Multi-Product Inventory Control under Stock-Out Based Substitution

by

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ABSTRACT

In the retail industry, substitution is a commonly observed customer response to stock-outs. The sales data of a particular product observed by a retailer is affected by substitutions from other products. In practice, retailers make their inventory decisions without the knowledge of the substitution effects on demand. These decisions can be improved if the substitution rates are known. In this thesis, an inventory management problem is considered in the retail industry with multiple items, stock-out based substitution and lost sales. The retailer employs a fixed review period, order-up-to level system to control the inventory. The purpose of this study is to develop a method to determine the optimal order up-to levels in the presence of stock-out based substitution. For given demand rates, substitution structure, length of review period, and the order-up-to levels, an analytical method to approximately evaluate the performance of the system is presented. The method computes the expected sales, inventory levels of each product, number of substitutions between all product pairs, service levels achieved for each product, and service level achieved by the system. In a computational analysis of the problem, the computed results are compared with values obtained with a simulation model to illustrate the effectiveness of approximation method. The approximation method is then used to determine the order-up-to levels that maximize the expected profit. It is observed that incorporating substitution information in retail inventory management can have a substantial effect to increase the profit.

ÖZETÇE

Perakendecilikte, ürünler arası ikame, talep edilen ürünün rafta bulunamaması durumunda sık rastlanan bir durumdur. Bir ürünün perakendeci tarafından gözlemlenen satış rakamları diğer ürünlerden bu ürüne yapılan ikamelerden etkilenmektedir. Pratikte, perakendeciler envanter yönetim kararlarını ürünler arası ikamenin ürünlerin talepleri üzerindeki etkisinin farkında olmadan vermektedirler. Bu tezde, perakendecilik sektöründe çok ürünlü, yoksatmaya dayalı ürün ikameli bir envanter yönetim problemi üzerinde çalışılmıştır. Perakendecinin envanter denetim sistemi olarak sabit gözden geçirme süreli, tamamlama seviyeli (R, S) sistemi kullandığı varsayılmaktadır. Bu çalışmanın amacı, yoksatmaya dayalı olasılıksal ürün ikameli durumlarda en iyi tamamlama seviyelerini bulan bir yöntem geliştirmektir. Ürünlerin talep oranları, ikame yapısı, tamamlama seviyeleri ve sistemin yeniden gözden geçirme süresi bilgileri verildiğinde sistemin performansını değerlendiren analitik bir metod geliştirilmiştir. Söz konusu yaklaşım metodu, her ürünün beklenen satış miktarını, beklenen envanter seviyesini, her ürün için sağlanan hizmet seviyelerini, her ürün çifti arasındaki beklenen ikame miktarlarını ve sistem tarafından ulaşılan hizmet seviyesini belirlemektedir. Sayısal bir analizle, tahmin metodu sonucu elde edilen sonuçlar aynı sistemin benzetiminden alınan sonuçlarla karşılaştırılmış ve bu yöntemin verimliliği hesaplanmıştır. Sonrasında, yaklaşım metodu, eniyi beklenen karı veren tamamlama seviyelerinin belirlenmesinde kullanılmıştır. Alınan sonuçlardan, perakendecilik sektöründe envanter yönetiminde ikame etkisini hesaba katmanın karı artırmada önemli etkisi olduğu gözlemlenmiştir.

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Chapter 1

INTRODUCTION

In retailing, it is commonly observed that when a customer looking for a particular product within a category is unable to find it, he may be willing to purchase a different product, which has a different attribute, such as brand, size, color, etc. This customer behavior is called substitution. There are two types of substitution; stock-out based substitution and assortment based substitution. In stock-out based substitution, a customer buys another item when his favorite product is carried in the assortment but stocked-out at the time of the shopping. In assortment based substitution, the customer switches to another product because his favorite product is not carried in the assortment of the store. Throughout this thesis we will focus on the stock-out based substitution. In the literature, the substitution types are also classified according to the decision maker of the substitution decision. In a manufacturer controlled substitution, supplier firm may choose to fill demand for an inferior product, which is stocked-out, with a superior one at hand, rather than producing the latter one. On the other hand, in retail settings, substitution decision is taken by the customers. In an environment of this type, retailer can only indirectly manipulate or affect customer choices through his inventory policy. In this study, we study the case with customer-driven substitution behavior.

In spite of many efforts and investments on decreasing the out of stock amounts, empirical studies show that the measured stock-out rates in retailing are quite high. Gruen et al. [8] define stock-out rate as the percentage of the SKUs unavailable on the shelf at a particular time. Based on this definition, they suggest that on a typical day, average product unavailability is 8.3% in retailing. There are 5 customer responses to stock-outs; do not purchase, purchase elsewhere, substitute - same brand, substitute - different brand and delay purchase. From the point of the retailer rather than losing the sale customer substitution is favorable. 40% of stock-out occasions results with substitution inside the store (Gruen et al. [8]). Other than taking actions on decreasing the stock-out rate, retailer can manipulate those 40% of customer choices on his favor with an effective inventory control policy. When determining the inventory policy about a product category, the existence of substitutable items makes the decision harder, because when a customer accepts a product in place of his first choice, the demands of both products and the inventory level of the second product will be affected. Then we can say, in a category with substitutable products, the realized demand of a particular item is also affected by the existence of substitutable items and their inventory levels. On the other hand, if the substitutability of the products in a category is accounted for in the inventory control strategy, inventory costs and the total number of the lost sales should be reduced. For example, when a retailer makes a decision about the inventory position of a particular product, independent from other products in the same category, he should keep some safety stock to respond to variability in demand. However, if there is at least one substitutable item in the same category, amount of the safety stock of that product may decrease because there is a possibility that the excess demand of the first product can be satisfied by its substitute. In this context, it is recommended that the inventory levels of substitutable products in an assortment should be optimized jointly by explicitly considering the substitution effect.

In this thesis, we consider a retailer that stocks a certain number of products in a category. When a customer arrives to the store, and if there is enough inventory of his first choice product, the sale is realized, otherwise either the customer chooses another product from the same product category with some probability to satisfy his need or the sale is lost. We utilize a probabilistic substitution model, in which when a preferred product is stocked-out, and the substitution decision has been taken, the product to be substituted for the first choice product is chosen according to a probability matrix. This substitution structure can be considered as a “multi-way demand substitution,” which enables the buyer to substitute to all other products in the assortment independent of their inventory levels, but in case the substituted product is out of stock too, the second substitution attempt is not allowed and the sale is lost.

One of the primary objectives of a retailer is to meet the demand. The customer service level can be defined as the probability that a customer's demand is satisfied with his first choice item. Setting this service level for a product too high forces companies to carry more inventory than needed, on the other hand setting the service level too low results in lost sales and reduces overall customer satisfaction. Our main research question can be stated as follows: does accounting for the substitution effects in the inventory control policy improve the profit of the retailer?

We develop a method to determine the average inventory levels of all products in a product category while taking substitutions between products into account. We then develop a model to estimate the direct sales of each product and substitution sales among all product pairs. Finally, by using these two methods, we build a mathematical programming model to find the optimal order up-to levels, which maximize the total profit, subject to minimum service levels for the customers of all products. Since the minimum service level is dependent on the customer portfolio of the retailer and the characteristics of the product category, it will be considered as an input to the proposed model. Similarly, in the model inventory holding costs, substitution costs, purchasing costs, substitution probabilities are considered as inputs. We present a computational comparison of our method with a simulation of the same problem setup. The results suggest that our approximation method performs quite well. Furthermore, we provide a numerical study to analyze the amount of added value on profit when substitution effects are considered.

The main contribution of this thesis is the development of a method, which can be used under general conditions of real life cases, to determine the optimal order-up-to levels that approximately maximize the retailer's profit.

The remainder of this thesis is organized as follows. In Chapter 2, an overview of the previous studies about inventory control under substitution is presented. Chapter 3 describes the problem setting and the proposed approximation methods. In Chapter 4, the optimization model, where the profit is maximized with respect to the order levels, is presented. In Chapter 5, we provide the computational results regarding the performance of our method, and concluding remarks are presented in Chapter 6.

Chapter 2

LITERATURE REVIEW

The literature on substitution deals with problems in which all products are stocked or a set of products to be stocked has been predetermined, and the basic issues are to determine the optimal stocking levels and substitution rules to be followed. We can analyze studies in the literature about inventory control under stock-out based substitution in two categories: inventory control under manufacturer-controlled substitution, and customer-driven substitution. We can also analyze them under subcategories according to their concerned substitution types, product variety levels, and planning horizons.

In the case of inventory control under manufacturer-controlled substitution, the firm chooses whether or not to fulfill the demand of a stocked-out product with another in order to avoid being stocked-out, prevent customer dissatisfaction, avoid carrying costs of excess inventory of substitute product, or simply minimize costs or maximize profits. A major part of the studies about inventory control under manufacturer-controlled substitution with multiple products considers the “one-way substitution structure,” in which products are classified into different grades, higher graded products can be used as substitute for lower graded products. Under this context, Drezner et al. [7] study a single period, two-product Economic Order Quantity problem with deterministic demand and substitution costs. They analyze and compare three types of product substitution settings: no substitution, partial substitution, and full substitution. They show that in a deterministic setting with proportional positive substitution costs, full substitution is never optimal. In the case of partial substitution, the substituted quantity decreases with the holding cost of the substitutable product and the transfer cost whereas it increases with the holding cost of the other product. Single period, multi product versions of this problem are studied by Drezner and Gurnani [6], Hsu and Bassok [9], Bassok et al. [2], Tibben-Lembke and Bassok [26], and Rao et al. [23]. The key differences among these papers are as follows. Drezner and Gurnani [6] extend the two-product analysis by Drezner et al. [7] to the case of n products. Under random demand and yields, Hsu and Bassok [9] develop a model to determine the optimal production amounts before the demand is realized. Bassok et al. [2] work on the same problem with different substitution costs and uncertain product demand. In their model, the manufacturer first determines the order quantities before the demand is realized, then the allocation of products to demands after demand is materialized. They show that the benefit of considering substitution at the ordering stage is higher when demand variability and salvage values are higher, substitution cost is lower, and products are similar in terms of prices and costs. Tibben-Lembke and Bassok [26] study a special case of the complete downward substitution model. In their problem, there is one product which faces no demand and can be used as substitute for any of the other products, and substitution by other products is not allowed. Tibben-Lembke and Bassok [26] consider the full substitution case. Rao et al. [23] deal with the same problem with substitution and setup costs. Bitran and Dasu [3] and Hsu et al. [10] extend the previous works with analyzing the same problem in a multi period setting. Bitran and Dasu [3] work on the problem with deterministic random yield, where the produced amounts vary from lot to lot due to the complexity of the produced item. Bitran and Dasu [3] assume that the production quantities are known and they develop an allocation model for products after demand is realized. Klein et al. [12] study the resource allocation model in a multi-product environment with deterministic demands and substitution. They model this problem as a minimax flow problem. In their problem there are limited amount of resources and the objective is to allocate the available resources to jobs according to their importance and minimize the maximum deviation from the targeted production amounts. Hsu et al. [10] analyze the same problem with and without substitution costs. Since the considered substitution settings are not customer-driven, papers mentioned so far are relevant mostly for production systems.

In a retail setting, customers who are not able to find their first-choice product on the shelf may accept a similar product in place, resulting in customer-driven substitution. Here, substitution decisions are not made by the retailer, instead they are made by a large number of self-interested customers. A retailer can only indirectly affect customers' decisions through his inventory policy. Our study is focused on the customer driven substitution. McGillivray and Silver [15], Pasternack and Drezner [18], Moinzadeh and Ingene [16], Avşar and Baykal-Gürsoy [1], Smith and Agrawal [25], Mahajan and van Ryzin [14], Netessine and Rudi [17] work on a single period inventory management problem under customer-driven demand substitution. McGillivray and Silver [15], Pasternack and Drezner [18], Moinzadeh and Ingene [16] and Avşar and Baykal-Gürsoy [1] study the two product version of the problem where both products are substitutable for each other. McGillivray and Silver [15] study the case with items similar in unit variable costs and shortage penalties, and assume that substitution is realized in deterministic proportions. They address the question whether the demand substitution has a considerable affect on inventory control policy or not, and they conclude that, when one of the related products' substitution probability is close to 1 the optimal inventory control policy becomes different than the case of independent item inventory control policy. They also analyze the effects of substitution on the order-up-to levels of the two products, and observe that the change between order-up to-levels from the independent item inventory control to substitutable item inventory control is of consequence when one of the substitution probabilities is above 0.75. They also conclude that the change becomes more pronounced as the number of substitutable items increase. Pasternack and Drezner [18] study the two product case with full substitution, in which if a customer is unable to buy his first choice product due to stock-out he will buy the other product instead, with probability 1. For this case, they prove that the expected profit function is concave in order-up-to levels. They find the optimal order up to levels for both products, and analyze the impact of the revenue differences between products on the optimal order quantities. They show that in the general case, if the substitution from one product becomes more profitable then the optimal order-up-to level of this product increases and the order-up-to level of the other product decreases. Avşar and Baykal-Gürsoy [1] analyze the two-product inventory management problem using the concepts of the game theory. Smith and Agrawal [25] study a single period, multi-item stochastic inventory model for one-way product substitution with stock-outs under both assortment-based and stock-out based substitution. They analyze the effects of product substitution on demand distributions, and observe that product substitution has a considerable effect on optimal inventory policy and the profit. They showed that in the presence of fixed unit costs, substitution decreases the optimal number of items to stock. Moreover, when items have different profit margins, substitution effects can reduce the optimal assortment size, even when the fixed costs are zero.But, the solution methodology proposed by Smith and Agrawal [25] has limited applicability for large problems. Mahajan and van Ryzin [14] study the same problem with Smith and Agrawal [25], but they focus on substitutions in retail settings, and consider the demand substitution in all directions with a probabilistic substitution pattern. They assume that the customers' choice process is based on a simple utility maximization idea. That is, each customer assigns a certain utility to purchasing a certain item and to no-purchase option. Then, based on the inventory level and the utility vector, the customer makes his decision to maximize his utility. Thus, the final decision would be either to purchase an item or not, depending on the utility vector. Mahajan and van Ryzin [14] show that under substitution, popular variants should be stocked relatively more, and unpopular variants relatively less when compared to what the traditional newsboy analysis indicates. Netessine and Rudi [17] consider the same problem with Mahajan and van Ryzin [14] and analyze it in retailing with and without competition. They develop a model to estimate near-optimal ordering quantities for both cases, and observe that as the product receives additional demand through substitution, the optimal order quantity of a product increases. They also analyze the effects of correlation between the demands of products on the optimal ordering quantities.

Rajaram and Tang [22] extend the simple newsvendor problem to include substitutability. They develop a heuristic to compute the order quantities and expected profits under substitution, and use this heuristic to analyze the effects of demand correlation and the degree of substitution on order quantities and expected profits.

The papers listed above under the context of customer driven substitution analyze a single period problem or a problem with two products, or they consider one way substitutability. We study a multi-product, multi-period problem, and employ multi-way substitution behavior.

The problem of finding the optimal assortment with substitution from products that are excluded from the assortment to the included ones has been widely studied. Smith and Agrawal [25], Rao et al. [23], Yücel et al. [28], Pentico [19], [20], [21], Chand et al. [5], van Ryzin and Mahajan [27], Cachon et al. [4] and Kök and Fisher [13] are among many. Yücel et al. [28] study inventory control under customer-driven substitution in retail operations. They study the multi period problem with deterministic demand, and single period problem with stochastic demand. In addition to the other studies in the literature, they allow for more than two substitution attempts, and consider shelf space limitations of the store and ordering quantity quotas for suppliers. They try to determine the optimal assortment together with the optimal ordering quantities of products in the assortment under both assortment-based and stock-out-based substitution settings. Pentico [19], [20], [21], and Chand et al. [5] consider a manufacturing problem where some components or parts may be used to substitute for others in an assembly process, which can be classified as a multi-product inventory management problem under deterministic, one-way substitution. They develop dynamic programming algorithms in order to find the optimal product assortment. van Ryzin and Mahajan [27] examine the structural properties of the optimal assortments in a stochastic single period problem. Cachon et al. [4] study different models of assortment planning in the presence of customer search (i.e., customers may search other retailers before buying the item). Smith and Agrawal [25] develop a model that determines which products will be included in the assortment, and then their respective inventory levels to achieve the maximum profit.

Our study is in the class of customer-driven, probabilistic, multi-way substitution problems with substitution and inventory holding costs. To the best of our knowledge, there is not any study which considers the multi-product, multi-period and multi way probabilistic substitution together in a model. In addition we employ a budget constraint for the selection of the optimal order up to levels while satisfying a proper service level for customers' first choices.

Chapter 3

MODEL DESCRIPTION AND APPROXIMATION METHOD

At a retail store, if a customer cannot find his first-choice product he might switch to another product in the same category. Therefore, the observed sales data of a product is affected by substitutions from other products in the same category. In practice, retailers make their inventory decisions without the knowledge of this substitution effect, and service levels or profits can be improved if the substitution effects are known. In this chapter, we will present three approximations, two to estimate the sales amounts and one to estimate the average inventory levels of substitutable products in a category. Also, we will evaluate the performance of these approximations by comparing the values of approximated performance measures with the ones calculated from a simulation of the system.

3.1 Model Description

We consider a retailer who stocks and sells N products in a category. The retailer employs a fixed review period, order-up-to level policy to control the inventory. The review period is denoted by RT. Demand for product i is a Poisson random variable with rate λ_i, the order-up-to level for product i is denoted by Q_i, total demand during the review period is denoted by D_i, where D_i is a Poisson random variable with rate $\lambda_i RT$, and $i = 1, 2, ..., N$.

In the considered system, arrival of a customer to the store will be concluded in one of the three cases. If there is enough inventory, the customer will buy his first choice of product, else he will either substitute to one of the other products in the same category of his choice or buy nothing, and the sale will be lost. In case his first choice product is stocked-out, the probability that he chooses to substitute to another product is δ. We utilize a probabilistic substitution model, in which if the substitution decision has been taken due to the stock-out of a product, the product to be substituted for is chosen according to a probability matrix, α. α_{ij}, $i, j = 1, 2, ..., N$, represents the probability of a product i customer substitute to product j due to the stock-out of product i. These probabilities can be determined via analysis of POS data with the method provided by Karabatı et al. [11] or by the methodology proposed by Kök and Fisher [13].

The considered substitution model has a "multi-way demand substitution" structure which enables the buyer to substitute to all other products in the assortment independent of the products' inventory levels. In our model, the second substitution attempt is not allowed. Limiting the number of substitution attempts is not very restrictive. Smith and Agrawal [25] show that, as the number of substitutable items increase, the effect of the number of allowed substitution attempts decreases, since the probability of finding the first substituted product increases.

For the considered system, it is assumed that;

- The retailer orders on a regular basis (RT days) and he orders all products at the same time and there is no lead time.
- There can not be more than one arrivals at the same time and each arrival is for one product only.
- As long as there is enough inventory of a product at stock, then customers can reach to that product at the time of shopping. Which means when there is enough inventory of a product at stock then there is no time that the shelf is empty.
- Customers make their purchase and substitution choices independent of the products' inventory levels or availability.

In our model, the objective is to determine the best order up-to levels according to a suitable objective function for all products in the assortment under stock-out based substitution. One possible objective of the inventory control policy can be the maximization of the total profit. Our optimization model can be formulated as presented with Equations (3.1) and (3.2):

$$Max_{Q_1,...,Q_N} \Pi \tag{3.1}$$

s.t.

$$SL_i \geq \gamma_i, \qquad i = 1, ..., N \tag{3.2}$$

In this problem, the objective is maximizing the profit under the service level constraints. The service level constraint of a product satisfies a desired direct sales rate for the customers of the product. In Equations (3.1) and (3.2), Π stands for total net profit, γ_i stands for the minimum service level for product i and SL_i stands for the achieved service level of product i after setting the order up to levels according to the optimization problem's solution. SL_i is defined as the ratio of direct sales of product i and total demand of product i.

If we denote the unit profit of product i with π_i, expected total sales amount of product i in one review period with S_i, average inventory of product i with $\bar{I}_i$, holding cost of product i per item per review period with h_i, unit substitution cost with s_i and substituted sales of product i with $\sum_{j\neq i} S_{ji}$, then total profit per time can be calculated as in Equation (3.3).

$$\Pi = \frac{1}{RT} \sum_{i=1}^{N} \left(\pi_i S_i - \bar{I}_i h_i - s_i \sum_{j\neq i} S_{ij} \right) \tag{3.3}$$

In the optimization problem modeled in Equations (3.1) and (3.2), the values for RT, π_i, h_i, s_i and γ_i, $i = 1, ..., N$, are given as inputs, and the values of S_i and $\bar{I}_i$ are not known and cannot be calculated directly. Therefore, we will use various approximation methods to calculate this values.

3.2 Approximate Solution of the Problem

As seen in Figure 3.1, in this section, first a deterministic approximation approach is presented to calculate the average inventory levels and the accuracy evaluation of this approximation is performed by comparing its results with the ones obtained from simulation. Then two approximations,one deterministic one stochastic, are developed to determine the expected direct and substitution sales, and later in a computational analysis, the performance of the these two methods are evaluated by comparing the values of approximated performance measures with the ones calculated from a simulation of the system.

3.2.1 Deterministic Approximation

In this section, we present a simple method to estimate inventory levels and sales and substitution amounts for a given set of order-up-to levels, i.e., Q_i values, by studying how the inventory changes in the review period. The depletion time of the first depleted product can be determined as the minimum of $\frac{Q_i}{\lambda_i}$. And in general, time period between two depletion times can be determined by calculating the minimum of the ratio of products', which have positive inventory at the beginning of the time period, order levels at the beginning of the time period and their arrival rates during this time period. Starting from the first depleted product the depletion times and the depletion orders of all products can be determined by following this approach. In this section, without any loss of generality, we assume that Product 1 is the first depleted product, Product 2 is the one which is depleted second and so on. T_i is the depletion time of product i. From the definition of

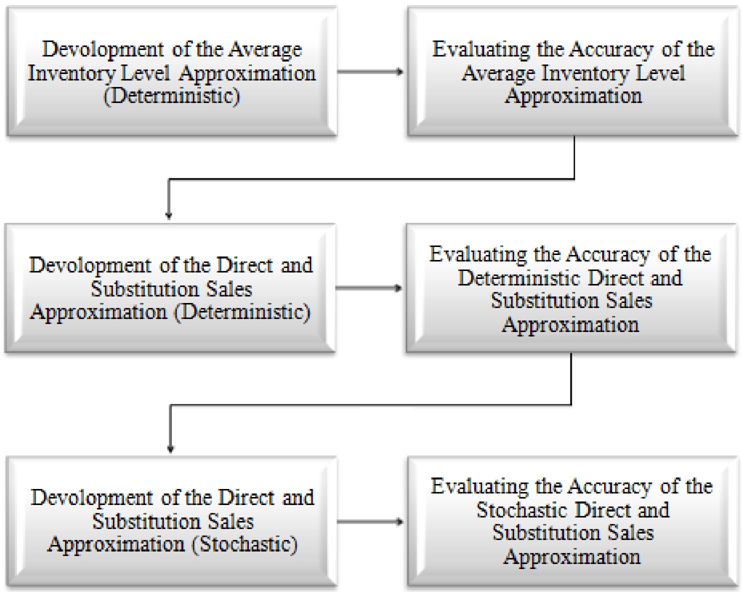

Figure 3.1: Overview of Section 3.2

T_i, it follows $T_1 < T_2 < ... < T_N$.

3.2.1.1 Average Inventory Calculations

In the deterministic setting, the average inventory levels can be determined. In this section we develop a formula in order to calculate the average inventory levels of substitutable products, by taking into account of substitution effects. For simplicity, it is assumed that $RT > T_i \ \ \forall\, i$, later we will discuss the case where the review time is smaller than at least one of the product depletion times.

For a four-product setting, the inventory level-time graph of four substitutable products is presented in Figure 3.2.

Let we denote the time period between depletion times $[T_{i-1}]$ and $[T_i]$ with TP_i. Then, we can develop a TP matrix whose components include the substitution time periods as

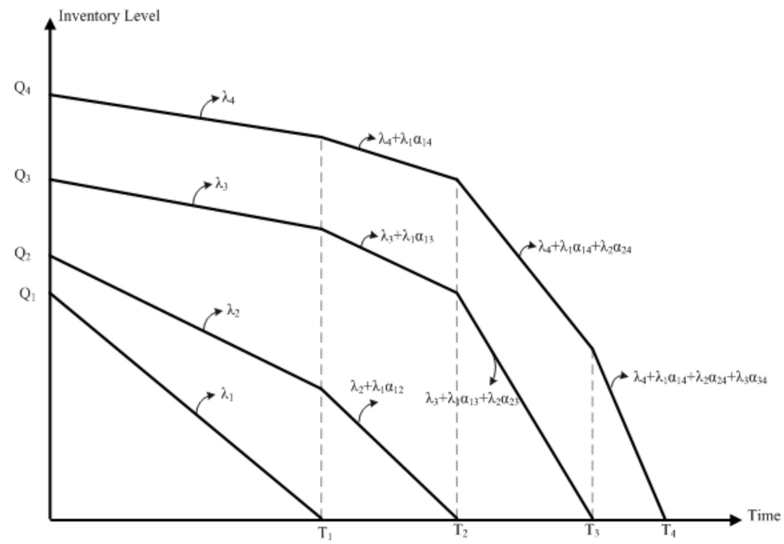

Figure 3.2: Inventory Level - Time Graph for 4 Substitutable Products

shown below:

$$TP = \begin{bmatrix} T_1 - T_0 \\ T_2 - T_1 \\ T_3 - T_2 \\ \vdots \\ T_N - T_{N-1} \end{bmatrix}$$

where $T_0 = 0$

As seen in Figure 3.2, arrival rates of all products change after each depletion time because of the substitutability of the depleted product with other products in the assortment. Then, we should consider the changes in the arrival rates of all products during our calculation of average inventory levels. For example, if we consider Product 3; after T_1, with an α_{13} probability the first product's customers will buy Product 3, and this will cause an increase in the expected demand rate of Product 3 in the amount of $\lambda_1\alpha_{13}$ after T_1. After T_2, because of the unavailability of Product 2, with an α_{23} probability, the customers of Product 2 will also buy Product 3, then the demand rate for Product 3 will be $\lambda_3 + \lambda_1\alpha_{13} + \lambda_2\alpha_{23}$ after T_2. In general, in period $[0, T_1]$ all products will have their individual demand rates since there is no depleted product demand to be substituted from, then for each time period TP_i the demand rate of all products having index greater than i will increase by $\lambda_i\alpha_{ij}$, (where j is the substituted product).

The following depletion rate matrix, O, can be developed for all products:

$$O = \begin{bmatrix} \lambda_1 & 0 & \dots & \dots & \dots & \dots & 0 \\ \lambda_2 & \lambda_2 + \lambda_1\alpha_{12} & 0 & \dots & \dots & \dots & 0 \\ \vdots & \dots & \dots & \dots & \dots & \dots & \vdots \\ \lambda_i & \lambda_i + \lambda_1\alpha_{1i} & \dots & \lambda_i + \lambda_1\alpha_{1i} + \dots + \lambda_{i-1}\alpha_{i-1i} & 0 & \dots & 0 \\ \vdots & \dots & \dots & \dots & \dots & \dots & \vdots \\ \lambda_n & \lambda_n + \lambda_1\alpha_{1n} & \dots & \dots & \dots & \dots & \lambda_n + \dots + \lambda_{n-1}\alpha_{n-1n} \end{bmatrix}$$

where O_{ij} is the depletion rate of product i in period TP_j.

Given the demand rates and order-up-to levels of all products, and substitution probabilities between products, we can calculate O and TP matrices as given in Equations (3.4), (3.5) and (3.7):

$$O_{ij} = \begin{cases} \lambda_i + \sum_{k=1}^{j-1} \lambda_k\alpha_{ki} & if \quad i > j \\ 0 & o/w \end{cases} \tag{3.4}$$

$$T_1 = min_{i=1,..,N}\left(\frac{Q_i}{\lambda_i}\right) \tag{3.5}$$

by definition

$$T_1 = \frac{Q_1}{\lambda_1} \tag{3.6}$$

$$TP_i = \frac{Q_i - \sum_{j=1}^{i-1} O_{ij} TP_j}{O_{ii}} \qquad i = 2, ..., N \tag{3.7}$$

Let's say inventory level of product i in the beginning of period $[T_{j-1}, T_j]$ is QN_{ij}. QN_{ij} can be calculated as given in Equations (3.8) and (3.9);

$$QN_{i1} = Q_i \qquad i = 1, ..., N \tag{3.8}$$

$$QN_{ij} = QN_{ij-1} - TP_{j-1} O_{ij-1} \qquad j = 2, ..., N \qquad i \geq j \tag{3.9}$$

In order to calculate the average inventory level of these products, first we should calculate the area under the graph. For instance, if we look at the inventory level-time graph of Product 2, (Figure 3.3), we can calculate the area under the curve in three parts; Part A, Part B and Part C. Part A and Part C are triangles with heights $O_{21}TP_1$ and $O_{22}TP_2$, and with bases TP_1 and TP_2, respectively. Part B is a rectangle with height QN_{22} and with base TP_1. Then, we can calculate the area under the curve as in Equation (3.10):

$$A + B + C = \frac{TP_1^2 O_{21}}{2} + QN_{22} TP_1 + \frac{TP_2^2 O_{22}}{2} \tag{3.10}$$

Finally, we will divide the total area under the curve to RT to find the average inventory level of Product 2. Then,

$$\bar{I}_2 = \frac{1}{RT} \left(\frac{TP_1^2 O_{21}}{2} + QN_{22} TP_1 + \frac{TP_2^2 O_{22}}{2} \right) \tag{3.11}$$

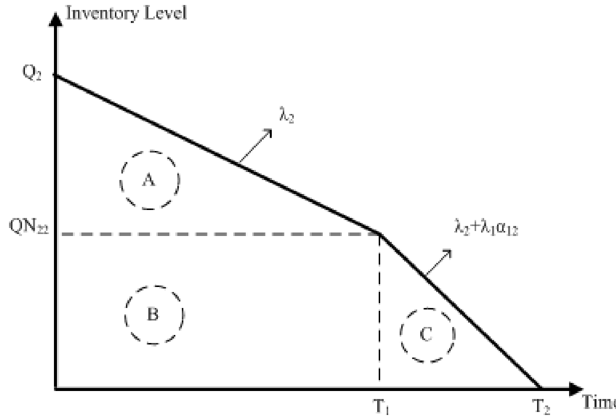

Figure 3.3: Inventory Level - Time Graph for Product 2

If we generalize Formula 3.11 for all products. We can calculate average inventory levels of all products by the formula given in Equation (3.12).

$$\bar{I}_i = \frac{1}{RT}\left(\sum_{j=1}^{i}\frac{TP_j^2 O_{ij}}{2} + \sum_{j=1}^{i-1} QN_{ij+1}TP_j\right) \qquad i = 1, ..., N \qquad (3.12)$$

The formulation in Equation (3.12) is valid for cases at which review time is greater than depletion times of all products. In real life applications this will not be the case all the time. For example, if we consider the case of a two substitutable products, where one of these products gains much more profit than the other, retailer will tend to keep the inventory level of the product with lower profit at its minimum amount, which will satisfy the minimum service level constraint for all customers of this product. Then he will keep more inventory for the higher-profited product in order to satisfy some of the unsatisfied customers of other products with this product. In such a case, the higher-profited product's inventory can be positive at the end of the review time.

In case the review time is less than at least one of the depletion times, we will equate the depletion times, which are greater than the review time, to the review time. Which means, if $RT < T_l$, $T_m = RT \quad \forall \quad m \geq l$. Then the formulation in Equation (3.12) will be valid for all cases.

3.2.1.1.1 Performance Evaluation

In order to determine the accuracy of the approximation method for average inventory levels, we compare the results of this approximation with the results we obtain from the simulation of 4 different cases, representing all possible scenarios in a 3-product setting. In Case 1, all products' depletion times are greater than the review time, in Case 2, one of the 3 products is depleted before the end of the review period, in Case 3 two of the 3 products are depleted before the review period where the other is not depleted until the end of the review period, and in Case 4 all products are depleted before the end of the review period. For the 3-product setting we consider in this section, with Product 1, Product 2 and Product 3, denoted by *P1*, *P2* and *P3*, respectively, demand rate matrix, λ, order-up-to levels matrix, Q, and length of the review time, RT, are given in Table 3.1 for the 4 cases. For all cases substitution probability matrix is calculated via a simple structure based on market shares of the products in a category (see Kök and Fisher [13]). According to this structure, the probability of substituting product i with product j is calculated as follows:

$$\alpha_{ij} = \delta \frac{\lambda_j}{\sum_{\substack{k=1 \\ k \neq i}}^{N} \lambda_k}. \tag{3.13}$$

For example, in a setting with 3 substitutable products in a category, A, B and C, the probability of substituting from A to B is calculated as:

$$\alpha_{AB} = \delta \frac{\lambda_B}{\lambda_B + \lambda_C}, \tag{3.14}$$

where δ is the probability that a customer chooses to substitute to another product.

We present the resulting average inventory level values of all products from the approximation and simulation for all cases in Table 3.2. Average inventory levels and depletion times we get from simulation given in Tables 3.2 and 3.4, are the average results of 50 simulation runs for 100 consecutive review time periods (for almost 6 years).

Table 3.1: Input Parameters of 4 Cases for 3 Products

Case #	***RT***	Q			λ		
		P1	***P2***	***P3***	***P1***	***P2***	***P3***
1	20	450	340	270	19	13	10
2	20	500	210	330	19	13	10
3	20	500	210	500	45	13	10
4	35	500	210	400	30	13	10

In Case 1, review time is taken as 20 days, and depletion times of all products are greater than the review period, which means all products have positive inventory until the end of the review period and so there is no substitution. In this case, the substitutions do not affect the inventory levels since there is no substitution. For this case, over the set of 50 simulation runs, the maximum absolute error of the approximation relative to the simulation results is 0.063% and average absolute error is 0.032%.

In Case 2, only depletion time of *P2* is smaller than the review time. Therefore, there is substitution only from *P2* to other products. The substitutions from *P2* to *P1* and *P3* increase the depletion rates of these two products, and their inventory levels will be overestimated when the substitution effect is not considered. In this particular case, the maximum absolute error of the approximation is 0.343% and the average absolute error is 0.209%.

In Case 3, depletion times of *P1* and *P2* are smaller than the review time. During the review period, there is substitution from *P1* to other products, and from *P2* to *P3*. In this case, after the depletion time of *P1*, with probability α_{13}, *P1*'s customers, and with probability α_{23}, *P2*'s customers and all customers of *P3* will buy *P3*, and this will decrease the inventory level of the third product at a higher rate. For this case, the maximum absolute error of the approximation is 0.253% and average absolute error is 0.104%.

In Case 4, depletion times of all products are smaller than the review time. During review period there is substitution from *P2* to other products, and from *P1* to *P3*. Since there is substitution to *P3* for a long period of time, the average inventory level of this product will be greatly affected by the substitutions. In this case, the maximum absolute error of the approximation is 0.423% and average absolute error is 0.228%.

Table 3.2: Average Inventory Level Results of 4 Cases from Simulation and Approximation for 3 Products

Case #	***Simulation***			***Approximation***		
	P1	***P2***	***P3***	***P1***	***P2***	***P3***
1	260.061	210.133	170.013	260.000	210.000	170.000
2	306.330	85.099	228.079	306.850	84.808	228.342
3	139.241	79.336	358.110	138.889	79.339	358.304
4	119.141	48.668	179.239	119.020	48.462	179.527

As the results presented in Tables 3.1 through 3.4 shows, the maximum absolute error of the approximation of average inventory levels is less than 1% (with the average of 0.66%, evaluated for 250 cases). Therefore, we can state that the average inventory levels computed through the above outlined approximation are quite accurate and can be used in the optimization problem.

Table 3.3: Resulted Absolute Percentage Errors of Approximation for 4 Cases with 3 Products

Case #	***Approximation Errors***		
	P1	***P2***	***P3***
1	0.023%	0.063%	0.008%
2	0.170%	0.343%	0.115%
3	0.253%	0.004%	0.054%
4	0.101%	0.423%	0.161%

Table 3.4: Simulated Depletion Times of 3 Products for 4 Cases

Case #	***P1***	***P2***	***P3***
1	>20.00	>20.00	>20.00
2	>20.00	16.10	>20.00
3	11.10	12.10	>20.00
4	16.40	16.10	25.34

3.2.1.2 Sales and Substitution Amounts Calculations

In this section, the direct sales and the substitution sales of all products are estimated using a deterministic approximation of the problem.

As we can see from Figure 3.2, until T_1, all customers of all products will be able to buy their first choice products. After T_1, the customers of *P1* will not be able to find their first choice product and they will either substitute to another product, say product i, with probability α_{1i} or buy nothing with probability $1-\delta$. In period $[\ 0\ ,\ T_1\ ]$, there will be no substitution between products and direct sales for product i can be approximated with $\lambda_i T_1$ in a deterministic setting. And also after T_1, product i's depletion rate will increase by the amount $\lambda_1\alpha_{1i}$, $i = 2, ..., N$. In period $[T_1, T_2]$, there will be substitution of

P1 from all other products with the amount of $\lambda_1 \alpha_{1i} TP_2$. In general, in period $[T_{i-1}, T_i]$ there will be substitution of all products with variable index k, where $k \leq i$, from other products having index greater than i, and the direct sales amount for product j will be $\lambda_j T_j$ and substitution sales from product k to product j will be $\lambda_k \alpha_{kj}(T_j - T_k)$ if $j > k$ and 0 otherwise.

If we define $Subs_{ijk}$ as the substitution amount from product i to product j in period $[T_{k-1}, T_k]$, it can be calculated as given in Equation (3.15):

$$Subs_{ijk} = \begin{cases} \lambda_i \alpha_{ij} TP_k & if \quad i < k \;\; \& \;\; k \leq j, \\ 0 & o/w. \end{cases} \tag{3.15}$$

If we denote the total direct sales of product i as S_{ii} and the substitution sales from i to j as S_{ij} then, we can calculate the total direct sales and substitution sales amount as shown in Equation (3.16).

$$S_{ij} = \begin{cases} \lambda_i T_i & if \quad i = j, \\ \sum_k Subs_{ijk} & o/w. \end{cases} \tag{3.16}$$

3.2.1.2.1 Performance Evaluation

In order to determine the accuracy of the approximation method, we compare the results of this approximation with the results we obtain from the simulation of 2 cases. In a 3-product setting, with Product 1, Product 2, and Product 3 denoted by *P1*, *P2*, and *P3*, respectively, demand rates, order-up-to levels and length of the review time values are given in Tables 3.5 and 3.9 for 2 different cases.

We present the resulting sales values of the three products for both cases from the approximation and simulation in Tables 3.6 and 3.10. In Table 3.6, the retailer sold 227.957 Product 3's, 199.981 of those products are sold to the direct customers of Product 3, 7.634

customers came to the store for Product 1 but were not able to buy it, then they switch to Product 3. The two cases include two different scenarios: in Case 1, the depletion times of all products are not close to each other, which means the depletion order of all products are constant and will not change because of the random behaviors of the customers, and in Case 2 the depletion times of products are close to each other such that the depletion order will change according to the random choice behavior and the random arrival of the customers. Sales and substitution amounts of simulation and depletion times of the three products, given in Tables 3.6, 3.10, 3.8 and 3.12, are the average results of 50 simulation runs with 100 consecutive review time periods.

In Case 1, as shown in the Table 3.7, the maximum absolute error in the approximation of direct sales is less than 0.5% with the average of 0.082% while the maximum absolute error in the approximation of substitution sales is less than 7% with the average of 1.091%. For this case, depletion times of products are not close, and the depletion order of the products is $P2-P1-P3$ all the time, for both simulation and the approximation and does not change because of the random behaviors of the customers. Under these circumstances, the deterministic approximation delivers a quite accurate performance.

Table 3.5: Input Parameters for Case 1

RT	Q			λ		
	P1	***P2***	***P3***	***P1***	***P2***	***P3***
20	395	201	262	19	13	10

According to the simulation of Case 2, depletion orders of products are close to the review time and to each other, and so the depletion order of the products changes according to the random choice and random arrivals of the customers, and can be different in each simulation run. Therefore, there is substitution among all products. But according to the deterministic approximation of Case 2, all product's depletion times are greater than

Table 3.6: Sales Amounts from Simulation and Approximation for Case 1

Case 1 (Sales Amounts)	***Simulation***			***Approximation***		
	P1	***P2***	***P3***	***P1***	***P2***	***P3***
P1	362.806	0.000	7.634	363.667	0.000	7.101
P2	30.545	201.000	20.342	31.333	201.000	20.345
P3	0.000	0.000	199.981	0.000	0.000	200.000
Total Sales	393.354	201.000	227.957	395.000	201.000	227.446

Table 3.7: Absolute Percentage Error and Error Difference Amounts of Approximation for Case 1

Case 1 (Error)	***% Error***			***Difference Error***		
	P1	***P2***	***P3***	***P1***	***P2***	***P3***
P1	0.237%	0.000%	6.974%	0.860	0.000	0.532
P2	2.582%	0.000%	0.012%	0.789	0.000	0.002
P3	0.000%	0.000%	0.010%	0.003	0.000	0.019
Total Sales	0.419%	0.000%	0.224%	1.646	0.000	0.511

Table 3.8: Depletion Times of 3 Products for Case 1

	Simulated Depletion Times	***Deterministically Calculated Depletion Times***
P1	18.071	19.140
P2	15.471	15.462
P3	19.058	>20

the review time. For such cases, when the depletion times are close to each other and/or the deterministically calculated depletion times are greater than the review time, the approximation approach will not foresee any substitutions between products as presented in the Table 3.10.

Because of the inefficiency of the deterministic approximation in some cases, we develop a stochastic model in order to implement for all cases, in Section 3.2.2.

Table 3.9: Input Parameters for Case 2

RT	*Q*			λ		
	P1	***P2***	***P3***	***P1***	***P2***	***P3***
20	404	263	207	19	13	10

Table 3.10: Sales Amounts from Simulation and Approximation for Case 2

Case 2 (Sales Amounts)	***Simulation***			***Approximation***		
	P1	***P2***	***P3***	***P1***	***P2***	***P3***
P1	378.300	0.385	0.370	380.000	0.000	0.000
P2	3.122	254.407	1.323	0.000	260.000	0.000
P3	1.686	0.814	196.755	0.000	0.000	200.000
Total Sales	383.108	255.606	198.449	380.000	260.000	200.000

Table 3.11: Absolute Percentage Error and Error Difference Amounts of Approximation for Case 2

Case 2 (Error)	***% Error***			***Difference Error***		
	P1	***P2***	***P3***	***P1***	***P2***	***P3***
P1	0.449%	100.000%	100.000%	1.700	0.385	0.370
P2	3.122%	2.198%	100.000%	3.122	5.593	1.323
P3	100.000%	100.000%	1.649%	1.686	0.814	3.245
Total Sales	0.811%	1.719%	0.782%	3.108	4.394	1.552

Table 3.12: Depletion Times of 3 Products for Case 2

	Simulated Depletion Times	***Deterministically Calculated Depletion Times***
P1	19.577	>20
P2	19.071	>20
P3	19.284	>20

3.2.2 Stochastic Approximation

The exact values of the direct and substitution sales amounts can be calculated by the use of Markov Chain method (see Appendix B.2). However, the Markov Chain model described in Appendix B.2 suffers from rapid increase of the number of states in the state space as the number of products and their order-up-to levels increase.

In this section, we present an approximate method to determine the expected direct and substitution sales of each product. The method is based on determining the expected duration of substitution between each product. Namely, if random variable Γ_{ij} is the length of period where product j is substituted for product i during one review period, then the expected substitution number from product i to product j is $\lambda_i \alpha_{ij} E[\Gamma_{ij}]$.

3.2.2.1 The Two Product Case

In order to explain the method, we first analyze the two product case. Let T_i be the time inventory of product i is depleted. Let us first assume that $T_1 < T_2$. Then *P1* is depleted before *P2*, and T_1 is the sum of Q_1 exponentially distributed random variables with rate λ_1. T_1 is a random variable with an Erlang distribution with Q_1 stages and rate λ_1 for each stage:

$$P[T_1 < t | T_1 < T_2] = 1 - \sum_{j=0}^{Q_1 - 1} \frac{(\lambda_1 t)^j}{j!} e^{-\lambda_1 t}, \tag{3.17}$$

and,

$$P[T_2 < t | T_1 < T_2] = P[T_1 + \tau_{12} < t | T_1 < T_2], \tag{3.18}$$

where τ_{12} is the time period where *P2* can be substituted for *P1*, when *P1* is depleted before *P2* and *P2* is depleted before the end of the review time.

During period $[0, T_1]$, the number of *P2*s sold has a Poisson distribution with rate λ_2.

Therefore,

$$P[I_2(T_1) = n_2] = \frac{(\lambda_2 T_1)^{Q_2 - n_2}}{(Q_2 - n_2)!} e^{-\lambda_2 T_1} \tag{3.19}$$

where $I_2(T_1)$ is the number of *P2*s sold during period $[0, T_1]$.

When *P1* is depleted but *P2* is still available, the demand rate of *P2* increases to $\lambda_2 + \lambda_1 \alpha_{12}$. Then the distribution τ_{12} is also Erlang with rate $\lambda_2 + \lambda_1 \alpha_{12}$ and $I_2(T_1)$ stages. Then

$$P[\tau_{12} < t | T_1 < T_2 < RT] = \int_0^{RT} \sum_{n_2=0}^{Q_2} \left(1 - \sum_{j=0}^{n_2-1} \frac{((\lambda_2 + \lambda_1 \alpha_{12})t)^j}{j!} e^{-(\lambda_2 + \lambda_1 \alpha_{12})t} \right) \frac{\lambda_2^{Q_2 - n_2} \lambda_1^{Q_1} t_1^{Q_1 + Q_2 - n_2}}{(Q_2 - n_2)!(Q_1)!} e^{-(\lambda_1 + \lambda_2)t_1} \, dt_1 \tag{3.20}$$

Since the distributions of T_1, T_2 and τ_{12} are given in Equations 3.17, 3.18 and 3.20, the substitution duration Γ_{12} can be expressed as

$$\Gamma_{12} = \begin{cases} RT - T_1 & if \quad T_1 < RT < T_2, \\ \tau_{12} & if \quad T_1 < T_2 < RT, \\ 0 & o/w. \end{cases} \tag{3.21}$$

The case $T_2 < T_1$ is similar and $E[\Gamma_{21}]$ can be explained in a similar manner.

Finally, we will calculate substitution and sales amounts of the two products as; $S_{12} = \lambda_1 \alpha_{12} E[\Gamma_{12}]$,$S_{21} = \lambda_2 \alpha_{21} E[\Gamma_{21}]$, $S_{11} = \lambda_1 E[min\{T_1, RT\}]$ and $S_{22} = \lambda_2 E[min\{T_2, RT\}]$.

Although this approach yields the expected direct and substituted sales numbers in closed form, extending this method to more than two products is not practical due to the increasing number of cases we need to consider in calculating substitution amounts.

In order to determine the substitution amounts for one product, we should calculate the substitution amounts from all other products to this product for each time period between two consecutive depletion times. For one product there are n depletion time periods, and for each depletion time period there are $n+1$ review time range scenarios. Where review time can be smaller than the first depletion time, or between the depletion times' of two products($n-1$ cases), or greater than all depletion times. Then for n products, calculating substitution amounts between each product pair requires time $O(n^3)$.

3.2.2.2 An Approximation for the Multi-product Case

We will use this approach to develop an approximation method for the multi-product case. We make the following assumptions:

- We assume that, while computing the substitution sales between two products, the depletion times of these two products are independent of the depletion times of other products in the assortment.

- Let (i, j) be the pair corresponding to the substituted product and the substitute product. Then, we know that $T_i < T_j$, and the distribution of T_i is an Erlang with rate λ_i and Q_i stages. This random variable can be approximated with a normal random variable with mean $\frac{Q_i}{\lambda_i}$ and variance $\frac{Q_i}{\lambda_i^2}$. (Note that this approximation follows the central limit theorem and quite accurate for large values of Q_i.)

- The depletion time of the substitute product depends on the order-up-to level and arrival rate of the substituted product, and the expected time between the depletion times of these two products.

From Figure 3.4, we can say that $\tau_{ij} = T_j - T_i$ is a random variable with an Erlang distribution with rate $\lambda_j + \lambda_i \alpha_{ij}$ and $Q_j - \frac{Q_i}{\lambda_i}\lambda_j$ stages, in other words τ_{ij} is a random variable with an Erlang distribution with mean $\frac{Q_j - \frac{Q_i}{\lambda_i}\lambda_j}{\lambda_j + \lambda_i \alpha_{ij}}$ and with standard deviation $\frac{\sqrt{Q_j - \frac{Q_i}{\lambda_i}\lambda_j}}{\lambda_j + \lambda_i \alpha_{ij}}$.

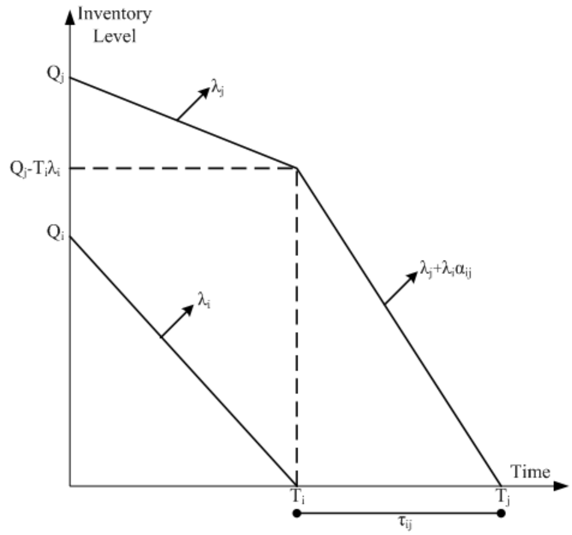

Figure 3.4: Inventory Level - Time Graph for Two Substitutable Products

We approximate this variable with a normal distribution with the same parameters. We know that T_i has been approximated with a normal distribution with mean $\frac{Q_i}{\lambda_i}$ and with standard deviation $\frac{\sqrt{Q_i}}{\lambda_i}$. With the assumption of independency between depletion times we can directly estimate the mean and variance of the random variable T_j. Since T_i and τ_{ij} are approximated with normal distribution then, T_j will be a normal random variable with mean $E[T_i] + E[\tau_{ij}]$ and with variance $Var(T_i) + Var(\tau_{ij})$. We will approximate T_j with a normal distribution with mean $\frac{Q_i}{\lambda_i} + \frac{Q_j - \frac{Q_i}{\lambda_i}\lambda_j}{\lambda_j + \lambda_i\alpha_{ij}}$ and variance $\frac{Q_i}{\lambda_i^2} + \frac{Q_j - \frac{Q_i}{\lambda_i}\lambda_j}{(\lambda_j + \lambda_i\alpha_{ij})^2}$.

Under these approximations, Γ_{ij} can be determined, with a normal random variable with mean $E[T_j] - E[T_i]$ and with variance $Var(T_i) + Var(T_j)$. Then, the mean and variance of $T_j - T_i$ are $\frac{Q_j - \frac{Q_i}{\lambda_i}\lambda_j}{\lambda_j + \lambda_i\alpha_{ij}}$ and $2\frac{Q_i}{\lambda_i^2} + \frac{Q_j - \frac{Q_i}{\lambda_i}\lambda_j}{(\lambda_j + \lambda_i\alpha_{ij})^2}$, respectively.

Under these assumptions, following Equation (3.21);

$$E[\Gamma_{ij}] \cong E[(T_j - T_i)^+]P(T_i < RT)P(T_j < RT) + E[(RT - T_i)^+]P(T_j > RT).$$

As a result, we can approximate S_{ij} as

$$S_{ij} \cong \tilde{S}_{ij} = (E[(T_j - T_i)^+]P(T_i < RT)P(T_j < RT) + E[(RT - T_i)^+]P(T_j > RT))\lambda_i \alpha_{ij} \tag{3.22}$$

With the normal approximation of T_i and T_j, we can determine $E[(T_j - T_i)^+]$, $P(T_i < RT)$, $P(T_j < RT)$, $E[(RT - T_i)^+]$ and $P[T_j > RT]$ directly.

Let $\eta(z)$ is equal to the expected number of units short of a standard normal random variable:

$$\eta(z) = \int_z^\infty \frac{1}{\sqrt{2\pi}}(x - z)e^{-\frac{1}{2}x^2}\, dx = \phi(z) - z\Phi(z) \tag{3.23}$$

where $\phi(z)$ and $\Phi(z)$ are the density function and cumulative distribution function of the standard normal given as $\phi(z) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}\, dx$ and $\Phi(z) = \int_z^\infty \phi(x)\, dx$.

Then,

$$E[(T_j - T_i)^+] = \sigma_{\Gamma_{ij}}\eta\left(-\frac{\mu_{\Gamma_{ij}}}{\sigma_{\Gamma_{ij}}}\right) + \sigma_{\Gamma_{ij}}\eta\left(\frac{RT - \mu_{\Gamma_{ij}}}{\sigma_{\Gamma_{ij}}}\right) - RT(1 - F_{\Gamma_{ij}}(RT)) \tag{3.24}$$

$$E[(RT - T_i)^+] = (RT - \mu_{T_i}) + \sigma_{T_i}\eta\left(\frac{RT - \mu_{T_i}}{\sigma_{T_i}}\right) \tag{3.25}$$

where μ_{T_i} is mean of the depletion time of product i and σ_{T_i} is its standard deviation. (See Appendix C.2 for detailed derivations.)

As a result, we can determine the approximate value of the expected number of substituted products from product i to j as

$$S_{ij} \cong \tilde{S}_{ij} = \left(\left(\sigma_{\Gamma_{ij}} \eta \left(-\frac{\mu_{\Gamma_{ij}}}{\sigma_{\Gamma_{ij}}} \right) + \sigma_{\Gamma_{ij}} \eta \left(\frac{RT - \mu_{\Gamma_{ij}}}{\sigma_{\Gamma_{ij}}} \right) - RT(1 - F_{\Gamma_{ij}}(RT)) \right) \right.$$
$$\left. P(T_i < RT)P(T_j < RT) + \left((RT - \mu_{T_i}) + \sigma_{T_i} \eta \left(\frac{RT - \mu_{T_i}}{\sigma_{T_i}} \right) \right) P(T_j > RT) \right) \lambda_i \alpha_{ij} \tag{3.26}$$

We assume that T_i, T_j and Γ_{ij} are normally distributed with means $\frac{Q_i}{\lambda_i}$, $\frac{Q_i}{\lambda_i} + \frac{Q_j - \frac{Q_i}{\lambda_i}\lambda_j}{\lambda_j + \lambda_i \alpha_{ij}}$, $\frac{Q_j - \frac{Q_i}{\lambda_i}\lambda_j}{\lambda_j + \lambda_i \alpha_{ij}}$ and with standard distributions $\frac{\sqrt{Q_i}}{\lambda_i}$, $\sqrt{\frac{Q_i}{\lambda_i^2} + \frac{Q_j - \frac{Q_i}{\lambda_i}\lambda_j}{(\lambda_j + \lambda_i \alpha_{ij})^2}}$, $\sqrt{2\frac{Q_i}{\lambda_i^2} + \frac{Q_j - \frac{Q_i}{\lambda_i}\lambda_j}{(\lambda_j + \lambda_i \alpha_{ij})^2}}$, respectively. We can rewrite Equation (3.26) as follows:

$$\tilde{S}_{ij} = \left[\left(\sqrt{2\frac{Q_i}{\lambda_i^2} + \frac{Q_j - \frac{Q_i}{\lambda_i}\lambda_j}{(\lambda_j+\lambda_i\alpha_{ij})^2}} \eta \left(- \frac{\frac{Q_j - \frac{Q_i}{\lambda_i}\lambda_j}{\lambda_j+\lambda_i\alpha_{ij}}}{\sqrt{2\frac{Q_i}{\lambda_i^2} + \frac{Q_j - \frac{Q_i}{\lambda_i}\lambda_j}{(\lambda_j+\lambda_i\alpha_{ij})^2}}} \right) + \sqrt{2\frac{Q_i}{\lambda_i^2} + \frac{Q_j - \frac{Q_i}{\lambda_i}\lambda_j}{(\lambda_j+\lambda_i\alpha_{ij})^2}} \eta \left(\frac{RT - \frac{Q_j - \frac{Q_i}{\lambda_i}\lambda_j}{\lambda_j+\lambda_i\alpha_{ij}}}{\sqrt{2\frac{Q_i}{\lambda_i^2} + \frac{Q_j - \frac{Q_i}{\lambda_i}\lambda_j}{(\lambda_j+\lambda_i\alpha_{ij})^2}}} \right) \right. \right.$$
$$\left. - RT\Phi \left(\frac{RT - \frac{Q_j - \frac{Q_i}{\lambda_i}\lambda_j}{\lambda_j+\lambda_i\alpha_{ij}}}{\sqrt{2\frac{Q_i}{\lambda_i^2} + \frac{Q_j - \frac{Q_i}{\lambda_i}\lambda_j}{(\lambda_j+\lambda_i\alpha_{ij})^2}}} \right) \right) \left(1 - \Phi \left(\frac{RT - \frac{Q_i}{\lambda_i}}{\frac{\sqrt{Q_i}}{\lambda_i}} \right) \right) \left(1 - \Phi \left(\frac{RT - \frac{Q_i}{\lambda_i} - \frac{Q_j - \frac{Q_i}{\lambda_i}\lambda_j}{\lambda_j+\lambda_i\alpha_{ij}}}{\sqrt{\frac{Q_i}{\lambda_i^2} + \frac{Q_j - \frac{Q_i}{\lambda_i}\lambda_j}{(\lambda_j+\lambda_i\alpha_{ij})^2}}} \right) \right)$$
$$\left. + \left((RT - \frac{Q_i}{\lambda_i}) + (\frac{\sqrt{Q_i}}{\lambda_i}) \eta \left(\frac{RT - \frac{Q_i}{\lambda_i}}{\frac{\sqrt{Q_i}}{\lambda_i}} \right) \right) \Phi \left(\frac{RT - \frac{Q_i}{\lambda_i} - \frac{Q_j - \frac{Q_i}{\lambda_i}\lambda_j}{\lambda_j+\lambda_i\alpha_{ij}}}{\sqrt{\frac{Q_i}{\lambda_i^2} + \frac{Q_j - \frac{Q_i}{\lambda_i}\lambda_j}{(\lambda_j+\lambda_i\alpha_{ij})^2}}} \right) \right] \lambda_i \alpha_{ij}, \quad i \neq j, \quad i,j = 1,...,N. \tag{3.27}$$

In order to determine an approximate value for the expected number of direct sales of product i to the customers of product i, we make the assumption that all the substitution from the other products to product i reduces the maximum number of items available for the customers of product i from Q_i to $Q_i - \sum_{j \neq i} \tilde{S}_{ji}$. Then the approximate value of the expected direct sales is

$$\tilde{S}_{ii} = E\left[min\{D_i, Q_i - \sum_{j\neq i}\tilde{S}_{ji}\}\right] = \lambda_i RT - \sqrt{\lambda_i RT}\eta\left(\frac{Q_i - \sum_{j\neq i}\tilde{S}_{ji} - \lambda_i RT}{\sqrt{\lambda_i RT}}\right) \quad (3.28)$$

where $\tilde{S}_{ij}$ is given in Equation (3.26).

3.2.2.2.1 Performance Evaluation

The expected direct and substitution sales numbers obtained from simulation and stochastic approximation for the 2 cases discussed in Section 3.2.1.2 are given in Table 3.13. The absolute percentage errors and absolute error differences are given in Table 3.14.

Table 3.13: Sales Amounts from Simulation and Stochastic Approximation for Case 1 of Section 3.2.1.2

Case 1 (Sales Amounts)	***Simulation***			***Approximation***		
	P1	***P2***	***P3***	***P1***	***P2***	***P3***
P1	362.806	0.000	7.634	359.494	0.099	1.368
P2	30.545	201.000	20.342	33.762	200.900	21.341
P3	0.003	0.000	199.981	0.000	0.000	199.988
Total Sales	393.354	201.000	227.957	393.256	201.000	222.697

As presented in Table 3.14, for the data in Table 3.5, the maximum absolute error in the approximation of direct sales is less than 1% with the average of 0.322%. Note that, when the simulated sales amounts are small, the absolute percentage error measure can be misleading. For example, in Table 3.14 the simulated substitution sales from *P3* to *P1* is 0.003 and the approximation yields 0.00002, this corresponds to an error of 99.320% while it is insignificant when its effect to the optimization problem is considered. In order to measure the effect of the error of the approximation on the inventory control problem, we

Table 3.14: Absolute Percentage Error and Error Difference Amounts of Stochastic Approximation for Case 1 of Section 3.2.1.2

Case 1 (Error)	***% Error***			***Difference Error***		
	P1	***P2***	***P3***	***P1***	***P2***	***P3***
P1	0.913%	0.000%	82.079%	3.312	0.000	6.266
P2	10.534%	0.050%	4.907%	3.218	0.100	0.998
P3	99.320%	0.000%	0.004%	0.003	0.000	0.008
Total Sales	0.000%	2.402%	0.023%	0.097	0.000	5.260

report the fraction of the difference of the simulated results and the approximated results with the total demand of the substituted product during the same period, i.e., $\frac{|\tilde{S}_{ij}-S_{ij}|}{\lambda_i RT}$. The resulting error measures are given in Table 3.18.

Table 3.15: $\frac{|\tilde{S}_{ij}-S_{ij}|}{\lambda_i RT}$ Error of Stochastic Approximation for Case 1 of Section 3.2.1.2

Case 1 (Error)	***% Error***		
	P1	***P2***	***P3***
P1	0.872%	0.038%	3.133%
P2	0.847%	0.038%	0.499%
P3	0.001%	0.000%	0.004%

The maximum effect of the substitution sales error of the approximation to the inventory control problem is less than 4% with the average of 0.753%.

For the data presented in Table 3.9, the approximated substitution and direct sales numbers are given in Table 3.16 together with the simulation results. The error values calculated as the rate of difference of the approximation and simulation results with the total demand of the substituted product are given in Table 3.16

For Case 2 of Section 3.2.1.2, the maximum effect of the substitution sales error of the approximation to the inventory control problem is less than 1% with the average of 0.087% while the maximum effect of the direct sales error of the approximation to the inventory

Table 3.16: Sales Amounts from Simulation and Stochastic Approximation for Case 2 of Section 3.2.1.2

Case 2 (Sales Amounts)	***Simulation***			***Approximation***		
	P1	***P2***	***P3***	***P1***	***P2***	***P3***
P1	378.300	0.385	0.370	378.225	0.518	0.462
P2	3.122	254.407	1.323	3.603	254.291	1.722
P3	1.686	0.814	196.755	1.820	0.979	196.442
Total Sales	383.108	255.606	198.449	383.649	255.788	198.626

Table 3.17: Absolute Percentage Error and Error Difference Amounts of Stochastic Approximation for Case 2 of Section 3.2.1.2

Case 2 (Error)	***% Error***			***Difference Error***		
	P1	***P2***	***P3***	***P1***	***P2***	***P3***
P1	0.020%	34.545%	24.865%	0.075	0.133	0.092
P2	15.407%	0.046%	30.159%	0.481	0.116	0.399
P3	7.948%	20.270%	0.159%	0.134	0.165	0.313
Total Sales	0.141%	0.071%	0.089%	0.541	0.182	0.177

Table 3.18: $\frac{|\tilde{S}_{ij}-S_{ij}|}{\lambda_i RT}$ Error of Stochastic Approximation for Case 2 of Section 3.2.1.2

Case 2 (Error)	***% Error***		
	P1	***P2***	***P3***
P1	0.020%	0.051%	0.046%
P2	0.127%	0.045%	0.199%
P3	0.035%	0.063%	0.156%

control problem is less than 1% with the average of 0.074%. This approximation evaluates the direct sales and substitutions that are relatively large quite accurately and therefore can be used in the optimization problem.

Chapter 4

OPTIMIZATION

Following Figure 4.1, two approximations are developed in Chapter 3 to determine the expected average inventory levels ($\bar{I}$), and direct and substitution sales (S_{ij}). In this chapter, by using these two approximations, we determine the approximated value of the expected profit in terms of the order levels. Then we construct a mixed integer nonlinear optimization formulation for the problem explained in Chapter 3 by using a mathematical programming approach. However, we were not able to get an integer solution for this mixed integer nonlinear optimization problem for all cases, then instead of dealing with a mixed integer nonlinear optimization problem, we build 2 nonlinear optimization problems. Since the integer variables are only needed to calculate the average inventory levels for determining the depletion orders and the position of the review period among depletion times, we first solve the optimization problem by ignoring inventory holding costs. From outputs of this optimization, we determine the values of integer values needed for the mixed integer nonlinear optimization problem, and then we use these variables as inputs for another run of the same optimization problem with inventory holding costs. For a number of cases, the optimum values of order levels and profit are obtained via a simulation based local search. And we evaluate the quality of the results of approximated optimization problem by comparing them with their optimum values. Later, in Chapter 5, the results of the approximated optimization problem are compared with the results obtained form a model which ignores the substitution effects to determine the effects of the input parameters on the profit and added value of calculating substitution effects.

In this section, we present the optimization problem explained in Chapter 3 by using a mathematical programming approach. The objective of the problem is to maximize the

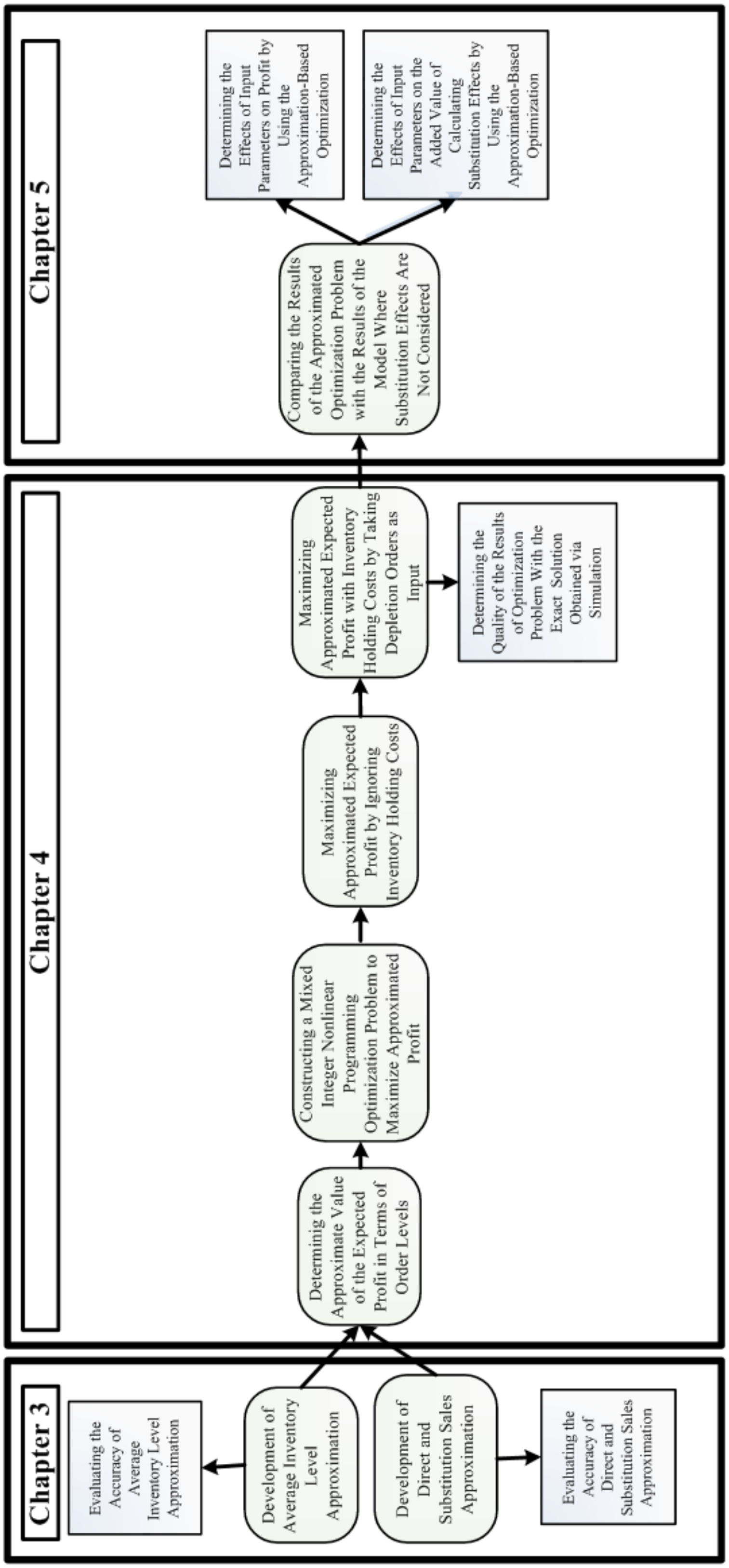

Figure 4.1: Flow Chart of the Concepts Considered Throughout This Thesis

total profit by finding the optimal order-up-to levels. In order to determine the optimal order levels, we constructed a mixed integer nonlinear programming problem. In this problem average inventory levels are calculated with the use of the approximation method explained in Chapter 3 and used to calculate the total inventory level cost, expected direct and substitution sales are calculated with the use of the approximation method explained in Chapter 3 and used to calculate the total sales revenue and total substitution cost.

The notation used in the formulation of the problem is as follows:

Parameters

N	number of substitutable products in a category
RT	review time
p_i	unit profit of product i per unit time, $i = 1, ..., N$
h_i	unit holding cost of product i, $i = 1, ..., N$
s_i	cost of substitution from product i, $i = 1, ..., N$
pp_i	unit purchase price of product i, $i = 1, ..., N$
λ_i	arrival rate of product i, $i = 1, ..., N$
DSL_i	minimum service level for product i, $i = 1, ..., N$
α_{ij}	substitution probability from product i to product j, $i, j = 1, ..., N$
MM	a sufficiently large number

Variables

X_{ij}	binary variable, 1 if product i is jth depleted product, 0 otherwise, $\quad i, j = 1, ..., N$
Y_i	binary variable, 1 if T_i is greater than review time, 0 otherwise, $\quad i = 1, ..., N$
$\overrightarrow{Q}_i$	order-up-to level of ith depleted product, $\quad i = 1, ..., N$
$\overrightarrow{\lambda}_i$	arrival rate of ith depleted product, $\quad i = 1, ..., N$
$\overrightarrow{\alpha}_{ij}$	substitution rate from ith depleted product to jth depleted product, $\quad i, j = 1, ..., N$
S_{ij}	substitution sales from product i to product j, $\quad i \neq j, \quad i, j = 1, ..., N$
S_{ii}	direct sales amount of product i, $\quad i = 1, ..., N$
TS_i	total sales amount of product i, $\quad i = 1, ..., N$
T_i	depletion time of ith depleted product, $\quad i = 1, ..., N$
TP_i	ith substitution time period, $\quad i = 1, ..., N$
O_{ij}	the depletion rate of jth depleted product in $[T_{i-1}, T_i]$ time period, $\quad i, j = 1, ..., N$
QN_{ij}	inventory level of ith depleted product at time T_{j-1}, $\quad i, j = 1, ..., N$
$\bar{I}_i$	average inventory level of product i, $\quad i = 1, ..., N$
$\overrightarrow{I}_i$	average inventory level of ith depleted product, $\quad i = 1, ..., N$

Decision Variables

Q_i order-up-to level of product i, $\quad i = 1, ..., N$

The formulation of the optimization problem is;

$$\text{maximize Profit} = \sum_{i=1}^{N} \left(p_i TS_i - h_i \bar{I}_i - \left(\sum_{\substack{j=1 \\ j \neq i}}^{N} S_{ij} \right) s_i \right)$$

subject to

$$TS_i = \sum_{j=1}^{N} S_{ji}, \quad \forall \quad i = 1, ..., N \tag{4.1}$$

$$S_{ii} = \lambda_i RT - \sqrt{\lambda_i RT}\eta \left(\frac{Q_i - \sum_{j \neq i}^{N} S_{ji} - \lambda_i RT}{\sqrt{\lambda_i RT}} \right), \quad \forall \quad i = 1, ..., N \tag{4.2}$$

$$\tilde{S}_{ij} = \Bigg[\Bigg(\sqrt{2\frac{Q_i}{\lambda_i^2} + \frac{Q_j - \frac{Q_i}{\lambda_i}\lambda_j}{(\lambda_j + \lambda_i \alpha_{ij})^2}}\,\eta \left(-\frac{\frac{Q_j - \frac{Q_i}{\lambda_i}\lambda_j}{\lambda_j + \lambda_i \alpha_{ij}}}{\sqrt{2\frac{Q_i}{\lambda_i^2} + \frac{Q_j - \frac{Q_i}{\lambda_i}\lambda_j}{(\lambda_j + \lambda_i \alpha_{ij})^2}}} \right) + \sqrt{2\frac{Q_i}{\lambda_i^2} + \frac{Q_j - \frac{Q_i}{\lambda_i}\lambda_j}{(\lambda_j + \lambda_i \alpha_{ij})^2}}\,\eta \left(\frac{RT - \frac{Q_j - \frac{Q_i}{\lambda_i}\lambda_j}{\lambda_j + \lambda_i \alpha_{ij}}}{\sqrt{2\frac{Q_i}{\lambda_i^2} + \frac{Q_j - \frac{Q_i}{\lambda_i}\lambda_j}{(\lambda_j + \lambda_i \alpha_{ij})^2}}} \right)$$

$$-RT\Phi \left(\frac{RT - \frac{Q_i}{\lambda_i} + \frac{Q_j - \frac{Q_i}{\lambda_i}\lambda_j}{\lambda_j + \lambda_i \alpha_{ij}}}{\sqrt{\frac{Q_i}{\lambda_i^2} + \frac{Q_j - \frac{Q_i}{\lambda_i}\lambda_j}{(\lambda_j + \lambda_i \alpha_{ij})^2}}} \right) \Bigg) \left(1 - \Phi \left(\frac{RT - \frac{Q_i}{\lambda_i}}{\frac{\sqrt{Q_i}}{\lambda_i}} \right) \right) \left(1 - \Phi \left(\frac{RT - \frac{Q_i}{\lambda_i} + \frac{Q_j - \frac{Q_i}{\lambda_i}\lambda_j}{\lambda_j + \lambda_i \alpha_{ij}}}{\sqrt{\frac{Q_i}{\lambda_i^2} + \frac{Q_j - \frac{Q_i}{\lambda_i}\lambda_j}{(\lambda_j + \lambda_i \alpha_{ij})^2}}} \right) \right)$$

$$+ \left((RT - \tfrac{Q_i}{\lambda_i})(\tfrac{\sqrt{Q_i}}{\lambda_i})\eta \left(\frac{RT - \frac{Q_i}{\lambda_i}}{\frac{\sqrt{Q_i}}{\lambda_i}} \right) \right) \Phi \left(\frac{RT - \frac{Q_i}{\lambda_i} + \frac{Q_j - \frac{Q_i}{\lambda_i}\lambda_j}{\lambda_j + \lambda_i \alpha_{ij}}}{\sqrt{\frac{Q_i}{\lambda_i^2} + \frac{Q_j - \frac{Q_i}{\lambda_i}\lambda_j}{(\lambda_j + \lambda_i \alpha_{ij})^2}}} \right) \Bigg] \lambda_i \alpha_{ij}, \quad i \neq j, \quad \forall \quad i, j = 1, ..., N \tag{4.3}$$

$$DSL_i \leq \frac{S_{ii}}{\lambda_i RT}, \quad \forall \quad i = 1, ..., N \tag{4.4}$$

$$\sum_{i} X_{ij} = 1, \quad \forall \quad j = 1, ..., N \tag{4.5}$$

$$\sum_{j} X_{ij} = 1, \quad \forall \quad i = 1, ..., N \tag{4.6}$$

$$\overrightarrow{Q}_i = \sum_{j=1}^{N} X_{ji} Q_j, \quad \forall \quad i = 1, ..., N \tag{4.7}$$

$$\overrightarrow{\lambda}_i = \sum_{j=1}^{N} X_{ji} \lambda_j, \quad \forall \quad i = 1, ..., N \tag{4.8}$$

$$\overrightarrow{\alpha}_{ij} = \sum_{k=1}^{N} \sum_{n=1}^{N} X_{ki} \alpha_{kn} X_{nj}, \quad \forall \quad i, j = 1, ..., N \tag{4.9}$$

$$T_1 = \frac{\overrightarrow{Q}_1}{\overrightarrow{\lambda}_1} \qquad\qquad TP_1 = T_1 \tag{4.10}$$

$$T_i = TP_i + T_{i-1}, \quad \forall \quad i = 2, ..., N \tag{4.11}$$

$$T_i \leq RT + MM(1 - Y_i), \quad \forall \quad i = 1, ..., N \tag{4.12}$$

$$T_i \geq RTY_i, \quad \forall \quad i = 1, ..., N \tag{4.13}$$

$$O_{i1} = \overrightarrow{\lambda}_i, \quad \forall \quad i = 1, ..., N \tag{4.14}$$

$$O_{ij} = O_{ij-1} + O_{j-11}\overrightarrow{\alpha}_{j-1i}, \quad j > 1 \ , \ i \geq j, \quad \forall \quad i, j = 1, ..., N \tag{4.15}$$

$$O_{ij} = 0, \quad i < j, \quad \forall \quad i, j = 1, ..., N \tag{4.16}$$

$$QN_{i1} = \overrightarrow{Q}_i, \quad \forall \quad i = 1, ..., N \tag{4.17}$$

$$QN_{ij} = QN_{ij-1} - TP_{j-1}O_{ij-1}, \quad j > 1 \ , \ i \geq j, \quad \forall \quad i, j = 1, ..., N \tag{4.18}$$

$$QN_{ij} = 0 \quad i < j, \quad \forall \quad i, j = 1, ..., N \tag{4.19}$$

$$\overrightarrow{I}_i = \frac{1}{RT}\left(\sum_{j=1}^{i}\frac{TP_j^2 O_{ij}}{2} + \sum_{j=1}^{i-1} QN_{ij+1}TP_j\right), \quad \forall \quad i = 1, ..., N \tag{4.20}$$

$$\bar{I}_i = \sum_{j=1}^{N} X_{ij}\overrightarrow{I}_j, \quad \forall \quad i = 1, ..., N \tag{4.21}$$

$$Q_i, \tilde{S}_{ij}, TP_i, \overrightarrow{I}_i, O_{ij}, QN_{ij} \geq 0 \qquad \forall \ \ i, j = 1, ..., N \tag{4.22}$$

$$X_{ij}, Y_i \ \ \epsilon \ \ \{0, 1\}, \quad i, j = 1, ..., N \tag{4.23}$$

The objective of this optimization problem is to maximize the profit. Profit is calculated as the difference between the total sales revenue and total costs. Revenue of a product is calculated by multiplying its total direct and substitution sales with its unit profit. The total revenue is the sum of the revenues generated from all products. Costs considered in this model are the total inventory holding costs and total substitution costs.

In the above formulation, total sales amount of product i is calculated in Equation 4.1. Estimated average direct sales of product i and estimated substitution sales of product j for product i are calculated in Equations 4.2 and 4.3, respectively. Equation 4.4 represents the service level constraint mentioned in Section 3.1. This constraint ensures that the service level, the rate of direct sales and the total demand of the product, is greater than the parameter DSL for; all products. In order to calculate $\bar{I}$, Equations 4.5 through 4.21 are added to the optimization problem. In the model, it is assumed that the probability of having two arrivals in the same period is very small, therefore two or more products cannot be depleted at the same time, and only one product can be the jth depleted product. These conditions are satisfied in Equations 4.5 and 4.6, respectively. In Equations 4.7 and 4.8, the order up to levels and demand rates are arranged respectively according to products' depletion orders. The substitution rate matrix is calculated according to the depletion order in Equation 4.9. Equations 4.10 stands for the initializations of matrices T and TP. Depletion times are calculated via substitution time periods in Equation 4.11. In Equations 4.12 and 4.13 it is determined that whether the depletion time of product is greater than the review period or not, and the depletion times greater than the review period are forced to be equal to the review period. In Equations 4.14, 4.15 and 4.16 depletion rate matrix O is calculated. In Equations 4.17, 4.18 and 4.19 the inventory level at depletion times matrix, QN, is calculated. Average inventory levels of all products are calculated according to their depletion order in Equation 4.20 and average inventory level values of each product is assigned to the products in Equation 4.21.

In order to evaluate the performance of the optimization model, the optimal order-up-to levels of the above optimization are used as inputs to the simulation program explained in Appendix A, and the results of this simulation runs are compared with the simulation of the case where order-up-to levels are calculated as explained Appendix E without taking into account of the substitution effects. The method explained in Appendix E does not take into account of the substitution effects and evaluates all products' inventory levels independently. In the optimization problem, service level constraint sets a lower limit for order-up-to levels, but there is no upper limit. And when the holding costs are small relative to the unit profits, stocking all products in very large quantities can become the optimum solution. But in real life applications, retailer should not be able to hold so much inventory because of the space and financial limitations. In order to compare the two cases, i.e. models with and without substitution, under the same circumstances, we should add a constraint to the optimization problem to set an upper limit for order-up-to levels, also this constraint should be valid for both cases. Adding a budget constraint to the optimization problem will limit the order amounts, and using the purchasing budget of the method in Appendix E will put the two methods under the same operating conditions.

If we denote the order-up-to levels of the case where substitution effects are not considered with SS_i and purchase price of the products with pp_i, we can calculate the budget limit for the optimization problem as in Equation (4.24):

$$OPL = \sum_{i=1}^{N} pp_i SS_i. \tag{4.24}$$

Then we can build the budget constraint of the optimization problem as in Equation (4.25).

$$OPL \geq \sum_{i=1}^{N} pp_i Q_i. \tag{4.25}$$

When we solve this MINLP problem, we were not able to get an integer solution for all cases. Since we have to know the depletion orders of products to calculate the average inventory levels of products, we add Equations 4.5 through 4.21 to the optimization problem. In order to determine the depletion orders of products and the position of the review time among depletion orders we define two integer variables (X and Y). In order to deal with these integer variables, we first evaluate the same optimization problem by ignoring inventory holding costs. In order to do this, we use Equations 4.1 through 4.4, 4.24 and 4.25 to form a NLP problem. By solving this NLP problem we determine the depletion orders of products and the position of the review time among depletion orders and develop X and Y variables. Later these determined X and Y variables are used as inputs to the MINLP problem and this problem then solved as a NLP problem. By this way, instead of solving the MINLP problem we solve two NLP problems.

Chapter 5

NUMERICAL ANALYSIS

In this chapter, we present the results of our computational study. In the first part, we explain the problem generation for the numerical study, and in the second part, we analyze the performance of the optimization problem formulated in Chapter 4 by comparing its solutions with the optimal solutions obtained from a local search of the simulation code explained in Appendix A. In the third part, we analyze the effects of changes in the input parameters on profit by comparing the results of the optimization problem discussed in Chapter 4 with the results of the method explained in Appendix E. In this method, the order-up-to level, in case of a periodic review system with lost sales and normal demand, is determined for a given desired service level β as follows (Silver et al. [24]):

$$S = \mu(RT + L) + z_\beta \sigma \sqrt{RT + L} \tag{5.1}$$

where μ is the mean and σ is the standard deviation of demand, RT is the length of the review period, L is the lead time and z_β is the safety stock multiplier which satisfies,

$$G_u(z_\beta) = \frac{(1-\beta)\mu RT}{\beta\sigma\sqrt{RT + L}}, \tag{5.2}$$

and

$$G_u(z_\beta) = \int_z^\infty (x - z) \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} dx. \tag{5.3}$$

5.1 Problem Generation

In the studied cases, the retailer uses a fixed review period order-up-to level (R, S) system to control the inventory. According to this system, at every R units of time (i.e. at every review point) an order is given to raise the inventory position to a predetermined level S. We assume that the lead time is zero without any loss of generality.

In the system, we assume that the customers arrive under a Poisson process, specifically the arrival rate of the customers of product i is assumed to be Poisson with rate λ_i. Since the interarrival times of the customers are independent from each other, and the assumption of arriving one customer at a time interval can be justified in a retail setting, then the assumption of Poisson arrivals is plausible. Upon arrival in the store, from a certain product category, the customer either buys one product or buys nothing. For the no purchase option, since we do not allow a second substitution attempt, there are only two scenarios:

1. the customer's first choice product is stocked-out at the time of the shopping and the customer did not substitute to another product (with probability δ)

2. the customer substituted to a product but that product was stocked-out.

For the purchase option, the customer either buys his favorite product i, or if it is not available at the store at the time of shopping, he substitutes to product j with probability α_{ij}, which is available at the store.

We employ a probabilistic model of substitution which is also used in Netessine and Rudi [17], Smith and Agrawal [25] and Kök and Fisher [13]. In this model, every customer has a first-choice product in a certain product category. If the first-choice product is not available due to a temporary stock-out, the second choice product is chosen according to the substitution probability matrix α. The substitution probability matrix is calculated as explained in Chapter 3, Equation (3.13).

5.2 Evaluation of the Optimization Problem Solutions

In order to evaluate the performance of the optimization model, we will compare the resulting optimum profit values of the optimization model with the real optimum value of the total profit under the same circumstances.

In order to determine the real optimum amounts of the order-up-to levels and the resulting profit values, we perform a simulation-based local search (see Appendix A for the details of the simulation model). In this search, arrival rates, unit profits, unit purchase prices, and substitution costs are given as inputs for all products in a category. Holding costs are calculated as a percentage of each product's purchase price and the percentage is also given as case input. Substitution rates are calculated by the use of the arrival rate values of products as in Equation 3.13. We take the review time as 20 days, and replenishments take place with the same periodicity.

We assume that, in the present situation, the retailer sets the inventory levels of all products according to a periodic review system with lost sales and normal demand for a given desired service level (as explained in Appendix E). In this system, the retailer examines each product independently and sets all products' order-up-to levels to achieve the proposed desired service level. We first determine the order-up-to levels of this system, then we determine the retailer's budget by calculating the required amount of money to have that much inventory.

In the local search of the overall optimal profit value, we set the minimum order-up-to level of product i to be 60% of its total demand, $\lambda_i RT$, which means at least 60% of all products' customers should be satisfied by their first choice products, and then we calculate its maximum order amount by setting the order-up-to levels of all other products in the assortment to their minimum levels, and allocating all remaining budget to product i. After determining the minimum and maximum order-up-to levels of all products, we take the average results of 10 simulations with 100 consecutive review periods (for almost 55 years) for all possible order-up-to level combinations for a 3 product environment. Finally, we call the order-up-to level combination which gives the maximum profit in this search as "real optimum order-up-to levels" and we call the resulting profit the "real optimum profit."

Although this search gives us optimum order-up-to levels and optimum profit values in a 3-product setting, it is not generatible because it takes too much CPU time. With a 8 GB RAM and Intel(R) Core(TM)2 Duo 3.2 GHz processor computer, a 3-product search takes almost 4 days, and as the number of products increases, the needed time to get optimum values increases exponentially. Because of this time restriction, we determine the optimum levels for 5 cases with 3 products in a category, and evaluate the same cases with our approximation method.

For the 5 cases we consider, the replenishment period is considered to be 20 days. Input parameters (arrival rates, unit purchase prices, unit profits, unit substitution costs from a product) used for the 5 cases are presented in Table 5.1. Holding cost per product per unit time is calculated as 30% of the purchase price of the product. The desired service level to be achieved for the without substitution method is set to be 85%, which will be used to determine the budget of the analyzed system.

Table 5.1: Input Parameters for 5 Cases

Input Parameters	***Arrival Rate***			***Purchase Price***			***Unit Profit***			***Substitution Cost***		
	P1	***P2***	***P3***	***P1***	***P2***	***P3***	***P1***	***P2***	***P3***	***P1***	***P2***	***P3***
Case 1	21	14	14	1	1	1	2	2	2	0.05	0.05	0.05
Case 2	14	14	14	1	1	1	4	2	2	0.05	0.05	0.05
Case 3	14	14	14	1	1	1	3	4	9	0.05	0.05	0.05
Case 4	11	15	19	1	1	1	2	2	2	0.05	0.05	0.05
Case 5	12	7	16	2	0.4	1.6	3	4	9	0.06	0.12	0.24

Using these input parameters, real optimum order-up-to levels and the optimum order-up-to levels obtained through the optimization problem are presented in Table 5.2.

After we obtain the results of the optimization and the optimum order-up-to levels from

Table 5.2: Optimum Order-up-to levels of the 2 Methods for 5 Cases

Order Levels	***Real Optimum***			***Approximation Optimization***		
	P1	***P2***	***P3***	***P1***	***P2***	***P3***
Case 1	388	258	253	390	254	255
Case 2	400	168	200	396	188	184
Case 3	168	180	420	172	172	418
Case 4	200	275	349	196	275	353
Case 5	144	178	374	144	174	375

simulation, we use these amounts as inputs and run 50 simulations with 100 consecutive review periods with the same random seeds and the resulting profits are reported in Table 5.3 for both methods.

Table 5.3: Optimum Profits of the 2 Methods for 5 Cases

Optimum Profits	***Real Optimum***	***Approximation Optimization***	***Optimization Error (%)***
Case 1	1789.672	1789.587	0.0048
Case 2	2242.521	2228.647	0.6187
Case 3	4706.625	4688.228	0.3909
Case 4	1640.325	1640.264	0.0038
Case 5	4430.293	4428.835	0.0329

As we can see from Table 5.3, the error of the approximation method is less than 1 %. According to the results we generate in this section, we see that the approximation method gives near optimum solutions and we can use this method with confidence to analyze problems with more products.

5.3 Analysis of Substitution Effects on Profit

In this section, we address the question of whether the effect of substitutions on the overall profit is substantial. In order to see the impact of substitutions on the profit, we compare the optimal profit obtained with optimization with the profit of the case where the order-up-to levels are determined without taking substitutions into account. We assume that the order-up-to levels, in case of a periodic review system with lost sales and normal demand (not affected by substitutions), is determined for a given desired service level β as explained in Appendix E. We designate this solution to be the base case, and refer to it as the "without substitution" case (WOS). And also we use the optimization model based on the approximation of substitutions, and refer to it as the "with substitution" case (WS). We analyze the effect of substitution on profit along with the effects of the input parameters in the following sections.

5.3.1 Data Generation

For the purpose of comparing the two methods, the same input data and the same amount of budget is used. The review period is taken as 20 days, and the holding cost per unit time of a product is taken as 30 % of the unit purchasing cost of that product. Arrival rates, desired service level for the WOS method, minimum service level to be achieved, total substitution probability, unit purchase prices, unit substitution costs and unit profits are given as inputs and set as indicated below.

⇒ *Arrival Rates:* Arrival rates are analyzed for 4 different scenarios with 5 products where the total arrival for the related product category is 50. In the first scenario all products have the same market share (i.e. $\lambda_i = 10,\ i = 1, ..., 5$). In the second scenario there is one market leader with a market share of 60% and other products share the remaining market share equally (i.e. $\lambda_1 = 30,\ \lambda_i = 5,\ i = 2, ..., 5$). In the third scenario there is one market leader with a market share of 40% and a main competitor which has

30% market share and remaining 3 products have equal market shares of 10% each (i.e. $\lambda_1 = 20$, $\lambda_2 = 15$, $\lambda_i = 5$, $i = 3, ..., 5$). In the fourth scenario there are three market leaders with equal shares of 30% and two minor competitors with the 5% of market share each (i.e. $\lambda_i = 15$, $i = 1, 2, 3$, $\lambda_j = 2.5$, $j = 4, 5$).

⇒ *Desired service level for the WOS method:* The desired service level for the WOS method is studied in three levels as 85%, 90% and 95%.

⇒ *Minimum service level:* Minimum service level to be achieved is studied in three levels as 40%, 50% and 60%.

⇒ *Total substitution probability:* The probability that a customer who is unable to find his first choice product on the shelf and willing to buy another item is set as 0.4, 0.6, 0.8 and 1.

⇒ *Unit purchase prices:* Unit purchase price of a product is calculated with the use of its market share rate as follows: Unit purchase price of product i=1−Market share of product $i * K$ and K is set as $-0.5, -0.25, 0, 0.25$ and 0.5.

⇒ *Unit substitution costs:* The cost of customer substitution from a product for the retailer is set as a percentage of that product's unit profit. This percentage is considered to be equal to 0%, 10%, and 20% and 30% .

⇒ *Unit profits:* The unit profit of a product is calculated as a percentage of its purchasing price. These percentages are taken as 10%, 15%, 20%, 40% and 30% for products 1, 2, 3, 4 and 5, respectively in one scenario, and 50%, 60%, 70%, 50% and 40% for products 1, 2, 3, 4 and 5, respectively in a second scenario.

By using the above outlined input data, all possible cases (5760 cases) are created. The resulting profits of the two methods (WS and WOS) for all scenarios are calculated, and used to analyze the effects of the input parameters on the profit when substitution effects are considered. The average profits from all scenarios of the two models are reported in Table 5.4.

Table 5.4: Resulting Profits for the Two Methods: WS and WOS

Profit (WS)	Profit (WOS)	Difference	% Increase
340.1637747	324.1910666	15.9727081	4.93%

As seen in Table 5.4, taking the substitution effects into consideration results in more profit then the case where substitution effects are ignored. When we consider the substitution affects, we observe that the profit increases 5% on average.

5.3.2 The effect of arrival rates

Table 5.5: The effect of arrival rates on profit

Arrival Rates	**Profit (WS)**	**Profit (WOS)**	**Difference**	**% Increase**
10-10-10-10-10	358.6506	341.0412	17.6094	5.16%
30- 5 - 5 - 5 - 5	323.4561	303.9033	19.5529	6.43%
20-15- 5 - 5 - 5	330.9853	317.1224	13.8630	4.37%
15-15-15-2.5-2.5	347.5631	334.6965	12.8666	3.84%

In Table 5.5, we can observe that the distribution of the arrival rates of products can have a significant impact on profit. When all products have the same arrival rates, all input parameters for these products become equal except their profit margins. With the same cost parameters and demand rates, considering substitution affects results in 5% more profit on average. This increase stems from the profit margin percentage differences between products. Therefore, in order to analyze the effects of arrival rates of the products we should also evaluate them by grouping them according to their profit margins. In Table 5.6, the results are presented in terms of profit margins and arrival rates.

Table 5.6: The effect of arrival rates and profit margins on profit

Profit Margin %	**Arrival Rates**	**Profit (WS)**	**Profit (WOS)**	**Difference**	**% Increase**
10-15-20-40-30	**10-10-10-10-10**	223.32505	201.4587127	21.8663	10.85%
	30-5-5-5-5	175.16662	143.7475127	31.4191	21.86%
	20-15-5-5-5	170.91309	152.5764162	18.3367	12.02%
	15-15-15-2.5-2.5	159.7866	147.8909156	11.8957	8.04%
50-60-70-50-40	**10-10-10-10-10**	493.97611	480.6254712	13.3506	2.78%
	30-5-5-5-5	471.74563	464.0590344	7.68659	1.66%
	20-15-5-5-5	491.0575	481.6683463	9.38915	1.95%
	15-15-15-2.5-2.5	535.33961	521.5021236	13.8375	2.65%

The highest profit increase is experienced in the case with the 1st case of unit profits and the second case of arrival rates. In this case, *P1* has the highest market share and lowest profit margin. Besides, the arrival rates of remaining products are the same and, the probability of substituting to those products is high and equal to each other. In this case, in the WS model the order-up-to level of the 1st product is set to its minimum level to make its customers to substitute to other products which have higher profit margins. For the 1st case of unit profits, the minimum profit increase is with 4th case of arrival rates. In this case, the two products with the lowest market share (products 4 and 5) have higher profit margins than the other products, but with the data generation method explained above, having lowest market share means having higher purchase prices and lower probability of being used as a substitute. Then, even though the profit margins of these two products are higher, together with the fact that the probabilities of substituting to these products are smaller and the purchase prices for these products are higher, substituting these products in place of products 1, 2 and 3 becomes unlikely. Moreover, since the arrival rates of first 3 products are equal, the purchasing prices of the first 3 products are equal, and, since their arrival rates are high, the probability of the substitution occurrence among these products is high. Then, in this case the majority of the profit increase will be gained from substituting to *P3*, rather than substituting to products 4 and 5, consequently the resulting profit increase will be lower than the other cases.

5.3.3 The effect of desired service level

Table 5.7: The effect of desired service levels on profit

Desired Service Level (%)	**Profit (WS)**	**Profit (WOS)**	**Difference**	**% Increase**
85	325.77339	305.88902	19.884366	9.77%
90	340.30446	323.83225	16.472202	7.79%
95	354.41348	342.85193	11.561556	5.50%

In Table 5.7, we observe that, as the desired service level for the WOS method increases, the profit increases. This is an expected behavior, because when the desired service level for the WOS method increases, the amount of available budget increases, and also the number of lost sales decreases. Although the substitution effects are not considered while determining the order-up-to levels in the WOS method, the number of direct sales increases abd the number of substitutions decreases so the resulting profit increases. In the WS method, the substitution numbers are increased together with the direct sales, and results an increase in profit, but this increase is not as much as the amount gained in WOS method from direct sales. Therefore, as the desired service level for the WOS method increases the profit difference between the two methods decreases.

5.3.4 The effect of the minimum service level

As seen in Table 5.8, the amount of the minimum service level does not have any effect on the WOS model. The major difference between these two methods is observed with the

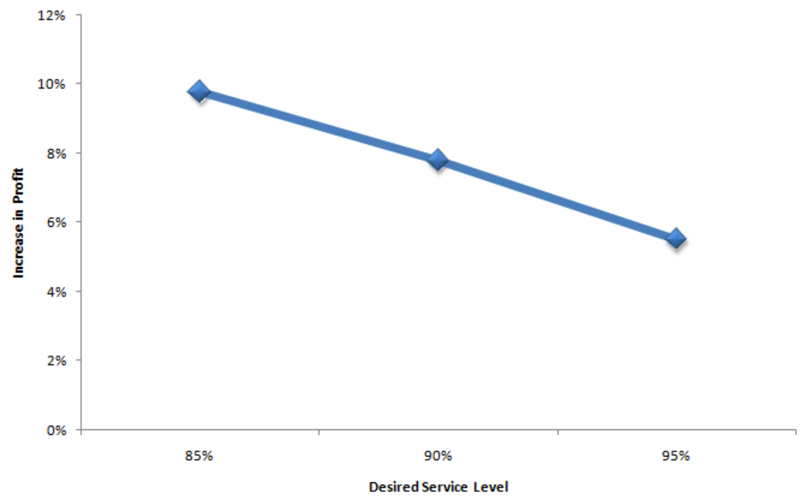

Figure 5.1: The Effect of Changes of Desired Service Levels on Profit When Taking Into Account of the Substitution Effects

Table 5.8: The effect of minimum service levels on profit

Minimum Service Level (%)	**Profit (WS)**	**Profit (WOS)**	**Difference**	**% Increase**
40	342.23110	324.19107	18.040035	8.66%
50	340.26919	324.19107	16.078118	7.65%
60	337.99104	324.19107	13.799972	6.50%

changing minimum service level. Since the substitution effects are not considered, in the WOS method the desired service level is achieved for all products. On the other hand, the WS method aims to satisfy a sufficient amount of all products' customers (minimum service level) then spends the rest of the budget on the most profitable product, and gain more profit from substitution to this product. In Table 5.8, we observe that as the minimum service level to be achieved increases, the profit in WS method decreases. This is reasonable because as the minimum service level to be achieved increases the minimum order-up-to levels to achieve for each product increases and consequently remaining budget to buy extra amounts of the more profitable items decreases. Since the number of customers satisfied with their first choice increases, and therefore the substitution amounts decreases, the WS method converges to the WOS method.

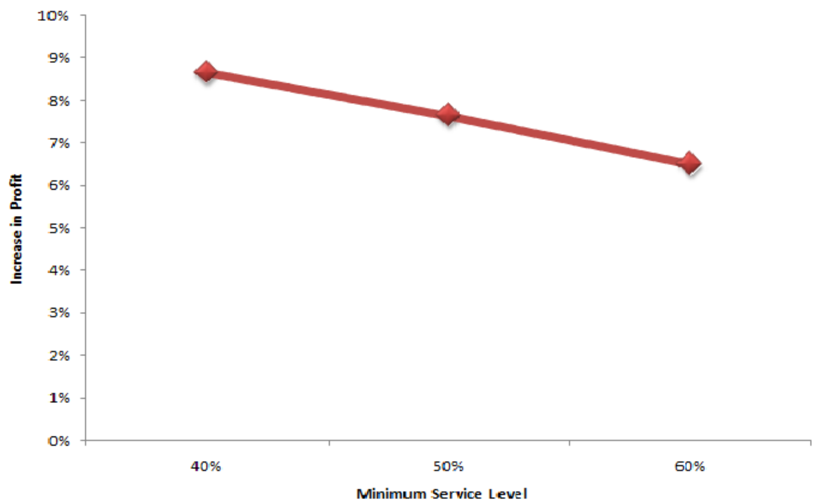

Figure 5.2: The Effect of Changes of Minimum Service Levels on Profit When Taking Into Account of the Substitution Effects

5.3.5 The effect of total substitution probability

Table 5.9: The effect of total substitution probability on profit

δ	**Profit (WS)**	**Profit (WOS)**	**Difference**	**% Increase**
0.4	334.1269537	324.4564314	9.67052231	4.51%
0.6	338.1020535	324.3103741	13.7916794	6.62%
0.8	342.2119812	324.1160631	18.0959181	8.68%
1	346.2141105	323.8813977	22.3327127	10.61%

In Table 5.9, we observe that as the total substitution probability increases the profit calculated from WS method increases. This is reasonable, because as the total substitution probability increases the lost sales decreases and the total substitution increases. On the other hand, since the order-up-to levels are calculated independently in the WOS method there are not so many substitutions between products and the increase in the total substitution probability does not affect the total profit of this model.

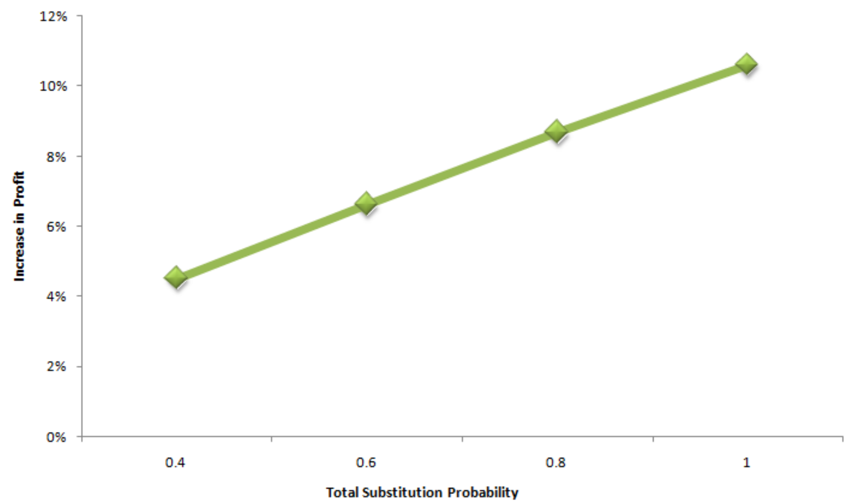

Figure 5.3: The Effect of Changes of Total Substitution Probability on Profit When Taking Into Account of the Substitution Effects

5.3.6 The effect of unit purchase prices

Table 5.10: The effect of unit purchase prices on profit

K	**Profit (WS)**	**Profit (WOS)**	**Difference**	**% Increase**
-0.5	201.9523277	180.1179381	21.8343895	% 12.12
-0.25	192.2038146	170.7681637	21.4356509	% 12.55
0	182.3977573	161.4183893	20.9793681	% 13.00
0.25	172.4919605	152.0686148	20.4233457	% 13.43
0.5	162.4433407	142.7188404	19.7245003	% 13.82

From the definition of the purchase prices explained in the previous section, we know that, as K increases, the purchase price decreases. And, since the unit profit is calculated as a percentage of the unit purchase prices, as the purchase price of a product decreases (K increases), its unit profit decreases. Therefore, the resulting profit values decrease in both methods as the value of K increases (see Table 5.10). (Also as K changes, the change in the profit increase is negligible, therefore we can conclude that changes in K does not have any effect on profits.) In the WOS model order-up-to levels of products are set by only taking into account of their arrival rates and the desired service level. Since none of these parameters change for this case, the order-up-to levels do not change and the decrease stems from only the decrease of the unit prices. The same effect is valid for the WS method, but in this method, since order-up-to levels are set to maximize the total profit by taking into account of all products' purchase prices and profits, the total decrease in this method is lower than that of the WOS method.

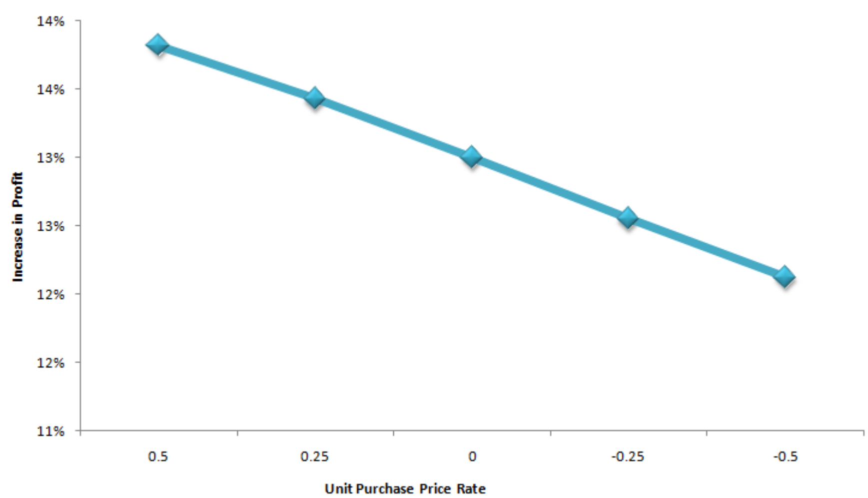

Figure 5.4: The Effect of Changes of Unit Purchase Price Rates on Profit When Taking Into Account of the Substitution Effects

5.3.7 The effects of unit substitution costs

Table 5.11: The effect of unit substitution costs on profit

Unit Substitution Cost Rates	**Profit (WS)**	**Profit (WOS)**	**Difference**	**% Increase**
0 %	184.4479808	162.2654578	22.182523	13.67%
10%	183.0090432	161.7007454	21.308297	13.18%
20%	181.5788113	161.1360331	20.442778	12.69%
30%	180.1555254	160.5713208	19.584205	12.20%
40%	178.7403888	160.0066084	18.733780	11.71%
50%	177.3266062	159.4418961	17.884710	11.22%
60%	175.9270633	158.8771837	17.049880	10.73%
70%	174.5618600	158.3124714	16.249389	10.26%
80%	173.1958002	157.7477590	15.448041	9.79%
90%	171.8571998	157.1830467	14.674153	9.34%

As explained in the previous section, the cost of substituting from a product is calculated as a percentage of its unit profit. In Table 5.11, we observe that, as the substitution cost percentage increases, the profit decreases in both methods. As the substitution costs are decreased, order-up-to levels of method WOS do not change. In the WS method as the substitution cost increases, the number of substitutions decreases in order to prevent the decrease in the profit. But since the avoided substitution cost decrease is very small comparing to the profit gained from the direct sales of that product, i.e. substitution sales of a product is much smaller than its direct sales, any change in the substitution costs does not have a significant effect on the profit.

5.3.8 The effect of unit profits

As observed in Table 5.12, the percentage profit increase when the 1st case of the profit margins is used is much higher than the case where the 2nd case of the profit margins is used. In the 2nd case, the profit margins are close to each other, and in the 1st case the

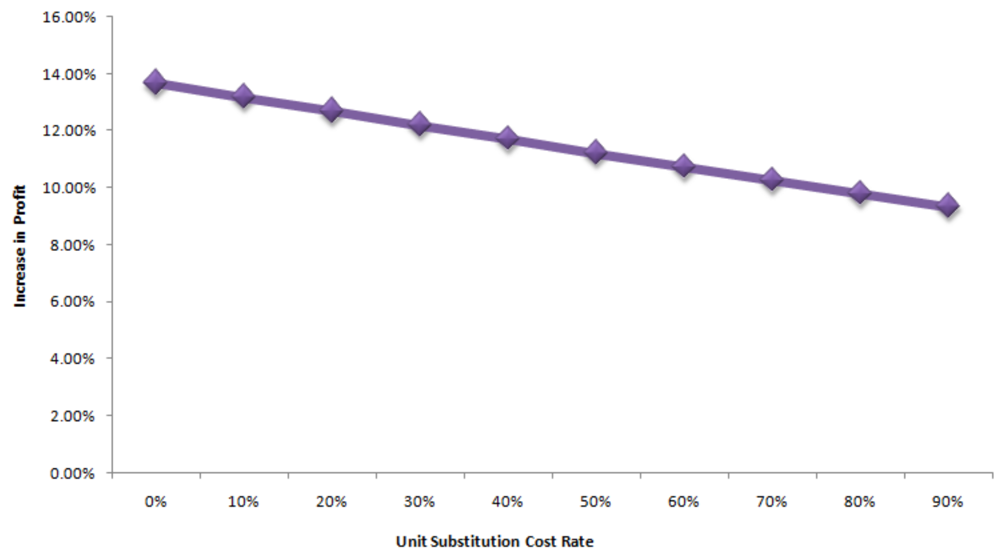

Figure 5.5: The Effect of Changes of Unit Unit Substitution Cost Rates on Profit When Taking Into Account of the Substitution Effects

Table 5.12: The effect of unit profits on profit

Profit Margin (%)	**Profit (WS)**	**Profit (WOS)**	**Difference**	**% Increase**
10-15-20-40-30	182.29784	161.41839	20.879451	% 12.94
50-60-70-50-40	498.02971	486.96374	11.065965	% 2.27

difference between the profit margin values of the 5 products are higher so the percentage profit increase is higher. In order to support this idea and see the corresponding effect, the effect of the profit margin values are analyzed in detail in the next section.

5.3.8.1 The detailed analysis of the effects of profit margins

In this section, we consider additional profit margin cases. The profit margin values are calculated proportionally with the arrival rates as follows. First, minimum and maximum profit margin levels are determined and, by using the idea that in retailing sector a product which has a higher market share might have lower profit margin, the profit margin of the product which has highest market share is set to the predetermined minimum profit margin value and the profit margin of the product which has lowest market share is set to the predetermined maximum profit margin value. Then the remaining products' profit margin values are calculated linearly according to their market shares. In order to cover all possible scenarios, 4 different cases are analyzed. In the 1st case the minimum profit margin value is low and the difference between the minimum and maximum profit margin levels is low (LL), in the 2nd case the minimum profit margin value is high and the difference between the minimum and maximum profit margin levels is low (HL), in the 3rd case the minimum profit margin value is low and the difference between the minimum and maximum profit margin levels is high (LH) and in the 4th case the minimum profit margin value is high and the difference between the minimum and maximum profit margin levels is high (HH). The low and high levels of the minimum profit margin are determined as 10% and 50%, respectively. The low and high levels of the minimum and maximum profit margin difference are determined as 10% and 30%, respectively. The profit margin cases according to the arrival rates of products are shown in Table 5.13

Table 5.13: The profit margin cases for 4 cases

Arrival Rates	**LL Case Profit Margin (%)**	**HL Case Profit Margin (%)**	**LH Case Profit Margin (%)**	**HH Case Profit Margin (%)**
10-10-10-10-10	10-10-10-10-10	50-50-50-50-50	10-10-10-10-10	50-50-50-50-50
30-5-5-5-5	10-20-20-20-20	50-60-60-60-60	10-40-40-40-40	50-80-80-80-80
20-15-5-5-5	10-13.3-20-20-20	50-53.3-60-60-60	10-20-40-40-40	50-60-80-80-80
15-15-15-2.5-2.5	10-10-10-20-20	50-50-50-60-60	10-10-10-40-40	50-50-50-80-80

In the analyzed cases, arrival rates, desired service level for the WOS method, minimum service level to be achieved, total substitution probability, unit substitution costs are assumed to take values explained in Section 5.3.1. Since the value of K does not have a significant effect on the profit it is set to be equal to 0. Together with the profit margin cases given in Table 5.13, these inputs are used to evaluate the profits of the WOS and WS methods for 576 scenarios, and the resulting profits are reported in Table 5.14.

Table 5.14: The effect of profit margins on profit

Profit Margin Scenario	**Profit (WS)**	**Profit (WOS)**	**Difference**	**% Increase**
LL	111.3028705	104.9419729	6.36089763	5.39 %
HL	468.0408462	465.7185021	2.32234414	0.49 %
LH	169.0792562	145.8711035	23.20815272	12.60 %
HH	521.6764235	506.6476327	15.02879079	2.77 %

In Table 5.14, we observe that, as the difference between the minimum and the maximum profit margin levels increases, the WS method yields higher profits.(LL vs LH cases) Furthermore, when the minimum profit margin value is at its low level, the resulting profit increase is more than the case where it is at its high level (LL vs HL cases), and when the minimum profit margin value is low and the difference between the minimum and the maximum profit margin levels is high the profit increase is at its maximum value (LH case). For the cases where minimum profit margin value is at its high level the relative increase in profit is low (HL and HH cases).

In the retailing sector, in a product category, the case where the profit margins are between 10% and 20% is a highly probable scenario, and in this case determining the order-up-to levels by considering the substitution effects gives 5.39% more profit (LL case). Moreover, if one of the products in the category is a product of the retailer, then the scenario where profit margins are between 10% and 40% may be the case. In this case, the profit margin of the retailer's product will be 40% and profit margins of other products in the category will change between 10% and 40% according to their market shares. In this case, WS method results in 12.6% more profit than the WOS method (LH case).

5.3.9 One-Item Substitution Case

In the above comparison of the two methods, the considered substitution structure is "multi-way substitution" in which all products in the same product category are substitutable with each other, and the substitution rates are calculated by the use of Equation 3.13. On the other hand, the case where only one product is substitutable for all products in the same product category (one-item substitution) can be observed in a retail setting. In order to see the effect of substitution on the profit for this case, we change the substitution matrix according to the corresponding case, and take all other inputs as explained in Section 5.3.1, and calculate the resulting profits for the WS and WOS methods. For example, for one-item substitution case, if the substitute product is *P2*, than the substitution probability matrix will be as follows;

$$\alpha = \begin{bmatrix} 0 & \delta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \delta & 0 & 0 & 0 \\ 0 & \delta & 0 & 0 & 0 \\ 0 & \delta & 0 & 0 & 0 \end{bmatrix}$$

The effects of desired service level for the WOS method, minimum service level to be achieved, total substitution probability, unit purchase prices, unit substitution costs and profit margins are analyzed and observed to be the same with the result of the multi-way substitution case (see Appendix E for detailed results). It is observed that the one-item substitution case gives different results under different arrival rates cases. The results are summarized in Table 5.15.

In Table 5.15, we observe that the percentage increase is very small for cases where only *P1* is substitutable for other products (i.e. $\alpha_{i1} = \gamma$, $i \neq 1$; $\alpha_{ij} = 0$, $j \neq 1$; $\alpha_{11} = 0$). Since *P1* has highest market share for all arrival rate scenarios, its profit margin is the

Table 5.15: The effect of arrival rates on profit for one item substitution

Substitute Product	**Arrival Rates**	**Profit (WS)**	**Profit (WOS)**	**Difference**	**% Increase**
	10-10-10-10-10	265.2469323	264.2269355	1.0199968	0.43%
1	**30-5-5-5-5**	340.2952986	336.9787442	3.3165544	1.54%
	20-15-5-5-5	340.2671557	336.7704969	3.4966588	1.58%
	15-15-15-2.5-2.5	283.7221197	282.6116204	1.1104993	0.64%
	10-10-10-10-10	265.2278916	264.2269355	1.0009561	0.42%
2	**30-5-5-5-5**	367.5610474	337.2574396	30.303608	13.06%
	20-15-5-5-5	343.6966068	336.930743	6.7658638	3.09%
	15-15-15-2.5-2.5	283.713012	282.6116204	1.1013917	0.64%
	10-10-10-10-10	265.2209001	264.2269355	0.9939646	0.41%
3	**30-5-5-5-5**	367.5425669	337.2574396	30.285127	13.06%
	20-15-5-5-5	362.5045682	337.192264	25.312304	10.96%
	15-15-15-2.5-2.5	283.7130465	282.6116204	1.1014261	0.64%
	10-10-10-10-10	265.1392767	264.2269355	0.9123412	0.38%
4	**30-5-5-5-5**	367.5468805	337.2574396	30.289441	13.05%
	20-15-5-5-5	364.4655756	337.192264	27.273312	11.87%
	15-15-15-2.5-2.5	326.3637974	283.1672632	43.196534	28.43%

lowest for all cases. Since the arrival rates are small for other products, the substitution numbers will be small and together with the fact that the substitutable product has the lowest profit margin, the WS method's profit will be close to that of the WOS method.

As we observe in Table 5.15, when the arrival rates are as in case 1, the percentage increases in the WS method profits are negligible. Since the arrival rates of all products are the same in this case, all input parameters are the same for all products, then at the optimum case, the substitution numbers will be small in order to prevent substitution costs in the WS method. Then WS method will converge to the WOS method and the effect of the substitution will be negligible. When arrival rates are 15, 15, 15, 2.5 and 2.5 and the substituted product is either *P2* or *P3*, the substitutions from the first 3 products to *P2* and *P3* will be small because of the same reason. And moreover, since the arrival rates are small for *P4* and *P5* comparing with other products, the substitution amounts from these to the first 3 products will be small. Consequently, the resulting percentage increase will be small.

For the case where only *P2* is substitutable for all other products in the category ($\alpha_{i2} = \gamma$, $i \neq 2$; $\alpha_{ij} = 0$, $j \neq 2$; $\alpha_{22} = 0$) and the arrival rates are as in either case 2 or 3, since the arrival rate of *P1* is high, there will be some substitution from *P1* to *P2*, and also because the unit profit of *P2* is higher than *P1* this will cause an increase in profit, and the increase in the profit is proportional to the difference of the arrival rates and the profit margins of the two products. The same case is valid for the same arrival rate scenarios where the substitutable product is *P3* or *P4*. When substituted product is *P4* and arrival rates are as in case 4, since this effect is valid for 3 products, the percentage increase is at its highest among all other scenarios.

Chapter 6

CONCLUSIONS

In this thesis, we develop a model to determine the optimum order-up-to levels which gives maximum profit, while satisfying a minimum service level for all products, in which customers may substitute one product with another when faced with stock-out at retail. In spite of many efforts and investments on decreasing the out of stock amounts, empirical studies show that the measured stock-out rates in retailing are quite high. Gruen et al. [8] define stock-out rate as the percentage of the SKUs unavailable on the shelf at a particular time. Based on this definition, they suggest that on a typical day, average product unavailability is 8.3% in retailing. There are 5 customer responses to stock-outs; do not purchase, purchase elsewhere, substitute - same brand, substitute - different brand and delay purchase. From the point of the retailer rather than losing the sale customer substitution is favorable. 40% of stock-out occasions results with substitution inside the store (Gruen et al. [8]). Other than taking actions on decreasing the stock-out rate, retailer can manipulate those 40% of customer choices on his favor with an effective inventory control policy.

In this thesis, we consider a retailer that stocks a certain number of products in a category. If customers are unable to buy their favorite item at the time of their shopping, the demand in question will be either directed to another product in the store or it will be lost. In the analyzed system, second substitution attempt is not allowed. A probabilistic, multi-way demand substitution type is considered. We first develop a method to determine the average inventory levels of all products in a product category while taking into account of the substitutions between products. This deterministic approach originated from the traditional deterministic average inventory calculation methods and this approximation is based on the depletion orders of products. This average inventory level calculation approximation works quiet well under all possible real case scenarios. Then we develop a stochastic model to determine the direct sales of each product and substitution sales among all product pairs. By using these two approximations, optimization model is solved to find the best order up-to levels, to achieve the maximum total profit, along with satisfying a minimum direct sales rate for the customers of all products.

Then we present a numerical study to measure the effectiveness of the optimization problem and to see effects of changes in the input parameters on profit. The results suggest that our approximation method performs quite well and accounting for substitution effects has substantial effects on profit. According to the numerical study, our approximation gives 5% more profit on average than the case where substitution effects are not considered. From the numerical study, we see that, the distribution of the arrival rates of products also has effects on profit and as the differences among rates of products in the same category increase, the added value of determining substitution effects increases.

We define two types of service levels in our computational study; desired service level, which is the rate of customers retailer want to satisfy with their first choices, and minimum service level, which is high enough to assure customer loyalty and also low enough to direct maximum number of low profited products' customers to substitute to higher profited products. In the numerical study we see that as the desired service level amount increases profit increases, but the added value of determining substitution effects decreases. And as the minimum service level increases both profit and the added value of determining substitution effects decreases. Our numerical study also shows that as the total substitution probability increases, profit increases and also the added value of determining substitution effects increases.

According to the numerical solutions, our approximation works best when the minimum profit margin value is low and the difference between the minimum and the maximum profit margins are high. For this case approximation method gives 12.6% more profit than the case where substitution effects are not considered. And together with the arrival rates distribution this profit difference among the two methods can reach to 22%.

A possible extension to this thesis is to extend this method to work with positive lead time and a further extension will be extending this work to work for different replenishment periods for all products in the category.

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Appendix A

SIMULATION

After building the model, we evaluate its performance of projecting the real case. To do this we first simulate the system and compare its results with the ones of our model. In this simulation; simulation time, arrival rates, unit costs, purchase prices, unit holding costs of all products, substitution rates between products, fixed order costs, substitution costs, replenishment time and the desired service level are given as inputs. And total sales, total demand, total direct sales, substitution sales, lost sales, direct loss sales, lost sales through substitution, total order amounts of all products, number of given orders and substitution amounts between products are taken as outputs for the given simulation time.

The logic of the simulation is based on the random number generation. The probability of two events occurring at the same time is assumed to be 0. In the system there are 3 types of events which could occur at a time. Those events are classified as a customer arrival, placing an order and arrival of a previous order. Since the replenishment time and the lead times are known, the time for placing an order and the time of the arrival of the order can be calculated directly in the system. In order to mimic a customer arrival in the system, an exponential random number is generated for all products to represent interarrival times of consecutive customers. By these iterative calculations, times of all possible events in the system can be determined. Program starts at time 0 and jumps to the time of first event and carries out the necessary adjustments about the event of the corresponding time and continues with the next event until the end of the simulation time.

If the "customer arrival" is the current event occurring at the system for product i, program checks the inventory level of this product. If there is enough inventory, the demand is satisfied by using that inventory. If there is not enough inventory to fulfill the demand, program generates a random number between 0 and 1. If the random number is smaller than the substitution probability, δ, it is decided to substitute from this product, else no substitution will occur and the sale will be lost. If the substitution decision has been taken, another random number is generated to determine which product to substitute. Let say the customer of product i is decided to substitute product i with product j. Then, the inventory amount of the product j is checked; if there is enough inventory, the demand will be fulfilled from product j's inventory. If there is not enough inventory to fulfill this demand the sale is lost. In the system of our problem, second substitutions are not allowed. In other words, if substitution decision from product i to a product j is made but there is not enough inventory to satisfy it with product j, the probability of substituting to another product is assumed to be 0. According to the action that takes place, the necessary adjustments for the demand, inventory level, total sales, direct sales, direct lost sales, total lost sales, substitution sales, lost sales through substitution amounts are made. After making all changes for arrival of a customer, the time of next arrival of a customer for product i is generated by a random number.

If the current event in the system is "placing an order", the inventory amount of that product is observed and then an order, is placed with the amount of difference between order-up-to level value and the current inventory level of the corresponding product. Order amount of that product is recorded, number of orders given for that product is updated and next order time is calculated by the use of the replenishment time information.

If the current event in the system is "arrival of a previous order", the inventory level of that product is increased by the order amount recorded at the time that order is placed and next order arrival time of this product is updated by the use of replenishment time information.

All these events are repeated until the end of the simulation time.

At the end, all data of recorded sales are used together with price and cost inputs to calculate the total revenue, costs and profit amounts. Using this simulation the cost and profit of a review cycle is determined while, simulation time, arrival rates, unit costs, purchase prices, unit holding costs of all products, substitution rates between products, fixed order costs, substitution costs, replenishment time and the desired service level and order up to level values are given as inputs.

Appendix B

MARKOV CHAIN MODEL FOR EXACT ANALYSIS

B.1 Determining the State Transition Matrix

We model the system as a discrete time-discrete state space stochastic process. The state of the system in period t is an N-tuple $\mathbf{I}(t) = (I_1(t), ..., I_N(t))$ where $I_i(t)$ is the inventory level of product i in period t.

The length of each period is Δ. Therefore there are T/Δ periods in each review cycle. We set Δ very short to ensure that the probability of having two arrivals in the same time period is very small. Therefore we assume that only one arrival can occur in each period.

Since the arrivals are Poisson, the probability that a demand for product i arrives in period t can be calculated by multiplying the arrival rate of product i with the length of the period,which is $\lambda_i\Delta$. Similarly, the probability that a customer substitutes product j for product i due to unavailability of product i in period t is the α_{ij} rate of the probability that a demand arrives for product j in period t,which is $\lambda_i\Delta\alpha_{ij}$.

A customer will substitute product j for product i only when product j is not available and product i is available in period t. Let the indicator variable $\xi_{I_i(t)}$ be defined to be equal to 1 when $I_i(t) > 0$ and 0 otherwise. Then the indicator variable $\xi_{I_i(t)}(1-\xi_{I_j(t)})$ will be 1 when a customer can substitute product j for product i in period t and 0 otherwise. Therefore, the state transitions are defined by the following equations:

$$P[I_i(t+1) = n-1 | I_i(t) = n] = \lambda_i\Delta + \sum_{j \neq i} \lambda_j \Delta \alpha_{ji} \xi_{I_i(t)} (1 - \xi_{I_j(t)}) \qquad n \geq 1, i = 1, ..., N \tag{B.1}$$

$$P[I_i(t+1) = 0 | I_i(t) = 0] = 1 \qquad i = 1, ..., N \tag{B.2}$$

In equation (B.1), the probability that the inventory level of product i at time $t+1$ is $n-1$ given that inventory level of product i at time t is n, which is the probability of selling product i at time t, is calculated. A product, say product i, is sold at time t if a customer of product i arrives to the store at time t or a customer for another product, say product j, arrives to the store at time t and unable to find product j at the store due to stockout and substitute to product i provided that it is available at the store at time t. Then, we can calculate the probability of selling product i at time t by summing these two events' probabilities.In equation (B.2), it is certain that inventory level of product i will be zero at time $t+1$, given that its inventory level is 0 at time t.

In order to analyze the system, we first generate the state transition matrix of the system automatically. Let $\mathbf{P}$ be the state transition probability matrix. Since the state of the system is an N-tuple $\mathbf{I}(t) = (I_1(t), ..., I_N(t))$ and $0 \leq I_i(t) \leq Q_i$, $i = 1, ..., N$, there will be $|\mathbf{I}| = \prod_{i=1}^{N}(Q_i + 1)$ states in the state space. Accordingly $\mathbf{P}$ is an $|\mathbf{I}|$x$|\mathbf{I}|$ sparse matrix. Note that since there are at most N transitions from each state, the number of non-zero elements will be less than $N|\mathbf{I}|$.

We start state-space generation at the state where all the inventory levels are at their order-up-to levels. Then we consider N possible changes that correspond to the decrease of one unit in the inventory level of one of the products from this state. If this new state is not included in the state space then it is included. Then, we store the index of the current state, the index of the next state, and the transition rate calculated from Equations (B.1) and (B.2). We repeat this process the process terminates, which is until we reach the state where all the inventory levels are zero. Once the non-diagonal elements of $\mathbf{P}$ are determined in this way, the diagonal elements are set to make the row sums equal to one.

B.2 Performance Evaluation

Let the state probability row vector be $\mathbf{p}(t) = p(t, i_1, ..., i_N)$ given as

$$\mathbf{p}(t) = p(t, i_1, ..., i_N) = P[I_1(t) = i_1, ..., I_N(t) = i_N] \tag{B.3}$$

Since the inventory levels are equal to the order-up-to levels at the beginning of each cycle, $p(0, Q_1, ..., Q_N) = 1$.

The state probability vector satisfies

$$\mathbf{p}(t+1) = \mathbf{p}(t)\mathbf{P} \tag{B.4}$$

Note that each periodic review cycle starts at state $(Q_1, ..., Q_N)$ and terminates at state $(I_1(T/\Delta), ..., I_N(T/\Delta))$ with $0 \leq I_i(T/\Delta) \leq Q_i, i = 1, ..., N$. The state probability vector at the end of the review cycle can be determined from Equation (B.4) starting with $p(0, Q_1, ..., Q_N) = 1$.

Once the probability vector at the end of the review cycle is determined, a number of performance measures can be determined directly.

Expected Sales The expected number of product is sold in each cycle can be determined from the state probabilities as

$$S_i = \sum_{n=0}^{Q_i} (Q_i - n) P[I_i(T/\Delta) = n] \tag{B.5}$$

Therefore the expected number of unsold items of product i is $Q_i - S_i$.

Expected Number of Substitutions The expected number of substitution sales from customers of product i to product j is

$$S_{ij} = \sum_{t=0}^{T/\Delta} P[I_i(t) = 0, I_j(t) > 0]\lambda_i \Delta \alpha_{ij} \tag{B.6}$$

Service Level The expected direct sales of product i to customers of product i can be determined from Equation (B.5) and Equation (B.6) as $S_{ii} = S_i - \sum_{j \neq i} S_{ij}$. Therefore the service level is $SL_i = \frac{S_{ii}}{\lambda_i T}$.

Inventory Level The expected inventory level can be determined as

$$\bar{I}_i = \frac{\Delta}{T} \sum_{t=0}^{T/\Delta} \sum_{n=0}^{Q_i} nP[I_i(T/\Delta) = n] \tag{B.7}$$

Appendix C

DERIVATION OF THE FORMULAS IN SECTION 3.3.2

C.1 Derivation of $E[(T_j - T_i)^+]$

$$E[(T_j - T_i)^+] = \int_0^{RT} \Gamma_{ij} f_{\Gamma_{ij}}(\Gamma_{ij}) d\Gamma_{ij} \tag{C.1}$$

$$\mu_{\Gamma_{ij}} = \int_{-\infty}^{\infty} \Gamma_{ij} f_{\Gamma_{ij}}(\Gamma_{ij}) d\Gamma_{ij} = \int_{-\infty}^{0} \Gamma_{ij} f_{\Gamma_{ij}}(\Gamma_{ij}) d\Gamma_{ij} + \int_{0}^{RT} \Gamma_{ij} f_{\Gamma_{ij}}(\Gamma_{ij}) d\Gamma_{ij} + \int_{RT}^{\infty} \Gamma_{ij} f_{\Gamma_{ij}}(\Gamma_{ij}) d\Gamma_{ij} \tag{C.2}$$

Then,

$$E[(T_j - T_i)^+] = \mu_{\Gamma_{ij}} - \int_{-\infty}^{0} \Gamma_{ij} f_{\Gamma_{ij}}(\Gamma_{ij}) d\Gamma_{ij} - \int_{RT}^{\infty} \Gamma_{ij} f_{\Gamma_{ij}}(\Gamma_{ij}) d\Gamma_{ij} \tag{C.3}$$

From equations, ...

$$E[(T_j - T_i)^+] = \mu_{\Gamma_{ij}} - \left(\mu_{\Gamma_{ij}} - \sigma_{\Gamma_{ij}} \eta \left(-\frac{\mu_{\Gamma_{ij}}}{\sigma_{\Gamma_{ij}}} \right) \right) - \left(\sigma_{\Gamma_{ij}} \eta \left(\frac{RT - \mu_{\Gamma_{ij}}}{\sigma_{\Gamma_{ij}}} \right) + RT[1 - F_{\Gamma_{ij}}(RT)] \right) \tag{C.4}$$

$$E[(T_j - T_i)^+] = \sigma_{\Gamma_{ij}} \eta \left(-\frac{\mu_{\Gamma_{ij}}}{\sigma_{\Gamma_{ij}}} \right) - \sigma_{\Gamma_{ij}} \eta \left(\frac{RT - \mu_{\Gamma_{ij}}}{\sigma_{\Gamma_{ij}}} \right) - RT[1 - F_{\Gamma_{ij}}(RT)] \tag{C.5}$$

C.1.1 Derivation of $\int_{-\infty}^{0} \Gamma_{ij} f_{\Gamma_{ij}}(\Gamma_{ij}) d\Gamma_{ij}$

$$\int_{-\infty}^{0} \Gamma_{ij} f_{\Gamma_{ij}}(\Gamma_{ij}) d\Gamma_{ij} = \int_{-\infty}^{0} \left(\frac{\Gamma_{ij} - \mu_{\Gamma_{ij}}}{\sigma_{\Gamma_{ij}}} \sigma_{\Gamma_{ij}} + \mu_{\Gamma_{ij}} \right) f_{\Gamma_{ij}}(\Gamma_{ij}) d\Gamma_{ij} \tag{C.6}$$

Let say $y = \frac{\Gamma_{ij} - \mu_{\Gamma_{ij}}}{\sigma_{\Gamma_{ij}}}$, since we know that Γ_{ij} is normally distributed with $\mu_{\Gamma_{ij}}$ and $\sigma_{\Gamma_{ij}}$, then y will be distributed with standard normal distribution.

Let say x is normally distributed with mean μ and standard deviation σ and z is distributed with standard normal distribution. As properties of standard normal and normal distribution we know that

- $f_x(x) = \frac{1}{\sigma} f_z(z)$
- $dx = \sigma dz$
- $f'_z(z) = -z f_z(z)$
- $f(-z) = f(z)$
- $F_z(z) = 1 - F(-z)$

Then,

$$\int_{-\infty}^{0} \left(\frac{\Gamma_{ij} - \mu_{\Gamma_{ij}}}{\sigma_{\Gamma_{ij}}} \sigma_{\Gamma_{ij}} + \mu_{\Gamma_{ij}} \right) f_{\Gamma_{ij}}(\Gamma_{ij}) d\Gamma_{ij} = \int_{-\infty}^{-\frac{\mu_{\Gamma_{ij}}}{\sigma_{\Gamma_{ij}}}} \left(y\sigma_{\Gamma_{ij}} + \mu_{\Gamma_{ij}} \right) f_y(y) dy \tag{C.7}$$

$$= \mu_{\Gamma_{ij}} \int_{-\infty}^{-\frac{\mu_{\Gamma_{ij}}}{\sigma_{\Gamma_{ij}}}} f_y(y)dy + \sigma_{\Gamma_{ij}} \int_{-\infty}^{-\frac{\mu_{\Gamma_{ij}}}{\sigma_{\Gamma_{ij}}}} y f_y(y)dy \quad \text{(C.8)}$$

$$= \mu_{\Gamma_{ij}} \int_{-\infty}^{-\frac{\mu_{\Gamma_{ij}}}{\sigma_{\Gamma_{ij}}}} f_y(y)dy + \sigma_{\Gamma_{ij}} \int_{-\infty}^{-\frac{\mu_{\Gamma_{ij}}}{\sigma_{\Gamma_{ij}}}} -f_y'(y)dy \quad \text{(C.9)}$$

$$= \mu_{\Gamma_{ij}} F_y\left(-\frac{\mu_{\Gamma_{ij}}}{\sigma_{\Gamma_{ij}}}\right) - \sigma_{\Gamma_{ij}} f_y\left(-\frac{\mu_{\Gamma_{ij}}}{\sigma_{\Gamma_{ij}}}\right) \quad \text{(C.10)}$$

We know that,

$$\eta(z) = \int_z^\infty \frac{1}{\sqrt{2\pi}}(x-z)e^{-\frac{1}{2}x^2}\,dx = \phi(z) - z\Phi(z) \quad \text{(C.11)}$$

where $\phi(z)$ and $\Phi(z)$ are the density function and cumulative distribution function of the standard normal given as $\phi(z) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}\,dx$ and $\Phi(z) = \int_z^\infty \phi(x)\,dx$.

Then,

$$\mu_{\Gamma_{ij}} F_y\left(-\frac{\mu_{\Gamma_{ij}}}{\sigma_{\Gamma_{ij}}}\right) - \sigma_{\Gamma_{ij}} f_y\left(-\frac{\mu_{\Gamma_{ij}}}{\sigma_{\Gamma_{ij}}}\right) = \mu_{\Gamma_{ij}}\left(1-\Phi\left(-\frac{\mu_{\Gamma_{ij}}}{\sigma_{\Gamma_{ij}}}\right)\right) - \sigma_{\Gamma_{ij}}\phi\left(-\frac{\mu_{\Gamma_{ij}}}{\sigma_{\Gamma_{ij}}}\right) \quad \text{(C.12)}$$

Finally,

$$\int_{-\infty}^0 \Gamma_{ij} f_{\Gamma_{ij}}(\Gamma_{ij})d\Gamma_{ij} = \mu_{\Gamma_{ij}} - \sigma_{\Gamma_{ij}}\left(\phi\left(-\frac{\mu_{\Gamma_{ij}}}{\sigma_{\Gamma_{ij}}}\right) + \frac{\mu_{\Gamma_{ij}}}{\sigma_{\Gamma_{ij}}}\Phi\left(-\frac{\mu_{\Gamma_{ij}}}{\sigma_{\Gamma_{ij}}}\right)\right) \quad \text{(C.13)}$$

$$= \mu_{\Gamma_{ij}} - \sigma_{\Gamma_{ij}}\eta\left(-\frac{\mu_{\Gamma_{ij}}}{\sigma_{\Gamma_{ij}}}\right) \quad \text{(C.14)}$$

C.1.2 Derivation of $\int_{RT}^{\infty} \Gamma_{ij} f_{\Gamma_{ij}}(\Gamma_{ij}) d\Gamma_{ij}$

$$\int_{RT}^{\infty} \Gamma_{ij} f_{\Gamma_{ij}}(\Gamma_{ij}) d\Gamma_{ij} = \int_{RT}^{\infty} (\Gamma_{ij} - RT) f_{\Gamma_{ij}}(\Gamma_{ij}) d\Gamma_{ij} + \int_{RT}^{\infty} RT f_{\Gamma_{ij}}(\Gamma_{ij}) d\Gamma_{ij} \quad \text{(C.15)}$$

Let say $x = \frac{RT - \mu_{\Gamma_{ij}}}{\sigma_{\Gamma_{ij}}}$ and $y = \frac{\Gamma_{ij} - \mu_{\Gamma_{ij}}}{\sigma_{\Gamma_{ij}}}$ then,

$$\begin{aligned}
\int_{RT}^{-\infty} (\Gamma_{ij} - RT) f_{\Gamma_{ij}}(\Gamma_{ij}) d\Gamma_{ij} &= \int_{x}^{\infty} (y\sigma_{\Gamma_{ij}} + \mu_{\Gamma_{ij}} - x\sigma_{\Gamma_{ij}} - \mu_{\Gamma_{ij}}) f_y(y) dy \\
&= \sigma_{\Gamma_{ij}} \int_{x}^{\infty} (y - x) f_y(y) dy \quad \text{(C.16)} \\
\int_{x}^{\infty} (y - x) f_y(y) dy &= \int_{x}^{\infty} y f_y(y) dy - x \int_{x}^{\infty} f_y(y) dy \quad \text{(C.17)} \\
&= -\int_{x}^{\infty} f_y'(y) dy - x(1 - \int_{-\infty}^{x} f_y(y) dy) \quad \text{(C.18)} \\
&= f_y(x) - x(1 - F_y(x)) \quad \text{(C.19)} \\
&= \phi(x) - x\Phi(x) \quad \text{(C.20)} \\
&= \eta(x) = \eta\left(\frac{RT - \mu_{\Gamma_{ij}}}{\sigma_{\Gamma_{ij}}}\right) \quad \text{(C.21)}
\end{aligned}$$

$$\int_{RT}^{-\infty}(\Gamma_{ij}-RT)f_{\Gamma_{ij}}(\Gamma_{ij})d\Gamma_{ij} = \sigma_{\Gamma_{ij}}\int_{x}^{\infty}(y-x)f_y(y)dy$$

$$= \sigma_{\Gamma_{ij}}\eta\left(\frac{RT-\mu_{\Gamma_{ij}}}{\sigma_{\Gamma_{ij}}}\right) \tag{C.22}$$

$$\int_{RT}^{\infty}RTf_{\Gamma_{ij}}(\Gamma_{ij})d\Gamma_{ij} = RT\left(1-\int_{-\infty}^{RT}f_{\Gamma_{ij}}(\Gamma_{ij})d\Gamma_{ij}\right) \tag{C.23}$$

$$= RT[1-F_{\Gamma_{ij}}(RT)] \tag{C.24}$$

$$\int_{RT}^{\infty}\Gamma_{ij}f_{\Gamma_{ij}}(\Gamma_{ij})d\Gamma_{ij} = \int_{RT}^{\infty}(\Gamma_{ij}-RT)f_{\Gamma_{ij}}(\Gamma_{ij})d\Gamma_{ij}+\int_{RT}^{\infty}RTf_{\Gamma_{ij}}(\Gamma_{ij})d\Gamma_{ij} \tag{C.25}$$

$$= \sigma_{\Gamma_{ij}}\eta\left(\frac{RT-\mu_{\Gamma_{ij}}}{\sigma_{\Gamma_{ij}}}\right)+RT[1-F_{\Gamma_{ij}}(RT)] \tag{C.26}$$

C.2 Derivation of $E[(RT-T_i)^+]$

$$E[(RT-T_i)^+] = \int_{-\infty}^{RT}(RT-t_i)f_{T_i}(t_i)dt_i \tag{C.27}$$

Let say $m = \frac{t_i-\mu_{T_i}}{\sigma_{T_i}}$ and $n = \frac{RT-\mu_{T_i}}{\sigma_{T_i}}$.

$$\int_{-\infty}^{RT} (RT - t_i) f_{T_i}(t_i) dt_i = \sigma_{T_i} \int_{-\infty}^{n} (n - m) f_m(m) dm \tag{C.28}$$

$$= -\sigma_{T_i} \left(nF_m(n) + f_m(n) \right) \tag{C.29}$$

$$= RT - \mu_{T_i} + \sigma_{T_i} \eta \left(\frac{RT - \mu_{T_i}}{\sigma_{T_i}} \right) \tag{C.30}$$

Appendix D

ANALYTICAL BEHAVIOR OF THE PROFIT FUNCTION

By using the approximation method, we are able to calculate the profit, explained in Section 3.1, only in terms of order levels of products in the category. But, since the formulation developed using this way is complicated it is hard to prove whether it is convex or concave or neither in terms of its variables. Instead, we evaluate the profit and graph its behavior according to all possible values of its variables in couples. We evaluate the profit behavior for a 3 products environment according to Q_1 and Q_2 (in Figure D.1), Q_1 and Q_3 (in Figure D.2), Q_2 and Q_3 (in Figure D.3), and then see that the profit function is neither convex nor concave according to Q_1, Q_2 and Q_3.

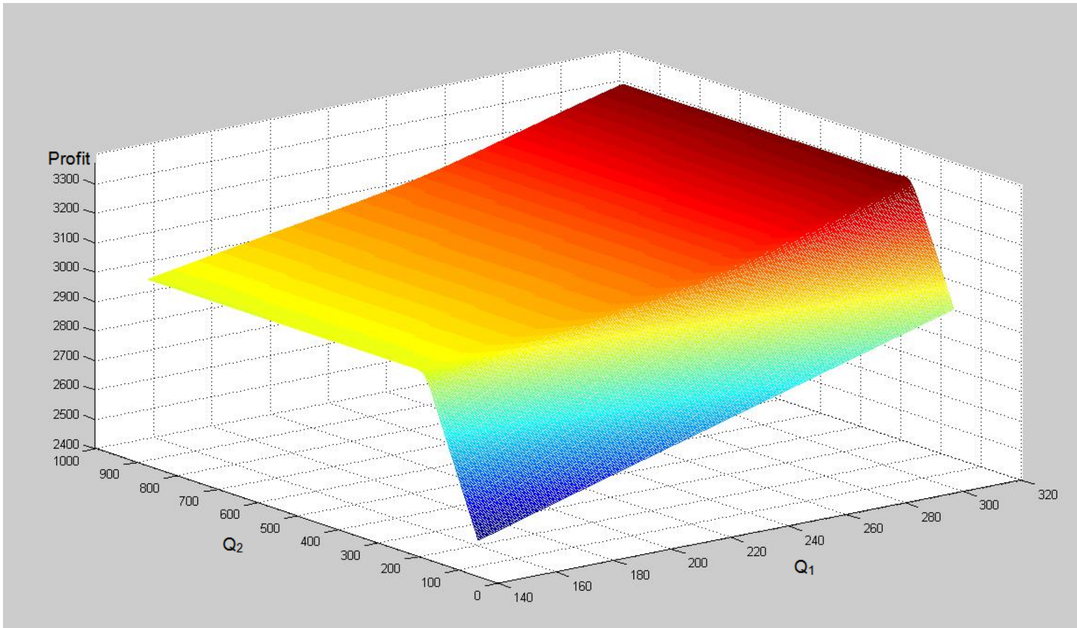

Figure D.1: Analytical Behavior of the Profit Function According to Q_1 and Q_2

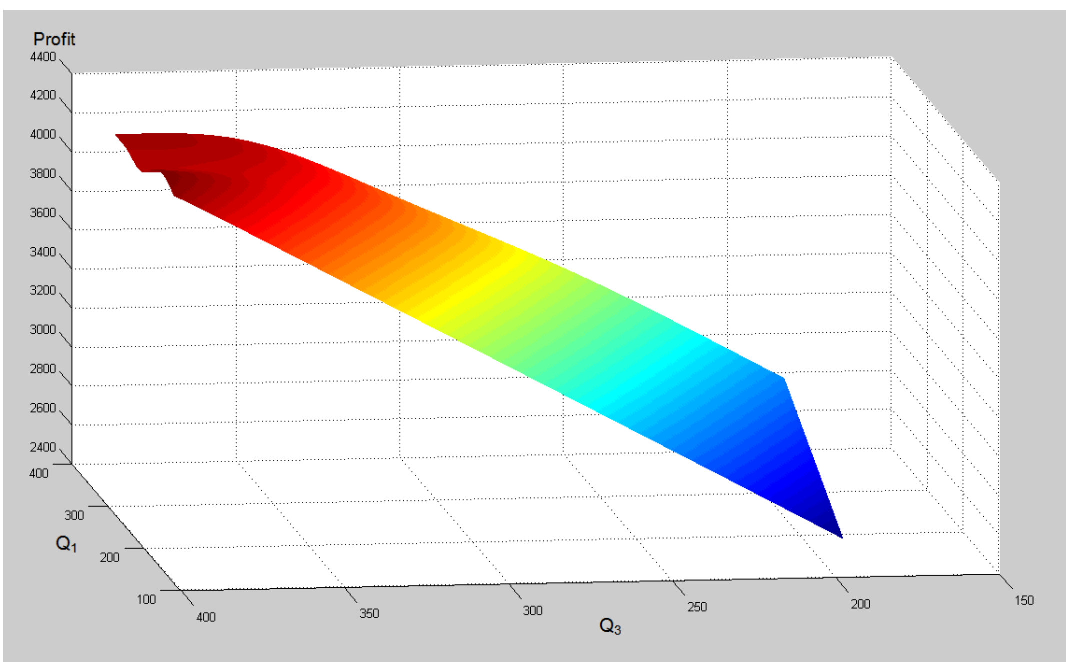

Figure D.2: Analytical Behavior of the Profit Function According to Q_1 and Q_3

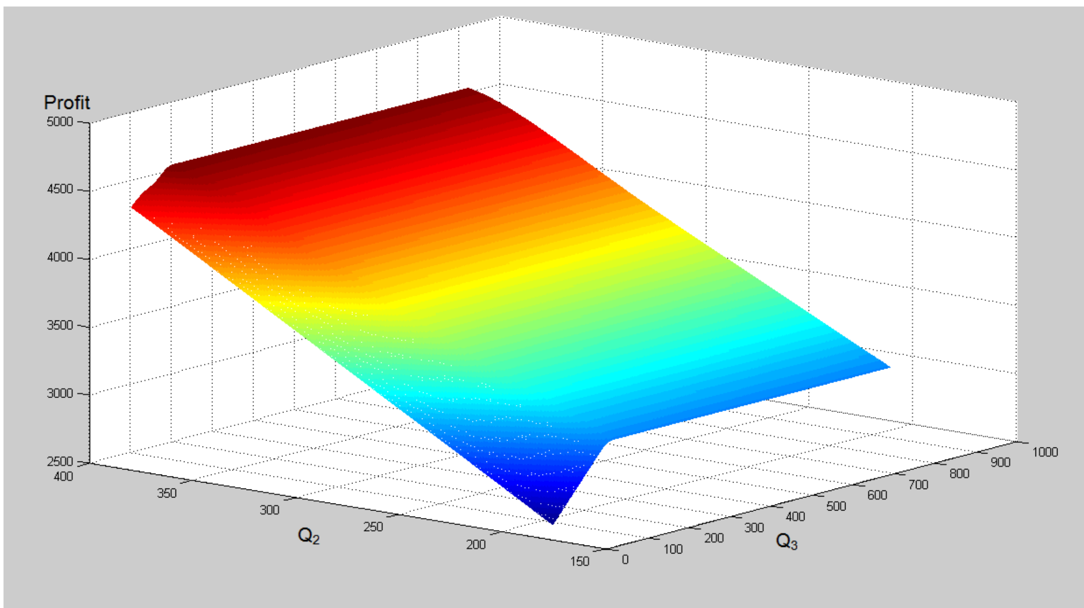

Figure D.3: Analytical Behavior of the Profit Function According to Q_2 and Q_3

Appendix E

INVENTORY CONTROL SYSTEM OF DETERMINING ORDER LEVELS BY IGNORING SUBSTITUTION EFFECTS

According to a fixed review period, order-up-to level (R, S), every R units of time (i.e. at every review point) an order is given to raise inventory position to a predetermined level S . The order-up-to level, in case of a periodic review system with lost sales and normal demand, is determined for a given desired service level β as follows (Silver et al. [24]):

Step 1: Select the safety factor z_β which satisfies

$$G_u(z_\beta) = \frac{(1-\beta)\mu R}{\beta\sigma\sqrt{R+L}} \tag{E.1}$$

where μ is the mean and σ is the standard deviation of demand. In addition,

$$G_u(z_\beta) = \int_z^\infty (x-z)\frac{1}{\sqrt{2\pi}}e^{\frac{-x^2}{2}}dx. \tag{E.2}$$

Here, we define the service level β as the fraction of demand satisfied directly from the shelf. Moreover, since demand is Poisson, normal approximation to Poisson can be used in calculating initial stock levels. That is, we take $\mu = \lambda$ and $\sigma = \sqrt{\lambda}$.

Step 2: Reorder level $S = \mu(R+L) + z_\beta\sigma\sqrt{R+L}$ (if not integer, increase to the next higher integer).

Appendix F

DETAILED RESULTS FOR ONE-ITEM SUBSTITUTION CASE

F.1 Substitution to Product 1

Arrival Rates	**Profit (WS)**	**Profit (WOS)**	**Difference**	**% Increase**
10-10-10-10-10	265.2469	264.2269	1.020	0.386%
30-5-5-5-5	340.2953	336.9787	3.317	0.984%
20-15-5-5-5	340.2672	336.7705	3.497	1.038%
15-15-15-2.5-2.5	283.7221	282.6116	1.110	0.393%

Table F.1: The effect of arrival rates on profit for one-item substitution

Minimum Service Level (%)	**Profit (WS)**	**Profit (WOS)**	**Difference**	**% Increase**
40	307.3786484	305.1469492	2.232	0.731%
50	307.3857215	305.1469492	2.239	0.734%
60	307.3842597	305.1469492	2.237	0.733%

Table F.2: The effect of minimum service levels on profit for one-item substitution

Desired Service Level (%)	**Profit (WS)**	**Profit (WOS)**	**Difference**	**% Increase**
85	292.4009373	288.5605587	3.840	1.331%
90	306.9459505	305.0724727	1.873	0.614%
95	322.8017419	321.8078163	0.994	0.309%

Table F.3: The effect of desired service levels on profit for one-item substitution

Profit Margin	**Profit (WS)**	**Profit (WOS)**	**Difference**	**% Increase**
LL	105.6391	104.6691	0.9700	0.927%
HL	466.2143	465.0362	1.1780	0.253%
LH	149.5989	145.2577	4.3412	2.989%
HH	508.0792	505.6247	2.4545	0.485%

Table F.4: The effect of profit margins on profit for one-item substitution

Substitution Cost (%)	**Profit (WS)**	**Profit (WOS)**	**Difference**	**% Increase**
0	308.3252	306.2198	2.1054	0.688%
10	307.4730	305.5046	1.9684	0.644%
20	307.0275	304.7893	2.2382	0.734%
30	306.7059	304.0741	2.6317	0.865%

Table F.5: The effect of substitution costs on profit for one-item substitution

ρ	**Profit (WS)**	**Profit (WOS)**	**Difference**	**% Increase**
0.4	307.4658	305.5696	1.8961	0.621%
0.6	307.4394	305.2975	2.1419	0.702%
0.8	307.3694	305.0092	2.3603	0.774%
1.0	307.2569	304.7115	2.5454	0.835%

Table F.6: The effect of substitution probability on profit for one-item substitution

F.2 Substitution to Product 2

Arrival Rates	Profit (WS)	Profit (WOS)	Difference	% Increase
10-10-10-10-10	265.2279	264.2269	1.001	0.379%
30-5-5-5-5	367.5610	337.2574	30.304	8.985%
20-15-5-5-5	343.6966	336.9307	6.766	2.008%
15-15-15-2.5-2.5	283.7130	282.6116	1.101	0.390%

Table F.7: The effect of arrival rates on profit for one-item substitution

Minimum Service Level (%)	Profit (WS)	Profit (WOS)	Difference	% Increase
40	316.8587415	305.2566846	11.602	3.801%
50	315.1817962	305.2566846	9.925	3.251%
60	313.1083807	305.2566846	7.852	2.572%

Table F.8: The effect of minimum service levels on profit for one-item substitution

Desired Service Level (%)	Profit (WS)	Profit (WOS)	Difference	% Increase
85	298.7034862	288.5217953	10.182	3.529%
90	314.9289935	305.1420751	9.787	3.207%
95	331.5164387	322.1061833	9.410	2.921%

Table F.9: The effect of desired service levels on profit for one-item substitution

Profit Margin	**Profit (WS)**	**Profit (WOS)**	**Difference**	**% Increase**
LL	109.8550	104.7245	5.1306	4.899%
HL	467.8405	465.0880	2.7525	0.592%
LH	164.2468	145.4254	18.8215	12.942%
HH	518.2562	505.7889	12.4673	2.465%

Table F.10: The effect of profit margins on profit for one-item substitution

Substitution Cost (%)	**Profit (WS)**	**Profit (WOS)**	**Difference**	**% Increase**
0	319.0479	306.4289	12.6190	4.118%
10	315.8732	305.6474	10.2258	3.346%
20	313.4797	304.8659	8.6138	2.825%
30	311.7977	304.0845	7.7133	2.537%

Table F.11: The effect of substitution costs on profit for one-item substitution

ρ	**Profit (WS)**	**Profit (WOS)**	**Difference**	**% Increase**
0.4	311.6173	305.7055	5.9119	1.934%
0.6	314.7464	305.4224	9.3241	3.053%
0.8	316.9173	305.1113	11.8061	3.869%
1.0	316.9174	304.7876	12.1298	3.980%

Table F.12: The effect of substitution probability on profit for one-item substitution

F.3 Substitution to Product 3

Arrival Rates	**Profit (WS)**	**Profit (WOS)**	**Difference**	**% Increase**
10-10-10-10-10	265.2209	264.2269	0.994	0.376%
30-5-5-5-5	367.5426	337.2574	30.285	8.980%
20-15-5-5-5	362.5046	337.1923	25.312	7.507%
15-15-15-2.5-2.5	283.7130	282.6116	1.101	0.390%

Table F.13: The effect of arrival rates on profit for one-item substitution

Minimum Service Level (%)	**Profit (WS)**	**Profit (WOS)**	**Difference**	**% Increase**
40	322.7922073	305.3220649	17.470	5.722%
50	320.0518351	305.3220649	14.730	4.824%
60	316.3917688	305.3220649	11.070	3.626%

Table F.14: The effect of minimum service levels on profit for one-item substitution

Desired Service Level (%)	**Profit (WS)**	**Profit (WOS)**	**Difference**	**% Increase**
85	303.8266538	289.8968996	13.930	4.805%
90	321.2554618	306.5834013	14.672	4.786%
95	409.9769038	392.9124968	17.064	4.343%

Table F.15: The effect of desired service levels on profit for one-item substitution

Profit Margin	**Profit (WS)**	**Profit (WOS)**	**Difference**	**% Increase**
LL	112.4495	104.7573	7.6922	7.343%
HL	468.7655	465.1206	3.6450	0.784%
LH	172.8580	145.5236	27.3344	18.783%
HH	524.9081	505.8868	19.0213	3.760%

Table F.16: The effect of profit margins on profit for one-item substitution

Substitution Cost (%)	**Profit (WS)**	**Profit (WOS)**	**Difference**	**% Increase**
0	325.1045	306.5642	18.5403	6.048%
10	320.9307	305.7361	15.1946	4.970%
20	317.7206	304.9080	12.8126	4.202%
30	315.2253	304.0799	11.1453	3.665%

Table F.17: The effect of substitution costs on profit for one-item substitution

ρ	**Profit (WS)**	**Profit (WOS)**	**Difference**	**% Increase**
0.4	314.1580	305.7993	8.3587	2.733%
0.6	319.4753	305.4989	13.9765	4.575%
0.8	323.3402	305.1666	18.1736	5.955%
1.0	322.0075	304.8234	17.1841	5.637%

Table F.18: The effect of substitution probability on profit for one-item substitution

F.4 Substitution to Product 4

Arrival Rates	Profit (WS)	Profit (WOS)	Difference	% Increase
10-10-10-10-10	265.1393	264.2269	0.912	0.345%
30-5-5-5-5	367.5469	337.2574	30.289	8.981%
20-15-5-5-5	364.4656	337.1923	27.273	8.088%
15-15-15-2.5-2.5	326.3638	283.1673	43.197	15.255%

Table F.19: The effect of arrival rates on profit for one-item substitution

Minimum Service Level (%)	Profit (WS)	Profit (WOS)	Difference	% Increase
40	334.7528228	305.4609756	29.292	9.589%
50	330.9462082	305.4609756	25.485	8.343%
60	326.9376166	305.4609756	21.477	7.031%

Table F.20: The effect of minimum service levels on profit for one-item substitution

Desired Service Level (%)	Profit (WS)	Profit (WOS)	Difference	% Increase
85	316.090743	289.979965	26.111	9.004%
90	332.0054107	306.7326452	25.273	8.239%
95	421.4088301	393.1471426	28.262	7.189%

Table F.21: The effect of desired service levels on profit for one-item substitution

Profit Margin	**Profit (WS)**	**Profit (WOS)**	**Difference**	**% Increase**
LL	118.4911	104.8210	13.6701	13.041%
HL	471.0677	465.2301	5.8375	1.255%
LH	194.2923	145.6918	48.6004	33.358%
HH	539.6645	506.1009	33.5636	6.632%

Table F.22: The effect of profit margins on profit for one-item substitution

Substitution Cost (%)	**Profit (WS)**	**Profit (WOS)**	**Difference**	**% Increase**
0	339.3129	306.7393	32.5736	10.619%
10	332.8741	305.8871	26.9870	8.823%
20	327.8959	305.0349	22.8611	7.495%
30	323.9327	304.1827	19.7500	6.493%

Table F.23: The effect of substitution costs on profit for one-item substitution

ρ	**Profit (WS)**	**Profit (WOS)**	**Difference**	**% Increase**
0.4	318.3486	305.9530	12.3956	4.051%
0.6	327.8807	305.6408	22.2399	7.276%
0.8	337.3933	305.2995	32.0938	10.512%
1.0	339.8930	304.9506	34.9424	11.458%

Table F.24: The effect of substitution probability on profit for one-item substitution

VITA

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