RELIABILITY AND MAINTENANCE OF SEMI-MARKOV MISSIONS

by

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A Thesis Submitted to the
Graduate School of Engineering
in Partial Fulfillment of the Requirements for
the Degree of

Doctor of Philosophy

in

Industrial Engineering and Operations Management

Koç University

August, 2009

Koç University

Graduate School of Sciences and Engineering

This is to certify that I have examined this copy of a doctoral dissertation by

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To my wife and son

ABSTRACT

In this thesis, we analyze the reliability and maintenance of a mission-based system where the mission process is the minimal semi-Markov process corresponding to a Markov renewal process. The system is a complex one with multiple components that have arbitrary lifetime distributions. The mission has different stages or phases with random sequence and durations. We assume that the failure parameters of the components and the configuration of the system change according to the phases of the mission. In other words, an external mission process modulates the deterioration or age process of the system. We analyze several performance measures under two repair policies: namely, maximal repair and no repair. We also discuss optimal maintenance policies minimizing the expected total discounted cost. We consider simpler models with Markovian mission and deterioration to obtain more explicit and computationally tractable results.

In the first two chapters, we give a brief review of the related literature, and introduce the notation, terminology, and symbols used through the thesis. The structure of the mission and age processes are discussed in detail as well as the failure probabilities of the components.

In Chapter 3, we present the reliability analysis of mission-based systems under both maximal and no repair policies. Three different reliability measures are characterized: namely, the probability of survival (system reliability), the probability of completing a number of phases (mission reliability), and the probability of completing a critical phase (phase reliability). We also give more explicit formulations and structural characterizations by making additional assumptions on the system structure, mission structure, and component lifetimes.

In Chapter 4, the mean time to failure of mission-based systems is analyzed under both maximal and no repair policies. In both cases, we first define a Poisson equation whose solution characterizes the mean time to failure and solve it under some reasonable assumptions which guarantee the existence and uniqueness of the solution. Our main assumption simply

states that the system may fail after a finite number of successfully completed phases with a positive probability whatever the initial phase is. We also analyze some interesting special cases by making additional assumptions on the mission structure.

Chapter 5 focuses on the availability of mission-based systems under both maximal and no repair policies. The availability is characterized by applying some limiting results for the solutions of Markov renewal equations. It is shown that irreducibility of the mission process is sufficient for the existence of the limit under some reasonable assumptions. We also provide a system of linear equations whose solution allows us to write the availability under the maximal repair policy in terms of the limiting probabilities of the mission process.

In Chapter 6, we analyze the reliability, mean time to failure and availability of systems working under a fixed phase with exponentially distributed component lifetimes. We show that the reliability and mean time to failure of coherent systems can be represented as a difference of two convex functions and we also obtain their explicit representations. It is also shown that the availability of coherent systems, and the mean time to failure and availability of series connection of standby redundant subsystems can be represented as a ratio of two polynomials with positive coefficients.

In Chapter 7, we consider the optimal maintenance problem of mission-based systems with multiple components whose lifetimes are generally distributed. The optimal replacement problem is first investigated and several monotonicity properties of the optimal policy minimizing the expected total discounted cost are proved under some increasing failure rate assumptions. We also study the optimal repair problem under similar assumptions by considering several cost structures and obtain interesting properties of the optimal policy.

In the last chapter, all of the previous analysis is repeated for the case where the mission and deterioration processes of the system have Markovian structures. The main incentive behind this simplification is to obtain more computationally tractable results. We obtain matrix exponential formulations for system reliability and phase reliability. We give explicit formulas for mission reliability, mean time to failure, and availability. We provide many numerical illustrations showing the applicability of our results. The optimal maintenance problem of such a system is also discussed. We prove that the optimal replacement policy has a control-limit structure under some monotonicity assumptions and provide many useful properties of the optimal repair policy under different cost structures. Detailed numerical examples are also given showing that our assumptions are really needed for the validity of our results on the structure of the optimal policies.

ÖZETÇE

Bu tezde görev-tabanlı sistemlerin güvenirliği ve bakımı ile ilgili problemler analiz edilmiştir. Ele alınan görev-tabanlı sistemin yerine getirdiği görev, bir Markov yenileme sürecinin en küçük yarı-Markov süreci ile temsil edilmektedir. Dolayısıyla, görevin aşamalarının sırası rassaldır ve aşamaların süreleri genel dağılıma sahip rassal değişkenlerdir. Sistem ise yaşam süreleri genel dağılımlara sahip birden fazla bileşenden oluşan kompleks bir yapıdır. Tez boyunca sistemin yapısının ve sistem bileşenlerine ait hasar parametrelerinin görevin aşamalarına göre değiştiği varsayılmıştır. Diğer bir ifade ile, sistemin bozulma veya yaşlanma sürecinin parametreleri, harici bir rassal sürece göre zaman içinde değişmektedir. Bu şekilde tanımlanmış bir sisteme ait çeşitli performans ölçütleri, maksimum bakım ve sıfır bakım politikaları altında tanımlanmış ve analiz edilmiştir. Ayrıca, toplam maliyetin şimdiki beklenen değerini enküçükleyen bakım politikalarının yapısı ortaya çıkarılmıştır. Son olarak, görev sürecinin ve görevin her aşamasındaki sistemin bozulma sürecinin birer Markov süreci olduğu varsayılarak daha önce yapılan tüm analizler tekrar edilmiştir. Bu son bölümdeki amaç, nümerik olarak daha kolay hesaplanabilen ifadeler elde etmektir.

Tezin ilk iki bölümünde öncelikle daha önce yapılmış ilgili akademik çalışmalar özetlenmiş ve tez boyunca kullanılan terimler, semboller ve notasyon açıklanmıştır. Görev ve yaşlanma süreçlerinin yapıları ve ayrıca bileşenlerin bozulma olasılıkları detaylı bir şekilde tarif edilmiştir.

Bölüm 3'de görev-tabanlı sistemlerin güvenirliği, maksimum ve sıfır bakım politikaları altında ele alınmıştır. Analiz edilen üç farklı güvenirlik ölçütü şunlardır: sistemin belli bir süre bozulmadan çalışma olasılığı (sistem güvenirliği), görevin belirli sayıda aşamasının başarıyla tamamlanma olasılığı (görev güvenirliği) ve kritik bir aşamanın belli bir süre içinde tamamlanma olasılığı (kritik aşama güvenirliği). Ayrıca, sistem yapısı, görev yapısı ve bileşen yaşam süreleri hakkında çeşitli varsayımlar yaparak daha açık formüller elde edilmiş ve bu formüllerin temel yapısal özellikleri ortaya çıkarılmıştır.

Bölüm 4'de görev-tabanlı sistemlerin beklenen yaşam süresi, maksimum ve sıfır bakım

politikaları altında analiz edilmiştir. Her iki politika için öncelikle, çözümü, beklenen yaşam süresini belirleyen bir Poisson denklemi tanımlanmıştır. Daha sonra bu denklem, çözümün varlığını ve tekliğini garanti eden varsayımlar altında çözülmüştür. Bu analizdeki en temel varsayım, sistemin herhangi bir aşamadan başlayarak sonlu sayıda, başarıyla tamamlanmış aşamadan sonra bozulma olasığının sıfırdan büyük olmasıdır. Bu bölümde de görev süreci ile ilgili çeşitli varsayımlar yapılarak oluşturulan çeşitli özel durumlar incelenmiştir.

Bölüm 5'de görev-tabanlı sistemlerin maksimum ve sıfır bakım politikaları altındaki kullanılabilirliğine odaklanılmıştır. Kullanılabilirlik, bir Markov yenileme denkleminin çözümünün limiti için geliştirilmiş neticelerin yardımıyla karakterize edilmiştir. Görev sürecinin indirgenemezliğinin, bazı makul varsayımlar altında, limitin varlığı için yeterli olduğu gösterilmiştir. Ayrıca, maksimum bakım politikası altında kullanılabilirliği, görev sürecinin limit olasılıkları cinsinden yazabilmeyi olanaklı kılan bir doğrusal denklem sistemi elde edilmiştir.

Bölüm 6, tek aşamalı bir görevi yapan ve yaşam süreleri üstel dağılan bileşenlerden oluşan sistemlerin güvenilirliği, beklenen yaşam süresi ve kullanılabilirliği üzerinedir. Tutarlı bir sistemin sistem güvenirliğinin ve beklenen yaşam süresinin, bileşenlerin hasar hızlarına göre dış bükey olan iki fonksiyonun farkı olarak yazılabileceği gösterilmiş ve bu gösterimin açık formülasyonu verilmiştir. Ayrıca, tutarlı yapıların kullanılabilirliğinin ve sıra dizili dönüşümlü koşut alt sistemlerin beklenen yaşam süresi ve kullanılabilirliğinin pozitif katsayılı iki polinomun oranı olarak yazılabileceği gösterilmiştir.

Bölüm 7'de yaşam süreleri genel dağılıma sahip birden fazla bileşenden oluşan görev-tabanlı sistemlerin eniyi bakım problemi incelenmiştir. İlk olarak eniyi değiştirme problemi analiz edilmiş ve toplam maliyetin şimdiki beklenen değerini enküçükleyen değiştirme politikasının birçok monotonluk özelliği, artan hasar hızı gibi çeşitli monotonluk varsayımları altında ispatlanmıştır. Daha sonra eniyi tamir problemi benzer varsayımlar altında analiz edilmiş ve çeşitli maliyetlendirme yapıları kullanılarak eniyi tamir politikasının bazı ilginç özellikleri ortaya çıkarılmıştır.

Son bölümde daha önce yapılmış olan tüm analizler, önceden ele alınan modelin basitleştirilmiş bir hali için tekrarlanmıştır. Bu daha basit modelde görev sürecinin ve görevin her aşamasındaki sistemin bozulma sürecinin birer Markov süreci olduğu varsayılmıştır. Bu basitleştirmeyle elde edilmek istenen, sayısal olarak daha kolay hesaplanabilen sonuçlar elde etmektir. Sistem güvenirliği ve kritik aşama güvenirliği için üstel matris yapısında formüller elde edilmiştir. Görev güvenirliği, beklenen yaşam süresi ve kullanılabilirlik için oldukça açık formülasyonlara ulaşılmıştır. Ayrıca, elde ettiğimiz sonuçların uygulanabilirliğini göstermek için birçok sayısal örnek verilmiştir. Daha sonra bu tarz bir sistem için eniyi bakım problemleri tanımlanmış ve ayrıntılı bir şekilde analiz edilmişlerdir. Eniyi değiştirme politikasının kontrol-sınır yapısına sahip olduğu gösterilmiş ve eniyi tamir politikasının birçok ilginç özelliği, faklı maliyetlendirme yapıları için ispatlanmıştır. Ayrıca, yaptığımız varsayımların gerçekten gerekli olduğunu gösteren sayısal örnekler üretilmiştir.

ACKNOWLEDGEMENTS

First, I would like to gratefully and sincerely thank my supervisor Prof. Süleyman Özekici who has been a great source of inspiration and provided the right balance of suggestions, criticism, and freedom. His mentorship was paramount in providing a well rounded experience consistent with my long-term career goals. Despite his excessive workload, whenever I needed to talk to him or to discuss something (I had too many), he was always accessible and kind. I thank him for everything that he did for me.

I am grateful to members of my thesis monitoring committee for critical reading of this thesis and their valuable comments. I am grateful to Prof. İ. Kuban Altınel and Assoc. Prof. Lerzan Örmeci for their valuable guidance and accessibility. I also would like to thank Assoc. Prof. Orhan Feyzioğlu and Assoc. Prof. Metin Türkay for taking part in my thesis comittee in spite of their excessive workload.

I am grateful to TÜBİTAK (The Scientific and Technological Research Council of TURKEY) for providing financial support through my Ph.D. study.

Finally, I would like to thank my beloved wife Bahar for her patience, encouragement, unwavering love and providing me morale support in hard days of my research. I thank her for every dinner table that she prepared after a full day at work. I thank her for every "Come on honey!" when I felt desperate. I thank her for every effort to make me smile. I am grateful to her for everything that she sacrificed for me.

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NOMENCLATURE

δ_i	:	Transition rate of Y during phase i in the Markovian deterioration model
$\lambda_i(a)$	:	Transition rate of $\mathcal{A}$ during phase i if the deterioration level of the system is a
$\lambda(i,a)$	:	Transition rate of $(Y,\mathcal{A})$ in state (i,a)
$\overline{\lambda}_i$	:	Total failure rate of the system during phase i
μ	:	Initial distribution of the mission process
ς_i	:	Rate of exponentially distributed repair time if the state of the system is i (or during phase i)
$\psi_i(a)$	:	Structure function of the system during phase i
Δ	:	State denoting system failure
$\triangledown C_i(a,b)$	:	Marginal repair cost ($\triangledown C_i(a,b) = C_i(a,b-1) - C_i(a,b)$)
$\mathbf{0}$	:	Column vector with all entries being equal to 0
$\mathbf{1}$	:	Column vector with all entries being equal to 1
$(1_k, x)$	:	Vector with kth entry being equal to 1 and all other entries being equal to the corresponding entries of x ($(1_k,x) = [x_1, x_2, \cdots, x_{k-1}, 1, x_{k+1}, \cdots, x_m]$)
A	:	Availability of the system
$A_t(k)$	:	Intrinsic age of component k at time t
A_t	:	Intrinsic age of the system at time t ($A = \{A_t(k); t \in \mathbb{R}_+, k \in S\}$)
$\mathcal{A}$	:	Deterioration process of the system in Markovian deterioration model
B	:	Binary set ($B = \{0,1\}$)
$B_n(k)$	:	Intrinsic age of component k at time T_n
B_k	:	Set of intrinsic age vectors whose kth entry is finite ($B_k = \{c \in \mathcal{F}; c(k) < +\infty\}$)

$\overline{B}_k$	:	Set of intrinsic age vectors whose kth entry is $+\infty$ ($B_k = \{c \in \mathcal{F}; c(k) < +\infty\}$)
B'_k	:	Set of intrinsic age vectors whose kth entry is 0 ($B_k = \{c \in \mathcal{F}; c(k) = 0\}$)
$\mathfrak{B}$	:	Set of all bounded nonnegative real-valued functions defined on $E \times \mathcal{F}$
$\mathcal{B}$	:	Set of all bounded nonnegative real-valued functions defined on $E \times F$
$c_i(a)$	:	Purchase cost of the system with age or deterioration level a
$c(i, a)$	:	Cost of performing phase i with an initial intrinsic age or deterioration level a
$c_m(i, a; r)$	:	Cost of applying the replacement policy r if the next phase is i and the intrinsic age vector of the system is a
C_i	:	Set of components used during phase i
$C_i(a, b)$	:	Cost of repairing system from age or deterioration level a to age or deterioration level b
$C_0(x)$	:	Set of failed components in state x ($C_0(x) = \{k; x_k = 0\}$)
$C_1(x)$	:	Set of working components in state x ($C_1(x) = \{k; x_k = 1\}$)
D_i	:	Duration of phase i
E	:	State space of (X, T)
$\widetilde{E}$	:	State space of $(\widetilde{X}, \widetilde{T})$ ($\widetilde{E} = E \cup \{\Delta\}$)
$\overline{E}$	:	State space of $(\overline{X}, \overline{T})$ ($\overline{E} = E \cup \{\Delta, S_j\}$)
$\widetilde{E \times \mathcal{F}}$	:	State space of $((\widetilde{X}, \widetilde{A}), \widetilde{T})$
$\overline{E \times \mathcal{F}}$	:	State space of $\left((\overline{X}, \overline{A}), \overline{T}\right)$
$E_i[\cdot]$	:	Expectation given that initial phase is i
$E_{ia}[\cdot]$	:	Expectation given that initial phase is i and initial intrinsic age or deterioration level of the system is a
f_i	:	Cost of failure during phase i
F	:	State space of $\mathcal{A}$ ($F = \{1, 2, \cdots, M\}$)
$F_i(t)$	:	Distribution function of the duration of phase i ($F_i(t) = 1 - \overline{F}_i(t)$)
F_j	:	Set of all failure states involving the failure of subsystem j

F_W	:	Set of states at which a coherent system with structure function ϕ is out of order but starts to work properly by changing the state of one of its failed components to 1 ($F_W = \{x \in \overline{W}; (1_i, x) \in W \text{ for some } i = 1, \cdots, m\}$)
$\mathcal{F}$	:	State space of A ($\mathcal{F} = \overline{\mathbb{R}}_+^m$)
$\mathcal{F}_i$	:	Set of intrinsic ages at which the system is out of order during phase i ($\mathcal{F}_i = \{b \in \mathcal{F}; \psi_i(b) = 0\}$)
G_i	:	Generator of $\mathcal{A}$ during phase i
$\mathcal{G}(i, a; j, b)$	:	Generator of $(Y, \mathcal{A})$
$\mathcal{G}^*(i, a; j, b)$	:	Remaining matrix after removing all absorbing states from $\mathcal{G}(i, a; j, b)$
$\widehat{\mathcal{G}}(i, a; j, b)$	:	Generator of $(\widehat{Y}, \widehat{\mathcal{A}})$
$\widetilde{\mathcal{G}}_j(i, a; k, b)$	:	Generator of Z
$G_\Delta(t)$	:	Distribution function of repair time under the maximal repair policy
$G_\Delta(i, a; t)$	:	Distribution function of repair time during phase i under the no repair policy if the system is at age a
$h_k(i, a, t)$	:	Intrinsic age of component k after t units of time during phase i if its initial age is a
$H(i, j)$	:	Generator of the mission process in Markovian deterioration model
$H_k(i, t)$	:	Intrinsic age of component k at time t provided that the system performs phase i throughout $[0, t]$
$H_k^{-1}(i, a)$	:	Time at which the intrinsic age of component k becomes a if the system performs phase i
$\boldsymbol{I}$	:	Identity matrix
$I_{\{condition\}}$	:	Indicator function which is 1 if *condition* holds, otherwise it is 0
$\mathcal{J}_i$	:	Set of subsystems used during phase i
L	:	Lifetime of the system
$\mathcal{L}_i(\alpha)$	:	Laplace transform of D_i
$L(k)$	:	Lifetime of component k
$\widehat{L}(k)$	:	Intrinsic lifetime of component k
m	:	Number of components in the system

$m(i)$	:	Expected duration of phase i
$m(\Delta)$	:	Mean repair time under the maximal repair policy
M	:	State denoting system failure in Markovian deterioration model
$n_i(k)$	:	Number of components in subsystem k during phase i
$n(A)$	:	Cardinality of A
N	:	State space representing the number of available components in each subsystem
$\mathbb{N}$	:	Set of natural numbers ($\mathbb{N} = \{1, 2, \cdots\}$)
$N_i(a)$	:	Intrinsic age vectors reachable from a in phase i
p_i	:	Preventive maintenance cost during phase i
$\overline{p}_i(t)$	:	Probability that the system will survive until time t during phase i ($\overline{p}_i(t) = 1 - p_i(t)$)
$\overline{p}_{ia}(s, db)$	:	Probability that the system is in working condition after s units of time during phase i and the new age is in db given that the initial age is a
$\overline{p}^k_{ia}(s, db)$	:	Probability that the new age of component k is in db after s units of time during phase i given that its initial age is a
$P_i(a, b)$	:	Transition probability matrix of $\mathcal{A}$ during phase i in Markovian deterioration model
$P(i, j)$	:	Transition probability matrix corresponding to (X, T)
$\widetilde{P}(i, j)$	:	Transition probability matrix corresponding to $(\widetilde{X}, \widetilde{T})$
$\widetilde{P}_\Delta(i, j)$	:	Matrix obtained by deleting the row and the column corresponding to state Δ from $\widetilde{P}(i, j)$
$\overline{P}(i, j)$	:	Transition probability matrix corresponding to $(\overline{X}, \overline{T})$
$\widehat{P}(i, j)$	:	Transition probability matrix corresponding to $(\widehat{X}, \widehat{T})$
$\widehat{P}_\Delta(i, j)$	:	Matrix obtained by deleting the row and the column corresponding to state Δ from the matrix $\widehat{P}(i, j)$
$\mathcal{P}(i, a; j, b)$	:	Transition probability matrix corresponding to $(Y, \mathcal{A})$
$\widehat{\mathcal{P}}(i, a; j, b)$	:	Transition probability matrix corresponding to $(\widehat{Y}, \widehat{\mathcal{A}})$
$\widetilde{P}(i, a; j, db)$	:	Transition kernel corresponding to $(\widetilde{X}, \widetilde{A})$

$\widetilde{P}_{\Delta}(i, a; j, db)$	:	Transition kernel obtained by deleting the row and column corresponding to the state Δ from $\widetilde{P}(i, a; j, db)$
$\overline{P}(i, a; k, db)$	:	Transition kernel corresponding to $(\overline{X}, \overline{A})$
$\widehat{P}(i, a; j, db)$	:	Transition kernel corresponding to $(\widehat{X}, \widehat{A})$
$P_i\{\cdot\}$	:	Conditional probability given that the initial phase is i
$P_{ia}\{\cdot\}$	:	Conditional probability given that the initial phase is i and the initial intrinsic age or deterioration level of the system is a
$Q(i, j)$	:	Transition probability matrix of the mission process in Markovian deterioration model
$Q(i, j, t)$	:	Semi-Markov kernel of (X, T)
$\widetilde{Q}(i, j, t)$	:	Semi-Markov kernel of $(\widetilde{X}, \widetilde{T})$
$\overline{Q}(i, k, t)$	:	Semi-Markov kernel of $(\overline{X}, \overline{T})$
$\widehat{Q}(i, j, t)$	:	Semi-Markov kernel of $(\widehat{X}, \widehat{T})$
$\widetilde{Q}(i, a; j, db; ds)$	:	Semi-Markov kernel of $((\widetilde{X}, \widetilde{A}), \widetilde{T})$
$\overline{Q}(i, a; k, db; ds)$	:	Semi-Markov kernel of $((\overline{X}, \overline{A}), \overline{T})$
$\widehat{Q}(i, a; j, db; ds)$	:	Semi-Markov kernel of $((\widehat{X}, \widehat{A}), \widehat{T})$
$r_k(i, a)$	:	Intrinsic aging rate of component k during phase i at age a
$\mathbb{R}$	:	Set of real numbers ($\mathbb{R} = (-\infty, +\infty)$)
$\mathbb{R}_+$	:	Set of nonnegative real numbers ($\mathbb{R}_+ = [0, +\infty)$)
$\overline{\mathbb{R}}_+$	:	Set of extended nonnegative real numbers ($\overline{\mathbb{R}}_+ = [0, +\infty]$)
$\widetilde{R}(i, j, t)$	:	Markov renewal kernel corresponding to $\widetilde{Q}(i, j, t)$
$\widetilde{R}_{\Delta}(i, j)$	:	Potential matrix corresponding to $\widetilde{P}_{\Delta}(i, j)$
$\overline{R}(i, k, t)$	:	Markov renewal kernel corresponding to $\overline{Q}(i, k, t)$
$\widehat{R}(i, j, t)$	:	Markov renewal kernel corresponding to $\widehat{Q}(i, j, t)$
$\widehat{R}_{\Delta}(i, j)$	:	Potential matrix corresponding to $\widehat{P}_{\Delta}(i, j)$
$\widetilde{R}(i, a; j, db; ds)$	:	Markov renewal kernel corresponding to $\widetilde{Q}(i, a; j, db; ds)$
$\widetilde{R}_{\Delta}(i, a; j, db)$	:	Potential kernel corresponding to $\widetilde{P}_{\Delta}(i, a; j, db)$
$\overline{R}(i, a; k, db; ds)$	:	Markov renewal kernel corresponding to $\overline{Q}(i, a; k, db; ds)$
$s_i(a)$	:	Salvage value of the system with intrinsic age or deterioration level a
S	:	Set of components ($S = \{1, \cdots, m\}$)

S_j	:	State denoting successful completion of critical phase j
$\mathcal{S}$	:	Set of all intrinsic ages at which the system is in working condition ($\mathcal{S} = \cup_{j \in E} \mathcal{S}_j$)
$\mathcal{S}_i$	:	Set of all intrinsic ages at which the system is in working condition in phase i ($\mathcal{S}_i = \{b \in \mathcal{F}; \psi_i(b) = 1\}$)
T_n	:	Time at which the nth phase starts
$\mathcal{T}_\Delta$	:	Number of transitions until system failure by $(\widehat{X}, \widehat{T})$
U_j	:	The first time that the mission process leaves state j ($U_j = \inf\{t \geq 0; Y_t \neq Y_{t-} = j\}$)
$v_{ia}(s)$	:	Probability that the system will work at most s units of time during phase i if the initial system age is a ($\overline{v}_{ia}(s) = 1 - v_{ia}(s)$)
$v(i, a)$	:	Minimum expected total discounted cost of maintaining the system if the initial phase is i, and the age or the deterioration level of the device is a
W	:	Set of states at which coherent system with structure ϕ is in working condition ($W = \{y \in B^m; \phi(y) = 1\}$)
$\overline{W}$	:	Set of states at which coherent system with structure ϕ is failed ($W = \{y \in B^m; \phi(y) = 0\}$)
(X, T)	:	Markov renewal process followed by the mission process
$(\widetilde{X}, \widetilde{T})$	:	Markov renewal process used to analyze system reliability, mission reliability and MTTF under the maximal repair policy
$(\overline{X}, \overline{T})$	:	Markov renewal process used to analyze phase reliability under the maximal repair policy
$(\widehat{X}, \widehat{T})$	:	Markov renewal process used to analyze availability under the maximal repair policy
$((\widetilde{X}, \widetilde{A}), \widetilde{T})$	:	Markov renewal process used to analyze system reliability, mission reliability and MTTF under the no repair policy
$\left((\overline{X}, \overline{A}), \overline{T}\right)$	:	Markov renewal process used to analyze phase reliability under the no repair policy

$((\widehat{X},\widehat{A}),\widehat{T})$	:	Markov renewal process used to analyze availability under the no repair policy
X_n	:	nth phase of the mission
Y	:	Mission process ($Y = \{Y_t; t \in \mathbb{R}_+\}$)
$\widetilde{Y}$	:	Minimal semi-Markov process corresponding to $(\widetilde{X},\widetilde{T})$
$\overline{Y}$	:	Minimal semi-Markov process corresponding to $(\overline{X},\overline{T})$
$\widehat{Y}$	:	Minimal semi-Markov process corresponding to $(\widehat{X},\widehat{T})$
$\widetilde{Y}^a$	:	Minimal semi-Markov process corresponding to $((\widetilde{X},\widetilde{A}),\widetilde{T})$
$\overline{Y}^a$	:	Minimal semi-Markov process corresponding to $\left(\left(\overline{X},\overline{A}\right),\overline{T}\right)$
$\widehat{Y}^a$	:	Minimal semi-Markov process corresponding to $((\widehat{X},\widehat{A}),\widehat{T})$
$(Y,\mathcal{A})$	:	Markov process used to analyze system reliability, mission reliability, and MTTF in Markovian deterioration model
$(\widehat{Y},\widehat{\mathcal{A}})$	:	Markov process used to analyze availability in Markovian deterioration model
Z	:	Markov process used to analyze phase reliability in Markovian deterioration model
CS	:	Coherent Systems
DC	:	Difference of Convex Functions
DPE	:	Dynamic Programming Equation
IFR	:	Increasing Failure Rate
MTTF	:	Mean Time to Failure
NBU	:	New Better than Used
RP	:	Ratio of Posynomials
RS	:	Series Connection of Standby Redundant Subsystems

Chapter 1

INTRODUCTION

This thesis focuses on the reliability and the maintenance of mission-based systems. Many missions which have to be accomplished by a complex system have different stages in which the deterioration of the components and the configuration of the system change dramatically from one stage to another. Such systems are called mission-based systems or phased-mission systems in the literature and each stage is called a phase. The sequence and the duration of the phases can be deterministic or random and the system can be repairable or non-repairable. Moreover, all stochastic and deterministic failure properties of the components of the system depend on the phase of the mission that is performed at a given time. These systems were introduced by Esary and Ziehms [1] and a vast literature has accumulated since then.

Most of the literature on phased-mission systems in the literature assume that the sequence of the phases is deterministic, and we will summarize some important papers on this type of phased-mission systems. Esary and Ziehms [1], Burdick et al. [2] and Veatch [3] analyze phased-mission systems with non-repairable components. Alam and Al-Saggaf [4] introduce a method for repairable systems with deterministic phase durations. Then, Kim and Park [5] extend this work to systems with generally distributed phase durations. An algorithm for non-repairable systems with general failure distribution, which is based on binary decision trees, is proposed by Zang et al. [6]. Vaurio [7] discusses a fault tree analysis for repairable systems with general repair and failure distributions. Xing and Dugan [8] analyze a more general class of systems which includes phased-mission systems with combinatorial phase requirement and imperfect coverage where the failure of the system is determined by the failures of the components by logical rules (and, or, k-out-of-m) and the failure of a component can be transient or permanent. Other important recent papers on generalized phased-mission systems are Xing and Dugan [9] and Xing and Dugan [10]. Furthermore, phased-mission systems with multimode failures in which different failure modes have different failure rates and effects are analyzed by Tang and Dugan [11] using binary decision diagrams. Tang et al. [12] use these diagrams to analyze the reliability of a phased-mission system with common cause failures (simultaneous failure of multiple components due to a common cause).

Phased-mission systems with random sequence of phases were first introduced by Mura and Bondavalli [13] assuming that the phase durations were deterministic. After that, a methodology to analyze the reliability of phased-mission systems with random phase sequence and generally distributed phase durations is provided by Mura and Bondavalli [14] using Markov regenerative stochastic petri nets. Bondavalli and Filippini [15] apply the methodology in Mura and Bondavalli [14] to scheduled maintenance systems. Bondavalli et al. [16] describe a solution procedure, DEEM, which can handle both deterministic and random sequence of phases.

The analysis and the structure of mission-based systems or phased-mission systems are quite similar to systems working under random environments. In reliability modeling, a device generally consists of a large number of components with stochastically dependent lifetimes. Random environments are used to provide a tractable model of dependence since this is taken as an external process that affects the deterioration, aging and failure of all of the components. Since all components are subjected to the same environmental conditions, their lifetimes are dependent via their common environmental process. Thus, the environmental process is actually a factor of variation in the failure structure of the components. An interesting model was introduced by Çınlar and Özekici [17] where stochastic dependence is introduced by a randomly changing common environment that all components of the system are subjected to. This model is based on the simple observation that the aging or deterioration process of any component depends very much on the environment that the component is operating in. They propose to construct an intrinsic clock which ticks differently in different environments to measure the intrinsic age of the device. The environment is modeled by a semi-Markov jump process and the intrinsic age is represented by the cumulative hazard accumulated in time during the operation of the device in the randomly varying environment. This is a rather stylish choice which envisions that the intrinsic lifetime of any device has an exponential distribution with parameter 1. The concept of random hazard functions is also used by Gaver [18] and Arjas [19]. The intrinsic aging model of Çınlar and Özekici [17] is studied further in Çınlar et al. [20] to determine the conditions that lead to associated component lifetimes, as well as multivariate increasing failure rate (IFR) and new better than used (NBU) life distribution characterizations. It was also extended in Shaked and Shanthikumar [21] by discussions on several different models with multicomponent replacement policies. Lindley and Singpurwalla [22] discuss the effects of the random environment on the reliability of a system consisting of components which share the same environment. Although the initial state of the environment is random, they assume that it remains constant in time and components have exponential life distributions in each possible environment. This model is also studied by Lefèvre and Malice [23] to determine partial orderings on the number of functioning components and the reliability of k-out-of-n systems, for different partial orderings of the probability distribution on the environmental state. The association of the lifetimes of components subjected to a randomly varying environment is discussed by Lefèvre and Milhaud [24]. Singpurwalla and Youngren [25] also discuss multivariate distributions that arise in models where a dynamic environment affects the failure rates of the components. In a recent article, Singpurvalla [26] provides a review by discussing hazard potentials in reliability modelling.

The use of random environments is not limited to applications in reliability models. There is now considerable amount of literature on modulation in a variety of applications. An example in queueing is provided by Prabhu and Zhu [27] where customer arrival and service rates are modulated by a Markov process. Erdem and Özekici [28] consider inventory models with a demand process that fluctuates with respect to stochastically changing economic conditions. A general discussion on the idea can be found in Özekici [29] who discusses the use of random environments in complex models in operations research with applications in reliability, inventory, and queueing. An interested reader is referred to Rolski et al. [30] for further applications in queueing, insurance and finance. More recently, Çelikyurt and Özekici [31] applied the idea to various multiperiod portfolio optimization problems. In their setting, the correlation among returns in different periods is formulated by a stochastic market representing underlying financial, economic, social, and other factors that affect returns on risky assets.

Reliability, mean time to failure (MTTF), and availability of complex systems are very important active research topics in operations research literature. There are many recent papers that consider various system structures. For example, Wang et al. [32] compare reliability and availability of four different system configurations with warm standby components and standby switching failures. They also obtain explicit expressions for MTTF and the steady state availability. A cold standby repairable system with one repairman and two dissimilar components that have different priorities in use is analyzed by Zhang and Wang [33]. They derive formulas to compute system availability, reliability, MTTF, rate of occurrence of failure and the idle probability of the repairman. Kharoufeh et al. [34] consider a periodically inspected system with hidden failures where a continuous-time Markov chain modulates the wear rate and damage is induced by a Poisson shock process. Explicit formulations for the system lifetime distribution, MTTF, and the limiting average availability are obtained. Several properties of the mean residual life function of a k-out-of-n system with independent and identical components are examined by Asadi and Bayramoglu [35]. Kiureghian et al. [36] analyze a general system where component failures are assumed to be homogeneous Poisson events in time and repair durations are exponentially distributed. They give closed form expressions for the steady-state availability, mean rate of failure, mean duration of downtime, and lower bound reliability.

Maintenance actions are vital for companies to increase reliability and availability of the production system and to decrease production costs. At the same time, Bevilacquaa and Bragliab [37] states that maintenance may require extensive expenditures which may vary from 15% to 70% of the total production cost depending on the industry. For instance, the total amount spent for maintenance is more than 200 billion dollars in the United States every year as observed by Chu et al. [38]. Moreover, a significant portion of the total work force in a company is employed in maintenance departments; Waeyenbergh and Pintelon [39] estimates that this is up to 30% or more in chemical process industries. These observations indicate that optimizing the obvious trade-off between maintenance costs and productivity will have a very significant impact on the total cost. This is why it is not surprising that an extensive body of literature on optimal maintenance problems has been accumulated. The review papers [40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55] survey hundreds of papers on optimal maintenance problems.

The primary purpose of this thesis is to analyze reliability and maintenance problems related to mission-based systems in a very general setting. We generally assume that the mission process is the semi-Markov process corresponding to a Markov renewal process. Therefore, the sequence of the phases follows a Markov chain and the durations of the phases are generally distributed. The system is a complex one with multiple components and generally distributed component lifetimes. The mission process modulates all deterministic and random failure properties of the components as well as the configuration of the system.

We first analyze several performance measures including system reliability (the probability of survival), mission reliability (the probability of completing n phases), phase reliability (the probability of completing a critical phase), MTTF, and availability under two repair policies: namely, maximal repair and no repair. In the former one, it is assumed that all components are replaced by brand new ones once a new phase starts. However, the worn system continues to work without any maintenance after completing a phase under the no repair policy. System reliability, mission reliability, and the phase reliability are analyzed using Markov renewal theory and intrinsic aging concepts described in Çınlar and Özekici [17]. In the MTTF analysis, we first define a Poisson equation whose solution characterizes the MTTF and solve it under some reasonable assumptions which guarantee the existence and uniqueness of the solution. Our main assumption simply states that the system may fail after a finite number of successfully completed phases with a positive probability whatever the initial phase is. We analyze the availability by applying some limiting results for the solutions of Markov renewal equations and show that the irreducibility of the mission process suffices for the existence of the limit under some reasonable assumptions.

We also discuss the optimal replacement and repair problems of mission-based systems. We obtain several monotonicity properties of the optimal replacement policy, minimizing the expected total discounted cost, under some increasing failure rate assumptions. We also analyze the optimal repair policy under similar assumptions by considering several cost structures.

In the last part of the thesis, all of the previous analyses are repeated for mission-based systems with Markovian mission and deterioration where the mission process is a Markov process, the deterioration process of the system is a Markov process modulated by another Markov process, and the deterioration level of the system can be classified into a finite number of states at any time. The main incentive behind this simplification is to obtain more explicit and computationally tractable results. We obtain matrix exponential formulations for the system reliability and the phase reliability. We give explicit results for the mission reliability, MTTF, and availability. We prove that the optimal replacement policy has a control-limit structure under some increasing failure rate assumptions and provide many useful properties of the optimal repair policy under some interesting cost structures.

The secondary purpose of this thesis is to derive some reliability functions which can be used in optimization models and have explicit forms in terms of the lifetime parameters. It is clear that the results obtained by using a very general model may not be computationally tractable and sufficiently explicit to be used directly in an optimization model which can be solved by using some numerical algorithms. Thus, after deriving the general results, we try to obtain more computationally tractable formulations by making some additional assumptions on the system structure, mission structure, and component lifetimes. We obtain many explicit formulations for some special cases where all component lifetimes are exponentially distributed. We also discuss some structural properties of the mission reliability functions in terms of the failure rates of the components. For example, we prove that the mission reliability of series connection of k-out-of-n subsystems is a linear combination of product of nonnegative, nonincreasing and convex functions provided that the component lifetimes are exponentially distributed. Furthermore, we study the reliability, MTTF, and availability of systems working under a fixed phase with exponentially distributed component lifetimes. It is shown that the reliability and MTTF of coherent systems (CS) are representable as a difference of convex (DC) functions, and the MTTF and availability of series connection of standby redundant subsystems (RS) can be represented as a ratio of two polynomials with positive coefficients.

In Chapter 2, the structure of the mission and age processes as well as the failure probabilities of the components in a fixed phase are described in detail. Chapter 3 is on the reliability analysis of mission-based systems under the maximal and no repair policies. The details of the MTTF analysis are presented in Chapter 4. Chapter 5 includes the results of the availability analysis. In Chapter 6, we analyze the reliability, MTTF, and availability of systems working under a fixed phase and obtain some structural properties. In Chapter 7, we consider the optimal maintenance of mission-based systems with a semi-

Markov structure. Chapter 8 is on the reliability and optimal maintenance of mission-based systems with Markovian mission and deterioration.

Chapter 2

MISSION AND AGE PROCESSES

2.1 Mission Process as an Environmental Process

Reliability models in random environments provide a perfect opportunity in order to analyze mission-based systems. These systems involve devices or machines that are designed to perform or assigned to missions consisting of a number of phases. The sequence as well as the durations of the phases may be random and, as in all random environment models, all stochastic and deterministic failure properties depend on the phases of the mission that is performed at a given time. Therefore, the mission process is the random environmental process in such models.

Let X_n denote the nth phase of the mission and T_n denote the time at which the nth phase starts with $T_0 \equiv 0$. The main assumption is that the process $(X, T) = \{(X_n, T_n); n \geq 0\}$ is a Markov renewal process on the countable state space E with some semi-Markov kernel Q. The state space E is actually that of the process X and it is implicitly understood that the process T always takes values in $\mathbb{R}_+ = [0, +\infty)$ since they denote times at which certain events occur. We refer the reader to Çınlar [56] for a more rigorous and detailed treatment of Markov renewal processes and theory. The Markov renewal property states that

$$P\left\{X_{n+1} = j, T_{n+1} - T_n \leq t | X_0, \cdots, X_n; T_0, \cdots, T_n\right\} = P\left\{X_{n+1} = j, T_{n+1} - T_n \leq t | X_n\right\} \tag{2.1}$$

where we suppose that the process is time-homogeneous with the semi-Markov kernel

$$Q(i, j, t) = P\{X_{n+1} = j, T_{n+1} - T_n \leq t | X_n = i\} \tag{2.2}$$

for all $i, j \in E$ and $t \in \mathbb{R}_+$. It is well-known that X is a Markov chain on E with transition matrix $P(i, j) = P\{X_{n+1} = j | X_n = i\} = Q(i, j, +\infty)$. We further assume that the Markov renewal process has infinite lifetime so that $\sup_n T_n = +\infty$. The probabilistic structure of

T is given by the conditional distributions

$$G(i,j,t) = P\{T_{n+1} - T_n \le t | X_n = i, X_{n+1} = j\} = \frac{Q(i,j,t)}{P(i,j)} \tag{2.3}$$

for $P(i,j) > 0$. Moreover, it is easy to see that

$$F_i(t) = 1 - \overline{F}_i(t) = P\{T_1 \le t | X_0 = i\} = \sum_{j \in E} Q(i,j,t) \tag{2.4}$$

denotes the distribution of the duration of phase i. We let

$$m(i) = \int_0^{+\infty} \overline{F}_i(s)\, ds$$

denote the mean sojourn time in phase $i \in E$. Finally, the mission process $Y = \{Y_t; t \in \mathbb{R}_+\}$ is the minimal semi-Markov process associated with (X,T) so that Y_t is the stage or phase of the mission at time t. More precisely, $Y_t = X_n$ whenever $T_n \le t < T_{n+1}$.

By saying that the mission process is a semi-Markov process, we implicitly mean that phase durations have general distributions depending on the phases, and the sequence of the phases follows a Markov chain. Most of the literature on phased-mission systems assume that the sequence of the phases are deterministic, and that the durations are either deterministic or exponentially distributed. The main reason is that random durations and sequence make the analysis more complex. There are only a few papers which consider dynamic phase sequences that are modified according to the states of the system. Therefore, our model provides some generalizations on classical phased-mission systems. The motivation for this generalization comes from previous work and real-life applications. Deterministic phase durations may be realistic for some applications, such as aerospace applications discussed by Mura and Bondavalli [14] where phases are preplanned on the ground. However, Alam and Al-Saggaf [4] state that random phase durations are more realistic in many systems such as real-time control for aircraft and space vehicles in which different sets of computational tasks are executed during different phases of the control process. Since the exponential distribution is not suitable to model all phase durations due to its long tail behavior, as stated by Mura et al. [57], phase durations should be modeled by general random variables. It is also clear that the sequence of the phases may be random. In NASA's Mars Exploration Rover Mission, for example, the mission consists of many phases like vehicle launch; cruise; approach; entry, descent and landing to Mars; rover egress; and a number of surface operations that involve scientific data collection and transmission to earth. What the rover actually does on the surface depends on complex calculations and evaluations performed by the science and engineering teams. The scientific investigations involve further phases with random sequence and durations. Moreover, systems may be affected by sources of randomness that are of an exogenous nature. For example, the performance of the rovers depends on various atmospheric conditions which can be defined as different phases affecting the system. These conditions can be evaluated statistically using past data and Markov renewal processes provide quite powerful tools to model such environmental variations.

Throughout this thesis, we consider a complex device with an arbitrary structure and m components. We let $S = \{1, \cdots, m\}$ denote the set of all components in the system and $L(k)$ denote the lifetime of component $k \in S$. The lifetime of the whole system is denoted by L. There is, of course, a relationship between the system and component failure times. For example, $L = \min\{L(k); k \in S\}$ for a series system and $L = \max\{L(k); k \in S\}$ for a parallel system. We suppose that the system structure is quite general unless otherwise indicated. Since all components perform the same phase at a given time, their lifetimes are dependent via their common mission process. We assume that the mission process explains all dependence among the component lifetimes and, hence, they are independent during any phase. Our model is motivated by Çınlar and Özekici [17] who analyze complex systems in random environments where the deterioration of system components all depend on the common environmental state that they all operate in. The environmental process not only modulates the reliability model, but it is also the source of stochastic dependence among the components. In our setting, the mission process is the environmental process.

Unless otherwise specified, $\mathbf{0}$ and $\mathbf{1}$ are column vectors with all entries being equal to 0 and 1 respectively, any vector c is a column vector and its transpose c^T is a row vector. If A is a square matrix, A^{-T} denotes the inverse of A^T. If a and b are vectors with the same size, we will use $a \geq (\leq)\, b$, $a \neq b$, and $a \lneqq b$ when $a\,(k) \geq (\leq)\, b\,(k)$ for every k, $a(k) \neq b(k)$ for at least one k, and $a \leq b$ with $a \neq b$ respectively. If b is a vector and x is a scalar, we will use $b \geq x$, and $b > x$ when $b\,(k) \geq x$, $b\,(k) > x$, respectively, for every k. Through the remainder of this thesis, Stieltjes integrals $\int_a^b$ are defined over the closed set $[a, b]$ for all

$a, b \in \mathbb{R}_+$, $\boldsymbol{I}$ denotes the identity matrix, and $B = \{0, 1\}$ denotes the binary set. We let

$$I_{\{condition\}} = \begin{cases} 1 & \text{if } condition \text{ holds} \\ 0 & \text{otherwise} \end{cases}$$

be the indicator function for any condition, e.g., $a < b$, $x \in A$, etc. We will use the standard symbols $\mathbb{N} = \{1, 2, \cdots\}$ and $\mathbb{R} = (-\infty, +\infty)$ to denote the sets of natural numbers and real numbers respectively. For $k \in \mathbb{N}$ and any set A, A^k is defined as k-fold Cartesian product $A \times \cdots \times A$. To simplify the notation, we write $P_i\{\cdot\} = P\{\cdot | X_0 = i\}$ and $E_i[\cdot] = E[\cdot | X_0 = i]$ to denote conditional probability and the conditional expectation given that $X_0 = i$ respectively. Moreover, for any pair of sets A and B, we let $A \backslash B = A \cap B^c$ where B^c is the complement of B.

2.2 Maximal Repair Model

Under the maximal repair policy, all of the system components are replaced by brand new ones at the end of each successfully completed phase. This implies that the system lifetime always has the same distribution for a given phase. We therefore let

$$\overline{p}_i(t) = 1 - p_i(t) = P\{L > t | Y = i\} \tag{2.5}$$

denote the survival probability of the brand new system in phase i for t units of time. In other words, $\overline{p}_i$ denotes the survival function in phase i where the condition $\{Y = i\}$ in (2.5) means $\{Y_t = i \text{ for all } t \geq 0\}$ so that the whole mission consists of phase i only. We therefore suppose that the system survival function is $\overline{p}_i$ whenever phase i of the mission is performed. As soon as a new phase j of the mission starts, the survival function changes accordingly to $\overline{p}_j$ with a brand new system.

If the system structure is coherent with structure function ϕ_i and component k has an exponentially distributed lifetime with parameter $\lambda_i(k)$ for all $k \in S$ during phase i, then we have

$$\overline{p}_i(t) = \phi_i(e^{-\lambda_i(1)t}, e^{-\lambda_i(2)t}, \cdots, e^{-\lambda_i(m)t}). \tag{2.6}$$

If the system is a series one, this becomes

$$\overline{p}_i(t) = e^{-\left(\sum_{k=1}^{m} \lambda_i(k)\right)t}. \tag{2.7}$$

If the system consists of a serial connection of m redundant subsystems where subsystem k has $n_i(k)$ active components during phase i, then

$$\overline{p}_i(t) = \prod_{k=1}^{m} \left(1 - \left(1 - e^{-\lambda_i(k)t}\right)^{n_i(k)}\right) \tag{2.8}$$

assuming that all components of subsystem k have independent and exponentially distributed lifetimes with parameter $\lambda_i(k)$ during phase i.

If the system consists of a serial connection of m standby redundant subsystems where subsystem k has $n_i(k)$ components, and all components of subsystem k have independent and exponentially distributed lifetimes with parameter $\lambda_i(k)$ during phase i, then we have

$$\overline{p}_i(t) = \prod_{k=1}^{m} \sum_{r_k=0}^{n_i(k)-1} \frac{e^{-\lambda_i(k)t} (\lambda_i(k)t)^{r_k}}{r_k!}.$$

The maximal repair policy may appear to be reasonable for a very limited set of real-life applications at first glance. However, we can find important examples which the maximal repair policy is very reasonable for and do not include a complete system replacement after each phase. The main point of the maximal repair policy is that the lifetime distribution of the system during a given phase is independent of the ages of the components and whenever the system performs that phase, it survives according to the same distribution. In this regard, if component lifetimes are exponentially distributed and all failed components are replaced before a new phase starts, then the maximal repair policy assumption holds by the memorylessness of the exponential distribution. Another example for which the maximal repair policy is reasonable is the case where different sets of components are used during different phases of a mission. Although the whole system is not brand new, the components used in a given phase are brand new and the maximal repair condition is still satisfied. The maximal repair policy is also reasonable for this case even if component lifetimes are not exponentially distributed provided that every phase is performed only once. It is obvious that the maximal repair policy is applicable for a series system with exponentially distributed component lifetimes. Moreover, it may provide approximations for series systems with many components and general component lifetimes. Drenick [58] shows that the lifetime of a series system with n components tends to be exponentially distributed as $n \to \infty$. This implies that the lifetime of a series system with many components is almost independent of the ages of its components and, hence, maximal repair is a very reasonable assumption for this type of systems. Furthermore, another important justification for the maximal repair policy is its mathematical and computational tractability. If the maximal repair policy is not applied, and after completing a phase, worn components are used during the next phase without any replacement, we need to define an aging function for each component and for each phase. Then, we may need to enlarge the state space enormously, which will increase the time needed to evaluate the solutions when we apply Markovian analysis techniques. It is also certain that the mathematical analysis will be more difficult, especially for the case where the component lifetimes are generally distributed.

2.3 No Repair Model

Under the no repair policy, the system will not experience any maintenance until system failure. Therefore, the worn system will continue to work after completing a phase and the survival probability of the system in a given phase depends on the deterioration level of the system at the beginning of that phase. Since the lifetimes of the components have general distributions, the concept of "aging" comes into consideration. For this purpose, we will use the "intrinsic aging" model introduced by Çınlar and Özekici [17].

Let $H_k(i,t)$ be the cumulative hazard of component k at time t in phase i which is assumed to be continuously differentiable in t. Then, we have the well-known equality

$$P\{L(k) > t | Y = i\} = e^{-H_k(i,t)}.$$

Note that if $L(k)$ has a continuously differentiable distribution function in phase i, then $H_k(i,t)$ is continuously differentiable in t. The intrinsic age of component k at time t is defined as $H_k(i,t)$ provided that the system performs phase i throughout $[0,t]$. The intrinsic aging rate of component k during phase i is defined as

$$r_k(i,a) = \frac{d}{dt} H_k(i,t) \mid_{t=H_k^{-1}(i,a)} \tag{2.9}$$

at any age $a \in \mathbb{R}_+$ where $H_k^{-1}(i,a)$ is the time at which the intrinsic age of component k becomes a if the system performs phase i; or

$$H_k^{-1}(i,a) = \inf\{t \in \mathbb{R}_+; H_k(i,t) > a\}.$$

It is known that $H_k(i,t)$ is increasing in t and, hence, $H_k\left(i, H_k^{-1}(i,a)\right) = a$.

Let $A(k)$ denote the intrinsic age process of component k for all $k \in S$. We assume that the intrinsic age process satisfies

$$\frac{dA_t(k)}{dt} = r_k(Y_t, A_t(k))$$

for $0 \leq t < \min\{L(k), L\}$. Therefore, if the intrinsic age of component k at time s when phase i starts is $A_s(k) = a$, then after t units of time its age becomes

$$A_{s+t}(k) = h_k(i, a, t) = H_k\left(i, H_k^{-1}(i, a) + t\right). \tag{2.10}$$

Note that this definition requires that both component k and the system are functioning at time $s + t$. Let $B_0(k) = A_0(k)$ and define an embedded process $B(k) = \{B_n(k); n = 0, 1, \cdots\}$ recursively through

$$B_{n+1}(k) = h_k(X_n, B_n(k), T_{n+1} - T_n)$$

for $n \geq 0$. The intrinsic aging process $A = \{A_t(k); t \in \mathbb{R}_+, k \in S\}$ of the whole system consists of the aging processes of the components. Note that $A_t \in \mathcal{F} = \overline{\mathbb{R}}_+^m = [0, +\infty]^m$ for all $t \in \mathbb{R}_+$ and $\mathcal{F}$ is the state space of A. The intrinsic age process of component k can be constructed recursively by

$$A_{T_n+t}(k) = h_k(X_n, B_n(k), t)$$

provided that $t \leq T_{n+1} - T_n$ and both the component and the whole system are functioning at time $T_n + t$. As soon as component k fails at some time $L(k)$, we set the intrinsic age to $A_{L(k)+t}(k) = +\infty$ for all $t \in \mathbb{R}_+$. Clearly, $+\infty$ denotes the failure state. We extend the definition of h_k in (2.10) such that $h_k(i, +\infty, t) = +\infty$ since a failed component remains failed.

Following the construction in Çınlar and Özekici [17], it is clear that component k is not in a failed state at time t if and only if $A_t(k) < \widehat{L}(k)$ where $\widehat{L}(k)$ is the intrinsic lifetime of component k. These also imply that component k fails at time

$$L(k) = \inf\{t \in \mathbb{R}_+; A_t(k) > \hat{L}(k)\}$$

when its intrinsic age exceeds its intrinsic lifetime. Furthermore, since the intrinsic lifetimes $\{\widehat{L}(k)\}$ are independent and identically distributed random variables that have the exponential distribution with rate 1, we can write

$$P_i\{L(k) > t | A_0(k) = a\} = P_i\left\{\widehat{L}(k) > A_t(k) | A_0(k) = a\right\} = E_i\left[e^{-(A_t(k)-a)} | A_0(k) = a\right]. \tag{2.11}$$

Let $h(i,a,t)$ denote the vector with elements $h_k(i,a(k),t)$. We let

$$H(i,a,t;db) = \begin{cases} 1 & \text{if } h(i,a,t) = b \\ 0 & \text{otherwise} \end{cases}$$

for all $i \in E$ and $a, b \in \mathcal{F}$.

Unless otherwise specified, we let ψ_i be the structure function of the system defined on $\mathcal{F}$ during phase i such that

$$\psi_i(a) = \begin{cases} 1 & \text{if the system is in working condition at intrinsic age } a \\ 0 & \text{otherwise} \end{cases}$$

for all $a \in \mathcal{F}$. It can be determined by using the reliability structure of the whole system. For instance, if the system is a series one with m components during phase i, then

$$\psi_i(b) = \prod_{k=1}^{m} I_{\{b(k)<+\infty\}}.$$

If the system is a parallel one during phase i, then

$$\psi_i(b) = 1 - \prod_{k=1}^{m} \left(1 - I_{\{b(k)<+\infty\}}\right).$$

More generally, if we have a coherent structure with some structure function ϕ_i defined on B^m during phase i, then it suffices to take

$$\psi_i(b) = \phi_i\left(I_{\{b(1)<+\infty\}}, I_{\{b(2)<+\infty\}}, \cdots, I_{\{b(m)<+\infty\}}\right).$$

If the system is at age a initially, the probability that the system is working at time s and the new age is in db is

$$\overline{p}_{ia}(s,db) = P\left\{A_s \in db, L > s | Y = i, A_0 = a\right\} = \psi_i(b) \prod_{k=1}^{m} \overline{p}^{k}_{ia(k)}(s, db(k)) \tag{2.12}$$

during phase i where

$$\overline{p}^{k}_{ia(k)}(s,db(k)) = \begin{cases} e^{-(b(k)-a(k))} & \text{if } a(k) < +\infty,\ b(k) = h_k(i,a(k),s) < +\infty \\ 1 - e^{-(h_k(i,a(k),s)-a(k))} & \text{if } a(k) < +\infty,\ b(k) = +\infty \\ 1 & \text{if } a(k) = +\infty,\ b(k) = +\infty \\ 0 & \text{otherwise.} \end{cases} \tag{2.13}$$

Note that (2.12) follows from our assumption that the aging of the components are independent during any given phase. As long as the phase of the mission is fixed, aging occurs deterministically according to $h(i, a, t)$ and any component fails as soon as the age exceeds the exponential threshold.

Through the remainder of this thesis, to simplify the notation, we write $P_{ia}\{\cdot\} = P\{\cdot|X_0 = i, A_0 = a, L > 0\}$ and $E_{ia}[\cdot] = E[\cdot|X_0 = i, A_0 = a, L > 0]$ to denote, respectively, the conditional probability and the conditional expectation given that the initial phase of the mission is i and the system is initially in working condition with intrinsic age a.

Chapter 3

RELIABILITY ANALYSIS

3.1 Models with Maximal Repair

Suppose that the system performs the mission such that at the beginning of each phase it is replaced by a brand new one. This simplifying assumption allows us to obtain various reliability measures quite easily by using the renewal property.

3.1.1 System Reliability

The probability that the system will function until time $t > 0$ will be determined using Markov renewal theory. Let $f(i,t) = P_i\{L > t\}$ denote the desired probability given that the initial phase is i. Then, conditioning on T_1,

$$\begin{aligned} f(i,t) &= P_i\{L > t, T_1 > t\} + P_i\{L > t, T_1 \le t\} \\ &= P_i\{L > t | T_1 > t\} P_i\{T_1 > t\} + \sum_{j\in E}\int_0^t P_i\{L > t, T_1 \in ds, X_1 = j\} \\ &= \overline{p}_i(t)\overline{F}_i(t) + \sum_{j\in E}\int_0^t Q(i,j,ds) P_i\{L > t | T_1 \in ds, X_1 = j\} \\ &= \overline{p}_i(t)\overline{F}_i(t) + \sum_{j\in E}\int_0^t Q(i,j,ds)\overline{p}_i(s) f(j,t-s) \\ &= g(i,t) + \widetilde{Q} * f(i,t) \end{aligned} \tag{3.1}$$

where $g(i,t) = \overline{p}_i(t)\overline{F}_i(t)$ and $\widetilde{Q}(i,j,ds) = Q(i,j,ds)\overline{p}_i(s)$ for all $i,j \in E$, and $s \in \mathbb{R}_+$. $\widetilde{Q}$ is the semi-Markov kernel of a new Markov renewal process which follows the mission process until a system failure and jumps to an absorbing state when the system fails. Let Δ denote this absorbing failure state. Clearly, $f(\Delta,t) = g(\Delta,t) = 0$. Then, (3.1) is a Markov renewal equation and, since the state space is finite, it has the unique solution $f = \widetilde{R} * g$ so that

$$P_i\{L > t\} = \sum_{j\in E}\int_0^t \widetilde{R}(i,j,ds)\, g(j,t-s) = \sum_{j\in E}\int_0^t \widetilde{R}(i,j,ds)\,\overline{p}_j(t-s)\,\overline{F}_j(t-s) \tag{3.2}$$

where $\widetilde{R} = \sum_{n=0}^{+\infty} \widetilde{Q}^n$ is the Markov renewal kernel corresponding to $\widetilde{Q}$. The uniqueness of the solution follows from the fact that $\lim_{n\to+\infty} T_n = +\infty$.

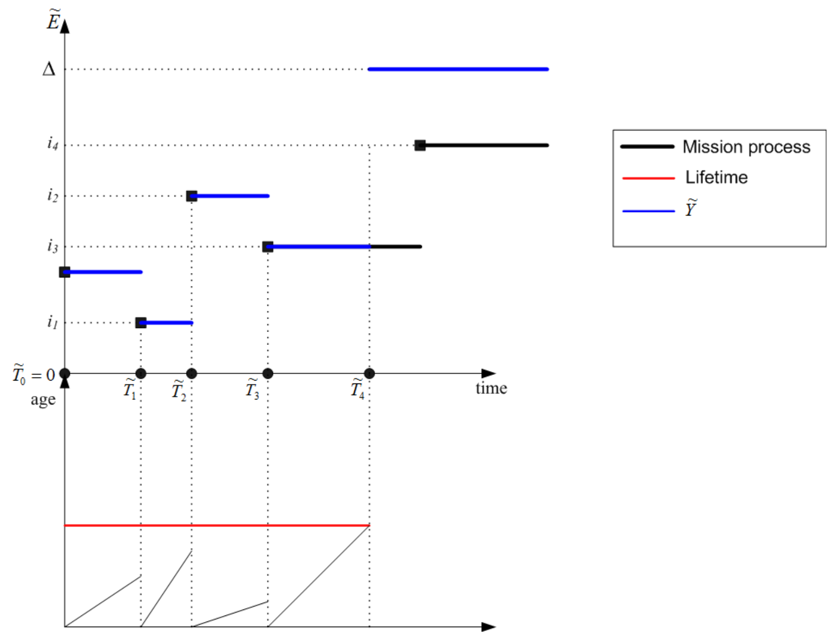

Figure 3.1: A typical representation of the structure of $\widetilde{Y}$.

We can actually obtain the Markov renewal processes $(\widetilde{X}, \widetilde{T})$ with semi-Markov kernel $\widetilde{Q}$ through its minimal semi-Markov process $\widetilde{Y}$ defined through

$$\widetilde{Y}_t = \begin{cases} Y_t & \text{if } t < L \\ \Delta & \text{if } t \geq L. \end{cases} \tag{3.3}$$

It is clear that $P_i\{L > t\} = 1 - P_i\{\widetilde{Y}_t = \Delta\}$. The structure of $\widetilde{Y}$ is as described in Figure 3.1.

We will describe the Markov renewal process $(\widetilde{X}, \widetilde{T})$ in more detail. The equation (3.3) implies that

$$\widetilde{T}_n = \begin{cases} 0 & \text{if } n = 0 \\ \inf\left\{t > \widetilde{T}_{n-1}; \widetilde{Y}_t \neq \widetilde{Y}_{\widetilde{T}_{n-1}}\right\} & \text{if } n \geq 1 \end{cases}$$

and $\widetilde{X}_n = \widetilde{Y}_{\widetilde{T}_n}$ for $n \geq 0$. Clearly, the state space of $(\widetilde{X}, \widetilde{T})$ is $\widetilde{E} = E \cup \{\Delta\}$ and its semi-Markov kernel is obtained by extending the definition of $\widetilde{Q}$ to $\widetilde{E}$ such that

$$\widetilde{Q}(i,j,ds) = \begin{cases} \overline{p}_i(s)\,Q(i,j,ds) & \text{if } i,j \in E \\ \overline{F}_i(s)\,p_i(ds) & \text{if } i \in E,\ j = \Delta \\ 0 & \text{if } i,j = \Delta \end{cases} \tag{3.4}$$

for all $i,j \in \widetilde{E}$. If $i,j \in E$ in (3.4), then this means that the system survives until time s and starts to perform phase j after completing phase i at time s. If $i \in E$ and $j = \Delta$, this means that the system fails at time s and the duration of phase i is larger than s. We can find the transition matrix of the Markov chain $\widetilde{X}$ as

$$\widetilde{P}(i,j) = \widetilde{Q}(i,j,+\infty) = \int_0^{+\infty} \widetilde{Q}(i,j,ds) = \int_0^{+\infty} \overline{p}_i(s)\,Q(i,j,ds) \tag{3.5}$$

for all $i,j \in E$, and

$$\widetilde{P}(i,\Delta) = \int_0^{+\infty} \overline{F}_i(s)\,p_i(ds) \tag{3.6}$$

for all $i \in E$. Note that $\widetilde{Q}(\Delta,j,t) = 0$ for all $t \in \mathbb{R}_+$, $j \in \widetilde{E}$ and $\widetilde{P}(\Delta,\Delta) = 1$ and $\widetilde{P}$ is a transition matrix since

$$\begin{aligned} \sum_{j\in\widetilde{E}} \widetilde{P}(i,j) &= \sum_{j\in E} \widetilde{P}(i,j) + \widetilde{P}(i,\Delta) = \sum_{j\in E} \int_0^{+\infty} \overline{p}_i(s)\,Q(i,j,ds) + \int_0^{+\infty} \overline{F}_i(s)\,p_i(ds) \\ &= \int_0^{+\infty} \overline{p}_i(s)\,F_i(ds) + \int_0^{+\infty} \overline{F}_i(s)\,p_i(ds) = 1. \end{aligned}$$

In plain words, the new process $\widetilde{Y}$ is obtained by "stopping" or "killing" the process Y whenever the system fails at time L. At the time of failure, the process is dumped to the absorbing state Δ that denotes system failure. The state space is therefore extended by including this new state Δ.

Both the semi-Markov kernel $\widetilde{Q}$ and the corresponding transition matrix $\widetilde{P}$ are possibly defective on E since $\sum_{j\in E} \widetilde{P}(i,j) = 1 - \widetilde{P}(i,\Delta) \leq 1$. As a matter of fact, we will suppose that they are indeed defective and there is an $i \in E$ such that $\widetilde{P}(i,\Delta) > 0$. Otherwise, we have a trivial situation and the system can never fail.

3.1.2 Mission Reliability

In a given application, it may be important to calculate the probability that the system will complete the first n phases successfully. In this part, we will show that this probability can be calculated using a recursive formula and then obtain an explicit solution. We first find the probability that the first phase will be completed without failure. Note that

$$P_i\{L > T_1\} = \sum_{j \in E} P_i\{L > T_1, X_1 = j\} \tag{3.7}$$

and

$$\begin{aligned} P_i\{L > T_1, X_1 = j\} &= \int_0^{+\infty} P_i\{L > s, X_1 = j, T_1 \in ds\} \\ &= \int_0^{+\infty} \overline{p}_i(s)\, Q(i, j, ds). \end{aligned} \tag{3.8}$$

Using (3.7) and (3.8), it easily follows that

$$\begin{aligned} P_i\{L > T_1\} &= \sum_{j \in E} \int_0^{+\infty} \overline{p}_i(s)\, G(i, j, ds)\, P(i, j) \\ &= \sum_{j \in E} \int_0^{+\infty} \overline{p}_i(s)\, Q(i, j, ds) = \int_0^{+\infty} \overline{p}_i(s)\, F_i(ds) \end{aligned} \tag{3.9}$$

which gives the probability that the first phase is completed successfully.

We trivially have

$$P_i\{L > T_{n+1}\} = P_i\{L > T_{n+1} \mid L > T_1\}\, P_i\{L > T_1\} \tag{3.10}$$

for any $n \geq 2$ and

$$\begin{aligned} P_i\{L > T_{n+1} \mid L > T_1\} &= \sum_{j \in E} P_i\{L > T_{n+1} \mid L > T_1, X_1 = j\}\, P_i\{X_1 = j \mid L > T_1\} \\ &= \sum_{j \in E} P_j\{L > T_n\}\left(P_i\{L > T_1, X_1 = j\} / P_i\{L > T_1\}\right). \end{aligned} \tag{3.11}$$

The first term in the right-hand side of (3.11) comes from the maximal repair assumption and the definition of conditional probability is used to obtain the second term. Using (3.8), (3.10), and (3.11), we have

$$\begin{aligned} P_i\{L > T_{n+1}\} &= \sum_{j \in E} P_j\{L > T_n\}\, P_i\{L > T_1, X_1 = j\} \\ &= \sum_{j \in E} P_j\{L > T_n\} \int_0^{+\infty} \overline{p}_i(s)\, Q(i, j, ds) \end{aligned} \tag{3.12}$$

which is a recursive relationship for $n \geq 0$ with the boundary condition $P_i\{L > T_0\} = 1$. Using (3.5) and (3.12), we have

$$P_i\{L > T_{n+1}\} = \sum_{j \in E} \widetilde{P}(i, j)\, P_j\{L > T_n\}. \tag{3.13}$$

Letting $f_n(i) = P_i\{L > T_n\}$, we can rewrite (3.13) as

$$f_{n+1} = \widetilde{P} f_n \tag{3.14}$$

for $n \geq 0$ with the boundary condition $f_0 = 1$. Then, it is clear that $f_1 = \widetilde{P} f_0, f_2 = \widetilde{P} f_1 = \widetilde{P}^2 f_0$, and more generally, $f_n = \widetilde{P} f_{n-1} = \widetilde{P}^n f_0$ so that the solution is

$$f_n(i) = \widetilde{P}^n f_0(i) = \sum_{j \in E} \widetilde{P}^n(i,j). \tag{3.15}$$

It is also possible to obtain the same solution by noting that

$$P_i\{L > T_n\} = P_i\left\{\widetilde{X}_n \in E\right\} = \sum_{j \in E} \widetilde{P}^n(i,j) = 1 - \widetilde{P}^n(i,\Delta) \tag{3.16}$$

since the nth phase of the mission is successfully completed if $\widetilde{X}_n \neq \Delta$.

The probability that the whole mission is completed without failure can be determined from

$$\lim_{n \to +\infty} P_i\{L > T_n\} = 1 - \lim_{n \to +\infty} \widetilde{P}^n(i,\Delta) \tag{3.17}$$

by using standard Markovian analysis since $\widetilde{P}$ is a Markov transition matrix. If all states $i \in E$ are transient, then this probability is 0 since the process will eventually be absorbed in state Δ. But, if there is another absorbing state $S \in E$ that is used to denote the successful termination of the whole mission, then mission reliability (3.17) is not necessarily equal to 0.

In a typical application, the mission will end as soon as the process Y enters a so-called success state, say $i_S \in E$, and Y is absorbed in this success state. Thus, one can think of entering this phase i_S as the successful completion of the mission. Then, letting $E_S = E \backslash \{i_S\}$ denote the set of all other operational phases excluding i_S, the probability that the whole mission will be completed successfully satisfies

$$\begin{aligned} P_i\{L = +\infty\} &= E_i\left[P_i\{L = +\infty | X_1, T_1\}\right] \\ &= \int_0^{+\infty} \overline{p}_i(s)\, Q(i, i_S, ds) + \sum_{j \in E_S} \int_0^{+\infty} \overline{p}_i(s)\, Q(i,j,ds)\, P_j\{L = +\infty\} \\ &= \widetilde{P}(i, i_S) + \sum_{j \in E_S} \widetilde{P}_S(i,j)\, P_j\{L = +\infty\} \end{aligned} \tag{3.18}$$

which is a system of linear equations of the form $h = g + \widetilde{P}_S h$ where $\widetilde{P}_S$ is matrix defined on E_S with $\widetilde{P}_S(i,j) = \widetilde{P}(i,j)$ for $i, j \in E_S$ and $h(i) = P_i\{L = +\infty\}$ is the mission reliability if the initial phase is $i \in E_S$ and $g(i) = \widetilde{P}(i, i_S)$. Clearly, $P_{i_S}\{L = +\infty\} = 1$ is a boundary condition. The solution is

$$h(i) = P_i\{L = +\infty\} = (\boldsymbol{I} - \widetilde{P}_S)^{-1} g(i) \tag{3.19}$$

which is an explicit expression for the mission reliability. Note that $(\boldsymbol{I} - \widetilde{P}_S)^{-1}$ exists since $\widetilde{P}_S$ is a defective transition matrix.

An example about how (3.16) and (3.19) can be applied will be given later. However, it may be interesting to give a more elementary example that does not include a probabilistic sequence of phases.

3.1.2.1 Deterministic Sequence of Phases with Exponential Lifetimes

In this special case, there is a fixed sequence of phases with random durations $\{D_i; i \in E\}$ where there are n phases so that $E = \{1, 2, \cdots, n\}$ and all phases must be completed without failure. Let λ denote the set of all failure rates. We will determine explicit expressions for the mission reliability $R(\lambda)$ and obtain some of its structural properties in terms of the failure rates of the components for different system structures.

Series System

Suppose that we have a series system in all phases and the lifetime of component j is exponentially distributed with rate $\lambda_i(j)$ during phase i for all $j \in C_i$ where C_i denotes the set of components in use during phase i. Then, we have

$$\overline{p}_i(s) = e^{-\overline{\lambda}_i s}$$

where $\overline{\lambda}_i = \sum_{j \in C_i} \lambda_i(j)$ is the total failure rate during phase i. To complete phase i, lifetimes of all components must be longer than the duration of phase i. So,

$$P_i\{L > D_i | D_i\} = e^{-\overline{\lambda}_i D_i}$$

and, using the memoryless property, we have

$$R(\lambda) = P_1\{L > T_n\} = E\left[\prod_{i=1}^{n} e^{-\overline{\lambda}_i D_i}\right] = \prod_{i=1}^{n} \mathcal{L}_i\left(\overline{\lambda}_i\right) \tag{3.20}$$

where $\mathcal{L}_i(\alpha) = E[\exp\{-\alpha D_i\}]$ is the Laplace transform of D_i and

$$\lambda = \{\lambda_i(j); i \in E, j \in C_i\}.$$

If the duration of phase i is exponentially distributed with parameter δ_i, then

$$R(\lambda) = P_i\{L > T_n\} = \prod\nolimits_{i=1}^{n} \left(\frac{\delta_i}{\delta_i + \overline{\lambda}_i} \right).$$

Lemma 3.1 *The transform* $\mathcal{L}_i^k\left(c^T\alpha\right) = E\left[e^{-\left(c^T\alpha\right)D_i} D_i^k\right]$ *is a nonnegative, convex and nonincreasing function of* $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{R}_+^n$, *for all* $c = (c_1, \cdots, c_n) \in \mathbb{R}_+^n$ *and* $k = 0, 1, \cdots$.

Proof. *Since* D_i *is a nonnegative random variable,* $\mathcal{L}_i^k\left(c^T\alpha\right) \geq 0$ *for all* k. *The gradient of* $\mathcal{L}_i^k\left(c^T\alpha\right)$ *is*

$$\frac{\partial \mathcal{L}_i^k\left(c^T\alpha\right)}{\partial \alpha_j} = -c_j E\left[e^{-\left(c^T\alpha\right)D_i} D_i^{k+1}\right] = -c_j \mathcal{L}_i^{k+1}\left(c^T\alpha\right) \leq 0$$

for all $j = 1, \cdots, n$. *Thus,* $\mathcal{L}_i^k\left(c^T\alpha\right)$ *is nonincreasing. The second order partial derivatives of* $\mathcal{L}_i^k\left(c\alpha^T\right)$ *are given by the Hessian matrix*

$$\left[H_i^k(\alpha)\right]_{jl} = \frac{\partial^2 \mathcal{L}_i^k\left(c^T\alpha\right)}{\partial \alpha_j \partial \alpha_l} = c_j c_l E\left[e^{-\left(c^T\alpha\right)D_i} D_i^{k+2}\right] = c_j c_l \mathcal{L}_i^{k+2}\left(c^T\alpha\right) \geq 0$$

for all $i, j = 1, \cdots, n$. *Take any* $z \in \mathbb{R}^n$ *and consider*

$$z^T H_i^k(\alpha) z = \mathcal{L}_i^{k+2}\left(c^T\alpha\right) \sum_{i=1}^{n} \sum_{j=1}^{n} (c_i z_i)(c_j z_j) = \mathcal{L}_i^{k+2}\left(c^T\alpha\right)(c_1 z_1 + \cdots + c_n z_n)^2 \geq 0.$$

Therefore, $H_i^k(\alpha)$ *is positive semidefinite and, hence,* $\mathcal{L}_i\left(c^T\alpha\right)$ *is convex.* ■

Taking $k = 0$ in Lemma 3.1, it follows that the reliability function $R(\lambda)$ in (3.20) for a series system is the product of nonnegative, nonincreasing and convex functions.

Series Connection of Redundant Subsystems

In this case, the system is a series of subsystems such that each subsystem must function during any phase of the mission. Moreover, all of the components in a subsystem are identical and work simultaneously in a parallel connection and a subsystem works if at least one component operates in the subsystem. Let $n_i(k)$ denote the number of components in use in subsystem k during phase i. We modify our notation slightly so that $\lambda_i(k)$ is the failure rate of any one of the $n_i(k)$ components in subsystem k during phase i. Then, we have

$$R(\lambda) = \prod\nolimits_{i=1}^{n} R_i(\lambda) \tag{3.21}$$

where

$$R_i(\lambda) = \left[\prod_{k \in \mathcal{J}_i} \left(1 - \left(1 - e^{-\lambda_i(k) D_i} \right)^{n_i(k)} \right) \right] \tag{3.22}$$

is the probability that phase i will be successfully completed, $\mathcal{J}_i$ denotes the set of subsystems in use during phase i and

$$\lambda = \{ \lambda_i(j) ; i \in E, j \in \mathcal{J}_i \} .$$

We label the subsystems so $\mathcal{J}_i = \{ j_{i1}, j_{i2}, \cdots, j_{il_i} \}$ where there are l_i subsystems used during phase i and j_{in} is the label of the nth one. The following lemma is useful to find an explicit formula for (3.21).

Lemma 3.2 *For any $a_0, a_1, \cdots, a_n \in \mathbb{R}$ and $n \geq 0$,*

$$\prod_{k=0}^{n} (1 - a_k) = \sum_{j_0=0}^{1} \cdots \sum_{j_n=0}^{1} (-1)^{j_0 + \cdots + j_n} a_0^{j_0} \cdots a_n^{j_n} .$$

Proof. *Suppose that $n = 0$. Then,*

$$\prod_{k=0}^{n} (1 - a_k) = \prod_{k=0}^{0} (1 - a_k) = 1 - a_0 = (-1)^0 a_0^0 + (-1)^1 a_0^1 = \sum_{j_0=0}^{1} (-1)^{j_0} a_0^{j_0}$$

trivially. To prove the result by induction, suppose that the hypothesis holds for $n - 1$ and consider it for n. Then,

$$\begin{aligned} \prod_{k=0}^{n} (1 - a_k) &= \prod_{k=0}^{n-1} (1 - a_k)(1 - a_n) = \prod_{k=0}^{n-1} (1 - a_k) - \prod_{k=0}^{n-1} (1 - a_k) a_n \\ &= \sum_{j_0=0}^{1} \cdots \sum_{j_{n-1}=0}^{1} (-1)^{j_0 + \cdots + j_{n-1} + 0} a_0^{j_0} \cdots a_{n-1}^{j_{n-1}} a_n^0 \\ &\quad + \sum_{j_0=0}^{1} \cdots \sum_{j_{n-1}=0}^{1} (-1)^{j_0 + \cdots + j_{n-1} + 1} a_0^{j_0} \cdots a_{n-1}^{j_{n-1}} a_n^1 \\ &= \sum_{j_0=0}^{1} \cdots \sum_{j_n=0}^{1} (-1)^{j_0 + \cdots + j_n} a_0^{j_0} \cdots a_n^{j_n} \end{aligned}$$

which completes the proof. ■

By using Lemma 3.2,

$$R_i(\lambda) = \sum_{j_1=0}^{1} \cdots \sum_{j_{l_i}=0}^{1} (-1)^{j_1 + \cdots + j_{l_i}} E \left[X_{j_{i1}}^{j_1} \cdots X_{j_{il_i}}^{j_{l_i}} \right] \tag{3.23}$$

where

$$X_k = \left(1 - e^{-\lambda_i(k)D_i}\right)^{n_i(k)} = \sum\nolimits_{r_k=0}^{n_i(k)} \binom{n_i(k)}{r_k} (-1)^{r_k} e^{-r_k\lambda_i(k)D_i}.$$

In the summation (3.23), if there are m subsystems with nonzero j values ($j_1+\cdots+j_{l_i} = m$) labeled as $s_1, s_2, \cdots, s_m$, then

$$\begin{aligned} E\left[X_{j_{i1}}^{j_1} \cdots X_{j_{il_i}}^{j_{l_i}}\right] &= \sum_{r_{s_1}=0}^{n_i(s_1)} \cdots \sum_{r_{s_m}=0}^{n_i(s_m)} \binom{n_i(s_1)}{r_{s_1}} \cdots \binom{n_i(s_m)}{r_{s_m}} (-1)^{r_{s_1}+\cdots+r_{s_m}} \\ &\times \mathcal{L}_i\left(r_{s_1}\lambda_i(s_1) + \cdots + r_{s_m}\lambda_i(s_m)\right). \end{aligned} \tag{3.24}$$

Here,we use the convention that (3.24) is equal to 1 when $m = 0$. Then,

$$\begin{aligned} R_i(\lambda) &= \sum_{j_1=0}^{1} \cdots \sum_{j_{l_i}=0}^{1} (-1)^{j_1+\cdots+j_{l_i}} \sum_{r_{s_1}=0}^{n_i(s_1)} \cdots \sum_{r_{s_m}=0}^{n_i(s_m)} \binom{n_i(s_1)}{r_{s_1}} \cdots \binom{n_i(s_m)}{r_{s_m}} \\ &\times (-1)^{r_{s_1}+\cdots+r_{s_m}} \mathcal{L}_i\left(r_{s_1}\lambda_i(s_1) + \cdots + r_{s_m}\lambda_i(s_m)\right) \end{aligned}$$

follows. If D_i is exponentially distributed with parameter δ_i, then (3.24) takes the explicit form

$$\begin{aligned} E\left[X_{j_{i1}}^{j_1} \cdots X_{j_{il_i}}^{j_{l_i}}\right] &= \sum_{r_{s_1}=0}^{n_i(s_1)} \cdots \sum_{r_{s_m}=0}^{n_i(s_m)} \binom{n_i(s_1)}{r_{s_1}} \cdots \binom{n_i(s_m)}{r_{s_m}} (-1)^{r_{s_1}+\cdots+r_{s_m}} \\ &\times \left(\frac{\delta_i}{\delta_i + r_{s_1}\lambda_i(s_1) + \cdots + r_{s_m}\lambda_i(s_m)}\right), \end{aligned}$$

resulting in

$$\begin{aligned} R_i(\lambda) &= \sum_{j_1=0}^{1} \cdots \sum_{j_{l_i}=0}^{1} (-1)^{j_1+\cdots+j_{l_i}} \sum_{r_{s_1}=0}^{n_i(s_1)} \cdots \sum_{r_{s_m}=0}^{n_i(s_m)} \binom{n_i(s_1)}{r_{s_1}} \cdots \binom{n_i(s_m)}{r_{s_m}} \\ &\times (-1)^{r_{s_1}+\cdots+r_{s_m}} \left(\frac{\delta_i}{\delta_i + r_{s_1}\lambda_i(s_1) + \cdots + r_{s_m}\lambda_i(s_m)}\right). \end{aligned} \tag{3.25}$$

By Lemma 3.1, the Laplace transform of a random variable is nonnegative, nonincreasing and convex. It can therefore be concluded that $R_i(\lambda)$ is a linear combination of nonnegative, nonincreasing and convex functions and $R(\lambda)$ is a product of linear combinations of nonnegative, nonincreasing convex functions and hence it is a linear combination of product of nonnegative, nonincreasing and convex functions.

Example 3.3 *Suppose that there are two subsystems with two and three parallel components. The mission consists of two stages, i.e.* $E = \{1, 2\}$*, and the phase set-ups are*

$\mathcal{J}_1 = \{1\}$ and $\mathcal{J}_2 = \{1, 2\}$. In other words, the mission requires only the first subsystem during the first phase, and both subsystems during the second phase. Thus, $l_1 = 1$, $l_2 = 2$, $n_1(1) = n_2(1) = 2$, and $n_1(2) = n_2(2) = 3$. Then,

$$\begin{aligned} R_1(\lambda) &= \sum_{j_1=0}^{1}\sum_{j_2=0}^{1}(-1)^{j_1+j_2} E\left[X_1^{j_1} X_2^{j_2}\right] = 1 - E[X_1] \\ &= 1 - \sum_{r_1=0}^{2}\binom{2}{r_1}(-1)^{r_1}\mathcal{L}_1(r_1\lambda_1(1)) \\ &= 2\mathcal{L}_1(\lambda_1(1)) - \mathcal{L}_1(2\lambda_1(1)). \end{aligned}$$

Similarly,

$$\begin{aligned} R_2(\lambda) &= \sum_{j_1=0}^{1}\sum_{j_2=0}^{1}(-1)^{j_1+j_2} E\left[X_1^{j_1} X_2^{j_2}\right] = 1 - E[X_1] - E[X_1] + E[X_1 X_2] \\ &= 1 - \sum_{r_1=0}^{2}\binom{2}{r_1}(-1)^{r_1}\mathcal{L}_2(r_1\lambda_2(1)) - \sum_{r_2=0}^{3}\binom{3}{r_2}(-1)^{r_2}\mathcal{L}_2(r_2\lambda_2(2)) \\ &\quad + \sum_{r_1=0}^{2}\sum_{r_2=0}^{3}\binom{2}{r_1}\binom{3}{r_2}(-1)^{r_1+r_2}\mathcal{L}_2(r_1\lambda_2(1) + r_2\lambda_2(2)) \\ &= 6\mathcal{L}_2(\lambda_2(1) + \lambda_2(2)) - 6\mathcal{L}_2(\lambda_2(1) + 2\lambda_2(2)) + 2\mathcal{L}_2(\lambda_2(1) + 3\lambda_2(2)) \\ &\quad -3\mathcal{L}_2(2\lambda_2(1) + \lambda_2(2)) + 3\mathcal{L}_2(2\lambda_2(1) + 2\lambda_2(2)) - \mathcal{L}_2(2\lambda_2(1) + 3\lambda_2(2)). \end{aligned}$$

If the durations of the first and second phases are both exponentially distributed with parameters δ_1 and δ_2 respectively, then

$$R_1(\lambda) = \left(\frac{2\delta_1}{\delta_1 + \lambda_1(1)}\right) - \left(\frac{\delta_1}{\delta_1 + 2\lambda_1(1)}\right) \tag{3.26}$$

and

$$\begin{aligned} R_2(\lambda) &= \left(\frac{6\delta_2}{\delta_2 + \lambda_2(1) + \lambda_2(2)}\right) - \left(\frac{6\delta_2}{\delta_2 + \lambda_2(1) + 2\lambda_2(2)}\right) && (3.27) \\ &\quad + \left(\frac{2\delta_2}{\delta_2 + \lambda_2(1) + 3\lambda_2(2)}\right) - \left(\frac{3\delta_2}{\delta_2 + 2\lambda_2(1) + \lambda_2(2)}\right) \\ &\quad + \left(\frac{3\delta_2}{\delta_2 + 2\lambda_2(1) + 2\lambda_2(2)}\right) - \left(\frac{\delta_2}{\delta_2 + 2\lambda_2(1) + 3\lambda_2(2)}\right). && (3.28) \end{aligned}$$

It is clear that mission reliability (3.21) is the product of $R_1(\lambda)$ and $R_2(\lambda)$, and the result is a linear combination of the product of Laplace transforms.

Series Connection of Standby Redundant Subsystems

In this model, the system has a series of subsystems as in the previous special case, but in a subsystem, only one component works at any time and whenever the working component fails, another component starts to work. For a phase to be completed successfully, the number of failures during the phase must be less than the number of components in all subsystems. Since the lifetime of each component is exponentially distributed, the number of failures in an interval has the Poisson distribution. Let $n_i(k)$ denote the number of components in use in subsystem k during phase i and l_i denote the number of subsystems in use during phase i. We assume that $\lambda_i(k)$ is the failure rate of any one of the $n_i(k)$ components in subsystem k during phase i. Then,

$$R(\lambda) = \prod\nolimits_{i=1}^{n} R_i(\lambda)$$

where

$$\begin{aligned} R_i(\lambda) &= E\left[\prod_{k\in\mathcal{J}_i}\sum_{r_k^i=0}^{n_i(k)-1}\frac{e^{-\lambda_i(k)D_i}\left(\lambda_i(k)D_i\right)^{r_k^i}}{r_k^i!}\right] \qquad (3.29) \\ &= \sum_{r_1^i=0}^{n_i(1)-1}\cdots\sum_{r_{l_i}^i=0}^{n_i(l_i)-1}\frac{\lambda_i(1)^{r_1^i}\cdots\lambda_i(l_i)^{r_{l_i}^i}}{r_1^i!\cdots r_{l_i}^i!}E\left[e^{-(\lambda_i(1)+\cdots+\lambda_i(l_i))D_i}D_i^{r_1^i+\cdots+r_{l_i}^i}\right] \end{aligned}$$

and this implies that

$$R(\lambda) = \prod\nolimits_{i=1}^{n}\sum_{r_1^i=0}^{n_i(1)-1}\cdots\sum_{r_{l_i}^i=0}^{n_i(l_i)-1}\frac{\lambda_i(1)^{r_1^i}\cdots\lambda_i(l_i)^{r_{l_i}^i}}{r_1^i!\cdots r_{l_i}^i!}\mathcal{L}_i^{r_1^i+\cdots+r_{l_i}^i}\left(\lambda_i(1)+\cdots+\lambda_i(l_i)\right). \qquad (3.30)$$

For an illustration, suppose that D_i is exponentially distributed with rate δ_i. Then, using the substitution $\alpha+\delta_i = y$,

$$\begin{aligned} \mathcal{L}_i^k(\alpha) &= \int_0^{+\infty} e^{-\alpha x}x^k\delta_i e^{-\delta_i x}dx = \delta_i\int_0^{+\infty} e^{-(\alpha+\delta_i)x}x^k dx \\ &= \frac{\delta_i}{\alpha+\delta_i}\int_0^{+\infty} e^{-y}\left(\frac{y}{\alpha+\delta_i}\right)^k dy = \frac{\delta_i}{(\alpha+\delta_i)^{k+1}}\int_0^{+\infty} e^{-y}y^k dy \\ &= \frac{\delta_i\Gamma(k+1)}{(\alpha+\delta_i)^{k+1}} = \frac{\delta_i k!}{(\alpha+\delta_i)^{k+1}}. \end{aligned}$$

This implies that

$$\begin{aligned}R_i(\lambda) &= \sum_{r_1^i=0}^{n_i(1)-1}\cdots\sum_{r_{l_i}^i=0}^{n_i(l_i)-1}\frac{\lambda_i(1)^{r_1^i}\cdots\lambda_i(l_i)^{r_{l_i}^i}}{r_1^i!\cdots r_{l_i}^i!}\left(\frac{\delta_i\left(r_1^i+\cdots+r_{l_i}^i\right)!}{\left(\lambda_i(1)+\cdots+\lambda_i(l_i)+\delta_i\right)^{r_1^i+\cdots+r_{l_i}^i+1}}\right)\\ &= \sum_{r_1^i=0}^{n_i(1)-1}\cdots\sum_{r_{l_i}^i=0}^{n_i(l_i)-1}\frac{\left(r_1^i+\cdots+r_{l_i}^i\right)!}{r_1^i!\cdots r_{l_i}^i!}\left(\frac{\delta_i}{\lambda_i+\delta_i}\right)\\ &\quad\times\left(\frac{\lambda_i(1)}{\lambda_i+\delta_i}\right)^{r_1^i}\cdots\left(\frac{\lambda_i(l_i)}{\lambda_i+\delta_i}\right)^{r_{l_i}^i} \qquad (3.31)\end{aligned}$$

where $\lambda_i=\lambda_i(1)+\cdots+\lambda_i(l_i)$.

Example 3.4 *Suppose that there are two subsystems with three parallel components each of which will perform two tasks. Then,* $E=\{1,2\}, n_1(1)=n_1(2)=n_2(1)=n_2(2)=3$ *and*

$$R_i(\lambda)=\sum_{r_1^i=0}^{2}\sum_{r_2^i=0}^{2}\frac{\lambda_i(1)^{r_1^i}\lambda_i(2)^{r_2^i}}{r_1^i!r_2^i!}\mathcal{L}_i^{r_1^i+r_2^i}\left(\lambda_i(1)+\lambda_i(2)\right)$$

for $i=1,2$, *and*

$$\begin{aligned}R(\lambda) &= \sum_{r_1^1=0}^{2}\sum_{r_2^1=0}^{2}\sum_{r_1^2=0}^{2}\sum_{r_2^2=0}^{2}\frac{\lambda_1(1)^{r_1^1}\lambda_1(2)^{r_2^1}\lambda_2(1)^{r_1^2}\lambda_2(2)^{r_2^2}}{r_1^1!r_2^1!r_1^2!r_2^2!}\\ &\quad\times\mathcal{L}_1^{r_1^1+r_2^1}\left(\lambda_1(1)+\lambda_1(2)\right)\mathcal{L}_2^{r_1^2+r_2^2}\left(\lambda_2(1)+\lambda_2(2)\right).\end{aligned}$$

If D_i *is exponentially distributed with parameter* δ_i, *then*

$$\begin{aligned}R(\lambda) &= \sum_{r_1^1=0}^{2}\sum_{r_2^1=0}^{2}\sum_{r_1^2=0}^{2}\sum_{r_2^2=0}^{2}\frac{\lambda_1(1)^{r_1^1}\lambda_1(2)^{r_2^1}\lambda_2(1)^{r_1^2}\lambda_2(2)^{r_2^2}}{r_1^1!r_2^1!r_1^2!r_2^2!}\\ &\quad\times\left(\frac{\delta_1\left(r_1^1+r_2^1\right)!}{\left(\delta_1+\lambda_1(1)+\lambda_1(2)\right)^{r_1^1+r_2^1+1}}\right)\left(\frac{\delta_2\left(r_1^2+r_2^2\right)!}{\left(\delta_2+\lambda_2(1)+\lambda_2(2)\right)^{r_1^2+r_2^2+1}}\right).\end{aligned}$$

Series Connection of k-out-of-n Subsystems

In this model, at least $k_i(j)$ out of $n_i(j)$ components must be in working condition in subsystem j during phase i to make the subsystem function and there are l_i subsystems used in phase i. We assume that $\lambda_i(j)$ is the failure rate of any one of the $n_i(j)$ components in subsystem j during phase i. So that we have

$$R(\lambda)=\prod_{i=1}^{n}R_i(\lambda)$$

where

$$R_i(\lambda) = \left[\prod_{j\in\mathcal{J}_i}\sum_{r_j^i=k_i(j)}^{n_i(j)}\binom{n_i(j)}{r_j^i}\left(e^{-\lambda_i(j)D_i}\right)^{r_j^i}\left(1-e^{-\lambda_i(j)D_i}\right)^{n_i(j)-r_j^i}\right] \tag{3.32}$$

$$= \sum_{r_1^i=k_i(1)}^{n_i(1)}\cdots\sum_{r_{l_i}^i=k_i(l_i)}^{n_i(l_i)}\binom{n_i(1)}{r_1^i}\cdots\binom{n_i(l_i)}{r_{l_i}^i}S_i\left(r_1^i,\cdots,r_{l_i}^i\right) \tag{3.33}$$

and

$$S_i\left(r_1^i,\cdots,r_{l_i}^i\right) = E\left[\left[e^{-D_i\sum_{j=1}^{l_i}r_j^i\lambda_i(j)}\right]\prod_{j=1}^{l_i}\left(1-e^{-\lambda_i(j)D_i}\right)^{n_i(j)-r_j^i}\right]$$

for phase i.

Using the binomial expansion,

$$\begin{aligned}S_i\left(r_1^i,\cdots,r_{l_i}^i\right) &= E\left[\left[e^{-D_i\sum_{j=1}^{l_i}r_j^i\lambda_i(j)}\right]\prod_{z_i=1}^{l_i}\sum_{s_{z_i}^i=0}^{n_i(z_i)-r_1^i}\binom{n_i(z_i)-r_{z_i}^i}{s_{z_i}^i}(-1)^{s_{z_i}^i}e^{-s_{z_i}^i\lambda_i(z_i)D_i}\right] \\ &= E\left[\prod_{z_i=1}^{l_i}\sum_{s_{z_i}^i=r_{z_i}^i}^{n_i(z_i)}\binom{n_i(z_i)-r_{z_i}^i}{s_{z_i}^i-r_{z_i}^i}(-1)^{s_{z_i}^i-r_{z_i}}e^{-s_{z_i}^i\lambda_i(z_i)D_i}\right] \\ &= \sum_{s_1^i=r_1^i}^{n_i(1)}\cdots\sum_{s_{l_i}^i=r_{l_i}^i}^{n_i(l_i)}\left[\prod_{j=1}^{l_i}\binom{n_i(j)-r_j^i}{s_j^i-r_j^i}\right](-1)^{s^i-r^i}\mathcal{L}_i\left(\sum_{j=1}^{l_i}s_j^i\lambda_i(j)\right)\end{aligned} \tag{3.34}$$

where $s^i = s_1^i+\cdots+s_{l_i}^i$ and $r^i = r_1^i+\cdots+r_{l_i}^i$. Thus, combining (3.32) and (3.34) and rearranging the combinations, we have

$$R_i(\lambda) = \sum_{r_1^i=k_i(1)}^{n_i(1)}\cdots\sum_{r_{l_i}^i=k_i(l_i)}^{n_i(l_i)}\sum_{s_1^i=r_1^i}^{n_i(1)}\cdots\sum_{s_{l_i}^i=r_{l_i}^i}^{n_i(l_i)}\left[\prod_{j=1}^{l_i}\binom{n_i(j)}{s_j^i}\binom{s_j^i}{r_j^i}\right](-1)^{s^i-r^i}\mathcal{L}_i\left(\sum_{j=1}^{l_i}s_j^i\lambda_i(j)\right). \tag{3.35}$$

If the duration of phase i is exponentially distributed with parameter δ_i, then

$$R_i(\lambda) = \sum_{r_1^i=k_i(1)}^{n_i(1)}\cdots\sum_{r_{l_i}^i=k_i(l_i)}^{n_i(l_i)}\sum_{s_1^i=r_1^i}^{n_i(1)}\cdots\sum_{s_{l_i}^i=r_{l_i}^i}^{n_i(l_i)}\frac{(-1)^{s^i-r^i}\delta_i\left[\prod_{j=1}^{l_i}\binom{n_i(j)}{s_j^i}\binom{s_j^i}{r_j^i}\right]}{\delta_i+\sum_{j=1}^{l_i}s_j^i\lambda_i(j)}. \tag{3.36}$$

Since the Laplace transform of a random variable is nonnegative, nonincreasing and convex; $R_i(\lambda)$ is a linear combination of nonnegative, nonincreasing convex functions, and

$R(\lambda)$ is a product of linear combinations of nonnegative, nonincreasing convex functions and, hence, it is a linear combination of product of nonnegative, nonincreasing and convex functions.

Example 3.5 *Suppose that there are two subsystems with three parallel components which will perform two tasks. At least two of three components must work to make the subsystems function. Then,* $E = \{1,2\}, \mathcal{J}_i = \{1,2\}$ *for* $i \in E, n_1(1) = n_1(2) = n_2(1) = n_2(2) = 3, k_1(1) = k_1(2) = k_2(1) = k_2(2) = 2$ *and*

$$R_i(\lambda) = \sum_{r_1^i=2}^{3} \sum_{r_2^i=2}^{3} \sum_{s_1^i=r_1^i}^{3} \sum_{s_2^i=r_2^i}^{3} \binom{3}{s_1^i}\binom{s_1^i}{r_1^i}\binom{3}{s_2^i}\binom{s_2^i}{r_2^i}(-1)^{s_1^i+s_2^i-r_1^i-r_2^i}\mathcal{L}_i\left(s_1^i\lambda_i(1)+s_2^i\lambda_i(2)\right)$$

for $i = 1,2$. *If* D_i *is exponentially distributed with parameter* δ_i *for every* i, *then*

$$\begin{aligned} R_i(\lambda) &= \sum_{r_1^i=2}^{3} \sum_{r_2^i=2}^{3} \sum_{s_1^i=r_1^i}^{3} \sum_{s_2^i=r_2^i}^{3} \binom{3}{s_1^i}\binom{s_1^i}{r_1^i}\binom{3}{s_2^i}\binom{s_2^i}{r_2^i}(-1)^{s_1^i+s_2^i-r_1^i-r_2^i} \\ &\quad \times \left(\frac{\delta_i}{\delta_i + s_1^i\lambda_i(1)+s_2^i\lambda_i(2)}\right). \end{aligned}$$

3.1.2.2 Markovian Mission with Exponential Lifetimes

Suppose that the mission process is a finite state Markov process with transition rates $\{\delta_i; i \in E\}$ and transition matrix P. We will show that how (3.16) can be used to calculate the mission reliability of such a mission-based system with different system structures.

To apply (3.16), each entry of the matrix $\widetilde{P}$ has to be determined using (3.5). Therefore, we should first find the semi-Markov kernel Q. Using the properties of Markov process,

$$\begin{aligned} Q(i,j,s) &= P\{X_1 = j, T_1 \le t | X_0 = i\} \\ &= P\{T_1 \le t | X_0 = i, X_1 = j\} P\{X_1 = j | X_0 = i\} \\ &= P\{T_1 \le t | X_0 = i\} P\{X_1 = j | X_0 = i\} \\ &= \left(1 - e^{-\delta_i s}\right) P(i,j) \end{aligned} \tag{3.37}$$

and

$$Q(i,j,ds) = \delta_i e^{-\delta_i s} P(i,j)\, ds. \tag{3.38}$$

Then, using (3.5) and (3.38),

$$\begin{aligned}\widetilde{P}(i,j) = \int_0^{+\infty} \overline{p}_i(s)\, Q(i,j,ds) &= \int_0^{+\infty} \overline{p}_i(s)\, \delta_i e^{-\delta_i s} P(i,j)\, ds \\ &= P(i,j) \int_0^{+\infty} \delta_i \overline{p}_i(s)\, e^{-\delta_i s} ds \\ &= P(i,j)\, E\left[\overline{p}_i(D_i)\right] \end{aligned} \tag{3.39}$$

for $i, j \neq \Delta$, and

$$\widetilde{P}(i,\Delta) = 1 - \sum_{j \in E} \widetilde{P}(i,j) = 1 - E\left[\overline{p}_i(D_i)\right].$$

It is clear that to calculate the mission reliability using (3.16), the function $\overline{p}_i(s)$ has to be determined. We will obtain it for different system structures.

Series System

Suppose that we have a series system in all phases and the lifetime of component j is exponentially distributed with rate $\lambda_i(j)$ during phase i for all $j \in C_i$. Since the minimum of exponentially distributed random variables is also exponentially distributed,

$$\overline{p}_i(s) = e^{-s\overline{\lambda}_i}. \tag{3.40}$$

So that using (3.39),

$$\widetilde{P}(i,j) = P(i,j)\, E\left[e^{-D_i \overline{\lambda}_i}\right] = P(i,j)\, \mathcal{L}_i\left(\overline{\lambda}_i\right) = \left(\frac{\delta_i}{\delta_i + \overline{\lambda}_i}\right) P(i,j) \tag{3.41}$$

for all $i, j \in E$, and

$$P_i\{L > T_1\} = \sum_{j \in E} \widetilde{P}(i,j) = \frac{\delta_i}{\delta_i + \overline{\lambda}_i} < 1.$$

It is clear that $\widetilde{P}(i,j)$ is a multiple of the Laplace transform of a random variable for $i, j \in E$. To calculate an entry of the squared matrix $\widetilde{P}^2$, each element of a row of $\widetilde{P}$ is multiplied by the corresponding elements of a column of $\widetilde{P}$. By Lemma 3.1, $\widetilde{P}^2(i,j)$ is a nonnegative combination of product of nonnegative, nonincreasing and convex functions for $i, j \in E$. Using this reasoning inductively, $\widetilde{P}^n(i,j)$ is a nonnegative combination of product of nonnegative, nonincreasing and convex functions for $i, j \in E$ and (3.16) shows that $P_i\{L > T_n\}$ has the same property.

Example 3.6 *Suppose that there are two components which will perform 2 tasks so that* $E = \{1, 2\}, m = 2$, *and the transition matrix is*

$$P = \begin{bmatrix} 0.50 & 0.50 \\ 0.60 & 0.40 \end{bmatrix}. \tag{3.42}$$

Then,

$$\widetilde{P} = \begin{bmatrix} \frac{0.50\delta_1}{\delta_1+\overline{\lambda}_1} & \frac{0.50\delta_1}{\delta_1+\overline{\lambda}_1} & 1-\frac{\delta_1}{\delta_1+\overline{\lambda}_1} \\ \frac{0.60\delta_2}{\delta_2+\overline{\lambda}_2} & \frac{0.40\delta_2}{\delta_2+\overline{\lambda}_2} & 1-\frac{\delta_2}{\delta_2+\overline{\lambda}_2} \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$\widetilde{P}^{(2)}(1,1) = 0.25\left(\frac{\delta_1}{\delta_1+\overline{\lambda}(1)}\right)^2 + 0.30\left(\frac{\delta_1}{\delta_1+\overline{\lambda}(1)}\right)\left(\frac{\delta_2}{\delta(2)+\overline{\lambda}(2)}\right)$$

$$\widetilde{P}^{(2)}(1,2) = 0.25\left(\frac{\delta_1}{\delta_1+\overline{\lambda}_1}\right)^2 + 0.20\left(\frac{\delta_1}{\delta_1+\overline{\lambda}_1}\right)\left(\frac{\delta_2}{\delta_2+\overline{\lambda}_2}\right)$$

$$\widetilde{P}^{(2)}(2,1) = 0.30\left(\frac{\delta_1}{\delta_1+\overline{\lambda}_1}\right)\left(\frac{\delta_2}{\delta_2+\overline{\lambda}_2}\right) + 0.24\left(\frac{\delta_2}{\delta_2+\overline{\lambda}_2}\right)^2$$

$$\widetilde{P}^{(2)}(2,2) = 0.30\left(\frac{\delta_1}{\delta_1+\overline{\lambda}_1}\right)\left(\frac{\delta_2}{\delta_2+\overline{\lambda}_2}\right) + 0.16\left(\frac{\delta_2}{\delta_2+\overline{\lambda}_2}\right)^2.$$

Finally, the mission reliability functions are

$$P_1\{L > T_2\} = 0.50\left(\frac{\delta_1}{\delta_1+\overline{\lambda}_1}\right)^2 + 0.50\left(\frac{\delta_1}{\delta_1+\overline{\lambda}_1}\right)\left(\frac{\delta_2}{\delta_2+\overline{\lambda}_2}\right)$$

and

$$P_2\{L > T_2\} = 0.60\left(\frac{\delta_1}{\delta_1+\overline{\lambda}_1}\right)\left(\frac{\delta_2}{\delta_2+\overline{\lambda}_2}\right) + 0.40\left(\frac{\delta_2}{\delta_2+\overline{\lambda}_2}\right)^2.$$

Example 3.7 *Suppose that there are two components which will perform 3 tasks where the third task is the success state so that* $E = \{1, 2, 3\}, m = 2, i_S = 3$ *and the transition matrix is*

$$P = \begin{bmatrix} 0.40 & 0.40 & 0.2 \\ 0.30 & 0.30 & 0.4 \\ 0 & 0 & 1 \end{bmatrix}. \tag{3.43}$$

Then, using (3.41),

$$\widetilde{P} = \begin{bmatrix} \frac{0.40\delta_1}{\delta_1+\overline{\lambda}_1} & \frac{0.40\delta_1}{\delta_1+\overline{\lambda}_1} & \frac{0.20\delta_1}{\delta_1+\overline{\lambda}_1} \\ \frac{0.30\delta_2}{\delta_2+\overline{\lambda}_2} & \frac{0.30\delta_2}{\delta_2+\overline{\lambda}_2} & \frac{0.40\delta_2}{\delta_2+\overline{\lambda}_2} \\ 0 & 0 & \frac{\delta_3}{\delta_3+\overline{\lambda}_3} \end{bmatrix}$$

from which we obtain

$$\widetilde{P}_S = \begin{bmatrix} \frac{0.40\delta_1}{\delta_1+\overline{\lambda}_1} & \frac{0.40\delta_1}{\delta_1+\overline{\lambda}_1} \\ \frac{0.30\delta_2}{\delta_2+\overline{\lambda}_2} & \frac{0.30\delta_2}{\delta_2+\overline{\lambda}_2} \end{bmatrix}$$

and

$$g = \begin{bmatrix} \frac{0.20\delta_1}{\delta_1+\overline{\lambda}_1} \\ \frac{0.40\delta_2}{\delta_2+\overline{\lambda}_2} \end{bmatrix}.$$

Finally, using (3.19), the reliability functions are

$$\begin{aligned} R_1(\lambda) = P_1\{L = +\infty\} &= (\boldsymbol{I} - \widetilde{P}_S)^{-1} g(1) \\ &= \frac{\delta_1\left(3\delta_2 + 2\overline{\lambda}_2\right)}{3\delta_1\delta_2 + 6\delta_1\overline{\lambda}_2 + 7\delta_2\overline{\lambda}_1 + 10\overline{\lambda}_1\overline{\lambda}_2} \end{aligned}$$

and

$$\begin{aligned} R_2(\lambda) = P_2\{L = +\infty\} &= (\boldsymbol{I} - \widetilde{P}_S)^{-1} g(2) \\ &= \frac{\delta_2\left(3\delta_1 + 4\overline{\lambda}_1\right)}{3\delta_1\delta_2 + 6\delta_1\overline{\lambda}_2 + 7\delta_2\overline{\lambda}_1 + 10\overline{\lambda}_1\overline{\lambda}_2}. \end{aligned}$$

Series Connection of Redundant Subsystems

For this structure, the system works properly if all subsystems work properly which implies that at least one component functions in each subsystem. Let $n_i(k)$ denote the number of components in use in subsystem k during phase i. We assume that $\lambda_i(k)$ is the failure rate of any one of the $n_i(k)$ components in subsystem k during phase i. Then, it is clear that

$$\overline{p}_i(s) = \prod_{k \in \mathcal{J}_i} \left(1 - \left(1 - e^{-\lambda_i(k)s}\right)^{n_i(k)}\right). \tag{3.44}$$

To calculate $\widetilde{P}$, the expected value of $\overline{p}_i(D_i)$ has to be determined. Since $E[\overline{p}_i(D_i)] = R_i(\lambda)$ in (3.22) for every i, the explicit formula (3.25) can be used to determine the matrix $\widetilde{P}$ via (3.39). The following example shows how.

By (3.25) and (3.39), $\widetilde{P}(i,j)$ is a multiple of the Laplace transform of a random variable for all $i, j \in E$. Using the same steps as in the previous subsection for series systems, it can be shown that $P_i\{L > T_n\}$ is a linear combination of product of nonnegative, nonincreasing and convex functions.

Example 3.8 *Consider Example 3.3 and suppose that the transition matrix of the sequence of phases mission process is (3.42). Then,*

$$\widetilde{P} = \begin{bmatrix} 0.50R_1(\lambda) & 0.50R_1(\lambda) \\ 0.60R_2(\lambda) & 0.40R_2(\lambda) \end{bmatrix}$$

where $R_1(\lambda)$ *and* $R_2(\lambda)$ *are as in (3.26) and (3.27). This further implies that*

$$\widetilde{P}^2 = \begin{bmatrix} 0.25R_1(\lambda)R_1(\lambda) + 0.3R_1(\lambda)R_2(\lambda) & 0.25R_1(\lambda)R_1(\lambda) + 0.2R_1(\lambda)R_2(\lambda) \\ 0.30R_1(\lambda)R_2(\lambda) + 0.24R_2(\lambda)R_2(\lambda) & 0.30R_1(\lambda)R_2(\lambda) + 0.16R_2(\lambda)R_2(\lambda) \end{bmatrix}.$$

Therefore,

$$P_1\{L > T_2\} = 0.50R_1(\lambda)(R_1(\lambda) + R_2(\lambda))$$

and

$$P_2\{L > T_2\} = R_2(\lambda)(0.60R_1(\lambda) + 0.40R_2(\lambda)).$$

The analysis can be repeated for any n *to compute the corresponding mission reliability.*

Example 3.9 *Suppose that there are two subsystems with two parallel components each of which perform three tasks where the third one is critical, so that* $E = \{1, 2, 3\}$, $\mathcal{J}_i = \{1, 2\}$ *for all* $i \in E$, $i_S = 3$ *and* $n_i(k) = l_i = 2$ *for any* k *and* i. *Assume that the transition matrix is (3.43). Then, we have*

$$\widetilde{P} = \begin{bmatrix} 0.40R_1(\lambda) & 0.40R_1(\lambda) & 0.2R_1(\lambda) \\ 0.30R_2(\lambda) & 0.30R_2(\lambda) & 0.4R_2(\lambda) \\ 0 & 0 & R_3(\lambda) \end{bmatrix}$$

where

$$\begin{aligned} R_i(\lambda) &= \sum_{j_1=0}^{1}\sum_{j_2=0}^{1}(-1)^{j_1+j_2}E\left[X_1^{j_1}X_2^{j_2}\right] = 1 - E[X_1] - E[X_2] + E[X_1X_2] \\ &= 1 - \sum_{r_1=0}^{2}\binom{2}{r_1}(-1)^{r_1}\mathcal{L}_i(r_1\lambda_i(1)) - \sum_{r_2=0}^{2}\binom{2}{r_2}(-1)^{r_2}\mathcal{L}_i(r_2\lambda_i(2)) \\ &\quad + \sum_{r_1=0}^{2}\sum_{r_2=0}^{2}\binom{2}{r_1}\binom{2}{r_2}(-1)^{r_1+r_2}\mathcal{L}_i(r_1\lambda_i(1) + r_2\lambda_i(2)) \\ &= 4\mathcal{L}_i(\lambda_i(1) + \lambda_i(2)) - 2\mathcal{L}_i(\lambda_i(1) + 2\lambda_i(2)) \\ &\quad - 2\mathcal{L}_i(2\lambda_i(1) + \lambda_i(2)) + \mathcal{L}_i(2\lambda_i(1) + 2\lambda_i(2)), \end{aligned}$$

which becomes

$$
\begin{aligned}
R_i(\lambda) \quad = \quad & \left(\frac{4\delta_i}{\delta_i+\lambda_i(1)+\lambda_i(2)}\right)-\left(\frac{2\delta_i}{\delta_i+\lambda_i(1)+\lambda_i(2)}\right) \\
& -\left(\frac{2\delta_i}{\delta_i+2\lambda_i(1)+\lambda_i(2)}\right)+\left(\frac{\delta_i}{\delta_i+2\lambda_i(1)+2\lambda_i(2)}\right)
\end{aligned}
$$

when phase durations are distributed exponentially with parameters δ_i *for* $i = 1, 2, 3$. *This further implies that*

$$
\widetilde{P}_S = \begin{bmatrix} 0.40R_1(\lambda) & 0.40R_1(\lambda) \\ 0.30R_2(\lambda) & 0.30R_2(\lambda) \end{bmatrix}
$$

and

$$
g = \begin{bmatrix} 0.2R_1(\lambda) \\ 0.4R_2(\lambda) \end{bmatrix}.
$$

Then, using the result (3.19), the reliability functions are

$$
P\{L=+\infty|Y_0=1\} = \left(\boldsymbol{I}-\widetilde{P}_S\right)^{-1} g(1) = \frac{R_1(\lambda)(R_2(\lambda)+2)}{10-4R_1(\lambda)-3R_2(\lambda)}
$$

$$
P\{L=+\infty|Y_0=2\} = \left(\boldsymbol{I}-\widetilde{P}_S\right)^{-1} g(2) = \frac{R_2(\lambda)(4-R_1(\lambda))}{10-4R_1(\lambda)-3R_2(\lambda)}.
$$

Series Connection of Standby Redundant Subsystems

For this structure, for a phase to be completed successfully, the number of failures during the phase must be less than the number of components in all subsystems. Let $n_i(k)$ denote the number of components in use in subsystem k during phase i. Suppose that $\lambda_i(k)$ is the failure rate of any one of the $n_i(k)$ components in subsystem k during phase i. Then, it is clear that

$$
\overline{p}_i(s) = \prod_{k\in\mathcal{J}_i} e^{-\lambda_i(k)s}\left(\sum_{r_k^i=0}^{n_i(k)-1}\frac{(\lambda_i(k)s)^{r_k^i}}{r_k^i!}\right). \tag{3.45}
$$

To calculate $\widetilde{P}$, the expected value of $\overline{p}_i(D_i)$ has to be determined. Since $E[\overline{p}_i(D_i)] = R_i(\lambda)$ in (3.31) for every i, the explicit formula (3.30) can be used to determine the matrix $\widetilde{P}$ via (3.39). The following example shows how.

Example 3.10 *Consider Example 3.4 and suppose that the transition matrix of the mission process is (3.42). Then, we have the same* $\widetilde{P}$ *and* $\widetilde{P}^{(2)}$ *as in Example 3.8 where*

$$\begin{aligned} R_i(\lambda) &= \sum_{r_1^i=0}^{2}\sum_{r_2^i=0}^{2} \frac{\lambda_i(1)^{r_1^i}\lambda_i(2)^{r_2^i}}{r_1^i!r_2^i!}\mathcal{L}_i^{r_1^i+r_2^i}(\lambda_i(1)+\lambda_i(2)) \\ &= \sum_{r_1^i=0}^{2}\sum_{r_2^i=0}^{2} \frac{(r_1^i+r_2^i)!}{r_1^i!r_2^i!}\left(\frac{\delta_i}{\lambda_i(1)+\lambda_i(2)+\delta_i}\right) \\ &\quad\times\left(\frac{\lambda_1(i)}{\lambda_i(1)+\lambda_i(2)+\delta_i}\right)^{r_1^i}\left(\frac{\lambda_2(i)}{\lambda_i(1)+\lambda_i(2)+\delta_i}\right)^{r_2^i} \end{aligned}$$

for $i = 1, 2$.

Example 3.11 *Consider Example 3.4 and suppose that the transition matrix of the mission process is (3.43). Then,* $\widetilde{P}$, $\widetilde{P}_S$, g *and the reliability functions will be the same as in Example 3.9 where*

$$\begin{aligned} R_i(\lambda) &= \sum_{r_1^i=0}^{2}\sum_{r_2^i=0}^{2} \frac{\lambda_i(1)^{r_1^i}\lambda_i(2)^{r_2^i}}{r_1^i!r_2^i!}\mathcal{L}_i^{r_1^i+r_2^i}(\lambda_i(1)+\lambda_i(2)) \\ &= \sum_{r_1^i=0}^{2}\sum_{r_2^i=0}^{2} \frac{(r_1^i+r_2^i)!}{r_1^i!r_2^i!}\left(\frac{\delta_i}{\lambda_i(1)+\lambda_i(2)+\delta_i}\right) \\ &\quad\times\left(\frac{\lambda_i(1)}{\lambda_i(1)+\lambda_i(2)+\delta_i}\right)^{r_1^i}\left(\frac{\lambda_i(2)}{\lambda_i(1)+\lambda_i(2)+\delta_i}\right)^{r_2^i} \end{aligned}$$

for $i = 1, 2, 3$.

Series Connection of k-out-of-n Subsystems

Suppose that each subsystem $j \in \mathcal{J}_i$ has $n_i(j)$ number of active parallel components and at least $k_i(j)$ of these components must be in working condition to make the subsystem function during phase i. Then,

$$\overline{p}_i(s) = \prod_{j\in\mathcal{J}_i}\sum_{r_j^i=k_i(j)}^{n_i(j)}\binom{n_i(j)}{r_j^i}\left(e^{-\lambda_i(j)s}\right)^{r_j^i}\left(1-e^{-\lambda_i(j)s}\right)^{n_i(j)-r_j^i} \tag{3.46}$$

and, using (3.46) and (3.39),

$$\widetilde{P}(i,l) = P(i,l)\,E\left[\prod_{j\in\mathcal{J}_i}\sum_{r_j^i=k_i(j)}^{n_i(j)}\binom{n_i(j)}{r_j^i}\left(e^{-\lambda_i(j)D_i}\right)^{r_j^i}\left(1-e^{-\lambda_i(j)D_i}\right)^{n_i(j)-r_j^i}\right]$$

for all $i, l \in E$. To calculate $\widetilde{P}$, the expected value of $\overline{p}_i(D_i)$ has to be determined. Since $E[\overline{p}_i(D_i)] = R_i(\lambda)$ in (3.32) for every i, the explicit formulas (3.35)-(3.36) can be used to compute the entries of the matrix $\widetilde{P}$ via (3.39). The following example shows how.

By (3.35) and (3.39), $\widetilde{P}(i, l)$ is a multiple of the Laplace transform of a random variable for all $i, l \in E$. Using the same steps as in the previous subsection for series systems, it can be shown that $P_i\{L > T_n\}$ is a linear combination of product of nonnegative, nonincreasing and convex functions.

Example 3.12 *Consider Example 3.5 and suppose that the transition matrix of the mission process is (3.42). Then, we have the same $\widetilde{P}$ and $\widetilde{P}^2$; hence $P_1\{L > T_2\}$ and $P_2\{L > T_2\}$ are as in Example 3.8 where*

$$\begin{aligned} R_i(\lambda) &= \sum_{r_1^i=2}^{3} \sum_{r_2^i=2}^{3} \sum_{s_1^i=r_1^i}^{3} \sum_{s_2^i=r_2^i}^{3} \binom{3}{s_1^i} \binom{s_1^i}{r_1^i} \binom{3}{s_2^i} \binom{s_2^i}{r_2^i} (-1)^{s_1^i+s_2^i-r_1^i-r_2^i} \\ &\quad \times \left(\frac{\delta_i}{\delta_i + s_1^i \lambda_i(1) + s_2^i \lambda_i(2)} \right) \end{aligned}$$

for $i = 1, 2$.

Example 3.13 *Suppose that there are two subsystems with three parallel components which will perform three tasks where the third one is critical. At least two of three components must work to make the subsystems function. Then, $E = \{1, 2, 3\}$, $\mathcal{J}_i = \{1, 2\}$ for all $i \in E$, $l_1 = l_2 = l_3 = 2$, $i_S = 3$, and $n_i(j) = 3$, $k_i(j) = 2$ for all $j = 1, 2$, $i = 1, 2, 3$. Assume that the transition matrix of the sequence of the phases is (3.43). Then, we have the same $\widetilde{P}$, $\widetilde{P}_S$, and g; hence $P\{L = +\infty | Y_0 = 1\}$ and $P\{L = +\infty | Y_0 = 2\}$ are as in Example 3.9 where*

$$R_i(\lambda) = \sum_{r_1^i=2}^{3} \sum_{r_2^i=2}^{3} \sum_{s_1^i=r_1^i}^{3} \sum_{s_2^i=r_2^i}^{3} \binom{3}{s_1^i} \binom{s_1^i}{r_1^i} \binom{3}{s_2^i} \binom{s_2^i}{r_2^i} (-1)^{s_1^i+s_2^i-r_1^i-r_2^i} \left(\frac{\delta_i}{\delta_i + s_1^i \lambda_{i1} + s_2^i \lambda_{i2}} \right)$$

for $i = 1, 2, 3$.

3.1.3 Phase Reliability

For a complex system, a given critical phase may be more important than the others due to the overall objective of the mission. Therefore, the probability that this phase will be completed in a fixed time period is an important measure to represent the reliability of the system. For instance, consider the NASA's Mars Exploration Rover Mission example given in Section 2.1 where the mission consists of phases like vehicle launch; cruise; approach; entry, descent and landing to Mars; rover egress; and a number of surface operations that involve scientific data collection and transmission to earth. Since one of the main aims of the whole mission is determining past water activity on the surface, scientific investigations and transmission of data are very critical towards this goal for the success of the mission. Therefore, reliability of such a critical phase is of extreme importance.

Suppose that one is interested in the successful completion of a given critical phase j of the mission. Letting

$$U_j = \inf\{t \geq 0; Y_t \neq Y_{t-} = j\}$$

denote the first time that the mission process leaves state j, we are interested in determining the phase reliability function $P_i\{U_j \leq t, L > U_j\}$ for phase j.

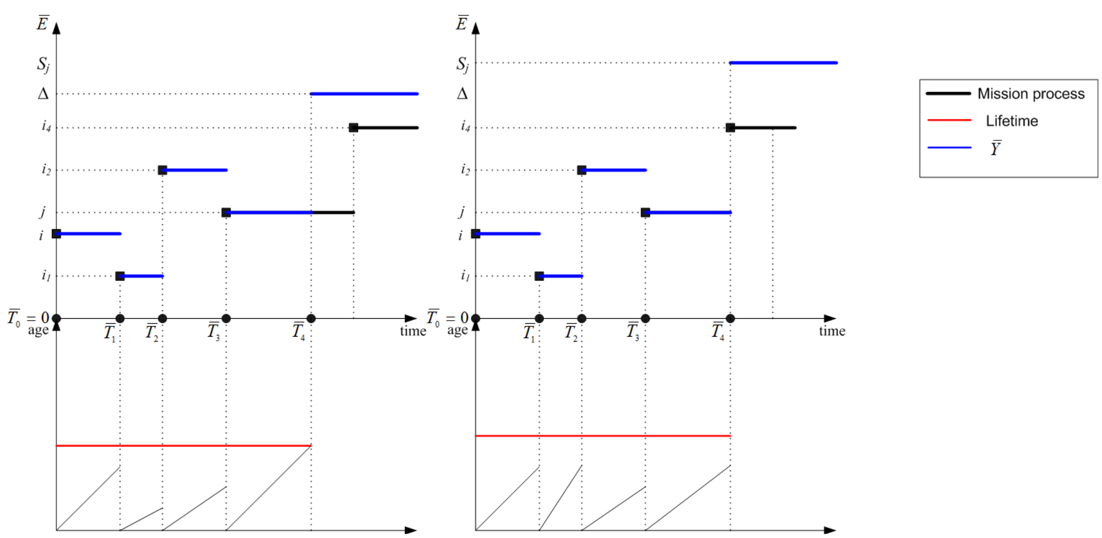

Figure 3.2: A typical representation of the structure of $\overline{Y}$.

We can now define another Markov renewal process $\left(\overline{X}, \overline{T}\right)$ as appropriate through its minimal semi-Markov process $\overline{Y}$. The new process is defined by

$$\overline{Y}_t = \begin{cases} Y_t & \text{if } t < \min\{L, U_j\} \\ \Delta & \text{if } L \leq \min\{t, U_j\} \\ S_j & \text{if } U_j \leq \min\{t, L\} \end{cases} \tag{3.47}$$

so that we now extend E to $\overline{E} = E \cup \{\Delta, S_j\}$. With this construction, we now let the semi-Markov mission process Y go to the absorbing state Δ as soon as the system fails before completing phase j, or to the absorbing state S_j as soon as phase j is completed without failure. Thus, if $\overline{X}_n = \Delta$ or $\overline{X}_n = S_j$, then $\overline{T}_{n+1} = +\infty$. The structure of $\overline{Y}$ is as described in Figure 3.2. The semi-Markov kernel of $\left(\overline{X}, \overline{T}\right)$ is

$$\overline{Q}\left(i,k,ds\right) = \begin{cases} \widetilde{Q}\left(i,k,ds\right) & \text{if } i \in E - \{j\}, k \in E \\ \widetilde{Q}\left(i,\Delta,ds\right) & \text{if } i \in E, k = \Delta \\ F_j\left(ds\right)\overline{p}_j\left(s\right) & \text{if } i = j \in E, k = S_j \\ 0 & \text{otherwise} \end{cases} \tag{3.48}$$

for $i,k \in \overline{E}$. Note that $\overline{Q}\left(S_j, j, t\right) = \overline{Q}\left(\Delta, j, t\right) = 0$ for all $t \geq 0$, $j \in \overline{E}$, and $\overline{P}\left(S_j, S_j\right) = \overline{P}\left(\Delta, \Delta\right) = 1$. If $i \in E - \{j\}$, and $k \in E$, then phase i is completed successfully before system failure and hence $\overline{Q}\left(i,k,ds\right) = \widetilde{Q}\left(i,k,ds\right)$. If $i \in E$ and $k = \Delta$, then $\overline{Q}\left(i,k,ds\right)$ represents the probability of failure during phase i in the vicinity of time s, which means that the duration of phase i is longer than s units of time and a failure will occur in the vicinity of time s. Therefore, this probability is equal to $\widetilde{Q}\left(i,\Delta,ds\right) = \overline{F}_i\left(s\right)p_i\left(ds\right)$. Moreover, $\overline{Q}\left(j,S_j,ds\right)$ represents the probability that the system will complete phase j in the vicinity of time s given that the system is in working condition at the beginning of phase j. This situation occurs if the duration of phase j is in the vicinity of time s and the system survives more than s units of time in these conditions; hence, $\overline{Q}\left(j,S_j,ds\right) = F_j\left(ds\right)\overline{p}_j\left(s\right)$. It is clear from the definition of the process $\left(\overline{X}, \overline{T}\right)$ that the states S_j and Δ are absorbing states. Therefore, if the process gets into states S_j or Δ, it remains there forever where S_j now represents the "success" state and Δ represents the "failure" state.

The transition matrix $\overline{P}$ of the Markov chain $\overline{X}$ is such that

$$\overline{P}\left(i,k\right) = \begin{cases} \widetilde{P}\left(i,k\right) & \text{if } i \in E - \{j\}, k \in E \\ \widetilde{P}\left(i,\Delta\right) & \text{if } i \in E, k = \Delta \\ \int_0^{+\infty} F_j\left(ds\right)\overline{P}_j\left(s\right) & \text{if } i = j \in E, k = S_j \\ 0 & \text{otherwise} \end{cases}$$

for $i,k \in \overline{E}$. Note, once more, that both the semi-Markov kernel $\overline{Q}$ and the corresponding transition matrix $\overline{P}$ are possibly defective on E and $\widetilde{E}$. We do assume that they are, in fact, defective to avoid trivialities in reliability analysis.

Using the Markov renewal process $\left(\overline{X}, \overline{T}\right)$, reliability of phase j is simply

$$P_i \left\{U_j \leq t, L > U_j\right\} = P_i \left\{\overline{Y}_t = S_j\right\} = \overline{R}\left(i, S_j, t\right) \tag{3.49}$$

where $\overline{R} = \sum_n \overline{Q}^n$ is the Markov renewal kernel corresponding to $\overline{Q}$. This follows trivially from Proposition (10.5.4) in Çınlar [59].

Note that $\overline{X}$ has two absorbing states and and all the other states are transient. If one wants to find the probability of ever completing phase j, this is equal to the probability of being absorbed at state S_j before Δ. Using first step analysis, this probability can be easily calculated.

3.1.3.1 Series System with Markovian Missions and Exponential Lifetimes

In this section, we identify the computational simplification provided by Markovian missions in order to compute the reliability measures discussed before. We further suppose that the system is a series one with m components where component k has an exponentially distributed lifetime with parameter $\lambda_i\left(k\right)$ during phase i. The mission process $\{Y_t; t \geq 0\}$ is a Markov process with transition rate vector δ, transition matrix P, and infinitesimal generator

$$G(i,j) = \begin{cases} -\delta_i & \text{if } j = i \\ \delta_i P(i,j) & \text{if } j \neq i. \end{cases}$$

In this special case, it is clear that

$$Q\left(i, j, s\right) = \left(1 - e^{-\delta_i s}\right) P\left(i, j\right) \tag{3.50}$$

so that

$$F_i(t) = 1 - e^{-\delta_i t}$$

and

$$\overline{p}_i\left(s\right) = e^{-\overline{\lambda}_i s} \tag{3.51}$$

where $\overline{\lambda}_i = \sum_{j=1}^{m} \lambda_i\left(j\right)$ since the minimum of exponentially distributed random variables is also exponentially distributed.

For the mission reliability analysis, note that (3.50) and (3.51) gives

$$\begin{aligned}\widetilde{P}(i,j) &= \int_0^{+\infty} \overline{p}_i(s)\, Q(i,j,ds) = \int_0^{+\infty} \overline{p}_i(s)\, \delta_i e^{-\delta_i s} P(i,j)\, ds \\ &= \left(\frac{\delta_i}{\delta_i + \overline{\lambda}_i}\right) P(i,j) \end{aligned} \tag{3.52}$$

for $i, j \neq \Delta$, and

$$\widetilde{P}(i,\Delta) = \int_0^{+\infty} \overline{F}_i(s)\, p_i(ds) = \int_0^{+\infty} e^{-\delta_i s} \overline{\lambda}_i e^{-\overline{\lambda}_i s} ds = \frac{\overline{\lambda}_i}{\delta_i + \overline{\lambda}_i} \tag{3.53}$$

in (3.5) and (3.6) . The explicit form of $\widetilde{P}$ can be used in (3.16) and (3.1.2) to compute mission reliability by simple linear algebra.

For the phase reliability analysis, note that $\overline{Y} = \left\{\overline{Y}_t; t \geq 0\right\}$ is also a Markov process with the infinitesimal generator

$$\overline{G}(i,k) = \begin{cases} \delta_i P(i,k) & \text{if } i,k \in E, i \neq j \text{ or } k \\ \overline{\lambda}_i & \text{if } i \in E, k = \Delta \\ \delta_i & \text{if } i = j \in E, k = S_j \\ -\left(\delta_i + \overline{\lambda}_i\right) & \text{if } i = k \in E \\ 0 & \text{otherwise} \end{cases} \tag{3.54}$$

for $i, k \in \overline{E}$. Then, the phase reliability has the matrix exponential form

$$\begin{aligned} P_i\{U_j \leq t, L > U_j\} &= P_i\left\{\overline{Y}_t = S_j\right\} = \exp\left(-\overline{G}t\right)(i, S_j) \\ &= \sum_{n=0}^{+\infty} \frac{t^n}{n!} \overline{G}^n (i, S_j) = \lim_{n \to +\infty} \left(\boldsymbol{I} + \frac{\overline{G}t}{n}\right)^n (i, S_j). \end{aligned}$$

There are various methods to compute the matrix exponential exactly or approximately and the reader is referred to Moler and Loan [60] for these computational issues. It is also well-known that these probabilities can be estimated by solving Kolmogorov backward equations (or forward equations).

Example 3.14 *Suppose that there are three phases and phase* 2 *is the critical one. Using*

Kolmogorov backward equations and (3.54), we have

$$\begin{aligned}
\overline{P}'_{1S_2}(t) &= \overline{G}(1,2)\,\overline{P}_{2S_2}(t)+\overline{G}(1,3)\,\overline{P}_{3S_2}(t)+\overline{G}(1,\Delta)\,\overline{P}_{\Delta S_2}(t)\\
&\quad+\overline{G}(1,S_2)\,\overline{P}_{S_2S_2}(t)-\overline{\delta}_1\overline{P}_{1S_2}(t)\\
\overline{P}'_{2S_2}(t) &= \overline{G}(2,1)\,\overline{P}_{1S_2}(t)+\overline{G}(2,3)\,\overline{P}_{3S_2}(t)+\overline{G}(2,\Delta)\,\overline{P}_{\Delta S_2}(t)\\
&\quad+\overline{G}(2,S_2)\,\overline{P}_{S_2S_2}(t)-\overline{\delta}_2\overline{P}_{2S_2}(t) \qquad (3.55)\\
\overline{P}'_{3S_2}(t) &= \overline{G}(3,1)\,\overline{P}_{1S_2}(t)+\overline{G}(3,2)\,\overline{P}_{2S_2}(t)+\overline{G}(3,\Delta)\,\overline{P}_{\Delta S_2}(t)\\
&\quad+\overline{G}(3,S_2)\,\overline{P}_{S_2S_2}(t)-\overline{\delta}_3\overline{P}_{3S_2}(t)
\end{aligned}$$

where $\overline{P}_{ik}(t)=P_i\{\overline{Y}_t=k\}$ *and* $\overline{\delta}_i=(\delta_i+\overline{\lambda}_i)$. *By the definition of* $\overline{Y}$, $\overline{G}(2,1)=\overline{G}(2,3)=\overline{G}(1,S_2)=\overline{G}(3,S_2)=\overline{P}_{\Delta S_2}(t)=0$ *and* $\overline{P}_{S_2S_2}(t)=1$. *Therefore, (3.55) simplifies to*

$$\begin{aligned}
\overline{P}'_{1S_2}(t) &= \overline{G}(1,2)\,\overline{P}_{2S_2}(t)+\overline{G}(1,3)\,\overline{P}_{3S_2}(t)-\overline{\delta}_1\overline{P}_{1S_2}(t)\\
\overline{P}'_{2S_2}(t) &= \overline{G}(2,S_2)-\overline{\delta}_2\overline{P}_{2S_2}(t) \qquad (3.56)\\
\overline{P}'_{3S_2}(t) &= \overline{G}(3,1)\,\overline{P}_{1S_2}(t)+\overline{G}(3,2)\,\overline{P}_{2S_2}(t)-\overline{\delta}_3\overline{P}_{3S_2}(t).
\end{aligned}$$

The second equation in (3.56) is a differential equation in the form of

$$y'+P(x)\,y=Q(x) \qquad (3.57)$$

where $y(t)=\overline{P}_{2S_2}(t), P(x)=\overline{\delta}_2,$ *and* $Q(x)=\overline{G}(2,S_2)$. *It is known that the solution of (3.57) is*

$$y=\frac{1}{v(t)}\int v(t)\,Q(t)\,dt,$$

where

$$v(t)=e^{\int P(t)dt}.$$

Therefore, in our case

$$v(t)=e^{\overline{\delta}_2 t}$$

and, hence,

$$\overline{P}_{2S_2}(t)=e^{-\overline{\delta}_2 t}\frac{\overline{G}(2,S_2)}{\overline{\delta}_2}\left(e^{\overline{\delta}_2 t}+C\right).$$

Since $\overline{P}_{2S_2}(0)=0$, $C=-1$ *and this implies that*

$$\overline{P}_{2S_2}(t)=\frac{\overline{G}(2,S_2)}{\overline{\delta}_2}\left(1-e^{-\overline{\delta}_2 t}\right). \qquad (3.58)$$

By using (3.58), (3.56) simplifies to

$$\begin{aligned} \overline{P}'_{1S_2}(t) &= -\overline{\delta}_1 \overline{P}_{1S_2}(t) + \overline{G}(1,3)\,\overline{P}_{3S_2}(t) + \overline{G}(1,2)\,\overline{P}_{2S_2}(t) \\ \overline{P}'_{3S_2}(t) &= \overline{P}_{1S_2}(t) - \overline{\delta}_3 \overline{P}_{3S_2}(t) + \overline{G}(3,2)\,\overline{P}_{2S_2}(t). \end{aligned} \tag{3.59}$$

We will use the standard procedure for solving such systems of differential equations. Let define

$$A = \begin{bmatrix} -\overline{\delta}_1 & \overline{G}(1,3) \\ \overline{G}(3,1) & -\overline{\delta}_3 \end{bmatrix}.$$

The eigenvalues of A are

$$\begin{aligned} r_1 &= -\frac{\overline{\delta}_1}{2} - \frac{\overline{\delta}_3}{2} + \frac{1}{2}\sqrt{\left(\overline{\delta}_1 - \overline{\delta}_3\right)^2 + 4\overline{G}(1,3)\,\overline{G}(3,1)} \\ r_2 &= -\frac{\overline{\delta}_1}{2} - \frac{\overline{\delta}_3}{2} - \frac{1}{2}\sqrt{\left(\overline{\delta}_1 - \overline{\delta}_3\right)^2 + 4\overline{G}(1,3)\,\overline{G}(3,1)} \end{aligned}$$

and it is clear that $r_1 \neq r_2$. By solving the equation $(A - r\boldsymbol{I})\,u = 0$ where r is a scalar and $u = [\ u_1 \quad u_2\ ]^T$, we have

$$u_1 = \frac{\overline{G}(1,3)}{\overline{\delta}_1 + r} u_2 = \frac{\overline{\delta}_3 + r}{\overline{G}(3,1)} u_2.$$

If $r = r_1$ and $r = r_2$, it can be shown that

$$\frac{\overline{G}(1,3)}{\overline{\delta}_1 + r} = \frac{\overline{\delta}_3 + r}{\overline{G}(3,1)}.$$

Then, choose

$$u^{(1)} = \begin{bmatrix} \frac{\overline{\delta}_3 + r_1}{\overline{G}(3,1)} \\ 1 \end{bmatrix} \text{ and } u^{(2)} = \begin{bmatrix} \frac{\overline{\delta}_3 + r_2}{\overline{G}(3,1)} \\ 1 \end{bmatrix}.$$

Without right hand side, we have the special solution

$$\begin{aligned} \overline{P}_{1S_2}(t) &= c_1 \frac{\overline{\delta}_3 + r_1}{\overline{G}(3,1)} e^{r_1 t} + c_2 \frac{\overline{\delta}_3 + r_2}{\overline{G}(3,1)} e^{r_2 t} \\ \overline{P}_{2S_2}(t) &= c_1 e^{r_1 t} + c_2 e^{r_2 t}. \end{aligned}$$

To find a solution for the right hand side, we need to solve

$$\begin{aligned} K'_1(t) \frac{\overline{\delta}_3 + r_1}{\overline{G}(3,1)} e^{r_1 t} + K'_2(t) \frac{\overline{\delta}_3 + r_2}{\overline{G}(3,1)} e^{r_2 t} &= \overline{G}(1,2)\,\overline{P}_{2S_2}(t) \\ K'_1(t)\, e^{r_1 t} + K'_2(t)\, e^{r_2 t} &= \overline{G}(3,2)\,\overline{P}_{2S_2}(t) \end{aligned}$$

in terms of K_1' *and* K_2'. *The solution is*

$$\begin{aligned} K_1'(t) &= \frac{\overline{P}_{2S_2}(t)\left(\overline{G}(1,2)\overline{G}(3,1)-\overline{G}(3,2)\overline{\delta}_3-\overline{G}(3,2)r_2\right)e^{-r_1 t}}{r_1-r_2} \\ K_2'(t) &= \frac{\overline{P}_{2S_2}(t)\left(-\overline{G}(1,2)\overline{G}(3,1)+\overline{G}(3,2)\overline{\delta}_3+\overline{G}(3,2)r_1\right)e^{-r_2 t}}{r_1-r_2} \end{aligned}$$

and this implies that

$$\begin{aligned} K_1(t) &= \frac{\overline{G}(2,S_2)\left(\overline{G}(1,2)\overline{G}(3,1)-\overline{G}(3,2)\overline{\delta}_3-\overline{G}(3,2)r_2\right)}{\overline{\delta}_2(r_1-r_2)}\left(\frac{e^{-\left(r_1+\overline{\delta}_2\right)t}}{r_1+\overline{\delta}_2}-\frac{e^{-r_1 t}}{r_1}\right) \\ K_2(t) &= \frac{\overline{G}(2,S_2)\left(-\overline{G}(1,2)\overline{G}(3,1)+\overline{G}(3,2)\overline{\delta}_3+\overline{G}(3,2)r_1\right)}{\overline{\delta}_2(r_1-r_2)}\left(\frac{e^{-\left(r_2+\overline{\delta}_2\right)t}}{r_2+\overline{\delta}_2}-\frac{e^{-r_2 t}}{r_2}\right). \end{aligned}$$

Therefore,

$$\begin{aligned} \overline{P}_{1S_2}(t) &= c_1\frac{\overline{\delta}_3+r_1}{\overline{G}(3,1)}e^{r_1 t}+c_2\frac{\overline{\delta}_3+r_2}{\overline{G}(3,1)}e^{r_2 t}+K_1(t)\frac{\overline{\delta}_3+r_1}{\overline{G}(3,1)}e^{r_1 t}+K_2(t)\frac{\overline{\delta}_3+r_2}{\overline{G}(3,1)}e^{r_2 t} \\ \overline{P}_{2S_2}(t) &= c_1 e^{r_1 t}+c_2 e^{r_2 t}+K_1(t)e^{r_1 t}+K_2(t)e^{r_2 t}. \end{aligned}$$

Since $\overline{P}_{1S_2}(0)=\overline{P}_{2S_2}(0)=0$,

$$\begin{aligned} c_1 &= \frac{-\overline{G}(2,S_2)\left(-\overline{G}(1,2)\overline{G}(3,1)+\overline{G}(3,2)\overline{\delta}_3+\overline{G}(3,2)r_2\right)}{(r_1-r_2)r_1\left(r_1+\overline{\delta}_2\right)} \\ c_2 &= \frac{\overline{G}(2,S_2)\left(-\overline{G}(1,2)\overline{G}(3,1)+\overline{G}(3,2)\overline{\delta}_3+\overline{G}(3,2)r_1\right)}{(r_1-r_2)r_2\left(r_2+\overline{\delta}_2\right)}. \end{aligned}$$

We also have $\overline{G}(2,S_2)=\delta_2$, $\overline{G}(1,2)=\delta_1 P(1,2)$, $\overline{G}(1,3)=\delta_1 P(1,3)$, $\overline{G}(3,1)=\delta_3 P(3,1)$, $\overline{G}(3,2)=\delta_3 P(3,2)$. *Then, using (3.54),*

$$\begin{aligned} \overline{P}_{1S_2}(t) &= c_1\frac{\left(\delta_3+\overline{\lambda}_3\right)+r_1}{\delta_3 P(3,1)}e^{r_1 t}+c_2\frac{\left(\delta_3+\overline{\lambda}_3\right)+r_2}{\delta_3 P(3,1)}e^{r_2 t} \\ &\quad +K_1(t)\frac{\left(\delta_3+\overline{\lambda}_3\right)+r_1}{\delta_3 P(3,1)}e^{r_1 t}+K_2(t)\frac{\left(\delta_3+\overline{\lambda}_3\right)+r_2}{\delta_3 P(3,1)}e^{r_2 t} \\ \overline{P}_{2S_2}(t) &= c_1 e^{r_1 t}+c_2 e^{r_2 t}+K_1(t)e^{r_1 t}+K_2(t)e^{r_2 t} \end{aligned}$$

where

$$\begin{aligned} K_1(t) &= \frac{\delta_2\left(\delta_1 P(1,2)\delta_3 P(3,1)-\delta_3 P(3,2)\left(\delta_3+\overline{\lambda}(3)\right)-\delta_3 P(3,2)r_2\right)}{\left(\delta_2+\overline{\lambda}_2\right)(r_1-r_2)} \\ &\quad \times\left(\frac{e^{-\left(r_1+\delta_2+\overline{\lambda}_2\right)t}}{r_1+\delta_2+\overline{\lambda}_2}-\frac{e^{-r_1 t}}{r_1}\right) \\ K_2(t) &= \frac{\delta_2\left(\delta_1 P(1,2)\delta_3 P(3,1)-\delta_3 P(3,2)\left(\delta_3+\overline{\lambda}_3\right)-\delta_3 P(3,2)r_1\right)}{\left(\delta_2+\overline{\lambda}_2\right)(r_1-r_2)} \\ &\quad \times\left(\frac{e^{-\left(r_2+\delta_2+\overline{\lambda}_2\right)t}}{r_2+\delta_2+\overline{\lambda}_2}-\frac{e^{-r_2 t}}{r_2}\right), \end{aligned}$$

$$\begin{aligned}
c_1 &= \frac{-\delta_2\left(-\delta_1 P(1,2)\,\delta_3 P(3,1) + \delta_3 P(3,2)\left(\delta_3 + \overline{\lambda}_3\right) + \delta_3 P(3,2)\, r_2\right)}{(r_1 - r_2)\, r_1 \left(r_1 + \delta_2 + \overline{\lambda}_2\right)} \\
c_2 &= \frac{\delta_2\left(-\delta_1 P(1,2)\,\delta_3 P(3,1) + \delta_3 P(3,2)\left(\delta_3 + \overline{\lambda}_3\right) + \delta_3 P(3,2)\, r_1\right)}{(r_1 - r_2)\, r_2 \left(r_2 + \delta_2 + \overline{\lambda}_2\right)},
\end{aligned}$$

and

$$\begin{aligned}
r_1 &= -\frac{\delta_1 + \overline{\lambda}_1}{2} - \frac{\delta_3 + \overline{\lambda}_3}{2} \\
&\quad + \frac{1}{2}\sqrt{\left(\delta_1 + \overline{\lambda}_1 - \delta_3 - \overline{\lambda}_3\right)^2 + 4\delta_1 P(1,2)\,\delta_3 P(3,1)} \\
r_2 &= -\frac{\delta_1 + \overline{\lambda}_1}{2} - \frac{\delta_3 + \overline{\lambda}_3}{2} \\
&\quad - \frac{1}{2}\sqrt{\left(\delta_1 + \overline{\lambda}_1 - \delta_3 - \overline{\lambda}_3\right)^2 + 4\delta_1 P(1,2)\,\delta_3 P(3,1)}.
\end{aligned}$$

3.2 Models with No Repair

In this section, we remove the simplifying assumption of the previous section that the system is repaired maximally so that it is brand new at the beginning of each phase. There is no repair now and, hence, the system will get deteriorate or get older in time. Since the lifetimes of the components have general distributions, the concept of "aging" comes into consideration. For this purpose, we will use the "intrinsic aging" model introduced by Çınlar and Özekici [17] (For details of the model and the related notation, see Section 2.3). The analysis will be presented for a general and arbitrary reliability system. In this regard, we extend Çınlar and Özekici [17] who considered the system reliability of a series system where the whole system fails as soon as any component fails.

3.2.1 Reliability of a Series System

3.2.1.1 Mission Reliability

Suppose that we have a series system. We focus on computing mission reliability involving the first n phases of the mission. We first find the probability that the first phase will be completed by conditioning on the next phase so that

$$P_{ia}\{L > T_1\} = \sum_{j \in E} P_{ia}\{L > T_1, X_1 = j\} = \sum_{j \in E} P_{ia}\{L > T_1 | X_1 = j\}\, P(i,j). \tag{3.60}$$

Note that

$$P_{ia}\left\{L > T_1 | X_1 = j\right\} = \int\limits_{\mathcal{F}\times\mathbb{R}_+} P_{ia}\left\{L > T_1, T_1 \in ds, B_1 \in db | X_1 = j\right\}$$
$$= \int\limits_{\mathcal{F}\times\mathbb{R}_+} e^{-\mathbf{1}^T(b-a)} G\left(i, j, ds\right) H\left(i, a, s; db\right). \quad (3.61)$$

Combining equations (3.60) and (3.61), we obtain

$$P_{ia}\left\{L > T_1\right\} = \sum_{j\in E} \int\limits_{\mathcal{F}\times\mathbb{R}_+} e^{-\mathbf{1}^T(b-a)} Q\left(i, j, ds\right) H\left(i, a, s; db\right). \quad (3.62)$$

Similarly, we can obtain a recursive relationship

$$P_{ia}\left\{L > T_{n+1}\right\} = \sum_{j\in E} P_{ia}\left\{L > T_{n+1} | X_1 = j\right\} P\left(i, j\right)$$
$$= \sum_{j\in E} \int\limits_{\mathcal{F}\times\mathbb{R}_+} P_{ia}\left\{L > T_{n+1}, T_1 \in ds, B_1 \in db | X_1 = j\right\} P\left(i, j\right)$$
$$= \sum_{j\in E} \int\limits_{\mathcal{F}\times\mathbb{R}_+} e^{-\mathbf{1}^T(b-a)} Q\left(i, j, ds\right) H\left(i, a, s; db\right) P_{jb}\left\{L > T_n\right\} \quad (3.63)$$

for $n \geq 0$. Let $\widehat{Q}_S\left(i, a; j, db; ds\right) = Q\left(i, j, ds\right) H\left(i, a, s; db\right) e^{-\mathbf{1}^T(b-a)}$, and define

$$\widehat{P}_S\left(i, a; j, db\right) = \widehat{Q}_S\left(i, a; j, db; +\infty\right) = P_{ia}\left\{X_1 = j, B_1 \in db\right\}$$

and

$$\widehat{P}_S^{n+1}\left(i, a; j, db\right) = \sum_{k\in E} \int_{\mathcal{F}} \widehat{P}_S^{n}\left(i, a; k, dc\right) \widehat{P}_S\left(k, c; j, db\right) = P_{ia}\left\{X_n = j, B_n \in db\right\} \quad (3.64)$$

for all $n \geq 1$.

Then, it follows from (3.62) that

$$P_{ia}\left\{L > T_1\right\} = \sum_{j\in E} \int\limits_{\mathcal{F}\times\mathbb{R}_+} \widehat{Q}_S\left(i, a; j, db; ds\right) = \sum_{j\in E} \int_{\mathcal{F}} \widehat{Q}_S\left(i, a; j, db; +\infty\right) \quad (3.65)$$
$$= \sum_{j\in E} \int_{\mathcal{F}} \widehat{P}_S\left(i, a; j, db\right). \quad (3.66)$$

Now, using induction, we will show that

$$P_{ia}\left\{L > T_n\right\} = \sum_{j\in E} \int_{\mathcal{F}} \widehat{P}_S^{n}\left(i, a; j, db\right) \quad (3.67)$$

for all $n \geq 1$. Note that $P_{ia}\{L > T_0\} = 1$ trivially. We have already shown that (3.67) holds for $n = 1$. Suppose that

$$P_{ia}\{L > T_k\} = \sum_{j \in E} \int_{\mathcal{F}} \widehat{P}_S^k (i, a; j, db)$$

for all $k \leq n$. Then, using (3.63) and (3.64),

$$\begin{aligned} P_{ia}\{L > T_{n+1}\} &= \sum_{j \in E} \int_{\mathcal{F} \times \mathbb{R}_+} \widehat{Q}_S (i, a; j, db; ds) P_{jb} \{L > T_n\} \\ &= \sum_{j \in E} \int_{\mathcal{F}} \widehat{P}_S (i, a; j, db) \sum_{k \in E} \int_{\mathcal{F}} \widehat{P}_S^n (j, b; k, dc) \qquad (3.68) \\ &= \sum_{j \in E} \int_{\mathcal{F}} \widehat{P}_S^{n+1} (i, a; j, db) . \qquad (3.69) \end{aligned}$$

3.2.1.2 Phase Reliability

Suppose that there is a critical phase j and we are interested in computing the probability $P_{ia}\{U_j \leq t, L > U_j\}$ that phase j will be completed successfully by time t. Using a similar approach as in Section 3.1.3, we note that the process $\overline{Y}$ defined by equation (3.47) is a semi-regenerative process. In this case, the semi-Markov kernel corresponding to the Markov renewal process $((\overline{X}, B), \overline{T})$ is given by

$$\overline{Q}_S(i, a; k, db; ds) = \begin{cases} Q(i, k, ds) H(i, a, s; db) e^{-\mathbf{1}^T (b-a)} & \text{if } i \in E - \{j\}, \ k \in E \\ \overline{F}_i (s) H(i, a, s; db) U_{ia}(ds) & \text{if } i \in E, \ k = \Delta \\ F_j (ds) H(i, a, s; db) e^{-\mathbf{1}^T (b-a)} & \text{if } i = j, \ k = S_j \\ 0 & \text{otherwise} \end{cases} \qquad (3.70)$$

for all $i, k \in \overline{E}$, and $a, b \in \mathcal{F}$ where

$$U_{ia}(s) = 1 - e^{-\mathbf{1}^T (h(i,a,s) - a)}. \qquad (3.71)$$

Note that $\overline{Q}_S (S_j, a; S_j, db; t) = \overline{Q}_S (\Delta, a; \Delta, db; t) = 0$ trivially for all $t \in \mathbb{R}_+$, $a, b \in \mathcal{F}$ and $\overline{P}_S (S_j, a; S_j, da) = \overline{P}_S (\Delta, a; \Delta, da) = 1$ for all $a \in \mathcal{F}$. It is clear that $((\overline{X}, B), \overline{T})$ is dumped to absorbing states (Δ, a) and (S_j, a) when the system fails and completes the critical phase successfully with age $a \in \mathcal{F}$ respectively. Using Markov renewal arguments,

$$\begin{aligned}
f(i,a,t) &= P_{ia}\left\{\overline{Y}_t = k, A_t \in db\right\} \\
&= P_{ia}\left\{\overline{Y}_t = k, A_t \in db, \overline{T}_1 > t\right\} + P_{ia}\left\{\overline{Y}_t = k, A_t \in db, \overline{T}_1 \le t\right\} \qquad (3.72)\\
&= I_{\{k=i\}} I_{\{b=a\}} q\left(i,a,t\right) + \sum_{l\in\overline{E}} \int_{\mathcal{F}\times[0,t]} \overline{Q}_S\left(i,a;l,dc;ds\right) P_{lc}\left\{\overline{Y}_{t-s} = k, A_{t-s} \in db\right\} \\
&= I_{\{k=i\}} I_{\{b=a\}} q\left(i,a,t\right) + \sum_{l\in\overline{E}} \int_{\mathcal{F}\times[0,t]} \overline{Q}_S\left(i,a;l,dc;ds\right) f\left(l,c,t-s\right) \qquad (3.73)
\end{aligned}$$

where

$$q\left(i,a,t\right) = P_{ia}\left\{\overline{T}_1 > t\right\} = 1 - \sum_{l\in\overline{E}} \int_{\mathcal{F}} \overline{Q}_S(i,a;l,db;t). \qquad (3.74)$$

We have a Markov renewal equation $f = g + \overline{Q}_S * f$ with

$$g\left(i,a,t\right) = I_{\{k=i\}} I_{\{b=a\}} q\left(i,a,t\right). \qquad (3.75)$$

Following Proposition A.2 in Çınlar and Özekici [17], (3.73) has a unique solution

$$\begin{aligned}
f(i,a,t) &= \sum_{l\in\overline{E}} \int_{\mathcal{F}\times[0,t]} \overline{R}_S\left(i,a;l,dc;ds\right) g\left(l,c,t-s\right) \\
&= \int_{[0,t]} \overline{R}_S\left(i,a;k,db;ds\right) q\left(k,b,t-s\right) \qquad (3.76)
\end{aligned}$$

where $\overline{R}_S = \sum_n \overline{Q}_S^n$ is the Markov renewal kernel corresponding to $\overline{Q}_S$.

Then, using (3.76),

$$\begin{aligned}
P_{ia}\left\{U_j \le t, L > U_j\right\} &= \int_{\mathcal{F}} P_{ia}\left\{\overline{Y}_t = S_j, A_t \in db\right\} \\
&= \int_{\mathcal{F}\times[0,t]} \overline{R}_S\left(i,a;S_j,db;ds\right) q\left(S_j,b,t-s\right) \qquad (3.77)\\
&= \overline{R}_S\left(i,a;S_j,\mathcal{F};t\right). \qquad (3.78)
\end{aligned}$$

3.2.2 Reliability of General Systems

In this section, we discuss the reliability of a more general system with structure function ψ_i during phase i by extending the work in the previous section.

We can determine the conditional lifetime distribution of the system during any phase using (2.12). Let $v_{ia}\left(s\right) = P\left\{L \le s | Y = i, A_0 = a\right\}$ be the probability that the system will work at most s units of time during phase i if the initial system age is a. Then,

$$v_{ia}(s) = 1 - \overline{v}_{ia}(s) = 1 - \int_{\mathcal{F}} \overline{p}_{ia}(s, db). \tag{3.79}$$

3.2.2.1 System Reliability

Using a Markov renewal argument for the system reliability function $f(i, a, t) = P_{ia}\{L > t\}$, we can write

$$\begin{aligned} f(i, a, t) &= P_{ia}\{L > t, T_1 > t\} + P_{ia}\{L > t, T_1 \leq t\} \\ &= \overline{v}_{ia}(t)\overline{F}_i(t) + \sum_{j \in E} \int_{\mathcal{F} \times [0,t]} Q(i, j, ds)\overline{p}_{ia}(s, db) P_{jb}\{L > t - s\} \\ &= \overline{v}_{ia}(t)\overline{F}_i(t) + \sum_{j \in E} \int_{\mathcal{F} \times [0,t]} \widetilde{Q}(i, a; j, db; ds) f(j, b, t - s). \end{aligned} \tag{3.80}$$

Thus, we have a Markov renewal equation $f = g + \widetilde{Q} * f$ where

$$g(i, a, t) = \overline{v}_{ia}(t)\overline{F}_i(t)$$

and

$$\widetilde{Q}(i, a; j, db; ds) = \overline{p}_{ia}(s, db) Q(i, j, ds).$$

Since $Q(i, j, ds)$ is nonnegative and $0 \leq \overline{p}_{ia}(s, \mathcal{F}) \leq 1$,

$$\widetilde{Q}(i, a; j, \mathcal{F}; ds) = \int_{\mathcal{F}} Q(i, j, ds)\overline{p}_{ia}(s, db) \leq Q(i, j, ds). \tag{3.81}$$

By Proposition *A.1* and Proposition *A.2* in Çınlar and Özekici [17], the Markov renewal equation (3.80) has a unique solution $f = \widetilde{R} * g$, or

$$P_{ia}\{L > t\} = \sum_{j \in E} \int_{\mathcal{F} \times [0,t]} \widetilde{R}(i, a; j, db; ds)\overline{F}_j(t - s)\overline{v}_{jb}(t - s)$$

where $\widetilde{R} = \sum_{n=0}^{+\infty} \widetilde{Q}^n$ is the Markov renewal kernel corresponding to $\widetilde{Q}$.

It is clear to observe that the process (Y, A) is, in fact, a semi-regenerative process on the Markov renewal process (X, T). We aggregate the 2 processes by defining a new semi-Markov process $Y^a = \{Y_t^a; t \geq 0\}$ so that

$$Y_t^a = (X_n, B_n)$$

whenever $T_n \leq t < T_{n+1}$. We can now obtain a new Markov renewal process $((\widetilde{X}, \widetilde{A}), \widetilde{T})$ through its minimal semi-Markov process $\widetilde{Y}^a$ defined through

$$\widetilde{Y}_t^a = \begin{cases} Y_t^a & \text{if } t < L \\ \Delta & \text{if } t \geq L \end{cases}$$

where Δ is the absorbing state denoting system failure. This also implies that

$$\widetilde{T}_n = \begin{cases} 0 & \text{if } n = 0 \\ \inf\left\{t > \widetilde{T}_{n-1}; \widetilde{Y}_t^a \neq \widetilde{Y}_{\widetilde{T}_{n-1}}^a\right\} & \text{if } n \geq 1 \end{cases}$$

and $(\widetilde{X}_n, \widetilde{A}_n) = \widetilde{Y}_{\widetilde{T}_n}^a$ for $n \geq 0$. Clearly, the state space of $((\widetilde{X}, \widetilde{A}), \widetilde{T})$ is $\widetilde{E \times \mathcal{F}} = E \times \mathcal{F} \cup \{\Delta\}$

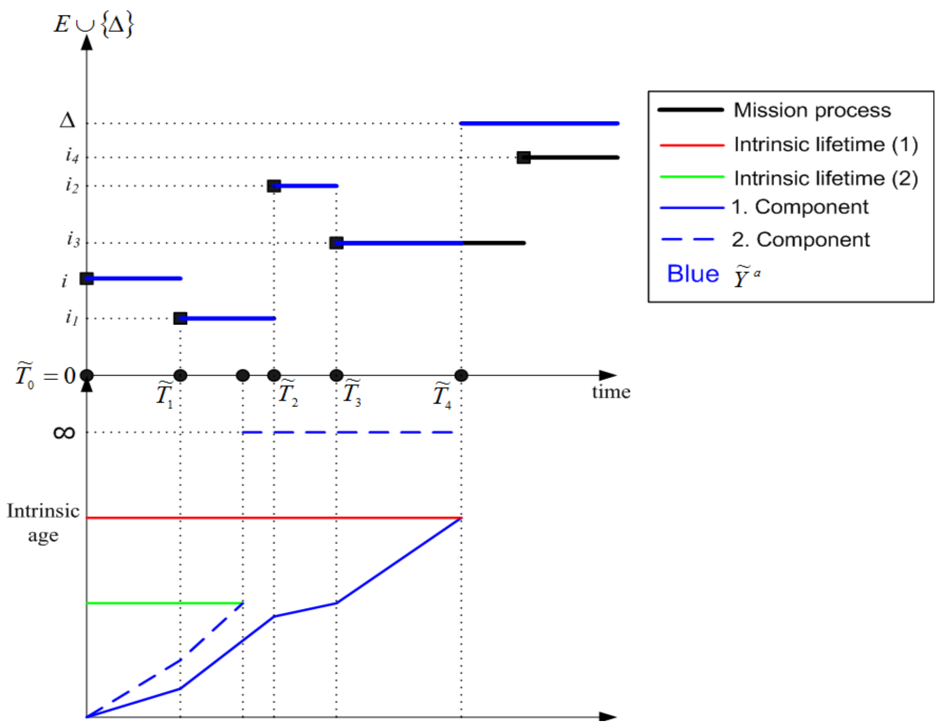

Figure 3.3: A typical representation of the structure of $\widetilde{Y}^a$ for a parallel system with two components.

and its semi-Markov kernel is obtained by extending the definition of $\widetilde{Q}$ to $\widetilde{E \times \mathcal{F}}$ such that

$$\widetilde{Q}\,(i, a; j, db; ds) = \begin{cases} Q\,(i, j, ds)\,\overline{p}_{ia}\,(s, db) & \text{if } i, j \in E, j \neq i \\ \overline{F}_i\,(s)\, v_{ia}\,(ds) & \text{if } i \in E,\ (j, b) = \Delta \\ 0 & \text{otherwise} \end{cases} \tag{3.82}$$

for $(i,a),(j,b) \in \widetilde{E \times \mathcal{F}}$. This new process $\widetilde{Y}^a$ follows the process Y^a until system failure and whenever the system fails at time L, it is dumped to the absorbing state Δ which denotes system failure. The structure of $\widetilde{Y}^a$ for a series system with two components is as described in Figure 3.3. We can find the transition kernel of the Markov chain $(\widetilde{X}, \widetilde{A})$ as

$$\begin{aligned} \widetilde{P}(i,a;j,db) &= \widetilde{Q}(i,a;j,db;+\infty) = \int_0^{+\infty} \widetilde{Q}(i,a;j,db;ds) \\ &= \int_0^{+\infty} Q(i,j,ds)\,\overline{p}_{ia}(s,db) \end{aligned} \tag{3.83}$$

for $(i,a),(j,b) \in E \times \mathcal{F}$, and

$$\widetilde{P}(i,a;\Delta) = \int_0^{+\infty} \overline{F}_i(s)\,v_{ia}(ds)$$

for $(i,a) \in E \times \mathcal{F}$. Note that $\widetilde{Q}(\Delta;j,db;t) = 0$ for all $t \in \mathbb{R}_+$, $(j,b) \in \widetilde{E \times \mathcal{F}}$ and $\widetilde{P}(\Delta;\Delta) = 1$. Both the semi-Markov kernel $\widetilde{Q}$ and the corresponding transition kernel $\widetilde{P}$ are possibly defective on $E \times \mathcal{F}$ since

$$\sum_{j\in E}\int_{\mathcal{F}} \widetilde{P}(i,a;j,db) = 1 - \widetilde{P}(i,a;\Delta) \leq 1.$$

As a matter of fact, we will suppose that they are indeed defective and there is an $(i,a) \in E \times \mathcal{F}$ such that $\widetilde{P}(i,a;\Delta) > 0$. Otherwise, we have a trivial situation and the system can never fail.

3.2.2.2 Mission Reliability

The mission reliability can be analyzed via a similar approach as in Section 3.1.2. The probability of survival for the first phase is

$$\begin{aligned} P_{ia}\{L > T_1\} &= \sum_{j\in E}\int_0^{+\infty} P_{ia}\{L > T_1, X_1 = j, T_1 \in ds\} \\ &= \sum_{j\in E}\int_0^{+\infty} P_{ia}\{L > s | X_1 = j, T_1 \in ds\}\,Q(i,j,ds) && (3.84) \\ &= \sum_{j\in E}\int_0^{+\infty} \overline{v}_{ia}(s)\,Q(i,j,ds) && (3.85) \\ &= \sum_{j\in E}\int_{\mathcal{F}\times\mathbb{R}_+} \overline{p}_{ia}(s,db)\,Q(i,j,ds) && (3.86) \\ &= \sum_{j\in E}\int_{\mathcal{F}\times\mathbb{R}_+} \widetilde{Q}(i,a;j,db;ds) = \sum_{j\in E}\int_{\mathcal{F}} \widetilde{P}(i,a;j,db). && (3.87) \end{aligned}$$

Similarly, we can find the probability of completing the nth phase as

$$\begin{aligned}
P_{ia}\{L > T_n\} &= \sum_{j\in E}\int_0^{+\infty} P_{ia}\{L > T_n, T_1 \in ds, X_1 = j\} \\
&= \sum_{j\in E}\int_{\mathcal{F}\times\mathbb{R}_+} \overline{p}_{ia}(s, db)\, P_{jb}\{L > T_{n-1}\}\, Q(i, j, ds) \\
&= \sum_{j\in E}\int_{\mathcal{F}\times\mathbb{R}_+} \widetilde{Q}(i, a; j, db; ds)\, P_{jb}\{L > T_{n-1}\} \qquad (3.88)
\end{aligned}$$

for $n \geq 1$ recursively. Using induction, we will show that

$$P_{ia}\{L > T_n\} = \sum_{j\in E}\int_{\mathcal{F}} \widetilde{P}^n(i, a; j, db) \qquad (3.89)$$

where

$$\widetilde{P}^n(i, a; j, db) = \sum_{k\in E}\int_{\mathcal{F}} \widetilde{P}(i, a; k, dc)\, \widetilde{P}^{n-1}(k, c; j, db)$$

defined recursively for $n \geq 1$. Due to (3.87), (3.89) holds if $n = 1$. Now, assume that (3.89) holds for $n = k$ and consider $P_{ia}\{L > T_{k+1}\}$. Then, using (3.88),

$$\begin{aligned}
P_{ia}\{L > T_{k+1}\} &= \sum_{j\in E}\int_{\mathcal{F}\times\mathbb{R}_+} \widetilde{Q}(i, a; j, db; ds)\, P_{jb}\{L > T_k\} \\
&= \sum_{j\in E}\int_{\mathcal{F}} \widetilde{P}(i, a; j, db)\left(\sum_{k\in E}\int_{\mathcal{F}} \widetilde{P}^k(j, b; k, dc)\right) \\
&= \sum_{j\in E}\int_{\mathcal{F}} \widetilde{P}^{k+1}(i, a; j, db).
\end{aligned}$$

The mission reliability can also be characterized through

$$\begin{aligned}
P_{ia}\{L > T_n\} &= P_i\left\{\left(\widetilde{X}_n, \widetilde{A}_n\right) \in E \times \mathcal{F}\right\} \\
&= \sum_{j\in E}\int_{\mathcal{F}} \widetilde{P}^n(i, a; j, db) = 1 - \widetilde{P}^n(i, a; \Delta)
\end{aligned}$$

and the probability that the whole mission is completed without failure can be determined from

$$\lim_{n\to+\infty} P_{ia}\{L > T_n\} = 1 - \lim_{n\to+\infty} \widetilde{P}^n(i, a; \Delta). \qquad (3.90)$$

If all states $(i, a) \in E \times \mathcal{F}$ are transient, then this probability is 0 since the process will eventually be absorbed in state Δ. But, if there is another absorbing state S that is used to denote the successful termination of the whole mission, then mission reliability (3.90) is not necessarily equal to 0.

3.2.2.3 Phase Reliability

Suppose that phase j is the critical one and we want to find the probability that phase j will be completed until time t given that the initial phase and age of the system are i and a respectively, $P_{ia}\{U_j \leq t, L > U_j\}$. As we did in the maximal repair case, we will analyze this probability by defining a new Markov renewal process $\left(\left(\overline{X}, \overline{A}\right), \overline{T}\right)$ through its minimal semi-Markov process $\overline{Y}^a$ such that

$$\overline{Y}_t^a = \begin{cases} Y_t^a & \text{if } t < \min\{L, U_j\} \\ \Delta & \text{if } L \leq \min\{t, U_j\} \\ S_j & \text{if } U_j \leq \min\{t, L\} \end{cases}$$

and, hence, we extend the state space to $\overline{E \times \mathcal{F}} = E \times \mathcal{F} \cup \{\Delta, S_j\}$. This process is very similar to the process $\left(\overline{X}, \overline{T}\right)$ defined in (3.47) and hence it follows the mission and the intrinsic age processes until system failure or successful completion of the critical phase j, whichever occurs first. If the system fails before completing phase j, then $\left(\overline{X}, \overline{A}\right)$ jumps to the absorbing state Δ at the instant of failure. On the other hand, if the system completes phase j without any failure, $\left(\overline{X}, \overline{A}\right)$ is dumped to another absorbing state S_j at the end of the phase which denotes the successful completion of the critical phase. The structure of

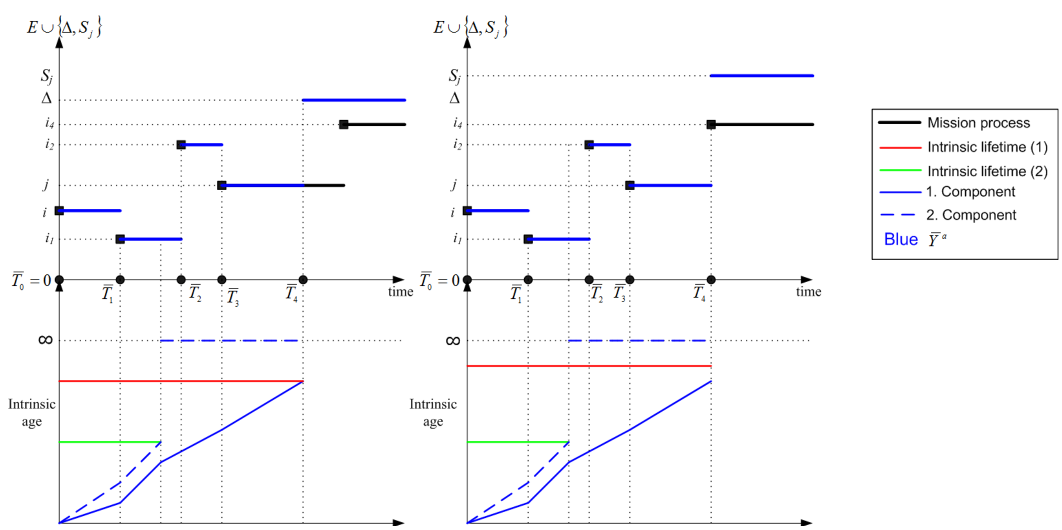

Figure 3.4: A typical representation of the structure of $\overline{Y}^a$ for a parallel system with two components.

$\overline{Y}^a$ for a series system with two components is as described in Figure 3.4. The semi-Markov kernel of $\left(\left(\overline{X}, \overline{A}\right), \overline{T}\right)$ is

$$\overline{Q}\left(i,a;k,db;ds\right)=\begin{cases}\widetilde{Q}\left(i,a;j,db;ds\right) & \text{if } i,k\in E, i\neq j\\ \widetilde{Q}\left(i,a;\Delta;ds\right) & \text{if } (k,b)=\Delta\\ F_j\left(ds\right)\overline{v}_{ja}\left(s\right) & \text{if } i=j \text{ and } (k,b)=S_j\\ 0 & \text{otherwise}\end{cases}$$

for all $(i,a),(k,b)\in\overline{E\times\mathcal{F}}$. We can find the transition kernel of the Markov chain $\left(\overline{X},\overline{A}\right)$ such that

$$\overline{P}\left(i,a;k,db\right)=\begin{cases}\widetilde{P}\left(i,a;k,db\right) & \text{if } i,k\in E, i\neq j\\ \widetilde{P}\left(i,a;\Delta\right) & \text{if } (k,b)=\Delta\\ \int_0^{+\infty}F_j\left(ds\right)\overline{v}_{ja}\left(s\right) & \text{if } i=j \text{ and } (k,b)=S_j\\ 0 & \text{otherwise}\end{cases}$$

for all $(i,a),(k,b)\in\overline{E\times\mathcal{F}}$. Note that $\overline{Q}\left(\Delta;k,db;t\right)=\overline{Q}\left(S_j;k,db;t\right)=0$ for all $t\in\mathbb{R}_+$, $(k,b)\in\overline{E\times\mathcal{F}}$ and $\overline{P}\left(\Delta;\Delta\right)=\overline{P}\left(S_j;S_j\right)=1$. Note, once more, that both the semi-Markov kernel $\overline{Q}$ and the corresponding transition matrix $\overline{P}$ are possibly defective on $E\times\mathcal{F}$ and $\widetilde{E\times\mathcal{F}}$. We do assume that they are, in fact, defective to avoid trivialities in reliability analysis.

It is clear that $P_{ia}\left\{U_j\leq t, L>U_j\right\}=P_{ia}\left\{\overline{Y}_t^a=S_j\right\}$. It is trivial that $P_{S_j}\left\{\overline{Y}_t^a=S_j\right\}=1-P_\Delta\left\{\overline{Y}_t^a=S_j\right\}=1$. Then, using a Markov renewal argument, we can write

$$\begin{aligned}f(i,a,t) &= P_{ia}\left\{\overline{Y}_t^a=S_j\right\}=P_{ia}\left\{\overline{Y}_t^a=S_j,\overline{T}_1>t\right\}+P_{ia}\left\{\overline{Y}_t^a=S_j,\overline{T}_1\leq t\right\}\\ &= I_{\{j=i\}}\int_0^t F_j\left(ds\right)\overline{v}_{ja}\left(s\right)+\int\limits_{\mathcal{F}\times[0,t]}\sum_{k\in E}\overline{Q}\left(i,a;k,dc;ds\right)f(k,c,t-s)\end{aligned} \quad (3.91)$$

for all $(i,a)\in E\times\mathcal{F}$ which is a Markov renewal equation $f=g+\overline{Q}*f$ with

$$g\left(i,a,t\right)=I_{\{j=i\}}\int_0^t F_j\left(ds\right)\overline{v}_{ja}\left(s\right).$$

We know that $\overline{Q}$ is defective on $E\times\mathcal{F}$ and $\overline{Q}\left(i,a;j,\mathcal{F};ds\right)\leq Q\left(i,j,ds\right)$ for all $(i,a)\in E\times\mathcal{F}, j\in E$. Therefore, using Proposition *A.1* and *A.2* in Çınlar and Özekici [17], there is a unique solution $f = \overline{R} * g$ for (3.91) such that

$$
\begin{aligned}
f(i,a,t) &= P_{ia}\left\{\overline{Y}_t^a = S_j\right\} = \int\limits_{\mathcal{F}\times[0,t]} \sum_{k\in E} \overline{R}(i,a;k,dc;ds)\, g(k,c,t-s) \\
&= \int\limits_{\mathcal{F}\times[0,t]} \sum_{k\in E} \overline{R}(i,a;k,dc;ds)\, I(k,j) \int_{[0,t-s]} F_j(du)\, \overline{v}_{jc}(u) \\
&= \int\limits_{\mathcal{F}\times[0,t]} \overline{R}(i,a;j,dc;ds)\, \overline{Q}(j,c;S_j;t-s) \qquad (3.92)
\end{aligned}
$$

where $\overline{R} = \sum_n \overline{Q}^n$ and

$$
\overline{Q}^n(i,a;l,db;t) = \int\limits_{\mathcal{F}\times[0,t]} \sum_{k\in E} \overline{Q}^{n-1}(i,a;k,dc;ds)\, \overline{Q}(k,c;l,db;t-s)
$$

for all $(i,a),(l,b) \in E\times\mathcal{F}$. Let $\overline{R}(i,a;S_j;t) = \sum_n \overline{Q}^n(i,a;S_j;t)$ where

$$
\begin{aligned}
\overline{Q}^n(i,a;S_j;t) &= \int\limits_{\mathcal{F}\times[0,t]} \sum_{k\in E} \overline{Q}^{n-1}(i,a;k,dc;ds)\, \overline{Q}(k,c;S_j;t-s) \\
&= \int\limits_{\mathcal{F}\times[0,t]} \overline{Q}^{n-1}(i,a;j,dc;ds)\, \overline{Q}(j,c;S_j;t-s).
\end{aligned}
$$

Then, using (3.92),

$$
\begin{aligned}
P_{ia}\left\{\overline{Y}_t^a = S_j\right\} &= \int\limits_{\mathcal{F}\times[0,t]} \overline{R}(i,a;j,dc;ds)\, \overline{Q}(j,c;S_j;t-s) \\
&= \sum_{n=0}^{+\infty} \int\limits_{\mathcal{F}\times[0,t]} \overline{Q}^n(i,a;j,dc;ds)\, \overline{Q}(j,c;S_j;t-s) \\
&= \sum_{n=0}^{+\infty} \overline{Q}^{n+1}(i,a;S_j;t) = \overline{R}(i,a;S_j;t) - I((i,a),S_j) \\
&= \overline{R}(i,a;S_j;t)
\end{aligned}
$$

where the last equality follows from the initial assumption that $(i,a) \in E\times\mathcal{F}$.

Chapter 4

MEAN TIME TO FAILURE ANALYSIS

4.1 Models with Maximal Repair

In this section, there is maximal repair so that the whole system is overhauled at the completion of each phase of the mission such that it becomes good as new before the next phase starts. The main purpose of this section is to characterize $E_i[L]$ for all $i \in E$, and to do this, we will use the Markov renewal process $(\widetilde{X}, \widetilde{T})$ defined by (3.4). Let $\widetilde{P}_\Delta$ denote the matrix obtained by deleting the row and the column corresponding to state Δ from the matrix $\widetilde{P}$. The potential matrix corresponding to $\widetilde{P}_\Delta$ is defined as

$$\widetilde{R}_\Delta = \sum_{n=0}^{+\infty} \widetilde{P}_\Delta^n.$$

The following is our basic assumption in this section.

Assumption 4.1 $\sup_{i \in E} \widetilde{P}_\Delta^k(i, E) = \sup_{i \in E} \sum_{j \in E} \widetilde{P}_\Delta^k(i, j) < 1$ *for some* $k \geq 1$.

This assumption simply states that whatever the initial phase is, the system may fail after a finite number of successfully completed phases with a positive probability. In other words, state Δ is reachable from any state $i \in E$. This trivially implies that $\widetilde{P}_\Delta$ is a defective transition probability matrix.

Note that

$$E_i[L] = E_i\left[L I_{\{L > T_1\}}\right] + E_i\left[L I_{\{L \leq T_1\}}\right]. \tag{4.1}$$

By applying the law of iterated expectations,

$$\begin{aligned} E_i\left[L I_{\{L > T_1\}}\right] &= \sum_{j \in E} \int_0^{+\infty} E_i\left[L I_{\{L > T_1\}} | X_1 = j, T_1 \in ds\right] P_i\{X_1 = j, T_1 \in ds\} \\ &= \sum_{j \in E} \int_0^{+\infty} (s + E_j[L])\, \overline{p}_i(s)\, Q(i, j, ds) \end{aligned} \tag{4.2}$$

$$
\begin{aligned}
&= \sum_{j\in E}\int_0^{+\infty} s\overline{p}_i(s)\,Q(i,j,ds) + \sum_{j\in E} E_j[L]\int_0^{+\infty}\overline{p}_i(s)\,Q(i,j,ds) \\
&= \int_0^{+\infty} s\overline{p}_i(s)\,F_i(ds) + \sum_{j\in E}\widetilde{P}_\Delta(i,j)\,E_j[L]. \qquad (4.3)
\end{aligned}
$$

In this derivation, (4.3) follows from (2.4) and (3.5), and (4.2) holds since using (3.4),

$$
\begin{aligned}
P_i\left\{I_{\{L>T_1\}}=1|X_1=j,T_1\in ds\right\} &= \frac{P_i\left\{I_{\{L>T_1\}}=1,X_1=j,T_1\in ds\right\}}{P_i\{X_1=j,T_1\in ds\}} \\
&= \frac{\widetilde{Q}(i,j,ds)}{Q(i,j,ds)} \\
&= \overline{p}_i(s).
\end{aligned}
$$

Suppose that the duration of the ith phase is $s_i>0$. Then,

$$
\begin{aligned}
P_i\left\{LI_{\{L\le s_i\}}\le t\right\} &= P_i\left\{LI_{\{L\le s_i\}}\le t|L\le s_i\right\}P_i\{L\le s_i\} \\
&\quad +P_i\left\{LI_{\{L\le s_i\}}\le t|L>s_i\right\}P_i\{L>s_i\} \\
&= \begin{cases} 1 & \text{if } t>s_i \\ p_i(t)+1-p_i(s_i) & \text{if } t\le s_i \end{cases}
\end{aligned}
$$

for all $t>0$. This implies that

$$
P_i\left\{LI_{\{L\le s_i\}}>t\right\} = \begin{cases} 0 & \text{if } t>s_i \\ p_i(s_i)-p_i(t) & \text{if } t\le s_i \end{cases}
$$

and

$$
\begin{aligned}
E_i\left[LI_{\{L\le s_i\}}\right] &= \int_0^{+\infty} P_i\left\{LI_{\{L\le s_i\}}>t\right\}dt = \int_0^{s_i}(p_i(s_i)-p_i(t))\,dt = \int_0^{s_i}\int_t^{s_i} p_i(ds)\,dt \\
&= \int_0^{s_i}\int_0^{s} dt\,p_i(ds) = \int_0^{s_i} s\,p_i(ds). \qquad (4.4)
\end{aligned}
$$

Then, using (4.4),

$$
\begin{aligned}
E_i\left[LI_{\{L\le T_1\}}\right] &= E_i\left[E_i\left[LI_{\{L\le T_1\}}|T_1\right]\right] = \int_0^{+\infty} E_i\left[LI_{\{L\le T_1\}}|T_1\in ds\right]F_i(ds) \\
&= \int_0^{+\infty} F_i(ds)\int_0^{s} t\,p_i(dt). \qquad (4.5)
\end{aligned}
$$

Now, (4.1), (4.3) and (4.5) lead to

$$
E_i[L] = \int_0^{+\infty} s\overline{p}_i(s)\,F_i(ds) + \int_0^{+\infty} F_i(ds)\int_0^{s} t\,p_i(dt) + \sum_{j\in E}\widetilde{P}(i,j)\,E_j[L]. \qquad (4.6)
$$

Let $f(i) = E_i[L]$ for every i and

$$\begin{aligned} h(i) &= \int_0^{+\infty} s\overline{p}_i(s)\, F_i(ds) + \int_0^{+\infty} F_i(ds) \int_0^s t p_i(dt) \\ &= \int_0^{+\infty} \left(s\overline{p}_i(s) + \int_0^s p_i(dt) \int_0^t du \right) F_i(ds) \qquad (4.7) \\ &= \int_0^{+\infty} \left(s\overline{p}_i(s) + \int_0^s du \int_u^s p_i(dt) \right) F_i(ds) \\ &= \int_0^{+\infty} \left(s\overline{p}_i(s) + \int_0^s (p_i(s) - p_i(u))\, du \right) F_i(ds) \qquad (4.8) \\ &= \int_0^{+\infty} \left(s - \int_0^s p_i(u)\, du \right) F_i(ds) = \int_0^{+\infty} \left(\int_0^s \overline{p}_i(u)\, du \right) F_i(ds) \\ &= \int_0^{+\infty} \overline{p}_i(u)\, du \int_u^{+\infty} F_i(ds) = \int_0^{+\infty} \overline{p}_i(u)\, \overline{F}_i(u)\, du \qquad (4.9) \end{aligned}$$

for every $i \in E$. Then, (4.6) can be written in compact form as the Poisson equation

$$f = h + \widetilde{P}_\Delta f. \qquad (4.10)$$

In our analysis, we exclude the case where $E_i[L]$ is unbounded and try to find a unique bounded f satisfying (4.10). From now on, we therefore suppose that f is bounded.

If E is finite, then the solution of (4.10) is

$$f = \left(\boldsymbol{I} - \widetilde{P}_\Delta \right)^{-1} h = \widetilde{R}_\Delta h. \qquad (4.11)$$

Since $\widetilde{P}_\Delta$ is a defective transition matrix, the matrix inverse in (4.11) exists.

If E is not finite, then (4.10) implies

$$f = h + \widetilde{P}_\Delta h + \widetilde{P}_\Delta^2 f$$

by replacing f on the right-hand side by $h + \widetilde{P}_\Delta f$, and repeating this argument we get

$$f = h + \widetilde{P}_\Delta h + \cdots + \widetilde{P}_\Delta^n h + \widetilde{P}_\Delta^{n+1} f$$

for any $n \geq 0$. Therefore, we have

$$f = \widetilde{R}_\Delta h + \lim_{n \to +\infty} \widetilde{P}_\Delta^n f.$$

It is clear that $f = \widetilde{R}_\Delta h$ is the unique bounded solution of (4.10) provided that $\lim_{n\to+\infty} \widetilde{P}_\Delta^n f = 0$, and $\widetilde{R}_\Delta h$ is bounded. The following gives a reasonable condition for this to be true.

Theorem 4.2 *If h is bounded and Assumption 4.1 holds, then $\widetilde{R}_\Delta h$ is bounded, and $f = \widetilde{R}_\Delta h$ is the unique solution of (4.10)*

Proof. Suppose

$$\widetilde{P}^k_\Delta (i, E) \le 1 - c \tag{4.12}$$

for some $c \in (0, 1)$ and $k \ge 1$. We now show by induction that

$$\widetilde{P}^{kn}_\Delta (i, E) \le (1 - c)^n \tag{4.13}$$

for all $n \in \mathbb{N}$. For $n = 1$, it is true by Assumption 4.1. Now, assume that it holds for $1, 2, \cdots, n$. Then,

$$\begin{aligned}
\widetilde{P}^{(n+1)k}_\Delta (i, E) &= \sum_{j \in E} \widetilde{P}^{nk}_\Delta (i, j)\, \widetilde{P}^k_\Delta (j, E) \\
&\le (1 - c) \sum_{j \in E} \widetilde{P}^{kn}_\Delta (i, j) = (1 - c)\, \widetilde{P}^{kn}_\Delta (i, E) \\
&\le (1 - c)^{n+1}. \qquad (4.14)
\end{aligned}$$

Then,

$$\begin{aligned}
\widetilde{R}_\Delta (i, E) &= \sum_{n=0}^{+\infty} \widetilde{P}^n_\Delta (i, E) = \sum_{m=0}^{+\infty} \sum_{n=0}^{k-1} \widetilde{P}^{km+n}_\Delta (i, E) \\
&= \sum_{m=0}^{+\infty} \sum_{n=0}^{k-1} \left(\sum_{j \in E} \widetilde{P}^n_\Delta (i, j)\, \widetilde{P}^{km}_\Delta (j, E) \right) \\
&\le \sum_{m=0}^{+\infty} \sum_{n=0}^{k-1} (1 - c)^m\, \widetilde{P}^n_\Delta (i, E) \\
&\le \sum_{m=0}^{+\infty} k (1 - c)^m = \frac{k}{c} < +\infty. \qquad (4.15)
\end{aligned}$$

This also implies that $\lim_{n \to +\infty} \widetilde{P}^n_\Delta (i, E) = 0$ since $\widetilde{R}_\Delta = \sum_{n=0}^{+\infty} \widetilde{P}^n_\Delta$. Moreover,

$$\begin{aligned}
\widetilde{R}_\Delta h (i) &= \sum_{j \in E} \widetilde{R}_\Delta (i, j)\, h (j) \le \sup_{j \in E} h (j) \sum_{j \in E} \widetilde{R}_\Delta (i, j) \\
&= \sup_{j \in E} h (j)\, \widetilde{R}_\Delta (i, E) < +\infty. \qquad (4.16)
\end{aligned}$$

Therefore, $\widetilde{R}_\Delta h$ exists and it is bounded and, hence, it is a solution of (4.10). To prove the uniqueness, we need to show that $\lim_{n \to +\infty} \widetilde{P}^n_\Delta f = 0$. But, this is trivially true since $\lim_{n \to +\infty} \widetilde{P}^n_\Delta (i, j) = 0$ for all $i, j \in E$ and f is bounded. ■

It follows from the definition of h in (4.9) that

$$h(i) = \int_0^{+\infty} P_i\{L > u, T_1 > u\}\, du$$

and we have the representation $h(i) = E_i[\min\{L, T_1\}]$. We observe that boundedness of h depends on the mean durations of the phases and the mean lifetime of the system under fixed phases. Suppose that

$$\sup_{i\in E} m(i) = \sup_{i\in E} E_i[T_1] < +\infty.$$

Then, using (4.9),

$$\overline{h} = \sup_{i\in E} h(i) = \sup_{i\in E} \int_0^{+\infty} \overline{p}_i(u)\, \overline{F}_i(u)\, du \le \sup_{i\in E} \int_0^{+\infty} \overline{F}_i(u)\, du = \sup_{i\in E} m(i) < +\infty.$$

Following similar steps, it can be shown that

$$\overline{h} \le \sup_{i\in E} E[L|Y = i]$$

to obtain another bound for h.

4.1.1 An Alternative Derivation

In this section, we will give an alternative derivation for $E_i\left[LI_{\{L>T_1\}}\right]$. Suppose that the duration of phase i is $s_i > 0$. Then,

$$\begin{aligned} P_i\left\{LI_{\{L>s_i\}} \le t\right\} &= P_i\left\{LI_{\{L>s_i\}} \le t | L > s_i\right\} P_i\{L > s_i\} \\ &\quad + P_i\left\{LI_{\{L>s_i\}} \le t | L \le s_i\right\} P_i\{L \le s_i\} \\ &= \begin{cases} p_i(s_i) & \text{if } t \le s_i \\ p_i(s_i) + \sum_{j\in E} P_i\{L \le t | X_1 = j, L > s_i\} & \\ \times P_i\{X_1 = j | L > s_i\} P_i\{L > s_i\} & \text{if } t > s_i \end{cases} \\ &= \begin{cases} p_i(s_i) & \text{if } t \le s_i \\ p_i(s_i) + \sum_{j\in E} P_j\{L \le t - s_i\} & \\ \times P_i\{X_1 = j | L > s_i\} P_i\{L > s_i\} & \text{if } t > s_i \end{cases} \end{aligned}$$

and

$$P_i\left\{LI_{\{L>s_i\}} > t\right\} = \begin{cases} \overline{p}_i(s_i) & \text{if } t \le s_i \\ \overline{p}_i(s_i) - \sum_{j\in E} P_j\{L \le t - s_i\} & \\ \times P_i\{X_1 = j | L > s_i\} P_i\{L > s_i\} & \text{if } t > s_i \end{cases} \tag{4.17}$$

for all $t > 0$. This implies that

$$
\begin{aligned}
E_i\left[LI_{\{L>T_1\}}\right] &= E_i\left[E_i\left[LI_{\{L>T_1\}}|T_1\right]\right] = \int_0^{+\infty} F_i(ds)\, E_i\left[LI_{\{L>T_1\}}|T_1 \in ds\right] \\
&= \int_0^{+\infty} F_i(ds) \int_0^{+\infty} P_i\left\{LI_{\{L>T_1\}} > t|T_1 \in ds\right\} \\
&= \int_0^{+\infty} F_i(ds)\, \overline{p}_i(s) \int_0^{s} dt + \int_0^{+\infty}\int_s^{+\infty} \Bigg(\overline{p}_i(s)\, F_i(ds) - \sum_{j\in E} F_i(ds) \\
&\quad \times P_j\{L \le t-s\}\, P_i\{X_1 = j|L > s, T_1 \in ds\}\, P_i\{L > s|T_1 \in ds\}\Bigg) \\
&= \int_0^{+\infty} s\overline{p}_i(s)\, F_i(ds) + \int_0^{+\infty}\int_s^{+\infty} \Bigg(\overline{p}_i(s)\, F_i(ds) \\
&\quad - \sum_{j\in E} P_j\{L \le t-s\}\, \widetilde{Q}(i,j,ds)\Bigg) \\
&= \int_0^{+\infty} s\overline{p}_i(s)\, F_i(ds) + \int_0^{+\infty}\int_s^{+\infty} \overline{p}_i(s) \Bigg(F_i(ds) - \sum_{j\in E} Q(i,j,ds) \\
&\quad + \sum_{j\in E} P_j\{L > t-s\}\, Q(i,j,ds)\Bigg) \\
&= \int_0^{+\infty} s\overline{p}_i(s)\, F_i(ds) + \sum_{j\in E} E_j[L] \int_0^{+\infty} \overline{p}_i(s)\, Q(i,j,ds) \\
&= \int_0^{+\infty} s\overline{p}_i(s)\, F_i(ds) + \sum_{j\in E} \widetilde{P}_\Delta(i,j)\, E_j[L]
\end{aligned}
$$

and this is the same result as in (4.3).

4.1.2 Numerical Illustration

Suppose that we have a parallel system with two identical components, which will perform a hypothetical space mission with 4 phases. The phases of the mission and the respective distribution functions of the durations are

- Phase 1 (Hibernation 1): Weibull distribution with $\alpha_1 = 2, \beta_1 = 1$,
- Phase 2 (Hibernation 2): Weibull distribution with $\alpha_2 = 5, \beta_2 = 1$,
- Phase 3 (Scientific Observation 1): Weibull distribution with $\alpha_3 = 1, \beta_3 = 2$,
- Phase 4 (Scientific Observation 2): Weibull distribution with $\alpha_4 = 2, \beta_4 = 2$

where the probability density function of the duration of phase i is

$$f_i(t) = \alpha_i \beta_i t^{\beta_i - 1} e^{-\alpha_i t^{\beta_i}}$$

for all $t \in \mathbb{R}_+$ and $i = 1, 2, 3, 4$. Note that the distribution is exponential with rate $\alpha_1 = 2$ and $\alpha_2 = 5$ for phase 1 and phase 2 respectively since $\beta_1 = \beta_2 = 1$. Moreover, the mean durations of the phases are given by

$$\alpha_i^{-1/\beta_i} \Gamma(1 + 1/\beta_i)$$

where $\Gamma(z) = \int_0^{+\infty} t^{z-1} e^{-t} dt$ is the Gamma function. Therefore, the mean durations are $0.5000, 0.2000, 0.8862$, and 0.6267 for phases $1, 2, 3$, and 4 respectively. The transition probability matrix of the mission process is

$$P = \begin{bmatrix} 0 & 0 & 0.3 & 0.7 \\ 0.2 & 0 & 0.4 & 0.4 \\ 0.2 & 0.8 & 0 & 0 \\ 0.8 & 0.2 & 0 & 0 \end{bmatrix}. \tag{4.18}$$

We assume that component lifetimes are exponentially distributed with rates $\lambda_1 = 10^{-2}, \lambda_2 = 0, \lambda_3 = 10^{-1}$, and $\lambda_4 = 0.8$ in phases 1, 2, 3, and 4 respectively. In the foregoing part of this section, these parameters will be used unless otherwise specified. Moreover, in all of the tabular representations through this section, the rows and the columns represent the phases. These further imply that the semi-Markov kernel of the mission process is

$$Q(t) = \begin{bmatrix} 0 & 0 & 0.3\left(1 - e^{-2t}\right) & 0.7\left(1 - e^{-2t}\right) \\ 0.2\left(1 - e^{-5t}\right) & 0 & 0.4\left(1 - e^{-5t}\right) & 0.4\left(1 - e^{-5t}\right) \\ 0.2\left(1 - e^{-t^2}\right) & 0.8\left(1 - e^{-t^2}\right) & 0 & 0 \\ 0.8\left(1 - e^{-4t^2}\right) & 0.2\left(1 - e^{-4t^2}\right) & 0 & 0 \end{bmatrix} \tag{4.19}$$

and

$$\overline{p}(t) = \begin{bmatrix} 1 - \left(1 - e^{-10^{-5} t}\right)^2 \\ 1 \\ 1 - \left(1 - e^{-10^{-3} t}\right)^2 \\ 1 - \left(1 - e^{-10^{-1} t}\right)^2 \end{bmatrix} \tag{4.20}$$

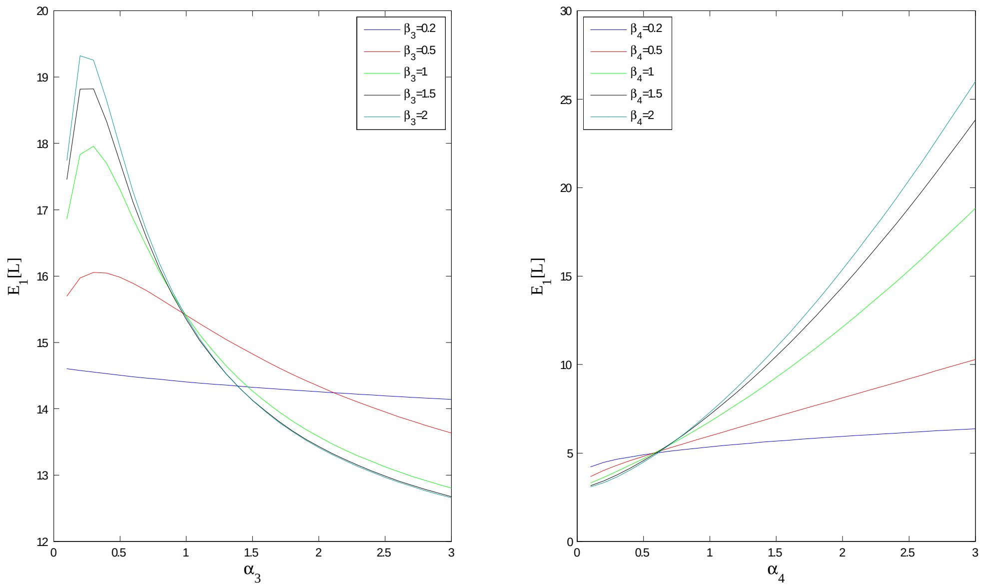

Figure 4.1: $E_1[L]$ vs. α_3 and α_4 for different values of β_3 and β_4.

for all $t \in \mathbb{R}_+$. Using (4.19), (4.20), (3.5), and (3.6), $\widetilde{P}$ can be calculated as

$$\widetilde{P} = \begin{bmatrix} 0 & 0 & 0.300 & 0.700 & 4.95 \times 10^{-5} \\ 0.200 & 0 & 0.400 & 0.400 & 0 \\ 0.198 & 0.793 & 0 & 0 & 0.009 \\ 0.722 & 0.180 & 0 & 0 & 0.098 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and then

$$E\left[L\right] = \begin{bmatrix} 15.3822 & 15.4866 & 16.2122 & 14.3132 \end{bmatrix} \tag{4.21}$$

using (4.11).

The behaviors of $E_1[L]$ vs. α_3 and α_4 for different values of β_3 and β_4 are shown in Figure 4.1. Since the failure rates of the components in phase 4 are the highest ($\lambda_4 = 0.8$), MTTF increases as the mean duration $\alpha_4^{-1/\beta_4}\Gamma(1 + 1/\beta_4)$ of phase 4 decreases. On the other hand, MTTF first increases and then decreases, especially for large values of β_3, as the mean duration $\alpha_3^{-1/\beta_3}\Gamma(1 + 1/\beta_3)$ of phase 3 decreases. When the mean duration of

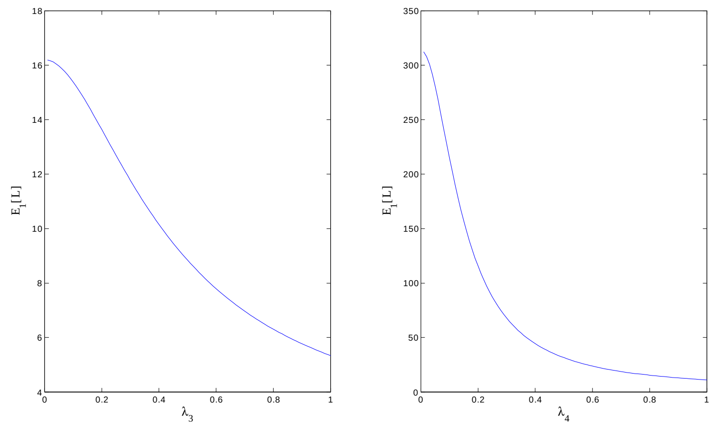

Figure 4.2: $E_1[L]$ vs. λ_3 and λ_4.

phase 3 decreases, the system starts to stay in the other phases for longer time intervals. If the system stays longer in Hibernation (phase 1 and 2), this will increase the MTTF since the failure rates are very low in these phases. However, if the system spends more time in Scientific Observation 2, this will decrease the MTTF since the failure rates are the highest in this state. As seen from Figure 4.1, for lower values of α_3 (or higher mean durations of phase 3), the effect of Hibernation phases to increase MTTF dominates the effect of Scientific Observation 2 to decrease MTTF. As α_3 increases more (or average duration of phase 3 decreases more), the effect of Scientific Observation 2 becomes more dominant resulting in the decrease of the average system lifetime. Similar graphs for $E_2[L]$, $E_3[L]$, $E_4[L]$ have the same structure.

The behaviors of $E_1[L]$ vs. λ_3 and λ_4 are shown in Figure 4.2. As expected, average system lifetime decreases as the failure rates of the components increase. Note that the failure rate in phase 4 has a greater effect on MTTF although the expected duration of phase 4 is shorter. This follows from the fact that the mission process visits phase 4 more frequently and spends more time in this phase, as it is clear from (4.18).

4.1.3 Special Cases

4.1.3.1 Markovian Mission

Suppose that the sequence of the phases follows a Markov chain with transition matrix P and duration of phase i is equal to a constant s_i. From (4.9) and (3.5), we trivially have

$$h(i) = \int_0^{s_i} \overline{p}_i(s)\, ds \tag{4.22}$$

and

$$\widetilde{P}(i,j) = P(i,j)\, \overline{p}_i(s_i). \tag{4.23}$$

4.1.3.2 Deterministic Sequence of Phases

Suppose that the mission has a deterministic sequence $\{1, 2, \cdots, n\}$ of phases. To find the MTTF, we can define an appropriate transition matrix and apply the result in Section 4.1.3.1. However, since taking the inverse of a matrix is computationally costly, we can find a more explicit solution by utilizing the special structure of this case. We assume that if the mission is completed successfully, then the system starts to perform the mission starting from the first phase instantaneously. In other words, if nth phase is completed without a failure, then the system starts to perform the first phase. Then, using (4.10), (4.9), (3.5), and (3.9) for $i = 1, \cdots, n-1$, we get

$$E_i[L] = \int_0^{+\infty} \overline{p}_i(u)\, \overline{F}_i(u)\, du + E_{i+1}[L]\, P_i\{L > T_1\} \tag{4.24}$$

and

$$E_n[L] = \int_0^{+\infty} \overline{p}_n(u)\, \overline{F}_n(u)\, du + E_1[L]\, P_n\{L > T_1\}. \tag{4.25}$$

Now, (4.24) and (4.25) can be solved to find the explicit solution

$$E_1[L] = \left(1 - \prod_{i=1}^{n} P_i\{L > T_1\}\right)^{-1} \left(h(1) + \sum_{i=2}^{n} \prod_{j=1}^{i-1} h(i)\, P_j\{L > T_1\}\right) \tag{4.26}$$

where

$$h(i) = \int_0^{+\infty} \overline{p}_i(t)\, \overline{F}_i(t)\, dt. \tag{4.27}$$

If the phase durations $\{s_1, s_2, \cdots, s_n\}$ are also deterministic, (4.24)-(4.27) reduce to

$$E_i[L] = \int_0^{s_i} \overline{p}_i(s)\, ds + \overline{p}_i(s_i)\, E_{i+1}[L] \tag{4.28}$$

and

$$E_n[L] = \int_0^{s_n} \overline{p}_n(s)\,ds + \overline{p}_n(s_n)\,E_1[L] \tag{4.29}$$

$$E_1[L] = \left(1 - \prod_{i=1}^{n} \overline{p}_i(s_i)\right)^{-1} \left(h(1) + \sum_{i=2}^{n} \prod_{j=1}^{i-1} h(i)\,\overline{p}_j(s_j)\right) \tag{4.30}$$

and

$$h(i) = \int_0^{s_i} \overline{p}_i(s)\,ds. \tag{4.31}$$

Example 4.3 *Suppose that we have a series system with 2 components that will perform a mission with two phases so that $n = m = 2$. Suppose that $s_1 = 5$ and $s_2 = 10$, and that component k has an exponentially distributed lifetime in phase i with parameter $\lambda_i(k)$. Then,*

$$\overline{p}_i(s_i) = e^{-s_i(\lambda_i(1)+\lambda_i(2))} = e^{-s_i\overline{\lambda}_i}$$

and, hence,

$$h(i) = \frac{1 - e^{-s_i\overline{\lambda}_i}}{\overline{\lambda}_i}.$$

Therefore,

$$E_1[L] = \left(1 - e^{-s_1\overline{\lambda}_1 - s_2\overline{\lambda}_2}\right)^{-1} \left(\frac{1 - e^{-s_1\overline{\lambda}_1}}{\overline{\lambda}_1} + \frac{\left(1 - e^{-s_2\overline{\lambda}_2}\right) e^{-s_1\overline{\lambda}_1}}{\overline{\lambda}_2}\right).$$

4.2 Models with No Repair

In this section, there is no repair so that all components age or deteriorate in time without system or component replacement after the completion of each phase. We suppose that the components age according to the intrinsic aging model of Çınlar and Özekici [17]. For details of the model and related notation, see Section 2.3.

The main purpose of this section is to characterize the MTTF $E_{ia}[L]$ for every $i \in E$ and $a \in \mathcal{F}$. We will use the equality

$$E_{ia}[L] = E_{ia}\left[LI_{\{L>T_1\}}\right] + E_{ia}\left[LI_{\{L\leq T_1\}}\right] \tag{4.32}$$

to compute the MTTF. By conditioning and using (3.82), and (3.83),

$$\begin{aligned} E_{ia}\left[LI_{\{L>T_1\}}\right] &= \sum_{j\in E}\int_{\mathcal{F}}\int_0^{+\infty} E_{ia}\left[LI_{\{L>T_1\}}|X_1=j, A_{T_1}\in db, T_1\in ds\right] \\ &\quad \times P_{ia}\left\{X_1=j, A_{T_1}\in db, T_1\in ds\right\} \qquad (4.33) \\ &= \sum_{j\in E}\int_{\mathcal{F}}\int_0^{+\infty} \left(s+E_{jb}\left[L\right]\right) Q\left(i,j,ds\right)\overline{p}_{ia}\left(s,db\right) \\ &= \sum_{j\in E}\int_{\mathcal{F}}\int_0^{+\infty} s\widetilde{Q}\left(i,a;j,db;ds\right)+\sum_{j\in E}\int_{\mathcal{F}}\widetilde{P}\left(i,a;j,db\right)E_{jb}\left[L\right]. \qquad (4.34) \end{aligned}$$

Suppose that the duration of phase i is a constant $s_i > 0$. Then,

$$\begin{aligned} P_{ia}\left\{LI_{\{L\le s_i\}}\le t\right\} &= P_{ia}\left\{LI_{\{L\le s_i\}}\le t|L\le s_i\right\}P_{ia}\left\{L\le s_i\right\} \\ &\quad +P_{ia}\left\{LI_{\{L\le s_i\}}\le t|L> s_i\right\}P_{ia}\left\{L> s_i\right\} \qquad (4.35) \\ &= \begin{cases} 1 & \text{if } t>s_i \\ P_{ia}\left\{L\le t\right\}+P_{ia}\left\{L>s_i\right\} & \text{if } t\le s_i \end{cases} \\ &= \begin{cases} 1 & \text{if } t>s_i \\ v_{ia}\left(t\right)+\overline{v}_{ia}\left(s_i\right) & \text{if } t\le s_i \end{cases} \\ &= \begin{cases} 1 & \text{if } t>s_i \\ v_{ia}\left(t\right)+1-v_{ia}\left(s_i\right) & \text{if } t\le s_i \end{cases} \qquad (4.36) \end{aligned}$$

and

$$P_{ia}\left\{LI_{\{L\le s_i\}}>t\right\}=\begin{cases} 0 & \text{if } t>s_i \\ v_{ia}\left(s_i\right)-v_{ia}\left(t\right) & \text{if } t\le s_i \end{cases}$$

for all $t > 0$. This implies that

$$\begin{aligned} E_{ia}\left[LI_{\{L\le s_i\}}\right] &= \int_0^{s_i}\left(v_{ia}\left(s_i\right)-v_{ia}\left(t\right)\right)dt=\int_0^{s_i}\int_t^{s_i}v_{ia}\left(ds\right)dt \\ &= \int_0^{s_i}\int_0^{s}dt v_{ia}\left(ds\right)=\int_0^{s_i}s v_{ia}\left(ds\right) \end{aligned}$$

and, hence,

$$E_{ia}\left[LI_{\{L\le T_1\}}\right]=E_{ia}\left[E_{ia}\left[LI_{\{L\le T_1\}}|T_1\right]\right]=\int_0^{+\infty}F_i\left(ds\right)\int_0^{s}t v_{ia}\left(dt\right). \qquad (4.37)$$

Then, combining (4.34) with (4.37) and using (4.32), we have

$$E_{ia}\left[L\right]=h\left(i,a\right)+\sum_{j\in E}\int_{\mathcal{F}}\widetilde{P}\left(i,a;j,db\right)E_{jb}\left[L\right] \qquad (4.38)$$

where

$$\begin{aligned}
h(i,a) &= \sum_{j\in E}\int_{\mathcal{F}}\int_0^{+\infty} s\widetilde{Q}(i,a;j,db;ds) + \int_0^{+\infty} F_i(ds)\int_0^s tv_{ia}(dt) \\
&= \sum_{j\in E}\int_{\mathcal{F}}\int_0^{+\infty} sQ(i,j,ds)\,\overline{p}_{ia}(s,db) + \int_0^{+\infty} F_i(ds)\int_0^s tv_{ia}(dt) \\
&= \int_0^{+\infty} sF_i(ds)\,\overline{v}_{ia}(s) + \int_0^{+\infty} F_i(ds)\int_0^s tv_{ia}(dt) \\
&= \int_0^{+\infty}\left(s\overline{v}_{ia}(s) + \int_0^s tv_{ia}(dt)\right)F_i(ds) \qquad (4.39)\\
&= \int_0^{+\infty}\left(s - sv_{ia}(s) + sv_{ia}(s) - \int_0^s v_{ia}(t)\,dt\right)F_i(ds) \qquad (4.40)\\
&= \int_0^{+\infty} \overline{v}_{ia}(t)\,\overline{F}_i(t)\,dt \qquad (4.41)
\end{aligned}$$

for all $(i,a)\in E\times\mathcal{F}$. By letting $f(i,a) = E_{ia}[L]$, we obtain

$$f = h + \widetilde{P}_\Delta f \qquad (4.42)$$

where $\widetilde{P}_\Delta$ is the transition kernel obtained by deleting the row and column corresponding to the state Δ from $\widetilde{P}$ defined by (3.83) so that

$$\widetilde{P}_\Delta f(i,a) = \sum_{j\in E}\int_{\mathcal{F}}\widetilde{P}(i,a;j,db)\,f(j,b) = \sum_{j\in E}\int_{\mathcal{F}}\widetilde{P}_\Delta(i,a;j,db)\,f(j,b)\,.$$

It is known that $\widetilde{P}$ is a transition kernel of a Markov chain with a general state space and, hence, (4.42) defines a Poisson equation. We will solve (4.42) under the assumption that the function f is bounded to make the analysis tractable. It is clear that if f is infinite at some point, then the expected lifetime of the system is infinite trivially. Moreover, the following lemma shows that this is not a very restrictive assumption.

Lemma 4.4 *If there exists $\lambda > 0$ such that*

$$h_k(i,a,s) \geq a + \lambda s$$

for every component $k\in S$, $i\in E$, $a\in\mathbb{R}_+$, and $s>0$, then f is bounded.

Proof. *It is clear that $f\geq 0$ since $L\geq 0$. To find an upper bound, choose arbitrary i, a, s, and k. If the initial intrinsic age of the component k is a, then $A_s(k)\geq a+\lambda s$ given that the component and the system work until time s. Then,*

$$P_{ia}\{L(k)>s\} = P_{ia}\left\{\widehat{L}(k) > A_k(s)\right\} \leq P_{ia}\left\{\widehat{L}(k) > a+\lambda s\right\} = e^{-\lambda s}$$

where the last equality follows from the facts that $A_0(k) = a \in \mathbb{R}_+$ implies that $\widehat{L}(k) > a$, and $\widehat{L}(k)$ is exponentially distributed with rate 1. Using this result, we get

$$E_{ia}[L(k)] = \int_0^{+\infty} P_{ia}\{L(k) > s\}\, ds \le \int_0^{+\infty} e^{-\lambda s} ds = \frac{1}{\lambda}.$$

It is clear that

$$f(i,a) = E_{ia}[L] \le E_{ia}\left[\max_k L(k)\right] \le \sum_k E_{ia}[L(k)] \le \frac{m}{\lambda}.$$

■

The assumption of Lemma 4.4 simply requires that the aging rate of a component is bounded below by a strictly positive constant λ. In other words, every working component deteriorates at a strictly positive rate during all phases. If each component works during every phase, this is a very reasonable assumption. We also remark that if $\sup_{i,a} E_{ia}[L(k)] < +\infty$, then $E_{ia}[L]$ is bounded trivially.

The boundedness of f for a semi-Markov mission with a coherent structure function can also be verified by stochastic comparison. Assume that we try to find $f(i,a) = E_{ia}\left[L^1\right]$ where L^1 represents the lifetime of the system under the no repair policy. Let $E_i\left[L^2\right]$ be the MTTF for the same system under the maximal repair policy. Consider $P_{i0}\left\{L^2(k) > t\right\}$ and $P_{ia}\left\{L^1(k) > t\right\}$ for some $k \in S$, $t > 0$ where $L^1(k)$ and $L^2(k)$ are the lifetimes of component k under the no repair and the maximal repair policies respectively. Suppose that $r_k(i,a)$ is increasing in a for every k and i. This implies that failure probability of each component increases as the intrinsic age of the component increases. Therefore, it is easy to see that $P_{ia}\left\{L^1(k) > t\right\} \le P_{i0}\left\{L^1(k) > t\right\}$ for all $a \ge 0$. Since all components are replaced with brand new ones in the maximal repair policy after completing a phase, and the intrinsic age of a brand new component is 0; $A_t^1(k) \ge A_t^2(k)$ for all $t \ge 0$ where $A_t^1(k)$ and $A_t^2(k)$ are the intrinsic ages of component k at time t under the no repair policy and under the maximal repair policy respectively with $A_0^1(k) = A_0^2(k) = 0$. Therefore,

$$\begin{aligned} P_{ia}\left\{L^1(k) > t\right\} &\le P_{i0}\left\{L^1(k) > t\right\} = P_{i0}\left\{\widehat{L}(k) > A_t^1(k)\right\} \\ &\le P_{i0}\left\{\widehat{L}(k) > A_t^2(k)\right\} = P_{i0}\left\{L^2(k) > t\right\}. \end{aligned}$$

In the maximal repair policy, the reliability of each component is higher and

$$P_{ia}\left\{L^1 > t\right\} \le P_{i0}\left\{L^2 > t\right\} = P_i\left\{L^2 > t\right\} \tag{4.43}$$

since the structure of the system is coherent. Therefore,

$$f(i,a) = E_{ia}\left[L^1\right] \leq E_i\left[L^2\right].$$

Thus, we can conclude that if the MTTF under the maximal repair policy is bounded, then MTTF under the no repair policy is also bounded provided that the system structure is coherent and the component lifetimes have increasing intrinsic aging rates.

In the foregoing analysis, it is always assumed that f is bounded. Now, if we take $f = \widetilde{R}_\Delta h = \sum_{k=0}^{+\infty} \widetilde{P}_\Delta^k h$, then

$$f = h + \widetilde{P}_\Delta\left(\widetilde{R}_\Delta h\right) = h + \left(\widetilde{P}_\Delta \widetilde{R}_\Delta\right) h = h + (\widetilde{R}_\Delta - \boldsymbol{I})h = \widetilde{R}_\Delta h$$

and $f = \widetilde{R}_\Delta h$ is a solution of (4.42) if $\widetilde{R}_\Delta h$ exists. If there is another solution of (4.42), it has the form

$$f = g + \widetilde{R}_\Delta h \tag{4.44}$$

where

$$g = \lim_{n\to+\infty} \widetilde{P}_\Delta^n f$$

provided that $\widetilde{R}_\Delta h$ exists by Riesz decomposition theorem in Revuz [61]. Therefore, if $\lim_{n\to+\infty} \widetilde{P}_\Delta^n f = 0$, then (4.42) has the unique solution $f = \widetilde{R}_\Delta h$. The following result shows that this is indeed true under reasonable assumptions.

Theorem 4.5 *If* $\sup_{(i,a)\in E\times\mathcal{F}} \widetilde{P}^k(i,a;E,\mathcal{F}) < 1$ *for some* $k \in \mathbb{N}$ *and* h *is bounded, then* $\widetilde{R}_\Delta h$ *is the unique solution of (4.42).*

Proof. Let

$$\widetilde{P}^k(i,a;E,\mathcal{F}) \leq 1 - c \tag{4.45}$$

for some $c \in (0,1)$. We now show by induction that

$$\widetilde{P}_\Delta^{kn}(i,a;E,\mathcal{F}) \leq (1-c)^n \tag{4.46}$$

for all $n \in \mathbb{N}$. For $n = 1$, it is true by the hypothesis. Now, assume that it holds for

$1, 2, \cdots, n$. Then,

$$\begin{aligned}
\widetilde{P}_{\Delta}^{(n+1)k}(i,a;E,\mathcal{F}) &= \sum_{j\in E}\int_{\mathcal{F}}\widetilde{P}_{\Delta}^{nk}(i,a;j,db)\,\widetilde{P}_{\Delta}^{k}(j,b;E,\mathcal{F}) \\
&\leq (1-c)\sum_{j\in E}\int_{\mathcal{F}}\widetilde{P}_{\Delta}^{kn}(i,a;j,db) \\
&= (1-c)\,\widetilde{P}_{\Delta}^{kn}(i,a;E,\mathcal{F}) \\
&\leq (1-c)^{n+1}. \qquad (4.47)
\end{aligned}$$

Then,

$$\begin{aligned}
\widetilde{R}_{\Delta}(i,a;E,\mathcal{F}) &= \sum_{n=0}^{+\infty}\widetilde{P}_{\Delta}^{n}(i,a;E,\mathcal{F}) = \sum_{m=0}^{+\infty}\sum_{n=0}^{k-1}\widetilde{P}_{\Delta}^{km+n}(i,a;E,\mathcal{F}) \\
&= \sum_{m=0}^{+\infty}\sum_{n=0}^{k-1}\left(\sum_{j\in E}\int_{\mathcal{F}}\widetilde{P}_{\Delta}^{n}(i,a;j,db)\,\widetilde{P}_{\Delta}^{km}(j,b;E,\mathcal{F})\right) \\
&\leq \sum_{m=0}^{+\infty}\sum_{n=0}^{k-1}(1-c)^{m}\,\widetilde{P}_{\Delta}^{n}(i,a;E,\mathcal{F}) \\
&\leq \sum_{m=0}^{+\infty}k(1-c)^{m} = \frac{k}{c} < +\infty. \qquad (4.48)
\end{aligned}$$

This also implies that $\lim_{n\to+\infty}\widetilde{P}_{\Delta}^{n}(i,a;E,\mathcal{F}) = 0$. Moreover,

$$\begin{aligned}
\widetilde{R}_{\Delta}h(i,a) &= \sum_{j\in E}\int_{\mathcal{F}}\widetilde{R}_{\Delta}(i,a;j,db)\,h(j,b) \leq \sup_{(j,b)\in E\times\mathcal{F}}\{h(j,b)\}\sum_{j\in E}\int_{\mathcal{F}}\widetilde{R}_{\Delta}(i,a;j,db) \\
&= \sup_{(j,b)\in E\times\mathcal{F}}\{h(j,b)\}\,\widetilde{R}_{\Delta}(i,a;E,\mathcal{F}) < +\infty. \qquad (4.49)
\end{aligned}$$

Therefore, $\widetilde{R}_{\Delta}h$ exists and, hence, it is a solution of (4.42). To prove the uniqueness, it suffices to show that $\lim_{n\to+\infty}\widetilde{P}_{\Delta}^{n}f = 0$. Note that

$$\begin{aligned}
\left|\widetilde{P}_{\Delta}^{n}f(i,a)\right| &\leq \sum_{j\in E}\int_{\mathcal{F}}\widetilde{P}_{\Delta}^{n}(i,a;j,db)\,|f(j,b)| \leq \sup_{(j,b)\in E\times\mathcal{F}}\{|f(j,b)|\}\sum_{j\in E}\int_{\mathcal{F}}\widetilde{P}_{\Delta}^{n}(i,a;j,db) \\
&= \sup_{(j,b)\in E\times\mathcal{F}}\{|f(j,b)|\}\,\widetilde{P}_{\Delta}^{n}(i,a;E,\mathcal{F}) \qquad (4.50)
\end{aligned}$$

and $\lim_{n\to+\infty}\widetilde{P}_{\Delta}^{n}f = 0$. ■

It is clear that the boundedness of h is significant for the existence of a unique solution to (4.42). This is a quite reasonable assumption since it is true, for example, if

$$\sup_{i\in E} m(i) < +\infty \qquad (4.51)$$

or

$$\sup_{(i,a)\in E\times\mathcal{F}}\int_0^{+\infty}\overline{v}_{ia}(s)\,ds<+\infty \tag{4.52}$$

by (4.41). This follows by noting that

$$h(i,a)=\int_0^{+\infty}P_{ia}\{L>u,T_1>u\}\,du=E_{ia}[\min\{L,T_1\}]. \tag{4.53}$$

In Theorem 4.5, we do not put any restriction on the age processes of the components or on the structure of the system, but we require that

$$\sup_{(i,a)\in E\times\mathcal{F}}\widetilde{P}^k(i,a;E,\mathcal{F})<1 \tag{4.54}$$

for some $k\in\mathbb{N}$. If component lifetimes have increasing intrinsic aging rates and the structure of the system is coherent, this condition will reduce to a simpler one. We prove that $\widetilde{R}_\Delta h$ is the unique solution under a more easily verifiable condition which simply states that the system may fail during each phase with a strictly positive probability and that these probabilities do not go to 0.

Theorem 4.6 *Suppose that the system is coherent, $r_k(i,a)$ is increasing in a for all $k\in S$ and $i\in E$, and h is bounded. If*

$$\sup_{i\in E}\int_0^{+\infty}F_i(ds)\,\overline{v}_{i0}(s)=\sup_{i\in E}P_{i0}\{L>T_1\}<1 \tag{4.55}$$

or

$$\inf_{i\in E}P_{i0}\{L\le T_1\}=\inf_{i\in E}\int_0^{+\infty}F_i(ds)\,v_{i0}(s)>0 \tag{4.56}$$

then $\widetilde{R}_\Delta h$ is the unique solution of (4.42).

Proof. The equivalence of (4.55) and (4.56) is trivial. Then, clearly,

$$\frac{dh_k(i,a,s)}{da}=\frac{r_k\left(i,H_k\left(i,H_k^{-1}(i,a)+s\right)\right)}{r_k(i,a)}=\frac{r_k\left(i,H_k\left(i,H_k^{-1}(i,a)+s\right)\right)}{r_k\left(i,H_k\left(i,H_k^{-1}(i,a)\right)\right)}\ge 1 \tag{4.57}$$

if component k and the system are in working condition since r_k is always positive and increasing. Therefore, $h_k(i,a,s)$ and $h_k(i,a,s)-a$ are both increasing in a.

We first show that $\overline{v}_{ib}(s)\le\overline{v}_{ia}(s)$ for all $a\le b$, $s\in\mathbb{R}_+$, and $i\in E$. Choose any $a,b\in\mathcal{F}$ such that $b(j)=a(j)$ for every $j\neq k$ for some k. Define

$$B_k=\{c\in\mathcal{F};c(k)<+\infty\} \tag{4.58}$$

so that its complement is

$$\overline{B}_k = \{c \in \mathcal{F}; c(k) = +\infty\}. \tag{4.59}$$

Moreover, let

$$B'_k = \{c \in \mathcal{F}; c(k) = 0\}. \tag{4.60}$$

Clearly, $B'_k \subset B_k$ since component k is brand new in B'_k. It is easy to see that if $c \in \overline{B}_k$ with $\psi_i(c) = 1$, then for any $c^* \in B_k$ with $c^*(j) = c(j)$ for every $j \neq k$, $\psi_i(c^*) = 1$ since the system is coherent. This implies that

$$\int_{B'_k} \psi_i(c) \prod_{j \neq k} \overline{p}^j_{ia(j)}(s, dc(j)) \geq \int_{\overline{B}_k} \psi_i(c) \prod_{j \neq k} \overline{p}^j_{ia(j)}(s, dc(j)). \tag{4.61}$$

It is clear that $\overline{v}_{ib}(s) - \overline{v}_{ia}(s) = 0$ if $a(k) = b(k)$. Consider first the case $a(k) < b(k) < +\infty$. Note that on B_k, $\overline{p}^k_{ia(k)}(s, dc(k)) = \exp(-(h_k(i, a(k), s) - a(k)))$ if $a(k) < +\infty$ and $c(k) = h_k(i, a(k), s)$ at time s; otherwise, it is zero. Then, using (3.79), (2.12), and (2.13),

$$\begin{aligned}
\overline{v}_{ib}(s) - \overline{v}_{ia}(s) &= \int_{B_k} (\overline{p}_{ib}(s, dc) - \overline{p}_{ia}(s, dc)) + \int_{\overline{B}_k} (\overline{p}_{ib}(s, dc) - \overline{p}_{ia}(s, dc)) \\
&= \int_{B_k} \psi_i(c) \prod_{j \neq k} \overline{p}^j_{ia(j)}(s, dc(j)) e^{-(c(k)-b(k))} I_{\{c(k)=h_k(i,b(k),s)\}} \\
&\quad - \int_{B_k} \psi_i(c) \prod_{j \neq k} \overline{p}^j_{ia(j)}(s, dc(j)) e^{-(c(k)-a(k))} I_{\{c(k)=h_k(i,a(k),s)\}} \\
&\quad + \int_{\overline{B}_k} \psi_i(c) \prod_{j \neq k} \overline{p}^j_{ia(j)}(s, dc(j)) \left(e^{-(h_k(i,a(k),s)-a(k))} - e^{-(h_k(i,b(k),s)-b(k))}\right) \\
&= \int_{B'_k} \psi_i(c) \prod_{j \neq k} \overline{p}^j_{ia(j)}(s, dc(j)) \left(e^{-(h_k(i,b(k),s)-b(k))} - e^{-(h_k(i,a(k),s)-a(k))}\right) \\
&\quad + \int_{\overline{B}_k} \psi_i(c) \prod_{j \neq k} \overline{p}^j_{ia(j)}(s, dc(j)) \left(e^{-(h_k(i,a(k),s)-a(k))} - e^{-(h_k(i,b(k),s)-b(k))}\right) \\
&= \left(e^{-(h_k(i,b(k),s)-b(k))} - e^{-(h_k(i,a(k),s)-a(k))}\right) \\
&\quad \times \left(\int_{B'_k} \psi_i(c) \prod_{j \neq k} \overline{p}^j_{ia(j)}(s, dc(j)) - \int_{\overline{B}_k} \psi_i(c) \prod_{j \neq k} \overline{p}^j_{ia(j)}(s, dc(j))\right) \\
&\leq 0
\end{aligned} \tag{4.62}$$

where the last inequality follows from the fact that $h_k(i, a, s) - a$ is increasing in a. Note also that the third equality holds since $\psi_i(c) = \psi_i(c^*)$ whenever $c(j) = c^*(j)$ for every $j \neq k$ and $c(k)$, $c^*(k) < +\infty$.

Now, suppose that $a(k) < b(k) = +\infty$. Then,

$$\begin{aligned}
\overline{v}_{ib}(s) - \overline{v}_{ia}(s) &= \int_{\overline{B}_k} \left(\overline{p}_{ib}(s,dc) - \overline{p}_{ia}(s,dc)\right) - \int_{B_k} \overline{p}_{ia}(s,dc) \\
&= \int_{\overline{B}_k} \psi_i(c) \prod_{j\neq k} \overline{p}^{j}_{ia(j)}(s,dc(j)) \left(e^{-(h_k(i,a(k),s)-a(k))}\right) \\
&\quad - \int_{B'_k} \psi_i(c) \prod_{j\neq k} \overline{p}^{j}_{ia(j)}(s,dc(j)) \left(e^{-(h_k(i,a(k),s)-a(k))}\right) \\
&= \left(e^{-(h_k(i,a(k),s)-a(k))}\right) \left(\int_{\overline{B}_k} \psi_i(c) \prod_{j\neq k} \overline{p}^{j}_{ia(j)}(s,dc(j)) \right. \\
&\quad \left. - \int_{B'_k} \psi_i(c) \prod_{j\neq k} \overline{p}^{j}_{ia(j)}(s,dc(j)) \right) \qquad (4.63) \\
&\leq 0. \qquad (4.64)
\end{aligned}$$

Therefore, $\overline{v}_{ib}(s) \leq \overline{v}_{ia}(s)$ and this trivially implies that $\overline{v}_{ia}(s) \leq \overline{v}_{i0}(s)$ for all $a \in \mathcal{F}$. Finally,

$$\begin{aligned}
\widetilde{P}(i,a;E,\mathcal{F}) &= \sum_{j\in E} \int_{\mathcal{F}} \widetilde{P}(i,a;j,db) = \sum_{j\in E} \int_{\mathcal{F}} \int_0^{+\infty} \widetilde{Q}(i,a;j,db;ds) \\
&= \sum_{j\in E} \int_{\mathcal{F}} \int_0^{+\infty} Q(i,j,ds)\, \overline{p}_{ia}(s,db) = \sum_{j\in E} \int_0^{+\infty} Q(i,j,ds)\, \overline{v}_{ia}(s) \\
&\leq \sum_{j\in E} \int_0^{+\infty} Q(i,j,ds)\, \overline{v}_{i0}(s) = \int_0^{+\infty} F_i(ds)\, \overline{v}_{i0}(s) \qquad (4.65)
\end{aligned}$$

and

$$\sup_{(i,a)\in E\times\mathcal{F}} \widetilde{P}(i,a;E,\mathcal{F}) < 1. \qquad (4.66)$$

Therefore, $\widetilde{R}_{\Delta} h$ is the unique solution of (4.42) by Theorem 4.5. ■

Chapter 5

AVAILABILITY ANALYSIS

In this chapter, we will analyze the system availability of mission-based systems with semi-Markov mission, which is defined as

$$A = \lim_{t \to +\infty} P\left\{\text{system is in working condition at rime } t\right\}.$$

It will be characterized by defining a new Markov renewal process, which now includes repair action on the mission process, and applying some limiting results for the solutions of Markov renewal equations. We also propose some sufficient conditions for the existence of the limit.

5.1 Models with Maximal Repair

In this section, there is maximal repair so that the whole system is overhauled at the completion of each phase of the mission such that it becomes good as new before the next phase starts. Therefore, there are two types of repairs. The first type of repair is preventive and it is done at the successful completion of a phase without system failure and requires the preventive replacement of all components with brand new ones. The second type of repair follows system failure and the whole system is replaced by a brand new one. In this study, we assume that the duration of the first type of repair is negligible and that the duration of the second type of repair has a general distribution with some probability distribution function G_Δ that has a finite mean $m(\Delta)$. Throughout the remainder of this section, repair activity is referred to as the second type or failure repair. We also assume that the first phase which will be performed initially and after each repair following system failure is determined according to some initial distribution μ on E.

In order to analyze the availability under the maximal repair policy, we define a new Markov renewal process $(\widehat{X}, \widehat{T})$, which includes the repair action on the mission process.

We obtain $(\widehat{X}, \widehat{T})$ through its minimal semi-Markov process $\widehat{Y}$ defined through

$$\widehat{Y}_t = \begin{cases} Y_t & \text{if the system is functioning at time } t \\ \Delta & \text{if the system is being repaired at time } t \end{cases}$$

where Δ denotes that the system is under repair. This also implies that

$$\widehat{T}_n = \begin{cases} 0 & \text{if } n = 0 \\ \inf\left\{t > \widehat{T}_{n-1}; \widehat{Y}_t \neq \widehat{Y}_{\widehat{T}_{n-1}}\right\} & \text{if } n \geq 1 \end{cases}$$

and $\widehat{X}_n = \widehat{Y}_{\widehat{T}_n}$ for $n \geq 0$. It is clear that $\widehat{Y}$ follows the mission process until system failure. Then, it jumps to Δ and stays there during repair for a random amount of time with distribution G_Δ. After the repair, it again starts to follow the mission process with the initial distribution μ. Therefore, availability can be defined as

$$A = \lim_{t \to +\infty} P_i\left\{\widehat{Y}_t \neq \Delta\right\} = 1 - \lim_{t \to +\infty} P_i\left\{\widehat{Y}_t = \Delta\right\}. \tag{5.1}$$

The semi-Markov kernel of $(\widehat{X}, \widehat{T})$ is

$$\widehat{Q}(i, j, ds) = \begin{cases} Q(i, j, ds)\,\overline{p}_i(s) & \text{if } i, j \in E \\ \overline{F}_i(s)\,p_i(ds) & \text{if } i \in E, j = \Delta \\ G_\Delta(ds)\,\mu(j) & \text{if } i = \Delta, j \in E \end{cases} \tag{5.2}$$

for all $i, j \in \widetilde{E}$. The transition matrix of the Markov chain $\widehat{X}$ is

$$\widehat{P}(i, j) = \begin{cases} \int_0^{+\infty} Q(i, j, ds)\,\overline{p}_i(s) & \text{if } i, j \in E \\ \int_0^{+\infty} \overline{F}_i(s)\,p_i(ds) & \text{if } i \in E, j = \Delta \\ \mu(j) & \text{if } i = \Delta, j \in E \end{cases} \tag{5.3}$$

and the Markov renewal function of $(\widehat{X}, \widehat{T})$ is

$$\widehat{R}(i, j, s) = \sum\nolimits_{n=0}^{+\infty} \widehat{Q}^n(i, j, s).$$

We let $\widehat{m}(i) = E_i[\widehat{T}_1]$ denote the mean sojourn time of phase i for the process $(\widehat{X}, \widehat{T})$. It is clear that $\widehat{m}(\Delta) = m(\Delta)$ and

$$\begin{aligned} \widehat{m}(i) &= E_i[\min\{T_1, L\}] = \int_0^{+\infty} P_i\{T_1 > s, L > s\}\,ds = \int_0^{+\infty} \overline{F}_i(s)\,\overline{p}_i(s)\,ds \\ &= \int_0^{+\infty} \overline{F}_i(s)\,ds - \int_0^{+\infty} \overline{F}_i(s)\,p_i(s)\,ds = m(i) - \int_0^{+\infty} \overline{F}_i(s)\,p_i(s)\,ds \end{aligned} \tag{5.4}$$

for all $i \in E$.

It is known that $\widehat{X}$ should be irreducible and recurrent with a positive invariant measure for the existence of the limit in (5.1). Therefore, we need to put some additional assumptions on the model to guarantee these conditions. Throughout the remainder of this section, we let

$$f(i) = P_i\left\{\widehat{X}_1 = \Delta\right\} = \widehat{P}(i,\Delta) = \int_0^{+\infty} \overline{F}_i(s)\, p_i(ds)$$

denote the probability that the system will fail during phase $i \in E$. We also let $\widehat{P}_\Delta$ be the matrix obtained by deleting the row and the column corresponding to state Δ from the matrix $\widehat{P}$. The potential matrix corresponding to $\widehat{P}_\Delta$ is defined as

$$\widehat{R}_\Delta = \sum\nolimits_{n=0}^{+\infty} \widehat{P}_\Delta^n.$$

Assumption 5.1 *X is irreducible.*

Assumption 5.2 *For every $i \in E$,*

$$0 < \sup_{j\in E} \inf\left\{s \in \mathbb{R}_+; G(i,j,s) > 0\right\} < \sup\left\{s \in \mathbb{R}_+; \overline{p}_i(s) > 0\right\}. \tag{5.5}$$

Assumption 5.3 $\sup_{i\in E} f(i) = \sup_{i\in E} \widehat{P}(i,\Delta) > 0.$

Assumption 5.4 $\mu^T \widehat{R}_\Delta f = 1.$

Assumption 5.2 simply states that the system is capable of completing every phase with a positive probability, i.e., for any t that satisfies $0 < \sup_{j\in E} \inf\{s \in \mathbb{R}_+; G(i,j,s) > 0\} < t < \sup\{s \in \mathbb{R}_+; \overline{p}_i(s) > 0\}$, we have $G(i,j,t)\,\overline{p}_i(t) > 0$ for all $j \in E$. This implies that

$$\begin{aligned} P_i\{L > T_1 | X_1 = j\} &= \int_0^{+\infty} P_i\{L > T_1 | T_1 \in ds, X_1 = j\}\, P_i\{T_1 \in ds | X_1 = j\} \\ &= \int_0^{+\infty} \overline{p}_i(s)\, G(i,j,ds) \geq \int_0^{t} \overline{p}_i(s)\, G(i,j,ds) \\ &\geq \overline{p}_i(t)\, G(i,j,t) > 0 \end{aligned}$$

where the last inequality follows from the fact that $\overline{p}_i$ is nonincreasing. This implies that $P_i\{L > T_1\} > 0$ for all $i \in E$. Assumption 5.3 states that $\widehat{P}(i,\Delta) = P_i\{L \leq T_1\} > 0$ for some $i \in E$ and, hence, eliminates the unrealistic case where the system can not fail during any phase. Clearly, these assumptions are very reasonable in real life applications. We will later show that Assumption 5.4 guarantees the recurrence of Δ. If it is not, it is transient and, hence, the availability will be 1 trivially. Moreover, even though Assumption 5.4 looks quite strict, it can be easily verified when E is finite. For instance, suppose that $\boldsymbol{I} - \widehat{P}_\Delta$ is invertible so that the potential matrix is

$$\widehat{R}_\Delta = \left(\boldsymbol{I} - \widehat{P}_\Delta\right)^{-1}.$$

It is clear that $f(i) = \widehat{P}(i, \Delta)$ satisfies

$$f = \left(\boldsymbol{I} - \widehat{P}_\Delta\right)\mathbf{1}.$$

These imply that

$$\mu^T \widehat{R}_\Delta f = \mu^T \left(\boldsymbol{I} - \widehat{P}_\Delta\right)^{-1} f = \mu^T \mathbf{1} = 1.$$

However, if E is countably infinite, the computational issues related to the potential matrix $\widehat{R}_\Delta$ may be difficult to deal with.

Proposition 5.5 *Suppose that Assumptions 5.1-5.3 hold. Then, $\widehat{X}$ is irreducible.*

Proof. *Choose arbitrary $i_0, j \in E$. Since X is irreducible, there is $n \in \mathbb{N}$ such that $P^n(i,j) > 0$ and we can find a path $i_0, i_1, i_2, \cdots, i_{n-1}, j \in E$ such that*

$$P(i_0, i_1) P(i_1, i_2) \cdots P(i_{n-1}, j) > 0. \tag{5.6}$$

Then, using (5.5), we can find $t_{i_0}, t_{i_1}, t_{i_2}, \cdots, t_{i_{n-1}}$ such that

$$G(i_k, i_{k+1}, t_{i_k}) \overline{p}_{i_k}(t_{i_k}) > 0$$

for all $k = 0, 1, \cdots, n-1$ where $i_n = j$. This result and (5.6) imply that

$$Q(i_k, i_{k+1}, t_{i_k}) \overline{p}_{i_k}(t_{i_k}) > 0$$

for all $k = 0, 1, \cdots, n-1$ where $i_n = j$. Then,

$$\begin{aligned} \widehat{P}(i_k, i_{k+1}) &= \int_0^{+\infty} Q(i_k, i_{k+1}, ds) \overline{p}_{i_k}(s) \geq \int_0^{t_{i_k}} Q(i_k, i_{k+1}, ds) \overline{p}_{i_k}(s) \\ &\geq \int_0^{t_{i_k}} Q(i_k, i_{k+1}, ds) \overline{p}_{i_k}(t_{i_k}) = Q(i_k, i_{k+1}, t_{i_k}) \overline{p}_{i_k}(t_{i_k}) > 0 \end{aligned} \tag{5.7}$$

for all $k = 0, 1, \cdots, n-1$ where $i_n = j$. The second inequality in the above derivation follows from the fact that $\overline{p}_i$ is nonincreasing. Then,

$$\widehat{P}^n(i_0, j) \geq \widehat{P}(i_0, i_1) \widehat{P}(i_1, i_2) \cdots \widehat{P}(i_{n-1}, j) > 0$$

using (5.7) so that j is reachable from i_0. Now, choose arbitrary $i \in E$ to show that Δ is also reachable from i. By Assumption 5.3, we have $\widehat{P}(j,\Delta) > 0$ for some $j \in E$. Then, since $i, j \in E$, there exists $n \in \mathbb{N}$ such that $\widehat{P}^n(i,j) > 0$ and, hence, $\widehat{P}^{n+1}(i,\Delta) > 0$. Now choose arbitrary $j \in E$ to show that j is reachable from Δ. It is known that $\mu_i > 0$ for at least one $i \in E$. Therefore,

$$\widehat{P}(\Delta, i) = \int_0^{+\infty} G_\Delta(ds)\,\mu(i) = \mu(i) > 0$$

where $G_\Delta(+\infty) = 1$ by the initial assumption that $m(\Delta) < +\infty$. Since $i, j \in E$, there exists $n \in \mathbb{N}$ such that $\widehat{P}^n(i,j) > 0$ and, hence, $\widehat{P}^{n+1}(\Delta, j) > 0$. ■

If we combine Proposition 5.5 and Assumption 5.4, we obtain that $\widehat{X}$ is irreducible and recurrent. Thus, $\widehat{X}$ has a strictly positive invariant measure $\widehat{v}$ under Assumption 5.1 - Assumption 5.4 by Theorem 6.2.25 in Çınlar [59]. This proves the first part of the following theorem which gives an explicit formula for the availability.

Theorem 5.6 *Suppose that Assumptions 5.1-5.4 hold. Then, $\widehat{X}$ is irreducible and recurrent with a positive invariant measure $\widehat{v}$. Moreover,*

$$A = 1 - \frac{\widehat{v}(\Delta)\, m(\Delta)}{\widehat{v}^T \widehat{m}}. \tag{5.8}$$

Proof. We will first show that

$$P_i\{\mathcal{T}_\Delta = k\} = \widehat{P}_\Delta^{k-1} f(i) \tag{5.9}$$

for all $k \in \mathbb{N}$ where $\mathcal{T}_\Delta$ is the number of transitions until system failure (or first passage time to state Δ by the Markov chain $\widehat{X}$). It is clear that (5.9) holds trivially for $k = 1$. Suppose that

$$P_i\{\mathcal{T}_\Delta = k\} = \widehat{P}_\Delta^{k-1} f(i)$$

and consider $P_i\{\mathcal{T}_\Delta = k+1\}$. Then,

$$\begin{aligned} P_i\{\mathcal{T}_\Delta = k+1\} &= \sum_{j\in E} \widehat{P}(i,j)\, P_j\{\mathcal{T}_\Delta = k\} = \sum_{j\in E} \widehat{P}_\Delta(i,j)\, \widehat{P}_\Delta^{k-1} f(j) \\ &= \widehat{P}_\Delta\left(\widehat{P}_\Delta^{k-1} f\right)(i) = \widehat{P}_\Delta^{k} f(i). \end{aligned}$$

Therefore,

$$P_\Delta\{\mathcal{T}_\Delta = k\} = \sum_{j\in E} \widehat{P}(\Delta, j)\, P_j\{\mathcal{T}_\Delta = k-1\} = \sum_{j\in E} \mu(j)\, \widehat{P}_\Delta^{k-2} f(j) = \mu^T \widehat{P}_\Delta^{k-2} f$$

for $k \geq 2$. This implies that

$$\begin{aligned} P_\Delta \{\mathcal{T}_\Delta < +\infty\} &= \sum_{k=1}^{+\infty} P_\Delta \{\mathcal{T}_\Delta = k\} = \sum_{k=2}^{+\infty} P_\Delta \{\mathcal{T}_\Delta = k\} \\ &= \sum_{k=0}^{+\infty} \mu^T \widehat{P}_\Delta^k f = \mu^T \widehat{R}_\Delta f = 1. \end{aligned} \tag{5.10}$$

Therefore, Δ is recurrent by Assumption 5.4. Moreover, $\widehat{X}$ is irreducible by Proposition 5.5 and, hence, $\widehat{X}$ is irreducible and recurrent. This implies that $\widehat{X}$ has a strictly positive invariant measure $\widehat{v}$ that satisfies $\widehat{v} = \widehat{v}\widehat{P}$ by Theorem 6.2.25 in Çınlar [59]. Now, it is sufficient to show that availability satisfies (5.8). We will use the stated results from Çınlar [59] in the remainder of the proof . Using Markov renewal theory and Proposition 10.5.4, we can write

$$A = 1 - \lim_{t \to +\infty} P \left\{ \widehat{Y}_t = \Delta \right\} = 1 - \lim_{t \to +\infty} \sum_{j \in \widetilde{E}} \int_0^t \widehat{R}(i, j, ds)\, g(j, t - s)$$

where $g(i,t) = I_{\{i=\Delta\}} P_i \left\{ \widehat{T}_1 > t \right\}$. Since $g(i,t)$ is monotone decreasing for each i,

$$\widehat{v}g(0) = \sum_{j \in \widetilde{E}} \widehat{v}(j)\, g(j, 0) = \widehat{v}(\Delta)(1 - G_\Delta(0)) < +\infty$$

and

$$\int_0^{+\infty} \widehat{v}g(t)\, dt = \int_0^{+\infty} \sum_{j \in \widetilde{E}} \widehat{v}(j)\, g(j,t)\, dt = \widehat{v}(\Delta)\, m(\Delta) < +\infty$$

g is directly Riemann integrable with respect to $\widehat{v}$ by Proposition 10.4.15. Then, applying Theorem 10.4.17, we get

$$\begin{aligned} A &= 1 - \frac{1}{\widehat{v}^T \widehat{m}} \int_0^{+\infty} \sum_{j \in \widetilde{E}} \widehat{v}(j)\, g(j,s)\, ds = 1 - \frac{1}{\widehat{v}^T \widehat{m}} \int_0^{+\infty} \widehat{v}(\Delta)\, \overline{G}_\Delta(s)\, ds \\ &= 1 - \frac{\widehat{v}(\Delta)\, m(\Delta)}{\widehat{v}^T \widehat{m}}. \end{aligned}$$

■

It is well-known that, if the state space of an irreducible Markov chain is finite, then it is also non-null recurrent and has a unique *invariant distribution* that is obtained by normalizing the invariant measure. We therefore have the following result.

Corollary 5.7 *Suppose that Assumption 5.1-5.3 hold and E is finite. Then, $\widehat{X}$ is irreducible, non-null recurrent and has a unique invariant distribution $\widehat{v}$. Moreover, (5.8) holds.*

Proof. Using Proposition 5.5, $\widehat{X}$ is irreducible. Then, since E is finite, $\widehat{X}$ is irreducible, non-null recurrent and has a unique invariant distribution trivially. The proof of (5.8) follows exactly the same steps of the proof of Theorem 5.6. ■

We have shown that $\widehat{X}$ is irreducible, recurrent and has an invariant measure under Assumption 5.1-5.4. However, this invariant measure may not be normalizable if E is not finite. The following gives a sufficient condition for the existence of an invariant distribution even if E is countably infinite.

Proposition 5.8 *Suppose that Assumption 5.1-5.4 hold. If*

$$\sup_{i \in E} \int_0^{+\infty} F_i\,(ds)\,\overline{p}_i\,(s) = \sup_{i \in E} P_i\,\{L > T_1\} < 1$$

then $\widehat{X}$ is non-null recurrent and has a unique invariant distribution.

Proof. *By Theorem 5.6, $\widehat{X}$ is irreducible recurrent and it is sufficient to establish the non-null property of any state. Suppose that*

$$\sup_{i \in E} \int_0^{+\infty} F_i\,(ds)\,\overline{p}_i\,(s) = 1 - c$$

for some $c \in (0, 1)$. For any $i \in E$, we can write

$$E_i\,[\mathcal{T}_\Delta] = 1 + E_i \left[\sum_{n=1}^{+\infty} I_{\left\{L > \widehat{T}_n\right\}} \right] = 1 + \sum_{n=1}^{+\infty} P_i \left\{ L > \widehat{T}_n \right\} = 1 + \sum_{n=1}^{+\infty} P_i\,\{L > T_n\}$$

where the last inequality follows from the fact that all transitions before the system failure occur at the end of the phases, and

$$E_i\,[\mathcal{T}_\Delta] = 1 + \sum_{n=1}^{+\infty} \widehat{P}^n_\Delta\,(i, E)$$

by using (3.16). We will show that $\widehat{P}^n_\Delta\,(i, E) \le (1 - c)^n$. If $n = 1$, then

$$\widehat{P}_\Delta\,(i, E) = \sum_{j \in E} \widehat{P}_\Delta\,(i, j) = \sum_{j \in E} \int_0^{+\infty} \overline{p}_i\,(s)\,Q\,(i, j, ds) = \int_0^{+\infty} \overline{p}_i\,(s)\,F_i\,(ds) \le 1 - c.$$

Suppose that $\widehat{P}^n_\Delta\,(i, E) \le (1 - c)^n$ holds and consider $\widehat{P}^{n+1}_\Delta\,(i, E)$. Then,

$$\widehat{P}^{n+1}_\Delta\,(i, E) = \sum_{j \in E} \widehat{P}_\Delta\,(i, j)\,\widehat{P}^n_\Delta\,(j, E) \le (1 - c)^n \sum_{j \in E} \widehat{P}_\Delta\,(i, j) = (1 - c)^n\,\widehat{P}_\Delta\,(i, E) \le (1 - c)^{n+1}.$$

Thus,

$$E_i\,[\mathcal{T}_\Delta] \le 1 + \sum_{n=1}^{+\infty} (1 - c)^n = \frac{1}{c} < +\infty.$$

Note that the right-hand side is independent of i and it is clear that

$$E_{\Delta}\left[\mathcal{T}_{\Delta}\right] = 1 + \sum_{i \in E} \mu(i) E_i\left[\mathcal{T}_{\Delta}\right] \leq 1 + \frac{1}{c} < +\infty.$$

Therefore, Δ is non-null recurrent. Moreover, $\widehat{X}$ is irreducible by Theorem 5.6, and $\widehat{X}$ is non-null recurrent trivially. The existence of the unique invariant distribution follows immediately. ■

Suppose that a more strict version of Assumption 5.3

$$\inf_{i \in E} P_i\{L \leq T_1\} > 0 \tag{5.11}$$

holds. This implies that

$$\sup_{i \in E} P_i\{L > T_1\} = 1 - \inf_{i \in E} P_i\{L \leq T_1\} < 1.$$

Therefore, Assumptions 5.1 - 5.2, Assumption 5.4 and (5.11) imply the non-null recurrence of $\widehat{X}$ by Proposition 5.8.

As a special case, suppose that Assumptions 5.1-5.2 hold and

$$\widehat{P}(i, \Delta) = \int_0^{+\infty} \overline{F}_i(s) p_i(ds) = q \tag{5.12}$$

for all $i \in E$ and for some $q > 0$. This implies that

$$P_i\{L \leq T_1\} = \int_0^{+\infty} F_i(ds) p_i(s) = \int_0^{+\infty} \overline{F}_i(s) p_i(ds) = q > 0$$

and Assumption 5.3 holds. Therefore, $\widehat{X}$ is irreducible by Proposition 5.5. Moreover, (5.12) also implies that

$$P_i\{\mathcal{T}_{\Delta} = k\} = (1-q)^{k-1} q$$

for all $k \in \mathbb{N}$. Then,

$$P_i\{\mathcal{T}_{\Delta} < +\infty\} = \sum_{k=1}^{+\infty} P_i\{\mathcal{T}_{\Delta} = k\} = \sum_{k=1}^{+\infty} (1-q)^{k-1} q = 1$$

and

$$P_{\Delta}\{\mathcal{T}_{\Delta} < +\infty\} = \sum_{i \in E} \mu(i) P_i\{\mathcal{T}_{\Delta} < +\infty\} = 1$$

which implies that Δ is a recurrent state and Assumption 5.4 holds by (5.10). Furthermore,

$$\int_0^{+\infty} F_i(ds) \overline{p}_i(s) = 1 - \int_0^{+\infty} F_i(ds) p_i(s) = 1 - \int_0^{+\infty} \overline{F}_i(s) p_i(ds) = 1 - q < 1$$

so that $\widehat{X}$ is irreducible and non-null recurrent by Proposition 5.8.

It is clear that Theorem 5.6 defines the availability in terms of the invariant measure $\widehat{v}$ of $\widehat{X}$. Let v be the invariant distribution of the process X. Then, the following result is helpful to compute availability in terms of v provided that both X and $\widehat{X}$ have invariant distributions v and $\widehat{v}$ respectively.

Theorem 5.9 *Let v and $\widehat{v}$ be the invariant distributions of the processes X and $\widehat{X}$ respectively. Then, for all $i \in E$*

$$\widehat{v}(i) = v(i) - \alpha(i)$$

and

$$\widehat{v}(\Delta) = \alpha^T \mathbf{1}$$

where the row vector α satisfies the system of linear equations

$$\left(\boldsymbol{I} - P + B + \mathbf{1}\mu^T\right)^T \alpha = B^T v \tag{5.13}$$

with

$$B(i,j) = \int_0^{+\infty} Q(i,j,ds)\, p_i(s) \tag{5.14}$$

for all i, $j \in E$.

Proof. *We know that $v^T = v^T P$ and $\widehat{v}^T = \widehat{v}^T \widehat{P}$. We can rewrite $\widehat{P}$ as*

$$\widehat{P} = \begin{bmatrix} P - B & B\mathbf{1} \\ \mu^T & 0 \end{bmatrix}$$

where μ is the initial distribution, and the last column and the last row correspond to the state Δ. This follows by noting that

$$P(i,j) - B(i,j) = \int_0^{+\infty} Q(i,j,ds) - \int_0^{+\infty} Q(i,j,ds)\, p_i(s) = \int_0^{+\infty} Q(i,j,ds)\, \overline{p}_i(s) = \widehat{P}(i,j)$$

and

$$\begin{aligned} B\mathbf{1}(i) &= \sum_{j\in E} B(i,j) = \sum_{j \in E} \int_0^{+\infty} Q(i,j,ds)\, p_i(s) \\ &= \int_0^{+\infty} F_i(ds)\, p_i(s) = \int_0^{+\infty} F_i(ds) \int_0^s p_i(du) \\ &= \int_0^{+\infty} p_i(du) \int_u^{+\infty} F_i(ds) = \int_0^{+\infty} \overline{F}_i(u)\, p_i(du) = \widehat{P}(i,\Delta) \end{aligned}$$

for all $i, j \in E$*. Let* $\alpha(i) = v(i) - \widehat{v}(i)$ *for all* $i \in E$*. Since* $\widehat{v}^T = \widehat{v}^T \widehat{P}$ *we have*

$$\begin{bmatrix} v^T - \alpha^T & \widehat{v}(\Delta) \end{bmatrix} \begin{bmatrix} P - B & B\mathbf{1} \\ \mu^T & 0 \end{bmatrix} = \begin{bmatrix} v^T - \alpha^T & \widehat{v}(\Delta) \end{bmatrix}.$$

Then,

$$\left(v^T - \alpha^T\right)(P - B) + \widehat{v}(\Delta)\mu^T = v^T - \alpha^T$$

and using the fact $\alpha^T\mathbf{1}= \left(v^T - \widehat{v}_{\Delta}^T\right)\mathbf{1} = v^T\mathbf{1} - \widehat{v}^T\mathbf{1} = 1 - \widehat{v}^T\mathbf{1} = \widehat{v}(\Delta)$ *where* $\widehat{v}_{\Delta}$ *is the vector obtained by removing the state* Δ *from the vector* $\widehat{v}$*,, we obtain*

$$v^T P - v^T B - \alpha^T P + \alpha^T B + \left(\alpha^T\mathbf{1}\right)\mu^T = v^T - \alpha^T.$$

Since $v^T = v^T P$*, this simplifies to*

$$\alpha^T - \alpha^T P + \alpha^T B + \alpha^T\left(\mathbf{1}\mu^T\right) = v^T B$$

and

$$\alpha^T\left(\boldsymbol{I} - P + B + \mathbf{1}\mu^T\right) = v^T B$$

which can be rewritten as (5.13). ■

Corollary 5.10 *Let* v *and* $\widehat{v}$ *be the invariant distributions of the processes* X *and* $\widehat{X}$ *respectively. If* E *is finite, then*

$$\widehat{v}(i) = \left(\boldsymbol{I} - \left(\boldsymbol{I} - P + B + \mathbf{1}\mu^T\right)^{-T} B^T\right) v(i) \tag{5.15}$$

for all $i \in E$ *and*

$$\widehat{v}(\Delta) = v^T B\left(\boldsymbol{I} - P + B + \mathbf{1}\mu^T\right)^{-1}\mathbf{1}. \tag{5.16}$$

Proof. *It follows from (5.13) that*

$$\alpha = \left(\boldsymbol{I} - P + B + \mathbf{1}\mu^T\right)^{-T} B^T v \tag{5.17}$$

and

$$\widehat{v}(i) = v(i) - \alpha(i) = \left(\boldsymbol{I} - \left(\boldsymbol{I} - P + B + \mathbf{1}\mu^T\right)^{-T} B^T\right) v(i).$$

Moreover, we have $\widehat{v}(\Delta) = \alpha^T\mathbf{1}$ *by Theorem 5.9 and (5.16) follows from (5.17).* ■

5.1.1 Numerical Illustration

Consider the example of Section 4.1.2. Suppose that the repair duration is uniformly distributed on $[1, 5]$ with $m(\Delta) = 3$, and $\mu^T = [\ 0.4 \quad 0.6 \quad 0 \quad 0\ ]$. Using (5.3),

$$\widehat{P} = \begin{bmatrix} 0 & 0 & 0.300 & 0.700 & 4.95 \times 10^{-5} \\ 0.200 & 0 & 0.400 & 0.400 & 0 \\ 0.198 & 0.793 & 0 & 0 & 0.009 \\ 0.722 & 0.180 & 0 & 0 & 0.098 \\ 0.400 & 0.600 & 0 & 0 & 0 \end{bmatrix}$$

and this implies that

$$\widehat{v}^T = \begin{bmatrix} 0.2982 & 0.2074 & 0.1724 & 0.2917 & 0.0301 \end{bmatrix}. \tag{5.18}$$

Then, using (5.4),

$$\widehat{m}^T = \begin{bmatrix} 0.5000 & 0.2000 & 0.8823 & 0.4197 & 3 \end{bmatrix}.$$

These further imply that $A = 0.8374$ by using Theorem 5.6 since Assumptions 5.1-5.4 hold for this case.

The same result can be obtained by applying Theorem 5.9. Note that

$$v^T = \begin{bmatrix} 0.3185 & 0.2017 & 0.1762 & 0.3036 \end{bmatrix}$$

using (4.18) and

$$B = \begin{bmatrix} 0 & 0 & 1.48 \times 10^{-5} & 3.45 \times 10^{-5} \\ 0 & 0 & 0 & 0 \\ 0.0018 & 0.0070 & 0 & 0 \\ 0.0784 & 0.0196 & 0 & 0 \end{bmatrix}$$

using (5.14). These imply that

$$\alpha^T = \begin{bmatrix} 0.0202 & -0.0058 & 0.0038 & 0.0119 \end{bmatrix}$$

using (5.13). Then, it is easy to see that (5.18) holds.

Note that the time points at which the repairs are completed form a renewal process. Then, the availability can also be obtained by using renewal arguments and (4.21) such that

$$A = \frac{\sum_{i=1}^{4} \mu_i E_i\left[L\right]}{\sum_{i=1}^{4} \mu_i E_i\left[L\right] + m(\Delta)} = 0.8374.$$

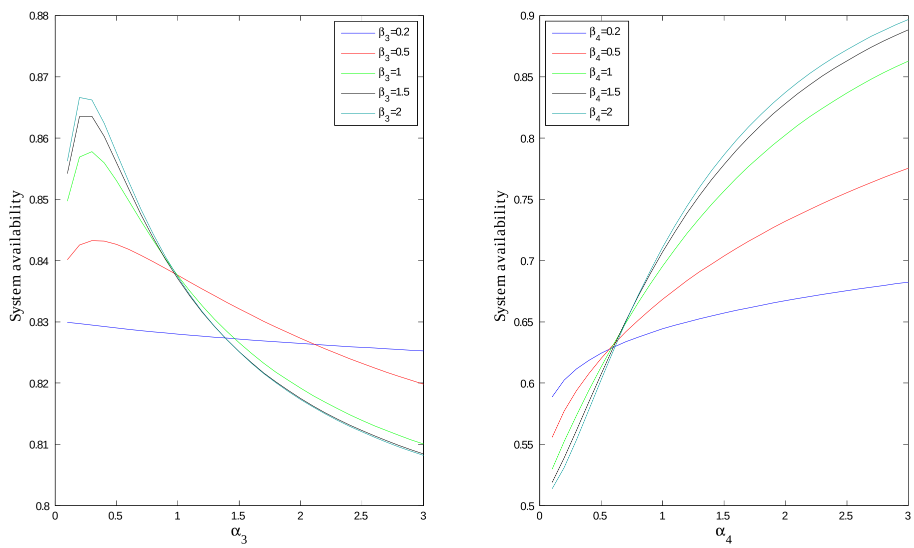

Figure 5.1: System availability vs. α_3 and α_4 for different values of β_3 and β_4.

The behaviors of the availability vs. α_3 and α_4 for different values of β_3 and β_4 are shown in Figure 5.1. Since the failure rates of the components in phase 4 are the highest, availability increases as the mean duration $\alpha_4^{-1/\beta_4}\Gamma(1+1/\beta_4)$ of phase 4 decreases. On the other hand, availability first increases and then decreases, especially for large values of β_3, as the mean duration $\alpha_3^{-1/\beta_3}\Gamma(1+1/\beta_3)$ of phase 3 decreases. When the mean duration of phase 3 decreases, the system starts to stay in the other phases for longer time intervals. If the system stays longer in Hibernation (phases 1 and 2), this will increase the system availability since the failure rates are very low in these phases. However, if the system spends more time in Scientific Observation 2, this will decrease the system availability since the failure rates are the highest in this phase. As seen from Figure 5.1, for lower values of α_3 (or higher mean durations of phase 3), the effect of Hibernation phases to increase availability dominates the effect of Scientific Observation 2 to decrease availability. As α_3 increases more (or average duration of phase 3 decreases more), the effect of Scientific Observation 2 becomes more dominant resulting in decrease of the system availability.

The behaviors of system availability vs. λ_3 and λ_4 are shown in Figure 5.2. As expected, system availability decreases as the failure rates of the components increase.

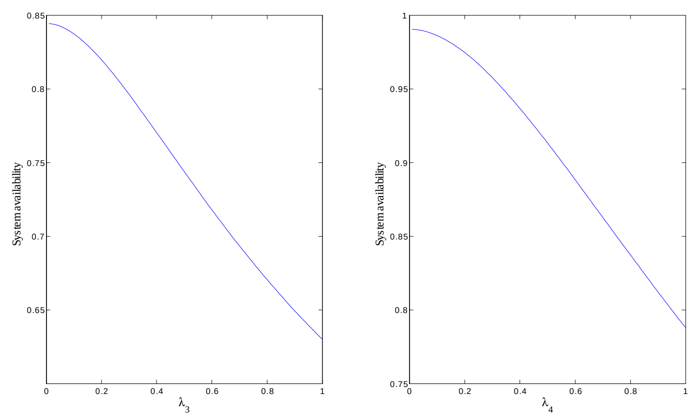

Figure 5.2: System availability vs. λ_3 and λ_4.

5.2 Models with No Repair

In this section, it is assumed that the system performs the mission under the no repair policy so that all components age or deteriorate in time without system or component replacement after the completion of each phase. Since the lifetimes of the components are generally distributed, ages of the components should be stored in the state space using an appropriate model. We use the intrinsic aging model introduced by Çınlar and Özekici [17] in which the intrinsic age of a component is defined as its cumulative hazard. For details of the intrinsic aging model and related notation, see Section 2.3.

We first need to introduce some new notation. It is clear that for a given initial intrinsic age $a \in \mathcal{F}$ with $\psi_i(a) = 1$, the system can not reach all intrinsic ages which are greater than or equal to a. Therefore, we define

$$N_i(a) = \{b \in \mathcal{F}; b \text{ is reachable from } a \text{ in phase } i\},$$

$$\mathcal{F}_i = \{b \in \mathcal{F}; \psi_i(b) = 0\},$$

and

$$\mathcal{S}_i = \{b \in \mathcal{F}; \psi_i(b) = 1\}.$$

Note that $\mathcal{S}_i \cup \mathcal{F}_i = \mathcal{F}$ for all $i \in E$.

The Markov renewal process $((\widetilde{X}, \widetilde{A}), \widetilde{T})$ with semi-Markov kernel (3.82) perfectly represents the state of the system until the system failure denoted by the absorbing state Δ. In Section 3.2, the failure state Δ is assumed to be absorbing because this is very useful for reliability analysis in which only the survival of the system until failure is considered. On the other hand, in availability analysis, repairs must be plugged into the process and, hence, the failure states are not absorbing any more. We will therefore define a new Markov renewal process extending the definition of $((\widetilde{X}, \widetilde{A}), \widetilde{T})$. We assume that the duration of repair has a general distribution with probability distribution function $G_\Delta(i, a; \cdot)$ if the system fails during phase i with intrinsic age a and the system is as good as a brand new one after each repair. We also assume that the system will start to perform phases according to an initial distribution after a repair. We let $\mu(i)$ denote the probability that the initial phase after repair is i with $\sum_{i \in E} \mu(i) = 1$. Note that the process $((\widetilde{X}, \widetilde{A}), \widetilde{T})$ does not have the information on what the intrinsic age of the system is when it fails. However, the repair duration depends on the intrinsic age of the system. Therefore, the extended process should also identify the intrinsic age of the system at the time of failure. For this purpose, we define a new Markov renewal process $((\widehat{X}, \widehat{A}), \widehat{T})$, by extending the definition of $((\widetilde{X}, \widetilde{A}), \widetilde{T})$ with semi-Markov kernel given in (3.32), through its minimal semi-Markov process $\widehat{Y}^a$ such that

i. $\widehat{Y}^a$ follows Y^a until system failure,

ii. If the system fails during phase i at age a, $\widehat{Y}^a$ will stay in state (i, a) during repair time with probability distribution function $G_\Delta(i, a; \cdot)$,

iii. After repair, $\widehat{Y}^a$ again starts to follow Y^a with initial state $(j, \mathbf{0})$ where the initial phase j is chosen according to the initial distribution μ.

Then, the system availability can be defined as

$$A = \lim_{t \to +\infty} P_{ia} \left\{ \widehat{Y}_t^a \in \mathcal{S} \right\} \tag{5.19}$$

where

$$\mathcal{S} = \bigcup_{j \in E} \mathcal{S}_j.$$

The semi-Markov kernel of $((\widehat{X}, \widehat{A}), \widehat{T})$ is

$$\widehat{Q}\left(i, a; j, db; ds\right) = \begin{cases} Q\left(i, j, ds\right) \overline{p}_{ia}\left(s, db\right) & \text{if } i, j \in E, a \in \mathcal{S}_i \\ \overline{F}_i\left(s\right) p_{ia}^L\left(ds, db\right) & \text{if } i \in E, a \in \mathcal{S}_i, b \in \mathcal{F}_i \cap N_i\left(a\right), j = i \\ G_\Delta\left(i, a; ds\right) \mu\left(j\right) & \text{if } i, j \in E, a \in \mathcal{F}_i, b = \mathbf{0} \\ 0 & \text{otherwise} \end{cases} \tag{5.20}$$

for all $i, j \in E$ and $a, b \in \mathcal{F}$ where $p_{ia}^L\left(s, db\right) = P\left\{L \leq s, A_L \in db | Y = i, A_0 = a\right\}$ is the probability that the system will fail before time s with intrinsic age in db given that the initial intrinsic age of the system is a and the initial phase is i.

It is clear that $p_{ia}^L\left(s, db\right)$ heavily depends on the structure of the system. Here, we will give an explicit form of this probability measure for CS. Consider a coherent system with structure function ϕ_i during phase i. Suppose that the system will immediately start to perform phase i with the initial intrinsic age $a \in \mathcal{F}$, and we want to find the probability that our system will fail in ds with the final intrinsic age $b \in \mathcal{F}_i \cap N_i\left(a\right)$. Define the binary vector

$$\theta\left(a\right) = \left[I_{\{a(1) < +\infty\}}, \cdots, I_{\{a(m) < +\infty\}}\right]$$

for every $a \in \mathcal{F}$,

$$C_0\left(x\right) = \left\{k; x_k = 0\right\}$$

for every $x \in B^m$ and $(1_k, x) = y$ where $y_k = 1$ and $y_j = x_j$ for every $j \neq k$. Since $b \in \mathcal{F}_i \cap N_i\left(a\right)$, the system can fail with intrinsic age vector b, and to make this possible, some of the components from the set $C_0\left(\theta\left(b\right)\right) \backslash C_0\left(\theta\left(a\right)\right)$ must fail at time s. However, if $\phi_i\left(\left(1_k, \theta\left(b\right)\right)\right) = 0$, then component k can not be the last one whose failure causes a system failure. Therefore,

$$p_{ia}^L\left(ds, db\right) = \sum_{k \in F_{i,a,b}^S} \overline{U}_{ia(k)}\left(ds\right) \prod_{j \neq k} \overline{p}_{ia(j)}^j\left(s, db\left(j\right)\right)$$

where

$$F^S_{i,a,b} = \{k; k \in C_0(\theta(b)) \backslash C_0(\theta(a)),\ \phi_i((1_k, \theta(b))) = 1\}$$

and

$$\overline{U}_{ia(k)}(s) = P_{ia(k)}\{L(k) \le s | T_1 > s\} = P_{ia(k)}\left\{\widehat{L}(k) \le h_k(i, a(k), s)\right\} = 1 - e^{-(h_k(i,a(k),s)-a(k))}.$$

Moreover, we trivially have

$$\overline{U}_{ia(k)}(ds) = h_k(i, a(k), ds)\, e^{-(h_k(i,a(k),s)-a(k))}$$

and

$$\int_{\mathcal{F}} p^L_{ia}(ds, db) = v_{ia}(ds).$$

We can find the transition kernel of the underlying Markov chain $(\widehat{X}, \widehat{A})$ as

$$\widehat{P}(i, a; j, db) = \begin{cases} \int_0^{+\infty} Q(i, j, ds)\, \overline{p}_{ia}(s, db) & \text{if } i, j \in E,\ a \in \mathcal{S}_i \\ \int_0^{+\infty} \overline{F}_i(s)\, p^L_{ia}(ds, db) & \text{if } i \in E,\ a \in \mathcal{S}_i,\ b \in \mathcal{F}_i \cap N_i(a),\ j = i \\ \mu(j) & \text{if } i,\ j \in E,\ a \in \mathcal{F}_i,\ b = \mathbf{0} \\ 0 & \text{otherwise.} \end{cases} \tag{5.21}$$

In this section, the availability of semi-Markov missions under the no repair policy will be analyzed under the following two assumptions.

Assumption 5.11 *There exists $\lambda_k > 0$ for every component k such that*

$$h_k(i, a, t) \ge a + \lambda_k t$$

for every $i \in E$, and a, $t \in \mathbb{R}_+$.

Assumption 5.12

$$\sup_{(i,a) \in E \times \mathcal{F}} \int_0^{+\infty} F_i(ds)\, \overline{v}_{ia}(s) = \sup_{(i,a) \in E \times \mathcal{F}} P_{ia}\{L > T_1\} < 1. \tag{5.22}$$

Assumption 5.11 states that the intrinsic aging rate of each component is bounded below by a strictly positive constant. If each component works during every phase, this is a very reasonable assumption. Assumption 5.12 guarantees that the system may fail during any phase with a positive probability and these probabilities do not go to zero. Note that if the system structure is coherent and $r_k(i,a)$ is increasing in a for every $k \in S$ and $i \in E$, then

$$\sup_{(i,a)\in E\times\mathcal{F}} \int_0^{+\infty} F_i(ds)\,\overline{v}_{ia}(s) \leq \sup_{i\in E} \int_0^{+\infty} F_i(ds)\,\overline{v}_{i0}(s)$$

since $\overline{v}_{ia}(s) \leq \overline{v}_{i0}(s)$ by the proof of Theorem 4.6.

Lemma 5.13 *Assumption 5.11 implies that*

$$P_{ia}\{L(k) = +\infty\} = 0 \tag{5.23}$$

for all $k \in S$, $i \in E$ *and* $a \in \mathbb{R}_+$.

Proof. *This follows trivially from*

$$\begin{aligned} P_{ia}\{L(k) = +\infty\} &= \lim_{t\to+\infty} P_{ia}\{L(k) > t\} = \lim_{t\to+\infty} P_{ia}\left\{\widehat{L}(k) > A_t(k)\right\} \\ &\leq \lim_{t\to+\infty} P_{ia}\left\{\widehat{L}(k) > a + \lambda_k t\right\} = \lim_{t\to+\infty} e^{-\lambda_k t} = 0 \end{aligned}$$

since $A_t(k) \geq a + \lambda_k t$ *by Assumption 5.11 if* $A_0(k) = a$. ■

Lemma 5.14 *Assumption 5.12 implies that*

$$\sup_{(i,a)\in E\times\mathcal{F}} E_{ia}[\mathcal{T}_{F^*}] < +\infty$$

where

$$\mathcal{T}_B = \inf\left\{k \geq 1; \left(\widehat{X}_k, \widehat{A}_k\right) \in B\right\}$$

and

$$F^* = \{(i,b); i \in E, b \in \mathcal{F}_i\}.$$

Proof. *Choose an arbitrary* $i \in E$ *and suppose that* $a \in \mathcal{S}_i$. *Then,*

$$E_{ia}[\mathcal{T}_{F^*}] = 1 + E_{ia}\left[\sum_{n=1}^{+\infty} 1_{\left\{\widehat{T}_n < L\right\}}\right] = 1 + \sum_{n=1}^{+\infty} P_{ia}\left\{L > \widehat{T}_n\right\} = 1 + \sum_{n=1}^{+\infty} P_{ia}\{L > T_n\}$$

where the last equality follows from the fact that all transitions before a failure occur at the end of the phases. Then, using the mission reliability formula in 3.89, we have

$$E_{ia}[\mathcal{T}_{F^*}] = 1 + \sum_{n=1}^{+\infty} \widetilde{P}^n(i,a;E,\mathcal{F}).$$

Let

$$\sup_{(i,a)\in E\times\mathcal{F}} \int_0^{+\infty} F_i\left(ds\right)\overline{v}_{ia}\left(s\right) = 1-c$$

for some $c \in (0,1)$. *Then,*

$$\begin{aligned} \widetilde{P}\left(i,a;E,\mathcal{F}\right) &= \sum_{j\in E}\int_{\mathcal{F}} \widetilde{P}\left(i,a;j,db\right) = \sum_{j\in E}\int_{\mathcal{F}}\left(\int_0^{+\infty} Q\left(i,j,ds\right)\overline{p}_{ia}\left(s,db\right)\right) \\ &= \int_0^{+\infty} F_i\left(ds\right)\overline{v}_{ia}\left(s\right) \le 1-c. \end{aligned} \tag{5.24}$$

Now, we will show that $\widetilde{P}^n\left(i,a;E,\mathcal{F}\right) \le (1-c)^n$ *for every* i, a, *and* n *by induction on* n. *It is clear from 5.24 that it holds for* $n=1$. *Now, assume that it holds for* $n=k$ *and consider it for* $n=k+1$. *Then,*

$$\widetilde{P}^{k+1}\left(i,a;E,\mathcal{F}\right) = \sum_{j\in E}\int_{\mathcal{F}} \widetilde{P}\left(i,a;j,db\right)\widetilde{P}^k\left(j,b;E,\mathcal{F}\right) \le (1-c)^k\,\widetilde{P}\left(i,a;E,\mathcal{F}\right) \le (1-c)^{k+1}.$$

This implies that

$$E_{ia}\left[\mathcal{T}_{F^*}\right] = 1 + \sum_{n=1}^{+\infty}\widetilde{P}^n\left(i,a;E,\mathcal{F}\right) \le 1 + \sum_{n=1}^{+\infty}(1-c)^n = \frac{1}{c} < +\infty.$$

Now, we will consider the case where $a \in \mathcal{F}_i$ *to complete the proof. Using (5.21),*

$$E_{ia}\left[\mathcal{T}_{F^*}\right] = 1 + \sum_{j\in E}\mu\left(j\right)E_{j0}\left[\mathcal{T}_{F^*}\right] \le 1 + \frac{1}{c}.$$

Therefore,

$$\sup_{(i,a)\in E\times\mathcal{F}} E_{ia}\left[\mathcal{T}_{F^*}\right] \le \frac{c+1}{c} < +\infty$$

which completes the proof. ■

We will now apply a limit theorem in Alsmeyer [62] to find the availability. The main assumption of this theorem is that $(\widehat{X},\widehat{A})$ is positive Harris recurrent. It is well-known that a Markov chain $\{X_n; n\ge 0\}$ is Harris recurrent with respect to a measure φ if

$$P\left\{X_n \in A \text{ for some } n \ge 0 | X_0 = x\right\} = 1$$

for all x whenever $\varphi\left(A\right) > 0$. We refer the interested reader to Meyn and Tweedie [63] for different definitions and characterizations of Harris recurrence. In this chapter, we will use the relationship between petite sets and Harris recurrence. The existence of a small set is actually sufficient for our purpose since each small set is also a petite set.

Definition 5.15 *(Meyn and Tweedie [63])A set C is called a small set if there exists $m > 0$, and a non-trivial measure v such that for all $x \in C$ and all measurable subsets A of the state space,*

$$P^m(x, A) \geq v(A)$$

where P is the transition kernel of the related Markov chain.

Define

$$A^i = \{(i, a); (i, a) \in A\}$$

for any $A \subset E \times \mathcal{F}$,

$$S^* = \{(i, \mathbf{0}); i \in E\}$$

and

$$\chi_i(A) = \begin{cases} 1 & \text{if } A^i \cap S^* \neq \varnothing \\ 0 & \text{otherwise} \end{cases}$$

for any $A \subset E \times \mathcal{F}$, and $i \in E$.

Lemma 5.16 *Let $A_0, A_1, \cdots$ be a disjoint sequence in $E \times \mathcal{F}$. Then,*

$$\chi_i(A) = \chi_i\left(\bigcup_n A_n\right) = \sum_n \chi_i(A_n)$$

for every $i \in E$.

Proof. *It is clear that*

$$A^i = \bigcup_n A_n^i$$

and $A_0^i, A_1^i, \cdots$ is a disjoint sequence for every i. If $A^i \cap S^* = \varnothing$, then $A_n^i \cap S^* = \varnothing$ for every n and, hence, $\chi_i(A_n) = \chi_i(A) = 0$. Suppose that $A^i \cap S^* \neq \varnothing$. Then, there exists k such that $A_k^i \cap S^* \neq \varnothing$. Since $A_0^i, A_1^i, \cdots$ is a disjoint sequence, $A_n^i \cap S^* = \varnothing$ for every $n \neq k$. This implies that

$$\chi_i(A) = 1 = \sum_n \chi_i(A_n) = \chi_i(A_k) = 1$$

and this completes the proof. ■

Proposition 5.17 *The set F^* is a petite set with respect to the probability measure γ defined as*

$$\gamma(A) = \sum_{j \in E} \mu(j) \chi_j(A)$$

for any $A \subset E \times \mathcal{F}$.

Proof. *We will first show that γ is a probability measure. It is trivial that $\chi_i(\varnothing) = 0$ and $\chi_i(E \times \mathcal{F}) = 1$ for every j. These imply that $\gamma(\varnothing) = 0$ and $\gamma(E \times \mathcal{F}) = 1$ since μ is a distribution function on E. Choose an arbitrary disjoint sequence $A_0, A_1, \cdots$ in $E \times \mathcal{F}$ and consider*

$$A = \bigcup_n A_n.$$

Then, using Lemma 5.16,

$$\gamma(A) = \sum_{j \in E} \mu(j)\chi_j(A) = \sum_{j \in E} \mu(j) \sum_n \chi_j(A_n) = \sum_n \sum_{j \in E} \mu(j)\chi_j(A_n) = \sum_n \gamma(A_n).$$

and this implies that γ is a probability measure on $E \times \mathcal{F}$. We will now show that F^* is a small set with respect to γ. Choose an arbitrary subset A of $E \times \mathcal{F}$ and an arbitrary element (i,a) of F^*. We need to show that

$$\widehat{P}(i,a;A) = \sum_{j;(j,0) \in A} \mu(j) \geq \gamma(A) = \sum_{j \in E} \mu(j)\chi_j(A). \tag{5.25}$$

If $\chi_j(A) = 0$ for every $j \in E$, then there is nothing to prove. Suppose that $\chi_j(A) = 1$. Then, $A^j \cap S^* \neq \varnothing$ and, hence, $(j,0) \in A$. This implies that (5.25) holds. Thus, F^* is a small set with respect to μ and, hence, F^* is a petite set since every small set is also a petite set by Meyn and Tweedie [63]. ■

We will utilize the following theorem to show the positive Harris recurrence of $(\widehat{X}, \widehat{A})$.

Theorem 5.18 *(Meyn and Tweedie [64])A Markov chain X is positive Harris recurrent if and only if a petite set A exists with $P_x\{\mathcal{T}_A < +\infty\} = 1$ for all x and $\sup_{x \in A} E_x[\mathcal{T}_A] < +\infty$ where*

$$\mathcal{T}_A = \inf\{k \geq 1; X_k \in A\}.$$

Theorem 5.19 *Suppose that Assumptions 5.11 - 5.12 hold. Then, the Markov chain $(\widehat{X}, \widehat{A})$ with the transition kernel (5.21) is positive Harris recurrent.*

Proof. *We know that F^* is a petite set and $\sup_{(i,a) \in E \times \mathcal{F}} E_{ia}[\mathcal{T}_{F^*}] < +\infty$ by Lemma 5.14 and Proposition 5.17. We need to show that*

$$P_{ia}\{\mathcal{T}_{F^*} < +\infty\} = 1.$$

Choose (i,a) *such that* $a \in \mathcal{S}_i$. *We know by Lemma 5.13 that*

$$P_{ia}\{L(k) = +\infty\} = 0. \tag{5.26}$$

This implies that

$$\begin{aligned} P_{ia}\{L = +\infty\} &\leq P_{ia}\left\{\max_k L(k) = +\infty\right\} = P_{ia}\{L(k) = +\infty \textit{ for some } k = 1,2,\cdots,m\} \\ &\leq \sum_{k=1}^{m} P_{ia}\{L(k) = +\infty\} = 0 \end{aligned} \tag{5.27}$$

where the first equality follows from the finiteness of the number of components and the second inequality follows from the sub-additivity of the probability measure. It is clear that if the system is initially in working condition, then

$$\mathcal{T}_{F^*} = \sup\{n+1; T_n \leq L\}.$$

Therefore,

$$\begin{aligned} P_{ia}\{\mathcal{T}_{F^*} < +\infty\} &= P_{ia}\{\sup\{n+1; T_n \leq L\} < +\infty\} \\ &= 1 - P_{ia}\{\sup\{n+1; T_n \leq L\} = +\infty\} \\ &\geq 1 - P_{ia}\{L = +\infty\} = 1 \end{aligned}$$

where the inequality follows from the assumption that $T_n \to +\infty$ *as* $n \to +\infty$ *and the last equality follows from (5.27). If* $a \in \mathcal{F}_i$, *then next state is* (i,b) *for some* $b \in \mathcal{F}$ *such that* $b \in \mathcal{S}_i$ *with probability* 1. *Then, trivially* $P_{ia}\{\mathcal{T}_{F^*} < +\infty\} = 1$ *for this case too.* ■

Theorem 5.20 *Suppose that Assumptions 5.11 - 5.12 hold. Then, the Markov chain* $(\widehat{X}, \widehat{A})$ *has a unique invariant probability measure* $\widehat{v}$ *and system availability is*

$$A = \frac{1}{\widehat{m}} \sum_{j \in E} \int_{\mathcal{S}_j} \widehat{m}(j,b)\, \widehat{v}(j,db)$$

where

$$\widehat{m}(i,a) = E_{ia}\left[\widehat{T}_1\right]$$

and

$$\widehat{m} = \sum_{i \in E} \int_{\mathcal{F}} \widehat{m}(i,a)\, \widehat{v}(i,da).$$

Proof. *It is well-known that if a Markov chain is positive Harris recurrent, then it has a finite invariant measure which is unique up to a multiplicative constant. Then, since the invariant measure is finite we can define a unique probability measure which is invariant by normalization. Suppose that* $\widehat{v}$ *is the invariant probability measure of the Markov chain* $(\widehat{X}, \widehat{A})$*. The underlying Markov chain* $(\widehat{X}, \widehat{A})$ *of* $\widehat{Y}^a$ *is Harris recurrent by Theorem 5.19. We also know that* $\widehat{v}(\mathcal{S}) < +\infty$*. Then, applying Corollary 1 in [62] and using (5.19), we get*

$$A = \frac{1}{\widehat{m}} \sum_{j \in E} \int_{\mathcal{S}_j} \widehat{m}(j, b)\, \widehat{v}(j, db).$$

■

Chapter 6

MTTF AND AVAILABILITY ANALYSIS OF RELIABILITY SYSTEMS WITH EXPONENTIAL LIFETIMES

6.1 Introduction

In this chapter, the primary objective is to analyze reliability, MTTF, and availability of CS and RS which work under a fixed phase, i.e., the systems under consideration are not mission-based. The lifetimes of all components and the repair times are exponentially distributed independent of each other. The maintenance policy is such that all failed components are replaced by brand new ones only when the whole system fails. Since the lifetimes are exponentially distributed, the system is as good as a brand new one after replacement. Moreover, we presume that the repair times depend on the state of the system (number and type of working and failed components) at the time of failure.

We want to point out that this chapter came out as a by-product of a research project on the component testing problem of mission-based systems. Component testing is done when it is very costly or often impossible to test the whole system. Aircrafts used in space missions or nuclear devices are typical examples where various performance measures associated with the devices can be predicted using data on component lifetimes. The component testing problem determines the optimal testing durations of the components at minimum cost while attaining desired levels of the performance measures. Among them, the primary focus has been the reliability of the system. In almost all of the literature, the components have exponential lifetimes and one has to find explicit expressions for the performance measure as a function of the unknown component failure rates. Using this explicit structure, one can make a semi-infinite linear programming formulation and solve it by an efficient algorithm to find the optimal solution. We refer the reader to Altınel et al. [65] and Altınel et al. [66] for details and examples regarding the component testing problem. The algorithms used in the solution stage use some structural properties like convexity of the performance measure as a function of the failure rates. Our effort in the present setting covers MTTF and availability in addition to a reliability based measure used in all of the literature. The objective is to find explicit functions for these measures and identify their structural properties that may be used in an optimization context. Another line of research in which the results in this chapter may be useful concerns Bayesian analysis of reliability systems. In Bayesian applications, the component failure rates are not assumed to be known; rather, they are random variables with some prior distributions. The explicit structure of the reliability and other functions as a function of the failure rates may be helpful in conducting posterior analysis.

The analysis first focuses on the reliability and MTTF of CS where we obtain DC representations of these measures. Then, we show that MTTF is a ratio of posynomials (RP) for RS. Furthermore, we give explicit formulas for series connection of k-out-of-n subsystems and RS assuming that all components in a subsystem are identical. Then, we discuss system availability for CS and RS to derive a system of linear equations to compute them. As special cases, we consider RS and series connection of k-out-of-n subsystems with identical components in a subsystem. Finally, we show that availability of CS and RS are both RP.

There is a huge amount of literature on CS. Most of these papers assume that whenever a component fails, it is repaired and all components are maintained separately. The distribution of time to failure is analyzed by Barlow and Proschan [67] and Brown [68], and formulas for interval availability, the expected number of failures and replacements in a fixed interval are given by Baxter [69]. In our setting, we analyze a different system where failed components wait for the failure of the system to be replaced.

Systems with k-out-of-n structure attract special attention in reliability literature because they have a very broad application area. MTTF for k-out-of-n systems is analyzed by Angus [70], and mean operating and repair times between two successive breakdowns, system availability and some mean first-passage times are studied by Iyer [71]. Moreover, Li et al. [72] give formulas for mean time between failures, mean working time in a failure-repair cycle and mean down time in a failure-repair cycle. In these studies, it is assumed that all lifetimes and repair times are exponentially distributed, there are enough repairmen for all components and replacement of a component starts immediately after failure.

A common approach to increase reliability is adding warm or cold spare components.

Availability and mean time between failures for k-out-of-n systems with M cold standby units that are identical to actives and different from actives are investigated by Wang and Loman [73]. Availability, the expected up-time and the expected down-time for a k-out-of-n system with general lifetimes and exponential repair times or vice versa are discussed by Frostig and Levikson [74] using Markov renewal processes. The references listed above assume that the repair of a malfunctioning component starts immediately after its failure. A k-out-of-n system in which failed components are not repaired until system failure is analyzed by Kouckỳ [75], and closed form reliability formula is derived for a quite general system. Moreover, de Smidt-Destombes et al. [76] analyze the availability of a k-out-of-n system with identical components whose maintenance starts when the number of failed components exceeds a critical level. In this chapter, we present MTTF and availability results for a model which extends the previous studies to series connection of k-out-of-n subsystems. However, for the sake of a computational tractability, we assume that the lifetimes and the repair times are exponentially distributed and independent of each other, and maintenance is done only when the system fails.

There is also an extensive body of literature on systems with standby redundancy. Natarajan [77] analyzes the time to failure and some limiting probabilities of a single unit system with $N-1$ spares and c repair facilities assuming that all repair times and lifetimes are exponentially distributed. Reliability and availability of a single unit system with cold standby components and general lifetimes is analyzed by Sarkar and Li [78]. Sridharan and Mohanavadivu [79] obtain time dependent and steady-state availability, reliability and MTTF numerically for a two-unit cold standby redundant system with two types of repairmen. Papageorgiou and Kokolakis [80] consider a two-unit parallel system supported by $(n-1)$ standbys with general and non-identical lifetimes and evaluate the system reliability by recursive relations. A series system with standby components is analyzed by Robinson and Neuts [81], and Prasad et al. [82] consider such a system in the spare allocation problem to maximize the system reliability using phase-type and general lifetimes respectively. Wang et al. [32] give explicit expressions for a series system with warm standby components assuming that all lifetimes and repair times are exponentially distributed, and that the repair of a failed component starts immediately after its failure. A similar system with general repair times is analyzed by Wang et al. [83], and explicit expressions for steady-state availability are developed for some special cases. Most of the papers on standby redundancy assume that the failed components are repaired immediately after the failure and they consider only a single standby module. Papers which investigate RS give explicit formulas for some special cases or they only analyze the reliability of the system. In this chapter, we give closed form expressions for the reliability and the MTTF of a series connection of a finite number cold standby redundant subsystems assuming that malfunctioning components wait for the system failure to be repaired, and all lifetimes and repair times are exponentially distributed. We also give a formula for the steady-state availability of these systems.

In Section 6.2, we present reliability and MTTF results, while availability of CS and RS is analyzed in Section 6.3.

6.2 Reliability and MTTF Analysis

Throughout this chapter, the lifetime of component k is exponentially distributed with parameter λ_k, and all component lifetimes are independent of each other. The states of the components are given by the binary processes

$$Z_t\left(k\right) = \begin{cases} 1 & \text{if component } k \text{ is working at time } t \\ 0 & \text{otherwise} \end{cases}$$

for $k \in S$ and $Z_t = \left(Z_t\left(1\right), \cdots, Z_t\left(m\right)\right)$ denotes the state of the system at time t. Clearly, $Z_t\left(k\right) \in B$ and $Z_t \in B^m$.

The analyses on reliability and MTTF are done for CS and RS separately and the results are presented in the following two sections. For any state $y \in B^m$, we let $C_1(y) = \{k; y_k = 1\}$ denote the set of functioning components and $C_0(y) = \{k; y_k = 0\}$ denote the set of failed components. It is clear that $C_0(y) \cap C_1(y) = \varnothing$ with $C_0(y) \cup C_1(y) = S$. For any finite set A, we let $n(A)$ be the cardinality of A.

6.2.1 Coherent Systems

Suppose that we have a coherent system with independent components and some structure function ϕ and reliability function h. Then, system reliability is

$$P\left\{L > t\right\} = P\left\{\phi\left(Z_t\right) = 1\right\} = E\left[\phi\left(Z_t\right)\right] = h(e^{-\lambda_1 t}, e^{-\lambda_2 t}, \cdots, e^{-\lambda_m t}). \tag{6.1}$$

Moreover, MTTF is

$$E[L] = \int_0^{+\infty} E[\phi(Z_t)]\,dt = \int_0^{+\infty} h(e^{-\lambda_1 t}, e^{-\lambda_2 t}, \cdots, e^{-\lambda_m t})dt. \tag{6.2}$$

Let $W = \{y \in B^m; \phi(y) = 1\} \subset B^m$ and $\overline{W} = \{y \in B^m; \phi(y) = 0\} \subset B^m$ denote the set of all path and cut vectors respectively. We now analyze (6.1) and (6.2) in more detail using the following well-known representation of the structure function that states

$$\phi(x) = \sum_{y \in B^m} \phi(y) \prod_{j=1}^{m} x_j^{y_j}(1-x_j)^{1-y_j} = \sum_{y \in W} \left(\prod_{i \in C_1(y)} x_i \right) \prod_{j \in C_0(y)} (1-x_j). \tag{6.3}$$

Note that we can write

$$\begin{aligned} \prod_{j \in C_0(y)} (1-x_j) &= \sum_{k=0}^{n(C_0(y))} (-1)^k \sum_{j_1, j_2, \cdots, j_k \in C_0(y)} x_{j_1} x_{j_2} \cdots x_{j_k} \\ &= \sum_{k=0}^{n(C_0(y))} (-1)^k \sum_{j_1, j_2, \cdots, j_k \in C_0(y)} \prod_{n=1}^{k} x_{j_n} \end{aligned} \tag{6.4}$$

where $j_1, j_2, \cdots, j_k \in C_0(y)$ denotes any combination of k distinct elements in $C_0(y)$. Here, we use the convention that (6.4) is equal to 1 when $k = 0$. Putting (6.3) and (6.4) together, we obtain

$$\begin{aligned} \phi(x) &= \sum_{y \in W} \sum_{k=0}^{n(C_0(y))} (-1)^k \sum_{j_1, j_2, \cdots, j_k \in C_0(y)} \left(\prod_{i \in C_1(y)} x_i \right) \prod_{n=1}^{k} x_{j_n} \\ &= \sum_{y \in W} \sum_{k=0}^{n(C_0(y))} (-1)^k \sum_{j_1, j_2, \cdots, j_k \in C_0(y)} \prod_{\substack{i \in C_1(y) \\ n=1,\cdots,k}} x_i x_{j_n} \end{aligned} \tag{6.5}$$

where $i \neq j_n$ for all $i \in C_1(y)$ and $j_n \in C_0(y)$.

Now, by combining (6.1) and (6.5), we have

$$\begin{aligned} P\{L > t\} &= \sum_{y \in W} \sum_{k=0}^{n(C_0(y))} (-1)^k \sum_{j_1, j_2, \cdots, j_k \in C_0(y)} \prod_{\substack{i \in C_1(y) \\ n=1,\cdots,k}} E[Z_t(i) Z_t(j_n)] \\ &= \sum_{y \in W} \sum_{k=0}^{n(C_0(y))} (-1)^k \sum_{j_1, j_2, \cdots, j_k \in C_0(y)} \prod_{\substack{i \in C_1(y) \\ n=1,\cdots,k}} e^{-(\lambda_i + \lambda_{j_n})t} \\ &= \sum_{y \in W} \sum_{k=0}^{n(C_0(y))} (-1)^k \sum_{j_1, j_2, \cdots, j_k \in C_0(y)} e^{-\left(\sum_{\substack{i \in C_1(y) \\ n=1,\cdots,k}} (\lambda_i + \lambda_{j_n}) \right) t} \end{aligned} \tag{6.6}$$

which is an explicit expression involving sums of exponential functions. Moreover,

$$E[L] = \sum_{y \in W} \sum_{k=0}^{n(C_0(y))} (-1)^k \sum_{j_1, j_2, \cdots, j_k \in C_0(y)} \left(\sum_{\substack{i \in C_1(y) \\ n=1,\cdots,k}} (\lambda_i + \lambda_{j_n}) \right)^{-1} \tag{6.7}$$

using (6.2) and (6.6). Note that both reliability and MTTF are explicit functions of the component failure rates $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_m)$. The structures of these functions play a critical role in Bayesian analysis of reliability systems and optimization studies involving these performance measures. Altınel et al. [65] and Altınel et al. [66] provide an example in the context of the component testing problem. The following result clarifies their structure.

Lemma 6.1 *The functions $f, g : \mathbb{R}_+^m \to \mathbb{R}$ defined as*

$$f(\lambda) = e^{-(a_1\lambda_1 + \cdots + a_m\lambda_m)}$$

and

$$g(\lambda) = \frac{1}{a_1\lambda_1 + \cdots + a_m\lambda_m}$$

are nonnegative, nonincreasing and convex functions for all $a \in \mathbb{R}_+^m$.

Proof. Non-negativity is obvious. The gradients of f and g are

$$\frac{\partial f(\lambda)}{\lambda_i} = -a_i f(\lambda) \leq 0, \ \frac{\partial g(\lambda)}{\lambda_i} = -a_i f(\lambda)^2 \leq 0$$

and f and g are both nonincreasing. The second order partial derivatives of f and g are given by the Hessian matrices

$$[H_f(\lambda)]_{ij} = a_i a_j f(\lambda), [Hg(\lambda)]_{ij} = 2a_i a_j f(\lambda)^3.$$

Take any $z \in \mathbb{R}^m$, then

$$\begin{aligned} z^T H_f z &= \sum_{i=1}^m \sum_{j=1}^m a_i z_i z_j a_j f(\lambda) = \left(\sum_{i=1}^m a_i z_i \right)^2 f(\lambda) \geq 0, \\ z^T H_g z &= 2 \sum_{i=1}^m \sum_{j=1}^m a_i z_i z_j a_j f(\lambda)^3 = 2 \left(\sum_{i=1}^m a_i z_i \right)^2 f(\lambda)^3 \geq 0. \end{aligned}$$

Therefore, H_f and H_g are both positive semidefinite and, hence, f and g are convex. ■

We can now conclude that both system reliability (6.6) and MTTF (6.7) can be represented as a difference of two nonnegative, nonincreasing and convex functions. Moreover,

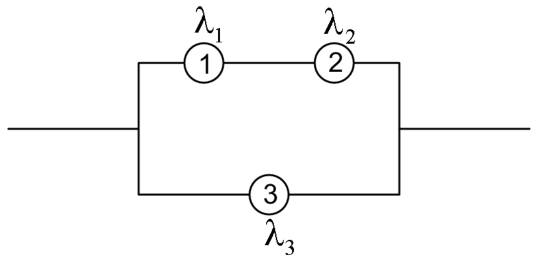

Figure 6.1: The structure of the system analyzed in Example 6.2.

(6.6) and (6.7) provide explicit DC representations of the reliability and MTTF of CS. Note that the two nonnegative, nonincreasing and convex functions of the DC representation are obtained trivially by grouping the terms for even and odd values of k separately.

Example 6.2 *Consider the system given in Figure 6.1. The lifetime of each component is exponentially distributed with the given parameters.The structure function of this system is*

$$\phi(x) = 1 - (1 - x_3)(1 - x_1 x_2) = x_1 x_2 + x_3 - x_1 x_2 x_3.$$

Then, using (6.1),

$$P\{L > t\} = e^{-(\lambda_1+\lambda_2)t} + e^{-\lambda_3 t} - e^{-(\lambda_1+\lambda_2+\lambda_3)t}$$

and this implies that

$$E[L] = \frac{1}{\lambda_1 + \lambda_2} + \frac{1}{\lambda_3} - \frac{1}{\lambda_1 + \lambda_2 + \lambda_3}.$$

6.2.1.1 Series Connection of k-out-of-n Subsystems

Suppose that we have a series system of m subsystems such that at least k_j out of n_j identical components must be in working condition in subsystem j to make the subsystem function. The identical components in subsystem j fail with exponential rates λ_j. This system is clearly a CS and the results presented in Section 6.2.1 hold. Moreover, due to the special structure of this type of systems, we can derive a more explicit formulation for reliability and MTTF. To make the system function every subsystem must work and we can write

$$
\begin{aligned}
P\{L>t\} &= \prod_{j=1}^{m}\sum_{r_j=k_j}^{n_j}\binom{n_j}{r_j}\left(e^{-\lambda_j t}\right)^{r_j}\left(1-e^{-\lambda_j t}\right)^{n_j-r_j} \\
&= \sum_{r_1=k_1}^{n_1}\cdots\sum_{r_m=k_m}^{n_m}\binom{n_1}{r_1}\cdots\binom{n_m}{r_m}S\left(r_1,\cdots,r_m;t\right) \qquad (6.8)
\end{aligned}
$$

where

$$
\begin{aligned}
S\left(r_1,\cdots,r_m;t\right) &= e^{-(r_1\lambda_1+\cdots+r_m\lambda_m)t}\left(1-e^{-\lambda_1 t}\right)^{n_1-r_1}\cdots\left(1-e^{-\lambda_m t}\right)^{n_m-r_m} \\
&= e^{-(r_1\lambda_1+\cdots+r_m\lambda_m)t}\prod_{z=1}^{m}\sum_{s_z=0}^{n_z-r_z}\binom{n_z-r_z}{s_z}(-1)^{s_z}e^{-s_z\lambda_z t} \\
&= \prod_{z=1}^{m}\sum_{s_z=r_z}^{n_z}\binom{n_z-r_z}{s_z-r_z}(-1)^{s_z-r_z}e^{-s_z\lambda_z t} \\
&= \sum_{s_1=r_1}^{n_1}\cdots\sum_{s_m=r_m}^{n_m}\binom{n_1-r_1}{s_1-r_1}\cdots \\
&\qquad\times\binom{n_m-r_m}{s_m-r_m}(-1)^{s-r}e^{-(s_1\lambda_1+\cdots+s_m\lambda_m)t} \qquad (6.9)
\end{aligned}
$$

and $s=s_1+\cdots+s_m$, $r=r_1+\cdots+r_m$. Thus, combining (6.8) and (6.9) and rearranging the combinations, we have

$$
\begin{aligned}
P\{L>t\} &= \sum_{r_1=k_1}^{n_1}\cdots\sum_{r_m=k_m}^{n_m}\sum_{s_1=r_1}^{n_1}\cdots\sum_{s_m=r_m}^{n_m}\binom{n_1}{s_1}\binom{s_1}{r_1}\cdots \\
&\qquad\times\binom{n_m}{s_m}\binom{s_m}{r_m}(-1)^{s-r}e^{-(s_1\lambda_1+\cdots+s_m\lambda_m)t}. \qquad (6.10)
\end{aligned}
$$

Then,

$$
\begin{aligned}
E[L] &= \sum_{r_1=k_1}^{n_1}\cdots\sum_{r_m=k_m}^{n_m}\sum_{s_1=r_1}^{n_1}\cdots\sum_{s_m=r_m}^{n_m}\binom{n_1}{s_1}\binom{s_1}{r_1}\cdots \\
&\qquad\times\binom{n_m}{s_m}\binom{s_m}{r_m}(-1)^{s-r}\left(\frac{1}{s_1\lambda_1+\cdots+s_m\lambda_m}\right). \qquad (6.11)
\end{aligned}
$$

Example 6.3 *Suppose that $m=2, k_1=n_1=1, k_2=2, n_2=3$ with failure rates λ_1 and λ_2. Then, using (6.11),*

$$
\begin{aligned}
E[L] &= \sum_{r_1=1}^{1}\sum_{r_2=2}^{3}\sum_{s_1=r_1}^{1}\sum_{s_2=r_2}^{3}\binom{1}{s_1}\binom{s_1}{r_1}\binom{3}{s_2}\binom{s_2}{r_2}(-1)^{s_1+s_2-r_1-r_2}\frac{1}{s_1\lambda_1+s_2\lambda_2} \\
&= \frac{3}{\lambda_1+2\lambda_2}-\frac{2}{\lambda_1+2\lambda_2}.
\end{aligned}
$$

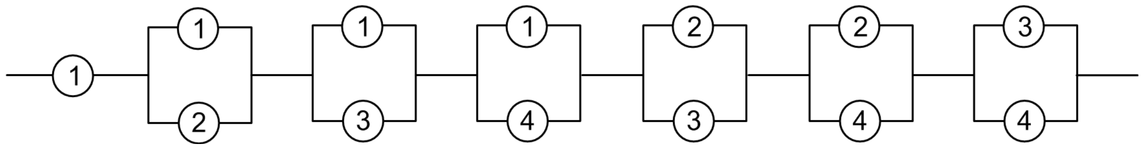

Figure 6.2: Minimal cut representation of the system analyzed in Example 6.3.

The same result can be obtained by using (6.1) and (6.2). Let $Z_1(t)$ *denote the state of the component in the first subsystem and* $Z_2(t), Z_3(t),$ *and* $Z_4(t)$ *denote the state of the components in the second subsystem respectively. The minimal cut sets for this reliability structure are*

$$\begin{aligned} K_1 &= \{1\} \quad K_2 = \{1,2\} \quad K_3 = \{1,3\} \quad K_4 = \{1,4\} \\ K_5 &= \{2,3\} \quad K_6 = \{2,4\} \quad K_7 = \{3,4\} \end{aligned}$$

and minimal cut representation of this system is given in the Figure 6.2.The structure function of this system is

$$\begin{aligned} \phi(x) &= \prod_{i=1}^{3} \prod_{j=i+1}^{4} x_1 (1 - (1 - x_i)(1 - x_j)) \\ &= x_1 x_2 x_3 + x_1 x_2 x_4 + x_1 x_3 x_4 - 2 x_1 x_2 x_3 x_4. \end{aligned}$$

This implies that

$$P\{L > t\} = E[\phi(Z_t)] = e^{-(\lambda_1 + 2\lambda_2)t} + e^{-(\lambda_1 + 2\lambda_2)t} + e^{-(\lambda_1 + 2\lambda_2)t} - 2e^{-(\lambda_1 + 3\lambda_2)t}$$

using (6.1), and

$$E[L] = \frac{3}{\lambda_1 + 2\lambda_2} - \frac{2}{\lambda_1 + 3\lambda_2}$$

using (6.2).

A series connection of parallel subsystems is a special case of series connection of k-out-of-n subsystems. To find the reliability and MTTF of a series connection of parallel subsystems with identical components in each subsystem, (6.10) and (6.11) can be used taking $k_j = 1$ for every j.

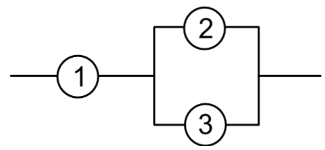

Figure 6.3: The structure of the system analyzed in Example 6.4.

Example 6.4 *Suppose that we have a series connection of 2 parallel subsystems, as given in Figure 6.3. There is only one component in the first subsystem with failure rate* λ_1 *and there are 2 components in the second subsystem with common failure rate* λ_2 *so that* $m = 2, n_1 = 1$ *and* $n_2 = 2$*. Then, using (6.11),*

$$\begin{aligned} E[L] &= \sum_{r_1=1}^{1}\sum_{r_2=1}^{2}\sum_{s_1=r_1}^{1}\sum_{s_2=r_2}^{2}\binom{1}{s_1}\binom{s_1}{r_1}\binom{2}{s_2}\binom{s_2}{r_2}(-1)^{s_1+s_2-r_1-r_2}\frac{1}{s_1\lambda_1+s_2\lambda_2} \\ &= \frac{2}{\lambda_1+\lambda_2}-\frac{1}{\lambda_1+2\lambda_2}. \end{aligned}$$

We can obtain the same result by using (6.1) and (6.2). The structure function of this system is

$$\phi(x) = x_1x_2 + x_1x_3 - x_1x_2x_3.$$

This implies that

$$P\{L > t\} = E[\phi(Z_t)] = e^{-(\lambda_1+\lambda_2)} + e^{-(\lambda_1+\lambda_2)} - e^{-(\lambda_1+2\lambda_2)}$$

and

$$E[L] = \frac{2}{\lambda_1+\lambda_2} - \frac{1}{\lambda_1+2\lambda_2}.$$

6.2.2 Series Connection of Standby Redundant Subsystems

It is well-known that the structure of this kind of systems is not coherent. Therefore, the results in the previous sections are not applicable. However, if we assume that all components have exponential lifetimes and the components in a subsystem are identical, then system reliability can be expressed explicitly. Suppose that there are m subsystems and

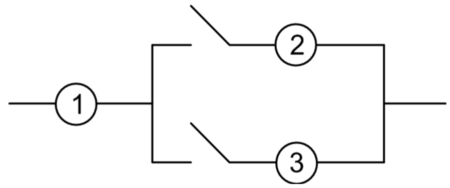

Figure 6.4: The structure of the system analyzed in Example 6.5.

subsystem k consists of n_k identical components with exponential failure rates λ_k. Then,

$$\begin{aligned} P\{L>t\} &= \prod_{k=1}^{m}\sum_{r_k=0}^{n_k-1}\frac{e^{-\lambda_k t}(\lambda_k t)^{r_k}}{r_k!} \\ &= \sum_{r_1=0}^{n_1-1}\cdots\sum_{r_m=0}^{n_m-1}\frac{\lambda_1^{r_1}\cdots\lambda_m^{r_m}}{r_1!\cdots r_m!}e^{-(\lambda_1+\cdots+\lambda_m)t}t^{r_1+\cdots+r_m} \end{aligned} \tag{6.12}$$

and

$$E[L]=\sum_{r_1=0}^{n_1-1}\cdots\sum_{r_m=0}^{n_m-1}\frac{\lambda_1^{r_1}\cdots\lambda_m^{r_m}(r_1+\cdots+r_m)!}{r_1!\cdots r_m!(\lambda_1+\cdots+\lambda_m)^{r_1+\cdots+r_m+1}}. \tag{6.13}$$

It is known that every function with continuous second order partial derivatives is DC. The reader is referred to Horst and Thoai [84] for an overview on the structure and properties of DC functions. Thus, reliability and MTTF of RS are DC, but finding their DC representations is quite difficult. However, we have more information on the structure of MTTF of RS. It is clear that (6.13) is a finite sum of a monomial over a posynomial with variables $\lambda_1,\cdots,\lambda_m$. Since the product of a monomial and a posynomial is a posynomial and posynomials are closed under summation and multiplication, MTTF of RS is RP. Furthermore, in (6.13) each power is a positive integer. Since the integers are closed under summation, we can conclude that MTTF of RS is RP with positive integer powers.

Example 6.5 *Suppose that $m=2, n_1=1,$ and $n_2=2$ with failure rates λ_1 and λ_2 and The structure of the system is as given in Figure 6.4.Then, using (6.13),*

$$E[L]=\sum_{r_1=0}^{1}\frac{\lambda_2^r r!}{r!(\lambda_1+\lambda_2)^{r+1}}=\frac{1}{\lambda_1+\lambda_2}+\frac{\lambda_2}{(\lambda_1+\lambda_2)^2}=\frac{\lambda_1+2\lambda_2}{(\lambda_1+\lambda_2)^2}.$$

6.3 Availability Analysis

We now suppose that the system is maintained by replacing all failed components when the whole system fails. Availability can be determined using Markovian analysis since all lifetimes and repair times are exponentially distributed. The states of the corresponding Markov process will depend on the system structure and we need to find the limiting distribution. We will demonstrate how this is done for CS and RS models. We introduce a new notation such that

$$F_W = \left\{x \in \overline{W}; (1_i, x) \in W \text{ for some } i = 1, \cdots, m\right\}$$

for any $x \in B^m$.

6.3.1 Coherent Systems

As stated earlier, there is no repair action unless the system fails. Repair starts when the system fails by entering some state $x \in F_W$, and we assume that it takes an exponentially distributed amount of time with some rate $\varsigma_x > 0$. After repair all components are in working condition. It is clear that the states of the system follow a Markov process with state space $E = W \cup F_W$ since all lifetimes and repairs are exponentially distributed. To find the limiting distribution, we need to solve the system of linear equations

$$\begin{aligned} \pi_{\mathbf{1}} \sum_{k=1}^{m} \lambda_k &= \sum_{x \in F_W} \pi_x \varsigma_x \\ \pi_x \sum_{j \in C_1(x)} \lambda_j &= \sum_{j \in C_0(x)} \pi_{(1_j, x)} \lambda_j, \quad x \in W \backslash \{\mathbf{1}\} \\ \pi_x \varsigma_x &= \sum_{j \in C_0(x), (1_j, x) \in W} \pi_{(1_j, x)} \lambda_j, \quad x \in F_W \\ \sum_{x \in W} \pi_x + \sum_{x \in F_W} \pi_x &= 1. \end{aligned} \tag{6.14}$$

Then, the availability of the system is

$$A = \sum_{x \in W} \pi_x. \tag{6.15}$$

Note that since $\varsigma_x > 0$ for all $x \in F_W$ and $\lambda_k > 0$ for all $k \in S$, the embedded Markov chain is irreducible with non-null recurrent states. Hence, the system of linear equations (6.14) has a unique solution.

System availability can also be analyzed using renewal theory since the times at which a repaired system starts working form a renewal process. In each renewal cycle, the system works properly for some amount of time until failure and then repair follows. Therefore, it is clear that

$$A = \frac{E[L]}{E[L] + E[R]} \tag{6.16}$$

where R is the repair time. The MTTF $E[L]$ for CS and some important special cases is analyzed in the previous section. Thus, we need to find $E[R]$ to find availability. We know that if the system fails at the state $x \in F_W$, then expected repair time is $1/\varsigma_x$. Therefore, we need to find the distribution of the state where the system fails. For this purpose, it suffices to consider a modification of the Markov process such that each state in F_W is an absorbing state. We then need to compute the probabilities that the process will eventually be absorbed in the absorbing states in F_W given that the initial state is $\mathbf{1}$. Let P be the transition probability matrix of the embedded Markov chain associated with our Markov process and the absorbing states in F_W are represented in the last rows and columns of P. Then, P has the form of

$$P = \begin{bmatrix} Q & M \\ 0 & \boldsymbol{I} \end{bmatrix}. \tag{6.17}$$

Then, it is well-known that the ijth entry of the matrix $(\boldsymbol{I} - Q)^{-1} M$ is the probability that the chain will be eventually absorbed in the absorbing state j given that the initially state is i. Then,

$$E[R] = \sum_{x \in F_W} (\boldsymbol{I} - Q)^{-1} M(\mathbf{1}, x)(1/\varsigma_x) \tag{6.18}$$

since expected repair duration is $1/\varsigma_x$ in state $x \in F_W$.

Example 6.6 *Consider Example 6.2. We have*

$$W = \{(1,1,1), (1,1,0), (1,0,1), (0,1,1), (0,0,1)\}$$

and

$$F_W = \{(0,1,0), (1,0,0), (0,0,0),\}.$$

Moreover, since the first and the second components are identical, we assume that $\varsigma_{100} =$

ς_{010}. *We need to solve the following system of linear equations:*

$$
\begin{aligned}
\pi_{111}(\lambda_1+\lambda_2+\lambda_3) &= \pi_{010}\varsigma_{100}+\pi_{100}\varsigma_{100}+\pi_{000}\varsigma_{000}\\
\pi_{110}(\lambda_1+\lambda_2) &= \pi_{111}\lambda_3\\
\pi_{101}(\lambda_1+\lambda_3) &= \pi_{111}\lambda_2\\
\pi_{011}(\lambda_2+\lambda_3) &= \pi_{111}\lambda_1\\
\pi_{001}\lambda_3 &= \pi_{101}\lambda_1+\pi_{011}\lambda_2\\
\pi_{010}\varsigma_{100} &= \pi_{110}\lambda_1+\pi_{011}\lambda_3\\
\pi_{100}\varsigma_{100} &= \pi_{110}\lambda_2+\pi_{101}\lambda_3\\
\pi_{000}\varsigma_{000} &= \pi_{001}\lambda_3\\
\pi_{111}+\pi_{110}+\pi_{101}+\pi_{011}+\pi_{001}+\pi_{010}+\pi_{100}+\pi_{000} &= 1.
\end{aligned}
$$

The solution is

$$
\begin{aligned}
\pi_{111} &= \frac{\varsigma_{000}\varsigma_{100}\lambda_3(\lambda_1+\lambda_2)(\lambda_1+\lambda_3)(\lambda_2+\lambda_3)}{X}\\
\pi_{110} &= \frac{\varsigma_{000}\varsigma_{100}\lambda_3^2(\lambda_1+\lambda_3)(\lambda_2+\lambda_3)}{X}\\
\pi_{101} &= \frac{\varsigma_{000}\varsigma_{100}\lambda_2\lambda_3(\lambda_1+\lambda_2)(\lambda_2+\lambda_3)}{X}\\
\pi_{011} &= \frac{\varsigma_{000}\varsigma_{100}\lambda_1\lambda_3(\lambda_1+\lambda_2)(\lambda_1+\lambda_3)}{X}\\
\pi_{001} &= \frac{\varsigma_{000}\varsigma_{100}\lambda_1\lambda_2(\lambda_1+\lambda_2)(\lambda_1+\lambda_2+2\lambda_3)}{X}\\
\pi_{010} &= \frac{\varsigma_{000}\lambda_1\lambda_3^2(\lambda_1+\lambda_3)(\lambda_1+2\lambda_2+\lambda_3)}{X}\\
\pi_{100} &= \frac{\varsigma_{000}\lambda_2\lambda_3^2(\lambda_2+\lambda_3)(2\lambda_1+\lambda_2+\lambda_3)}{X}\\
\pi_{000} &= \frac{\varsigma_{100}\lambda_1\lambda_2\lambda_3(\lambda_1+\lambda_2)(\lambda_1+\lambda_2+2\lambda_3)}{X}
\end{aligned}
$$

where

$$
\begin{aligned}
X \;=\; & \varsigma_{000}\Big[\lambda_3^2(\lambda_1+\lambda_2)\Big(\lambda_1^2+(\lambda_2+\lambda_3)^2+\lambda_1(\lambda_2+2\lambda_3)\Big)+\varsigma_{100}(\lambda_1+\lambda_3)(\lambda_2+\lambda_3)\\
& \Big((\lambda_1+\lambda_2)^2+\lambda_3(\lambda_1+\lambda_2)+\lambda_3^2\Big)\Big]+\lambda_1\lambda_2\lambda_3\varsigma_{100}(\lambda_1+\lambda_2)(\lambda_1+\lambda_2+2\lambda_3)
\end{aligned}
$$

and the availability becomes

$$
A=\frac{\varsigma_{100}\varsigma_{000}(\lambda_1+\lambda_3)(\lambda_2+\lambda_3)\Big((\lambda_1+\lambda_2)^2+\lambda_3(\lambda_1+\lambda_2)+\lambda_3^2\Big)}{X}. \tag{6.19}
$$

For this case, we also have

$$P = \begin{bmatrix} 0 & \frac{\lambda_3}{\lambda_1+\lambda_2+\lambda_3} & \frac{\lambda_2}{\lambda_1+\lambda_2+\lambda_3} & \frac{\lambda_1}{\lambda_1+\lambda_2+\lambda_3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{\lambda_1}{\lambda_1+\lambda_2} & \frac{\lambda_2}{\lambda_1+\lambda_2} & 0 \\ 0 & 0 & 0 & 0 & \frac{\lambda_1}{\lambda_1+\lambda_3} & 0 & \frac{\lambda_3}{\lambda_1+\lambda_3} & 0 \\ 0 & 0 & 0 & 0 & \frac{\lambda_2}{\lambda_2+\lambda_3} & \frac{\lambda_3}{\lambda_2+\lambda_3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$E\left[L\right] = \frac{1}{\lambda_1+\lambda_2} + \frac{1}{\lambda_3} - \frac{1}{\lambda_1+\lambda_2+\lambda_3}. \tag{6.20}$$

Moreover, using (6.18),

$$\begin{aligned} E\left[R\right] \quad = \quad & \frac{1}{\varsigma_{100}}\left(\frac{\lambda_1}{\lambda_1+\lambda_2} - \frac{\lambda_1\lambda_2}{\left(\lambda_2+\lambda_3\right)\left(\lambda_1+\lambda_2+\lambda_3\right)}\right) \\ & +\frac{1}{\varsigma_{100}}\left(\frac{\lambda_2}{\lambda_1+\lambda_2} - \frac{\lambda_1\lambda_2}{\left(\lambda_1+\lambda_3\right)\left(\lambda_1+\lambda_2+\lambda_3\right)}\right) \\ & +\frac{\lambda_1\lambda_2\left(\lambda_1+\lambda_2+2\lambda_3\right)}{\varsigma_{000}\left(\lambda_1+\lambda_3\right)\left(\lambda_2+\lambda_3\right)\left(\lambda_1+\lambda_2+\lambda_3\right)}. \end{aligned} \tag{6.21}$$

Using (6.20) and (6.21), we can also obtain the result in (6.19).

We now analyze some important special cases using a different and computationally efficient modelling approach.

6.3.1.1 Series Connection of k-out-of-n Subsystems

Suppose that we now have a series system of m subsystems such that at least k_j out of n_j identical components with exponential failure rates λ_j must be in working condition in subsystem j to make the subsystem function. Let the state space represent the number of available components in each subsystem such that

$$N = \left\{(i_1, \cdots, i_m); i_j = k_j - 1, \cdots, n_j, j = 1, \cdots, m \text{ where } i_j = k_j - 1 \text{ for only one } j\right\}.$$

The system will start in the initial state $\bar{n} = (n_1, \cdots, n_m)$ and it will be repaired whenever it enters a state with $i_j = k_j - 1$ for some failed subsystem j. It is clear that the process can not enter a state with $i_j = k_j - 1$ for more that one j. Therefore, we let

$$F_j = \{x \in N; x_j = k_j - 1 \text{ and } x_i \geq k_i \text{ for every } i \neq j\}$$

be the set of all failure states involving the failure of subsystem j. The system fails whenever it enters a state $x \in F_W = \cup_{j=1,\cdots,m} F_j$ and it takes an exponentially distributed amount of time with some rate $\varsigma_x > 0$. Now, $W = N \backslash F_W$ is the set of all functioning states. It is clear that states of the system follow a Markov process with state space $N = W \cup F_W$ since all lifetimes and repair durations are exponentially distributed and the limiting distribution can be found by solving the system of linear equations

$$\begin{aligned} \pi_{\bar{n}} \sum_{i=1}^{m} n_i \lambda_i &= \sum_{x \in F_W} \pi_x \varsigma_x \\ \pi_x \sum_{i=1}^{m} x_i \lambda_i &= \sum_{j \in O(x)} \pi_{\left(1_j^+, x\right)} \left(1 + x_j\right) \lambda_j, \quad x \in W \backslash \{\bar{n}\} \\ \pi_x \varsigma_x &= \pi_{\left(1_j^+, x\right)} k_j \lambda_j, \quad x \in F_j, j = 1, \cdots, m \\ \sum_{x \in W} \pi_x + \sum_{x \in F_W} \pi_x &= 1 \end{aligned} \tag{6.22}$$

where

$$O(x) = \{j; x_j < n_j\}$$

and

$$\left(1_j^+, x\right) = \left(x_1, \cdots, x_{j-1}, x_j + 1, x_{j+1}, \cdots, x_m\right)$$

for $x \in N$. System availability is given by (6.15). Note that the embedded Markov chain is irreducible with non-null recurrent states, and the system of linear equations (6.22) has a unique solution.

A series connection of redundant subsystems is a special case of series connection of k-out-of-n subsystems. To find the availability of a series connection of redundant subsystem with identical components in each subsystem, it suffices to take $k_j = 1$ for every j.

Example 6.7 *Consider Example 6.3. We will first analyze this example by using the results for CS. For this case, we have*

$$W = \{(1,1,1,1), (1,1,1,0), (1,1,0,1), (1,0,1,1)\}$$

and

$$F_W = \{(0,1,1,1),(0,1,1,0),(1,1,0,0),(1,0,1,0),(0,1,0,1),(1,0,0,1),(0,0,1,1)\}.$$

Moreover, since all components in the second subsystem are identical, we can assume that $\varsigma_{0110} = \varsigma_{0101} = \varsigma_{0011}$ *and* $\varsigma_{1100} = \varsigma_{1010} = \varsigma_{1001}$*. We need to solve the following system of linear equations:*

$$\begin{aligned}
\pi_{1111}(\lambda_1+3\lambda_2) &= \varsigma_{0111}\pi_{0111} + \varsigma_{0110}(\pi_{0110}+\pi_{0101}+\pi_{0011}) + \varsigma_{1100}(\pi_{1100}+\pi_{1010}+\pi_{1001}) \\
\pi_{1110}(\lambda_1+2\lambda_2) &= \lambda_2\pi_{1111} \\
\pi_{1101}(\lambda_1+2\lambda_2) &= \lambda_2\pi_{1111} \\
\pi_{1011}(\lambda_1+2\lambda_2) &= \lambda_2\pi_{1111} \\
\pi_{0111}\varsigma_{0111} &= \lambda_1\pi_{1111} \\
\pi_{0110}\varsigma_{0110} &= \lambda_1\pi_{1110} \\
\pi_{1100}\varsigma_{1100} &= \lambda_2\pi_{1101} + \lambda_2\pi_{1110} \\
\pi_{1010}\varsigma_{1100} &= \lambda_2\pi_{1110} + \lambda_2\pi_{1011} \\
\pi_{0101}\varsigma_{0110} &= \lambda_1\pi_{1101} \\
\pi_{1001}\varsigma_{1100} &= \lambda_2\pi_{1101} + \lambda_2\pi_{1011} \\
\pi_{0011}\varsigma_{0110} &= \lambda_1\pi_{1011} \\
1 &= \sum_{x\in W\cup F_W}\pi_x.
\end{aligned}$$

The solution is

$$\begin{aligned}
\pi_{1111} &= \frac{\varsigma_{0111}\varsigma_{0110}\varsigma_{1100}(\lambda_1+2\lambda_2)}{6\lambda_2^2\varsigma_{0111}\varsigma_{0110} + \varsigma_{1100}\left(3\lambda_1\lambda_2\varsigma_{0111} + \varsigma_{0110}\left(\lambda_1^2 + 5\lambda_2\varsigma_{0111} + \lambda_1(2\lambda_2+\varsigma_{0111})\right)\right)} \\
\pi_{1110} &= \frac{\varsigma_{0111}\varsigma_{0110}\varsigma_{1100}\lambda_2}{6\lambda_2^2\varsigma_{0111}\varsigma_{0110} + \varsigma_{1100}\left(3\lambda_1\lambda_2\varsigma_{0111} + \varsigma_{0110}\left(\lambda_1^2 + 5\lambda_2\varsigma_{0111} + \lambda_1(2\lambda_2+\varsigma_{0111})\right)\right)} \\
\pi_{1101} &= \frac{\varsigma_{0111}\varsigma_{0110}\varsigma_{1100}\lambda_2}{6\lambda_2^2\varsigma_{0111}\varsigma_{0110} + \varsigma_{1100}\left(3\lambda_1\lambda_2\varsigma_{0111} + \varsigma_{0110}\left(\lambda_1^2 + 5\lambda_2\varsigma_{0111} + \lambda_1(2\lambda_2+\varsigma_{0111})\right)\right)} \\
\pi_{1011} &= \frac{\varsigma_{0111}\varsigma_{0110}\varsigma_{1100}\lambda_2}{6\lambda_2^2\varsigma_{0111}\varsigma_{0110} + \varsigma_{1100}\left(3\lambda_1\lambda_2\varsigma_{0111} + \varsigma_{0110}\left(\lambda_1^2 + 5\lambda_2\varsigma_{0111} + \lambda_1(2\lambda_2+\varsigma_{0111})\right)\right)} \\
\pi_{0111} &= \frac{\lambda_1\varsigma_{0110}\varsigma_{1100}(\lambda_1+2\lambda_2)}{6\lambda_2^2\varsigma_{0111}\varsigma_{0110} + \varsigma_{1100}\left(3\lambda_1\lambda_2\varsigma_{0111} + \varsigma_{0110}\left(\lambda_1^2 + 5\lambda_2\varsigma_{0111} + \lambda_1(2\lambda_2+\varsigma_{0111})\right)\right)} \\
\pi_{0110} &= \frac{\lambda_1\lambda_2\varsigma_{0111}\varsigma_{1100}}{6\lambda_2^2\varsigma_{0111}\varsigma_{0110} + \varsigma_{1100}\left(3\lambda_1\lambda_2\varsigma_{0111} + \varsigma_{0110}\left(\lambda_1^2 + 5\lambda_2\varsigma_{0111} + \lambda_1(2\lambda_2+\varsigma_{0111})\right)\right)}
\end{aligned}$$

$$\begin{aligned}
\pi_{1100} &= \frac{2\lambda_2^2\varsigma_{0111}\varsigma_{0110}}{6\lambda_2^2\varsigma_{0111}\varsigma_{0110}+\varsigma_{1100}\left(3\lambda_1\lambda_2\varsigma_{0111}+\varsigma_{0110}\left(\lambda_1^2+5\lambda_2\varsigma_{0111}+\lambda_1\left(2\lambda_2+\varsigma_{0111}\right)\right)\right)}\\
\pi_{1010} &= \frac{2\lambda_2^2\varsigma_{0111}\varsigma_{0110}}{6\lambda_2^2\varsigma_{0111}\varsigma_{0110}+\varsigma_{1100}\left(3\lambda_1\lambda_2\varsigma_{0111}+\varsigma_{0110}\left(\lambda_1^2+5\lambda_2\varsigma_{0111}+\lambda_1\left(2\lambda_2+\varsigma_{0111}\right)\right)\right)}\\
\pi_{0101} &= \frac{\lambda_1\lambda_2\varsigma_{0111}\varsigma_{1100}}{6\lambda_2^2\varsigma_{0111}\varsigma_{0110}+\varsigma_{1100}\left(3\lambda_1\lambda_2\varsigma_{0111}+\varsigma_{0110}\left(\lambda_1^2+5\lambda_2\varsigma_{0111}+\lambda_1\left(2\lambda_2+\varsigma_{0111}\right)\right)\right)}\\
\pi_{1001} &= \frac{2\lambda_2^2\varsigma_{0111}\varsigma_{0110}}{6\lambda_2^2\varsigma_{0111}\varsigma_{0110}+\varsigma_{1100}\left(3\lambda_1\lambda_2\varsigma_{0111}+\varsigma_{0110}\left(\lambda_1^2+5\lambda_2\varsigma_{0111}+\lambda_1\left(2\lambda_2+\varsigma_{0111}\right)\right)\right)}\\
\pi_{0011} &= \frac{\lambda_1\lambda_2\varsigma_{0111}\varsigma_{1100}}{6\lambda_2^2\varsigma_{0111}\varsigma_{0110}+\varsigma_{1100}\left(3\lambda_1\lambda_2\varsigma_{0111}+\varsigma_{0110}\left(\lambda_1^2+5\lambda_2\varsigma_{0111}+\lambda_1\left(2\lambda_2+\varsigma_{0111}\right)\right)\right)}
\end{aligned}$$

and

$$A=\frac{\varsigma_{0111}\varsigma_{0110}\varsigma_{1100}\left(\lambda_1+5\lambda_2\right)}{X} \tag{6.23}$$

where

$$\begin{aligned}
X &= \lambda_1^2\varsigma_{0110}\varsigma_{1100}+\lambda_1\varsigma_{0110}\varsigma_{1100}\varsigma_{0111}+\lambda_1\lambda_2\left(2\varsigma_{0110}\varsigma_{1100}+3\varsigma_{1100}\varsigma_{0111}\right)\\
&\quad +5\lambda_2\varsigma_{0110}\varsigma_{1100}\varsigma_{0111}+6\lambda_2^2\varsigma_{0110}\varsigma_{0111}.
\end{aligned}$$

Using (6.16), we can obtain the same result. For this case, we have

$$P=\begin{bmatrix}
0 & \frac{\lambda_2}{\lambda_1+3\lambda_2} & \frac{\lambda_2}{\lambda_1+3\lambda_2} & \frac{\lambda_2}{\lambda_1+3\lambda_2} & \frac{\lambda_1}{\lambda_1+3\lambda_2} & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & \frac{\lambda_1}{\lambda_1+2\lambda_2} & \frac{\lambda_2}{\lambda_1+2\lambda_2} & \frac{\lambda_2}{\lambda_1+2\lambda_2} & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & \frac{\lambda_2}{\lambda_1+2\lambda_2} & 0 & \frac{\lambda_1}{\lambda_1+2\lambda_2} & \frac{\lambda_2}{\lambda_1+2\lambda_2} & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\lambda_2}{\lambda_1+2\lambda_2} & 0 & \frac{\lambda_2}{\lambda_1+2\lambda_2} & \frac{\lambda_1}{\lambda_1+2\lambda_2}\\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}$$

$$E\left[L\right]=\frac{3}{\lambda_1+2\lambda_2}-\frac{2}{\lambda_1+3\lambda_2}$$

and

$$E\left[R\right]=\frac{\lambda_1}{\varsigma_{0111}\left(\lambda_1+3\lambda_2\right)}+6\frac{\lambda_2^2}{\varsigma_{1100}\left(\lambda_1^2+5\lambda_1\lambda_2+6\lambda_2^2\right)}+3\lambda_1\frac{\lambda_2}{\varsigma_{0110}\left(\lambda_1^2+5\lambda_1\lambda_2+6\lambda_2^2\right)}.$$

These imply that

$$A = \frac{\varsigma_{0111}\varsigma_{0110}\varsigma_{1100}\left(\lambda_1 + 5\lambda_2\right)}{Y} \tag{6.24}$$

which coincides with (6.23). The same result can be obtained by solving the system of linear equations in (6.22). In the first analysis at the beginning of this example, it is assumed that the repair of the component in the first subsystem is exponentially distributed with rate ς_{0111}*, the repair of two components one in the first subsystem and one in the second subsystem is exponentially distributed with parameter* ς_{0110}*, and the repair of two components in the second subsystem is exponentially distributed with parameter* ς_{1100}*. Therefore, to use (6.22), we can assume that ,* $\varsigma_{03} = \varsigma_{0111}, \varsigma_{02} = \varsigma_{0110}$ *and* $\varsigma_{11} = \varsigma_{1100}$*. Then, the following system of linear equations needs to be solved:*

$$\begin{aligned}
\pi_{13}\left(\lambda_1 + 3\lambda_2\right) &= \varsigma_{03}\pi_{03} + \varsigma_{02}\pi_{02} + \varsigma_{11}\pi_{11} \\
\pi_{12}\left(\lambda_1 + 2\lambda_2\right) &= 3\lambda_2\pi_{13} \\
\varsigma_{03}\pi_{03} &= \lambda_1\pi_{13} \\
\varsigma_{02}\pi_{02} &= \lambda_1\pi_{12} \\
\varsigma_{11}\pi_{11} &= 2\lambda_2\pi_{12} \\
\pi_{13} + \pi_{12} + \pi_{03} + \pi_{02} + \pi_{11} &= 1.
\end{aligned}$$

The solution is

$$\begin{aligned}
\pi_{13} &= \frac{\varsigma_{03}\varsigma_{02}\varsigma_{11}\left(\lambda_1 + 2\lambda_2\right)}{6\lambda_2^2\varsigma_{03}\varsigma_{02} + \varsigma_{11}\left(3\lambda_1\lambda_2\varsigma_{03} + \varsigma_{02}\left(\lambda_1^2 + 5\lambda_2\varsigma_{03} + \lambda_1\left(2\lambda_2 + \varsigma_{03}\right)\right)\right)} \\
\pi_{12} &= \frac{3\lambda_2\varsigma_{03}\varsigma_{02}\varsigma_{11}}{6\lambda_2^2\varsigma_{03}\varsigma_{02} + \varsigma_{11}\left(3\lambda_1\lambda_2\varsigma_{03} + \varsigma_{02}\left(\lambda_1^2 + 5\lambda_2\varsigma_{03} + \lambda_1\left(2\lambda_2 + \varsigma_{03}\right)\right)\right)} \\
\pi_{03} &= \frac{\lambda_1\left(\lambda_1 + 2\lambda_2\right)\varsigma_{02}\varsigma_{11}}{6\lambda_2^2\varsigma_{03}\varsigma_{02} + \varsigma_{11}\left(3\lambda_1\lambda_2\varsigma_{03} + \varsigma_{02}\left(\lambda_1^2 + 5\lambda_2\varsigma_{03} + \lambda_1\left(2\lambda_2 + \varsigma_{03}\right)\right)\right)} \\
\pi_{02} &= \frac{3\lambda_1\lambda_2\varsigma_{03}\varsigma_{11}}{6\lambda_2^2\varsigma_{03}\varsigma_{02} + \varsigma_{11}\left(3\lambda_1\lambda_2\varsigma_{03} + \varsigma_{02}\left(\lambda_1^2 + 5\lambda_2\varsigma_{03} + \lambda_1\left(2\lambda_2 + \varsigma_{03}\right)\right)\right)} \\
\pi_{11} &= \frac{6\lambda_2^2\varsigma_{03}\varsigma_{02}}{6\lambda_2^2\varsigma_{03}\varsigma_{02} + \varsigma_{11}\left(3\lambda_1\lambda_2\varsigma_{03} + \varsigma_{02}\left(\lambda_1^2 + 5\lambda_2\varsigma_{03} + \lambda_1\left(2\lambda_2 + \varsigma_{03}\right)\right)\right)}
\end{aligned}$$

and

$$A = \frac{\varsigma_{0111}\varsigma_{0110}\varsigma_{1100}\left(\lambda_1 + 5\lambda_2\right)}{X}$$

which coincides with (6.23) and (6.24).

6.3.2 Series Connection of Standby Redundant Subsystems

Let the state space represent the number of available components in each subsystem so that

$$N = \{(i_1, \cdots, i_m); i_j = 0, 1, \cdots, n_j, j = 1, \cdots, m \text{ where } i_j = 0 \text{ for only one } j\}.$$

The system will start processing in the initial state $\bar{n}$ and it will be repaired whenever it fails by entering a state with $i_j = 0$ when subsystem j fails. It is clear that the process can not enter a state with $i_j = 0$ for more that one j. Therefore, we now let

$$F_j = \{x \in N; x_j = 0 \text{ and } x_i \geq 1 \text{ for every } i \neq j\}$$

be the set of all failure states involving the failure of subsystem j. Repair starts whenever the system enters a state $x \in F_W = \cup_{j=1,\cdots,m} F_j$ and it takes an exponentially distributed amount of time with some rate $\varsigma_x > 0$. Once again, $W = N \backslash F_W$ is the set of all functioning states. It is clear that states of the system follow a Markov process with state space $N = W \cup F_W$ since all lifetimes and repair durations are exponentially distributed and the limiting distribution can be found by solving the system of linear equations

$$\begin{aligned} \pi_{\bar{n}} \sum_{i=1}^{m} \lambda_i &= \sum_{x \in F_W} \pi_x \varsigma_x \\ \pi_x \sum_{i=1}^{m} \lambda_i &= \sum_{j \in O(x)} \pi_{(1_j^+, x)} \lambda_j, \qquad x \in W \backslash \{\bar{n}\} \\ \pi_x \varsigma_x &= \pi_{(1_j^+, x)} \lambda_j, \qquad x \in F_j, j = 1, \cdots, m \\ \sum_{x \in W} \pi_x + \sum_{x \in F_W} \pi_x &= 1. \end{aligned} \tag{6.25}$$

Once again, system availability is given by (6.15). In this case, the imbedded Markov chain is also irreducible with non-null recurrent states and the system of linear equations (6.25) has a unique solution.

For CS, (6.16) and (6.18) define an alternative formula for the system availability using renewal theory. The same formulation can be used to compute the availability of RS since we use Markovian analysis. It is sufficient to replace $\mathbf{1}$ by $\bar{n}$ in (6.18).

Example 6.8 *Consider Example 6.5. To find the limiting probabilities, the following sys-*

tem of linear equations needs to be solved:

$$\begin{aligned}
\pi_{12}(\lambda_1+\lambda_2) &= \varsigma_{02}\pi_{02}+\varsigma_{01}\pi_{01}+\varsigma_{10}\pi_{10}\\
\pi_{11}(\lambda_1+\lambda_2) &= \lambda_2\pi_{12}\\
\varsigma_{02}\pi_{02} &= \lambda_1\pi_{12}\\
\varsigma_{01}\pi_{01} &= \lambda_1\pi_{11}\\
\varsigma_{10}\pi_{10} &= \lambda_2\pi_{11}\\
\pi_{12}+\pi_{11}+\pi_{02}+\pi_{01}+\pi_{10} &= 1.
\end{aligned}$$

The solution is

$$\begin{aligned}
\pi_{12} &= \frac{\varsigma_{02}\varsigma_{01}\varsigma_{10}(\lambda_1+\lambda_2)}{\lambda_2^2\varsigma_{02}\varsigma_{01}+\lambda_1\varsigma_{01}\varsigma_{10}(\lambda_1+\varsigma_{02})+\lambda_2\varsigma_{10}(2\varsigma_{02}\varsigma_{01}+\lambda_1(\varsigma_{02}+\varsigma_{01}))}\\
\pi_{11} &= \frac{\varsigma_{02}\varsigma_{01}\varsigma_{10}\lambda_2}{\lambda_2^2\varsigma_{02}\varsigma_{01}+\lambda_1\varsigma_{01}\varsigma_{10}(\lambda_1+\varsigma_{02})+\lambda_2\varsigma_{10}(2\varsigma_{02}\varsigma_{01}+\lambda_1(\varsigma_{02}+\varsigma_{01}))}\\
\pi_{02} &= \frac{\varsigma_{01}\varsigma_{10}\lambda_1(\lambda_1+\lambda_2)}{\lambda_2^2\varsigma_{02}\varsigma_{01}+\lambda_1\varsigma_{01}\varsigma_{10}(\lambda_1+\varsigma_{02})+\lambda_2\varsigma_{10}(2\varsigma_{02}\varsigma_{01}+\lambda_1(\varsigma_{02}+\varsigma_{01}))}\\
\pi_{01} &= \frac{\varsigma_{02}\varsigma_{10}\lambda_1\lambda_2}{\lambda_2^2\varsigma_{02}\varsigma_{01}+\lambda_1\varsigma_{01}\varsigma_{10}(\lambda_1+\varsigma_{02})+\lambda_2\varsigma_{10}(2\varsigma_{02}\varsigma_{01}+\lambda_1(\varsigma_{02}+\varsigma_{01}))}\\
\pi_{10} &= \frac{\varsigma_{02}\varsigma_{01}\lambda_2^2}{\lambda_2^2\varsigma_{02}\varsigma_{01}+\lambda_1\varsigma_{01}\varsigma_{10}(\lambda_1+\varsigma_{02})+\lambda_2\varsigma_{10}(2\varsigma_{02}\varsigma_{01}+\lambda_1(\varsigma_{02}+\varsigma_{01}))}
\end{aligned}$$

and the availability of the system is

$$\begin{aligned}
A &= \frac{\varsigma_{02}\varsigma_{01}\varsigma_{10}(\lambda_1+2\lambda_2)}{\lambda_2^2\varsigma_{02}\varsigma_{01}+\lambda_1\varsigma_{01}\varsigma_{10}(\lambda_1+\varsigma_{02})+\lambda_2\varsigma_{10}(2\varsigma_{02}\varsigma_{01}+\lambda_1(\varsigma_{02}+\varsigma_{01}))}\\
&= \frac{\varsigma_{02}\varsigma_{01}\varsigma_{10}(\lambda_1+2\lambda_2)}{\lambda_1^2\varsigma_{01}\varsigma_{10}+\lambda_1\varsigma_{01}\varsigma_{10}\varsigma_{02}+\lambda_1\lambda_2\varsigma_{10}(\varsigma_{01}+\varsigma_{02})+2\lambda_2\varsigma_{01}\varsigma_{10}\varsigma_{02}+\lambda_2^2\varsigma_{01}\varsigma_{02}}. \quad (6.26)
\end{aligned}$$

For this example,

$$P=\begin{bmatrix}
0 & \frac{\lambda_2}{\lambda_1+\lambda_2} & \frac{\lambda_1}{\lambda_1+\lambda_2} & 0 & 0\\
0 & 0 & 0 & \frac{\lambda_1}{\lambda_1+\lambda_2} & \frac{\lambda_2}{\lambda_1+\lambda_2}\\
0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 1
\end{bmatrix},$$

$$E[L]=\frac{\lambda_1+2\lambda_2}{(\lambda_1+\lambda_2)^2},$$

and

$$E[R]=\frac{\lambda_1}{\varsigma_{02}(\lambda_1+\lambda_2)}+\frac{\lambda_2^2}{\varsigma_{10}(\lambda_1+\lambda_2)^2}+\lambda_1\frac{\lambda_2}{\varsigma_{01}(\lambda_1+\lambda_2)^2}.$$

These imply that

$$A = \frac{\varsigma_{02}\varsigma_{01}\varsigma_{10}\left(\lambda_1 + 2\lambda_2\right)}{\lambda_1^2\varsigma_{01}\varsigma_{10} + \lambda_1\varsigma_{01}\varsigma_{10}\varsigma_{02} + \lambda_1\lambda_2\varsigma_{10}\left(\varsigma_{01} + \varsigma_{02}\right) + 2\lambda_2\varsigma_{01}\varsigma_{10}\varsigma_{02} + \lambda_2^2\varsigma_{01}\varsigma_{02}}$$

which coincides with (6.26).

6.3.3 Structure of the Availability Function

In this section, we focus on the structure of the availability function for CS and RS. In Section 6.3.1 and 6.3.2, we obtained two systems of linear equations ((6.14) and (6.25)) to find availability where one of the equations can be represented as a linear combination of the others. Therefore, we eliminate the first equation in both systems and the remaining equations can be represented in the matrix form

$$A\pi = b$$

where A is the coefficient matrix, π is a vector representing the limiting distribution, and $b = [0, 0, \cdots, 0, 1]^T$ is the right-hand side vector. It is clear that each entry of A is linear in $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_m)$. Then, the limiting distribution satisfies

$$\pi = A^{-1}b$$

where the existence of A^{-1} follows form the fact that (6.14) and (6.25) have unique solutions under our assumptions. It is well-known that the ijth entry of A^{-1} is equal to the cofactor of the jith entry of A divided by $\det A$, and the cofactor of the jith entry of A is $(-1)^{i+j}$ times the determinant of the submatrix obtained from A by deleting the jth row and the ith column. To find the structure of the availability function, the structures of the determinant of A and the determinant of a submatrix obtained from A is crucial. At this point, recall the Leibniz formula for determinants

$$\det B = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} B_{i,\sigma(i)} \tag{6.27}$$

where B is an $n \times n$ square matrix, S_n is the set of all permutations of the columns, and $\operatorname{sgn}(\sigma)$ denotes the sign of the permutation σ. Then, since each entry of A is linear in λ, we can conclude using (6.27) that the determinant of A and the determinant of a submatrix obtained from A is a polynomial in λ. Thus, each entry of A^{-1} is a ratio of two polynomials in λ and, hence, the limiting probability for each state is a ratio of two polynomials in λ due to the special structure of b. Since availability is a finite sum of some limiting probabilities and polynomials are closed under multiplication and summation, availability is always a ratio of two polynomials in λ.

A polynomial's coefficients need not be positive in general. However, in the previous numerical examples, we observe that availability is a ratio of two polynomials with positive coefficients. In other words, it is a ratio of two posynomials with positive integer powers. This is a more special structure and we will show that this structure holds for the availability of CS and RS.

For CS, it is clear that the number of states in the linear equation system (6.14)-(6.15) is finite. If the initial state is $\mathbf{1}$, then we can reach all states in $W \cup F_W$. Consider the transition rate diagram of the Markovian state process where each state in F_W is made absorbing. Then, we have an acyclic graph since the system always deteriorates to reach a state in F_W. Therefore, we can partition all states into subsets by defining $R(x)$ to be the set of all states that are reachable from $x \in W$ in only one transition, and $R(x) = \emptyset$ for $x \in F_W$. Now, let $D_1 = \{\mathbf{1}\}$ and define sets $D_k = \{x \in R(y)\,; y \in D_{k-1}\}$ recursively. Using this approach, we define a finite sequence of sets $D_1, \cdots, D_N$ for some finite integer N until we reach the empty set $D_N = \emptyset$. The sets $\{D_1, \cdots, D_N\}$ form a disjoint partition of all states since we will always reach a state in F_W after a finite number of steps. Now, we will show by induction that the limiting probability of every state can be written in terms of $\pi_{\mathbf{1}}$ multiplied by a coefficient in the form of RP. It is clear that the only state in $D_1 = \{\mathbf{1}\}$ has that form. Suppose that every state in D_{k-1} has the desired property and consider the states in D_k. The states in D_k may be in W or F_W. If $x \in D_k \cap W$, then it follows from (6.14) that

$$\pi_x = \frac{\sum\limits_{j \in C_0(x)} \pi_{(1_j,x)} \lambda_j}{\sum\limits_{j \in C_1(x)} \lambda_y} \tag{6.28}$$

where $(1_j, x) \in D_{k-1}$ for $j \in C_0(x)$, and π_x can be written in terms of $\pi_{\mathbf{1}}$ multiplied by a coefficient in the form of RP since posynomials are closed under summation and multiplication. If $x \in D_k \cap F_W$, then (6.14) implies

$$\pi_x = \frac{\sum\limits_{j \in C_0(x), (1_j,x) \in W} \pi_{(1_j,x)} \lambda_j}{\varsigma_x} \tag{6.29}$$

Table 6.1: Structure of reliability, MTTF, and availability functions for systems with exponential component lifetimes.

Measure\Structure	**CS**	**RS**
Reliability	DC	DC
MTTF	DC	RP
Availability	RP	RP

where $(1_j, x) \in D_{k-1}$ for $j \in C_0(x)$. Similarly, π_x satisfies the desired property since every state in D_{k-1} has the RP form and posynomials are closed under summation and multiplication.

Since the sets $D_1, \cdots, D_N$ form a disjoint partition of all states, we can conclude that the limiting probability of each state is equal to $\pi_{\mathbf{1}}$ times an RP coefficient. Then, the normalizing condition $\sum_{x \in W} \pi_x + \sum_{x \in F_W} \pi_x = 1$ can be written as $h(\lambda)\pi_{\mathbf{1}} = 1$ where h is some RP function. Therefore, $\pi_{\mathbf{1}} = 1/h(\lambda)$ is RP and hence, π_x is RP for every x since posynomials are closed under summation and multiplication. Therefore, the availability function (6.15) is RP. Furthermore, we also know that availability of CS can be expressed as a ratio of polynomials. Thus, it can be concluded that availability of CS is RP with positive integer powers.

Using the same approach, we can also conclude that the system availability of a series connection of standby redundant subsystem is RP with positive integer powers. It suffices to replace $\mathbf{1}$ and (6.14) by $\bar{n}$ and (6.25) respectively in the analysis. The structure of reliability, MTTF, and availability functions for CS and RS systems are summarized in Table 6.1. Furthermore, (6.6) and (6.7) provide explicit DC representations of reliability and MTTF of CS.

6.4 Illustration

Suppose that we have a series connection of $m = 2$ k-out-of-n subsystems with $k_1 = n_1 = 1, k_2 = 2, n_2 = 3$, and failure rates λ_1 and λ_2 for the components in subsystems 1 and 2 respectively. The components are label as 1 (for subsystem 1), and 2, 3, 4 (for the 3 identical components in subsystem 2). It is clear that $W = \{(1,1,1,1), (1,1,1,0), (1,1,0,1), (1,0,1,1)\}$, and $F_W = \{(0,1,1,1),(0,1,1,0),(0,1,0,1),(0,0,1,1),(1,1,0,0),(1,0,1,0),(1,0,0,1)\}$. The representation (6.5) gives

$$\phi(x) = x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 - 2x_1x_2x_3x_4$$

and the reliability function is

$$P\{L > t\} = E[\phi(Z_t)] = e^{-(\lambda_1+2\lambda_2)t} + e^{-(\lambda_1+2\lambda_2)t} + e^{-(\lambda_1+2\lambda_2)t} - 2e^{-(\lambda_1+3\lambda_2)t}$$

using (6.6), and MTTF is

$$E[L] = \frac{3}{\lambda_1+2\lambda_2} - \frac{2}{\lambda_1+3\lambda_2}$$

using (6.7). Both reliability and MTTF are explicit DC functions of λ.

To compute the availability function, suppose that we arbitrarily set the repair rate at any state $x \in F_W$ equal to the inverse of number of components replaced without loss of generality. Then, using the renewal theoretic approach (6.16)-(6.18),

$$Q = \begin{bmatrix} 0 & \frac{\lambda_2}{\lambda_1+3\lambda_2} & \frac{\lambda_2}{\lambda_1+3\lambda_2} & \frac{\lambda_2}{\lambda_1+3\lambda_2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M = \begin{bmatrix} \frac{\lambda_1}{\lambda_1+3\lambda_2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\lambda_1}{\lambda_1+2\lambda_2} & \frac{\lambda_2}{\lambda_1+2\lambda_2} & \frac{\lambda_2}{\lambda_1+2\lambda_2} & 0 & 0 & 0 \\ 0 & 0 & \frac{\lambda_2}{\lambda_1+2\lambda_2} & 0 & \frac{\lambda_1}{\lambda_1+2\lambda_2} & \frac{\lambda_2}{\lambda_1+2\lambda_2} & 0 \\ 0 & 0 & 0 & \frac{\lambda_2}{\lambda_1+2\lambda_2} & 0 & \frac{\lambda_2}{\lambda_1+2\lambda_2} & \frac{\lambda_1}{\lambda_1+2\lambda_2} \end{bmatrix}$$

$$(\boldsymbol{I} - Q)^{-1} M(\mathbf{1}, i) = \begin{cases} \frac{\lambda_1}{\lambda_1+3\lambda_2} & i = (0,1,1,1) \\ \frac{\lambda_1\lambda_2}{\lambda_1^2+5\lambda_1\lambda_2+6\lambda_2^2} & i = (0,1,1,0),(0,1,0,1),(0,0,1,1) \\ \frac{2\lambda_2^2}{\lambda_1^2+5\lambda_1\lambda_2+6\lambda_2^2} & i = (1,1,0,0),(1,0,1,0),(1,0,0,1) \end{cases}$$

and

$$\varsigma_{0111} = 1, \ \varsigma_{0110} = \varsigma_{0101} = \varsigma_{0011} = \varsigma_{1100} = \varsigma_{1010} = \varsigma_{1001} = 0.5.$$

Therefore,

$$A = \frac{\lambda_1 + 5\lambda_2}{\lambda_1^2 + \lambda_1 + 8\lambda_1\lambda_2 + 5\lambda_2 + 12\lambda_2^2}$$

which is clearly RP with positive integer powers.

We can obtain the same result by applying the formulation in Section 6.3.1.1. In this case,

$$N = \{(1,3), (0,3), (1,2), (0,2), (1,1)\}$$

with $W = \{(1,3), (1,2)\}, F_1 = \{(0,3), (0,2)\}$ and $F_2 = \{(1,1)\}$. Applying the formulation in Section 6.3.1.1, we have

$$\begin{aligned}
\pi_{13}(\lambda_1 + 3\lambda_2) &= \pi_{03} + 0.5\pi_{02} + 0.5\pi_{11} \\
\pi_{12}(\lambda_1 + 2\lambda_2) &= 3\lambda_2\pi_{13} \\
\pi_{03} &= \lambda_1\pi_{13} \\
0.5\pi_{02} &= \lambda_1\pi_{12} \\
0.5\pi_{11} &= 2\lambda_2\pi_{12} \\
\pi_{13} + \pi_{12} + \pi_{03} + \pi_{02} + \pi_{11} &= 1.
\end{aligned}$$

The solution is

$$\begin{aligned}
\pi_{13} &= (\lambda_1 + 2\lambda_2) / \left(\lambda_1^2 + \lambda_1 + 8\lambda_1\lambda_2 + 5\lambda_2 + 12\lambda_2^2\right) \\
\pi_{12} &= 3\lambda_2 / \left(\lambda_1^2 + \lambda_1 + 8\lambda_1\lambda_2 + 5\lambda_2 + 12\lambda_2^2\right) \\
\pi_{03} &= \lambda_1(\lambda_1 + 2\lambda_2) / \left(\lambda_1^2 + \lambda_1 + 8\lambda_1\lambda_2 + 5\lambda_2 + 12\lambda_2^2\right) \\
\pi_{02} &= 6\lambda_1\lambda_2 / \left(\lambda_1^2 + \lambda_1 + 8\lambda_1\lambda_2 + 5\lambda_2 + 12\lambda_2^2\right) \\
\pi_{11} &= 12\lambda_2^2 / \left(\lambda_1^2 + \lambda_1 + 8\lambda_1\lambda_2 + 5\lambda_2 + 12\lambda_2^2\right)
\end{aligned}$$

and

$$A = \pi_{13} + \pi_{12} = \frac{\lambda_1 + 5\lambda_2}{\lambda_1^2 + \lambda_1 + 8\lambda_1\lambda_2 + 5\lambda_2 + 12\lambda_2^2}$$

which coincides with the renewal solution.

Chapter 7

OPTIMAL MAINTENANCE OF SEMI-MARKOV MISSIONS

In this chapter, we consider the optimal maintenance of mission-based systems with multiple components under the usual assumptions requiring IFR life distributions and reasonable cost structures. Our results are valid for any lifetime distribution which can be chosen arbitrarily for each component. We use the intrinsic aging model introduced by Çınlar and Özekici [17] in which the intrinsic age of a component is defined as its cumulative hazard. For details of the intrinsic aging model and related notation, see Section 2.3. Özekici [85] analyzed optimal replacement and repair problems for a single unit working under a randomly changing environment. In that study, intrinsic aging concepts are used to show that optimal replacement policy is a control-limit policy. Moreover, some characterizations for the optimal repair policy under different cost structures are proposed. In this chapter, we actually extend the study of Özekici [85] to multi-component case since we define the mission process as an environmental process which is not affected by the deterioration levels of the components.

In this chapter, for any vectors $x, y \in \mathcal{F}$ with $x = (x(1), \cdots, x(m))$ and $y = (y(1), \cdots, y(m))$, the arithmetic operations xy, $x+y$, $x-y$, and x/y define the vectors whose ith entries are given by $x(i)\,y(i)$, $x(i)+y(i)$, $x(i)-y(i)$, and $x(i)/y(i)$ respectively. We also assume that $r_k(i,a)$ is increasing in a and it is strictly positive. Finally, all costs are discounted at some rate $\alpha > 0$. For a technical reason which will be clear shortly, we further assume that $K = \sup_{i \in E} E\left[e^{-\alpha T_1} | X_0 = i\right] < 1$.

7.1 Optimal Replacement Problem

In this section, we will analyze a quite complex maintenance problem for a mission-based system with some structure function ψ_i during phase $i \in E$. We assume that the system is observed only at the beginning of each phase. After an observation, a decision is made for each component to replace or not to replace it by considering the intrinsic age vector of the system and then the system starts to perform a new phase. We also assume that the duration of the replacement activity is negligible (or included in the phase durations) and $\psi_i(a) = \psi_i(a_1, a_2, \cdots, a_m)$ is nonincreasing in a_k for every k. Note that if the system structure is coherent in all phases, the former condition is satisfied trivially.

We let B^m be the set of all replacement policies so that for any $r \in B^m$, if $r(k) = 1(0)$, component k will (not) be replaced. If the next phase is i and the intrinsic age vector of the system is a, then the cost of applying the replacement policy r is $c_m(i, a; r)$. The cost of performing phase i with an initial intrinsic age a is $c(i, a)$ which is increasing in a and if the system fails during phase i, the failure cost f_i is incurred. We assume that $\sup_{i\in E, a\in\mathcal{F}} c(i, a) = C < +\infty$ and $\sup_{i\in E} f_i = f < +\infty$.

Assumption 7.1 *The maintenance cost function $c_m : E \times \mathcal{F} \times B^m \to \mathbb{R}_+$ satisfies*

i. $c_m(i, a; \mathbf{0}) = 0$,

ii. $r, s \in B^m$ *with* $r \geq s$ *implies* $c_m(i, a; r) \geq c_m(i, a; s)$,

iii. $r, s \in B^m$ *with* $rs = \mathbf{0}$ *implies that* $c_m(i, a; r + s) \leq c_m(i, a; r) + c_m(i, a; s)$,

iv. $c_m(i, a; r)$ *is independent of* a_k *if* $r_k = 0$ *for all* k,

v. $\sup_{i\in E, a\in\mathcal{F}} c_m(i, a; \mathbf{1}) = C_m < +\infty$,

vi. $c_m(i, a; r)$ *is increasing in* a_k *for all* k.

The conditions imposed on c_m by Assumption 7.1 are quite important and interesting. Conditions (*i*) and (*ii*) simply state that no cost is incurred if there is no replacement and the replacement cost increases as more components are replaced. By condition (*iii*), if we consider two replacement policies which do not replace the same components, the cost of applying both policies at the same time is less than the sum of the individual costs. This is very reasonable if there is a fixed cost associated with each replacement activity. Condition (*iv*) asserts that the cost of a replacement policy is not affected by the age of a component that is not replaced. The cost of replacing older components is higher by condition (*vi*). This is also reasonable since the salvage value of older components is lower.

We let $\widetilde{p}(i,a,s,db) = P\{A_s \in db | Y = i, A_0 = a\}$ denote the probability that the age of the system will be in db after s units of time during phase i given that the initial age of the system is a. Using intrinsic aging concepts and the independence of the component lifetimes during a given phase, $\widetilde{p}(i,a,s,db)$ can be written explicitly as

$$\widetilde{p}\left(i,a,s,db\right) = \prod_{k=1}^{m} P\{A_s(k) \in db\left(k\right)|Y = i, A_0\left(k\right) = a\left(k\right)\} = \prod_{k=1}^{m} \overline{p}^{k}_{ia(k)}(s, db(k))$$

where $\overline{p}^{k}_{ia(k)}$ is given by (2.13).

Our purpose is to find a replacement policy which minimizes the expected total discounted cost. Let $v\left(i,a\right)$ denote the minimum expected total discounted cost if the initial phase is i, and the device is at age a. Then, v satisfies the dynamic programming equation (DPE)

$$v\left(i,a\right) = \min_{r \in B^m} \left\{c_m\left(i,a;r\right) + c\left(i,a\left(1-r\right)\right) + \Gamma v\left(i,a\left(1-r\right)\right)\right\} \tag{7.1}$$

where the operator $\Gamma : \mathfrak{B} \to \mathfrak{B}$ is defined by

$$\Gamma g\left(i,a\right) = \sum_{j\in E}\int_0^{+\infty} Q\left(i,j,ds\right) e^{-\alpha s}\left\{\int_{\mathcal{F}} \widetilde{p}\left(i,a,s,db\right)\left[g\left(j,b\right) + \left(1-\psi_i\left(b\right)\right) f_i\right]\right\} \tag{7.2}$$

for any function g in the set $\mathfrak{B}$ of all bounded nonnegative real-valued functions defined on $E \times \mathcal{F}$. Note that we implicitly assume in (7.2) that if the system fails during a phase, the failure cost is incurred at the end of the phase. Otherwise, the analysis becomes much more complicated.

For any $g \in \mathfrak{B}$, we define the operator $\Upsilon : \mathfrak{B} \to \mathfrak{B}$ so that

$$\Upsilon g\left(i,a\right) = \min_{r \in B^m} \left\{c_m\left(i,a;r\right) + c\left(i,a\left(1-r\right)\right) + \Gamma g\left(i,a\left(1-r\right)\right)\right\} \tag{7.3}$$

for all $i \in E, a \in \mathcal{F}$.

Theorem 7.2 *There is a unique function v^* in $\mathfrak{B}$ which satisfies the DPE (7.1).*

Proof. *We will use Banach's contraction mapping theorem. Choose two functions $f, g \in \mathfrak{B}$ and suppose $\|\cdot\|$ is the usual supremum norm on $\mathfrak{B}$ such that $\|g\| = \sup_{i\in E, a\in\mathcal{F}} |g\left(i,a\right)|$. Note that*

$$\begin{aligned}\Upsilon g\left(i,a\right) - \Upsilon f\left(i,a\right) \quad &= \quad \min_{r \in B^m} \left\{c_m\left(i,a;r\right) + c\left(i,a\left(1-r\right)\right) + \Gamma g\left(i,a\left(1-r\right)\right)\right\} \\ &\quad - \min_{r \in B^m} \left\{c_m\left(i,a;r\right) + c\left(i,a\left(1-r\right)\right) + \Gamma f\left(i,a\left(1-r\right)\right)\right\}.\end{aligned} \tag{7.4}$$

Let $\overline{r}$ be the replacement policy which minimizes the second term on the right hand side of (7.4). Then,

$$\begin{aligned}
\Upsilon g\left(i,a\right)-\Upsilon f\left(i,a\right) &= \min_{r\in B^m}\left\{c_m\left(i,a;r\right)+c\left(i,a\left(1-r\right)\right)+\Gamma g\left(i,a\left(1-r\right)\right)\right\} \\
&\quad -c_m\left(i,a;\overline{r}\right)-c\left(i,a\left(1-\overline{r}\right)\right)-\Gamma f\left(i,a\left(1-\overline{r}\right)\right) \\
&\leq c_m\left(i,a;\overline{r}\right)+c\left(i,a\left(1-\overline{r}\right)\right)+\Gamma g\left(i,a\left(1-\overline{r}\right)\right) \\
&\quad -c_m\left(i,a;\overline{r}\right)-c\left(i,a\left(1-\overline{r}\right)\right)-\Gamma f\left(i,a\left(1-\overline{r}\right)\right) \\
&= \Gamma g\left(i,a\left(1-\overline{r}\right)\right)-\Gamma f\left(i,a\left(1-\overline{r}\right)\right) \\
&= \sum_{j\in E}\int_0^{+\infty}Q\left(i,j,ds\right)e^{-\alpha s}\int_{\mathcal{F}}\widetilde{p}\left(i,a\left(1-\overline{r}\right),s,db\right)\left[g\left(j,b\right)-f\left(j,b\right)\right] \\
&\leq \sum_{j\in E}\int_0^{+\infty}Q\left(i,j,ds\right)e^{-\alpha s}\int_{\mathcal{F}}\widetilde{p}\left(i,a\left(1-\overline{r}\right),s,db\right)\left\|g-f\right\| \\
&\leq K\left\|g-f\right\|.
\end{aligned}$$

Similarly, it can be shown that $\Upsilon f\left(i,a\right)-\Upsilon g\left(i,a\right)\leq K\left\|g-f\right\|$ for any $i\in E$ and $a\in\mathcal{F}$. Thus, we have $\left\|\Upsilon g-\Upsilon f\right\|\leq K\left\|g-f\right\|$. Since $K<1$, Υ is a contraction mapping on $\mathfrak{B}$ and it has a unique fixed point $v^*=\Upsilon v^*$ which is the unique solution of DPE (7.1). ■

Lemma 7.3 *If $g\left(i,a\right)$ is increasing in a for every $i\in E$, then*

$$\int_{\overline{B}_k}\prod_{j\neq k}\overline{p}^j_{ia(j)}(s,dc(j))f\left(i,j,c\right)\geq\int\limits_{\substack{B_k;\\ c(k)=h_k(i,a(k),s)}}\prod_{j\neq k}\overline{p}^j_{ia(j)}(s,dc(j))f\left(i,j,c\right)$$

for all $i\in E,k\in S,a\in\mathcal{F}$, and $s\in\mathbb{R}_+$ where B_k and $\overline{B}_k$ are given by (4.58) and (4.59) respectively, and

$$f\left(i,j,c\right)=g\left(j,c\right)+\left(1-\psi_i\left(c\right)\right)f_i.$$

Proof. *Choose arbitrary $c\in B_k$ such that $c\left(k\right)=h_k\left(i,a\left(k\right),s\right)$. Then, there exists $c^*\in\overline{B}_k$ such that $c^*\left(k\right)=+\infty$ and $c^*\left(j\right)=c\left(j\right)$ for every $j\neq k$. If $\psi_i\left(c\right)=0$, then $\psi_i\left(c^*\right)=0$ since ψ_i is nonincreasing in c_k. Then, $f\left(i,j,c\right)=f_i+g\left(j,c\right)$, $f\left(i,j,c^*\right)=f_i+g\left(j,c^*\right)$ and, hence, $f\left(i,j,c^*\right)\geq f\left(i,j,c\right)$. Now, suppose that $\psi_i\left(c\right)=1$. Then,*

$$f\left(i,j,c\right)=g\left(j,c\right)\leq g\left(j,c^*\right)+\left(1-\psi_i\left(c^*\right)\right)f_i=f\left(i,j,c^*\right).$$

Thus, for every $c\in B_k$ with $c\left(k\right)=h_k\left(i,a\left(k\right),s\right)$, we can find $c^*\in\overline{B}_k$ such that $f\left(i,j,c^*\right)\geq f\left(i,j,c\right)$ and $c^*\left(j\right)=c\left(j\right)$ for every $j\neq k$. This completes the proof trivially. ■

Theorem 7.4 *Let v^* be the optimal return function of Theorem 7.2, then*

i. $0 \leq v^* \leq (C_m + C + Kf)/(1-K)$,

ii. $v^*(i,a)$ *is increasing in* a.

Proof. *It suffices to show that* Υg *is increasing in* a *and* $0 \leq \Upsilon g \leq (C_m + C + Kf)/(1-K)$ *if* $0 \leq g \leq (C_m + C + Kf)/(1-K)$ *and* g *is increasing in* a. *It is clear that* $0 \leq \Gamma g \leq K(C_m + C + f)/(1-K)$. *Then, using (7.3) we have* $0 \leq \Upsilon g \leq (C_m + C + Kf)/(1-K)$. *It is clear that*

$$\begin{aligned}
\frac{dh_k(i,a_k,s)}{da_k} &= \frac{dH_k\left(i, H_k^{-1}(i,a_k)+s\right)}{da_k} = \frac{dH_k(i,t)}{dt}\Big|_{t=H_k^{-1}(i,a_k)+s} \frac{d\left(H_k^{-1}(i,a_k)+s\right)}{da_k} \\
&= r_k\left(i, H_k\left(i, H_k^{-1}(i,a_k)+s\right)\right) \frac{dH_k^{-1}(i,a_k)}{da_k} \\
&= r_k\left(i, H_k\left(i, H_k^{-1}(i,a_k)+s\right)\right) \frac{1}{\frac{dH_k(i,t)}{dt}\big|_{t=H_k^{-1}(i,a_k)}} \\
&= \frac{r_k\left(i, H_k\left(i, H_k^{-1}(i,a_k)+s\right)\right)}{r_k(i,a_k)} \geq 1
\end{aligned}$$

if component k *and the system are in working condition since* r_k *is always positive and increasing. Therefore,* $h(i,a,s)$ *and* $h(i,a,s)-a$ *are increasing in* a. *Choose* $a,b \in \mathcal{F}$ *such that* $a(k) < b(k)$ *and* $b(j) = a(j)$ *for every* $j \neq k$ *for some* k. *We need to show that* $\Upsilon g(i,b) \geq \Upsilon g(i,a)$. *Define*

$$\widetilde{p}^k(i,a,s,dc) = \prod_{j\neq k} \overline{p}^j_{ia(j)}(s, dc(j)).$$

Then, since $c_m(i,a;r)$ *and* $c(i,a(1-r))$ *are increasing in* a *for every* r, *it is sufficient to show that*

$$\int_{\mathcal{F}} \widetilde{p}(i,b,s,dc)\, f(i,j,c) \geq \int_{\mathcal{F}} \widetilde{p}(i,a,s,dc)\, f(i,j,c)$$

for a given s *where* $f(i,j,c) = g(j,c) + (1-\psi_i(c))\, f_i$. *Let*

$$q = \int_{\mathcal{F}} \left(\widetilde{p}(i,b,s,dc) - \widetilde{p}(i,a,s,dc)\right) f(i,j,c).$$

Then, we need to show that $q \geq 0$*. Suppose that* $b(k) < +\infty$*. Then,*

$$\begin{aligned}
q &= \int_{B_k} \left(\widetilde{p}(i,b,s,dc) - \widetilde{p}(i,a,s,dc)\right) f(i,j,c) \\
&\quad + \int_{\overline{B}_k} \left(\widetilde{p}(i,b,s,dc) - \widetilde{p}(i,a,s,dc)\right) f(i,j,c) \\
&= \int\limits_{\substack{B_k; \\ c(k)=h_k(i,b(k),s)}} \widetilde{p}^k(i,a,s,dc)\, e^{-(h_k(i,b(k),s)-b(k))} f(i,j,c) \\
&\quad - \int\limits_{\substack{B_k; \\ c(k)=h_k(i,a(k),s)}} \widetilde{p}^k(i,a,s,dc)\, e^{-(h_k(i,a(k),s)-a(k))} f(i,j,c) \\
&\quad + \int_{\overline{B}_k} \widetilde{p}^k(i,a,s,dc) \left(e^{-(h_k(i,a(k),s)-a(k))} - e^{-(h_k(i,b(k),s)-b(k))}\right) f(i,j,c) \\
&\geq \int\limits_{\substack{B_k; \\ c(k)=h_k(i,a(k),s)}} \widetilde{p}^k(i,a,s,dc) \left(e^{-(h_k(i,b(k),s)-b(k))} - e^{-(h_k(i,a(k),s)-a(k))}\right) f(i,j,c) \\
&\quad + \int_{\overline{B}_k} \widetilde{p}^k(i,a,s,dc) \left(e^{-(h_k(i,a(k),s)-a(k))} - e^{-(h_k(i,b(k),s)-b(k))}\right) f(i,j,c) \\
&= \left(e^{-(h_k(i,b(k),s)-b(k))} - e^{-(h_k(i,a(k),s)-a(k))}\right) \\
&\quad \times \left[\int\limits_{\substack{B_k; \\ c(k)=h_k(i,a(k),s)}} \widetilde{p}^k(i,a,s,dc) f(i,j,c) - \int_{\overline{B}_k} \widetilde{p}^k(i,a,s,dc) f(i,j,c) \right] \\
&\geq 0
\end{aligned}$$

where the last inequality follows from Lemma 7.3.

Now, suppose that $b(k) = +\infty$*. Then,*

$$\begin{aligned}
q &= \int_{\overline{B}_k} \left(\widetilde{p}(i,b,s,dc) - \widetilde{p}(i,a,s,dc)\right) f(i,j,c) - \int_{B_k} \widetilde{p}(i,a,s,dc) f(i,j,c) \\
&= \int_{\overline{B}_k} \widetilde{p}^k(i,a,s,dc) \left(e^{-(h_k(i,a(k),s)-a(k))}\right) f(i,j,c) \\
&\quad - \int\limits_{\substack{B_k; \\ c(k)=h_k(i,a(k),s)}} \widetilde{p}^k(i,a,s,dc)\, e^{-(h_k(i,a(k),s)-a(k))} f(i,j,c)
\end{aligned}$$

$$
\begin{aligned}
&= e^{-(h_k(i,a(k),s)-a(k))} \\
&\quad \times \left[\int_{\overline{B}_k} \widetilde{p}^k (i,a,s,dc)\, f(i,j,c) - \int_{\substack{B_k;\\ c(k)=h_k(i,a(k),s)}} \widetilde{p}^k (i,a,s,dc)\, f(i,j,c) \right] \\
&\geq 0
\end{aligned}
$$

where the last inequality follows from Lemma 7.3. ■

We introduce some new notation for simplicity. If r^* is the optimal policy, we let

$$C(i,a) = \{k; r_k^*(i,a) = 1\}$$

$$R(i,A) = \{a; C(i,a) = A\}$$

for every $i \in E, a \in \mathcal{F}$, and $A \subset S$. Here, $C(i,a)$ denotes the set of components which are optimally replaced if the age of the system is a during phase i, and $R(i,A)$ denotes the set of ages at which the optimal decision is to replace the components in A during phase i.

Theorem 7.5 *There is an optimal replacement policy satisfying DPE (7.1) such that*

i. $r_k^*(i,a) = 0$ *if* $a_k = 0$,

ii. $a(1 - r^*(i,a)) \in R(i,\varnothing)$.

Proof. The proof of the first statement trivially follows from (7.1) since $c_m(i,a;r)$ is increasing in r and kth entry of $a(1-r)$ is 0 independent of r. To prove the second statement, first note that $r_k^*(i,a) = 1$ implies that

$$r_k^*(i, a(1 - r^*(i,a))) = 0$$

for any $k \in C(i,a)$ by the first statement. By taking the contrapositive, $r_k^*(i, a(1 - r^*(i,a))) = 1$ implies that $r_k^*(i,a) = 0$. Therefore,

$$r^*(i,a)\, r^*(i, a(1 - r^*(i,a))) = \mathbf{0}$$

and

$$c_m(i,a; r^*(i, a(1 - r^*(i,a)))) = c_m(i, a(1 - r^*(i,a)); r^*(i, a(1 - r^*(i,a))))$$

by Assumption 7.1. By defining

$$\widehat{r}(i,a) = r^*(i,a) + r^*(i,a(1-r^*(i,a)))$$

we have

$$\begin{aligned}
v^*(i,a) &\leq c_m(i,a;\widehat{r}(i,a)) + c(i,a(1-\widehat{r}(i,a))) + \Gamma v^*(i,a(1-\widehat{r}(i,a))) \\
&\leq c_m(i,a;r^*(i,a)) + c(i,a(1-\widehat{r}(i,a))) + \Gamma v^*(i,a(1-\widehat{r}(i,a))) \\
&\quad + c_m(i,a(1-r^*(i,a));r^*(i,a(1-r^*(i,a)))) \\
&= c_m(i,a;r^*(i,a)) + c_m(i,a(1-r^*(i,a));r^*(i,a(1-r^*(i,a)))) \\
&\quad + c(i,a(1-r^*(i,a))(1-r^*(i,a(1-r_k^*(i,a))))) \\
&\quad + \Gamma v^*(i,a(1-r^*(i,a))(1-r^*(i,a(1-r_k^*(i,a))))) \\
&= c_m(i,a;r^*(i,a)) + v^*(i,a(1-r^*(i,a))) \\
&\leq c_m(i,a;r^*(i,a)) + c(i,a(1-r^*(i,a))) + \Gamma v^*(i,a(1-r^*(i,a))) \\
&= v^*(i,a).
\end{aligned}$$

In this chain of implications, the first inequality directly follows from (7.1) since $\widehat{r} \in B^m$. The second inequality follows from (iii) in Assumption 7.1. The first equality follows from the facts that

$$\begin{aligned}
1-\widehat{r}(i,a) &= 1-r^*(i,a) - r^*(i,a(1-r^*(i,a))) + r^*(i,a)\,r^*(i,a(1-r^*(i,a))) \\
&= 1-r^*(i,a) - (1-r^*(i,a))\,r^*(i,a(1-r^*(i,a))) \\
&= (1-r^*(i,a))(1-r^*(i,a(1-r^*(i,a))))
\end{aligned}$$

since

$$r^*(i,a)\,r^*(i,a(1-r^*(i,a))) = \mathbf{0}.$$

The second equality follows from the fact that

$$\begin{aligned}
v^*(i,a(1-r^*(i,a))) &= c_m(i,a(1-r^*(i,a));r^*(i,a(1-r^*(i,a)))) \\
&\quad + c(i,a(1-r^*(i,a))(1-r^*(i,a(1-r_k^*(i,a))))) \\
&\quad + \Gamma v^*(i,a(1-r^*(i,a))\,1-r^*(i,a(1-r_k^*(i,a))))
\end{aligned}$$

and the third inequality follows from the fact that $c_m(i,a,x;\mathbf{0}) = 0$ and

$$v^*(i,a(1-r^*(i,a))) \leq c(i,a(1-r^*(i,a))) + \Gamma v^*(i,a(1-r^*(i,a))). \tag{7.5}$$

Finally, the last equality is trivial since $r^*(i,a)$ minimizes the right side of (7.1). Therefore, all of these inequalities must be equalities which means that

$$c_m(i,a;\widehat{r}(i,a)) = c_m(i,a;r^*(i,a)) + c_m(i,a(1-r^*(i,a));r^*(i,a(1-r^*(i,a))))$$

and (7.5) is also an equality. But this implies that we can define r^* at $a(1-r^*(i,a))$ such that $r^*(i,a(1-r^*(i,a))) = 0$ and $\widehat{r}(i,a) = r^*(i,a)$ with $a(1-r^*(i,a)) \in R(i,\varnothing)$. ■

Theorem 7.6 *Suppose that $c_m(i,a;r)$ is independent of a. Then, there is an optimal replacement policy satisfying DPE (7.1) such that*

i. If $b_k \geq a_k$ for $k \in A \subset C(i,a)$ and $b_k = a_k$ for $k \notin A$, then $r^*(i,b) = r^*(i,a)$

ii. If $b_k < a_k$ for $k \in A \subset S \backslash C(i,a)$ and $b_k = a_k$ for $k \notin A$, then there exists $k \in A$ such that $r_k^*(i,b) = 0$.

Proof. Using Theorem 7.4, we have $v^*(i,b) \geq v^*(i,a)$ and, hence,

$$\begin{aligned} &c_m(i,b;r^*(i,b)) + c(i,b(1-r^*(i,b))) + \Gamma v^*(i,b(1-r^*(i,b,x))) \\ &\quad \geq c_m(i,a;r^*(i,a)) + c(i,a(1-r^*(i,a))) + \Gamma v^*(i,a(1-r^*(i,a))) \\ &\quad = c_m(i,b;r^*(i,a)) + c(i,b(1-r^*(i,a))) + \Gamma v^*(i,b(1-r^*(i,a))). \end{aligned}$$

The last equality follows from the main hypothesis. This result implies that at age b, if we apply the optimal policy at age a, we have the same optimal cost. Therefore, in the optimal policy at age b, we can apply the optimal replacement policy at age a and this proves (i). To prove (ii) by contradiction, suppose that $r_k^*(i,b) = 1$ for every $k \in A$. Then, $a(k) \geq b(k)$ for $k \in A \subset C(i,b)$ and $a(k) = b(k)$ for $k \notin A$. Applying Theorem 7.6 (i), we have $r^*(i,a) = r^*(i,b)$. This implies that $r_k^*(i,a) = 1$ for every $k \in A \subset S \backslash C(i,a)$. Clearly, this is a contradiction. ■

Corollary 7.7 *Let r^* be the optimal policy of Theorem 7.6. Then,*

i. $r_k^*(i,a)$ is increasing in a_k for all $k \in S$,

ii. If $a \in R(i,S)$, then $b \in R(i,S)$ for all $b \geq a$, *iii.* *If* $b(k) < a(k)$ *and* $b(j) = a(j)$ *for every* $j \neq k$ *with* $r_k^*(i,a) = 0$, *then* $r_k^*(i,b) = 0$,

iv. *If* $a \in R(i, \varnothing)$, $b(k) < a(k)$ *and* $b(j) = a(j)$ *for every* $j \neq k$, *then* $r_k^*(i,b) = 0$.

Proof. To prove (i), it suffices to show that if $r_k^*(i,a) = 1$ for some $a \in \mathcal{F}$, then $r_k^*(i,b) = 1$ for $b \in \mathcal{F}$ with $b(k) \geq a(k)$ and $b(j) = a(j)$ for every $j \neq k$. This follows from Theorem 7.6 trivially. To prove the second statement suppose that $a \in R(i,S)$ and choose $b \in \mathcal{F}$ such that $b \geq a$. Then, since $r_k^*(i,a) = 1$ for every $k \in S$, $r^*(i,b) = r^*(i,a)$ using Theorem 7.6. This trivially implies that $b \in R(i,S)$. The proofs of (iii) and (iv) follow trivially from (ii) in Theorem 7.6 by taking $A = \{k\}$. ■

7.2 Optimal Repair Problem

In the previous section, there are only two decision alternatives at each decision epoch: to replace a component by a brand new one or to let it operate during the next phase. In many applications, however, it is also possible to repair a component so that its age is decreased to a lower level by some technical maintenance operations or by simply replacing the old component by one that is younger, if not brand new. We will use the settings and probabilities constructed in the previous section once more.

The decision maker observes the system at the beginning of each phase and makes a repair decision. If the next phase is i and the intrinsic age of the system is a at the end of a phase, then the decision maker chooses an action $y(i,a)$ from the set $\{b \in \mathcal{F}; b \leq a\}$. The cost of repairing the system from age a to b during phase i is $C_i(a;b)$ where $b \leq a$.

Assumption 7.8 *The repair cost function* $C_i : \left\{(a,b) \in \mathcal{F}^2; b \leq a\right\} \to \mathbb{R}_+$ *satisfies*

i. $C_i(a;b)$ *is increasing in* a_k *and decreasing in* b_k *for every* $k \in S$,

ii. $C_i(a;a) = 0$,

iii. $\sup_{i \in E, a \in \mathcal{F}} C_i(a;\mathbf{0}) = C_r < +\infty$.

Condition (i) simply states that the cost of repair increases as the amount of improvement increases. Condition (ii) asserts that if the system does not experience any maintenance, then no cost will be incurred. It is clear that these are very reasonable assumptions.

We also suppose that if there are more than one optimal repair action, then the alternative with lower final age will be chosen. Our purpose is to find a repair policy which minimizes the expected total discounted cost. Let $v(i,a)$ denote the minimum expected total discounted cost if the initial phase is i, and the device is at age a. Then, v satisfies the DPE

$$v(i,a) = \inf_{b;b\leq a} \{C_i(a;b) + c(i,b) + \Gamma v(i,b)\} \tag{7.6}$$

where the operator $\Gamma : \mathfrak{B} \to \mathfrak{B}$ is defined by

$$\Gamma g(i,a) = \sum_{j\in E} \int_0^{+\infty} Q(i,j,ds)\, e^{-\alpha s} \left\{ \int_{\mathcal{F}} \widetilde{p}(i,a,s,db)\, [g(j,b) + (1-\psi_i(b)) f_i] \right\} \tag{7.7}$$

for any function g in $\mathfrak{B}$.

For any $g \in \mathfrak{B}$, we define the operator $\Upsilon : \mathfrak{B} \to \mathfrak{B}$ so that

$$\Upsilon g(i,a) = \inf_{b;b\leq a} \{C_i(a;b) + c(i,b) + \Gamma g(i,b)\} \tag{7.8}$$

for all $i \in E, a \in \mathcal{F}$.

Theorem 7.9 *There is a unique function v^* in $\mathfrak{B}$ which satisfies the DPE (7.6).*

Proof. *We will use Banach's contraction mapping theorem. Choose two functions $f, g \in \mathfrak{B}$ and suppose that $\|\cdot\|$ is the usual supremum norm on $\mathfrak{B}$ such that $\|g\| = \sup_{i\in E, a\in\mathcal{F}} |g(i,a)|$. It suffices to show that Υ is a contraction mapping and note that*

$$\Upsilon g(i,a) - \Upsilon f(i,a) = \inf_{b;b\leq a} \{C_i(a;b) + c(i,b) + \Gamma g(i,b)\} - \inf_{b;b\leq a} \{C_i(a;b) + c(i,b) + \Gamma f(i,b)\}.$$

Let $\overline{b}$ be the intrinsic age which minimizes the second term on the right hand side in the equation above. Then,

$$\begin{aligned}
\Upsilon g(i,a) - \Upsilon f(i,a) &= \inf_{b;b\leq a} \{C_i(a;b) + c(i,b) + \Gamma g(i,b)\} - C_i\left(a;\overline{b}\right) - c\left(i,\overline{b}\right) - \Gamma f\left(i,\overline{b}\right) \\
&\leq C_i\left(a;\overline{b}\right) + c\left(i,\overline{b}\right) + \Gamma g\left(i,\overline{b}\right) - C_i\left(a;\overline{b}\right) - c\left(i,\overline{b}\right) - \Gamma f\left(i,\overline{b}\right) \\
&= \Gamma g\left(i,\overline{b}\right) - \Gamma f\left(i,\overline{b}\right) \\
&= \sum_{j\in E} \int_0^{+\infty} Q(i,j,ds)\, e^{-\alpha s} \int_{\mathcal{F}} \widetilde{p}\left(i,\overline{b},s,dc\right) [g(j,c) - f(j,c)] \\
&\leq \sum_{j\in E} \int_0^{+\infty} Q(i,j,ds)\, e^{-\alpha s} \int_{\mathcal{F}} \widetilde{p}\left(i,\overline{b},s,dc\right) \|g-f\| \\
&\leq K\|g-f\|.
\end{aligned}$$

Similarly, it can be shown that $\Upsilon f(i,a) - \Upsilon g(i,a) \leq K\|g-f\|$ *for any* $i \in E$ *and* $a \in \mathcal{F}$. *Thus, we have* $\|\Upsilon g - \Upsilon f\| \leq K\|g-f\|$. *Since* $K<1$, Υ *is a contraction mapping on* $\mathfrak{B}$ *and it has a unique fixed point* $v^* = \Upsilon v^*$ *which is the unique solution of DPE (7.6).* ■

To simplify the notation, we let $(a_k, b) = c$ where $c(k) = a(k)$ and $c(j) = b(j)$ for every $j \neq k$.

Theorem 7.10 *Let* v^* *be the optimal value function of Theorem 7.9. Then,*

i. $0 \leq v^* \leq (C_r + C + Kf)/(1-K)$,

ii. If $C_i(a;b) \geq C_i(\overline{a};\overline{b})$ *whenever* $b_k = a_k \geq \overline{a}_k = \overline{b}_k$ *for some* k *and* $\overline{a}_j = a_j$, $\overline{b}_j = b_j$ *for every* $j \neq k$, *then* $v^*(i,a)$ *is increasing in* a_k *for every* k.

Proof. *We need to show that* Υg *is increasing in* a *and* $0 \leq \Upsilon g \leq (C_r + C + Kf)/(1-K)$ *if* $0 \leq g \leq (C_r + C + Kf)/(1-K)$ *and* g *is increasing in* a. *Following the same steps as in the proof of Theorem 7.4, it can be shown that* $0 \leq v^* \leq (C_r + C + Kf)/(1-K)$ *and* $\Gamma g(i,a)$ *is increasing in* a. *Now choose* $a, c \in \mathcal{F}$ *such that* $c(k) > a(k)$ *and* $c(j) = a(j)$ *for every* $j \neq k$. *We need to show that* $\Upsilon g(i,c) \geq \Upsilon g(i,a)$. *Choose* $b \leq a$. *Then, trivially* $b \leq c$ *and, hence,*

$$C_i(c;b) + c(i,b) + \Gamma g(i,b) \geq C_i(a;b) + c(i,b) + \Gamma g(i,b) \geq \Upsilon g(i,a)$$

since $C_i(a;b)$ *is increasing in* a. *Now, choose* $b \leq c$ *with* $b(k) > a(k)$. *Then,*

$$\begin{aligned}
C_i(c;b) + c(i,b) + \Gamma g(i,b) &\geq C_i((b_k,c);b) + c(i,b) + \Gamma g(i,b) \\
&\geq C_i((a_k,(b_k,c));(a_k,b)) + c(i,(a_k,b)) + \Gamma g(i,(a_k,b)) \\
&= C_i(a;(a_k,b)) + c(i,(a_k,b)) + \Gamma g(i,(a_k,b)) \\
&\geq \Upsilon g(i,a)
\end{aligned}$$

where the first inequality follows from the fact that $c \geq (b_k, c)$ *and the second inequality follows from the main hypothesis and* $b \geq (a_k, b)$. *Then,*

$$\begin{aligned}
\Upsilon g(i,c) &= \min\left\{ \inf_{b;b\leq a} \{C_i(c;b) + c(i,b) + \Gamma g(i,b)\}, \right. \\
&\qquad \left. \inf_{b;b\leq c, b(k)>a(k)} \{C_i(c;b) + c(i,b) + \Gamma g(i,b)\} \right\} \\
&\geq \min\{\Upsilon g(i,a), \Upsilon g(i,a)\} = \Upsilon g(i,a).
\end{aligned}$$

■

We will let $y^*(i,a)$ denote the optimal decision at state (i,a) which provides the minimum to the right hand side of (7.6). The structure of the repair cost function C is too general to obtain useful characterizations of the optimal policy. We will therefore impose additional restrictions on C which lead to some simplifications.

Theorem 7.11 *If $C_i(a;b) \le C_i(a;c) + C_i(c;b)$ for all $a \ge c \ge b \ge \mathbf{0}$ and $i \in E$, then there is an optimal policy such that $y^*(i, y^*(i,a)) = y^*(i,a)$.*

Proof. *Choose arbitrary $i \in E$, and $a \in \mathcal{F}$. Suppose that $y^*(i,a) = \alpha$, and choose some $b \le \alpha$. Using the main hypothesis, we have*

$$C_i(a;b) \le C_i(a;\alpha) + C_i(\alpha;b)$$

and

$$C_i(a;b) - C_i(a;\alpha) \le C_i(\alpha;b). \tag{7.9}$$

Since $y^*(i,a) = \alpha$,

$$C_i(a;\alpha) + c(i,\alpha) + \Gamma v^*(i,\alpha) \le C_i(a;b) + c(i,b) + \Gamma v^*(i,b)$$

and

$$c(i,\alpha) + \Gamma v^*(i,\alpha) \le C_i(a;b) - C_i(a;\alpha) + c(i,b) + \Gamma v^*(i,b).$$

This implies that

$$c(i,\alpha) + \Gamma v^*(i,\alpha) \le C_i(\alpha;b) + c(i,b) + \Gamma v^*(i,b) \tag{7.10}$$

by using (7.9). Since b is an arbitrary value satisfying $b \le \alpha$, using (7.10), we have

$$c(i,\alpha) + \Gamma v^*(i,\alpha) \le \inf_{b;b\le\alpha} \{C_i(\alpha;b) + c(i,b) + \Gamma v^*(i,b)\}.$$

Since $C_i(\alpha;\alpha) = 0$, we can conclude that we can choose $y^*(i,\alpha) = \alpha$ in the optimal repair policy and, hence, $y^*(i, y^*(i,a)) = y^*(i,a)$. ■

Theorem 7.12 *Choose some $a \in \mathcal{F}$ and $i \in E$. Suppose that $C_i(b;d) = C_i(b;c) + C_i(c;d)$ for all $a + u \ge b \ge c \ge d \ge \mathbf{0}$ for some $u \ge \mathbf{0}$. Then, $y^*(i, a+u) \le a$ implies that there is an optimal policy such that $y^*(i, a+z) = y^*(i,a)$ for all $0 \le z \le u$.*

Proof. Choose arbitrary $z \neq \mathbf{0}$. By the main hypothesis, it is clear that there is a repair decision at $(i, a+u)$ and, hence,

$$\begin{aligned} v^*(i, a+u) &= \inf_{b; b \leq a+u} \{C_i(a+u; b) + c(i, b) + \Gamma v^*(i, b)\} \\ &= \inf_{b; b \leq a+z} \{C_i(a+u; b) + c(i, b) + \Gamma v^*(i, b)\} \\ &= \inf_{b; b \leq a} \{C_i(a+u; b) + c(i, b) + \Gamma v^*(i, b)\}. \end{aligned}$$

This implies that

$$\begin{aligned} \inf_{b; b \leq a+z} \{C_i(a+u; b) + c(i, b) + \Gamma v^*(i, b)\} &= C_i(a+u; a+z) \\ &\quad + \inf_{b; b \leq a+z} \{C_i(a+z; b) + c(i, b) + \Gamma v^*(i, b)\} \\ &= C_i(a+u; a+z) \\ &\quad + \inf_{b; b \leq a} \{C_i(a+z; b) + c(i, b) + \Gamma v^*(i, b)\} \end{aligned}$$

and

$$v^*(i, a+z) = \inf_{b; b \leq a+z} \{C_i(a+z; b) + c(i, b) + \Gamma v^*(i, b)\} \tag{7.11}$$

$$= \inf_{b; b \leq a} \{C_i(a+z; b) + c(i, b) + \Gamma v^*(i, b)\}. \tag{7.12}$$

Therefore, there is an optimal policy with $y^*(i, a+z) \leq a$. Now, choose $b \leq a$. Then,

$$\begin{aligned} & C_i(a+z; y^*(i, a)) + c(i, y^*(i, a)) + \Gamma v^*(i, y^*(i, a)) \\ &= C_i(a+z; a) + C_i(a; y^*(i, a)) + c(i, y^*(i, a)) + \Gamma v^*(i, y^*(i, a)) \\ &\leq C_i(a+z; a) + C_i(a; b) + c(i, b) + \Gamma v^*(i, b) \\ &= C_i(a+z; b) + c(i, b) + \Gamma v^*(i, b). \end{aligned}$$

This and (7.11) imply that

$$\begin{aligned} & C_i(a+z; y^*(i, a)) + c(i, y^*(i, a)) + \Gamma v^*(i, y^*(i, a)) \\ &\leq \inf_{b; b \leq a} \{C_i(a+z; b) + c(i, b) + \Gamma v^*(i, b)\} \\ &= \inf_{b; b \leq a+z} \{C_i(a+z; b) + c(i, b) + \Gamma v^*(i, b)\} \end{aligned}$$

and there is an optimal policy such that $y^*(i, a+z) = y^*(i, a)$. ■

Corollary 7.13 *Suppose that* $C_i(a;b) = C_i(a;c) + C_i(c;b)$ *for every* $a \geq c \geq b \geq \mathbf{0}$ *and* $i \in E$*. Then, there is an optimal policy such that* $y_k^*(i,a)$ *is increasing in* a_k*.*

Proof. Suppose that $y^*(i,a) = c$ and $y^*(i,a+u) = b$ where $u_k > 0$ and $u_j = 0$ for every $j \neq k$. If $b_k \geq a_k$, then $b_k \geq c_k$ since $c_k \leq a_k$ trivially. Now, suppose that $b_k < a_k$. Then, we have $b \leq a$ and $y^*(i,a) = y^*(i,a+u)$ using Theorem 7.12 and this completes the proof. ■

Proposition 7.14 *Suppose that*

$$C_i(a;b) + c(i,b) \geq C_i(a;c) + c(i,c) \tag{7.13}$$

whenever $b \geq c$ *and* $b \neq a$*. Then,* $v^*(i,a)$ *is increasing in* a *and there is an optimal policy such that* $y^*(i,a) \in \{\mathbf{0}, a\}$*. If (7.13) also holds for* $b = a$*, then there is an optimal policy such that* $y^*(i,a) = \mathbf{0}$*.*

Proof. Following the same steps as in the proof of Theorem 7.4, it can be shown that $\Gamma g(i,a)$ is increasing in a. Choose $a, c \in \mathcal{F}$ such that $c(k) > a(k)$ and $c(j) = a(j)$ for every $j \neq k$. We need to show that $\Upsilon g(i,c) \geq \Upsilon g(i,a)$. Choose $b \leq a$. Then, trivially $b \leq c$ and, hence,

$$C_i(c;b) + c(i,b) + \Gamma g(i,b) \geq C_i(a;b) + c(i,b) + \Gamma g(i,b) \geq \Upsilon g(i,a)$$

since $C_i(a;b)$ is increasing in a. Now, choose $b \leq c$ with $c(k) > b(k) > a(k)$. Define $\bar{b}$ such that $\bar{b}(k) = a(k)$ and $\bar{b}(j) = b(j)$ for every $j \neq k$. Then, $\bar{b} \leq b$ and $\bar{b} \leq a$. This implies that

$$\begin{aligned} C_i(c;b) + c(i,b) + \Gamma g(i,b) &\geq C_i\left(c;\bar{b}\right) + c\left(i,\bar{b}\right) + \Gamma g\left(i,\bar{b}\right) \\ &\geq C_i\left(a;\bar{b}\right) + c\left(i,\bar{b}\right) + \Gamma g\left(i,\bar{b}\right) \\ &\geq \Upsilon g(i,a). \end{aligned}$$

If $b = c$,

$$\begin{aligned} C_i(c;b) + c(i,b) + \Gamma g(i,b) &= c(i,c) + \Gamma g(i,c) \\ &\geq c(i,a) + \Gamma g(i,a) \\ &\geq \Upsilon g(i,a). \end{aligned}$$

Then,

$$\begin{aligned}\Upsilon g(i,c) &= \min\left\{\inf_{b;b\leq a}\{C_i(c;b)+c(i,b)+\Gamma g(i,b)\},\right.\\ &\qquad \left.\inf_{b;b\leq c,b(k)>a(k)}\{C_i(c;b)+c(i,b)+\Gamma g(i,b)\}\right\}\\ &\geq \min\{\Upsilon g(i,a),\Upsilon g(i,a)\}=\Upsilon g(i,a).\end{aligned}$$

Using the main hypothesis,

$$C_i(a;\mathbf{0})+c(i,\mathbf{0})+\Gamma v(i,\mathbf{0})\leq C_i(a;b)+c(i,b)+\Gamma v(i,b)$$

for every $b \lneqq a$. Therefore,

$$\begin{aligned}v^*(i,a) &= \inf_{b;b\leq a}\{C_i(a;b)+c(i,b)+\Gamma v^*(i,b)\}\\ &= \min\left\{\inf_{b\lneqq a}\{C_i(a;b)+c(i,b)+\Gamma v^*(i,b)\},c(i,a)+\Gamma v^*(i,a)\right\}\\ &= \min\{C_i(a;\mathbf{0})+c(i,\mathbf{0})+\Gamma v^*(i,\mathbf{0}),c(i,a)+\Gamma v^*(i,a)\}.\end{aligned}$$

This trivially implies that there is an optimal policy such that $y^*(i,a)\in\{\mathbf{0},a\}$. Suppose that (7.13) also holds for $b=a$. It is sufficient to show that $C_i(a;\mathbf{0})+c(i,\mathbf{0})+\Gamma v^*(i,\mathbf{0})\leq c(i,a)+\Gamma v^*(i,a)$. This follows from $\Gamma v^*(i,a)\geq\Gamma v^*(i,\mathbf{0})$ and

$$c(i,a)=C_i(a;a)+c(i,a)\geq C_i(a;\mathbf{0})+c(i,\mathbf{0}).$$

■

This result is very intuitive since repairing the device to a smaller age is always cheaper under this cost structure.

An interesting special case is when the repair action corresponds to selling the old device at hand and replacing it with a younger one purchased from the market. Let $c_i(a)$ and $s_i(a)$ be the purchase cost and salvage value, respectively, of a device with intrinsic age a. Then, $C_i(a;b)=c_i(b)-s_i(a)$ whenever $b\leq a$ with $a\neq b$ and, as usual, $C_i(a;a)=0$. We assume that c_i and s_i are both decreasing in a_k for every k with $c_i\geq s_i$. It is easy to show that the hypothesis of Theorem 7.11 is satisfied under this cost structure and hence $y^*(i,y^*(i,a))=y^*(i,a)$ for every a.

Under this cost structure, (7.6) simplifies to

$$v(i,a)=\min\left\{c(i,a)+\Gamma v(i,a)+s_i(a),\inf_{b\lneqq a}\{c_i(b)+c(i,b)+\Gamma v(i,b)\}\right\}-s_i(a) \quad (7.14)$$

for all $i\in E$ and $a\in\mathcal{F}$.

Theorem 7.15 *Let $y^*(i,a)$ be the optimal repair policy of DPE (7.14). Then, if $y^*(i,a) \neq a$, $y^*(i,a+u) \lneqq a$ for some $u \geq \mathbf{0}$ and $y^*(i,a+z) \neq a+z$ for some $\mathbf{0} \leq z \leq u$, then there is an optimal policy such that $y^*(i,a+z) = y^*(i,a)$.*

Proof. *Since $y^*(i,a) \neq a$,*

$$v^*(i,a) = \inf_{b \lneqq a} \{c_i(b) + \Gamma v^*(i,b)\} - s_i(a).$$

If $u = \mathbf{0}$, then there is nothing to prove. Suppose that $u \neq \mathbf{0}$. By the main hypothesis, it is clear that there is a repair decision at $(i,a+u)$ and, hence,

$$\begin{aligned} v^*(i,a+u) &= \inf_{b \lneqq a+u} \{c_i(b) + c(i,b) + \Gamma v^*(i,b)\} - s_i(a+u) \\ &= \inf_{b \lneqq a} \{c_i(b) + c(i,b) + \Gamma v^*(i,b)\} - s_i(a+u) \end{aligned} \tag{7.15}$$

where the last equality follows from $y^*(i,a+u) \lneqq a$. Now, choose arbitrary $z \neq \mathbf{0}$ such that $y^*(i,a+z) \neq a+z$. Then, we have

$$v^*(i,a+z) = \inf_{b \lneqq a+z} \{c_i(b) + c(i,b) + \Gamma v^*(i,b)\} - s_i(a+z)$$

and

$$\{b; b \lneqq a\} \subset \{b; b \lneqq a+z\} \subset \{b; b \lneqq a+u\}.$$

This implies that

$$\begin{aligned} \inf_{b \lneqq a} \{c_i(b) + c(i,b) + \Gamma v^*(i,b)\} &\geq \inf_{b \lneqq a+z} \{c_i(b) + c(i,b) + \Gamma v^*(i,b)\} \\ &\geq \inf_{b \lneqq a+u} \{c_i(b) + c(i,b) + \Gamma v^*(i,b)\} \end{aligned}$$

and using (7.15),

$$\inf_{b \lneqq a+z} \{c_i(b) + c(i,b) + \Gamma v^*(i,b)\} = \inf_{b \lneqq a} \{c_i(b) + c(i,b) + \Gamma v^*(i,b)\}$$

which implies that we can choose $y^*(i,a+z) = y^*(i,a)$. ■

Corollary 7.16 *Let $y^*(i,a)$ be the optimal repair policy of DPE (7.14). Then, there is an optimal policy such that $y_k^*(i,\overline{a}) \geq y_k^*(i,a)$ provided that $\overline{a}_k > a_k$ for some k, $\overline{a}_j = a_j$ for every $j \neq k$ and $y^*(i,a) \neq a$.*

Proof. *Note that if $y_k^*(i,\overline{a}) = \overline{a}$, then there is nothing to prove. Choose a, $\overline{a}$ and k such that $\overline{a}_k > a_k$, $\overline{a}_j = a_j$ for every $j \neq k$, $y^*(i,a) = c \neq a$, and $y^*(i,\overline{a}) = b \neq \overline{a}$. If $b_k \geq a_k$,* *then* $b_k \geq c_k$ *since* $c_k \leq a_k$. *Now, assume that* $b_k < a_k$. *This implies that* $b \leq a$ *and* $b \neq a$. *By taking* $u = \overline{a} - a$, *Theorem 7.15 implies that* $y^*(i,a) \neq a$, $y^*(i,a+u) \lneqq a$ *and* $y^*(i,a+u) \neq a+u$. *This implies that* $y^*(i,a+u) = y^*(i,a)$ *and, hence,* $y^*(i,\overline{a}) = y^*(i,a)$ *which completes the proof.* ■

In some cases, the purchase cost and the salvage value of a system may be equal. Then, $C_i(a;b) = c_i(b) - c_i(a)$ and (7.6) simplifies to

$$v(i,a) = \inf_{b;b\leq a} \{c_i(b) + c(i,b) + \Gamma v(i,b)\} - c_i(a). \tag{7.16}$$

Theorem 7.17 *Let* $y^*(i,a)$ *be the optimal repair policy of DPE (7.16). Then, there is an optimal policy such that*

i. If $y^*(i,a+u) \leq a$, *for some* $u \geq 0$, *then* $y^*(i,a+z) = y^*(i,a)$ *for all* $\mathbf{0} \leq z \leq u$,

ii. $y_k^*(i,a)$ *is increasing in* a_k.

Proof. *Choose arbitrary* $a \geq c \geq b \geq \mathbf{0}$. *Then,*

$$\begin{aligned} C_i(a;c) + C_i(c;b) &= c_i(c) - c_i(a) + c_i(b) - c_i(c) \\ &= c_i(b) - c_i(a) = C_i(a;b). \end{aligned}$$

Then, the results trivially follow from Theorem 7.12 and Corollary 7.13. ■

In addition, if there is no salvage value, i.e., $s_i = 0$, the DPE (7.6) can be rewritten as

$$v(i,a) = \min\left\{c(i,a) + \Gamma v(i,a), \inf_{b\lneqq a}\{c_i(b) + c(i,b) + \Gamma v(i,b)\}\right\}. \tag{7.17}$$

Theorem 7.18 *Let* $y^*(i,a)$ *be the optimal repair policy of DPE (7.17). Then, there is an optimal policy such that*

i. If $y^*(i,a) \neq a$, $y^*(i,a+u) = b$ *with* $b \lneqq a$ *for some* $u \geq \mathbf{0}$ *and* $y^*(i,a+z) \neq a+z$ *for some* $\mathbf{0} \leq z \leq u$, *then* $y^*(i,a+z) = y^*(i,a)$,

ii. $y_k^*(i,\overline{a}) \geq y_k^*(i,a)$ *provided that* $\overline{a}_k > a_k$ *for some* k, $\overline{a}_j = a_j$ *for every* $j \neq k$ *and* $y^*(i,a) \neq a$.

Proof. *The results follow trivially from Theorem 7.15 and Corollary 7.16 since* $c_i \geq s_i = 0$.
■

Chapter 8

RELIABILITY AND OPTIMAL MAINTENANCE UNDER MARKOVIAN MISSION AND DETERIORATION

In this section, we consider a mission-based system with a mission process governed by a finite state Markov process. In other words, the system under consideration performs a mission whose successive phases follow a Markov chain and all phase durations are exponentially distributed. We also assume that the system is subject to Markovian deterioration. In other words, successive deterioration levels of the system follow a Markov chain and holding times in each deterioration level are exponentially distributed during any phase. The most important point is that the generator of the deterioration process of the system (the transition probability matrix and rates of the holding times) depends on the mission process. This implies that the deterioration process is a Markov process modulated by another Markov process, i.e., the mission process. Our setting is much more simple than the ones used in the previous sections. The main incentive behind this simplification is the desire to find more computationally tractable results. We analyze reliability, MTTF, availability, and optimal maintenance and provide numerical illustrations. We also show how our reliability, MTTF, and availability results can be applied to any coherent system with independent and exponentially distributed component lifetimes.

8.1 Mission and Deterioration Processes

Let Y_t be the phase of the mission which is performed at time t. We assume that the mission process $Y = \{Y_t; t \geq 0\}$ is a Markov process with a finite state space E, infinitesimal generator H, transition probability matrix Q, and transition rate vector δ. We suppose that the deterioration level or age of the system takes values in some finite set $F = \{0, 1, \cdots, M\}$ where 0 stands for a brand new system and M represents system failure. The deterioration process of the system is $\mathcal{A} = \{\mathcal{A}_t; t \geq 0\}$ with state space F. Since the survival properties of the system change depending on the phases of the mission process, we assume that $\mathcal{A}$ is modulated by Y. The deterioration process follows a Markov process with state space F, generator G_i, transition probability matrix P_i, and transition rate vector λ_i during phase i. We also assume that M is an absorbing state for every i unless otherwise specified, i.e., $P_i(M,M)=1$ and $\lambda_i(M)=0$. In this chapter, unless otherwise specified, we do not put any assumption which says that the deterioration process is increasing although $\mathcal{A}$ shows the deterioration of the system. Therefore, P_i does not have to be upper triangular. We just distinguish between the best state 0 and the worst state M. It should also be clear that $\mathcal{A}$ does not really measure "real age" in time with respect to some continuous clock, it indicates the deterioration level the system.

Considering the dependence between the age process and the mission process, we will use the bivariate process $(Y,\mathcal{A})=\{(Y_t,\mathcal{A}_t);t\geq 0\}$ which is more suitable for our purpose in the foregoing analysis. It is clear that $(Y,\mathcal{A})$ is also a Markov process with state space $E\times F$ and infinitesimal generator

$$\mathcal{G}(i,a;j,b)=\begin{cases} G_i(a,b) & \text{if } a\neq M, j=i, b\neq a \\ H(i,j) & \text{if } a\neq M, j\neq i, b=a \\ G_i(a,a)+H(i,i) & \text{if } a\neq M, j=i, b=a \\ 0 & \text{otherwise} \end{cases} \tag{8.1}$$

for all $i,j\in E$ and $a,b\in F$. Note that (i,M) is absorbing for all $i\in E$. Using $\mathcal{G}$, the transition probability matrix of the imbedded Markov chain and the transition rates of $(Y,\mathcal{A})$ can be obtained, respectively, as

$$\mathcal{P}(i,a;j,b)=\begin{cases} \left(\frac{\lambda_i(a)}{\lambda_i(a)+\delta_i}\right)P_i(a,b) & \text{if } a\neq M, j=i, b\neq a \\ \left(\frac{\delta_i}{\lambda_i(a)+\delta_i}\right)Q(i,j) & \text{if } a\neq M, j\neq i, b=a \\ 1 & \text{if } b=a=M, j=i \\ 0 & \text{otherwise} \end{cases}$$

and

$$\lambda(i,a)=\begin{cases} \lambda_i(a)+\delta_i & \text{if } a\neq M \\ 0 & \text{if } a=M. \end{cases}$$

To simplify the notation, for any event C and any random variable Z, we will let $P_{ia}(C)=P(C|(Y_0,\mathcal{A}_0)=(i,a))$ and $E_{ia}[Z]=E[Z|(Y_0,\mathcal{A}_0)=(i,a)]$.

8.2 Reliability

The lifetime of the system is

$$L = \inf\{t \geq 0; \mathcal{A}_t = M\}$$

which is clearly a first passage time of the Markov process $(Y, \mathcal{A})$. For mission-based systems, one may be interested in the 3 different reliability measures discussed next in this section.

8.2.1 System Reliability

System reliability function is given by $P_{ia}\{L > t\}$ which is the survival probability until time $t \in \mathbb{R}_+$ given that the initial phase and deterioration level are i and a respectively. Since L is the first passage time of $\mathcal{A}$ to the absorbing state M, it is clear that

$$P_{ia}\{L > t\} = P_{ia}\{\mathcal{A}_t \neq M\} = \sum_{j\in E}\sum_{b=0}^{M-1} P_{ia}\{(Y_t, \mathcal{A}_t) = (j, b)\}.$$

It is well-known that

$$P_{ia}\{(Y_t, \mathcal{A}_t) = (j, b)\} = e^{t\mathcal{G}}(i, a; j, b)$$

for any $j \in E$ and $b \in F$ where $e^{t\mathcal{G}}$ is the matrix exponential

$$e^{t\mathcal{G}} = \sum_{n=0}^{+\infty} \frac{t^n}{n!}\mathcal{G}^n = \lim_{n\to+\infty}\left(\boldsymbol{I} + \frac{\mathcal{G}}{n}\right)^n. \tag{8.2}$$

Using this fact, we have the explicit representation for the system reliability

$$P_{ia}\{L > t\} = \sum_{j\in E}\sum_{b=0}^{M-1} e^{t\mathcal{G}}(i, a; j, b). \tag{8.3}$$

The computation of the matrix exponential (8.2) can be done in many ways and we refer the interested reader to Moler and Loan [60] for various methods. As a matter of fact, most of our results can be stated using the matrix exponential (8.2) and there are many efficient methods to compute it. The computations on the illustrations in Section 8.6 are made using MATLAB which uses the scaling and squaring method employing Padé approximants.

8.2.2 Mission Reliability

Let $T_0, T_1, T_2, \cdots$ be the transition times of the mission process so that T_n denotes the time at which the nth phase ends. In a given application, it may be important to determine the probability that the system will complete the first n phases successfully, or $P_{ia}\{L > T_n\}$ where $a \neq M$. We now focus on this issue and show that this probability can be calculated using a recursive formula that yields an explicit solution. Note that

$$\begin{aligned} P_{ia}\{L > T_1\} &= \sum_{j\in E}\int_0^{+\infty} P_{ia}\{L > T_1, T_1 \in ds, Y_{T_1} = j\} \\ &= \sum_{j\in E}\int_0^{+\infty} P_{ia}\{L > s | T_1 \in ds, Y_{T_1} = j\} Q(i,j)\,\delta_i e^{-\delta_i s} ds \\ &= \sum_{j\in E}\int_0^{+\infty} \sum_{b=0}^{M-1} e^{sG_i}(a,b)\, Q(i,j)\,\delta_i e^{-\delta_i s} ds \\ &= \sum_{j\in E}\sum_{b=0}^{M-1} E\left[e^{D_i G_i}(a,b)\right] Q(i,j) \end{aligned}$$

where D_i is the duration of phase i which is exponentially distributed with rate δ_i. Let $P^*(i,a;j,b) = E\left[e^{D_i G_i}(a,b)\right] Q(i,j)$ for all $i,j \in E$ and $a,b \in F\backslash\{M\}$. This further implies that

$$\begin{aligned} P^*(i,a;j,b) &= \int_0^{+\infty} e^{sG_i}(a,b)\, Q(i,j)\,\delta_i e^{-\delta_i s} ds \\ &= \sum_{n=0}^{+\infty} \frac{G_i^n(a,b)}{n!} Q(i,j) \left(\int_0^{+\infty} \delta_i s^n e^{-\delta_i s} ds\right) \\ &= \sum_{n=0}^{+\infty} \frac{G_i^n(a,b)}{\delta_i^n} Q(i,j) = \left(\boldsymbol{I} - \frac{1}{\delta_i} G_i\right)^{-1}(a,b)\, Q(i,j)\,. \end{aligned} \tag{8.4}$$

This is a computationally tractable solution which can be calculated by taking a matrix inverse for every phase.

Then, letting $f_{ia}^{(n)} = P_{ia}\{L > T_n\}$, we have

$$f_{ia}^{(1)} = P_{ia}\{L > T_1\} = \sum_{j\in E}\sum_{b=0}^{M-1} P^*(i,a;j,b) = P^*\mathbf{1}(i,a)$$

for $n = 1$. Conditioning on the state after the first transition, we have

$$\begin{aligned} P_{ia}\{L > T_{n+1} | L > T_1\} &= \sum_{j\in E}\sum_{b=0}^{M-1} P_{ia}\{L > T_{n+1}, (Y_{T_1}, \mathcal{A}_{T_1}) = (j,b) | L > T_1\} \\ &= \sum_{j\in E}\sum_{b=0}^{M-1} P_{ia}\{L > T_{n+1} | (Y_{T_1}, \mathcal{A}_{T_1}) = (j,b), L > T_1\} \\ &\quad \times P_{ia}\{(Y_{T_1}, \mathcal{A}_{T_1}) = (j,b) | L > T_1\} \end{aligned}$$

$$= \sum_{j \in E} \sum_{b=0}^{M-1} P_{jb}\{L > T_n\} \frac{P_{ia}\{(Y_{T_1}, \mathcal{A}_{T_1}) = (j,b), L > T_1\}}{P_{ia}\{L > T_1\}}$$
$$= \sum_{j \in E} \sum_{b=0}^{M-1} P_{jb}\{L > T_n\} \frac{P_{ia}\{(Y_{T_1}, \mathcal{A}_{T_1}) = (j,b)\}}{P_{ia}\{L > T_1\}}. \tag{8.5}$$

This implies that

$$f_{ia}^{(n+1)} = P_{ia}\{L > T_{n+1}\} = P_{ia}\{L > T_{n+1} | L > T_1\} P_{ia}\{L > T_1\}$$
$$= \sum_{j \in E} \sum_{b=0}^{M-1} P_{jb}\{L > T_n\} P^*(i,a;j,b)$$

and, hence,

$$f^{(n+1)} = P^* f^{(n)}$$

with the boundary condition $f_{ia}^{(0)} = \mathbf{1}$. Then, using induction, it can be concluded that the mission reliability for the first n phases is simply

$$f^{(n)} = P_{ia}\{L > T_n\} = (P^*)^n \, \mathbf{1}(i,a) \tag{8.6}$$

which is actually the row sum of the nth power of the matrix P^* on $E \times (F \backslash \{M\})$.

8.2.3 Phase Reliability

Depending on the overall objective of the mission, a given critical phase may be more important than the others for a complex system. Therefore, an important measure to represent the reliability of the system may be the probability that this critical phase will be completed in a fixed time period. For instance, consider NASA's Mars Exploration Rover Mission example discussed in Chapter 2. Since one of the main aims of the whole mission is determining past water activity on the surface, scientific investigations and transmission of data towards this goal are most critical for the success of the mission. Therefore, reliability of such a critical phase is of extreme importance.

Suppose that one is interested in the successful completion of a given critical phase $j \in E$ of the mission. We will determine the phase reliability $P_{ia}\{U_j \leq t, L > U_j\}$ which is the probability that phase j is successfully completed before time t. In this analysis, we define a new Markov process $Z = \{Z_t; t \geq 0\}$ by stopping the Markov process $(Y, \mathcal{A}) = \{(Y_t, \mathcal{A}_t); t \geq 0\}$ such that if the critical phase is completed without any failure while the age of the unit is a, then the process Z will jump to an absorbing success state (S_j, a) where S_j denotes successful completion of phase j. It is clear that Z is also a Markov process on the extended state space $(E \cup \{S_j\}) \times F$ with infinitesimal generator

$$\widetilde{\mathcal{G}}_j(i,a;k,b) = \begin{cases} \delta_j & \text{if } i=j, a\neq M, k=S_j, b=a \\ G_i(a,b) & \text{if } k=i, a\neq M, b\neq a \\ H(i,k) & \text{if } i\neq j, a\neq M, k\neq i, b=a \\ G_i(a,a)+H(i,i) & \text{if } a\neq M, k=i, b=a \\ 0 & \text{otherwise.} \end{cases} \tag{8.7}$$

Then, it is clear that the phase reliability of phase j is

$$P_{ia}\{U_j \le t, L > U_j\} = \sum_{b=0}^{M-1} P_{ia}\{Z_t = (S_j, b)\} = \sum_{b=0}^{M-1} e^{t\widetilde{\mathcal{G}}_j}(i,a;S_j,b) \tag{8.8}$$

which is also a matrix exponential solution.

8.3 Mean Time to Failure

In this section, we are interested in computing the MTTF or $E[L]$. It is known that

$$E_{ia}[L] = \int_0^{+\infty} P_{ia}\{L > t\}\, dt.$$

Using (8.3),

$$\begin{aligned} E_{ia}[L] &= \int_0^{+\infty} \sum_{j\in E}\sum_{b=0}^{M-1} e^{t\mathcal{G}^*}(i,a;j,b)\,dt = \sum_{j\in E}\sum_{b=0}^{M-1}\int_0^{+\infty}\sum_{n=0}^{+\infty}\frac{t^n(\mathcal{G}^*)^n(i,a;j,b)}{n!}dt \\ &= \sum_{j\in E}\sum_{b=0}^{M-1}\left[\lim_{k\to+\infty}\sum_{n=0}^{+\infty}\frac{(\mathcal{G}^*)^n(i,a;j,b)}{n!}\int_0^k t^n dt\right] \\ &= \sum_{j\in E}\sum_{b=0}^{M-1}\left[\lim_{k\to+\infty}\sum_{n=0}^{+\infty}\frac{k^{n+1}(\mathcal{G}^*)^n(i,a;j,b)}{(n+1)!}\right] \end{aligned}$$

where $\mathcal{G}^*$ is the remaining matrix after removing all absorbing states from $\mathcal{G}$. Let α be the initial distribution of $(Y, \mathcal{A})$ so that $\alpha(i,a) = P\{Y_0 = i, \mathcal{A}_0 = a\}$. Then, using Theorem 2.3.1 in Neuts [86],

$$\begin{aligned} E[L] &= \alpha^T\left(\lim_{k\to+\infty}\sum_{n=0}^{+\infty}\frac{k^{n+1}(\mathcal{G}^*)^n}{(n+1)!}\right)\mathbf{1} = \alpha^T\left(\lim_{k\to+\infty}\sum_{n=0}^{+\infty}\frac{k^{n+1}(\mathcal{G}^*)^{n+1}}{(n+1)!}(\mathcal{G}^*)^{-1}\right)\mathbf{1} \\ &= \alpha^T\left(\lim_{k\to+\infty}\left(e^{k\mathcal{G}^*}-\boldsymbol{I}\right)(\mathcal{G}^*)^{-1}\right)\mathbf{1} \\ &= \alpha^T\left(\lim_{k\to+\infty}\left(e^{-\eta k}M + o\left(e^{-\eta k}\right)\right)-\boldsymbol{I}\right)(\mathcal{G}^*)^{-1}\mathbf{1} = -\alpha^T(\mathcal{G}^*)^{-1}\mathbf{1} \end{aligned} \tag{8.9}$$

where $-\eta$ is the eigenvalue of $\mathcal{G}^*$ with largest real part. M is a matrix given by $M_{ij} = v_i u_j$ where v is the positive right eigenvector of $\mathcal{G}^*$ associated with $-\eta$ and u is a left eigenvector of $\mathcal{G}^*$. This results is consistent with (2.2.7) in Neuts [86] since the distribution of the time until absorption is a PH-type (phase type) distribution with representation $(\alpha, \mathcal{G}^*)$.

8.4 Availability

In this section, we will analyze the availability of mission-based systems and provide computationally tractable formulas. In the previous analysis, it was assumed that the state (i, M) was absorbing for all $i \in E$. However, in this section, we assume that the state (i, M) is not absorbing since the failed system is repaired. The repair duration is exponentially distributed with rate ς_i during phase i and after the repair the system is as good as a brand new one. We also assume that when the system fails during a phase, it will start to reperform the same phase after the repair. This is common in applications since a failure of the system will stop the mission.

Let $(\widehat{Y}, \widehat{\mathcal{A}}) = \{(\widehat{Y}_t, \widehat{\mathcal{A}}_t); t \in \mathbb{R}_+\}$ denote this new modified Markov process which also includes the repair activity. Its infinitesimal generator now becomes

$$\widehat{\mathcal{G}}(i, a; j, b) = \begin{cases} \varsigma_i & \text{if } a = M, b = 0, j = i \\ -\varsigma_i & \text{if } b = a = M, j = i \\ \mathcal{G}(i, a; j, b) & \text{otherwise.} \end{cases} \tag{8.10}$$

The transition probability matrix $\widehat{\mathcal{P}}$ of the Markov chain imbedded in the Markov process $(\widehat{Y}, \widehat{\mathcal{A}})$ can be easily obtained as

$$\widehat{\mathcal{P}}(i, a; j, b) = \begin{cases} \left(\frac{\lambda_i(a)}{\lambda_i(a)+\delta_i}\right) P_i(a, b) & \text{if } a \neq M, j = i, b \neq a \\ \left(\frac{\delta_i}{\lambda_i(a)+\delta_i}\right) Q(i, j) & \text{if } a \neq M, j \neq i, b = a \\ 1 & \text{if } a = M, b = 0, j = i \\ 0 & \text{otherwise.} \end{cases} \tag{8.11}$$

It is possible to compute the availability using a conventional renewal theoretic approach. It is clear that time points at which $(\widehat{Y}, \widehat{\mathcal{A}})$ enters the state $(i^*, 0)$ from (i^*, M) form a renewal process where, without loss of generality, $i^* \in E$ is any phase of the mission . The expected cycle length is the sum of mean time until absorption to state (i^*, M) given that the initial state is $(i^*, 0)$, and expected repair duration in state (i^*, M). Therefore, we further modify

$(\widehat{Y}, \widehat{\mathcal{A}})$ with generator $\widehat{\mathcal{G}}$ such that the state (i^*, M) is now an absorbing state. The mean time until absorption to state (i^*, M) given that the initial state is $(i^*, 0)$ is $-\alpha^T \overline{\mathcal{G}}^{-1} \mathbf{1}$ using (2.2.7) in Neuts [86], where α is a vector with all zero entries except the entry for state $(i^*, 0)$ being equal to 1, and $\overline{\mathcal{G}}_{i^*}$ is the matrix obtained by removing the absorbing state (i^*, M) from the matrix $\widehat{\mathcal{G}}$. Let f_{ia}^j be the expected number of visits to the failure state (j, M) until absorption for each $j \neq i^*$ given that the initial state is (i, a) for $(i, a) \in (E \times F) \backslash \{(i^*, M)\}$. Using Markovian analysis for a fixed $j \neq i^*$, it is easy to see that f_{ia}^j satisfies

$$f_{i^*a}^j = \sum_{b \neq M} \widehat{\mathcal{P}}(i^*, a; i^*, b) f_{i^*b}^j + \sum_{k \neq i^*} \widehat{\mathcal{P}}(i^*, a; k, a) f_{ka}^j \tag{8.12}$$

for every $a \neq M$,

$$f_{ia}^j = \sum_{b \in F} \widehat{\mathcal{P}}(i, a; i, b) f_{ib}^j + \sum_{k \neq i} \widehat{\mathcal{P}}(i, a; k, a) f_{ka}^j \tag{8.13}$$

for every $i \neq i^*$ and $i \neq j$, and

$$f_{ja}^j = \widehat{\mathcal{P}}(j, a; j, M) + \sum_{b \in F} \widehat{\mathcal{P}}(j, a; j, b) f_{jb}^j + \sum_{k \neq j} \widehat{\mathcal{P}}(j, a; k, a) f_{ka}^j \tag{8.14}$$

for $i = j$. Using (8.11) and (8.12) - (8.14), we obtain

$$\overline{\mathcal{P}}_{i^*} f^j = g^j$$

so that $f^j = \overline{\mathcal{P}}_{i^*}^{-1} g^j$ where $\overline{\mathcal{P}}_{i^*}$ is the matrix obtained by removing the state (i^*, M) from the matrix $-\widehat{\mathcal{P}}$ and setting the diagonal entries to 1 so that $\overline{\mathcal{P}}_{i^*}(i, a; i, a) = 1$, and

$$g_{ia}^j = \begin{cases} \widehat{\mathcal{P}}(i, a; i, M) & \text{if } j = i, a \neq M \\ 0 & \text{otherwise} \end{cases}$$

for all $(i, a) \in (E \times F) \backslash \{(i^*, M)\}$ and $j \neq i^*$. Then, using renewal arguments, system availability becomes

$$A = \frac{-\alpha^T \overline{\mathcal{G}}_{i^*}^{-1} \mathbf{1} - \sum_{j \neq i^*} \left(\overline{\mathcal{P}}_{i^*}^{-1} g^j (i^*, 0) / \varsigma_j \right)}{-\alpha^T \overline{\mathcal{G}}_{i^*}^{-1} \mathbf{1} + (1/\varsigma_{i^*})}. \tag{8.15}$$

Note that the availability formula (8.15) is true for any state $i^* \in E$.

8.4.1 A Special Case

Note that there are only 2 states in $F = \{0, 1\}$ when $M = 1$, and the system is either functioning (state 0) or failed (state 1). This simplifications allows us to obtain an explicit formula for availability. Suppose that the mission process Y has a limiting distribution β that satisfies $\beta H = 0$ and let

$$v_0^i = 1 - v_1^i = \frac{\varsigma_i}{\lambda_i(0) + \varsigma_i} \tag{8.16}$$

for every i. For any fixed phase i, it is clear that v_0^i is indeed the availability of a system that fails after an exponential lifetime with rate $\lambda_i(0)$ which is then replaced by a new one after an exponential repair time with rate ς_i. The limiting distribution of $(\widehat{Y}, \widehat{\mathcal{A}})$ can be computed explicitly in terms of the limiting distribution β of the mission process Y and system availability vector v_0.

Theorem 8.1 *If the mission process Y has a limiting distribution β, then the limiting distribution π of $(\widehat{Y}, \widehat{\mathcal{A}})$ is given explicitly by*

$$\pi_{ia} = \lim_{t \to +\infty} P\left\{\widehat{Y}_t = i, \widehat{\mathcal{A}}_t = a\right\} = \frac{\beta_i v_a^i \prod\limits_{k \neq i} v_0^k}{\sum\limits_{i \in E} \beta_i \prod\limits_{k \neq i} v_0^k}. \tag{8.17}$$

Proof. *Since $E \times F$ is finite, the limiting distribution π can be determined by solving $\pi\widehat{\mathcal{G}} = 0$ and $\sum_{(i,a) \in E \times F} \pi_{ia} = 1$. For $a = 0$, we have the balance equation*

$$-(\lambda_i(0) + \delta_i)\pi_{i0} + \sum_{k \neq i} \delta_k Q(k,i)\pi_{k0} + \varsigma_i \pi_{i1} = 0 \tag{8.18}$$

for all $i \in E$. We need to show that (8.17) satisfies (8.18). Let

$$V = \frac{1}{\sum\limits_{i \in E} \beta_i \prod\limits_{k \neq i} v_0^k}.$$

Then,

$$\begin{aligned}
-(\lambda_i(0) + \delta_i)\pi_{i0} + &\sum_{k \neq i} \delta_k Q(k,i)\pi_{k0} + \varsigma_i \pi_{i1} \\
&= V\left[-\lambda_i(0)\beta_i v_0^i \prod_{k \neq i} v_0^k - \delta_i \beta_i v_0^i \prod_{k \neq i} v_0^k + \varsigma_i \beta_i v_1^i \prod_{k \neq i} v_0^k\right. \\
&\quad \left. + \sum_{k \neq i} \delta_k Q(k,i)\beta_k v_0^k \prod_{j \neq k} v_0^j\right] \\
&= V\left[\beta_i\left(-\lambda_i(0)v_0^i + \varsigma_i v_1^i\right)\prod_{k \neq i} v_0^k\right. \\
&\quad \left. + \left(-\delta_i \beta_i + \sum_{k \neq i} \delta_k Q(k,i)\beta_k\right)\prod_{j \in E} v_0^j\right] = 0
\end{aligned}$$

since

$$-\lambda_i(0)\, v_0^i + \varsigma_i v_1^i = 0 \tag{8.19}$$

by (8.16), and

$$-\delta_i \beta_i + \sum_{k \neq i} \delta_k Q(k,i)\, \beta_k = 0$$

since $\beta H = 0$.

Now, for $a = 1$, *we have the balance equation*

$$-\varsigma_i \pi_{i1} + \lambda_i(0)\, \pi_{i0} = 0 \tag{8.20}$$

for all $i \in E$. *We need to show that (8.17) satisfies (8.20). Then,*

$$\begin{aligned} -\varsigma_i \pi_{i1} + \lambda_i(0)\, \pi_{i0} &= V\left[-\varsigma_i \beta_i v_1^i \prod_{k \neq i} v_0^k + \lambda_i(0)\, \beta_i v_0^i \prod_{k \neq i} v_0^k\right] \\ &= V\left[\beta_i \left(-\varsigma_i v_1^i + \lambda_i(0)\, v_0^i\right) \prod_{k \neq i} v_0^k\right] \\ &= 0 \end{aligned}$$

since

$$-\varsigma_i v_1^i + \lambda_i(0)\, v_0^i = 0$$

by (8.16). It is also clear that the solution (8.17) satisfies $\sum_{(i,a) \in E \times F} \pi_{ia} = 1$ *and this completes the proof.* ■

Theorem 8.1 trivially implies that system availability is

$$A = 1 - \sum_{i \in E} \pi_{i1} = \frac{\prod_{k \in E} \left(\frac{\varsigma_k}{\lambda_k(0) + \varsigma_k}\right)}{\sum_{i \in E} \beta_i \prod_{k \neq i} \left(\frac{\varsigma_k}{\lambda_k(0) + \varsigma_k}\right)}.$$

8.5 Coherent Systems with Multiple Components

In the previous sections, the condition of the system was classified into $M+1$ states. The system was considered as a whole and no consideration was given to its specific structure or components. Therefore, it is implicitly assumed that it consists of a single component. The system states represent the deterioration levels, but they do not necessarily get worse by increasing from 0 to M. The only restriction is that state M is the one representing the failure of the system and state 0 represents a brand new system. By making use of this fact, we can actually use the results obtained in the previous sections for any coherent system with independent and exponentially distributed component lifetimes. We now show how this can be done and give a simple example.

Suppose that we have a coherent system with structure function ϕ_i during any phase $i \in E$. We assume that the lifetime of component k is exponentially distributed with rate α_k^i during phase i. The states of the components is represented by $x = (x_1, x_2, \cdots, x_m)^T \in B^m$ where

$$x_k = \begin{cases} 1 & \text{if component } k \text{ is in working condition} \\ 0 & \text{otherwise.} \end{cases}$$

Suppose that there are M states in W and label them as $y_0, y_1, \cdots, y_{M-1}$ with $y_0 = \mathbf{1}$ (brand new system). Note that $y_0 = \mathbf{1} \in W$ since $\phi_i(\mathbf{1}) = 1$. To make our analysis consistent with the previous results, we relabel the states so that $0 \approx y_0, 1 \approx y_1, \cdots, y_{M-1} \approx M-1$. Moreover, we let state $M \in F$ represent all failure states in $\overline{W}$. This completes the definition of F in this generalization with states $0, 1, \cdots, M-1$ corresponding to states $y_0, y_1, \cdots, y_{M-1} \in W$ and state M corresponding to all states $y \in \overline{W}$.

Let

$$N(x) = \{\ z \in W; x = (1_k, z) \text{ for some } k\}$$

denote the set of all states to which the system may jump from state x due to a component failure. Then, the age process $\mathcal{A}$ of this system during phase i is a Markov process with the generator

$$G_i(j,k) = \begin{cases} -\sum\limits_{r \in C_1(y_j)} \alpha_r^i & \text{if } k = j \neq M \\ \sum\limits_{r \in C_1(y_j) \cap C_0(y_k)} \alpha_r^i & \text{if } y_k \in N(y_j) \\ \sum\limits_{r \in C_1(y_j)} \alpha_r^i - \sum\limits_{x \in N(y_j)} \sum\limits_{r \in C_1(y_j) \cap C_0(x)} \alpha_r^i & \text{if } k = M, j \neq M \\ 0 & \text{otherwise} \end{cases} \tag{8.21}$$

and all of the results in the previous sections can be applied trivially using the generators G_i in (8.21) and the generator $\mathcal{G}$ of the mission process.

Example 8.2 *Consider the coherent system given in Figure 8.1 for some phase i. Then, the structure function is $\phi_i(x) = x_1x_2 + x_1x_3 - x_1x_2x_3$. It is clear that $W = \{(1,1,1), (1,1,0),$*

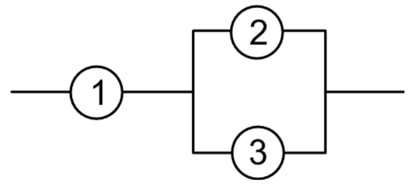

Figure 8.1: The structure of the system analyzed in Example 8.2.

$(1,0,1)\}$ *and* $\overline{W} = B^3 \backslash W$. *We first define states* $0,1,2,3 \in F$ *by relabeling* $0 \approx (1,1,1)$, $1 \approx (1,1,0), 2 \approx (1,0,1)$, *and state* $M = 3$ *represent all the other failure states in* $\overline{W}$. *Then, the age process of this system during phase i is a Markov process with the generator*

$$G_i = \begin{bmatrix} -\left(\alpha_1^i + \alpha_2^i + \alpha_3^i\right) & \alpha_3^i & \alpha_2^i & \alpha_1^i \\ 0 & -\left(\alpha_1^i + \alpha_2^i\right) & 0 & \alpha_1^i + \alpha_2^i \\ 0 & 0 & -\left(\alpha_1^i + \alpha_3^i\right) & \alpha_1^i + \alpha_3^i \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

8.6 Numerical Illustration for Reliability, MTTF, and Availability

Consider a hypothetical mission with 3 phases performed by a system with 3 deterioration levels so that $E = \{1,2,3\}$ and $F = \{0,1,2\}$ where $M = 2$ denotes failure. Suppose arbitrarily that $\delta = (1,2,1.5)$, $\lambda_1 = (0.5,1)$, $\lambda_2 = (1.5,2)$, $\lambda_3 = (1,1.5)$ and

$$Q = \begin{bmatrix} 0 & 0.5 & 0.5 \\ 0.4 & 0 & 0.6 \\ 0.7 & 0.3 & 0 \end{bmatrix}, P_1 = \begin{bmatrix} 0 & 0.7 & 0.3 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, P_2 = \begin{bmatrix} 0 & 0.8 & 0.2 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, P_3 = \begin{bmatrix} 0 & 0.6 & 0.4 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

These imply that

$$H = \begin{bmatrix} -1 & 0.50 & 0.50 \\ 0.80 & -2 & 1.20 \\ 1.05 & 0.45 & -1.50 \end{bmatrix}, G_1 = \begin{bmatrix} -0.50 & 0.35 & 0.15 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

$$G_2 = \begin{bmatrix} -1.5 & 1.2 & 0.3 \\ 0 & -2 & 2 \\ 0 & 0 & 0 \end{bmatrix}, G_3 = \begin{bmatrix} -1 & 0.6 & 0.4 \\ 0 & -1.5 & 1.5 \\ 0 & 0 & 0 \end{bmatrix}$$

and using (8.1),

$$\mathcal{G} = \begin{array}{c} (1,0) \\ (1,1) \\ (1,2) \\ (2,0) \\ (2,1) \\ (2,2) \\ (3,0) \\ (3,1) \\ (3,2) \end{array} \begin{bmatrix} -1.50 & 0.35 & 0.15 & 0.50 & 0 & 0 & 0.50 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0.50 & 0 & 0 & 0.50 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.80 & 0 & 0 & -3.50 & 1.20 & 0.30 & 1.20 & 0 & 0 \\ 0 & 0.80 & 0 & 0 & -4 & 2 & 0 & 1.20 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1.05 & 0 & 0 & 0.45 & 0 & 0 & -2.50 & 0.60 & 0.40 \\ 0 & 1.05 & 0 & 0 & 0.45 & 0 & 0 & -3 & 1.50 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then, for $t = 2$

$$e^{2\mathcal{G}} = \begin{bmatrix} 0.1352 & 0.0740 & 0.2783 & 0.0352 & 0.0283 & 0.1560 & 0.0625 & 0.0424 & 0.1881 \\ 0 & 0.0498 & 0.5709 & 0 & 0.0129 & 0.1633 & 0 & 0.0230 & 0.1802 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.0843 & 0.0580 & 0.1585 & 0.0231 & 0.0222 & 0.3493 & 0.0426 & 0.0351 & 0.2268 \\ 0 & 0.0310 & 0.1702 & 0 & 0.0085 & 0.5691 & 0 & 0.0157 & 0.2055 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0.1031 & 0.0623 & 0.1654 & 0.0277 & 0.0239 & 0.1386 & 0.0521 & 0.0383 & 0.3888 \\ 0 & 0.0379 & 0.2127 & 0 & 0.0102 & 0.1357 & 0 & 0.0192 & 0.5843 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and, using (8.3), we can easily determine the system reliability $P_{10}\{L > 2\} = 0.1352 + 0.0740 + 0.0352 + 0.0283 + 0.0625 + 0.0424 = 0.3776$. The matrix exponential is calculated by MATLAB 7.6.0 (R2008a) which uses the scaling and squaring method employing Padé approximants.

To obtain the mission reliability, we first calculate

$$\left(\boldsymbol{I} - \frac{1}{\delta_1} G_1\right)^{-1} = \begin{bmatrix} 0.6667 & 0.1167 & 0.2167 \\ 0 & 0.5000 & 0.5000 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\left(\boldsymbol{I} - \frac{1}{\delta_2} G_2\right)^{-1} = \begin{bmatrix} 0.5714 & 0.1714 & 0.2571 \\ 0 & 0.5000 & 0.5000 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\left(\boldsymbol{I} - \frac{1}{\delta_3} G_3\right)^{-1} = \begin{bmatrix} 0.6000 & 0.1200 & 0.2800 \\ 0 & 0.5000 & 0.5000 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then, using (8.4),

$$P^* = \begin{matrix} (1,0) \\ (1,1) \\ (2,0) \\ (2,1) \\ (3,0) \\ (3,1) \end{matrix} \begin{bmatrix} 0 & 0 & 0.3333 & 0.0583 & 0.3333 & 0.0583 \\ 0 & 0 & 0 & 0.2500 & 0 & 0.2500 \\ 0.2286 & 0.0686 & 0 & 0 & 0.3428 & 0.1028 \\ 0 & 0.2000 & 0 & 0 & 0 & 0.3000 \\ 0.4200 & 0.0840 & 0.1800 & 0.0360 & 0 & 0 \\ 0 & 0.3500 & 0 & 0.1500 & 0 & 0 \end{bmatrix}$$

and using (8.6), we can calculate the mission reliabilities $P_{10}\{L > T_1\} = 0.3333 + 0.0583 + 0.3333 + 0.0583 = 0.7832$, and $P_{10}\{L > T_2\} = 0.5459$, and $P_{20}\{L > T_3\} = 0.3468$ after taking the second and the third powers of P^*.

Now, we will find the probability that the critical phase $j = 2$ will be completed until time 1.5. Using (8.7), we have

$$\widetilde{\mathcal{G}}_2 = \begin{matrix} (1,0) \\ (1,1) \\ (1,2) \\ (2,0) \\ (2,1) \\ (2,2) \\ (3,0) \\ (3,1) \\ (3,2) \\ (S_2,0) \\ (S_2,1) \end{matrix} \begin{bmatrix} -1.50 & 0.35 & 0.15 & 0.50 & 0 & 0 & 0.50 & 0 & 0 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0.50 & 0 & 0 & 0.50 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3.50 & 1.20 & 0.30 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & -4 & 2 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1.05 & 0 & 0 & 0.45 & 0 & 0 & -2.50 & 0.60 & 0.40 & 0 & 0 \\ 0 & 1.05 & 0 & 0 & 0.45 & 0 & 0 & -3 & 1.50 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

so that the matrix exponential is

$$e^{1.5\widetilde{\mathcal{G}}_2} = \begin{bmatrix} 0.149 & 0.058 & 0.196 & 0.039 & 0.026 & 0,104 & 0.049 & 0.024 & 0.089 & 0.190 & 0.075 \\ 0 & 0.070 & 0.505 & 0 & 0.019 & 0.134 & 0 & 0.023 & 0.115 & 0 & 0.134 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.005 & 0.007 & 0.253 & 0 & 0 & 0 & 0.568 & 0.167 \\ 0 & 0 & 0 & 0 & 0.003 & 0.499 & 0 & 0 & 0 & 0 & 0.499 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0.103 & 0.047 & 0.096 & 0.031 & 0.023 & 0.094 & 0.051 & 0.029 & 0.299 & 0.157 & 0.070 \\ 0 & 0.049 & 0.161 & 0 & 0.015 & 0.112 & 0 & 0.024 & 0.528 & 0 & 0.112 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

This implies that critical phase reliabilities are $P_{10}\{U_2 \leq 1.5, L > U_2\} = 0.190 + 0.075 = 0.265$ and $P_{21}\{U_2 \leq 1.5, L > U_2\} = 0.499$ by using (8.8).

Suppose that $\alpha^T = (0.2,\ 0,\ 0.3,\ 0,\ 0.5,\ 0)$. Then,

$$\mathcal{G}^* = \begin{matrix} (1,0) \\ (1,1) \\ (2,0) \\ (2,1) \\ (3,0) \\ (3,1) \end{matrix} \begin{bmatrix} -1.50 & 0.35 & 0.50 & 0 & 0.50 & 0 \\ 0 & -2 & 0 & 0.50 & 0 & 0.50 \\ 0.80 & 0 & -3.50 & 1.20 & 1.20 & 0 \\ 0 & 0.80 & 0 & -4 & 0 & 1.20 \\ 1.05 & 0 & 0.45 & 0 & -2.50 & 0.60 \\ 0 & 1.05 & 0 & 0.45 & 0 & -3 \end{bmatrix}$$

and using (8.9), the MTTF is

$$E\,[L] = -\alpha^T \left(\mathcal{G}^*\right)^{-1} \mathbf{1} = 1.6586.$$

In order to calculate availability by (8.15), we further suppose that $\varsigma = [0.50, 0.75, 0.85]^T$.

Then, choosing $i^* = 1$ and using (8.10),

$$\widehat{\mathcal{G}} = \begin{array}{c} (1,0) \\ (1,1) \\ (1,2) \\ (2,0) \\ (2,1) \\ (2,2) \\ (3,0) \\ (3,1) \\ (3,2) \end{array} \begin{bmatrix} -1.50 & 0.35 & 0.15 & 0.50 & 0 & 0 & 0.50 & 0 & 0 \\ 0 & -2 & 1 & 0 & 0.50 & 0 & 0 & 0.50 & 0 \\ 0.50 & 0 & -0.50 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.80 & 0 & 0 & -3.50 & 1.20 & 0.30 & 1.20 & 0 & 0 \\ 0 & 0.80 & 0 & 0 & -4 & 2 & 0 & 1.20 & 0 \\ 0 & 0 & 0 & 0.75 & 0 & -0.75 & 0 & 0 & 0 \\ 1.05 & 0 & 0 & 0.45 & 0 & 0 & -2.50 & 0.60 & 0.40 \\ 0 & 1.05 & 0 & 0 & 0.45 & 0 & 0 & -3 & 1.50 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.85 & 0 & -0.85 \end{bmatrix} \quad (8.22)$$

which implies that $-\alpha^T \overline{\mathcal{G}}_1^{-1} \mathbf{1} = 7.8878$ in (8.15). Moreover, we have

$$\overline{\mathcal{P}}_1 = \begin{array}{c} (1,0) \\ (1,1) \\ (2,0) \\ (2,1) \\ (2,2) \\ (3,0) \\ (3,1) \\ (3,2) \end{array} \begin{bmatrix} 1 & -0.233 & -0.333 & 0 & 0 & -0.333 & 0 & 0 \\ 0 & 1 & 0 & -0.250 & 0 & 0 & -0.250 & 0 \\ -0.223 & 0 & 1 & -0.343 & -0.086 & -0.343 & 0 & 0 \\ 0 & -0.200 & 0 & 1 & -0.500 & 0 & -0.300 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ -0.420 & 0 & -0.180 & 0 & 0 & 1 & -0.240 & -0.160 \\ 0 & -0.350 & 0 & -0.150 & 0 & 0 & 1 & -0.500 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix}$$

$$g^2 = \begin{bmatrix} 0 & 0 & 0.3/3.5 & 0.5 & 0 & 0 & 0 & 0 \end{bmatrix}^T$$

and

$$g^3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0.4/2.5 & 1.5/3 & 0 \end{bmatrix}^T.$$

These imply that $\overline{\mathcal{P}}_1^{-1} g^2 (1,0) = 0.8967$, $\overline{\mathcal{P}}_1^{-1} g^3 (1,0) = 1.2137$, and using (8.15), we finally obtain

$$A = \frac{7.8878 - \left(\frac{0.8967}{0.75}\right) - \left(\frac{1.2137}{0.85}\right)}{7.8878 + 2} = 0.5324.$$

8.7 Optimal Replacement Problem

Optimal replacement problem for systems whose condition can be classified into a finite set has been extensively studied in the literature under the title of "Condition-Based Maintenance (CBM)". The following section gives a detailed review of the literature on CBM.

8.7.1 Literature Review

8.7.1.1 Introduction

Maintenance actions are vital for companies to increase reliability and availability of the production system and to decrease production costs. At the same time, Bevilacquaa and Bragliab [37] states that maintenance may require extensive expenditures which may vary from 15% to 70% of the total production cost depending on the industry. For instance, the total amount of money spent for maintenance is more than 200 billion dollars in the United States every year as observed by Chu et al. [38]. Moreover, a significant portion of total work force in a company is employed in maintenance departments; Waeyenbergh and Pintelon [39] estimates that this is up to 30% or more in chemical process industries. These observations indicate that optimizing the obvious trade-off between maintenance costs and productivity will have a very significant impact on the total cost. This is why it is not surprising that an extensive body of literature on optimal maintenance has accumulated in the last 50 years. The review papers [40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55] survey hundreds of papers in chronological order on optimal maintenance problems.

In general, there are two types of maintenance considered in the literature: corrective maintenance (CM) and preventive maintenance (PM). CM involves actions performed after a failure to restore the system to a better condition. Sim and Endrenyi [87] define PM as the actions performed regularly at preselected times (not necessarily identical) to reduce or eliminate the accumulated deterioration. PM can be further classified into two main classes: time-based preventive maintenance (TBPM) and condition-based maintenance (CBM). System lifetime is considered as a random variable in TBPM and its distribution is determined by statistical analysis. Then, optimal preventive maintenance actions are planned according to a mathematical model developed using the failure distribution of the system and related maintenance costs. On the other hand, Wang et al. [88] states that maintenance decisions are given according to the actual state or condition of the system under CBM policies. The state of the system may take either discrete values such as real or intrinsic age (e.g., number of flights for planes) or predefined deterioration levels, or continuous values such as temperature, vibration, cumulative wear, etc. In the former one, multi-state Markov decision processes are generally used to determine the states at which a preventive replacement decision is optimal to minimize a cost criteria. In the latter one, the system is generally subject to continuous wear or deterioration. The general purpose of the models developed to investigate such systems is to find an optimal treshold above which a preventive replacement decision is optimal. In this review, we focus on condition-based maintenance models under Markovian deterioration where the state of the system can be classified into one of a finite number of states. For more information on condition-based maintenance models with continuous deterioration, we refer the interested reader to [89, 90, 91, 92, 38, 93, 94, 95, 96, 97, 98, 88] and the references cited in these papers .

CBM models are further classified by the time points at which the state of the system is observed by the decision maker. There are three types of inspection policies applied in CBM literature: continuous inspection policy (CIP), periodic inspection policy (PIP), and sequential inspection policy (SIP). Systems are always monitored or the state of the system is always known by the decision maker under CIP. On the other hand, the condition of the system under PIP or SIP is known only at some discrete time points. The main difference between PIP and SIP is that in PIP, the system is inspected and its state is observed at equal time intervals, but these intervals do not have to be equal in SIP. If we apply PIP or SIP, the time points at which the system will be inspected must be determined carefully, since more frequent inspections will increase the related inspection cost and less frequent inspections will decrease our ability to maintain the survival of the system. We will also review some important optimal inspection and replacement models for which a control-limit policy is optimal.

Although they may not be optimal even under very intuitive conditions (as we will illustrate by examples later), control-limit policies are studied extensively in the literature. The significance of control-limit policies is that they are very easy to understand and implement. Similarly, there are many CBM policies defined using several thresholds in the literature. These models are important because they are also easy to apply and they can reasonably describe the deterioration-maintenance process of some systems in real-world applications. We will also review some of the important papers which analyze maintenance policies defined via several thresholds.

In Section 8.7.1.2, the general assumptions and notation used in CBM models with Markovian deterioration are given. Section 8.7.1.3 and Section 8.7.1.4 are on optimal replacement, and optimal inspection/replacement models respectively where optimality of control-limit policies are established. We conclude the review with Section 8.7.1.5 by summarizing some other important maintenance models where policies are described by a few tresholds.

8.7.1.2 General Assumptions

We shall concentrate specifically on optimal policies for CBM models with Markovian deterioration. In the most generic sense, these models satisfy the following main assumptions:

i. The system can be observed to be in one of the $M+1$ states from the set $F = \{0, 1, \cdots, M-1, M\}$ where state 0 represents a brand new system, states $1, 2, \cdots, M-1$ represent intermediate deterioration levels in ascending order, and M denotes system failure.

ii. Transitions among the deterioration levels at successive decision times follow an increasing Markov chain with an upper-triangular transition probability matrix $P = [P_{ab}]$ where P_{ab} is the probability that the next deterioration level of the system will be b given that current level is a for every $a, b \in F$.

iii. Holding time in each level is a random variable with parameters which may or may not depend on the deterioration level. Let t_a be the holding time in state $a \in F$ with mean $\bar{t}_a$. We suppose that t_a is exponentially distributed for every a so that the deterioration process is a Markov process. Otherwise, it is a semi-Markov process.

iv. Replacement durations may be negligible or random. Let r_a denote the replacement duration with mean $\bar{r}_a$ if the replacement decision is given when the system is in state $a \in F$. There is always a replacement cost c_a if the system is replaced when its deterioration level is $a \in F$.

v. The system may be inspected continuously, periodically, or sequentially.

vi. A state occupancy cost h_a may be incurred when the system is occupying level $a \in F$. When the system fails, a failure cost K is incurred.

vii. At each decision epoch, there are two alternative decisions: the system will be replaced ($s_a = 1$) or not replaced ($s_a = 0$) respectively if the system is in state $a \in F$ at that decision epoch. The replacement action is assumed to be perfect so that the system state is restored to 0 after the replacement.

viii. The objective of the problem is to minimize the expected total discounted cost or the average cost per time.

These type of maintenance problems have been studied in the literature since 1960's. In general, there are two types of research papers where the first type defines a mathematical model of the maintenance problem and obtains a policy (usually a control-limit policy) which solves the problem optimally. The second type investigates the problems using a given simplified policy that is not necessarily optimal and finds the optimum parameters of this class of policies that minimize a cost function. In the following two sections, we will review papers of the first type for replacement and inspection/replacement models respectively. The final section focuses on papers of the second type.

8.7.1.3 Optimal Replacement Models

One of the earliest and basic cases where the deterioration process is described by a Markov chain is analyzed by Derman [99]. It is assumed that the system is inspected at equally spaced points in time and the system is classified into one of the deterioration levels after each inspection. The holding times are not considered and the decision model is formulated based on the deterioration levels of the system observed at each inspection time. It is assumed that the successive levels follow a Markov chain whose transition matrix is monotone so that the cumulative matrix

$$\bar{P}_{ab} = \sum_{k=b}^{M} P_{ak} \tag{8.23}$$

is nondecreasing in a for every $b \in F$. The other assumptions of this model are that the replacement duration is equal to one inspection interval, replacement costs do not depend on the deterioration level ($c_a = c$ for every a and some c), and there is no state occupancy cost $(h_a = 0)$. The DPE is

$$v(a) = \begin{cases} c + K + \alpha \sum_{b=0}^{M} P_{0b} v(b) & \text{if } a = M \\ \min \left\{ \alpha \sum_{b=0}^{M} P_{ab} v(b), c + \alpha \sum_{b=0}^{M} P_{0b} v(b) \right\} & \text{if } a \neq M \end{cases} \tag{8.24}$$

where $0 \leq \alpha < 1$ is the periodic discount factor and $v(a)$ is the total expected discounted cost. In our discussions, we will present details primarily on $v(a)$ with the understanding that the average cost can be obtained in a similar fashion. It is proven that the optimal policy has a control-limit structure. In particular, there exists $a^* \in F$ such that

$$s_a^* = \begin{cases} 1 & \text{if } a \geq a^* \\ 0 & \text{if } a < a^* \end{cases} \tag{8.25}$$

for every $a \in F$ where s^* denotes the optimal policy. The same model with sate occupancy costs incurred each time that the system is inspected is analyzed by Kolesar [100]. The DPE now becomes

$$v(a) = \begin{cases} c + h_M + \alpha \sum_{b=0}^{M} P_{0b} v(b) & \text{if } a = M \\ \min \left\{ h_a + \alpha \sum_{b=0}^{M} P_{ab} v(b), c + h_a + \alpha \sum_{b=0}^{M} P_{0b} v(b) \right\} & \text{if } a \neq M \end{cases} \tag{8.26}$$

and the optimal policies minimizing both total expected discounted cost and average cost are control-limit type if h_a and $\bar{P}_{ab}$ are nondecreasing in a.

Kawai et al. [101] consider another model where state occupancy costs are not paid during replacement, and all costs are state dependent. The DPE is

$$v(a) = \min \left\{ h_a + \alpha \sum_{b=0}^{M} P_{ab} v(b), c_a + \alpha v(0) \right\} \tag{8.27}$$

and it is shown that the optimal policy still has a control-limit structure even when c_a is increasing in a provided that $h_a, h_a - c_a$, and $\bar{P}_{ab}$ are increasing in a.

A generalization of the model in [100] is analyzed by Wood [102] by considering the case where the replacement action may fail with a probability $1-p$ and the occupancy costs are not paid during replacement. The standard recursion for this model can be formulated as

$$v(a) = \begin{cases} c + K + \alpha p v_\alpha(0) + \alpha(1-p) v_\alpha(M) & \text{if } a = M \\ \min \left\{ h_a + \alpha \sum_{b=0}^{M} P_{ab} v_\alpha(b), c + \alpha p v_\alpha(0) + \alpha(1-p) v_\alpha(a) \right\} & \text{if } a \neq M. \end{cases} \tag{8.28}$$

For this model, the optimality of a control-limit policy minimizing the total expected discounted cost and the average cost is proven under the same assumptions used by Kolesar [100]. The model where the occupancy costs are paid during replacement has the DPE

$$v(a) = \begin{cases} c + K + h_M + \alpha p v(0) + \alpha (1-p) v(M) & \text{if } a = M \\ \min\left\{ h_a + \alpha \sum_{b=0}^{M} P_{ab} v(b), c + h_a + \alpha p v(0) + \alpha (1-p) v(a) \right\} & \text{if } a \neq M. \end{cases} \tag{8.29}$$

It is shown that the control-limit rule may not be optimal for this case by a counterexample. In the same paper, a constantly monitored system is also investigated with the assumptions that the replacement duration and holding times are exponentially distributed, and replacement decisions are allowed only when a transition occurs in the deterioration process of the system. It is assumed that the holding times depend on the deterioration level with rate λ_a for level a, but the replacement duration does not with a constant rate λ. The analysis is done by applying uniformization techniques by which a continuous-time Markov decision process can be converted into an equivalent discrete-time Markov decision process. Wood [102] concludes that the optimal policy is control-limit type for both total expected discounted cost and average cost criteria provided that h_a and $\sum_{k=b}^{M} P_{ak}\lambda_k$ are nondecreasing in a, and occupancy costs are not paid during replacement. It is also proven that the same result holds for the case where the occupancy costs are also paid during replacement if the replacement duration is stochastically smaller than the holding time in each state ($\lambda \geq \lambda_a$). Özekici and Günlük [103] propose some sufficient conditions which make the lifetime of a system with Markovian deterioration increasing failure rate on average (IFRA), and also show that these conditions imply the optimality of a control-limit policy if the replacement cost does not depend on the deterioration level of the system.

In all of the papers discussed so far, the holding times are either negligible or exponentially distributed. However, the optimality of a control-limit policy may be obtained for the models with different (even more general) holding time distributions satisfying some monotonicity properties. A model where replacement and holding times are discrete random variables and the system is monitored continuously is analyzed by Kao [104]. It is assumed that a replacement decision can only be given after a transition of the deterioration level of the system. The author proves that the optimal policy which minimizes the total discounted expected cost has control-limit structure provided that h_a is nondecreasing in a, $\bar{t}_a$ is nonincreasing in a, P is monotone, and the cost and time for replacement are independent of the deterioration level. In this model, the deterioration process is actually a semi-Markov process where sojourn times are discrete random variables. Following this line of research, So [105] used semi-Markov decision processes to analyze the model where replacement duration has a general distribution which is independent of the deterioration level ($\bar{r}_a = \bar{r}$), holding times are independent and identically distributed random variables where $\bar{t}_a$ is nonincreasing in a, h_a and c_a are nondecreasing in a, and $\sum_{b=a}^{M} P_{ab}c_b - c_a$ is nondecreasing in a for $1 \leq a < M$. The last assumption might be strict, but it is shown that this condition can be easily verified in some important special cases. It is also assumed that a fixed charge β is incurred when the system is occupying level 0 to do parametric analysis on β. The optimality of a control-limit policy minimizing average cost is proven under some monotonicity assumptions for every β; in particular, for $\beta = 0$. The author also extends this result to the case where replacement durations are dependent on the deterioration level under the assumptions that $\sum_{b=a}^{M} P_{ab}c_b - c_a$ is nondecreasing in a for $1 \leq a < M$, $\bar{r}_a$ is nondecreasing in a, $h_a\bar{t}_a$ is nondecreasing in a, and $\sum_{b=a}^{M} P_{ab}\bar{r}_b - \bar{r}_a + \bar{t}_a$ is nonincreasing in a.

Another study using a semi-Markov process with continuous sojourn times to model the deterioration process of a system is presented by Lam and Yeh [106]. In this model, the holding time in level a has a general distribution F_a with hazard rate function f_a; state occupancy costs, replacement costs and times are state dependent, and from level a, the deterioration process will make a direct transition either to level $a+1$ with probability p_a or level M with probability $1-p_a$. It is assumed that the system is monitored continuously and a decision is given when the system enters a new deterioration level. When the system enters level a, the decision maker takes a decision to replace the system t_a units of time later if it remains in level a. If $t_a = 0$, then the system is replaced as soon as it enters level a and if $t_a = +\infty$, then the system will not be replaced as long as it stays in level a. The model has the following monotonicity assumptions:

i. The state occupancy cost rate, the replacement cost rate, the expected replacement time, the marginal replacement cost, and the marginal replacement time increase as the system deteriorates,

ii. F_a is an increasing failure rate distribution for every a and $f_a(t)$ increases in a for every t,

iii. p_a is nondecreasing in a.

Under these conditions, there exist h^* and k^* with $0 \leq h^* \leq k^* \leq M$, such that

$$t_a^* = \begin{cases} +\infty & \text{if } a < h^* \\ s_a & \text{if } h^* \leq a < k^* \\ 0 & \text{if } k^* \leq a \leq M \end{cases}$$

where t_a^* is the optimal decision in level a, and $0 < s_{h^*} \leq s_{h^*+1} \leq \cdots \leq s_{k^*-1} < +\infty$. In other words, the system is replaced immediately as soon as it enters one of the states $\{\ k^*, k^*+1, \cdots, M\}$, and it is never replaced in states $\{0, 1, \cdots, h^*-1\}$. However, in any state $a \in \{h^*, h^*+1, \cdots, k^*\}$, it is replaced after s_a units of time in that state.

8.7.1.4 Optimal Inspection/Replacement Models

Another interesting research problem involves optimal inspection and replacement where the state of the system can only be observed via inspections performed at selected times. A fixed cost is incurred whenever the system is inspected and then, either a replacement occurs or the time until the next inspection is determined. Such a problem with negligible replacement and inspection times is analyzed by Ohnishi et al. [107] who consider a system with Markovian deterioration. It is assumed that holding times are state dependent exponential random variables, state occupancy and replacement costs are dependent on the deterioration level, and from state a, a direct transition can occur only to state $a+1$ or state M. Under some monotonicity assumptions on costs and transition rates, it is shown that the optimal policy minimizing the average cost has a control-limit structure and the optimal time interval between successive inspections becomes shorter as the deterioration level of the system increases. Similar results are obtained by Lam and Yeh [108] for an identical model where replacement and inspection times are not negligible. It is clear that in real-world applications, numerical procedures are necessary to find optimal policies even if it has a control-limit structure. Iterative algorithms are derived for the optimal inspection and replacement problem under different maintenance strategies by Lam and Yeh [108].

These maintenance strategies include failure replacement, age replacement, sequential inspection, periodic inspection, and continuous inspection. These algorithms are valid for a model where the deterioration process is a continuous time Markov process and from state a, a direct transition can occur only to state $a+1$ or state M. Numerical procedures for a more general model are proposed by Yeh [109]. In this model, the deterioration process is a semi-Markov process where the holding time in each level follows a general distribution that depends on the level and from level a, direct transitions to levels $a+1, a+2, \cdots, M-1, M$ are allowed. Iterative algorithms minimizing the average cost rate are provided to derive the optimal state dependent and state-age dependent inspection/replacement policies. In a state-age dependent policy, once the state of the system is identified, the maintenance decision is made according to the deterioration level of the system and the time spent in the current state. However, if we apply the state dependent policy, each maintenance action is determined only according to the deterioration level of the system no matter how long the system has been in that state.

8.7.1.5 Optimal Maintenance Using Thresholds

Besides the papers which investigate the optimality of control-limit policies, there is also abundant literature that focus on a given special class of policies where maintenance decision is made within that class. This type of models can be useful especially when a special policy reasonably describes the deterioration-maintenance process of the system. For instance, an optimal preventive maintenance model suitable for (not limited to) especially coal pulverizers, circuit breakers and transformers is proposed by Sim and Endrenyi [87]. In this model, the system is subject to two types of failure: Poisson failures and deterioration failures. The deterioration process is an increasing Markov process where the holding times are exponentially distributed with a constant rate and $P_{ij} = 1$ where $j = i+1$. The times to Poisson failures are exponentially distributed with a constant rate independent of the deterioration level of the system. The system is removed from operation periodically for preventive minimal maintenance which restores the deterioration level to the previous level (i.e., from level i to level $i-1$ if the deterioration level is i when the preventive maintenance starts). The duration between two successive minimal preventive maintenance actions has an Erlang-r distribution with mean $1/\lambda_m$. If a failure occurs, the system is restored to level

0 and the time to repair depends on the type of the failure. The steady-state equations for this model and a simple algorithm to find the steady-state probabilities when $r = 1$ are proposed. The authors also analyze the optimal preventive maintenance problem to minimize unavailability with respect to λ_m. An extension of this model where $r = 1$ is investigated by Sim and Endrenyi [110]. This paper considers the systems which can be restored to "as good as new" status preventively. It is assumed that the first $s - 1$ preventive maintenance actions are minimal (i.e., the system is restored to the previous deterioration level), then the following preventive maintenance is major maintenance where the system is restored to level 0. This system is also subject to Poisson failures; but after a Poisson failure, the system experiences a minimal repair which is exponentially distributed (i.e., the system is restored to the operable state it was in just before the failure). After a deterioration failure, the system is again overhauled to state 0. A recursive algorithm is proposed to find steady-state probabilities, and closed-form expressions for steady-state probabilities in the case where $s \to +\infty$ are given. The optimal values of λ_m which minimize unavailability and average cost respectively are also discussed.

This line of research is also followed by Chen and Trivedi [111] who consider a model where the holding times are dependent on the deterioration level and each inspection takes an exponentially distributed amount of time. It is assumed that the system is inspected after a random period which is exponentially distributed with rate λ_{in}. The applied preventive maintenance policy can be summarized by the two tresholds (g, b) as follows: if the observed deterioration level is i with $i \leq g$ after an inspection, then no maintenance occurs. If the system deterioration level is i with $g < i \leq b$, then the system is restored to level $i - 1$ by minimal maintenance. The system experiences major maintenance when the deterioration level is found to be in i with $b < i \leq M - 1$, by which the system is restored to level 0. If a deterioration failure occurs, the system is overhauled to level 0. Moreover, the system is subject to Poisson failures after which a minimal repair is performed which restores the system to the level it was in just before the failure. For this model, the authors give closed-form expressions for steady-state probabilities, steady-state availability, and MTTF. They also numerically analyze the optimal inspection intervals (λ_{in}) minimizing unavailability and average cost respectively, and maximizing MTTF under a target availability constraint. The optimality of such a threshold policy is shown by Chen and Trivedi [112] who analyze numerical examples for the case where the deterioration rate at each level is the same. A similar model where $g = 0$ is analyzed by Amari and McLaughlin [113]. Closed form expressions for steady-state probabilities and availability are presented and algorithms to solve three optimization problems maximizing the system availability are given. These problems are formulated to find optimal λ_{in} for a given value of b, to find optimal b for a given value of λ_{in}, and to find optimal values of b and λ_{in}.

Minimal repair action may also be applied at any deterioration level of the system. Moustafa et al. [114] analyze a model where each holding time follows a general distribution and, at each state transition, one of three possible actions can be chosen: do-nothing, minimal maintenance, and replacement. To derive the optimal policy minimizing the expected long-run cost rate, two different approaches are followed. In the first approach, a control-limit policy with two thresholds is determined. The second approach uses the conventional policy iteration algorithm. By numerical examples, it is shown that the optimal policy may not be control-limit and minimal maintenance may not be optimal in any state when the cost and the time of minimal maintenance increase relative to the cost and the time of replacement respectively.

Minimal maintenance restores the system to the previous deterioration level in all previously cited papers. An extension of this idea can be repair by which the system can be restored to any better deterioration level. For example, a rather general policy $R_{ij}(T, N, \alpha)$ for a continuous time Markovian deteriorating system is proposed and analyzed by Chiang and Yuan [115]. Under this policy, the system is inspected at times $T, 2T, 3T, \cdots$ to identify the current deterioration level a of the system. Let m be the number of repairs already undertaken until the inspection time. The maintenance decision will be do-nothing if $a \leq i-1$, or $i \leq a \leq j-1$ and $m = N$. The system is repaired to a better state if $i \leq a \leq j-1$ and $m < N$. The next deterioration level of the system after the repair will be determined by the probability matrix α (i.e., the system will be restored to state r with probability α_{ar}). The maintenance decision will be replacement if $j \leq a \leq M$. An algorithm is also proposed to derive the optimal values of i, j, and T for given N and α.

8.7.2 Optimal Policy Characterization

In this section, the optimal replacement problem of a mission-based system with Markovian mission and deterioration is analyzed. The mission process and the age or deterioration process of the system have the same structure as given at the beginning of Chapter 8. The mission process is a Markov process and the deterioration process is a Markov process modulated by the mission process. We analyze the optimal replacement policy under some IFR assumptions on the deterioration process. It is assumed that P_i is a stochastically monotone upper triangular matrix and failure rate of the system increases as the deterioration level of the system increases. In other words,

$$\sum_{b=k}^{M} P_i(a,b) \geq \sum_{b=k}^{M} P_i(a-1,b) \tag{8.30}$$

and

$$\lambda_i(a) \geq \lambda_i(a-1) \tag{8.31}$$

for any $a = 1, \cdots, M$ and $k \in F$. We also assume that all replacements are instantaneous.

We will characterize the optimal replacement policy which minimizes the expected total discounted cost under the assumption that the decision maker is allowed to make a decision only when a change occurs in the mission process or the deterioration process. There are three costs associated with our problem. The preventive replacement and the failure costs during phase i are p_i and f_i respectively with $f_i \geq p_i$ and $\sup_{i\in E} f_i = f < +\infty$. A state occupancy cost $c(i,a)$ with $\sup_{i,a} c(i,a) = C < +\infty$ is incurred if the system starts to perform phase i with the initial deterioration level a. It is assumed that $c(i,a)$ is increasing in a for all $i \in E$ and $\alpha > 0$ is the continuous discount rate. Although we assume that $c(i,a)$ is a fixed lump-sum cost incurred at the beginning of each decision epoch, it is possible to obtain it explicitly when the state occupancy costs are incurred continuously. If the state occupancy cost rate is $c_r(i,a)$ for the system performing phase i with deterioration level a, then

$$c(i,a) = \int_0^{+\infty} \lambda(i,a)\, e^{-\lambda(i,a)t} \int_0^t c_r(i,a) e^{-\alpha s} ds dt = \frac{c_r(i,a)}{\lambda(i,a)+\alpha}.$$

We need to solve the DPE

$$v(i,a) = \min_{s\in A_a} \{r_s(i,a) + \Gamma_s v(i,a)\} \tag{8.32}$$

where $s = 1$ implies replacement, $s = 0$ implies do nothing, $A_0 = \{0\}, A_M = \{1\}, A_a = \{0, 1\}$, $r_0(i, a) = c(i, a)$, $r_1(i, a) = c(i, 0) + p_i$, $r_1(i, M) = c(i, 0) + f_i$,

$$\Gamma_0 g(i,a) = \frac{\delta_i + \lambda_i(a)}{\delta_i + \lambda_i(a) + \alpha} \left[\frac{\delta_i}{\delta_i + \lambda_i(a)} \sum_{j \in E} Q(i,j)\, g(j,a) + \frac{\lambda_i(a)}{\delta_i + \lambda_i(a)} \sum_{b=0}^{M} P_i(a,b)\, g(i,b) \right]$$

for $a = 0, \cdots, M-1$ and

$$\Gamma_1 g(i,a) = \frac{\delta_i + \lambda_i(0)}{\delta_i + \lambda_i(0) + \alpha} \left[\frac{\delta_i}{\delta_i + \lambda_i(0)} \sum_{j \in E} Q(i,j)\, g(j,0) + \frac{\lambda_i(0)}{\delta_i + \lambda_i(0)} \sum_{b=0}^{M} P_i(0,b)\, g(i,b) \right] \tag{8.33}$$

for $a = 1, \cdots, M$.

Now, we can use uniformization technique by applying the procedure in Puterman [116]. The original model has

$$P_s(i,a;j,b) = \begin{cases} \mathcal{P}(i,a;j,b) & \text{if } s = 0 \\ \mathcal{P}(i,0;j,b) & \text{if } s = 1 \end{cases}$$

and

$$\lambda_s(i,a) = \begin{cases} \lambda(i,a) & \text{if } s = 0 \\ \lambda(i,0) & \text{if } s = 1. \end{cases}$$

In the above formulation, the subscripts after P and λ mean that the related formula is valid if the replacement decision denoted by the subscript is applied.

To apply uniformization, we need to assume that there exists a positive constant such that

$$\overline{\lambda}_s(i,a) \leq c$$

for all $i \in E$, $a \in F$ and $s \in B$. Then, define

$$\widetilde{r}_s(i,a) = r_s(i,a) \left(\frac{\alpha + \overline{\lambda}_s(i,a)}{\alpha + c} \right)$$

and

$$\widetilde{P}_s(i,a;j,b) = \begin{cases} 1 - \frac{\left(1 - \overline{P}_s(i,a;j,b)\right)\overline{\lambda}_s(i,a)}{c} & \text{if } (j,b) = (i,a) \\ \frac{\overline{P}_s(i,a;j,b)\overline{\lambda}_s(i,a)}{c} & \text{if } (j,b) \neq (i,a). \end{cases}$$

Then, by Proposition 11.5.1. in Puterman [116], we have $v(i,a) = \widetilde{v}(i,a)$ for every stationary policy where $\widetilde{v}$ satisfies the DPE

$$\widetilde{v}(i,a) = \min_{s \in A_a} \left\{ \widetilde{r}_s(i,a) + \widetilde{\Gamma}_s \widetilde{v}(i,a) \right\} \tag{8.34}$$

for all $i \in E$ and $a \in F$ where

$$\begin{aligned}\widetilde{r}_0(i,a) &= c(i,a)\left(\frac{\alpha+\delta_i+\lambda_i(a)}{\alpha+c}\right)\\ \widetilde{r}_1(i,a) &= (c(i,0)+p_i)\left(\frac{\alpha+\lambda_i(0)+\delta_i}{\alpha+c}\right)\\ \widetilde{r}_1(i,M) &= (c(i,0)+f_i)\left(\frac{\alpha+\lambda_i(0)+\delta_i}{\alpha+c}\right)\end{aligned}$$

for all $a \in F\backslash\{M\}$,

$$\widetilde{\Gamma}_0 g(i,a) = \frac{c}{c+\alpha}\left[\frac{\delta_i}{c}\sum_{j\in E}Q(i,j)\,g(j,a)+\frac{\lambda_i(a)}{c}\sum_{b=0}^{M}P_i(a,b)\,g(i,b)\right. \tag{8.35}$$

$$\left.+\left(1-\frac{\delta_i+\lambda_i(a)}{c}\right)g(i,a)\right] \tag{8.36}$$

and

$$\begin{aligned}\widetilde{\Gamma}_1 g(i,a) = \frac{c}{c+\alpha}&\left[\frac{\delta_i}{c}\sum_{j\in E}Q(i,j)\,g(j,0)+\frac{\lambda_i(0)}{c}\sum_{b=0,b\neq a}^{M}P_i(0,b)\,g(i,b)\right.\\ &\left.+\left(1-\frac{\left(1-\frac{\lambda_i(0)}{\delta_i+\lambda_i(0)}P_i(0,a)\right)(\delta_i+\lambda_i(0))}{c}\right)g(i,a)\right]\end{aligned}$$

or

$$\widetilde{\Gamma}_1 g(i,a) = \frac{c}{c+\alpha}\left[\frac{\delta_i}{c}\sum_{j\in E}Q(i,j)\,g(j,0)+\frac{\lambda_i(0)}{c}\sum_{b=0}^{M}P_i(0,b)\,g(i,b)\right. \tag{8.37}$$

$$\left.+\left(1-\frac{\delta_i+\lambda_i(0)}{c}\right)g(i,a)\right]. \tag{8.38}$$

Let $\mathcal{B}$ denote the set of all real-valued bounded functions defined on $E \times F$. For any $f \in \mathcal{B}$, we define the operator Υ so that

$$\Upsilon f(i,a) = \min_{s\in A_a}\left\{\widetilde{r}_s(i,a)+\widetilde{\Gamma}_s f(i,a)\right\} \tag{8.39}$$

for all $i \in E$, and $a \in F$.

Lemma 8.3 *If $g(i,a) = f(i,a)+h$ for some constant h, then $\Upsilon g(i,a) = \Upsilon f(i,a)+(\frac{c}{c+\alpha})h$.*

Proof. *Using (8.36) and (8.38),*

$$\begin{aligned}\widetilde{\Gamma}_0 g(i,a) &= \frac{c}{c+\alpha}\left[\frac{\delta_i}{c}\sum_{j\in E}Q(i,j)\,g(j,a)+\frac{\lambda_i(a)}{c}\sum_{b=0}^{M}P_i(a,b)\,g(i,b)\right.\\ &\qquad\left.+\left(1-\frac{\delta_i+\lambda_i(a)}{c}\right)g(i,a)\right]\\ &= \Gamma_0 f(i,a)+\left(\frac{c}{c+\alpha}\right)h\end{aligned}$$

and

$$\begin{aligned}\widetilde{\Gamma}_1 g(i,a) &= \frac{c}{c+\alpha}\left[\frac{\delta_i}{c}\sum_{j\in E} Q(i,j)\, g(j,0) + \frac{\lambda_i(0)}{c}\sum_{b=0}^{M} P_i(0,b)\, g(i,b)\right. \\ &\quad \left. \times\left(1-\frac{\delta_i+\lambda_i(0)}{c}\right) g(i,a)\right] \\ &= \Gamma_1 f(i,a) + \left(\frac{c}{c+\alpha}\right) h.\end{aligned}$$

This implies that

$$\Upsilon g(i,a) = \min_{s\in A_a}\left\{\widetilde{r}_s(i,a) + \widetilde{\Gamma}_s f(i,a) + \left(\frac{c}{c+\alpha}\right) h\right\} = \Upsilon f(i,a) + \left(\frac{c}{c+\alpha}\right) h.$$

■

Theorem 8.4 *There exists a unique $\widetilde{v}^*$ in $\mathcal{B}$ which satisfies the DPE (8.34).*

Proof. *We will use Banach's contraction mapping theorem. Choose two functions $f, g \in \mathcal{B}$ and suppose that $\|\cdot\|$ is the usual supremum norm. Let*

$$h = \|f-g\|.$$

Then,

$$g(i,a) - h \le f(i,a) \le g(i,a) + h \tag{8.40}$$

for all $i \in E$ and $a \in F$. It is easy to see that

$$\Upsilon(g(i,a) - h) \le \Upsilon f(i,a) \le \Upsilon(g(i,a)+h).$$

Then, using Lemma 8.3,

$$\Upsilon g(i,a) - h\left(\frac{c}{c+\alpha}\right) \le \Upsilon f(i,a) \le \Upsilon g(i,a) + h\left(\frac{c}{c+\alpha}\right)$$

and this further implies that

$$|\Upsilon f(i,a) - \Upsilon g(i,a)| \le h\left(\frac{c}{c+\alpha}\right)$$

and

$$\|\Upsilon f - \Upsilon g\| \le \left(\frac{c}{c+\alpha}\right)\|f-g\|.$$

Then, since

$$\frac{c}{c+\alpha} < 1$$

Υ is a contraction mapping and it has a unique fixed point using Banach's contraction mapping theorem. ■

By Theorem 8.4, we know that there is a stationary policy which solves the DPE (8.34). Since, $\widetilde{v} = v$ for every stationary policy, they are equal for the optimal replacement policy. Therefore, if we have a characterization for the transformed process, the same characterization will be valid for the original process. From now on, v and $\widetilde{v}$ will represent the expected total discounted cost under the optimal replacement policy for the original process and the transformed process respectively. Our aim is to characterize the optimal policy. First, we need some preliminary results.

Lemma 8.5 *If $\widetilde{v}(i,a)$ is increasing in a, then*

$$\frac{\lambda_i(a)}{c}\sum_{b\geq a+1} P_i(a,b)\,\widetilde{v}(i,b) + \left(1-\frac{\delta_i+\lambda_i(a)}{c}\right)\widetilde{v}(i,a)$$

is also increasing in a for all $i \in E$

Proof. *Define two vectors as*

$$u_1 = \begin{bmatrix} \frac{c-\lambda_i(a)-\delta_i}{c} & \frac{\lambda_i(a)P_i(a,a+1)}{c} & \frac{\lambda_i(a)P_i(a,a+2)}{c} & \cdots & \frac{\lambda_i(a)P_i(a,M)}{c} \end{bmatrix}$$

and

$$u_2 = \begin{bmatrix} 0 & \frac{c-\lambda_i(a+1)-\delta_i}{c} & \frac{\lambda_i(a+1)P_i(a+1,a+2)}{c} & \frac{\lambda_i(a+1)P_i(a+1,a+3)}{c} & \cdots & \frac{\lambda_i(a+1)P_i(a+1,M)}{c} \end{bmatrix}.$$

We first show that

$$\sum_{i=k}^{M-a+1} u_2(i) \geq \sum_{i=k}^{M-a+1} u_1(i)$$

for all $k = 1,\cdots,M-a+1$. If $k=1$, then

$$\sum_{i=1}^{M-a+1} u_1(i) = \frac{c-\delta_i}{c} = \sum_{i=1}^{M-a+1} u_2(i) = \frac{c-\delta_i}{c}.$$

If $k=2$, then

$$\sum_{i=2}^{M-a+1} u_1(i) = \frac{\lambda_i(a)}{c} \leq \sum_{i=2}^{M-a+1} u_2(i) = \frac{c-\delta_i}{c}$$

since $\lambda_i(0)+\delta_i \leq c$.

If $k=j$ for some $j = 3,\cdots,M-a+1$, then using (8.30) and (8.31),

$$\sum_{i=j}^{M-a+1} u_1(i) = \frac{\lambda_i(a)}{c}\sum_{b=a+j-1}^{M} P_i(a,b) \leq \sum_{i=j}^{M-a+1} u_2(i) = \frac{\lambda_i(a+1)}{c}\sum_{b=a+j-1}^{M} P_i(a+1,b).$$

Thus, we have

$$\sum_{i=k}^{M-a+1} u_2(i) \geq \sum_{i=k}^{M-a+1} u_1(i)$$

for all $k = 1, \cdots, M-a+1$. *Moreover, since* $\widetilde{v}$ *is increasing in* a, *using Lemma 1 on page 123 in Derman [99], we have*

$$\sum_{j=a}^{M} u_2(j-a+1)\,\widetilde{v}(i,j) \geq \sum_{j=a}^{M} u_1(j-a+1)\,\widetilde{v}(i,j)$$

and this completes the proof. ■

Theorem 8.6 $\widetilde{v}$ *is increasing in* a *and bounded by* $(c+\alpha)(C+f)/\alpha$.

Proof. *It is sufficient to show that* $\widetilde{r}_s(i,a) + \widetilde{\Gamma}_s\widetilde{v}(i,a)$ *is increasing in* a *and bounded by* $(c+\alpha)(C+f)/\alpha$ *from above for each value of* s *assuming that* $\widetilde{v}(i,a)$ *is increasing in* a *and bounded by* $(c+\alpha)(C+f)/\alpha$ *from above. It is trivial that*

$$\widetilde{r}_s(i,a) \leq C + f$$

and

$$\widetilde{\Gamma}_s\widetilde{v}(i,a) \leq \left(\frac{c}{c+\alpha}\right)\frac{(c+\alpha)(C+f)}{\alpha}$$

for each value of s. *These imply that*

$$\widetilde{r}_s(i,a) + \widetilde{\Gamma}_s\widetilde{v}(i,a) \leq (c+\alpha)(C+f)/\alpha.$$

Choose arbitrary $a \in F\backslash\{M\}$ *and first assume that* $s = 0$. *Since,* $\widetilde{r}_0(i,a+1) \geq \widetilde{r}_0(i,a)$, *it is sufficient to show that* $\widetilde{\Gamma}_0\widetilde{v}(i,a+1) \geq \widetilde{\Gamma}_0\widetilde{v}(i,a)$. *We have* $\widetilde{v}(j,a+1) \geq \widetilde{v}(j,a)$ *for every* a *by the main hypothesis and using Lemma 8.5, we have the desired result. Now assume that* $s = 1$. *It is trivial that* $\widetilde{\Gamma}_1\widetilde{v}(i,a+1) \geq \widetilde{\Gamma}_1\widetilde{v}(i,a)$ *by the main hypothesis and* $\widetilde{r}_1(i,a+1) \geq \widetilde{r}_1(i,a)$. *This implies that* $\widetilde{r}_1(i,a) + \widetilde{\Gamma}_1\widetilde{v}(i,a)$ *is increasing in* a *and this completes the proof.* ■

An immediate corollary of this theorem is the following.

Corollary 8.7 $v(i,a)$ *is increasing in* a *and bounded by* $(c+\alpha)(C+f)/\alpha$.

Proof. *This follows simply from Proposition 11.5.1. in Puterman [116].* ■

Corollary 8.8 *Suppose that s^* is the optimal replacement policy. Then, there is a_i^* for all $i \in E$ such that*

$$s^*(i,a) = \begin{cases} 1 & \text{if } a \geq a_i^* \\ 0 & \text{if } a < a_i^*. \end{cases}$$

Proof. *Choose arbitrary $i \in E$. It suffices to show that if $s^*(i,a) = 1$, then $s^*(i,b) = 1$ for all $b \geq a$ since $s^*(i,M) = 1$. Assume that $s^*(i,a) = 1$ for some $a < M$ and, for a contradiction, suppose that there exists $b > a$ such that $s^*(i,b) = 0$. Note that if we can not find such an a, there is nothing to prove. Since, $s^*(i,a) = 1$, $r_1(i,a) + \Gamma_1 v(i,a) \leq r_0(i,a) + \Gamma_0 v(i,a)$ and $v(i,a) = r_1(i,a) + \Gamma_1 v(i,a)$. Since $s^*(i,b) = 0$, $v(i,b) = r_0(i,b) + \Gamma_0 v(i,b) < r_1(i,b) + \Gamma_1 v(i,b) = r_1(i,a) + \Gamma_1 v(i,a) = v(i,a)$. But, this result is a contradiction by Corollary 8.7 and the proof is completed.* ■

Thus, we have proved that the optimal replacement policy is a phase dependent control-limit policy on the deterioration level of the system. The critical replacement levels depend on the phases of the mission. The optimal replacement policy must be as depicted in Figure 8.2 for a system performing a mission with two phases.

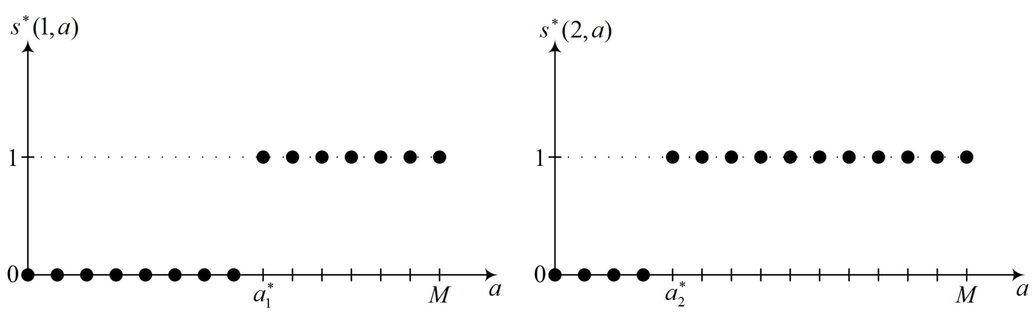

Figure 8.2: A typical optimal replacement policy for a system performing a mission with two phases.

8.7.2.1 Numerical Examples

We will show by some counterexamples that the main assumptions on our model are really necessary to guarantee the optimality of a policy with a control-limit structure. As mentioned earlier, we assume that $p_i \leq f_i$ for all $i \in E$, p_i does not depend on the deterioration level of the system and the deterioration process satisfies some monotonicity conditions. In the following examples, we show that if these assumptions do not hold, then we can find a case such that the optimal replacement policy is not control-limit. In this thesis, all dynamic programming equations are solved by transforming them into appropriate linear programming models.

In the following examples, it is assumed that $M = 6$ and the system performs a mission with three phases so that $E = \{1, 2, 3\}$ and $F = \{0, 1, \cdots, 6\}$. The transition probability matrix and the transition rates of the mission process are

$$Q = \begin{bmatrix} 0 & 0.3 & 0.7 \\ 0.2 & 0 & 0.8 \\ 0.5 & 0.5 & 0 \end{bmatrix} \tag{8.41}$$

and

$$\delta = \begin{bmatrix} 8 & 1 & 4 \end{bmatrix}. \tag{8.42}$$

The transition probability matrices of the deterioration process for each phase are

$$P_1 = \begin{bmatrix} 0 & 0.1 & 0.2 & 0.2 & 0.3 & 0.1 & 0.1 \\ 0 & 0 & 0.1 & 0.15 & 0.2 & 0.25 & 0.3 \\ 0 & 0 & 0 & 0.2 & 0.23 & 0.22 & 0.35 \\ 0 & 0 & 0 & 0 & 0.3 & 0.3 & 0.4 \\ 0 & 0 & 0 & 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{8.43}$$

$$P_2 = \begin{bmatrix} 0 & 0.1 & 0.2 & 0.2 & 0.2 & 0.1 & 0.2 \\ 0 & 0 & 0.05 & 0.13 & 0.22 & 0.25 & 0.35 \\ 0 & 0 & 0 & 0.17 & 0.22 & 0.23 & 0.38 \\ 0 & 0 & 0 & 0 & 0.24 & 0.32 & 0.44 \\ 0 & 0 & 0 & 0 & 0 & 0.4 & 0.6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{8.44}$$

and

$$P_3 = \begin{bmatrix} 0 & 0.1 & 0.1 & 0.1 & 0.15 & 0.2 & 0.35 \\ 0 & 0 & 0.05 & 0.08 & 0.22 & 0.25 & 0.4 \\ 0 & 0 & 0 & 0.05 & 0.24 & 0.26 & 0.45 \\ 0 & 0 & 0 & 0 & 0.18 & 0.32 & 0.5 \\ 0 & 0 & 0 & 0 & 0 & 0.45 & 0.55 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{8.45}$$

and the related transition rates are

$$\lambda_1 = \begin{bmatrix} 4 & 5 & 5.5 & 9 & 80,000 & 90,000 \end{bmatrix}$$

$$\lambda_2 = \begin{bmatrix} 2 & 3 & 5 & 6 & 620,000 & 650,000 \end{bmatrix}$$

$$\lambda_3 = \begin{bmatrix} 4 & 5 & 6 & 7 & 80,000 & 10,0000 \end{bmatrix}.$$

The maintenance and failure costs are $p_1 = 200$, $p_2 = 10$, $p_3 = 30$, $f_1 = 300$, $f_2 = 50$ and $f_3 = 80$. The discount rate is $\alpha = 0.8$ and all state occupancy costs are 0. Unless otherwise specified, these parameters will be used in the following examples. This is our base case and we will produce the counterexamples by changing some parameters in the base case. In all of the tabular representations through this and the following sections, if it is not clear from the context, we suppose that the rows correspond to the phases of the mission while the columns represent deterioration levels. Then, the optimal replacement policy and the optimal costs are

$$s^* = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$v^* = \begin{bmatrix} 211.790 & 251.023 & 265.112 & 311.449 & 411.790 & 411.790 & 511.790 \\ 122.279 & 132.279 & 132.279 & 132.279 & 132.279 & 132.279 & 172.279 \\ 177.180 & 193.980 & 201.191 & 207.180 & 207.180 & 207.180 & 257.180 \end{bmatrix}$$

in the base case. It is clear that this is a control-limit policy and the critical thresholds are $a_1^* = 4$, $a_2^* = 1$, $a_3^* = 3$.

Example 8.9 *In this example, we show that if $f_i < p_i$ for some $i \in E$, then the optimal replacement policy does not have to be control-limit and the value function does not have* *to be increasing in the deterioration level of the system. Suppose that* $p_1 = 200$, $p_2 = 10$, $p_3 = 1$, $f_1 = 210$, $f_2 = 11$ *and* $f_3 = 0.8$. *The discount rate is* $\alpha = 0.99$. *Using these parameters, the optimal replacement policy and optimal costs are*

$$r^* = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$v^* = \begin{bmatrix} 104.522 & 129.644 & 139.025 & 180.017 & 304.522 & 304.522 & 314.522 \\ 42.876 & 46.638 & 48.754 & 50.627 & 52.876 & 52.876 & 53.876 \\ 59.768 & 60.768 & 60.768 & 60.768 & 60.575 & 60.572 & 60.658 \end{bmatrix}.$$

It is obvious that the optimal cost function is not increasing and the optimal policy is not control-limit for phase 3.

Example 8.10 *In this example, we show that if* p_i *depends on the deterioration level of the system, then the optimal policy does not have to be control-limit. Suppose that*

$$Q = \begin{bmatrix} 0 & 0.4 & 0.6 \\ 0.3 & 0 & 0.7 \\ 0.5 & 0.5 & 0 \end{bmatrix},$$

$$\delta = \begin{bmatrix} 5 & 8 & 4 \end{bmatrix},$$

$$\lambda_1 = \begin{bmatrix} 1 & 2 & 3 & 3.1 & 3.2 & 3.3 \end{bmatrix},$$

$$\lambda_2 = \begin{bmatrix} 2 & 3 & 5 & 80 & 1000 & 1200 \end{bmatrix},$$

$$\lambda_3 = \begin{bmatrix} 1 & 1.5 & 2.5 & 4 & 4.5 & 5 \end{bmatrix},$$

$$p = \begin{bmatrix} 15 & 20 & 25 & 30 & 35 & 40 \\ 15 & 20 & 25 & 30 & 35 & 40 \\ 15 & 20 & 25 & 30 & 35 & 40 \end{bmatrix},$$

and

$$f = \begin{bmatrix} 76 & 45 & 41 \end{bmatrix},$$

and $\alpha = 0.75$. *Then, the optimal replacement policy and optimal costs are*

$$r^* = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$v^* = \begin{bmatrix} 51.709 & 67.839 & 76.709 & 81.709 & 86.487 & 91.709 & 127.709 \\ 54.330 & 68.141 & 76.872 & 84.330 & 89.330 & 94.330 & 99.330 \\ 51.340 & 64.182 & 72.452 & 78.269 & 81.443 & 85.516 & 92.340 \end{bmatrix}.$$

It is clear that even if the cost function is increasing, the optimal policy is not control-limit for phase 1.

Example 8.11 *In this example, it is shown that if deterioration process of the system for a given phase does not satisfy (8.30) and (8.31), then the optimal policy does not have to be control-limit. Suppose that* $p_1 = 20,\ p_2 = 10,\ p_3 = 30,\ f_1 = 300,\ f_2 = 150,\ f_3 = 180,$

$$\lambda_1 = \begin{bmatrix} 4 & 5 & 150 & 180 & 3 & 5 \end{bmatrix},$$

$$\lambda_2 = \begin{bmatrix} 2 & 3 & 200 & 220 & 3 & 5 \end{bmatrix},$$

$$\lambda_3 = \begin{bmatrix} 4 & 5 & 300 & 350 & 3 & 5 \end{bmatrix},$$

and

$$P_1 = P_2 = P_3 = \begin{bmatrix} 0 & 0.1 & 0.1 & 0.1 & 0.15 & 0.2 & 0.35 \\ 0 & 0 & 0.4 & 0.08 & 0.1 & 0.1 & 0.32 \\ 0 & 0 & 0 & 0.55 & 0.13 & 0.12 & 0.2 \\ 0 & 0 & 0 & 0 & 0.7 & 0.2 & 0.1 \\ 0 & 0 & 0 & 0 & 0 & 0.8 & 0.2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then, the optimal policy and optimal costs are

$$r^* = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}$$

and

$$v^* = \begin{bmatrix} 270.753 & 283.442 & 290.753 & 281.886 & 241.782 & 290.753 & 570.753 \\ 208.036 & 218.036 & 218.036 & 218.036 & 203.278 & 218.036 & 358.036 \\ 258.427 & 270.443 & 288.427 & 266.345 & 236.590 & 288.427 & 438.427 \end{bmatrix}.$$

It is clear that the optimal policy is not control-limit and the optimal costs are not increasing in the deterioration level of the system for each phase.

8.8 Optimal Repair Problem

In the previous section, we analyze the optimal replacement problem in which a decision maker observes the system at the beginning of each decision epoch and then makes a decision of replacing the system or not. However, repairing the system to a better state is also possible in real life applications in addition to replacement option. In this section, we assume that after any change in the deterioration process or in the mission process, a decision maker observes the system and then decides to repair the system to a better state immediately or to do nothing. The mission and deterioration processes have the same structure as used in Section 8.7.2 and all repair activities are instantaneous.

Let $C_i(a,b)$ be the cost of repairing the system from deterioration level a to deterioration level b during phase i. It is assumed that $C_i(a,b)$ is increasing in a for a fixed b, decreasing in b for a given a, and $C_i(a,a)=0$. It is also assumed that for any initial deterioration level $a<M$, we can repair the system to any deterioration level in the set $\{0,1,\cdots,a\}$ with the option of doing nothing. However, a failed system will be replaced. In other words, a failed system (with deterioration level M) can only be repaired to the deterioration level 0. We let $v(i,a)$ denote the expected total discounted cost given that the initial phase is $i\in E$ and the initial deterioration level of the system is $a\in F$.

We need to solve the DPE

$$v(i,a)=\min_{b\in A(a)}\{C_i(a,b)+c(i,b)+\Gamma v(i,b)\} \tag{8.46}$$

where

$$\Gamma v(i,a)=\frac{\delta_i+\lambda_i(a)}{\delta_i+\lambda_i(a)+\alpha}\left[\frac{\delta_i}{\delta_i+\lambda_i(a)}\sum_{j\in E}Q(i,j)\,v(j,a)+\frac{\lambda_i(a)}{\delta_i+\lambda_i(a)}\sum_{b=0}^{M}P_i(a,b)\,v(i,b)\right] \tag{8.47}$$

and

$$A(a)=\begin{cases}\{0,1,\cdots,a\} & \text{if } a<M\\ \{0\} & \text{if } a=M.\end{cases}$$

It is assumed that in the existence of a tie, the decision maker chooses the smaller deterioration level to which the system will be repaired. We also assume that

$$\sup_{i\in E,a\in F}\{\delta_i+\lambda_i(a)\}=c<+\infty \tag{8.48}$$

and

$$\sup_{i\in E}\{C_i(M,0)\} = C_r < +\infty.$$

Theorem 8.12 *There is a unique function in $\mathcal{B}$ that satisfies the DPE (8.46).*

Proof. *Define the operator Υ so that*

$$\Upsilon f(i,a) = \min_{b\in A(a)}\{C_i(a,b) + c(i,b) + \Gamma f(i,b)\}$$

for any $f\in\mathcal{B}$. We will use Banach's contraction mapping theorem. Choose two functions g and h from $\mathcal{B}$ and assume that $\|\cdot\|$ is the usual supremum norm. Consider

$$\Upsilon g(i,a) - \Upsilon h(i,a) = \min_{b\in A(a)}\{C_i(a,b) + c(i,b) + \Gamma g(i,b)\} - \min_{b\in A(a)}\{C_i(a,b) + c(i,b) + \Gamma h(i,b)\}. \tag{8.49}$$

Suppose that $\bar{b}\in A(a)$ minimizes the second term in the right hand side of (8.49). Then,

$$\begin{aligned}
\Upsilon g(i,a) - \Upsilon h(i,a) &\le C_i\left(a,\bar{b}\right) + c\left(i,\bar{b}\right) + \Gamma g\left(i,\bar{b}\right) - C_i\left(a,\bar{b}\right) - c\left(i,\bar{b}\right) - \Gamma h\left(i,\bar{b}\right) \\
&\le \frac{\delta_i + \lambda_i\left(\bar{b}\right)}{\delta_i + \lambda_i\left(\bar{b}\right) + \alpha}\left[\frac{\delta_i}{\delta_i + \lambda_i\left(\bar{b}\right)}\sum_{j\in E} Q(i,j)\left(g\left(j,\bar{b}\right) - h\left(j,\bar{b}\right)\right)\right. \\
&\quad \left. + \frac{\lambda_i\left(\bar{b}\right)}{\delta_i + \lambda_i\left(\bar{b}\right)}\sum_{b=0}^{M} P_i\left(\bar{b},b\right)\left(g(i,b) - h(i,b)\right)\right] \\
&\le \frac{\delta_i + \lambda_i\left(\bar{b}\right)}{\delta_i + \lambda_i\left(\bar{b}\right) + \alpha}\left[\frac{\delta_i}{\delta_i + \lambda_i\left(\bar{b}\right)}\sum_{j\in E} Q(i,j)\|g-h\|\right. \\
&\quad \left. + \frac{\lambda_i\left(\bar{b}\right)}{\delta_i + \lambda_i\left(\bar{b}\right)}\sum_{b=0}^{M} P_i\left(\bar{b},b\right)\|g-h\|\right] \\
&\le \sup_{i\in E, a\in F}\left\{\frac{\delta_i + \lambda_i(a)}{\delta_i + \lambda_i(a) + \alpha}\right\}\|g-h\|.
\end{aligned}$$

Similarly, it can be shown that

$$\Upsilon h(i,a) - \Upsilon g(i,a) \le \sup_{i\in E, a\in F}\left\{\frac{\delta_i + \lambda_i(a)}{\delta_i + \lambda_i(a) + \alpha}\right\}\|g-h\|.$$

Thus, we have

$$\|\Upsilon g - \Upsilon h\| \le \sup_{i\in E, a\in F}\left\{\frac{\delta_i + \lambda_i(a)}{\delta_i + \lambda_i(a) + \alpha}\right\}\|g-h\|.$$

Then, using (8.48),

$$\begin{aligned}\sup_{i\in E,a\in F}\left\{\frac{\delta_i+\lambda_i(a)}{\delta_i+\lambda_i(a)+\alpha}\right\} &= \sup_{i\in E,a\in F}\left\{1-\frac{\alpha}{\delta_i+\lambda_i(a)+\alpha}\right\}\\ &= 1-\frac{\alpha}{\sup_{i\in E,a\in F}\{\delta_i+\lambda_i(a)\}+\alpha}\\ &= 1-\frac{\alpha}{c+\alpha}=\frac{c}{c+\alpha}<1\end{aligned}$$

and this implies that Υ is a contraction mapping. Using Banach's contraction mapping theorem, Υ has a unique fixed point and this completes the proof. ■

Now, we can use uniformization technique by applying the procedure in Puterman [116]. The DPE (8.46) can be rewritten using the notation in Puterman [116] as

$$v(i,a)=\min_{s\in A(a)}\{r_s(i,a)+\Gamma_s v(i,a)\} \tag{8.50}$$

where $r_s(i,a)=C_i(a,s)+c(i,s)$,

$$\Gamma_s v(i,a)=\frac{\delta_i+\lambda_i(s)}{\delta_i+\lambda_i(s)+\alpha}\left[\frac{\delta_i}{\delta_i+\lambda_i(s)}\sum_{j\in E}Q(i,j)\,v(j,s)+\frac{\lambda_i(s)}{\delta_i+\lambda_i(s)}\sum_{b=s+1}^{M}P_i(s,b)\,v(i,b)\right].$$

The original model has $\lambda_s(i,a)=\delta_i+\lambda_i(s)$ and

$$P_s(i,a;j,b)=\begin{cases}\frac{\delta_i}{\delta_i+\lambda_i(s)}Q(i,j) & \text{if } b=s, j\neq i\\ \frac{\lambda_i(s)}{\delta_i+\lambda_i(s)}P_i(s,b) & \text{if } b\neq s, j=i.\end{cases}$$

In the above formulation, the subscripts after P and λ mean that the related formula is valid if the repair decision denoted by the subscript is applied. Moreover, we have

$$\lambda_s(i,a)\le c$$

for all $i\in E$, $a\in F$, and $s\in A(a)$ by (8.48), which is a necessary condition for the uniformization technique.

Then, define

$$\widetilde{r}_s(i,a)=r_s(i,a)\left(\frac{\alpha+\lambda_s(i,a)}{\alpha+c}\right)$$

and

$$\widetilde{P}_s(i,a;j,b)=\begin{cases}1-\frac{(1-P_s(i,a;j,b))\lambda_s(i,a)}{c} & \text{if } (j,b)=(i,a)\\ \frac{P_s(i,a;j,b)\lambda_s(i,a)}{c} & \text{if } (j,b)\neq(i,a).\end{cases}$$

Then, by Proposition 11.5.1 in Puterman [116], we have $v(i,a) = \widetilde{v}(i,a)$ for every stationary policy where $\widetilde{v}$ satisfies the DPE

$$\widetilde{v}(i,a) = \min_{s \in A(a)} \left\{ \widetilde{r}_s(i,a) + \widetilde{\Gamma}_s \widetilde{v}(i,a) \right\}$$

for all $i \in E$ and $a \in F$, and

$$\begin{aligned} \widetilde{\Gamma}_s \widetilde{v}(i,a) &= \frac{c}{c+\alpha} \left[\frac{\delta_i}{c} \sum_{j \in E} Q(i,j)\, \widetilde{v}(j,s) + \frac{\lambda_i(s)}{c} \sum_{b=s+1}^{M} P_i(s,b)\, \widetilde{v}(i,b) \right. \\ &\quad \left. + \left(1 - \frac{\delta_i + \lambda_i(s)}{c}\right) \widetilde{v}(i,a) \right]. \end{aligned} \tag{8.51}$$

By Theorem 8.12, we know that there is a stationary policy which solve the DPE (8.46). Since $\widetilde{v} = v$ for every stationary policy, they are equal for the optimal repair policy. Therefore, if we have a characterization for the transformed process, the same characterization will be valid for the original process. From now on, v and $\widetilde{v}$ will represent the expected total discounted cost under the optimal repair policy for the original process and the transformed process respectively. Our aim is to characterize the optimal policy. First we need some preliminary results.

Lemma 8.13 *If $\widetilde{v}(i,a)$ is increasing in a for all $i \in E$, then $\widetilde{\Gamma}_a \widetilde{v}(i,a)$ is also increasing in a for all $i \in E$.*

Proof. *Choose arbitrary $a \in \{0, 1, \cdots, M-2\}$. Then, we have*

$$\begin{aligned} \widetilde{\Gamma}_{a+1} \widetilde{v}(i,a+1) &= \frac{c}{c+\alpha} \left[\frac{\delta_i}{c} \sum_{j \in E} Q(i,j)\, \widetilde{v}(j,a+1) + \frac{\lambda_i(a+1)}{c} \sum_{b \geq a+2} P_i(a+1,b)\, \widetilde{v}(i,b) \right. \\ &\quad \left. + \left(1 - \frac{\delta_i + \lambda_i(a+1)}{c}\right) \widetilde{v}(i,a+1) \right] \end{aligned}$$

and

$$\begin{aligned} \widetilde{\Gamma}_a \widetilde{v}(i,a) &= \frac{c}{c+\alpha} \left[\frac{\delta_i}{c} \sum_{j \in E} Q(i,j)\, \widetilde{v}(j,a) + \frac{\lambda_i(a)}{c} \sum_{b \geq a+1} P_i(a,b)\, \widetilde{v}(i,b) \right. \\ &\quad \left. + \left(1 - \frac{\delta_i + \lambda_i(a)}{c}\right) \widetilde{v}(i,a) \right]. \end{aligned}$$

Since $\widetilde{v}(j,a+1) \geq \widetilde{v}(j,a)$ for every j and $\lambda_i(a+1) \geq \lambda_i(a)$ we have

$$\frac{\lambda_i(a)}{c} \sum_{b \geq a+1} P_i(a,b)\, \widetilde{v}(i,b) + \left(1 - \frac{\delta_i + \lambda_i(a)}{c}\right) \widetilde{v}(i,a)$$

$$\leq \frac{\lambda_i(a+1)}{c} \sum_{b \geq a+2} P_i(a+1,b)\,\widetilde{v}(i,b) + \left(1 - \frac{\delta_i + \lambda_i(a+1)}{c}\right)\widetilde{v}(i,a+1)$$

using Lemma 8.5 and this completes the proof. ■

Theorem 8.14 *$\widetilde{v}$ is increasing in a and bounded by $(c+\alpha)(C_r+C)/\alpha$.*

Proof. *Define the operator $\widetilde{\Upsilon}$ so that*

$$\widetilde{\Upsilon} f(i,a) = \min_{s \in A(a)} \left\{ \widetilde{r}_s(i,a) + \widetilde{\Gamma}_s \widetilde{v}(i,a) \right\}$$

for any $f \in \mathcal{B}$. It suffices to show that if $\widetilde{v}(i,a)$ is increasing in a and bounded by $(c+\alpha)(C_r+C)/\alpha$, then $\widetilde{\Upsilon}\widetilde{v}(i,a)$ is increasing in a and bounded by $(c+\alpha)(C_r+C)/\alpha$. It is clear that $\widetilde{r}_s(i,a) \leq C_r + C$ and

$$\Gamma_s \widetilde{v}(i,a) \leq \frac{c}{c+\alpha} \frac{(c+\alpha)(C_r+C)}{\alpha} = \frac{c(C_r+C)}{\alpha}.$$

These trivially imply that

$$\widetilde{r}_s(i,a) + \Gamma_s \widetilde{v}(i,a) \leq \frac{(c+\alpha)(C_r+C)}{\alpha}$$

and this proves that $\widetilde{\Upsilon}\widetilde{v}$ is bounded by $(c+\alpha)(C_r+C)/\alpha$. Since $\widetilde{r}_s(i,a)$ and $\widetilde{v}(i,a)$ are increasing in a, using (8.51), we have

$$\widetilde{r}_s(i,a+1) + \widetilde{\Gamma}_s \widetilde{v}(i,a+1) \geq \widetilde{r}_s(i,a) + \widetilde{\Gamma}_s \widetilde{v}(i,a) \tag{8.52}$$

for any $s \in \{0,1,\cdots,a\}$ if $a \in \{0,1,\cdots,M-2\}$ and for $s=0$ if $a=M-1$. Then, if

$$\widetilde{\Upsilon}\widetilde{v}(i,a+1) = \widetilde{r}_{s^*_{a+1}}(i,a+1) + \widetilde{\Gamma}_{s^*_{a+1}} \widetilde{v}(i,a+1)$$

for some $s^*_{a+1} \in \{0,1,\cdots,a\}$, using (8.52),

$$\widetilde{\Upsilon}\widetilde{v}(i,a+1) = \widetilde{r}_{s^*_{a+1}}(i,a+1) + \widetilde{\Gamma}_{s^*_{a+1}} \widetilde{v}(i,a+1) \geq \widetilde{r}_{s^*_{a+1}}(i,a) + \widetilde{\Gamma}_{s^*_{a+1}} \widetilde{v}(i,a) \geq \widetilde{\Upsilon}\widetilde{v}(i,a).$$

If

$$\widetilde{\Upsilon}\widetilde{v}(i,a+1) = \widetilde{r}_{a+1}(i,a+1) + \widetilde{\Gamma}_{a+1} \widetilde{v}(i,a+1)$$

where $a \in \{0,1,\cdots,M-2\}$ necessarily, then using Lemma 8.13,

$$\widetilde{\Upsilon}\widetilde{v}(i,a+1) = \widetilde{r}_{a+1}(i,a+1) + \widetilde{\Gamma}_{a+1} \widetilde{v}(i,a+1) \geq \widetilde{r}_a(i,a) + \widetilde{\Gamma}_a \widetilde{v}(i,a) \geq \widetilde{\Upsilon}\widetilde{v}(i,a)$$

where the first inequality follows from

$$\widetilde{r}_{a+1}\left(i,a+1\right)=\frac{c\left(i,a+1\right)\left(\alpha+\delta_i+\lambda_i\left(a+1\right)\right)}{\alpha+c}\geq\frac{c\left(i,a\right)\left(\alpha+\delta_i+\lambda_i\left(a\right)\right)}{\alpha+c}=\widetilde{r}_a\left(i,a\right)$$

since $c\left(i,a\right)$ *and* $\lambda_i\left(a\right)$ *are increasing in* a. *Since* $\widetilde{\Upsilon}\widetilde{v}\left(i,a+1\right)\geq\widetilde{\Upsilon}\widetilde{v}\left(i,a\right)$ *for any* $a\in F\backslash\{M\}$ *in all possible cases, the proof is completed.* ■

An immediate corollary of this theorem is the following.

Corollary 8.15 $v(i,a)$ *is increasing in* a *for all* $i\in E$ *and bounded by* $\left(c+\alpha\right)\left(C_r+C\right)/\alpha$.

From now on, we analyze the structure of the optimal repair policy. Let $r_i\left(a\right)$ be the optimal repair decision during phase i if the deterioration level of the system is a. We define the marginal repair cost $\triangledown C_i\left(a,b\right)$, for $b\leq a$, as

$$\triangledown C_i\left(a,b\right)=C_i\left(a,b-1\right)-C_i\left(a,b\right).$$

Then, we have

$$C_i\left(a,b\right)=\sum_{k=b+1}^{a}\triangledown C_i\left(a,k\right)\tag{8.53}$$

for all $b<a$.

We will characterize the optimal repair policy by making some additional assumptions on the repair costs.

Assumption 8.16 *For a given* $i\in E$, $C_i\left(a,b\right)\leq C_i\left(a,k\right)+C_i\left(k,b\right)$, *for all* b, k, $a\in F$ *such that* $b\leq k\leq a$.

Assumption 8.17 *For a given* $i\in E$, $\triangledown C_i\left(a,b\right)$ *is increasing in* a *on* $\{k\in F;k\geq b\}$, *for all fixed* $b\in F\backslash\{M\}$.

Assumption 8.18 *For a given* $i\in E$, $\triangledown C_i\left(a,b\right)$ *is increasing in* a *on* $\{k\in F;k>b\}$, *for all fixed* $b\in F\backslash\{M\}$.

Assumption 8.16 states that the cost of repairing the system from deterioration level a to deterioration level b is less than or equal to the cost of applying two successive repair actions which take the deterioration level of the system first from a to an intermediate deterioration level k and then from k to b. If there exists a fixed cost associated with each repair action, then this assumption is quite reasonable. Assumption 8.17 and Assumption 8.18 state that the marginal cost of repairing the system to a fixed state is increasing in the deterioration level of the system. This assumption is also quite reasonable, since in real life the cost of making the same amount of improvement in the state of a system generally increases as the deterioration level of the system increases. It is clear that Assumption 8.17 is stronger than Assumption 8.18, since its requires that the condition must hold when $k = b$. Thus, besides what Assumption 8.18 states, Assumption 8.17 additionally states that $\triangledown C_i(a,a) \leq \triangledown C_i(a+1,a)$ and hence

$$C_i(a+1,a) + C_i(a,a-1) \leq C_i(a+1,a-1).$$

This result clearly contradicts Assumption 8.16.

Proposition 8.19 *Assumption 8.16 and Assumption 8.17 cannot hold simultaneously unless* $\triangledown C_i(a,b)$ *is constant in* $a \in F$ *for all fixed* $b \in F$.

Proof. *Suppose that Assumption 8.16 holds. Then, we have*

$$C_i(a,b) \leq C_i(a,k) + C_i(k,b)$$

for $a > k > b$. *Using (8.53),*

$$\sum_{j=b+1}^{a} \triangledown C_i(a,j) \leq \sum_{j=k+1}^{a} \triangledown C_i(a,j) + \sum_{j=b+1}^{k} \triangledown C_i(k,j).$$

This implies that

$$\sum_{j=b+1}^{k} \triangledown C_i(a,j) \leq \sum_{j=b+1}^{k} \triangledown C_i(k,j)$$

and

$$\sum_{j=b+1}^{k} \left[\triangledown C_i(a,j) - \triangledown C_i(k,j)\right] \leq 0.$$

The last result clearly contradicts Assumption 8.17 unless $\triangledown C_i(a,b)$ *is constant in* $a \in F$.

■

Assumption 8.16 simply states that for the same amount of improvement, a direct repair is better than successive repairs. An immediate consequence of this assumption is the following theorem.

Theorem 8.20 *If Assumption 8.16 holds, then* $r_i(r_i(a)) = r_i(a)$ *for all* $a \in F$ *and* $i \in E$.

Proof. *If* $r_i(a) = a$*, then the result is trivial. Suppose that* $r_i(a) = b < a$ *and* $r_i(b) = c < b$ *for a contradiction. We have*

$$v(i,a) = C_i(a,b) + c(i,b) + \Gamma v(i,b) < C_i(a,c) + c(i,c) + \Gamma v(i,c) \tag{8.54}$$

and

$$v(i,b) = C_i(b,c) + c(i,c) + \Gamma v(i,c) \leq c(i,b) + \Gamma v(i,b). \tag{8.55}$$

Then, using (8.54),

$$C_i(a,b) - C_i(a,c) < c(i,c) + \Gamma v(i,c) - c(i,b) - \Gamma v(i,b)$$

and using (8.55),

$$\Gamma v(i,c) - \Gamma v(i,b) \leq c(i,b) - c(i,c) - C_i(b,c).$$

These imply that

$$C_i(a,b) + C_i(b,c) < C_i(a,c).$$

This result clearly contradicts Assumption 8.16. ■

Theorem 8.21 *If Assumption 8.17 holds, then the optimal repair policy* r_i *is increasing on* $F \backslash \{M\}$ *for all* $i \in E$.

Proof. *For a contradiction, suppose that* $r_i(a_1) = b$ *and* $r_i(a_2) = c < b$ *where* $a_2 > a_1$. *Then, we have*

$$v(i,a_1) = C_i(a_1,b) + c(i,b) + \Gamma v(i,b) < C_i(a_1,c) + c(i,c) + \Gamma v(i,c)$$

$$v(i,a_2) = C_i(a_2,c) + c(i,c) + \Gamma v(i,c) \leq C_i(a_2,b) + c(i,b) + \Gamma v(i,b)$$

and these imply that

$$C_i(a_2,c) - C_i(a_2,b) < C_i(a_1,c) - C_i(a_1,b).$$

Using (8.53),

$$\sum_{j=c+1}^{a_2} \nabla C_i(a_2,j) - \sum_{j=b+1}^{a_2} \nabla C_i(a_2,j) < \sum_{j=c+1}^{a_1} \nabla C_i(a_1,j) - \sum_{j=b+1}^{a_1} \nabla C_i(a_1,j)$$

and

$$\sum_{j=c+1}^{b} \triangledown C_i(a_2, j) < \sum_{j=c+1}^{b} \triangledown C_i(a_1, j)$$

and, hence,

$$\sum_{j=c+1}^{b} [\triangledown C_i(a_2, j) - \triangledown C_i(a_1, j)] < 0. \tag{8.56}$$

This is a contradiction since every term in (8.56) is nonnegative by Assumption 8.17. ■

Theorem 8.22 *If Assumption 8.18 holds, then the optimal repair policy r_i is increasing on*

$$\{b \in F \backslash \{M\}; r_i(b) < b\}$$

for all $i \in E$, i.e., if $r_i(a) = b < a$, then $r_i(c) \geq b$ for all $c > a$.

Proof. *For a contradiction, suppose that $r_i(a_1) = b < a_1$ and $r_i(a_2) = c < b$ where $a_2 > a_1$. Then, we have*

$$v(i, a_1) = C_i(a_1, b) + \Gamma v(i, b) < C_i(a_1, c) + \Gamma v(i, c)$$

$$v(i, a_2) = C_i(a_2, c) + \Gamma v(i, c) \leq C_i(a_2, b) + \Gamma v(i, b)$$

and these imply that

$$C_i(a_2, c) - C_i(a_2, b) < C_i(a_1, c) - C_i(a_1, b).$$

Using (8.53),

$$\sum_{j=c+1}^{a_2} \triangledown C_i(a_2, j) - \sum_{j=b+1}^{a_2} \triangledown C_i(a_2, j) < \sum_{j=c+1}^{a_1} \triangledown C_i(a_1, j) - \sum_{j=b+1}^{a_1} \triangledown C_i(a_1, j)$$

and

$$\sum_{j=c+1}^{b} \triangledown C_i(a_2, j) < \sum_{j=c+1}^{b} \triangledown C_i(a_1, j)$$

and, hence,

$$\sum_{j=c+1}^{b} [\triangledown C_i(a_2, j) - \triangledown C_i(a_1, j)] < 0. \tag{8.57}$$

This is a contradiction since every term in (8.57) is nonnegative by Assumption 8.18. ■

The main difference between Theorem 8.21 and Theorem 8.22 is the following. Theorem 8.21 holds when $r_i(a_1) = a_1$, but Theorem 8.22 may not hold in this case. If $r_i(a_1) = a_1$, following the same steps as in the proof of Theorem 8.22, we can achieve the result

$$\sum_{j=c+1}^{b} \left[\triangledown C_i(a_2, j) - \triangledown C_i(a_1, j)\right] < 0. \tag{8.58}$$

Since $a_1 = b$, $\triangledown C_i(a_2, b) - \triangledown C_i(a_1, b)$ may be negative according to Assumption 8.18 and (8.58) may hold. The following example shows that if neither Assumption 8.17 nor Assumption 8.18 holds, then r_i does not have to be increasing in the deterioration level of the system.

Example 8.23 *Consider the base problem in Section 8.7.2.1. Suppose that the transition rates of the deterioration process are*

$$\lambda_1 = \begin{bmatrix} 4 & 5 & 5.5 & 9 & 9200 & 9500 \end{bmatrix},$$
$$\lambda_2 = \begin{bmatrix} 2 & 3 & 5 & 6 & 7000 & 8000 \end{bmatrix},$$
$$\lambda_3 = \begin{bmatrix} 4 & 5 & 6 & 7 & 8000 & 9000 \end{bmatrix},$$

and the cost matrix is

$$C_i = \begin{bmatrix} 0 & - & - & - & - & - & - \\ 700 & 0 & - & - & - & - & - \\ 800 & 300 & 0 & - & - & - & - \\ 900 & 330 & 100 & 0 & - & - & - \\ 1000 & 500 & 120 & 100 & 0 & - & - \\ 2000 & 1140 & 1130 & 1100 & 1100 & 0 & - \\ 2500 & 1900 & 1800 & 1600 & 1400 & 1200 & 0 \end{bmatrix}$$

for all $i \in E$ and $\alpha = 0.80$. For these parameters, the optimal repair policy is

$$r = \begin{bmatrix} 0 & 1 & 2 & 2 & 2 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 2 & 1 & 0 \end{bmatrix}.$$

In this example, $\nabla C_i(1,1) = 700$, $\nabla C_i(2,1) = 500$, $\nabla C_i(3,1) = 570$, $\nabla C_i(4,1) = 500$ and, hence, both Assumption 8.17 and Assumption 8.18 do not hold. Moreover, it is clear that r_i is not increasing.

Theorem 8.24 *If Assumption 8.16 and Assumption 8.17 hold,* $r_i(a) < a$ *implies that* $r_i(a) = r_i(a-1)$ *for all* $a \in F \backslash \{M\}$.

Proof. *If* $r_i(a-1) = a-1$, *then* $r_i(a) = a-1 = r_i(a-1)$ *trivially using Theorem 8.21. Now, suppose that* $r_i(a-1) = k < a-1$. *Choose arbitrary* b *such that* $k+1 \leq b \leq a-1$. *If* $r_i(a) = b$, *then* $r_i(b) = b > k$ *using Theorem 8.20. However, this contradicts Theorem 8.21 since* $b \leq a-1$ *and* $r_i(b) > r_i(a-1)$. *Thus, we have* $r_i(a) \notin \{k+1, k+2, \cdots, a-1\}$. *Since* $k = r_i(a-1) \leq r_i(a) < a$ *by Theorem 8.21,* $r_i(a) = k = r_i(a-1)$ *and this completes the proof.* ■

An immediate corollary of this theorem is the following.

Corollary 8.25 *If Assumption 8.16 and Assumption 8.17 hold, then* $r_i(a+1) \in \{r_i(a), a+1\}$ *for* $a = 0, 1, \cdots, M-2$.

Proof. Suppose that $r_i(a+1) < a+1$. If $r_i(a) = a$, then $r_i(a+1) = r_i(a) = a$ since $r_i(a+1) \geq r_i(a)$ by Theorem 8.21. If $r_i(a) < a$, then $r_i(a+1) = r_i(a)$ by Theorem 8.24. ■

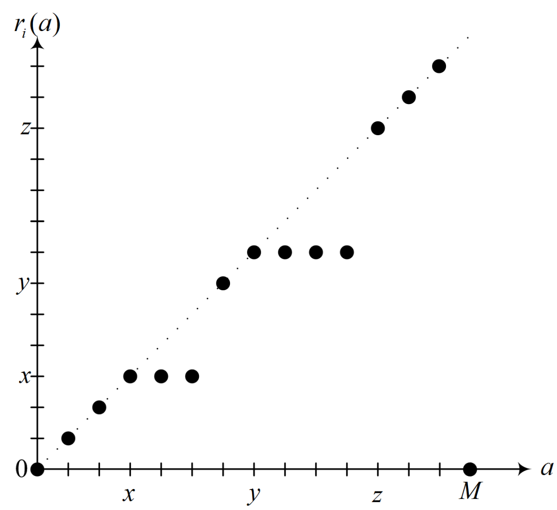

Figure 8.3: A typical optimal repair policy under a cost structure for which Assumption 8.16 and Assumption 8.17 hold.

Theorem 8.24 and Corollary 8.25 imply that the optimal repair policy for a given phase i must be as depicted by Figure 8.3 under a cost structure for which Assumption 8.16 and

Assumption 8.17 hold. Observe that the optimal policy has the form of an increasing step function. For levels $x, x+1$, and $x+2$, the optimal policy is to repair to level x, for levels $y, y+1, y+2$, and $y+3$, the optimal policy is to repair to level y. Also, note that the optimal policy is do nothing in $z, z+1$, and $z+2$.

At first glance, intuition may say that if $C_i(a,b)$ is increasing in a and decreasing in b, and if Assumption 8.16 and Assumption 8.17 hold, then $r_i(a) < a$ implies that $r_i(a+1) < a+1$. However, the following example shows that this is not always true.

Example 8.26 *Consider Example 8.23. Suppose that the transition rates of the deterioration process are*

$$\lambda_1 = \begin{bmatrix} 4 & 5 & 5.5 & 9 & 9.2 & 9.5 \end{bmatrix},$$
$$\lambda_2 = \begin{bmatrix} 2 & 3 & 5 & 6 & 7 & 8 \end{bmatrix},$$
$$\lambda_3 = \begin{bmatrix} 4 & 5 & 6 & 7 & 8 & 9 \end{bmatrix},$$

and the cost matrix is

$$C_i = \begin{bmatrix} 0 & - & - & - & - & - & - \\ 10 & 0 & - & - & - & - & - \\ 30 & 20 & 0 & - & - & - & - \\ 50 & 40 & 20 & 0 & - & - & - \\ 1000 & 990 & 970 & 950 & 0 & - & - \\ 2000 & 1990 & 1970 & 1950 & 1000 & 0 & - \\ 2500 & 2490 & 2470 & 2450 & 1500 & 500 & 0 \end{bmatrix}$$

for all $i \in E$ and $\alpha = 0.80$. For these parameters, the optimal repair policy is

$$r = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 & 0 \end{bmatrix}.$$

In this example, it is clear that $C_i(a,b)$ is increasing in a and decreasing in b, and Assumption 8.16 and Assumption 8.17 hold. However,we have $r_1(4) = 0 < 4$ and $r_1(5) = 5$.

8.8.1 Some Interesting Repair Cost Models

In this section, some different repair cost matrices will be analyzed and optimal repair policy will be characterized for each cost model.

8.8.1.1 Linear Repair Cost Model 1

Suppose that

$$C_i(a,b) = \begin{cases} K_i(b) + s_i(b)(a-b) & \text{if } b < a \\ 0 & \text{otherwise} \end{cases} \tag{8.59}$$

where $s_i(b) \geq 0$ is the marginal cost and $K_i(b) \geq 0$ is the fixed cost of repairing the system to deterioration level b during phase i.

Lemma 8.27 *If s_i is increasing on $F \backslash \{M\}$ for all $i \in E$, then Assumption 8.16 holds.*

Proof. *Choose arbitrary $b < c < a$. Then,*

$$\begin{aligned} C_i(a,c) + C_i(c,b) - C_i(a,b) &= K_i(c) + s_i(c)(a-c) + K_i(b) \\ &\quad + s_i(b)(c-b) - K_i(b) - s_i(b)(a-b) \\ &= K_i(c) + s_i(c)(a-c) + s_i(b)(c-b-a+b) \\ &= K_i(c) + (a-c)(s_i(c) - s_i(b)) \\ &\geq 0. \end{aligned}$$

■

Lemma 8.28 *If s_i is decreasing on $F \backslash \{M\}$ for all $i \in E$, then Assumption 8.18 holds.*

Proof. It suffices to show that $\triangledown C_i(a,b) \leq \triangledown C_i(a+1,b)$ for arbitrary $a > b$. Then,

$$\begin{aligned} \triangledown C_i(a+1,b) - \triangledown C_i(a,b) &= C_i(a+1,b-1) - C_i(a+1,b) - C_i(a,b-1) + C_i(a,b) \\ &= s_i(b-1)(a+1-b+1-a+b-1) + s_i(b)(a-b-a-1+b) \\ &= s_i(b-1) - s_i(b) \geq 0 \end{aligned}$$

since s_i is decreasing. ■

Assumption 8.17 may not hold for this cost structure since

$$\begin{aligned} \triangledown C_i(a+1,a) - \triangledown C_i(a,a) &= K_i(a-1) + 2s_i(a-1) - K_i(a) \\ &\quad - s_i(a) - K_i(a-1) - s_i(a-1) \\ &= s_i(a-1) - s_i(a) - K_i(a) \end{aligned} \tag{8.60}$$

and we do not have any information about the sign of the last term.

Lemma 8.29 *If*

$$K_i(a) \leq s_i(a-1) - s_i(a)$$

for all $a \in F \backslash \{M\}$, then Assumption 8.17 holds.

Proof. Since $K_i(a) \geq 0$, Assumption 8.18 holds. It suffices to show that $\bigtriangledown C_i(a+1,a) - \bigtriangledown C_i(a,a) \geq 0$. Using (8.60) and the main hypothesis,

$$\bigtriangledown C_i(a+1,a) - \bigtriangledown C_i(a,a) = s_i(a-1) - s_i(a) - K_i(a) \geq 0.$$

■

Corollary 8.30 *If s_i is constant and $K_i = 0$, then both Assumption 8.16 and Assumption 8.17 hold.*

The previous results characterize the optimal repair policy through Theorem 8.20, Theorem 8.21, Theorem 8.22, Theorem 8.24, and Corollary 8.25.

Note that all of the results in this section hold if

$$C_i(a,b) = \begin{cases} K_i(a) + s_i(b)(a-b) & \text{if } b < a \\ 0 & \text{otherwise.} \end{cases} \tag{8.61}$$

8.8.1.2 Linear Repair Cost Model 2

Suppose that

$$C_i(a,b) = \begin{cases} K_i(a) + s_i(a)(a-b) & \text{if } b < a \\ 0 & \text{otherwise} \end{cases} \tag{8.62}$$

where $s_i(a) \geq 0$ is the marginal cost and $K_i(a) \geq 0$ is the fixed cost of repairing the system with deterioration level b during phase i.

Lemma 8.31 *If s_i is decreasing on F for all $i \in E$, then Assumption 8.16 holds.*

Proof. *Choose arbitrary $b < c < a$. Then,*

$$\begin{aligned} C_i(a,c) + C_i(c,b) - C_i(a,b) &= K_i(a) + s_i(a)(a-c) + K_i(c) \\ &\quad + s_i(c)(c-b) - K_i(a) - s_i(a)(a-b) \\ &= K_i(c) + s_i(c)(c-b) + s_i(a)(a-c-a+b) \\ &= K_i(c) + (c-b)(s_i(c) - s_i(a)) \\ &\geq 0. \end{aligned}$$

■

Lemma 8.32 *If s_i is increasing on F for all $i \in E$, then Assumption 8.18 holds.*

Proof. It suffices to show that $\triangledown C_i(a,b) \leq \triangledown C_i(a+1,b)$ for arbitrary $a > b$. Then,

$$\begin{aligned}\triangledown C_i(a+1,b) - \triangledown C_i(a,b) &= C_i(a+1,b-1) - C_i(a+1,b) - C_i(a,b-1) + C_i(a,b) \\ &= s_i(a+1)(a+1-b+1-a+b-1) \\ &\quad + s_i(a)(a-b-a-1+b) \\ &= s_i(a+1) - s_i(a) \geq 0\end{aligned}$$

since s_i is increasing. ■

Assumption 8.17 may not hold for this cost structure since

$$\begin{aligned}\triangledown C_i(a+1,a) - \triangledown C_i(a,a) &= K_i(a+1) + 2s_i(a+1) - K_i(a+1) \\ &\quad - s_i(a+1) - K_i(a) - s_i(a) \\ &= s_i(a+1) - s_i(a) - K_i(a) \end{aligned} \tag{8.63}$$

and we do not have any information about the sign of the last term.

Lemma 8.33 *If*

$$K_i(a) \leq s_i(a+1) - s_i(a)$$

for all $a \in F \backslash \{M\}$, then Assumption 8.17 holds.

Proof. Since $K_i(a) \geq 0$, Assumption 8.18 holds. It suffices to show that $\triangledown C_i(a+1,a) - \triangledown C_i(a,a) \geq 0$. Using (8.63) and the main hypothesis,

$$\triangledown C_i(a+1,a) - \triangledown C_i(a,a) = s_i(a+1) - s_i(a) - K_i(a) \geq 0.$$

■

Corollary 8.34 *If s_i is constant and $K_i = 0$, then both Assumption 8.16 and Assumption 8.17 hold.*

The previous results characterize the optimal repair policy through Theorem 8.20, Theorem 8.21, Theorem 8.22, Theorem 8.24, and Corollary 8.25.

Note that all of the results in this section hold if

$$C_i(a,b) = \begin{cases} K_i(b) + s_i(a)(a-b) & \text{if } b < a \\ 0 & \text{otherwise.} \end{cases} \tag{8.64}$$

8.8.1.3 Sell-Purchase Model 1

In this model, if the decision maker gives a decision to repair, then the old device is sold and a better devise is purchased. Let $s_i(a)$ and $c_i(a)$ be the salvage value and the purchase cost of a system with deterioration level a during phase i respectively. It is assumed that s_i and c_i are decreasing functions such that $c_i \geq s_i$ for all $i \in E$. Then,

$$C_i(a,b) = \begin{cases} c_i(b) - s_i(a) & \text{if } b < a \\ 0 & \text{if } b = a. \end{cases} \tag{8.65}$$

Proposition 8.35 *If s_i and c_i are decreasing functions such that $c_i \geq s_i$, then Assumption 8.16 and Assumption 8.18 hold.*

Proof. *Choose arbitrary a, b, k such that $b \leq k \leq a$. Then,*

$$\begin{aligned} C_i(a,k) + C_i(k,b) - C_i(a,b) &= c_i(k) - s_i(a) + c_i(b) - s_i(k) - c_i(b) + s_i(a) \\ &= c_i(k) - s_i(k) \\ &\geq 0 \end{aligned}$$

and this implies Assumption 8.16. Choose arbitrary a, b such that $a > b$. Then,

$$\begin{aligned} \triangledown C_i(a+1,b) - \triangledown C_i(a,b) &= C_i(a+1,b-1) - C_i(a+1,b) - C_i(a,b-1) + C_i(a,b) \\ &= c_i(b-1) - s_i(a+1) - c_i(b) + s_i(a+1) \\ &\quad -c_i(b-1) + s_i(a) + c_i(b) - s_i(a) \\ &= 0 \end{aligned}$$

and this implies Assumption 8.18. ■

It is easy to see that Assumption 8.17 does not have to hold for this cost structure since

$$\begin{aligned} \triangledown C_i(a+1,a) - \triangledown C_i(a,a) &= C_i(a+1,a-1) - C_i(a+1,a) - C_i(a,a-1) \\ &= c_i(a-1) - s_i(a+1) - c_i(a) + s_i(a+1) - c_i(a-1) + s_i(a) \\ &= s_i(a) - c_i(a) \\ &\leq 0. \end{aligned}$$

Theorem 8.36 *If $r_i(a) < a$ and $r_i(a+1) < a+1$, then $r_i(a+1) = r_i(a)$ for all $a \in F \backslash \{M\}$.*

Proof. *Since $r_i(a) < a$, we have*

$$\begin{aligned} v(i,a) &= \min\left\{\min_{b\le a-1}\{c_i(b)+c(i,b)+\Gamma v(i,b)\}-s_i(a), c(i,a)+\Gamma v(i,a)\right\} \\ &= \min_{b\le a-1}\{c_i(b)+c(i,b)+\Gamma v(i,b)\}-s_i(a). \end{aligned}$$

Suppose that $r_i(a) = k < a$ and, hence,

$$\min_{b\le a-1}\{c_i(b)+c(i,b)+\Gamma v(i,b)\} = c_i(k)+c(i,k)+\Gamma v(i,k).$$

Since $r_i(a+1) < a+1$,

$$\begin{aligned} v(i,a+1) &= \min\left\{\min_{b\le a}\{c_i(b)+c(i,b)+\Gamma v(i,b)\}-s_i(a+1), c(i,a+1)+\Gamma v(i,a+1)\right\} \\ &= \min_{b\le a}\{c_i(b)+c(i,b)+\Gamma v(i,b)\}-s_i(a+1). \end{aligned}$$

If $r_i(a+1) \neq a$, then

$$\min_{b\le a-1}\{c_i(b)+c(i,b)+\Gamma v(i,b)\} \le c_i(a)+c(i,a)+\Gamma v(i,a)$$

and hence $r_i(a+1) = k$. Therefore, if $r_i(a+1) \neq k$, then $r_i(a+1) = a$. However, this leads to a contradiction since Assumption 8.16 holds for this cost structure using Proposition 8.35 and, hence, $r_i(a) = a$ by Theorem 8.20. ■

Theorem 8.36 implies that the optimal repair policy for a given phase i must be as depicted by Figure 8.4 under this special cost structure. Observe that the optimal policy does not have to be increasing. For levels $x, x+1$, and $x+2$, the optimal policy is to repair to level x. For levels $y, y+2, y+3$, and $y+4$, the optimal policy is to repair to level y while the optimal decision is do nothing in $y+1$. Also, note that the optimal policy is do nothing in $z, z+1$, and $z+2$.

Example 8.37 *Consider Example 8.23 with the following cost functions*

$$c = \begin{bmatrix} 10000 & 8000 & 6000 & 2000 & 760 & 750 & 500 \\ 10000 & 4100 & 4000 & 1600 & 1500 & 1250 & 1000 \\ 10000 & 3000 & 2500 & 2000 & 1000 & 750 & 500 \end{bmatrix},$$

$$s = \begin{bmatrix} 9000 & 3500 & 3000 & 2000 & 755 & 740 & 490 \\ 8000 & 4000 & 3900 & 1010 & 1000 & 750 & 600 \\ 7000 & 2500 & 2250 & 1500 & 750 & 650 & 400 \end{bmatrix}.$$

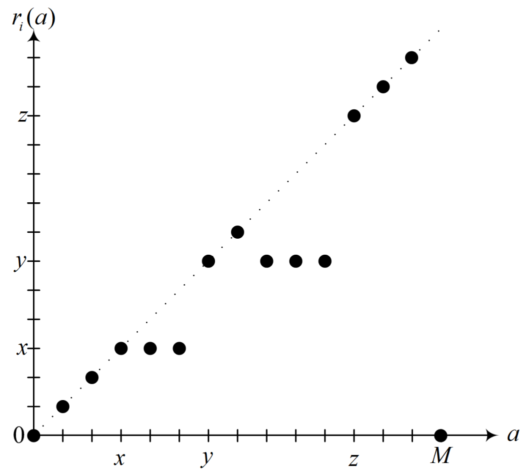

Figure 8.4: A typical optimal repair policy for the sell-purchase model 1.

Then, the optimal repair policy is

$$r = \begin{bmatrix} 0 & 1 & 2 & 3 & 3 & 3 & 0 \\ 0 & 1 & 1 & 3 & 3 & 3 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

This example shows that for a given phase, if the optimal decision is to repair in two different deterioration levels, then the system can be repaired to different deterioration levels optimally. It also shows that if $r_i(a) < a$, then $r_i(a+1)$ does not have to be strictly less than $a+1$.

Example 8.38 *Consider Example 8.23 with the following cost functions*

$$c = \begin{bmatrix} 10000 & 4000 & 3500 & 2500 & 1000 & 750 & 500 \\ 10000 & 5000 & 4000 & 3000 & 1500 & 1250 & 1000 \\ 10000 & 3000 & 2500 & 2000 & 1000 & 750 & 500 \end{bmatrix},$$

$$s = \begin{bmatrix} 9000 & 3500 & 3000 & 2000 & 755 & 500 & 250 \\ 8000 & 4000 & 3000 & 2000 & 1000 & 750 & 600 \\ 7000 & 2500 & 2000 & 1000 & 750 & 500 & 250 \end{bmatrix}.$$

Then, the optimal repair policy is

$$r = \begin{bmatrix} 0 & 1 & 2 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 3 & 1 & 1 & 0 \\ 0 & 1 & 1 & 3 & 1 & 1 & 0 \end{bmatrix}.$$

This example shows that if $r_i(a) < a$*, then* $r_i(a)$ *does not have to be equal to* $r_i(a-1)$*.*

8.8.1.4 Sell-Purchase Model 2

This model is a special case of the previous model. The only difference between them is that in this model, selling price and purchase price of the system are equal for the same deterioration levels. Then,

$$C_i(a,b) = c_i(b) - c_i(a)$$

where c_i is a decreasing nonnegative function on F which is selling and purchase price of the system. Özekici and Günlük [103] shows that Assumption 8.16, Assumption 8.17 and Assumption 8.18 all hold for this cost structure.

Theorem 8.39 *If* $r_i(a) < a$*, then* $r_i(a) = r_i(a-1)$ *for all* $a \in F\backslash\{M\}$*.*

Proof. *Although this is an immediate corollary of Theorem 8.24, we provide another proof. Suppose that* $r_i(a-1) = k$ *and* $r_i(a) < a$*. We need to show that* $r_i(a) = k$*. Since* $C_i(a,b) = c_i(b) - c_i(a)$*, we have*

$$\begin{aligned} v(i,a-1) &= \min_{b \le a-1} \{c_i(b) - c_i(a) + c(i,b) + \Gamma v(i,b)\} \\ &= \min_{b \le a-1} \{c_i(b) + c(i,b) + \Gamma v(i,b)\} - c_i(a). \end{aligned}$$

Then, since $r_i(a-1) = k$

$$\min_{b \le a-1} \{c_i(b) + c(i,b) + \Gamma v(i,b)\} = c_i(k) + c(i,k) + \Gamma v(i,k). \tag{8.66}$$

We know that $r_i(a) < a$ *and, hence,*

$$v(i,a) = \min_{b \le a-1} \{c_i(b) + c(i,b) + \Gamma v(i,b)\} - c_i(a).$$

Using (8.66), $r_i(a) = k$ *trivially.* ■

Since Assumption 8.16 and Assumption 8.17 hold for this cost structure, the optimal repair policy under this cost structure has the form described in Figure 8.3.

Example 8.40 *Consider Example 8.23 with the following cost function*

$$c = \begin{bmatrix} 10000 & 8000 & 6000 & 2000 & 760 & 750 & 500 \\ 10000 & 4100 & 4000 & 1600 & 1500 & 1250 & 1000 \\ 10000 & 3000 & 2500 & 2000 & 1000 & 750 & 500 \end{bmatrix}.$$

Then, the optimal repair policy is

$$r = \begin{bmatrix} 0 & 1 & 2 & 3 & 3 & 3 & 0 \\ 0 & 1 & 1 & 3 & 3 & 3 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

This example shows that if $r_i(a) < a$, then $r_i(a+1)$ does not have to be strictly less than $a+1$.

8.8.1.5 A Purchase Model

This cost model is similar to the previous one, but now the salvage value of the old system is zero. Therefore, the cost of a repair is equal to the purchase cost of the better system, i.e., $C_i(a,b) = c_i(b)$ if $b < a$ and $C_i(a,a) = 0$ for every phase $i \in E$. Özekici and Günlük [103] shows that Assumption 8.16, and Assumption 8.18 hold, but Assumption 8.17 does not hold for this cost structure.

Theorem 8.41 *If c is a nonnegative decreasing function on $F\backslash\{M\}$, then there exists $k_i \in F$ and $l_i \in F\backslash\{M\}$ such that $r_i(a) = a$ for all $a < k_i$ and $r_i(a) = l_i$ for all $a \in \{k_i, k_i+1, \cdots, M-1\}$ for all $i \in E$.*

Proof. *Choose arbitrary phase $i \in E$. Let k_i be the first deterioration level at which the decision maker decides to repair the system, i.e., $k_i = \inf\{a; r_i(a) < a\}$. Then, trivially if $a < k_i$, then $r_i(a) = a$. Suppose that $r_i(k_i) = l_i$. Then, we need to show that $r_i(a) = l_i$ for all $a > k_i$. Choose arbitrary $a > k_i$. We have*

$$\begin{aligned} v(i,k_i) &= \min\left\{\min_{b \le k_i - 1}\{c_i(b) + c(i,b) + \Gamma v(i,b)\}, c(i,k_i) + \Gamma v(i,k_i)\right\} \\ &= c_i(l_i) + c(i,l_i) + \Gamma v(i,l_i) \end{aligned}$$

and

$$\min_{b \le k_i - 1}\{c_i(b) + c(i,b) + \Gamma v(i,b)\} = c_i(l_i) + c(i,l_i) + \Gamma v(i,l_i). \tag{8.67}$$

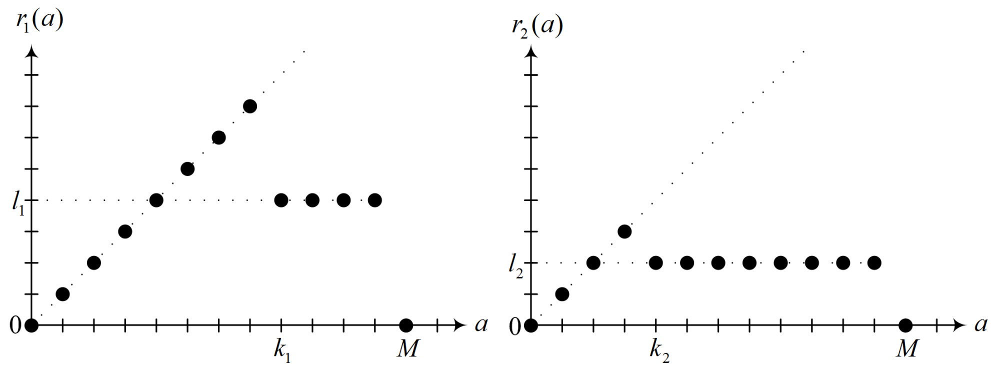

Figure 8.5: A typical optimal replacement policy for a system performing a mission with two phases.

Since, $v(i,a)$ is increasing in a

$$c_i(l_i) + c(i,l_i) + \Gamma v(i,l_i) = v(i,k_i) \le v(i,a) \le c(i,a) + \Gamma v(i,a). \tag{8.68}$$

Moreover,

$$\begin{aligned} v(i,a) &= \min\left\{\min_{b\le a-1}\{c_i(b)+c(i,b)+\Gamma v(i,b)\}, c(i,a)+\Gamma v(i,a)\right\} \\ &= \min\{c_i(l_i)+c(i,l_i)+\Gamma v(i,l_i), \\ &\qquad \left.\min_{k_i\le b\le a-1}\{c_i(b)+c(i,b)+\Gamma v(i,b)\}, c(i,a)+\Gamma v(i,a)\right\} && (8.69)\\ &= \min\{c_i(l_i)+c(i,l_i)+\Gamma v(i,l_i), && (8.70)\\ &\qquad \left.\min_{k_i\le b\le a-1}\{c_i(b)+c(i,b)+\Gamma v(i,b)\}\right\} && (8.71)\end{aligned}$$

where the last equality follows from (8.68). Now, choose arbitrary b such that $k_i \le b \le a-1$. Then, since $v(i,a)$ is increasing in a,

$$c_i(l_i)+c(i,l_i)+\Gamma v(i,l_i) = v(i,k_i) \le v(i,b) \le c(i,b)+\Gamma v(i,b) \le c_i(b)+c(i,b)+\Gamma v(i,b).$$

Thus, we have

$$v(i,a) = c_i(l_i)+c(i,l_i)+\Gamma v(i,l_i)$$

and, hence, $r_i(a) = l_i$. ■

Theorem 8.41 implies that the optimal repair policy for a system performing a mission with two phases must be as depicted by Figure 8.5 under this special cost structure. Observe that the optimal policy has a phase dependent control-limit structure specified by the pairs (k_1, l_1) and (k_2, l_2). The optimal policy is do nothing for levels $a < k_i$, and to repair to level l_i for levels $a \geq k_i$, for both $i = 1, 2$.

BIBLIOGRAPHY

[1] J. Esary, H. Ziehms, Reliability analysis of phased missions, in: Proceedings of the Conference on Reliability and Fault Tree Analysis, SIAM, 1975, pp. 213–236.

[2] G. Burdick, J. Fussell, D. Rasmuson, J. Wilson, Phased mission analysis: A review of new developments and an application, IEEE Transactions on Reliability 26 (1977) 43–49.

[3] M. Veatch, Reliability of periodic, coherent, binary systems, IEEE Transactions on Reliability 35 (1986) 504–507.

[4] M. Alam, U. Al-Saggaf, Quantitative reliability evaluation of repairable phased-mission systems using Markov approach, IEEE Transactions on Reliability 35 (1986) 498–503.

[5] K. Kim, K. Park, Phased-mission system reliability under Markov environment, IEEE Transactions on Reliability 43 (1994) 301–309.

[6] X. Zang, H. Sun, K. Trivedi, A bdd-based algorithm for reliability analysis of phased-mission systems, IEEE Transactions on Reliability 48 (1999) 50–60.

[7] J. Vaurio, Fault tree analysis of phased-mission systems with repairable and non-repairable components, Reliability Engineering and System Safety 74 (2001) 169–180.

[8] L. Xing, J. Dugan, Analysis of general phased-mission system reliability, performance and sensitivity, IEEE Transactions on Reliability 51 (2002) 199–211.

[9] L. Xing, J. Dugan, Generalized imperfect coverage phased-mission analysis, in: Proceedings of Annual Reliability and Maintainability Symposium, 2002, pp. 112–119.

[10] L. Xing, J. Dugan, A separable ternary decision diagram based analysis of generalized phased-mission reliability, IEEE Transactions on Reliability 53 (2004) 174–184.

[11] Z. Tang, J. Dugan, BDD-based reliability analysis of phased-mission systems with multimode failures, IEEE Transactions on Reliability 55 (2006) 350–360.

[12] Z. Tang, H. Xu, J. Dugan, BDD-based reliability analysis of phased-mission systems with multimode failures, in: Proceedings of Annual Reliability and Maintainability Symposium, 2005, pp. 313–318.

[13] I. Mura, A. Bondavalli, Hierarchical modelling and evaluation of phased-mission systems, IEEE Transactions on Reliability 48 (1999) 360–368.

[14] I. Mura, A. Bondavalli, Markov regenerative stochastic Petri nets to model and evaluate phased mission systems dependability, IEEE Transactions on Computers 50 (2001) 1337–1351.

[15] A. Bondavalli, R. Filippini, Modeling and analysis of a scheduled maintenance system: a DSPN approach, Computer Journal 47 (2004) 634–650.

[16] A. Bondavalli, S. Chiaradonna, F. D. Giandomenico, I. Mura, Dependability modeling and evaluation of multiple-phased systems using DEEM, IEEE Transactions on Reliability 53 (2004) 509–522.

[17] E. Çınlar, S. Özekici, Reliability of complex devices in random environments, Probability in the Engineering and Informational Sciences 1 (1987) 97–115.

[18] D. Gaver, Random hazard in reliability problems, Technometrics 5 (1963) 211–226.

[19] E. Arjas, The failure and hazard process in multivariate reliability systems, Mathematics of Operations Research 6 (1981) 551–562.

[20] E. Çınlar, M. Shaked, J. Shanthikumar, On lifetimes influenced by a common environment, Stochastic Processes and their Applications 33 (1989) 347–359.

[21] M. Shaked, J. G. Shanthikumar, Some replacement policies in a random environment, Probability in the Engineering and Informational Sciences 3 (1989) 117–134.

[22] D. Lindley, N. Singpurwalla, Multivariate distributions for the lifelengths of components of a system sharing a common environment, Journal of Applied Probability 23 (1986) 418–431.

[23] C. Lefèvre, M.-P. Malice, On a system of components with joint lifetimes distributed as a mixture of exponential laws, Journal of Applied Probability 26 (1989) 202–208.

[24] C. Lefèvre, X. Milhaud, On the association of the lifelenghts of components subjected to a stochastic environment, Advances in Applied Probability 22 (1990) 961–964.

[25] N. D. Singpurwalla, M. A. Youngren, Multivariate distributions induced by dynamic environments, Scandinavian Journal of Statistics 20 (1993) 251–261.

[26] N. Singpurwalla, The hazard potential: Introduction and overview, Journal of the American Statistical Association 101 (2006) 1705–1717.

[27] N. Prabhu, Y. Zhu, Markov-modulated queueing systems, Queueing Systems 5 (1989) 215–246.

[28] A. Erdem, S. Özekici, Inventory models with random yield in a random environment, International Journal of Production Economics 78 (2002) 239–253.

[29] S. Özekici, Complex systems in random environments, in: S. Özekici (Ed.), Reliability and Maintenance of Complex Systems, Springer-Verlag, Berlin, 1996, pp. 137–157, nATO ASI Series Volume F154.

[30] T. Rolski, H. Schmidli, V. Schmidt, J. Teugels, Stochastic Processes for Insurance and Finance, John Wiley & Sons, Chichester, 1999.

[31] U. Çelikyurt, S. Özekici, Multiperiod portfolio optimization models in stochastic markets using the mean-variance approach, European Journal of Operational Research 179 (2007) 186–202.

[32] K. Wang, W. Dong, J. Ke, Comparison of reliability and the availability between four systems with warm standby components and standby switching failures, Applied Mathematics and Computation 183 (2006) 1310–1322.

[33] Y. Zhang, G. Wang, A deteriorating cold standby repairable system with priority in use, Eurpean Journal of Operational Research 183 (2007) 278–295.

[34] J. P. Kharoufeh, D. Finkelstein, D. Mixon, Availability of periodically inspected systems with Markovian wear and shocks, Journal of Applied Probability 43 (2006) 303–317.

[35] M. Asadi, I. Bayramoglu, The mean residual life function of a k-out-of-n structure at the system level, IEEE Transactions on Reliability 55 (2006) 314–318.

[36] A. Kiureghian, O. Ditlevsen, J. Song, Availability, reliability and downtime of systems with repairable components, Reliability Engineering and System Safety 92 (2007) 231–242.

[37] M. M. Bevilacquaa, M. Bragliab, The analytic hierarchy process applied to maintenance strategy selection, Reliability Engineering and System Safety 70 (2000) 71–83.

[38] C. Chu, J.-M. Proth, P. Wolff, Predictive maintenance: The one-unit replacement model, Int. J. Production Economics 54 (1998) 285–295.

[39] G. Waeyenbergh, L. Pintelon, A framework for maintenance concept development, Int. J. Production Economics 77 (2002) 299–313.

[40] J. J. McCall, Maintenance policies for stochastically failing equipment: A survey, Management Science 11 (1965) 493–524.

[41] R. E. Barlow, F. Proschan, Mathematical Theory of Reliability, SIAM, 1965.

[42] R. E. Barlow, F. Proschan, Statistical Theory of Reliability and Life Testing: Probability Models, Holt, Rinehart and Winston, INC., 1975.

[43] W. Pierskalla, J. Voelker, A survey of maintenance models: the control and surveillance of deteriorating systems, Naval Research Logistics Quarterly 23 (1976) 353–388.

[44] S. Osaki, T. Nakagawa, Bibliography for reliability and availability of stochastic systems, IEEE Transactions on Reliability 25 (1976) 284–287.

[45] Y. Sherif, M. Smith, Optimal maintenance models for systems subject to failure: A review, Naval Research Logistics Quarterly 28 (1981) 47–74.

[46] A. Jardine, J. Buzacott, Equipment reliability and maintenance, European Journal of Operational Research 19 (1985) 285–296.

[47] C. Valdez-Flores, R. Feldman, A survey of preventive maintenance models for stochastically deteriorating single-unit systems, Naval Research Logistics 36 (1989) 419–446.

[48] D. I. Cho, M. Parlar, A survey of maintenance models for multi-unit systems, European Journal of Operational Research 51 (1991) 1–23.

[49] U. Jensen, Stochastic models of reliability and maintenance: an overview, in: S. Ozekici (Ed.), Reliability and maintenance of complex systems., Vol. 154 of NATO ASI, NATO Advanced Study Institute on Current Issues and Challenges in the Reliability and Maintenance of Complex Systems, Kemer-Antalya, Turkey, Springer, Berlin, 1995, pp. 3–36.

[50] R. Dekker, Applications of maintenance optimization models: A review and analysis, Reliability Engineering and System Safety 51 (1996) 229–240.

[51] H. Pham, H. Wang, Imperfect maintenance, European Journal of Operational Research 94 (1996) 425–438.

[52] F. Van der Duyn Schouten, Maintenance policies for multicomponent systems: An overview, in: S. Özekici (Ed.), Reliability and Maintenance of Complex Systems, Springer-Verlag, Berlin, 1996, pp. 117–136, NATO ASI Series Volume F154.

[53] R. Dekker, R. E. Wildeman, F. A. Van der Duyn Schouten, A review of multi-component maintenance models with economic dependence, Mathematical Methods of Operations Research 45 (1997) 411–435.

[54] H. Wang, A survey of maintenance policies of deteriorating systems, European Journal of Operational Research 139 (2002) 469–489.

[55] R. P. Nicolai, R. Dekker, Optimal maintenance of multi-component systems: A review, Complex System Maintenance Handbook: Blending Theory with Practice (forthcoming), Springer Verlag, Berlin.

[56] E. Çınlar, Markov renewal theory, Advances in Applied Probability 1 (1969) 123–187.

[57] I. Mura, A. Bondavalli, X. Zang, K. Trivedi, Dependability modeling and evaluation of phased mission systems: A DSPN approach, in: Proc. IFIP Int'l Conf. Dependable Computing for Critical Applications (DCCA-7), 1999, pp. 299–318.

[58] R. F. Drenick, The failure law of complex equipment, Journal of the Society for Industrial and Applied Mathematics 8 (1960) 680–690.

[59] E. Çınlar, Introduction to Stochastic Processes, Prentice-Hall, Englewood Cliffs, NJ, 1975.

[60] C. Moler, C. Loan, Nineteen dubious ways to compute the exponential of a matrix, twenty-five years later, SIAM Review 45 (1) (2003) 3–49.

[61] D. Revuz, Markov Chains, Elsevier Science Publihers, North Holland, Amsterdam, 1984.

[62] G. Alsmeyer, The Markov renewal theorem and related results, Markov Processes and Related Fields 3 (1997) 103–127.

[63] S. Meyn, R. Tweedie, Markov Chains and Stochastic Stability. Control and Communication in Engineering, Springer-Verlag, Berlin, 1992.

[64] S. Meyn, R. Tweedie, Stability of Markovian processes I: Criteria for discrete-time chains, Advances in Applied Probability 24 (1992) 542–574.

[65] I. Altınel, O. Feyzioğlu, S. Özekici, Component testing of a series system in a random mission, Reliability Engineering and System Safety 78 (2002) 33–43.

[66] I. Altınel, O. Feyzioğlu, S. Özekici, Component testing of a repairable system in multistage missions, Journal of the Operational Research Society 52 (2001) 937–944.

[67] R. Barlow, F. Proschan, Theory of maintained systems: Distribution of time to first system failure, Mathematics of Operations Research 1 (1976) 32–42.

[68] M. Brown, On the reliability of repairable systems, Operations Research 32 (3) (1984) 607–615.

[69] L. Baxter, Availability measures for coherent systems of separately maintained components, Journal of Applied Probability 20 (3) (1983) 627–636.

[70] J. Angus, On computing MTBF for a k-out-of-n: G repairable system, IEEE Transactions on Reliability 37 (3) (1988) 312–313.

[71] S. Iyer, Mean first-passage times for a k-out-of-n system with repair, The Journal of the Operational Research Society 40 (10) (1989) 889–894.

[72] X. Li, M. Zuo, R. Yam, Reliability analysis of a repairable k-out-of-n system with some components being suspended when the system is down, Reliability Engineering and System Safety 91 (3) (2006) 305–310.

[73] W. Wang, J. Loman, Reliability/availability of K-out-of-N system with M cold standby units, in: Proceedings of Reliability and Maintainability Symposium, 2002, pp. 450–455.

[74] E. Frostig, B. Levikson, On the availability of r out of n repairable systems, Naval Research Logistics 49 (5) (2002) 483–498.

[75] M. Kouckỳ, Exact reliability formula and bounds for general k-out-of-n systems, Reliability Engineering and System Safety 82 (2) (2003) 229–231.

[76] K. de Smidt-Destombes, M. C van der Heijden, A. van Harten, On the availability of a k-out-of-n system given limited spares and repair capacity under a condition based maintenance strategy, Reliability Engineering and System Safety 83 (3) (2004) 287–300.

[77] R. Natarajan, A reliability problem with spares and multiple repair facilities, Operations Research 16 (5) (1968) 1041–1057.

[78] J. Sarkar, F. Li, Limiting average availability of a system supported by several spares and several repair facilities, Statistics and Probability Letters 76 (18) (2006) 1965–1974.

[79] V. Sridharan, P. Mohanavadivu, Stochastic Behaviour of a Two-Unit Standby System with Two Types of Repairmen and Patience Time, Mathematical and Computer Modelling 28 (9) (1998) 63–71.

[80] E. Papageorgiou, G. Kokolakis, A two-unit parallel system supported by (n-2) standbys with general and non-identical lifetimes, International Journal of Systems Science 35 (2004) 1–12.

[81] D. Robinson, M. Neuts, An algorithmic approach to increased reliability through standby redundancy, Reliability, IEEE Transactions on 38 (4) (1989) 430–435.

[82] V. Prasad, W. Kuo, K. Kim, Optimal allocation of s-identical, multi-functional spares in a series system, IEEE Transactions on Reliability 48 (2) (1999) 118–126.

[83] K. Wang, Y. Liu, W. Pearn, Cost benefit analysis of series systems with warm standby components and general repair time, Mathematical Methods of Operations Research (ZOR) 61 (2) (2005) 329–343.

[84] R. Horst, N. Thoai, DC programming: Overview, Journal of Optimization Theory and Applications 103 (1) (1999) 1–43.

[85] S. Özekici, Optimal maintenance policies in random environments, European Journal of Operational Research 82 (1995) 283–294.

[86] M. Neuts, Matrix-Geometric Solutions in Stochastic Models, An Algorithic Approach, Dover Publications, Inc., New York, 1994.

[87] S. H. Sim, J. Endrenyi, Optimal preventive maintenance with repair, IEEE Transactions on Reliability 37 (1988) 92–96.

[88] L. Wang, J. Chu, W. Mao, A condition-based replacement and spare provisioning policy for deteriorating systems with uncertain deterioration to failure, European Journal of Operational Research 194 (2009) 184–205.

[89] K. S. Park, Optimal continuous-wear limit replacement under periodic inspections, IEEE Transactions on Reliability 37 (1988) 97–102.

[90] K. S. Park, Optimal wear-limit replacement with wear-dependent failures, IEEE Transactions on Reliability 37 (1988) 293–294.

[91] C. Pellegrin, Choice of a periodic on-condition maintenance policy, International Journal of Production Research 30 (1992) 1153–1173.

[92] F. Barbera, H. Schneider, P. Kelle, A condition based maintenance model with exponential failures and fixed inspection intervals, The Journal of the Operational Research Society 47 (1996) 1037–1045.

[93] F. Barbera, H. Schneider, E. Watson, A condition based maintenance model for a two-unit series system, European Journal of Operational Research 116 (1999) 281–290.

[94] A. Grall, C. Bérenguer, L. Dieulle, A condition-based maintenance policy for stochastically deteriorating systems, Reliability Engineering and System Safety 76 (2002) 167–180.

[95] A. Grall, C. Bérenguer, L. Dieulle, M. Roussignol, Continuous-time predictive-maintenance scheduling for a deteriorating system, IEEE Transactions on Reliability 51 (2002) 141–150.

[96] L. Dieulle, C. Bérenguer, A. Grall, M. Roussignol, Sequential condition-based maintenance scheduling for a deteriorating system, European Journal of Operational Research 150 (2003) 451–461.

[97] W. Wang, Modelling condition monitoring intervals: A hybrid of simulation and analytical approaches, Journal of the Operational Research Society 54 (2003) 273–282.

[98] B. Castanier, A. Grall, C. Bérenguer, A condition-based maintenance policy with non-periodic inspections for a two-unit series system, Reliability Engineering and System Safety 87 (2005) 109–120.

[99] C. Derman, Finite State Markovian Decision Processes, Academic Press, London, 1970.

[100] P. Kolesar, Minimum cost replacement under Markovian deterioration, Management Science 12 (1966) 684–706.

[101] H. Kawai, J. Koyanagi, M. Ohnishi, Stochastic Models in Reliability and Maintenance, Springer, 2002, Ch. Optimal Maintenance Problems for Markovian Deteriorating Systems, pp. 193–218.

[102] A. P. Wood, Optimal maintenance policies for constantly monitored systems, Naval Research Logistics 35 (1988) 461–471.

[103] S. Özekici, N. O. Günlük, Maintenance of a device with age-dependent exponential failures, Naval Research Logistics 39 (1992) 699–714.

[104] E. P. C. Kao, Optimal replacement rules when changes of state are semi-Markovian, Operations Research 21 (1973) 1231–1249.

[105] K. So, Optimality of control limit policies in replacement models, Naval Research Logistics 39 (1992) 685–697.

[106] C. T. Lam, R. H. Yeh, Optimal replacement policies for multistate deteriorating systems, Naval Research Logistics 41 (1994) 303–315.

[107] M. Ohnishi, H. Kawai, H. Mine, An optimal inspection and replacement policy for a deteriorating system, Journal of Applied Probability 23 (1986) 973–988.

[108] C. T. Lam, R. H. Yeh, Optimal maintenance-policies for deteriorating systems under various maintenance strategies, IEEE Transactions on Reliability 43 (1994) 423–430.

[109] R. H. Yeh, Optimal inspection and replacement policies for multi-state deteriorating systems, European Journal of Operational Research 96 (1996) 248–259.

[110] S. H. Sim, J. Endrenyi, A failure-repair model with minimal & major maintenance, IEEE Transactions on Reliability 42 (1993) 134–140.

[111] D. Chen, K. Trivedi, Closed-form analytical results for condition-based maintenance, Reliability Engineering and System Safety 76 (2002) 43–51.

[112] D. Chen, K. Trivedi, Optimization for condition-based maintenance with semi-Markov decision process, Reliability Engineering and System Safety 90 (2005) 25–29.

[113] S. Amari, L. McLaughlin, Optimal design of a condition-based maintenance model, in: Reliability and Maintainability, 2004 Annual Symposium-RAMS, 2004, pp. 528–533.

[114] M. Moustafa, E. A. Maksoudb, S. Sadekb, Optimal major and minimal maintenance policies for deteriorating systems, Reliability Engineering and System Safety 83 (2004) 363–368.

[115] J. Chiang, J. Yuan, Optimal maintenance policy for a Markovian system under periodic inspection, Reliability Engineering and System Safety 71 (2001) 165–172.

[116] M. Puterman, Markov Decision Processes, John Wiley and Sons, New York, 1994.

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