision of accepting or denying an arriving request is not so simple, and is based on the factors such as the remaining time to go till the end of the season, the available inventory on hand, relative profitability of demand classes, and projected volume of the future demand.

In this problem, the selling period is split into a number of decision periods in which at most one type of requests can arrive, and we use the following assumptions to model the seller's problem:

- Requests for different fare classes are independent, hence the demand for one class does not affect the demand for another class.

- A rejected request is lost sale.
- Request probabilities vary with time.
- Requests from only one class of customers are allowed during a decision period.
- An accept/deny decision has to be made each time a request arrives.

3.2.1 Problem Formulation

Suppose that time is divided into decision periods, as in [20], such that at most one request is received in any given period (though the customer can demand more than one unit of the product). Let K be the number of decision periods. Time is indexed by k in our model, where $k = K$ is the first period and $k = 1$ is the last period after which all inventories perish. There are n demand classes, with class i offering to pay R_i, $i = 1, 2, \ldots, n$, for a unit of the product. Assume that $R_1 \geq R_2 \geq R_3 \geq \ldots \geq R_n$, without loss of generality. Let p_{ibk} be the probability that a customer belonging to demand class i (referred to as a class-i customer) requests b units of inventory in period k, and p_{0k} be the probability that no customers arrive in period k. We assume B_i is an upper bound on the batch demand size for class-i customers. Note that $p_{0k} + \sum_{i=1}^{n} \sum_{b=1}^{B_i} p_{ibk} = 1$ for all $k = 1, 2, \ldots, K$. The decision-maker's problem of maximizing expected revenues over the entire finite time horizon can be modelled using a dynamic programming formulation. Let $v_k(x)$ be the *expected maximum revenue-to-go* in period k when there are x units of inventory are available. We can express $v_k(x)$ as

$$v_k(x) = \sum_{i=1}^{n} \sum_{b=1}^{B_i} p_{ibk} \left(\max_{\kappa_i \in \{0,1,\ldots,\min(b,x)\}} \kappa_i R_i + v_{k-1}(x - \kappa_i) \right) + p_{0k} v_{k-1}(x) \qquad (3.1)$$

with boundary conditions $v_k(0) = 0$ for all k and $v_0(x) = 0$ for all x. In the above formulation, κ_i is the inventory assigned to the class-i customer, requesting b units of the product. Note that κ_i is an integer between 0 and $\min(x, b)$. We can rewrite the value function in (3.1) as a combination of the fictitious and rationing *event operators* defined in [2]. The *batch rationing* *operator* $T_{b\_RT_i}$ determines the number of inventory units assigned to class-i customers and the *fictitious operator* T_{FIC} represents the fictitious event. These two operators when applied on a function $f(x)$ yield

$$T_{FIC}f(x) = f(x)$$

and

$$T_{b\_RT_i}f(x) = \max_{\kappa_i \leq \min\{x,b\}} \{\kappa_i R_i + f(x - \kappa_i)\}.$$

Hence, we have

$$v_k(x) = \sum_{i=1}^{n} \sum_{b=1}^{B_i} p_{ibk} T_{b\_RT_i} v_{k-1}(x) + p_{0k} T_{FIC} v_{k-1}(x) \tag{3.2}$$

3.3 Structural Properties

3.3.1 Basic Properties

Proposition 1 *$T_{b\_RT_i}$ and T_{FIC} event operators have the following properties*

1. *If $f(x)$ is non-decreasing in x, then the $T_{b\_RT_i}$ and T_{FIC} preserve the monotonicity of $f(x)$.*

   (a)

   $$Tf(x) \leq Tf(x+1) \tag{3.3}$$

   $Tf(x)$ is an increasing function of x.

   (b)

   $$Tf(k) \leq Tf(k+1) \tag{3.4}$$

   $Tf(k)$ is an increasing function of k.

Proof: (a) By the definition of T_{FIC}, it preserves all the structural properties of a function. Therefore, it is enough to show that the batch rationing operator $T_{b\_RT_i}$ preserves the monotonicity of the value function with respect to x and k. Let κ_x and κ_{x+1} be the optimal number of class-i customers to be admitted from an arriving batch, then we can write (3) as follows:

$$\kappa_x R_i + f(x - \kappa_x) \leq \kappa_{x+1} R_i + f(x + 1 - \kappa_{x+1}) \tag{3.5}$$

Since κ_{x+1} is the optimal action for the state x+1 and f(x) is non-decreasing in x, we have

$$\kappa_{x+1} R_i + f(x + 1 - \kappa_{x+1}) \geq \kappa_x R_i + f(x + 1 - \kappa_x) \quad \textit{and}$$

$$\kappa_x R_i + f(x - \kappa_x) \leq \kappa_x R_i + f(x + 1 - \kappa_x)$$

Hence we have $\kappa_x R_i + f(x - \kappa_x) \leq \kappa_{x+1} R_i + f(x + 1 - \kappa_{x+1})$

Proof of (b) is similar.

2. *Class-1 customers should be always admitted.*

$$v_k(x) - v_k(x-1) \leq R_1 \quad \forall\ x,k \tag{3.6}$$

Proof: In [2], it is proved that $T_{b\_RT_i}$ preserves lower-bound difference (LBD) property so if we know that $v_k(x+1) - v_k(x) \leq R_1$ where R_1 is the reward associated with class-1, then we have

$$T_{b\_RT_i} f(x) - T_{b\_RT_i} f(x+1) \geq -R_1$$

Now for the initial step, we need to check

$$v_0(x+1) - v_0(x) \leq R_1 \ \text{ and we see that} \Rightarrow 0 \leq R_1 \ \surd$$

Assume $v_k(x+1) - v_k(x) \leq R_1$ is true, then as $v_{k+1}(x) = T_{b\_RT_i} v_k(x)$ and $T_{b\_RT_i}$ preserves the LBD property, the proof is completed.

3. If $f(x)$ is concave in x, then the $T_{b\_RT_i}$ and T_{FIC} preserve the concavity of $f(x)$.

   Proof: It is easy to see that T_{FIC} preserves all structural properties, so checking $T_{b\_RT_i}$ is enough.
   Let $\bar{\kappa^*} = (\kappa_x^*, \kappa_{x+1}^*, \kappa_{x+2}^*)$ be an optimal action vector and κ_x^* be optimal number of customers admitted from an arriving batch in state x. Assume that $f(x)$ is a concave function in x.

We will show that $\forall\ \bar{\kappa^*}$

$$\kappa_x^* R_i + f(x - \kappa_x^*) - \kappa_{x+1}^* R_i - f(x + 1 - \kappa_{x+1}^*)$$

$$\leq \ ?$$

$$\kappa_{x+1}^* R_i + f(x + 1 - \kappa_{x+1}^*) - \kappa_{x+2}^* R_i - f(x + 2 - \kappa_{x+2}^*)$$

Since concavity of $f(x)$ implies that the optimal number of customers to be admitted in

Cases	$\bar{\kappa *} = (\kappa *_x, \kappa *_{x+1}, \kappa *_{x+2})$	Rewritten form of the inequality
Case I	(a, a, a)	$f(x-a) - f(x+1-a) \leq f(x+1-a) - f(x+2-a)$
Case II	$(a, a+1, a+1)$	$f(x+1-a) - f(x-a) \leq R_i$
Case III	$(a, a, a+1)$	$R_i \leq f(x+1-a) - f(x-a)$
Case IV	$(a, a+1, a+2)$	$-R_i \leq -R_i$

Table 3.1: Possible optimal actions

states x and $x+1$ can differ at most by 1, all possible cases are as shown in Table 3.1.

Concavity of f guarantees the first case. Case II is true as we admit a customer in state *x+1*. Case III holds since a customer is rejected in state *x+1*. Hence, $T_{b\_RT_i}$ preserves concavity in x.

Proposition 2 *The maximum expected revenue-to-go function,* $v_k(x)$ *is*

1. *a non-decreasing function of the inventory level,* x*,*

2. *a non-decreasing function of the time remaining till the end of the finite time horizon,* k*,*

   Proof: We have proved that $T_{b\_RT_i}$ preserves monotonicity of $v_k(x)$ with respect to x and k. Now, by using this property of $T_{b\_RT_i}$ and applying the mathematical induction proof method, we will show that maximal value function $v_k(x)$ is non-decreasing in x and k. The initial step for the monotonicity in x holds because of the boundary conditions. (i.e. $v_0(x) \leq v_0(x+1)$ holds.)Now, assume that it is true for k. (i.e.$v_k(x) \leq v_k(x+1)$), then as

$$v_{k+1}(x) = \sum_{i=1}^{n} \sum_{B=1} p_{iB} T_{b\_RT_i} v(x) + p_{n+1} T_{FIC}\ v_k(x)$$

and both operators preserves monotonicity, we are done.
Note that the proof of monotonicity in k is similar.

3. a concave function of the inventory level, x.

   Proof: By using mathematical induction method, we first check the initial conditions. Knowing the specification of the boundary condition on $v_0(x)$, we assume that $v_k(x)$ is concave in x and show that $v_{k+1}(x)$ is also concave in x. In other words, we show that the following inequality is true.

$$\sum_{i=1}^{n}\sum_{B=1} p_{iB}\Delta T_{b\_RT_i} v_k(x-1) + p_{n+1}\Delta T_{FIC}\ v_k(x-1)$$

$$\leq$$

$$\sum_{i=1}^{n}\sum_{B=1} p_{iB}\Delta T_{b\_RT_i} v(x) + p_{n+1}\Delta T_{FIC}\ v_k(x)$$

   But, we have already shown that $T_{b\_RT_i}$ and T_{FIC} preserves concavity. Hence, $v_k(x)$ is concave in x.

Concavity of the value function $v_k(x)$ means that marginal value of an inventory is non-increasing with the current inventory level, x. Let ℓ^*_{ik} be defined as follows

$$\ell^*_{ik} = max\{x : v_k(x) - v_k(x-1) > R_i\}$$

More explicitly, ℓ^*_{ik} is the maximum number of inventory on hand such that if the current inventory on hand, x, is less than or equal to ℓ^*_{ik}, it is optimal to reject the whole class-i batch. Similarly, if the current inventory level, x, is greater than or equal to $\ell^*_{ik} + 1$, it is optimal to satisfy class-i demand until either the inventory level drops down to ℓ^*_{ik} or the whole batch is satisfied. Here, ℓ^*_{ik} is the optimal threshold value for class-i demand such that the optimal policy will reject the whole class-i batch if $x < \ell^*_{ik}$, partially satisfy the demand if $\ell^*_{ik} < x < \ell^*_{ik} + b$, and satisfy the entire batch if $x \geq \ell^*_{ik} + b$. Therefore, threshold policy is the optimal policy in our model. It is obvious that if the reward of a class-i customer is higher than the reward of a class-j customer, then the optimal threshold value of class-i will be lower than that of class-j as shown below figure 3.1

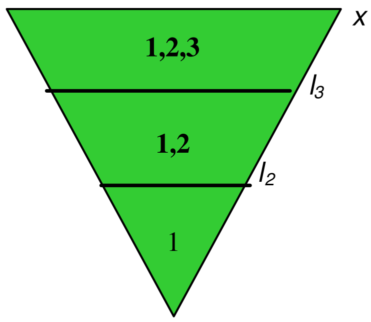

Figure 3.1: Threshold values

Hence, the following proposition follows as a natural consequence.

Proposition 3 *Given a batch of* b *units of class-1 demand in period* k*, the optimal policy assigns* $\kappa_1 = \min\{x, b\}$ *units of inventory to this demand, i.e., the optimal policy accepts as much class-1 demand as possible.*

3.3.2 More Complicated Properties

Proposition 4 $v_k(x)$ *is a non-decreasing function of* p_{ibk}*.*

Proof. Consider two systems, system 1 and system 2. All model parameters of these two systems, as well as their demand distributions are identical except for some period t, where $1 \leq t \leq K$. In the t^{th} period, the arrival probability of a particular class-j customer with a batch demand of size $\tilde{b}$ units is given by $p_{j\tilde{b}t}$ in system 1, whereas the likelihood of the same event in system 2 is given by $p_{j\tilde{b}t} + \varepsilon$. Let $v_k(x)$ be the optimal value function of system 1 in period k and $v_k^\varepsilon(x)$ be the optimal value function of system 2. We prove $v_k^\varepsilon(x) \geq v_k(x)$ by using induction on k. For $k = 0, 1, \ldots, t-1$, the proposition is trivially true since the two systems are identical. Now, let $k = t$. We can express the value functions in the two systems as

$$\begin{aligned} v_t(x) &= \sum_{\substack{i=1 \\ i \neq j}}^{n} \sum_{b=1}^{B_i} p_{ibt} T_{b\_RT_i} v_{t-1}(x) + \sum_{\substack{b=1 \\ b \neq \tilde{b}}}^{B_j} p_{jbt} T_{b\_RT_j} v_{t-1}(x) \\ &+ \; p_{j\tilde{b}t} T_{\tilde{b}\_RT_j} v_{t-1}(x) + p_{0t} T_{FIC} \; v_{t-1}(x) \end{aligned}$$

and

$$\begin{aligned} v_t^\varepsilon(x) &= \sum_{\substack{i=1 \\ i \neq j}}^{n} \sum_{b=1}^{B_i} p_{ibt} T_{b\_RT_i} v_{t-1}^\varepsilon(x) + \sum_{\substack{b=1 \\ b \neq \tilde{b}}}^{B_j} p_{jbt} T_{b\_RT_j} v_{t-1}^\varepsilon(x) \\ &+ \; (p_{j\tilde{b}t} + \epsilon) T_{\tilde{b}\_RT_j} v_{t-1}^\varepsilon(x) + (p_{0t} - \varepsilon) T_{FIC} \; v_{t-1}^\varepsilon(x) \end{aligned}$$

Since $v_{t-1}(x) = v_{t-1}^\varepsilon(x)$, one can show $v_t(x) \leq v_t^\varepsilon(x)$ if

$$T_{\tilde{b}\_RT_j} v_{t-1}^\varepsilon(x) \geq T_{FIC} v_{t-1}^\varepsilon(x)$$

or equivalently

$$\max_{\kappa_j \leq \min\{\tilde{b}, x\}} \{\kappa_j R_j + v_{t-1}^\varepsilon(x - \kappa_j)\} \geq v_{t-1}^\varepsilon(x).$$

The above inequality is true due to the optimality of the number of units of inventory allocated to class-j demand in system 2 in period t. Hence, $v_t(x) \leq v_t^\varepsilon(x)$. Finally, for $k > t$, the proposition holds if

$$\max_{\kappa_j \leq \min\{\tilde{b}, x\}} \{\kappa_j R_j + v_{k-1}^\varepsilon(x - \kappa_j)\} \geq v_{k-1}^\varepsilon(x),$$

which is true due to the optimality of the number of units of inventory allocated to class-j demand in system 2 in period k. ■

Proposition 5 *$v_k(x)$ is a supermodular function of p_{jbk} and x.*

Proof. Consider two systems, system 1 and system 2. All model parameters of these two systems, as well as their demand distributions are identical except for some period t, where $1 \leq t \leq K$. In the t^{th} period, the arrival probability of a particular class-j customer with a batch demand of size $\tilde{b}$ units is given by $p_{j\tilde{b}t}$ in system 1, whereas the likelihood of the same event in system 2 is given by $p_{j\tilde{b}t} + \varepsilon$. Let $v_k(x)$ be the optimal value function of system 1 in period k and $v_k^{\varepsilon}(x)$ be the optimal value function of system 2. From the definition of supermodularity, we need to show

$$v_k^{\varepsilon}(x) - v_k^{\varepsilon}(x-1) \geq v_k(x) - v_k(x-1).$$

Let us define the marginal value function $\Delta f = f(x) - f(x-1)$. Hence, the above expression can be written as

$$\Delta v_k^{\varepsilon}(x) \geq \Delta v_k(x).$$

For $k = 0, 1, \ldots, t-1$, supermodularity holds trivially since $v_k(x) = v_k^{\varepsilon}(x)$. Hence, we next verify $\Delta v_t^{\varepsilon}(x) \geq \Delta v_t(x)$, i.e., $k = t$, which can be written explicitly as follows

$$\sum_{\substack{i=1 \\ i \neq j}}^{n} \sum_{b=1}^{B_i} p_{ibt} \Delta T_{b\_RT_i} v_{t-1}^{\varepsilon}(x) + \sum_{\substack{b=1 \\ b \neq \tilde{b}}}^{B_j} p_{jbt} \Delta T_{b\_RT_j} v_{t-1}^{\varepsilon}(x)$$

$$+(p_{j\tilde{b}t} + \epsilon) \Delta T_{\tilde{b}\_RT_j} v_{t-1}^{\varepsilon}(x) + (p_{0t} - \varepsilon) \Delta T_{FIC}\ v_t^{\varepsilon}(x)$$

$$\geq$$

$$\sum_{\substack{i=1\\ i\neq j}}^{n}\sum_{b=1}^{B_i} p_{ibt}\Delta T_{b\_RT_i} v_{t-1}(x) + \sum_{\substack{b=1\\ b\neq \tilde{b}}}^{B_j} p_{jbt}\Delta T_{b\_RT_j} v_{t-1}(x)$$

$$+p_{j\tilde{b}t}\Delta T_{\tilde{b}\_RT_j} v_{t-1}(x) + p_{0t}\Delta T_{FIC}\ v_t(x)$$

Note that $\Delta T_{b\_RT_i} v_{t-1}(x) = \Delta T_{b\_RT_i} v^{\varepsilon}_{t-1}(x)$ since $v_{t-1}(x) = v^{\varepsilon}_{t-1}(x)$. Therefore, the above expression can be simplified to

$$\Delta T_{\tilde{b}\_RT_j} v^{\varepsilon}_{t-1}(x) - \Delta T_{FIC}\ v^{\varepsilon}_t(x) \geq 0$$

which can also be written as

$$\max_{\kappa_j \leq \min\{x,\tilde{b}\}} \{\kappa_j R_j + v^{\varepsilon}_{t-1}(x-\kappa_j)\} - v^{\varepsilon}_{t-1}(x) \geq 0 \tag{3.7}$$

Let κ^*_j be the optimal number of units of class-j demand filled out of a batch of size $\tilde{b}$ in period t. If $\kappa^*_j = 0$, then obviously Equation (3.7) is true. If $\kappa^*_j > 0$ then we know that $\kappa^*_j R_j + v^{\varepsilon}_{t-1}(x-\kappa_j) > v^{\varepsilon}_{t-1}(x)$ due to the optimality of κ^*_j, which also implies Equation (3.7). As a result, we have $\Delta v^{\varepsilon}_t(x) \geq \Delta v_t(x)$.

Now, let $k = t+1$. The statement in the proposition can be stated as

$$\sum_{\substack{i=1\\ i\neq j}}^{n}\sum_{b=1}^{B_i} p_{ib(t+1)}\Delta T_{b\_RT_i} v^{\varepsilon}_t(x) + \sum_{\substack{b=1\\ b\neq \tilde{b}}}^{B_j} p_{jb(t+1)}\Delta T_{b\_RT_j} v^{\varepsilon}_t(x)$$

$$+p_{j\tilde{b}(t+1)}\Delta T_{\tilde{b}\_RT_j} v^{\varepsilon}_t(x) + p_{0(t+1)}\Delta T_{FIC}\ v^{\varepsilon}_t(x)$$

$$\geq$$

$$\sum_{\substack{i=1 \\ i \neq j}}^{n} \sum_{b=1}^{B_i} p_{ib(t+1)} \Delta T_{b\_RT_i} v_t(x) + \sum_{\substack{b=1 \\ b \neq \tilde{b}}}^{B_j} p_{jb(t+1)} \Delta T_{b\_RT_j} v_t(x)$$

$$+ p_{j\tilde{b}(t+1)} \Delta T_{\tilde{b}\_RT_j} v_t(x) + p_{0(t+1)} \Delta T_{FIC}\ v_t(x)$$

Since supermodularity holds in period for period t, we know that $\Delta v_t^{\varepsilon}(x) \geq \Delta v_t(x)$. Hence, the above inequality would hold if

$$\Delta T_{b\_RT_i} v_t^{\varepsilon}(x) \geq \Delta T_{b\_RT_i} v_t(x) \quad \text{for all } i = 1, \ldots, n,$$

which can also be written as

$$T_{b\_RT_i} v_t(x-1) + T_{b\_RT_i} v_t^{\varepsilon}(x) \quad \geq \quad T_{b\_RT_i} v_t^{\varepsilon}(x-1) + T_{b\_RT_i} v_t(x)$$

Based on the definition of the batch rationing operator, this expression is equivalent to

$$\max_{\kappa_i \leq \ \min\{x-1,b\}} \{\kappa_i R_i + v_t(x-1-\kappa_i)\} + \max_{\kappa_i \leq \min\{x,b\}} \{\kappa_i R_i + v_t^{\varepsilon}(x-\kappa_i)\}$$
$$\geq$$
$$\max_{\kappa_i \leq \min\{x-1,b\}} \{\kappa_i R_i + v_t^{\varepsilon}(x-1-\kappa_i)\} + \max_{\kappa_i \leq \min\{x,b\}} \{\kappa_i R_i + v_k(x-\kappa_i x)\}$$

Let κ_{ix} be the optimal number of units of inventory allocated to class-i demand in system 1, and $\kappa_{ix}^{\varepsilon}$ be the optimal number of units of inventory allocated to class-i demand in system 2, with x units of available inventory in period $t+1$ in both systems. Consequently,

$$\kappa_{i(x-1)} R_i + v_t(x-1-\kappa_{i(x-1)}) + \kappa_{ix}^{\varepsilon} R_i + v_t^{\varepsilon}(x-\kappa_{ix}^{\varepsilon})$$
$$\geq$$
$$\kappa_{i(x-1)}^{\varepsilon} R_i + v_t^{\varepsilon}(x-1-\kappa_{i(x-1)}^{\varepsilon}) + \kappa_{ix} R_i + v_t(x-\kappa_{ix})$$

Next, we prove the validity of the above inequality by considering all possible values for κ_{ix} and $\kappa^{\varepsilon}_{ix}$. First note that κ_{ix} and $\kappa_{i(x-1)}$ can differ at most by 1 unit due to the concavity of the value function $v_t(x)$. Further, if $\kappa_{ix} = \kappa_{i(x-1)}$, then it should be true that either $\kappa_{ix} = \kappa_{i(x-1)} = 0$ or $\kappa_{ix} = \kappa_{i(x-1)} = b$ (same property holds for $\kappa^{\varepsilon}_{ix}$). Also, due to the optimality of κ_{ix} and $\kappa^{\varepsilon}_{ix}$, and our hypothesis in period t, we have

$$\begin{aligned} R_i \geq v^{\varepsilon}_t(x) - v^{\varepsilon}_t(x-1) &\geq v_t(x) - v_t(x-1) \\ R_i \geq v^{\varepsilon}_t(x-1) - v^{\varepsilon}_t(x-2) &\geq v_t(x-1) - v_t(x-2) \\ &\vdots \\ R_i \geq v^{\varepsilon}_t(x-\kappa^{\varepsilon}_{ix}+1) - v^{\varepsilon}_t(x-\kappa^{\varepsilon}_{ix}) &\geq v_t(x-\kappa^{\varepsilon}_{ix}+1) - v_t(x-\kappa^{\varepsilon}_{ix}) \end{aligned}$$

Hence, in the first system, the optimal number of units of inventory allocated to class-i demand in period $t+1$ with x units of available inventory, κ_{ix}, is at least $\kappa^{\varepsilon}_{ix}$, i.e. $\kappa^{\varepsilon}_{ix} \leq \kappa_{ix}$. Now for any two integers w_1 and w_2, such that $0 \leq w_1 \leq b-1$ and $w_1 \leq w_2 \leq b-1$, consider the following cases

Case	$(\kappa_{ix}, \kappa_{i(x-1)}, \kappa^{\varepsilon}_{ix}, \kappa^{\varepsilon}_{i(x-1)})$	Supermodularity Inequality
1	$(0,0,0,0)$	$v_t(x-1) + v^{\varepsilon}_t(x) \geq v_t(x) + v^{\varepsilon}_t(x-1)$
2	$(w_1+1, w_1, 0, 0)$	$v^{\varepsilon}_t(x) - v^{\varepsilon}_t(x-1) \geq R_i$
3	$(b,b,0,0)$	$v^{\varepsilon}_t(x) - v^{\varepsilon}_t(x-1) \geq v_t(x-b) - v_t(x-1-b)$
4	(w_1+1, w_1, w_2+1, w_2)	$R_i \geq R_i$
5	(b,b,w_2+1,w_2)	$R_i \geq v_t(x-b) - v_t(x-1-b)$
6	(b,b,b,b)	$v^{\varepsilon}_t(x-b) + v_t(x-1-b) \geq v_k(x-b) + v^{\varepsilon}_k(x-1-b)$

Cases 1 and 6 are true due to the supermodularity of $v_k(x)$ in period k and x. Case 2 is satisfied since no class-i demand is filled in the second system. In Case 3, the left hand side of the inequality is greater than or equal to than R_i, whereas the left hand side is less than or equal to R_i, hence is true. Case 4 trivially holds. In Case 5, the inequality is true since all type-i demand is filled in the first system. As a result, $v_{t+1}(x)$ is supermodular with respect in x and

$p_{ib(t+1)}$. Clearly, the supermodularity property is also valid for any $k > t+1$. ■

Proposition 6 *$v_k(x)$ is neither concave nor convex in p_{ibk}.*

Proof. We prove this proposition by a counterexample. Consider a problem with two demand classes ($n = 2$) and assume that each demand is for a single unit of inventory ($B_1 = B_2 = 1$). Further, let $k = 30$, $x = 14$, $R_1 = 25$ and $R_2 = 14$. Recall that p_{ibk} denotes the likelihood of receiving a class-i demand for b units of inventory in period k. When $p_{1,1,30} = 0.08$ and $p_{2,1,30} = 0.8$, we compute $v_{30}(14) = 221.3$. When $p_{1,1,30} = 0.1$ and $p_{2,1,30} = 0.8$, then $v_{30}(14) = 227.7$. When $p_{1,1,30} = 0.18$ and $p_{2,1,30} = 0.8$, we have $v_{30}(14) = 254.8$. These three observations suggest that $v_k(x)$ is a non-decreasing convex function of p_{1bk}. On the other hand, for $p_{1,1,30} = 0.08$ and $p_{2,1,30} = 0.2$, we find $v_{30}(14) = 143.8$. When $p_{1,1,30} = 0.1$ and $p_{2,1,30} = 0.2$, we compute $v_{30}(14) = 158.85$. When $p_{1,1,30} = 0.18$ and $p_{2,1,30} = 0.2$, then $v_{30}(14) = 214.8$. Contrary to our previous statement, these three observations indicate that $v_k(x)$ is a non-decreasing concave function of p_{1bk}. Therefore, we can conclude that $v_k(x)$ is neither concave nor convex in p_{1bk}. We can also show that $v_k(x)$ is neither concave nor convex in p_{2bk} in a similar manner. ■

Proposition 7 *$v_k(x)$ is a supermodular function of x and k.*

Proof. We would like to show that

$$v_k(x) - v_k(x-1) \geq v_{k-1}(x) - v_{k-1}(x-1) \tag{3.8}$$

Recall that the optimal value function $v_k(x)$ is defined as

$$v_k(x) = \sum_{i=1}^{n} \sum_{b=1}^{B_i} p_{ibk} \left(\max_{\kappa_i \leq \min(b,x)\}} \kappa_i R_i + v_{k-1}(x - \kappa_i) \right) + p_{0k} v_{k-1}(x)$$

Let $\Delta v_k^m(x)$ be defined as the *marginal value per unit* of inventory when m units out of x units of available inventory is allocated to demand in period k, i.e.,

$$\Delta v_k^m(x) = \frac{v_k(x) - v_k(x-m)}{m} \tag{3.9}$$

Based on this definition, we can express $\Delta v_k(x)$ as follows

$$\begin{aligned}\Delta v_k(x) &= v_k(x) - v_k(x-1)\\ &= \sum_{i=1}^{n}\sum_{b=1}^{B_i} p_{ibk}\left(\max_{\kappa_i \le \min(b,x)} \kappa_i R_i + v_{k-1}(x-\kappa_i)\right) + p_{0k} v_{k-1}(x)\\ &- \sum_{i=1}^{n}\sum_{b=1}^{B_i} p_{ibk}\left(\max_{\kappa_i \le \min(b,x-1)} \kappa_i R_i + v_{k-1}(x-\kappa_i-1)\right) + p_{0k} v_{k-1}(x-1)\end{aligned}$$

Substituting $p_{0k} = 1 - \sum_{i=1}^{n}\sum_{b=1}^{B_i} p_{ibk}$ into the above equation, we can simplify $\Delta v_k(x)$ as

$$\begin{aligned}\Delta v_k(x) &= \Delta v_{k-1}(x) + \sum_{i=1}^{n}\sum_{b=1}^{B_i} p_{ibk}\Bigg\{\max_{\kappa_i \le \min(b,x)} \kappa_i R_i - \kappa_i \Delta v_{k-1}^{\kappa_i}(x)\\ &\qquad - \max_{\kappa_i \le \min(b,x-1)} \kappa_i R_i - \kappa_i \Delta v_{k-1}^{\kappa_i}(x-1)\Bigg\}\end{aligned} \tag{3.10}$$

Note that, in order to show the supermodularity of $v_k(x)$ as a function of k and x, we need to prove $\Delta v_k(x) \ge \Delta v_{k-1}(x)$, which is equivalent to the definition in (3.8). Hence, it suffices to show in Equation (3.10) that

$$\sum_{i=1}^{n}\sum_{b=1}^{B_i} p_{ibk}\left\{\max_{\kappa_i \le \min(b,x)} \kappa_i R_i - \kappa_i \Delta v_{k-1}^{\kappa_i}(x) - \max_{\kappa_i \le \min(b,x-1)} \kappa_i R_i - \kappa_i \Delta v_{k-1}^{\kappa_i}(x-1)\right\} \ge 0 \tag{3.11}$$

Due to concavity of $v_k(x)$ as a function of x, we know that

$$v_{k-1}(x) - v_{k-1}(x-\kappa_i) \le v_{k-1}(x-1) - v_{k-1}(x-\kappa_i-1) \;\;\Rightarrow\;\; \Delta v_{k-1}^{\kappa_i}(x) \le \Delta v_{k-1}^{\kappa_i}(x-1). \tag{3.12}$$

Let κ_i^* be the optimal number of units of inventory assigned to class-i in period k with x units of inventory available. Similarly, let $\tilde{\kappa}_i{}^*$ be the optimal number of units of inventory assigned to class-i in period k with $x-1$ units of inventory available. Then, we can rewrite (3.11) as follows

$$\sum_{i=1}^{n}\sum_{b=1}^{B_i} p_{ibk}\left\{[\kappa_i^* R_i - \kappa_i^* \Delta v_{k-1}^{\kappa_i^*}(x)] - [\tilde{\kappa}_i^* R_i - \tilde{\kappa}_i^* \Delta v_{k-1}^{\tilde{\kappa}_i^*}(x-1)]\right\} \geq 0 \tag{3.13}$$

Since we already know that $v_k(x)$ is concave in x, it should be true that either $\kappa_i^* = \tilde{\kappa_i}^*$ or $\kappa_i^* = \tilde{\kappa_i}^* + 1$ for all $i = 1, \ldots, n$. If $\kappa_i^* = \tilde{\kappa_i}^*$, then

$$[\kappa_i^* R_i - \kappa_i^* \Delta v_{k-1}^{\kappa_i^*}(x)] - [\tilde{\kappa}_i^* R_i - \tilde{\kappa}_i^* \Delta v_{k-1}^{\tilde{\kappa}_i^*}(x-1)] = \kappa_i^*(\Delta v_{k-1}^{\kappa_i^*}(x-1) - \Delta v_{k-1}^{\kappa_i^*}(x))$$

which is nonnegative due to (3.12). On the other hand, if $\kappa_i^* = \tilde{\kappa_i}^* + 1$, we have

$$[\kappa_i^* R_i - \kappa_i^* \Delta v_{k-1}^{\kappa_i^*}(x)] - [\tilde{\kappa}_i^* R_i - \tilde{\kappa}_i^* \Delta v_{k-1}^{\tilde{\kappa}_i^*}(x-1)] = R_i + (\kappa_i^* - 1)\Delta v_{k-1}^{\kappa_i^*-1}(x-1) - \kappa_i^* \Delta v_{k-1}^{\kappa_i^*}(x).$$

We can simplify $(\kappa_i^* - 1)\Delta v_{k-1}^{\kappa_i^*-1}(x-1) - \kappa_i^* \Delta v_{k-1}^{\kappa_i^*}(x)$, using the definition in (3.9), as

$$\begin{aligned}
(\kappa_i^* - 1)\Delta v_{k-1}^{\kappa_i^*-1}(x-1) - \kappa_i^* \Delta v_{k-1}^{\kappa_i^*}(x) \quad &= \quad (\kappa_i^* - 1)\frac{v_{k-1}(x-1) - v_{k-1}(x-(\kappa_i^*-1)-1)}{\kappa_i^* - 1} \\
&\quad\quad -\kappa_i^* \frac{v_{k-1}(x) - v_{k-1}(x-\kappa_i^*)}{\kappa_i^*} \\
&= \quad v_{k-1}(x-1) - v_{k-1}(x).
\end{aligned}$$

Therefore, $R_i + (\kappa_i^* - 1)\Delta v_{k-1}^{\kappa_i^*-1}(x-1) - \kappa_i^* \Delta v_{k-1}^{\kappa_i^*}(x) = R_i + v_{k-1}(x-1) - v_{k-1}(x)$. Clearly, $R_i + v_{k-1}(x-1) - v_{k-1}(x)$ is nonnegative since at least one unit of inventory is assigned to class-i demand, i.e., $\kappa_i^* > 0$. As a result, the inequality in (3.13) is satisfied, which implies supermodularity of $v_k(x)$ in terms of k and x. ■

Proposition 8 *$v_k(x)$ is a convex function of R_i.*

Proof. We can write the linear programming formulation equivalent (see [**?**]) of $v_k(x)$, given in (3.1), as follows

$$
\begin{aligned}
&\min \ v_k(x) \\
&\quad s.t. \\
&\quad v_t(w) \ \geq \ \sum_{i=1}^{n}\sum_{b=1}^{B_i} p_{ibt}\left(\kappa_{itwb}R_i + v_{t-1}(w-\kappa_{itwb})\right) + p_{0t}v_{t-1}(w), \text{ for } t=1,\ldots,k, \quad w=1,\ldots,x \\
&\quad v_0(w) \ = \ 0, \quad \text{for } w=0,\ldots,x \\
&\quad v_t(0) \ = \ 0, \quad \text{for } t=1,\ldots,k \\
&\quad \kappa_{itwb} \ \leq \ w, \quad \text{for } \ t=1,\ldots,k, \ i=1,\ldots,n, \quad w=0,\ldots,x \ \ b=1,\ldots,B_i \\
&\quad \kappa_{itwb} \ \leq \ b, \quad \text{for } \ t=1,\ldots,k, \ i=1,\ldots,n, \quad w=0,\ldots,x \ \ b=1,\ldots,B_i \\
&\quad v_t(w) \ \geq \ 0, \ t=1,\ldots,k, \quad w=1,\ldots,x \\
&\quad \kappa_{itwb} \ \geq \ 0, \quad \text{for } \ t=1,\ldots,k, \ i=1,\ldots,n, \quad w=0,\ldots,x \ \ b=1,\ldots,B_i
\end{aligned}
$$

In the above formulation, $v_t(w)$ and κ_{itwb} are the decision variables, where $v_t(w)$ is the value in period t with w units of inventory, and κ_{itwb} is the number of inventory units allocated to class-i demand in period t when demand batch size is b and available inventory is w. We can rewrite the first constraint as follows

$$v_t(w) - p_{0t}v_{t-1}(w) - \sum_{i=1}^{n}\sum_{b=1}^{B_i} p_{ibt}v_{t-1}(w-\kappa_{itwb}) \geq \sum_{i=1}^{n}\sum_{b=1}^{B_i} p_{ibt}\kappa_{itwb}R_i$$

Since the optimal value of such a linear programming model is a convex function of its right-hand-side coefficients [**?**], we conclude that $v_k(x)$ is convex in R_i, for all $i=1,\ldots,n$. ∎

Proposition 9 *$v_k(x)$ is supermodular with respect to R_j and x.*

Proof. Consider two systems, system 1 and system 2. All model parameters of these two systems are identical except the reward of a particular class-j customer. In system 1, the reward of a particular class-j customer is given by R_j, whereas the reward of the same class of customer in system 2 is given by $R_j + \varepsilon$. Let $v_k(x)$ be the optimal value function of system 1 in period k and $v_k^\varepsilon(x)$ be the optimal value function of system 2. From the definition of supermodularity, we need to show

$$v_k^\varepsilon(x) - v_k^\varepsilon(x-1) \geq v_k(x) - v_k(x-1).$$

Let us define the marginal value function $\Delta f = f(x) - f(x-1)$. Hence, the above expression can be written as

$$\Delta v_k^\varepsilon(x) \geq \Delta v_k(x).$$

For $k = 0$, supermodularity holds trivially since $v_0(x) = v_0^\varepsilon(x) = 0\ \forall x$. Assume that for $k = t-1$, $\Delta v_{t-1}^\varepsilon(x) \geq \Delta v_{t-1}(x)$ is true. Hence, we next need to verify for $k = t$. i.e. $\Delta v_t^\varepsilon(x) \geq \Delta v_t(x)$ is true. Note that $\Delta v_t^\varepsilon(x)$ can be written explicitly as follows

$$\begin{aligned}
\Delta v_t^\varepsilon(x) &= v_t^\varepsilon(x) - v_t^\varepsilon(x-1) && (3.14)\\
&= \sum_{\substack{i=1\\ i\neq j}}^{n} \sum_{b=1}^{B_i} p_{ibt} \max_{\kappa_i \in \{0,1,\ldots,\min(b,x)\}} \{\kappa_i R_i + v_{t-1}^\varepsilon(x-\kappa_i)\} && (3.15)\\
&+ \sum_{b=1}^{B_j} p_{jbt} \max_{\kappa_j \in \{0,1,\ldots,\min(b,x)\}} \{\kappa_j (R_j+\varepsilon) + v_{t-1}^\varepsilon(x-\kappa_j)\} && (3.16)\\
&+ p_{0t} v_{t-1}^\varepsilon(x) && (3.17)\\
&- \sum_{\substack{i=1\\ i\neq j}}^{n} \sum_{b=1}^{B_i} p_{ibt} \max_{\kappa_i \in \{0,1,\ldots,\min(b,x-1)\}} \{\kappa_i R_i + v_{t-1}^\varepsilon(x-1-\kappa_i)\} && (3.18)\\
&- \sum_{b=1}^{B_j} p_{jbt} \max_{\kappa_j \in \{0,1,\ldots,\min(b,x-1)\}} \{\kappa_j (R_j+\varepsilon) + v_{t-1}^\varepsilon(x-1-\kappa_j)\} && (3.19)\\
&- p_{0t} v_{t-1}^\varepsilon(x-1) && (3.20)\\
& && (3.21)
\end{aligned}$$

Assume that $\min\{B_i : i \in \{1,\ldots,n\}\} > x$ then write

$$\begin{aligned}(11)+(14) &= \sum_{\substack{i=1\\i\neq j}}^{n}\sum_{b=1}^{x-1} p_{ibt}\left(\max_{\kappa_i\in\{0,1,\ldots,b\}}\{\kappa_i R_i + v^{\varepsilon}_{t-1}(x-\kappa_i)\} - \max_{\kappa_i\in\{0,1,\ldots,b\}}\{\kappa_i R_i + v^{\varepsilon}_{t-1}(x-1-\kappa_i)\}\right)\\ &+ \sum_{\substack{i=1\\i\neq j}}^{n}\sum_{b=x}^{B_i} p_{ibt}\left(\max_{\kappa_i\in\{0,1,\ldots,x\}}\{\kappa_i R_i + v^{\varepsilon}_{t-1}(x-\kappa_i)\} - \max_{\kappa_i\in\{0,1,\ldots,x-1\}}\{\kappa_i R_i + v^{\varepsilon}_{t-1}(x-1-\kappa_i)\}\right)\end{aligned}$$

Let κ_i^* be the optimal number of units of inventory assigned to class-i in period t with x units of inventory available. Similarly, let $\tilde{\kappa_i}^*$ be the optimal number of units of inventory assigned to class-i in period t with $x-1$ units of inventory available.First note that κ_i^* and $\tilde{\kappa_i}^*$ can differ at most by 1 unit due to the concavity of the value function $v_t(x)$. Further, if $\kappa_i^* = \tilde{\kappa_i}^*$, then it should be true that either $\kappa_i^* = \tilde{\kappa_i}^* = 0$ or $\kappa_i^* = \tilde{\kappa_i}^* = b$.

It is easily seen that if $\kappa_i^* = \tilde{\kappa_i}^*$ and if they are equal to either 0 or b, (11) and (14) sum up to 0. If $\tilde{\kappa_i}^*$ is substituted with $\kappa_i^* - 1$,

$$(11)+(14) = \sum_{\substack{i=1\\i\neq j}}^{n}\sum_{b=1}^{x-1} p_{ibt}R_i + \sum_{\substack{i=1\\i\neq j}}^{n}\sum_{b=x}^{B_i} p_{ibt}R_i = \sum_{\substack{i=1\\i\neq j}}^{n}\sum_{b=1}^{B_i} p_{ibt}R_i$$

Using the similar approach, (12) and (15) sum up to $\sum_{b=1}^{B_j} p_{jbt}(R_j+\varepsilon)$
Hence, we can write

$$\Delta v^{\varepsilon}_k(x) = \sum_{\substack{i=1\\i\neq j}}^{n}\sum_{b=1}^{B_i} p_{ibt}R_i + \sum_{b=1}^{B_j} p_{jbt}(R_j+\varepsilon) + p_{0t}\Delta v^{\varepsilon}_{t-1}(x) \tag{3.22}$$

And similarly we can write,

$$\Delta v_k(x) = \sum_{\substack{i=1 \\ i \neq j}}^{n} \sum_{b=1}^{B_i} p_{ibt} R_i + \sum_{b=1}^{B_j} p_{jbt} R_j + p_{0t} \Delta v_{t-1}(x) \tag{3.23}$$

Now, using (18) and (19), the following inequality

$$\sum_{\substack{i=1 \\ i \neq j}}^{n} \sum_{b=1}^{B_i} p_{ibt} R_i + \sum_{b=1}^{B_j} p_{jbt}(R_j + \varepsilon) + p_{0t} \Delta v^{\varepsilon}_{t-1}(x) \quad \geq \quad \sum_{\substack{i=1 \\ i \neq j}}^{n} \sum_{b=1}^{B_i} p_{ibt} R_i + \sum_{b=1}^{B_j} p_{jbt} R_j + p_{0t} \Delta v_{t-1}(x)$$

implies $\Delta v^{\varepsilon}_k(x) \geq \Delta v_k(x)$, but it is easily seen that (20) is true by the hypothesis. Thus, if $\min\{B_i : i \in \{1, \ldots, n\}\} > x$, supermodularity with respect to R_j and x holds.
Now, assume that $\max\{B_i : i \in \{1, \ldots, n\}\} \leq x$. If $\kappa_i^* = \tilde{\kappa_i}^*$ and if they are equal to either 0 or b, summation of (11) and (14) gives 0 as in the previous case. If instead of $\tilde{\kappa_i}^*$, $\kappa_i^* - 1$ is used,

$$\begin{aligned} (11) + (14) \quad &= \quad \sum_{\substack{i=1 \\ i \neq j}}^{n} \sum_{b=1}^{B_i} p_{ibt} \left(\max_{\kappa_i \in \{0,1,\ldots,b\}} \{\kappa_i R_i + v^{\varepsilon}_{t-1}(x - \kappa_i)\} - \max_{\kappa_i \in \{0,1,\ldots,b\}} \{\kappa_i R_i + v^{\varepsilon}_{t-1}(x - 1 - \kappa_i)\} \right) \\ &= \quad \sum_{\substack{i=1 \\ i \neq j}}^{n} \sum_{b=1}^{B_i} p_{ibt} R_i \end{aligned}$$

Using similar approach, we get

$$(12) + (15) = \sum_{b=1}^{B_j} p_{jbt}(R_j + \varepsilon)$$

Thus, again

$$\Delta v^{\varepsilon}_k(x) \quad \geq \quad \Delta v_k(x)$$

holds following the similar manner in the case of $\min\{B_i : i \in \{1, \ldots, n\}\} > x$.
Hence, $v_k(x)$ is supermodular with respect to R_j and x.

3.4 Numerical Example

Consider that the seller observes requests from two different demand classes, (i.e. $n = 2$), and assume that once a customer arrives, he requests three units of inventory regardless of the class that he belongs to, (i.e. $B_1 = B_2 = 3$). In addition, assume that the seller has 5 periods to go till the end of the selling season, (i.e. $k = 5$) and 5 units of initial inventory, (i.e. $x = 5$). If a customer from the first class arrives, he offers $R_1 = 7$ per item. On the other hand, a customer from the second class offers $R_2 = 2$ for the same item. Recall that p_{ibk} denotes the likelihood of receiving a class-i demand for b units of inventory in period k, assume that $p_{i,b,k}$ is given as follows:

$$p_{ibk} = \begin{pmatrix} 0.25 & 0.50 & 0.10 \\ 0.05 & 0.05 & 0.05 \end{pmatrix}$$

The total expected revenue to go till the end of the selling season, $v_k(x)\ \ \forall\ k$ and x, is computed in Matlab Version 7.0 by solving the proposed DPP problem. Then the marginal revenues, seen in Table 3.2, are calculated.

	$\Delta v_k(1)$	$\Delta v_k(2)$	$\Delta v_k(3)$	$\Delta v_k(4)$	$\Delta v_k(5)$
k=0	0	0	0	0	0
k=1	6.25	4.40	0.80	0	0
k=2	6.88	6.42	5.22	3.26	0.98
k=3	6.98	6.88	6.53	5.69	4.37
k=4	6.99	6.97	6.89	6.62	6.03
k=5	7.00	6.99	6.97	6.90	6.70

Table 3.2: Concavity in x and supermodularity with respect to x and k

Table 3.2 validates Proposition 2.3 and Proposition 7. It is seen that the marginal revenue of an additional inventory is non-increasing in x by concavity in x shown in Proposition 2.3. Moreover, the marginal revenue of an additional inventory in the system is non-decreasing in k by supermodularity with respect to x and k shown in Proposition 7.

R_2	$v_k(x)$	slope
2	19.71	1.57
3	21.29	1.70
4	22.99	1.71
5	24.70	1.71
6	26.41	1.73
7	28.14	

Table 3.3: $v_k(x)$ is convex in R_2

For a fixed level of inventory, and number of periods to go till the end of the selling season, we compute the expected revenue $\forall R_2 \in \{2,3,4,5,6,7 = R_1\}$. We see that the slopes between the points are non-decreasing in R_2. Hence, Table 3.3 suggests that the value function, $v_k(x)$, is a convex function of R_2 as suggested in Proposition 8.

Possible Extensions

As possible extensions of this problem, we can also handle cases with salvage cost and holding cost. Since we use induction method to prove the desired properties, properties must hold in the last period. So if salvage cost and holding cost are chosen linear or at least concave functions of inventory level, then all properties that we have proven are still valid. In the case of adding a holding cost, there is one thing that should be noted. If holding cost is too big, then an additional time may not increase the value of the available inventory. Since adding a holding cost function would be like adding a different event operator to the model that should also preserve the desired property, Proposition 7 should be studied carefully.

3.5 Conclusion

In this chapter, we studied a discrete-time single product inventory control problem with several demand classes. This was a fairly general model, first introduced in [19]. We developed a dynamic programming model for this partial admission problem aiming to optimize the expected revenue of the seller who is making acceptance-rejection decisions. Since our objective was to understand the behavior of the optimal policies when system parameters change, we used event-based dynamic programming as an approach to prove the structure of the models and optimal policies. To use this approach, we first defined certain event operators to represent the events occurring in our model. Then we used these individual operators to show the desired properties by first showing that the operators preserve monotonicity, concavity, and supermodularity properties. Some basic structural properties of the optimal policy of our model were established in [19]. We employed the same model, but we contributed to the model by examining the effects of varying system parameters on the optimal policy. We established that for any class-i, the optimal threshold value that determines the maximum number of inventory on hand below which it is optimal to reject the whole class-i batch is non-decreasing in time. We showed that the optimal thresholds are non-decreasing in the arrival rates of demands. We investigated that the thresholds are non-decreasing in revenues of demand classes, and the total expected revenue is convex in revenue of a demand class and also in a batch size. Shortly, our work was a guideline for predicting the response of the optimal policy to the changes in random parameters and varying parameters of the problem.

Chapter 4

PRICING OF PERISHABLE PRODUCTS WITH RANDOM QUALITY

4.1 Introduction

Pricing decisions are critical to improve the expected revenues of firms, and should reflect the level of inventory, time remaining till the end of the selling season, and the information about customers' valuations of the products. Learning about customers' preferences and projecting their reservation prices for a product with an uncertain quality is not so difficult by analyzing the sales data, collected by advances in information technologies, of similar products or the same products with similar quality. In other words, a seller can monitor the past sales and demand, and can update prices approximately to enhance the expected revenue even if he cannot perceive the exact quality of the available products.

For instance, a seller might have a pool of single-type inventories that differ in their product qualities. In this case, if the seller cannot perceive the exact quality of each product or cannot discriminate between the products (due to the difficulty of quality inspection or the lack of expert knowledge about the product quality), he might use a subjective and maybe an unrealistic probability distribution for the quality of products. Then, he updates this distribution as learning occurs about the customers' valuations to products by monitoring the past sales data and demands. As a result, a rational seller in this situation, must understand the choices that customers make when facing a product with a random quality q, and must determine a fair single-price for the products with different qualities considering the number of available inventory on hand and the distribution of the product quality. Motivated by this pricing problem of the seller, this chapter studies the dynamic and static pricing strategies of a seller who sells given initial inventories of perishable products with random quality over a finite selling season.

The models associated with the static and dynamic pricing strategies are referred as the *"No information update scenario"* and the *"Information update scenario"*, respectively. We formulate a dynamic optimization problem for the information update scenario in which the seller updates the distribution of the product quality once at a point in time during the selling season, and accordingly adjusts the sales price of the unsold products for the remaining part of the season. We solve a static pricing problem in the no information update scenario in which the seller is not allowed to update the distribution of the product quality, and so the sales price.

For the information update scenario, consider a seller who sells a set of single-type products with random quality, and in the beginning of the selling season, has no or a little perception about the quality of available products. Although each of the products is with a random quality, they are priced at the same. The problem is formulated in a finite time horizon considering a single selling season. The potential demand during the selling season is assumed to be a Poisson random variable with mean λ. The seller updates the price only once during the selling season. The time at which the seller makes the pricing decision can be seen as the beginning of a new period. In other words, the seller splits the season into two periods where the first period is called the *"regular period"*, and the second period that begins with updating the sales price and lasts till the end of the season is called the *"clearance period"*. Thus, the seller updates the price only once at the start of the clearance period and once selected, this price is not changed until the end of the selling season.

One of the key assumptions of the model is that, from the customers' point of view, inventory is considered as being ordered on the basis of its quality from the highest to the lowest. Therefore, the unsold items are identified as low-quality products compared to the ones which are sold in the regular period. We assume that customers are homogeneous which means that the customers have the same willingness to pay to the products with the same quality, and that customers who purchase early prefer high-quality products. Therefore, higher quality of products are sold out at the beginning of the regular period, and then unsold products with lower quality remain for the clearance period. Another critical part of the problem is that after monitoring sales in the regular period, and learning about the customers' sensitivity to product quality, the seller updates its quality distribution in order to price the remaining inventories which can be identified as low-quality products. The seller's goal is to follow a dynamic pricing policy that brings the maximum expected total revenue over the selling season. This task is challenging since the seller needs to update the cumulative distribution function of the product quality. In addition, he also needs to consider the way this subjective distribution function affects his ability to set a realistic price to the remaining products for the remaining part of the season.

The content of this scenario is relevant to clearance pricing. Clearance pricing is an important issue for sellers who sell perishable products only during the selling season [27]. Sellers prefer not to change prices frequently during a specified selling season since fewer price changes are easier to implement and help them to avoid the cost of relabelling products [16]. Therefore, a dynamic pricing policy in which the clearance price is set once during the selling season and left unchanged for the remainder of the clearance period is reasonable.

In the no information update scenario, all the modelling assumptions except updating the distribution of the product quality during the selling season, are valid. However, this is the assumption that makes the model a dynamic pricing problem. So, when we replaced this assumption with the assumption of no updating the distribution of the product quality, the seller's problem turns to be a static pricing problem. As in all static pricing problems, we formulate a model for this scenario in which the seller sets the sales price in the beginning of the selling season, and once he sets the price, he is not allowed to change it during the season. The optimal price set in the no information update scenario, and the optimal revenue that corresponds to this price are used to assess the benefits of the information updating to the seller in the numerical results.

The organization of this chapter is as follows: in section 4.2, as a general case the information update scenario is presented under random demand and unlimited demand settings, respectively. Then, the subsequent sections are given as special cases of the information update scenario. The no information update scenario is studied in section 4.3. As in the information update scenario in section 4.2, the model presented here is also studied under random demand and unlimited demand settings. In addition, in section 4.3, we formulate the total expected revenue in two different ways: 1) applying the binomial model, 2) applying the order statistics. Next, in section 4.4, we investigate some analytical results assuming that the seller has a single product. In section 4.5, we determine the optimal pricing policy for the case in which the product quality is constant. In section 4.6, we illustrate our problem with numerical examples, compare the performance of the information update scenario with no information update scenario and investigate the effects of problem parameters on the optimal policy. Finally, 4.7 sums up the chapter.

4.2 General Case: Information Update Scenario

This section provides a detailed description of the general setting, assumptions on the model and the motivation behind the modelling assumptions of the information update scenarios under a random demand model and an unlimited demand model, respectively. After giving the problem description, we formulate the seller's problem as a two-period dynamic optimization model. The formulations of Order Statistics are applied to construct the purchase probabilities in the pricing model in this thesis. A short review of Order Statistics can be found in the Appendix A.

Consider the seller in the beginning of the season with a subjective cumulative distribution function for the product quality to represent the seller's lack of information about the customers' valuations to products. After starting with a subjective cumulative distribution for quality of initial inventories on hand, the seller continuously observes the sales, and learns about customers' valuations to the quality of products. Then, he adjusts the current distribution of product quality according to the number of the remaining inventory, so updates the optimal price at some point in time to enhance the total expected revenue for the rest of the selling season. We assume that all customers have an identical reservation price (customers are homogeneous).

In the model presented here, our main concern is not how much inventory we should start with at the beginning of the selling season, but instead, our focus is on determining optimal prices for the regular and the clearance period. However, later in section 4.6, we investigate the optimal starting inventory level through numerical examples.

4.2.1 Random Demand

In this subsection, our problem is formulated in a finite time horizon considering a single selling season where the potential demand is assumed to be a Poisson process with rate λ. As mentioned before, we study the case in which the seller updates the price once at a point in time during the entire season. In other words, we begin with a regular period and after the point in time at which the decision maker updates the price, we continue with a clearance period. Since the customers have the same willingness to pay and the customers who purchase early prefer high-quality products, if the inventory is considered as being ordered on the basis of its quality from the highest to the lowest, the unsold items left to the clearance period are low-quality products compared to the ones which are sold in the regular period. Reasonably, products of higher quality are sold out at the higher initial price in the regular period, so the seller sets a discount price in the clearance period.

In the model studied in this section, the quality of inventory is assumed to be distributed uniformly over the interval $[\underline{q}, \bar{q}]$. Since we order the inventory according to the product quality from the highest to lowest, the quality ratings can also be ordered as $q_x > q_{x-1} > \ldots > q_1$. Moreover, remember the modelling assumptions that products are assumed to be with random quality and the seller cannot identify the exact quality of each item on hand. In other words, the customers' valuation of a product is not perceived by the seller. Because of the initially unknown quality of the products, the seller begins the problem with a subjective cumulative distribution function associated with quality. Learning about a customer's purchase behavior occurs as the seller monitors the customer's response to its pricing decision made in the beginning of the selling season. By observing sales in the regular period, the seller gets a signal about the quality of its inventory and updates his quality distribution function at the beginning of the clearance period according to the number of the remaining inventory. Refining the characterization of the quality distribution leads the seller to manipulate his price. As stated before the seller's goal is to follow a dynamic pricing policy that brings the maximum expected total revenue over the selling season.

Notations and Assumptions

In the problem studied in this section, it is assumed that we begin with a regular period and after the time point at which the decision maker makes the pricing decision, we continue with a clearance period. For convenience, we use the subscript "r" for the *regular period* and the subscript "c" for the *clearance period*, hereafter.

Then, let p_r and p_c denote the net revenue associated with selling a unit of product in the regular and the clearance period, respectively. Let ρ be the relative length of the regular period (length of the regular period \ length of the season), and be determined exogenously. Since demand during the season arrives according to a Poisson process with rate λ, D_r is a Poisson random variable with mean $\lambda_r = \rho\lambda$ and D_c is a Poisson random variable with mean $\lambda_c = (1-\rho)\lambda$. As a result of randomness in demand, excess demand may be observed and an unsatisfied demand is lost.

Because of the initially unknown quality of the products, the seller begins the problem with a subjective cumulative distribution function associated with the quality of items. Learning about a customer's purchase behavior occurs as the seller monitors the customer's response to its pricing decisions. By observing sales in the regular period, the seller gets a signal about the quality of its inventory and updates its quality distribution function according to the unsold inventory that we have at the beginning of the clearance period. Refining the characterization of the quality distribution may lead the seller to manipulate its price, since we know by the modelling assumption that early customers buy the higher quality products in the store and the unsold items are the ones with lower quality.

We should note that not all customer arrivals end up with a purchase. If the quality of a product is sufficient with respect to its sales price, then an arriving demand can be called as a *"sale"*. Here, "sufficient quality" purports a quality which makes the utility of an arriving customer non-negative, and it will be using frequently also in the subsequent sections. To model consumer choice in this problem, we use similar approach presented in [1], but in our model customers are homogenous. A consumer's sensitivity to quality is assumed to be parameterized by a scalar θ, and θ is a constant. Then, when a customer buys a product with quality q, at price p, his utility is represented by the following function:

$$U(p,q) = \theta q - p$$

Considering the product quality, our goal is to set a price that makes consumers' utility functions non-negative. In other words, setting a price for which a customer arrival is converted into a purchase is our primary purpose. Hence, the below inequality must be satisfied for a sale to take place:

$$U(p,q) = \theta q - p \geq 0 \tag{4.1}$$

$$q \geq \frac{p}{\theta} \tag{4.2}$$

To summarize, the modelling assumptions of this problem are motivated by the above discussion and are as follows:

1. The problem is formulated in a finite time horizon.
2. Inventories perish once the sales season is over.
3. Demand is modelled as a Poisson process, i.e., the demand distributions are assumed to be known.
4. In a given period, products may vary in quality, but not in price.
5. Customers place the same valuations on the same quality of product, and prefer purchasing the highest quality-product first.
6. Seller updates distribution of the product quality at the end of the regular period; hence, makes the pricing decision only once during the selling season.
7. Seller dynamically adjusts the sales price according the number of the remaining inventories.

Problem Formulation and Description

We will now formulate a general version of the problem analyzed in subsequent sections. Let $P(S = s_r \mid \pi_x^r)$ denote the probability of selling s_r units of the product in the regular period given that the seller starts the regular period with x units of inventory and the quality distribution function $\pi_x^r(.)$ associated to it. Similarly, let $P(S = s_c \mid \pi_{x,s_r}^c)$ denote the probability of selling s_c units of the product in the clearance period given that the seller continues with an updated cumulative distribution function of quality, π_{x,s_r}^c, when he has sold s_r of x units of the inventory in the regular period. Given that the process starts with a (subjective) cumulative distribution function π_x^r and x units of inventory, the *expected maximum revenue-to-go* is denoted by $v(\pi_x)$ and is explicitly written as follows:

$$v(\pi_x) = \sum_{s_r=0}^{x} P(S = s_r|\pi_x^r) \left[s_r p_r(x) + \sum_{s_C=0}^{x-s_r} P(S = s_c|\pi_{x,s_r}^c) \; s_c \; p_c(x, s_r) \right] \tag{4.3}$$

Here, the value function $v(\pi_x)$ says that the seller starts the selling season with x inventory. The products are of random quality and the seller uses a subjective cumulative distribution function, π_x, associated with the quality of the inventory on hand. Moreover, the above value function calculates the total expected profit. Firstly, the seller considers the probability of selling $s_r \in \{0, 1, \ldots, x\}$ units of good if he assumes to have π_x as the cdf of product quality of the inventory on hand and accordingly charge $p_r(x)$ as the initial price. In this case, he expects to earn $s_r p_r(x)$ units of dollar. However, the seller is aware of that in this regular period, with price $p_r(x)$ and cdf π_x he may not sell all the inventory in stock. If this is the case, he needs to update the cdf of product quality according to the number of unsold inventory. In fact, considering the unsold inventory as low-quality products, the seller needs to use an updated distribution of the product quality, π_{x,s_r}^c, which is defined in the interval with a reduced upper bound. In order to sell the unsold items and to maximize the total expected revenue, he reduces the price from $p_r(x)$ to $p_c(x, s_r)$. By doing so, in the clearance period, he expects to earn $s_c \; p_c(x, s_r)$ units of dollar under the probability of selling $s_c \in \{0, 1, \ldots, x - s_r\}$ items.

Note that while we are describing the construction of the total expected profit above, we use the probability of selling a number of products given that the cdf of the quality is predicted by the seller according to the quantity of sales and inventory on hand. It is a fact that a sale of a product occurs if and only if there is at least one customer demanding that product, and that a potential demand is converted into a purchase if and only if there is at least one product in the store with a sufficient quality that generates a non-negative utility when purchasing it. Hence, the potential sales probabilities are modelled by considering both the number of demand arrivals to the store and the number of items with sufficient quality, and it is seen that the number of sales is equal to the minimum of these two quantities. Formally, let Q_r and Q_c be the number of items with sufficient quality in the regular and the clearance period, respectively. Note that the number of products with sufficient quality in the regular period, Q_r, is determined out of x items, but the number of products with sufficient quality in the clearance period, Q_c, is determined out of $x - s_r$ items by successively checking the products, which are ordered on the basis of the quality from the highest to the lowest. Then, the quantity of sales in the regular period, s_r is said to be equal to $min(D_r, Q_r)$ where D_r is the number of potential demand arriving in this period. Similarly, the quantity of sales in the clearance period, s_c is said to be equal to $min(D_c, Q_c)$ where D_c is the number of potential demand arriving in this period.

Therefore, the probability of selling s_r units in the regular period given that π_x^r is assumed to be the cdf of the product quality, $P(S = s_r|\pi_x^r)$, is composed of two independent parts, $\forall\ s_r \in \{0, 1, \ldots, x\}$.

$$P(S = s_r|\pi_x^r) = \begin{cases} \sum_{z_r=s_r}^{x} P(D_r = s_r)P(Q_r = z_r) & \text{if } \min(D_r, Q_r) = D_r \\ P(D_r > s_r)P(Q_r = s_r) & \text{otherwise} \end{cases}$$

Hence, the probability of selling s_r units of product in the regular period can be expressed as follows $\forall\ s_r \in \{0, 1, \ldots, x\}$:

$$P(S = s_r|\pi_x^r) = \sum_{z_r=s_r}^{x} P(D_r = s_r)P(Q_r = z_r) + P(D_r > s_r)P(Q_r = s_r)$$

Similarly, the probability of selling s_c units of good in the clearance period given that π^c_{x,s_r} is assumed to be the cdf of the product quality when s_r units of x items were sold in the regular period, $P(S = s_c|\pi^c_{x,s_r})$, is composed of two independent parts, $\forall\ s_c \in \{0, 1, \ldots, x - s_r\}$, and depending on the sales in the regular period $P(S = s_c|\pi^c_{x,s_r})$, can be expressed in two different ways.

1. If $s_r = min(Q_r, D_r) = D_r$

$$P(S = s_c|\pi^c_{x,s_r}) = \begin{cases} \sum_{z_c=s_c}^{x-s_r} P(D_c = s_c)P(Q_c = z_c \mid Q_r \geq s_r, D_r = s_r) & \text{if } \min(D_c, Q_c) = D_c \\ P(D_c > s_c)P(Q_c = s_c \mid Q_r \geq s_r, D_r = s_r) & \text{otherwise} \end{cases}$$

2. If $s_r = min(Q_r, D_r) = Q_r$

$$P(S = s_c|\pi^c_{x,s_r}) = \begin{cases} \sum_{z_c=s_c}^{x-s_r} P(D_c = s_c)P(Q_c = z_c \mid Q_r = s_r, D_r > s_r) & \text{if } \min(D_c, Q_c) = D_c \\ P(D_c > s_c)P(Q_c = s_c \mid Q_r = s_r, D_r > s_r) & \text{otherwise} \end{cases}$$

Hence, the probability of selling s_c units of product in the clearance period when the cdf of product quality is π^c_{x,s_r} can be expressed for each of the above two cases as follows:
$\forall\ s_c \in \{0, 1, \ldots, x - s_r\}$,

1. If $s_r = min(Q_r, D_r) = D_r$

$$\begin{aligned} P(S = s_c|\pi^c_{x,s_r}) &= \sum_{z_c=s_c}^{x-s_r} P(D_c = s_c)P(Q_c = z_c \mid Q_r \geq s_r, D_r = s_r) \\ &+ P(D_c > s_c)P(Q_c = s_c \mid Q_r \geq s_r, D_r = s_r) \end{aligned}$$

2. If $s_r = min(Q_r, D_r) = D_r$

$$\begin{aligned} P(S = s_c|\pi^c_{x,s_r}) &= \sum_{z_c=s_c}^{x-s_r} P(D_c = s_c)P(Q_c = z_c \mid Q_r = s_r, D_r > s_r) \\ &+ P(D_c > s_c)P(Q_c = s_c \mid Q_r = s_r, D_r > s_r) \end{aligned}$$

Period Name	Case	Purchase Probability
Regular		
s_r	$min(D_r, Q_r) = D_r$	$\sum_{z_r=s_r}^{x} P(D_r = s_r)P(Q_r = z_r)$
Clearance		
s_c	$min(D_c, Q_c) = D_c$	$\sum_{s_c=0}^{x-s_r} \sum_{z_c=s_c}^{x-s_r} P(D_c = s_c)P(Q_c = z_c \mid Q_r \geq s_r, D_r = s_r)$
s_c	$min(D_c, Q_c) = Q_c$	$\sum_{s_c=0}^{x-s_r} P(D_c > s_c)P(Q_c = s_c \mid Q_r \geq s_r, D_r = s_r)$
Regular		
s_r	$min(D_r, Q_r) = Q_r$	$P(D_r > s_r)P(Q_r = s_r)$
Clearance		
s_c	$min(D_c, Q_c) = D_c$	$\sum_{s_c=0}^{x-s_r} \sum_{z_c=s_c}^{x-s_r} P(D_c = s_c)P(Q_c = z_c \mid Q_r = s_r, D_r > s_r)$
s_c	$min(D_c, Q_c) = Q_c$	$\sum_{s_c=0}^{x-s_r} P(D_c > s_c)P(Q_c = s_c \mid Q_r = s_r, D_r > s_r)$

Table 4.1: Summary of the purchase probabilities' construction

Table 4.1 sums up the construction of the purchase probabilities explained above. The table can be easily used to find the formulations of the purchase probabilities for all cases. For example, the upper part of the table is designed to find the probability of selling s_r items in the regular period when $D_r = min(D_r, Q_r)$, and if this is the case, the third line of the upper part of the table gives the probability of selling s_c items in the clearance period when the number of products with sufficient quality is less than the number of arriving demand in this period, i.e. when $Q_c = min(D_c, Q_c)$.

At this point one can easily follow that the total expected revenue that has been expressed in the following form so far,

$$
\begin{aligned}
v(\pi_x) &= \sum_{s_r=0}^{x} P(S = s_r|\pi_x^r) \left[s_r p_r(x) + \sum_{s_c=0}^{x-s_r} P(S = s_c|\pi_{x,s_r}^c)\ s_c\ p_c(x, s_r) \right] &(4.4)\\
&= \sum_{s_r=0}^{x} P(S = s_r|\pi_x^r) s_r p_r(x) + \sum_{s_r=0}^{x} \left[P(S = s_r|\pi_x^r) \sum_{s_c=0}^{x-s_r} P(S = s_c|\pi_{x,s_r}^c)\ s_c\ p_c(x, s_r) \right] &(4.5)
\end{aligned}
$$

can be written considering the two different forms of sales in the regular and the clearance period individually as follows:

$$= \sum_{s_r=0}^{x} \quad \sum_{z_r=s_r}^{x} P(D_r = s_r)P(Q_r = z_r)\Bigg) s_r p_r(x) \tag{4.6}$$

$$+ \sum_{s_r=0}^{x} \Bigg[\quad \sum_{z_r=s_r}^{x} P(D_r = s_r)P(Q_r = z_r)\Bigg) \sum_{s_c=0}^{x-s_r} \sum_{z_c=s_c}^{x-s_r} P(D_c = s_c)P(Q_c = z_c \mid Q_r = z_r, D_r = s_r) s_c p_c(x, s_r)\Bigg] \tag{4.7}$$

$$+ \sum_{s_r=0}^{x} \Bigg[\quad \sum_{z_r=s_r}^{x} P(D_r = s_r)P(Q_r = z_r)\Bigg) \sum_{s_c=0}^{x-s_r} P(D_c > s_c)P(Q_c = s_c \mid Q_r = z_r, D_r = s_r) s_c p_c(x, s_r)\Bigg] \tag{4.8}$$

$$+ \sum_{s_r=0}^{x} P(D_r > s_r)P(Q_r = s_r) s_r p_r(x) \tag{4.9}$$

$$+ \sum_{s_r=0}^{x} P(D_r > s_r)P(Q_r = s_r) \left[\sum_{s_c=0}^{x-s_r} \sum_{z_c=s_c}^{x-s_r} P(D_c = s_c)P(Q_c = z_c \mid Q_r = s_r, D_r > s_r) s_c p_c(x, s_r)\right] \tag{4.10}$$

$$+ \sum_{s_r=0}^{x} P(D_r > s_r)P(Q_r = s_r) \left[\sum_{s_c=0}^{x-s_r} P(D_c > s_c)P(Q_c = s_c \mid Q_r = s_r, D_r > s_r) s_c p_c(x, s_r)\right] \tag{4.11}$$

In the above formulation of the value function, we have just used the forementioned purchase probabilities for the regular and the clearance period which are constructed one by one comparing the volume of the arriving demand with the quantity of the remaining products with sufficient quality. In order to make the above formulation of the value function more comprehensible, note that the first three terms are for the case in which the number of sales in the regular period is equal to the number of arriving demand. In other words, the sum of the first three terms gives us the total expected profit of the seller if the seller has $min(D_r, Q_r) = D_r$. On the other hand, the last three terms contributes to the total expected profit if the seller has a number of demands in the regular period that is bigger than the number of products with sufficient quality as to the price p_r, i.e., if $min(D_r, Q_r) = Q_r$. Note that in both cases, we sum the terms in (4.6) and (4.9) over s_r where $s_r \in \{0, 1, \ldots, x\}$ to calculate the total expected profit of the seller in the regular period. As mentioned in the construction of the purchase probabilities for the clearance period above, a similar comparison of the demand volume and the quantity of products with sufficient quality depending on p_c is used to calculate the total expected profit of the seller in the clearance period. According to the minimum of the number of demand and the number qualified products, the terms (4.7), (4.8), (4.10), (4.11) contributes the seller's expected profit in that period. The terms (4.7) and (4.10) are written for the case where $min(D_c, Q_c) = D_c$, and

(4.8), (4.11) are for the case $min(D_c, Q_c) = Q_c$. Since the seller updates the product quality distribution considering the number of sales in the regular period and determine a clearance price according to the new characterization of that distribution, for each $s_r \in \{0, 1, \ldots, x\}$, the seller's optimal clearance price is going to be different. This is why we use $p_c(x, s_r)$ notation for the clearance price in the formulation of the value function and why we sum all the terms in (4.7), (4.8), (4.10), (4.11) over s_r to find the expected profit of the seller in the clearance period.

At this point, in order to be able to calculate the value of the total expected profit of the seller, we need to find the following probabilities:

- $P(D_r = s_r)$, $P(D_c = s_c)$

  The probability that the number of arriving demand is equal to the number of sales in the regular and the clearance period, respectively.

- $P(D_r > s_r)$, $P(D_c > s_c)$

  The probability that the number arriving demand is bigger than the number of sales in the corresponding period.

- $P(Q_r = z_r), \ \forall z_r \in \{s_r, s_r + 1, \ldots, x\}$

  The probability that the number of the products with sufficient quality as to the price set in the regular period is equal to z_r where $z_r \in \{s_r, s_r + 1, \ldots, x\}$

- $P(Q_c = z_c \mid Q_r \geq s_r, D_r = s_r), \ \forall z_c \in \{s_c, s_c + 1, \ldots, x\}$

  The probability that the number of products with sufficient quality is equal to z_c in the clearance period given that while there were at least s_r units of good with sufficient quality in the regular period, only s_r units of demand had arrived. (i.e. The probability that the number of products with sufficient quality is equal to z_c in the clearance period given that there were unsold products with sufficient quality in the regular period carried to the clearance period.)

- $P(Q_c = z_c \mid Q_r = s_r, D_r > s_r), \forall z_c \in \{s_c, s_c + 1, \ldots, x\}$

  The probability that the number of products with sufficient quality is equal to z_c in the clearance period given that while there were only s_r units of good with sufficient quality in the regular period, the number of arriving demand was bigger than s_r. (i.e. The probability that the number of products with sufficient quality is equal to z_c in the clearance period given that all of the products with sufficient quality in the regular period were sold in that period.)

The first two of the above probabilities are easy to obtain since the demands, D_r and D_c, are Poisson random variables with means $\lambda_r = \rho\lambda$ and $\lambda_c = (1-\rho)\lambda$, respectively.

To calculate the remaining probabilities above, we will make use of the order statistics. Consider the probability that the number of the products with sufficient quality as a function of the price set in the regular period is equal to z_r, then we need to follow the below steps to calculate the value of this probability.

$$
\begin{aligned}
P(Q_r = z_r) &= P\left(q_{x-z_r+1} \geq \frac{p_r(x)}{\theta}, q_{x-z_r} < \frac{p_r(x)}{\theta}\right) \qquad \forall z_r \in \{s_r, s_r+1, \ldots, x\} \\
&= \left(1 - \frac{P\left(q_{x-z_r+1} \leq \frac{p_r(x)}{\theta}, q_{x-z_r} < \frac{p_r(x)}{\theta}\right)}{P\left(q_{x-z_r} < \frac{p_r(x)}{\theta}\right)}\right) P\left(q_{x-z_r} < \frac{p_r(x)}{\theta}\right) \\
&= P\left(q_{x-z_r} < \frac{p_r(x)}{\theta}\right) - P\left(q_{x-z_r+1} \leq \frac{p_r(x)}{\theta}, q_{x-z_r} < \frac{p_r(x)}{\theta}\right) \\
&= \sum_{k=x-z_r}^{x} \binom{x}{k} \pi_x^k\left(\frac{p_r(x)}{\theta}\right)\left[1 - \pi_x\left(\frac{p_r(x)}{\theta}\right)\right]^{x-k} \\
&- \sum_{k=x-z_r+1}^{x} \binom{x}{k} \pi_x^k\left(\frac{p_r(x)}{\theta}\right)\left[1 - \pi_x\left(\frac{p_r(x)}{\theta}\right)\right]^{x-k} \\
&= \binom{x}{z_r} \pi_x^{x-z_r}\left(\frac{p_r(x)}{\theta}\right)\left[1 - \pi_x\left(\frac{p_r(x)}{\theta}\right)\right]^{z_r}
\end{aligned}
$$

Now, as far as the probability of being with z_c units of good with sufficient quality in the clearance period given that $s_r \leq Q_r = z_r$ units of good were sold in the regular period and $Q_r - s_r$ items were carried to the clearance period is concerned, we could formulate and then calculate it case by case.

In its general form this probability can be written as,

$$
P(Q_c = z_c \mid Q_r = z_r, S = s_r) = \frac{P\left(q_{x-s_r-z_c+1} \geq \frac{p_c(x,s_r)}{\theta}, q_{x-s_r-z_c} < \frac{p_c(x,s_r)}{\theta}, q_{x-z_r+1} \geq \frac{p_r(x)}{\theta}, q_{x-z_r} < \frac{p_r(x)}{\theta}\right)}{P\left(q_{x-z_r+1} \geq \frac{p_r(x)}{\theta}, q_{x-z_r} < \frac{p_r(x)}{\theta}\right)} \tag{4.12}
$$

where $S = \min(Q_r, D_r)$; in other words, it is the number of sales in the regular period.

First of all, note that we need to know the value of the above probability for all s_r such that $s_r \in \{0, 1, \ldots, x\}$, for all $Q_r = z_r$ where $z_r \in \{s_r, s_r + 1, \ldots, x\}$, and for all z_c where $z_c \in \{0, 1, \ldots, x - s_r\}$, since we would like to know the probability of being with any number of products with sufficient quality in the clearance period for all possible sales quantities observed in the regular period and for any number of inventory which had to be carried to the clearance period although they might have been sold in the regular period unless there was a scarce demand. Therefore, we need to reformulate it for all cases which creates special conditions. We present the formulations of these special cases with their implications in the following:

For all $s_r \in \{0, 1, \ldots, x\}$, for all $Q_r = z_r \in \{s_r, s_r+1, \ldots, x\}$, and for all $z_c \in \{0, 1, \ldots, x-s_r\}$, *Case1:* If $z_c < z_r - s_r$

$$P(Q_c = z_c \mid Q_r = z_r, S = s_r) = 0$$

Consider that the seller sold s_r items in the regular period just because of scarcity of demand. It is to say that although the number of products with sufficient quality was more than s_r, there was only s_r units of demand arriving to the store in the regular period. Then to the next period, it is left $x - s_r$ units of unsold items such that the first $z_r - s_r$ of them were already with sufficient qualities in the previous period. So, in any case, those first $z_r - s_r$ products provide a non-negative utility when the seller reduces the price from $p_r(x)$ to $p_c(x, s_r)$ in the clearance period in order to be able to sell the remaining items. Thus, it is impossible to have less than $z_r - s_r$ units of good satisfying the utility constraint in the clearance period, and it is reasonable that the above probability is equal to zero if $z_c < z_r - s_r$. Hence, from now on we only consider the cases where $z_c \geq z_r - s_r$.

Case 2: If $s_r = 0, z_r = 0$, and $z_c = 0$

$$P(Q_c = z_c \mid Q_r = z_r, S = s_r) = \frac{P\left(q_x < \frac{p_c(x,s_r)}{\theta}\right)}{P\left(q_x < \frac{p_r(x)}{\theta}\right)} = \frac{[\pi_x^r(q_x)]^x}{[\pi_{x,s_r}^c(q_x)]^x}$$

Consider that the seller could not sell anything in the regular period since there was no item with sufficient quality, then the probability that there isn't any good with sufficient quality despite the reduced price set in the clearance period is as given above, since no item satisfies the utility constraint of the customers during the selling season.

Case 3: If $s_r = 0, z_r = 0$, and $z_c \neq 0$ and also $z_c \neq x$

$$\begin{aligned} P(Q_c = z_c \mid Q_r = z_r, S = s_r) &= \frac{P\left(q_{x-z_c+1} \geq \frac{p_c(x,s_r)}{\theta}, q_{x-z_c} < \frac{p_c(x,s_r)}{\theta}, q_x < \frac{p_r(x)}{\theta}\right)}{P\left(q_x < \frac{p_r(x)}{\theta}\right)} \\ &= \frac{P\left(q_{x-z_c} < \frac{p_c(x,s_r)}{\theta}, q_x < \frac{p_r(x)}{\theta}\right) - P\left(q_{x-z_c+1} < \frac{p_c(x,s_r)}{\theta}, q_x < \frac{p_r(x)}{\theta}\right)}{P\left(q_x < \frac{p_r(x)}{\theta}\right)} \end{aligned}$$

Consider that the seller could not sell anything in the regular period since there was no item with sufficient quality, then the probability of being with any number of product which is not equal to zero or to x is given above, since in the above formulation there are $z_c \in \{1, 2, \ldots, x-1\}$ units of good satisfying the customers' utility at price $p_c(x, s_r)$ in the clearance period, but none of the starting inventories satisfy the utility constraint of the customers in the regular period at price $p_r(x)$.

Case 4: If $s_r = 0, z_r = 0$, and $z_c = x$

$$\begin{aligned} P(Q_c = z_c \mid Q_r = z_r, S = s_r) &= \frac{P\left(q_1 \geq \frac{p_c(x,s_r)}{\theta}, q_x < \frac{p_r(x)}{\theta}\right)}{P\left(q_x < \frac{p_r(x)}{\theta}\right)} \\ &= \frac{P\left(q_x < \frac{p_r(x)}{\theta}\right) - P\left(q_1 < \frac{p_c(x,s_r)}{\theta}, q_x < \frac{p_r(x)}{\theta}\right)}{P\left(q_x < \frac{p_r(x)}{\theta}\right)} \end{aligned}$$

Consider that the seller could not sell anything in the regular period since there was no item with sufficient quality, then the probability that all of the products satisfy a non-negative utility *to the customers at the reduced price $p_c(x, s_r)$ in the clearance period is as given above, since in the above formulation even the lowest-quality product satisfies the utility constraint at price $p_c(x, s_r)$, while none of the starting inventories are qualified enough to be sold at price $p_r(x)$.*

Case 5: If $s_r \neq 0, z_r \neq 0$, and $z_c = z_r - s_r$

$$\begin{aligned} P(Q_c = z_c \mid Q_r = z_r, S = s_r) &= \frac{P\left(q_{x-z_r+1} \geq \frac{p_r(x)}{\theta}, q_{x-s_r-z_c} < \frac{p_c(x,s_r)}{\theta}\right)}{P\left(q_{x-z_r+1} \geq \frac{p_r(x)}{\theta}, q_{x-z_r} < \frac{p_r(x)}{\theta}\right)} \\ &= \frac{P\left(q_{x-s_r-z_c} < \frac{p_c(x,s_r)}{\theta}\right) - P\left(q_{x-z_r+1} < \frac{p_r(x)}{\theta}, q_{x-s_r-z_c} < \frac{p_c(x,s_r)}{\theta}\right)}{P\left(q_{x-z_r} < \frac{p_r(x)}{\theta}\right) - P\left(q_{x-z_r+1} < \frac{p_r(x)}{\theta}\right)} \end{aligned}$$

Consider that the seller had z_r items with sufficient quality in the regular period, and he sold $s_r \leq z_r$ units of them. Moreover, consider that the seller sets a price $p_c(x, s_r)$ in the clearance period provided that only the unsold items carried from the regular period, but nothing else are with sufficient quality at that price. Then, the above formula gives the probability of the situation in which the seller has just $z_c = z_r - s_r$ units of good with sufficient quality in the clearance period.

Case 6: If $s_r \neq 0, z_r \neq 0$, and $z_c > z_r - s_r$, but $z_c \neq x - s_r$

$$P(Q_c = z_c \mid Q_r = z_r, S = s_r) = \frac{P\left(q_{x-s_r-z_c+1} \geq \frac{p_c(x,s_r)}{\theta}, q_{x-s_r-z_c} < \frac{p_c(x,s_r)}{\theta}, q_{x-z_r+1} \geq \frac{p_r(x)}{\theta}, q_{x-z_r} < \frac{p_r(x)}{\theta}\right)}{P\left(q_{x-z_r+1} \geq \frac{p_r(x)}{\theta}, q_{x-z_r} < \frac{p_r(x)}{\theta}\right)}$$

$$= \frac{P\left(q_{x-s_r-z_c} < \frac{p_c(x,s_r)}{\theta}, q_{x-z_r} < \frac{p_r(x)}{\theta}\right) - P\left(q_{x-s_r-z_c+1} < \frac{p_c(x,s_r)}{\theta}, q_{x-z_r} < \frac{p_r(x)}{\theta}\right)}{P\left(q_{x-z_r} < \frac{p_r(x)}{\theta}\right) - P\left(q_{x-z_r+1} < \frac{p_r(x)}{\theta}\right)}$$

$$+ \frac{P\left(q_{x-s_r-z_c} < \frac{p_c(x,s_r)}{\theta}, q_{x-z_r+1} < \frac{p_r(x)}{\theta}\right) - P\left(q_{x-s_r-z_c+1} < \frac{p_c(x,s_r)}{\theta}, q_{x-z_r+1} < \frac{p_r(x)}{\theta}\right)}{P\left(q_{x-z_r} < \frac{p_r(x)}{\theta}\right) - P\left(q_{x-z_r+1} < \frac{p_r(x)}{\theta}\right)}$$

Consider that the seller had z_r units of good with sufficient in the regular period, and he sold $s_r \leq z_r$ units of them. Moreover, consider that reducing the price from $p_r(x)$ to $p_c(x, s_r)$ in the clearance period is not enough to make ***all*** *the remaining products satisfy the customers' utility. Then the probability of this situation is as given above where the seller has less than $x - s_r$ units of good with sufficient quality in the clearance period.*

Case 7: If $s_r \neq 0, z_r \neq 0$, and $z_c > z_r - s_r$, $z_c = x - s_r$

$$P(Q_c = z_c \mid Q_r = z_r, S = s_r) = \frac{P\left(q_1 \geq \frac{p_c(x,s_r)}{\theta}, q_{x-z_r+1} \geq \frac{p_r(x)}{\theta}, q_{x-z_r} < \frac{p_r(x)}{\theta}\right)}{P\left(q_{x-z_r+1} \geq \frac{p_r(x)}{\theta}, q_{x-z_r} < \frac{p_r(x)}{\theta}\right)}$$

$$= \frac{P\left(q_{x-z_r} < \frac{p_r(x)}{\theta}\right) - P\left(q_{x-z_r+1} < \frac{p_r(x)}{\theta}\right)}{P\left(q_{x-z_r} < \frac{p_r(x)}{\theta}\right) - P\left(q_{x-z_r+1} < \frac{p_r(x)}{\theta}\right)}$$

$$- \frac{P\left(q_1 < \frac{p_c(x,s_r)}{\theta}, q_{x-z_r} < \frac{p_r(x)}{\theta}\right) - P\left(q_1 < \frac{p_c(x,s_r)}{\theta}, q_{x-z_r+1} < \frac{p_r(x)}{\theta}\right)}{P\left(q_{x-z_r} < \frac{p_r(x)}{\theta}\right) - P\left(q_{x-z_r+1} < \frac{p_r(x)}{\theta}\right)}$$

Consider that the seller had z_r units of good with sufficient quality in the regular period, and he sold $s_r \leq z_r$ units of them. Moreover, consider that reducing the price from $p_r(x)$ to $p_c(x, s_r)$ in the clearance period makes ***all*** *the remaining products satisfy the customers' utility. Then the probability of this situation is as given above where the seller has exactly $x - s_r$ units of good with sufficient quality in the clearance period.*

As can be seen we have seven special cases where we restructure the general form of the formula that gives us the probability of being with z_c units of good with sufficient quality in the clearance period when the price $p_c(x, s_r)$ is set by the seller provided that s_r units of z_r items with sufficient quality in the regular period were purchased by the customers at price $p_r(x)$. Here, note that we did not give a formula that explicitly depends on the volume of the potential demand, D_r arriving in the regular period for this probability. However, it should be noted that the number of sales in the regular period appears as $S = s_r$ in the formulas and we know that $S = s_r$ is the minimum of Q_r and D_r. Hence, as long as you know the number of sales in the regular period and the current case: $s_r = \min(Q_r, D_r) = D_r$, or $s_r = \min(Q_r, D_r) = Q_r$, the above formulas can be taken and applied for all cases regardless of the volume of the potential demand. In the next subsection, we will study the scenario with unlimited demand, so we will remove the demand restriction from the model.

4.2.2 No Demand Restriction Scenario

In this subsection, we investigate the problem with information update scenario in its simpler form with no demand restriction. Consider that we have an unlimited potential demand among which some them are converted into a purchase and the seller updates the distribution of the product quality once during the selling season. If the seller is left with unsold items at the end of the regular period or even at the end of the clearance period, this is not because of the deficiency of demand for these products in the corresponding period, but because the quality of them does not satisfy the utility constraint of the arriving demands at the predetermined price. As mentioned before, a customer's decides to purchase a product with quality q_i at price p if $q_i \geq \frac{p}{\theta}$, otherwise he does not purchase it. Thus, in this subsection, the only criteria for selling a product with quality q_i is the following:

The seller sells the product with quality q_i if

$$\begin{array}{ll} q_i \geq \frac{p_r(x)}{\theta} & \textit{in the regular period} \\ q_i \geq \frac{p_c(x,s_r)}{\theta} & \textit{in the clearance period} \end{array}$$

Therefore, the probability of selling s_r units of good in the regular period given that π_x^r is assumed to be the cdf of the product quality, $P(S = s_r|\pi_x^r)$, $\forall\ s_r \in \{0, 1, \ldots, x\}$, is the following

$$P(S = s_r|\pi_x^r) = P(Q_r = s_r)$$

Similarly, the probability of selling s_c units of good in the clearance period given that π_{x,s_r}^c is assumed to be the cdf of the product quality, $P(S = s_c|\pi_{x,s_r}^c)$, $\forall\ s_c \in \{0, 1, \ldots, x - s_r\}$, is the following

$$P(S = s_c|\pi_{x,s_r}^c) = P(Q_c = s_c \mid Q_r = s_r)$$

Hence, the total expected revenue to go till the end of the selling season can be written as follows:

$$\begin{aligned} v(\pi_x) &= \sum_{s_r=0}^{x} P(S = s_r|\pi_x^r) \left[s_r p_r(x) + \sum_{s_c=0}^{x-s_r} P(S = s_c|\pi_{x,s_r}^c)\ s_c\ p_c(x, s_r) \right] \\ &= \sum_{s_r=0}^{x} P(Q_r = s_r) s_r p_r(x) + \sum_{s_r=0}^{x} \left[P(Q_r = s_r) \sum_{s_c=0}^{x-s_r} P(Q_c = s_c \mid Q_r = s_r)\ s_c\ p_c(x, s_r) \right] \end{aligned}$$

where $P(Q_r = s_r)$ is as given in the previous subsection. Note that putting s_c instead of z_c in the general form of $P(Q_c = z_c \mid Q_r = z_r, S = s_r)$ given in 4.12 generates

$$P(Q_c = s_c \mid Q_r = s_r) = \frac{P\left(q_{x-s_r-s_c+1} \geq \frac{p_c(x,s_r)}{\theta}, q_{x-s_r-s_c} < \frac{p_c(x,s_r)}{\theta}, q_{x-s_r+1} \geq \frac{p_r(x)}{\theta}, q_{x-s_r} < \frac{p_r(x)}{\theta}\right)}{P\left(q_{x-s_r+1} \geq \frac{p_r(x)}{\theta}, q_{x-s_r} < \frac{p_r(x)}{\theta}\right)}$$

and the closed form of this conditional probability has been shown before in the subsection 4.2.1. Since, all purchase probabilities can be calculated, the seller can maximize his total expected profit easily under no demand restriction scenario by solving the above dynamic optimization problem.

Remark 10 *This problem might be studied under simpler demand assumptions. For instance, instead of stochastic demand model, we might have used observable demand. Since the number of demand observed in each period was known, construction of the sales probabilities would be simpler.*

Remark 11 *The random choice scenario where customers choose randomly among qualified items can be studied for the information update scenario. However, construction of the purchase probabilities will be very difficult. The reason of this difficulty will be explained later in section 4.3.*

In the subsequent sections, in order to gain additional insight about the problem that we have presented so far, we will provide 3 special cases of the model under some simplifying assumptions.

4.3 Special Case 1: No Information Update Scenario

In this subsection, our problem is formulated in a finite time horizon, in a single selling season where the seller does not update the distribution of the product quality throughout the entire season. Since the seller does not update the distribution of the product quality as consumers' valuations on products become to reveal, there is no clearance period in this scenario. The seller begins the selling season with a subjective distribution function for the product quality and then never comes with an idea of updating the distribution or price by monitoring the consumers' purchase behaviors. Hence, there is a single selling period in which the seller sets a single price.

All the problem assumptions, except the assumption of updating the distribution of the product quality, so the sales price at a point in time during the selling season, are valid in this scenario. This section is studied in order to determine the optimal price and optimal expected revenue of the seller in no information scenario, thus to generate the marginal benefit of information updating to the seller. This comparison is made numerically in section 4.6. This section also studies the no information update scenario under two different demand models as in the information update scenario. The no information update scenario is presented with the demand models by order of unlimited demand model and random demand model.

4.3.1 No Demand Restriction

In this subsection, we employ the problem with no information update in its simplest form: with no demand restriction. Assume that the seller does not update the distribution of the product quality during the selling season and we have unlimited potential demand. As far as the choice between buying and not buying is concerned, a customer's decision could be explained as follows. A customer purchases a product with quality $q(i)$ at price p among the set of products with random quality if $q(i) \geq \frac{p}{\theta}$, otherwise he does not purchase it. This is true for all customers since remember that they all have the same ranking over the same product. Then, it is clear that the product with quality $q(i)$ remains unsold if $q(i) \leq \frac{p}{\theta}$.

The expected revenue to go derived from this scenario can be read as follows:

$$v(\pi_x) = \sum_{s=0}^{x} P(S = s|\pi_x)\ s\ p(x) \tag{4.13}$$

The only unknown in the above expression of the value function is the purchase probabilities of the customers given that the quality of x units inventory on hand is distributed with a cdf π_x. Before applying "Order Statistics", which is suggested in the previous section, to calculate the purchase probabilities, for this scenario, we consider an alternative way to find those probabilities. The approach we use to find the purchase probabilities is explained in the following and is named as *"Binomial Model"*, since the purchase probabilities correspond to classical binomial probabilities in this scenario after making particular assumptions.

4.3.2 Applying Binomial Model

Consider that customers do not see the highest quality product in the set of products at once when they arrive the store. They select products randomly from the set and question the utility function for those products. Since we are looking for the number of products that provide non-negative utility to a customer when being sold at price p (i.e. the number of success in a sequence of x independent quality levels), the purchase probabilities in the model, $P(S = s|\pi_x)$, can be thought as a classical *"binomial probability"* in this scenario.

Now, if $q(i)$ denotes the quality of randomly selected i^{th} item, the probability for which a potential demand is converted into a purchase corresponds to $P(q(i) > \frac{p}{\theta})$. Since the trials are independent and identical, and also $P(q(i) > \frac{p}{\theta})$ is constant from trial to trial, the purchase probability is in fact a classical success probability in a binomial experiment.

Being motivated by the above discussion, the total expected revenue can be written as follows:

$$v(\pi_x) = p\ P(q(1) > \frac{p}{\theta}) + p\ P(q(2) > \frac{p}{\theta}) + \ldots + p\ P(q(x_0) > \frac{p}{\theta}) \tag{4.14}$$

Moreover, the seller's optimal price can be determined in closed form regardless of the initial level of inventory. The following proposition verifies that there is a closed form of the optimal price for this scenario which is independent of x.

Proposition 12 *The optimal price for no information update scenario with no demand restriction is independent of the starting level of inventory.*

Proof: As the product quality is identically and independently distributed with π_x, the above expression 4.14 turns out to be

$$v(\pi_x) = x\ p\ (1 - \pi_x(\frac{p}{\theta})) \tag{4.15}$$

Since the goal of the seller is to maximize the total expected revenue, he needs an optimal price p^* for his inventory with random quality waiting to be sold till the end of the season. We can find a closed form for an optimal price p^* by just taking the derivative of equation 4.15 with *respect to p and then setting it equal to zero.*

$$\begin{aligned} \frac{\partial(x\ p - x\ p\ \pi_x(\frac{p}{\theta}))}{\partial p} &= 0 \\ x - x\ \pi_x(\frac{p}{\theta}) - x\ \frac{p}{\theta}\ \pi_x'(\frac{p}{\theta}) &= 0 \end{aligned}$$

Here, we assume that π_x is continuous and differentiable. Also note that the second derivative is negative, so it satisfies the concavity condition. Hence, we can write a closed form for the optimal price p^* as follows:

$$p^* = \frac{(1 - \pi_x(\frac{p}{\theta}))\ \theta}{\pi_x'(\frac{p}{\theta})} \tag{4.16}$$

Proposition 12 refers to the following corollary about the optimal price set for the entire selling season where the seller does not update the distribution of the product quality and observes an unlimited demand.

Corollary 13 *The optimal price depends only on the distribution of product quality π_x if the seller does not update the distribution of the product quality and observes unlimited demand during the selling season.*

4.3.3 Applying Order Statistics

Definitely, we get the same solution for the optimal price if the purchase probabilities are calculated by applying order statistics. To see this, consider that the products are sorted according to their quality levels as $q_x > q_{x-1} > \ldots > q_1$ before an arrival of a customer to the store. Therefore, when a customer arrives to the store, he will see the best quality product with quality index q_x and since the higher quality is always preferable if a single price is charged to all of the products, the customer will purchase it definitely. Selling the products successively in quality index order will continue until the quality of a product does not provide a non-negative utility to a customer at price p.

Since, under an infinite-demand assumption, selling i products corresponds to having exactly i items that make customers' utility function non-negative at price p, the probability of selling i items can be written as follows:

$$P(S = i|\pi_x) = P(q_{x-i+1} \geq \frac{p}{\theta}, q_{x-i} < \frac{p}{\theta}) \tag{4.17}$$

where

$$\begin{aligned}
P(q_{x-i+1} \geq \frac{p}{\theta}, q_{x-i} < \frac{p}{\theta}) &= \left(1 - \frac{P(q_{x-i+1} \leq \frac{p}{\theta}, q_{x-i} < \frac{p}{\theta})}{P(q_{x-i} < \frac{p}{\theta})}\right) P(q_{x-i} < \frac{p}{\theta}) \\
&= P(q_{x-i} < \frac{p}{\theta}) - P(q_{x-i+1} \leq \frac{p}{\theta}, q_{x-i} < \frac{p}{\theta}) \\
&= \sum_{k=x-i}^{x} \begin{pmatrix} x \\ k \end{pmatrix} \pi_x^k \left(\frac{p}{\theta}\right) \left[1 - \pi_x \left(\frac{p}{\theta}\right)\right]^{x-k} \\
&- \sum_{k=x-i+1}^{x} \begin{pmatrix} x \\ k \end{pmatrix} \pi_x^k \left(\frac{p}{\theta}\right) \left[1 - \pi_x \left(\frac{p}{\theta}\right)\right]^{x-k} \\
&= \begin{pmatrix} x \\ i \end{pmatrix} \pi_x^{x-i} \left(\frac{p}{\theta}\right) \left[1 - \pi_x \left(\frac{p}{\theta}\right)\right]^{i}
\end{aligned}$$

Note that the last term in the above formulation is the binomial probability. In other words, it is the probability for having exactly $(x - i)$ ratings from the rating set $\{q_x, q_{x-1}, \ldots, q_1\}$ which are less than or equal to $\frac{p}{\theta}$. Hence, when the total expected revenue to go is expressed by using the above formulation for the purchase probabilities, we have the following:

$$v(\pi_x) = \sum_{i=0}^{x} \begin{pmatrix} x \\ i \end{pmatrix} \pi_x^{x-i} \left(\frac{p}{\theta}\right) \left[1 - \pi_x \left(\frac{p}{\theta}\right)\right]^{i} i \; p \tag{4.18}$$

This is the expected value and it is equal to $x(1 - \pi_x(\frac{p}{\theta}))p$. As expected this is the same as what we have stated for the value function in equation 4.15. Hence, the optimal price that maximizes the total expected profit found by using the order statistics approach will be the same as the optimal price given in equation 4.16.

4.3.4 Random Demand

As in section 4.3.1, the seller does not update the distribution of the product quality during the selling season. However, here we replaced the unrestricted-demand assumption by the assumption of random demand and restructure the model in the context of random demand scenario.

The main difference from the previous scenario occurs in the origination of a *"sale"*. In the previous scenario with unrestricted demand, since there is always an unlimited demand to the products, the necessary and sufficient condition for a *sale* is just making available the product which is qualified enough to make a customer's utility non-negative at the predetermined price p. Hence, the number of sales in that scenario is said to be equal to the number of items with a sufficient product quality as to its price. Since it is guaranteed that all the products with sufficient quality are going to be sold, an arriving demand can be considered as being indifferent to the product quality as long as it satisfies a non-negative utility. Hence, the product quality order does not play a role in determining the purchase probabilities and this is why we could treat the previous scenario as a classical binomial experiment.

However, in this section, a " sale" is observed if there is at least one customer arrival to the store. In other words, beside the product quality, volume of the potential demand is now added to the problem as a determining factor of a " sale". If we continue with the classical binomial model, the probability of selling $i \leq x$ units of good is going to be underestimated because of the discussion below. But, first of all, note that in this scenario we observe a random demand D where $D \leq x$; otherwise, we had observed an unlimited demand as in the previous section. Furthermore, the reader should detect that $D \leq x$ units of potential demand is converted into a purchase of $i \leq x$ units of good either if there are at least i units of inventory in the store with a product quality that satisfies a non-negative utility to a customer at the predetermined price p and $D = i$, or if there are precisely i units of inventory with sufficient quality as to its price in the store and $D > i$.

Recall that in the previous section the probability for which a potential demand is converted into a purchase corresponds to $P(q(i) > \frac{p}{\theta})$ where $q(i)$ denotes the quality of randomly selected item i. Here, similarly, if we had continued with the classical binomial model, because of the demand restriction warmly added to the problem, the purchase probability of the item i with quality $q(i)$ could be expressed by the product of the probability of that item satisfies a non-negative utility at price p, $P(q(i) > \frac{p}{\theta})$, and the probability of a demand arrival precisely for that product. The reader can find the details on *"Binomial Model"* for this scenario in the Appendix B.

Since the expression of the value function for this scenario derived by using binomial probabilities (see Appendix B) is not so tractable, it is preferable to express the value function by applying the order statistics to find the purchase probabilities. Remember that the inventory is ordered according to the product quality from the highest to the lowest as $q_x > q_{x-1} > \ldots > q_i > q_{i-1} > \ldots > q_1$. Now for a random demand D, the probability of selling $i \leq x$ items during the selling season where the seller does not update the distribution of the product quality corresponds to the sum of the two following probabilities:

1.
$$P(D = i)\left(P(q_1 \geq \frac{p}{\theta}) + \sum_{j=2}^{x-i+1} P(q_j \geq \frac{p}{\theta}, q_{j-1} < \frac{p}{\theta})\right)$$

While observing $i \leq x$ units of demand, the Probability of making available at least the best i products that satisfy customers' utility.

2.
$$P(D > i)\left(P(q_{x-i+1} \geq \frac{p}{\theta}, q_{x-i} < \frac{p}{\theta})\right)$$

While observing more than i units of demand, the probability of making available precisely the best i products that satisfy customers' utility.

In order to make things more comprehensible, consider the probability of making available precisely the best i products that satisfy customers' utility, $P(q_{x-i+1} \geq \frac{p}{\theta}, q_{x-i} < \frac{p}{\theta})$. Since the products are ordered according to their quality from the highest to the lowest, we can ensure that the i^{th} product quality provides a non-negative utility by $q_{x-i+1} \geq \frac{p}{\theta}$, and that the $(i+1)^{th}$ product does not satisfy a customer's utility by $q_{x-i} < \frac{p}{\theta}$. Hence, the probability of making available precisely the best i products that satisfy customers' utility can be represented by the joint probability, $P(q_{x-i+1} \geq \frac{p}{\theta}, q_{x-i} < \frac{p}{\theta})$.

Now, for a random demand scenario with no information update, we can formulate the purchasing probability during the selling season $\forall i \in \{0, 1, \ldots, x\}$ as follows:

$$\begin{aligned} P(S = i \mid \pi_x) &= P(D = i)\left(P(q_1 \geq \frac{p}{\theta}) + \sum_{j=2}^{x-i+1} P(q_j \geq \frac{p}{\theta}, q_{j-1} < \frac{p}{\theta})\right) \\ &+ P(D > i)\left(P(q_{x-i+1} \geq \frac{p}{\theta}, q_{x-i} < \frac{p}{\theta})\right) \end{aligned}$$

To calculate the value of the above joint probabilities, we need follow the steps below.

$$
\begin{aligned}
P\left(q_{x-i+1} \geq \frac{p}{\theta}, q_{x-i} < \frac{p}{\theta}\right) &= P\left(q_{x-i+1} \geq \frac{p}{\theta} \mid q_{x-i} < \frac{p}{\theta}\right) P\left(q_{x-i} < \frac{p}{\theta}\right) \\
&= \left(1 - \frac{P(q_{x-i+1} \leq \frac{p}{\theta}, q_{x-i} < \frac{p}{\theta})}{P(q_{x-i} < \frac{p}{\theta})}\right) P(q_{x-i} < \frac{p}{\theta}) \\
&= P(q_{x-i} < \frac{p}{\theta}) - P(q_{x-i+1} \leq \frac{p}{\theta}, q_{x-i} < \frac{p}{\theta}) \\
&= \sum_{k=x-i}^{x} \begin{pmatrix} x \\ k \end{pmatrix} \pi_x^k \left(\frac{p}{\theta}\right) \left[1 - \pi_x\left(\frac{p}{\theta}\right)\right]^{x-k} \\
&- \sum_{k=x-i+1}^{x} \begin{pmatrix} x \\ k \end{pmatrix} \pi_x^k \left(\frac{p}{\theta}\right) \left[1 - \pi_x\left(\frac{p}{\theta}\right)\right]^{x-k} \\
&= \begin{pmatrix} x \\ i \end{pmatrix} \pi_x^{x-i} \left(\frac{p}{\theta}\right) \left[1 - \pi_x\left(\frac{p}{\theta}\right)\right]^{i}
\end{aligned}
$$

Now, the seller knows how to calculate the purchase probabilities, so the total expected profit if he does not update the distribution of the product quality during the selling season.

Remark 14 *Note that the binomial model is not applicable to construct the purchase probabilities if the seller updates the distribution of the product quality. In such a case, the seller is considered as if he had two periods to go till the end of the selling season. So, it is important which products of what quality are left to be sold in the second period. Therefore, we have to turn back to our main assumption that the products are ordered according to their quality from the highest to the lowest and customers arriving to the store prefer first the product with the highest quality. Hence, the order statistics approach has to be applied to construct the purchase probabilities in the information update scenario.*

4.4 Special Case 2: Single Item Scenario

In order to illustrate the problems of pricing perishable products with random quality described in the information update and the no information update scenarios, this section studies the simplest case where the seller has a single product to sell. Without loss of generality, we assume that this product is with a quality q such that q is uniformly distributed between 0 and 1, and investigate the following analytical results.

Proposition 15 *For the no information update scenario with unlimited demand, the optimal price that gives the maximum value of the total expected profit is $\theta/2$.*

Proof: Assume that the seller does not update the distribution of the product quality during the selling season and we have an unlimited potential demand among which some of them are converted into a purchase. Recall that the probability of selling a product is equal to the probability of making a customer's utility non-negative, i.e. $P(\theta q - p \geq 0) = Pr(q \geq \frac{p}{\theta})$. So, the probability of selling the product is $1 - \frac{p}{\theta}$. Since, the seller wants to set a price maximizes his profit, he needs to find an optimal price, p^* that maximizes the total expected profit which is equal to $(1 - \frac{p}{\theta})p$. Concavity of the total expected profit in p verifies that the optimal price that gives the maximum value of the total expected profit is $\theta/2$.

Proposition 16 *For the information update scenario with unlimited demand, the optimal regular price is $\frac{2\theta}{3}$, and the optimal clearance price is $\frac{p_r}{2}$.*

Proof: Assume that the seller updates the distribution of the product quality once during the selling season and we have an unlimited potential demand among of which some of them are converted into a purchase. Note that in this case, as the seller gets the signal from the customers about their valuations on the products, he knows more about the quality of the products on hand and so decreases the sales price at one point point in time during the selling season. Therefore, in this case we have a regular period with a regular price p_r and a clearance period with a markdown price p_c. In the regular period, if the seller sets a price p_r then he sells the product with probability $(1 - \frac{p_r}{\theta})$. The discounting decision means the seller knows that the product quality is not distributed between 0 and 1 anymore, but takes its value from the interval $[0, \frac{p_r}{\theta}]$.

Therefore, after updating the characteristics of distribution of the product quality, the probability of selling an item in the clearance period corresponds to $(1-\frac{p_c}{p_r})$*. Hence, the expected profit of the clearance period that needs be maximized is equal to* $(1-\frac{p_c}{p_r})p_c$*, and simply taking the derivative with respect to* p_c *gives us the optimal clearance price as* $p_c^* = \frac{p_r}{2}$*. Note that the overall expected profit for the selling season can be written as* $v(\pi_1) = (1-\frac{p_r}{\theta})p_r + \frac{p_r}{\theta}(1-\frac{p_c}{p_r})p_c$*. After substituting* $p_c^* = \frac{p_r}{2}$ *instead of* p_c *in* TP *and taking its derivative with respect to* p_r *gives us the optimal regular price that the seller charges to his product at the beginning of the selling season, and it is* $p_r^* = \frac{2\theta}{3}$*.*

Now, we can compare the expected total profit of the seller in the no information update scenario with the expected total profit that he earns in the information update scenario when he observes unlimited demand in both cases. Thus, we can see the ratio of the performance of the static pricing to the dynamic pricing. The total expected profit which is to be maximized in the no information update scenario is $(1-\frac{p_r}{\theta})p_r$, and the optimal regular price $p_r^* = \frac{\theta}{2}$ gives its maximum value. Hence, the seller's optimal profit is $\frac{\theta}{4}$. On the other hand, the total expected profit of the season in which the seller updates the distribution of the product quality is $TP = (1-\frac{p_r}{\theta})p_r + \frac{p_r}{\theta}(1-\frac{p_c}{p_r})p_c$, and by the proposition 15, the optimal prices of the regular period and the clearance period that maximizes TP are $p_r^* = \frac{2\theta}{3}$ and $p_c^* = \frac{\theta}{3}$, respectively. Hence, the optimal total profit for the information update scenario turns out to be $\frac{\theta}{3}$. The following corollary sums up the implication derived from this comparison.

Corollary 17 *Under unlimited demand model, there is the following ratio between the total expected profits of the no information update (static pricing)and the information update (dynamic pricing) scenarios.*

$$\frac{\textit{Expected profit of static pricing}}{\textit{Expected profit of dynamic pricing}} = \frac{3}{4} \tag{4.19}$$

This ratio is used a benchmark statistics in our numerical results in section 4.6.

Proposition 18 *For the no information scenario with random demand, the optimal price that* *maximizes the total expected profit is,* $\frac{\theta}{2}$ *which is the same price as the optimal price for that scenario with unlimited demand.*

Proof: As in the proof of proposition 15, assume that the seller does not update the distribution of the product quality, but differently, random demand is observed during the selling season. In this case, a sale occurs if there is at least one customer arriving to the store and the product quality makes the customer utility non-negative at the predetermined price. Thus, the purchase probability can be written as $P(D > 0)Pr(q \geq \frac{p}{\theta})$*. Here, note that since the demand is independent of price, the optimal price is* $\frac{\theta}{2}$ *as in proposition 15.*

Proposition 19 *For the information update scenario with random demand the total expected revenue which is to be maximized is*

$$v(\pi_1) = p\ (1 - \frac{p_r}{\theta})\ p_r + \left((1-p)(1 - \frac{p_r}{\theta}) + \frac{p_r}{\theta}\right)\left(p\ \frac{\frac{p_r - p_c}{\theta}}{\theta - p_r} + p(1-p)\right)\ p_c$$

Proof: Assume that the seller updates once the distribution of the product quality during the selling season as he observes the sales thus far. Random demand assumption is still valid, so a sale occurs if and only if there is both a demand for the product and quality of that product that satisfies the customer's utility. In its general form, the total expected profit to be maximized can be expressed as follows:

$$v(\pi_1) = P(S = 1|\pi_1^r)\ p_r + P(S = 0|\pi_1^r)\ P(S = 1|\pi_{1,0}^c)\ p_c$$

Note that since there is a single product, a single customer arrival to the store is enough for the demand side of the problem. Considering the potential single demand that may be observed during the selling season as a Bernoulli random variable and the product quality as a standard uniform random variable, the purchase probabilities in the above expression are written explicitly *as follows:*

$$
\begin{aligned}
P(S=0|\pi_1^r) &= P(D_r=0)P(q \geq \frac{p_r}{\theta}) + P(D_r \geq 0)P(q < \frac{p_r}{\theta}) \\
&= (1-p)(1-\frac{p_r}{\theta}) + \frac{p_r}{\theta}
\end{aligned}
$$

$$
\begin{aligned}
P(S=1|\pi_1^r) &= P(D_r > 0)P(q \geq \frac{p_r}{\theta}) \\
&= p\ (1-\frac{p_r}{\theta})
\end{aligned}
$$

$$
\begin{aligned}
P(S=1|\pi_{1,0}^c) &= P(D_r=0)P(D_c>0)\frac{P(q \geq \frac{p_c}{\theta}, q \geq \frac{p_r}{\theta})}{P(q \geq \frac{p_r}{\theta})} \\
&+ P(D_r=0)P(D_c>0)\frac{P(q \geq \frac{p_c}{\theta}, q < \frac{p_r}{\theta})}{P(q < \frac{p_r}{\theta})} \\
&+ P(D_r>0)P(D_c>0)\frac{P(q \geq \frac{p_c}{\theta}, q < \frac{p_r}{\theta})}{P(q < \frac{p_r}{\theta})} \\
&= P(D_r \geq 0)P(D_c>0)\frac{P(\frac{p_c}{\theta} < q \leq \frac{p_r}{\theta})}{P(q < \frac{p_r}{\theta})} + P(D_r=0)P(D_c>0) \\
&= p\ \frac{\frac{p_r-p_c}{\theta}}{\theta - p_r} + p(1-p)
\end{aligned}
$$

Here, p is used to represent the success probability in the Bernoulli distribution of demand and note that both p_r and p_c should be less or equal to the quality parameter of the customer, θ, in order to be able to sell the product.

Now, the above value function can be written again but now using the purchase probabilities in their explicit form as follows:

$$
v(\pi_1) = p\ (1-\frac{p_r}{\theta})\ p_r + \left((1-p)(1-\frac{p_r}{\theta}) + \frac{p_r}{\theta}\right)\left(p\ \frac{\frac{p_r-p_c}{\theta}}{\theta - p_r} + p(1-p)\right)\ p_c
$$

.

4.5 Special Case 3: Constant Quality Scenario

This section investigates the special case where the qualities of inventories are known in advance and all product qualities are assumed to be the same. Since the inventories do not vary in quality, and all of the customers have the same willingness to pay till the end of the selling season, there would be no need to update the price once it was set at the beginning of the season according to the product quality.

As the quality ratings of the products are constant, an arriving demand is converted into a sale if the customer's utility when purchasing a product with quality q at price p is non-negative. In this set up, since all products have the same quality ratings, the seller and even a customer is indifferent to which products to be purchased. Therefore, instead of a purchase probability of an item, $P(q > \frac{p}{\theta})$, we can use a binary variable η such that

$$\eta = \begin{cases} 1 & \text{if} \quad q > \frac{p}{\theta} \\ 0 & \text{otherwise} \end{cases}$$

Now, if the volume of incoming demand D exceeds the available inventory x and at price p if the quality of a product is sufficient to make a customer's utility non-negative, the seller expects to earn $P(D > x)xp$ where η is taken as 1. Besides, while the product quality q satisfies a customer's utility function at price p, if the number of arriving demands D is less than or equal to the number of available inventory x, this time the seller expects to earn $\sum_{j=0}^{x} P(D = j)jp$. Hence, expected total profit when there are x units of inventory available with product quality q can be written as follows:

$$v(x) = \begin{cases} P(D > x)xp + \sum_{j=0}^{x} P(D = j)jp & \text{if} \quad \eta = 1 \\ 0 & \text{otherwise} \end{cases} \tag{4.20}$$

Proposition 20 *The optimal price is independent of the initial inventory level if the seller has inventories with constant quality which is known in advance.*

Proof: Assume that the seller has inventories with constant quality which is known in advance, then it is clear that as long as the product quality q satisfies the utility of customers, he will certainly sell as much as the arriving demand. Therefore, the seller charges the maximum price $p = q\theta$ which gives a non-negative utility; and, it is basically the optimal price that can be charged for all levels of inventory.

Proposition 20 leads to the following corollary.

Corollary 21 *Keeping the customers' sensitivity to quality constant, and multiplying the quality of the inventory by a positive scalar a will lead the seller set a new optimal price $p^*_{new} = ap^*$*

4.6 Numerical Results

In this section, we will discuss the information update scenario and its comparison with no information update scenario further with the help of numerical examples. The objective of the illustrations is to offer some insight into how the optimal prices and the optimal profit of the models with and without information update could respond to the changes of the following system parameters:

- x: starting inventory level
- $x - s$: remaining inventory level in the clearance period
- ρ: coefficient that determines the starting time of the clearance period
- θ: customers' sensitivity to quality
- h: holding cost

The presented models were programmed in Matlab Version 7.0. Several numerical experiments were constructed to run the program with different sets of input to verify the models and the numerical results. The optimal procedure was validated by a separate simulation study.

The most challenging part of this work was the calculations of the sales probabilities for each decision period, since the probabilities were dependent on two unknowns: volume of the arriving demand and the quality of inventories. We used the purchase probability formulations given in the previous sections.

We constructed 5 problems which only differ in the product quality distributions. We start with a uniform distribution between 0 and 10, and then continue with uniform distributions supported by the intervals with different lower bounds. In other words, each time, we contract the intervals of the distributions. The distributions used in the example problems are abbreviated U(0,10), U(2,10), U(4,10), U(6,10), U(8,10).

In the subsection 4.6.1, we compare the performance of the information update scenario with no information update scenario for the base case problem. Then in the subsection 4.6.2, we analyze the effect of initial inventory level x on the net revenue and the optimal prices for both no information update scenario and the information update scenario, respectively. Next, the subsection 4.6.3 follows to see the effect of remaining inventory level on the optimal clearance price. In addition, the comparison of the optimal clearance price with the optimal regular price for the same amount of inventory is discussed in this subsection. The effect of the coefficient ρ that determines the starting time of the clearance period is studied in the following subsection 4.6.5, and the starting time of the clearance period to maximize the expected profit is found. The subsection 4.6.5 is added to validate our modelling assumption about θ which is the customers' sensitivity to the product quality. Finally, in the subsection 4.6.6, we extend the base case problem by including a factor of holding cost, and the subsection 4.6.7 is aiming to give a brief summary of our numerical results.

4.6.1 Performance comparison between the information update and the no information update scenarios

Assume that there is no holding cost for the seller, and the value of θ, the parameter denoting the customers' sensitivity to product quality, is 1. Demand process is considered as a Poisson process with rate 8, and it is also assumed that there is a unit purchase cost of 2. Then it

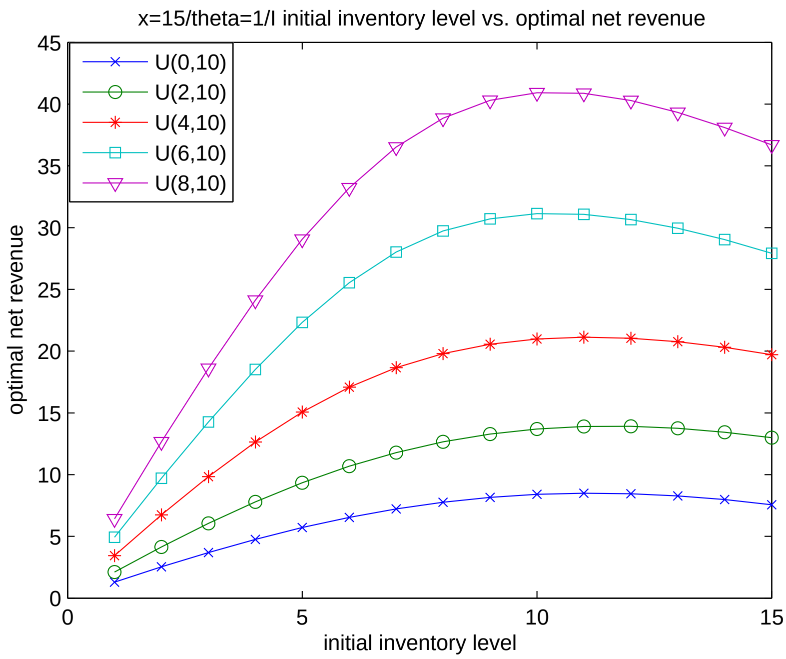

Figure 4.1: Initial inventory level vs. net revenue in the information update scenario

may be meaningful to choose the starting inventory level as 10 units, since as seen in figure 4.1, the inventory level that maximizes the net revenue of the seller for all five problems is around that value. Thus, the base case to be considered consists of 10 units of starting inventories with random product qualities distributed uniformly between 0 and 10. Table 4.2 shows the the

x=10	*No Information Update*				
theta=1	q ~ U(0,10)	q ~ U(2,10)	q ~ U(4,10)	q ~ U(6,10)	q ~ U(8,10)
Total Expected Revenue	*23.73*	*28.76*	*35.95*	*46.35*	*60.55*
Optimal Regular Price	*5.29*	*5.47*	*5.82*	*6.52*	*8.01*

Table 4.2: Optimal expected profit and optimal price

optimal price and the optimal total expected revenue of the seller for each of five problems in the no information update scenario with 10 units of initial inventory. Recall that the customers' valuations to the product quality is parameterized by a constant θ, and θ is taken as 1 in each of the problem. In order to verify our calculations given in table 4.2, consider Proposition 12 which says that the optimal price for the no information update scenario with an unlimited demand is $p^* = \frac{\theta}{2}$ when the product quality ranges uniformly in the interval [0,1]. Now, consider the case where the distribution of the quality is a Uniform distribution between 0 and 10. In this case, by Proposition 12, the optimal price will be $p^* = \frac{10\theta}{2} = 5$ It is seen from table 4.2, the optimal price for the no information update scenario with random demand is determined as 5.29 for $U(0,10)$ which is bigger than the optimal price of the scenario with an unlimited demand. This is an intuitive result in the presence of demand restriction. In addition, table 4.2 shows that as the products tend to be more qualified, i.e. as the distribution of the product quality is defined in smaller intervals, the seller gains more total expected profit by charging a higher optimal price for the entire selling season.

For each problem, we generate the optimal total expected revenue, and determine the optimal

x=10	*Information Update*				
theta=1	q ~ U(0,10)	q ~ U(2,10)	q ~ U(4,10)	q ~ U(6,10)	q ~ U(8,10)
Total Expected Revenue	28.40	33.69	40.98	51.13	62.45
Optimal Regular Price	6.89	6.98	7.17	7.65	8.62

Table 4.3: Optimal expected profit and optimal regular price

regular price for the information update scenario as in table 4.3. Similar to the no information update scenario, if the qualities of the products close up, then it is sure that the seller will set a higher regular price for his products in the beginning of the selling season. For instance, if the qualities of the products begin to take values from the continuous interval [8,10] rather than the interval [0,10], then the seller will set a higher price for his products in the regular period. Contracting the distribution interval towards 10 means that the seller is assuming that his products are at least at the quality level 8. The optimal profit corresponds to the optimal regular prices is also increasing as the quality of products increases.

Another interesting result is the following. The seller earns the total expected profit of 35.95 by applying the static pricing strategy when he orders products within the quality interval of $[4, 10]$ from the manufacturer. However, if he had used the dynamic pricing strategy, he would have earned even more total expected revenue, 36.14, by ordering products within the quality interval of $[2.75, 10]$.

The above discussion implies the following. The seller may accept to pay more per unit to the manufacturer, and may have her make the quality control of the products and supply him the ones with quality ratings at least 4. In such a case, the seller will not monitor the sales during the selling season to learn about the quality of his products, and will not update the distribution of the product quality. Thus, he will apply the static pricing. On the other hand, the seller may not want to have the manufacturer make the quality control. He may purchase the products at lower quality ratings at a lower price, and may start with a subjective quality distribution $U[2.75, 10]$. He prefers to learn about the quality of the products by observing the customers' response to the price which implies that the quality controllers are the customers. In this case, although the seller pays less to the manufacturer for the products, he earns the same profit by applying the dynamic pricing strategy.

marginal benefit of using information update rather than no information update (%)				
q ~ U(0,10)	q ~ U(2,10)	q ~ U(4,10)	q ~ U(6,10)	q ~ U(8,10)
19.68	*17.17*	*14.00*	*10.32*	*3.14*

Table 4.4: Marginal benefit of information updating

Considering the difference between the performance of the information update scenario and the performance of the no information update scenario, table 4.4 can be seen as a brief summary of the comparison of table 4.2 and table 4.3 which definitely says that the marginal benefit of updating the quality distribution of the remaining products at some point in time during the selling season, and charging a clearance price for the remaining part of the season is not negligible. When the quality of the products ranges in a relatively wide intervals; for instance, when the quality of the products takes its value from the interval [0,10], then using the information update scenario instead of the no information update scenario provides almost 20% increase in the total expected revenue of the seller. This value of the benefit to the seller decreases as the interval of the product quality contracts. This is an intuitive result, since if the gap between the quality of products gets smaller and smaller and approaches to 0, then the problem converges to the constant quality scenario presented in section 4.5, and recall that in the that case it optimal to use static pricing strategy for the entire selling season.

To compare the performance of the static pricing with the dynamic pricing further, we use the benchmark statistics given in Corollary 17. Recall that under unlimited demand model, if the seller starts with a single product with quality ranking $q\ U(0, 1)$, there is the ratio $3/4$ between

Ratio of the performance of the static pricing to the dynamic pricing					
x=1 q ~ U(0,1) Benchmark Statistics = 0.75	**x=1** **q~U(0,10)**	**x=1** **q~U(2,10)**	**x=1** **q~U(4,10)**	**x=1** **q~U(6,10)**	**x=1** **q~U(8,10)**
	0.75	0.75	0.76	0.86	0.94
	x=10 **q~U(0,10)**	**x=10** **q~U(2,10)**	**x=10** **q~U(4,10)**	**x=10** **q~U(6,10)**	**x=10** **q~U(8,10)**
	0.84	0.85	0.88	0.91	0.97

Table 4.5: Performance of the static pricing compared to the dynamic pricing

the total expected profits of the no information update (static pricing)and the information update (dynamic pricing) scenarios. As seen in the table 4.5, when we contract the quality interval for the single item case, this ratio tends to approach to 1. This means that when the product quality of the inventories close up, the performance of the static pricing approaches to the performance of the dynamic pricing. In addition, in the table 4.5, the benchmark statistics is seen as the lower bound of the performance ratio. Then, we investigate this performance ratio for the base case and also for the other 4 problems with different quality distributions. As seen in the table 4.5, when the initial inventory level is increased to 10, the gap between the performance of the static and the dynamic pricing gets smaller for a given interval of the product quality. This is an intuitive result. As we have discussed before, for a given interval of the product quality when there are more inventory on hand, the quality effect weakens. However, one of the main reasons is the quality effect that stimulates the seller to adjust the sales price dynamically. Therefore, when the quality effect weakens, the performance of the static pricing approaches to the performance of the dynamic pricing.

4.6.2 Effects of initial inventory level

- ***on the net revenue***:

In order to better understand the impact of initial inventory level on the net revenue, so as to determine the optimal inventory level to start with, we first consider the net profit of the seller as the inventory level increases. For each level of inventory, five problems are constructed such that all model parameters in these problems, as well as their demand distributions are identical except the cumulative distribution of their product quality.

It is seen in figure 4.1, for each problem that the net revenue of the seller is concave in the starting inventory level, x which means that the value of an additional inventory is decreasing with the initial inventory level x. Concavity with respect to the starting level of inventory is critical to determine the optimal level of initial inventory.

- ***on the optimal prices***:

  1. ***No information update scenario***:

     We first start with the simpler setting to determine an optimal price for the available inventory with random quality at the beginning of the selling season and not to allow the seller update the distribution of product quality, and so change this price throughout the season. Hence, there is a single price for the inventories for the entire season.

     Recall that, this problem is called "No information update scenario" in the section 4.3. Consider the probability of a sale. In previous sections, it is demonstrated that a potential demand to a product is converted into a purchase if and only if $q \geq \frac{p}{\theta}$, so the probability of a sale can be written as $P(q \geq \frac{p}{\theta}) = 1 - P(q < \frac{p}{\theta})$ which expresses that the seller sells a product with quality q if and only if this sale is going make a customer's utility non-negative.

     Now, consider each problem where the product quality of an available inventory uniformly takes a value from the continuous intervals $[0, 10], [2, 10], [4, 10], [6, 10]$, $[8, 10]$. Then, the corresponding probability of selling a product with the worst quality is going to be $(1 - (p - a)/(10 - a))$ for each problem when the seller sets price p for

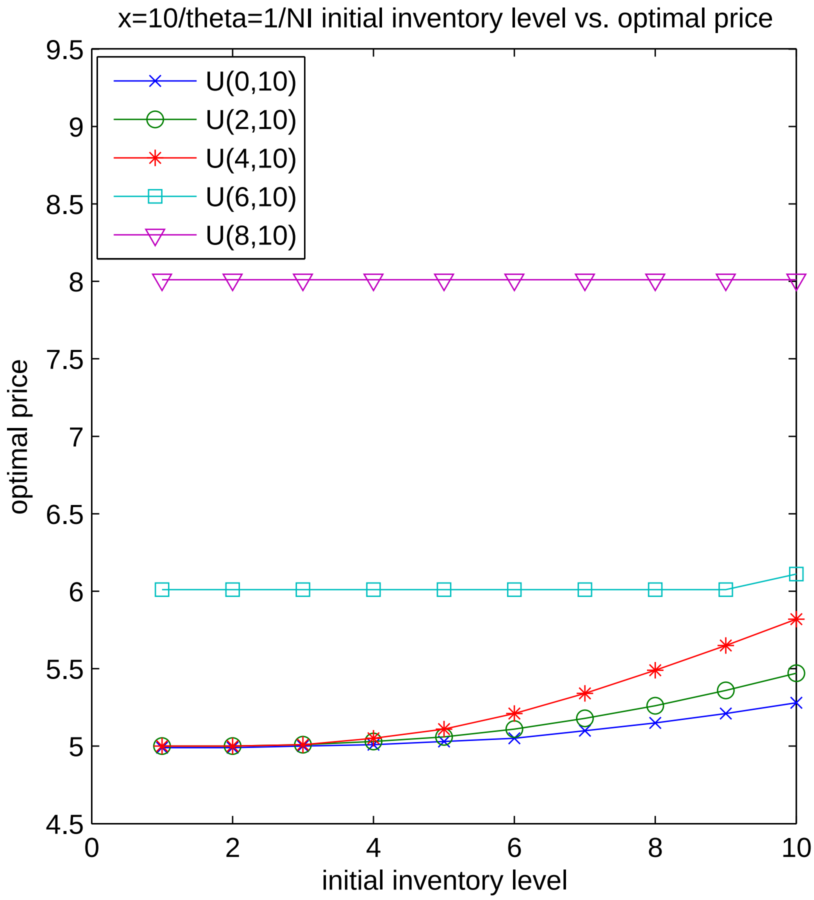

Figure 4.2: Initial inventory level vs. optimal price

the products, and where a is the lower bound of the support of the distribution of the corresponding problem such that $a \in \{0, 2, 4, 6, 8\}$. Moreover, multiplying these probabilities with price p and the number of units sold, s, will give the expected revenue to go till the end of the selling season. As the seller is a profit maximizer, and as the optimal price that gives the highest revenue is $p^* = 5$ in each problem when the seller is assumed to have only the products with the worst quality, it is reasonable that the seller will never set a price p which is lower than $p^* = 5$.

Note that $p^* = 5$ is the optimal price to be set when the seller has the worst products in his stock. For the quality distributions, $U(0, 10), U(2, 10), U(4, 10)$ of the first three problem, in fact the utility constraint implies that if the seller wants to sell all of his products, i.e. if he wants to set a price that provides a non-negative utility to the customers even when the worst products with quality $0, 2, 4$, respectively, are purchased, then he should set a price which is less than or equal to $0, 2, 4$ in the corresponding problems. However, the aim of the seller is to maximize the expected revenue, but not to sell all the inventories at a lower price than he should charge. Hence, as seen in figure 4.2, the optimal price to be set is at least 5, that is, the optimal price for the case in which the seller is assumed to have the products with worst quality.

For both of the last two problems, where the product qualities take values from the continuous intervals, $U(6, 10)$ and $U(8, 10)$, respectively, the optimal price, p^*, that should be set when the seller has products with the lowest quality in the quality interval is still 5. On the other hand, if the aim of the seller were to provide a non-negative utility even from a purchase of the worst product, the utility constraint would lead him to set a price which less than or equal to 6 and 8, respectively. These two constraints do not contradict with each other, and they together imply that a profit-maximizer seller should set a price which is at least 6, and 8, respectively, in the corresponding problems. Figure 4.2 verifies the discussion that we have made. As seen, the optimal prices for the problems with the distribution of product quality

$U(6, 10)$ and $U(8, 10)$ are found at least 6 and 8, respectively.

In addition, we see in figure 4.2 that as the inventory level increases, the quality effect becomes to be more dominant, since the likelihood of having more qualified products increases on the average, so the likelihood of having more items which are in the purchasing region increases. Therefore, it is reasonable that the seller sets a higher price when there are more inventories on hand at the beginning of the selling season. In addition, it is clear that as the product quality tends to raise, for the same amount of starting inventory, the seller should set a higher price.

The last point that should be mentioned in this case is, as seen again in figure 4.2, as the seller contracts the quality interval, variations in the product quality will decrease. Therefore, the effect of having more initial inventory will decrease as the qualities of products close up. This argument is validated in figure 4.2, when the product qualities take values from the continuous interval [8,10], the seller starts to increase the optimal price if the initial inventory level is increased to 9; however, he begins to set a higher optimal price in the base if the initial inventory level is increased to 2. We see that there is a significant effect of one additional product on the optimal price increase if the product qualities of the inventories vary in a bigger interval.

2. ***Information update scenario***:

   The results of numerical computations given in the figure below represents optimal regular prices, set in the beginning of the selling season, for different levels of starting inventory level from 1 to 10 for each starting cumulative distribution of the product quality: $U(0, 10), U(2, 10), U(4, 10), U(6, 10), U(8, 10)$. Here, we see two effects acting together for determining the optimal regular price as to the level of initial inventory on hand, demand effect and the quality effect. If an imaginary vertical line is drawn starting from the point corresponding to the inventory level 6 on the x-axis, on the left hand side of this imaginary line, the demand effect dominates the quality effect, and on the right hand side of this line, the leading effect is the quality effect.

   As the interval for the product quality is contracted, i.e. as the qualities of the products close up, the quality effect weakens, and the problem turns into a traditional pricing problem where only the demand effect determines the optimal price. Demand effect has been studied in traditional pricing models with no quality component. Gallego and van Ryzin [14] and Zhao and Zheng [36] show that the optimal price decreases with increasing inventory and when there is less time.

In this study, we verify that this intuition is not always correct: the optimal price may be non-decreasing in the starting level of inventory when the products are with random quality by the dominance of the quality effect over the demand effect.

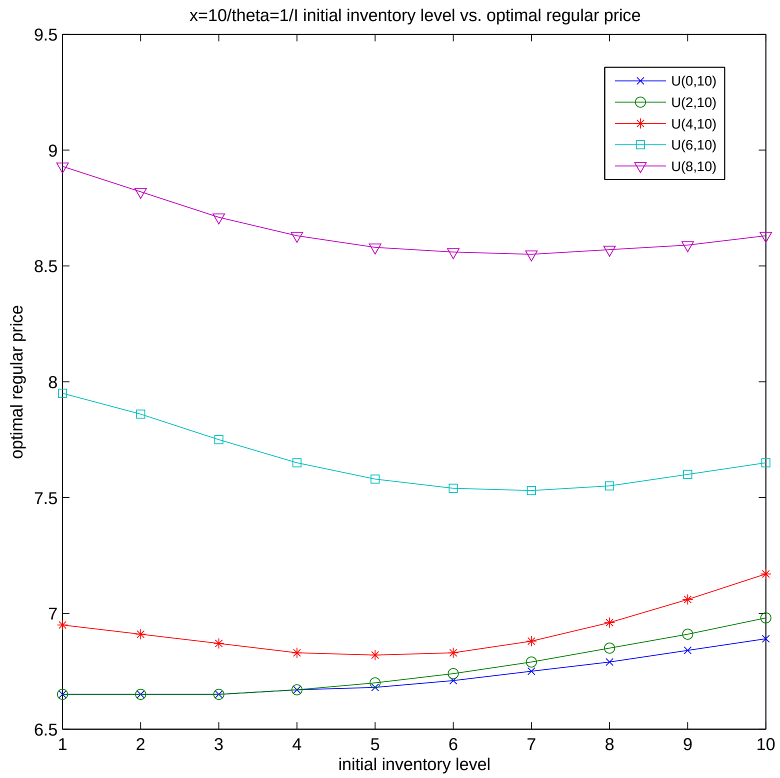

Figure 4.3: Initial inventory level vs. optimal regular price

In order to better understand the quality effect in figure 4.3, consider two scenarios, scenario 1 and scenario 2. All model parameters of these two scenarios are identical except the initial inventory level in the beginning of the regular period. While, in the first scenario, the seller starts with one unit of inventory, in the second scenario, he has 10 units of products in his stock. Consider selling one unit of product in both scenarios, i.e. consider the case where the seller will be observing only one unit of demand for his products, so will be selling at most one unit of inventory depending on whether the utility constraint of an arriving customer is satisfied or not. Under these circumstances, in scenario 1 for all quality distributions, $U(0,10), U(2,10), U(4,10), U(6,10), U(8,10)$, the probability of the event that the quality of the product on hand is lower than $\frac{p_r}{\theta}$. In other words, the likelihood of the event that the product with the highest quality in scenario 2 will be sold in the regular period is higher than the likelihood of the event that the product in scenario 1 will be sold in that period, since it is more likely that maximum product quality in the second scenario is bigger than the product quality of the single item in the first scenario. Therefore, the practice of raising the regular price as the starting inventory level increases is reasonable.

Moreover, in figure 4.3, it is seen that at the same inventory level, when the product quality of the inventories is higher, the seller will set a higher optimal regular price for this fixed amount of inventory. Hence, we can conclude that as the quality of the products increases, the optimal regular price set for these products also increases.

4.6.3 Effects of remaining inventory level on the optimal clearance period

In this subsection, we investigate the behavior of the clearance price as the remaining inventory increases, i.e. as the unsold inventory in the regular period increases.

As seen in figure 4.4, the clearance price is non-decreasing in the number of remaining inventory. In order to understand the intuition behind the monotonicity of clearance price in the number of remaining inventory, consider the simplest case where the seller has 1 unit of

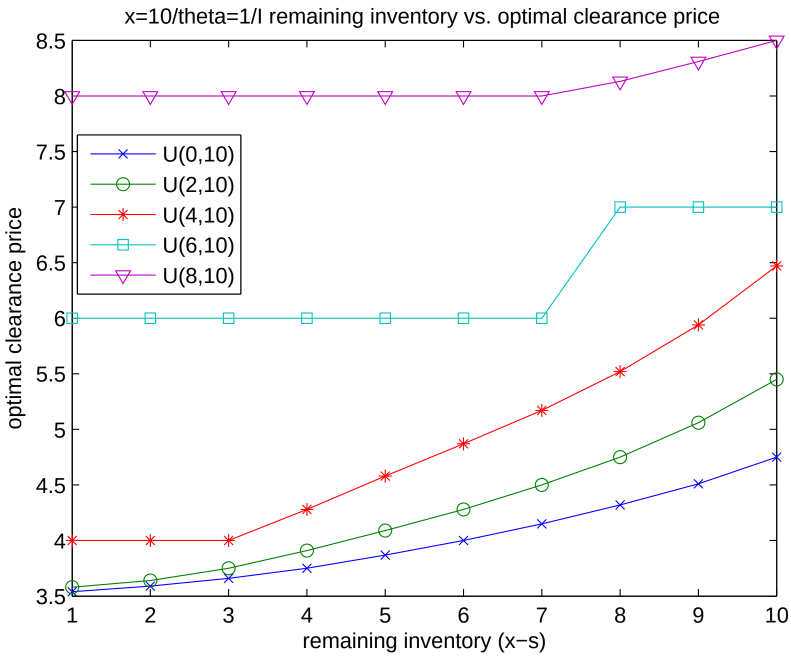

Figure 4.4: Remaining inventory level vs. optimal clearance price

inventory left when the product quality of the starting inventories is assumed to be distributed uniformly in $[0, 10]$. By our modelling assumptions, note that this is the product with lowest quality in the seller's hand since the customers prefer first the products with higher qualities. The price to be set in the clearance period for this remaining product is 3.5 as can be seen in figure 4.4. However, if the seller is left with 2 units of inventories in the clearance period, it means that the seller has the product with the lowest quality together with the second worst quality of product. Compared to the first case, now he is left with a relatively better product in quality, and aiming to sell just one unit of item in the clearance period will lead him to set a higher price. This is an illustrative example of why the optimal clearance price is higher than the one set in the first case.

In addition, starting with more qualified initial inventories leads the seller set a higher clearance price for a fixed amount of remaining inventory. In other words, consider that the seller starts with 10 units of inventories in two parallel systems which have identical parameters, but different quality distributions. Let the quality distribution of the system 1 and the system 2 be $U(0, 10)$ and $U(8, 10)$, respectively. Then, as expected the quality of the products will be higher on average in the system 2. The seller might be left with the same number of remaining inventories in the clearance period, and if this is the case, he will set a higher clearance price in the system 2, since the quality of the remaining inventory on average is expected to be higher in the second system.

Now, drawing the attention to the last point of the graph of the base case with $U(0, 10)$ that corresponds to the remaining inventory level 10 in figure 4.4, it is seen that there is a drastic drop in the optimal price compared to the optimal regular price set for the same amount of inventory in figure 4.3. This is a good point to explain the reason of the significant difference in the optimal prices set to the same amount of inventory in the regular and the clearance period. In fact, having 10 units of remaining inventories in the clearance period implies that the seller could not sell anything in the regular period. Recall that there may be two reasons for not being able to sell the inventories: 1) absence of sufficient number of demand or 2) shortage of sufficient number of qualified products. Assuming the seller carries all of the starting inventories to the clearance period because of the absence of demand in the regular period means that the seller could not receive any signal about customers' preferences in the regular period. In such a case, diminishing time will force the seller ignore the effect of inventory level on price, and the seller will determine a clearance price as if he had carried the entire inventory to the next period, so the prices should go down in the clearance period. Another reason of this drastic drop may be the misperception of the product quality by the seller in the beginning of the season which means that the seller might overestimate the product quality in the beginning and set an unrealistically high price for the regular period.

4.6.4 Effects of the coefficient ρ

Recall that the mean of the demand process in the base case is 8. This means that the seller expects to see 8 units of demand on the average during the entire selling season. Moreover, we know that the seller monitors the customers to be able to update fairly the distribution of the product quality, and so the price for the remaining inventories at some point in time during the selling season. Since the seller is a profit-maximizer, it is acceptable that he will be willing to know the best time for the updates in the quality distribution of the products and the optimal price in order to maximize his expected profit. The idea of dividing the total demand of the entire season into two parts, and finding the best time for it corresponds to finding the starting time of the clearance period that maximizes the seller's expected profit. Hence, this subsection is aiming to find answers to the following critical question: "When should the seller start the clearance period in order to optimize the total expected profit? (i.e. when should he update the distribution of product quality and set a discount (clearance) price to the remaining inventories?)"

Let ρ be the coefficient that determines the best starting time of the clearance period which is the best time of dividing the total demand into two parts. So, if the total demand of an entire season is a Poisson random variable with mean λ, then the demand of the regular and the clearance periods are also Poisson random variables with means $\rho\lambda$, and $(1-\rho)\lambda$, respectively.

As seen in figure 4.5, regardless of the product quality, the optimal value of ρ is around 0.5.

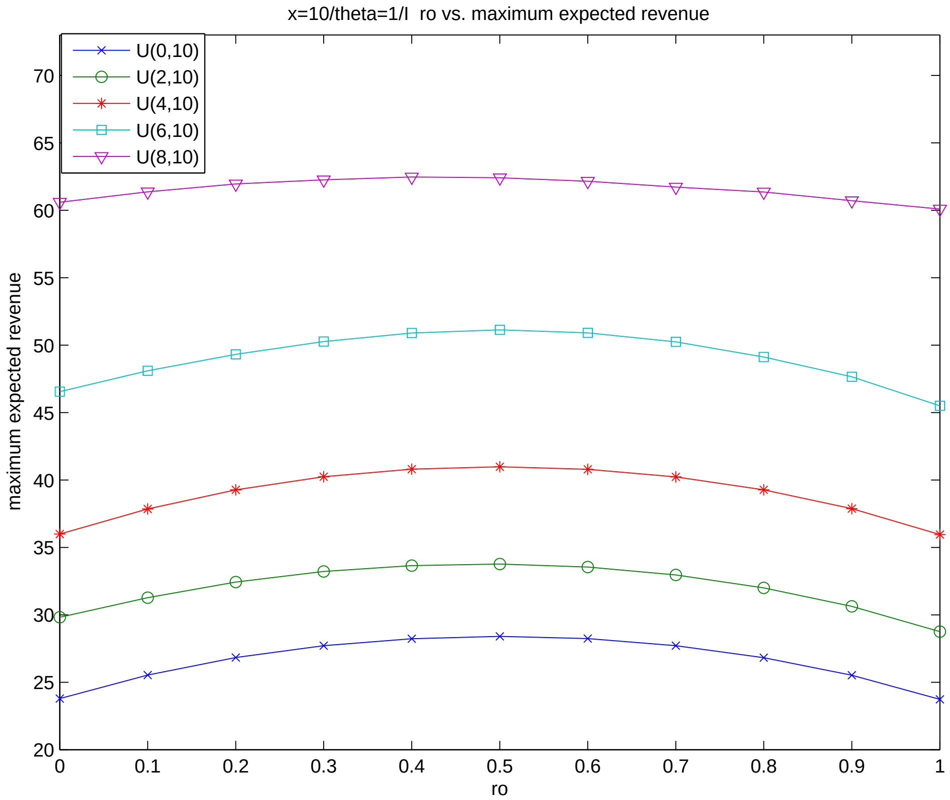

Figure 4.5: ρ vs maximum expected revenue

This is intuitively reasonable, since the decision of when to update the quality distribution, so as to set a discount price depends on a core tradeoff. Naturally, the longer the regular period, the more time there is to monitor the customers preferences and thus the greater the value of the collected information, but the shorter the clearance period where the seller will apply this information. In other words, if the seller waits for a long time to gather more information about the customers' valuations to the products to ensure that he will be charging the best price for the remaining inventory, he foregoes the opportunity of applying this information due to the lack of time in the clearance period. On the other hand, if he starts the clearance period before, he will be updating the product quality distribution inaccurately and setting an unfair price to the remaining inventories in the clearance period due to the lack of information.

4.6.5 Effects of the parameter θ

In this subsection, we validate what we have claimed about the necessary and sufficient condition for converting a demand arrival into a purchase in the previous sections. Recall that if a price set for a product with quality q is less than or equal to the customers' valuation to that quality, then the sale of that product occurs. Measurement of the customers' valuations to products with different quality is not an easy work even though it is known that all customers are homogenous, i.e. their valuations are identical to a product quality, and it is denoted by a constant θ. So, as mentioned before in the problem description, the seller is assumed to measure customers' sensitivity to quality by multiplying the quality scale by this constant parameter θ and then compare it to the price, p, set for the products. For the customers, if the value of the product quality, in currency unit, is bigger than the price set, then it means that the customers' utility will be non-negative from a purchase of this product. Thus, a sale will occur.

Based on the above discussion, it is seen that figure 4.6 is consistent, since it shows that the optimal regular price linearly increases if the customers' valuations to quality is increased. In fact, a change in θ has an effect on the optimal regular price as similar to multiplication by a constant scalar. In addition, we see the same effect of θ on the optimal regular price in each of the example problems regardless of the differentiation in the quality distributions.

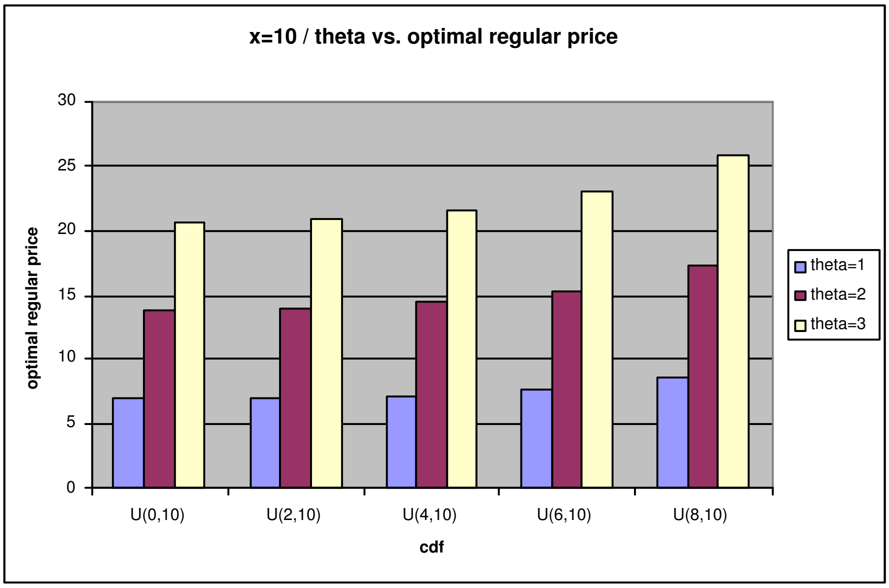

Figure 4.6: θ vs. optimal regular price

4.6.6 Effects of holding cost

In order to provide an additional insight about the problem with information update scenario, we include a factor of holding cost to the model. We use the base case problem to see the effect of having a holding cost per unit. Recall that the seller has a purchase cost of 2 per item regardless of the quality rating of the product that is purchased. The cost of holding the stock is based on the annual interest rate. If the annual interest rate, r, is taken as 10%, the annual holding cost of per item will be 0.2. Since our problem is defined for seasonal products, it is meaningful to assume that the selling season is shorter than one year. For instance, let the selling season take 6 months, so the seasonal holding cost of per item will be 0.1. According to figure 4.5, each of the periods will take 3 months. Therefore, we assume that the holding cost of per item in the regular period, h_r, and the holding cost of per item in the clearance period, h_c are the same and are equal to 0.05 in the base case.

We investigate the effect of the holding cost $h = h_r = h_c$ on the optimal regular price and on the optimal total expected revenue in the base case problem. Then, we show the effect of the initial inventory level on the optimal expected revenues in the presence of different values of holding costs.

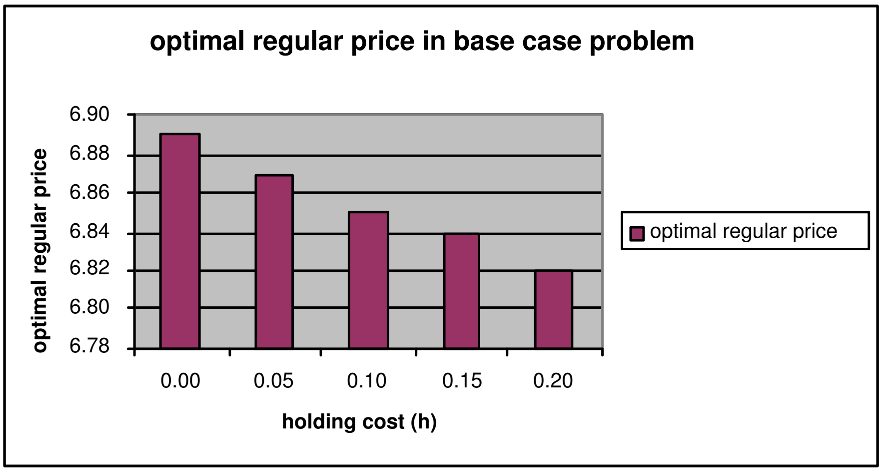

Figure 4.7: Optimal regular prices for different holding costs in base case

Considering the base case problem, the figure 4.7 shows that the optimal regular price takes its highest value when the seller does not have a holding cost. As the holding cost per item increases, the seller will set a lower regular price for his products. This is an intuitive result, since when it is more costly to hold the inventory on hand, the seller will be more willing to sell his products. Thus, he will set a lower price for the inventory as the holding cost increases.

Figure 4.8 shows the effect of the holding cost on the optimal expected profit for the base case. Recall that when the seller has no holding cost, he earns 28.4 from the base case scenario. However, based on the discussion in the beginning of this subsection, if the holding cost is set as 0.05 for each period, he will earn 27.52. So, there will be a 3% decrease in the seller's expected profit when a holding cost of 0.05 per item for each period is added to the problem. In fact, this is an expected result, considering that the seller holds in his stock $x = 10$ units of inventory in

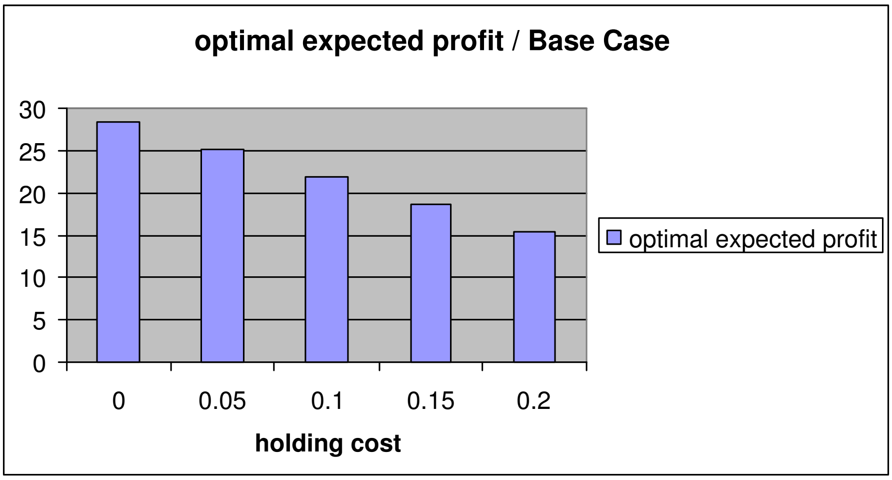

Figure 4.8: Holding cost vs. optimal expected profit

the regular period, and $x - s$ units of inventory in the clearance period where $s \in \{0, 1, \ldots, x\}$. Figure 4.8 also verifies another expected result, that is, the optimal expected profit of the seller decreases as the holding cost per item increases.

Finally, we investigate the effect of the initial inventory level on the optimal expected profit of the seller for different values of holding costs in the base case scenario. Figure 4.9 verifies that the concavity of the optimal profit with respect to the initial inventory level is preserved in the presence of the holding cost.

Hence, based on the three discussion above, we see that adding a holding cost factor to the problem may not create a big difference from what we have shown for this problem before.

4.6.7 Summary

In this subsection, we give a summary of the work presented and the main results of the study. We illustrate our numerical results by analyzing the optimal policy for five examples which differ in the product quality distribution. We compare the information update strategy with no information update strategy under the same settings to prove that updating the cumulative

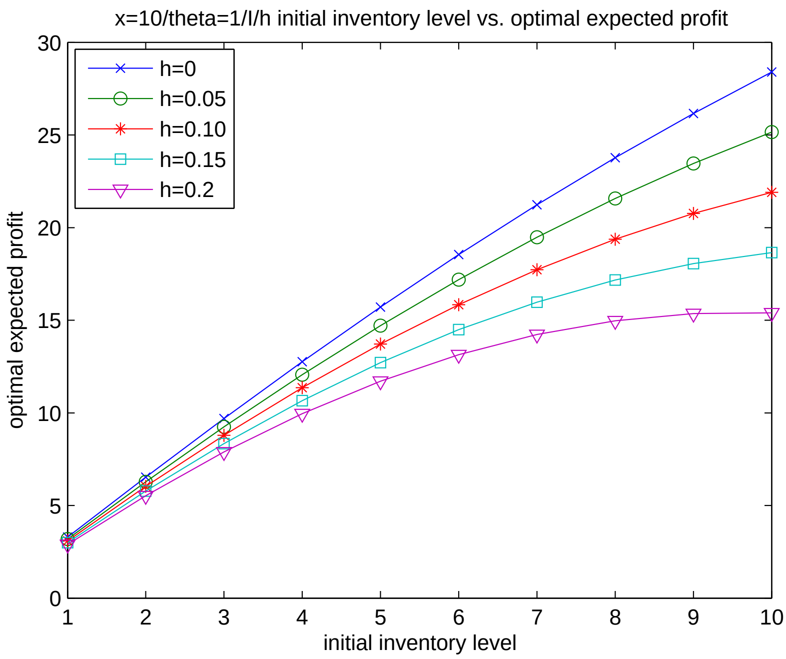

Figure 4.9: Optimal expected profit vs. inventory level in the presence of holding cost

distribution function of product quality is valuable. We examine the effects of the system parameters on the optimal policy. The parameters that we consider are the initial inventory level, remaining inventory level, the starting time of the clearance period, the customers' valuations to the product quality and the holding cost. It is ensured that when the initial product quality is assumed to be higher, the seller sets a higher initial price to maximize his revenue, and gains more profit. In addition, it is shown that the maximum expected revenue is non-decreasing and concave in the initial inventory. We verify that when the initial inventory level is high, the seller will post a higher price, since the chance of having qualified goods will increase if he starts with more inventories. The results also show that when the quality of the products are taken very close to each other, the changes in the optimal price returns to its traditional form, i.e. a decrease in the inventory level is observed. Moreover, the optimal price for the clearance period is non-decreasing in the remaining inventory, and the best time to start the clearance period is found as midpoint of the selling season. Customers' valuations to quality are analyzed and it is shown that for more quality sensitive customers, the seller will set a higher regular price. Finally, we consider for the base case problem, the effect of an additional cost, namely the holding cost, on the optimal policy.

4.7 Conclusion

In this chapter, we aim to determine the optimal pricing policy when the seller has perishable products which are at random quality. To our knowledge, this was the first attempt to solve the pricing problem of perishable products in the presence of an uncertain quality component. The main idea was to find the best strategy given uncertainty on the product quality when the goal of the seller is to determine a pricing policy for the finite selling season that maximizes the total expected revenue. We formulated the problem as a dynamic optimization problem. The difficulty of solving the problem under this setting was essentially the construction of the sales probabilities. Part of the task in formulating the model was to design these probabilities in such a way that numerical calculations will be tractable.

We first presented the model with the information update scenario where the distribution of the product quality was updated according to the number of unsold inventory at some point in time during the selling season. Then, the models with the no information update scenario, with the single-product scenario and with the constant quality scenario were presented. All of the modelling scenarios were examined under both a random demand and an unlimited demand settings. We determined the optimal pricing strategies for these scenarios.

Finally, we presented our numerical results aiming to illustrate the behavior of the model and to assess the impact of varying problem parameters on the optimal expected revenue and the optimal prices. We compared the information update strategy with no information update strategy under the same settings to prove that updating the cumulative distribution function of product quality is valuable. The results have shown that the information update scenario performs significantly better than no information update scenario.

Chapter 5

CONCLUSION

In the scope of this thesis, we considered two different dynamic inventory allocation problems with stochastic demand. In the first problem, the seller applies a capacity allocation rule that controls which and how many of the randomly arriving requests should be accepted. On the other hand, the second problem uses dynamic pricing policy for the products with random quality and determines how to adjust the price to enhance the expected total revenue during a finite selling season by updating the distribution of the product quality of the remaining inventory for the rest of the selling season.

In the first part, we aimed to study a well-known revenue management problem first introduced in [19]. Since our objective was to understand the behavior of the optimal policy when the system parameters change, we used event-based dynamic programming as an approach to prove the structural properties of the model, by first showing that the certain event operators preserve monotonicity, concavity, and supermodularity properties. In fact, basic structural properties of the optimal policy of our model were established in the earlier studies, but little is known about the effects of the system parameters on the optimal policy. In this thesis, the effects of varying system parameters were examined for this model, guidelines for predicting the response of the optimal policy to changes in random system parameters were given.

In the second part of this thesis, we aimed to determine the optimal pricing policy when the seller has perishable products which have random qualities. To our knowledge, this was the first attempt to solve the pricing problem of perishable products with random quality. The main idea was to find the best pricing strategy for the finite selling season that maximizes the total expected revenue given an uncertainty on the product quality. We presented our numerical results aiming to illustrate the behavior of the model and to assess the impact of varying problem parameters on the optimal expected revenue and the optimal prices. We compared the dynamic pricing strategy with the static pricing strategy under the same settings. The results verified that the dynamic pricing strategy performs significantly better the static pricing strategy.

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Appendix A

REVIEW OF ORDER STATISTICS

A.1 Distribution of a Single Order Statistics

Order statistics study the properties and applications of ordered random variables. We refer the reader to [11] for a comprehensive survey of order statistics and its applications.

Let $X_1, X_2, \ldots, X_n$ be random variables arranged in order of magnitude as follows:

$$X_1 \leq X_2 \leq \ldots \leq X_n$$

Hence,

$$\begin{aligned} X_1 &= \min\{X_1, X_2, \ldots, X_n\} \\ X_2 &= \min\{\{X_1, X_2, \ldots, X_n\} - \{X_1\}\} \\ X_n &= \max\{X_1, X_2, \ldots, X_n\} \end{aligned}$$

The ordered values of independently and identically distributed sample are known as the order statistics, and X_i is called the i^{th} order statistics for all $i \in \{1, 2, \ldots, n\}$. Together with rank statistics, order statistics are among the most fundamental tools in non-parametric statistics and inference.

In this thesis, $X_1, X_2, \ldots, X_n$ are n independent ordered variates, each with a cumulative distribution function $F(x)$. Let $F_r(x), r = 1, \ldots, n$ denote the cdf of the r^{th} order statistics X_r. Then the cdf of the largest order statistics X_n is given by.

$$F_n(x) = Pr(X_n \leq x) = Pr(all\ X_i \leq x) = [F(x)]^n$$

Likewise we have

$$F_1(x) = Pr(X_1 \leq x) = 1 - Pr(X_1 > x) = 1 - Pr(all\ X_i > x) = 1 - [1 - F(x)]^n$$

These are important special cases of the general result for $F_r(x)$:

$$\begin{aligned} F_r(x) &= Pr(X_r \leq x) \\ &= 1 - Pr(X_r > x) \\ &= Pr(\text{at least } r \text{ of the } X_i \text{ are less than or equal to } x) \\ &= \sum_{i=r}^{n} \binom{n}{i} F_i(x)[1 - F(x)]^{n-i} \end{aligned}$$

Note that the term in the summation is the binomial probability that exactly i of $X_1, X_2, \ldots, X_n$ are less than or equal to x.

A.2 Joint Distribution of Two Order Statistics

The joint cumulative function of X_r and X_s where $1 \leq r < s \leq n$ is denoted by $F_{r,s}(x, y)$. For $x < y$, we can write

$$\begin{aligned} F_{r,s}(x, y) &= Pr(\text{at least } r\ X_i \leq x, \text{ at least } s\ X_i \leq y) \\ &= \sum_{j=s}^{n} \sum_{i=r}^{j} Pr(\text{exactly } i\ X_i \leq x, \text{ exactly } j\ X_i \leq y) \\ &= \sum_{j=s}^{n} \sum_{i=r}^{j} \frac{n!}{i!(j-i)!(n-j)!} F_i(x)[F(y) - F(x)]^{j-i}[1 - F(y)]^{n-j} \end{aligned}$$

Also for $x \geq y$, the inequality $X_s \leq y$ implies $X_r \leq x$, thus

$$F_{r,s}(x, y) = F_s(y)$$

Appendix B

BINOMIAL MODEL FOR NO INFORMATION UPDATE WITH RANDOM DEMAND

Let Q be the number of items with sufficient product quality as to the price p and let $X = 1$ be the event that among D arriving customers, one customer precisely demands the product with quality $q(i)$. Then the distribution we derived for the event X is called the hypergeometric distribution. We can define observing a demand for the product with quality $q(i)$ for some $i \in \{1, 2, \ldots x\}$ as success and the rest coming for the other products as failure. Since we have x products that can be sold at price p, $(N = x)$, with 1 successes in the population ($M = 1$ a demand for the product i with quality $q(i)$) and D customers arriving to the store $(n = D)$, the probability of observing the event $X = 1$ is the following:

$$P(X=1) \quad = \quad \frac{\begin{pmatrix} M \\ 1 \end{pmatrix}\begin{pmatrix} N-M \\ n-k \end{pmatrix}}{\begin{pmatrix} N \\ n \end{pmatrix}} = \frac{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\begin{pmatrix} x-1 \\ D-1 \end{pmatrix}}{\begin{pmatrix} x \\ D \end{pmatrix}} = \frac{D}{x} \tag{B.1}$$

So, the probability of observing a demand for an item which belongs to the set of sufficiently qualified products would be independent of the price charged at the beginning of the season and more significantly, the quality of that product. If the seller observed a deterministic demand D, where $Q \leq D \leq x$, then, D could be considered as an unrestricted demand, since the demand constraint did not prevent a sale of a product with a sufficient quality. In this case, the total expected revenue could have been written as follows:

$$v(\pi_x) = p\ \frac{D}{x}P(q(1) > \frac{p}{\theta}) + p\ \frac{D}{x}P(q(2) > \frac{p}{\theta}) + \ldots + p\ \frac{D}{x}P(q(x) > \frac{p}{\theta}) \tag{B.2}$$

As the product quality is identically and independently distributed with π_x, the above expression turns out to be

$$v(\pi_x) = p\ D\ \left(1 - \pi_x\left(\frac{p}{\theta}\right)\right) \tag{B.3}$$

Moreover, even for a stochastic demand D where again $Q \leq D \leq x$, the total expected revenue could have been written as follows:

$$\begin{aligned} v(\pi_x) &= p\ P(q(1) > \frac{p}{\theta}) \left(\sum_{d=Q}^{x} P(D = d)\ \frac{d}{x}\right) \\ &+ p\ P(q(2) > \frac{p}{\theta}) \left(\sum_{d=Q}^{x} P(D = d)\ \frac{d}{x}\right) \\ &\vdots \\ &+ p\ P(q(x) > \frac{p}{\theta}) \left(\sum_{d=Q}^{x} P(D = d)\ \frac{d}{x}\right) \\ &= p\ x\ (1 - \pi_x(\frac{p}{\theta})) \left(\sum_{d=Q}^{x} P(D = d)\ \frac{d}{x}\right) \end{aligned}$$

Now, consider that the seller observes a stochastic demand D where $D < Q \leq x$. In such a case, since the arriving demand is less than the number of products with sufficient quality, Q, and the seller does not update the price during the season according to the quality of the remaining inventory on hand, considering that arriving customers randomly select a product; but, they select it among the items that satisfy a non-negative utility is reasonable to find the purchase probabilities. In the above formulation, we just need to make the arriving demand come to the product with sufficient quality so N is now equal to Q; thus, the probability of observing the event $X = 1$ is equal to $\frac{D}{Q}$ anymore. The total expected revenue could have been written explicitly as follows if the demand was deterministic and equal to D.

$$v(\pi_x) = p\ \frac{D}{Q} P(q(1) > \frac{p}{\theta}) + p\ \frac{D}{Q} P(q(2) > \frac{p}{\theta}) + \ldots + p\ \frac{D}{Q} P(q(x) > \frac{p}{\theta}) \tag{B.4}$$

As the product quality is identically and independently distributed with π_x, the above expression turns out to be

$$v(\pi_x) = p\ x\ \frac{D}{Q}\ \left(1 - \pi_x \left(\frac{p}{\theta}\right)\right) \tag{B.5}$$

Moreover, even for a stochastic demand D where again $D \leq Q$, the total expected revenue could have been written as follows:

$$\begin{aligned}
v(\pi_x) &= p\ P(q(1) > \frac{p}{\theta})\ \left(\sum_{d=0}^{Q} P(D = d)\ \frac{d}{Q}\right) \\
&+ p\ P(q(2) > \frac{p}{\theta})\ \left(\sum_{d=0}^{Q} P(D = d)\ \frac{d}{Q}\right) \\
&\vdots \\
&+ p\ P(q(x) > \frac{p}{\theta})\ \left(\sum_{d=0}^{Q} P(D = d)\ \frac{d}{Q}\right) \\
&= p\ x\ (1 - \pi_x(\frac{p}{\theta}))\ \left(\sum_{d=0}^{Q} P(D = d)\ \frac{d}{Q}\right)
\end{aligned}$$

VITA

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