Portfolio Selection in Stochastic Markets: Utility Based Approach

by

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ABSTRACT

In this thesis, we consider the optimal portfolio selection problem in multiple period and continuous time settings where the investor maximizes the expected utility of the terminal wealth in a stochastic market. The utility function has the structure of the HARA family and the market states change according to a Markov chain. The states of the market describe the prevailing economic, financial, social and other conditions that affect the deterministic and probabilistic parameters of the model.

In first part we assumed a discrete time market and discuss the stochastic structure of the wealth process under the optimal policy and determine various quantities of interest including its Fourier transform. The exponential, power and logarithmic return-risk frontiers of the terminal wealth is shown to have a linear form.

In the second part we investigated the case where the investor does not have perfect information about the market. The unobserved stochastic market is a Markov chain and it emits signals, or provides information, that is observed by the market players. The optimal portfolio policy under imperfect information is constructed and the differences between the perfect and imperfect information cases are presented.

In the last part, we analyzed a Black-Scholes type continuous time models where the market parameters are driven by Markov processes. The problem of maximizing the expected utility from terminal wealth is investigated. We found explicit solutions for optimal policy and the associated value functions. We also constructed the optimal wealth process explicitly and discussed some of its properties.

ÖZETÇE

Bu tezde çoklu zamanda ve sürekli zamanda rassal markette dönem sonu servetininin beklenen değerininin fayda fonksiyonunu en büyükleyen bir yatırımcının en iyi portföy seçimi problemi incelenmiştir. Fayda fonksiyonunun tipi HARA olarak alınmış ve marketin durumları bir Markov zincirine bağlı olarak değişmektedir. Marketin değişik durumları karşılaşılan ekonomik, finansal, sosyal, ve diğer şartları göstererek modelin belirli ve olasılıksal parametrelerini etkilemektedir.

Tezin ilk kısmında ayrık zamanlı bir market varsayılmıştır ve problem olarak dönem sonundaki servetininin beklenen fayda değerini enbüyükleyen bir yatırımcı ele alınmıştır. Eniyi yatırım politikası kullanıldığında oluşan varlık sürecinin yapısı ile Fourier transformu da dahil olmak üzere değişik özellikleri bulunmuştur. Üstel, güç, ve logaritmik kazanç-risk eğrileri hesaplanmış ve bunların doğrusal olduğu sonucu bulunmuştur. Normal ve üstel dağılımlar gibi bazıözel durumlar incelenerek sayısal örnekler verilmiştir.

Tezin ikinci kısmında yatırımcının marketin durumu hakkında tam bilgi sahibi olmadığı durum incelenmiştir. Gözlenemeyen marketin durumu bir Markov zincirine bağlı hareket eder ve yatırımcılara bazı sinyaller gönderir. Rassal marketin durumu ile gözlemlerin arasındaki ilişki iki değişik method ile açıklanmıştır. Birincisi saklı Markov modelleri, ikincisi ise yeterli istatistik yöntemidir. Kısmi gözlemlere dayanan durum için en iyi portföy yönetimi politikaları çıkarılmış ve bu durumun tam bilgi akışının olduğu durum ile farkları gösterilmiştir.

Tezin son kısmında ise Black-Scholes modeli kullanılarak sürekli zamanda market parametrelerinin bir Markov sürecine bağlı olduğu portföy eniyileme problemi incelenmiştir. En iyi yatırım politikasının açık çözümleri bulunmuş ve karşılık gelen değer fonksiyonu incelenmiştir. Ayrıca en iyi politikaya karşılık gelen servet süreci açık olarak hesaplanmış ve bazı özellikleri açıklanmıştır. Özellikle risk-kazanç eğrilerinin doğrusallığı gösterilmiştir.

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NOMENCLATURE

HMM	:	Hidden Markov Model
$U(i,x)$	:	Utility function given wealth is x and market state is i
HARA	:	Hyperbolic Absolute Risk Aversion
CRRA	:	Constant Relative Risk Aversion
CARA	:	Constant Absolute Risk Aversion
Z_n	:	State of the real market at time n
Y_n	:	State of the observed market at time n
$Q(a,b)$	:	Transition matrix of real market process
Z	:	Real market process $(Z = \{Z_n; n = 0,1,2,3,\dots\})$
F	:	State space of Z $\big(F = \{a,b,\dots\}\big)$
Y	:	Observed market process $(Y = \{Y_n; n = 0,1,2,3,\dots\})$
E	:	State space of Y $\big(E = \{i,j,\dots\}\big)$
$\bar{Z}_n$	:	All real states up to time n $\big(\bar{Z}_n = (Z_0, Z_1, \cdots, Z_n)\big)$
F^{n+1}	:	State space of $\bar{Z}_n$ $\big(F^{n+1} = F \times F \times \cdots \times F = \{(a_0, a_1, \cdots, a_n) : a_m \in F\}\big)$
$\bar{Y}_n$	:	All observations up to time n $\big(\bar{Y}_n = (Y_0, Y_1, \cdots, Y_n)\big)$
E^{n+1}	:	State space of $\bar{Y}_n$ $\big(E^{n+1} = E \times E \times \cdots \times E = \{(i_0, i_1, \cdots, i_n) : i_m \in E\}\big)$
$R(i)$	:	Vector of return of random assets given that the market state is i
$r(i)$	:	Expected value of $R(i)$
r_f	:	Return of the riskless asset
$R_k^e(a)$	:	Excess return of the kth asset in state a
$r_k^e(a)$	:	Expected value of excess return of the kth asset in state a
$\sigma_{kl}(i)$	:	Covariance between kth and lth asset returns given the market state is i
X_n	:	Wealth level of the investor at period n
$E_i[\bullet]$	:	Expectation given that the initial market state is i
$Var_i(\bullet)$	:	Variance given that the initial state is i
P	:	Probability

u	:	Vector representing the amounts invested in risky assets
u^*	:	Vector representing the optimal amounts invested in risky assets
β	:	Risk aversion factor
β_n	:	Risk aversion factor discounted to period n
$\alpha_n(i)$	:	Optimal investment ratio in period n if the market is in state i
$v_n(i,x)$	:	Value function given that the market is in state i and the amount of money available for investment is x at period n
$A(i)$	:	Random variable $R^e(i)'\alpha(i)$
$\bar{\alpha}(i)$	:	Expected value of $A(i)$
$\tilde{\alpha}(i)$	:	Second moment of $A(i)$
$G_n(a,i)$	:	Time dependant emission matrix $P\{Y_n = i \| Z_n = a\}$
$O_n(\bar{i}_n, a)$	:	$P\{Z_n = a \| \bar{Y}_n = \bar{\imath}_n\}$ for HMM formulation
π_n^a	:	$P\{Z_n = a \| \bar{Y}_n = \bar{\imath}_n\}$ for sufficient statistics formulation
π_n	:	The conditional distribution of Z
$(\Omega, F, \mathcal{F}, P)$	:	Filtered probability space
$B(t)$	:	Bond process
$S_k(t)$	:	Stock process for kth asset
$\mu_k(i,t)$	:	Drift rate of kth asset
$\sigma_{kj}(i,t)$	:	Volatility of kth asset with respect to jth Wiener process
$A(i,j)$	:	Transition rate matrix of Markov process
$\mathcal{F}$	:	minimal σ-algebra generated by $(W_1, W_2, \cdots, W_m, Y)$
$\bar{r}_f(s,t)$	:	Total compound factor from time s to time t
$\mu^e(i,t)$	:	Excess drift rate of the risky assets in state i
$\pi_k(t)$	:	The ratio of current wealth invested on asset k at time t
$u_k(t)$	:	The money invested on asset k at time t
X_t^π	:	Wealth level at time t if the policy π is used
X_t^u	:	Wealth level at time t if the policy u is used
X_t^*	:	Wealth level at time t if optimal policy is used
$E_{i,x,t}[R]$	:	Expectation $E[R \| Y_t = i, X_t = x]$
$\hat{E}_{i,x,t}[R]$	:	Expectation $E[R \| Y_0 = i, X_0 = x, Z_0 = t]$

Chapter 1

INTRODUCTION

Portfolio selection problem seeks the best allocation of wealth among different investment opportunities in a market consisting of risky assets. Determination of optimal portfolios is a rather complex problem depending on the objective of the investor. The classical mean-variance model build by Markowitz [43] is undoubtedly the most celebrated one within the vast area of portfolio management where the objective is to minimize the variance of the terminal wealth for a desired level of expected return. In a survey paper, Steinbach [65] reviews the mean-variance models in financial portfolio analysis. This survey refers to 208 papers which shows the diversity of different models and approaches used to analyze this problem for both single period and multi-period cases. There are many that consider the multiperiod problem including, for example, Mossin [50], Samuelson [60], Chen et al. [12], Elton and Gruber [27], Bodily and White [8], Dumas and Luciano [21], Ehrlich and Hamlen [23], and Li and Ng [41], among many others.

In most of the multiperiod problems, the rates of return of the assets during consecutive periods are assumed to be uncorrelated. In a realistic setting, this is not correct and the dependence among the rates of return in consecutive periods should also be considered. This dependence or correlation is often achieved through a stochastic market process that affects all deterministic and probabilistic parameters of the model. A tractable and realistic approach is provided by using a Markov chain that represents the economic, financial, social, political and other factors which affect the returns of the assets. The use of a modulating stochastic process as a source of variation in the model parameters and of dependence among the model components has proved to be quite useful in operations research and management science applications. The concept was introduced by Çınlar and Özekici [13] in a reliability setting where the failure rate and hazard functions of a device depend on the prevailing environmental conditions. There is now considerable amount of literature on modulation in a variety of applications. An example in queueing is provided by Prabhu and Zhu [55] where customer arrival and service rates are modulated by a Markov process. Song and Zipkin [63] consider an inventory model with a demand process that fluctuates with respect to stochastically changing economic conditions. A general discussion on the idea can be found in Özekici [54]. The interested reader is referred to Asmussen [2] and Rolski et al. [59] for further applications in queueing, insurance and finance. Çakmak and Özekici [10] have applied the idea to multiperiod mean-variance portfolio optimization problem. In their setting, the correlation among returns in different periods is formulated by a stochastic market representing the underlying factors that form a Markov chain. Considering a market with one riskless and m risky assets, a multiperiod mean-variance formulation is developed. An auxiliary problem generating the same efficient frontier is used to eliminate nonseparability in the sense of dynamic programming. The analytical optimal solution is obtained for the auxiliary problem using dynamic programming. Following their work, Çelikyurt and Özekici [11] analyze the multiperiod mean-variance model by considering the safety-first approach, coefficient of variation of the terminal wealth and quadratic utility functions. Using dynamic programming, efficient frontiers and optimal portfolio management policies are obtained.

Another line of research in portfolio optimization follows utility theory and expected utility maximization. In this setting, the objective of the investor is to maximize the expected value of a utility function of the terminal wealth. The risk preferences of the investor is given and measured by the utility function. The most widely used measures of risk-aversion were introduced by Pratt [56] and Arrow [1]. Mossin [50] examined some utility functions and discovered the utility functions that leads to myopic policies. Bertsekas [5] also examines a special cases of utility functions and derives the multiperiod optimal policies for these cases. Merton [46] considered special utility functions with logarithmic and power structures. Hakansson [32], in discrete time setting, investigates the optimization of logarithmic and power utility in a random market. More recently, Dokuchaev [20] considers a model where the expected utility of the terminal wealth with power and logarithmic utility functions are maximized in a discrete-time market with serial correlations. Also, Breuer and Gürtler [9] investigate the performance of funds using different utility functions. In our work we will extend the utility based approach to multiperiod portfolio optimization by considering an investor with exponential utility where we suppose that the asset returns all depend on a stochastic market depicted by a Markov chain.

Even though hidden Markov models (HMM) are one of the important tools used in areas like speech recognition, bioinformatics, and gene prediction; they have not been used in portfolio optimization until very recently. Elliott et al. [25] use a HMM to describe stock price movements in order to find optimal portfolio trading strategy that maximizes the expected terminal wealth. Dericioğlu and Özekici [18] has applied the HMM to mean-variance portfolio selection problem in a Markovian market. They solved the problem with dynamic programming and obtained an explicit optimal solution to represent the efficient frontier.

1.1 Preliminaries

In this section we will give some preliminary information about the model and outline the tools that we will use for the solution. Analysis and solution to the model will be provided in later sections.

1.1.1 Utility Functions

A utility function is a non-decreasing real valued function U defined on the real numbers. Once a utility function is defined, all alternative random wealth levels can be ranked by evaluating their expected utility values. For a given utility function U, certainty equivalent CE is defined as a certain amount of money that is equivalent to the uncertain payout such that

$$U(CE) = E\left[U\left(X\right)\right].$$

An investor compares two random wealths X and Y by comparing the corresponding certainty equivalents $CE(X)$ and $CE(Y)$ and prefers the larger one. Utility functions describe risk preferences of the investor. An investor is called risk averse (risk seeking) if the certainty equivalent of the uncertain payout is smaller (larger) than the expected income from payout. Between these two stands the risk neutral behavior where $CE = E[X]$. The utility function of risk averse investor is concave, risk seeking investor is convex and risk neutral investor is linear.

Pratt [56] and Arrow [1] suggests the risk aversion function

$$r(x) = -\frac{U''(x)}{U'(x)} \tag{1.1}$$

which is called the Pratt-Arrow ratio as a measure of absolute risk aversion. Notice that $r(x) > 0$ if U is monotonically increasing and strictly concave. Naturally, $r(x) = 0$ for the risk-neutral individual with a linear utility function, and $r(x) > 0$ for the risk seeking individual with a strictly convex utility function.

As we can see, the Arrow-Pratt measure of absolute risk aversion cannot capture a situation as the agent switches from risk averse, to risk seeking and then back to risk averse for different wealth levels. Thus, an alternative would be to weigh the measure of risk aversion by the level of wealth x. In this case we obtain the Arrow-Pratt measure of relative risk aversion, which is defined as

$$r^*(x) = -\frac{U''(x)x}{U'(x)}. \tag{1.2}$$

In both of these measures if the ratio is equal to zero, then the second derivative of the utility function need to be zero, which means the utility function is linear

$$U(x) = c_1 + c_2 x.$$

If the utility function is linear then the risk preference of the investor is defined as risk neutral behavior and expected utility maximization for linear utility is the same as maximization of the expected terminal wealth. Therefore risk neutral behavior is not interesting and will not be investigated.

If the ratio in (1.1) is constant, meaning $r(x) = c \neq 0$, the utility function is called constant absolute risk aversion(CARA) type function and this can be interpreted as

$$-\frac{U''(x)}{U'(x)} = c$$

which can be solved to find

$$U(x) = c_1 + c_2 \exp(-cx).$$

So the CARA type functions are the exponential functions. On the other hand if the ratio in (1.2) is constant, meaning $r^*(x) = c \neq 0$, the utility function is called constant relative risk aversion(CRRA) type function and this type of functions satisfy

$$-\frac{U''(x)x}{U'(x)} = c$$

	$a = 0$ (CRRA)	$a \neq 0$
$b = 0$ (CARA)	$U(x) = c$	$U(x) = c_1 + c_2 \exp(-x/a)$
$b = 1$	$U(x) = c_2 + c_1 \ln(x)$	$U(x) = c_1 + c_2 \ln(x + a)$
$b \neq 0, b \neq 1$	$U(x) = c_1 + c_2 \left(\frac{x^{1-(1/b)}}{1-(1/b)} \right)$	$U(x) = c_1 + c_2 \left(\frac{(x+(a/b))^{1-(1/b)}}{1-(1/b)} \right)$

Table 1.1: Utility functions for different values of a, b

which can be solved to find out

$$U(x) = \begin{cases} c_1 + c_2 \ln(x) & c = 1 \\ c_1 + c_2 \left(\frac{x^{1-c}}{1-c} \right) & c \neq 1 \end{cases}.$$

A more general case can be used if we define the ratio as

$$-\frac{U''(x)}{U'(x)} = \frac{1}{a + bx}$$

where it can be seen that $a = 0$ refers to the CRRA case where $b = 1/c$, and $b = 0$ refers to the CARA case where $a = 1/c$. This general case of utility functions are called the hyperbolic absolute risk aversion (HARA) type function. In this thesis we will concentrate on HARA functions. If we analyze the case where both $a \neq 0$ and $b \neq 0$, we can see that

$$-\frac{U''(x)}{U'(x)} = \frac{1}{a + bx} \tag{1.3}$$

leads to

$$(\ln U'(x))' = -\frac{1}{b} \left(\frac{1}{x + (a/b)} \right)$$

which after integration, gives

$$U'(x) = \frac{c_2}{(x + (a/b))^{1/b}}$$

and this can be solved to find

$$U(x) = \begin{cases} c_1 + c_2 \left(\frac{(x+(a/b))^{1-(1/b)}}{1-(1/b)} \right) & b \neq 1 \\ c_1 + c_2 \ln(x + (a/b)) & b = 1 \end{cases}$$

as summarized at Table 1.1.

Bertsekas [5] has proven that in a deterministic environment where the market parameters are known, if the utility function satisfies (1.3) then the optimal portfolio is given by the linear policy

$$u^*(x_0) = \alpha(a + bsx_0)$$

where s is the interest rate, α is some constant and x_0 is the initial wealth.

1.1.2 Return Distributions

In this section we will summarize some of the return distributions that we used in our research.

Multivariate Normal Distributions

The multivariate normal distribution is a specific probability distribution which is the generalization to higher dimensions of the one-dimensional normal distribution. A random vector $X = [X_1, \cdots, X_n]$ follows a multivariate normal distribution if it has the following joint probability density function

$$f_X(x_1, \cdots, x_n) = \frac{1}{(2\pi)^{n/2} |\sigma|^{1/2}} \exp\left(-\frac{1}{2}(x-\mu)'\sigma^{-1}(x-\mu)\right)$$

where μ is the mean vector, σ is the covariance matrix and $|\sigma|$ is the determinant of σ Unless otherwise stated a vector y is a column vector so that its transpose, denoted by y', is always a row vector. If X is a random vector with multivariate normal distribution then every linear combination $Y = a_1X_1 + \cdots + a_mX_m$ is normally distributed. The Fourier transform of X is

$$E\left[\exp\left(jz'X\right)\right] = \exp\left(jz'\mu - \frac{1}{2}z'\sigma z\right)$$

for any vector $z = (z_1, z_2, \cdots, z_n)$ of real numbers where $j = \sqrt{-1}$.

Let $Z = [Z_1, \cdots, Z_n]$ be a random vector whose components are independent standard normal random variables and A be the Cholesky decomposition of the symmetric, positive semidefinite matrix σ. Then, the random vector $X = \mu + AZ$ has the multivariate normal distribution with mean vector μ and covariance matrix σ.

Multivariate Exponential Distribution

A number of multivariate exponential distributions are known. A trivial case is the one where the random variables are independent. Also there are other cases where the marginal distributions are exponential but the random variables are not independent. Here we will concentrate on two of them for which meaningful moment generating functions can be obtained. The first one has been suggested by Marshall and Olkin [44] and the second one is the generalized case of the distribution suggested by Gumbel [31].

Marshall-Olkin's Bivariate Exponential Distribution Marshall and Olkin [44] defined a bivariate exponential distribution where the survival function is defined as

$$\bar{F}(s,t) = P\{X > s, Y > t\} = \exp\left[-\lambda_1 s - \lambda_2 t - \lambda_{12}\max(s,t)\right]. \tag{1.4}$$

The underlying idea of the model is explained with a "fatal shock" model where λ_1 is the rate of shocks effecting the first component, λ_2 is the rate of shocks effecting the second component, and λ_{12} is the rate of shocks effecting both components. Using the survival function in (1.4), the Laplace transform can be found as

$$E\left[\exp\left(-\left(sX + tY\right)\right)\right] = \frac{(\lambda + s + t)(\lambda_1 + \lambda_{12})(\lambda_2 + \lambda_{12}) + st\lambda_{12}}{(\lambda + s + t)(\lambda_1 + \lambda_{12} + s)(\lambda_2 + \lambda_{12} + t)} \tag{1.5}$$

where $\lambda = \lambda_1 + \lambda_2 + \lambda_{12}$ is the total shock rate. Using the marginal distributions, it can be easily computed that

$$E[X] = \frac{1}{\lambda_1 + \lambda_{12}}, \quad \mathrm{Var}(X) = \frac{1}{(\lambda_1 + \lambda_{12})^2},$$
$$E[Y] = \frac{1}{\lambda_2 + \lambda_{12}}, \quad \mathrm{Var}(Y) = \frac{1}{(\lambda_2 + \lambda_{12})^2}$$

and

$$E[XY] = \frac{1}{\lambda}\left(\frac{1}{\lambda_1 + \lambda_{12}} + \frac{1}{\lambda_2 + \lambda_{12}}\right).$$

Hence, the covariance is given by

$$\mathrm{Cov}(X,Y) = \frac{\lambda_{12}}{\lambda(\lambda_1 + \lambda_{12})(\lambda_2 + \lambda_{12})},$$

and the correlation is $\rho(X,Y) = \lambda_{12}/\lambda$. Note that $0 \le \rho(X,Y) \le 1$. This is an unwanted case in our model since we want our assets be negatively as well as positively correlated.

Marshall-Olkin's Multivariate Exponential Distribution The multivariate extension of the bivariate exponential distribution defined above is given by

$$\bar{F}(x_1, \cdots, x_n) = \exp\left[\begin{array}{c} -\sum_{i=1}^{n} \lambda_i x_i - \sum_{i<j} \lambda_{ij} \max(x_i, x_j) - \sum_{i<j<k} \lambda_{ijk} \max(x_i, x_j, x_k) \\ - \cdots - \lambda_{12\cdots n} \max(x_1, x_2, \cdots, x_n) \end{array}\right].$$

It is possible but too complex to calculate the Laplace transform of the multivariate case. So we will use two asset models with bivariate exponential distribution in our analysis.

Gumbel's Bivariate Exponential Distribution Gumbel [31] suggested the use of a bivariate exponential distribution with the joint survival function

$$\bar{F}(s, t) = P\{X > s, Y > t\} = \exp(-s - t - \delta st).$$

But, in this model the expected returns for both X and Y are equal. So we are suggesting using a bivariate exponential distribution where the joint survival function is

$$\bar{F}(s, t) = \exp[-\lambda_1 s - \lambda_2 t - \lambda_{12} st].$$

Using the marginal distributions

$$P\{X > s\} = \exp(-\lambda_1 s), \quad P\{Y > t\} = \exp(-\lambda_2 t)$$

we can calculate

$$\begin{array}{c} E[X] = \frac{1}{\lambda_1}, \quad \mathrm{Var}(X) = \frac{1}{\lambda_1^2}, \\ E[Y] = \frac{1}{\lambda_2}, \quad \mathrm{Var}(Y) = \frac{1}{\lambda_2^2} \end{array}.$$

1.1.3 The Stochastic Market

The returns of risky assets in a market are random. While we do not know the exact distribution of the returns in general, we are often aware of the factors of variation affecting their distributions, means, variances and covariances with each other. These are the underlying economic, social, political and other factors that affect the parameters in one way or another. As the state of a market changes over time, the returns will change accordingly. It is fair to say that in today's financial markets most of the risks, or variances of asset returns, are due to the changes in local or global factors. Investment decisions are affected by these factors as well as the correlation among asset returns. Modeling a stochastic financial market by a Markov chain is a reasonable approach and this idea dates back to Pye [57]. In the continuous time setting, Norberg [53] considers an interest rate model that is modulated by a Markov process. Recently there is growing interest in the literature to use a stochastic market process in order to modulate various parameters of the financial model to make it more realistic. Hernández-Hernández and Marcus [34], Bielecki et al. [6], Bielecki and Pliska [7], Di Massi and Stettner [45], Stettner ([66], [67]), and Nagai and Peng [51] provide examples on risk-sensitive portfolio optimization with observed, unobserved and partially observed states in Markovian markets. Continuous-time Markov chains with a discrete state space are used in a number of papers including, for example, Bäuerle and Rieder [3], Yin and Zhou [72], and Zhang [74] to modulate model parameters in portfolio selection and stock trading problems. Zariphopoulou [73], Fleming and Hernández-Hernández [28] use diffusion processes for modulating purposes. There are also models where only one of the parameters in modulated. Models of stochastic interest rates with some sort of a Markovian structure are also quiet common as in Korn and Kraft [39] and Elliott and Mamon [26], among others.

Let $R(i)$ denote the random vector of asset returns in any period given that the stochastic market is in state i. The means, variances and covariances of asset returns depend only on the current state of the stochastic market. The market consists of one riskless asset with known return $r_f(i)$ and standard deviation $\sigma_f(i) = 0$ and m risky assets with random returns $R(i) = (R_1(i), R_2(i), \cdots, R_m(i))$ in state i. We let $r_k(i) = E[R_k(i)]$ denote the mean return of the kth asset in state i and $\sigma_{kj}(i) = \text{Cov}(R_k(i), R_j(i))$ denote the covariance between kth and jth asset returns in state i. The excess return of the kth asset in state i is $R^e_k(i) = R_k(i) - r_f(i)$. It follows that

$$r^e_k(i) = E[R^e_k(i)] = r_k(i) - r_f(i) \tag{1.6}$$

$$\sigma_{kj}(i) = \text{Cov}(R^e_k(i), R^e_j(i)). \tag{1.7}$$

Our notation is such that $r(i) = (r_1(i), r_2(i), \cdots, r_m(i))$ and $r^e(i) = (r^e_1(i), r^e_2(i), \cdots, r^e_m(i))$ are column vectors and $r_f(i)$ is a scalar for all i. For any column vector z, z' denotes the row vector representing its transpose.

We define the matrix

$$V(i) = E\left[R^e(i)R^e\,(i)'\right] = \sigma\,(i) + r^e\,(i)\,r^e\,(i)' \tag{1.8}$$

for any state i. Note that the covariance matrix $\sigma\,(i)$ is positive definite for all i so one can easily see that $V(i)$ is also positive definite.

We define X_n as the amount of investor's wealth at period n and correspondingly X_T denotes the final wealth at the end of the investment horizon. The vector $u = (u_1, u_2, \cdots, u_m)$ gives the amounts invested in risky assets $(1, 2, \cdots, m)$. Given any investment policy, the stochastic evolution of the investor's wealth follows the so-called wealth dynamics equation

$$\begin{aligned} X_{n+1}\,(u) &= R\,(Y_n)'\,u + \left(X_n - 1'u\right) r_f\,(Y_n) \\ &= r_f\,(Y_n)\,X_n + R^e\,(Y_n)'\,u \end{aligned} \tag{1.9}$$

where $1 = (1, 1, \cdots, 1)$ is the column vector consisting of 1's.

We will use the notation $E_i[Z] = E\,[Z \mid Y_0 = i]$ and $\mathrm{Var}_i(Z) = E_i[Z^2] - E_i[Z]^2$ to denote the conditional expectation and variance of any random variable Z given that the initial market state is i.

The assumptions regarding the model formulation can be summarized as follows: (a) There is unlimited borrowing and lending at the prevailing return of the riskless asset in any period, (b) Short selling is allowed for all assets in all periods, (c) No capital additions or withdrawals are allowed throughout the investment horizon, and (d) Transaction costs and fees are negligible.

1.1.4 Dynamic Programming Formulation

Dynamic programming is the method used in the derivation of the optimal solution of the multiperiod portfolio selection problem

$$\max_u \; E_i\left[U(Y_T, X_T)\right]$$

where the investor maximizes his expected utility of the terminal wealth X_T at some terminal time T. Let $g_n\,(i, x, u)$ denote the expected utility using the investment policy u in period n and the optimal policies from period $n+1$ to period T given that the market is in state i and the amount of money available for investment is x at period n. Define

$$v_n(i, x) = \max_u g_n(i, x, u)$$

as the optimal expected utility using the optimal policy given that the market is in state i and the amount of money available for investment is x at period n. Then, according to the dynamic programming principle

$$g_n(i,x,u_n) = E\left[v_{n+1}\left(Y_{n+1}, X_{n+1}(u_n)\right)\right]$$

and we can write the dynamic programming equation (DPE) as

$$v_n(i,x) = \max_u E\left[v_{n+1}\left(Y_{n+1}, X_{n+1}(u)\right)\right]$$

which can be rewritten as

$$v_n(i,x) = \max_u \sum_{j\in E} Q\left(i,j\right) E\left[v_{n+1}\left(j, r_f\left(i\right)x + R^e\left(i\right)' u\right)\right] \tag{1.10}$$

for $n = 0, 1, \cdots, T-1$ with the boundary condition $v_T(i,x) = U(i,x)$ for all i. The solution for this problem is found by solving the DPE recursively.

Chapter 2

MODELS WITH PERFECT INFORMATION

In this chapter we will assume that the state of the world as well as the transition probabilities are known by the investor. We analyzed the cases where the utility of the investor is HARA function. Both the exponential utility case and the other HARA utility cases are completed and are summarized in two distinct papers.

2.1 Exponential Utility Function

We assume that the utility of the investor in state i is given by the exponential function

$$U(i, x) = K(i) - C(i) \exp(-x/\beta) \tag{2.1}$$

with $\beta > 0$, $C(i) > 0$ where we can easily see that Pratt-Arrow's measure of absolute risk aversion is simply equal to the constant $-U''(i, x)/U'(i, x) = 1/\beta$ for all i. So, if we check Table 1.1, this is the case where $b = 0$, $a = \beta$. The exponential utility function is one of the most widely used ones to represent investors attitude towards risk in portfolio optimization. It has constant absolute risk aversion given by $1/\beta$ which means that the investor has the same risk preferences for random outcomes independent of his wealth. In continuous time, Merton [46] addressed the problem of utility maximization and showed that if the utility of the investor is exponential, then the value function for any time is also exponential. Samuelson [60] worked on the discrete time version of Merton [46] and showed that similar conclusions apply for the discrete time market. Bertsekas [5] analyzed the multiperiod portfolio optimization problem and showed that, for well-known utility functions, the value function in the dynamic programming algorithm is the same type as the utility function of the investor. In a more recent paper, Tehranchi [71] shows that similar results apply for the exponential utility optimization problem in an incomplete market. Hu, Imkeller and Muller [36] examines the exponential utility maximization problem in an incomplete market when there is a liability to be paid at terminal time and shows that a similar result can be found.

Note from (2.1) that β is that same for all market states so that risk classification of the investor does not depend on the stochastic market. Similarly, we assume that the return for the riskless asset is same for all market states so that $r_f(i) = r_f$ for all i.

Theorem 1 *Let the utility function of the investor be the exponential function (2.1) and suppose that the riskless asset return does not depend on the market state. Then, the optimal solution of the dynamic programming equation (1.10) is*

$$v_n(i,x) = K_n(i) - C_n(i)e^{-x/\beta_n} \tag{2.2}$$

and the optimal portfolio is

$$u_n^*(i,x) = \alpha(i)\beta_{n+1} \tag{2.3}$$

where

$$\beta_n = \frac{\beta}{r_f^{T-n}},\ K_n(i) = Q^{T-n}K(i),\ C_n(i) = \hat{Q}^{T-n}C(i) \tag{2.4}$$

and

$$\hat{Q}(i,j) = Q(i,j)E\left[\exp(-R^e(i)'\alpha(i))\right] \tag{2.5}$$

for all $n = 0, 1, \cdots, T-1$; and $\alpha(i)$ satisfies

$$E\left[R_k^e(i)\exp(-R^e(i)'\alpha(i))\right] = 0 \tag{2.6}$$

for all assets $k = 1, 2, \cdots, m$ and all i.

Proof. We use induction starting with the boundary condition for exponential utility as $v_T(i,x) = K(i) - C(i)\exp(-x/\beta)$ and obtain

$$\begin{aligned} g_{T-1}(i,x,u) &= \sum_{j\in E} Q(i,j)E[U(j, r_f x + R^e(i)'u)] \\ &= -\exp\left(-r_f x/\beta\right)QC(i)E[\exp(-R^e(i)'u/\beta)] + QK(i). \end{aligned}$$

Let $u^* = (u_1^*, u_2^*, \cdots, u_m^*)$ be the optimal amount of money that should be invested in the risky asset so that

$$v_{T-1}(i,x) = \max_u g_{T-1}(i,x,u) = g_{T-1}(i,x,u^*).$$

Taking the derivative of g_{T-1} with respect to u_k we obtain the gradient vector with entries

$$\nabla_k g_{T-1}(i,x,u) = \frac{\partial g_{T-1}(i,x,u)}{\partial u_k} = \exp(-r_f x/\beta)QC(i)\,E\left[R_k^e(i)\exp(-R^e(i)'u/\beta)\right]/\beta \tag{2.7}$$

for all k. If we take the second derivatives of g_{T-1}, we can find the Hessian matrix with entries

$$H_{k,l}(i,x,u) = \frac{\partial^2 g_{T-1}(i,x,u)}{\partial u_k \partial u_l} = -\exp(-r_f x/\beta) QC\,(i)\, E\left[R_k^e(i) R_l^e(i) \exp(-R^e(i)' u/\beta)\right]/\beta^2.$$

Let $z = (z_1, \cdots, z_m)$ be any non-zero column vector where z_i's are real numbers. Then, one can see that $z^T H\,(i)\, z$ is equal to

$$z^T H\,(i)\, z = -\frac{QC(i)}{\beta^2} E\left[(z_1 R_1^e\,(i) + z_2 R_2^e\,(i) + \cdots + z_m R_m^e\,(i))^2 \exp(-(r_f x + R^e(i)' u)/\beta)\right]$$

which is always smaller than or equal to zero since all $C(i)$ are positive. Thus, $H\,(i)$ is negative semi-definite and we can find the optimal solution by setting the gradient (2.7) equal to zero to obtain the optimality condition

$$E\left[R^e(i) \exp(-R^e(i)' u^*(i,x)/\beta)\right] = 0. \tag{2.8}$$

Since there is no dependence on x in (2.8), $u^*(i,x)$ does not depend on x and $u^*(i,x) = u^*(i)$. Letting $\alpha\,(i) = u^*\,(i)\,/\beta$, we obtain $u^*_{T-1}(i,x) = \alpha\,(i)\,\beta$. When the value function at time $T-1$ is rewritten for the optimal policy, we obtain

$$\begin{aligned} v_{T-1}(i,x) &= QC(i) \exp(-r_f x/\beta) E[-\exp(-R^e(i)' u^*(i,x)/\beta)] + QK(i) \\ &= K_{T-1}(i) - C_{T-1}(i) \exp(-x/\beta_{T-1}). \end{aligned}$$

and the value function is still exponential like the utility function and $C_{T-1}(i)$ is positive for all values of i. This shows that the induction hypothesis holds for $n = T-1$.

Suppose now that the induction hypothesis holds for periods $T, T-1, T-2, \cdots, n$. Then, for period $n-1$,

$$\begin{aligned} g_{n-1}(i,x,u) &= \sum_{j \in E} Q(i,j) E[v_n(j, r_f x + R^e(i)' u)] \\ &= -\exp\left(-r_f x/\beta_n\right) QC_n(i) E\left[\exp\left(-R^e(i) u/\beta_n\right)\right] + QK_n(i). \end{aligned}$$

One can easily see that the Hessian matrix of $g_{n-1}\,(i,x,u)$ is negative semi-definite as for $g_{T-1}\,(i,x,u)$. Letting $u^*_{n-1}(i,x)$ be the optimal policy such that

$$v_{n-1}(i,x) = \max_u g_{n-1}(i,x,u) = g_{n-1}(i,x,u^*).$$

If we take the derivative of $g_{n-1}(i,x,u)$ with respect to u_k and set it equal to 0, we get the optimality condition

$$E\left[R^e(i) \exp(-R^e(i)' u^*_{n-1}(i,x)/\beta_n)\right] = 0. \tag{2.9}$$

Since there is no dependence on x in (2.9), $u^*_{n-1}(i,x)$ does not depend on x and $u^*_{n-1}(i,x) = u^*_{n-1}(i)$. Letting $\alpha(i) = u^*_{n-1}(i)/\beta_n$ we obtain $u^*_{n-1}(i,x) = \alpha(i)\beta_n$ and

$$E\left[R^e_k(i)\exp\left(-R^e(i)\alpha(i)\right)\right] = 0$$

If we insert the optimal policy in the value function, we can see that

$$\begin{aligned} v_{n-1}(i,x) &= \sum_{j\in E} Q(i,j)E[-C_n(j)\exp\left(-(r_f x/\beta_n + R^e(i)\alpha(i))\right) + K_n\left(j\right)] \\ &= -\exp\left(-r_f x/\beta_n\right) QC_n(i)E\left[\exp\left(-R^e(i)\alpha(i)\right)\right] + QK_n(i) \\ &= K_{n-1}\left(i\right) - C_{n-1}(i)\exp(-x/\beta_{n-1}) \end{aligned}$$

and this completes the proof. ■

In Theorem 1, we have found a closed-form solution for the optimal portfolio. We can further characterize the optimal policy by noting from (2.6) that the optimal solution satisfies

$$E\left[\left(R_k\left(i\right) - r_f\right)\exp\left(-\left(R\left(i\right) - r_f\right)'\alpha\left(i\right)\right)\right] = 0$$

which implies

$$E\left[\left(R_k\left(i\right) - r_f\right)\exp\left(-R\left(i\right)'\alpha\left(i\right)\right)\right] = 0$$

and

$$E\left[R_k\left(i\right)\exp\left(-R\left(i\right)'\alpha\left(i\right)\right)\right] = r_f E\left[\exp\left(-R\left(i\right)'\alpha(i)\right)\right]$$

or

$$\frac{E\left[R_k\left(i\right)\exp\left(-R\left(i\right)'\alpha\left(i\right)\right)\right]}{E\left[\exp\left(-R\left(i\right)'\alpha\left(i\right)\right)\right]} = r_f \tag{2.10}$$

for all assets $k = 1, 2, \cdots, m$.

A significant characterization implied by the optimal solution (2.3) is that the optimal distribution of wealth invested on the risky assets depend only on the state of the market independent of time. Moreover, it is quite amazing that it is also independent of the wealth level. If the market is in state i in period n, then the total amount of money invested on the risky assets is

$$1'u^*_n(i,x) = 1'\alpha(i)\beta_{n+1} = \frac{\beta}{r_f^{T-(n+1)}}\sum_{k=1}^{m}\alpha_k(i)$$

which does not depend on the current wealth level x. Moreover, the proportion on wealth allocated for asset k is

$$w_k(i) = \frac{\alpha_k(i)}{\sum\limits_{k=1}^{m}\alpha_k(i)} \tag{2.11}$$

which is totally independent of both time n and wealth x. The exponential investor therefore decides by considering the state of the market only. The intuition in this amazing result is in the exponential utility function. Like the memorylessness property of the exponential distribution that is associated with time, the exponential utility function implies a similar property associated with the wealth of the investor. The investor is memoryless in the sense that his current wealth level does not affect how he chooses to allocate his money among the risky assets. However, note that there is randomness involved in this choice due to the randomly changing market conditions. Our results are of course consistent with similar work in the literature on exponential utility functions, but our stochastic market approach makes our model more realistic without causing substantial difficulty in the analysis. Another important observation is that the structure of the optimal portfolio is not affected by the transition matrix Q of the stochastic market. It only depends on the joints distribution of the risky asset returns as prescribed by (2.10). This further implies that the exponential investor is not only memoryless about his wealth, he is also myopic since he does not care much about future states of the market in choosing his portfolios.

2.1.1 Evolution of Wealth and the Exponential Frontier

The evolution of the wealth process X using the optimal policy can be analyzed by the wealth dynamics equation

$$\begin{aligned} X_{n+1} &= r_f X_n + R^e(Y_n)' u_n^*(Y_n, X_n) \\ &= r_f X_n + R^e(Y_n)'\alpha(Y_n)\beta_{n+1} \\ &= r_f X_n + r_f^{n+1-T} R^e(Y_n)'\alpha(Y_n)\beta. \end{aligned} \tag{2.12}$$

Define $A(i)$ as the random variable

$$A(i) = R^e(i)'\alpha(i) \tag{2.13}$$

with mean

$$\bar{\alpha}(i) = E[A(i)] = E\left[R^e(i)'\alpha(i)\right] = r^e(i)'\alpha(i) = r(i)' a(i) - r_f 1'\alpha(i) \tag{2.14}$$

and second moment

$$\tilde{\alpha}(i) = E\left[A(i)^2\right] = E\left[\alpha(i)' R^e(i) R^e(i)'\alpha(i)\right] = \alpha(i)' V(i)\alpha(i) \tag{2.15}$$

which gives the variance

$$\mathrm{Var}\left(A\left(i\right)\right) = \tilde{\alpha}\left(i\right) - \bar{\alpha}\left(i\right)^2. \tag{2.16}$$

The Fourier transform of the random vector $R^e\left(i\right) = \left(R_1^e\left(i\right), R_2^e\left(i\right), \cdots, R_n^e\left(i\right)\right)$ is denoted by

$$\mathcal{F}_i\left(z\right) = E\left[\exp\left(jz'R^e\left(i\right)\right)\right] \tag{2.17}$$

for $z = \left(z_1, z_2, \cdots, z_n\right)$.

Now, we will show that the wealth process is given by

$$X_n = r_f^n X_0 + r_f^{n-T}\beta \sum_{k=0}^{n-1} A\left(Y_k\right) \tag{2.18}$$

using the induction method where the sum on the right-hand side is set to zero when $n = 0$. The induction hypothesis holds trivially for $n = 0$. Suppose (2.18) holds for some $n \geq 0$. If we write X_{n+1} using the wealth dynamics equation (2.12)

$$\begin{aligned} X_{n+1} &= r_f X_n + r_f^{n+1-T} A(Y_n)\beta \\ &= r_f^{n+1} X_0 + r_f^{n+1-T}\beta \sum_{k=0}^{n-1} A\left(Y_k\right) + r_f^{n+1-T} A(Y_n)\beta \\ &= r_f^{n+1} X_0 + r_f^{n+1-T}\beta \sum_{k=0}^{n} A\left(Y_k\right) \end{aligned}$$

we see that the induction hypothesis also holds for $n+1$. So, we can conclude that the wealth process can be written as in (2.18) and for $n = T$ we can find the terminal wealth as

$$X_T = r_f^T X_0 + \beta \sum_{k=0}^{T-1} A\left(Y_k\right). \tag{2.19}$$

Given $X_0 = x_0$, the expected value of the terminal wealth satisfies

$$E_i\left[X_T\right] = r_f^T x_0 + m\left(i, T\right)\beta \tag{2.20}$$

where

$$m\left(i, T\right) = \sum_{k=0}^{T-1} Q^k\left(i, j\right)\bar{\alpha}\left(j\right) = \sum_{k=0}^{T-1} Q^k \bar{\alpha}\left(i\right) \tag{2.21}$$

and the variance of the terminal wealth satisfies

$$\mathrm{Var}_i\left(X_T\right) = \beta^2 \,\mathrm{Var}_i\left(\sum_{k=0}^{T-1} A\left(Y_k\right)\right) = v^2\left(i, T\right)\beta^2 \tag{2.22}$$

where

$$
\begin{aligned}
v^2(i,T) &= \sum_{k=0}^{T-1}\sum_{m=0}^{T-1} \mathrm{Cov}_i\left(A(Y_k), A(Y_m)\right) \\
&= \sum_{k=0}^{T-1} \mathrm{Var}_i\left(A(Y_k)\right) + 2\sum_{k=0}^{T-1}\sum_{m=k+1}^{T-1} \mathrm{Cov}_i\left(A(Y_k), A(Y_m)\right) \\
&= \sum_{k=0}^{T-1} \left(E_i\left[A(Y_k)^2\right] - E_i\left[A(Y_k)\right]^2\right) \\
&\quad +2\sum_{k=0}^{T-1}\sum_{m=k+1}^{T-1} \left(E_i\left[A(Y_k)A(Y_m)\right] - E_i\left[A(Y_k)\right]E_i\left[A(Y_m)\right]\right) \qquad (2.23) \\
&= \sum_{k=0}^{T-1} \left(Q^k\tilde{\alpha}(i) - \left(Q^k\bar{\alpha}(i)\right)^2\right) \qquad (2.24) \\
&\quad +2\sum_{k=0}^{T-1}\sum_{m=k+1}^{T-1} \left(\sum_{j\in E}\sum_{s\in E} Q^k(i,j)\,Q^{m-k}(j,s)\,\bar{\alpha}(j)\,\bar{\alpha}(s) - Q^k\bar{\alpha}(i)\,Q^m\bar{\alpha}(i)\right).
\end{aligned}
$$

We can see that both the return and the standard deviation of X_T depends linearly on β. This shows that the exponential frontier is a line with formula

$$E_i\left[X_T\right] = r_f^T x_0 + \left(\frac{m(i,T)}{v(i,T)}\right)\mathrm{SD}_i\left(X_T\right) \qquad (2.25)$$

where $\mathrm{SD}_i\left(X_T\right)) = \sqrt{\mathrm{Var}_i\left(X_T\right)}$. Also, we can see it cuts the zero-risk line at $E_i\left[X_T\right] = r_f^T x_0$ as expected. The reason for this is that for zero-risk level investor puts all of his money on the risk free asset. The return of the risk free asset until the terminal time T is r^T. So the wealth at the terminal time will be $r_f^T x_0$ for sure. The risk premium for the exponential investor is given by the ratio $m(i,T)/v(i,T)$.

We also calculated the efficient frontier for the mean-variance problem discussed in Çakmak and Özekici [10] where the problem is solved by considering an alternative formulation with the linear-quadratic objective function

$$\mathrm{Max}\ E_i\left[-X_T^2 + \gamma X_T\right]$$

defined parametrically for real γ. They use the stochastic market where both the risky and the riskless assets return distributions depend on the underlying market states that change according to a Markov chain. Since riskless asset returns are independent of the market state in our model, we have taken the returns of the riskless asset the same for all of the market states in their model as a special case. With this additional restriction, it is possible to show that in mean-variance models

$$E_i\left[X_T\right] = r_f^T c\left(i,T\right) x_0 + 0.5\left(1 - c(i,T)\right)\gamma = r_f^T x_0 + \left(0.5\gamma - r_f^T x_0\right)\left(1 - c(i,T)\right) \tag{2.26}$$

and

$$\mathrm{Var}_i\left(X_T\right) = c(i,T)\left(1 - c(i,T)\right)\left(0.5\gamma - r_f^T x_0\right)^2 \tag{2.27}$$

where

$$c\left(i,T\right) = E_i\left[\prod_{k=0}^{T-1}\left(1 - h(Y_k)\right)\right] \tag{2.28}$$

and $h\left(i\right) = r^e\left(i\right)' V(i)^{-1} r^e\left(i\right)$. Putting (2.26) and (2.27) together we obtain the mean-variance efficient frontier

$$E_i\left[X_T\right] = r_f^T x_0 + \left(\frac{\sqrt{1 - c(i,T)}}{\sqrt{c(i,T)}}\right)\mathrm{SD}_i(X_T). \tag{2.29}$$

From (2.29), it can be seen that the mean of the final wealth depends linearly on the standard deviation and, for the zero-risk level, the expected wealth is equal to $E_i\left[X_T\right] = r_f^T x_0$ as expected. Note that the risk premium is now given by the ratio $\sqrt{1 - c(i,T)}/\sqrt{c(i,T)}$. Both exponential and efficient frontiers are linear with the same intercept $r_f^T x_0$, but with different risk premiums.

The distribution of the final wealth other than just the mean and variance is also important. To derive this distribution we will find the Fourier transform

$$E_i\left[\exp\left(j\gamma X_T\right)\right] = \exp\left(j\gamma r_f^T x_0\right) E_i\left[\exp\left(j\gamma\beta\sum_{k=0}^{T-1} A\left(Y_k\right)\right)\right]$$

using the fact that

$$\begin{aligned} E\left[\exp\left(j\gamma\beta\sum_{k=0}^{T-1} A\left(Y_k\right)\right)\middle| Y_0, Y_1, \cdots, Y_{T-1}\right] &= \prod_{k=0}^{T-1} E\left[\exp\left(j\gamma\beta A\left(Y_k\right)\right)|Y_k\right] & (2.30)\\ &= \prod_{k=0}^{T-1} \mathcal{F}_{Y_k}\left(\gamma\beta\alpha\left(Y_k\right)\right). & (2.31)\end{aligned}$$

Therefore, the Fourier transform of the final wealth can be written as

$$\begin{aligned} E_i\left[\exp\left(j\gamma X_T\right)\right] &= \exp\left(j\gamma r_f^T x_0\right)\mathcal{F}_i\left(\gamma\beta\alpha\left(i\right)\right) & (2.32)\\ &\cdot \sum_{i_1, i_2, \cdots, i_{T-1}} Q\left(i, i_1\right) Q\left(i_1, i_2\right)\cdots Q\left(i_{T-2}, i_{T-1}\right)\prod_{k=1}^{T-1}\mathcal{F}_{i_k}\left(\gamma\beta\alpha\left(i_k\right)\right). \end{aligned}$$

In order to get some insight and demonstrate how our results can be used, we consider special cases where asset return distributions are multivariate normal and exponential.

Multivariate Normal Asset Returns

Suppose that portfolio returns have a multivariate normal distribution with mean vector $\mu(i)$ and covariance matrix $\sigma(i)$ in market state i. The Fourier transform of $R^e(i) \sim N(\mu(i) - r_f, \sigma(i))$ is

$$\mathcal{F}_i(z) = E\left[\exp\left(jz'R^e(i)\right)\right] = \exp\left(jz'\left(\mu(i) - r_f\right) - \frac{1}{2}z'\sigma(i)\,z\right)$$

so that

$$E\left[\exp\left(R^e(i)'\alpha(i)\right)\right] = \exp\left(-\left(\mu(i) - r_f\right)'\alpha(i) + \frac{1}{2}\alpha(i)'\sigma\alpha(i)\right). \tag{2.33}$$

It follows from (2.19) and (2.30) that, given the stochastic market process Y, the terminal wealth X_T has the normal distribution. This is also obviously the case if Y changes deterministically so that the market has a dynamic structure as already known from other models in the literature. However, if the market is stochastic as in our case, then (2.30) implies that the terminal wealth distribution is no longer normal; it is a now a mixture of normal distributions.

If we take the derivative of each side of (2.33) with respect to $\alpha_k(i)$ we get

$$E\left[R_k^e(i)\exp\left(R^e(i)'\alpha(i)\right)\right] = \left(\sum_{j\in E}\sigma_{kj}(i)\,\alpha_j(i) - \left(\mu_k(i) - r_f\right)\right) \tag{2.34}$$

$$\exp\left(-\left(\mu(i) - r_f\right)'\alpha(i) + \frac{1}{2}\alpha(i)'\sigma\alpha(i)\right) \tag{2.35}$$

and, since left-hand side of (2.34) is equal to zero for optimality, we can write the optimality condition

$$\sum_{j\in E}\sigma_{kj}(i)\,\alpha_j(i) = \mu_k(i) - r_f$$

or

$$\alpha(i) = \sigma(i)^{-1}(\mu(i) - r_f) = \sigma(i)^{-1}r^e(i) \tag{2.36}$$

and the optimality policy is

$$u_n^*(i,x) = \alpha(i)\,\beta_{n+1} = \sigma(i)^{-1}(\mu(i) - r_f)\frac{\beta}{r_f^{T-n-1}} \tag{2.37}$$

in period n. We can see that the ratios of wealth invested in risky assets in (2.36) are equal to the ratios in the one-fund theorem for the single period problem with $T = 1$ and $n = 0$.

This tells us that if the returns are normally distributed, the portfolios we generate are on the efficient frontier and the portfolio of risky assets is equal to the fund for each period and in any market state. The ratio that is put in the risk-free asset depends on the environment as well as the number of periods till period T and β.

For the multivariate normal distribution case, we can calculate

$$\bar{\alpha}(i) = r^e(i)'\alpha(i) = r^e(i)'\sigma(i)^{-1}r^e(i) \tag{2.38}$$

and

$$\begin{aligned}
\tilde{\alpha}(i) &= \alpha(i)' V(i)\alpha(i) = \left(\sigma(i)^{-1}r^e(i)\right)' \left(\sigma(i) + r^e(i) r^e(i)'\right)\sigma(i)^{-1}r^e(i) \\
&= r^e(i)'\sigma(i)^{-1}r^e(i) + r^e(i)'\sigma(i)^{-1}r^e(i) r^e(i)'\sigma(i)^{-1}r^e(i) \\
&= \bar{\alpha}(i)(1+\bar{\alpha}(i))
\end{aligned} \tag{2.39}$$

where $r^e(i) = \mu(i) - r_f$.

Multivariate Exponential Returns

A number of multivariate exponential distributions are known. A trivial case is the one where the random variables are independent. If the returns of the assets are independent and exponential with parameters $\lambda_k(i)$ for asset k in state i, then

$$\begin{aligned}
\frac{E\left[R_k(i)\exp\left(-R(i)'\alpha(i)\right)\right]}{E\left[\exp\left(-R(i)'\alpha(i)\right)\right]} &= \frac{E\left[R_k(i)\exp\left(-R_k(i)\alpha_k(i)\right)\right]\prod_{j\neq k} E\left[\exp\left(-R_j(i)\alpha_j(i)\right)\right]}{\prod_{j=1}^{m} E\left[\exp\left(-R_j(i)\alpha_j(i)\right)\right]} \\
&= \frac{E\left[R_k(i)\exp\left(-R_k(i)\alpha_k(i)\right)\right]}{E\left[\exp\left(-R_k(i)\alpha_k(i)\right)\right]} = r_f
\end{aligned} \tag{2.40}$$

so that the optimality condition (2.10) can be written as

$$\frac{\mathcal{L}_k'(\alpha_k(i))}{\mathcal{L}_k(\alpha_k(i))} = -r_f$$

where

$$\mathcal{L}_k(\alpha_k(i)) = E\left[\exp\left(-R_k(i)\alpha_k(i)\right)\right] = \frac{\lambda_k(i)}{\alpha_k(i)+\lambda_k(i)}$$

is the Laplace transform of $R_k(i)$, and

$$\mathcal{L}_k'(\alpha_k(i)) = -E\left[R_k(i)\exp\left(-R(i)'\alpha(i)\right)\right] = -\frac{\lambda_k(i)}{\left(\alpha_k(i)+\lambda_k(i)\right)^2}.$$

Therefore,

$$\frac{\mathcal{L}_k'(\alpha_k(i))}{\mathcal{L}_k(\alpha_k(i))} = -\frac{1}{\alpha_k(i)+\lambda_k(i)} = -r_f.$$

and the optimality policy is given as

$$\alpha_k(i) = \frac{\mu_k(i) - r_f}{\mu_k(i)\, r_f} \tag{2.41}$$

where $\mu_k(i) = 1/\lambda_k(i)$ is the mean return.

There are other cases of the multivariate exponential distribution where the marginal distributions are exponential but the random variables are not independent. Marshall and Olkin [44] defined a bivariate exponential distribution where the survival function is defined as

$$\bar{F}(x_1, x_2) = P\{X_1 > x_1, X_2 > x_2\} = \exp\left[-\lambda_1 x_1 - \lambda_2 x_2 - \lambda_{12}\max(x_1, x_2)\right]. \tag{2.42}$$

The underlying idea of the model is explained with a "fatal shock" model where λ_1 is the rate of shocks effecting the first component, λ_2 is the rate of shocks effecting the second component, and λ_{12} is the rate of shocks effecting both components. Using the survival function in (2.42), the Laplace transform can be found as

$$E\left[\exp\left(-\left(sX_1 + tX_2\right)\right)\right] = \frac{(\lambda + s + t)(\lambda_1 + \lambda_{12})(\lambda_2 + \lambda_{12}) + st\lambda_{12}}{(\lambda + s + t)(\lambda_1 + \lambda_{12} + s)(\lambda_2 + \lambda_{12} + t)} \tag{2.43}$$

where $\lambda = \lambda_1 + \lambda_2 + \lambda_{12}$ is the total shock rate. If we assume that there are two assets and the returns of the assets have the Marshall and Olkin [44] bivariate exponential distribution in each state, then using (2.43) we can write

$$\begin{aligned} & E\left[\exp\left(-R(i)'\alpha(i)\right)\right] \\ = & \frac{\left[(\lambda(i) + \alpha_1(i) + \alpha_2(i))(\lambda_1(i) + \lambda_{12}(i))(\lambda_2(i) + \lambda_{12}(i))\right] + \alpha_1(i)\alpha_2(i)\lambda_{12}(i)}{(\lambda(i) + \alpha_1(i) + \alpha_2(i))(\lambda_1(i) + \lambda_{12}(i) + \alpha_1(i))(\lambda_2(i) + \lambda_{12}(i) + \alpha_2(i))} \end{aligned}$$

and taking the derivative with respect to $\alpha_k(i)$

$$\begin{aligned} & E\left[R_k(i)\exp\left(-R(i)'\alpha(i)\right)\right] \\ = & \frac{\left[(\lambda_1(i) + \lambda_{12}(i))(\lambda_2(i) + \lambda_{12}(i)) + \alpha_2(i)\alpha_2(i)\lambda_{12}(i)/\alpha_k(i)\right]}{(\lambda(i) + \alpha_1(i) + \alpha_2(i))(\lambda_1(i) + \lambda_{12}(i) + \alpha_1(i))(\lambda_2(i) + \lambda_{12}(i) + \alpha_2(i))} \\ & - \frac{\left[\lambda(i) + \alpha_1(i) + \alpha_2(i) + \alpha_k(i) + \lambda_k(i) + \lambda_{12}(i)\right]\prod_{j=1,2}(\lambda_1(i) + \lambda_{12}(i))}{(\lambda(i) + \alpha_1(i) + \alpha_2(i))\prod_{j=1,2,k}(\lambda_j(i) + \lambda_{12}(i) + \alpha_j(i))} \\ & - \frac{\alpha_1(i)\alpha_2(i)\lambda_{12}(i)\left[\lambda(i) + \alpha_1(i) + \alpha_2(i) + \alpha_i(i) + \lambda_k(i) + \lambda_{12}(i)\right]}{(\lambda(i) + \alpha_1(i) + \alpha_2(i))^2\prod_{j=1,2,k}(\lambda_j(i) + \lambda_{12}(i) + \alpha_j(i))}. \end{aligned}$$

for $k = 1, 2$ and solve the system of nonlinear equations using (2.10).

The multivariate extension of the bivariate exponential distribution is given by

$$\bar{F}(x_1, \cdots, x_n) = \exp\left[\begin{array}{c} -\sum_{i=1}^{n} \lambda_i x_i - \sum_{i<j} \lambda_{ij} \max(x_i, x_j) - \sum_{i<j<k} \lambda_{ijk} \max(x_i, x_j, x_k) \\ - \cdots - \lambda_{12\cdots n} \max(x_1, x_2, \cdots, x_n) \end{array}\right]$$

and one can use a similar approach to obtain a complex system of nonlinear equations in order to determine the optimal policy.

2.2 Logarithmic Utility Function

In this section, we assume that the utility of the investor in state i is given by the logarithmic function

$$U(i, x) = \begin{cases} K(i) + C(i) \log(x + \beta) & x + \beta > 0 \\ -\infty & x + \beta \leq 0 \end{cases} \tag{2.44}$$

with $C(i) > 0$ where we can easily see that Pratt-Arrow's measure of absolute risk aversion is simply equal to $r(x) = 1/(\beta + x) > 0$ for all i so that $b = 1$ and $a = \beta$ in Table 1.1. Note that β is the same for all market states so that risk classification of the investor does not depend on the stochastic market. Similarly, we assume that the return for the riskless asset is the same for all market states so that $r_f(i) = r_f$ for all i.

We will first consider an optimization problem of the form

$$\max_u E\left[\log\left(R^{e\prime} u + c\right)\right] \tag{2.45}$$

where $c > 0$ is any constant and R^e is any random vector. Now, let

$$A(c) = \left\{u : P\left\{R^{e\prime} u + c > 0\right\} = 1\right\}$$

be the set of all investment policies that gives finite expected utility so that $|E[\log(R^{e\prime} u + c)]| < +\infty$ for $u \in A(c)$. It can be seen that $u = (u_1, u_2, \cdots, u_n) = (0, 0, \cdots, 0) \in A(c)$ satisfies this condition trivially for all $c > 0$. So, $A(c)$ is not empty. Also, let u, $w \in A(c)$, then $R^{e\prime} u + c > 0$, and $R^{e\prime} w + c > 0$ implies that

$$\lambda R^{e\prime} u + (1 - \lambda) R^{e\prime} w + c > 0$$

so that $\lambda u + (1 - \lambda) w \in A(c)$ for all $0 \leq \lambda \leq 1$. Therefore, the solution set $A(c)$ is nonempty and convex. The gradient vector of the objection function $g(u) = E[\log(R^{e\prime} u + c)]$ is given by

$$\bigtriangledown_k g(u) = \frac{\partial g(u)}{\partial u_k} = E\left[\frac{R_k^e}{R_k^{e\prime} u + c}\right]$$

while the Hessian matrix is

$$\bigtriangledown_{k,l}^2 g(u) = \frac{\partial^2 g(u)}{\partial u_k \partial u_l} = -E\left[\frac{R_k^e R_l^e}{\left(R_k^{e\prime} u + c\right)^2}\right]$$

for all k, l.

The first order optimality condition to find the optimal solution of (2.45) is obtained by setting the gradient vector equal to zero so that

$$E\left[\frac{R_k^e}{R^{e\prime} u + c}\right] = 0 \tag{2.46}$$

for all k.

Let $z = (z_1, \cdots, z_m)$ be any non-zero column vector where z_i's are real numbers. Then, one can see that

$$z' \bigtriangledown^2 g(u) z = -E\left[\left(z_1 R_1^e + z_2 R_2^e + \cdots + z_m R_m^e\right)^2 / \left(R^{e\prime} u + c\right)^2\right] \leq 0.$$

Thus, the Hessian matrix $\bigtriangledown^2 g(u)$ is negative semi-definite and if there is a solution $u \in A(c)$ satisfying the first order condition (2.46), it must be optimal. Throughout this section, we assume that the excess returns are such that there is a solution of the first order condition (2.46) in $A(c)$ for all $\{R^e(i)\}$ and $c > 0$.

We shall not dwell with the implications of our assumption on asset returns; but to get some insight, we consider the case when there is only a single risky asset. Let

$$m_l = \sup\{y; P\{R^e \leq y\} = 0\}$$

and

$$m_h = \inf\{y; P\{R_k^e \leq y\} = 1\}$$

so that $P\{R^e \in [m_l, m_h]\} = 1$. This also implies that the condition $R^e u + c > 0$ is satisfied if and only if $u \in (-c/m_h, -c/m_l)$. It should be noted that $m_l \leq 0 \leq m_h$ must be satisfied; otherwise, there exists arbitrage opportunity in the market either by shortselling the riskless asset (if $m_l > 0$) or by shortselling the risky asset (if $m_h < 0$). We know that $\bigtriangledown^2 g$ is always negative and g is concave on $A(c)$. Another observation is that the optimal solution found from the first order condition is $u = 0$ if and only if $r^e = 0$. So, if $r^e = 0$ we can always solve the optimization problem trivially. In the following analysis we will consider the cases when $r^e \neq 0$. There are four possible cases depending on the support of the distribution of R^e as analyzed below.

Case 1 $(m_l = -\infty, m_h = +\infty)$**:** In this case $u \neq 0$ implies that $P\{R^{e\prime}u + c < 0\} > 0$ and $E[\log(R^{e\prime}u + c)] = -\infty$, thus $A(c) = \{0\}$ and the only solution with finite objective function value is $u = 0$. Therefore, $u = 0$ is also the optimal solution which does not necessarily satisfy the first order condition (2.46) except for the case $r^e = 0$ as mentioned earlier.

Case 2 $(m_l = -\infty, m_h < +\infty)$**:** In this case $u > 0$ or $u \leq -c/m_h$ implies that $P\{R^{e\prime}u + c < 0\} > 0$ and $E[\log(R^{e\prime}u + c)] = -\infty$. So, $A(c) = (-c/m_h, 0]$ and for the solution of the first order condition (2.46) to be in the interior of $A(c)$, we need to have $\nabla g(0) < 0$ and $\nabla g(-c/m_h) > 0$. This requires $r^e < 0$ and $E[R^e/(m_h - R^e)] > 0$. However, if $r^e \geq 0$, then the optimal solution is at the boundary $u = 0$. Similarly, if $E[R^e/(m_h - R^e)] \leq 0$, then the optimal solution is at the other boundary $u = -c/m_h$.

Case 3 $(m_l > -\infty, m_h = +\infty)$**:** In this case $u < 0$ or $u \geq -c/m_l$ implies that $P\{R^{e\prime}u + c < 0\} > 0$ and $E[\log(R^{e\prime}u + c)] = -\infty$. So, $A(c) = [0, -c/m_l)$ and for the solution of the first order condition (2.46) to be in the interior of $A(c)$, we need to have $\nabla g(0) > 0$ and $\nabla g(-c/m_l) < 0$. This requires $r^e > 0$ and $E[R^e/(m_l - R^e)] < 0$. However, if $r^e < 0$, then the optimal solution is at the boundary $u = 0$. Similarly, if $E[R^e/(m_h - R^e)] \geq 0$, then the optimal solution is at the other boundary $u = -c/m_l$.

Case 4 $(m_l > -\infty, m_h < +\infty)$**:** In this case $u \leq -c/m_h$ or $u \geq -c/m_l$ implies that $P\{R^{e\prime}u + c < 0\} > 0$ and $E[\log(R^{e\prime}u + c)] = -\infty$. So, $A(c) = (-c/m_h, -c/m_l)$ for the solution of the first order condition to be interior of $A(c)$, we need to have $\nabla g(-c/m_h) > 0$ and $\nabla g(-c/m_l) < 0$. This requires $E[R^e/(m_h - R^e)] > 0$ and $E[R^e/(m_l - R^e)] < 0$. However, if $E[R^e/(m_h - R^e)] \leq 0$, then the optimal solution is at the boundary $u = -c/m_h$. Similarly, if $E[R^e/(m_h - R^e)] \geq 0$, then the optimal solution is at the other boundary $u = -c/m_l$.

We will now show that the log utility function is meaningful for an investor with $x_n + \beta/r_f^{T-n} > 0$ at period n where x_n is the wealth in period n. Suppose $x_n + \beta/r_f^{T-n} \leq 0$; then using the strategy of only buying risk-free bonds in each period, the investor will have a terminal wealth of $r_f^{T-n}x_n$ with utility equal to $-\infty$ since $r_f^{T-n}x_n + \beta = r_f^{T-n}(x_n + \beta/r_f^{T-n}) \leq$

0. For any other strategy, the final wealth should satisfy

$$P\{X_T \leq r_f^{T-n} x_n\} > 0$$

according to no arbitrage condition. Otherwise, if the probability $P\left\{X_T \leq r_f^{T-n} x_n\right\} = 0$ (or $P\left\{X_T > r_f^{T-n} x_n\right\} = 1$), then an arbitrage opportunity exists by selling bonds. We can therefore write

$$P\{X_T + \beta \leq r_f^{T-n} x_n + \beta\} > 0$$

and

$$P\{X_T + \beta \leq 0\} > 0$$

which means that the investor has $-\infty$ terminal utility for any investment strategy and any policy is therefore optimal.

At the beginning, if $x_0 + \beta/r_f^T \leq 0$, then any policy leads to $-\infty$ utility. We therefore suppose that $x_0 + \beta/r_f^T > 0$. Then, the policy of investing only on the risk-free asset for n periods leads to $x_n = x_0 r_f^n$ and

$$x_n + \frac{\beta}{r_f^{T-n}} = x_0 r_f^n + \frac{\beta}{r_f^{T-n}} = r_f^n \left(x_0 + \frac{\beta}{r_f^T} \right) > 0.$$

Since the investor selects a policy optimally to maximize the expected terminal utility, we can assume without loss of generality that $x_n + \beta/r_f^{T-n} > 0$ (or $r_f^{T-n} x_n + \beta > 0$) for any $n = 0, 1, \cdots, T$.

Theorem 2 *Let the utility function of the investor be the logarithmic function (2.44) and suppose that the riskless asset return does not depend on the market state. Then, the optimal solution of the dynamic programming equation (1.10) is*

$$v_n(i, x) = K_n(i) + C_n(i) \log(x + \beta_n)$$

and the optimal portfolio is

$$u_n^*(i, x) = \alpha(i)(r_f x + \beta_{n+1}) \tag{2.47}$$

where

$$\beta_n = \frac{\beta}{r_f^{T-n}}, \; K_n = Q^{T-n} K + \left(\sum_{m=0}^{T-n-1} Q^m \hat{Q}_\alpha Q^{T-n-1-m} \right) C, C_n = Q^{T-n} C \tag{2.48}$$

and

$$\hat{Q}_\alpha(i,j) = E\left[\log(r_f\left(1 + R^e(i)'\alpha(i)\right))\right] Q(i,j)$$

for $n = 0, 1, \cdots, T-1$*; where* $\alpha(i)$ *satisfies*

$$E\left[\frac{R_k^e(i)}{1 + R^e(i)'\alpha(i)}\right] = 0 \tag{2.49}$$

for all assets $k = 1, 2, \cdots, m$ *and all* i*.*

Proof. We use induction starting with the boundary condition $v_T(i,x) = C(i)\log(x+\beta) + K(i)$ and obtain

$$\begin{aligned} g_{T-1}(i,x,u) &= \sum_{j\in E} Q(i,j)E[U(j, r_f x + R^e(i)'u)] \\ &= QK(i) + QC(i)E[\log(r_f x + R^e(i)'u + \beta)] \end{aligned}$$

for all available investment strategies. Let u^* be the optimal amount of money that should be invested in the risky asset so that

$$v_{T-1}(i,x) = \max_u g_{T-1}(i,x,u) = g_{T-1}(i,x,u^*).$$

One can see that the objection function $g_{T-1}(i,x,u)$ is in the form of the objection function in (2.45) where $c = r_f x + \beta > 0$. So, the objective function is concave since $QC(i) = \sum_{j\in E} Q(i,j)C(j) > 0$ and, with our assumption on $\{R^e(i)\}$, the optimal policy can be found using the first order condition

$$E\left[\frac{R_k^e(i)}{r_f x + R^e(i)'u^*_{T-1}(i,x) + \beta}\right] = 0$$

for all $k = 1, 2, \cdots, m$. Defining the vector function $\alpha(i,x) = (\alpha_1(i,x), \alpha_2(i,x), \cdots, \alpha_m(i,x))$ such that $\alpha(i,x) = u^*(i,x)/(r_f x + \beta)$ we obtain $u^*_{T-1}(i,x) = \alpha(i,x)(r_f x + \beta)$ so the optimality condition can be rewritten as

$$E\left[\frac{R^e(i)}{r_f x + R^e(i)'\alpha(i,x)(r_f x + \beta) + \beta}\right] = E\left[\frac{R^e(i)}{(r_f x + \beta)(1 + R^e(i)'\alpha(i,x))}\right] = 0$$

and, since $r_f x + \beta > 0$, we have

$$E\left[\frac{R_k^e(i)}{1 + R^e(i)'\alpha(i,x)}\right] = 0. \tag{2.50}$$

Since (2.50) holds for every x we can say that α does not depend on x and $\alpha_k(i,x) = \alpha_k(i)$ for all $k = 1, 2, \cdots, m$. We can write the optimal policy as $u^*_{T-1}(i,x) = \alpha(i)\,(r_f x + \beta)$ where $\alpha(i)$ satisfies

$$E\left[\frac{R^e_k(i)}{1 + R^e(i)'\alpha(i)}\right] = 0$$

for all $k = 1, 2, \cdots, m$. When the value function at time $T-1$ is rewritten for the optimal policy, we obtain

$$\begin{aligned}
v_{T-1}(i,x) &= \sum_{j\in E} Q(i,j)E[K(j) + C(j)\log(r_f x + R^e(i)'\alpha(i)\,(r_f x + \beta) + \beta)] \\
&= QK(i) + QC(i)\left[E\left[\log\left(r_f(1 + R^e(i)'\alpha(i))\right)\right] + \log(x + \beta/r_f)\right] \\
&= QK(i) + \hat{Q}_\alpha C(i) + QC(i)\log(x + \beta/r_f) \\
&= K_{T-1}(i) + C_{T-1}(i)\log(x + \beta_{T-1})
\end{aligned}$$

and the value function is still logarithmic like the utility function. This follows by noting that $K_{T-1} = QK + \hat{Q}_\alpha C$ and $C_{T-1} = QC$ in (2.48). This completes the proof for $n = T-1$.

Suppose now that the induction hypothesis holds for periods $T, T-1, T-2, \cdots, n$. Then, for period $n-1$,

$$\begin{aligned}
g_{n-1}(i,x,u) &= \sum_{j\in E} Q(i,j)E[v_n(j, r_f x + R^e(i)'u + \beta_n)] \\
&= QK_n(i) + QC_n\,(i)\,E[\log(r_f x + R^e(i)'u + \beta_n)]. \qquad (2.51)
\end{aligned}$$

Let u^* be the optimal policy such that

$$v_{n-1}(i,x) = \max_u g_{n-1}(i,x,u) = g_{n-1}(i,x,u^*).$$

It is clear, once again, that the objective function $g_{n-1}(i,x,u)$ is in the form of the objection function in (2.45) with $c = r_f x_n + \beta_n > 0$ and it is concave since $QC_n = Q^{T-n+1}C > 0$. The optimal solution can be found by using the first order condition

$$E\left[\frac{R^e_k(i)}{r_f x + R^e(i)'u^*_{n-1}(i,x) + \beta_n}\right] = 0$$

for $k = 1, 2, \cdots, m$.

Letting $\alpha(i,x) = u^*_{n-1}(i,x)/\,(r_f x + \beta_n)$ we obtain $u^*_{n-1}(i,x) = \alpha(i,x)\ (r_f x + \beta_n)$ and

$$E\left[\frac{R^e_k(i)}{1 + R^e(i)'\alpha(i,x)}\right] = 0$$

where $\alpha(i,x)$ does not depend on the period n and on x as in equation (2.50). Therefore, we can write $\alpha(i,x) = \alpha(i)$ and the optimal policy is $u^*_{n-1}(i,x) = \alpha(i)\,(r_f x + \beta_n)$ where $\alpha(i)$ satisfies

$$E\left[\frac{R^e_k(i)}{1+R^e(i)'\alpha(i)}\right] = 0$$

for all $k = 1, 2, \cdots, m$. If we insert the optimal policy in the value function using (2.51), we can see that

$$\begin{aligned}
v_{n-1}(i,x) &= QK_n(i) + QC_n(i)E[\log(r_f x + R^e(i)'\alpha(i)\,(r_f x + \beta_n) + \beta_n)] \\
&= QK_n(i) + QC_n(i)\left(E\left[\log\left(r_f(1 + R^e(i)'\alpha(i))\right)\right] + \log(x + \beta_n/r_f)\right) \\
&= QK_n(i) + \hat{Q}_\alpha C_n(i) + QC_n(i)\log(x + \beta_n/r_f) \\
&= K_{n-1}(i) + C_{n-1}(i)\log(x + \beta_{n-1})
\end{aligned}$$

and the value function is still logarithmic. Note that the recursions $K_{n-1} = QK_n + \hat{Q}_\alpha C_n$ and $C_{n-1} = QC_n$ with boundary values $K_T = K$ and $C_T = C$ give the explicit solutions in (2.48). This completes the proof. ■

In Theorem 2, we have found a closed-form solution (2.47) for the optimal portfolio. We can further characterize the optimal policy if the return distributions are discrete. Suppose that there is a single risky asset and the return distribution is such that excess return of the asset is equal to $u\,(i)$ with probability $p\,(i)$ and $d\,(i)$ with probability $(1 - p\,(i))$ with $d\,(i) < 0 < u\,(i)$. This assumption is required for the no-arbitrage requirement to hold. From (2.49), the optimality condition is

$$p\,(i)\left(\frac{u\,(i)}{1 + u\,(i)\,\alpha\,(i)}\right) + (1 - p\,(i))\left(\frac{d\,(i)}{1 + d\,(i)\,\alpha\,(i)}\right) = 0$$

which can be solved to find

$$\alpha\,(i) = -\frac{d\,(i) + p\,(i)\,(u\,(i) - d\,(i))}{d\,(i)\,u\,(i)} = -\frac{r^e\,(i)}{d\,(i)\,u\,(i)}.$$

Then, we can see that $\alpha\,(i) = 0$ if and only if $r^e\,(i) = 0$. Also, $\alpha\,(i) > 0$ if $r^e\,(i) > 0$ and $\alpha\,(i) < 0$ if $r^e\,(i) < 0$.

Suppose that there is a single risky asset and the excess return of the asset is equal to $a_k\,(i)$ with probability $p_k\,(i)$ for $k = 1, 2, \cdots, n_i$ in state i where $\sum_{k=1}^{n_i} p_k\,(i) = 1$. Then, from (2.49), the optimality condition is

$$\sum_{k=1}^{n_i} p_k\,(i)\left(\frac{a_k\,(i)}{1 + a_k\,(i)\,\alpha\,(i)}\right) = 0. \tag{2.52}$$

It is clear that (2.52) is a polynomial equation with power n_i, and solving this will give the optimal $\alpha(i)$ values.

Note that the structure of the optimal solution in (2.47) is such that the optimal distribution of wealth invested in the risky assets depend only on the state of the market independent of time. If the market is in state i in period n, then the total amount of money invested on the risky assets is

$$1'u_n^*(i,x) = 1'\alpha(i)(r_f x + \beta_{n+1}) = \left(r_f x + \frac{\beta}{r_f^{T-(n+1)}}\right)\sum_{k=1}^{m}\alpha_k(i)$$

and the proportion on wealth allocated for asset k in the risky portfolio is

$$w_k(i) = \frac{\alpha_k(i)}{\sum_{k=1}^{m}\alpha_k(i)} \tag{2.53}$$

which is totally independent of both time n and wealth x. The optimal policy specified by (2.47) is not static in time since it depends on n, and it is not memoryless in wealth since it depends on x. However, (2.53) clearly indicates that the composition of the risky part of the optimal portfolio only depends on the market state. The risky portfolio composition is both static and memoryless. It satisfies the separation property in the sense that it represents the single fund of risky assets that logarithmic investors choose. The amount of total wealth allocated for risky assets depend on the level of wealth, but the composition of the risky assets depend only on the market state. This composition, however, is random due to the randomly changing market conditions in time. Our results are of course consistent with similar work in the literature on logarithmic utility functions, but the stochastic market approach makes our model more realistic without causing substantial difficulty in the analysis. Another important observation is that the structure of the optimal portfolio is not affected by the transition matrix Q of the stochastic market. It only depends on the joint distribution of the risky asset returns as prescribed by (2.49) in a given market state, irrespective of future expectations on the stochastic market.

2.2.1 Evolution of Wealth and the Logarithmic Frontier

The evolution of the wealth process X using the optimal policy can be analyzed by the wealth dynamics equation

$$\begin{aligned} X_{n+1} &= r_f X_n + R^e(Y_n)' u_n^*(Y_n, X_n) \\ &= r_f X_n + R^e(Y_n)'\alpha(Y_n)\left(r_f X_n + \beta_{n+1}\right) \\ &= r_f X_n \left(1 + A(Y_n)\right) + r_f^{n+1-T} A(Y_n)\beta \end{aligned} \tag{2.54}$$

where we define $A(i)$ as the random variable

$$A(i) = R^e(i)'\alpha(i) = \sum_{k=1}^{m} \alpha_k(i) R_k^e(i) \tag{2.55}$$

for any state i.

Note that the wealth process satisfies

$$X_{n+1} + \beta_{n+1} = \left(1 + A(Y_n)\right) r_f \left(X_n + \beta_n\right)$$

since $\beta_{n+1} = r_f \beta_n$. Recall that we initially assumed that $X_0 + \beta_0 = x_0 + \beta / r_f^T > 0$ since the objective function value is $-\infty$ for any policy otherwise. Now, suppose that $X_n + \beta_n > 0$ for some n. According to our assumption on excess returns we know that the optimal investment policy $u^* \in A(c)$ satisfies the condition

$$P\left\{R^{e'}(Y_n)u^* + c > 0\right\} = 1$$

with $c = r_f X_n + \beta_{n+1} = r_f\left(X_n + \beta_n\right) > 0$. Since $u^* = \alpha(Y_n)\left(r_f X_n + \beta_{n+1}\right)$, we get

$$P\left\{R^{e'}(Y_n)\,\alpha(Y_n)\left(r_f X_n + \beta_{n+1}\right) + r_f X_n + \beta_{n+1} > 0\right\} = 1$$

and

$$P\left\{\left(1 + A(Y_n)\right) r_f \left(X_n + \beta_n\right) > 0\right\} = P\{X_{n+1} + \beta_{n+1} > 0 = 1.$$

This argument clearly shows that if $X_0 + \beta_0 > 0$ as we initially assume, then $X_n + \beta_n > 0$ for all n using the optimal policy. We are therefore justified in supposing implicitly that this condition is always satisfied before the statement of Theorem 2.

It is clear that $A(i)$ is a linear combination of the excess returns of the risky assets with mean

$$a(i) = E\left[A(i)\right] = r^e(i)'\alpha(i) \tag{2.56}$$

and second moment

$$s(i) = E\left[A(i)^2\right] = E\left[\alpha(i)' R^e(i) R^e(i)' \alpha(i)\right] = \alpha(i)' V(i) \alpha(i) \qquad (2.57)$$

which gives the variance

$$\mathrm{Var}(A(i)) = \alpha(i)' V(i)\alpha(i) - \alpha(i)' r^e(i) r^e(i)' \alpha(i) = \alpha(i)' \sigma(i) \alpha(i).$$

As a computational formula that we will use frequently in the following analysis, we define

$$E_i[h_n(g(Y_0), g(Y_1), \cdots, g(Y_n))] = \sum_{i_1, \cdots, i_n \in E} Q(i, i_1) \cdots Q(i_{n-1}, i_n) h_n(g(i), g(i_1), \cdots, g(i_n)) \qquad (2.58)$$

which provides an explicit expression to compute expectations for any deterministic functions h_n and g of the random vector $\overline{Y}_n = (Y_0, Y_1, \cdots, Y_n)$ of Markovian states. For notational simplification in our analysis, we will let

$$g\left(\overline{Y}_n\right) = (g(Y_0), g(Y_1), \cdots, g(Y_n))$$

for any function g defined on E. We will use the representation (2.58) whenever necessary to economize on the notation and note that this provides an exact computational formula. In particular, if $h_n(x_0, x_1, \cdots, x_n) = \prod_{k=0}^n x_k$, then letting $f_n(i) = E_i\left[h_n\left(g\left(\overline{Y}_n\right)\right)\right]$ we obtain

$$\begin{aligned} f_n(i) &= E_i\left[\prod_{k=0}^n g\left(\overline{Y}_k\right)\right] = g(i) \sum_{j \in E} Q(i,j) f_{n-1}(j) \\ &= \sum_{j \in E} Q_g(i,j) f_{n-1}(j) \\ &= Q_g f_{n-1}(j) \end{aligned} \qquad (2.59)$$

where we define the matrix Q_g such that $Q_g(i,j) = g(i) Q(i,j)$ for all i, j. Using the boundary condition $f_0(i) = g(i)$ and the recursion (2.59), the explicit solution is

$$f_n(i) = E_i\left[\prod_{k=0}^n g(Y_k)\right] = Q_g^n g(i) \qquad (2.60)$$

and $f_n = Q_g^n g$ is simply the product of the matrix Q_g^n by the vector g. For computational analysis we use (2.60) whenever appropriate.

Define

$$\mathbb{C}_n\left(x_1, x_2, \cdots, x_n\right) = \prod_{k=1}^{n}\left(1 + x_k\right) - 1$$

as the sum of all combinations of the products of n variables for $n \geq 1$, and set $\mathbb{C}_0 = 0$. Note that using (2.60) we can compute

$$E_i\left[\mathbb{C}_n\left(h\left(\overline{Y}_{n-1}\right)\right)\right] = E_i\left[\mathbb{C}_n\left(h\left(Y_0\right), h\left(Y_1\right), \cdots, h\left(Y_{n-1}\right)\right)\right] = Q_g^{n-1} g\left(i\right) - 1$$

explicitly for $n \geq 1$ and any function h by setting $g\left(i\right) = 1 + h\left(i\right)$.

Now, we will show that the wealth process is

$$X_n = r_f^n X_0 \prod_{k=0}^{n-1}\left(1 + A\left(Y_k\right)\right) + r_f^{n-T} \beta \mathbb{C}_n\left(A\left(\overline{Y}_{n-1}\right)\right) \tag{2.61}$$

using induction where the product on the right hand side is set to 1 when $n = 0$. The induction hypothesis holds trivially for $n = 0$. Suppose (2.61) holds for some $n \geq 0$. If we write X_{n+1} using the wealth dynamics equation (2.54)

$$\begin{aligned} X_{n+1} &= r_f X_n\left(1 + A(Y_n)\right) + r_f^{n+1-T} A(Y_n)\beta \\ &= r_f^{n+1} X_0 \prod_{k=0}^{n}\left(1 + A\left(Y_k\right)\right) + r_f^{n+1-T}\beta\left[\left(1 + A(Y_n)\right)\mathbb{C}_n\left(A\left(\overline{Y}_{n-1}\right)\right) + A(Y_n)\right] \\ &= r_f^{n+1} X_0 \prod_{k=0}^{n}\left(1 + A\left(Y_k\right)\right) + r_f^{n+1-T}\beta\mathbb{C}_{n+1}\left(A\left(\overline{Y}_n\right)\right) \end{aligned}$$

and we see that the induction hypothesis also holds for $n+1$. So, we conclude that the wealth process can be written as in (2.61) and, for $n = T$, we can find the terminal wealth as

$$\begin{aligned} X_T &= r_f^T X_0 \prod_{k=0}^{T-1}\left(1 + A\left(Y_k\right)\right) + \beta\mathbb{C}_T\left(A\left(\overline{Y}_{T-1}\right)\right) \\ &= r_f^T X_0 + \left(r_f^T X_0 + \beta\right)\mathbb{C}_T\left(A\left(\overline{Y}_{T-1}\right)\right). \end{aligned} \tag{2.62}$$

It is clear from (2.61) and (2.62) that the random variables $\{A(i)\}$ will play a key role in any probabilistic analysis involving the wealth process X. Given $X_0 = x_0$, the expected value of the terminal wealth satisfies

$$E_i\left[X_T\right] = r_f^T x_0 + \left(r_f^T x_0 + \beta\right) m_l\left(i, T\right) \tag{2.63}$$

where

$$m_l\left(i, T\right) = E_i\left[\mathbb{C}_T\left(A\left(\overline{Y}_{T-1}\right)\right)\right] \tag{2.64}$$

and the variance of the terminal wealth satisfies

$$\mathrm{Var}_i\left(X_T\right) = \left(r_f^T x_0 + \beta\right)^2 v_l^2\left(i, T\right) \tag{2.65}$$

where

$$v_l^2\left(i, T\right) = \mathrm{Var}_i\left(\mathbb{C}_T\left(A\left(\overline{Y}_{T-1}\right)\right)\right). \tag{2.66}$$

Using the fact that Y is a Markov chain with transition matrix Q and the distributions of the random variables $\{A(i)\}$, one can easily obtain computational formulas. In particular,

$$m_l\left(i, T\right) = E_i\left[\prod_{k=0}^{T-1}\left(1 + A\left(Y_k\right)\right) - 1\right] = E_i\left[E_i\left[\prod_{k=0}^{T-1}\left(1 + A\left(Y_k\right)\right) - 1 \middle| Y_1, \cdots, Y_{T-1}\right]\right]$$

and, since the returns in different periods are independent given the market states, we obtain

$$\begin{aligned} m_l\left(i, T\right) &= E_i\left[\prod_{k=0}^{T-1}\left(1 + a\left(Y_k\right)\right) - 1\right] = E_i\left[\mathbb{C}_T\left(a\left(\overline{Y}_{T-1}\right)\right)\right] \\ &= Q_g^{T-1} g\left(i\right) - 1 \end{aligned} \tag{2.67}$$

with $g\left(i\right) = 1 + a\left(i\right)$.

To determine $v_l^2\left(i, T\right)$, we first calculate the second moment

$$\begin{aligned} E_i\left[\mathbb{C}_T\left(A\left(\overline{Y}_{T-1}\right)\right)^2\right] &= E_i\left[E_i\left[\left(\prod_{k=0}^{T-1}\left(1 + A\left(Y_k\right)\right) - 1\right)^2 \middle| Y_1, \cdots, Y_{T-1}\right]\right] \\ &= E_i\left[E_i\left[\prod_{k=0}^{T-1}\left(1 + A\left(Y_k\right)\right)^2\right.\right. \\ &\qquad \left.\left. - 2\prod_{k=0}^{T-1}\left(1 + A\left(Y_k\right)\right) + 1 \middle| Y_1, \cdots, Y_{T-1}\right]\right] \\ &= E_i\left[\prod_{k=0}^{T-1}\left(1 + 2a\left(Y_k\right) + s\left(Y_k\right)\right) - 1 - 2\left(\prod_{k=0}^{T-1}\left(1 + a\left(Y_k\right)\right) - 1\right)\right] \\ &= E_i\left[\mathbb{C}_T\left(2a\left(\overline{Y}_{T-1}\right) + s(\overline{Y}_{T-1})\right)\right] - 2E_i\left[\mathbb{C}_T\left(a\left(\overline{Y}_{T-1}\right)\right)\right] \end{aligned} \tag{2.68}$$

since the returns in different periods are independent given the market states. Therefore, we can write

$$E_i\left[\mathbb{C}_T\left(A\left(\overline{Y}_{T-1}\right)\right)^2\right] = E_i\left[\mathbb{C}_T\left(2a\left(\overline{Y}_{T-1}\right) + s\left(\overline{Y}_{T-1}\right)\right)\right] - 2E_i\left[\mathbb{C}_T\left(a\left(\overline{Y}_{T-1}\right)\right)\right]$$

and the variance can be found as

$$\begin{aligned} v_l^2(i,T) &= E_i\left[\mathbb{C}_T\left(2a\left(\overline{Y}_{T-1}\right)+s\left(\overline{Y}_{T-1}\right)\right)\right] \\ &\qquad -2E_i\left[\mathbb{C}_T\left(a\left(\overline{Y}_{T-1}\right)\right)\right]-E_i\left[\mathbb{C}_T\left(a\left(\overline{Y}_{T-1}\right)\right)\right]^2 \qquad (2.69) \\ &= Q_{g_1}^{T-1}g_1(i)-\left(Q_g^{T-1}g(i)\right)^2 \qquad (2.70) \end{aligned}$$

where $g_1(i) = 1+2a(i)+s(i)$ and $g(i) = 1+a(i)$. The mean (2.63) and variance (2.65) of the terminal wealth can thus be computed explicitly using (2.67) and (2.70) where $\{(a(i), s(i))\}$ are determined from (2.56) and (2.57). The distribution of the final wealth other than just the mean and variance is also important. Using (2.62), this distribution can be characterized through its Fourier transform

$$\begin{aligned} E_i\left[\exp\left(j\lambda X_T\right)\right] &= \exp\left(j\lambda r_f^T x_0\right)E_i\left[\exp\left(j\lambda\left(r_f^T x_0+\beta\right)\mathbb{C}_T\left(A\left(\overline{Y}_{T-1}\right)\right)\right)\right] \\ &= \exp\left(j\lambda r_f^T x_0\right)E_i\left[\mathcal{F}_T\left(Y_0,Y_1,\cdots,Y_{T-1};\lambda\left(r_f^T x_0+\beta\right)\right)\right] \end{aligned}$$

where

$$\mathcal{F}_T(i,i_1,\cdots,i_{T-1};\gamma) = E\left[\exp\left(j\gamma\mathbb{C}_T\left(A(i),A(i_1),\cdots,A(i_{T-1})\right)\right)\right]$$

is the Fourier transform of $\mathbb{C}_T(A(i), A(i_1), \cdots, A(i_{T-1}))$ for independent random variables $A(i), A(i_1), \cdots, A(i_{T-1})$. When $T = 2$, for example, this transform becomes

$$E_i\left[\exp\left(j\lambda X_2\right)\right] = \exp\left(j\lambda r_f^2 x_0\right)\sum_{k\in E}Q(i,k)\mathcal{F}_2\left(i,k;\lambda\left(r_f^2x_0+\beta\right)\right)$$

where $\mathcal{F}_2(i,k;\gamma) = E[\exp(j\gamma(A(i)+A(k)+A(i)A(k)))]$ for independent random variables $A(i)$ and $A(k)$. The mean, variance and Fourier transform of the final wealth can be computed once the means, variances and Fourier transforms of the product of any combination of independent random variables in $\{A(i)\}$ are known.

We can clearly see from (2.63) and (2.65) that both the return and the standard deviation of X_T depends linearly on β. This shows that the logarithmic frontier is the straight line

$$E_i[X_T] = r_f^T x_0 + \left(\frac{m_l(i,T)}{v_l(i,T)}\right)\mathrm{SD}_i(X_T) \qquad (2.71)$$

where $\mathrm{SD}_i(X_T)) = \sqrt{\mathrm{Var}_i(X_T)}$. In other words, the expected value and standard deviation of the terminal wealth fall on this straight line when they are calculated and plotted for different values of β. Also, it cuts the zero-risk line at $E_i[X_T] = r_f^T x_0$ as expected. The reason for this is that for zero-risk level investor puts all of his money on the riskless asset. The return of the riskless asset until the terminal time T is r_f^T, and the wealth at the terminal time will be $r_f^T x_0$ for sure. The risk premium for the logarithmic investor is given by the ratio $m_l(i,T)/v_l(i,T)$.

The case with exponential utility is considered in 2.1.1 where the utility function $U(i,x) = K(i) - C(i)\exp(-x/\beta)$ and the optimal solution has the simpler structure

$$u_n^*(i,x) = \alpha(i)\beta_{n+1} \tag{2.72}$$

where $\beta_n = \beta / r_f^{T-n}$ and $\alpha(i)$ satisfies

$$E\left[R^e(i)\exp(-R^e(i)'\alpha(i))\right] = 0. \tag{2.73}$$

The optimal portfolio is separable in the sense that the amounts of money invested in the risky assets by exponential investors are independent of their wealth levels. For computational purposes we only need to find $\alpha(i)$ for any market state i to determine the single fund of risky assets. The total investment also depends only on the period n in a simple way as prescribed by (2.72). They also show that the terminal wealth is on the exponential frontier represented by the straight line

$$E_i[X_T] = r_f^T x_0 + \left(\frac{m_e(i,T)}{v_e(i,T)}\right)\mathrm{SD}_i(X_T) \tag{2.74}$$

where

$$m_e(i,T) = E_i\left[\sum_{k=0}^{T-1} A(Y_k)\right] = \sum_{k=0}^{T-1}\sum_{j\in E} Q^k(i,j)a(i) = \sum_{k=0}^{T-1} Q^k a(i) \tag{2.75}$$

and

$$\begin{aligned} v_e^2(i,T) &= \sum_{k=0}^{T-1}\left(Q^k s(i) - \left(Q^k a(i)\right)^2\right) \qquad (2.76)\\ &\quad +2\sum_{k=0}^{T-1}\sum_{m=k+1}^{T-1}\left(\sum_{j\in E}\sum_{l\in E} Q^k(i,j)\,Q^{m-k}(j,l)\,a(j)\,a(l) - Q^k a(i)\,Q^m a(i)\right). \end{aligned}$$

Therefore, in all cases involving logarithmic, and exponential utility functions the relationship between the expected value and standard deviation of the terminal wealth is represented by a linear frontier. These are given by (2.71), and (2.74) respectively for these two cases.

2.2.2 Simple Logarithmic Utility Function

We now consider the special case of a simple logarithmic utility function with $\beta = 0$ so that

$$U(i,x) = C(i)\log(x) + K(i) \tag{2.77}$$

with $C(i) > 0$ where we can easily see that $r(x) = 1/x$. However, we remove the restriction that $r_f(i) = r_f$ and the riskless return depends on the market state.

Theorem 3 *Let the utility function of the investor be the simple logarithmic function (2.77). Then, the optimal solution of the dynamic programming equation (1.10) is*

$$v_n(i,x) = K_n(i) + C_n(i)\log(x)$$

and the optimal portfolio is

$$u_n^*(i,x) = \alpha(i) r_f(i) x \tag{2.78}$$

where

$$C_n = Q^{T-n}C, K_n = Q^{T-n}K + \left(\sum_{m=0}^{T-n-1} Q^m \hat{Q}_\alpha Q^{T-n-1-m}\right) C$$

and

$$\hat{Q}_\alpha(i,j) = E\left[\log(r_f(i)\left(1 + R^e(i)'\alpha(i)\right))\right] Q(i,j)$$

for all $n = 0, 1, \cdots, T-1$; *and* $\alpha(i)$ *satisfies*

$$E\left[\frac{R_k^e(i)}{1 + R^e(i)'\alpha(i)}\right] = 0 \tag{2.79}$$

for all assets $k = 1, 2, \cdots, m$ *independent of period* n *and all* i.

Proof. We use induction starting with the boundary condition $v_T(i,x) = C(i)\log(x) + K(i)$ and obtain

$$\begin{aligned} g_{T-1}(i,x,u) &= \sum_{j\in E} Q(i,j)E[U(j, r_f(i)x + R^e(i)'u)] \\ &= QC(i)E[\log(r_f(i)x + R^e(i)'u)] + QK(i) \end{aligned}$$

where

$$(r_f(i)x + R^e(i)'u) > 0$$

for all available investment strategies. Let $u^* = (u_1^*, u_2^*, \cdots, u_m^*)$ be the optimal amount of money that should be invested in the risky asset so that

$$v_{T-1}(i,x) = \max_u g_{T-1}(i,x,u) = g_{T-1}(i,x,u^*).$$

Taking the derivative of g_{T-1} with respect to u_k we obtain the gradient vector with entries

$$\nabla_k g_{T-1}(i,x,u) = \frac{\partial g_{T-1}(i,x,u)}{\partial u_k} = QC(i)E\left[\frac{R_k^e(i)}{r_f\,(i)\,x + R^e(i)'u}\right] \tag{2.80}$$

for all k. If we take the second derivatives of g_{T-1}, we can find the Hessian matrix with entries

$$H_{k,l}(i,x,u) = \nabla_k^2 g_{T-1}(i,x,u) = \frac{\partial^2 g_{T-1}(i,x,u)}{\partial u_k \partial u_l} = -QC(i)E\left[\frac{R_k^e(i)R_l^e(i)}{r_f\,(i)\,x + R^e(i)'u}\right].$$

Let $z = (z_1, \cdots, z_m)$ be any non-zero column vector where z_i's are real numbers. Then one can see that $z^T H\,(i)\,z$ is equal to

$$-QC(i)E\left[\left(z_1 R_1^e\,(i) + z_2 R_2^e\,(i) + \cdots + z_m R_m^e\,(i)\right)^2 / \left(r_f\,(i)\,x + R^e(i)'u\right)\right]$$

which is always smaller than or equal to zero since all $C(i)$ are positive and $(r_f\,(i)\,x + R^e(i)'u)$ is positive. Thus, $H\,(i)$ is negative semi-definite and we can find the optimal solution by setting the gradient (2.80) equal to zero to obtain the optimality condition

$$E\left[\frac{R_k^e(i)}{r_f\,(i)\,x + R^e(i)'u^*\,(i,x)}\right] = 0$$

for any asset $k = 1, 2, \cdots, m$. Defining the vector function $\alpha(i,x) = (\alpha_1(i,x), \cdots, \alpha_m(i,x))$ such that $\alpha(i,x) = u^*(i,x)/r_f\,(i)\,x$ we obtain $u^*(i,x) = \alpha(i,x) r_f\,(i)\,x$ so the optimality condition can be rewritten as

$$E\left[\frac{R^e(i)}{r_f\,(i)\,x + R^e(i)'\alpha(i,x) r_f x}\right] = E\left[\frac{R^e(i)}{r_f\,(i)\,x\,(1 + R^e(i)'\alpha(i,x))}\right] = 0 \tag{2.81}$$

and since $r_f\,(i)\,x \neq 0$, we have

$$E\left[\frac{R_k^e(i)}{1 + R^e(i)'\alpha(i,x)}\right] = 0. \tag{2.82}$$

Since the equation (2.82) holds for every x we can say that α does not depend on x which can be concluded as $d\alpha_k(i,x)/dx = 0$ for all $k = 1, 2, \cdots, m$. So we can write the optimal policy as $u^*(i,x) = \alpha(i) r_f\,(i)\,x$ where $\alpha(i)$ satisfies

$$E\left[\frac{R_k^e(i)}{1 + R^e(i)'\alpha(i)}\right] = 0 \tag{2.83}$$

for all $k = 1, 2, \cdots, m$. When the value function at time $T-1$ is rewritten for the optimal policy, we obtain

$$\begin{aligned} v_{T-1}(i,x) &= \sum_{j\in E} Q(i,j)E[C(j)\log(r_f x + R^e(i)'\alpha(i)r_f(i)\,x) + K(j)] \\ &= QC(i)\log(x) + QC(i)\left(\log(r_f(i)) + E\left[\log(1+R^e(i)'\alpha(i))\right]\right) + QK(i) \\ &= C_{T-1}(i)\log(x) + K_{T-1}(i) \end{aligned}$$

and the value function is still logarithmic like the utility function and $C_{T-1}(i)$ is positive for all values of i. This completes the proof for $n = T-1$.

Suppose now that the induction hypothesis holds for periods $T, T-1, T-2, \cdots, n$. Then, for period $n-1$,

$$\begin{aligned} g_{n-1}(i,x,u) &= \sum_{j\in E} Q(i,j)E[v_n(j, r_f(i)\,x + R^e(i)'u)] \\ &= QC_n(i)\,E[\log(r_f(i)\,x + R^e(i)'u)] + QK_n(i). \end{aligned} \tag{2.84}$$

One can easily see that the Hessian matrix of $g_{n-1}(i,x,u)$ is negative semi-definite as for $g_{T-1}(i,x,u)$. Letting $u^*_{n-1}(i,x)$ be the optimal policy such that

$$v_{n-1}(i,x) = \max_u g_{n-1}(i,x,u) = g_{n-1}(i,x,u^*).$$

If we take the derivative of $g_{n-1}(i,x,u)$ with respect to u_k and set it equal to 0, we get the optimality condition

$$E\left[\frac{R^e_k(i)}{r_f(i)\,x + R^e(i)'u^*(i,x)}\right] = 0. \tag{2.85}$$

Letting $\alpha(i,x) = u^*_{n-1}(i,x)/r_f(i)\,x$ we obtain $u^*_{n-1}(i,x) = \alpha(i,x)r_f(i)\,x$ and

$$E\left[\frac{R^e_k(i)}{1 + R^e(i)'\alpha(i,x)}\right] = 0$$

where $\alpha_{n-1}(i,x)$ does not depend on the period and on x as in equation (2.50) so that we can write $\alpha_{n-1}(i,x) = \alpha(i)$ and we can write the optimal policy as $u^*(i,x) = \alpha(i)r_f(i)\,x$ where $\alpha(i)$ satisfies

$$E\left[\frac{R^e_k(i)}{1 + R^e(i)'\alpha(i)}\right] = 0 \tag{2.86}$$

for all $k = 1, 2, \cdots, m$. If we insert the optimal policy in the value function, we can see that

$$\begin{aligned} v_{n-1}(i,x) &= \sum_{j\in E} Q(i,j)E[C(j)\log(r_f(i)\,x + R^e(i)'\alpha(i)r_f(i)\,x) + K_n(j)] \\ &= QC_n(i)\log(x) + QC_n(i)(\log(r_f(i)) + E\left[\log(1+R^e(i)'\alpha(i))\right]) + QK_n(i) \\ &= C_{n-1}(i)\log(x) + K_{n-1}(i) \end{aligned}$$

and this completes the proof. ■

In this special case with $\beta = 0$, it is clear that the optimal policy in (2.78) is myopic since there is no dependence on n. At any time n, the total amount of money invested in the risky assets depends only on the market state i and wealth x. Since the total risky investment is $1'u_n^*(i,x) = 1'\alpha(i)r_f(i)x$, it follows that $r_f(i)\sum_{k=1}^m \alpha_k(i)$ is the proportion of total wealth that is invested in the risky assets if the market is in state i. Moreover, as in the general logarithmic case, the composition of the risky portfolio (2.53) also depends only on the market state i independent of the available wealth x.

The evolution of the wealth process X using the optimal policy can be analyzed by the wealth dynamics equation

$$\begin{aligned} X_{n+1} &= r_f(Y_n)X_n + R^e(Y_n)'u^*(Y_n, X_n) \\ &= X_n r_f(Y_n)\left(1 + A(Y_n)\right) = X_n B(Y_n) \end{aligned}$$

where $B(i) = r_f(i)\left(1 + A(i)\right)$. Clearly, the solution is

$$X_n = X_0 \prod_{k=0}^{n-1} B(Y_k) \tag{2.87}$$

for $n \geq 1$, and this simple structure can be exploited to analyze the terminal wealth X_T. In particular, given $X_0 = x_0$

$$E_i[X_T] = x_0\left(1 + E_i\left[\mathbb{C}_T\left(b\left(Y_0\right) - 1, b\left(Y_1\right) - 1, \cdots, b\left(Y_{T-1}\right) - 1\right)\right]\right) = x_0 Q_g^{T-1} g\left(i\right) \tag{2.88}$$

where $b(i) = r_f(i)\left(1 + a(i)\right)$ and $g\left(i\right) = b\left(i\right) - 1$. The second moment is

$$E_i[X_T^2] = x_0^2\left(1 + E_i\left[\mathbb{C}_T\left(b_2\left(Y_0\right) - 1, b_2\left(Y_1\right) - 1, \cdots, b_2\left(Y_{T-1}\right) - 1\right)\right]\right) = x_0^2 Q_f^{T-1} f\left(i\right) \tag{2.89}$$

where $b_2(i) = r_f(i)^2 E[(1 + A(i))^2] = r_f(i)^2(1 + 2a(i) + s(i))$, $f\left(i\right) = b_2\left(i\right) - 1$ and $\mathrm{Var}_i(X_T)$ is the difference of (2.89) and the square of (2.88).

The log-return at the terminal time T is

$$\ln\left(X_T/X_0\right) = \sum_{k=0}^{T-1} \ln(B(Y_k))$$

so that the mean is

$$E_i\left[\ln\left(X_T/X_0\right)\right] = \sum_{k=0}^{T-1} Q^k(i,j) E\left[\ln(B(j))\right] = \sum_{k=0}^{T-1} Q^k(i,j)\left(\ln\left(r_f(j)\right) + E\left[\ln(1 + A(j))\right]\right)$$

which can be determined using the distributions of $\{A(i) = R^e(i)'\alpha(i)\}$. The simple structure of (2.87) can be exploited to determine various quantities of interest associated with the terminal wealth.

2.3 Power Utility Function

Suppose that the utility function is the power function

$$U(i,x) = K(i) + C(i)\frac{(x-\beta)^\gamma}{\gamma} \tag{2.90}$$

and Pratt-Arrow ratio can be calculated as $r(x) = (1-\gamma)/(x-\beta)$ for all i so that $b = 1/(1-\gamma)$ and $a = \beta/(\gamma-1)$ in Table 1.1. In this chapter, we assume that the utility function (2.90) is well-defined for all possible values of x. For example, if $(x-\beta) < 0$ is possible, then we exclude $\gamma = 1/2$ in our analysis. If we need to include these values of γ, we can define the utility function to be $-\infty$ whenever (2.90) is not well-defined and make appropriate assumptions on excess returns $\{R^e(i)\}$ as in Section 2.2. For $U(i,x)$ to be a legitimate utility function some additional restrictions may be imposed, but we do not dwell with such technical issues here. Note that γ and β is the same for all market states so that risk classification of the investor does not depend on the stochastic market. Similarly, we assume that the return for the riskless asset is the same for all market states so that $r_f(i) = r_f$ for all i.

We will first consider an optimization problem of the form

$$\max_u c_0 E\left[\frac{(R^{e\prime}u - c)^\gamma}{\gamma}\right] \tag{2.91}$$

where R^e is any random vector. The gradient vector of the objection function $g(u)$ is given by

$$\triangledown_k g(u) = \frac{\partial g(u)}{\partial u_k} = c_0 E\left[R_k^e \left(R^{e\prime}u - c\right)^{\gamma-1}\right]$$

while the Hessian matrix is

$$\triangledown_{k,l}^2 g(u) = \frac{\partial^2 g(u)}{\partial u_k \partial u_l} = (\gamma-1)c_0 E\left[R_k^e R_l^e \left(R^{e\prime}u - c\right)^{\gamma-2}\right]$$

for all k, l.

The first order optimality condition to find the optimal solution of (2.91) is obtained by setting the gradient vector equal to zero so that

$$E\left[R_k^e \left(R^{e\prime}u - c\right)^{\gamma-1}\right] = 0 \tag{2.92}$$

for all $k = 1, 2, \cdots, m$. Let $z = (z_1, \cdots, z_m)$ be any non-zero column vector where z_i's are real numbers. Then, one can see that

$$z' \bigtriangledown^2 g(u) z = (\gamma - 1) c_0 E \left[(z_1 R_1^e + z_2 R_2^e + \cdots + z_m R_m^e)^2 \left(R^{e\prime} u - c \right)^{\gamma - 2} \right]. \tag{2.93}$$

Throughout this chapter, we assume that the excess returns $\{R^e(i)\}$ and the parameters of the utility function are such that there is always an optimal solution of (2.91) that satisfies the first order conditions (2.92). Note that this requirement does not necessarily impose concavity restriction on the objective function. We only require that the optimal solution is at an interior point which satisfies the necessary conditions of optimality (2.92). Our purpose is to identify the structure of the optimal policy and we will not dwell will these technical details on optimization. This is of course an important issue and we do not intend to undermine its significance. We now consider some possible cases to illustrate how one can approach this technical problem. If $\gamma - 2$ is even, then the Hessian matrix $\bigtriangledown^2 g$ in (2.93) is negative semi-definite provided that $(\gamma - 1) c_0 \leq 0$ and the optimal solution satisfies (2.92) since we have an unconstrained concave maximization problem. If $\gamma - 2$ is not even and $(\gamma - 1) c_0 \leq 0$, then the objective function is concave over the set

$$A(c) = \left\{ u : P \left\{ \left(R^{e\prime} u - c \right)^{\gamma - 2} \geq 0 \right\} = 1 \right\} \tag{2.94}$$

and we need additional restrictions on the excess returns $\{R^e(i)\}$; like the existence of a solution of the first order condition (2.92) in $A(c)$ for all c. In case $(\gamma - 1) c_0 \geq 0$, it suffices to reverse the inequality in (2.94).

Theorem 4 *Let the utility function of the investor be the power utility function (2.90) and suppose that the riskless asset return does not depend on the market state. Then, the optimal solution of the dynamic programming equation (1.10) is*

$$v_n(i, x) = K_n(i) + C_n(i) \frac{(x - \beta_n)^\gamma}{\gamma}$$

and the optimal portfolio is

$$u_n^*(i, x) = \alpha(i) \left(r_f x - \beta_{n+1} \right) \tag{2.95}$$

where

$$\beta_n = \frac{\beta}{r_f^{T-n}}, \; C_n = \hat{Q}_\alpha^{T-n} C, K_n = Q^{T-n} K \tag{2.96}$$

and

$$\hat{Q}_\alpha(i,j) = E\left[(r_f\left(1 + R^e(i)'\alpha(i)\right))^\gamma\right] Q(i,j)$$

for all $n = 0, 1, \cdots, T-1$*; and* $\alpha(i)$ *satisfies*

$$E\left[R_k^e(i)\left(1 + R^e(i)'\alpha(i)\right)^{\gamma-1}\right] = 0 \tag{2.97}$$

for all assets $k = 1, 2, \cdots, m$ *and all* i*.*

Proof. We will use dynamic programming for the proof of the theorem. We use induction starting with the boundary condition $v_T(i,x) = K(i) + C(i)(x-\beta)^\gamma/\gamma$ and obtain

$$\begin{aligned} g_{T-1}(i,x,u) &= \sum_{j\in E} Q(i,j) E[U(j, r_f x + R^e(i)'u)] \\ &= QK(i) + \frac{QC(i)\, E[(r_f x + R^e(i)'u - \beta)^\gamma]}{\gamma}. \end{aligned}$$

Let u^* be the optimal amount of money that should be invested in the risky asset so that

$$v_{T-1}(i,x) = \max_u g_{T-1}(i,x,u) = g_{T-1}(i,x,u^*).$$

One can see that $g_{T-1}(i,x,u)$ is in the form of (2.91) where $c = \beta - r_f x$ and $c_0 = QC(i)/\gamma$. Our assumption implies that the optimal policy can be found using the first order condition

$$E\left[R_k^e(i)(r_f x + R^e(i)'u^*_{T-1}(i,x) - \beta)^{\gamma-1}\right] = 0$$

for all $k = 1, 2, \cdots, m$. Defining the vector function $\alpha(i,x) = (\alpha_1(i,x), \alpha_2(i,x), \cdots, \alpha_m(i,x))$ such that $\alpha(i,x) = u^*_{T-1}(i,x)/(r_f x - \beta)$ we obtain $u^*_{T-1}(i,x) = \alpha(i,x)(r_f x - \beta)$ so that the optimality condition can be rewritten as

$$\begin{aligned} E\left[R_k^e(i)(r_f x + R^e(i)'\alpha(i,x)(r_f x - \beta) - \beta)^{\gamma-1}\right] &= 0 \\ (r_f x - \beta)^{\gamma-1} E\left[R_k^e(i)(1 + R^e(i)'\alpha(i,x))^{\gamma-1}\right] &= 0 \end{aligned}$$

or

$$E\left[R_k^e(i)(1 + R^e(i)'\alpha(i,x))^{\gamma-1}\right] = 0. \tag{2.98}$$

Since (2.98) holds for every x we can say that α does not depend on x and $\alpha_k(i,x) = \alpha_k(i)$ for all $k = 1, 2, \cdots, m$. So, we can write the optimal policy as $u^*_{T-1}(i,x) = \alpha(i)(r_f x - \beta)$ where $\alpha(i)$ satisfies

$$E\left[R_k^e(i)\left(1 + R^e(i)'\alpha(i)\right)^{\gamma-1}\right] = 0$$

for all $k = 1, 2, \cdots, m$. When the value function at time $T-1$ is rewritten for the optimal policy, we obtain

$$\begin{aligned} v_{T-1}(i,x) &= QK(i) + \frac{QC(i)\, E[(r_f x + R^e(i)'\alpha(i)\,(r_f x - \beta) - \beta)^\gamma]}{\gamma} \\ &= QK(i) + QC(i)\left(\frac{E\left[(r_f(1+R^e(i)'\alpha(i)))^\gamma\right](x-\beta/r_f)^\gamma}{\gamma}\right) \\ &= QK(i) + \hat{Q}_\alpha C(i)\,\frac{(x-\beta/r_f)^\gamma}{\gamma} \\ &= K_{T-1}(i) + C_{T-1}(i)\frac{\left(x-\beta_{T-1}\right)^\gamma}{\gamma} \end{aligned}$$

and the value function is still a power function like the utility function. This follows by noting that $K_{T-1} = QK$ and $C_{T-1} = \hat{Q}_\alpha C$ in (2.96). This completes the proof for $n = T-1$.

Suppose now that the induction hypothesis holds for periods $T, T-1, T-2, \cdots, n$. Then, for period $n-1$,

$$\begin{aligned} g_{n-1}(i,x,u) &= \sum_{j\in E} Q(i,j)E[v_n(j, r_f x + R^e(i)'u)] \\ &= QK_n(i) + \frac{QC_n(i)\, E[(r_f x + R^e(i)'u - \beta_n)^\gamma]}{\gamma}. \end{aligned} \tag{2.99}$$

Once again, the objective function $g_{n-1}(i,x,u)$ is in the form of (2.91) where $c = \beta_n - r_f x$ and $c_0 = QC_n(i)/\gamma$. Our assumption implies that the optimal policy can be found using the first order condition

$$E\left[R^e_k(i)(r_f x + R^e(i)'u^*_{n-1}(i,x) - \beta_n)^{\gamma-1}\right] = 0$$

and, letting $\alpha(i,x) = u^*_{n-1}(i,x)/\left(r_f x - \beta_n\right)$ or $u^*_{n-1}(i,x) = \alpha(i,x)\left(r_f x - \beta_n\right)$,we obtain

$$\left(r_f x - \beta_n\right) E\left[R^e_k(i)(1 + R^e(i)'\alpha(i,x))^{\gamma-1}\right] = 0$$

which implies

$$E\left[R^e_k(i)(1 + R^e(i)'\alpha(i,x))^{\gamma-1}\right] = 0$$

for all $k = 1, 2, \cdots, m$.. Since $\alpha(i,x)$ does not depend on the period n and on x as in equation (2.98), we can write $\alpha(i,x) = \alpha(i)$ and the optimal policy is $u^*_{n-1}(i,x) = \alpha(i)\left(r_f x - \beta_n\right)$ where $\alpha(i)$ satisfies

$$E\left[R^e_k(i)(1 + R^e(i)'\alpha(i))^{\gamma-1}\right] = 0$$

for all $k = 1, 2, \cdots, m$. If we insert the optimal policy in the value function, we can see that

$$\begin{aligned} v_{n-1}(i,x) &= QK_n(i) + \frac{QC_n(i)\, E[(r_f x + R^e(i)'\alpha(i)\,(r_f x - \beta_n) - \beta_n)^\gamma]}{\gamma} \\ &= QK_n(i) + QC_n(i) \left(\frac{E[(r_f\,(1 + R^e(i)'\alpha(i)))^\gamma]\,(x - \beta_n/r_f))^\gamma}{\gamma} \right) \\ &= QK_n(i) + \hat{Q}_\alpha C_n(i)\, \frac{(x - \beta_{n-1})^\gamma}{\gamma} \\ &= K_{n-1}(i) + C_{n-1}(i) \frac{(x - \beta_{T-1})^\gamma}{\gamma} \end{aligned}$$

and the value function is a power function. Note that the recursions $K_{n-1} = QK_n$ and $C_{n-1} = \hat{Q}_\alpha C_n$ with boundary values $K_T = K$ and $C_T = C$ give the explicit solutions in (2.96). This completes the proof. ■

Note that the wealth dynamics equation for the power utility function is not the same as the wealth dynamics equation (2.54) for the logarithmic case since the structure of the optimal policy in (2.47) and (2.95) are different where for the latter β has a minus sign. However, using a similar analysis as in Section 2.2.1 we can easily determine

$$E_i[X_T] = r_f^T x_0 + \left(\beta - r_f^T x_0\right) m_\gamma(i,T) \tag{2.100}$$

and

$$\mathrm{Var}_i(X_T) = \left(\beta - r_f^T x_0\right)^2 v_\gamma^2(i,T) \tag{2.101}$$

where

$$m_\gamma(i,T) = -E_i\left[\mathbb{C}_T\left(A\left(\overline{Y}_{T-1}\right)\right)\right] \tag{2.102}$$

and

$$v_\gamma^2(i,T) = \mathrm{Var}_i\left(\mathbb{C}_T\left(A\left(\overline{Y}_{T-1}\right)\right)\right). \tag{2.103}$$

Likewise, similar interpretations can be made on the structure of the optimal policy. In particular, the optimal policy is not myopic, but the risky composition of the portfolio is both myopic and memoryless. Moreover, this composition only depends on the state of the market. Although we obtain similar characterizations and interpretations, note that the optimal policies for logarithmic and power cases are not identical since (2.49) and (2.97) have different solutions. In particular, the solution of (2.97) clearly depends on the risk aversion coefficient γ.

For the power utility case, (2.100) and (2.101) imply that we can also write

$$E_i[X_T] = r_f^T x_0 + \left(\frac{m_\gamma(i,T)}{v_\gamma(i,T)}\right) \mathrm{SD}_i(X_T) \tag{2.104}$$

to represent the power frontier with slope or risk premium $m_\gamma(i,T)/v_\gamma(i,T)$.

If $\gamma = 1$, then the utility function (2.90) becomes linear and the investor tries to maximize the expected terminal wealth. The optimal solution then is uninteresting and trivial since the investor will invest an infinite amount of money on the asset (including the riskless asset) with the highest expected return in any market state.

More interestingly, when $\gamma = 2$, the utility function (2.90) has a quadratic form. In this case, the assumption is satisfied and there is a unique solution satisfying the first order condition (2.97), which simplifies to

$$E[R^e(i)] + E\left[R^e(i)R^e(i)'\alpha(i)\right] = 0$$

and the optimal solution can be found explicitly as

$$\alpha(i) = -V(i)^{-1} r^e(i) \tag{2.105}$$

where $V(i) = E\left[R^e(i)R^e(i)'\right] = \sigma(i) + r^e(i) r^e(i)'$ is the matrix of second moments, and $r^e(i) = E[R^e(i)]$ is the expected value of the return vector in state i. Furthermore, it follows form (2.55) that $A(i) = -R^e(i)'V(i)^{-1}r^e(i)$ which gives

$$a(i) = -E[R^e(i)'V(i)^{-1}r^e(i)] = -r^e(i)'V(i)^{-1}r^e(i) \tag{2.106}$$

and $m_2(i,T)$ can be computed using (2.106) in (2.67) with $g(i) = 1 - r^e(i)' V(i)^{-1} r^e(i)$. Note from (2.57) that

$$s(i) = \alpha(i)' V(i) \alpha(i) = r^e(i)'V(i)^{-1}V(i)V(i)^{-1}r^e(i)] = r^e(i)'V(i)^{-1}r^e(i) = -a(i)$$

and (2.68) becomes

$$\begin{aligned} E_i\left[\mathbb{C}_T\left(A\left(\overline{Y}_{T-1}\right)\right)^2\right] &= E_i\left[\mathbb{C}_T(a(\overline{Y}_{T-1}))\right] - 2E_i\left[\mathbb{C}_T\left(a\left(\overline{Y}_{T-1}\right)\right)\right] \\ &= -E_i\left[\mathbb{C}_T(a(\overline{Y}_{T-1}))\right] = m_2(i,T) \end{aligned}$$

so that the variance term is

$$v_2^2(i,T) = m_2(i,T) - m_2(i,T)^2 = m_2(i,T)(1 - m_2(i,T)).$$

Therefore, for the quadratic model with $\gamma = 2$, we obtain the mean-variance efficient frontier using (2.104) given by the straight line

$$E_i\left[X_T\right] = r_f^T x_0 + \left(\sqrt{\frac{m_2(i,T)}{1-m_2(i,T)}}\right)\mathrm{SD}_i(X_T) \tag{2.107}$$

where the slope, or the risk premium, is $m_2(i,T)/v_2(i,T)$. Çakmak and Özekici [10] discussed the mean-variance problem where the objective is to maximize the linear-quadratic objective function $E_i\left[-X_T^2 + \beta X_T\right]$ parametrized by β.

When $\gamma = 3$, (2.97) implies that the optimal $\alpha(i)$ satisfies $E\left[R_k^e(i)\left(1 + R^e(i)'\alpha(i)\right)^2\right] = 0$, or

$$E\left[R_k^e(i) + 2R_k^e(i)\,R^e(i)'\,\alpha(i) + R_k^e(i)\left(R^e(i)'\,\alpha(i)\right)^2\right] = 0 \tag{2.108}$$

for all $k = 1, 2, \cdots, m$. This is a system of m quadratic equations with m unknowns which can be solved using numerical methods. Note that the solution of (2.108) is not necessarily unique, but the optimal value can be found by calculating the objective function one at this point.

2.3.1 Quadratic Utility Function

Quadratic utility function is a special case of the power utility function where $\gamma = 2$ and $\beta = -\lambda/2\omega$. Then, if $C(i) = -2\omega$ and $K(i) = \lambda^2/4\omega$, the utility function (2.90) take the linear quadratic form

$$U(i,x) = -\omega x^2 + \lambda x$$

which is the utility function of the auxiliary problem defined by Çakmak and Özekici [10]. They have found that the value functions for periods $n = 1, 2, \cdots, T-1$ the value function also has the linear quadratic form. In their study the risk free rate of return r_f depends on the market state. To compare our solution with theirs, if we take the risk-free return to be same at all market conditions so that $r_f(i) = r_f$ for quadratic optimization problem, then one can show that

$$v_n(i,x) = -\frac{1}{2}C_n(i)\,x^2 + \beta_n C_n(i)\,x + K_n(i) + \frac{\beta_n^2}{2}C_n(i) \tag{2.109}$$

where C_n, β_n, and K_n are as defined in Theorem 4 and the corresponding optimal policy is

$$u_n(i,x) = \left[\frac{1}{2}\left(\frac{\lambda}{\omega}\right)\frac{1}{r_f^{T-n-1}} - r_f(i)\,x\right]V^{-1}(i)\,r^e(i)\,. \tag{2.110}$$

If we compare the solutions found by them and the solution of our power utility function we see that they are the same for r_f independent of the market condition. A strange observation is that if r_f is independent of the market condition then the optimal portfolio policy only depends on the current market state - not on the market dynamics matrix Q. Also if we check the efficient frontier we see that for r_f independent of market condition the efficient frontier is a linear line which cuts zero risk level at $E\left[X_T\right] = r_f^T$. The formula for the efficient frontier is found at section 2.1.1 at (2.107).

As a special case, suppose now that the utility function is the CRRA (constant relative risk aversion) function with $\beta = 0$ so that

$$U(i,x) = \frac{C\left(i\right)}{\gamma} x^{\gamma} + K\left(i\right) \tag{2.111}$$

We can easily see that $-U'(i,x)/U''(i,x) = (1-\gamma)x$. We remove the restriction that $r_f(i) = r_f$ and the riskless return depends on the market state.

2.3.2 CRRA Utility Function

As a special case, suppose now that the utility function is the CRRA (constant relative risk aversion) function with $\beta = 0$ so that

$$U(i,x) = K\left(i\right) + C\left(i\right)\left(\frac{x^{\gamma}}{\gamma}\right) \tag{2.112}$$

We can easily see that $r(x) = (1-\gamma)/x$ where $a = 0, b = 1/(1-\gamma)$. in Table 1.1. We remove the restriction that $r_f(i) = r_f$ and the riskless return depends on the market state.

Theorem 5 *Let the utility function of the investor be the CRRA function (2.112). Then, the optimal solution of the dynamic programming equation (1.10) is*

$$v_n(i,x) = \frac{C_n(i)}{\gamma} x^{\gamma} + K_n\left(i\right)$$

and the optimal portfolio is

$$u_n^*(i,x) = \alpha(i) r_f(i) x \tag{2.113}$$

where

$$K_n = Q^{T-n} K,\ C_n = Q^{T-n} C \tag{2.114}$$

and

$$\hat{Q}(i,j) = Q(i,j) r_f(i)^{\gamma} E\left[\left(1 + R^e(i)'\alpha(i)\right)^{\gamma}\right]$$

for all $n = 0, 1, \cdots, T-1$; *and* $\alpha(i)$ *satisfies*

$$E\left[R_k^e(i)\left(1 + R^e(i)'\alpha(i)\right)^{\gamma-1}\right] = 0 \tag{2.115}$$

for all assets $k = 1, 2, \cdots, m$ *independent of period* n *and all* i.

Proof. We use induction starting with the boundary condition $V_T(i,x) = C(i)(x)^\gamma/\gamma + K(i)$. and obtain

$$\begin{aligned} g_{T-1}(i,x,u) &= \sum_{j\in E} Q(i,j)E[U(j, r_f(i)x + R^e(i)'u)] \\ &= QC(i)E[(r_f(i)x + R^e(i)'u)^\gamma]/\gamma + QK(i)/\gamma \end{aligned}$$

where

$$(r_f(i)x + R^e(i)'u) > 0$$

for all available investment strategies. Let $u^* = (u_1^*, u_2^*, \cdots, u_m^*)$ be the optimal amount of money that should be invested in the risky asset so that

$$v_{T-1}(i,x) = \max_u g_{T-1}(i,x,u) = g_{T-1}(i,x,u^*).$$

Taking the derivative of g_{T-1} with respect to u_k we obtain the gradient vector with entries

$$\nabla_k g_{T-1}(i,x,u) = \frac{\partial g_{T-1}(i,x,u)}{\partial u_k} = QC(i)E\left[(r_f(i)x + R^e(i)'u(i,x))^{\gamma-1}R_k^e(i)\right] \tag{2.116}$$

for all k and if we take the second derivatives of g_{T-1}, we can find the Hessian matrix as with entries

$$\begin{aligned} \nabla_k^2 g_{T-1}(i,x,u) &= \frac{\partial^2 g_{T-1}(i,x,u)}{\partial u_k \partial u_l} \\ &= QC(i)(\gamma-1)E\left[(r_f(i)x + R^e(i)'u(i,x))^{\gamma-2}R_k^e(i)R_l^e(i)\right]. \end{aligned}$$

Let $z = (z_1, \cdots, z_m)$ be any non-zero column vector where z_i's are real numbers. Then one can see that $z^T H(i) z$ is equal to

$$QC(i)(\gamma-1)E\left[(z_1R_1^e(i) + z_2R_2^e(i) + \cdots + z_mR_m^e(i))^2(r_f(i)x + R^e(i)'u(i,x))^{\gamma-1}\right]$$

which is always negative since all $C(i)(\gamma-1)$ are negative and $(r_f(i)x + R^e(i)'u(i,x))$ is positive. Thus, $H(i)$ is negative semi-definite and we can find the optimal solution by setting the gradient 2.116 equal to zero to obtain the optimality condition

$$E\left[(r_f(i)x + R^e(i)'u^* + \beta)^{\gamma-1}R_k^e(i)\right] = 0$$

for any asset $k = 1, 2, \cdots, m$. Defining the vector function $\alpha(i,x) = (\alpha_1(i,x), \cdots, \alpha_m(i,x))$ such that $\alpha(i,x) = u^*(i,x)/r_f(i)\,x$ we obtain $u^*(i,x) = r_f(i)\,x\alpha(i,x)$ so the optimality condition can be rewritten as

$$\begin{aligned} E\left[(r_f x + R^e(i)'\alpha(i,x) r_f(i)\,x)^{\gamma-1} R^e_k(i)\right] &= 0 \\ (r_f(i)\,x)^{\gamma-1} E\left[(1 + R^e(i)'\alpha(i,x))^{\gamma-1} R^e_k(i)\right] &= 0 \end{aligned} \tag{2.117}$$

and since $r_f(i)\,x \neq 0$, we have

$$E\left[(1 + R^e(i)'\alpha(i,x))^{\gamma-1} R^e_k(i)\right] = 0. \tag{2.118}$$

Since the equation (2.118) holds for every x we can say that α does not depend on x which can be concluded as $d\alpha_k(i,x)/dx = 0$ for all $k = 1, 2, \cdots, m$. So we can write the optimal policy as $u^*(i,x) = \alpha(i) r_f(i)\,x$ where $\alpha(i)$ satisfies

$$E\left[R^e_k(i)\left(1 + R^e(i)'\alpha(i)\right)^{\gamma-1}\right] = 0 \tag{2.119}$$

for all $k = 1, 2, \cdots, m$. When the value function at time $T-1$ is rewritten for the optimal policy, we obtain

$$\begin{aligned} v_{T-1}(i,x) &= \sum_{j \in E} Q(i,j) E\left[\frac{C(j)}{\gamma}(r_f(i)\,x + R^e(i)'\alpha(i) r_f x)^{\gamma} + K(j)\right] \\ &= QC(i)\,\frac{r_f(i)^{\gamma}\, E\left[(1 + R^e(i)'\alpha(i))^{\gamma}\right]}{\gamma} x^{\gamma} + QK(i) \\ &= C_{T-1}(i)\frac{x^{\gamma}}{\gamma} + K_{T-1}(i) \end{aligned} \tag{2.120}$$

and the value function is still power function like the utility function and $C_{T-1}(i)$ has the same sign with $C(i)$ for all values of i. This completes the proof for $n = T-1$.

Suppose now that the induction hypothesis holds for periods $T, T-1, T-2, \cdots, n$. Then, for period $n-1$,

$$\begin{aligned} g_{n-1}(i,x,u) &= \sum_{j \in E} Q(i,j) E[v_n(j, r_f(i)\,x + R^e(i)'u)] \\ &= QC_n(i)\, E[(r_f(i)\,x + R^e(i)'u)^{\gamma}]/\gamma + QK_n(i)/\gamma \end{aligned} \tag{2.121}$$

One can easily see that the Hessian matrix $g_{n-1}(i,x,u)$ is positive definite as for $g_{T-1}(i,x,u)$. Letting $u^*_{n-1}(i,x)$ be the optimal policy such that

$$v_{n-1}(i,x) = \max_u g_{n-1}(i,x,u) = g_{n-1}(i,x,u^*).$$

If we take the derivative of $g_{n-1}(i,x,u)$ with respect to u_k and set it equal to 0, we get the optimality condition

$$E\left[(r_f(i)\,x + R^e(i)'u^*(i,x))^{\gamma-1}R^e_k(i)\right] = 0 \tag{2.122}$$

and letting $\alpha(i,x) = u^*_{n-1}(i,x)/r_f(i)\,x$ we obtain $u^*_{n-1}(i,x) = \alpha(i,x)r_f(i)\,x$ and

$$(r_f(i)\,x)^{\gamma}\,E\left[(1 + R^e(i)'\alpha(i,x))^{\gamma-1}R^e_k(i)\right] = 0$$

for all k where $\alpha(i,x)$ does not depend on the period and on x as in equation (2.118) so that we can write $\alpha(i,x) = \alpha(i)$ and we can write the optimal policy as $u^*(i,x) = \alpha(i)r_f(i)\,x$ where $\alpha(i)$ satisfies

$$E\left[(1 + R^e(i)'\alpha(i,x))^{\gamma-1}R^e_k(i)\right] = 0 \tag{2.123}$$

for all $k = 1, 2, \cdots, m$. If we insert the optimal policy in the value function, we can see that

$$\begin{aligned} v_{n-1}(i,x) &= \sum_{j\in E} Q(i,j)E\left[\frac{C_k(j)}{\gamma}(r_f(i)\,x + R^e(i)'\alpha(i)r_f(i)\,x)^{\gamma}\right] + \sum_{j\in E} Q(i,j)K_n(j) \\ &= QC_k(i)r_f(i)^{\gamma}\,E[(1 + R^e(i)'\alpha(i))^{\gamma}]\frac{x^{\gamma}}{\gamma} + QK_n(i) \\ &= \frac{C_{n-1}(i)}{\gamma}x^{\gamma} + K_{n-1}(i) \end{aligned}$$

and this completes the proof. ■

Note that the structure of the optimal policy (2.113) is identical to (2.78). Therefore, the results and interpretations presented for the simple logarithmic case also hold. The optimal policies are of course different since the solutions of (2.79) and (2.115) are not identical.

2.4 Illustrations for Discrete Time

In this section three different illustrations for

2.4.1 Example 1

Consider a market with three risky assets and one riskless asset where the returns of the risky assets follow the multivariate normal distribution. Assume that the market is modulated by a Markov chain that has four states. Suppose the return of the riskless asset and the expected return of each risky asset for each state are as given in the following table.

i	r_f	$\mu_1(i)$	$\mu_2(i)$	$\mu_3(i)$
1	1.0028	0.9162	0.8558	0.8751
2	1.0028	0.9690	0.9970	0.9691
3	1.0028	1.0318	1.0668	1.0802
4	1.0028	1.1160	1.1704	1.1297

Assume further that the covariance matrix for each state is given as follows

$$\sigma(1) = \begin{bmatrix} 2.927 & -0.513 & -0.361 \\ -0.513 & 8.979 & 1.304 \\ -0.361 & 1.304 & 4.365 \end{bmatrix}, \sigma(2) = \begin{bmatrix} 9.762 & -2.506 & -1.553 \\ -2.506 & 9.461 & -2.309 \\ -1.553 & -2.309 & 6.649 \end{bmatrix}$$

$$\sigma(3) = \begin{bmatrix} 12.641 & -3.664 & -3.492 \\ -3.664 & 14.714 & 8.258 \\ -3.492 & 8.258 & 15.136 \end{bmatrix}, \sigma(4) = \begin{bmatrix} 8.202 & 3.119 & 2.282 \\ 3.119 & 18.438 & 5.821 \\ 2.282 & 5.821 & 10.355 \end{bmatrix}.$$

Note that these values are obtained by multiplying the actual values by 1,000 to simplify the values. The expected returns and covariance matrices are estimated using data obtained during October 1997 to October 2002 from monthly return information of three assets (IBM, Dell and Microsoft) traded in New York Stock Exchange; and yield information of 3-month US treasury bonds where the expected return for the bonds at all market states are assumed to be constant. The states of the Markov chain are classified according to the number of assets whose values increased in a given period. If the values of all three assets increase in a given period the state for this period is taken to be 4, if the values of exactly two assets increase the state for this period is assumed to be 3, and so on. Using the historical data the transition probability matrix Q of the Markov chain is obtained as

$$Q = \begin{bmatrix} 0.23 & 0.18 & 0.12 & 0.47 \\ 0.23 & 0.23 & 0.08 & 0.46 \\ 0.30 & 0 & 0.30 & 0.40 \\ 0.37 & 0.37 & 0.21 & 0.05 \end{bmatrix}.$$

We consider the problem of an investor whose current wealth is $x_0 = 1$ and who wants to maximize the expected value of terminal utility where he has an exponential utility function and the time horizon is $T = 4$ periods. We calculated both the exponential and mean-variance efficient frontiers. Numerical values $m(i,T), v(i,T)$, and $c(i,T)$ are found to be

$$m(i,T) = \begin{bmatrix} 19.601 & 11.724 & 12.050 & 13.885 \end{bmatrix}$$
$$v(i,T) = \begin{bmatrix} 12.955 & 18.367 & 19.230 & 18.144 \end{bmatrix}$$
$$c(i,T) = \begin{bmatrix} 0.0045 & 0.0304 & 0.0232 & 0.0134 \end{bmatrix}$$

and the slopes, or the risk premiums, are

$$\frac{m(i,T)}{v(i,T)} = \begin{bmatrix} 1.5130 & 0.6383 & 0.6266 & 0.7653 \end{bmatrix}$$
$$\frac{\sqrt{1-c(i,T)}}{\sqrt{c(i,T)}} = \begin{bmatrix} 14.8884 & 5.6453 & 6.4876 & 8.5648 \end{bmatrix}.$$

As expected, the risk premiums for the mean-variance frontier are higher. The exponential frontier and the efficient frontier faced by an investor at time zero is given in Figure 2.1 for $Y_0 = 1$.

In order to determine the optimal portfolio, we first compute

$$\alpha(i) = \begin{bmatrix} -35.5436 & -5.7427 & 4.3883 & 9.8410 \\ -14.3492 & -4.0410 & 2.8887 & 5.1560 \\ -27.9083 & -7.8131 & 4.5500 & 7.1878 \end{bmatrix}$$

which implies that it is optimal to short sell the risky assets in market state 1 while just the opposite is true in state 4. By the way that the market states are classified, the market becomes more attractive to invest in the risky assets as its state increases from 1 to 4. This is clearly reflected in the optimal portfolio since the investment amounts increase from negative (shortselling) to positive values. Furthermore, the proportions of the risky assets in the risky part of the portfolio are

$$w(i) = \begin{bmatrix} 0.46 & 0.33 & 0.37 & 0.44 \\ 0.18 & 0.23 & 0.24 & 0.23 \\ 0.36 & 0.44 & 0.39 & 0.33 \end{bmatrix}$$

obtained by normalizing $\alpha(i)$ values. The exact amounts to be invested can easily be determined using (2.37) by simply multiplying the $\alpha(i)$ values by the discounted value of β.

Illustrations for Power Utility

We address the computational issues and demonstrate how our results can be put to work by considering a numerical example for the power utility case. The numerical case is the same as the one used in Section 2.4.1. We consider the problem of an investor whose current wealth is $x_0 = 1$ and who wants to maximize the expected value of terminal utility where he has an power utility function with $\gamma = 3$ and the time horizon is $T = 4$ periods. Here, since the returns are multivariate normal, the excess returns are also multivariate normal. For any multivariate normal vector $(X_1, X_2, \cdots, X_n)$ with mean vector μ and correlation matrix σ we can write

$$E\left[X_i X_j\right] = \sigma_{ij} + \mu_i \mu_j \tag{2.124}$$

and

$$E\left[X_i X_j X_k\right] = \sigma_{ij}\mu_k + \sigma_{jk}\mu_i + \sigma_{ik}\mu_j + \mu_i\mu_j\mu_k. \tag{2.125}$$

Note that (2.124) follows trivially and (2.125) is obtained using the fact that

$$E\left[(X_1 - \mu_1)^{k_1}(X_2 - \mu_2)^{k_2}\cdots(X_n - \mu_n)^{k_n}\right] = 0$$

if $k_1 + k_2 + \cdots + k_n$ is odd. It suffices to take $k_1 = k_2 = k_3 = 1$ in our case with $n = 3$ assets. Using (2.124) and (2.125) with (2.108) we determined the optimal $\alpha(i)$ values numerically using MATLAB. The optimal solution is

$$\alpha(i) = \begin{bmatrix} -27.41 & -6.96 & 2.58 & 6.96 \\ -21.46 & -3.04 & 1.53 & 7.66 \\ -36.34 & -7.15 & 6.13 & 10.23 \end{bmatrix}$$

which implies that it is optimal to shortsell the risky assets in market state 1 while just the opposite is true in state 4. By the way that the market states are classified, the market becomes more attractive to invest in the risky assets as its state increases from 1 to 4. This is clearly reflected in the optimal portfolio since the investment amounts increase from negative (shortselling) to positive values. Furthermore, the proportions of the risky assets in the risky part of the portfolio are

$$w(i) = \begin{bmatrix} 0.32 & 0.40 & 0.25 & 0.28 \\ 0.25 & 0.18 & 0.15 & 0.31 \\ 0.43 & 0.42 & 0.60 & 0.41 \end{bmatrix}$$

obtained by normalizing $\alpha(i)$ values. The exact amounts to be invested can easily be determined using (2.95) by simply multiplying the $\alpha(i)$ values by the discounted value of β.

We determined the power frontier (2.104) with $\gamma = 3$ using the coefficients $m_3\,(i, T)$ and $v_3(i, T)$, and the explicit formulas (2.67) and (2.70). Note that the mean-variance efficient frontier is also the power frontier with the quadratic utility function with $\gamma = 2$. Numerical values are

$$
\begin{aligned}
m_3\,(i,T) &= \begin{bmatrix} 0.9978 & 0.9561 & 0.9431 & 0.9750 \end{bmatrix} \\
v_3\,(i,T) &= \begin{bmatrix} 0.1473 & 0.2048 & 0.2316 & 0.1561 \end{bmatrix} \\
m_2\,(i,T) &= \begin{bmatrix} 0.9955 & 0.9696 & 0.9768 & 0.9866 \end{bmatrix} \\
v_2\,(i,T) &= \begin{bmatrix} 0.0669 & 0.1717 & 0.1506 & 0.1152 \end{bmatrix} \\
m_e\,(i,T) &= \begin{bmatrix} 19.601 & 11.724 & 12.050 & 13.885 \end{bmatrix} \\
v_e\,(i,T) &= \begin{bmatrix} 12.955 & 18.367 & 19.230 & 18.144 \end{bmatrix}
\end{aligned}
$$

and the slopes, or the risk premiums, are

$$
\begin{aligned}
m_3\,(i,T)\,/v_3(i,T) &= \begin{bmatrix} 6.7739 & 4.6688 & 4.0724 & 6.2481 \end{bmatrix} \\
m_2\,(i,T)\,/v_2(i,T) &= \begin{bmatrix} 14.8884 & 5.6453 & 6.4876 & 8.5648 \end{bmatrix} \\
m_e\,(i,T)\,/v_e(i,T) &= \begin{bmatrix} 1.5130 & 0.6383 & 0.6266 & 0.7653 \end{bmatrix}.
\end{aligned}
$$

The exponential frontier, the efficient frontier and the power frontier for $\gamma = 3$ faced by an investor at time zero is given in Figure 2.1 for $Y_0 = 4$.

Investors with quadratic, logarithmic, and power (with $\gamma = 3$) utility functions will have differing risk preferences measured by β in their utility functions. The return and risk of the terminal wealth for these investors will be on the respective frontier in Figure 2.1. The slopes measure the risk premiums and, as expected, the risk premiums for the mean-variance frontier are highest.

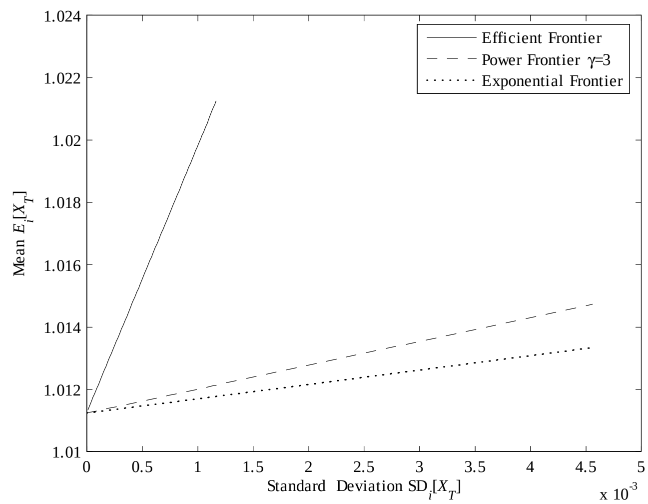

Figure 2.1: Efficient and Exponential Frontiers for $i = 4, T = 4$

2.4.2 Example 2

Exponential Utility

Consider a market with three risky assets and one riskless asset where the returns of the risky assets follow the multivariate normal distribution. Assume that the market is modulated by a Markov chain that has four states. Suppose the return of the riskless asset and the expected return of each risky asset for each state are as given in the following table.

i	r_f	$\mu_1(i)$	$\mu_2(i)$	$\mu_3(i)$
1	1.0008	1.0105	1.0094	0.9997
2	1.0008	1.0071	1.0097	1.0061
3	1.0008	1.0039	1.0114	1.0052
4	1.0008	1.0010	1.0038	0.9994

Assume further that the covariance matrix for each state is given as follows

$$\sigma(1) = \begin{bmatrix} 2.408 & 1.797 & 0.602 \\ 1.797 & 5.952 & 0.676 \\ 0.602 & 0.676 & 1.883 \end{bmatrix}, \sigma(2) = \begin{bmatrix} 2.046 & 1.310 & 0.542 \\ 1.310 & 4.855 & 0.906 \\ 0.542 & 0.906 & 1.657 \end{bmatrix}$$

$$\sigma(3) = \begin{bmatrix} 2.109 & 1.417 & 1.074 \\ 1.417 & 4.663 & 1.169 \\ 1.074 & 1.169 & 1.982 \end{bmatrix}, \sigma(4) = \begin{bmatrix} 1.738 & 1.353 & 0.445 \\ 1.353 & 4.375 & 0.494 \\ 0.445 & 0.494 & 1.499 \end{bmatrix}.$$

Note that these values are obtained by multiplying the actual numbers by 1,000 for simplification. The expected returns and covariance matrices are estimated using data obtained during January 1991 to December 2006 from weekly return information of three assets (IBM, Dell and Microsoft) traded in New York Stock Exchange; and the daily effective federal funds rate. The states of the market are classified by considering whether the SP500 index went up or down during the previous 2 weeks. Therefore, there are 4 states labeled as $1 \equiv$ (down, down), $2 \equiv$ (down, up), $3 \equiv$ (up, down), and $4 \equiv$ (up, up). The weekly interest rates for all states were approximately equal to 0.08% and our assumption is satisfied. Using the historical data the transition probability matrix Q of the Markov chain is obtained as

$$Q = \begin{bmatrix} 0.410 & 0 & 0.590 & 0 \\ 0.388 & 0 & 0.612 & 0 \\ 0 & 0.445 & 0 & 0.555 \\ 0 & 0.492 & 0 & 0.508 \end{bmatrix}.$$

We consider the problem of an investor whose current wealth is $x_0 = 1$ and who wants to maximize the expected value of terminal utility where he has an exponential utility function and the time horizon is $T = 4$ periods. We calculated both the exponential and mean-variance efficient frontiers. Numerical values $m(i,T), v(i,T)$, and $c(i,T)$ are found to be

$$m(i,T) = \begin{bmatrix} 0.1304 & 0.1138 & 0.0925 & 0.0725 \end{bmatrix}$$
$$v(i,T) = \begin{bmatrix} 0.3716 & 0.3519 & 0.3143 & 0.2868 \end{bmatrix}$$
$$c(i,T) = \begin{bmatrix} 0.8803 & 0.8945 & 0.9134 & 0.9315 \end{bmatrix}$$

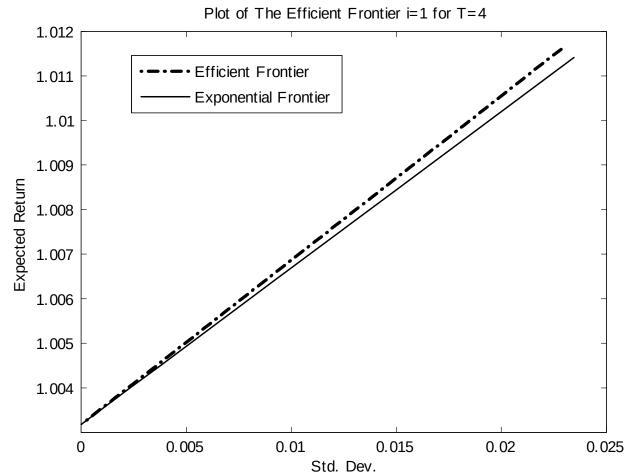

Figure 2.2: Efficient and Exponential Frontiers for $i = 1, T = 4$

and the slopes, or the risk premiums, are

$$\frac{m(i,T)}{v(i,T)} = \begin{bmatrix} 0.3509 & 0.3234 & 0.2942 & 0.2530 \end{bmatrix}$$

$$\sqrt{\frac{1-c(i,T)}{c(i,T)}} = \begin{bmatrix} 0.3687 & 0.3434 & 0.3080 & 0.2712 \end{bmatrix}.$$

As expected, the risk premiums for the mean-variance frontier are higher. The exponential frontier and the efficient frontier faced by an investor at time zero is given in Figure 2.2 for $Y_0 = 1$.

In order to determine the optimal portfolio, we first compute

$$\alpha(i) = \begin{bmatrix} 4.2455 & 1.9527 & -0.6093 & -0.2846 \\ 0.4107 & 0.9175 & 2.1325 & 0.9073 \\ -2.1075 & 2.0552 & 1.2760 & -1.1765 \end{bmatrix}$$

and, furthermore, the proportions of the risky assets in the risky part of the portfolio are

$$w(i) = \begin{bmatrix} 1.67 & 0.40 & -0.22 & 0.51 \\ 0.16 & 0.19 & 0.76 & -1.64 \\ -0.83 & 0.42 & 0.46 & 2.12 \end{bmatrix}$$

obtained by normalizing $\alpha(i)$ values. The exact amounts to be invested can easily be determined using (2.37) by simply multiplying the $\alpha(i)$ values by the discounted value of β.

Note that the total amount invested in the risky assets is positive for state $1 \equiv$ (down, down), while it is negative for state $4 \equiv$ (up, up). This implies that shortselling stocks 1 and 3 is optimal if the index increased in the previous two weeks while no stock is sold short in state $2 \equiv$ (down, up). These results clearly indicate that the composition of the risky portfolio depends very much on the market state. While it is optimal to shortsell the third stock in favor of the first one in state $1 \equiv$ (down, down), it is optimal to shortsell the second stock in favor of the third one in state $4 \equiv$ (up, up).

Other HARA Cases

In this section, we address the computational issues and demonstrate how our results can be put to work by considering a numerical illustration for the logarithmic, power ($\gamma = 0.5, 2$ (quadratic), and 4) cases. Consider a market with three risky assets and one riskless asset where the returns of the risky assets follow an arbitrary multivariate distribution. The illustration is based on data obtained during January 1991 to December 2006 from weekly return information of three assets (IBM, Dell and Microsoft) traded in New York Stock Exchange; and the daily effective federal funds rate. The states of the market are classified by considering whether the SP500 index went up or down during the previous 2 weeks. Therefore, there are 4 states labeled as $1 \equiv$ (down, down), $2 \equiv$ (down, up), $3 \equiv$ (up, down), and $4 \equiv$ (up, up). The weekly interest rates for all states were approximately equal to 0.08% and our assumption is satisfied. Using historical data the transition probability matrix Q of the Markov chain is obtained as

$$Q = \begin{bmatrix} 0.410 & 0 & 0.590 & 0 \\ 0.388 & 0 & 0.612 & 0 \\ 0 & 0.445 & 0 & 0.555 \\ 0 & 0.494 & 0 & 0.506 \end{bmatrix}.$$

The return of the riskless asset and the expected return of each risky asset for each state are

i	r_f	$\mu_1(i)$	$\mu_2(i)$	$\mu_3(i)$
1	1.0008	1.0105	1.0096	0.9995
2	1.0008	1.0071	1.0097	1.0061
3	1.0008	1.0039	1.0114	1.0052
4	1.0008	1.0011	1.0033	0.9990

and the covariance matrices for each state are

$$\sigma(1) = \begin{bmatrix} 2.425 & 1.809 & 0.607 \\ 1.809 & 5.990 & 0.684 \\ 0.607 & 0.684 & 1.893 \end{bmatrix}, \sigma(2) = \begin{bmatrix} 2.046 & 1.310 & 0.542 \\ 1.310 & 4.855 & 0.906 \\ 0.542 & 0.906 & 1.657 \end{bmatrix}$$

$$\sigma(3) = \begin{bmatrix} 2.109 & 1.417 & 1.074 \\ 1.417 & 4.663 & 1.169 \\ 1.074 & 1.169 & 1.982 \end{bmatrix}, \sigma(4) = \begin{bmatrix} 1.607 & 1.229 & 0.430 \\ 1.229 & 4.556 & 0.486 \\ 0.430 & 0.486 & 1.446 \end{bmatrix}.$$

Note that these values are obtained by multiplying the actual numbers by 1,000 for simplification.

We consider the problem of investors with initial wealth $x_0 = 1$ who want to maximize the expected utility of terminal wealth. We consider cases with logarithmic, power ($\gamma = 0.5$, 2 and 4) and exponential utility functions where the time horizon is $T = 4$ periods.

It is difficult to calculate optimal α values numerically for an arbitrary distribution using (2.49), (2.6) and (2.40). Our approach is to use Taylor series expansion of the utility function around the expected value $\bar{W} = E[W]$ of the terminal wealth $W = X_T$. The reader is referred to Jondeau and Rockinger [37] for a detailed discussion on the benefits, advantages and disadvantages of using Taylor series expansion in optimal portfolio allocation. In particular, they give a convincing argument for using the first 4 moments in the approximation. When we checked our data we recognized that the return distributions have non-zero skewness and excess kurtosis, so we decided to use the first four moments. Taylor series expansion is

$$U(W) = \sum_{j=0}^{+\infty} U^{(j)}\left(\bar{W}\right) \frac{\left(W - \bar{W}\right)^j}{j!}$$

where $U^{(j)}\left(\bar{W}\right)$ is the jth derivative of the utility function at $\bar{W}$. Taking expectations we can write

$$E\left[U(W)\right] = U\left(\bar{W}\right) + \frac{1}{2!}U^{(2)}\left(\bar{W}\right)\mu_p^2 + \frac{1}{3!}U^{(3)}\left(\bar{W}\right)\mu_p^3 + \frac{1}{4!}U^{(4)}\left(\bar{W}\right)\mu_p^4 + E\left[R_4\left(W, \bar{W}\right)\right] \tag{2.126}$$

where $R_4\left(W, \bar{W}\right)$ is the remainder for the first 4 moments and μ_p^n is the nth moment of the portfolio defined as

$$\mu_p^n = E\left[\left(W - \bar{W}\right)^n\right].$$

Using the definitions in Jondeau and Rockinger [37] for any market state, the second moment can be expressed as

$$\mu_p^2 = \alpha' M_2 \alpha$$

where $M_2 = \sigma$ is the covariance matrix. Similarly,

$$\mu_p^3 = \alpha' M_3 (\alpha \otimes \alpha)$$

where $\otimes$ is the Kronecker product, and M_3 is the 3×9 co-skewness matrix defined as

$$M_3 = \left[\begin{array}{ccc|ccc|ccc} s_{111} & s_{112} & s_{113} & s_{211} & s_{212} & s_{213} & s_{311} & s_{312} & s_{313} \\ s_{121} & s_{122} & s_{123} & s_{221} & s_{222} & s_{223} & s_{321} & s_{322} & s_{323} \\ s_{131} & s_{132} & s_{133} & s_{231} & s_{232} & s_{233} & s_{331} & s_{332} & s_{333} \end{array}\right]$$

with

$$s_{ijk} = E\left[(R_i - \mu_i)\left(R_j - \mu_j\right)(R_k - \mu_k)\right]$$

for $i, j, k = 1, 2, 3$. Finally,

$$\mu_p^4 = \alpha' M_4 (\alpha \otimes \alpha \otimes \alpha)$$

where M_4 is the 3×27 co-kurtosis matrix with elements

$$k_{ijkl} = E\left[(R_i - \mu_i)\left(R_j - \mu_j\right)(R_k - \mu_k)(R_l - \mu_l)\right]$$

for $i, j, k, l = 1, 2, 3$.

For the logarithmic utility function $U(x) = \log(x)$, we can write (2.126) as

$$E[U(W)] \cong \log\left(\bar{W}\right) - \frac{1}{2\bar{W}^2}\mu_p^2 + \frac{1}{3\bar{W}^3}\mu_p^3 - \frac{1}{4\bar{W}^4}\mu_p^4.$$

According to [42], Taylor series for power and logarithmic functions converge for $0 < W < 2\bar{W}$ and we suppose that this is indeed the case here. For the logarithmic utility case in Theorem 2, the optimal policy α has the same solution for the maximization problem $\max E\left[\log\left(1 + R^{e'}\alpha(i)\right)\right]$. Therefore, it suffices to take $W = 1 + R^{e\prime}\alpha$ in the Taylor series expansion (2.126). If we check our data, both covariances and expected excess returns are in the order of 0.01. So, for $W = \left(1 + R^{e'}\alpha(i)\right)$, we can suppose that $0 < W < 2\bar{W}$ and the series converges most of the time. We can therefore use the Taylor series expansion

$$E\left[U\left(1 + R^{e'}\alpha\right)\right] \cong \log\left(1 + r^{e'}\alpha\right) - \frac{1}{2\left(1 + r^{e'}\alpha\right)^2}\mu_p^2 + \frac{1}{3\left(1 + r^{e'}\alpha\right)^3}\mu_p^3 - \frac{1}{4\left(1 + r^{e'}\alpha\right)^4}\mu_p^4. \tag{2.127}$$

If we take the gradient of (2.127) with respect to α, and set it equal to zero, we find the first order condition

$$\frac{r^e}{(1+r^{e'}\alpha)}+\frac{r^e}{(1+r^{e'}\alpha)^3}\mu_p^2-\frac{r^e}{(1+r^{e'}\alpha)^4}\mu_p^3+\frac{r^e}{(1+r^{e'}\alpha)^5}\mu_p^4$$
$$-\frac{1}{(1+r^{e'}\alpha)^2}M_2\alpha+\frac{1}{(1+r^{e'}\alpha)^3}M_3(\alpha\otimes\alpha)-\frac{1}{(1+r^{e'}\alpha)^4}M_4(\alpha\otimes\alpha\otimes\alpha)=0.$$

We determined the optimal α values numerically using MATLAB for each market state and the optimal solution is

$$\alpha_l=\begin{bmatrix}4.258 & 1.968 & -0.590 & 0.033\\ 0.528 & 0.931 & 2.069 & 0.771\\ -2.196 & 2.053 & 1.406 & -1.469\end{bmatrix}$$

where the rows correspond to 3 assets and the columns correspond to 4 market states. Furthermore, the proportions of the risky assets in the risky part of the portfolio are

$$w_l=\begin{bmatrix}1.644 & 0.397 & -0.205 & -0.050\\ 0.203 & 0.188 & 0.717 & -1.161\\ -0.847 & 0.415 & 0.488 & 2.211\end{bmatrix}$$

obtained by normalizing α values. The exact amounts to be invested can easily be determined using (2.37) by simply multiplying the α values by the discounted value of β.

When we make a similar analysis through Taylor series approximation (2.126) for the power utility function $U(x)=x^{0.5}$, we obtain

$$E\left[U\left(1+R^{e'}\alpha\right)\right]\cong\left(1+r^{e'}\alpha\right)^{0.5}-\frac{1}{8}\left(1+r^{e'}\alpha\right)^{-1.5}\mu_p^2+\frac{3}{48}\left(1+r^{e'}\alpha\right)^{-2.5}\mu_p^3$$
$$-\frac{15}{384}\left(1+r^{e'}\alpha\right)^{-3.5}\mu_p^4 \qquad (2.128)$$

which give the optimality condition

$$\frac{1}{2}r^e\left(1+r^{e'}\alpha\right)^{-0.5}+\frac{3}{16}r^e\left(1+r^{e'}\alpha\right)^{-2.5}\mu_p^2-\frac{1}{4}\left(1+r^{e'}\alpha\right)^{-1.5}M_2\alpha$$
$$-\frac{15}{96}r^e\left(1+r^{e'}\alpha\right)^{-3.5}\mu_p^3+\frac{9}{48}\left(1+r^{e'}\alpha\right)^{-2.5}M_3(\alpha\otimes\alpha)$$
$$+\frac{105}{768}\left(1+r^{e'}\alpha\right)^{-4.5}\mu_p^4-\frac{15}{96}\left(1+r^{e'}\alpha\right)^{-3.5}M_4(\alpha\otimes\alpha\otimes\alpha)=0$$

by setting the gradient of (2.128) equal to zero.

The optimal α values are computed numerically using MATLAB so that

$$\alpha_{0.5} = \begin{bmatrix} 7.886 & 3.655 & -1.092 & 0.116 \\ 1.115 & 1.785 & 3.874 & 1.486 \\ -4.031 & 3.789 & 2.809 & -2.859 \end{bmatrix}.$$

For the power utility function $U(x) = x^2$ case, Taylor series expansion (2.126) is exact with

$$E\left[U\left(1 + R^{e'}\alpha\right)\right] = \left(1 + r^{e'}\alpha\right)^2 + \mu_p^2$$

which now gives the optimality condition

$$2r^e\left(1 + r^{e'}\alpha\right) + 2M_2\alpha = 0$$

or

$$\alpha = -V^{-1}r^e$$

which is equal to (2.105) since $V = M_2 + r^e r^{e\prime}$. The optimal solution is

$$\alpha_2 = \begin{bmatrix} -4.032 & -1.893 & 0.594 & 0.022 \\ -0.431 & -0.893 & -2.075 & -0.803 \\ 2.090 & -1.990 & -1.244 & 1.481 \end{bmatrix}.$$

For the power utility function $U(x) = x^4$, we can write

$$E\left[U\left(1 + R^{e'}\alpha\right)\right] = \left(1 + r^{e'}\alpha\right)^4 + 6\left(1 + r^{e'}\alpha\right)^2\mu_p^2 + 4\left(1 + r^{e'}\alpha\right)\mu_p^3 + \mu_p^4$$

and, by taking the gradient, the first order conditions are

$$4r^e\left(1 + r^{e'}\alpha\right)^3 + 12r^e\left(1 + r^{e'}\alpha\right)\mu_p^2 + 12\left(1 + r^{e'}\alpha\right)^2 M_2\alpha +$$
$$12r^e\mu_p^3 + 12\left(1 + r^{e'}\alpha\right)M_3(\alpha \otimes \alpha) + 4M_4(\alpha \otimes \alpha \otimes \alpha) = 0.$$

We determined the optimal α values numerically using MATLAB. The optimal solution is

$$\alpha_4 = \begin{bmatrix} -0.727 & -0.334 & 0.103 & 0.002 \\ -0.076 & -0.156 & -0.360 & -0.133 \\ 0.377 & -0.352 & -0.218 & 0.249 \end{bmatrix}.$$

For the exponential utility function $U(x) = \exp(-x)$, the Taylor series approximation becomes

$$E\left[U\left(1 + R^{e'}\alpha\right)\right] \cong \exp\left(-\left(1 + r^{e'}\alpha\right)\right)\left(1 + \frac{1}{2}\mu_p^2 + -\frac{1}{6}\mu_p^3 + \frac{1}{24}\mu_p^4\right)$$

and the optimality condition is

$$\exp\left(-\left(1+r^{e'}\alpha\right)\right)-r^{e}\left(1+\frac{1}{2}\mu_p^2-\frac{1}{6}\mu_p^3+\frac{1}{24}\mu_p^4\right)$$
$$\left.+M_2\alpha-\frac{1}{2}M_3\left(\alpha\otimes\alpha\right)+\frac{1}{6}M_4\left(\alpha\otimes\alpha\otimes\alpha\right)\right]=0$$

or

$$-r^{e}\left(1+\frac{1}{2}\mu_p^2-\frac{1}{6}\mu_p^3+\frac{1}{24}\mu_p^4\right)+M_2\alpha-\frac{1}{2}M_3\left(\alpha\otimes\alpha\right)+\frac{1}{6}M_4\left(\alpha\otimes\alpha\otimes\alpha\right)=0.$$

Using MATLAB, the optimal solution is

$$\alpha_e=\begin{bmatrix}4.409 & 2.031 & -0.615 & 0.005\\ 0.474 & 0.940 & 2.144 & 0.791\\ -2.292 & 2.133 & 1.352 & -1.489\end{bmatrix}.$$

We determined the logarithmic frontier, power frontier (2.104) with $\gamma = 0.5$, 2, and 4, and the exponential frontier using the explicit formulas (2.67), (2.70), (2.21), (2.22), (2.102), and (2.103). Note that the mean-variance efficient frontier is also the power frontier with the quadratic utility function with $\gamma = 2$. Numerical values are computed to be

$$m_l\left(i,T\right)=\begin{bmatrix}0.140 & 0.121 & 0.097 & 0.076\end{bmatrix}$$
$$v_l\left(i,T\right)=\begin{bmatrix}0.418 & 0.383 & 0.338 & 0.294\end{bmatrix}$$
$$m_{0.5}\left(i,T\right)=\begin{bmatrix}0.276 & 0.236 & 0.189 & 0.147\end{bmatrix}$$
$$v_{0.5}\left(i,T\right)=\begin{bmatrix}0.906 & 0.817 & 0.709 & 0.603\end{bmatrix}$$
$$m_2\left(i,T\right)=\begin{bmatrix}0.120 & 0.106 & 0.087 & 0.069\end{bmatrix}$$
$$v_2\left(i,T\right)=\begin{bmatrix}0.326 & 0.308 & 0.282 & 0.254\end{bmatrix}$$
$$m_4\left(i,T\right)=\begin{bmatrix}0.022 & 0.019 & 0.016 & 0.012\end{bmatrix}$$
$$v_4\left(i,T\right)=\begin{bmatrix}0.061 & 0.057 & 0.051 & 0.046\end{bmatrix}$$
$$m_e\left(i,T\right)=\begin{bmatrix}0.137 & 0.118 & 0.096 & 0.076\end{bmatrix}$$
$$v_e\left(i,T\right)=\begin{bmatrix}0.387 & 0.365 & 0.324 & 0.297\end{bmatrix}$$

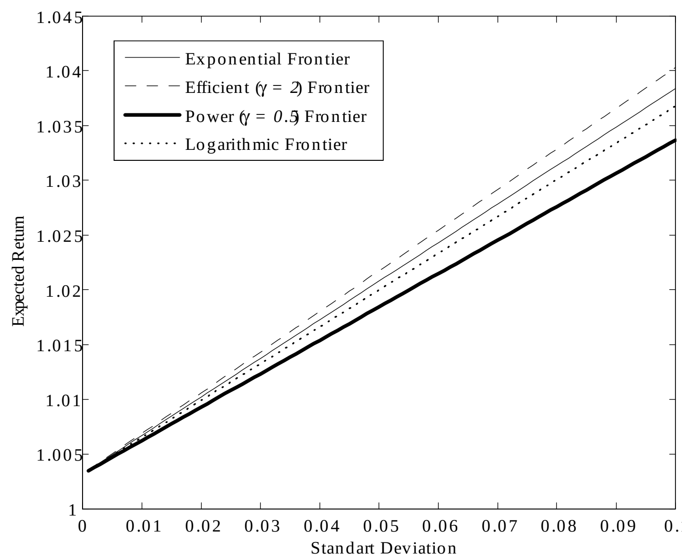

Figure 2.3: HARA frontiers for $i = 1$, $T = 4$

and the slopes, or the risk premiums, are

$$m_l\,(i,T)\,/v_l(i,T) = \left[\begin{array}{cccc} 0.336 & 0.316 & 0.287 & 0.258 \end{array}\right]$$
$$m_{0.5}\,(i,T)\,/v_{0.5}(i,T) = \left[\begin{array}{cccc} 0.305 & 0.289 & 0.267 & 0.244 \end{array}\right]$$
$$m_2\,(i,T)\,/v_2(i,T) = \left[\begin{array}{cccc} 0.371 & 0.344 & 0.309 & 0.273 \end{array}\right]$$
$$m_4\,(i,T)\,/v_4(i,T) = \left[\begin{array}{cccc} 0.364 & 0.339 & 0.305 & 0.270 \end{array}\right]$$
$$m_e\,(i,T)\,/v_e(i,T) = \left[\begin{array}{cccc} 0.352 & 0.324 & 0.294 & 0.254 \end{array}\right].$$

Investors with different utility functions will have differing risk preferences measured by β in their utility functions. The return and risk of the terminal wealth for these investors will be on the respective frontier in Figure 2.3. The slopes measure the risk premiums and, as expected, the risk premiums for the mean-variance frontier are highest.

We considered the exponential utility case and obtained the exponential frontier by solving the optimality condition (2.6) directly under the assumption that the asset returns follow a multivariate normal distribution. They obtained the frontier to be

$$m_e\,(i,T)\,/v_e\,(i,T) = \left[\begin{array}{cccc} 0.353 & 0.324 & 0.295 & 0.255 \end{array}\right]$$

which is very close to our approximate values. However, note that Taylor series approximation does not require knowledge on the asset return distributions.

Chapter 3

MODELS WITH IMPERFECT INFORMATION

In this chapter, the market still depends on the environment driven by a Markov chain, however, the market process is hidden. Imperfect information flow is set up through a probabilistic relationship between the observed and unobserved market processes. The unobserved stochastic market is a Markov chain and it emits signals, or provides information, that are observed by the market players. Models of this type, where the random market environment is represented by a Markov chain and the true state of this Markov chain cannot be observed directly (however, there is another process which gives partial information about the true state) are called "Partially Observed Markov Decision Processes" (POMDP). A general discussion about these models can be found in Elliott et al. [24].

Partially observed Markov decision processes in portfolio optimization have been used in the last ten years. More recently, Elliott et al. [25] use a hidden market model (HMM) to describe stock price movements in order to find optimal portfolio trading strategy that maximizes the expected terminal wealth. Even though HMMs are one of the important tools used in areas like speech recognition, bioinformatics, and gene prediction; they have not been used in portfolio optimization until quite recently. Sass and Haussmann [61] discuss a model in continuous time where the interest rate and rates of returns of the risky assets depend on a continuous time Markov process. Rieder and Bauerle [58] extended their research where the drift rate of the stock depends on the Markov process and the only observation is the stock prices. Corsi et al. [14] study a numerical approximation method to solve hidden Markov models using quantization methods. Dericioğlu and Özekici [18] applied the imperfect information concept to mean-variance portfolio selection problem in a Markovian market. They solved the problem with dynamic programming and obtained an explicit optimal solution to represent the efficient frontier.

Imperfect information is also a rather interesting concept in finance like HMMs. This concept appears in game theory and is used for sequential games where a player does not know exactly what actions other players take. Stiglitz [69] focuses on imperfect information and emphasizes that obtaining information is imperfect, costly and there are major information asymmetries. Moreover, he believes that understanding imperfect information is one of the most important breaks from the past, and provides explanations to some of the basic characteristics of a market economy. The study on credit rationing by Stiglitz and Weiss [70] presents the first theoretical justification of true credit rationing by considering the effect of imperfect information in markets. In addition, Stiglitz [68]explains the observed phenomena of price dispersions and advertising effects at the equilibrium of product markets, which cannot be explained by traditional models of competition with perfect information.

3.1 Hidden Markov Model

We let Z_n denote the state of the stochastic market in period n, and assume that $Z = \{Z_n;\ n = 0, 1, 2, \cdots\}$ is a Markov chain with some possibly time-dependent transition matrix

$$Q_n(a, b) = P\{Z_{n+1} = b | Z_n = a\}$$

and discrete state space $F = \{a, b, c, \cdots\}$. The states of the market are not observable and the Markov chain Z is hidden. This implies that information available to investors is not perfect. The imperfect observations on the state of the market are given by an observation process $Y = \{Y_n;\ n = 0, 1, 2, \cdots\}$ with some discrete state space $E = \{i, j, \cdots\}$ where Y_n is the information available in period n. The market functions according to the unobserved process Z whose states depend on various economic factors; however, investors in the market can only see the observed process Y. Moreover, they base their decisions on what they observe.

The relationship between the stochastic market and the random returns is such that the distribution of the return of risky assets in a period depends only on the unobserved state of the market in that period. The market consists of m risky assets with random returns $R(n, a) = (R_1(n, a), R_2(n, a), \cdots, R_m(n, a))$ whenever the state of the market is a in period n. We let $r_k(n, a) = E[R_k(n, a)]$ denote the mean return of the kth asset and $\sigma_{kl}(n, a) = \text{Cov}(R_k(n, a), R_l(n, a))$ denote the covariance between kth and lth asset returns in state a and period n. There is also a riskless asset which is typically a cash bond and the return of the cash bond depends on the observed state of the market since it is known to the investor with certainty. This allows us to assume that riskless lending or borrowing is possible with return $r_f(i)$ if the observed market state is i. In such a case, the excess return of the kth asset is

$$R_k(n, Z_n) - r_f(Y_n) \tag{3.1}$$

in period n. As in perfect information case, X_n gives the wealth and the vector $u = (u_1, u_2, \cdots, u_m)$ gives the amounts invested in risky assets $(1, 2, \cdots, m)$ at period n. As a result the wealth dynamics equation

$$X_{n+1}(u) = r_f(Y_n) X_n + R_n^e(Y_n, Z_n)' u$$

is still valid where

$$R_n^e(i, a) = R(n, a) - r_f(i).$$

We let

$$r_n^e(i, a) = E[R_n^e(i, a)] = r(n, a) - r_f(i) \tag{3.2}$$

denote the mean vector of the excess return and

$$V_n(i, a) = E\left[R_n^e(i, a) R_n^e(i, a)'\right] = \sigma(n, a) + r^e(i, a) r^e(i, a)'$$

denote the matrix of second moments as before. Note that the covariance matrix $\sigma(n, a)$ is positive definite so one can easily see that $V_n(i, a)$ is also positive definite.

We will use the notation $E_i[Z] = E[Z \mid Y_0 = i]$ and $\mathrm{Var}_i(Z) = E_i[Z^2] - E_i[Z]^2$ to denote the conditional expectation and variance of any random variable Z given that the initial market state is i.

It is clear that the observed process Y is not necessarily a Markov chain and the state of the stochastic market Z depends on all of the past observations of Y. The relationship between the two processes Y and Z is made formal by enlarging the state spaces E and F so that $E^{n+1} = E \times E \times \cdots \times E = \{(i_0, i_1, i_2, \cdots, i_n) : i_k \in E\}$ and $F^{n+1} = F \times F \times \cdots \times F = \{(a_0, a_1, a_2, \cdots, a_n) : a_k \in F\}$. We also simplify our notation by letting $\bar{\imath}_n = (i_0, i_1, i_2, \cdots, i_n) \in E^{n+1}$, $\bar{a}_n = (a_0, a_1, a_2, \cdots, a_n) \in F^{n+1}$ denote elements of E^{n+1} and F^{n+1}; and $\bar{Y}_n = (Y_0, Y_1, Y_2, \cdots, Y_n)$ and $\bar{Z}_n = (Z_0, Z_1, Z_2, \cdots, Z_n)$ denote the information gathered and the past history of the market until period n respectively. The probabilistic evolution of Y depends purely on the state of Z such that

$$P\{Y_n = i | Z_n = a\} = G_n(a, i)$$

independent of all previous states of Z and Y in any period n. Here, G is often called the emission matrix as in the signal processing context from which the idea of HMM is driven.

Simple probabilistic arguments give

$$\begin{aligned} O_n(\bar{\imath}_n, a) &= P\{Z_n = a | \bar{Y}_n = \bar{\imath}_n\} \\ &= \frac{\sum\limits_{\bar{b}_{n-1} \in F^n} P\{Z_0 = b_0\} G_0(b_0, i_0) Q_0(b_0, b_1) G_1(b_1, i_1) \cdots Q_{n-1}(b_{n-1}, a) G_n(a, i_n)}{\sum\limits_{\bar{b}_n \in F^{n+1}} P\{Z_0 = b_0\} G_0(b_0, i_0) Q_0(b_0, a_1) G_1(b_1, i_1) \cdots Q_{n-1}(b_{n-1}, b_n) G_n(b_n, i_n)} \end{aligned}$$

for $n \geq 1$, while

$$O_0(i, a) = P\{Z_0 = a | Y_0 = i_0\} = \frac{P\{Z_0 = a\} G_0(a, i_0)}{\sum\limits_{b_0 \in F} P\{Z_0 = b_0\} G_0(b_0, i_0)}$$

for $n = 0$. We can now write

$$\begin{aligned} P_n(\bar{\imath}_n, j, a) &= P\left\{Y_{n+1} = j, Z_n = a | \bar{Y}_n = \bar{\imath}_n\right\} \\ &= O_n(\bar{\imath}_n, a) \sum_{b \in F} Q_n(a, b) G_n(b, j). \end{aligned}$$

Note that $\{P_n\}$, and $\{O_n\}$ can easily be determined once the transition matrices $\{Q_n\}$, emission matrices $\{G_n\}$ and the initial distribution of the true state of the market are known.

3.1.1 Dynamic Programming Formulation

In order to solve the portfolio selection problem, we define $g_n(\bar{\imath}_n, x, u)$ as the expected utility using the investment policy u in period n and the optimal policies from period $n+1$ to period T given that the observed market state is $\bar{\imath}_n$ and the amount of money available for investment is x at period n. Then,

$$v_n(\bar{\imath}_n, x) = \max_u g_n(\bar{\imath}_n, x, u)$$

is the optimal expected utility using the optimal policy given observation vector $\bar{\imath}_n$ and the amount of money x available for investment at period n. According to the dynamic programming principle

$$g_n(\bar{\imath}_n, x, u) = E\left[v_{n+1}\left(\bar{Y}_{n+1}, X_{n+1}(u)\right) \middle| \bar{Y}_n = \bar{\imath}_n, X_n = x\right]$$

and we can write the dynamic programming equation (DPE) as

$$v_n(\bar{\imath}_n, x) = \max_u E\left[v_{n+1}\left(\bar{Y}_{n+1}, X_{n+1}(u)\right)\left|\bar{Y}_n = \bar{\imath}_n, X_n = x\right.\right] \tag{3.3}$$

which can be rewritten as

$$v_n(\bar{\imath}_n, x) = \max_u \sum_{a\in F}\sum_{j\in E} P_n(\bar{\imath}_n, j, a)\, E\left[v_{n+1}\left((\bar{\imath}_n, j), r_f(i_n)\, x + R_n^e(i_n, a)' u\right)\right] \tag{3.4}$$

for $n = 0, 1, \cdots, T-1$ with the boundary condition $v_T(\bar{\imath}_T, x) = U(i_T, x)$ for all $\bar{\imath}_T \in E^{T+1}$. The solution for this problem is found by solving the DPE recursively.

3.1.2 Exponential Utility

In this section, we assume that the utility of the investor in state i is given by the exponential function

$$U(i, x) = K(i) - C(i)\exp(-x/\beta) \tag{3.5}$$

with $C(i) > 0$. Note that as in the case of perfect information β is that same for all market states so that risk classification of the investor does not depend on the stochastic market. Similarly, we assume that the return for the riskless asset is same for all market states so that $r_f(i) = r_f$ for all i. Then,

$$R_n^e(i, a) = R_n^e(a) = R(n, a) - r_f$$

so that both vector r_n^e and matrix V_n do not depend on the observed state. To simplify the notation, we can write $r_n^e(i, a) = r_n^e(a) = r(n, a) - r_f$ and $V_n(i, a) = V_n(a) = \sigma(n, a) + r_n^e(a)\, r_n^e(a)'$.

Theorem 6 *Let the utility function of the investor be the exponential function (3.5) and suppose that the riskless asset return does not depend on the market state. Then, the optimal solution of the dynamic programming equation (3.4) is*

$$v_n(\bar{\imath}_n, x) = K_n(\bar{\imath}_n) - C_n(\bar{\imath}_n) e^{-x/\beta_n}$$

and the optimal portfolio is

$$u_n^*(\bar{\imath}_n, x) = \alpha_n(\bar{\imath}_n)\beta_{n+1} \tag{3.6}$$

where

$$\begin{aligned}\beta_n &= \frac{\beta}{r_f^{T-n}},\ K_n(\bar{\imath}_n) = \sum_{a\in F}\sum_{j\in E} P_n(\bar{\imath}_n, j, a) K_{n+1}((\bar{\imath}_n, j)),\\ C_n(\bar{\imath}_n) &= \sum_{a\in F}\sum_{j\in E} P_n(\bar{\imath}_n, j, a) E\left[\exp\left(-R_n^e(a)'\alpha_n(\bar{\imath}_n)\right)\right] C_{n+1}((\bar{\imath}_n, j))\end{aligned}$$

and $\alpha_n(\bar{\imath}_n)$ *satisfies*

$$\sum_{a\in F}\sum_{j\in E} P_n(\bar{\imath}_n, j, a) C_{n+1}((\bar{\imath}_n, j)) E\left[R_{n,k}^e(a)\exp\left(-R_n^e(a)'\alpha_n(\bar{\imath}_n)\right)\right] = 0 \tag{3.7}$$

for all assets $k = 1, 2, \cdots, m$, $\bar{\imath}_n \in E^{n+1}$, *and* $n = 0, 1, \cdots, T-1$ *with boundary conditions*

$$K_T(\bar{\imath}_T) = K(i_T),\quad C_T(\bar{\imath}_T) = C(i_T).$$

Proof. We use induction starting with the boundary condition $v_T(\bar{\imath}_T, x) = K(i_T) - C(i_T)\exp(-x/\beta)$ and obtain

$$\begin{aligned} g_{T-1}(\bar{\imath}_{T-1}, x, u) &= \sum_{a\in F}\sum_{j\in E} P_{T-1}(\bar{\imath}_{T-1}, j, a) E[U(j, r_f x + R_{T-1}^e(a)'u)]\\ &= -\exp(-r_f x/\beta)\sum_{a\in F}\sum_{j\in E} P_{T-1}(\bar{\imath}_{T-1}, j, a) C(j) E[\exp(-R_{T-1}^e(a)'u/\beta)]\\ &\quad + \sum_{a\in F}\sum_{j\in E} P_{T-1}(\bar{\imath}_{T-1}, j, a) K(j).\end{aligned}$$

Let $u^* = (u_1^*, u_2^*, \cdots, u_m^*)$ be the optimal amount of money that should be invested in the risky assets so that

$$v_{T-1}(\bar{\imath}_{T-1}, x) = \max_u g_{T-1}(\bar{\imath}_{T-1}, x, u) = g_{T-1}(\bar{\imath}_{T-1}, x, u^*)$$

as in the perfect information case. The entries of the gradient vector of the objective function are

$$\begin{aligned}\frac{\partial g_{T-1}(\bar{\imath}_{T-1}, x, u)}{\partial u_k} &= \exp(-r_f x/\beta)\sum_{a\in F}\sum_{j\in E} P_{T-1}(\bar{\imath}_{T-1}, j, a) C(j)\\ &\quad \cdot E\left[R_{T-1,k}^e(a)\exp(-R_{T-1}^e(a)'u/\beta)\right]/\beta \end{aligned}\tag{3.8}$$

and the Hessian matrix $H(\bar{\imath}_{T-1}, x, u)$ entries are

$$\begin{aligned}\frac{\partial^2 g_{T-1}(\bar{\imath}_{T-1}, x, u)}{\partial u_k \partial u_l} &= -\frac{1}{\beta^2}\exp(-r_f x/\beta)\sum_{a\in F}\sum_{j\in E} P_{T-1}(\bar{\imath}_{T-1}, j, a) C(j)\\ &\quad \cdot E\left[R_{T-1,k}^e(a) R_{l,T-1}^e(a)\exp(-R_{T-1}^e(a)'u/\beta)\right].\end{aligned}$$

Let $z = (z_1, \cdots, z_m)$ be any non-zero column vector where z_i's are real numbers. Then, one can see that

$$\begin{aligned} z^T H\left(\bar{\imath}_{T-1}, x, u\right) z &= -\frac{1}{\beta^2} \exp\left(-r_f x/\beta\right) \sum_{a\in F} \sum_{j\in E} P_{T-1}\left(\bar{\imath}_{T-1}, j, a\right) C(j) \\ &\quad \cdot E\left[\left(\sum_{k=1}^{m} z_k R^e_{T-1,k}\left(a\right)\right)^2 \exp(-(r_f x + R^e_{T-1}(a)'u)/\beta)\right] \end{aligned}$$

which is always less than or equal to zero since all $P_{T-1}\left(\bar{\imath}_{T-1}, j, a\right)$ and $C(j)$ are all positive. Thus, $H\left(\bar{\imath}_{T-1}, x, u\right)$ is negative semi-definite and we can find the optimal solution by setting the gradient (3.8) equal to zero to obtain the optimality condition

$$\sum_{a\in F} \sum_{j\in E} P_{T-1}(\bar{\imath}_{T-1}, j, a) C(j) E\left[R^e_{T-1,k}\left(a\right) \exp(-R^e_{T-1}\left(a\right)' u^*/\beta)\right]/\beta = 0 \qquad (3.9)$$

for all $k = 1, 2, \cdots, m$. Since there is no dependence on x in (3.9), $u^*_{T-1}(\bar{\imath}_{T-1}, x)$ does not depend on x and $u^*_{T-1}(\bar{\imath}_{T-1}, x) = u^*_{T-1}(\bar{\imath}_{T-1})$. Letting $\alpha_{T-1}\left(\bar{\imath}_{T-1}\right) = u^*_{T-1}\left(\bar{\imath}_{T-1}\right)/\beta$, we obtain $u^*_{T-1}(\bar{\imath}_{T-1}, x) = \alpha_{T-1}\left(\bar{\imath}_{T-1}\right)\beta$ and this gives optimality condition (3.7). When the value function at time $T-1$ is rewritten for the optimal policy, we obtain

$$\begin{aligned} v_{T-1}(\bar{\imath}, x) &= \sum_{a\in F} \sum_{j\in E} P_{T-1}(\bar{\imath}_{T-1}, j, a) C(j) \exp(-r_f x/\beta) E[-\exp\left(-R^e_{T-1}\left(a\right)' \alpha_{T-1}\left(\bar{\imath}_{T-1}\right)\right)] \\ &\quad + \sum_{a\in F} \sum_{j\in E} P_{T-1}\left(\bar{\imath}_{T-1}, j, a\right) K(j) \\ &= K_{T-1}(\bar{\imath}_{T-1}) - C_{T-1}(\bar{\imath}_{T-1}) \exp(-x/\beta_{T-1}) \end{aligned}$$

and the value function is still exponential like the utility function. This shows that the induction hypothesis holds for $n = T-1$.

Suppose now that the induction hypothesis holds for periods $T, T-1, T-2, \cdots, n$. Then, for period $n-1$,

$$\begin{aligned} g_{n-1}(\bar{\imath}_{n-1}, x, u) &= \sum_{a\in F} \sum_{j\in E} P_{n-1}\left(\bar{\imath}_{n-1}, j, a\right) E[v_n(\left(\bar{\imath}_{n-1}, j\right), r_f x + R^e_{n-1}\left(a\right)' u)] \\ &= -\exp\left(-r_f x/\beta_n\right) \sum_{a\in F} \sum_{j\in E} P_{n-1}\left(\bar{\imath}_{n-1}, j, a\right) C_n(\left(\bar{\imath}_{n-1}, j\right)) \\ &\quad \cdot E\left[\exp\left(-R^e_{n-1}\left(a\right)' u/\beta_n\right)\right] \\ &\quad + \sum_{a\in F} \sum_{j\in E} P_{n-1}\left(\bar{\imath}_{n-1}, j, a\right) K_n\left(\left(\bar{\imath}_{n-1}, j\right)\right). \end{aligned}$$

One can easily see that the Hessian matrix of $g_{n-1}(\bar{\imath}_{n-1}, x, u)$ is negative semi-definite like the Hessian matrix of $g_{T-1}(\bar{\imath}_{T-1}, x, u)$. Let $u^*_{n-1}(\bar{\imath}_{n-1}, x)$ be the optimal policy such that

$$v_{n-1}(\bar{\imath}_{n-1}, x) = \max_u g_{n-1}(\bar{\imath}_{n-1}, x, u) = g_{n-1}(\bar{\imath}_{n-1}, x, u^*_{n-1}).$$

If we take the gradient of $g_{n-1}(\bar{\imath}_{n-1}, x, u)$ with respect to u and set it equal to 0, we get the optimality condition

$$\sum_{a \in F} \sum_{j \in E} P_{n-1}(\bar{\imath}_n, j, a) C_n((\bar{\imath}_{n-1}, j)) E\left[R^e_{n-1,k}(a) \exp(-R^e_{n-1}(a)' u^*_{n-1}/\beta_n)\right] = 0. \quad (3.10)$$

Since there is no dependence on x in (3.10), $u^*_{n-1}(\bar{\imath}_{n-1}, x)$ does not depend on x and $u^*_{n-1}(\bar{\imath}_{n-1}, x) = u^*_{n-1}(\bar{\imath}_{n-1})$. Letting $\alpha_{n-1}(\bar{\imath}_n) = u^*_{n-1}(\bar{\imath}_{n-1})/\beta_n$ we obtain $u^*_{n-1}(\bar{\imath}_{n-1}, x) = \alpha_{n-1}(\bar{\imath}_{n-1})\beta_n$ and this gives optimality condition (3.7). If we insert the optimal policy in the value function, we can see that

$$\begin{aligned}
v_{n-1}(\bar{\imath}_{n-1}, x) &= -\sum_{a \in F} \sum_{j \in E} P_{n-1}(\bar{\imath}_{n-1}, j, a) C_n((\bar{\imath}_{n-1}, j)) \\
&\qquad \cdot E\left[\exp\left(-\left(r_f x/\beta_n + R^e_{n-1}(a)' \alpha_{n-1}(\bar{\imath}_{n-1})\right)\right)\right] \\
&\qquad + \sum_{a \in F} \sum_{j \in E} P_{n-1}(\bar{\imath}_{n-1}, j, a) K_n((\bar{\imath}_{n-1}, j)) \\
&= -\exp(-r_f x/\beta_n) \sum_{a \in F} \sum_{j \in E} P_{n-1}(\bar{\imath}_{n-1}, j, a) C_n((\bar{\imath}_{n-1}, j)) \\
&\qquad \cdot E\left[\exp\left(-R^e_{n-1}(a)' \alpha_{n-1}(\bar{\imath}_{n-1})\right)\right] \\
&\qquad + \sum_{a \in F} \sum_{j \in E} P_{n-1}(\bar{\imath}_{n-1}, j, a) K_n((\bar{\imath}_{n-1}, j)) \\
&= K_{n-1}(\bar{\imath}_{n-1}) - C_{n-1}(\bar{\imath}_{n-1}) \exp(-x/\beta_{n-1})
\end{aligned}$$

and this completes the proof. ■

In Theorem 6, we have found a closed-form solution for the optimal portfolio. We can further characterize the optimal policy by noting from (3.7) that the optimal solution satisfies

$$\sum_{a \in F} \sum_{j \in E} P_n(\bar{\imath}_n, j, a) C_{n+1}((\bar{\imath}_n, j)) E\left[(R_{n,k}(a) - r_f) \exp\left(-(R_n(a) - r_f)' \alpha_n(\bar{\imath}_n)\right)\right] = 0$$

which implies

$$\sum_{a \in F} \sum_{j \in E} P_n(\bar{\imath}_n, j, a) C_{n+1}((\bar{\imath}_n, j)) E\left[(R_{n,k}(a) - r_f) \exp\left(-R_n(a)' \alpha_n(\bar{\imath}_n)\right)\right] = 0$$

and

$$\frac{\sum_{a\in F}\sum_{j\in E}P_n(\bar{\imath}_n,j,a)C_{n+1}((\bar{\imath}_n,j))E\left[R_{n,k}\left(a\right)\exp\left(-R_n\left(a\right)'\alpha_n(\bar{\imath}_n)\right)\right]}{\sum_{a\in F}\sum_{j\in E}P_n(\bar{\imath}_n,j,a)C_{n+1}((\bar{\imath}_n,j))E\left[\exp\left(-R_n\left(a\right)'\alpha_n(\bar{\imath}_n)\right)\right]}=r_f \tag{3.11}$$

for all assets $k=1,2,\cdots,m$.

A significant characterization implied by the optimal solution (3.6) is that the optimal distribution of wealth invested on the risky assets depend on the state of the market, but it is independent of the wealth level. If the market is in state $\bar{\imath}_n$ in period n, then the total amount of money invested on the risky assets is

$$1'u_n^*(\bar{\imath}_n,x)=1'\alpha_n(\bar{\imath}_n)\beta_{n+1}=\frac{\beta}{r_f^{T-(n+1)}}\sum_{k=1}^{m}\alpha_{n,k}(\bar{\imath}_n)$$

which does not depend on the current wealth level x. Moreover, the proportion on wealth allocated for asset k is

$$w_{n,k}(\bar{\imath}_n)=\frac{\alpha_{n,k}(\bar{\imath}_n)}{\sum\limits_{k=1}^{m}\alpha_{n,k}(\bar{\imath}_n)} \tag{3.12}$$

which is also totally independent of wealth x. But if we examine the optimality condition (3.7) we see that the optimal portfolio policy depends on the $\{C_n\}$ values contrarily to the perfect information case. This is a very interesting observation. In the perfect information case, the optimal policy of the investor is independent of the transition matrix $\{Q_n\}$ of the stochastic market, but if the state of the market cannot be observed the investor must take the transition matrix into consideration.

The memorylessness property of the exponential utility function is still valid. Like the memorylessness property of the exponential distribution that is associated with time, the exponential utility function implies a similar property associated with the wealth of the investor. The investor is memoryless in the sense that his current wealth level does not affect how he chooses to allocate his money among the risky assets. However, note that there is randomness involved in this choice due to the randomly changing market conditions.

Evolution of Wealth

The evolution of the wealth process X using the optimal policy can be analyzed by the wealth dynamics equation

$$\begin{aligned} X_{n+1} &= r_f X_n + R_n^e(Z_n)' u_n^*(\bar{Y}_n, X_n) \\ &= r_f X_n + R_n^e(Z_n)' \alpha_n(\bar{Y}_n) \beta_{n+1} \\ &= r_f X_n + r_f^{n+1-T} R_n^e(Z_n)' \alpha_n(\bar{Y}_n) \beta. \end{aligned} \tag{3.13}$$

Define the random variable

$$A_n\left(\bar{\imath}_n, a\right) = R_n^e(a)' \alpha_n\left(\bar{\imath}_n\right) \tag{3.14}$$

with mean

$$\begin{aligned} \bar{\alpha}_n\left(\bar{\imath}_n, a\right) &= E\left[A_n\left(\bar{\imath}_n, a\right)\right] = E\left[R_n^e(a)' \alpha_n\left(\bar{\imath}_n\right)\right] \\ &= r_n^e(a)' \alpha_n\left(\bar{\imath}_n\right) \end{aligned} \tag{3.15}$$

and second moment

$$\begin{aligned} \tilde{\alpha}_n\left(\bar{\imath}_n, a\right) &= E\left[A_n\left(\bar{\imath}_n, a\right)^2\right] = E\left[\alpha_n\left(\bar{\imath}_n\right)' R_n^e(a) R_n^e(a)' \alpha_n\left(\bar{\imath}_n\right)\right] \\ &= \alpha_n\left(\bar{\imath}_n\right)' V_n\left(a\right) \alpha_n\left(\bar{\imath}_n\right) \end{aligned} \tag{3.16}$$

which gives the variance

$$\operatorname{Var}\left(A_n\left(\bar{\imath}_n, a\right)\right) = \tilde{\alpha}_n\left(\bar{\imath}_n, a\right) - \bar{\alpha}_n\left(\bar{\imath}_n, a\right)^2. \tag{3.17}$$

Now, we will show that the wealth process is given by

$$X_n = r_f^n X_0 + r_f^{n-T} \beta \sum_{k=0}^{n-1} A_k\left(\bar{Y}_k, Z_k\right) \tag{3.18}$$

using the induction method where the sum on the right-hand side is set to zero when $n = 0$. The induction hypothesis holds trivially for $n = 0$. Suppose (3.18) holds for some $n \geq 0$. If we write X_{n+1} using the wealth dynamics equation (3.13)

$$\begin{aligned} X_{n+1} &= r_f X_n + r_f^{n+1-T} A_n\left(\bar{Y}_n, Z_n\right) \beta \\ &= r_f^{n+1} X_0 + r_f^{n+1-T} \beta \sum_{k=0}^{n-1} A_k\left(\bar{Y}_k, Z_k\right) + r_f^{n+1-T} A_n\left(\bar{Y}_n, Z_n\right) \beta \\ &= r_f^{n+1} X_0 + r_f^{n+1-T} \beta \sum_{k=0}^{n} A_k\left(\bar{Y}_k, Z_k\right) \end{aligned}$$

we see that the induction hypothesis also holds for $n+1$. So, we can conclude that the wealth process can be written as in (3.18), and the terminal wealth is

$$X_T = r_f^T X_0 + \beta \sum_{k=0}^{T-1} A_k\left(\bar{Y}_k, Z_k\right) \tag{3.19}$$

for $n = T$.

Given $X_0 = x_0$, the expected value of the terminal wealth satisfies

$$E_i\left[X_T\right] = r_f^T x_0 + m\left(i, T\right)\beta$$

where

$$m\left(i, T\right) = \sum_{k=0}^{T-1} E_i\left[A_k\left(\bar{Y}_k, Z_k\right)\right] \tag{3.20}$$

and the variance of the terminal wealth satisfies

$$\mathrm{Var}_i\left(X_T\right) = v^2\left(i, T\right)\beta^2$$

where

$$v^2\left(i, T\right) = \mathrm{Var}_i\left(\sum_{k=0}^{T-1} A_k\left(\bar{Y}_k, Z_k\right)\right). \tag{3.21}$$

We see that both the return and the standard deviation of X_T depends linearly on β. This shows that the exponential frontier is the line

$$E_i\left[X_T\right] = r_f^T x_0 + \left(\frac{m\left(i, T\right)}{v\left(i, T\right)}\right)\mathrm{SD}_i\left(X_T\right)$$

where $\mathrm{SD}_i\left(X_T\right)) = \sqrt{\mathrm{Var}_i\left(X_T\right)}$. Note that $m\left(i, T\right)$ and $v\left(i, T\right)$ can easily be computed through linear operations. The computational process is outlined next.

Computational Formulas

Define transition probabilities of the process $\left(\bar{Y}, Z\right)$ such that

$$T_{n,k}\left(\left(\bar{\imath}_n, a\right), \left(\bar{\imath}_{n+k}, b\right)\right) = P\left\{\bar{Y}_{n+k} = \bar{\imath}_{n+k}, Z_{n+k} = b | \bar{Y}_n = \bar{\imath}_n, Z_n = a\right\}$$

where $\bar{\imath}_{n+k} = \left(\bar{\imath}_n, i_{n+1}, i_{n+2}, \cdots, i_{n+k}\right)$. We can determine $T_{n,k}$ using a recursive algorithm with initial condition

$$\begin{aligned} T_{n,1}\left(\left(\bar{\imath}_n, a\right), \left(\left(\bar{\imath}_n, i_{n+1}\right), b\right)\right) &= P\left\{Y_{n+1} = i_{n+1}, Z_{n+1} = b | \bar{Y}_n = \bar{\imath}_n, Z_n = a\right\} \\ &= Q_n\left(a, b\right) G_{n+1}\left(b, i_{n+1}\right) \end{aligned}$$

and the recursion

$$\begin{aligned} T_{n,k}\left(\left(\bar{\imath}_n, a\right), \left(\bar{\imath}_{n+k}, b\right)\right) &= \sum_{b_{k-1} \in F} T_{n,k-1}\left(\left(\bar{\imath}_n, a\right), \left(\bar{\imath}_{n+k-1}, b_{k-1}\right)\right) \\ &\quad \cdot Q_{n+k-1}\left(b_{k-1}, b\right) G_{n+k}\left(b, i_{n+k}\right) \end{aligned}$$

for $k \geq 2$.

It follows that these transition probabilities can be computed directly using

$$\begin{aligned} T_{n,k}\left(\left(\bar{\imath}_n, a\right), \left(\bar{\imath}_{n+k}, b\right)\right) &= \sum_{b_1, b_2, \cdots, b_{k-1} \in F} Q_n\left(a, b_1\right) G_{n+1}\left(b_1, i_{n+1}\right) Q_{n+1}\left(b_1, b_2\right) \\ &\quad G_{n+2}\left(b_2, i_{n+2}\right) \cdots Q_{n+k-1}\left(b_{k-1}, b\right) G_{n+k}\left(b, i_{n+k}\right) \end{aligned}$$

for $\bar{\imath}_{n+k} = \left(\bar{\imath}_n, i_{n+1}, \cdots i_{n+k}\right)$.

We can also determine the distribution of $\left(\bar{Y}_k, Z_k\right)$ using

$$\begin{aligned} \mathcal{T}_k\left(i, \left(\bar{\imath}_k, b\right)\right) &= P\left\{\bar{Y}_k = \bar{\imath}_k, Z_k = b | Y_0 = i\right\} \\ &= \sum_{a \in F} P\left\{\bar{Y}_k = \bar{\imath}_k, Z_k = b \,|Y_0 = i, Z_0 = a\right\} P\left\{Z_0 = a \,|Y_0 = i\right\} \\ &= \sum_{a \in F} T_{0,k}\left(\left(i, a\right), \left(\bar{\imath}_k, b\right)\right) O_0\left(i, a\right) \\ &= \sum_{a, b_1, b_2, \cdots, b_{k-1} \in F} O_0\left(i, a\right) Q_0\left(a, b_1\right) G_1\left(b_1, i_1\right) Q_1\left(b_1, b_2\right) G_2\left(b_2, i_2\right) \\ &\quad \cdots Q_{k-1}\left(b_{k-1}, b\right) G_k\left(b, i_k\right) \end{aligned}$$

for $k \geq 1$ and $\bar{\imath}_k = \left(i, i_1, \cdots, i_k\right)$. Note that $T_{0,0}\left(\left(i, a\right), \left(i_0, b\right)\right) = I\left(i, i_0\right) I\left(a, b\right)$ trivially.

Using these results we can rewrite (3.20) as

$$\begin{aligned} m\left(i, T\right) &= \sum_{k=0}^{T-1} \sum_{i_1, i_2, \cdots, i_k \in E, b \in F} P\left\{Y_k = \bar{\imath}_k, Z_k = b | Y_0 = i\right\} \bar{\alpha}_k\left(\bar{\imath}_k, b\right) \\ &= \sum_{k=0}^{T-1} \sum_{i_1, i_2, \cdots, i_k \in E, b \in F} \mathcal{T}_k\left(i, \left(\bar{\imath}_k, b\right)\right) \bar{\alpha}_k\left(\bar{\imath}_k, b\right). \end{aligned}$$

where $i_0 = i$.

Similarly, (3.21) becomes

$$\begin{aligned}
v^2(i,T) &= \mathrm{Var}_i\left(\sum_{k=0}^{T-1} A_k\left(\bar{Y}_k, Z_k\right)\right) \\
&= \sum_{k=0}^{T-1}\sum_{m=0}^{T-1} \mathrm{Cov}_i\left(A\left(\bar{Y}_k, Z_k\right), A\left(\bar{Y}_m, Z_m\right)\right) \\
&= \sum_{k=0}^{T-1} \mathrm{Var}_i\left(A_k\left(\bar{Y}_k, Z_k\right)\right) + 2\sum_{k=0}^{T-1}\sum_{m=k+1}^{T-1} \mathrm{Cov}_i\left(A_k\left(\bar{Y}_k, Z_k\right), A_m\left(\bar{Y}_m, Z_m\right)\right) \\
&= \sum_{k=0}^{T-1}\left(E_i\left[\left(A_k\left(\bar{Y}_k, Z_k\right)\right)^2\right] - E_i\left[A_k\left(\bar{Y}_k, Z_k\right)\right]^2\right) \\
&\quad +2\sum_{k=0}^{T-1}\sum_{m=k+1}^{T-1}\Big(E_i\left[A_k\left(\bar{Y}_k, Z_k\right) A_m\left(\bar{Y}_m, Z_m\right)\right] \\
&\qquad -E_i\left[A_k\left(\bar{Y}_k, Z_k\right)\right] E_i\left[A_m\left(\bar{Y}_m, Z_m\right)\right]\Big)
\end{aligned}$$

and we can write

$$\begin{aligned}
v^2(i,T) &= \sum_{k=0}^{T-1}\left(\sum_{i_1,i_2,\cdots,i_k\in E, b\in F} \mathcal{T}_k\left(i,(\bar{\imath}_k, b)\right)\tilde{\alpha}_k\left(\bar{\imath}_k, b\right)\right. \\
&\qquad \left. -\left(\sum_{i_1,i_2,\cdots,i_k\in E, b\in F} \mathcal{T}_k\left(i,(\bar{\imath}_k, b)\right)\bar{\alpha}_k\left(\bar{\imath}_k, b\right)\right)^2\right) \\
&\qquad +2\sum_{k=0}^{T-1}\sum_{m=k+1}^{T-1-n}\left(\sum_{i_1,i_2,\cdots,i_k\in E, b\in F} \mathcal{T}_k\left(i,(\bar{\imath}_k, b)\right)\bar{\alpha}_k\left(\bar{\imath}_k, b\right)\right. \\
&\qquad\qquad \left.\cdot \sum_{i_{k+1},\cdots i_m\in E, c\in F} T_{k,m-k}\left(\left(\bar{\imath}_k, b\right),\left(\bar{\imath}_m, c\right)\right)\bar{\alpha}_m\left(\bar{\imath}_m, c\right)\right) \\
&\qquad -2\sum_{k=0}^{T-1}\sum_{m=k+1}^{T-1}\left(\sum_{i_1,i_2,\cdots,i_k\in E, b\in F} \mathcal{T}_k\left(i,(\bar{\imath}_k, b)\right)\bar{\alpha}_k\left(\bar{\imath}_k, b\right)\right) \\
&\qquad\qquad \cdot\left(\sum_{i_1,i_2,\cdots,i_m\in E, b\in F} \mathcal{T}_m\left(i,(\bar{\imath}_m, b)\right)\bar{\alpha}_m\left(\bar{\imath}_m, b\right)\right)
\end{aligned}$$

where $i_0 = i$.

3.1.3 Logarithmic Utility

In this section, we assume that the utility of the investor in state i is given by the logarithmic function

$$U(i,x)=\begin{cases} K(i)+C(i)\log(x+\beta) & x+\beta>0 \\ -\infty & x+\beta\leq 0 \end{cases} \tag{3.22}$$

with $C(i)>0$ where we can easily see that Pratt-Arrow's measure of absolute risk aversion is simply equal to $r(x)=1/(\beta+x)>0$ for all i so that $b=1$ and $a=\beta$ in Table 1.1. Note that β is the same for all market states so that risk classification of the investor does not depend on the stochastic market. Similarly, we assume that the return for the riskless asset is the same for all market states so that $r_f(i)=r_f$ for all i.

We will first consider a generic optimization problem of the form

$$\max_u \sum_{a\in F}\mu_a E\left[\log\left(R^e(a)'u+c\right)\right] \tag{3.23}$$

where $c>0$ is any constant, μ is any measure on F, and $R^e(a)$ is a random vector for any $a\in F$. Now, let

$$A(c)=\left\{u:P\left\{R^e(a)'u+c>0\right\}=1\right\}$$

be the set of all possible investment policies that gives finite expected utility so that $\left|E\left[\log\left(R^e(a)'u+c\right)\right]\right|<+\infty$ for $u\in A(c)$. It can be seen that $u=(u_1,u_2,\cdots,u_m)=(0,0,\cdots,0)\in A(c)$ satisfies this condition trivially for all $c>0$. So, $A(c)$ is not empty. Also, let u, $w\in A(c)$, then $R^e(a)'u+c>0$, and $R^e(a)'w+c>0$ implies that

$$\lambda R^e(a)'u+(1-\lambda)R^e(a)'w+c>0$$

so that $\lambda u+(1-\lambda)w\in A(c)$ for all $0\leq\lambda\leq 1$. Therefore, the solution set $A(c)$ is nonempty and convex. The gradient vector of the objection function $g(u)$ can be defined as $g(u)=\sum_{a\in F}\mu_a E\left[\log\left(R^e(a)'u+c\right)\right]$ and is given by

$$\nabla_k g(u)=\frac{\partial g(u)}{\partial u_k}=\sum_{a\in F}\mu_a E\left[\frac{R^e_k(a)}{R^e(a)'u+c}\right]$$

while the Hessian matrix is

$$\nabla^2_{k,l}g(u)=\frac{\partial^2 g(u)}{\partial u_k\partial u_l}=-\sum_{a\in F}\mu_a E\left[\frac{R^e_k(a)R^e_l(a)}{\left(R^e(a)'u+c\right)^2}\right]$$

for all k,l.

The first order optimality condition to find the optimal solution of (3.23) is obtained by setting the gradient vector equal to zero so that

$$\sum_{a \in F} \mu_a E \left[\frac{R_k^e(a)}{R^e(a)' u + c} \right] = 0 \tag{3.24}$$

for all k.

Let $z = (z_1, \cdots, z_m)$ be any non-zero column vector where z_i's are real numbers. Then, one can see that

$$z' \bigtriangledown^2 g(u) z = -\sum_{a \in F} \mu_a E \left[\frac{\left(\sum_{k=1}^{m} z_k R_k^e(a) \right)^2}{\left(R^e(a)' u + c \right)^2} \right] \leq 0.$$

Thus, the Hessian matrix $\bigtriangledown^2 g(u)$ is negative semi-definite and if there is a solution $u \in A(c)$ satisfying the first order condition (3.24), it must be optimal. Throughout this chapter, we assume that the excess returns are such that there is a solution of the first order condition (3.24) in $A(c)$ for all $\{R_n^e(a)\}$ and $c > 0$.

Theorem 7 *Let the utility function of the investor be the logarithmic function (3.22) and suppose that the riskless asset return does not depend on the market state. Then, the optimal solution of the dynamic programming equation (3.4) is*

$$v_n(\bar{\imath}_n, x) = K_n(\bar{\imath}_n) + C_n(\bar{\imath}_n) \log(x + \beta_n)$$

and the optimal portfolio is

$$u_n^*(\bar{\imath}_n, x) = \alpha_n(\bar{\imath}_n) \left(r_f x + \beta_{n+1} \right) \tag{3.25}$$

where

$$\begin{aligned} \beta_n &= \frac{\beta}{r_f^{T-n}}, \; C_n(\bar{\imath}_n) = \sum_{a \in F} \sum_{j \in E} P_n(\bar{\imath}_n, j, a) C_{n+1}((\bar{\imath}_n, j)), \\ K_n(\bar{\imath}_n) &= \sum_{a \in F} \sum_{j \in E} P_n(\bar{\imath}_n, j, a) \Big(K_{n+1}((\bar{\imath}_n, j)) \\ &\quad + C_{n+1}(\bar{\imath}_n, j) E\left[\log\left(r_f \left(1 + R_n^e(a)' \alpha_n(\bar{\imath}_n) \right) \right) \right] \Big) \end{aligned}$$

and $\alpha_n(\bar{\imath}_n)$ *satisfies*

$$\sum_{a \in F} \sum_{j \in E} P_n(\bar{\imath}_n, j, a) C_{n+1}((\bar{\imath}_n, j)) E\left[\frac{R_{n,k}^e(a)}{1 + R_n^e(a)' \alpha_n(\bar{\imath}_n)} \right] = 0 \tag{3.26}$$

for all assets $k = 1, 2, \cdots, m$, $\bar{\imath}_n \in E^{n+1}$, *and* $n = 0, 1, \cdots, T-1$ *with boundary conditions*

$$K_T(\bar{\imath}_T) = K(i_T), \quad C_T(\bar{\imath}_T) = C(i_T).$$

Proof. We use induction starting with the boundary condition $v_T(\bar{\imath}_T, x) = K(i_T) + C(i_T)\log(x+\beta)$ and obtain

$$\begin{aligned} g_{T-1}(\bar{\imath}_{T-1}, x, u) &= \sum_{a\in F}\sum_{j\in E} P_{T-1}(\bar{\imath}_{T-1}, j, a)\, E[U(j, r_f x + R^e_{T-1}(a)' u)] \\ &= \sum_{a\in F}\sum_{j\in E} P_{T-1}(\bar{\imath}_{T-1}, j, a)\, C(j)\, E[\log(r_f x + R^e_{T-1}(a)' u + \beta)] \\ &\quad + \sum_{a\in F}\sum_{j\in E} P_{T-1}(\bar{\imath}_{T-1}, j, a) K(j). \end{aligned}$$

Let $u^* = (u_1^*, u_2^*, \cdots, u_m^*)$ be the optimal amount of money that should be invested in the risky assets so that

$$v_{T-1}(\bar{\imath}_{T-1}, x) = \max_u g_{T-1}(\bar{\imath}_{T-1}, x, u) = g_{T-1}(\bar{\imath}_{T-1}, x, u^*)$$

as in the perfect information case. One can see that the optimization problem of maximizing the objective function $g_{T-1}(\bar{\imath}_{T-1}, x, u)$ is similar to the optimization problem (3.23) where the coefficients $\mu_a = \sum_{j\in E} P_{T-1}(\bar{\imath}_{T-1}, j, a)\, C(j)$ and $c = r_f x + \beta$. Therefore, using our assumption, we can write the optimality condition (3.24) as

$$\sum_{a\in F}\sum_{j\in E} P_{T-1}(\bar{\imath}_{T-1}, j, a)\, C(j)\, E\left[\frac{R^e_{T-1}(a)}{r_f x + R^e_{T-1}(a)' u^* + \beta}\right] = 0$$

for all $k = 1, 2, \cdots, m$. Define $\alpha_{T-1}(\bar{\imath}_{T-1}, x) = u^*_{T-1}(\bar{\imath}_{T-1}, x) / (r_f x + \beta)$. The optimality condition can now be rewritten as

$$\sum_{a\in F}\sum_{j\in E} P_{T-1}(\bar{\imath}_{T-1}, j, a)\, C(j)\, E\left[\frac{R^e_{T-1}(a)}{1 + R^e_{T-1}(a)' \alpha_{T-1}(\bar{\imath}_{T-1}, x)}\right] = 0$$

which is the condition (3.26) for $n = T-1$ since there is no dependence on x and $\alpha_{T-1}(\bar{\imath}_{T-1}, x) = \alpha_{T-1}(\bar{\imath}_{T-1})$.

When the value function at time $T-1$ is rewritten for the optimal policy, we obtain

$$\begin{aligned} v_{T-1}(\bar{\imath}_{T-1}, x) = &\sum_{a\in F}\sum_{j\in E} P_{T-1}(\bar{\imath}_{T-1}, j, a)\Big(K(j) \\ &\quad + C(j) E[\log\left(r_f\left(1 + R^e_{T-1}(a)' \alpha_{T-1}(\bar{\imath}_{T-1})\right)\right)]\Big) \\ &\qquad + \sum_{a\in F}\sum_{j\in E} P_{T-1}(\bar{\imath}_{T-1}, j, a) C(j)\log(x + \beta/r_f) \\ = &\, K_{T-1}(\bar{\imath}_{T-1}) + C_{T-1}(\bar{\imath}_{T-1})\log(x + \beta_{T-1}) \end{aligned}$$

and the value function is still logarithmic like the utility function. This completes the proof for $n = T - 1$.

Suppose now that the induction hypothesis holds for periods $T, T-1, T-2, \cdots, n$. Then, for period $n-1$,

$$\begin{aligned} g_{n-1}(\bar{\imath}_{n-1}, x, u) &= \sum_{a\in F}\sum_{j\in E} P_{n-1}\left(\bar{\imath}_{n-1}, j, a\right) E[v_n((\bar{\imath}_{n-1}, j), r_f x + R^e_{n-1}(a)' u)] \\ &= \sum_{a\in F}\sum_{j\in E} P_{n-1}\left(\bar{\imath}_{n-1}, j, a\right) C_n((\bar{\imath}_{n-1}, j)) E[\log(r_f x + R^e_{n-1}(a)' u + \beta_n)] \\ &\quad + \sum_{a\in F}\sum_{j\in E} P_{n-1}\left(\bar{\imath}_{n-1}, j, a\right) K_n((\bar{\imath}_{n-1}, j)). \end{aligned}$$

Let $u^*_{n-1}(\bar{\imath}_{n-1}, x)$ be the optimal policy such that

$$v_{n-1}(\bar{\imath}_{n-1}, x) = \max_u g_{n-1}(\bar{\imath}_{n-1}, x, u) = g_{n-1}(\bar{\imath}_{n-1}, x, u^*_{n-1}).$$

It is clear, once again, that the objective function $g_{n-1}(\bar{\imath}_{n-1}, x, u)$ is in the form of the generic objective function in (3.23) with $\mu_a = \sum_{j\in E} P_{n-1}\left(\bar{\imath}_{n-1}, j, a\right) C_n((\bar{\imath}_{n-1}, j))$ and $c = r_f x + \beta_n$. Therefore, the optimal solution can be found using the first order condition

$$\sum_{a\in F}\sum_{j\in E} P_{n-1}(\bar{\imath}_{n-1}, j, a) C_n((\bar{\imath}_{n-1}, j)) E\left[\frac{R^e_{n-1}(a)}{r_f x + R^e_{n-1}(a)' u^* + \beta}\right] = 0$$

and defining $\alpha_{n-1}\left(\bar{\imath}_{n-1}, x\right) = u^*\left(\bar{\imath}_{n-1}, x\right) / \left(r_f x + \beta\right)$ we get

$$\sum_{a\in F}\sum_{j\in E} P_{n-1}(\bar{\imath}_{n-1}, j, a) C_n((\bar{\imath}_{n-1}, j)) E\left[\frac{R^e_{n-1}(a)}{1 + R^e_{n-1}(a)' \alpha_{n-1}\left(\bar{\imath}_{n-1}, x\right)}\right] = 0. \tag{3.27}$$

Since there is no dependence on x in (3.27), $\alpha\left(\bar{\imath}_{n-1}, x\right)$ does not depend on x and $\alpha\left(\bar{\imath}_{n-1}, x\right) = \alpha(\bar{\imath}_{n-1})$. So, $u^*_{n-1}(\bar{\imath}_{n-1}, x) = \alpha(\bar{\imath}_{n-1})\left(r_f x + \beta_n\right)$ and this gives the optimality condition (3.26).

If we insert the optimal policy in the value function, we can see that

$$\begin{aligned} v_{n-1}(\bar{\imath}_{n-1}, x) &= \sum_{a\in F}\sum_{j\in E} P_{n-1}\left(\bar{\imath}_{n-1}, j, a\right) \left(K_n\left((\bar{\imath}_{n-1}, j)\right)\right. \\ &\qquad \left. + C_n\left((\bar{\imath}_{n-1}, j)\right) E\left[\log\left(r_f\left(1 + R^e_{n-1}(a)' \alpha_{n-1}(\bar{\imath}_{n-1})\right)\right)\right]\right) \\ &\qquad + \sum_{a\in F}\sum_{j\in E} P_{n-1}\left(\bar{\imath}_{n-1}, j, a\right) C_n\left((\bar{\imath}_{n-1}, j)\right) \log(x + \beta_n / r_f) \\ &= K_{n-1}\left(\bar{\imath}_{n-1}\right) + C_{n-1}(\bar{\imath}_{n-1}) \log(x + \beta_{n-1}) \end{aligned}$$

and this completes the proof. ■

Note that the structure of the optimal solution in (3.25) is such that the optimal distribution of wealth invested in the risky assets depend only on the state of the market independent of time. If the market is in state i in period n, then the total amount of money invested on the risky assets is

$$1'u_n^*(\bar{\imath}_n, x) = 1'\alpha_n(\bar{\imath}_n)(r_f x + \beta_{n+1}) = \left(r_f x + \frac{\beta}{r_f^{T-(n+1)}} \right) \sum_{k=1}^{m} \alpha_{n,k}(\bar{\imath}_n)$$

and the proportion on wealth allocated for asset k in the risky portfolio is

$$w_{n,k}(\bar{\imath}_n) = \frac{\alpha_{n,k}(\bar{\imath}_n)}{\sum_{k=1}^{m} \alpha_{n,k}(\bar{\imath}_n)} \tag{3.28}$$

which is totally independent of wealth x. The optimal policy specified by (3.25) is not static in time since it depends on n, and it is not memoryless in wealth since it depends on x. However, (3.28) clearly indicates that the composition of the risky part of the optimal portfolio only depends on the market state and time. The risky portfolio composition is memoryless. It satisfies the separation property in the sense that it represents the single fund of risky assets that logarithmic investors choose. The amount of total wealth allocated for risky assets depend on the level of wealth, but the composition of the risky assets depend only on the market state and time. This composition, however, is random due to the randomly changing market conditions in time. Our results are of course consistent with similar work in the literature on logarithmic utility functions, but the stochastic market approach makes our model more realistic without causing substantial difficulty in the analysis. Another important observation is that the structure of the optimal portfolio is not affected by the transition matrix $\{Q_n\}$ of the stochastic market. It only depends on the joint distribution of the risky asset returns as prescribed by (3.26) in a given market state, irrespective of future expectations on the stochastic market.

Evolution of Wealth

The evolution of the wealth process X using the optimal policy can be analyzed by the wealth dynamics equation

$$\begin{aligned} X_{n+1} &= r_f X_n + R^e(Z_n)' u_n^*(\bar{Y}_n, X_n) \\ &= r_f X_n + R^e(Z_n)' \alpha_n(\bar{Y}_n) \left(r_f X_n + \beta_{n+1}\right) \\ &= r_f X_n \left(1 + A_n\left(\bar{Y}_n, Z_n\right)\right) + r_f^{n+1-T} A_n\left(\bar{Y}_n, Z_n\right)\beta \end{aligned} \tag{3.29}$$

where $A_n\left(\bar{\imath}_n, a\right)$ is defined in (3.14).

Define

$$\mathbb{C}_n\left(x_1, x_2, \cdots, x_n\right) = \prod_{k=1}^{n}\left(1 + x_k\right) - 1$$

as the sum of all combinations of the products of n variables for $n \geq 1$, and set $\mathbb{C}_0 = 0$.

Now, we will show that the wealth process is

$$X_n = r_f^n X_0 \prod_{k=0}^{n-1}\left(1 + A_k\left(\bar{Y}_k, Z_k\right)\right) + r_f^{n-T}\beta\mathbb{C}_n\left(A_0\left(\bar{Y}_0, Z_0\right), \cdots, A_{n-1}\left(\bar{Y}_{n-1}, Z_{n-1}\right)\right) \tag{3.30}$$

using induction where the product on the right hand side is set to 1 when $n = 0$. The induction hypothesis holds trivially for $n = 0$. Suppose (3.30) holds for some $n \geq 0$. If we write X_{n+1} using the wealth dynamics equation (3.29)

$$\begin{aligned} X_{n+1} &= r_f X_n \left(1 + A_n(\bar{Y}_n, Z_n)\right) + r_f^{n+1-T} A_n(\bar{Y}_n, Z_n)\beta \\ &= r_f^{n+1} X_0 \prod_{k=0}^{n}\left(1 + A_k\left(\bar{Y}_k, Z_k\right)\right) + r_f^{n+1-T}\beta\left[\left(1 + A_n(\bar{Y}_n, Z_n)\right)\right. \\ &\qquad \left. \cdot\, \mathbb{C}_n\left(A_0\left(\bar{Y}_0, Z_0\right), A_1\left(\bar{Y}_1, Z_1\right), \cdots, A_{n-1}\left(\bar{Y}_{n-1}, Z_{n-1}\right)\right) + A_n(\bar{Y}_n, Z_n)\right] \\ &= r_f^{n+1} X_0 \prod_{k=0}^{n}\left(1 + A_k\left(\bar{Y}_k, Z_k\right)\right) + r_f^{n+1-T}\beta\mathbb{C}_{n+1}\left(A_0\left(\bar{Y}_0, Z_0\right), \cdots, A_n\left(\bar{Y}_n, Z_n\right)\right) \end{aligned}$$

and we see that the induction hypothesis also holds for $n + 1$. So, we conclude that the wealth process can be written as in (3.30) and, for $n = T$, we can find the terminal wealth as

$$\begin{aligned} X_T &= r_f^T X_0 \prod_{k=0}^{T-1}\left(1 + A_k\left(\bar{Y}_k, Z_k\right)\right) + \beta\mathbb{C}_T\left(A_0\left(\bar{Y}_0, Z_0\right), \cdots, A_{T-1}\left(\bar{Y}_{T-1}, Z_{T-1}\right)\right) \\ &= r_f^T X_0 + \left(r_f^T X_0 + \beta\right)\mathbb{C}_T\left(A_0\left(\bar{Y}_0, Z_0\right), A_1\left(\bar{Y}_1, Z_1\right), \cdots, A_{T-1}\left(\bar{Y}_{T-1}, Z_{T-1}\right)\right). \end{aligned}$$

Given $X_0 = x_0$, the expected value of the terminal wealth satisfies

$$E_i[X_T] = r_f^T x_0 + \left(r_f^T x_0 + \beta\right) m_l(i,T) \tag{3.31}$$

where

$$m_l(i,T) = E_i\left[\mathbb{C}_T\left(A_0\left(\bar{Y}_0, Z_0\right), A_1\left(\bar{Y}_1, Z_1\right), \cdots, A_{T-1}\left(\bar{Y}_{T-1}, Z_{T-1}\right)\right)\right] \tag{3.32}$$

and the variance of the terminal wealth satisfies

$$\mathrm{Var}_i(X_T) = \left(r_f^T x_0 + \beta\right)^2 v_l^2(i,T) \tag{3.33}$$

where

$$v_l^2(i,T) = \mathrm{Var}_i\left(\mathbb{C}_T\left(A_0\left(\bar{Y}_0, Z_0\right), A_1\left(\bar{Y}_1, Z_1\right), \cdots, A_{T-1}\left(\bar{Y}_{T-1}, Z_{T-1}\right)\right)\right). \tag{3.34}$$

We can clearly see from (3.31) and (3.33) that both the return and the standard deviation of X_T depends linearly on β. This shows that the logarithmic frontier is the straight line

$$E_i[X_T] = r_f^T x_0 + \left(\frac{m_l(i,T)}{v_l(i,T)}\right) \mathrm{SD}_i(X_T)$$

where $\mathrm{SD}_i(X_T)) = \sqrt{\mathrm{Var}_i(X_T)}$. In other words, the expected value and standard deviation of the terminal wealth fall on this straight line when they are calculated and plotted for different values of β. Also, it cuts the zero-risk line at $E_i[X_T] = r_f^T x_0$ as expected. The reason for this is that for zero-risk level investor puts all of his money on the riskless asset. The return of the riskless asset until the terminal time T is r_f^T, and the wealth at the terminal time will be $r_f^T x_0$ for sure. The risk premium for the logarithmic investor is given by the ratio $m_l(i,T)/v_l(i,T)$.

Computational Formulas

The computation of $m_l(i,T)$ and $v_l(i,T)$ are possible although they are not as simple as their counterparts for the exponential utility case. We will use the definition (2.60) whenever appropriate. Note that,

$$\begin{aligned} m_l(i,T) &= E_i\left[\prod_{k=0}^{T-1}\left(1 + A_k\left(\bar{Y}_k, Z_k\right)\right) - 1\right] \\ &= E_i\left[E_i\left[\prod_{k=0}^{T-1}\left(1 + A_k\left(\bar{Y}_k, Z_k\right)\right) - 1 \,\middle|\, \bar{Z}_{T-1}, \bar{Y}_{T-1}\right]\right] \end{aligned}$$

and, since the returns in different periods are independent given the market states, we obtain

$$\begin{aligned} m_l(i,T) &= E_i\left[\prod_{k=0}^{T-1}\left(1+\bar{\alpha}_k\left(\bar{Y}_k,Z_k\right)\right)-1\right] \\ &= E_i\left[\mathbb{C}_T\left(\bar{\alpha}_0\left(\bar{Y}_0,Z_0\right),\bar{\alpha}_1\left(\bar{Y}_1,Z_1\right),\cdots,\bar{\alpha}_{T-1}\left(\bar{Y}_{T-1},Z_{T-1}\right)\right)\right] \end{aligned} \tag{3.35}$$

Given $Y_0 = i$, the conditional joint distribution of $\bar{Y}_{T-1}, \bar{Z}_{T-1}$ is

$$\begin{aligned} P\left\{\bar{Y}_{T-1}=\bar{\imath}_{T-1},\bar{Z}_{T-1}=\bar{a}_{T-1}\middle|\, Y_0=i\right\} &= I\left(i,i_0\right)O_0\left(i,a_0\right) \\ &\quad \cdot\prod_{k=0}^{T-2}Q_k\left(a_k,a_{k+1}\right)G_{k+1}\left(a_{k+1},i_{k+1}\right) \end{aligned} \tag{3.36}$$

where the product on the right-hand side of (3.36) is set to be equal to 1 when $T \equiv 1$, and the expected return of terminal wealth(3.35) can be found by using this distribution so that

$$\begin{aligned} m_l(i,T) &= \sum_{\bar{a}_{T-1}\in F^T}\sum_{\bar{\imath}_{T-1}\in E^T} I\left(i,i_0\right)O_0\left(i,a_0\right)\prod_{k=0}^{T-2}Q_k\left(a_k,a_{k+1}\right)G_{k+1}\left(a_{k+1},i_{k+1}\right) \\ &\quad \cdot\mathbb{C}_T\left(\bar{\alpha}_0\left(i,a_0\right),\bar{\alpha}_1\left(\bar{\imath}_1,a_1\right),\cdots,\bar{\alpha}_{T-1}\left(\bar{\imath}_{T-1},a_{T-1}\right)\right). \end{aligned} \tag{3.37}$$

To determine the variance $v_l^2(i,T)$, we first calculate the second moment as

$$\begin{aligned} s_l(i,T) &= E_i\left[\prod_{k=0}^{T-1}\left(\left(1+A_k\left(\bar{Y}_k,Z_k\right)\right)-1\right)^2\right] \\ &= E_i\left[E_i\left[\left(\prod_{k=0}^{T-1}\left(1+A_k\left(\bar{Y}_k,Z_k\right)\right)-1\right)^2\middle|\,\bar{Z}_{T-1},\bar{Y}_{T-1}\right]\right] \\ &= E_i\left[E_i\left[\prod_{k=0}^{T-1}\left(1+A_k\left(\bar{Y}_k,Z_k\right)\right)^2-\right.\right. \\ &\qquad \left.\left. 2\prod_{k=0}^{T-1}\left(1+A_k\left(\bar{Y}_k,Z_k\right)\right)+1\middle|\,\bar{Z}_{T-1},\bar{Y}_{T-1}\right]\right] \\ &= E_i\left[\prod_{k=0}^{T-1}\left(1+2\bar{\alpha}_k\left(\bar{Y}_k,Z_k\right)+\tilde{\alpha}_k\left(\bar{Y}_k,Z_k\right)\right)\right. \\ &\qquad \left. -1-2\prod_{k=0}^{T-1}\left(1+\bar{\alpha}_k\left(\bar{Y}_k,Z_k\right)\right)-2\right] \\ &= E_i\left[\mathbb{C}_T\left(\begin{array}{c} 2\bar{\alpha}_0\left(\bar{Y}_0,Z_0\right)+\tilde{\alpha}_0\left(\bar{Y}_0,Z_0\right), \\ \cdots,2\bar{\alpha}_{T-1}\left(\bar{Y}_{T-1},Z_{T-1}\right)+\tilde{\alpha}_{T-1}\left(\bar{Y}_{T-1},Z_{T-1}\right) \end{array}\right)\right] \\ &\qquad -2E_i\left[\mathbb{C}_T\left(\bar{\alpha}_0\left(\bar{Y}_0,Z_0\right),\bar{\alpha}_1\left(\bar{Y}_1,Z_1\right),\cdots,\bar{\alpha}_{T-1}\left(\bar{Y}_{T-1},Z_{T-1}\right)\right)\right] \end{aligned}$$

which can be found using the conditional distribution (3.36) as in (3.37). Finally, the variance becomes $v_l^2(i,T) = s_l(i,T) - m_l(i,T)^2$.

3.1.4 Simple Logarithmic Utility

In this section, we assume that the utility of the investor in state i is given by the simple logarithmic function

$$U(i,x) = \begin{cases} K(i) + C(i)\log(x) & x > 0 \\ -\infty & x \leq 0 \end{cases} \tag{3.38}$$

with $C(i) > 0$ and $\beta = 0$. In this part, we can relax the assumption on risk free rate as $r_f(i)$ now depends on the observed market state. Note that this structure implies $R_n^e(i,a) = R(n,a) - r_f(i)$ which depends on a as well as i. We still need the assumptions on the return distributions as in Section 3.1.3. Therefore, for an optimization problem of the form

$$\max_u \sum_{a\in F} \mu_a E\left[\log\left(R^e(i,a)' u + c\right)\right] \tag{3.39}$$

we assume that the excess returns are such that the solution to the first order condition is in the feasible set. We therefore suppose that for any measure μ on F, any state $i \in E$, and any period n, the following equation

$$\sum_{a\in F} \mu_a E\left[\frac{R_n^e(i,a)}{1 + R_n^e(i,a)'\alpha}\right] = 0 \tag{3.40}$$

has a unique solution $\alpha \in R^m$.

Theorem 8 *Let the utility function of the investor be the logarithmic function (3.38). Then, the optimal solution of the dynamic programming equation (3.4) is*

$$v_n(\bar{\imath}_n, x) = K_n(\bar{\imath}_n) + C_n(\bar{\imath}_n)\log(x)$$

and the optimal portfolio is

$$u_n^*(\bar{\imath}_n, x) = \alpha_n(\bar{\imath}_n) r_f(i_n)\, x$$

where

$$\begin{aligned} C_n(\bar{\imath}_n) &= \sum_{a\in F}\sum_{j\in E} P_n(\bar{\imath}_n, j, a)\, C_{n+1}((\bar{\imath}_n, j)), \\ K_n(\bar{\imath}_n) &= \sum_{a\in F}\sum_{j\in E} P_n(\bar{\imath}_n, j, a)\Big(K_{n+1}((\bar{\imath}_n, j)) \\ &\quad + C_{n+1}(\bar{\imath}_n, j)\, E\left[\log\left(r_f(i_n)\left(1 + R_n^e(i_n,a)'\alpha_n(\bar{\imath}_n)\right)\right)\right]\Big) \end{aligned}$$

and $\alpha_n(\bar{\imath}_n)$ *satisfies*

$$\sum_{a\in F}\sum_{j\in E} P_n(\bar{\imath}_n, j, a) C_{n+1}((\bar{\imath}_n, j)) E\left[\frac{R^e_{n,k}(i_n, a)}{1 + R^e_n(i_n, a)'\alpha_n(\bar{\imath}_n)}\right] = 0 \qquad (3.41)$$

for all assets $k = 1, 2, \cdots, m$, $\bar{\imath}_n \in E^{n+1}$, *and* $n = 0, 1, \cdots, T-1$ *with boundary conditions*

$$K_T(\bar{\imath}_T) = K(i_T), \quad C_T(\bar{\imath}_T) = C(i_T).$$

Proof. We will show that the recursion is true by induction starting with the boundary condition $v_T(\bar{\imath}_T, x) = K(i_T) + C(i_T)\log(x)$ and obtain

$$\begin{aligned} g_{T-1}(\bar{\imath}_{T-1}, x, u) &= \sum_{a\in F}\sum_{j\in E} P_{T-1}(\bar{\imath}_{T-1}, j, a)\, E[U(j, r_f(i_{T-1})\, x + R^e_{T-1}(i_{T-1}, a)'\, u)] \\ &= \sum_{a\in F}\sum_{j\in E} P_{T-1}(\bar{\imath}_{T-1}, j, a) C(j) E[\log(r_f(i_{T-1})\, x + R^e_{T-1}(i_{T-1}, a)'\, u)] \\ &\quad + \sum_{a\in F}\sum_{j\in E} P_{T-1}(\bar{\imath}_{T-1}, j, a) K(j). \end{aligned}$$

Let $u^* = (u^*_1, u^*_2, \cdots, u^*_m)$ be the optimal amount of money that should be invested in the risky assets so that

$$v_{T-1}(\bar{\imath}_{T-1}, x) = \max_u g_{T-1}(\bar{\imath}_{T-1}, x, u) = g_{T-1}(\bar{\imath}_{T-1}, x, u^*)$$

as in the perfect information case.

Note that the optimization problem of maximizing the objective function $g_{T-1}(\bar{\imath}_{T-1}, x, u)$ is similar to the optimization problem of (3.39) where $\mu_a = \sum_{j\in E} P_{T-1}(\bar{\imath}_{T-1}, j, a)\, C(j)$ and $c = r_f(i_{T-1})\, x$. Therefore, similar to Theorem 7, we can find the optimal portfolio by setting the gradient equal to zero so that (3.41) becomes

$$\sum_{a\in F}\sum_{j\in E} P_{T-1}(\bar{\imath}_{T-1}, j, a) C(j) E\left[\frac{R^e_{T-1,k}(i_{T-1}, a)}{r_f(i_{T-1})\, x + R^e_{T-1}(i_{T-1}, a)'\, u^*}\right] = 0$$

for all $k = 1, 2, \cdots, m$. Define $\alpha_{T-1}(\bar{\imath}_{T-1}, x) = u^*_{T-1}(\bar{\imath}_{T-1}, x)/r_f(i_{T-1})\, x$. The optimality condition can now be rewritten as

$$\sum_{a\in F}\sum_{j\in E} P_{T-1}(\bar{\imath}_{T-1}, j, a) C(j) E\left[\frac{R^e_{T-1,k}(i_{T-1}, a)}{1 + R^e_{T-1}(i_{T-1}, a)'\, \alpha(\bar{\imath}_{T-1}, x)}\right] = 0. \qquad (3.42)$$

Since there is no dependence on x in (3.42), $\alpha(\bar{\imath}_{T-1}, x)$ does not depend on x and therefore $\alpha(\bar{\imath}_{T-1}, x) = \alpha(\bar{\imath}_{T-1})$. So, we obtain $u^*_{T-1}(\bar{\imath}_{T-1}, x) = \alpha_{T-1}(\bar{\imath}_{T-1})\, r_f(i_{T-1})\, x$ where α_{T-1}

satisfies (3.41). When the value function at time $T-1$ is rewritten for the optimal policy, we obtain

$$\begin{aligned} v_{T-1}(\bar{\imath}_{T-1},x) &= \sum_{a\in F}\sum_{j\in E} P_{T-1}(\bar{\imath}_{T-1},j,a)C(j)\log(x) + \sum_{a\in F}\sum_{j\in E} P_{T-1}(\bar{\imath}_{T-1},j,a)\left(K(j)\right. \\ &\quad \left. +C(j)E\left[\log\left(r_f\left(i_{T-1}\right)\left(1+R^e_{T-1}\left(i_{T-1},a\right)'\alpha\left(\bar{\imath}_{T-1}\right)\right)\right)\right]\right) \\ &= K_{T-1}\left(\bar{\imath}_{T-1}\right) + C_{T-1}(\bar{\imath}_{T-1})\log(x) \end{aligned}$$

and the value function is still logarithmic like the utility function. This completes the proof for $n = T-1$.

Suppose now that the induction hypothesis holds for periods $T, T-1, T-2, \cdots, n$. Then, for period $n-1$,

$$\begin{aligned} g_{n-1}(\bar{\imath}_{n-1},x,u) &= \sum_{a\in F}\sum_{j\in E} P_{n-1}\left(\bar{\imath}_{n-1},j,a\right)E[v_n((\bar{\imath}_{n-1},j), r_f\left(i_{n-1}\right)x + R^e_{n-1}\left(i_{n-1},a\right)'u)] \\ &= \sum_{a\in F}\sum_{j\in E} P_{n-1}\left(\bar{\imath}_{n-1},j,a\right)C_n\left(\left(\bar{\imath}_{n-1},j\right)\right)E[\log(r_f\left(i_{n-1}\right)x \\ &\quad + R^e_{n-1}\left(i_{T-1},a\right)'u)] + \sum_{a\in F}\sum_{j\in E} P_{n-1}\left(\bar{\imath}_{n-1},j,a\right)K_n\left(\left(\bar{\imath}_{n-1},j\right)\right). \end{aligned}$$

Let $u^*_{n-1}(\bar{\imath}_{n-1},x)$ be the optimal policy such that

$$v_{n-1}(\bar{\imath}_{n-1},x) = \max_u g_{n-1}(\bar{\imath}_{n-1},x,u) = g_{n-1}(\bar{\imath}_{n-1},x,u^*_{n-1}). \tag{3.43}$$

Note that this optimization problem defined in (3.43) is similar to the problem in (3.39) with $\mu_a = \sum_{j\in E} P_{n-1}\left(\bar{\imath}_{n-1},j,a\right)C_n\left(\left(\bar{\imath}_{n-1},j\right)\right)$ and $c = r_f\left(i_{n-1}\right)x$. If we take the gradient of $g_{n-1}(\bar{\imath}_{n-1},x,u)$ with respect to u and set it equal to 0, we get the optimality condition

$$\sum_{a\in F}\sum_{j\in E} P_{n-1}(\bar{\imath}_{n-1},j,a)C_n\left(\left(\bar{\imath}_{n-1},j\right)\right)E\left[\frac{R^e_{n-1}\left(i_{n-1},a\right)}{1+R^e_{n-1}\left(i_{n-1},a\right)'\alpha\left(\bar{\imath}_{n-1},x\right)}\right] = 0 \tag{3.44}$$

after taking $u^*_{n-1}(\bar{\imath}_{n-1},x) = \alpha(\bar{\imath}_{n-1},x)r_f\left(i_{n-1}\right)x$. Since there is no dependence on x in (3.44), $\alpha\left(i_{T-1},x\right)$ does not depend on x and $\alpha\left(i_{n-1},x\right) = \alpha(\bar{\imath}_{n-1})$ and we obtain $u^*_{n-1}(\bar{\imath}_{n-1},x) = \alpha(\bar{\imath}_{n-1})r_f\left(i_{n-1}\right)x$ which gives (3.41). If we insert the optimal policy in the value function, we can see that

$$\begin{aligned} v_{n-1}(\bar{\imath}_{n-1},x) &= \sum_{a\in F}\sum_{j\in E} P_{n-1}(\bar{\imath}_{n-1},j,a)C_n((i_{n-1,}j))\log(x) \\ &\quad + \sum_{a\in F}\sum_{j\in E} P_{T-1}(\bar{\imath}_{n-1},j,a)\left(K_n((i_{n-1},j))\right. \\ &\quad \left. + C_n((i_{n-1},j))E\left[\log\left(r_f(i_{n-1})\left(1+R^e_{n-1}(i_{n-1},a)'\alpha(\bar{\imath}_{n-1})\right)\right)\right]\right) \\ &= K_{n-1}(\bar{\imath}_{n-1}) + C_{n-1}(\bar{\imath}_{n-1})\log(x) \end{aligned}$$

and this completes the proof. ■

In this special case with $\beta = 0$, at any time n, the total amount of money invested in the risky assets depends on the observed market states $\bar{\imath}_n$ and wealth x. Since the total risky investment is $1'u^*_n(\bar{\imath}_n,x) = 1'\alpha_n(\bar{\imath}_n)r_f(i_n)x$, it follows that $1'\alpha_n(\bar{\imath}_n)r_f(i_n)$ is the proportion of total wealth that is invested in the risky assets. Moreover, as in the general logarithmic case, the composition of the risky portfolio also depends only on $\bar{\imath}_n$ independent of the available wealth x.

The evolution of the wealth process X using the optimal policy can be analyzed by the wealth dynamics equation

$$\begin{aligned} X_{n+1} &= r_f(Y_n)X_n + R^e_n(Y_n,Z_n)'u^*_n(\bar{Y}_n,X_n) \\ &= X_n r_f(Y_n)\left(1+A_n\left(\bar{Y}_n,Z_n\right)\right) = X_n B_n\left(\bar{Y}_n,Z_n\right) \end{aligned}$$

where $B_k(\bar{\imath}_k,a) = r_f(i_k)\left(1+A_k(\bar{\imath}_k,a)\right)$. Clearly, the solution is

$$X_n = X_0 \prod_{k=0}^{n-1} B_k\left(\bar{Y}_k,Z_k\right) \tag{3.45}$$

for $n \geq 1$, and this simple structure can be exploited to analyze the terminal wealth X_T. In particular, given $X_0 = x_0$

$$m_l(i,T) = E_i[X_T] = x_0\left(1+E_i\left[\mathbb{C}_T\left(b_0\left(\bar{Y}_0,Z_0\right)-1,\cdots,b_{T-1}\left(\bar{Y}_{T-1},Z_{T-1}\right)-1\right)\right]\right) \tag{3.46}$$

where $b_k(\bar{\imath}_k,a) = E\left[B_k(\bar{\imath}_k,a)\right] = E\left[r_f(i_k)\left(1+A_k(\bar{\imath}_k,a)\right)\right] = r_f(i_k)\left(1+\bar{\alpha}_k(\bar{\imath}_k,a)\right)$. Note that the expected value of terminal wealth (3.46) can be calculated using the joint distribution (3.36) as in (3.37). The second moment is

$$s_l(i,T) = E_i[X_T^2] = x_0^2\left(1+E_i\left[\mathbb{C}_T\left(\bar{b}_0\left(\bar{Y}_0,Z_0\right)-1,\cdots,\bar{b}_{T-1}\left(\bar{Y}_{T-1},Z_{T-1}\right)-1\right)\right]\right) \tag{3.47}$$

where $\bar{b}_k(\bar{\imath}_k, a) = E\left[B_k(\bar{\imath}_k, a)^2\right] = r_f(i_k)^2 E[(1 + A_k(\bar{\imath}_k, a))^2] = r_f(i_k)^2(1 + 2\bar{\alpha}(\bar{\imath}_k, a) + \tilde{\alpha}(\bar{\imath}_k, a))$, which can also be calculated using the joint distribution (3.36.) Finally, the variance is $\mathrm{Var}_i(X_T) = v_l^2(i, T) = s_l(i, T) - m_l(i, T)^2$.

The log-return at the terminal time T is

$$\ln(X_T/X_0) = \sum_{k=0}^{T-1} \ln(B_k(\bar{Y}_k, Z_k))$$

so that the mean is

$$\begin{aligned} E_i[\ln(X_T/X_0)] &= \sum_{k=0}^{T-1} \sum_{\bar{a}_{T-1} \in F^T} \sum_{\bar{\imath}_{T-1} \in E^T} I(i, i_0) O_0(i, a_0) \\ &\quad \cdot \prod_{n=0}^{k-1} Q_n(a_n, a_{n+1}) G_{n+1}(a_{n+1}, i_{n+1}) E[\ln(B_k(\bar{\imath}_k, a_k))]. \end{aligned}$$

3.1.5 Power Utility

In this section, we assume that the utility of the investor in state i is given by the logarithmic function

$$U(i, x) = K(i) + C(i) \frac{(x + \beta)^\gamma}{\gamma} \tag{3.48}$$

with $C(i) > 0$ and $r_f(i) = r_f$ still holds. Note that Pratt-Arrow ratio can be calculated as $r(x) = (1 - \gamma)/(x - \beta)$ for all i so that $b = 1/(1 - \gamma)$ and $a = \beta/(\gamma - 1)$ in Table 1.1. In this chapter, we assume that the utility function (3.48) is well-defined for all possible values of x. For example, if $(x - \beta) < 0$ is possible, then we exclude $\gamma = 1/2$ in our analysis. If we need to include these values of γ, we can define the utility function to be $-\infty$ whenever (3.48) is not well-defined and make appropriate assumptions on excess returns $\{R_n^e(a)\}$ as in Section 3.1.3. For $U(i, x)$ to be a legitimate utility function some additional restrictions may be imposed, but we do not dwell with such technical issues here. Note that γ and β is the same for all market states so that risk classification of the investor does not depend on the stochastic market.

We will first consider an optimization problem of the form

$$\max_u \sum_{a \in F} \mu_a E\left[\frac{(R^e(a)' u + c)^\gamma}{\gamma}\right] \tag{3.49}$$

where $R^e(a)$ is a random vector, and μ is any measure on F for any a. The gradient vector of the objection function $g(u)$ is given by

$$\bigtriangledown_k g(u) = \frac{\partial g(u)}{\partial u_k} = \sum_{a \in F} \mu_a E\left[R_k^e(a)\left(R^e(a)' u + c\right)^{\gamma-1}\right]$$

while the Hessian matrix is

$$\bigtriangledown_{k,l}^2 g(u) = \frac{\partial^2 g(u)}{\partial u_k \partial u_l} = (\gamma - 1) \sum_{a \in F} \mu_a E\left[R_k^e(a) R_l^e(a)\left(R^e(a)' u + c\right)^{\gamma-2}\right]$$

for all k, l.

The first order optimality condition to find the optimal solution of (3.49) is obtained by setting the gradient vector equal to zero so that

$$\sum_{a \in F} \mu_a E\left[R_k^e(a)\left(R^e(a)' u + c\right)^{\gamma-1}\right] = 0 \tag{3.50}$$

for all $k = 1, 2, \cdots, m$. Let $z = (z_1, \cdots, z_m)$ be any non-zero column vector where z_i's are real numbers. Then, one can see that

$$z' \bigtriangledown^2 g(u) z = (\gamma - 1) \sum_{a \in F} \mu_a E\left[\left(\sum_{k=1}^{m} z_k R^e(a)\right)^2 \left(R^e(a)' u + c\right)^{\gamma-2}\right]. \tag{3.51}$$

Throughout this chapter, we assume that the excess returns $\{R_n^e(a)\}$ and the parameters of the utility function are such that there is always an optimal solution of (3.49) that satisfies the first order conditions (3.50). Note that this requirement does not necessarily impose concavity restriction on the objective function. We only require that the optimal solution is at an interior point which satisfies the necessary conditions of optimality (3.50). Our purpose is to identify the structure of the optimal policy and we will not dwell will these technical details on optimization. This is of course an important issue and we do not intend to undermine its significance.

We now consider some possible cases to illustrate how one can approach this technical problem. If $\gamma - 2$ is even, then the Hessian matrix $\bigtriangledown^2 g$ in (3.51) is negative semi-definite provided that $\gamma \leq 1$ and the optimal solution satisfies (3.50) since we have an unconstrained concave maximization problem. If $\gamma - 2$ is not even and $\gamma \leq 1$, then the objective function is concave over the set

$$A(c) = \left\{u : P\left\{\left(R^e(a)' u + c\right)^{\gamma-2} \geq 0\right\} = 1\right\} \tag{3.52}$$

and we need additional restrictions on the excess returns $\{R^e(a)\}$; like the existence of a solution of the first order condition (3.50) in $A(c)$ for all c. In case $\gamma \geq 1$, it suffices to reverse the inequality in (3.52).

Theorem 9 *Let the utility function of the investor be the power function (3.48) and suppose that the riskless asset return does not depend on the market state. Then, the optimal solution of the dynamic programming equation (3.4) is*

$$v_n(\bar{\imath}_n, x) = K_n(\bar{\imath}_n) + C_n(\bar{\imath}_n)\frac{(x+\beta_n)^\gamma}{\gamma}$$

and the optimal portfolio is

$$u_n^*(\bar{\imath}_n, x) = \alpha_n(\bar{\imath}_n)\left(r_f x + \beta_{n+1}\right) \tag{3.53}$$

where

$$\begin{aligned}
\beta_n &= \frac{\beta}{r_f^{T-n}},\ K_n(\bar{\imath}_n) = \sum_{a\in F}\sum_{j\in E} P_n(\bar{\imath}_n, j, a) K_{n+1}((\bar{\imath}_n, j)),\\
C_n(\bar{\imath}_n) &= \sum_{a\in F}\sum_{j\in E} P_n(\bar{\imath}_n, j, a) C_{n+1}((\bar{\imath}_n, j)) E\left[\left(r_f\left(1+R_n^e(a)'\alpha_n(\bar{\imath}_n)\right)\right)^\gamma\right]
\end{aligned}$$

and $\alpha_n(\bar{\imath}_n)$ *satisfies*

$$\sum_{a\in F}\sum_{j\in E} P_n(\bar{\imath}_n, j, a) C_{n+1}((\bar{\imath}_n, j)) E\left[R_{n,k}^e(a)\left(1+R_n^e(a)'\alpha_n(\bar{\imath}_n)\right)^{\gamma-1}\right] = 0 \tag{3.54}$$

for all assets $k = 1, 2, \cdots, m$, $\bar{\imath}_n \in E^{n+1}$ *and* $n = 0, 1, \cdots, T-1$ *with boundary conditions*

$$K_T(\bar{\imath}_T) = K(i_T), \quad C_T(\bar{\imath}_T) = C(i_T).$$

Proof. We use induction starting with the boundary condition $v_T(\bar{\imath}_T, x) = K(i_T) + C(i_T)(x+\beta)^\gamma/\gamma$ and obtain

$$\begin{aligned}
g_{T-1}(\bar{\imath}_{T-1}, x, u) &= \sum_{a\in F}\sum_{j\in E} P_{T-1}(\bar{\imath}_{T-1}, j, a) E[U(j, r_f x + R_{T-1}^e(a)' u)]\\
&= \sum_{a\in F}\sum_{j\in E} P_{T-1}(\bar{\imath}_{T-1}, j, a) C(j) E[(r_f x + R_{T-1}^e(a)' u + \beta)^\gamma/\gamma]\\
&\quad + \sum_{a\in F}\sum_{j\in E} P_{T-1}(\bar{\imath}_{T-1}, j, a) K(j).
\end{aligned}$$

Let $u^* = (u_1^*, u_2^*, \cdots, u_m^*)$ be the optimal amount of money that should be invested in the risky assets so that

$$v_{T-1}(\bar{\imath}_{T-1}, x) = \max_u g_{T-1}(\bar{\imath}_{T-1}, x, u) = g_{T-1}(\bar{\imath}_{T-1}, x, u^*)$$

as in the perfect information case. One can see that $g_{T-1}(\bar{\imath}_{T-1}, x, u)$ is in the form of (3.49) where $\mu_a = \sum_{j \in E} P_{T-1}(\bar{\imath}_{T-1}, j, a)C(j)$ and $c = r_f x + \beta$. Our assumption implies that the optimal policy can be found using the first order condition (3.50) which can be rewritten as

$$\sum_{a \in F} \sum_{j \in E} P_{T-1}(\bar{\imath}_{T-1}, j, a)C(j)E\left[R^e_{T-1,k}(a)\left(r_f x + R^e_{T-1}(a)' u^* + \beta\right)^{\gamma-1}\right] = 0$$

for all $k = 1, 2, \cdots, m$. Defining $\alpha_{T-1}(\bar{\imath}_{T-1}, x) = u^*_{T-1}(\bar{\imath}_{T-1}, x)/\left(r_f x + \beta\right)$ one can see that $u^*_{T-1}(\bar{\imath}_{T-1}, x) = \alpha_{T-1}(\bar{\imath}_{T-1}, x)\left(r_f x + \beta\right)$. The optimality condition can now be rewritten as

$$\sum_{a \in F} \sum_{j \in E} P_{T-1}(\bar{\imath}_{T-1}, j, a)C(j)\, E\left[R^e_{T-1,k}(a)\left(1 + R^e_{T-1}(a)' \alpha_{T-1}(\bar{\imath}_{T-1})\right)^{\gamma-1}\right] = 0 \qquad (3.55)$$

which is the condition (3.54) for $n = T - 1$ since there is no dependence on x and $\alpha_{T-1}(\bar{\imath}_{T-1}, x) = \alpha_{T-1}(\bar{\imath}_{T-1})$. When the value function at time $T - 1$ is rewritten for the optimal policy, we obtain

$$\begin{aligned} v_{T-1}(\bar{\imath}_{T-1}, x) &= \sum_{a \in F} \sum_{j \in E} P_{T-1}(\bar{\imath}_{T-1}, j, a)\, K(j) + \sum_{a \in F} \sum_{j \in E} P_{T-1}(\bar{\imath}_{T-1}, j, a)C(j) \\ &\quad \cdot E\left[\left(r_f\left(1 + R^e_{T-1}(a)' \alpha_{T-1}(\bar{\imath}_{T-1})\right)\right)^{\gamma}\right](x + \beta/r_f)^{\gamma} \\ &= K_{T-1}(\bar{\imath}_{T-1}) + C_{T-1}(\bar{\imath}_{T-1})(x + \beta_{T-1})^{\gamma}/\gamma \end{aligned}$$

and the value function is still logarithmic like the utility function. This completes the proof for $n = T - 1$.

Suppose now that the induction hypothesis holds for periods $T, T-1, T-2, \cdots, n$. Then, for period $n - 1$,

$$\begin{aligned} g_{n-1}(\bar{\imath}_{n-1}, x, u) &= \sum_{a \in F} \sum_{j \in E} P_{n-1}(\bar{\imath}_{n-1}, j, a)\, E[v_n((\bar{\imath}_{n-1}, j), r_f x + R^e_{n-1}(a)' u)] \\ &= \sum_{a \in F} \sum_{j \in E} P_{n-1}(\bar{\imath}_{n-1}, j, a)\, C_n((\bar{\imath}_{n-1}, j))E[(r_f x + R^e_{n-1}(a)' u + \beta_n)^{\gamma}] \\ &\quad + \sum_{a \in F} \sum_{j \in E} P_{n-1}(\bar{\imath}_{n-1}, j, a)\, K_n((\bar{\imath}_{n-1}, j)). \end{aligned}$$

Let $u^*_{n-1}(\bar{\imath}_{n-1}, x)$ be the optimal policy such that

$$v_{n-1}(\bar{\imath}_{n-1}, x) = \max_u g_{n-1}(\bar{\imath}_{n-1}, x, u) = g_{n-1}(\bar{\imath}_{n-1}, x, u^*_{n-1}).$$

Note that this optimization problem is also similar to (3.49) where the parameter $\mu_a = \sum_{j \in E} P_{n-1}(\bar{\imath}_{n-1}, j, a)\, C_n((\bar{\imath}_{n-1}, j))$ and $c = r_f x + \beta$. If we take the gradient of $g_{n-1}(\bar{\imath}_{n-1}, x, u)$

with respect to u and set it equal to 0, we get the optimality condition

$$\sum_{a\in F}\sum_{j\in E} P_{n-1}(\bar{\imath}_{n-1},j,a)C_n((\bar{\imath}_{n-1},j))E\left[R^e_{n-1}(a)\left(r_f x + R^e_{n-1}(a)' u^* + \beta\right)^{\gamma-1}\right] = 0$$

and defining $\alpha_{n-1}(\bar{\imath}_{n-1},x) = u^*(\bar{\imath}_{n-1},x)/(r_f x + \beta)$ we get

$$\sum_{a\in F}\sum_{j\in E} P_{n-1}(\bar{\imath}_{n-1},j,a)C_n((\bar{\imath}_{n-1},j))E\left[R^e_{n-1}(a)\left(1 + R^e_{n-1}(a)' \alpha_{n-1}(\bar{\imath}_{n-1},x)\right)^{\gamma-1}\right] = 0. \tag{3.56}$$

Since there is no dependence on x in (3.56), $\alpha(\bar{\imath}_{n-1},x)$ does not depend on x and $\alpha(\bar{\imath}_{n-1},x) = \alpha(\bar{\imath}_{n-1})$ and we obtain $u^*_{n-1}(\bar{\imath}_{n-1},x) = \alpha(\bar{\imath}_{n-1})(r_f x + \beta_n)$ and this gives the optimality condition (3.54). If we insert the optimal policy in the value function, we can see that

$$\begin{aligned} v_{n-1}(\bar{\imath}_{n-1},x) = &\sum_{a\in F}\sum_{j\in E} P_{n-1}(\bar{\imath}_{n-1},j,a)K_n((\bar{\imath}_{n-1},j)) \\ &+ \sum_{a\in F}\sum_{j\in E} P_{n-1}(\bar{\imath}_{n-1},j,a)C_n((\bar{\imath}_{n-1},j)) \\ &\cdot E\left[\left(r_f\left(1 + R^e_{n-1}(a)'\alpha_{n-1}(\bar{\imath}_{n-1})\right)\right)^{\gamma}\right](x+\beta_n/r_f)^{\gamma}/\gamma \\ = &K_{n-1}(\bar{\imath}_{n-1}) + C_{n-1}(\bar{\imath}_{n-1})(x+\beta_{n-1})^{\gamma}/\gamma \end{aligned}$$

and this completes the proof. ■

Note that the wealth dynamics equation for the power utility case is the same as the wealth dynamics equation (3.29) for the logarithmic utility case although the structure of the optimal policy in (3.23) and (3.49) are different. Therefore, using a similar analysis as in Section 3.1.3 we can easily determine the evolution of the wealth process. Likewise, similar interpretations can be made on the structure of the optimal policy. In particular, the optimal policy is not myopic, but the risky composition of the portfolio is myopic. Moreover, this composition only depends on the state of the market. Although we obtain similar characterizations and interpretations, note that the optimal policies for logarithmic and power cases are not identical.

3.2 Sufficient Statistics

The main problem about our formulation of the problem with imperfect information in Section 3.1 is that the state space is of increasing dimension. A new observation at period n causes an increase in the dimension of the information vector $\bar{Y}_n$. This clearly creates some problems as the number of periods increases since it is computationally difficult to keep track of this much information. A common approach used under these circumstances is to use sufficient statistics or a process that represents the probabilistic structure of the information flow.

Define $\Pi_n^a = P\left[Z_n = a \,\middle|\, \bar{Y}_n\right]$ so that Π_n^a is the probability that the true state of the environment at time n is a given all observations $\bar{Y}_n$ until that time. The vector $\Pi_n = \left[\Pi_n^a, \Pi_n^b, \cdots\right]$ denotes the conditional distribution of Z_n given $\bar{Y}_n$ where $\sum_{a \in F} \Pi_n^a = 1$ and $\Pi_n^a \geq 0$ for all $a \in F$. Additional information that we obtain at each period is the new state of the observed process Y. Therefore, information at time $n+1$ is information at time n plus Y_{n+1} so that $\bar{Y}_{n+1} = \left(\bar{Y}_n, Y_{n+1}\right)$. The conditional distribution of the true state of the market at time $n+1$ is specified by

$$\Pi_{n+1}^b = P\left\{Z_{n+1} = b | \bar{Y}_{n+1}\right\} = \mathcal{T}_n^b\left(\Pi_n, Y_{n+1}\right)$$

where we can write

$$\mathcal{T}_n^b\left(\pi, j\right) = \frac{\sum_{a \in F} \pi^a Q_n(a, b) G_{n+1}(b, j)}{\sum_{a, c \in F} \pi^a Q_n(a, c) G_{n+1}(c, j)} \tag{3.57}$$

for $n \geq 0$ using Bayesian updating. Note that Π_0 is either known at the beginning or it can be determined from

$$\Pi_0^b = \frac{P\left[Z_0 = b\right] G_0(b, Y_0)}{\sum_{a \in F} P\left[Z_0 = a\right] G_0(a, Y_0)} \tag{3.58}$$

using the initial observation Y_0. The most important property of (3.57) is that calculation of the conditional distribution of the true state of the environment after time $n+1$ requires only Π_n, conditional probability of the true state of the environment after time n, and Y_{n+1}, the new observation on the true state of the environment at time $n+1$. Therefore, Π_n summarizes the information up to time n and represents a sufficient statistic for the complete past history of $\bar{Y}_n$. This result is also stated in Smallwood and Sondik [62], Monahan [49], and Bell [4]. Moreover, it is stated in Monahan [49] that $\Pi = \{\Pi_n; n \geq 0\}$ is a Markov chain. As a result, our problem can be modeled as a completely observable Markov decision chain where Π_n is the state of this Markov chain at time n. The unobservable environmental process Z is defined on the finite state space F whereas the Markov chain Π is defined on a continuous state space $\mathcal{D}_F$ which is the set of all probability distributions on F.

Expression in (3.57) is a transformation from Π_n to Π_{n+1} if the observations $\bar{Y}_n$ become $\bar{Y}_{n+1} = (\bar{Y}_n, Y_{n+1})$, and the transformation is $\Pi_{n+1} = \mathcal{T}_n\left(\Pi_n, Y_{n+1}\right)$ for $n \geq 0$. Clearly,

$\mathcal{T}_n(\pi, j) = \left\{\mathcal{T}_n^b(\pi, j); b \in F\right\}$ is a probability distribution with $\sum_{b\in F} \mathcal{T}_n^b(\pi, j) = 1$ and $\mathcal{T}_n^b(\pi, j) \geq 0$ for all $b \in F$. Note that Π_0 is again either known as an initial condition or it can be found by (3.58). In this case, we assume that $P[Z_0 = a]$ is externally specified so that it is initially known. In practice, $P[Z_0 = a]$ can be determined via preliminary analysis of the unobserved environment. Then, using $P[Z_0 = a]$, we can determine Π_0 by (3.58) since we already know G_0.

The evolution of Y is now described probabilistically by

$$P_n^k(\pi_n, j) = P\left[Y_{n+k} = j \,\middle|\, \bar{Y}_n\right] = \sum_{a\in F} \pi_n^a \Psi_n^k(a, j) \tag{3.59}$$

for $k \geq 1$, where Ψ_n^k is defined as

$$\begin{aligned} \Psi_n^k(a, j) &= P[Y_{n+k} = j | Z_n = a] \\ &= \sum_{b_{n+1},\cdots,b_{n+k}\in F} Q_n(a, b_{n+1}) \cdots Q_{n+k-1}(b_{n+k-1}, b_{n+k}) G_{n+k}(b_{n+k}, j) \end{aligned} \tag{3.60}$$

for $k \geq 1$. Moreover, we use P_n and Ψ_n instead of P_n^k and Ψ_n^k respectively when $k = 1$. We then have the simpler representation

$$\Psi_n(a, j) = \sum_{b\in F} Q_n(a, b)\, G_{n+1}(b, j)$$

and

$$P_n(\pi, j) = \sum_{a,b\in F} \pi^a Q_n(a, b)\, G_{n+1}(b, j)\,.$$

If the hidden Markov chain is stationary such that $Q_n = Q$ and $G_n = G$ for any $n = 0, 1, \cdots T-1$, then

$$\Psi_n = \Psi = QG$$

and

$$P_n = P = \pi QG.$$

In addition, we can write the recursion

$$P_n^{k+1}(\pi, j) = \sum_{l\in E} P_n(\pi, l)\, P_{n+1}^k(\mathcal{T}_n(\pi, l), j) \tag{3.61}$$

for all π and j.

For any function f on $\mathcal{D}_F$, let

$$\Psi_n f(\pi, a) = \sum_{j \in E} \Psi_n(a, j) f(\mathcal{T}_n(\pi, j))$$

for $n = 0, 1, \cdots, T-2$; and similarly, for any function g on E, define

$$\Psi_{T-1} g(a) = \sum_{j \in E} \Psi_{T-1}(a, j) g(j).$$

Note that $\{P_n\}$ can easily be determined once the conditional distribution of true state of environment $\{\Pi_n\}$, the transition matrices $\{Q_n\}$, emission matrices $\{G_n\}$ and the initial distribution of true state of the environment are known. Here, we assume that $\{Q_n\}$, $\{E_n\}$ and $\{\Psi_n\}$ are time-dependent for single and multiple period analyses.

3.2.1 Dynamic Programming Formulation

In order to solve the portfolio selection problem, we define $g_n(\pi, x, u)$ as the expected utility using the investment policy u in period n and the optimal policies from period $n+1$ to period T given that the probability distribution about the state of the market state is π, and the amount of money available for investment is x at period n. Then,

$$v_n(\pi, x) = \max_u g_n(\pi, x, u)$$

is the optimal expected utility using the optimal policy given that the probability distribution about the state of the market state is π, and the amount of money available for investment is x at period n. According to the dynamic programming principle

$$g_n(\pi, x, u) = E\left[v_{n+1}(\Pi_{n+1}, X_{n+1}(u)) \,|\Pi_n = \pi, X_n = x\right]$$

and we can write the dynamic programming equation (DPE) as

$$v_n(\pi, x) = \max_u E\left[v_{n+1}(\Pi_{n+1}, X_{n+1}(u)) \,|\Pi_n = \pi, X_n = x\right] \tag{3.62}$$

which can be rewritten as

$$v_n(\pi, x) = \max_u E\left[v_{n+1}\left(\mathcal{T}_n(\pi, Y_{n+1}), r_f x + R_n^e(Z_n)' u\right) |\Pi_n = \pi, X_n = x\right]$$

and

$$v_n(\pi, x) = \max_u \sum_{a \in F} \sum_{j \in E} \pi^a \Psi_n(a, j) E\left[v_{n+1}\left(\mathcal{T}_n(\pi, j), r_f x + R_n^e(a)' u\right)\right] \tag{3.63}$$

for $n = 0, 1, \cdots, T-1$ with the boundary condition $v_T\left(\mathcal{T}_{T-1}(\pi, j), x\right) = U(j, x)$ for all $\pi \in \mathcal{D}_F$ and $j \in E$. The solution for this problem is found by solving the DPE recursively. It should be noted that in this analysis we assume r_f does not depend neither on real market state nor observed market state.

3.2.2 Exponential Utility

In this section, we assume that the utility of the investor in state i is given by the exponential function

$$U(i, x) = K(i) - C(i)\exp(-x/\beta) \tag{3.64}$$

with $C(i) > 0$. Note that as in the case of perfect information β is that same for all market states so that risk classification of the investor does not depend on the stochastic market. Similarly, we assume that the return for the riskless asset is same for all market states so that $r_f(i) = r_f$ for all i. Then,

$$R_n^e(i, a) = R_n^e(a) = R(n, a) - r_f$$

so that both vector r_n^e and matrix V_n do not depend on the observed state. To simplify the notation, we can write $r_n^e(i, a) = r_n^e(a) = r(n, a) - r_f$ and $V_n(i, a) = V_n(a) = \sigma(n, a) + r_n^e(a)\, r_n^e(a)'$.

Theorem 10 *Let the utility function of the investor be the exponential function (3.64) and suppose that the riskless asset return does not depend on the market state. Then, the optimal solution of the dynamic programming equation (3.63) is*

$$v_n(\pi, x) = K_n(\pi) - C_n(\pi) e^{-x/\beta_n}$$

and the optimal portfolio is

$$u_n^*(\pi) = \alpha_n(\pi)\, \beta_n$$

where

$$\begin{aligned}
\beta_n &= \frac{\beta}{r_f^{T-n}}, \; K_n(\pi) = \sum_{a \in F} \pi^a \Psi_n K_{n+1}(\pi, a), \\
C_n(\pi) &= \sum_{a \in F} \pi^a \Psi_n C_{n+1}(\pi, a)\, E\left[\exp\left(-R_n^e(a)'\, \alpha_n(\pi)\right)\right],
\end{aligned}$$

and $\alpha_n(\pi)$ satisfies

$$\sum_{a\in F}\pi^a\Psi_n C_{n+1}(\pi,a)\,E\left[R_n^e(a)\exp\left(-R_n^e(a)'\,\alpha_n(\pi)\right)\right]=0 \tag{3.65}$$

for all $n=0,1,\cdots,T-2$ and

$$\sum_{a\in F}\pi^a\Psi_{T-1}C(a)\,E\left[R_{T-1}^e(a)\exp\left(-R_{T-1}^e(a)'\,\alpha_{T-1}(\pi)\right)\right]=0$$

for $n=T-1$ with boundary conditions

$$K_{T-1}(\pi)=\sum_{a\in F}\pi^a\Psi_{T-1}K(a)\,,$$
$$C_{T-1}(\pi)=\sum_{a\in F}\pi^a\Psi_{T-1}C(a)\,E\left[\exp\left(-R_{T-1}^e(a)'\,\alpha_{T-1}(\pi)\right)\right].$$

Proof. We will show that the recursion is true by induction starting with the fact that $U(i,x)=K(i)-C(i)\exp(-x/\beta)$. Note that

$$\begin{aligned}
g_{T-1}(\pi,x,u) &= \sum_{a\in F}\pi^a\sum_{j\in E}\Psi_{T-1}(a,j)\,E\left[U(j,r_fx+R_{T-1}^e(a)'\,u)\right]\\
&= \sum_{a\in F}\pi^a\sum_{j\in E}\Psi_{T-1}(a,j)\,K(j)\\
&\quad -\sum_{a\in F}\pi^a\sum_{j\in E}\Psi_{T-1}(a,j)\,C(j)\,E\left[\exp\left(-\left(R_{T-1}^e(a)'\,u/\beta+r_fx\right)/\beta\right)\right].
\end{aligned}$$

Let $u^*=(u_1^*,u_2^*,\cdots,u_m^*)$ be the optimal amount of money that should be invested in the risky assets so that

$$v_{T-1}(\pi,x)=\max_u g_{T-1}(\pi,x,u)=g_{T-1}(\pi,x,u^*)$$

as in the perfect information case. The gradient of g_{T-1} is

$$\frac{\partial g_{T-1}}{\partial u_k}=\exp(-r_fx/\beta)\sum_{a\in F}\pi^a\sum_{j\in E}\Psi_{T-1}(a,j)\,C(j)\,E\left[R_{T-1,k}^e(a)\exp\left(-R_{T-1}^e(a)'\,u/\beta\right)\right]/\beta \tag{3.66}$$

and the entries of the Hessian matrix $H(\pi,x,u)$ are

$$\begin{aligned}
\frac{\partial^2 g_{T-1}}{\partial u_k\partial u_l} &= -\exp(-r_fx/\beta)\sum_{a\in F}\pi^a\sum_{j\in E}\Psi_{T-1}(a,j)\,C(j)\\
&\quad\cdot E\left[R_{T-1,k}^e(a)\,R_{T-1,l}^e(a)\exp\left(-R_{T-1}^e(a)'\,u/\beta\right)\right]/\beta^2.
\end{aligned} \tag{3.67}$$

Let $z = (z_1, \cdots, z_m)$ be any non-zero column vector where z_i's are real numbers. Then, one can see that $z^T H(\pi, x, u) z$ is equal to

$$\begin{aligned} z^T H(\pi, x, u) z &= -\frac{1}{\beta^2} \exp(-r_f x/\beta) \sum_{a \in F} \pi^a \sum_{j \in E} \Psi_{T-1}(a, j) C(j) \\ & \quad \cdot E\left[\left(\sum_{k=1}^{m} z_k R^e_{T-1,k}(a)\right)^2 \exp\left(-R^e_{T-1}(a)' u/\beta\right)\right] \end{aligned}$$

which is always less than or equal to zero since $\pi^a, \Psi_{T-1}(a, j), C(j)$ are all positive. Thus, $H(\pi, x, u)$ is negative semi-definite and we can find the optimal solution by setting the gradient (3.66) equal to zero to obtain the optimality condition

$$\exp(-r_f x/\beta) \sum_{a \in F} \pi^a \sum_{j \in E} \Psi_{T-1}(a, j) C(j) E\left[R^e_{T-1,k}(a) \exp\left(-R^e_{T-1}(a)' u^*/\beta\right)\right]/\beta = 0$$

which can be rewritten as

$$\sum_{a \in F} \pi^a \Psi_{T-1} C(a) E\left[R^e_{T-1,k}(a) \exp\left(-R^e_{T-1}(a)' u^*/\beta\right)\right] = 0. \qquad (3.68)$$

Since there is no dependence on x in (3.68), $u^*_{T-1}(\pi, x)$ does not depend on x and the policy $u^*_{T-1}(\pi, x) = u^*_{T-1}(\pi)$. Letting $\alpha_{T-1}(\pi) = u^*_{T-1}(\pi)/\beta$, we obtain $u^*_{T-1}(\pi, x) = \alpha_{T-1}(\pi)\beta$ and this gives optimality condition (3.65). When the value function at time $T-1$ is rewritten for the optimal policy, we obtain

$$\begin{aligned} v_{T-1}(\pi, x) &= \sum_{a \in F} \pi^a \Psi_{T-1} K(a) - \exp(-r_f x/\beta) \sum_{a \in F} \pi^a \Psi_{T-1} C(a) \\ & \quad \cdot E\left[\exp\left(-R^e_{T-1}(a)' \alpha_{T-1}(\pi)\right)\right] \\ &= K_{T-1}(\pi) - C_{T-1}(\pi) \exp(-x/\beta_{T-1}) \end{aligned}$$

and the value function is still exponential like the utility function. This completes the proof for $n = T - 1$.

Suppose now that the induction hypothesis holds for periods $T, T-1, T-2, \cdots, n$. Then, for period $n - 1$,

$$\begin{aligned} g_{n-1}(\pi, x, u) &= \sum_{a \in F} \pi^a \sum_{j \in E} \Psi_{n-1}(a, j) E\left[v_n(\mathcal{T}_{n-1}(\pi, j), r_f x + R^e_{n-1}(a)' u)\right] \\ &= \sum_{a \in F} \pi^a \sum_{j \in E} \Psi_{n-1}(a, j) (K_n(\mathcal{T}_{n-1}(\pi, j)) \\ & \quad - C_n(\mathcal{T}_{n-1}(\pi, j)) E\left[\exp\left(-\left(r_f x + R^e_{n-1}(a)' u\right)/\beta_n\right)\right]). \end{aligned}$$

One can easily see that the Hessian matrix of $g_{n-1}(\bar{\imath}_{n-1}, x, u)$ is negative semi-definite as for $g_{T-1}(\bar{\imath}_{T-1}, x, u)$. Let $u^*_{n-1}(\bar{\imath}_{n-1}, x)$ be the optimal policy such that

$$v_{n-1}(\pi, x) = \max_u g_{n-1}(\pi, x, u) = g_{n-1}(\pi, x, u^*).$$

If we take the gradient of $g_{n-1}(\pi, x, u)$ with respect to u and set it equal to 0, we get the optimality condition

$$-\exp(-r_f x/\beta_n)\sum_{a\in F}\pi^a\sum_{j\in E}\Psi_{n-1}(a,j)\,C_n(\mathcal{T}_{n-1}(\pi,j)) \cdot E\left[R^e_{n-1,k}(a)\exp\left(-R^e_{n-1}(a)' u^*/\beta_n\right)\right]/\beta_n = 0$$

which can be rewritten as

$$\sum_{a\in F}\pi^a\Psi_{n-1}C_n(\pi,a)\,E\left[R^e_{n-1,k}(a)\exp\left(-R^e_{n-1}(a)' u^*/\beta_n\right)\right] = 0 \tag{3.69}$$

for all $k = 1, 2, \cdots, m$. Since there is no dependence on x in (3.69), $u^*_{n-1}(\pi, x)$ does not depend on x and $u^*_{n-1}(\pi, x) = u^*_{n-1}(\bar{\imath}_{n-1})$. Letting $\alpha_{n-1}(\pi) = u^*_{n-1}(\pi)/\beta_n$ we obtain $u^*_{n-1}(\pi, x) = \alpha_{n-1}(\pi)\beta_n$ and this gives optimality condition (3.65). If we insert the optimal policy in the value function, we can see that

$$\begin{aligned} v_{n-1}(\pi,x) = {} & \sum_{a\in F}\pi^a\sum_{j\in E}\Psi_{n-1}(a,j)\,K_n(\mathcal{T}_{n-1}(\pi,j)) - \exp(-r_f x/\beta_n)\sum_{a\in F}\pi^a\sum_{j\in E}\Psi_{n-1}(a,j) \\ & \cdot C_n(\mathcal{T}_{n-1}(\pi,j))\,E\left[\exp\left(-R^e_{n-1}(a)'\alpha_{n-1}(\pi)\right)\right] \\ = {} & K_{n-1}(\pi) - C_{n-1}(\pi)\exp(-x/\beta_{n-1}). \end{aligned}$$

and this completes the proof. ■

Evolution of Wealth

The evolution of the wealth process X using the optimal policy can be analyzed by the wealth dynamics equation

$$\begin{aligned} X_{n+1} &= r_f X_n + R^e_n(Z_n)' u^*_n(\Pi_n, X_n) \\ &= r_f X_n + R^e_n(Z_n)'\alpha_n(\Pi_n)\beta_{n+1} \\ &= r_f X_n + r_f^{n+1-T} R^e_n(Z_n)'\alpha_n(\Pi_n)\beta. \end{aligned} \tag{3.70}$$

Define the random variable

$$A_n(\pi, a) = R_n^e(a)'\alpha_n(\pi) \tag{3.71}$$

with mean

$$\begin{aligned} \bar{\alpha}_n(\pi, a) &= E[A_n(\pi, a)] = E\left[R_n^e(a)'\alpha_n(\pi)\right] \\ &= r_n^e(a)'\alpha_n(\pi) \end{aligned} \tag{3.72}$$

and second moment

$$\begin{aligned} \tilde{\alpha}_n(\pi, a) &= E\left[A_n(\pi, a)^2\right] = E\left[\alpha_n(\pi)' R_n^e(a) R_n^e(a)'\alpha_n(\pi)\right] \\ &= \alpha_n(\pi)' V_n(a)\,\alpha_n(\pi) \end{aligned} \tag{3.73}$$

which gives the variance

$$\mathrm{Var}(A_n(\pi, a)) = \tilde{\alpha}_n(\pi, a) - \bar{\alpha}_n(\pi, a)^2. \tag{3.74}$$

Now, we will show that the wealth process is given by

$$X_n = r_f^n X_0 + r_f^{n-T}\beta \sum_{k=0}^{n-1} A_k(\Pi_k, Z_k) \tag{3.75}$$

using the induction method where the sum on the right-hand side is set to zero when $n = 0$. The induction hypothesis holds trivially for $n = 0$. Suppose (3.75) holds for some $n \geq 0$. If we write X_{n+1} using the wealth dynamics equation (3.70)

$$\begin{aligned} X_{n+1} &= r_f X_n + r_f^{n+1-T} A_n(\Pi_n, Z_n)\,\beta \\ &= r_f^{n+1} X_0 + r_f^{n+1-T}\beta \sum_{k=0}^{n-1} A_k(\Pi_k, Z_k) + r_f^{n+1-T} A_n(\Pi_n, Z_n)\,\beta \\ &= r_f^{n+1} X_0 + r_f^{n+1-T}\beta \sum_{k=0}^{n} A_k(\Pi_k, Z_k) \end{aligned}$$

and we see that the induction hypothesis also holds for $n+1$. So, we can conclude that the wealth process can be written as in (3.75) and for $n = T$ we can find the terminal wealth as

$$X_T = r_f^T X_0 + \beta \sum_{k=0}^{T-1} A_k(\Pi_k, Z_k). \tag{3.76}$$

Given $X_0 = x_0$, the expected value of the terminal wealth satisfies

$$E[X_T | \Pi_0 = \pi] = r_f^T x_0 + m_e(\pi, T)\,\beta \tag{3.77}$$

where

$$m_e(\pi, T) = \sum_{k=0}^{T-1} E\left[A_k(\Pi_k, Z_k)\,|\Pi_0 = \pi\right] \tag{3.78}$$

and the variance of the terminal wealth satisfies

$$\operatorname{Var}(X_T|\Pi_0 = \pi) = \beta^2 \operatorname{Var}\left(\sum_{k=0}^{T-1} A_k(\Pi_k, Z_k|\Pi_0 = \pi)\right) = v_e^2(\pi, T)\,\beta^2. \tag{3.79}$$

We see that both the return and the standard deviation of X_T depends linearly on β. This shows that the exponential frontier is the line

$$E\left[X_T|\Pi_0 = \pi\right] = r_f^T x_0 + \left(\frac{m_e(\pi, T)}{v_e(\pi, T)}\right) \operatorname{SD}(X_T|\Pi_0 = \pi) \tag{3.80}$$

where $\operatorname{SD}(X_T)) = \sqrt{\operatorname{Var}(X_T)}$. We now show how $m(\pi, T)$ and $v(\pi, T)$ can be computed.

Computational Formulas

Define the transition densities of the process (Π, Z) such that

$$T_{n,k}\left((\pi_n, a), (d\pi_{n+k}, b)\right) = P\left\{\Pi_{n+k} \in d\pi_{n+k}, Z_{n+k} = b|\Pi_n = \pi_n, Z_n = a\right\}.$$

We can determine $T_{n,k}$ using a recursive algorithm with initial condition

$$\begin{aligned} T_{n,1}\left((\pi_n, a), (d\pi_{n+1}, b)\right) &= P\left\{\Pi_{n+1} \in d\pi_{n+1}, Z_{n+1} = b|\Pi_n = \pi_n, Z_n = a\right\} \\ &= \sum_{j \in E} Q_n(a, b)\, G_{n+1}(b, j)\, 1_{\{\mathcal{T}_n(\pi_n, j) \in d\pi_{n+1}\}} \end{aligned}$$

and the recursion

$$T_{n,k}\left((\pi_n, a), (d\pi_{n+k}, b)\right) = \sum_{b_{k-1} \in F} \int_{\mathcal{D}_F} T_{n,k-1}\left((\pi_n, a), (d\pi, b_{k-1})\right) 1_{\{\mathcal{T}_{n+k-1}(\pi, j) \in d\pi_{n+k}\}}$$

for $k \geq 2$. This can be solved to find

$$\begin{aligned} T_{n,k}\left((\pi_n, a), (d\pi_{n+k}, b)\right) &= \sum_{a_1, \cdots, a_k \in F} \sum_{j_1, \cdots, j_k \in E} Q_n(a, a_1)\, G_{n+1}(a_1, j_1) \cdots Q_n(a_{k-1}, a_k) \\ &\qquad \cdot G_{n+k}(a_k, j_k)\, 1_{\{\mathcal{T}_{n+k}(\mathcal{T}_{n+k-1}(\cdots(\mathcal{T}_n(\pi_n, j_1))\cdots, j_k)) \in d\pi_{n+k}\}}. \end{aligned}$$

We can also determine

$$\begin{aligned} P\left\{\Pi_k = d\pi_k, Z_k = b|\Pi_0 = \pi_0\right\} &= \sum_{a \in F} P\left\{\Pi_k \in d\pi_k, Z_k = b|\Pi_0 = \pi_0, Z_0 = a\right\} \\ &\qquad \cdot P\left\{Z_0 = a|\Pi_0 = \pi_0\right\} \\ &= \sum_{a \in F} T_{0,k}\left((\pi_0, a), (d\pi_k, b)\right) \pi_0^a. \end{aligned}$$

Using these relationships, we can write (3.78) as

$$\begin{aligned} m_e(\pi,T) &= \sum_{k=0}^{T-1}\int_{\mathcal{D}_F}\sum_{b\in F} P\left\{\Pi_k\in d\pi_k, Z_k=b|\Pi_0=\pi\right\}\bar{\alpha}_k(\pi_k,b) \\ & \sum_{k=0}^{T-1}\int_{\mathcal{D}_F}\sum_{a,b\in F} T_{0,k}\left((\pi,a),(d\pi_k,b)\right)\pi^a\bar{\alpha}_k(\pi_k,b) \end{aligned}$$

can be calculated using matrix operations.

Similarly,

$$\begin{aligned} v_e^2(\pi,T) &= \operatorname{Var}\left(\sum_{k=0}^{T-1} A_k(\Pi_k,Z_k)\middle|\Pi_0=\pi\right) \\ &= \sum_{k=0}^{T-1}\sum_{m=0}^{T-1}\operatorname{Cov}\left(A_k(\Pi_k,Z_k),A_m(\Pi_m,Z_m)|\Pi_0=\pi\right) \\ &= \sum_{k=0}^{T-1}\operatorname{Var}\left(A_k(\Pi_k,Z_k)|\Pi_0=\pi\right) \\ &\qquad +2\sum_{k=0}^{T-1}\sum_{m=k+1}^{T-1}\operatorname{Cov}\left(A_k(\Pi_k,Z_k),A_m(\Pi_m,Z_m)|\Pi_0=\pi\right) \\ &= \sum_{k=0}^{T-1}\left(E\left[A_k(\Pi_k,Z_k)^2|\Pi_0=\pi\right]-E\left[A_k(\Pi_k,Z_k)|\Pi_0=\pi\right]^2\right) \\ &\qquad +2\sum_{k=0}^{T-1}\sum_{m=k+1}^{T-1}\left(E\left[A_k(\Pi_k,Z_k)A_m(\Pi_m,Z_m)|\Pi_0=\pi\right]\right. \\ &\qquad\qquad \left.-E\left[A_k(\Pi_k,Z_k)|\Pi_0=\pi\right]E\left[A_m(\Pi_m,Z_m)|\Pi_0=\pi\right]\right) \end{aligned}$$

and we can write

$$
\begin{aligned}
v_e^2(\pi,T) \;=\; & \sum_{k=0}^{T-1}\Bigg(\int_{\mathcal{D}_F}\sum_{a,b\in F} T_{0,k}\left((\pi,a),(d\pi_k,b)\right)\pi^a\tilde{\alpha}_k(\pi_k,b) \\
& -\left(\int_{\mathcal{D}_F}\sum_{a,b\in F} T_{0,k}\left((\pi,a),(d\pi_k,b)\right)\pi^a\bar{\alpha}_k(\pi_k,b)\right)^2\Bigg) \\
& +2\sum_{k=0}^{T-1}\sum_{m=k+1}^{T-1}\Bigg(\int_{\mathcal{D}_F}\sum_{a,b\in F} T_{0,k}\left((\pi,a),(d\pi_k,b)\right)\pi^a\bar{\alpha}_k(\pi_k,b) \\
& \cdot\int_{\mathcal{D}_F}\sum_{c\in F} T_{k,m-k}\left((\pi_k,b),(d\pi_m,c)\right)\bar{\alpha}_m(\pi_m,c)\Bigg) \\
& -2\sum_{k=0}^{T-1}\sum_{m=k+1}^{T-1}\left(\int_{\mathcal{D}_F}\sum_{a,b\in F} T_{0,k}\left((\pi,a),(d\pi_k,b)\right)\pi^a\bar{\alpha}_k(\pi_k,b)\right) \\
& \cdot\left(\int_{\mathcal{D}_F}\sum_{a,b\in F} T_{0,m}\left((\pi_0,a),(d\pi_m,b)\right)\pi_0^a\bar{\alpha}_m(\pi_m,b)\right)
\end{aligned}
$$

3.2.3 Logarithmic Utility

In this section, we assume that the utility of the investor in state i is given by the logarithmic function

$$
U(i,x)=\begin{cases} K(i)+C(i)\log(x+\beta) & x+\beta>0 \\ -\infty & x+\beta\leq 0 \end{cases} \tag{3.81}
$$

with $C(i)>0$ where we can easily see that Pratt-Arrow's measure of absolute risk aversion is simply equal to $r(x)=1/(\beta+x)>0$ for all i so that $b=1$ and $a=\beta$ in Table 1.1. Note that β is the same for all market states so that risk classification of the investor does not depend on the stochastic market. Similarly, we assume that the return for the riskless asset is the same for all market states so that $r_f(i)=r_f$ for all i.

We will first consider a generic optimization problem of the form

$$
\max_u \sum_{a\in F}\mu_a E\left[\log\left(R^e(a)'u+c\right)\right] \tag{3.82}
$$

where $c>0$ is any constant, μ is any measure on F, and $R^e(a)$ is a random vector for any $a\in F$. Now, let

$$
A(c)=\left\{u : P\left\{R^e(a)'u+c>0\right\}=1\right\}
$$

be the set of all policies with finite expected utility so that $\left|E\left[\log\left(R^e(a)' u + c\right)\right]\right| < +\infty$ for $u \in A(c)$. It can be seen that $u = (u_1, u_2, \cdots, u_m) = (0, 0, \cdots, 0) \in A(c)$ satisfies this condition trivially for all $c > 0$. So, $A(c)$ is not empty. Also, let u, $w \in A(c)$, then $R^e(a)' u + c > 0$, and $R^e(a)' w + c > 0$ implies that

$$\lambda R^e(a)' u + (1-\lambda) R^e(a)' w + c > 0$$

so that $\lambda u + (1-\lambda) w \in A(c)$ for all $0 \le \lambda \le 1$. So, the solution set $A(c)$ is nonempty and convex. The gradient vector of the objection function $g(u) = \sum_{a \in F} \mu_a E\left[\log\left(R^e(a)' u + c\right)\right]$ is given by

$$\bigtriangledown_k g(u) = \frac{\partial g(u)}{\partial u_k} = \sum_{a \in F} \mu_a E\left[\frac{R^e_k(a)}{R^e(a)' u + c}\right]$$

while the Hessian matrix is

$$\bigtriangledown^2_{k,l} g(u) = \frac{\partial^2 g(u)}{\partial u_k \partial u_l} = -\sum_{a \in F} \mu_a E\left[\frac{R^e_k(a) R^e_l(a)}{\left(R^e(a)' u + c\right)^2}\right]$$

for all k, l.

The first order optimality condition to find the optimal solution of (3.82) is obtained by setting the gradient vector equal to zero so that

$$\sum_{a \in F} \mu_a E\left[\frac{R^e_k(a)}{R^e(a)' u + c}\right] = 0 \tag{3.83}$$

for all k.

Let $z = (z_1, \cdots, z_m)$ be any non-zero column vector where z_i's are real numbers. Then, one can see that

$$z' \bigtriangledown^2 g(u) z = -\sum_{a \in F} \mu_a E\left[\frac{\left(\sum_{k=1}^{m} z_k R^e_k(a)\right)^2}{\left(R^e(a)' u + c\right)^2}\right] \le 0.$$

Thus, the Hessian matrix $\bigtriangledown^2 g(u)$ is negative semi-definite and if there is a solution $u \in A(c)$ satisfying the first order condition (3.83), it must be optimal. Throughout this chapter, we assume that the excess returns are such that there is a solution of the first order condition (3.83) in $A(c)$ for all $\{R^e_n(a)\}$ and $c > 0$.

Theorem 11 *Let the utility function of the investor be the logarithmic function (3.81) and suppose that the riskless asset return does not depend on the market state. Then the optimal solution of the dynamic programming equation (3.63) is*

$$v_n(\pi, x) = K_n(\pi) + C_n(\pi)\log(x+\beta)$$

and the optimal portfolio is

$$u_n^*(\pi, x) = \alpha_n(\pi)\left(r_f x + \beta_{n+1}\right) \tag{3.84}$$

where

$$\begin{aligned}
\beta_n &= \frac{\beta}{r_f^{T-n}},\ C_{n-1}(\pi) = \sum_{a\in F}\pi^a \sum_{j\in E}\Psi_{n-1}(a,j)\, C_n(\mathcal{T}_n(\pi,j)) \\
K_{n-1}(\pi) &= \sum_{a\in F}\pi^a \left(\sum_{j\in E}\Psi_{n-1}(a,j)\, K_n(\mathcal{T}_n(\pi,j))\right. \\
&\quad \left. + \sum_{j\in E}\Psi_{n-1}(a,j)\, C_n(\mathcal{T}_n(\pi,j))\, E\left[\log\left(1+R_{n-1}^e(a)'\alpha(\pi)\right)\right]\right),
\end{aligned}$$

and $\alpha_n(i)$ *satisfies*

$$\sum_{a\in F}\pi^a \Psi_{n-1} C_n(a)\, E\left[\frac{R_{n,k}^e(a)}{1+R_n^e(a)'\alpha(\pi_n, x)}\right] = 0 \tag{3.85}$$

for all $n = 0, 1, \cdots, T-1$ *with boundary conditions*

$$\begin{aligned}
K_{T-1}(\pi) &= \sum_{a\in F}\pi^a \left(\sum_{j\in E}\Psi_{T-1}(a,j)\, K(j)\right. \\
&\quad \left. + E\left[\log\left(1+R_{n-1}^e(a)'\alpha(\pi)\right)\right]\sum_{j\in E}\Psi_{T-1}(a,j)\, C(j)\right) \\
C_{T-1}(\pi) &= \sum_{a\in F}\pi^a \sum_{j\in E}\Psi_{T-1}(a,j)\, C(j)
\end{aligned}$$

for all i.

Proof. We will show that the recursion is true by induction starting with the boundary condition $U(i,x) = K(i) + C(i)\log(x+\beta)$. Note that

$$\begin{aligned}
g_{T-1}(\pi, x, u) &= E\left[\sum_{a\in F}\pi^a \sum_{j\in E}\Psi_{T-1}(a,j)\, U(j, r_f x + R_{T-1}^e(a)'u)\right] \\
&= E\left[\sum_{a\in F}\pi^a \sum_{j\in E}\Psi_{T-1}(a,j)\left(K(j) + C(j)\log\left(R_{T-1}^e(a)'u + r_f x + \beta\right)\right)\right]
\end{aligned}$$

Let $u^* = (u_1^*, u_2^*, \cdots, u_m^*)$ be the optimal amount of money that should be invested in the risky assets so that

$$v_{T-1}(\pi, x) = \max_u g_{T-1}(\pi, x, u) = g_{T-1}(\pi, x, u^*)$$

as in the perfect information case.

Note that the optimization problem of maximizing the objective function $g_{T-1}(\pi, x, u)$ is similar to the optimization problem (3.82) where $\mu_a = \pi^a \sum_{j \in E} \Psi_{T-1}(a, j) C(j)$ and $c = r_f x + \beta$. Therefore, using the assumption on return distributions we can find the optimal portfolio by setting the gradient equal to zero so that

$$\frac{\partial g_{T-1}}{\partial u_k} = \sum_{a \in F} \pi^a \sum_{j \in E} \Psi_{T-1}(a, j) C(j) E\left[\frac{R^e_{T-1,k}(a)}{R^e_{T-1}(a)' u^* + r_f x + \beta}\right] = 0$$

for any asset $k = 1, 2, \cdots, m$ where $\Psi_{T-1} C(a) = \sum_{j \in E} \Psi_n(a, j) C(j)$. Defining the vector function $\alpha\left(\pi_{T-1}, x\right) = \left(\alpha_1\left(\pi_{T-1}, x\right), \alpha_2\left(\pi_{T-1}, x\right), \cdots, \alpha_m\left(\pi_{T-1}, x\right)\right)$ such that $\alpha\left(\pi_{T-1}, x\right) = u^*\left(\pi_{T-1}, x\right) / (r_f x + \beta)$ we obtain $u^*(a, x) = \alpha(a, x)(r_f x + \beta)$ so the optimality condition can be rewritten as

$$\sum_{a \in F} \pi^a \Psi_{T-1} C(a) E\left[\frac{R^e_{T-1,k}(a)}{1 + R^e_{T-1}(a)' \alpha\left(\pi_{T-1}, x\right)}\right] = 0 \tag{3.86}$$

for all $k = 1, 2, \cdots m$. Since there is no dependence on x in (3.86), $\alpha(\pi, x)$ does not depend on x and $\alpha(\pi, x) = \alpha(\pi)$. When the value function at time $T-1$ is rewritten for the optimal policy, we obtain

$$\begin{aligned} v_{T-1}(\pi, x) = &\sum_{a \in F} \pi^a \Psi_{T-1} K(a) \\ &+ \sum_{a \in F} \pi^a \Psi_{T-1} C(a) \left(\log(r_f x + \beta) + E\left[\log\left(1 + R^e_{T-1}(a)' \alpha(\pi)\right)\right]\right) \\ = &\, K_{T-1}(\pi) - C_{T-1}(\pi) \log(r_f x + \beta). \end{aligned}$$

and the value function is still exponential like the utility function. This completes the proof for $n = T-1$.

Suppose now that the induction hypothesis holds for periods $T, T-1, T-2, \cdots, n$. Then, for period $n-1$,

$$\begin{aligned} g_{n-1}(\pi,x,u) &= E\left[\sum_{a\in F}\pi^a\sum_{j\in E}\Psi_n(a,j)\,v_n(\mathcal{T}_{n-1}(\pi,j)\,,r_f x+R^e_{n-1}(a)'u)\right] \\ &= \sum_{a\in F}\pi^a\sum_{j\in E}\Psi_n(a,j)\,C_n(\mathcal{T}_{n-1}(\pi,j))E\left[\log\left(R^e_{n-1}(a)'u+r_f x+\beta_n\right)\right] \\ &\quad+\sum_{a\in F}\pi^a\sum_{j\in E}\Psi_n(a,j)\,K_n(\mathcal{T}_{n-1}(\pi,j)) \end{aligned}$$

Let $u^*_{n-1}(\bar{\imath}_{n-1},x)$ be the optimal policy such that

$$v_{n-1}(\pi,x)=\max_u g_{n-1}(\pi,x,u)=g_{n-1}(\pi,x,u^*)\,.$$

It is clear, once again, that the objective function $g_{n-1}(\pi_{n-1},x,u)$ is in the form of the generic objective function in (3.82) with $\mu_a=\pi^a\sum_{j\in E}\Psi_n(a,j)\,C_n(\mathcal{T}_{n-1}(\pi,j))$ and $c=r_f x+\beta_n$. If we take the gradient of $g_{n-1}(\pi,x,u)$ with respect to u and set it equal to 0, we get the optimality condition

$$\frac{\partial g_{n-1}}{\partial u_k}=\sum_{a\in F}\pi^a\sum_{j\in E}\Psi_{n-1}(a,j)\,C_n\left(\mathcal{T}_{n-1}(\pi_{n-1},j)\right)E\left[\frac{R^e_{n-1,k}(a)}{R^e_{n-1}(a)'u^*+r_f x+\beta}\right]=0 \quad (3.87)$$

and

$$\sum_{a\in F}\pi^a\Psi_{n-1}C_n\left(\mathcal{T}_{n-1}(\pi_{n-1},j)\right)E\left[\frac{R^e_{n-1,k}(a)}{R^e_{n-1}(a)'u^*+r_f x+\beta}\right]=0 \quad (3.88)$$

for all $k=1,2,\cdots m$ are the optimality conditions. Defining the vector function $\alpha(\pi_{n-1},x)$ such that $\alpha(\pi_{n-1},x)=u^*(\pi_{n-1},x)/(r_f x+\beta)$ we obtain $u^*_{n-1}(a,x)=\alpha(a,x)(r_f x+\beta)$ so the optimality condition can be rewritten as

$$\sum_{a\in F}\pi^a\Psi_{n-1}C_n\left(\mathcal{T}_{n-1}(\pi_{n-1},j)\right)E\left[\frac{R^e_{n-1,k}(a)}{1+R^e_{n-1}(a)'\alpha(\pi_{n-1},x)}\right]=0. \quad (3.89)$$

for all $k=1,2,\cdots m$ are the optimality conditions. Since there is no dependence on x and i in (3.89), α does not depend on x and i and $u^*_{n-1}=\alpha_{n-1}(\pi_{n-1})(r_f x+\beta)$. When the value function at time $n-1$ is rewritten for the optimal policy, we obtain

$$\begin{aligned} v_{n-1}(\pi,i,x) &= \sum_{a\in F}\pi^a\sum_{j\in E}\Psi_{n-1}K_n\left(\mathcal{T}_{n-1}(\pi,j)\right)\log\left(r_f x+\beta_n\right) \\ &\quad-\sum_{a\in F}\pi^a\sum_{j\in E}\Psi_{n-1}C_n\left(\mathcal{T}_{n-1}(\pi,j)\right)\left(\log\left(r_f x+\beta_n\right)\right. \\ &\quad\left.+E\left[\log\left(1+R^e_{n-1}(a)'\alpha(\pi)\right)\right]\right) \\ &= K_{n-1}(\pi)-C_{n-1}(\pi)\log\left(r_f x+\beta_n\right). \end{aligned}$$

and this completes the proof. ■

Note that the structure of the optimal solution in (3.84) is such that the optimal distribution of wealth invested in the risky assets depend only on the state of the market independent of time. If the market is in state i in period n, then the total amount of money invested on the risky assets is

$$1'u_n^*(\pi, x) = 1'\alpha_n(\pi)(r_f x + \beta_{n+1}) = \left(r_f x + \frac{\beta}{r_f^{T-(n+1)}}\right)\sum_{k=1}^{m}\alpha_{n,k}(\pi)$$

and the proportion on wealth allocated for asset k in the risky portfolio is

$$w_{n,k}(\pi) = \frac{\alpha_{n,k}(\pi)}{\sum_{k=1}^{m}\alpha_{n,k}(\pi)} \tag{3.90}$$

which is totally independent of wealth x. The optimal policy specified by (3.84) is not static in time since it depends on n, and it is not memoryless in wealth since it depends on x. However, (3.90) clearly indicates that the composition of the risky part of the optimal portfolio only depends on the market state and time. The risky portfolio composition is memoryless. It satisfies the separation property in the sense that it represents the single fund of risky assets that logarithmic investors choose. The amount of total wealth allocated for risky assets depend on the level of wealth, but the composition of the risky assets depend only on the market state and time. This composition, however, is random due to the randomly changing market conditions in time. Our results are of course consistent with similar work in the literature on logarithmic utility functions, but the stochastic market approach makes our model more realistic without causing substantial difficulty in the analysis. Another important observation is that the structure of the optimal portfolio is not affected by the transition matrix $\{Q_n\}$ of the stochastic market. It only depends on the joint distribution of the risky asset returns as prescribed by (3.85) in a given market state, irrespective of future expectations on the stochastic market.

Evolution of Wealth

The evolution of the wealth process X using the optimal policy can be analyzed by the wealth dynamics equation

$$\begin{aligned} X_{n+1} &= r_f X_n + R^e(Z_n)' u_n^*(\Pi_n, X_n) \\ &= r_f X_n + R^e(Z_n)' \alpha_n(\Pi_n) \left(r_f X_n + \beta_{n+1} \right) \\ &= r_f X_n \left(1 + A_n \left(\Pi_n, Z_n \right) \right) + r_f^{n+1-T} A_n \left(\Pi_n, Z_n \right) \beta \end{aligned} \tag{3.91}$$

where $A_n(\pi, a)$ is defined in (3.71).

Define

$$\mathbb{C}_n \left(x_1, x_2, \cdots, x_n \right) = \prod_{k=1}^{n} \left(1 + x_k \right) - 1$$

as the sum of all combinations of the products of n variables for $n \geq 1$, and set $\mathbb{C}_0 = 0$.

Now, we will show that the wealth process is

$$X_n = r_f^n X_0 \prod_{k=0}^{n-1} \left(1 + A_k \left(\Pi_k, Z_k \right) \right) + r_f^{n-T} \beta \mathbb{C}_n \left(A_0 \left(\Pi_0, Z_0 \right), \cdots, A_{n-1} \left(\Pi_{n-1}, Z_{n-1} \right) \right) \tag{3.92}$$

using induction where the product on the right hand side is set to 1 when $n = 0$. The induction hypothesis holds trivially for $n = 0$. Suppose (3.92) holds for some $n \geq 0$. If we write X_{n+1} using the wealth dynamics equation (3.91)

$$\begin{aligned} X_{n+1} &= r_f X_n \left(1 + A_n(\Pi_n, Z_n) \right) + r_f^{n+1-T} A_n(\Pi_n, Z_n) \beta \\ &= r_f^{n+1} X_0 \prod_{k=0}^{n} \left(1 + A_k \left(\Pi_k, Z_k \right) \right) + r_f^{n+1-T} \beta \left[\left(1 + A_n(\Pi_n, Z_n) \right) \right. \\ &\qquad \left. \cdot \, \mathbb{C}_n \left(A_0 \left(\Pi_0, Z_0 \right), A_1 \left(\Pi_1, Z_1 \right), \cdots, A_{n-1} \left(\Pi_{n-1}, Z_{n-1} \right) \right) + A_n(\Pi_n, Z_n) \right] \\ &= r_f^{n+1} X_0 \prod_{k=0}^{n} \left(1 + A_k \left(\Pi_k, Z_k \right) \right) + r_f^{n+1-T} \beta \mathbb{C}_{n+1} \left(A_0 \left(\Pi_0, Z_0 \right), \cdots, A_n \left(\Pi_n, Z_n \right) \right) \end{aligned}$$

and we see that the induction hypothesis also holds for $n + 1$. So, we conclude that the wealth process can be written as in (3.92) and, for $n = T$, we can find the terminal wealth as

$$\begin{aligned} X_T &= r_f^T X_0 \prod_{k=0}^{T-1} \left(1 + A_k \left(\Pi_k, Z_k \right) \right) + \beta \mathbb{C}_T \left(A_0 \left(\Pi_0, Z_0 \right), \cdots, A_{T-1} \left(\Pi_{T-1}, Z_{T-1} \right) \right) \\ &= r_f^T X_0 + \left(r_f^T X_0 + \beta \right) \mathbb{C}_T \left(A_0 \left(\Pi_0, Z_0 \right), A_1 \left(\Pi_1, Z_1 \right), \cdots, A_{T-1} \left(\Pi_{T-1}, Z_{T-1} \right) \right). \end{aligned}$$

Given $X_0 = x_0$, the expected value of the terminal wealth satisfies

$$E \left[X_T | \Pi_0 = \pi \right] = r_f^T x_0 + \left(r_f^T x_0 + \beta \right) m_l \left(\pi, T \right) \tag{3.93}$$

where

$$m_l(\pi,T) = E\left[\mathbb{C}_T\left(A_0(\Pi_0,Z_0), A_1(\Pi_1,Z_1), \cdots, A_{T-1}(\Pi_{T-1},Z_{T-1})\right)|\Pi_0=\pi\right] \tag{3.94}$$

and the variance of the terminal wealth satisfies

$$\operatorname{Var}(X_T|\Pi_0=\pi) = \left(r_f^T x_0+\beta\right)^2 v_l^2(\pi,T) \tag{3.95}$$

where

$$v_l^2(\pi,T) = \operatorname{Var}\left(\mathbb{C}_T\left(A_0(\Pi_0,Z_0), A_1(\Pi_1,Z_1), \cdots, A_{T-1}(\Pi_{T-1},Z_{T-1})\right)|\Pi_0=\pi\right). \tag{3.96}$$

We can clearly see from (3.93) and (3.95) that both the return and the standard deviation of X_T depends linearly on β. This shows that the logarithmic frontier is the straight line

$$E[X_T|\Pi_0=\pi] = r_f^T x_0 + \left(\frac{m_l(\pi,T)}{v_l(\pi,T)}\right)\operatorname{SD}(X_T|\Pi_0=\pi)$$

where $\operatorname{SD}(X_T|\Pi_0=\pi)) = \sqrt{\operatorname{Var}(X_T|\Pi_0=\pi)}$. In other words, the expected value and standard deviation of the terminal wealth fall on this straight line when they are calculated and plotted for different values of β. Also, it cuts the zero-risk line at $E[X_T|\Pi_0=\pi]=r_f^T x_0$ as expected. The reason for this is that for zero-risk level investor puts all of his money on the riskless asset. The return of the riskless asset until the terminal time T is r_f^T, and the wealth at the terminal time will be $r_f^T x_0$ for sure. The risk premium for the logarithmic investor is given by the ratio $m_l(\pi,T)/v_l(\pi,T)$.

Computational Formulas

The computation of $m_l(\pi,T)$ and $v_l(\pi,T)$ are possible although they are not as simple as their counterparts for the exponential utility case. We will use the definition (2.60) whenever appropriate. Note that,

$$\begin{aligned} m_l(\pi,T) &= E\left[\prod_{k=0}^{T-1}\left(1+A_k(\Pi_k,Z_k)\right)-1\middle|\Pi_0=\pi\right] \\ &= E\left[E\left[\prod_{k=0}^{T-1}\left(1+A_k(\Pi_k,Z_k)\right)-1\middle|\bar{Z}_{T-1},\Pi_1,\cdots\Pi_{T-1},\Pi_0=\pi\right]\right] \end{aligned}$$

and, since the returns in different periods are independent given (Π, Z), we obtain

$$\begin{aligned} m_l(\pi,T) &= E\left[\prod_{k=0}^{T-1}\left(1+\bar{\alpha}_k(\Pi_k,Z_k)\right)-1\middle|\Pi_0=\pi\right] \\ &= E\left[\mathbb{C}_T\left(\bar{\alpha}_0(\Pi_0,Z_0), \bar{\alpha}_1(\Pi_1,Z_1), \cdots, \bar{\alpha}_{T-1}(\Pi_{T-1},Z_{T-1})\right)|\Pi_0=\pi\right]. \end{aligned} \tag{3.97}$$

Given $\Pi_0 = \pi$, the conditional joint distribution of $\Pi_1, \cdots, \Pi_{T-1}, \bar{Z}_{T-1}$ is

$$P\left\{\Pi_1 \in d\pi_1, \Pi_2 \in d\pi_2, \cdots, \Pi_{T-1} \in d\pi_{T-1}, \bar{Z}_{T-1} = \bar{a}_{T-1} \middle| \Pi_0 = \pi\right\} \tag{3.98}$$
$$= \pi^{a_0} T_{0,1}\left(\left(\pi, a_0\right), \left(d\pi_1, a_1\right)\right) \cdots T_{T-2,1}\left(\left(\pi_{T-2}, a_{T-2}\right), \left(d\pi_{T-1}, a_{T-1}\right)\right)$$

and the expected return of terminal wealth (3.97) can found by using this distribution so that

$$\begin{aligned} m_l\left(\pi, T\right) &= \sum_{\bar{a}_{T-1} \in F^T} \int_{\mathcal{D}_F} \cdots \int_{\mathcal{D}_F} \pi^{a_0} T_{0,1}\left(\left(\pi, a_0\right), \left(d\pi_1, a_1\right)\right) \cdots \\ &\quad \cdot T_{T-2,1}\left(\left(\pi_{T-2}, a_{T-2}\right), \left(d\pi_{T-1}, a_{T-1}\right)\right) \\ &\quad \cdot \mathbb{C}_T\left(\bar{\alpha}_0\left(\pi, a_0\right), \bar{\alpha}_1\left(\pi_1, a_1\right), \cdots, \bar{\alpha}_{T-1}\left(\pi_{T-1}, a_{T-1}\right)\right). \end{aligned} \tag{3.99}$$

To determine the variance $v_l^2\left(\pi, T\right)$, we first calculate the second moment as

$$\begin{aligned} s_l\left(\pi, T\right) &= E\left[\prod_{k=0}^{T-1} \left(\left(1 + A_k\left(\Pi_k, Z_k\right)\right) - 1\right)^2 \middle| \Pi_0 = \pi\right] \\ &= E\left[E\left[\prod_{k=0}^{T-1} \left(\left(1 + A_k\left(\Pi_k, Z_k\right)\right) - 1\right)^2 \middle| \bar{Z}_{T-1}, \Pi_1, \cdots \Pi_{T-1}, \Pi_0 = \pi\right]\right] \\ &= E\left[E\left[\prod_{k=0}^{T-1} \left(1 + A_k\left(\Pi_k, Z_k\right)\right)^2 - 2\prod_{k=0}^{T-1} \left(1 + A_k\left(\Pi_k, Z_k\right)\right)\right.\right. \\ &\quad \left.\left. +1 \middle| \bar{Z}_{T-1}, \Pi_1, \cdots \Pi_{T-1}, \Pi_0 = \pi\right]\right] \\ &= E\left[\prod_{k=0}^{T-1} \left(1 + 2\bar{\alpha}_k\left(\Pi_k, Z_k\right) + \tilde{\alpha}_k\left(\Pi_k, Z_k\right)\right) - 1\right. \\ &\quad \left. -2\left(\prod_{k=0}^{T-1} \left(1 + \bar{\alpha}_k\left(\Pi_k, Z_k\right)\right) - 1\right) \middle| \Pi_0 = \pi\right] \\ &= E[\mathbb{C}_T(2\bar{\alpha}_0\left(\Pi_0, Z_0\right) + \tilde{\alpha}_0\left(\Pi_0, Z_0\right), \cdots, \\ &\quad 2\bar{\alpha}_{T-1}\left(\Pi_{T-1}, Z_{T-1}\right) + \tilde{\alpha}_{T-1}\left(\Pi_{T-1}, Z_{T-1}\right))|\Pi_0 \pi] \\ &\quad -2E\left[\mathbb{C}_T\left(\bar{\alpha}_0\left(\Pi_0, Z_0\right), \bar{\alpha}_1\left(\Pi_1, Z_1\right), \cdots, \bar{\alpha}_{T-1}\left(\Pi_{T-1}, Z_{T-1}\right)\right)|\Pi_0 = \pi\right] \end{aligned}$$

which can be found using the conditional distribution (3.98) as in (3.99). Finally, the variance becomes $v_l^2\left(\pi, T\right) = s_l\left(\pi, T\right) - m_l\left(\pi, T\right)^2$.

3.2.4 Simple Logarithmic Utility

In this section, we assume that the utility of the investor in state i is given by the simple logarithmic function

$$U(i,x)=\begin{cases} K(i)+C(i)\log(x) & x>0 \\ -\infty & x\le 0 \end{cases} \tag{3.100}$$

with $C(i)>0$ and $\beta=0$. In this part, we can relax the assumption on risk free rate as $r_f(i)$ now depends on the observed market state. Note that this structure implies $R_n^e(i,a)=R(n,a)-r_f(i)$ which depends on a as well as i. We still need the assumptions on the return distributions as in Section 3.2.3. Therefore for an optimization problem of the form

$$\max_u \sum_{a\in F}\mu_a E\left[\log\left(R^e(i,a)'u+c\right)\right] \tag{3.101}$$

we assume that the excess returns are such that the solution to the first order condition is in the feasible set. We therefore suppose that for any measure μ on F, any state $i\in E$, and any period n, the following equation

$$\sum_{a\in F}\mu_a E\left[\frac{R_n^e(i,a)}{1+R_n^e(i,a)'\alpha}\right]=0 \tag{3.102}$$

has a unique solution $\alpha\in R^m$.

Theorem 12 *Let the utility function of the investor be the logarithmic function (3.100). Then, the optimal solution of the dynamic programming equation (3.63) is*

$$v_n(\pi,x)=K_n(\pi)+C_n(\pi)\log(x)$$

and the optimal portfolio is

$$u_n^*(\pi,i_n,x)=\alpha_n(\pi)r_f(i_n)x$$

where

$$\begin{aligned}
C_{n-1}(\pi) &= \sum_{a\in F}\pi^a\sum_{j\in E}\Psi_{n-1}(a,j)\,C_n(\mathcal{T}_{n-1}(\pi,j)) \\
K_{n-1}(\pi) &= \sum_{a\in F}\pi^a\left(\sum_{j\in E}\Psi_{n-1}(a,j)\,K_n(\mathcal{T}_{n-1}(\pi,j))\right. \\
&\quad \left.+\sum_{j\in E}\Psi_{n-1}(a,j)\,C_n(\mathcal{T}_{n-1}(\pi,j))\,E\left[\log\left(1+R_{n-1}^e(i_n,a)'\alpha_{n-1}(\pi)\right)\right]\right)
\end{aligned}$$

and $\alpha_n(\pi)$ *satisfies*

$$\sum_{a\in F}\pi^a\Psi_{n-1}C_n(a)\,E\left[\frac{R^e_{n,k}(i_n,a)}{1+R^e_n(i_n,a)'\,\alpha_n(\pi)}\right]=0 \tag{3.103}$$

for all assets $k=1,2,\cdots,m$, $\bar{\imath}_n\in E^{n+1}$, *and* $n=0,1,\cdots,T-1$ *with boundary conditions*

$$\begin{aligned}K_{T-1}(\pi) &= \sum_{a\in F}\pi^a\left(\sum_{=j\in E}\Psi_{T-1}(a,j)\,K(j)\right.\\ &\qquad\left.+E\left[\log\left(1+R^e_{n-1}(a)'\,\alpha_{T-1}(\pi)\right)\right]\sum_{j\in E}\Psi_{T-1}(a,j)\,C(j)\right)\\ C_{T-1}(\pi) &= \sum_{a\in F}\pi^a\sum_{j\in E}\Psi_{T-1}(a,j)\,C(j)\,.\end{aligned}$$

Proof. We will show that the recursion is true by induction starting with the boundary condition $v_T(\bar{\imath}_T,x)=K(i_T)+C(i_T)\log(x)$ and obtain

$$\begin{aligned}g_{T-1}(\pi,x,u) &= \sum_{a\in F}\pi^a\sum_{j\in E}\Psi_{T-1}(a,j)\,E[U(j,r_f(i_{T-1})\,x+R^e_{T-1}(i_{T-1},a)'\,u)]\\ &= \sum_{a\in F}\pi^a\sum_{j\in E}\Psi_{T-1}(a,j)\,C(j)E[\log(r_f(i_{T-1})\,x+R^e_{T-1}(i_{T-1},a)'\,u)]\\ &\quad+\sum_{a\in F}\pi^a\sum_{j\in E}\Psi_{T-1}(a,j)\,K(j).\end{aligned}$$

Let $u^*=(u_1^*,u_2^*,\cdots,u_m^*)$ be the optimal amount of money that should be invested in the risky assets so that

$$v_{T-1}(\pi,x)=\max_u g_{T-1}(\pi,x,u)=g_{T-1}(\pi,x,u^*)$$

as in the perfect information case.

Note that the optimization problem of maximizing the objective function $g_{T-1}(\pi,x,u)$ is similar to the optimization problem of (3.101) where $\mu_a=\pi^a\sum_{j\in E}\Psi_{T-1}(a,j)\,C(j)$ and $c=r_f(i_{T-1})\,x$. Therefore similar to Theorem 11, we can find the optimal portfolio by setting the gradient equal to zero so that

$$\sum_{a\in F}\pi^a\sum_{j\in E}\Psi_{T-1}(a,j)\,C(j)E\left[\frac{R^e_{T-1,k}(i_{T-1},a)}{r_f(i_{T-1})\,x+R^e_{T-1}(i_{T-1},a)'\,u^*}\right]=0$$

for all $k=1,2,\cdots,m$. Defining $\alpha_{T-1}(\pi,x)=u^*_{T-1}(\pi,i_{T-1},x)/r_f(i_{T-1})\,x$ one can see that $u^*_{T-1}(\pi,i_{T-1},x)=\alpha_{T-1}(\pi,x)r_f(i_{T-1})\,x$. The optimality condition can now be rewritten as

$$\sum_{a\in F}\pi^a\sum_{j\in E}\Psi_{T-1}(a,j)\,C(j)E\left[\frac{R^e_{T-1,k}(i_{T-1},a)}{1+R^e_{T-1}(i_{T-1},a)'\,\alpha_{T-1}(\pi,x)}\right]=0. \tag{3.104}$$

Since there is no dependence on x in (3.104), $\alpha_{T-1}(\pi, x)$ does not depend on x and $\alpha_{T-1}(\pi, x) = \alpha_{T-1}(\pi)$. So, we obtain $u^*_{T-1}(\pi, i_{T-1}, x) = \alpha_{T-1}(\pi) r_f(i_{T-1}) x$ where α_{T-1} satisfies (3.103). When the value function at time $T-1$ is rewritten for the optimal policy, we obtain

$$\begin{aligned} v_{T-1}(\bar{\imath}_{T-1}, x) &= \sum_{a\in F} \pi^a \sum_{j\in E} \Psi_{T-1}(a,j)\, C(j) \log(x) + \sum_{a\in F} \pi^a \sum_{j\in E} \Psi_{T-1}(a,j)\, (K(j) \\ &\qquad + C(j) E\left[\log\left(r_f(i_{T-1})\left(1 + R^e_{T-1}(i_{T-1}, a)' \alpha_{T-1}(\pi)\right)\right)\right]) \\ &= K_{T-1}(\pi) + C_{T-1}(\pi)\log(x) \end{aligned}$$

and the value function is still logarithmic like the utility function. This completes the proof for $n = T-1$.

Suppose now that the induction hypothesis holds for periods $T, T-1, T-2, \cdots, n$. Then, for period $n-1$,

$$\begin{aligned} g_{n-1}(\pi, x, u) &= \sum_{a\in F} \pi^a \sum_{j\in E} \Psi_n(a,j)\, v_n(\mathcal{T}_{n-1}(\pi, j), r_f(i_{n-1}) x + R^e_{n-1}(i_{n-1}, a)' u)] \\ &= \sum_{a\in F} \pi^a \sum_{j\in E} \Psi_n(a,j)\, C_n(\mathcal{T}_{n-1}(\pi, j)) E[\log(r_f(i_{n-1}) x + R^e_{n-1}(i_{T-1}, a)' u)] \\ &\qquad + \sum_{a\in F} \pi^a \sum_{j\in E} \Psi_n(a,j)\, K_n(\mathcal{T}_{n-1}(\pi, j)). \end{aligned}$$

One can easily see that the Hessian matrix of $g_{n-1}(\bar{\imath}_{n-1}, x, u)$ is negative semi-definite as for $g_{T-1}(\pi, x, u)$. Let $u^*_{n-1}(\pi, x)$ be the optimal policy such that

$$v_{n-1}(\pi, x) = \max_u g_{n-1}(\pi, x, u) = g_{n-1}(\pi, x, u^*_{n-1}).$$

If we take the gradient of $g_{n-1}(\pi, x, u)$ with respect to u and set it equal to 0, we get the optimality condition

$$\frac{\partial g_{n-1}}{\partial u_k} = \sum_{a\in F} \pi^a \sum_{j\in E} \Psi_{n-1}(a,j)\, C_n(\mathcal{T}_{n-1}(\pi, j))\, E\left[\frac{R^e_{n-1,k}(i_{n-1}, a)}{R^e_{n-1}(i_{n-1}, a)' u^* + r_f(i_{n-1}) x}\right] = 0 \tag{3.105}$$

and

$$\sum_{a\in F} \pi^a \Psi_{n-1} C_n(\mathcal{T}_{n-1}(\pi, j))\, E\left[\frac{R^e_{n-1,k}(i_{n-1}, a)}{R^e_{n-1}(i_{n-1}, a)' u^* + r_f(i_{n-1}) x}\right] = 0 \tag{3.106}$$

for all $k = 1, 2, \cdots m$. Defining the vector function $\alpha_{n-1}(\pi_{n-1}, x)$ such that $\alpha_{n-1}(\pi_{n-1}, x) = u^*(\pi_{n-1}, i_{n-1}, x) / r_f(i_{n-1}) x$ we obtain $u^*_{n-1}(a, i_{n-1}, x) = \alpha_{n-1}(a, x) r_f(i_{n-1}) x$ so the optimality condition can be rewritten as

$$\sum_{a\in F}\pi^a\Psi_{n-1}C_n\left(\mathcal{T}_{n-1}\left(\pi,j\right)\right)E\left[\frac{R^e_{n-1,k}\left(i_{n-1},a\right)}{1+R^e_{n-1}\left(i_{n-1},a\right)'\alpha_{n-1}\left(\pi_{n-1},x\right)}\right]=0. \tag{3.107}$$

for all $k=1,2,\cdots m$. Since there is no dependence on x and i in (3.107), α_{n-1} does not depend on x and i and $u^*_{n-1}=\alpha_{n-1}\left(\pi\right)r_f\left(i_{n-1}\right)x$. When the value function at time $n-1$ is rewritten for the optimal policy, we obtain

$$\begin{aligned} v_{n-1}\left(\pi,i,x\right) &= \sum_{a\in F}\pi^a\sum_{j\in E}\Psi_{n-1}K_n\left(a\right)\log\left(r_f\left(i_{n-1}\right)x\right)-\sum_{a\in F}\pi^a\sum_{j\in E}\Psi_{n-1} \\ &\quad\cdot C_n\left(\mathcal{T}_{n-1}\left(\pi,j\right)\right)\left(\log\left(r_f\left(i_{n-1}\right)x\right)+E\left[\log\left(1+R^e_{n-1}\left(i_{n-1},a\right)'\alpha_{n-1}\left(\pi\right)\right)\right]\right) \\ &= K_{n-1}(\pi)-C_{n-1}(\pi)\log\left(r_f\left(i_{n-1}\right)x\right). \end{aligned}$$

and this completes the proof. ■

In this special case with $\beta=0$, at any time n, the total amount of money invested in the risky assets depends only on the information vector π_n and wealth x. Since the total risky investment is $1'u^*_n(\pi,i_n,x)=1'\alpha_n(\pi)r_f(i_n)x$, it follows that $r_f(i)\sum_{k=1}^m\alpha_{n,k}(\bar{\imath}_k)$ is the proportion of total wealth that is invested in the risky assets if the sufficient statistics is π_n. Moreover, as in the general logarithmic case, the composition of the risky portfolio also depends only on π_n independent of the available wealth x.

The evolution of the wealth process X using the optimal policy can be analyzed by the wealth dynamics equation

$$\begin{aligned} X_{n+1} &= r_f(Y_n)X_n+R^e_n(Z_n)'u^*_n(\Pi_n,Y_n,X_n) \\ &= X_nr_f(Y_n)\left(1+A_n\left(\Pi_n,Z_n\right)\right)=X_nB_n\left(\Pi_n,Z_n\right) \end{aligned}$$

where $B_k(\pi_k,a)=r_f(i_k)\left(1+A_k(\pi_k,a)\right)$. Clearly, the solution is

$$X_n=X_0\prod_{k=0}^{n-1}B_k\left(\Pi_k,Z_k\right) \tag{3.108}$$

for $n\geq 1$, and this simple structure can be exploited to analyze the terminal wealth X_T. In particular, given $X_0=x_0$

$$E[X_T]=x_0\left(1+E\left[\mathbb{C}_T\left(b_0\left(\Pi_0,Z_0\right)-1,b_1\left(\Pi_1,Z_1\right)-1,\cdots,b_{T-1}\left(\Pi_{T-1},Z_{T-1}\right)-1\right)\right]\right) \tag{3.109}$$

where $b_k(\pi_k, a) = r_f(i_k)(1 + a_k(\pi_k, a))$.

The log-return at the terminal time T is

$$\ln(X_T/X_0) = \sum_{k=0}^{T-1} \ln(B_k(\Pi_k, Z_k))$$

so that the mean is

$$E[\ln(X_T/X_0) | \Pi_0 = \pi] = \sum_{k=0}^{T-1} \pi^a Q_0 Q_1 \cdots Q_{k-1}(a,b) E[\ln(B_k(\pi_k, b))].$$

The simple structure of (3.108) can be exploited to determine various quantities of interest associated with the terminal wealth.

3.2.5 Power Utility

In this section, we assume that the utility of the investor in state i is given by the logarithmic function

$$U(i,x) = K(i) + C(i) \frac{(x+\beta)^\gamma}{\gamma} \tag{3.110}$$

with $C(i) > 0$ and $r_f(i) = r_f$ still holds. Note that Pratt-Arrow ratio can be calculated as $r(x) = (1-\gamma)/(x+\beta)$ for all i so that $b = 1/(1-\gamma)$ and $a = \beta/(\gamma-1)$ in Table 1.1. In this chapter, we assume that the utility function (3.110) is well-defined for all possible values of x. For example, if $(x-\beta) < 0$ is possible, then we exclude $\gamma = 1/2$ in our analysis. If we need to include these values of γ, we can define the utility function to be $-\infty$ whenever (3.110) is not well-defined and make appropriate assumptions on excess returns $\{R_n^e(a)\}$ as in Section 3.2.3. For $U(i,x)$ to be a legitimate utility function some additional restrictions may be imposed, but we do not dwell with such technical issues here. Note that γ and β is the same for all market states so that risk classification of the investor does not depend on the stochastic market.

We will first consider an optimization problem of the form

$$\max_u \sum_{a\in F} \mu_a E\left[\frac{\left(R^e(a)' u + c\right)^\gamma}{\gamma}\right] \tag{3.111}$$

where $R_n^e(a)$ is any random vector, and μ is any measure on F with $\mu_a \geq 0$ for any a. The gradient vector of the objection function $g(u)$ is given by

$$\nabla_k g(u) = \frac{\partial g(u)}{\partial u_k} = \sum_{a\in F} \mu_a E\left[R_k^e(a)\left(R^e(a)' u + c\right)^{\gamma-1}\right]$$

while the Hessian matrix is

$$\nabla^2_{k,l} g(u) = \frac{\partial^2 g(u)}{\partial u_k \partial u_l} = (\gamma - 1) \sum_{a \in F} \mu_a E\left[R^e_k(a) R^e_l(a) \left(R^e(a)' u + c\right)^{\gamma-2}\right]$$

for all k, l.

The first order optimality condition to find the optimal solution of (3.111) is obtained by setting the gradient vector equal to zero so that

$$\sum_{a \in F} \mu_a E\left[R^e_k(a) \left(R^e(a)' u + c\right)^{\gamma-1}\right] = 0 \tag{3.112}$$

for all $k = 1, 2, \cdots, m$. Let $z = (z_1, \cdots, z_m)$ be any non-zero column vector where z_i's are real numbers. Then, one can see that

$$z' \nabla^2 g(u) z = (\gamma - 1) E\left[\left(\sum_{k=1}^{m} z_k R^e_k(a)\right)^2 \left(R^e_n(a)' u + c\right)^{\gamma-2}\right]. \tag{3.113}$$

Throughout this chapter, we assume that the excess returns $\{R^e_n(a)\}$ and the parameters of the utility function are such that there is always an optimal solution of (3.111) that satisfies the first order conditions (3.112). Note that this requirement does not necessarily impose concavity restriction on the objective function. We only require that the optimal solution is at an interior point which satisfies the necessary conditions of optimality (3.112). Our purpose is to identify the structure of the optimal policy and we will not dwell will these technical details on optimization. This is of course an important issue and we do not intend to undermine its significance. We now consider some possible cases to illustrate how one can approach this technical problem. If $\gamma - 2$ is even, then the Hessian matrix $\nabla^2 g$ in (3.113) is negative semi-definite provided that $\gamma \leq 1$ and the optimal solution satisfies (3.112) since we have an unconstrained concave maximization problem. If $\gamma - 2$ is not even and $\gamma \leq 1$, then the objective function is concave over the set

$$A(c) = \left\{u : P\left\{\left(R^e_n(a)' u + c\right)^{\gamma-2} \geq 0\right\} = 1\right\} \tag{3.114}$$

and we need additional restrictions on the excess returns $\{R^e(i)\}$; like the existence of a solution of the first order condition (3.112) in $A(c)$ for all c. In case $\gamma \geq 1$, it suffices to reverse the inequality in (3.114).

Theorem 13 *Let the utility function of the investor be the power function (3.110) and suppose that the riskless asset return does not depend on the market state. Then the optimal solution of the dynamic programming equation (3.63) is*

$$v_n(\pi, x) = K_n(\pi) + C_n(\pi)\frac{(x+\beta_n)^\gamma}{\gamma}$$

and the optimal portfolio is

$$u_n^*(\pi, x) = \alpha_n(\pi)\left(r_f x + \beta_{n+1}\right)$$

where

$$\begin{aligned} \beta_n &= \frac{\beta}{r_f^{T-n}}, \ K_{n-1}(\pi) = \sum_{a\in F}\pi^a \sum_{j\in E}\Psi_{n-1}(a,j) K_n(\mathcal{T}_{n-1}(\pi,j)), \\ C_{n-1}(\pi) &= \sum_{a\in F}\pi^a \sum_{j\in E}\Psi_{n-1}(a,j) C_n(\mathcal{T}_{n-1}(\pi,j)) E\left[r_f\left(1+R_n^e(a)'\alpha(\pi_n, x)\right)^\gamma\right] \end{aligned}$$

and $\alpha_n(i)$ satisfies

$$\sum_{a\in F}\pi^a \Psi_{n-1} C_n(a) E\left[R_{n,k}^e(a)\left(1+R_n^e(a)'\alpha(\pi_n, x)\right)^{\gamma-1}\right] = 0$$

for all $n = 0, 1, \cdots, T-1$ with boundary conditions

$$K_{T-1}(\pi) = \sum_{a\in F}\pi^a \sum_{=j\in E}\Psi_{T-1}(a,j) K(j)$$

$$C_{T-1}(\pi) = \sum_{a\in F}\pi^a \sum_{j\in E}\Psi_{T-1}(a,j) C(j) E\left[r_f\left(1+R_{T-1}^e(a)'\alpha(\pi_n, x)\right)^\gamma\right]$$

for all i.

Proof. We will show that the recursion is true by induction starting with the boundary condition $v_T(i_T, x) = K(i_T) + C(i_T)(x+\beta)^\gamma/\gamma$. Note that

$$\begin{aligned} g_{T-1}(\pi, x, u) &= E\left[\sum_{a\in F}\pi^a \sum_{j\in E}\Psi_{T-1}(a,j) U(j, r_f x + R_{T-1}^e(a)' u)\right] \\ &= E\left[\sum_{a\in F}\pi^a \sum_{j\in E}\Psi_{T-1}(a,j)\left(K(j) + C(j)\left(R_{T-1}^e(a)' u + r_f x + \beta\right)^\gamma/\gamma\right)\right] \end{aligned}$$

Let $u^* = (u_1^*, u_2^*, \cdots, u_m^*)$ be the optimal amount of money that should be invested in the risky assets so that

$$v_{T-1}(\pi, x) = \max_u g_{T-1}(\pi, x, u) = g_{T-1}(\pi, x, u^*)$$

as in the perfect information case. The gradient of g_{T-1} is

$$\frac{\partial g_{T-1}}{\partial u_k} = \sum_{a\in F} \pi^a \sum_{j\in E} \Psi_{T-1}(a,j)\, C(j)\, E\left[R^e_{T-1,k}(a)\left(R^e_{T-1}(a)' u + r_f x + \beta\right)^{\gamma-1}\right]$$

and the entries of the Hessian matrix $H(\pi,x,u)$ as

$$\frac{\partial^2 g_{T-1}}{\partial u_k \partial u_l} = \sum_{a\in F} \pi^a \sum_{j\in E} \Psi_{T-1}(a,j)\, C(j)\, (\gamma-1) \tag{3.115}$$

$$\cdot E\left[R^e_{T-1,k}(a)\, R^e_{T-1,l}(a)\left(R^e_{T-1}(a)' u + r_f x + \beta\right)^{\gamma-2}\right]. \tag{3.116}$$

Let $z = (z_1, \cdots, z_m)$ be any non-zero column vector where z_i's are real numbers. Then, one can see that $z^T H(\pi,x,u)\, z$ is equal to

$$\sum_{a\in F} \pi^a \sum_{j\in E} \Psi_{T-1}(a,j)\, C(j)\, E\left[\left(\sum_{k=1}^{m} z_k R^e_{T-1,k}(i_{T-1},a)\right)^2 \left(R^e_{T-1}(a)' u + r_f x + \beta\right)^{\gamma-2}\right]. \tag{3.117}$$

In (3.117) $\pi^a, \Psi_n(a,j), C(j)$ are positive. Inside of the expectation is also positive. So the Hessian matrix is negative semidefinite because of the negative sign and we can find the optimal portfolio policy by taking the derivative of g_{T-1}. So if we set the gradient equal to zero,

$$\frac{\partial g_{T-1}}{\partial u_k} = \sum_{a\in F} \pi^a \sum_{j\in E} \Psi_{T-1}(a,j)\, C(j)\, E\left[R^e_{T-1,k}(a)\left(R^e_{T-1}(a)' u^* + r_f x + \beta\right)^{\gamma-1}\right] = 0$$

for any asset $k = 1, 2, \cdots, m$ where $\Psi_{T-1}C(a) = \sum_{j\in E} \Psi_n(a,j)\, C(j)$. Defining the vector function $\alpha(\pi_{T-1}, x) = (\alpha_1(\pi_{T-1}, x), \alpha_2(\pi_{T-1}, x), \cdots, \alpha_m(\pi_{T-1}, x))$ so that $\alpha(\pi_{T-1}, x) = u^*(\pi_{T-1}, x) / (r_f x + \beta)$ we obtain $u^*(a,x) = \alpha(a,x)(r_f x + \beta)$ so the optimality condition can be rewritten as

$$\sum_{a\in F} \pi^a \Psi_{T-1} C(a)\, E\left[R^e_{T-1,k}(a)\left(1 + R^e_{T-1}(a)' \alpha(\pi_{T-1}, x)\right)^{\gamma-1}\right] = 0. \tag{3.118}$$

for all $k = 1, 2, \cdots m$ are the optimality conditions. Since there is no dependence on x in (3.118), $\alpha(\pi,x)$ does not depend on x and $\alpha(\pi,x) = \alpha(\pi)$. When the value function at time $T-1$ is rewritten for the optimal policy, we obtain

$$\begin{aligned} v_{T-1}(\pi,x) &= \sum_{a\in F} \pi^a \Psi_{T-1} K(a) + \sum_{a\in F} \pi^a \Psi_{T-1} C(a) \left((r_f x + \beta)^\gamma E\left[\left(R^e_{T-1}(a)' \alpha(a,x) + 1\right)^\gamma\right] / \gamma\right) \\ &= K_{T-1}(\pi) + C_{T-1}(\pi)\left(x + \beta_{T-1}\right)^\gamma / \gamma. \end{aligned}$$

and the value function is still exponential like the utility function. This completes the proof for $n = T - 1$.

Suppose now that the induction hypothesis holds for periods $T, T-1, T-2, \cdots, n$. Then, for period $n-1$,

$$\begin{aligned} g_{n-1}(\pi, x, u) &= E\left[\sum_{a\in F} \pi^a \sum_{j\in E} \Psi_n(a,j)\, v_n(\mathcal{T}_{n-1}(\pi, j), r_f x + R^e_{n-1}(a)' u)\right] \\ &= E\left[\begin{array}{c} \sum_{a\in F} \pi^a \sum_{j\in E} \Psi_n(a,j)\, K_n(\mathcal{T}_{n-1}(\pi, j)) \\ +C_n(\mathcal{T}_{n-1}(\pi,j)) \left(R^e_{n-1}(a)' u + r_f x + \beta_n\right)^\gamma / \gamma \end{array}\right] \end{aligned}$$

One can easily see that the Hessian matrix of $g_{n-1}(\pi, x, u)$ is negative semi-definite as for $g_{T-1}(\pi, x, u)$. Let $u^*_{n-1}(\bar{\imath}_{n-1}, x)$ be the optimal policy such that

$$v_{n-1}(\pi, x) = \max_u g_{n-1}(\pi, x, u) = g_{n-1}(\pi, x, u^*).$$

If we take the gradient of $g_{n-1}(\pi, x, u)$ with respect to u and set it equal to 0, we get the optimality condition

$$\begin{aligned} \frac{\partial g_{n-1}}{\partial u_k} = \sum_{a\in F} \pi^a \sum_{j\in E} \Psi_{n-1}(a,j)\, C_n(\mathcal{T}_{n-1}(\pi, j)) \\ \cdot E\left[R^e_{T-1,k}(a) \left(R^e_{T-1}(a)' u^* + r_f x + \beta\right)^{\gamma-1}\right] = 0 \end{aligned}$$

and

$$\sum_{a\in F} \pi^a \Psi_{n-1} C(a)\, E\left[R^e_{T-1,k}(a) \left(R^e_{T-1}(a)' u^* + r_f x + \beta\right)^{\gamma-1}\right] = 0$$

for all $k = 1, 2, \cdots m$ are the optimality conditions. Defining the vector function $\alpha\left(\pi_{n-1}, x\right)$ such that $\alpha\left(\pi_{n-1}, x\right) = u^*\left(\pi_{n-1}, x\right) / (r_f x + \beta)$ we obtain $u^*_{n-1}(a, x) = \alpha(a, x)(r_f x + \beta)$ so the optimality condition can be rewritten as

$$\sum_{a\in F} \pi^a \Psi_{n-1} C(a)\, E\left[R^e_{T-1,k}(a) \left(1 + R^e_{T-1}(a)' \alpha\left(\pi_{n-1}, x\right)\right)^{\gamma-1}\right] = 0. \quad (3.119)$$

for all $k = 1, 2, \cdots m$ are the optimality conditions. Since there is no dependence on x and i in (3.119), α does not depend on x and i and $u^*_{n-1} = \alpha_{n-1}(\pi)(r_f x + \beta)$. When the value function at time $n-1$ is rewritten for the optimal policy, we obtain

$$\begin{aligned} v_{n-1}(\pi, i, x) &= \sum_{a \in F} \pi^a \sum_{j \in E} \Psi_{n-1} K_n(a) \\ &\quad + \sum_{a \in F} \pi^a \sum_{j \in E} \Psi_{n-1} C(a) (r_f x + \beta_n)^{\gamma} \\ &\qquad \left(E\left[\left(1 + R^e_{T-1}(a)' \alpha\left(\pi_{n-1}, x\right)\right)^{\gamma} \right] / \gamma \right) \\ &= K_{n-1}(\pi) + C_{n-1}(\pi). \end{aligned}$$

and this completes the proof. ■

Note that the wealth dynamics equation for the power utility case is the same as the wealth dynamics equation (3.91) for the logarithmic utility case although the structure of the optimal policy in (3.82) and (3.111) are different. Therefore, using a similar analysis as in Section 3.2.3 we can easily determine the evolution of the wealth process. Likewise, similar interpretations can be made on the structure of the optimal policy. In particular, the optimal policy is not myopic, but the risky composition of the portfolio is myopic. Moreover, this composition only depends on the state of the market. Although we obtain similar characterizations and interpretations, note that the optimal policies for logarithmic and power cases are not identical.

Chapter 4

CONTINUOUS-TIME PORTFOLIO OPTIMIZATION

Beginning with the fundamental work by Merton [47], a number of very sophisticated stochastic control models have been proposed for making optimal investment decisions. A typical approach takes diffusion process models of securities and looks for the trading strategy which maximizes the expected utility of consumption and/or terminal wealth over a finite planning horizon. The optimal strategy is obtained by solving the dynamic programming equation, which is a PDE that is also called the Hamilton-Jacobi-Bellman equation in stochastic control theory. There have been many research papers related to Merton's classical portfolio optimization problem. Bielecki and Pliska [7], and Fleming and Sheu [29] considered the cases in which there is no consumptions and the goal is to maximize the long term growth rate of the utility based on the wealth. There are also some models involving stochastic volatility instead of constant volatility, such as Fleming and Hernandez-Hernandez [28] and Zariphopoulou [73]. Bäuerle and Rieder [3] uses continuous-time Markov chains with a discrete state space as a market driving process in their research. Recently, Detemple and Rindisbacher [19] discussed the optimal portfolio selection problem with stochastic interest rate and investment constraints. The problem they considered is also defined on a finite time horizon and the utility function is the power utility function based on the terminal wealth. Sotomayor and Cadenillas [64] examined the consumption-investment problem in a regime switching market where they assumed the utility function, as well as the market parameters, depend on the regime of the market. Honda [35] studied an optimal consumption problem in a setting where the mean asset returns depend on a regime variable which is observed by the investor using past and current market prices. Sass and Haussmann [**?**] solved the utility of terminal wealth optimization problem in continuous-time setting numerically. They considered a case where the investor observes the stock prices rather than the market regime which depends on the continuous-time Markov chain. Nagai and Runggaldier [52] also studied utility maximization in regime switching models.

In this chapter, we will analyze optimal portfolio strategies for HARA utility cases which basically includes exponential, logarithmic and power functions. This will also extend the work done by Bäuerle and Rieder [3] to the multiasset case.

4.1 The Stochastic Market

The returns of risky assets in a financial market are random and there are various underlying economic, financial, social, political and other factors that affect their distributions in one way or another. As the state of a market changes over time, the returns will change accordingly. It is fair to say that in today's financial markets most of the risks, or variances of asset returns, are due to the changes in local or global factors. Investment decisions are affected by these factors as well as the correlation among asset returns. The previous studies on stochastic processes effecting the variables of financial market are usually under the concept of regime switching in financial markets.

Modeling a stochastic financial market by a Markov chain is a reasonable approach and this idea dates back to Pye [57]. Hamilton [33] is one of the first papers that suggested use of regime switching for explaining business cycles. He suggests that the state of the business can be described by a state variable which can be parameterized as a first-order Markov process. Gray [30] uses the regime switching approach for short term interest rates. He first examines different models with both single regime and multiple regimes and proposes a new model called generalized regime switching. It is concluded in the paper that generalized regime switching model outperforms simple single-regime models in an out-of-sample forecasting experiment. In a more recent work, Costa and Araujo [15] worked on the continuous-time generalized mean-variance optimization in the case of a Markov modulated market. They have defined the problem as optimization of a utility function of mean and variance of the terminal wealth. They derived both the necessary and sufficient conditions for the optimal solution.

We consider a financial market with one riskless asset (bond) and m risky assets. Let $(\Omega, F, \mathcal{F}, P)$ be a filtered probability space with filtration $\mathcal{F} = \{\mathcal{F}_t; 0 \leq t \leq T\}$ and $F = \mathcal{F}_T$. The bond process $B = \{B(t); t \geq 0\}$ evolves according to

$$dB(t) = r_f(Y_t, t) B(t) dt$$

where $r_f(i,t)$ is the risk free interest rate at time t given that the state of the market is i. The stock price process for the kth asset $S_k = \{S_k(t); t \geq 0\}$ evolves according to

$$dS_k(t) = \mu_k(Y_t, t) S_k(t) dt + S_k(t) \sum_{j=1}^{d} \sigma_{kj}(Y_t, t) dW_j(t)$$

where $\{W_1, W_2, \cdots, W_m\}$ are independent Wiener processes. Here for the remaining of the thesis we will suppose that the market is complete and $d = m$. Further discussion about the complete markets can be found in Karatzas and Shreve [38].

Here, we suppose that $Y = \{Y_t; t \geq 0\}$ is a Markov process with a finite state space E and infinitesimal generator, or transition rate matrix,

$$A(i,j) = \begin{cases} -\lambda(i) & j = i \\ \lambda(i) Q(i,j) & j \neq i \end{cases}.$$

Moreover, Y is assumed to be independent of the Wiener processes $\{W_1, W_2, \cdots, W_m\}$. The process Y represents the stochastic evolution of the financial, economic, and other factors that affect the prices of all assets in a market. Define the filtration $\mathcal{Y} = \sigma(Y)$ as the minimal σ-algebra generated by Y. We further suppose that the filtration $\mathcal{F} = \sigma(W_1, W_2, \cdots, W_m, Y)$ is the minimal σ-algebra generated by $(W_1, W_2, \cdots, W_m, Y)$. Note that $\mathcal{Y} \subseteq \mathcal{F}$.

Denote the times of the jumps of the Markov processes Y as $0 = T_0 < T_1 < T_2 < \cdots$, and it is well-known that

$$P\{T_{n+1} - T_n \leq t | Y_{T_n} = i\} = 1 - e^{-\lambda(i)t}$$

so that the amount of time spent in state i is exponentially distributed with rate $\lambda(i)$. The sequence of states $\{Y_{T_n}\}$ visited by Markov process Y is a Markov chain with transition matrix Q. Define $\mathbb{R} = [-\infty, \infty)$ to denote the set of real numbers, and $\mathbb{R}_+ = (0, \infty)$ to denote the set of positive real numbers.

Let $\bar{r}_f(s,t)$ denote the total compound factor from time s to time t defined as

$$\bar{r}_f(s,t) = e^{\int_s^t r_f(Y_z, z) dz}$$

for $s \leq t$. Note that $\bar{r}_f(s,t)$ is random since r_f depends on Y. However, if $r_f(i,t) = r_f(t)$ independent of i, then

$$\bar{r}_f(s,t) = e^{\int_s^t r_f(z) dz}$$

is a deterministic quantity. Furthermore, if the interest rate is constant $r_f(i,t) = r_f$, then $\bar{r}_f(s,t) = e^{r_f(t-s)}$ trivially.

We assume that all of the market functions μ, σ, and r_f are bounded. We also define

$$V(i,t) = \sigma'(i,t)\,\sigma(i,t)$$

and

$$\mu^e(i,t) = \mu(i,t) - r_f(i,t)$$

for all $i \in E$, and $0 \le t \le T$. We further suppose without loss of generality that σ is positive definite. It also follows that V is also positive definite, the inverses σ^{-1} and V^{-1} both exist and they are also positive definite.

In this chapter, we let class $\mathcal{H}^2$ denote the set of all measurable adapted functions or random variables $f : \Omega \times [0,T] \to \mathbb{R}$ that satisfy the integrability constraint

$$E\left[\int_0^T f^2(t)\,dt\right] = E\left[\int_\Omega P(d\omega)\int_0^T f^2(\omega,t)\,dt\right] < \infty.$$

Note that while defining a portfolio policy, it is possible to either decide on the ratios of current wealth π that will be invested in each asset or the amount of money u that will be invested in each asset. If x is the wealth level, the dependence between the two can be written as $u = \pi x$ provided that $x \neq 0$. However, for a given policy u, the π values become bigger and bigger as x approaches to zero. Therefore, we have constructed the optimality conditions and wealth dynamics equations for both π and u separately in the following sections.

4.2 Dynamic Programming Formulation I

Our objective is to find the policy that maximizes the expected utility of the terminal wealth at time T. A portfolio management policy is denoted by $\pi = \{\pi(t) = (\pi_1(t), \pi_2(t), \ldots, \pi_m(t)); 0 \le t \le T\}$ where $\pi_k(t)$ is the ratio of current wealth invested on asset k at time t. For any admissible policy π, we let $X^\pi = \{X_t^\pi, 0 \le t \le T\}$ denote the corresponding wealth process.

Definition 14 *A portfolio management policy π is called admissible if*

i. π takes values in a given measurable subset of $\mathbb{R}^m$, *ii.* π *is measurable and adapted to* $\mathcal{F}$,

iii. $\int_0^t \pi(s)' V(Y_s, s)\pi(s)\,ds < \infty$, *and* $\int_0^t \pi(s)' \mu^e(Y_s, s)\,ds < \infty$ *almost surely for any* $0 \le t \le T$.

iv. $P\{X_T^\pi > X_0^\pi \bar{r}_f(0,T)\} < 1$.

Note that condition (iv) of Definition 14 implies that there is no free lunch or arbitrage opportunity. In other words, the return X_T^π / X_0^π during $[0,T]$ cannot exceed the risk free return $\bar{r}_f(0,T)$ with certainty. It is clear that this condition excludes the optimality of so called doubling strategies which yield terminal wealth levels that exceed any desired value almost surely. It is quite common to put additional restrictions on admissibility policies that create arbitrage opportunities. This may be achieved by putting bounds on policies, line of credit, or lower bounds on wealth levels. Dybvig and Huang [22] show that a lower bound on wealth precludes arbitrage opportunities. Kreps [40], on the other hand, uses bounds on the portfolio positions while Delbaen and Schachermayer [17] imposes bounds on line of credit. We prefer condition (iv) of Definition 14 since it is quite intuitive, practical, and easy to verify for the HARA class of utility functions that we consider in this chapter. More details about the doubling strategies can be found in Karatzas and Shreve [38, Chap. 1].

Let the set $\mathcal{A}$ denote the set of all admissible policies which satisfy Definition 14. For a self financing policy $\pi \in \mathcal{A}$, the wealth process satisfies the wealth dynamics equation

$$
\begin{aligned}
dX_t^\pi &= X_t^\pi \sum_{k=1}^{m} \pi_k(t) \frac{dS_k(t)}{S_k(t)} + X_t^\pi \left(1 - \sum_{k=1}^{m} \pi_k(t)\right) \frac{dB(t)}{B(t)} \\
&= X_t^\pi \left[r_f(Y_t, t) + \sum_{k=1}^{m} \pi_k(t)\left(\mu_k(Y_t, t) - r_f(Y_t, t)\right)\right] dt \\
&\quad + X_t^\pi \sum_{k=1}^{m} \sum_{j=1}^{m} \pi_k(t)\,\sigma_{kj}(Y_t, t)\,dW_j(t) \\
&= X_t^\pi \left[r_f(Y_t, t) + \pi(t)' \mu^e(Y_t, t)\right] dt + X_t^\pi \pi(t)' \sigma(Y_t, t)\,dW(t) \qquad (4.1)
\end{aligned}
$$

with $X_0^\pi = x > 0$ being the initial wealth.

For any admissible policy π, the stochastic differential equation (4.1) has a unique solution. Using Itô calculus, we can solve the stochastic differential equation (4.1) as

$$\begin{aligned} X_t^\pi &= X_0^\pi \exp\left(\int_0^t \left(r_f\left(Y_s,s\right)+\pi\left(s\right)'\mu^e\left(Y_s,s\right)-\frac{1}{2}\pi\left(s\right)'V\left(Y_s,s\right)\pi\left(s\right)\right)ds \right. \\ &\quad \left. + \int_0^t \pi\left(s\right)'\sigma\left(Y_s,s\right)dW\left(s\right)\right). \end{aligned} \tag{4.2}$$

Note from (4.2), that X_t^π and X_0^π will have the same sign, which implies that if the starting wealth of the investor is positive, then $X_t^\pi > 0$ for all $0 \le t \le T$. Using π as the portfolio policy therefore implies that $X_t^\pi > 0$ for all $0 \le t \le T$. One example where this is handy is the case where the investor has simple logarithmic or power utility. The detailed analysis is given in Section 4.5.

Our optimization problem is to find a policy π^* over all admissible policies such that

$$E\left[U\left(Y_{T,}X_T^{\pi^*}\right)|Y_0=i,X_0=x\right]=\sup_{u\in\mathcal{A}}E\left[U\left(Y_{T,}X_T^{\pi}\right)|Y_0=i,X_0^\pi=x\right] \tag{4.3}$$

where $U\left(i,x\right)$ is the utility function.

To keep track of time properly, let $Z=\{Z_t;t\ge 0\}$ be the process defined trivially by $Z_t=Z_0+t$. Then (4.1) becomes

$$dX_t^\pi = X_t^\pi\left[r_f\left(Y_t,Z_t\right)+\pi\left(Z_t\right)'\mu^e\left(Y_t,Z_t\right)\right]dt+X_t^\pi\pi\left(Z_t\right)'\sigma\left(Y_t,Z_t\right)dW\left(t\right)$$

for any given policy $\pi\in\mathcal{A}$. Throughout this section, we will set $X=X^\pi$ for the given policy π to simplify the notation. We will also let

$$E_{i,x,t}\left[R\right]=E\left[R\,|Y_t=i,X_t=x\right]$$

and

$$\hat{E}_{i,x,t}\left[R\right]=E\left[R\,|Y_0=i,X_0=x,Z_0=t\right]$$

denote the conditional expectations of any random variable R given the stated conditions.

We let $C_p^{2,1}$ denote the set of all continuous functions $f\left(i,x,t\right):E\times\mathbb{R}_+\times[0,T]\to\mathbb{R}$ with polynomial growth in x that are continuously differentiable in t and twice continuously differentiable in x with a finite first order derivative with respect to x. Therefore, for $f\in C^{2,1}$, we let $f_t\left(i,x,t\right)=\partial f\left(i,x,t\right)/\partial t$, $f_x\left(i,x,t\right)=\partial f\left(i,x,t\right)/\partial x$, and $f_{xx}\left(i,x,t\right)=\partial^2 f\left(i,x,t\right)/\partial x^2$ denote partial derivatives, and we further impose the finiteness condition $|f_x\left(i,x,t\right)|<\infty$ for any $i\in E$, $x\in\mathbb{R}_+$ and $0\le t\le T$.

The generator for the trivariate Markov process (Y, X, Z) is defined as

$$Gf(i,x,t) = \lim_{s\downarrow 0}\left(\frac{\hat{E}_{i,x,t}\left[f(Y_s, X_s, Z_s) - f(i,x,t)\right]}{s}\right) \tag{4.4}$$

for any $f \in C_p^{2,1}$.

Note that for any policy $\pi \in \mathcal{A}$, $\pi(t) \in \mathcal{F}_t$ implies that $\pi(t) = \pi(i,x,t)$ on $\{Y_t = i, X_t = x, Z_t = t\} \in \mathcal{F}_t$ since (Y, X, Z) satisfies the Markov property. Therefore, on the set $\{Y_t = i, X_t = x, Z_t = t\}$, the policy $\pi(t) = \pi(i,x,t)$ is in fact not a random variable, but a function of i, x, and t. The reader should keep this in mind throughout the remainder of this chapter. To identify the generator G, if we set

$$h(i,x,t;s) = \hat{E}_{i,x,t}\left[f(Y_s, X_s, Z_s)\right]$$

then, it satisfies

$$\begin{aligned} h(i,x,t;s) &= e^{-\lambda(i)s}\hat{E}_{i,x,t}\left[f(i, X_s, Z_s)\right] \\ &\quad + \int_0^s \lambda(i)\, e^{-\lambda(i)u} du \sum_{j\in E} Q(i,j)\, \hat{E}_{i,x,t}\left[h(j, X_u, Z_u; s-u)\right] \\ &= e^{-\lambda(i)s}\hat{E}_{i,x,t}\left[f(i, X_s, Z_s)\right] \qquad (4.5) \\ &\quad + e^{-\lambda(i)s}\int_0^s \lambda(i)\, e^{-\lambda(i)(u-s)} du \sum_{j\in E} Q(i,j)\, \hat{E}_{i,x,t}\left[h(j, X_u, Z_u; s-u)\right] \end{aligned}$$

by conditioning on the time of the first jump of Y. Note that X satisfies the stochastic differential equation (4.1) with $Y_t = i$ for $t \le T_1 \wedge T$. From the definition of the generator $Gf(i,x,t)$ in (4.4), we see that both the denominator and numerator go to zero as s approaches zero. If we use Itô's formula, for $f \in C^{2,1}$, we obtain

$$\begin{aligned} f(i, X_s, Z_s) &= f(i, X_0, Z_0) + \int_0^s f_t(i, X_v, Z_v)\, dv + \int_0^s f_x(i, X_v, Z_v)\, dX_v \\ &\quad + \frac{1}{2}\int_0^s f_{xx}(i, X_v, Z_v)\,(dX_v)^2 \\ &= f(i, X_0, Z_0) + \int_0^s f_x(i, X_v, Z_v)\, X_v \pi(Z_v)' \sigma(i, Z_v)\, dW(v) \\ &\quad + \int_0^s \left[f_t(i, X_v, Z_v) + f_x(i, X_v, Z_v)\, X_v \left(r_f(i, Z_v) + \pi(Z_v)' \mu^e(i, Z_v)\right)\right] dv \\ &\quad + \int_0^s \frac{1}{2} f_{xx}(i, X_v, Z_v)\, X_v^2 \pi(Z_v)' V(i, Z_v)\, \pi(Z_v)\, dv. \end{aligned} \tag{4.6}$$

In applying Itô's formula, note from (4.1) that $(dX_v)^2 = X_v^2 \pi(Z_v)' V(i, Z_v) \pi(Z_v) dv$. In (4.6), the stochastic integral is a martingale using $f \in C_p^{2,1}$ and Assumption 14. Therefore,

$$\begin{aligned}\hat{E}_{i,x,t}[f(i, X_s, Z_s)] &= \hat{E}_{i,x,t}[f(i, X_0, Z_0)] + \hat{E}_{i,x,t}\left[\int_0^s (f_t(i, X_v, Z_v)\right.\\ &\quad + f_x(i, X_v, Z_v) X_v \left(r_f(i, Z_v) + \pi(Z_v)' \mu^e(i, Z_v)\right)) dv\\ &\quad \left. + \frac{1}{2}\int_0^s f_{xx}(i, X_v, Z_v) X_v^2 \pi(Z_v)' V(i, Z_v) \pi(Z_v) dv\right]\end{aligned}$$

Now, it follows that

$$\begin{aligned}\lim_{s\downarrow 0} \frac{d\hat{E}_{i,x,t}[f(i, X_s, Z_s)]}{ds} &= \lim_{s\downarrow 0} \hat{E}_{i,x,t}[f_t(i, X_s, Z_s)\\ &\quad + f_x(i, X_s, Z_s) X_s \left(r_f(i, Z_s) + \pi(Z_s)' \mu^e(i, Z_s)\right)\\ &\quad \left. + \frac{1}{2} f_{xx}(i, X_s, Z_s) X_s^2 \pi(Z_s)' V(i, Z_s) \pi(Z_s)\right]\\ &= f_t(i, x, t) + f_x(i, x, t) x \left(r_f(i, t) + \pi(t)' \mu^e(i, t)\right)\\ &\quad + \frac{1}{2} f_{xx}(i, x, t) x^2 \pi(t)' V(i, t) \pi(t). \end{aligned} \tag{4.7}$$

If we differentiate (4.5) and evaluate it at $s = 0$,

$$\begin{aligned}\lim_{s\downarrow 0} \frac{dh(i, x, t; s)}{ds} &= -\lambda(i) f(i, x, t) + \lim_{s\downarrow 0} \frac{d\hat{E}_{i,x,t}[f(i, X_s, Z_s)]}{ds}\\ &\quad + \lambda(i) \sum_{j\in E} Q(i, j) \hat{E}_{i,x,t}[f(j, x, t)]\\ &= \lim_{s\downarrow 0} \frac{d\hat{E}_{i,x,t}[f(i, X_s, Z_s)]}{ds} + \sum_{j\in E} A(i, j) f(j, x, t)\end{aligned}$$

and, using (4.7), we get

$$\begin{aligned}\lim_{s\downarrow 0} \frac{dh(i, x, t; s)}{ds} &= f_t(i, x, t) + f_x(i, x, t) x \left(r_f(i, t) + \pi(t)' \mu^e(i, t)\right)\\ &\quad + \frac{1}{2} f_{xx}(i, x, t) x^2 \pi(t)' V(i, t) \pi(t) + \sum_{j\in E} A(i, j) f(j, x, t).\end{aligned}$$

In (4.4), both the denominator and numerator are zero at $s = 0$, so if we apply L'Hopital's rule, the denominator is equal to one and taking the derivative of the numerator with respect to s, and taking the limit as s goes to 0, we obtain

$$\begin{aligned}Gf(i, x, t) &= \lim_{s\downarrow 0} \frac{dh(i, x, t; s)}{ds}\\ &= f_t(i, x, t) + f_x(i, x, t) x \left(r_f(i, t) + \pi(t)' \mu^e(i, t)\right)\\ &\quad + \frac{1}{2} f_{xx}(i, x, t) x^2 \pi(t)' V(i, t) \pi(t) + \sum_{j\in E} A(i,j) f(j, x, t).\end{aligned}$$

For a given policy $\pi \in \mathcal{A}$, we let

$$v^{\pi}(i,x,t) = E_{i,x,t}\left[U\left(Y_T, X_T^{\pi}\right)\right]$$

where U is the utility function and the optimization problem becomes

$$v(i,x,t) = \sup_{u \in \mathcal{A}} v^{\pi}(i,x,t).$$

Note that we can also write

$$v^{\pi}(i,x,t) = E\left[v^{\pi}\left(Y_{t+h}, X_{t+h}^{\pi}, Z_{t+h}\right) | Y_t = i, X_t^{\pi} = x, Z_t = t\right]$$

for all $t + h < T$. Rearranging the terms and dividing by h, we obtain

$$\frac{1}{h} E\left[v^{\pi}\left(Y_{t+h}, X_{t+h}, Z_{t+h}\right) - v^{\pi}(i,x,t) | Y_t = i, X_t^{\pi} = x, Z_t = t\right] = 0.$$

Taking the limit as h goes down to zero, we obtain

$$Gv^{\pi}(i,x,t) = 0$$

which implies that v^{π} can now be characterized as the solution of the linear system of second order partial differential equations

$$\begin{aligned} &v_t^{\pi}(i,x,t) + v_x^{\pi}(i,x,t)\, r_f(i,t)\, x + \pi(t)' \mu^e(i,t)\, x \\ &\quad + \frac{1}{2} v_{xx}^{\pi}(i,x,t)\, \pi(t)' V(i,t)\, \pi(t)\, x^2 + \sum_{j \in E} A(i,j)\, v^{\pi}(j,x,t) = 0 \end{aligned}$$

for $0 \leq t < T$ with the boundary condition $v^{\pi}(i,x,T) = U(i,x)$.

The optimal portfolio selection problem can therefore be formulated by the dynamic programming equation

$$\begin{aligned} v_t(i,x,t) + \sup_{\pi \in \mathbb{R}^m} &\left\{ v_x(i,x,t)\, x \left(r_f(i,t) + \pi' \mu^e(i,t)\right) \right. \\ &\left. + \frac{1}{2} v_{xx}(i,x,t)\, x^2 \pi' V(i,t)\, \pi + \sum_{j \in E} A(i,j)\, v(j,x,t) \right\} = 0. \end{aligned} \tag{4.8}$$

for $0 \leq t < T$ with the boundary condition $v(i,x,t) = U(i,x)$.

4.2.1 Verification Theorem

The optimality condition of the dynamic programming equation (4.8) is not sufficient if a verification theorem is not proven. The verification theorem connects the dynamic programming equation (4.8) to the control problem of maximizing the expected value of the terminal wealth defined in (4.3).

Theorem 15 (Verification) *Suppose $g \in C_p^{2,1}$ is a solution of (4.8), then*

a) $g(i,x,t) \geq v(i,x,t)$ for all $i \in E$, $x \in \mathbb{R}_+$, and $0 \leq t \leq T$,

b) If $\pi^* \in \mathcal{A}$ is a maximizer of (4.8), then $g(i,x,t) = v(i,x,t) = v^{\pi^*}(i,x,t)$ for all $i \in E$, $x \in \mathbb{R}_+$, and $0 \leq t \leq T$. In particular, π^* is an optimal portfolio policy.

Proof. Let $\pi \in \mathcal{A}$ be a policy and X^π be the corresponding wealth process. Using Itô's formula for local martingales with discrete parts, we can write $g(Y_T, X_T^\pi, T)$ as

$$\begin{aligned} g(Y_T, X_T^\pi, T) &= g(Y_t, X_t^\pi, t) + \int_t^T [g_t(Y_s, X_s^\pi, s) + g_x(Y_s, X_s^\pi, s) X_s^\pi (r_f(Y_s, s) \\ &\quad + \pi(s)' \mu^e(Y_s, s)\big) + \frac{1}{2} g_{xx}(Y_s, X_s^\pi, s)(X_s^\pi)^2 \left(\pi(s)' V(Y_s, s)\pi(s)\right)\Big] ds \\ &\quad + \int_t^T g_x(Y_s, X_s^\pi, s)\pi(s)'\sigma(Y_s, s) dW(s) \\ &\quad + \sum_{t \leq T_n < T} \left[g\left(Y_{T_n}, X_{T_n}^\pi, T_n\right) - g\left(Y_{T_n-}, X_{T_n-}^\pi, T_n-\right)\right]. \end{aligned} \tag{4.9}$$

Since g satisfies (4.8), we can write

$$\begin{aligned} &g_t(i,x,t) + g_x(i,x,t)\, x\,(r_f(i,t) + \pi'\mu^e(i,t)) \\ &\quad + \tfrac{1}{2} g_{xx}(i,x,t)\, x^2 \pi' V(i,t)\pi + \textstyle\sum_{j \in E} A(i,j)\, g(j,x,t) \leq 0 \end{aligned} \tag{4.10}$$

for all $i \in E$, $x \in \mathbb{R}$, $0 \leq t \leq T$, and $\pi \in \mathbb{R}^m$. This implies that we can also write

$$\begin{aligned} &g_t(Y_s, X_s^\pi, s) + g_x(Y_s, X_s^u, s) X_s^\pi \left(r_f(Y_s, s) + \pi(s)'\mu^e(Y_s, s)\right) \\ &\quad + \tfrac{1}{2} g_{xx}(Y_s, X_s^u, s)(X_s^\pi)^2 \pi(s)' V(Y_s, s)\pi(s) + \textstyle\sum_{j \in E} A(Y_s, j)\, g(j, X_s^\pi, s) \leq 0 \end{aligned} \tag{4.11}$$

for all $s \in [0,T]$. Therefore, the integral of the left-hand side of (4.11) from t to T is nonpositive. If we subtract this integral from the right-hand side of (4.9), we get

$$\begin{aligned} g(Y_T, X_T, T) &\leq g(Y_t, X_t^\pi, t) + \int_t^T g_x(Y_s, X_s^\pi, s) X_s^\pi \pi(s)'\sigma(Y_s, s) dW(s) \\ &\quad + \sum_{t \leq T_n < T} \left[g\left(Y_{T_n}, X_{T_n}^\pi, T_n\right) - g\left(Y_{T_n-}, X_{T_n-}^\pi, T_n-\right)\right] \\ &\quad - \int_t^T \sum_{j \in E} A(Y_s, j)\, g(j, X_s^\pi, s)\, ds \end{aligned}$$

and taking the conditional expectation given $\{Y_t = i, X_t^\pi = x\} \in \mathcal{F}_t$ we obtain

$$\begin{aligned} E_{i,x,t}\left[g\left(Y_T, X_T^\pi, T\right)\right] \leq\; & g\left(i,x,t\right) + E_{i,x,t}\left[\int_t^T g_x\left(Y_s, X_s^\pi, s\right) X_s^\pi \pi\left(s\right)' \sigma\left(Y_s, s\right) dW\left(s\right)\right] \\ & + E_{i,x,t}\left[\sum_{t \leq T_n < T}\left[g\left(Y_{T_n}, X_{T_n}^\pi, T_n\right) - g\left(Y_{T_n-}, X_{T_n-}^\pi, T_n-\right)\right]\right. \\ & \left. - \int_t^T \sum_{j \in E} A\left(Y_s, j\right) g\left(j, X_s^\pi, s\right) ds\right] \end{aligned}$$

where $E_{i,x,t}\left[g\left(Y_T, X_T^\pi, T\right)\right] = E_{i,x,t}\left[U\left(Y_T, X_T^\pi\right)\right]$. Since $g \in C_p^{2,1}$ and $\pi \in \mathcal{A}$, we have $g_x\left(Y_s, X_s^\pi, s\right) X_s^\pi \pi\left(s\right)' \sigma\left(s, Y_s\right) \in \mathcal{H}^2$. Therefore, the first integral is a martingale and

$$E_{i,x,t}\left[\int_t^T g_x\left(Y_s, X_s^\pi, s\right) X_s^\pi \pi\left(s\right)' \sigma\left(s, Y_s\right) dW\left(s\right)\right] = 0.$$

Due to the growth condition

$$E_{i,x,t}\left[\sum_{t \leq T_n < T}\left|g\left(Y_{T_n}, X_{T_n}^\pi, T_n\right)\right|\right] < \infty$$

and the last part is equal to zero since it is also a martingale according to Theorem 26.12 in Davis [16, Chap. 2]. So, we see that

$$g\left(i,x,t\right) \geq E_{i,x,t}\left[g\left(Y_T, X_T^\pi, T\right)\right] = E_{i,x,t}\left[U\left(Y_T, X_T^\pi\right)\right] = v^\pi\left(i,x,t\right) \tag{4.12}$$

for all $\pi \in \mathcal{A}$ which further implies that

$$g\left(i,x,t\right) \geq \sup_{\pi \in \mathcal{A}} v^\pi\left(i,x,t\right) = v\left(i,x,t\right). \tag{4.13}$$

If $\pi^* \in \mathcal{A}$ is a maximizer of (4.8), then it satisfies

$$\begin{aligned} & g_t\left(i,x,t\right) + g_x\left(i,x,t\right) x\left(r_f\left(i,t\right) + \pi^*\left(t\right)' \mu^e\left(i,t\right)\right) \\ & \quad + \frac{1}{2} g_{xx}\left(i,x,t\right) x^2 \pi^*\left(t\right)' V\left(i,t\right) \pi^*\left(t\right) + \sum_{j \in E} A\left(i,j\right) g\left(j,x,t\right) = 0 \end{aligned}$$

and (4.13) is satisfied with equality so that

$$g\left(i,x,t\right) = v\left(i,x,t\right) = v^{\pi^*}\left(i,x,t\right) = E_{i,x,t}\left[U\left(Y_T, X_T^{\pi^*}\right)\right].$$

■

4.3 Dynamic Programming Formulation II

In this section, a portfolio management policy is denoted by $u = \{u(t) = (u_1(t), \ldots, u_m(t)); 0 \leq t \leq T\}$ where $u_k(t)$ is the amount of wealth invested on asset k at time t. For any admissible policy u, we let $X^u = \{X_t^u; 0 \leq t \leq T\}$ denote the corresponding wealth process.

Definition 16 *A portfolio management policy u is called admissible if*

i. u takes values in a given measurable subset of $\mathbb{R}^m$,

ii. u is measurable and adapted to $\mathcal{F}$,

iii. $\int_0^t u(s)' V(Y_s, s) u(s) ds < \infty$, and $\int_0^t u(s)' \mu^e(Y_s, s) < \infty$ almost surely for any $0 \leq t \leq T$.

iv. $P\{X_T^u > X_0^u \bar{r}_f(0, T)\} < 1$.

Condition (iv) of Definition 16 is analogous to the condition (iv) of Definition 14 and it implies that it is not possible to exceed the risk free return with certainty. Let the set $\mathcal{A}$ denote the set of all admissible policies. For a self financing policy $u \in \mathcal{A}$, the wealth process satisfies the wealth dynamics equation

$$\begin{aligned} dX_t^u &= \sum_{k=1}^m u_k(t) \frac{dS_k(t)}{S_k(t)} + \left(X_t^u - \sum_{k=1}^m u_k(t) \right) \frac{dB(t)}{B(t)} \\ &= \left[X_t^u r_f(Y_t, t) + \sum_{k=1}^m u_k(t) \left(\mu_k(Y_t, t) - r_f(Y_t, t) \right) \right] dt \\ &\quad + \sum_{k=1}^m \sum_{j=1}^m u_k(t) \sigma_{kj}(Y_t, t) dW_j(t) \\ &= \left[X_t^u r_f(Y_t, t) + u(t)' \mu^e(Y_t, t) \right] dt + u(t)' \sigma(Y_t, t) dW(t) \end{aligned} \tag{4.14}$$

with X_0^u being the initial wealth.

For any admissible policy u, the stochastic differential equation (4.14) has unique solution. Using Itô calculus, we can write the wealth process as

$$\begin{aligned} X_t^u &= X_0 e^{\int_0^t r_f(Y_s,s)ds} + e^{\int_0^t r_f(Y_s,s)ds} \int_0^t e^{-\int_0^s r_f(Y_z,z)dz} u(s)' \mu^e(Y_s,s)\, ds \\ &\quad + e^{\int_0^t r_f(Y_s,s)ds} \int_0^t e^{-\int_0^s r_f(Y_z,z)dz} u(s)' \sigma(Y_s,s)\, dW(s) \\ &= X_0 \bar{r}_f(0,t) + \int_0^t \bar{r}_f(s,t)\, u(s)' \mu^e(Y_s,s)\, ds \\ &\quad + \int_0^t \bar{r}_f(s,t)\, u(s)' \sigma(Y_s,s)\, dW(s). \end{aligned}$$

Our optimization problem is to find a policy u^* over all admissible policies such that

$$E\left[U\left(Y_T, X_T^{u^*}\right) | Y_0 = i, X_0 = x\right] = \sup_{u \in \mathcal{A}} E\left[U\left(Y_T, X_T^u\right) | Y_0 = i, X_0 = x\right] \tag{4.15}$$

where $U(i,x)$ is the utility function. Note that for any policy $u \in \mathcal{A}$, $u(t) \in \mathcal{F}_t$ implies that $u(t) = u(i,x,t)$ on $\{Y_t = i, X_t = x, Z_t = t\} \in \mathcal{F}_t$ since (Y,X,Z) satisfies the Markov property. Therefore, on the set $\{Y_t = i, X_t = x, Z_t = t\}$, the policy $u(t) = u(i,x,t)$ is in fact not a random variable, but a function of i, x, and t. The reader should keep this in mind throughout the remainder of this chapter. It is clear that $X_t^u \in \mathbb{R}$ and the wealth level is not necessarily positive as it was in Section 4.2. We therefore extend the definition of $C_p^{2,1}$ to include functions $f(i,x,t) : E \times \mathbb{R} \times [0,T] \to \mathbb{R}$ with the same properties. The domain $\mathbb{R}_+$ of x is simply extended to $\mathbb{R}$ now since $X_t^u \in \mathbb{R}$.

We define the generator as in (4.4), and recall the definition of h in (4.5) to obtain

$$\begin{aligned} f(i,X_s,Z_s) &= f(i,X_0,Z_0) + \int_0^s f_t(i,X_v,Z_v)\, dv + \int_0^s f_x(i,X_v,Z_v)\, dX_v \\ &\quad + \frac{1}{2}\int_0^s f_{xx}(i,X_v,Z_v)\,(dX_v)^2 \\ &= f(i,X_0,Z_0) + \int_0^s f_x(i,X_v,Z_v)\, u(Z_v)' \sigma(i,Z_v)\, dW(v) \\ &\quad + \int_0^s \left[f_t(i,X_v,Z_v) + f_x(i,X_v,Z_v)\left(X_v r_f(i,Z_v) + u(Z_v)' \mu^e(i,Z_v)\right)\right] dv \\ &\quad + \int_0^s \frac{1}{2} f_{xx}(i,X_v,Z_v)\, u(Z_v)' V(i,Z_v)\, u(Z_v)\, dv. \end{aligned} \tag{4.16}$$

In applying Itô's formula, note from (4.14) that $(dX_v)^2 = u(Z_v)' V(i, Z_v) u(Z_v) dv$. In (4.16), the stochastic integral is a martingale using $f \in C_p^{2,1}$ and Assumption 16. Therefore,

$$\begin{aligned} \hat{E}_{i,x,t}[f(i, X_s, Z_s)] &= \hat{E}_{i,x,t}[f(i, X_0, Z_0)] + \hat{E}_{i,x,t}\left[\int_0^s (f_t(i, X_v, Z_v) \right. \\ &\quad + f_x(i, X_v, Z_v)\left(X_v r_f(i, Z_v) + u(Z_v)' \mu^e(i, Z_v)\right)) dv \\ &\quad \left. + \frac{1}{2}\int_0^s f_{xx}(i, X_v, Z_v) u(Z_v)' V(i, Z_v) u(Z_v) dv \right] \end{aligned}$$

Now, it follows that

$$\begin{aligned} \lim_{s \downarrow 0} \frac{d\hat{E}_{i,x,t}[f(i, X_s, Z_s)]}{ds} &= \lim_{s \downarrow 0} \hat{E}_{i,x,t}\left[f_t(i, X_s, Z_s) \right. \\ &\quad + f_x(i, X_s, Z_s)\left(X_s r_f(i, Z_s) + u(Z_s)' \mu^e(i, Z_s)\right) \\ &\quad \left. + \frac{1}{2} f_{xx}(i, X_s, Z_s) u(Z_s)' V(i, Z_s) u(Z_s) \right] \\ &= f_t(i, x, t) + f_x(i, x, t)\left(x r_f(i, t) + u(t)' \mu^e(i, t)\right) \\ &\quad + \frac{1}{2} f_{xx}(i, x, t) u(t)' V(i, t) u(t). \end{aligned} \tag{4.17}$$

If we differentiate (4.5) and evaluate it at $s = 0$, and use (4.17), we now get

$$\begin{aligned} \lim_{s \downarrow 0} \frac{dh(i, x, t; s)}{ds} &= f_t(i, x, t) + f_x(i, x, t)\left(x r_f(i, t) + u(t)' \mu^e(i, t)\right) \\ &\quad + \frac{1}{2} f_{xx}(i, x, t) u(t)' V(i, t) u(t) + \sum_{j \in E} A(i, j) f(j, x, t). \end{aligned}$$

If we apply L'Hopital's rule to the generator, the denominator is equal to one and taking the derivative of the numerator with respect to s, and taking the limit as s goes to 0, we obtain

$$\begin{aligned} Gf(i, x, t) &= \lim_{s \downarrow 0} \frac{dh(i, x, t; s)}{ds} \\ &= f_t(i, x, t) + f_x(i, x, t)\left(r_f(i, t) x + u(t)' \mu^e(i, t)\right) \\ &\quad + \frac{1}{2} f_{xx}(i, x, t) u(t)' V(i, t) u(t) + \sum_{j \in E} A(i,j) f(j, x, t). \end{aligned}$$

For a given policy $u \in \mathcal{A}$, we let

$$v^u(i, x, t) = E_{i,x,t}[U(Y_T, X_T^u)]$$

where U is the utility function and the optimization problem becomes

$$v(i, x, t) = \sup_{u \in \mathcal{A}} v^u(i, x, t).$$

Note that we can write

$$v^u(i,x,t) = E\left[v^u\left(Y_{t+h}, X^u_{t+h}, Z_{t+h}\right)|Y_t = i, X^u_t = x, Z_t = t\right]$$

for all $t+h<T$. Rearranging the terms and dividing by h, we obtain

$$\frac{1}{h}E\left[v^u\left(Y_{t+h}, X_{t+h}, Z_{t+h}\right) - v^u(i,x,t)|Y_t = i, X^u_t = x, Z_t = t\right] = 0.$$

Taking the limit as h goes down to zero, we obtain

$$Gv^u(i,x,t) = 0$$

which implies that v^u can now be characterized as the solution of the linear system of second order partial differential equations

$$\begin{aligned} &v^u_t(i,x,t) + v^u_x(i,x,t)\, r_f(i,t)\, x + u(t)'\mu^e(i,t) \\ &\quad + \frac{1}{2}v^u_{xx}(i,x,t)\, u(t)' V(i,t)\, u(t) + \sum_{j\in E} A(i,j)\, v^u(j,x,t) = 0 \end{aligned}$$

for $0 \le t < T$ with the boundary condition $v^u(i,x,T) = U(i,x)$.

The optimal portfolio selection problem can therefore be formulated by the dynamic programming equation

$$\begin{aligned} v_t(i,x,t) + \sup_{u\in\mathbb{R}^m} &\Bigg\{ v_x(i,x,t)\left(r_f(i,t)\, x + u'\mu^e(i,t)\right) \\ &+ \frac{1}{2}v_{xx}(i,x,t)\, u'V(i,t)\, u + \sum_{j\in E} A(i,j)\, v(j,x,t) \Bigg\} = 0 \end{aligned} \tag{4.18}$$

for $0 \le t < T$ with the boundary condition $v(i,x,t) = U(i,x)$.

4.3.1 Verification Theorem

The optimality condition of the dynamic programming equation (4.18) is not sufficient if a verification theorem is not proven. The verification theorem connects the dynamic programming equation (4.18) to the control problem of maximizing the expected value of the terminal wealth defined in (4.15).

Theorem 17 (Verification) *Suppose $g \in C^{2,1}_p$ is a solution of (4.18), then*

a) $g(i,x,t) \ge v(i,x,t)$ for all $i \in E$, $x \in \mathbb{R}$, and $0 \le t \le T$,

b) If $u^* \in \mathcal{A}$ is a maximizer of (4.18), then $g(i,x,t) = v(i,x,t) = v^{u^*}(i,x,t)$ for all $i \in E$, $x \in \mathbb{R}$, and $0 \le t \le T$. In particular, u^* is an optimal portfolio policy.

Proof. Let $u \in \mathcal{A}$ be any policy and X^u be the corresponding wealth process. Using Itô's formula for local martingales with discrete parts, we can write $g\left(Y_T, X_T^u, T\right)$ as

$$\begin{aligned} g\left(Y_T, X_T^u, T\right) &= g\left(Y_t, X_t^\pi, t\right) + \int_t^T \left[g_t\left(Y_s, X_s^u, s\right) + g_x\left(Y_s, X_s^u, s\right)\left(r_f\left(Y_s, s\right) X_s^u\right.\right. \\ &\left.\left. + u\left(s\right)' \mu^e\left(Y_s, s\right)\right) + \frac{1}{2} g_{xx}\left(Y_s, X_s^u, s\right)\left(u\left(s\right)' V\left(Y_s, s\right) u\left(s\right)\right)\right] ds \\ &+ \int_t^T g_x\left(Y_s, X_s^u, s\right) u\left(s\right)' \sigma\left(Y_s, s\right) dW\left(s\right) \\ &+ \sum_{t \leq T_n < T} \left[g\left(Y_{T_n}, X_{T_n}^u, T_n\right) - g\left(Y_{T_n-}, X_{T_n-}^u, T_n-\right)\right]. \end{aligned} \tag{4.19}$$

Since g satisfies (4.18), we can write

$$\begin{aligned} & g_t\left(i, x, t\right) + g_x\left(i, x, t\right) r_f\left(i, t\right) x + u' \mu^e\left(i, t\right) \\ & + \frac{1}{2} g_{xx}\left(i, x, t\right) u' V\left(i, t\right) u + \sum_{j \in E} A\left(i, j\right) g\left(j, x, t\right) \leq 0 \end{aligned} \tag{4.20}$$

for all $i \in E$, $x \in \mathbb{R}$, $0 \leq t \leq T$, and $u \in \mathbb{R}^m$. This implies that we can also write

$$\begin{aligned} & g_t\left(Y_s, X_s^u, s\right) + g_x\left(Y_s, X_s^u, s\right)\left(r_f\left(Y_s, s\right) X_s^u + u\left(s\right)' \mu^e\left(Y_s, s\right)\right) \\ & + \tfrac{1}{2} g_{xx}\left(Y_s, X_s^u, s\right) u\left(s\right)' V\left(Y_s, s\right) u\left(s\right) + \textstyle\sum_{j \in E} A\left(Y_s, j\right) g\left(j, X_s^u, s\right) \leq 0 \end{aligned} \tag{4.21}$$

for all $s \in [0, T]$. Therefore, the integral of the left-hand side of (4.21) from t to T is nonpositive. If we subtract this integral from the right-hand side of (4.19), we get

$$\begin{aligned} g\left(Y_T, X_T, T\right) &\leq g\left(Y_t, X_t^\pi, t\right) + \int_t^T g_x\left(Y_s, X_s^u, s\right) u\left(s\right)' \sigma\left(Y_s, s\right) dW\left(s\right) \\ &+ \sum_{t \leq T_n < T} \left[g\left(Y_{T_n}, X_{T_n}^u, T_n\right) - g\left(Y_{T_n-}, X_{T_n-}^u, T_n-\right)\right] \\ &- \int_t^T \sum_{j \in E} A\left(Y_s, j\right) g\left(j, X_s^u, s\right) ds \end{aligned}$$

and taking the conditional expectation given $\{Y_t = i, X_t^u = x\} \in \mathcal{F}_t$ we obtain

$$\begin{aligned} E_{i,x,t}\left[g\left(Y_T, X_T^u, T\right)\right] &\leq g\left(i, x, t\right) + E_{i,x,t}\left[\int_t^T g_x\left(Y_s, X_s^u, s\right) u\left(s\right)' \sigma\left(Y_s, s\right) dW\left(s\right)\right] \\ &+ E_{i,x,t}\left[\sum_{t \leq T_n < T} \left[g\left(Y_{T_n}, X_{T_n}^u, T_n\right) - g\left(Y_{T_n-}, X_{T_n-}^u, T_n-\right)\right]\right. \\ &\left. - \int_t^T \sum_{j \in E} A\left(Y_s, j\right) g\left(j, X_s^u, s\right) ds\right] \end{aligned}$$

where $E_{i,x,t}\left[g\left(Y_T, X_T^u, T\right)\right] = E_{i,x,t}\left[U\left(Y_T, X_T^u\right)\right]$. Since $g \in C_p^{2,1}$ and $u \in \mathcal{A}$, we therefore get $g_x\left(Y_s, X_s^\pi, s\right) X_s^\pi \pi\left(s\right)' \sigma\left(s, Y_s\right) \in \mathcal{H}^2$. Therefore, the first integral is a martingale and

$$E_{i,x,t}\left[\int_t^T g_x\left(Y_s, X_s^u, s\right) u\left(s\right)' \sigma\left(s, Y_s\right) dW\left(s\right)\right] = 0.$$

Due to the growth condition

$$E_{i,x,t}\left[\sum_{t \le T_n < T} g\left(Y_{T_n}, X_{T_n}^u, T_n\right)\right] < \infty$$

and the last part is equal to zero since it is also a martingale according to Theorem 26.12 in Davis [16, Ch. 2]. So, we see that

$$g\left(i, x, t\right) \ge E_{i,x,t}\left[g\left(Y_T, X_T^u, T\right)\right] = E_{i,x,t}\left[U\left(Y_T, X_T^u\right)\right] = v^u\left(i, x, t\right) \tag{4.22}$$

for all $u \in \mathcal{A}$ which further implies that

$$g\left(i, x, t\right) \ge \sup_{u \in \mathcal{A}} v^u\left(i, x, t\right) = v\left(i, x, t\right). \tag{4.23}$$

If $u^* \in \mathcal{A}$ is a maximizer of (4.18), then it satisfies

$$\begin{aligned} & g_t\left(i, x, t\right) + g_x\left(i, x, t\right) r_f\left(i, t\right) x + u^*\left(t\right)' \mu^e\left(i, t\right) \\ & \quad + \frac{1}{2} g_{xx}\left(i, x, t\right) u^*\left(t\right)' V\left(i, t\right) u^*\left(t\right) + \sum_{j \in E} A\left(i, j\right) g\left(j, x, t\right) = 0 \end{aligned}$$

and (4.23) is satisfied with equality so that

$$g\left(i, x, t\right) = v\left(i, x, t\right) = v^{u^*}\left(i, x, t\right) = E_{i,x,t}\left[U\left(Y_T, X_T^{u^*}\right)\right].$$

■

At this point, we want to point out that the utility function that we will consider later in this chapter will require certain restriction on the wealth level. In some cases, the set $C_p^{2,1}$ will consist of function $f\left(i, x, t\right)$ defined for all $i \in E$, $0 \le t \le T$, and $x \in \mathbb{R}_t \subseteq \mathbb{R}$ where $\mathbb{R}_t$ is some subinterval of $\mathbb{R}$. Theorem 17 still holds provided that $\mathbb{R}$ is replaced by $\mathbb{R}_t$ in a) and b)

4.4 Exponential Utility Model

We assume that the utility of the investor in state i is given by the exponential function

$$U\left(i, x\right) = K(i) - C(i) \exp(-x/\beta) \tag{4.24}$$

with $\beta > 0$, and $C(i) > 0$ where we can easily see that Pratt-Arrow's measure of absolute risk aversion is simply equal to the constant $-U''(i,x)/U'(i,x) = 1/\beta$ for all i.

In this section we will assume that the risk free interest rate r_f does not depend on the state of the market i. Therefore, $r_f(i,t) = r_f(t)$ for all $i \in E$ and $\bar{r}_f(s,t) = \exp\left(\int_s^t r_f(z)\,dz\right)$ is not random.

Note that for the optimization problem with the exponential utility function (4.24), we can write the value function as

$$\begin{aligned} v(i,x,t) &= \sup_{u\in\mathcal{A}} E_{i,x,t}\left[K(Y_T) - C(Y_T)\,e^{-X_T^u/\beta}\right] \\ &= E_{i,x,t}\left[K(Y_T)\right] + E_{i,x,t}\left[\sup_{u\in\mathcal{A}} -C(Y_T)\,e^{-X_T^u/\beta}\right]. \end{aligned}$$

It is clear that the optimal policy u^* is independent of K. Let $\hat{U}(i,x) = -C(i)\exp(-x/\beta) = U(i,x) - K(i)$, then the optimal policy for $U(i,x)$ will also be optimal for $\hat{U}(i,x)$ since u^* does not depend on K. Therefore, if $\hat{v}(i,x,t)$ is defined as

$$\begin{aligned} \hat{v}(i,x,t) &= \sup_{u\in\mathcal{A}} E_{i,x,t}\left[-C(Y_T)\,e^{-X_T^u/\beta}\right] \\ &= \sup_{u\in\mathcal{A}} E_{i,x,t}\left[\hat{U}(Y_T, X_T^u)\right] \end{aligned} \tag{4.25}$$

then

$$v(i,x,t) = E_{i,x,t}\left[K(Y_T)\right] + \hat{v}(i,x,t). \tag{4.26}$$

If we can find $\hat{v}(i,x,t)$, we can easily determine $v(i,x,t)$ using (4.26) and the fact that

$$E_{i,x,t}\left[K(Y_T)\right] = E\left[K(Y_T)\,|Y_t = i\right] = K(i,t) = e^{A(T-t)}K(i) \tag{4.27}$$

independent of x. Here, e^{Ms} is the matrix exponential

$$e^{Ms} = \sum_{n=0}^{+\infty} \frac{s^n}{n!} M^n = \lim_{n\to\infty}\left(I + \frac{Ms}{n}\right)^n$$

defined for any matrix M and scalar $s \in \mathbb{R}_+$. Therefore, $K(\cdot,t)$ denotes the multiplication of the matrix exponential $\exp(A(T-t))$ by the vector K. The matrix exponential can be computed in many ways. Nineteen different techniques are summarized by Moler and Van Loan [48]. The matrix exponential will take an important part throughout this chapter. It is quite clear that the derivative is

$$\frac{d}{ds}e^{M_s} = Me^{Ms} = e^{Ms}M.$$

Moreover, we can also show that (4.27) yields the differential equation

$$K_t(i,t) = \frac{d}{ds}K(i,t) = -Ae^{A(T-t)}K(i) = -e^{A(T-t)}AK(i)$$

or equivalently

$$K_t(i,t) = -AK(i,t) = -\sum_{j\in E} A(i,j)\,K(j,t)$$

with the boundary condition $K(i,T) = K(i)$. These representations will be used throughout this chapter.

Theorem 18 *Let the utility function of the investor be the exponential function (4.24) and suppose that the riskless asset return does not depend on the market state so that $r_f(i,t) = r_f(t)$. Then, the optimal solution of the dynamic programming equation (4.18) is*

$$v(i,x,t) = K(i,t) - C(i,t)\,e^{-x/\beta_t} \tag{4.28}$$

and the optimal portfolio is

$$u^*(i,x,t) = \beta_t V(i,t)^{-1}\mu^e(i,t) \tag{4.29}$$

where

$$\beta_t = \frac{\beta}{\bar{r}_f(t,T)},\ K(i,t) = e^{A(T-t)}K(i), \tag{4.30}$$

and $C(i,t)$ is the solution of the linear system of first order differential equations

$$\frac{dC(i,t)}{dt} = C_t(i,t) = \rho(i,t)\,C(i,t) - \sum_{j\in E} A(i,j)\,C(j,t)$$

with the boundary condition $C(i,T) = C(i)$ where

$$\rho(i,t) = \frac{1}{2}\mu^e(i,t)'\,V(i,t)^{-1}\mu^e(i,t)$$

for all $i \in E$ and $0 \le t \le T$.

Proof. We will focus on the optimization problem (4.25) by examining $\hat{v}(i,x,t)$. For any $\lambda \in \mathbb{R}$, if $u \in \mathcal{A}$ is an admissible policy for $x_0 = x$, $i_0 = i$ then it is also admissible for $x_0 = x + \lambda$, $i_0 = i$. If the same policy u is used for both $x_0 = x$ and $x_0 = x + \lambda$ then the difference between the terminal wealths will be $\lambda\bar{r}_f(t,T)$ since the excess amount in the second case is invested in the risk free asset. This observation can be expressed as

$$E\left[-C(Y_T)\,e^{-X_T^u/\beta}\,|Y_t = i, X_t = x+\lambda\right] = E\left[-C(Y_T)\,e^{-\left(X_T^u+\lambda\bar{r}_f(t,T)\right)/\beta}\,|Y_t = i, X_t = x\right]$$

and

$$E\left[-C\left(Y_T\right)e^{-X_T^u/\beta}|Y_t=i,X_t=x+\lambda\right]=e^{(-\lambda/\beta)\bar{r}_f(t,T)}E\left[C\left(Y_T\right)e^{-X_T^u/\beta}|Y_t=i,X_t=x\right]$$

which means

$$E_{i,x+\lambda,t}\left[-C\left(Y_T\right)e^{-X_T^u/\beta}\right]=e^{(-\lambda/\beta)\bar{r}_f(t,T)}E_{i,x,t}\left[-C\left(Y_T\right)e^{-X_T^u/\beta}\right] \tag{4.31}$$

for any $\lambda\in\mathbb{R}$ and $u\in\mathcal{A}$.

Let $\bar{u}$ be the optimal policy when $x_0=x$, $i_0=i$, then (4.31) implies

$$E_{i,x+\lambda,t}\left[-C\left(Y_T\right)e^{-X_T^{\bar{u}}/\beta}\right]=e^{(-\lambda/\beta)\bar{r}_f(t,T)}E_{i,x,t}\left[-C\left(Y_T\right)e^{-X_T^{\bar{u}}/\beta}\right] \tag{4.32}$$

and, since $\bar{u}$ is optimal,

$$E_{i,x,t}\left[-C\left(Y_T\right)e^{-X^{\bar{u}}(T)/\beta}\right]\geq E_{i,x,t}\left[-C\left(Y_T\right)e^{-X^{u}(T)/\beta}\right] \tag{4.33}$$

for all $u\in\mathcal{A}$. If we multiply both sides with $e^{(-\lambda/\beta)\bar{r}(t)}$, then using (4.31) we can write

$$E_{i,x+\lambda,t}\left[-C\left(Y_T\right)e^{-X_T^{\bar{u}}/\beta}\right]\geq E_{i,x+\lambda,t}\left[-C\left(Y_T\right)e^{-X_T^{u}/\beta}\right]$$

for any admissible policy $u\in\mathcal{A}$. So, $\bar{u}$ is also optimal for $x_0=x+\lambda$, $i_0=i$. We therefore conclude that

$$\begin{aligned}\hat{v}\left(i,x,t\right) &= e^{-x\bar{r}_f(t,T)/\beta}\sup_{u\in\mathcal{A}}E_{i,0,t}\left[-C\left(Y_T\right)e^{-X_T^u/\beta}\right]\\ &= e^{-x/\beta_t}\hat{v}\left(i,0,t\right).\end{aligned}$$

by taking $x=0$, $\lambda=x$ in (4.31). Define $C\left(i,t\right)=-\hat{v}\left(i,0,t\right)$ so that $\hat{v}\left(i,x,t\right)=-e^{-x/\beta_t}C\left(i,t\right)$.

Therefore,

$$\hat{v}_x\left(i,x,t\right)=\left(1/\beta_t\right)e^{-x/\beta_t}C\left(i,t\right) \tag{4.34}$$

and

$$\hat{v}_{xx}\left(i,x,t\right)=-\left(1/\beta_t\right)^2e^{-x/\beta_t}C\left(i,t\right). \tag{4.35}$$

Also, if we take the derivative with respect to t

$$\hat{v}_t\left(i,x,t\right)=-r_f\left(t\right)\left(x/\beta_t\right)e^{-x/\beta_t}C\left(i,t\right)-e^{-x/\beta_t}C_t\left(i,t\right). \tag{4.36}$$

Using (4.18), the optimal policy can be found by

$$\begin{aligned} u^*(t) &= \arg\max_u \Big(\hat{v}_x(i,x,t)\big(r_f(t)x + u'\mu^e(i,t)\big) + \frac{1}{2}\hat{v}_{xx}(i,x,t)u'V(i,t)u \\ &\qquad + \sum_{j\in E} A(i,j)\hat{v}(j,x,t)\Big) \\ &= \arg\max_u \left(\hat{v}_x(i,x,t)u'\mu^e(i,t) + \frac{1}{2}\hat{v}_{xx}(i,x,t)u'V(i,t)u \right) \end{aligned}$$

and setting the gradient equal to zero, we obtain

$$u^*(t) = -\frac{\hat{v}_x(i,x,t)}{\hat{v}_{xx}(i,x,t)}V(i,t)^{-1}\mu^e(i,t)$$

and finally,

$$u^*(t) = \beta_t V(i,t)^{-1}\mu^e(i,t). \tag{4.37}$$

If we plug in the optimal policy in (4.18)

$$\begin{aligned} &\hat{v}_t(i,x,t) + \hat{v}_x(i,x,t)r_f(t)x + u^*(t)'\mu^e(i,t) \\ &\quad + \frac{1}{2}\hat{v}_{xx}(i,x,t)u^*(t)'V(i,t)u^*(t) + \sum_{j\in E} A(i,j)\hat{v}(j,x,t) = 0 \end{aligned}$$

and, using (4.34), (4.35), and (4.36), we have

$$\begin{aligned} &-r_f(t)\frac{x}{\beta_t}e^{-x/\beta_t}C(i,t) - e^{-x/\beta_t}C_t(i,t) + \frac{1}{\beta_t}e^{-x/\beta_t}C(i,t)\big(r_f(t)x + u^*(t)'\mu^e(i,t)\big) \\ &\qquad - \frac{1}{2}\frac{1}{\beta_t^2}e^{-x/\beta_t}C(i,t)u^*(t)'V(i,t)u^*(t) - e^{-x/\beta_t}\sum_{j\in E} A(i,j)C(j,t) = 0. \end{aligned}$$

If we cancel e^{-x/β_t} and insert the policies $u^*(t)'\mu^e(i,t) = \beta_t\mu^e(i,t)'V(i,t)^{-1}\mu^e(i,t)$ and $u^*(t)'V(i,t)u^*(t) = \beta_t^2\mu^e(i,t)'V(i,t)^{-1}\mu^e(i,t)$, we have

$$\begin{aligned} &-C_t(i,t) + C(i,t)\mu^e(i,t)'V(i,t)^{-1}\mu^e(i,t) \\ &\qquad -\frac{1}{2}C(i,t)\mu^e(i,t)'V(i,t)^{-1}\mu^e(i,t) - \sum_{j\in E} A(i,j)C(j,t) = 0. \end{aligned}$$

After rearranging the terms we obtain

$$C_t(i,t) = \rho(i,t)C(i,t) - \sum_{j\in E} A(i,j)C(j,t) \tag{4.38}$$

with the boundary condition $C(i,T) = -\hat{v}(i,0,T) = C(i)$ where $\rho(i,t)$ is defined as $\rho(i,t) = \frac{1}{2}\mu^e(i,t)'V(i,t)^{-1}\mu^e(i,t)$.

To complete the proof, it now suffices to show that the optimal policy (4.29) is admissible and the exponential value function (4.28) satisfies the condition of the verification Theorem 17. Note that

$$|v_x(i,x,t)| = \frac{C(i,t)}{\beta_t} e^{-x/\beta_t} < \infty$$

for all $x \in \mathbb{R}$ and $v \in C_p^{2,1}$. The optimal policy (4.29) is clearly admissible in view of the fact that the optimal wealth process is given by (4.41) since the sum of the two integrals in (4.41) is not necessarily positive. ■

If one examines the differential equation (4.38), it can be rewritten as

$$C_t(i,t) = \sum_{j \in E} \tilde{A}(i,j,t)\, C(j,t) \tag{4.39}$$

where

$$\tilde{A}(i,j,t) = \begin{cases} -\lambda(i)\, Q(i,j) & i \neq j \\ \rho(i,t) + \lambda(i) & i = j \end{cases}.$$

If the market functions μ, σ, and r_f do not depend on time so that $\mu(i,t) = \mu(i)$, $\sigma(i,t) = \sigma(i)$, and $r_f(t) = r_f$, then

$$\rho(i,t) = \rho(i) = \frac{1}{2}(\mu(i) - r_f)' V(i)^{-1} (\mu(i) - r_f)$$

is independent of time and the differential equation (4.39) can be rewritten as

$$C_t(i,t) = \sum_{j \in E} \tilde{A}(i,j)\, C(j,t) \tag{4.40}$$

where

$$\tilde{A}(i,j) = \begin{cases} -\lambda(i)\, Q(i,j) & i \neq j \\ \rho(i) + \lambda(i) & i = j \end{cases}.$$

The solution of (4.40) is the matrix exponential

$$C(i,t) = e^{-\tilde{A}(T-t)} C(i).$$

4.4.1 Evolution of Wealth

The wealth dynamics equation is

$$\begin{aligned} X_t^u &= X_0^u \bar{r}_f(0,t) + \int_0^t \bar{r}_f(s,t)\, u(s)' \mu^e(Y_s,s)\, ds \\ &\quad + \int_0^t \bar{r}_f(s,t)\, u(s)' \sigma(Y_s,s)\, dW(s) \end{aligned}$$

and if we insert the optimal policy

$$u^*(i,x,t) = \beta_t V(i,t)^{-1} \mu^e(i,t)$$

we see that the optimal wealth process X^* is given explicitly by

$$\begin{aligned} X_t^* &= X_0^* \bar{r}_f(0,t) + \beta_t \left(\int_0^t \mu^e(Y_s,s)' V(Y_s,s)^{-1} \mu^e(Y_s,s)\, ds \right. \\ &\quad \left. + \int_0^t \mu^e(Y_s,s)' \sigma(Y_s,s)^{-1} dW(s) \right). \end{aligned} \tag{4.41}$$

Note that the Riemann integral in (4.41) is nonnegative since $V(i,t)^{-1}$ is positive definite for all $i \in E$ and $0 \le t \le T$. However, the stochastic integral in (4.41) takes negative as well as positive values except for trivial cases. Therefore, one can easily see that the optimal policy is admissible since X_T^* is not greater than $X_0^* \bar{r}_f(0,T)$ almost surely.

To simplify the notation we now let $E_i[R] = E[R|Y_0 = i]$, and $\mathrm{Var}_i(R) = \mathrm{Var}(R|Y_0 = i)$ for any random variable R given $Y_0 = i$. Supposing $X_0^* = x$ is given, if we take the expectation

$$\begin{aligned} E_i[X_t^*] &= x\bar{r}_f(0,t) + \beta_t \left(E_i \left[\int_0^t \mu^e(Y_s,s)' V(Y_s,s)^{-1} \mu^e(Y_s,s)\, ds \right] \right. \\ &\quad \left. + E_i \left[\int_0^t \mu^e(Y_s,s)' \sigma(Y_s,s)^{-1} dW(s) \right] \right) \end{aligned}$$

we see that the last part is a martingale; therefore,

$$E_i[X_t^*] = x\bar{r}_f(0,t) + \beta_t E_i \left[\int_0^t \mu^e(Y_s,s)' V(Y_s,s)^{-1} \mu^e(Y_s,s)\, ds \right]. \tag{4.42}$$

It clearly follows from (4.42) that $E_i[X_t^*] \ge E_i[X_0^* \bar{r}_f(0,t)]$ for all $0 \le t \le T$ since $\beta_t > 0$ and

$$E_i \left[\int_0^t \mu^e(Y_s,s)' V(Y_s,s)^{-1} \mu^e(Y_s,s)\, ds \right] \ge 0$$

because $V(i,t)^{-1}$ is positive definite for all $i \in E$ and $0 \le t \le T$.

For the terminal time T, we can write the expected value of optimal wealth as

$$E_i[X_T^*] = x\bar{r}_f(0,T) + \beta m_e(i,T)$$

where

$$m_e(i,T) = E_i \left[\int_0^T \mu^e(Y_s,s)' V(Y_s,s)^{-1} \mu^e(Y_s,s)\, ds \right]$$

since $\beta_T = \beta$ as in the discrete time case.

The numerical calculations can be made by using the transition probability function

$$P_{ij}(t) = P\{Y_t = j | Y_0 = i\}$$

of the Markov process Y so that

$$m_e(i, T) = \sum_{j \in E} \int_0^T P_{ij}(s)\, \mu^e(j, s)' V(j, s)^{-1} \mu^e(j, s)\, ds.$$

Note that if the market functions μ, σ, and r_f do not depend on time, then

$$\begin{aligned} m_e(i, T) &= \sum_{j \in E} \int_0^T P_{ij}(s)\, \mu^e(j)' V(j)^{-1} \mu^e(j)\, ds \\ &= \sum_{j \in E} \mu^e(j)' V(j)^{-1} \mu^e(j) \int_0^T P_{ij}(s)\, ds \end{aligned}$$

where $\int_0^T P_{ij}(s)\, ds$ is the expected time spent in state j until time T given that the initial state is i. Finally, note that the transition function $P(t)$ is given by the matrix exponential

$$P(t) = e^{At}$$

which is quite handy for computational purposes.

In (4.41) if we want to calculate the variance of the optimal wealth we can see that

$$\mathrm{Var}_i(X_T^*) = \beta^2 v_e^2(i, T)$$

where

$$\begin{aligned} v_e^2(i, T) = \mathrm{Var}_i \Bigg(&\int_0^t \mu^e(Y_s, s)' V(Y_s, s)^{-1} \mu^e(Y_s, s)\, ds \\ &+ \int_0^t \mu^e(Y_s, s)' \sigma(Y_s, s)^{-1}\, dW(s) \Bigg) \end{aligned}$$

and the exponential frontier is a line with formula

$$E_i[X_T^*] = x\bar{r}_f(0, T) + \left(\frac{m_e(i, T)}{v_e(i, T)} \right) \mathrm{SD}_i(X_t^*)$$

as in the discrete time case, where $\mathrm{SD}_i(X_T^*) = \sqrt{\mathrm{Var}_i\left(X_T^*\right)} = \beta v_e(i, T)$.

4.5 Simple Logarithmic Utility Model

We now assume that the utility of the investor in state i is given by the logarithmic function

$$U(i,x)=\begin{cases} K(i)+C(i)\log(x) & x>0 \\ -\infty & x\leq 0 \end{cases} \tag{4.43}$$

with $C(i)>0$ where we can easily see that Pratt-Arrow's measure of absolute risk aversion is simply equal to $-U''(i,x)/U'(i,x)=1/x>0$ for all i.

For convenience, we will be using π rather than u as the policy that is used by the investor. Since $u=\pi x$, it is always possible to determine one from the other provided that $x>0$. One should note the important difference between using u and π as the policy. Using u allows negative values for the wealth process, whereas using π as a policy guarantees the that the wealth always stays positive provided that $X_0^\pi>0$. For the logarithmic utility case, it is intuitive to observe that the optimal policy should result with positive terminal wealth since if $X(T)<0$ with positive probability then the utility function is $-\infty$.

For the optimization problem with the logarithmic utility function (4.43), we can write the value function as

$$\begin{aligned} v(i,x,t) &= \sup_{\pi\in\mathcal{A}} E_{i,x,t}\left[K(Y_T)+C(Y_T)\log(X_T^\pi)\right] \\ &= E_{i,x,t}\left[K(Y_T)\right]+\sup_{\pi\in\mathcal{A}} E_{i,x,t}\left[C(Y_T)\log(X_T^\pi)\right]. \end{aligned}$$

It is clear that the optimal policy π^* is independent of K. Let $\hat{U}(i,x)=C(i)\log(x)=U(i,x)-K(i)$, then the optimal policy for $U(i,x)$ will also be optimal for $\hat{U}(i,x)$ since π^* does not depend on K. Therefore, if $\hat{v}(i,x,t)$ is defined as

$$\begin{aligned} \hat{v}(i,x,t) &= \sup_{\pi\in\mathcal{A}} E_{i,x,t}\left[C(Y_T)\log(X_T^\pi)\right] \\ &= \sup_{\pi\in\mathcal{A}} E_{i,x,t}\left[\hat{U}(Y_T,X_T^\pi)\right] \end{aligned} \tag{4.44}$$

then

$$v(i,x,t)=E_{i,x,t}\left[K(Y_T)\right]+\hat{v}(i,x,t) \tag{4.45}$$

and, if we can find $\hat{v}(i,x,t)$, we can easily determine $v(i,x,t)$ using (4.45) and the fact that

$$E_{i,x,t}\left[K(Y_T)\right]=E_i\left[K(Y_T)\right]=e^{A(T-t)}K(i).$$

Theorem 19 *Let the utility function of the investor be the logarithmic function (4.43).. Then, the optimal solution of the dynamic programming equation (4.8) is*

$$v(i,x,t) = K(i,t) + C(i,t)\log(x) \tag{4.46}$$

and the optimal portfolio is

$$\pi^*(i,x,t) = V(i,t)^{-1}\mu^e(i,t) \tag{4.47}$$

where

$$C(i,t) = e^{A(T-t)}C(i),\ \ K(i,t) = e^{A(T-t)}K(i) + P(i,t)$$

and $P(i,t)$ is the solution of the linear system of first order differential equations

$$\frac{dP(i,t)}{dt} = P_t(i,t) = -\rho(i,t) - \sum_{j\in E} A(i,j)P(j,t)$$

with the boundary condition $P(i,T) = 0$, where

$$\rho(i,t) = \left(r_f(i,t) + \frac{1}{2}\mu^e(i,t)' V(i,t)^{-1}\mu^e(i,t)\right)C(i,t) \tag{4.48}$$

for all $i \in E$ and $0 \le t \le T$.

Proof. We will focus on the optimization problem (4.44) by examining $\hat{v}(i,x,t)$. For any $\lambda \in \mathbb{R}$, if $\pi \in \mathcal{A}$ is an admissible policy for $x_0 = x$, $i_0 = i$ then it is also admissible for $x_0 = \lambda x$, $i_0 = i$. So, we can write

$$E\left[C(Y_T)\log(X_T^\pi)\,|Y_t = i, X_t^\pi = \lambda x\right] = E\left[C(Y_T)\log(\lambda X_T^\pi)\,|Y_t = i, X_t^\pi = x\right]$$

and

$$\begin{aligned} E\left[C(Y_T)\log(X_T^\pi)\,|Y_t = i, X_t^\pi = \lambda x\right] &= E\left[\log(\lambda)C(Y_T)\,|Y_t = i, X_t^\pi = x\right] \\ &\quad + E\left[C(Y_T)\log(X_T^\pi)\,|Y_t = i, X_t^\pi = x\right] \end{aligned}$$

which means

$$E_{i,\lambda x,t}\left[C(Y_T)\log(X_T^\pi)\right] = E_{i,x,t}\left[\log(\lambda)C(Y_T)\right] + E_{i,x,t}\left[C(Y_T)\log(X_T^\pi)\right] \tag{4.49}$$

for any $\lambda \in \mathbb{R}$ and $\pi \in \mathcal{A}$.

Let π^* be the optimal policy when $x_0 = x$, $i_0 = i$, then (4.49) implies

$$E_{i,\lambda x,t}\left[C\left(Y_T\right)\log\left(X_T^{\pi^*}\right)\right] = E_{i,x,t}\left[\log\left(\lambda\right)C\left(Y_T\right)\right] + E_{i,x,t}\left[C\left(Y_T\right)\log\left(X_T^{\pi^*}\right)\right] \tag{4.50}$$

and, since π^* is optimal,

$$E_{i,x,t}\left[C\left(Y_T\right)\log\left(X_T^{\pi^*}\right)\right] \geq E_{i,x,t}\left[C\left(Y_T\right)\log\left(X_T^{\pi}\right)\right] \tag{4.51}$$

for all $\pi \in \mathcal{A}$. If we add $E_{i,x,t}\left[\log\left(\lambda\right)C\left(Y_T\right)\right]$ to both sides of (4.51), then using (4.49) we can write

$$E_{i,\lambda x,t}\left[C\left(Y_T\right)\log\left(X_T^{\pi^*}\right)\right] \geq E_{i,\lambda x,t}\left[C\left(Y_T\right)\log\left(X_T^{\pi}\right)\right]$$

for any admissible policy $\pi \in \mathcal{A}$. So, π^* is also optimal for $x_0 = \lambda x$, $i_0 = i$. We therefore conclude that

$$\begin{aligned}\hat{v}\left(i,x,t\right) &= E_{i,x,t}\left[\log\left(x\right)C\left(Y_T\right)\right] + \sup_{\pi\in\mathcal{A}} E_{i,1,t}\left[C\left(Y_T\right)\log\left(X_T^{\pi}\right)\right] \\ &= E_{i,x,t}\left[\log\left(x\right)C\left(Y_T\right)\right] + \hat{v}\left(i,1,t\right)\end{aligned}$$

by taking $x = 1$, $\lambda = x$ in (4.49). Note that the first term is given by

$$E_{i,x,t}\left[\log\left(x\right)C\left(Y_T\right)\right] = E_i\left[C\left(Y_T\right)\right]\log\left(x\right) = e^{A(T-t)}C\left(i\right)\log\left(x\right) = C\left(i,t\right)\log\left(x\right).$$

Define the function $P\left(i,t\right) = \hat{v}\left(i,1,t\right)$ so that

$$\hat{v}\left(i,x,t\right) = C\left(i,t\right)\log\left(x\right) + P\left(i,t\right).$$

Note that this clearly implies (4.46) via (4.45) and $P\left(i,T\right) = 0$. Therefore,

$$\hat{v}_x\left(i,x,t\right) = \frac{1}{x}C\left(i,t\right) \tag{4.52}$$

and

$$\hat{v}_{xx}\left(i,x,t\right) = -\frac{1}{x^2}C\left(i,t\right). \tag{4.53}$$

Also, if we take the derivative with respect to t

$$\hat{v}_t\left(i,x,t\right) = -\log\left(x\right)e^{A(T-t)}AC\left(i\right) + P_t\left(i,t\right). \tag{4.54}$$

Using (4.8), the optimal policy can be found by

$$\begin{aligned}\pi^*\left(t\right) &= \arg\max_{\pi}\Bigg(\hat{v}_x\left(i,x,t\right)x\left(r_f\left(i,t\right) + \pi'\mu^e\left(i,t\right)\right) + \frac{1}{2}\hat{v}_{xx}\left(i,x,t\right)x^2\pi'V\left(i,t\right)\pi \\ &\qquad + \sum_{j\in E} A\left(i,j\right)\hat{v}\left(j,x,t\right)\Bigg) \\ &= \arg\max_{\pi}\left(\hat{v}_x\left(i,x,t\right)x\pi'\mu^e\left(i,t\right) + \frac{1}{2}\hat{v}_{xx}\left(i,x,t\right)x^2\pi'V\left(i,t\right)\pi\right)\end{aligned}$$

and setting the gradient equal to zero, we obtain

$$\pi^*(t) = -\frac{\hat{v}_x(i,x,t)}{x\hat{v}_{xx}(i,x,t)} V(i,t)^{-1} \mu^e(i,t)$$

and, finally,

$$\pi^*(t) = V(i,t)^{-1} \mu^e(i,t). \tag{4.55}$$

If we plug in the optimal policy in (4.8)

$$\hat{v}_t(i,x,t) + \hat{v}_x(i,x,t)\, x \left(r_f(i,t) + \pi^*(t)' \mu^e(i,t)\right) + \frac{1}{2}\hat{v}_{xx}(i,x,t)\, x^2 \pi^*(t)' V(i,t)\, \pi^*(t) + \sum_{j \in E} A(i,j)\, \hat{v}(j,x,t) = 0$$

and, using (4.52),(4.53), and (4.54), we have

$$-\log(x)\, e^{A(T-t)} AC(i) + P_t(i,t) + C(i,t) \left(r_f(i,t) + \pi^*(t)' \mu^e(i,t)\right) - \frac{1}{2} C(i,t)\, \pi^*(t)' V(i,t)\, \pi^*(t) + \log(x)\, e^{A(T-t)} AC(i) + \sum_{j \in E} A(i,j)\, P(j,t) = 0.$$

If we cancel similar terms and insert $\pi^*(t)' \mu^e(t,i) = \mu^e(i,t)' V(i,t)^{-1} \mu^e(i,t)$ and $\pi^*(t)' V(i,t)\, u^*(t) = \mu^e(i,t)' V(i,t)^{-1} \mu^e(i,t)$, we have

$$P_t(i,t) + C(i,t) \left(r_f(i,t) + \mu^e(i,t)' V(i,t)^{-1} \mu^e(i,t)\right) - \frac{1}{2} C(i,t)\, \mu^e(i,t)' V(i,t)^{-1} \mu^e(i,t) + \sum_{j \in E} A(i,j)\, P(j,t) = 0.$$

After rearranging the terms, we obtain

$$P_t(i,t) = -\rho(i,t) - \sum_{j \in E} A(i,j)\, P(j,t) \tag{4.56}$$

with the boundary condition $P(i,T) = 0$ where $\rho(i,t)$ is defined by (4.48).

To complete the proof, it now suffices to show that the optimal policy (4.47) is admissible and the logarithmic value function (4.46) satisfies the condition of the verification Theorem 17. Recall that $X_t^\pi > 0$ for all $\pi \in \mathcal{A}$ and $0 \le t \le T$. It is therefore clear that $v \in C_p^{2,1}$ since

$$|\hat{v}_x(i,x,t)| = \frac{1}{x} C(i,t) < \infty$$

for all $i \in E$, $x \in \mathbb{R}_+$, and $0 \le t \le T$. Moreover, $\pi \in \mathcal{A}$ trivially since it satisfies condition (iv) of Definition 14 by (4.57). ■

4.5.1 Evolution of Wealth

If we analyze the evolution of the wealth process X^* when the optimal policy (4.47) is used we see that

$$\begin{aligned}
X_t^* &= X_0^* \exp\left(\int_0^t \left(r_f(Y_s,s) + \pi(s)' \mu^e(Y_s,s) - \frac{1}{2}\pi(s)' V(Y_s,s)\pi(s)\right) ds \right. \\
&\quad \left. + \int_0^t \pi(s)' \sigma(Y_s,s)\, dW(s)\right) \\
&= X_0^* \exp\left(\int_0^t \left(r_f(Y_s,s) + \mu^e(Y_s,s)' V(Y_s,s)^{-1} \mu^e(Y_s,s) \right.\right. \\
&\quad \left.\left. -\frac{1}{2}\mu^e(Y_s,s)' V(Y_s,s)^{-1} \mu^e(Y_s,s)\right) ds + \int_0^t \mu^e(Y_s,s)' \sigma(Y_s,s)^{-1} dW(s)\right) \\
&= X_0^* \exp\left(\int_0^t \left(r_f(Y_s,s) + \frac{1}{2}\mu^e(Y_s,s)' V(Y_s,s)^{-1} \mu^e(Y_s,s)\right) ds \right. \\
&\quad \left. + \int_0^t \mu^e(Y_s,s)' \sigma(Y_s,s)^{-1} dW(s)\right) \\
&= X_0^* \bar{r}_f(0,t) \exp\left(\int_0^t \frac{1}{2}\mu^e(Y_s,s)' V(Y_s,s)^{-1} \mu^e(Y_s,s)\, ds \right. \\
&\quad \left. + \int_0^t \mu^e(Y_s,s)' \sigma(Y_s,s)^{-1} dW(s)\right). \qquad (4.57)
\end{aligned}$$

Note that once again the Riemann integral in (4.57) is nonnegative but the stochastic integral in (4.57) takes negative as well as positive values except for trivial cases. Therefore, the optimal policy is admissible.

The expected value of the wealth satisfies

$$\begin{aligned}
E_i[X_t^*] &= x E_i\left[\bar{r}_f(0,t) \exp\left(\int_0^t \frac{1}{2}\mu^e(Y_s,s)' V(Y_s,s)^{-1} \mu^e(Y_s,s)\, ds \right.\right. \\
&\quad \left.\left. + \int_0^t \mu^e(Y_s,s)' \sigma(Y_s,s)^{-1} dW s\right)\right]
\end{aligned}$$

when $X_0^* = x$ and, given $\mathcal{Y}$,

$$\begin{aligned}
E[X_t^* | \mathcal{Y}] &= x\bar{r}_f(0,t) \exp\left(\int_0^t \mu^e(Y_s,s)' V(Y_s,s)^{-1} \mu^e(Y_s,s)\, ds\right) \\
&\quad \cdot E\left[\exp\left(\int_0^t \mu^e(Y_s,s)' \sigma(Y_s,s)^{-1} dW s \right.\right. \\
&\quad \left.\left. -\frac{1}{2}\int_0^t \mu^e(Y_s,s)' V(Y_s,s)^{-1} \mu^e(Y_s,s)\, ds\right) \Big| \mathcal{Y}\right] \\
&= x\bar{r}_f(0,t) \exp\left(\int_0^t \mu^e(Y_s,s)' V(Y_s,s)^{-1} \mu^e(Y_s,s)\, ds\right)
\end{aligned}$$

since $\exp\left(\int_0^t \mu^e(Y_s,s)'\sigma(Y_s,s)^{-1}dWs - \frac{1}{2}\int_0^t \mu^e(Y_s,s)'V(Y_s,s)^{-1}\mu^e(Y_s,s)\,ds\right)$ is a martingale given $\mathcal{Y}$. Therefore,

$$E_i[X_t^*] = xE_i\left[\bar{r}_f(0,t)\exp\left(\int_0^t \mu^e(Y_s,s)'V(Y_s,s)^{-1}\mu^e(Y_s,s)\,ds\right)\right] \tag{4.58}$$

It clearly follows from (4.58) that $E_i[X_t^*] \geq E_i[X_0^*\bar{r}_f(0,t)]$ for all $0 \leq t \leq T$ since $\beta_t > 0$ and

$$E_i\left[\int_0^t \mu^e(Y_s,s)'V(Y_s,s)^{-1}\mu^e(Y_s,s)\,ds\right] \geq 0$$

because $V(i,t)^{-1}$ is positive definite for all $i \in E$ and $0 \leq t \leq T$.

Similarly, if we consider the square of the wealth

$$\begin{aligned}(X_t^*)^2 &= (X_0^*)^2\bar{r}_f(0,t)^2\exp\left(\int_0^t \mu^e(Y_s,s)'V(Y_s,s)^{-1}\mu^e(Y_s,s)\,ds\right.\\ &\qquad \left.+2\int_0^t \mu^e(Y_s,s)'\sigma(Y_s,s)^{-1}dWs\right)\end{aligned}$$

and, when $X_0^* = x$,

$$\begin{aligned}E\left[(X_t^*)^2|\mathcal{Y}\right] &= x^2\bar{r}_f(0,t)^2\exp\left(\int_0^t 3\mu^e(Y_s,s)'V(Y_s,s)^{-1}\mu^e(Y_s,s)\,ds\right)\\ &\qquad E\left[\exp\left(\int_0^t 2\mu^e(Y_s,s)'\sigma(Y_s,s)\,dWs\right.\right.\\ &\qquad\qquad \left.\left.-2\int_0^t \mu^e(Y_s,s)'V(Y_s,s)^{-1}\mu^e(Y_s,s)\,ds\right)\middle|\mathcal{Y}\right]\\ &= x^2\bar{r}_f(0,t)^2\exp\left(\int_0^t 3\mu^e(Y_s,s)'V(Y_s,s)^{-1}\mu^e(Y_s,s)\,ds\right)\end{aligned}$$

since $\exp\left(\int_0^t 2\mu^e(Y_s,s)'\sigma(Y_s,s)\,dWs - 2\mu^e(Y_s,s)'V(Y_s,s)^{-1}\mu^e(Y_s,s)\,ds\right)$ is a martingale given $\mathcal{Y}$. Therefore,

$$E_i\left[(X_t^*)^2\right] = x^2E_i\left[\bar{r}_f(0,t)^2\exp\left(\int_0^t 3\mu^e(Y_s,s)'V(Y_s,s)^{-1}\mu^e(Y_s,s)\,ds\right)\right].$$

The variance can be found through the formula

$$\mathrm{Var}_i(X_t^*) = E_i\left[(X_t^*)^2\right] - E_i[X_t^*]^2.$$

4.6 Logarithmic Utility Model

We now assume that the utility of the investor in state i is given by the logarithmic function

$$U(i,x) = \begin{cases} K(i) + C(i)\log(x+\beta) & x > -\beta \\ -\infty & x \leq \beta \end{cases} \tag{4.59}$$

with $\beta > 0$, and $C(i) > 0$ where we can easily see that Pratt-Arrow's measure of absolute risk aversion is simply equal to $-U''(i,x)/U'(i,x) = 1/(x+\beta) > 0$ for all i. Note that in this section we will assume that the risk free interest rate does not depend on the market state i.

We first make some intuitive analysis. One can see that the utility of the investor will be finite if the terminal wealth is greater than $-\beta$. If $X_T^u \leq -\beta$, then the utility will be $-\infty$. If the wealth level at time t is $X_t > -\beta_t = -\beta/\bar{r}_f(t,T)$, then the policy $u(s) = 0$ for $t \leq s \leq T$ will result with $X_T^u > -\beta$ and the utility $U(Y_T, X_T)$ will be finite. Therefore, for an investor with wealth level greater than $-\beta_t$, it is always possible to have finite utility. Actually, according to our analysis in the discrete time model, $-\beta_t$ seems to be a critical level under which there is infinite disutility. Similar to the discrete case, if $X_t < -\beta_t$ then $P\{X_T^u < -\beta\} > 0$ according to the no arbitrage principle for any policy u.

To see this, assume that when $X_t = x < -\beta_t$ and there is a policy $\bar{u}$ such that $X_T^{\bar{u}} \geq -\beta$. For an investor with wealth equal to 0 at time t, consider the policy of borrowing x at the risk free rate and applying the policy $\bar{u}$ up to time T. The wealth level at time T will be $X_T^{\bar{u}} - x\bar{r}_f(t,T)$ and

$$\begin{aligned} X_T^{\bar{u}} - x\bar{r}_f(t,T) &\geq -\beta - x\bar{r}_f(t,T) \\ &> -\beta + \beta_t\bar{r}_f(t,T) = 0 \end{aligned}$$

so that $X_T^{\bar{u}} > x\bar{r}_f(t,T)$. But the investor had wealth equal to 0 at time t and this is a contradiction to condition (iv) of Definition 16. We therefore suppose that $X_0 > -\beta_0 = -\beta/\bar{r}_f(0,T)$.

Since the wealth level X_t^u at any time t satisfies $X_t^u > \beta_t$, we modify $C_p^{2,1}$ in verification Theorem 16 so that it now includes functions $f(i,x,t)$ defined for $i \in E$, $0 \leq t \leq T$, and $x > -\beta_t$ or $x \in \mathbb{R}_t = (\beta_t, +\infty)$.

Theorem 20 *Let the utility function of the investor be the logarithmic function (4.59) and suppose that $r_f(i,t) = r_f(t)$ for all $i \in E$, and $0 \leq t \leq T$. Then, the optimal solution of the dynamic programming equation (4.8) is*

$$v(i,x,t) = K(i,t) + C(i,t)\log(x+\beta_t) \tag{4.60}$$

and the optimal portfolio is

$$u^*(i,x,t) = (x+\beta_t)V(i,t)^{-1}\mu^e(i,t) \tag{4.61}$$

where

$$C(i,t) = e^{A(T-t)} C(i), \; K(i,t) = e^{A(T-t)} K(i) + P(i,t) \tag{4.62}$$

and $P(i,t)$ is the solution of the linear system of first order differential equations

$$\frac{dP(i,t)}{dt} = P_t(i,t) = -\rho(i,t) - \sum_{j \in E} A(i,j) P(j,t)$$

with the boundary condition $P(i,T) = 0$, where

$$\rho(i,t) = \left(r_f(t) + \frac{1}{2} \mu^e(t,i)' V(i,t)^{-1} \mu^e(i,t) \right) C(i,t)$$

for all $i \in E$ and $0 \leq t \leq T$.

Proof. Suppose that a candidate solution for the dynamic programming equation can be written as

$$g(i,x,t) = K(i,t) + C(i,t) \log(x + \beta_t)$$

where $K(i,t)$, and $C(i,t)$ are as defined at (4.62). Then, we can calculate the partial derivatives as

$$g_t(i,x,t) = K_t(i,t) + C_t(i,t) \log(x + \beta_t) + \frac{C(i,t) \beta_t r_f(t)}{x + \beta_t}$$

$$g_x(i,x,t) = \frac{C(i,t)}{x + \beta_t}$$

and

$$g_{xx}(i,x,t) = -\frac{C(i,t)}{(x + \beta_t)^2}.$$

Using the dynamic programming equation (4.18)

$$g_t(i,x,t) + \sup_{u \in \mathbb{R}^m} \Bigg\{ g_x(i,x,t) \left(r_f(t) x + u' \mu^e(i,t) \right) + \frac{1}{2} g_{xx}(i,x,t) u' V(i,t) u + \sum_{j \in E} A(i,j) g(j,x,t) \Bigg\} = 0$$

we see that

$$\begin{aligned} u^*(t) &= -V^{-1}(i,x,t) \mu^e(i,t) \frac{g_x(i,x,t)}{g_{xx}(i,x,t)} \\ &= (x + \beta_t) V^{-1}(i,x,t) \mu^e(i,t) \end{aligned}$$

is the maximizer of the equation. If we insert the optimal policy in the dynamic programming equation then

$$
\begin{aligned}
& g_t(i,x,t) + g_x(i,x,t)\left(r_f(t)\,x + u(t)^{*\prime}\mu^e(i,t)\right) \\
& \quad + \frac{1}{2} g_{xx}(i,x,t)\,u(t)^{*\prime}V(i,t)\,u(t)^* + \sum_{j\in E} A(i,j)\,g(j,x,t) \\
& \quad = K_t(i,t) + C_t(i,t)\log(x+\beta_t) + \frac{C(i,t)}{x+\beta_t}\beta_t r_f(t) + \frac{C(i,t)}{x+\beta_t} r_f(t)\,x \\
& \qquad + C(i,t)\,\frac{1}{2}\mu^e(i,t)\,V^{-1}(i,x,t)\,\mu^e(i,t) + \sum_{j\in E} A(i,j)\left(K(j,t) + C(j,t)\log(x+\beta_t)\right) \\
& \quad = -e^{A(T-t)}AK(i) + P_t(i,t) - e^{A(T-t)}AC(i)\log(x+\beta_t) + C(i,t)\,\rho(i,t) \\
& \qquad + e^{A(T-t)}AK(i) + \sum_{j\in E} A(i,j)\,P(j,t) + e^{A(T-t)}AC(i)\log(x+\beta_t) \\
& \quad = P_t(i,t) + C(i,t)\,\rho(i,t) + \sum_{j\in E} A(i,j)\,P(j,t) \\
& \quad = 0.
\end{aligned}
$$

Therefore, $g(i,x,t)$ solves the dynamic programming equation (4.18) and the corresponding optimal policy is given by (4.61).

To complete the proof, we will need to show that the optimal policy (4.61) is admissible and the logarithmic value function (4.60) satisfies the conditions of the verification Theorem 16. Note again that $X_t^u > -\beta_t$ for all $u \in \mathcal{A}$ and $0 \le t \le T$. It therefore follows that $v \in C_p^{2,1}$ since

$$
|v_x(i,x,t)| = \frac{C(i,t)}{x+\beta_t} < \infty
$$

for all $i \in E$, $x > \beta_t$, and $0 \le t \le T$. Moreover, the optimal policy (4.61) is admissible since condition (iv) of Definition 16 is satisfied by (4.63). ■

4.6.1 Evolution of wealth

If we write the wealth dynamics equation

$$
dX_t^u = \left[X_t^u r_f(t) + u(t)'\mu^e(Y_t,t)\right]dt + u(t)'\sigma(Y_t,t)\,dW(t)
$$

and insert $u^*(t) = (X_t^* + \beta_t)\,V(Y_t,t)^{-1}\mu^e(Y_t,t)$, we get

$$
\begin{aligned}
dX_t^* \quad = \quad & X_t^* r_f(t)\,dt + (X_t^* + \beta_t)\,\mu^e(Y_t,t)'\,V(Y_t,t)^{-1}\mu^e(Y_t,t)\,dt \\
& + (X_t^* + \beta_t)\,\mu^e(Y_t,t)'\,\sigma(Y_t,t)^{-1}dW(t)\,.
\end{aligned}
$$

Adding $d\beta_t = \beta_t r_f(t)\,dt$ to both sides

$$\begin{aligned} dX_t^* + d\beta_t &= (X_t^* + \beta_t)\left[r_f(t) + \mu^e(Y_t,t)' V(Y_t,t)^{-1}\mu^e(Y_t,t)\right]dt \\ &\quad + (X_t^* + \beta_t)\,\mu^e(Y_t,t)'\sigma(Y_t,t)^{-1}\,dW(t) \end{aligned}$$

or

$$\begin{aligned} d(X_t^* + \beta_t) &= (X_t^* + \beta_t)\Big[\Big(r_f(t) + \mu^e(Y_t,t)' V(Y_t,t)^{-1}\mu^e(Y_t,t)\Big)dt \\ &\quad + \mu^e(Y_t,t)'\sigma(Y_t,t)^{-1}\,dW(t)\Big]. \end{aligned}$$

Using Itô calculus, we can determine the wealth process explicitly as

$$\begin{aligned} X_t^* + \beta_t &= (X_0^* + \beta_0)\exp\left(\int_0^t \left(r_f(s) + \frac{1}{2}\mu^e(Y_s,s)' V(Y_s,s)^{-1}\mu^e(Y_s,s)\right)ds \right. \\ &\quad \left. + \int_0^t \mu^e(Y_s,s)'\sigma(Y_s,s)^{-1}\,dW(s)\right) \\ &= (X_0^* + \beta_0)\,\bar{r}_f(0,t)\left(\exp\left(\int_0^t \left(\frac{1}{2}\mu^e(Y_s,s)' V(Y_s,s)^{-1}\mu^e(Y_s,s)\right)ds \right.\right. \\ &\quad \left.\left. + \int_0^t \mu^e(Y_s,s)'\sigma(Y_s,s)\,dW(s)\right)\right). \end{aligned} \tag{4.63}$$

We can rewrite the wealth process as

$$\begin{aligned} X_t^* &= X_0^* r_f(0,t) + (X_0^* r_f(0,t) + \beta_t)\left(\exp\left(\int_0^t \left(\frac{1}{2}\mu^e(Y_s,s)' V(Y_s,s)^{-1}\mu^e(Y_s,s)\right)ds \right.\right. \\ &\quad \left.\left. + \int_0^t \mu^e(Y_s,s)'\sigma(Y_s,s)\,dW(s)\right) - 1\right) \end{aligned} \tag{4.64}$$

since $\beta_t = \beta_0 \bar{r}_f(0,t)$. In (4.64) the Riemann integral is nonnegative but the stochastic integral takes negative as well as positive values except for trivial cases. This implies the admissibility of the optimal policy.

If we take the conditional expectation given $X_0^* = x$ and $\mathcal{Y}$, we obtain

$$\begin{aligned} E[X_t^*|\mathcal{Y}] &= (x + \beta_0)\,\bar{r}_f(0,t)\exp\left(\int_0^t \mu^e(Y_s,s)' V(Y_s,s)^{-1}\mu^e(Y_s,s)\,ds\right) \\ &\quad E\left[\exp\left(\int_0^t \mu^e(Y_s,s)'\sigma(Y_s,s)^{-1}\,dWs \right.\right. \\ &\quad \left.\left. - \frac{1}{2}\mu^e(Y_s,s)' V(Y_s,s)^{-1}\mu^e(Y_s,s)\,ds\right)\middle|\,\mathcal{Y}\right] - \beta_t \\ &= (x + \beta_0)\,\bar{r}_f(0,t)\exp\left(\int_0^t \mu^e(Y_s,s)' V(Y_s,s)^{-1}\mu^e(Y_s,s)\,ds\right) - \beta_t \end{aligned}$$

since the term $\exp\left(\int_0^t \mu^e(Y_s,s)'\sigma(Y_s,s)^{-1}dWs - \frac{1}{2}\int_0^t \mu^e(Y_s,s)'V(Y_s,s)^{-1}\mu^e(Y_s,s)\,ds\right)$ is a martingale given $\mathcal{Y}$. Therefore,

$$E_i[X_t^*] = x\bar{r}_f(0,t) + (x\bar{r}_f(0,t)+\beta_t)\,E_i\left[\exp\left(\int_0^t \mu^e(Y_s,s)'V(Y_s,s)^{-1}\mu^e(Y_s,s)\,ds\right)-1\right].$$

It can easily be observed that $E_i[X_t^*] \geq E_i[X_0^*\bar{r}_f(0,t)]$ for all $0 \leq t \leq T$ since $\beta_t > 0$ and V^{-1} is positive definite. If we define

$$m_l(i,t) = E_i\left[\exp\left(\int_0^t \mu^e(Y_s,s)'V(Y_s,s)^{-1}\mu^e(Y_s,s)\,ds\right)-1\right]$$

we get

$$E_i[X_t^*] = x\bar{r}_f(0,t) + (x\bar{r}_f(0,t)+\beta_t)\,m_l(i,t)$$

Using (4.64), the variance of X_t^* can be written as

$$\mathrm{Var}_i(X_t^*) = (xr_f(0,t)+\beta_t)^2\,v_l^2(i,t)$$

where

$$\begin{aligned} v_l^2(i,t) &= \mathrm{Var}_i\left(\exp\left(\int_0^t \left(\frac{1}{2}\mu^e(Y_s,s)'V(Y_s,s)^{-1}\mu^e(Y_s,s)\right)ds\right.\right. \\ &\quad \left.\left. + \int_0^t \mu^e(Y_s,s)'\sigma(Y_s,s)\,dW(s)\right)-1\right). \end{aligned}$$

Therefore, the logarithmic frontier is a line with formula

$$E_i[X_T^*] = x\bar{r}_f(0,T) + \left(\frac{m_l(i,T)}{v_l(i,T)}\right)\mathrm{SD}_i(X_t^*).$$

as in the discrete time case.

4.7 CRRA Utility Model

We now assume that the utility of the investor in state i is given by the power function

$$U(i,x) = K(i) + C(i)\frac{x^\gamma}{\gamma} \tag{4.65}$$

with $\gamma < 1$, and $C(i) > 0$ where we can easily see that Pratt-Arrow's measure of absolute risk aversion is simply equal to $-U''(i,x)/U'(i,x) = (1-\gamma)/x > 0$ for all i.

As in the Section 4.5, we will be using π rather than u as the policy that is used by the investor. Later in Theorem 21, it is verified that this policy is convenient since it will result with a wealth process where $X_t^\pi > 0$ provided that $X_0^\pi > 0$. This is actually necessary since for negative wealth levels the utility function could be undefined for some values of γ. The structure of the optimal solution prevents problems that can result from negative wealth levels in the utility function.

For the optimization problem with the power utility function (4.65), we can write the value function as

$$\begin{aligned} v(i,x,t) &= \sup_{\pi\in\mathcal{A}} E_{i,x,t}\left[K(Y_T) + C(Y_T)(X_T^\pi)^\gamma/\gamma\right] \\ &= E_{i,x,t}\left[K(Y_T)\right] + \sup_{\pi\in\mathcal{A}} E_{i,x,t}\left[C(Y_T)(X_T^\pi)^\gamma/\gamma\right]. \end{aligned}$$

It is clear that the optimal policy π^* is independent of K. Let $\hat{U}(i,x) = C(i)\,x^\gamma/\gamma = U(i,x) - K(i)$, then the optimal policy for $U(i,x)$ will also be optimal for $\hat{U}(i,x)$ since π^* does not depend on K. Therefore, if $\hat{v}(i,x,t)$ is defined as

$$\begin{aligned} \hat{v}(i,x,t) &= \sup_{\pi\in\mathcal{A}} E_{i,x,t}\left[C(Y_T)(X_T^\pi)^\gamma/\gamma\right] \\ &= \sup_{u\in\mathcal{A}} E_{i,x,t}\left[\hat{U}(Y_T, X_T^\pi)\right] \end{aligned} \tag{4.66}$$

then

$$v(i,x,t) = E_{i,x,t}\left[K(Y_T)\right] + \hat{v}(i,x,t) \tag{4.67}$$

and, if we can find $\hat{v}(i,x,t)$, we can easily determine $v(i,x,t)$ using (4.67) and the fact that

$$E_{i,x,t}\left[K(Y_T)\right] = E_i\left[K(Y_T)\right] = e^{A(T-t)}K(i).$$

Theorem 21 *Let the utility function of the investor be the power function (4.65).. Then, the optimal solution of the dynamic programming equation (4.8) is*

$$v(i,x,t) = K(i,t) + P(i,t)\,x^\gamma/\gamma \tag{4.68}$$

and the optimal portfolio is

$$\pi^*(i,x,t) = \frac{1}{1-\gamma} V(i,t)^{-1}\mu^e(i,t) \tag{4.69}$$

where

$$K(i,t) = e^{A(T-t)}K(i)$$

and $P(i,t)$ is the solution of the linear system of first order differential equations

$$\frac{dP(i,t)}{dt} = P_t(i,t) = -\rho(i,t)\,P(i,t) - \sum_{j\in E} A(i,j)\,P(j,t)$$

with the boundary condition $P(i,T) = C(i)$, *where*

$$\rho(i,t) = \frac{\gamma}{1-\gamma}\left((1-\gamma)\, r_f(i,t) + \frac{1}{2}\mu^e(i,t)' V(i,t)^{-1} \mu^e(i,t)\right) \tag{4.70}$$

for all $i \in E$ *and* $0 \leq t \leq T$.

Proof. We will focus on the optimization problem (4.66) by examining $\hat{v}(i,x,t)$. For any $\lambda \in \mathbb{R}$, if $\pi \in \mathcal{A}$ is an admissible policy for $x_0 = x$, $i_0 = i$ then it is also admissible for $x_0 = \lambda x$, $i_0 = i$. So, we can write

$$E\left[C(Y_T)(X_T^\pi)^\gamma/\gamma\,|Y_t = i, X_t^\pi = \lambda x\right] = E\left[C(Y_T)(\lambda X_T^\pi)^\gamma/\gamma|Y_t = i, X_t^\pi = x\right]$$

and

$$E\left[C(Y_T)(X_T^\pi)^\gamma/\gamma\,|Y_t = i, X_t^\pi = \lambda x\right] = \frac{\lambda^\gamma}{\gamma}E\left[C(Y_T)(X_T^\pi)^\gamma/\gamma\,|Y_t = i, X_t^\pi = x\right]$$

which means

$$E_{i,\lambda x,t}\left[C(Y_T)(X_T^\pi)^\gamma/\gamma\right] = \frac{\lambda^\gamma}{\gamma}E_{i,x,t}\left[C(Y_T)(X_T^\pi)^\gamma/\gamma\right] \tag{4.71}$$

for any $\lambda \in \mathbb{R}$ and $\pi \in \mathcal{A}$.

Let π^* be the optimal policy when $x_0 = x$, $i_0 = i$, then (4.71) implies

$$E_{i,\lambda x,t}\left[C(Y_T)\left(X_T^{\pi^*}\right)^\gamma/\gamma\right] = \frac{\lambda^\gamma}{\gamma}E_{i,x,t}\left[C(Y_T)\left(X_T^{\pi^*}\right)^\gamma/\gamma\right] \tag{4.72}$$

and, since π^* is optimal,

$$E_{i,x,t}\left[C(Y_T)\left(X_T^{\pi^*}\right)^\gamma/\gamma\right] \geq E_{i,x,t}\left[C(Y_T)(X_T^\pi)^\gamma/\gamma\right] \tag{4.73}$$

for all $\pi \in \mathcal{A}$. If we multiply both sides of (4.73) withλ^γ/γ then using (4.71) we can write

$$E_{i,\lambda x,t}\left[C(Y_T)\left(X_T^{\pi^*}\right)^\gamma/\gamma\right] \geq E_{i,\lambda x,t}\left[C(Y_T)(X_T^\pi)^\gamma/\gamma\right]$$

for any admissible policy $\pi \in \mathcal{A}$. So, π^* is also optimal for $x_0 = \lambda x$, $i_0 = i$. We therefore conclude that

$$\begin{aligned}\hat{v}(i,x,t) &= \frac{x^\gamma}{\gamma}\sup_{\pi\in\mathcal{A}} E_{i,1,t}\left[C(Y_T)(X_T^\pi)^\gamma/\gamma\right] \\ &= \frac{x^\gamma}{\gamma}\hat{v}(i,1,t)\end{aligned}$$

by taking $x = 1$, $\lambda = x$ in (4.71). Define the function $P(i,t) = \hat{v}(i,1,t)$ so that

$$\hat{v}(i,x,t) = \frac{x^\gamma}{\gamma}P(i,t).$$

Note that this clearly implies (4.68) via (4.67) and $P(i,T) = C(i)$. Therefore,

$$\hat{v}_x(i,x,t) = x^{\gamma-1} P(i,t) \tag{4.74}$$

and

$$\hat{v}_{xx}(i,x,t) = (\gamma-1)\, x^{\gamma-2} P(i,t)\,. \tag{4.75}$$

Also, if we take the derivative with respect to t

$$\hat{v}_t(i,x,t) = \frac{x^\gamma}{\gamma} P_t(i,t)\,. \tag{4.76}$$

Using (4.8), the optimal policy can be found by

$$\begin{aligned}
\pi^*(t) &= \arg\max_\pi \Bigg(\hat{v}_x(i,x,t)\, x \left(r_f(i,t) + \pi' \mu^e(i,t)\right) + \frac{1}{2} \hat{v}_{xx}(i,x,t)\, x^2 \pi' V(i,t)\, \pi \\
&\qquad + \sum_{j \in E} A(i,j)\, \hat{v}(j,x,t) \Bigg) \\
&= \arg\max_\pi \left(\hat{v}_x(i,x,t)\, x \pi' \mu^e(i,t) + \frac{1}{2} \hat{v}_{xx}(i,x,t)\, x^2 \pi' V(i,t)\, \pi \right)
\end{aligned}$$

and setting the gradient equal to zero, we obtain

$$\pi^*(t) = -\frac{\hat{v}_x(i,x,t)}{x \hat{v}_{xx}(i,x,t)} V(i,t)^{-1} \mu^e(i,t)$$

and finally,

$$\pi^*(t) = \frac{1}{1-\gamma} V(t,i)^{-1} \mu^e(i,t)\,. \tag{4.77}$$

If we plug in the optimal policy in (4.8)

$$\begin{aligned}
&\hat{v}_t(i,x,t) + \hat{v}_x(i,x,t)\, x \left(r_f(i,t) + \pi^*(t)' \mu^e(i,t)\right) \\
&\quad + \frac{1}{2} \hat{v}_{xx}(i,x,t)\, x^2 \pi^*(t)' V(i,t)\, \pi^*(t) + \sum_{j \in E} A(i,j)\, \hat{v}(j,x,t) = 0
\end{aligned}$$

and, using (4.74),(4.75), and (4.76), we have

$$\begin{aligned}
&\frac{x^\gamma}{\gamma} P_t(i,t) + x^\gamma \left(r_f(i,t) + \pi^*(t)' \mu^e(i,t)\right) P(i,t) \\
&\quad + \frac{1}{2} x^\gamma (\gamma-1)\, \pi^*(t)' V(i,t)\, \pi^*(t) + \frac{x^\gamma}{\gamma} \sum_{j \in E} A(i,j)\, P(j,t) = 0.
\end{aligned}$$

If we cancel similar terms and insert $\pi^*(t)' \mu^e(t,i) = 1\mu^e(i,t)' V(i,t)^{-1} \mu^e(i,t) / (1-\gamma)$ and $\pi^*(t)' V(i,t) u^*(t) = \mu^e(i,t)' V(i,t)^{-1} \mu^e(i,t) / (1-\gamma)^2$, we have

$$\begin{aligned} P_t(i,t) + \frac{\gamma}{1-\gamma} \Big((1-\gamma) r_f(i,t) + \mu^e(i,t)' V(i,t)^{-1} \mu^e(i,t) \Big) P(i,t) \\ -\frac{1}{2}\frac{\gamma}{1-\gamma} \mu^e(i,t)' V(i,t)^{-1} \mu^e(i,t) P(i,t) + \sum_{j\in E} A(i,j) P(j,t) = 0. \end{aligned}$$

After rearranging the terms we obtain

$$P_t(i,t) = -\rho(i,t) P(i,t) - \sum_{j\in E} A(i,j) P(j,t) \tag{4.78}$$

with the boundary condition $P(i,T) = 0$ where $\rho(i,t)$ is given by (4.70).

To complete the proof, it now suffices to show that the optimal policy (4.69) is admissible and the logarithmic value function (4.68) satisfies the condition of the verification Theorem 17. Recall that $X_t^\pi > 0$ for all $\pi \in \mathcal{A}$ and $0 \le t \le T$. It is therefore clear that $v \in C_p^{2,1}$ since

$$|\hat{v}_x(i,x,t)| = x^{\gamma-1} P(i,t) < \infty$$

for all $i \in E$, $x \in \mathbb{R}_+$, and $0 \le t \le T$. Moreover, $\pi \in \mathcal{A}$ trivially since it satisfies condition (iv) of Definition 14 by (4.80). ■

4.7.1 Evolution of wealth

If we analyze the evolution of the wealth process X^* when the optimal policy (4.69) is used we see that

$$\begin{aligned} X_t^* &= X_0^* \exp\Bigg(\int_0^t \Big(r_f(Y_s,s) + \pi(s)' \mu^e(Y_s,s) - \frac{1}{2}\pi(s)' V(Y_s,s) \pi(s) \Big) ds \\ &\quad + \int_0^t \pi(s)' \sigma(Y_s,s) dW(s) \Bigg) \\ &= X_0^* \exp\Bigg(\int_0^t \Big(r_f(Y_s,s) + \frac{1}{1-\gamma} \mu^e(Y_s,s)' V(Y_s,s)^{-1} \mu^e(Y_s,s) \\ &\quad - \frac{1}{2}\frac{1}{(1-\gamma)^2} \mu^e(Y_s,s)' V(Y_s,s)^{-1} \mu^e(Y_s,s) \Big) ds \\ &\quad + \int_0^t \frac{1}{1-\gamma} \mu^e(Y_s,s)' \sigma(Y_s,s)^{-1} dW(s) \Bigg) \qquad (4.79) \\ &= X_0^* \bar{r}_f(0,t) \exp\Bigg(\int_0^t \frac{1-2\gamma}{2(1-\gamma)^2} \mu^e(Y_s,s)' V(Y_s,s)^{-1} \mu^e(Y_s,s) ds \\ &\quad + \int_0^t \frac{1}{1-\gamma} \mu^e(Y_s,s)' \sigma(Y_s,s)^{-1} dW(s) \Bigg). \qquad (4.80) \end{aligned}$$

Here, in (4.80), the stochastic integral can take any positive or negative value. Therefore, the exponent can be smaller or greater than 1 with positive probabilities. So, the optimal policy is admissible.

The expected value of the wealth satisfies

$$\begin{aligned} E_i\left[X_t^*\right] &= xE_i\left[\bar{r}_f(0,t)\exp\left(\int_0^t \frac{1-2\gamma}{2(1-\gamma)^2}\mu^e(Y_s,s)' V(Y_s,s)^{-1}\mu^e(Y_s,s)\,ds \right.\right. \\ &\quad \left.\left. + \int_0^t \frac{1}{1-\gamma}\mu^e(Y_s,s)'\sigma(Y_s,s)^{-1}dWs\right)\right] \end{aligned} \tag{4.81}$$

when $X_0^* = x$ and, given $\mathcal{Y}$,

$$\begin{aligned} E\left[X_t^*|\mathcal{Y}\right] &= x\bar{r}_f(0,t)\exp\left(\int_0^t \frac{1}{1-\gamma}\mu^e(Y_s,s)' V(Y_s,s)^{-1}\mu^e(Y_s,s)\,ds\right) \\ &\quad \cdot E\left[\exp\left(\int_0^t \frac{1}{1-\gamma}\mu^e(Y_s,s)'\sigma(Y_s,s)^{-1}dWs \right.\right. \\ &\quad \left.\left. -\frac{1}{2}\frac{1}{(1-\gamma)^2}\mu^e(Y_s,s)' V(Y_s,s)^{-1}\mu^e(Y_s,s)\,ds\right)\middle|\mathcal{Y}\right] \\ &= x\bar{r}_f(0,t)\exp\left(\int_0^t \frac{1}{1-\gamma}\mu^e(Y_s,s)' V(Y_s,s)^{-1}\mu^e(Y_s,s)\,ds\right) \end{aligned}$$

since $\exp\left(\frac{1}{1-\gamma}\int_0^t \mu^e(Y_s,s)'\sigma(Y_s,s)^{-1}dWs - \frac{1}{2(1-\gamma)^2}\int_0^t \mu^e(Y_s,s)' V(Y_s,s)^{-1}\mu^e(Y_s,s)\,ds\right)$ is a martingale given $\mathcal{Y}$. Therefore,

$$E_i\left[X_t^*\right] = xE_i\left[\bar{r}_f(0,t)\exp\left(\frac{1}{1-\gamma}\int_0^t \mu^e(Y_s,s)' V(Y_s,s)^{-1}\mu^e(Y_s,s)\,ds\right)\right]. \tag{4.82}$$

Note that in (4.82), since $\gamma < 1$ the exponent is positive and we can see that $E_i\left[X_t^*\right] \geq E_i\left[X_0^*\bar{r}_f(0,t)\right]$.

If we take the square of the wealth process

$$\begin{aligned} X_t^* &= (X_0^*)^2\,\bar{r}_f(0,t)^2\exp\left(\int_0^t \frac{1-2\gamma}{(1-\gamma)^2}\mu^e(Y_s,s)' V(Y_s,s)^{-1}\mu^e(Y_s,s)\,ds \right. \\ &\quad \left. + \int_0^t \frac{2}{(1-\gamma)}\mu^e(Y_s,s)'\sigma(Y_s,s)^{-1}dW(s)\right) \\ &= (X_0^*)^2\,\bar{r}_f(0,t)^2\exp\left(\int_0^t \frac{3-2\gamma}{(1-\gamma)^2}\mu^e(Y_s,s)' V(Y_s,s)^{-1}\mu^e(Y_s,s)\,ds \right. \\ &\quad - \int_0^t \frac{2}{(1-\gamma)^2}\mu^e(Y_s,s)' V(Y_s,s)^{-1}\mu^e(Y_s,s)\,ds \\ &\quad \left. + \int_0^t \frac{2}{(1-\gamma)}\mu^e(Y_s,s)'\sigma(Y_s,s)^{-1}dW(s)\right) \end{aligned}$$

and taking the expectation given $X_0^* = x$ and $\mathcal{Y}$ we obtain

$$E\left[(X_t^*)^2|\mathcal{Y}\right] = x^2\bar{r}_f(0,t)^2\exp\left(\int_0^t \frac{3-2\gamma}{(1-\gamma)^2}\mu^e(Y_s,s)' V(Y_s,s)^{-1}\mu^e(Y_s,s)\,ds\right)$$

since the second part is martingale. Therefore,

$$E_i\left[(X_t^*)^2\right] = x^2 E_i\left[\bar{r}_f(0,t)^2 \exp\left(\frac{3-2\gamma}{(1-\gamma)^2}\int_0^t \mu^e(Y_s,s)' V(Y_s,s)^{-1}\mu^e(Y_s,s)\,ds\right)\right] \quad (4.83)$$

and $\mathrm{Var}_i(X_t^*) = E_i\left[(X_t^*)^2\right] - E_i\left[(X_t^*)\right]^2$ can be found by using (4.81) and (4.83).

4.8 Power Utility Model

We now assume that the utility of the investor in state i is given by the power function

$$U(i,x) = K(i) + C(i)\frac{(x+\beta)^\gamma}{\gamma} \quad (4.84)$$

with $\beta > 0$, $\gamma < 1$, and $C(i) > 0$ where we can easily see that Pratt-Arrow's measure of absolute risk aversion is simply equal to $-U''(i,x)/U'(i,x) = (1-\gamma)/(x+\beta) > 0$ for all i. Note that in this section we will assume that the risk free interest rate does not depend on the market state i.

Similar to our analysis in Section 4.6, if $X_t < -\beta_t$ then $P\{X_T^u < -\beta\} > 0$ and the utility function can be undefined. Moreover, $X_t < -\beta_t$ clearly implies the existence of arbitrage opportunities. We therefore suppose that $X_0 > -\beta_0 = -\beta/\bar{r}_f(0,T)$ which further implies that $X_t > -\beta_t$ for all $0 \le t \le T$ and $u \in \mathcal{A}$.

Since the wealth level X_t^u at any time t satisfies $X_t^u > \beta_t$, we modify $C_p^{2,1}$ in verification Theorem 16 so that it now includes functions $f(i,x,t)$ defined for $i \in E$, $0 \le t \le T$, and $x > -\beta_t$ or $x \in \mathbb{R}_t = (\beta_t, +\infty)$.

Theorem 22 *Let the utility function of the investor be the logarithmic function (4.84) and* $r_f(i,t) = r_f(t)$.. *Then, the optimal solution of the dynamic programming equation (4.8) is*

$$v(i,x,t) = K(i,t) + P(i,t)(x+\beta_t)^\gamma/\gamma \quad (4.85)$$

and the optimal portfolio is

$$u^*(i,x,t) = \frac{(x+\beta_t)}{1-\gamma}V(i,t)^{-1}\mu^e(i,t) \quad (4.86)$$

where

$$K(i,t) = e^{A(T-t)}K(i) \quad (4.87)$$

and $P(i,t)$ is the solution of the linear system of first order differential equations

$$\frac{dP(i,t)}{dt} = P_t(i,t) = -\rho(i,t) P(i,t) - \sum_{j \in E} A(i,j) P(j,t) \tag{4.88}$$

with the boundary condition $P(i,T) = 0$, where

$$\rho(i,t) = \frac{\gamma}{1-\gamma} \left((1-\gamma) r_f(t) + \frac{1}{2} \mu^e(i,t)' V(i,t)^{-1} \mu^e(i,t) \right)$$

for all $i \in E$ and $0 \leq t \leq T$.

Proof. Suppose that a candidate solution for the dynamic programming equation can be written as

$$g(i,x,t) = K(i,t) + P(i,t)(x+\beta_t)^{\gamma}/\gamma$$

where $K(i,t)$, and $P(i,t)$ are as defined at (4.87) and (4.88). Then, we can calculate the partial derivatives as

$$g_t(i,x,t) = K_t(i,t) + P_t(i,t)(x+\beta_t)^{\gamma}/\gamma + P(i,t)\beta_t r_f(t)(x+\beta_t)^{\gamma-1}$$

$$g_x(i,x,t) = P(i,t)(x+\beta_t)^{\gamma-1}$$

and

$$g_{xx}(i,x,t) = -(1-\gamma) P(i,t)(x+\beta_t)^{\gamma-2}$$

where $K_t(i,t) = dK(i,t)/dt = -AK(i,t)$. Using the dynamic programming equation (4.8)

$$g_t(i,x,t) + \sup_{u \in \mathbb{R}^m} \left\{ g_x(i,x,t)\left(r_f(t)x + u'\mu^e(i,t)\right) + \frac{1}{2} g_{xx}(i,x,t) u' V(i,t) u + \sum_{j \in E} A(i,j) g(j,x,t) \right\} = 0$$

we see that

$$\begin{aligned} u^*(t) &= -V^{-1}(i,x,t)\mu^e(i,t)\frac{g_x(i,x,t)}{g_{xx}(i,x,t)} \\ &= \frac{(x+\beta_t)}{1-\gamma} V^{-1}(i,x,t)\mu^e(i,t) \end{aligned}$$

is the maximizer of the equation. It is the policy that we have analyzed before. If we insert u in the dynamic programming equation then

$$\begin{aligned}
& g_t(i,x,t) + g_x(i,x,t)\left(r_f(t)x + u(t)'\mu^e(i,t)\right) \\
& \quad + \frac{1}{2} g_{xx}(i,x,t)\, u(t)' V(i,t)\, u(t) + \sum_{j\in E} A(i,j)\, g(j,x,t) \\
& \quad = K_t(i,t) + P_t(i,t)\frac{(x+\beta_t)^\gamma}{\gamma} + P(i,t)\beta_t r_f(t)(x+\beta_t)^{\gamma-1} \\
& \qquad + P(i,t)(x+\beta_t)^{\gamma-1} r_f(t)x + P(i,t)\frac{(x+\beta_t)^\gamma}{1-\gamma}\mu^e(i,t)V^{-1}(i,x,t)\mu^e(i,t) \\
& \qquad - \frac{1}{2}P(i,t)\frac{(x+\beta_t)^\gamma}{1-\gamma}\mu^e(i,t)V^{-1}(i,x,t)\mu^e(i,t) \\
& \qquad + \sum_{j\in E} A(i,j)\left(K(j,t) + P(j,t)\frac{(x+\beta_t)^\gamma}{\gamma}\right) \\
& \quad = \left(\frac{P_t(i,t)}{\gamma} + P(i,t)r_f(t) + \frac{P(i,t)}{2(1-\gamma)}\mu^e(i,t)V^{-1}(i,x,t)\mu^e(i,t)\right. \\
& \qquad \left. + \sum_{j\in E} A(i,j)\frac{P(j,t)}{\gamma}\right)(x+\beta_t)^\gamma \\
& \quad = 0
\end{aligned}$$

by (4.88). Therefore, $g(i,x,t)$ solves the dynamic programming equation and the corresponding optimal policy is given by (4.86).

To complete the proof, we will need to show that the optimal policy (4.86) is admissible and the logarithmic value function (4.85) satisfies the conditions of the verification Theorem 16. Note again that $X_t^u > -\beta_t$ for all $u \in \mathcal{A}$ and $0 \le t \le T$. It therefore follows that $v \in C_p^{2,1}$ since

$$|v_x(i,x,t)| = P(i,t)(x+\beta_t)^{\gamma-1} < \infty$$

for all $i \in E$, $x > \beta_t$, and $0 \le t \le T$. Moreover, the optimal policy (4.86) is admissible since condition (iv) of Definition 16 is satisfied by (4.89). ■

4.8.1 Evolution of wealth

If we write the wealth dynamics equation

$$dX_t^u = \left[X_t^u r_f(t) + u(t)'\mu^e(Y_t,t)\right]dt + u(t)'\sigma(Y_t,t)\,dW(t)$$

and insert $u^*(t) = (x + \beta_t) V(Y_t, t)^{-1} \mu^e(Y_t, t) / (1 - \gamma)$, we get

$$\begin{aligned} dX_t^* &= X_t^* r_f(t)\, dt + \frac{(X_t^* + \beta_t)}{1-\gamma} \mu^e(Y_t, t)' V(Y_t, t)^{-1} \mu^e(Y_t, t)\, dt \\ &\quad + \frac{(X_t^* + \beta_t)}{1-\gamma} \mu^e(Y_t, t)' \sigma(Y_t, t)^{-1} dW(t). \end{aligned}$$

Adding $d\beta_t = \beta_t r_f(t)\, dt$ to both sides

$$\begin{aligned} dX_t^* + d\beta_t &= (X_t^* + \beta_t) r_f(t)\, dt + \frac{(X_t^* + \beta_t)}{1-\gamma} \mu^e(Y_t, t)' V(Y_t, t)^{-1} \mu^e(Y_t, t)\, dt \\ &\quad + \frac{(X_t^* + \beta_t)}{1-\gamma} \mu^e(Y_t, t)' \sigma(Y_t, t)^{-1} dW(t) \\ &= (X_t^* + \beta_t) \left[\left(r_f(t) + \frac{\mu^e(Y_t, t)' V(Y_t, t)^{-1} \mu^e(Y_t, t)}{1-\gamma} \right) dt \right. \\ &\quad \left. + \frac{\mu^e(Y_t, t)' \sigma(Y_t, t)^{-1}}{1-\gamma} dW(t) \right]. \end{aligned}$$

Using Itô calculus, we can determine the wealth process explicitly as

$$\begin{aligned} X_t^* + \beta_t &= (X_0^* + \beta_0) \bar{r}_f(0, t) \exp\left(\int_0^t \frac{1-2\gamma}{2(1-\gamma)^2} \mu^e(Y_s, s)' V(Y_s, s)^{-1} \mu^e(Y_s, s)\, ds \right. \\ &\quad \left. + \int_0^t \frac{1}{(1-\gamma)} \mu^e(Y_s, s)' \sigma(Y_s, s)\, dW(s) \right). \end{aligned}$$

We can also rewrite the wealth process as

$$\begin{aligned} X_t^* &= (X_0^* + \beta_0) \bar{r}_f(0, t) \exp\left(\int_0^t \frac{1-2\gamma}{2(1-\gamma)^2} \mu^e(Y_s, s)' V(Y_s, s)^{-1} \mu^e(Y_s, s)\, ds \right. \\ &\quad \left. + \int_0^t \frac{1}{(1-\gamma)} \mu^e(Y_s, s)' \sigma(Y_s, s)\, dW(s) \right) - \beta_t \qquad (4.89) \\ &= X_0^* r_f(0, t) + (X_0^* r_f(0, t) + \beta_t) \\ &\quad \cdot \left(\exp\left(\int_0^t \frac{1-2\gamma}{2(1-\gamma)^2} \mu^e(Y_s, s)' V(Y_s, s)^{-1} \mu^e(Y_s, s)\, ds \right. \right. \\ &\quad \left. \left. + \int_0^t \frac{1}{(1-\gamma)} \mu^e(Y_s, s)' \sigma(Y_s, s)\, dW(s) \right) - 1 \right). \qquad (4.90) \end{aligned}$$

Here, in (4.90), the stochastic integral can take any positive or negative value. Therefore, the exponent can be smaller or greater than 1 with positive probabilities. So, the optimal policy is admissible.

If we take the expectation given $X_0^* = x$ and $\mathcal{Y}$, we obtain

$$\begin{aligned} E\left[X_t^*|\mathcal{Y}\right] + \beta_t &= (x+\beta_0)\,\bar{r}_f(0,t)\exp\left(\int_0^t \frac{1}{(1-\gamma)}\mu^e(Y_s,s)'\,V(Y_s,s)^{-1}\mu^e(Y_s,s)\,ds\right) \\ &\quad E\left[\exp\left(\int_0^t \frac{1}{(1-\gamma)}\mu^e(Y_s,s)'\,\sigma(Y_s,s)\,dWs \right.\right. \\ &\quad \left.\left. -\frac{1}{2(1-\gamma)^2}\mu^e(Y_s,s)'\,V(Y_s,s)^{-1}\mu^e(Y_s,s)\,ds\right)\Big|\,\mathcal{Y}\right] \\ &= (x+\beta_0)\,\bar{r}_f(0,t)\exp\left(\int_0^t \frac{1}{(1-\gamma)}\mu^e(Y_s,s)'\,V(Y_s,s)^{-1}\mu^e(Y_s,s)\,ds\right) \end{aligned}$$

since $\exp\left(\int_0^t \frac{1}{(1-\gamma)}\mu^e(Y_s,s)'\,\sigma(Y_s,s)\,dWs - \frac{1}{2(1-\gamma)^2}\int_0^t \mu^e(Y_s,s)'\,V(Y_s,s)^{-1}\mu^e(Y_s,s)\,ds\right)$ is a martingale given $\mathcal{Y}$. Therefore,

$$\begin{aligned} E\left[X_t^*\right] &= x\bar{r}_f(0,t) + (x\bar{r}_f(0,t)+\beta_t) \\ &\quad \cdot E_i\left[\exp\left(\int_0^t \frac{1}{(1-\gamma)}\mu^e(Y_s,s)'\,V(Y_s,s)^{-1}\mu^e(Y_s,s)\,ds\right) - 1\right]. \end{aligned} \tag{4.91}$$

Note that in (4.91), since $\gamma < 1$ the exponent is positive and we can see that $E_i\left[X_t^*\right] \geq E_i\left[X_0^*\bar{r}_f(0,t)\right]$. If we define

$$m_p(i,t) = E_i\left[\exp\left(\int_0^t \frac{1}{(1-\gamma)}\mu^e(Y_s,s)'\,V(Y_s,s)^{-1}\mu^e(Y_s,s)\,ds\right) - 1\right]$$

we get

$$E_i\left[X_t^*\right] = x\bar{r}_f(0,t) + (x\bar{r}_f(0,t)+\beta_t)\,m_p(i,t)\,.$$

We also see from (4.90) that the variance of X_t^* can be written as

$$\mathrm{Var}_i\left(X_t^*\right) = (xr_f(0,t)+\beta_t)^2\,v_p^2(i,t)$$

where

$$\begin{aligned} v_p^2(i,t) &= \mathrm{Var}_i\left(\exp\left(\int_0^t \frac{1-2\gamma}{2(1-\gamma)^2}\mu^e(Y_s,s)'\,V(Y_s,s)^{-1}\mu^e(Y_s,s)\,ds \right.\right. \\ &\quad \left.\left. + \int_0^t \frac{1}{(1-\gamma)}\mu^e(Y_s,s)'\,\sigma(Y_s,s)\,dW(s)\right) - 1\right). \end{aligned}$$

Therefore, the power frontier is a line with formula

$$E_i\left[X_T^*\right] = x\bar{r}_f(0,T) + \left(\frac{m_p(i,T)}{v_p(i,T)}\right)\mathrm{SD}_i\left(X_t^*\right).$$

as in the discrete time case.

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