ASYMPTOTIC FREEDOM OF NONCOMMUTATIVE ϕ^3 THEORY IN SIX DIMENSIONS

by

UMUT BÜYÜKÇAM

A Thesis Submitted to the
Graduate School of Sciences and Engineering
in Partial Fulfillment of the Requirements for
the Degree of

Master of Science

in

Physics

Koç University

August, 2009

Koç University

Graduate School of Sciences and Engineering

This is to certify that I have examined this copy of a master's thesis by

UMUT BÜYÜKÇAM

and have found that it is complete and satisfactory in all respects,

and that any and all revisions required by the final

examining committee have been made.

Committee Members:

Prof. Tekin Dereli

Prof. Ayşe Hümeyra Bilge

Dr. Alkan Kabakçıoğlu

Date:

To all whom I love and am loved by

ÖZETÇE

6 boyutlu komutatif olmayan düz uzay-zamanda reel, kütleli ϕ^3 etkileşmeli skaler alanın kuantizasyonu gözden geçirilecektir. 1 halkada saçılma genliklerinin pertürbatif renormalizasyonu ve renormalizasyon grup denklemlerinin çıkarılmasıaçıkça gösterilecektir.Bu denklemlerin incelenmesi bu kuantum alan teorisinin asimptotik özgürlüğünü verecektir. Pertürbasyon hesaplarından elde edilen nicel sonuçlar kesin hesaplarla karşılaştırılacaktır.

ABSTRACT

Quantization of a real, massive scalar field with ϕ^3 interactions in D=6 dimensional noncommutative flat space-time will be reviewed. 1- loop perturbative renormalization of the scattering amplitudes and the derivation of the renormalization group equations will be shown explicitly. Analysis of these equations will show the asymptotic freedom of this particular quantum field theory. Quantitative results from perturbative calculations will be compared to those from exact calculations.

ACKNOWLEDGMENTS

In July 2001, after I took the National University Entrance Exam and before I made my choices, my father and I visited Koç University to see around and meet some of the faculty to hear what they would suggest for my further education. Unfortunately none of the physics people were then present but I had the chance to talk with Prof. Mete Soner. He told me that Prof. Tekin Dereli would join the Physics Department of Koç University and I can still remember my astonishment upon hearing that news. I first met him and Prof. Nejat Bulut in the orientation week and when they asked me what I expected of physics I told I would like to study quantum chromodynamics; most probably it was just a fancy word I read from some popular physics book.

Some professors just take classes and teach you invaluable things in the lectures. Some others are not good teachers in the class but are really friendly and closely interested in your studies. I would like to give my endless thanks to Prof. Tekin Dereli for letting me see how these two can be combined and know closely what we call in Turkish a real 'hoca' is. I apologize for all my laziness and appreciate his patience in building a physicist out of nothing. I just hope to be able to apply his teachings on my own students in some future.

Also, I would like to thank my family for their belief in their son. Without their trust and support I would not be able to finish this thesis.

My special thanks goes to Prof. Giovanni Landi and Dr. Gherardo Piacitelli for invaluable comments and sharing their precious time with me on this project.

Finally, I would like to thank TUBITAK for its financial support during my M.S. in Physics studies.

TABLE OF CONTENTS

LIST OF FIGURES

Chapter 1

INTRODUCTION

Naïvely, the theory of both small and fast is known as *quantum field theory*. More rigorously, it is an attempt at incorporating special relativity of Einstein which explains the dynamics of bodies moving close to speed of light into quantum mechanics which is the theory of the tiniest constituents of matter surrounding us. Thanks to the ingenius method of *path integral quantization* invented by Feynman [1], for free theory we are able to solve exactly the contribution of a path to the transition amplitude from an initial configuration to a final one. However, due to the highly complicated nature of the interactions we resort to a perturbative approach based on our exact result for the free theory. Then, even at first order, one realizes that for every analytic expression coming from our calculations of the expansion we can draw some diagrams to represent that particular contribution. These so-called *Stueckelberg-Feynman diagrams* come along with some rules assigned to each component of the graph(in graph-theoretical sense) which are finally embodied in some integral representation. Then what one needs to do is close his eyes, imagine all topologically distinct possible diagrams at each order foreseen by the previously derived rules and write down the corresponding integrals which are yet to be evaluated. However, it was realized as early as the 1930's by Dirac, Jordan, Heisenberg and Born that even at the classical level (Abraham-Lorentz theory) some of those expressions were ill-defined. In principle the singular and the finite parts of the Feynman integrals can be separated adopting some regularization scheme like Pauli-Villars, cut-off method, analytic or lattice regularization among which *dimensional regularization* due to 't Hooft [2] is the most suitable in many contexts (as it preserves the gauge as well as Lorentz symmetries of the theory at each intermediate step of computation). In this method, the dimension d of the space-time is treated as a continuous parameter of the theory in which the infinite integrals now converge. Upon analytically continuing to physical space-time dimension it is seen that the singular behavior shows itself as the simple poles arising from the Gamma functions resulting from the dimensional regularization which can be traced back to the arbitrary splitting of the Lagrangian as the free and the interacting parts. Hence we can modify the Lagrangian by adding what are called *counterterms* which serve to cancel those infinities, to wit $\infty - \infty$ =finite. Effectively this corresponds to a redefinition of the terms in the original Lagrangian and we can sweep the divergences into the unobservable elements of our theory through this redefinition, rendering the observables finite. In a strict sense, the theories which allow such a procedure at each order of the perturbative expansion without any need to introduce new terms into the Lagrangian are called *renormalizable* [3]. But, it is true that there is seemingly an ambiguity in 'how much' of the remaining finite part to put into the swept away infinity since ∞ + finite=∞. Technically, this is nothing more than a choice of a *renormalization scheme* among which there is the on-shell renormalization (momentum subtraction) and the minimal (and generalized minimal) subtraction (MS and $\overline{MS}$ resp.) schemes. In MS only the singular part (in the form of a simple pole of the Gamma function from the previously employed dimensional regularization) is thrown away, whereas in $\overline{MS}$ one also gets rid of the irrelevant unphysical finite part which insists at each order. Although renormalizability is one of the most efficient tools currently at hand to make physical sense out of a model, obviously non-renormalizable theories constitute a much greater subset of all mathematically sensible field theories. So the notion is of utmost importance for high-energy physicists. It was observed as early as [4] that introducing a noncommutative character on the space-time coordinates would serve as a UV cut-off at very short lengths for field theories built upon such a space which would subsequently improve the renormalizability of the theory. Elaboration of this point for a specific scalar model to be defined below on a particular type on noncommutative space-time will be the main aim of this project. (For a review of string theory or condensed matter related topics in noncommutative field theories, NCQFT, see [5, 6]) More explicitly, non-commutative field theories are mathematical models for the usual scalar, fermionic and gauge fields whose space-time arguments get replaced by Hermitian operators upon quantization which satisfy the following non-trivial commutation relation:

$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu} \qquad\qquad \mu, \nu = 0, ..., d-1 \tag{1.1}$$

where $\theta^{\mu\nu}$ is a constant, second-rank, real, anti-symmetric tensor of units length squared, d is the dimension of the space-time, and i is there to ensure that commutator of two Hermitian operators is again Hermitian. As clearly seen this commutation relation very much resembles the Heisenberg commutation relations for the Hermitian operators corresponding to the conjugate position-momentum variables of standard quantum mechanics and as such it implies the similar uncertainty relation, this time among the space-time points:

$$\Delta x^{\mu} \Delta x^{\nu} \geq \frac{\theta^{\mu\nu}}{2}. \tag{1.2}$$

This in turn results in the discretization of space-time into hypercubes of dimensions of order Planck scale and hence we substitute an appropriate Hilbert space for the space-time manifold. It is in this context we construct our field theory. However, at this point it is appropriate to inform the reader that such noncommutativity in its full content ($\theta^{0j} \neq 0$) implies that the interaction term involves infinite number of time derivatives which signals the theory being non-local in time and hence has been argued in many papers ([7], [8] and references therein) to violate unitarity and also causality. These results were in contrast with the earlier remark [9] that if proper time ordering of the field operators are taken into account even in space-time noncommutative case unitarity could be established owing to the Hamiltonian being (formally) self-adjoint. This comment was mostly ignored until when the connection was revived in [10] where also the two-point function of noncommutative ϕ^3 theory was shown to preserve unitarity at one loop in the Yang-Feldman[11] sense. Motivated by this work, Liao and Sibold showed that given the explicit Hermiticity of the (nonlocal) Lagrangian for a noncommutative scalar field theory the standard time-ordered perturbation theory suitably extended to noncommutative realm ensures the unitarity of the model [12]. As we will see in Chapter 4, some rules can be invented to simplify the appearance of a Feynman graph while leaving its topological properties invariant. In [13] it was shown that such naive considerations of these rules are just a consequence of Gell-Mann Low formula still valid for a space-time commutative theory but need to be modified in a space-time noncommutative context as in their framework the interaction vertices are dependent on on-shell momenta of the particles involved in the process which will change drastically the character of the Green's function as the complex contour integration over k^0 is now totally different. To circumvent such complications in our construction we will consider purely space-space non-commutativity for which $\theta^{0j} = 0$. Hence the outline of the paper will be as follows. In the following section we give the basic formal aspects of non-commutative algebras and its consequences for the construction of a field theory in a six-dimensional Minkowskian space-time with signature $(+,-,-,-,-,-)$ with a self interaction of the form $-g\phi^3/3!$ at the classical level and deduce some results concerning classical symmetries of the theory. In Chapter 3 we go on to quantizing this theory path-integral methods and as in the commutative case we consider a quantum effective action and its diagrammatic expansion up to one loop. Then on we can derive Feynman rules for the interaction under consideration and examine the structures of the corresponding Feynman diagrams. A review of the standard methods of dimensional analysis and power counting will be given in Chapter 4 which will be followed by the extension to noncommutative theories. In Chapter 5 using dimensional regularization expressions for all the diagrams contributing to the UV divergent N-point functions of the theory up to one loop will be calculated explicitly and we will discover a new phenomena called UV-IR mixing. We will adopt the MS scheme for our renormalization conditions and as a result in Chapter 6 accordingly set up our renormalization group equations whose solution at one loop will yield the expected qualitative behavior of asymptotic freedom of the theory. However, in Chapter 7 we will quote some recent results and comment on the disagreement of the perturbative result with the scale dependence of the exact beta function and draw some conclusions.

1.1 Mathematical Preliminaries

It is obvious that the perturbative approach we physicists are used take is to use real valued functions rather than operator valued ones satisfying equation (1.1). To proceed in this direction, in this section we follow a procedure, due to Hermann Weyl [14], of assigning a function of space-time variables (or probably of momentum upon Fourier transforming) to a quantum operator. This in essence allows one to work with the usual commutative ring of real numbers on which lives the algebra of functions obeying a deformed multiplication law. We first define the Weyl symbol for Schwartz class of real functions[1] $f(x)$:

$$\hat{W}[f] := \frac{1}{(2\pi)^d} \int d^d k \; \tilde{f}(k) \; e^{ik_i \hat{x}^i} \tag{1.3}$$

where $\tilde{f}(k)$ is the usual Fourier transform of $f(x)$. We may introduce a Hermitian operator in terms of which we write the above definition allowing us to give $f(x)$ the meaning of the

[1]functions whose position and momentum space derivatives vanish to all orders at infinity

position space representation of the Weyl symbol:

$$\hat{\Delta}(x) = \frac{1}{(2\pi)^d} \int d^d k \; e^{-ik_i x^i} \; e^{ik_i \hat{x}^i} \tag{1.4}$$

so that equation (1.3) becomes

$$\hat{W}[f] = \int d^d x \; f(x) \; \hat{\Delta}(x). \tag{1.5}$$

Since any field theory, whether conventional or non-commutative, is defined through an action functional of fields and their derivatives, we now continue by defining some abstract properties of this linear operation:

$$[\hat{\partial}_i, \hat{\partial}_j] = 0, \tag{1.6}$$

$$[\hat{\partial}_i, \hat{x}^j] = \delta_i^j. \tag{1.7}$$

It can be easily seen that this operator is the generator for translations, i.e.

$$e^{ia_i \hat{\partial}^i} \; \hat{\Delta}(x) \; e^{-ia_i \hat{\partial}^i} = \hat{\Delta}(x + a) \tag{1.8}$$

explicitly suggested from the following identity which comes out automatically when we take the commutator of this operator with $\hat{\Delta}(x)$ given by (1.4):

$$[\hat{\partial}_i \; , \; \hat{\Delta}(x)] = -\partial_i \, \hat{\Delta}(x) \; . \tag{1.9}$$

Again taking the commutator of $\hat{\partial}_i$ with the Weyl symbol as in (1.5) turns

$$[\hat{\partial}_i, \hat{W}[f]] = \int d^d k \; f(x) \; [\hat{\partial}_i, \hat{\Delta}(x)] = -\int d^d k \; f(x) \; \partial_i \hat{\Delta}(x) \tag{1.10}$$

and recalling the assumptions made on the commutative algebra of functions on $\mathbf{R}^d$ integration by parts gives us the next important result we will need to use in constructing our model:

$$[\hat{\partial}_i \; , \; \hat{W}[f]] = \int d^d x \; \partial_i f(x) \; , \hat{\Delta}(x) = \hat{W}[\partial_i f] \; . \tag{1.11}$$

Equation (1.8) reveals yet another important property for the algebra. Upon taking the trace of both sides and using the cyclic property for the trace we discover that $Tr(\hat{\Delta}(x))$ does not depend on the particular space-time point it is evaluated and hence we are free to choose a conventional normalization of $Tr(\hat{\Delta}(x)) = 1$ Hence taking the trace of equation (1.5) we obtain:

$$Tr(\hat{W}[f]) = \int d^d x \; f(x) \; . \tag{1.12}$$

Indeed this relation can be inverted and the mapping $f(x) \leftrightarrow \hat{W}[f]$ mediated by $\hat{\Delta}(x)$ is known as Weyl-Wigner correspondence. To see how this works in practice, consider first the product of two such intermediary operators:

$$\begin{aligned}
\hat{\Delta}(x)\,\hat{\Delta}(y) &= \iint \frac{d^dk}{(2\pi)^d}\,\frac{d^dk'}{(2\pi)^d}\; e^{i(k+k')_i\hat{x}^i}\; e^{-\frac{i}{2}\theta^{ij}k_ik'_j}\; e^{-ik_ix^i-ik'_iy^i} \\
&= \iiint d^dz\frac{d^dk}{(2\pi)^d}\,\frac{d^dk'}{(2\pi)^d}\;\hat{\Delta}(z)\; e^{i(k+k')_iz^i}\; e^{-\frac{i}{2}\theta^{ij}k_ik'_j}\; e^{-ik_ix^i-ik'_iy^i} \\
&= \frac{1}{\pi^d|\det\theta|}\int d^dz\;\hat{\Delta}(z)\; e^{-2i(x-z)^i(\theta^{-1})_{ij}(y-z)^j} \qquad (1.13)
\end{aligned}$$

where in the first equality we made use of Baker-Campbell-Hausdorf (BCH) formula, inverse-Fourier transforming on equation (1.4) and exchanging the order of integration in the second, and finally doing out the Gaussian integration on k and k' in getting the last equality. Now, taking the trace of both sides of (1.13) and using the previously agreed normalization of $Tr(\hat{\Delta}) = 1$ we immediately obtain

$$Tr(\hat{\Delta}(x)\hat{\Delta}(y)) = \delta^d(x-y) \qquad (1.14)$$

When combined with (1.5), this gives the sought after Wigner distribution function:

$$f(x) = Tr(\hat{W}[f]\hat{\Delta}(x)) \qquad (1.15)$$

Since while we write down actions for our field theories it includes some powers of the fields, and now we know product of at least two operators connecting the Weyl operators to the Wigner functions, we need to calculate some sort of product of two Weyl symbols:

$$\begin{aligned}
Tr(\hat{W}[f]\hat{W}[g]\hat{\Delta}(x)) &= Tr\iint d^dyd^dz f(y)g(z)\hat{\Delta}(y)\hat{\Delta}(z)\hat{\Delta}(x) \\
&= \frac{1}{\pi^d|\det\theta|}Tr\iiint d^dyd^dzd^dz' f(y)g(z)\hat{\Delta}(z')\hat{\Delta}(x)\; e^{-2i(y-z')^i(\theta^{-1})_{ij}(z-z')^j} \\
&= \frac{1}{\pi^d|\det\theta|}\iiint d^dyd^dzd^dz' f(y)g(z)\delta^d(z'-x)\; e^{-2i(y-z')^i(\theta^{-1})_{ij}(z-z')^j} \\
&= \frac{1}{\pi^d|\det\theta|}\iint d^dyd^dz f(y)g(z)\; e^{-2i(x-y)^i(\theta^{-1})_{ij}(x-z)^j} \qquad (1.16)
\end{aligned}$$

where in the first line we used the definition (1.5), in the second line the result (1.13), in the third line equation (1.14). The above equation serves as the position space representation of the product of two Weyl symbols. We can also explicitly calculate this product in momentum space and assign it a unique Weyl symbol:

$$\hat{W}[f]\hat{W}[g] = \iint \frac{d^dk}{(2\pi)^d}\frac{d^dk'}{(2\pi)^d}\;\tilde{f}(k)\tilde{g}(k')\; e^{ik_i\hat{x}^i}e^{ik'_j\hat{x}^j}$$

$$
\begin{aligned}
&= \iint \frac{d^d k}{(2\pi)^d} \frac{d^d k'}{(2\pi)^d}\ \tilde{f}(k)\tilde{g}(k')\ e^{-\frac{i}{2}k_i\theta^{ij}k'_j}\ e^{i(k_i+k'_i)\hat{x}^i} \\
&=:\ \hat{W}[f\star g] \qquad (1.17)
\end{aligned}
$$

where we used the definition (1.3) in the first equality and BCH formula in the second. The so-called Groenewold-Moyal star product as defined in the last equality above can be given a coordinate space representation:

$$
f(x)\star g(x) = f(x)\ \exp\left(\frac{i}{2}\overleftarrow{\partial_i}\,\theta^{ij}\,\overrightarrow{\partial_j}\right)\ g(x) \qquad (1.18)
$$

The arrows on the partial derivatives are self-explanatory and they are there to account for which momentum contribution to the contraction with the θ tensor in the exponential came from the Fourier expansion of which (f or g) function in the second line of (1.17). There is another way to keep track of this: Shift the space coordinates of the two functions to be star multiplied by two different parameters (say α and β) and differentiate with respect to the respective parameter. When these parameters are set to zero at the end, it will have the same effect of the arrows on the partial derivatives:

$$
(f\star g)(x) \equiv [e^{\frac{i}{2}\theta_{\mu\nu}\partial_{\alpha_\mu}\partial_{\beta_\nu}} f(x+\alpha)g(x+\beta)]_{\alpha=\beta=0} \qquad (1.19)
$$

where we again used the full Lorentz indices to keep the covariance although we will consider (see below) only space-space noncommutativities. As a special case of the above identity take the two functions to be coordinate basis and take their commutator with the standard multiplication replaced by the star product. This is called the Moyal bracket and amounts to the position space representation of expression (1.1):

$$
[x^\mu, x^\nu]_{MB} := x^\mu \star x^\nu - x^\nu \star x^\mu = i\theta^{\mu\nu} \qquad (1.20)
$$

As a side-remark notice that the exponential factor in equation (1.16) plays the role of an integral kernel, $K(y,z;x)$, which can be written as the star product of two Dirac-delta functions, $\delta^d(z-x)\star\delta^d(z-y)$. We will have something more to say about this result when we consider the dynamics of our quantum field theory. At this point, we can list some properties of this product which will be useful in our discussion of the field theory:

1. Since the simplest and usually the most convenient basis to expand our physical fields is the plane wave basis the star product of two exponentials is rewarding to know:

$$
e^{ik_\mu x^\mu} \star e^{iq_\nu x^\nu} = e^{i(k+q)_\sigma x^\sigma} e^{-\frac{i}{2}k_\rho\theta^{\rho\tau}k_\tau} \qquad (1.21)
$$

2. The star product is associative as easily seen from the momentum space representation:

$$[f(x) \star g(x)] \star h(x) = f(x) \star [g(x) \star h(x)] \tag{1.22}$$

3. The star product obeys a cyclic property based on the cyclicity of the operator trace:

$$\begin{aligned} \int d^d x (f_1(x) \star f_2(x) \star ... \star f_m(x)) &= Tr(\hat{W}[f_1]\hat{W}[f_2]...\hat{W}[f_m]) \\ &= Tr(\hat{W}[f_m]\hat{W}[f_1]...\hat{W}[f_{m-1}]) \\ &= \int d^d x (f_m(x) \star f_1(x) \star ... \star f_{m-1}(x)) \end{aligned} \tag{1.23}$$

   In particular the star product of two functions under integral sign behaves as the usual commutative pointwise multiplication:

$$\int d^d x f(x) \star g(x) = \int d^d x f(x).g(x) \tag{1.24}$$

4. Complex conjugation:

$$[f(x) \star g(x)]^* = g^*(x) \star f^*(x) \tag{1.25}$$

   Specifically, $f \star f$ is real if f is real.

All these mathematical tools at hand we can now proceed to building some physical models. In the next chapter we demonstrate this for a simple toy model.

Chapter 2

CONSTRUCTION OF CLASSICAL NONCOMMUTATIVE FIELD THEORY

In this section we first write the action for a massive, scalar ϕ^3 theory in 6 dimensional Minkowski space-time in terms of the Weyl symbol of the real, scalar function $\phi(x)$. However, it should be noted that at the classical level its energy is not bounded from below due to the interaction term. In the next chapter, upon quantization of the classical theory, this will translate into the fact that the theory does not have any stable vacua and that the states of the theory will decay beyond any limit. We do study this model just for its simplicity and hence having the advantage to convey the basic computational tools needed for noncommutative field theories escaping further conceptual difficulties faced with while dealing with gauge theories or theories involving fermionic fields. The action functional in its operator form is written as below:

$$\mathcal{S}[\phi] = Tr\left(\frac{1}{2}\Big[\hat{\partial}_i, \hat{W}[\phi]\Big]^2 - \frac{m^2}{2}\hat{W}[\phi]^2 - \frac{\lambda}{3!}\hat{W}[\phi]^3\right) \tag{2.1}$$

Using the Weyl-Wigner correspondence derived in the previous chapter, we can write this action in terms of the star product as:

$$\mathcal{S}[\phi] = \int d^6x \left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 - \frac{\lambda}{3!}\phi\star\phi\star\phi\right] \tag{2.2}$$

where we used equations (1.11), (1.12), (1.23), (1.24), respectively. In the following we will refer to the interaction Lagrangian as $V_\star(\phi)$ and the action defined by this part of the Lagrangian as $\mathcal{S}_{int}$. As immediately read from the action, the free part is the same as in the conventional scalar field theory and hence will receive no changes to the Feynman rules in the quantization procedure on the top of the commutative case. This we will review in the next chapter for the sake of being self-contained. Now, we are interested in the classical equations of motion which is formally calculated as in the commutative case. The only difference we can trace is due to the interaction:

$$\begin{aligned}
\int d^6x\Big(V_\star(\phi+\delta\phi)-V_\star(\phi)\Big) &= \frac{\lambda}{3!}\int d^6x\Big[(\phi+\delta\phi)\star(\phi+\delta\phi)\star(\phi+\delta\phi)-\phi\star\phi\star\phi\Big] \\
&= \frac{\lambda}{3!}\int d^6x[\phi\star\phi\star\phi+\phi\star\phi\star\delta\phi+\phi\star\delta\phi\star\phi+\delta\phi\star\phi\star\phi \\
&\qquad -\phi\star\phi\star\phi] \\
&= \frac{\lambda}{2}\int d^6x[\phi\star\phi\star\delta\phi] \\
&= \frac{\lambda}{2}\int d^6x[(\phi\star\phi)\delta\phi] =: \int d^6x\frac{\delta V_\star(\phi)}{\delta\phi(x)}\delta\phi(x) \qquad (2.3)
\end{aligned}$$

where we ignored the second order variations in the field. Thus the equations of motion are as below:

$$\frac{\delta\mathcal{S}}{\delta\phi(x)}=0 \quad \Rightarrow \quad (\Box+m^2)\phi_c(x)=-\frac{\lambda}{2}\phi_c(x)\star\phi_c(x) \qquad (2.4)$$

Under our assumption that we only consider theories with $\theta^{0j}=0$ it is obvious that the conjugate momentum to the field is as in the commutative case since the only time derivative appears in the kinetic Lagrangian:

$$\pi(x)=\frac{\partial\mathcal{L}}{\partial\dot{\phi}(x)}=\dot{\phi}(x) \qquad (2.5)$$

For theories with non-zero time-like noncommutativities, interaction part brings in an infinite number of time derivatives from the expansion of the exponential, leading to a theory non-local in time and hence, as mentioned before, has acausal behavior. As we have been discussing the ϕ^3 model at the classical level it is instructive to examine its invariances under classical symmetries. Since it does not have any internal global symmetries(nor even discrete Z_2 is applicable here), we proceed with an application of external symmetry such as space-time translations. That is we want our action to be invariant under the following infinitesimal translations:

$$\begin{aligned}
x^\mu &\rightarrow x'^\mu = x^\mu + a^\mu \\
\phi(x) &\rightarrow \phi(x') = \phi(x) + a^\mu\partial_\mu\phi := \phi+\delta\phi \qquad (2.6)
\end{aligned}$$

For the classical path ϕ_c we know that the variation in the action vanishes and we are left with the expression below:

$$\begin{aligned}
\delta\mathcal{S}|_{\phi_c}=0 &= \mathcal{S}(\phi_c+\delta\phi_c)-\mathcal{S}(\phi_c) \\
&= \int d^6x\partial^\mu\Big[\frac{1}{2}(\partial_\mu\phi_c\star\partial_\nu\phi_c+\partial_\nu\phi_c\star\partial_\mu\phi_c)-\eta_{\mu\nu}\mathcal{L}\Big]a^\nu \qquad (2.7)
\end{aligned}$$

from which it can be deduced that, for an arbitrary shift in the space-time variable, if we make the below definition

$$T_{\mu\nu} = \frac{1}{2}(\partial_\mu \phi_c \star \partial_\nu \phi_c + \partial_\nu \phi_c \star \partial_\mu \phi_c) - \eta_{\mu\nu}\mathcal{L} \tag{2.8}$$

then under the integral sign we get the following identity:

$$\int d^6x \; \partial^\mu T_{\mu\nu} = 0 \tag{2.9}$$

There are two observations to make from the above discussion. First, it is remarkable that the assignment (2.8) we made for the energy-momentum tensor is automatically symmetric as in the case of the conventional scalar theory. However, this need not be always true for noncommutative field theories. After all the rotational sector of Lorentz symmetry which, through the local conservation of angular momentum density, implies the symmetry of the energy-momentum tensor is broken by the constant (noncovariantly transforming) noncommutativity parameter θ (see [15] for a discussion Noether procedure at operator level.) Second, although in (2.9) we found that under the integral sign $T_{\mu\nu}$ is divergenceless (we still have to check if it is conserved. In order to do this we explicitly begin by differentiating (2.8) and use equations of motion wherever necessary:

$$\begin{aligned}
\partial^\mu T_{\mu\nu} &= \frac{1}{2}\Big(\Box\phi \star \partial_\nu \Phi + \partial_\mu \phi \star \partial^\mu \partial_\nu \phi + \partial^\mu \partial_\nu \phi \star \partial_\mu \phi + \partial_\nu \phi \star \Box\phi\Big) \\
&\quad -\frac{1}{2}\Big(\partial_\nu \partial^\mu \phi \star \partial_\mu \phi + \partial^\mu \phi \star \partial_\mu \partial_\nu \phi\Big) + \frac{m^2}{2}\Big(\partial_\nu \phi \star \phi + \phi \star \partial_\nu \phi\Big) \\
&\quad +\frac{\lambda}{3!}\Big(\partial_\nu \phi \star \phi^{\star 2} + \phi \star \partial_\nu \phi \star \phi + \phi^{\star 2} \star \partial_\nu \phi\Big) \\
&= -\frac{\lambda}{2 \cdot 2}\Big(\phi^{\star 2} \star \partial_\nu \phi + \partial_\nu \phi \star \phi^{\star 2}\Big) \\
&\quad +\frac{\lambda}{3!}\Big(\partial_\nu \phi \star \phi^{\star 2} + \phi \star \partial_\nu \phi \star \phi + \phi^{\star 2} \star \partial_\nu \phi\Big) \\
&= -\frac{\lambda}{12}\partial_\nu \phi \star \phi^{\star 2} + \frac{\lambda}{6}\phi \star \partial_\nu \phi \star \phi - \frac{\lambda}{12}\phi^{\star 2} \star \partial_\nu \phi \\
&= \frac{\lambda}{12}\Big[[\phi, \partial_\nu \phi]_{MB}, \phi\Big]_{MB}
\end{aligned} \tag{2.10}$$

where $\phi^{\star 2} = \phi \star \phi$ and we made used of (2.4) in the second equality above. Although the concept of a conserved current is modified [1], the Moyal bracket vanishes in the $\theta \to 0$

[1] In [16] it is discussed that for a massless noncommutative field theory the improved energy-momentum tensor in the sense of [17] can be brought to a locally conserved form. However, this will not be traceless which is not problematic because of the non-scale invariance of the theory due to the dimensionful noncommutativity parameter.

limit and we recover the standard Noether's theorem. This result can be better analyzed by considering some generalizations to Noether's theorem which states that to every continuous global symmetry of the action there corresponds a conserved current and a conserved charge derived from this current. This translates into the action being invariant whether evaluated along the original path or the infinitesimally shifted path $\mathcal{S}[\phi] = \mathcal{S}[\phi + \epsilon\delta\phi]$, where ϵ is constant. However, if we let the small deformation parameter be space-time dependent we have no other choice but the following identity to hold:

$$\mathcal{S}[\phi + \epsilon(x)\delta\phi] - \mathcal{S}[\phi] = \delta\mathcal{S} = \int d^d x \mathcal{J}^\mu \partial_\mu \epsilon(x) \tag{2.11}$$

because we expect the difference give zero when x-dependence of ϵ is taken away and because this is the way to construct a Lorentz invariant since the action is a scalar. Using the first line of (2.7) and integrating by parts gives the following result:

$$\int d^d x \; \epsilon(x) \partial_\mu \mathcal{J}^\mu[\phi] = 0 \tag{2.12}$$

In commutative algebra of fields this would read as for an arbitrary $\epsilon(x)$ the current is conserved. However, since for any two functions $f(x)$ and $g(x)$ we know that using (1.24)

$$\begin{aligned} 0 = \int d^d x \Big(f(x) \cdot g(x) - g(x) \cdot f(x) \Big) &= \int d^d x \Big(f(x) \star g(x) - g(x) \star f(x) \Big) \\ &= \int d^d x [f(x), g(x)]_{MB} \end{aligned} \tag{2.13}$$

and hence for noncommutative theories in any space-time dimensions the following modification takes place:

$$\partial_\mu \mathcal{J}^\mu[\phi(x)] = \Big[F_1[\phi(x)], F_2[\phi(x)] \Big]_{MB} \tag{2.14}$$

where F_1 and F_2 are some functionals of the fields (and most probably of their derivatives) since on the right hand the current depends on ϕ explicitly. These functionals are to be determined specifically from the theory under consideration and for the global symmetry whose current is looked for. For example, for our model (2.2) we deduced in (2.10) that

$$\begin{aligned} F_1[\phi] &\equiv [\phi, \partial_\nu \phi]_{MB} \\ F_2[\phi] &\equiv \phi \end{aligned} \tag{2.15}$$

It should be noted that although we use the word "current vector" for the conserved quantity it need not be of a truly vector nature as in the case for space-time translational invariance since the deformation parameter of the path itself had an extra Lorentz index as opposed to internal symmetries of a theory. Then this can be understood as a vector equation for each of the six uncontracted indices ν in the expression $\partial^{\mu}T_{\mu\nu} \propto [[\phi, \partial_{\nu}\phi]_{MB}, \phi]_{MB}$. Although the way the current is conserved has been changed, for the purely space-space noncommutative theories we are considering there still exists some conserved quantity as easily can be seen by the argument below:

$$\begin{aligned}
\int d^5x[f,g]_{MB} &= \iiint d^5x d^6k d^6q \tilde{f}(k)\tilde{g}(q)\Big(e^{-\frac{i}{2}k_{\rho}\theta^{\rho\sigma}q_{\sigma}} - e^{-\frac{i}{2}q_{\rho}\theta^{\rho\sigma}k_{\sigma}}\Big)e^{i(k+q)_{\mu}x^{\mu}} \\
&= \iint d^6k d^6q \tilde{f}(k)\tilde{g}(q)\Big(e^{-\frac{i}{2}k_i\theta^{ij}q_j} - e^{-\frac{i}{2}q_i\theta^{ij}k_j}\Big)e^{i(k+q)_0x^0}\int d^5x e^{-i(k+q)_ix^i} \\
&= \iint d^6k d^6q \tilde{f}(k)\tilde{g}(q)\Big(e^{-\frac{i}{2}k_i\theta^{ij}q_j} - e^{-\frac{i}{2}q_i\theta^{ij}k_j}\Big)e^{i(k+q)_0x^0}\delta^5(k-q) \\
&= 0 \qquad (2.16)
\end{aligned}$$

where the last equality follows from the antisymmetry of the θ. Then using the right hand side of equation (2.14):

$$\begin{aligned}
\int d^5x(\partial_0\mathcal{J}^0 + \partial_i\mathcal{J}^i) &= 0 \\
&= \partial_0\int d^5x\mathcal{J}^0 + \int d^5x\vec{\nabla}.\vec{\mathcal{J}} \\
&= \frac{\partial}{\partial t}Q \qquad (2.17)
\end{aligned}$$

where the second term in second to the last equation is a total divergence and vanishes under appropriate boundary conditions and we defined the conserved charge Q corresponding to our symmetry transformation.

Chapter 3

QUANTIZATION OF THE NONCOMMUTATIVE SCALAR FIELD THEORY

As we usually consider in the commutative field theory there are mainly two basic quantizations: canonical and path integral methods. In this section we will start considering mainly path integral quantization for noncommutative scalar field theories and make some comments on canonical quantization wherever suitable. We have seen that the free part of the action is exactly the same as the commutative theory, so upon quantization it will give us the same Feynman rules for the propagator. Hence, let us review this procedure for the free theory and then continue with the self-interacting model of our interest.

3.1 Path Integral Approach to the Free Theory

In this section, we follow a more heuristic approach to deriving the Feynman propagator whereas the same treatment would be done in terms of functional analogue of the usual Gaussian type integrands [18]. We set forth by defining Z_0 , the vacuum-vacuum transition amplitude in the absence of a genuine interaction:

$$Z_0[J] = \int \mathcal{D}\phi \; e^{i\int d^6x[\mathcal{L}_0 + J(x)\phi(x) + \frac{i}{2}\varepsilon\phi^2]} \tag{3.1}$$

where $\mathcal{D}\phi$ denotes integration over all possible histories in the space of field configurations of the process under consideration and the integral in the exponent is over a six dimensional Minkowski space-time. We also included the small real, positive constant ε to ensure the convergence of the path integral but we always keep in mind to set it to zero at the end of our calculations. Z_0 is a functional of the source J which in turn depends on the coordinates x themselves and the Lagrangian of a single, scalar, non-interacting field is given by:

$$\mathcal{L}_0 = \frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}m^2\phi^2 \tag{3.2}$$

After integrating by parts, converting the first term to a surface term by 6 dimensional Gauss theorem and assuming that $\phi \to 0$ as $x \to \infty$, the $(\partial_\mu\phi)(\partial^\mu\phi)$ term in the integral above can be written as:

$$\int (\partial_\mu \phi)(\partial^\mu \phi) d^6x = -\int \phi \Box \phi \; d^6x \tag{3.3}$$

hence leading to the expression

$$Z_0[J] = \int \mathcal{D}\phi \; e^{-i\int d^6x[\frac{1}{2}\phi(\Box + m^2 - i\varepsilon)\phi + J(x)\phi(x)]} \tag{3.4}$$

Now if we define a new field $\phi' = \phi + \phi_0$ with ϕ_0 satisfying the same assumption as before and the relation

$$(\Box + m^2 - i\varepsilon)\phi_0(x) = J(x) \tag{3.5}$$

the integrand of the exponent in (3.4) in terms of the new field

$$\int d^6x \; [\frac{1}{2}\phi(\Box + m^2 - i\varepsilon)\phi + \phi(\Box + m^2 - i\varepsilon)\phi_0 + \frac{1}{2}\phi_0(\Box + m^2 - i\varepsilon)\phi_0 - J(x)\phi(x) - J(x)\phi_0(x)] \tag{3.6}$$

boils down to the simpler expression

$$\int d^6x \; \frac{1}{2}[\phi(\Box + m^2 - i\varepsilon)\phi - J(x)\phi_0(x)] \tag{3.7}$$

Now, we can suggest an integral representation for ϕ_0 of the following form in terms of the so-called Feynman propagator $\Delta(x)$

$$\phi_0(x) = -\int d^6x_1 \; \Delta(x - x_1)J(x_1) \tag{3.8}$$

where $\Delta(x)$ is a Green's function for the Klein-Gordon equation:

$$(\Box + m^2 - i\varepsilon)\Delta(x) = -\delta^6(x) \tag{3.9}$$

As a result, Z_0 becomes

$$Z_0[J] = \int \mathcal{D}\phi \; e^{-\frac{i}{2}\int d^6x \phi(\Box + m^2 - i\varepsilon)\phi} \times e^{-\frac{i}{2}\int d^6x_1 d^6x_2 J(x_1)\Delta(x_1 - x_2)J(x_2)} \tag{3.10}$$

However, the first term being integrated over the whole configuration space of fields gives a **C**-number. Although this might be severely divergent it will not cause a problem for us since we will be interested in appropriately normalized transition amplitudes. Hence, our final result for vacuum-to-vacuum transition amplitude in the presence of an external source becomes

$$Z_0[J] = \mathcal{N} \; e^{-\frac{i}{2}\int d^6x_1 d^6x_2 J(x_1)\Delta(x_1 - x_2)J(x_2)} \tag{3.11}$$

We usually work in momentum space so it is useful to consider the Fourier transform of equation (3.9):

$$\Delta(x) = \frac{1}{(2\pi)^6}\int d^6k \frac{e^{-ikx}}{k^2 - m^2 + i\varepsilon} \tag{3.12}$$

We see that the extra small pure imaginary part we added to the Lagrangian to improve its convergence properties at the beginning, now defines a contour in the k_0 plane where the poles of the propagator are at $k_0 = \mp(\vec{k}^{\ 2} + m^2)^{1/2} \pm i\delta = \mp E_k \pm i\delta$. However as we said before, it is always understood that $\varepsilon \to 0$ limit shall be taken at the end. We illustrate the pole structure of the propagator in Figure 3.1, and how the path of integration gets deformed accordingly when this limit is realized in Figure 3.2 (see [19] for a comparison of the deformed contours for the advanced and the retarded propagators).

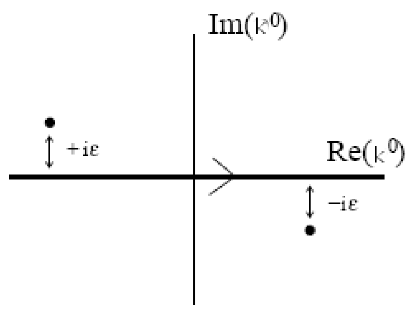


Figure 3.1: The poles of the propagator and integration path along real k^0

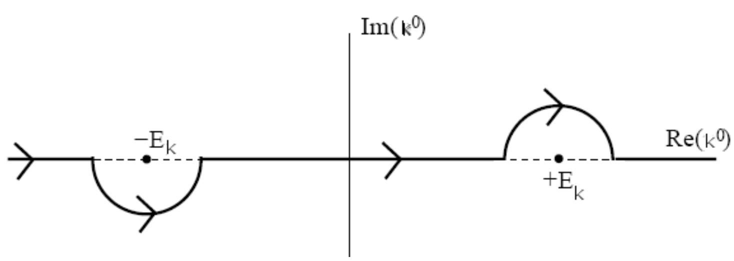


Figure 3.2: The choice of contour for $\varepsilon \to 0$

The free theory being the same at the classical level and hence having the same quantization procedure both in commutative and noncommutative scalar field theories allows us to use the same Fock space and its vacuum state in both theories and hence expand our fields in terms of the usual creation and annihilation operators:

$$\phi(x) = \sum_k a(k)e^{-ikx} + a^\dagger(k)e^{ikx} \tag{3.13}$$

The path integral measure in (3.4) can then be understood better if we define a new but natural measure for our noncommutative theory as:

$$(\mathcal{D}\phi(x))_\star := \lim_{N\to\infty} d\phi(x_1) \star d\phi(x_2) \star ... \star d\phi(x_N) \tag{3.14}$$

If we make use of equation (3.13) here, then all the star products in (3.14) will combine to give a momentum dependent phase factor which can be redefined into $\mathcal{N}$ of (3.11). It is worth mentioning at the end of this section that with the conjugate momentum we found in (2.5) we can impose the usual equal time commutation relations(see [20] for a suggestion of the method on space-time noncommutativities)

$$[\phi(\vec{x},t), \pi(\vec{y},t)] = i\delta^3(\vec{x}-\vec{y}) \tag{3.15}$$

3.2 A Diagrammatic Approach to the Quantum Effective Action

We know from the conventional field theory that $Z[J]$ embodies much more information than indeed needed in calculations yielding physical results. In particular, it involves the disconnected diagrams which do not talk to each other and hence do not contribute to the off diagonal entries of the scattering matrix [21]. Hence we define the generating functional $W[J]$ for connected diagrams as

$$Z[J] = e^{iW[J]} \tag{3.16}$$

Even this object has excess data in the sense that it gives rise to diagrams which can be built from some smaller basic diagrams. Thus we Legendre transform $W[J]$ to obtain the effective action $\Gamma[\phi]$, or equivalently the generating functional for those connected diagrams which cannot be separated into two disconnected diagrams by cutting an internal propagator, the so-called 1-particle irreducible (1PI) graphs.

$$\Gamma[\phi_c] = W[J] - \int d^6x J(x)\phi_c(x) \tag{3.17}$$

where the classical path satisfies

$$\phi_c(x) = \frac{\delta W[J]}{\delta J(x)} \tag{3.18}$$

From here it is easy to derive the effective field equations:

$$\frac{\delta\Gamma[\phi_c]}{\delta\phi_c(x)} = -J(x) \tag{3.19}$$

Our aim now is to express these field equations as a set of functional differential equations and try to solve them approximating through a series expansion in $\hbar$. We assume appropriate boundary conditions so that the functional generalization of the property that the integral of a total derivative vanishes also holds:

$$\begin{aligned} 0 &= \int \mathcal{D}\phi\ \frac{\hbar}{i}\frac{\delta}{\delta\phi(x)}\ e^{\frac{i}{\hbar}\mathcal{S}(\phi)+\frac{i}{\hbar}\int J(y)\phi(y)\,dy]} \\ &= \int \mathcal{D}\phi\ \left(\frac{\delta\mathcal{S}}{\delta\phi(x)} + J(x)\right)\ e^{\frac{i}{\hbar}\mathcal{S}(\phi)+\frac{i}{\hbar}\int J(y)\phi(y)\,dy} \end{aligned} \tag{3.20}$$

Conventionally, to do the integration on the first term of the above expression we replace the fields down with the functional derivatives with respect to the external source J so that they will bring down as many ϕ's as needed in the first derivative of the action. However, star product among the interaction term complicates things a little more, but we will now show, particularly for our ϕ^3 theory, that the results from both commutative and noncommutative calculations agree at least at a formal level. Plugging from (2.3) into the problematic first term of (3.20):

$$\begin{aligned} &\int \mathcal{D}\phi\ (\phi(x)\star\phi(x))\ e^{\frac{i}{\hbar}\mathcal{S}(\phi)+\frac{i}{\hbar}\int J(y)\phi(y)\,dy} \\ =& \left[e^{\frac{i}{2}\theta_{\mu\nu}\partial_{\alpha_\mu}\partial_{\beta_\nu}}\int \mathcal{D}\phi\ \phi(x+\alpha)\phi(x+\beta)e^{\frac{i}{\hbar}\mathcal{S}(\phi)+\frac{i}{\hbar}\int J(y)\phi(y)\,dy}\right]_{\alpha=\beta=0} \\ =& \frac{\hbar^3}{i^2}\left[e^{\frac{i}{2}\theta_{\mu\nu}\partial_{\alpha_\mu}\partial_{\beta_\nu}}\frac{\delta}{\delta J(x+\alpha)}\frac{\delta}{\delta J(x+\beta)}e^{\frac{i}{\hbar}W[J]}\right]_{\alpha=\beta=0} \\ =& \left(\frac{\hbar}{i}\right)^2\ \left(\frac{\delta}{\delta J(x)}\star\frac{\delta}{\delta J(x)}\right)(x)\ e^{\frac{i}{\hbar}W[J]} \end{aligned} \tag{3.21}$$

The exact meaning embedded in this star product of functional derivatives will become clear when we try to deduce Feynman rules for the interaction vertex of our theory. For now, we can continue from the step we were left with at (3.20), by using the functional version of the well-known trick:

$$F(\partial_x)\ e^{g(x)} = e^{g(x)}\ F(g'(x)+\partial_x) \tag{3.22}$$

Hence (3.20) becomes

$$0 = \left[\left(\frac{\delta\mathcal{S}}{\delta\phi(x)}\right)_{\phi(x)\to\frac{\hbar}{i}\frac{\delta}{\delta J(x)}} + J(x)\right]e^{\frac{i}{\hbar}W[J]} = e^{\frac{i}{\hbar}W[J]}\left[\left(\frac{\delta\mathcal{S}}{\delta\phi(x)}\right)_{\phi(x)\to\frac{\delta W}{\delta J(x)}+\frac{\hbar}{i}\frac{\delta}{\delta J(x)}} + J(x)\right] \tag{3.23}$$

so that

$$\left.\frac{\delta\mathcal{S}}{\delta\phi(x)}\right|_{\phi(x)\to\frac{\delta W}{\delta J(x)}+\frac{\hbar}{i}\frac{\delta}{\delta J(x)}} + J(x) = 0 \tag{3.24}$$

or equivalently upon using (3.18) and functional generalization of the chain rule we get

$$\frac{\delta\Gamma}{\delta\phi_c(x)} = \left(\frac{\delta\mathcal{S}}{\delta\phi(x)}\right)_{\phi(x)\to\phi_c(x)+\frac{\hbar}{i}\int d^6y G_{\phi_c}(x,y)\frac{\delta}{\delta\phi_c(y)}} \tag{3.25}$$

where $G_{\phi_c}(x,y)$ and its inverse satisfy the following identities:

$$G_{\phi_c}(x,y) = \frac{\delta\phi_c(y)}{\delta J(x)} = \frac{\delta^2 W[J]}{\delta J(x)\delta J(y)}$$

$$G^{-1}_{\phi_c}(x,y) = \frac{\delta J(x)}{\delta\phi_c(y)} = -\frac{\delta^2\Gamma[\phi_c]}{\delta\phi_c(x)\delta\phi_c(y)}$$

$$\int d^6z G_{\phi_c}(x,z)\frac{\delta^2\Gamma[\phi_c]}{\delta\phi_c(z)\delta\phi_c(y)} = -\delta^6(x-y) \tag{3.26}$$

Knowing these we can now compute the 1-loop effective action; that is we take the already derived result (2.3) and make the substitution in (3.25):

$$\begin{aligned}\left(\frac{\delta\mathcal{S}_{int}}{\delta\phi(x)}\right)_{\phi\to\phi_c+\frac{\hbar}{i}\int G\frac{\delta}{\delta\phi_c}} &= -\frac{\lambda}{2}\left(\phi_c(x)+\frac{\hbar}{i}\int d^6y\, G(x,y)\frac{\delta}{\delta\phi_c(y)}\right)\star\phi_c(x) \\ &= -\frac{\lambda}{2}\left(\phi_c(x)\star\phi_c(x)\right) - \frac{\lambda}{2}\frac{\hbar}{i}\int d^6y\, G(x,y)\frac{\delta}{\delta\phi_c(y)}\star\phi_c(x)\end{aligned} \tag{3.27}$$

The first term is the classical part and the second term includes the quantum corrections coming from the loop effects. Hence computing the star products on the second term, we explicitly see the 1-loop contribution to the effective action:

$$\begin{aligned}-\frac{\lambda}{2}\frac{\hbar}{i}\int d^6y\, G(x,y)\frac{\delta}{\delta\phi_c(y)}\star\phi_c(x) &\propto \left[e^{\frac{i}{2}\theta_{\mu\nu}\partial_{\alpha_\mu}\partial_{\beta_\nu}}\int d^6yG(x+\alpha,y)\frac{\delta}{\phi_c(y)}\phi(x+\beta)\right]_{\alpha=\beta=0} \\ &\propto \left[e^{\frac{i}{2}\theta_{\mu\nu}\partial_{\alpha_\mu}\partial_{\beta_\nu}}\int d^6yG(x+\alpha,y)\delta^6(y-x-\beta)\right]_{\alpha=\beta=0} \\ &\propto \left[e^{\frac{i}{2}\theta_{\mu\nu}\partial_{\alpha_\mu}\partial_{\beta_\nu}}G(x+\alpha,x+\beta)\right]_{\alpha=\beta=0} \\ &\propto \left[e^{\frac{i}{2}\theta_{\mu\nu}\partial_{\alpha_\mu}\partial_{\beta_\nu}}\int d^6k\tilde{G}(k)e^{ik(\alpha-\beta)}\right]_{\alpha=\beta=0} \\ &\propto \int d^6k\tilde{G}(k) = G(0)\end{aligned} \tag{3.28}$$

where in the first line we employed the definition of star product, in the fourth line we used the translational invariance of the propagator and its Fourier transform and in the last line the anti-commutativity of the θ. So, if we assume an expansion of the effective action Γ and the two point function $G(x,y)$ in increasing powers of $\hbar$[1], comparison of $\hbar$'s in (3.27) with respect to the ones in (3.25) gives the following result:

$$\frac{\delta\Gamma_1}{\delta\phi_c(x)} = -\frac{\lambda}{2}G_0(0) \tag{3.29}$$

To find the 1-loop effective action this equality needs to be integrated. However, there is a nicer and simpler argument concerning diagrammatics of the theory. After all, since the free theory is the same as the commutative one, we can still identify the classical propagator $G(x,y)$ with a straight line with the ends labeled the space-time points x and y. To this end we first derive the 3-vertex rule given by the third functional derivative of the classical action:

$$\begin{aligned}\frac{\delta^2\mathcal{S}_{int}}{\delta\phi(x_1)\delta\phi(x_2)} = -\frac{\lambda}{2}\, Re\Big[e^{\frac{i}{2}\theta_{\mu\nu}\partial_{\alpha_\mu}\partial_{\beta_\nu}} \quad [\quad & \delta^6(x_1+\alpha-x_2)\;\phi(x_1+\beta) \\ + \quad & \delta^6(x_1+\beta-x_2)\;\phi(x_1+\alpha)]\Big]\end{aligned} \tag{3.30}$$

$$\begin{aligned}\frac{\delta^3\mathcal{S}_{int}}{\delta\phi(x_1)\delta\phi(x_2)\phi(x_3)} &= -\frac{\lambda}{2}\, Re\Big[e^{\frac{i}{2}\theta_{\mu\nu}\partial_{\alpha_\mu}\partial_{\beta_\nu}}\;[\delta^6(x_1+\alpha-x_2)\;\delta^6(x_1+\beta-x_3) \\ &\qquad + \delta^6(x_1+\beta-x_2)\;\delta^6(x_1+\alpha-x_3)\Big] \\ &= -\frac{\lambda}{2}\, Re\Big[\int d^6k_2 d^6k_3 e^{\frac{i}{2}\theta_{\mu\nu}\partial_{\alpha_\mu}\partial_{\beta_\nu}}\;[e^{ik_2(x_1+\alpha-x_2)}\;e^{ik_3(x_1+\beta-x_3)} \\ &\qquad + e^{ik_2(x_1+\beta-x_2)}\;e^{ik_3(x_1+\alpha-x_3)}\Big] \\ &= -\lambda\; Re\Big[\int d^6k_2 d^6k_3 e^{ik_2(x_1-x_2)}\;e^{ik_3(x_1-x_3)}\cos\frac{k_2\theta k_3}{2}\Big] \\ &= -\frac{\lambda}{3}\, Re\Big[\int d^6k_1 d^6k_2 d^6k_3 (2\pi)^6\delta^6(k_1+k_2+k_3)e^{-ik_1x_1-ik_2x_2-ik_3x_3}\times \\ &\qquad \left(\cos\frac{k_1\theta k_2}{2}+\cos\frac{k_1\theta k_3}{2}+\cos\frac{k_2\theta k_3}{2}\right)\Big]\end{aligned} \tag{3.31}$$

where in the second equality we wrote the delta functions in momentum space and in the last equality we added a delta function from out by hand which is indeed needed to symmetrize over the momenta entering the vertex. Hence we see that on the top of the commutative Feynman rules for the vertex we have to multiply with a symmetric combination of cosine functions of the momenta coming to the interaction vertex. As a side note it is obvious

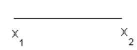
[1]For a proof of the fact that powers of $\hbar$ in this expansion actually counts the number of loops in that term of the action, see [22].

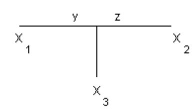
that some of the terms arising from such vertex factors can heavily change the behavior of the Feynman integrals. We will come to this point in the next chapter. For now, as usual we can also differentiate the propagator and using the last identity of (3.26) at the zeroth order we obtain:

$$0 = \int d^6 z \left[\frac{\delta G_0(x_1, z)}{\delta\phi_c(x_3)} \frac{\delta^2 \Gamma_0[\phi_c]}{\delta\phi_c(z)\delta\phi_c(x_2)} + G_0(x_1, z) \frac{\delta^3 \Gamma_0[\phi_c]}{\delta\phi_c(z)\delta\phi_c(x_2)\delta\phi_c(x_3)} \right]$$

$$\Rightarrow \quad \frac{\delta G_0(x_1, x_2)}{\delta\phi_c(x_3)} = \int d^6 y d^6 z G_0(x_1, y) \frac{\delta^3 \mathcal{S}[\phi_c]}{\delta\phi_c(y)\delta\phi_c(x_3)\delta\phi_c(z)} G_0(z, x_2) \tag{3.32}$$

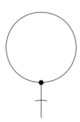
which in the light of the above rules can be diagrammatically represented as:

$$\frac{\delta}{\delta\phi_c(x_3)} \Bigg(\quad$$ 

$$\quad \Bigg) \quad = $$ 

(3.33)

The effect of differentiating the free propagator with respect to a field at space-time point x_3 has been to add an external line labeled with the same space-time point. Keeping these in mind we will now give an ansatz for the diagrammatic representation of derivative of the one-loop effective action and show that it gives the same expression as we found in (3.29):

$$\frac{\delta\Gamma_1}{\delta\phi_c(x)} = \frac{1}{2}$$ 

$$\begin{aligned}
&= \frac{1}{2} \int d^6 y \, d^6 z \; G_0(y, z) \frac{\delta^3 \mathcal{S}}{\delta\phi(y)\delta\phi(z)\delta\phi(x)} \\
&= -\frac{\lambda}{4} \int d^6 y \, d^6 z \; G_0(y, z) \left[e^{\frac{i}{2}\theta_{\mu\nu}\partial_{\alpha_\mu}\partial_{\beta_\nu}} \; [\delta^6(x + \alpha - y) \, \delta^6(x + \beta - z) \right. \\
&\qquad \left. + \, \delta^6(x + \beta - y) \, \delta^6(x + \alpha - z)] \right] \\
&= -\frac{\lambda}{4} e^{\frac{i}{2}\theta_{\mu\nu}\partial_{\alpha_\mu}\partial_{\beta_\nu}} [G_0(x + \alpha, x + \beta) \; + G_0(x + \beta, x + \alpha)] \\
&= -\frac{\lambda}{2} e^{\frac{i}{2}\theta_{\mu\nu}\partial_{\alpha_\mu}\partial_{\beta_\nu}} \int d^6 k \tilde{G}_0(k) e^{ik(\alpha - \beta)} \\
&= -\frac{\lambda}{2} \int d^6 k \tilde{G}_0(k) e^{ik\theta k} \\
&= -\frac{\lambda}{2} G_0(0)
\end{aligned} \tag{3.34}$$

where in the second line we used the first equality of (3.31), in the third line we evaluated the six-dimensional delta functions, in the fourth we Fourier transformed the position space propagator together with the property that it is translational invariant and symmetric in its arguments, and finally the anti-symmetry of the noncommutativity tensor in the last line.

This result is the same as the one found in (3.29) and thus this suggests that we can assign a diagram for the one loop effective action to our theory as below:

$$\Gamma[\phi_c] \quad = \quad \bullet + \left(\frac{\hbar}{i}\right)\frac{1}{2} \quad \bigcirc \quad + \cdots \tag{3.35}$$

where $\cdots$ denotes the higher order loops in the effective action. We see that the effective action for the noncommutative theory matches with that of commutative at least at one-loop order.

Chapter 4

INTERLUDE:
EXPOSITION OF DIVERGENCES IN N-POINT FUNCTIONS, DIMENSIONAL ANALYSIS, POWER COUNTING AND ALL THAT

To begin this chapter, we will first review the power counting method in the usual conventional field theory which allows one to make conclusions about the renormalizabilty of the theory following mainly [23]. For applications to dimensional regularization method that will be elaborated later on, we take our space-time of the generic dimensions $d = 6 - \epsilon$ from the beginning. After that we will make the necessary connections with the noncommutative theory. In conventional field theory we consider an alternative way to improve the convergence properties of the oscillating behavior of the generating functional Z which was tried to be cured by including a small imaginary part in the action in the exponential of (3.1). This new method allows one to make a so-called Wick rotation, that is define a new imaginary time coordinate $x_0 = -ix_D$ so that the generating functional can be written as

$$Z[J] = \mathcal{N} \int \mathcal{D}\phi \; e^{-\mathcal{S}_E^0 + \int d^dx J(x)\phi_0(x)} \tag{4.1}$$

where

$$\mathcal{S}_E^0 = \int d^dx \left(\frac{1}{2}\partial^\mu \phi_0 \partial_\mu \phi_0 + \frac{m_0^2}{2}\phi_0^2 + \frac{\lambda_0}{3!}\phi_0^3 \right) \tag{4.2}$$

We denoted the field, coupling and mass with the subscript zero in reference to bare parameters whose meaning will become clear shortly, and the integrals are considered over a d dimensional Euclidean space. However, there is a problem with this procedure in noncommutative field theories due to the nonlocality inherent in the formalism. For noncommutative space-times the twist factor $e^{ik_0\theta^{0j}k_j}$ blows up on rotating in the momentum space and not allowing analytic continuation. (For example see [24] for an example where a graph converges in Minkowski space and diverges upon performing a Wick rotation.) For our case of pure space-space noncommutativity the situation may be said to be better: the Osterwalder-Schrader positivity which ensures the positivity of transition amplitude is lost upon analytic continuation. But we go on with performing it because it allow to discover some new physics called UV/IR mixing whose Minkowskian counterpart has not been discovered yet and not expected much to be found. Of course this rotation into the Euclidean plane will cause some changes in the Feynman rules (such as pole structure of the propagator). First thing to notice from the above expression is that since the action is in the exponential it should be dimensionless to admit a meaningful series expansion. This in turn means that Lagrangian must have a mass dimension d since it is being integrated over a d dimensional space and we know that $[m] = [x]^{-1} = 1$. Hence $[\frac{\partial}{\partial x}] = 1$ and $[\phi_0] = [m]^{\frac{(d-2)}{2}}$ which, upon considering the interaction term, implies $[\lambda_0] = [m]^{3-\frac{d}{2}}$ and call this power δ_3 for future references. We observe that the coupling is dimensionless in $d = 6$ dimensions. This result will have important consequences for the renormalizability of our model. However, outside 6 dimensions it is not dimensionless anymore and we would like to introduce a dimensionless coupling g defined as $\lambda = \mu^\epsilon g$. Then together with this definition, we can separate the above given bare action in terms of what is called a renormalized action and a counter term, $\mathcal{S}_E^0 = \mathcal{S}_E^R + \mathcal{S}_E^{(c.t)}$, where

$$\begin{aligned} \mathcal{S}_E^R &= \int d^d x \left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{m^2\phi^2}{2} + \frac{\mu^\epsilon g}{3!}\phi\star\phi\star\phi \right] & (4.3) \\ \mathcal{S}_E^{(c.t)} &= \int d^d x \left[\frac{1}{2}(Z_3-1)\partial_\mu\phi\partial^\mu\phi + \frac{\delta m^2\phi^2}{2} + \frac{\mu^\epsilon g}{3!}(Z_1-1)\phi\star\phi\star\phi \right] & (4.4) \end{aligned}$$

Now the main entity which allows a model's validity to be tested in the laboratory is the scattering amplitude $\mathcal{A}$. This amplitude for a graph with L number of loops, V number of vertices and I number of internal propagators, N number of external legs can be schematically be expressed as

$$\mathcal{A} = \lambda^V F(p_1, p_2, \cdots, p_N) \tag{4.5}$$

So this amplitude for a particular process contributing to some N-point function of the theory will have the same dimensions as $\tilde{\Gamma}^N(p_1, p_2, \cdots, p_N)$ since the function F is a purely numerical factor specific to the process under consideration resulting from the integrations over loop momenta. Hence we can write

$$[F] := [m]^\delta = [m]^{[\tilde{\Gamma}^N]-V\frac{6-d}{2}} \tag{4.6}$$

To obtain the mass dimension of $\tilde{\Gamma}^N(p_1, p_2, \cdots, p_N)$, which is the Fourier transform of $\Gamma^N(x_1, x_2, \cdots, x_N)$, we have to recall that they are the expansion coefficients of the effective action in the basis of fields:

$$\Gamma[\phi] = \sum_{N=0}^{\infty} \frac{1}{N!} \int d^d x_1 \; d^d x_2 \; ...d^d x_N \; \Gamma^N(x_1, x_2, ..., x_N)\phi(x_1)...\phi(x_N) \quad (4.7)$$

$$(2\pi)^d \delta^d(p_1 + ... + p_N)\tilde{\Gamma}^N(p_1, ..., p_N) = \int d^d x_1 ... d^d x_N e^{i(p_1 x_1 + ... + p_N x_N)} \Gamma(x_1, ..., x_N) \quad (4.8)$$

From equation (4.7) we see that the dimension of the proper N-vertex is $[m]^{\frac{Nd}{2}+N}$ because $\Gamma[\phi]$ being an action is dimensionless. Together with this, using the fact that $[\delta^d(p)] = [m]^{-d}$, equation (4.8) gives the mass dimensions of its Fourier transform as $[\tilde{\Gamma}^N(p_1, p_2, \cdots, p_N)] = [m]^{d-N(\frac{d}{2}-1)}$. Thus, plugging this expression in (4.6) gives the following equation for ϕ^3 theory in a d-dimensional space-time:

$$\delta = d - N(\frac{d}{2} - 1) - V\delta_3 \quad (4.9)$$

Let us now try to extract the physical meaning out of this expression. F is the contribution of the internal kinematical factors of the graph under consideration. Since Feynman rules dictate using a propagator for each internal line, we get a power of 2I in denominator, and each independent loop momenta brings an integration of dimension d hence in total a power of dL in numerator. However, since momentum conservation at each vertex tells us to write down a delta function, not all of the I internal momenta are independent but only L=I-V+1, where the extra 1 is added due to the overall momentum conservation in (4.8). Since in our theory from every vertex 3 legs are emanated and N of them are external whereas internal lines are connected pairwise to each vertex, holds the relation $3V = N+2I$ which can be used to eliminate I. Hence we can deduce from this simple power counting that in the presence of an ultraviolet cut-off for the upper boundary of a Feynman integral corresponding to a particular process of an arbitrary order, its high-energy behavior is determined by the above derived δ, the so-called superficial degree of divergence. This means if the powers of loop momenta in numerator dominates then $\delta > 0$ and this particular integral certainly diverges. But, this does not mean that it will ensure the convergence of integrals with $\delta < 0$ because the graph may consist of some smaller parts called subgraphs which has themselves non-negative superficial degree of divergences. To see this more explicitly for ϕ^3 theory in 6 dimensions, plug in d=6 in (4.9) and it gives that the critical value of $\delta_c = 0$ separating the diverging and seemingly converging N point functions of the theory occurs for $N = 3$.

Hence we should expect that any loop correction to the propagator (2-point function) and the proper vertex (3-point function) will diverge. These are called primitive divergences and the fact that there are finitely many of them in a theory is a very important aspect of its renormalizability. We will return to the renormalization of these expressions in the next chapter. But the graphs with 4 or more external lines do not necessarily yield automatically finite answers as seen from the Figure 4.1 that the subgraph contributes to the divergent 1-loop correction of the 3-point vertex of the theory.

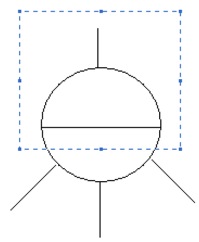


Figure 4.1: Superficially convergent 4-point function of ϕ^3 theory with a divergent subgraph

With this in mind we can focus our attention closer to search for infinities out of a smaller subset of all diagrams of a theory with the help the following theorem (for a proof see [25],[26])

Theorem 1 (Weinberg's) *A Feynman diagram for any kind of quantum field theory at an arbitrary order will converge if the superficial degree of divergence of the graph itself together with those of all of its subgraphs (the graphs obtained from the original one by cutting some internal lines) are negative.*

Up to this point what we all considered was the review of the results for the old, conventional commutative field theories. To understand the situation for the noncommutative case, we would better alter a bit our perception of the newly introduced vertex factors in (3.31). (Since we can trace noncommutativity only throughout interactions consideration of just such factors is enough.) Indeed we know that from (1.21) the star product of two plane waves will bring the non-local momentum dependent factor and hence even without symmetrizing in the momenta flowing into a vertex as in (3.31), momentum space expansion of $\mathcal{S}_{int}$

would introduce a phase of the form $e^{-\frac{i}{2}\sum_{1\leq i<j}^{3} k_i\theta^{ij}k_j}$ which is only cyclically symmetric due to momentum conservation ensuring delta function coming from the x-integration of the other part from BCH formula. Since it is not arbitrarily symmetric under exchanges of the momenta flowing into the vertex, the order of the momenta appearing in such a vertex should be somehow taken care of. One of the most efficient ways to do this is to use ribbon graphs introduced by 't Hooft to handle matrix field theory models [27]. Then using what are called Filk moves [28] one can reduce the complexity of a graph constituted from such lines.

Filk Move 1 *Two vertices V_1 and V_2 of a graph that are connected by an internal line carrying momentum q can be contracted(keeping the order and orientation of the original lines) to give a single vertex V as follows:*

$$V_1(k_1, ..., k_{n_1}, q)V_2(-q, k_{n_1+1}, ..., k_{n_2}) = V(k_1, ..., k_{n_2})\delta(k_1 + ... + k_{n_1} + q) \tag{4.10}$$

Filk Move 2 *A loop which does not cross other lines can be eliminated as follows:*

$$V_1(k_1, ..., k_{n_1}, q, k_{n_1+1}, ..., k_{n_2}, -q) = V(k_1, ..., k_{n_2}), \quad \sum_{i=n_1+1}^{n_2} k_i = 0 \tag{4.11}$$

After the usage of these moves any diagram can be reduced to a generic vertex with the same external lines as the original ones along with some closed loops connected to this vertex. This vertex can contribute to the structure of the Feynman amplitude in two ways: either it may involve a phase factor which is a function of only external momenta in which case the original graph is called a planar graph, or it may also have dependence on the internal loop momenta which could not be eliminated from the original diagram for which it is called a non-planar diagram. From the topological point of view this ribbon diagrams allows us to define the Euler characteristic of a graph with V number of vertices, E number of edges and F number of faces[1] as $\chi = V - E + F$. Then planar graphs have Euler characteristic 2 through the relation $\chi = 2 - 2g$ i.e. they can be drawn without any crossings of the ribbon propagators on the plane whose genus is $g = 0$, and non-planar graphs have $g \geq 1$ and can be drawn on a surface of genus g without crossings [6]. It is known that nonplanar graphs will be finite [2] unless they host divergent planar subgraphs [30].

[1]Faces are the closed single lines of a graph that can be traced continuously.

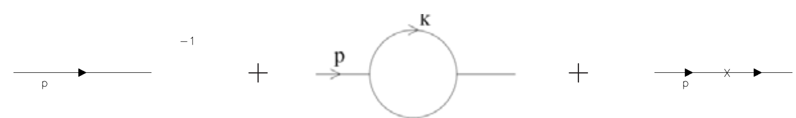
[2]For a general convergence theorem depending on power counting in terms of the topology of the 2 dimensional surface on which the genus-g graph is drawn, see [29].

Chapter 5

DIMENSIONAL REGULARIZATION AND 1-LOOP RENORMALIZATION OF NONCOMMUTATIVE ϕ^3 THEORY

As we saw in the previous chapter that the only graphs contributing to the divergences of the scattering amplitude of ϕ^3 theory in six dimensions are the two point and three point functions. Hence we begin by considering the two point function at one loop whose diagrammatic representation is given below in Figure 5.1. (for a similar investigation of noncommutative ϕ^4_4 see [31, 32])

$\Gamma_2 =$



(5.1)

Figure 5.1: $\Gamma^{(2)}$ at one loop

The first graph is the usual free propagator inverse which does not get any modifications in noncommutative theory as we already argued and hence gives the contribution p^2+m^2. The third graph is the graphical representation for the yet to be determined counterterms up to one loop appearing in (4.4). The second one is the 1 loop contribution to the self energy and is indeed the symmetric vertex representative of the planar and nonplanar graphs of the theory as shown in Figure 5.2.

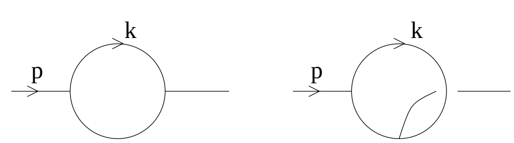
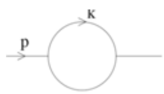


Figure 5.2: Contributions from planar and nonplanar graphs to the mass correction of noncommutative ϕ^3 theory at one loop

By considering the Feynman rules for the Euclidean action we derived earlier we can separate the contributions coming from each type of graph as below. Here we consider the integrals over a $d = 6 - \epsilon$ dimensional Euclidean space:



$$\begin{aligned} &= \frac{\lambda^2}{9}\int \frac{d^dk}{(2\pi)^d}\left[\cos\frac{p\theta k}{2} + \cos\frac{p\theta(p-k)}{2} + \cos\frac{(p-k)\theta k}{2}\right]^2 \frac{1}{(k^2+m^2)[(p-k)^2+m^2]} \\ &= \lambda^2 \int \frac{d^dk}{(2\pi)^d}\cos^2\frac{p\theta k}{2}\frac{1}{(k^2+m^2)[(p-k)^2+m^2]} \\ &= \lambda^2 \int \frac{d^dk}{(2\pi)^d}\frac{\cos p\theta k + 1}{2}\frac{1}{(k^2+m^2)[(p-k)^2+m^2]} \\ &= \frac{\lambda^2}{2(2\pi)^d}\int_0^1 dx \int d^dk(\cos p\theta k + 1)\frac{1}{[k^2+m^2+x(1-x)p^2]^2} \\ &= \frac{\lambda^2}{2(2\pi)^d}\int_0^\infty d\alpha\ \alpha \int_0^1 dx \int d^dk e^{ik\theta p - \alpha[k^2+m^2+x(1-x)p^2]} \\ &+ \frac{\lambda^2}{2(2\pi)^d}\int_0^1 dx \int d^dk \frac{1}{[k^2+m^2+x(1-x)p^2]^2} \equiv \Gamma_2^{non-planar} + \Gamma_2^{planar} \end{aligned} \tag{5.2}$$

where in the third equality above we used the trigonometric identity $\cos 2\beta = 2\cos^2\beta - 1$, in the fourth equality the Feynman parametrization

$$\frac{1}{AB} = \int_0^1 \frac{dx}{xA + (1-x)B} \tag{5.3}$$

and in the first line of the last equality of (5.2) the Schwinger parametrization

$$\frac{1}{k^2+m^2} = \int_0^\infty d\alpha\ e^{-\alpha(k^2+m^2)} \tag{5.4}$$

being differentiated with respect to the Euclidean propagator leading to:

$$\frac{\partial(k^2+m^2)^{-1}}{\partial(k^2+m^2)} = -(k^2+m^2)^{-2} = -\int_0^\infty d\alpha\ \alpha\ e^{-\alpha(k^2+m^2)} \tag{5.5}$$

We identify the first term of the result with nonplanar contribution and even at a qualitative level we can argue that it will be finite at high loop momenta due to the rapidly oscillating behavior of the exponential unless the noncommutativity parameter or the external momenta vanishes. To extract more physics out of the discussion we will consider the momentum integral for this term directly in d=6 dimensions and first by using Gaussian integrals we reduce the non-planar(N.P.) part to

$$N.P. = \frac{\lambda^2}{2^7\pi^3}\int_0^\infty d\alpha\ \alpha \int_0^1 dx \frac{e^{-\alpha[m^2+x(1-x)p^2] - \frac{p\circ p}{4\alpha}}}{\alpha^3} \tag{5.6}$$

where we introduced the positive definite inner product $p \circ p = p_i \theta^{ik} \theta^{jk} p_j$. As we see the UV behavior of the momentum integrals manifests itself as a small α divergence which we will regularize by $e^{-1/4\alpha\Lambda^2}$. Then the integral over the α's give modified Bessel functions [33]:

$$\begin{aligned} N.P. &= \frac{\lambda^2}{2^7\pi^3}\int_0^1 dx \int_0^\infty d\alpha \frac{e^{-\alpha[m^2+x(1-x)p^2]-\frac{p\circ p}{4\alpha}-\frac{1}{4\alpha\Lambda^2}}}{\alpha^2} \\ &= \frac{\lambda^2\Lambda_{eff}}{2^8\pi^3}\int_0^1 dx \sqrt{m^2+x(1-x)p^2}\; K_1\Big(\sqrt{[p^2x(1-x)+m^2]\Lambda_{eff}^{-2}}\Big) \end{aligned} \tag{5.7}$$

where we introduced the effective cut-off $\Lambda_{eff}^{-2} = p \circ p + \Lambda^{-2}$. First of all, one has to observe that the above representation is valid for the non-zero arguments of the function. So, the integral remains finite in the $\Lambda \to \infty$ limit which is a quantitative translation of the argument that the wild behavior of the integrand was smoothed by the oscillatory phase. As a note, we keep in mind that both $\theta \to 0$ and $p_{nc} \to 0$ result in $\Lambda_{eff}^2 = 1/p \circ p$ blowing up. For for small arguments of the irregular modified Bessel function, the below expansion holds [34]:

$$K_1(z) \approx \frac{1}{z} + \frac{1}{2} z \ln(z) + (\frac{1}{2}\gamma - \frac{1}{4} - \frac{1}{2}\ln(2))z + \mathcal{O}(z^3) \tag{5.8}$$

Applying this expansion non-planar and the planar (which is of the same functional form with $\theta = 0$, i.e. $\Lambda_{eff} = \Lambda$) are found to be:

$$\Gamma_2^{non-planar} = -\frac{\lambda^2}{2^8\pi^3}\Lambda_{eff}^2 + \frac{\lambda^2}{2^9 3\pi^3}(p^2+6m^2)\ln(\frac{\Lambda_{eff}^2}{m^2}) + \mathcal{O}(1) \tag{5.9}$$

$$\Gamma_2^{planar} = -\frac{\lambda^2}{2^8\pi^3}\Lambda^2 + \frac{\lambda^2}{2^9 3\pi^3}(p^2+6m^2)\ln(\frac{\Lambda^2}{m^2}) + \mathcal{O}(1) \tag{5.10}$$

where we simply set $\theta = 0$ from passing to non-planar to planar expressions (although we will calculate explicitly the planar graphs below, it is useful to write them like this here for the clarity of the coming discussion). Hence the 1-loop effective action which yields the above 1PI 2-point functions is written as:

$$\mathcal{S}_2 = \int d^6p \phi_R(p)\phi_R(-p)\frac{1}{2}\left(p^2 + m_R^2 - \frac{\lambda^2\Lambda_{eff}^2}{2^8\pi^3} + \frac{\lambda^2}{2^9 3\pi^3}(p^2+6m_R^2)\ln(\frac{\Lambda_{eff}^2}{m_R^2}) + ...\right) \tag{5.11}$$

where m_R is the planar renormalized mass and ϕ_R the renormalized scalar field to be determined using dimensional regularization shortly. Now we would like to consider two different limits on this action:

1. $p \circ p << 1/\Lambda^2$ or effectively $p \to 0$ for which $\Lambda_{eff} \approx \Lambda$:

$$\mathcal{S}_2 \approx \int \frac{1}{2}(p^2 + m_R'^2)\phi_R(p)\phi_R(-p)$$

which diverges as $\Lambda \to \infty$ if m_R is cut-off independent by choice.

2. $p \circ p >> 1/\Lambda^2$ or effectively $\Lambda \to \infty$ for which $\Lambda_{eff}^2 \approx 1\; p \circ p$:

$$\mathcal{S}_2 \approx \int \frac{1}{2}\left(p^2 + m_R^2 - \frac{\lambda^2}{2^8\pi^3 p \circ p} + \frac{\lambda^2}{2^9 3\pi^3}(p^2 + 6m_R^2)\ln(\frac{1}{m_R^2\; p \circ p})\right)\phi_R(p)\phi_R(-p)$$

The divergence above being non-local does not allow for a definition of standard mass counter term and hence spoils the otherwise renormalizability of the theory. From the discussion above it is clearly seen that the two limits considered are not interchangable. This is the so-called UV/IR mixing first realized by Seiberg et.al. in [35]. More explicitly, the high frequency integration region gives rise to the zero momentum pole in the above scalar propagator [5]. This kind of high energy internal dynamics effecting the low momenta particle dynamics has no counterpart in the commutative case. After all, facing with IR divergences in a massive theory is at best surprising to the conventional mind. A very important remark is in order here. The vacuum for our model, being already unstable due to the unboundedness from below of the corresponding classical theory as we discussed, possesses another unstability because the scalar field has a non-zero expectation value for small momanta as to satisfy $(p^2 + m_R^2)p \circ p < \mathcal{O}(g^2)$.[1]

However, there is a way out of this mixing phenomena as Grosse and Wulkenhaar [37, 38] showed by adding a harmonic potential term to the action which serves as a IR cut-off and hence defining a new 'vulcanized' action:

$$\mathcal{S}[\phi] = \int d^6x \left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 + \Omega^2\; \tilde{x}_\mu\phi\; \tilde{x}^\mu\phi - \frac{\lambda}{3!}\phi\star\phi\star\phi\right] \tag{5.12}$$

where $\tilde{x} = 2\theta^{-1}x$. Although they worked on ϕ^4 specifically, the same procedure can be applied to our model and as a result, since it decouples the different scales of the theory, the perturbative renormalizability of the scalar theory could be achieved as will be shown considering renormalization group method. Furthermore, this term restores the symmetry what

[1]This may sound trivial, but as stressed in [35] ϕ_4^4 theory suitably coupled to fermions through Yukawa interaction exhibits a similar instability whose low energy character requires detailed examination

is called the Langmann-Szabo (L-S) duality. In [39] it was realized that the action written in direct space and the momentum phase space respected the following transformation which transformed one into the other:

$$\tilde{\phi}(p) \leftrightarrow \pi^2 \sqrt{|\det \theta|}\phi(x); \qquad p^\mu \leftrightarrow \tilde{x}^\mu \tag{5.13}$$

Then the vulcanization implies the following mapping for the action itself [40]:

$$\mathcal{S}[\phi; m, \lambda, \Omega] \leftrightarrow \mathcal{S}\left[\phi; \frac{m}{\Omega}, \frac{\lambda}{\Omega^2}, \frac{1}{\Omega}\right] \tag{5.14}$$

which for $\Omega = 1$ gets mapped onto itself; in some sense the theory for this value is called self-dual. Having taken care of IR divergences, we will be mainly interested with the UV behavior of the theory. The divergences of the second term in (5.2) are identified using dimensional regularization as in the conventional case [41, 42]. Since the momentum integral has explicit rotational symmetry, the angular integrals are easily evaluated yielding the $d-1$ dimensional sphere surface area S^{d-1} embedded in the d dimensional Euclidean space-time:

$$I_\epsilon \equiv \frac{S^{d-1}}{(2\pi)^d} \int_0^1 dx \int_0^\infty \frac{k^{d-1} dk}{(k^2+\alpha)^2} \tag{5.15}$$

where we wrote for

$$\begin{aligned} \alpha &:= x(1-x)p^2 + m^2 \\ S^{d-1} &= \frac{(2\pi)^{d/2}}{\Gamma(d/2)} \end{aligned} \tag{5.16}$$

To evaluate the integral in (5.15) we now make a change of variables as $u = k^2$ so that its differential becomes $dk = du/2u^{1/2}$ and use the identity below:

$$\int_0^\infty \frac{u^w du}{(u+a)^v} = a^{w+1-v} \frac{\Gamma(w+1)\Gamma(v-w-1)}{\Gamma(v)} \tag{5.17}$$

Hence the radial integration in (5.15) becomes

$$\int_0^\infty \frac{k^{d-1} dk}{(k^2+\alpha)^2} = \frac{1}{2}\int_0^\infty \frac{u^{\frac{d}{2}-1} du}{(u+\alpha)^2} = \frac{\alpha^{\frac{d}{2}-2}}{2} \frac{\Gamma(\frac{d}{2})\Gamma(2-\frac{d}{2})}{\Gamma(2)} \tag{5.18}$$

As a result, upon using (5.16), (5.15) becomes

$$I_\epsilon = \frac{S^{d-1}}{2(2\pi)^d}\Gamma(\frac{d}{2})\Gamma(\frac{4-d}{2}) \int_0^1 dx \, [x(1-x)p^2 + m^2]^{\frac{d-4}{2}}$$

$$= \frac{(2\pi)^{d/2}}{2}\Gamma(\frac{4-d}{2})\int_0^1 dx\ [x(1-x)p^2+m^2]^{\frac{d-4}{2}} \tag{5.19}$$

Using $d = 6-\epsilon$, we can expand the d dependent terms at each step of our calculation entering the above expression up to order such that at the end it will not give $\mathcal{O}(\epsilon)$ terms or higher since we will take the $\epsilon \to 0$ limit. Using the Gamma function property

$$\left(\frac{\epsilon}{2}-1\right)\Gamma(\frac{\epsilon}{2}-1) = \Gamma(\frac{\epsilon}{2}) = \frac{2}{\epsilon}+\gamma+\mathcal{O}(\epsilon)$$

$$\Rightarrow \quad \Gamma(\frac{\epsilon}{2}-1) = -\left(\frac{2}{\epsilon}+\gamma+\mathcal{O}(\epsilon)\right)\left(\frac{\epsilon}{2}+1+\mathcal{O}(\epsilon^2)\right) = \left(\frac{2}{\epsilon}+\gamma+1\right)+\mathcal{O}(\epsilon^2) \tag{5.20}$$

and expanding the other terms as below

$$(2\pi)^{-d/2} = (2\pi)^{-3}\left(1+\frac{\epsilon}{2}\ln 2\pi\right) \tag{5.21}$$

$$\mu^{2\epsilon} = 1+\epsilon\ln\mu^2 \tag{5.22}$$

$$\begin{aligned}
(x(1-x)p^2+m^2)^{1-\frac{\epsilon}{2}} &= (x(1-x)p^2+m^2)(1-\frac{\epsilon}{2}\ln(x(1-x)p^2+m^2)) \\
&= m^2(1+x(1-x)\frac{p^2}{m^2})(1-\frac{\epsilon}{2}\ln m^2-\frac{\epsilon}{2}\ln(1+x(1-x)\frac{p^2}{m^2})) \\
&= m^2(1+x(1-x)\frac{p^2}{m^2}) \\
&\quad -\frac{\epsilon}{2}m^2(1+x(1-x)\frac{p^2}{m^2})\ln m^2 \\
&\quad -\frac{\epsilon}{2}m^2(1+x(1-x)\frac{p^2}{m^2})\ln(1+x(1-x)\frac{p^2}{m^2}))
\end{aligned} \tag{5.23}$$

The first term above appearing in (5.19) under integral can be easily evaluated to give

$$\int_0^1 dx(1+x(1-x)\frac{p^2}{m^2}) = 1+\frac{p^2}{m^2}\left(\frac{1}{2}-\frac{1}{3}\right) = 1+\frac{p^2}{6m^2} \tag{5.24}$$

Thus one loop correction to the propagator becomes

$$\begin{aligned}
\mu^{2\epsilon}g^2 I_\epsilon &= \frac{g^2}{2}(2\pi)^{-3}\times(1+\epsilon\ln\mu^2)\times(1+\epsilon\ln 2\pi+\mathcal{O}(\epsilon^2))\times\left(-\frac{2}{\epsilon}+\gamma-1+\mathcal{O}(\epsilon)\right) \\
&\quad \times\left[m^2\left(1+\frac{p^2}{6m^2}\right)-\frac{\epsilon}{2}\left(1+\frac{p^2}{6m^2}\right)m^2\ln m^2-\frac{\epsilon}{2}m^2\int_0^1 dx\ F(x,p^2/m^2)\right] \\
&= \frac{g^2}{16\pi^3}(1+\epsilon\ln\mu^2)(1+\epsilon\ln 2\pi)\left(-\frac{2}{\epsilon}m^2(1+\frac{p^2}{6m^2})+(\gamma-1)m^2(1+\frac{p^2}{6m^2})\right. \\
&\qquad \left.+m^2\ln m^2(1+\frac{p^2}{6m^2})+m^2\int_0^1 dx\ F(x,p^2/m^2)\right)
\end{aligned}$$

where F given by $(+x(1-x)p^2+m^2)\ln(+x(1-x)p^2+m^2)$ yields a finite value being integrated over $[0,1]$. Hence the two point function becomes

$$\begin{aligned}\Gamma_2 \quad &= \quad p^2+m^2+\delta m_1^2-\frac{m^2g^2}{8\pi^3}\frac{1}{\epsilon}(1+\frac{p^2}{6m^2})-\frac{m^2g^2}{8\pi^3}(1+\frac{p^2}{6m^2})\ln(\frac{2\pi\mu^2}{m^2}) \\ &+\frac{(\gamma-1)m^2g^2}{16\pi^3}(1+\frac{p^2}{6m^2})+\frac{m^2g^2}{16\pi^3}\int_0^1 dxF \end{aligned} \tag{5.25}$$

The fourth term in the above expression gives the pole term. In the minimal subtraction scheme we just take away the poles and do not touch any already finite parts. The first part of this term can be identified with a mass counterterm and the second momentum dependent one need wavefunction renormalization to be carried away. So upon making the choice for our counterterms as

$$\delta m_1^2 = \frac{m^2g^2}{8\pi^3}\frac{1}{\epsilon} \tag{5.26}$$

$$Z_3^{(1)}-1=\frac{g^2}{48\pi^3\epsilon} \tag{5.27}$$

the renormalized full propagator up to one loop becomes

$$\begin{aligned}\Gamma_2 \quad &= \quad p^2+m^2-\frac{m^2g^2}{8\pi^3}(1+\frac{p^2}{6m^2})\ln(\frac{2\pi\mu^2}{m^2}) \\ &+\frac{(\gamma-1)m^2g^2}{16\pi^3}(1+\frac{p^2}{6m^2})+\frac{m^2g^2}{16\pi^3}\int_0^1 dxF \end{aligned} \tag{5.28}$$

This completes the renormalization of the two point function at one loop. Another expression to be renormalized is the planar part of the 1-loop contribution to the 3-vertex. In Figure 5.3 the full 3-point function is given.

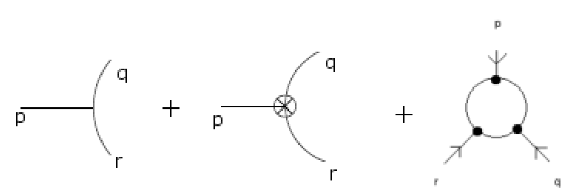
$\Gamma_3 =$  (5.29)

Figure 5.3: $\Gamma^{(3)}$ at one loop

The first term is the bare vertex contributing a factor of $-\mu^\epsilon g$. The second one is the 1-loop order vertex counter term $-\mu^\epsilon g(Z_1^{(1)} - 1)$ of the $S_R^{(c.t.)}$ of equation (4.4). The third term is the one loop correction to the interaction vertex whose symmetric factor, getting contributions from the planar and non-planar diagrams in Figure 5.4, is a little more complicated than that of 1-loop 2-point graph calculated previously so we compute it below separately.

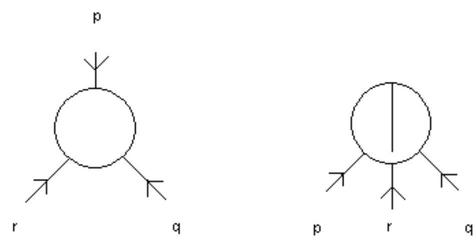


Figure 5.4: Contributions from planar and nonplanar graphs to the interaction vertex of noncommutative ϕ^3 theory at one loop

Calling the external momenta p, q, r and the independent loop momentum k, we see that due to the overall momentum conservation $q + r = -p \equiv P$ we simplify the vertex factor $V(p, q, r; k)$ as

$$
\begin{aligned}
V(p,q,r;k) &= \left(\frac{1}{3}\right)^3 \left(cos\frac{r\theta(k+q)}{2} + cos\frac{r\theta(k+P)}{2} + cos\frac{(k+P)\theta(k+q)}{2} \right) \\
&\quad \times \left(cos\frac{k\theta(k+q)}{2} + cos\frac{k\theta q}{2} + cos\frac{q\theta(k+q)}{2} \right) \\
&\quad \times \left(cos\frac{k\theta p}{2} + cos\frac{k\theta(k+P)}{2} + cos\frac{p\theta(k+P)}{2} \right) \\
&= cos\frac{r\theta(k+q)}{2} \times cos\frac{k\theta q}{2} \times cos\frac{k\theta p}{2} \\
&= \frac{1}{2} cos\frac{r\theta(k+q)}{2} \times \left(cos\frac{k\theta(q+p)}{2} + cos\frac{k\theta(q-p)}{2} \right) \\
&= \frac{1}{2} \left(cos\frac{r\theta(k+q)}{2} cos\frac{k\theta(q+p)}{2} + cos\frac{r\theta(k+q)}{2} cos\frac{k\theta(q-p)}{2} \right)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{4}\Bigg(cos\frac{r\theta k + r\theta q + k\theta(-r)}{2} + cos\frac{r\theta k + r\theta q - k\theta(-r)}{2} \\
&+ \ cos\frac{r\theta k + r\theta q + k\theta q - k\theta p}{2} + cos\frac{r\theta k + r\theta q - k\theta q + k\theta p}{2}\Bigg) \\
&= \frac{1}{4}\left(cos\left(r\theta k + \frac{r\theta q}{2}\right) + cos\left(k\theta q + \frac{r\theta q}{2}\right) + cos\left(k\theta p + \frac{r\theta q}{2}\right) + cos\left(\frac{r\theta q}{2}\right)\right)
\end{aligned}
$$

The last term being solely dependent on the external momenta can be easily symmetrized using the overall momentum conserving delta to yield

$$
\frac{1}{12}cos\left(\frac{p\theta q}{2}\right) + cos\left(\frac{r\theta p}{2}\right) + cos\left(\frac{r\theta q}{2}\right) \equiv \mathcal{P}(r,p,q) \tag{5.30}
$$

which contributes to planar graph yet to be renormalized shortly. This symmetrization procedure applies equally well to the first three terms also, but it is more convenient to write this contribution as an exponential phase as below for the calculation of the non-planar digrams

$$
\sum_j a^j e^{ic^j(p) + ib^j(p)\theta k} \tag{5.31}
$$

For now we return to the calculation of the planar diagrams which will be considered modulo the external momenta dependent phase factor:

$$
Planar \propto -\frac{\mu^{3\epsilon} g^3}{3!3!3!} 9.6.3.4.2. \frac{1}{(2\pi)^d} \int d^d k \frac{1}{k^2 + m^2} \frac{1}{(q+k)^2 + m^2} \frac{1}{(P+k)^2 + m^2} \tag{5.32}
$$

The integrand can be simplified as before by using Feynman parametrization; this time a more general identity holds:

$$
\frac{1}{ABC} = 2\int_0^1\int_0^1 \frac{dx_1\ dx_2}{[Ax_1 + Bx_2 + C(1-x_1-x_2)]^3} \tag{5.33}
$$

If we make a change of variables $l \equiv k + x_2 q + (1 - x_1 - x_2)P$ so that $d^d k = d^d l$ then the momentum integral becomes neglecting the numerical symmetry factor for now

$$
\frac{S^{d-1}}{(2\pi)^d} \int_0^1\int_0^1 dx_1 dx_2 \int_0^\infty \frac{l^{d-1} dl}{(l^2 + \beta)^3} \tag{5.34}
$$

where $\beta = x_2(1-x_2)q^2 + (1-x_1-x_2)(x_1+x_2)P^2 - 2x_2(1-x_1-x_2)Pq + m^2$. Proceeding as before and using (5.17) with $u = l^2$ the momentum integral becomes

$$\frac{1}{2}\int_0^\infty \frac{u^{\frac{d}{2}-1}du}{(u+\beta)^3} = \frac{\beta^{\frac{d-6}{2}}}{2}\frac{\Gamma(d/2)\Gamma(3-d/2)}{\Gamma(3)} \tag{5.35}$$

Hence equation (5.34) becomes

$$\frac{1}{4}\frac{S^{d-1}}{(2\pi)^d}\Gamma(\frac{d}{2})\Gamma(\frac{6-d}{2})\int_0^1\int_0^1 dx_1 dx_2 \beta^{\frac{d-6}{2}} \tag{5.36}$$

As we are in $d = 6-\epsilon$ dimensions on using (5.16) and since in MS scheme we are only interested in the exact form of only the pole we obtain

$$\begin{aligned}\Gamma_3 &= -\mu^\epsilon g - \mu^\epsilon g[Z_1 - 1 + 6\mu^{2\epsilon}g^2\left(\frac{1}{16\pi^3\epsilon} + finite\right)\mathcal{P}(r,p,q)] \\ &= -\mu^\epsilon g - \mu^\epsilon g[Z_1 - 1 + \left(\frac{3g^2}{8\pi^3\epsilon} + finite\right)\mathcal{P}(r,p,q)]\end{aligned} \tag{5.37}$$

As we choose the pole parts for counter terms in MS scheme we have

$$Z_1^{(1)} = \frac{1}{12}(1 - \frac{3g^2}{8\pi^3\epsilon})\left(cos\left(\frac{p\theta q}{2}\right) + cos\left(\frac{r\theta p}{2}\right) + cos\left(\frac{r\theta q}{2}\right)\right) \tag{5.38}$$

Chapter 6

RENORMALIZATION GROUP EQUATIONS AND ASYMPTOTIC FREEDOM OF NONCOMMUTATIVE ϕ^3 THEORY IN SIX DIMENSIONS

Our choice of the renormalization scheme as minimal subtraction involves some arbitrariness in the sense that also the finite part including the Euler constant and the logarithmic part would also be taken into the definition of counterterm. However, this seemingly drawback of renormalization program indeed turns out to be an invaluable outcome upon realizing that any two renormalized N-point functions $\tilde{\Gamma}^N = Z_3^{-N/2}\tilde{\Gamma}^N_R = Z_3'^{-N/2}\tilde{\Gamma}'^N_R$ are equivalent up to another finite renormalization $Z = Z_3'/Z_3$ so that $\tilde{\Gamma}'^N_R = Z^{N/2}\tilde{\Gamma}^N$. First to understand why the renormalization of the N-point function we have to go back to (4.4) and realize that it actually amounts to the redefinitions below

$$\lambda_0 = Z_1\, Z_3^{-2}\, \lambda \tag{6.1}$$

$$\phi_0 = Z_3^{1/2}\, \phi \tag{6.2}$$

$$m_0^2 = (m^2 - \delta m^2)/Z_3 \tag{6.3}$$

Since he N-point function involves N fields, the renormalized N-point function is related to the bare one with the $Z_3^{-N/2}$ behavior above. And as Z is just the ratio of two renormalized quantities it remains finite in the limiting behavior of the regulator. Thus, the supposedly arbitrary choice of the renormalization scheme is indeed intimately restricted by the collection of all such finite transformations which takes one configuration in the parameter space of the theory to another one. They constitute a semigroup (because it lacks a unique inverse) called renormalization group. μ being an intermediate parameter used as a tool for computation, no physical entity should depend on it at the end. Also from the above additive renormalization definition we can turn to equivalent multiplicative renormalization by defining

$$\lambda_0 = Z_g \mu^\epsilon \, g, \qquad Z_g := Z_1 \, Z_3^{-2} \tag{6.4}$$

$$m_0^2 = Z_m^{1/2} \, m \tag{6.5}$$

Naturally the bare parameters are not supposed to be dependent on this mass parameter. Hence from this consideration we try to deduce the dependences of the renormalized ones by differentiating the bare parameters at their constant values as follows:

$$\mu \frac{d\lambda_0}{d\mu} = 0 = \epsilon \, g + \mu \frac{d \ln Z_g}{d\mu} \, g + \mu \frac{dg}{d\mu} \tag{6.6}$$

$$\mu \frac{dm_0}{d\mu} = 0 = \frac{\mu}{m} \frac{dm}{d\mu} + \mu \frac{d \ln Z_m^{1/2}}{d\mu} \tag{6.7}$$

The second of which can be solved in terms of the dimensionless function γ_m

$$\mu \frac{dm}{d\mu} = m\gamma_m \quad ; \qquad \gamma_m = \mu \frac{d \ln Z_m^{-1/2}}{d\mu} \tag{6.8}$$

and in (6.6) it is customary to define the beta function of the theory as

$$\beta\left(g, \frac{m}{\mu}\right) = \mu \, \frac{dg}{d\mu} \tag{6.9}$$

which along with γ_m are well behaved in the limiting behavior. Furthermore, being dimensionless they are expected to depend on the dimensionless parameters g and m/μ, but as calculated above in the minimal subtraction since the wavefunction, mass and coupling constant renormalizations depend only on g, we will have $\beta = \beta(g)$ and $\gamma_m(g)$ which implies

$$\mu \frac{d}{d\mu} \ln Z_g = \beta(g) \frac{d}{dg} \ln Z_g \tag{6.10}$$

hence 6.6 will read

$$\mu \frac{dg}{d\mu} = -\epsilon g \left(1 + g \frac{d}{dg} \ln Z_g\right)^{-1} \tag{6.11}$$

Now we defined Z_g up in 6.4 which we need up to one loop and we have $Z_3^{(1)}$ and $Z_1^{(1)}$ from (5.27) and (5.38) respectively. Thus up to that order Z_g is given by

$$\begin{aligned} Z_g = (Z_1)(Z_3)^{-2} &= (1 - \frac{3g^2}{8\pi^3\epsilon})(1 + \frac{g^2}{48\pi^3\epsilon})^{-2}\mathcal{P}(r,p,q) \\ &= (1 - \frac{3g^2}{8\pi^3\epsilon})(1 - \frac{2g^2}{48\pi^3\epsilon})\mathcal{P}(r,p,q) = (1 - \frac{5g^2}{12\pi^3})\mathcal{P}(r,p,q) \end{aligned} \tag{6.12}$$

upon the use of which (6.11) in the $\epsilon \to 0$ limit has the form

$$\mu \frac{dg}{d\mu} = -\frac{5g^3}{6\pi^3} \tag{6.13}$$

We can make use of this equation by considering the proper vertices renormalization mentioned at the beginning of this chapter

$$\tilde{\Gamma}^N(p_i; \lambda_0, m_0, \epsilon) = Z_3^{-N/2} \tilde{\Gamma}^N_R(p_i; g, m, \mu) \tag{6.14}$$

As above for the bare coupling constant and the bare mass, the bare N-point function does not depend on the intermediate mass scale hence

$$\left(\mu \frac{\partial}{\partial \mu} + \beta \frac{\partial}{\partial g} + m\gamma_m \frac{\partial}{\partial m} - \frac{N}{2}\gamma\right) \tilde{\Gamma}^N_R(p_i; g, m, \mu) = 0 \tag{6.15}$$

where we defined the new function

$$\gamma\left(g, \frac{m}{\mu}, \epsilon\right) = \mu \frac{d}{d\mu} \ln Z_3 \tag{6.16}$$

However, this form is not amenable to an easy solution and the newly defined function can be made absent from 6.15 by a redefinition as below:

$$F^{(N)}(p_i; g, m, \mu) = e^{-\frac{N}{2}\int_0^g dx \frac{\gamma(x)}{\beta(x)}} \tilde{\Gamma}^N_R(p_i; g, m, \mu) \tag{6.17}$$

written in terms of which 6.15 becomes

$$\left(\mu \frac{\partial}{\partial \mu} + \beta \frac{\partial}{\partial g} + m\gamma_m \frac{\partial}{\partial m}\right) F^{(N)}(p_i; g, m, \mu) = 0 \tag{6.18}$$

Then if we can solve

$$s \frac{d}{ds} F^{(N)}(p_i; \bar{g}(s), \bar{m}(s), \bar{\mu}(s)) = 0 \tag{6.19}$$

where the initial conditions with the dimensionless parameter s are given for the equation above:

$$\bar{\mu}(s) = s\mu \tag{6.20}$$

$$\bar{g}(1, g) = g \tag{6.21}$$

$$\bar{m}(1, m) = m \tag{6.22}$$

Applying chain rule with respect to these variables now in 6.19 and together with 6.18 we obtain

$$\frac{d\bar{g}}{d\ln s} = \beta(\bar{g}(s)) \tag{6.23}$$

$$\frac{d\bar{m}}{\bar{m}d\ln s} = \gamma_m(\bar{g}(s)) \tag{6.24}$$

As a result for F^N to be s-independent, we should solve the above subject to the conditions given before so that the basic equation we need in terms of N-point functions becomes:

$$\tilde{\Gamma}_R^{(N)}(p_i; g, m, \mu) = e^{-\frac{N}{2}\int_g^{\bar{g}(s)} dx \frac{\gamma(x)}{\beta(x)}}\ \tilde{\Gamma}_R^{(N)}(p_i; \bar{g}(s), \bar{m}(s), s\mu) \tag{6.25}$$

As we know a scaling argument for external momenta like $p_i \to sp_i$ through dimensional analysis transforms the arbitrariness on the arbitrary mass scale μ to the behavior of the N-point function under different momenta:

$$\tilde{\Gamma}_R^{(N)}(e^t p_i; g, m, \mu) \quad = \quad e^{(d-Nd_0)t - \frac{N}{2}\int_g^{\bar{g}(t)} dx \frac{\gamma(x)}{\beta(x)}}\ \tilde{\Gamma}_R^{(N)}(p_i; \bar{g}(t), e^{-t}\bar{m}(t), \mu) \tag{6.26}$$

$$\frac{d\bar{g}(t)}{dt} \quad = \quad \beta(\bar{g}(t))\ , \qquad \bar{g}(0) = g \tag{6.27}$$

$$\frac{d\bar{m}(t)}{\bar{m}(t)dt} \quad = \quad \gamma_m(\bar{g}(t))\ , \quad \bar{m}(0) = m \tag{6.28}$$

where $s = e^t$. Then the asymptotic freedom of the theory means, since as we found in (6.13) that this function starts as negative and has to pass through zero, upon the consideration of (6.27) that as $t \to \infty$, $g \to 0$ should hold for the $\beta(\bar{g}(t))$ to remain negative.

Chapter 7

CONCLUSIONS: VALIDITY OF 1-LOOP β FUNCTION

Time permitting, the discussion in this paper considered the perturbative expansion of the six-dimensional noncommutative ϕ^3 theory up to one loop order. The approach taken mostly serves to question the internal consistency of field theoretical concepts without giving any reference for theoretical predictions or experimental limits on the parameters of the theory. After all the model we examined does not represent reality in its full truth and the interested reader can check the articles given in the bibliography and the references therein for phenomenological considerations coming from more realistic models and concentrating on the Lorentz violation bounds [43, 44, 45, 46]. We began our discussion by considering a particular type of noncommutative space and then on translated the inherent properties of the space into the algebra of our fields satisfying a rather modified product called Moyal product. Having constructed the classical theory and found the differences in the consequances of the classical symmetries to Noether currents by the noncommutativity we went on to quantize the classical theory. The quantization for purely space-space noncommutativities left us with a new kind of graph called non-planar graphs which upon the use of Filk moves supplied some phase factors which helped to regularize some of the graphs. We found out that the planar contributions obey exactly the same divergent behaviors as the commutative case and hence the renormalization program. The standard analysis of the RGE gave us the asymptotically free behavior of the theory embodied in the 1 loop beta function. The two loop calculations would be similarly carried out; just with a lot more complicated internal momenta dependent vertex factors for the non-planar graphs. Below, in Figure 7.1, we just give the two loop contribution to the two point function of the theory. But still regardless of the form of such a trigonometric polynomial, these factors would regularize the divergence of the momentum integrals. And yet the planar part would still admit the usual renormalization program at two loops. Even at first order we derived the qualitative conclusion that our model is asymptotically free. However, there is big catch regarding the quantitative

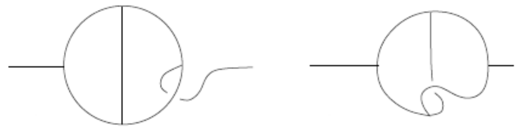


Figure 7.1: Possible non-planar contributions to the 2-point function of noncommutative ϕ^3 theory

result above due to H. Grosse and H. Steinacker which would force the mathematically rigorous physicist to reconsider the degree of carelessness in the applicability of the naive perturbative approaches. We would like to conclude by summarizing their work below and extracting the result in which we are most interested. They used the similarity between the matrix models and NCQFT through the already mentioned ribbon graphs leading to genus expansions(another application appears in [47]). In particular through a series of papers [48, 49, 50] they considered the mapping from a noncommutative euclidean self-dual ($\Omega = 1$) ϕ^3 theory to a Kontsevich model [51] in 2, 4, and finally 6 space-time dimensions which can always be done for even space-time dimensions for some of its eigenvalues generated by an external source. In particular, making use of inner derivations, the derivations satisfying (1.6),(1.7) and further can be written as $\partial_i\phi = -i[\tilde{x}_i, \phi]$, they could write the vulcanized action for the special self-dual point $\Omega = 1$ and with an imaginary coupling constant $i\tilde{\lambda}$, which admits a sensible analytic continuation to real $\tilde{\lambda}'$, as follows:

$$\tilde{\mathcal{S}} = \int \tilde{x}_i\tilde{x}_i\phi\phi - i\tilde{\lambda}\tilde{a}\phi + \frac{\mu^2\phi^2}{2} + \frac{i\tilde{\lambda}\phi^3}{3!} \tag{7.1}$$

where a linear ϕ term was included to allow quantization with real $\tilde{a}$. Then replacing the direct space representation of trace by the operator Tr itself and making some intermediate change of variables, the above scalar model can be written in terms of a matrix model coupled to an external source:

$$\mathcal{S} = Tr\left(\frac{1}{2}MX^2 + \frac{i\lambda}{3!}X^3 - \frac{1}{3(i\lambda)^2}M^3\right) \tag{7.2}$$

with X, M, a, λ defined as

$$X = \phi + \frac{J - M}{i\lambda} \tag{7.3}$$

$$M = \sqrt{J^2 + 2a} \tag{7.4}$$

$$a = -(i\lambda)^2 \tilde{a} \tag{7.5}$$

$$\lambda = 2\pi\theta\tilde{\lambda} \tag{7.6}$$

This has the form of a Kontsevich model [52] with the source given by the operator

$$J = 4\pi(\theta \sum_i \tilde{x}_i \tilde{x}_i + \frac{\mu^2\theta}{2}) \tag{7.7}$$

whose eigenvalues

$$J|n\rangle = 4\pi(n + \frac{1+\mu^2\theta}{2})|n\rangle \tag{7.8}$$

suggest that it is actually the quantum mechanical harmonic oscillator Hamiltonian. As usual the quantization of the model is given by the partition function

$$Z[J] = \int \mathcal{D}\phi\ e^{-S} \tag{7.9}$$

with the classical action given above and where the path integral measure is defined over all now $N \times N$ Hermitian matrices ϕ. The rest of the paper involves rather complicated calculations and we will suffice to outline the procedure. First n-point correlation functions are defined from the above generating functional. Then making use of the already established results of Kontsevich model renormalization of those functions in terms of a genus expansion for all genera is given. The functions are finite except the genus-0 contribution which yields well-defined contributions for finite nonzero coupling. While in 2 and four dimensions where the theory is superrenormalizable only mass renormalization is required, wavefunction and coupling constant renormalizations need to be performed in 6 dimensions for which the corresponding commutative counterpart is known to be renormalizable and asymptotically free. The result from their analysis of the exact renormalization group equations that the flow governing the running of the coupling constant behaves differently from the one expected from our perturbative one loop beta function although both predict asymptotic freedom. Specifically, their coupling constant dies more sharply with the scale dependence $(\ln N)^{-2}$ in comparison to the perturbative $(\ln N)^{-1}$ behavior for large N. Translated into the terms we calculated our running above, their $(\ln m)^{-2}$ is in remarkably severe disagreement[1] with the $(\ln m)^{-1}$ of first order beta. This result suggests that noncommutative field theories, being

[1]For a comparison of the coupling constant behavior at one loop in the ϕ_4^4 context, see [53]

more friendly to analytical explanations in terms of admitting exact solutions as above, may be exploited as to put some restrictions on our notions of standard perturbation theory and renormalization.

BIBLIOGRAPHY

[1] R.P. Feynman, *The Space-Time Formulation of Nonrelativistic Quantum Mechanics*, Rev. of Mod. Phys. **20** (1948) 367-387.

[2] G. 'tHooft, M. Veltman, Nucl. Phys. **B50** (1972) 318.

[3] T. Muta, *Foundations of Quantum Chromodynamics*, World Scientific, 1987.

[4] H.S. Snyder, *Quantized Space-Time*, Phys. Rev. **71** (1947) 38.

[5] R.J. Szabo, *Quantum Field Theory on Noncommutative Spaces*, hep-th/0109162v4.

[6] M.R. Douglas, A.N. Nekrasov, *Noncommutative Field Theory*, hep-th/0106048v4.

[7] N. Seiberg, L. Susskind, N. Toumbas, *Space/Time Noncommutativity and Causality*, JHEP **06** (2000) 044. hep-th/0005015.

[8] J. Gomis, T. Mehen, *Space-Time Noncommutative Field Theories and Unitarity*, hep-th/0005129v2.

[9] S. Doplicher, K. Fredenhagen, J.E. Roberts, *The Quantum Structure on Spacetime at the Planck Scale and Quantum Fields*, Comm. Math. Phys. **172**, (1995) 187-220. hep-th/0303037v1.

[10] D. Bahns et.al., *On the Unitarity Problem in Space/Time Noncommutative Theories*, hep-th/0201222v2.

[11] C.N. Yang, D. Feldman, *The S Matrix in the Heisenberg Representation*, Phys. Rev. **79** (1950) 72.

[12] Y. Liao, K. Sibold, *Time-ordered Perturbation Theory on Noncommutative Spacetime: Unitarity*, hep-th/0206011v2.

[13] Y. Liao, K. Sibold, *Time-ordered Perturbation Theory on Noncommutative Spacetime: Basic Rules*, hep-th/0205269v2.

[14] H. Weyl, *The Theory of Groups and Quantum Mechanics*, Dover Publications, NY, 1931.

[15] A. Gerhold et. al., *The Energy Momentum Tensor on Noncommutative Spaces*, hep-th/0012112v1

[16] M. Abou-Zeid, H. Dorn, *Comments on the Energy-Momentum Tensor in Non-commutative Field Theories*, hep-th/0104244v3

[17] S. Coleman, R. Jackiw, *Why Dilatation Generators Do Not Generate Dilatations?*, Ann. Phys. **67** (1971) 552.

[18] L.H. Ryder, *Quantum Field Theory*, Cambridge University Press, 2nd ed., 1996.

[19] M.E. Peskin, D.V. Schroeder, *An Introduction to Quantum Field Theory*, Addison-Wesley, 1996.

[20] Y. Abe, *Noncommutative Quantization for Noncommutative Field Theory*, hep-th/0606183v3.

[21] M. Srednicki, *Quantum Field Theory*, Cambridge University Press, 2007.

[22] A. Das, *Field Theory-A Path Integral Approach*, World Scientific, 2006.

[23] G. Thompson, *ICTP Lecture notes, Reprint of HEP-QFT 2003-04 (3)*, 1998.

[24] D. Bahns, *On the Inequivalence of Euclidean and Minkowskian Methods on Moyal Plane*, http://www.mat.uniroma2.it/08QSTNG/Bahns/.

[25] S. Weinberg, *High-Energy Behavior in Quantum Field Theory*, Phys. Rev. **118**, (1960).

[26] W. Zimmermann, *The Power Counting Theorem for Minkowski Metric*, Comm. Math. Phys. **11** (1968).

[27] G. 'tHooft, *A Planar Diagram Theory for Strong Interactions*, Nucl. Phys. **B72**, (1974) 461.

[28] T. Filk, *Divergences in a Field Theory on Quantum Space*, Phys.Lett. **B376** (1996) 53.

[29] I. Chepelev, R. Roiban, *Renormalization of Quantum Field Theories on Noncommutative R^d*, JHEP **05** (2000) 37.

[30] D. Bigatti, L. Susskind, *Magnetic Fields, Branes and Noncommutative Geometry*, hep-th/9908056v4

[31] I.Ya. Aref'eva, D.M. Belov, A.S. Koshelev, *Two-Loop Diagrams in Noncommutative ϕ^4_4 Theory*, Phys. Lett. **B476** (2000) 431-436

[32] A. Micu, M.M. Sheikh-Jabbari, *Noncommutative ϕ^4 Theory at Two Loops*, JHEP **01** (2001) 025.

[33] I.S. Gradshteyn, I.M. Ryzhik, *Table of Integrals, Series and Products*, Academic Press, NY, 2000.

[34] I.Ya. Aref'eva et. al., *Noncommutative Field Theories*, hep-th/0111208.

[35] S. Minwalla, M.V. Raamsdonk, N. Seiberg, *Noncommutative Perturbative Dynamics*, hep-th/9912072v2.

[36] M.V. Raamsdonk, N. Seiberg, *Comments on Noncommutative Perturbative Dynamics*, hep-th/0002186v2.

[37] H. Grosse, R. Wulkenhaar, *Renormalisation of ϕ^4 theory on noncommutative R^2 in the matrix base*, hep-th/0307017.

[38] H. Grosse, R. Wulkenhaar, *Renormalisation of ϕ^4 theory on noncommutative R^4 in the matrix base*, hep-th/0401128.

[39] E. Langmann, R.J. Szabo, *Duality in Scalar Field Theory on Noncommutative Phase Spaces*, Phys. Lett. **B533** (2002) 168. hep-th/0202039.

[40] V. Rivasseau, *Non-commutative Renormalization*, arxiv:0705.0705v1 [hep-th].

[41] G. Sterman, *Introduction to Quantum Field Theory*, Cambridge University Press, 1993.

[42] F. Gross, *Relativistic Quantum Mechanics and Field Theory*, Wiley-VCH, 1999.

[43] G. Amelino-Camelia et.al., *On the IR/UV Mixing and Experimental Limits on the Paramaters of Canonical Noncommutative Spacetimes*, hep-th/0209254.

[44] C.E. Carlson, C.D. Carone, *Supersymmetric Noncummutative QED and Lorentz Violation*, hep-th/0209077v2.

[45] C.E. Carlson, C.D. Carone, R.F. Lebed, *Bounding Noncommutative QCD*, hep-th/0107291v2.

[46] S.M. Carroll et.al., *Noncommutative Field Theory and Lorentz Violation*, hep-th/0105082v1.

[47] E. Langmann, R.J. Szabo, K. Zarembo, *Exact Solution of Quantum Field Theory on Noncommutative Phase Spaces*, hep-th/0308043v1.

[48] H. Grosse, H. Steinacker, *Renormalization of the Noncommutative ϕ^3 Model through the Kontsevich Model*, hep-th/0512203v1.

[49] H. Grosse, H. Steinacker, *A Nontrivial Solvable Noncommutative ϕ^3 Model in 4 Dimensions*, hep-th/0603052v2.

[50] H. Grosse, H. Steinacker, *Exact Renormalization of the Noncommutative ϕ^3 Model in 6 Dimensions*, hep-th/0607235v2.

[51] M. Kontsevich, *Intersection theory on the moduli space of curves and the matrix Airy function*, Comm. Math. Phys. **147** (1992) 1.

[52] C. Itzykson, J.-B. Zuber, *Combinatorics of the Modular Group II-The Kontsevich Integrals*, hep-th/9201001v1.

[53] H. Grosse, R. Wulkenhaar, *The Beta function in the Duality Covariant Noncommutative ϕ^4 Theory*, hep-th/0402093.