Portfolio Selection with Random Risk Preference

by

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This is to certify that I have examined this copy of a master's thesis by

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and have found that it is complete and satisfactory in all respects,

and that any and all revisions required by the final

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To my family

ABSTRACT

In this thesis, we analyze a single-period portfolio selection problem where the investor maximizes the expected utility of the terminal wealth. The utility function is exponential, but the Pratt-Arrow measure of absolute risk aversion or risk tolerance is random. This is due to the random variations in individual's decisions concerning stochastic choice. It is well-known that the investor is memoryless in wealth for exponential utility functions with a constant risk tolerance. In other words, the investment portfolio consisting of risky stocks does not depend on the level of wealth. However, we show that this is no longer true if risk tolerance is random. We obtain a number of interesting characterizations on the structure of the optimal policy.

In the first part of the thesis, we analyzed the characteristics of the optimal policy when the return of the risky asset has an arbitrary distribution. We considered the single asset case and showed that the decision on buying or short selling the risky asset depends on the sign of the mean excess return. We also showed that the optimal decision is bounded, and it increases in wealth when mean excess return is positive and decreases otherwise.

In the second part of the thesis, we analyzed a specific case where the distributions of the returns of the risky assets are normal. Normal and multivariate normal cases are discussed separately. We proved that in this setting, the multivariate case can be reduced to the single asset case. Moreover, the decision on buying or short selling the assets depends on the "adjusted mean excess return".

In the last part, we considered some of the other results that we have found during the research. For the exponential distribution case, we used a direct approach to obtain the similar results we found in the general distribution case. We finally extended the exponential utility functions to arbitrary concave utility functions and obtained some characterizations on the optimal policy.

ÖZETÇE

Bu tezde tekli zamanda dönem sonu servetinin beklenen değerini fayda fonksiyonunun en büyükleyen bir yatırımcının en iyi portföy seçimi problemi incelenmiştir. Fayda fonksiyonu üstel, fakat Prat-Arrow mutlak riskten kaçınma katsayısı yada risk toleransı rassaldır. Bu stokastik seçim durumunda bireyin yaşadığı rassal varyasyonlardan kaynaklanmaktadır. Yatırımcının fayda fonksiyonu üstel ve sabit risk toleranslı olduğunda yatırımcının serveti açısından unutkan olduğu iyi bilinmektedir. Diğer bir değişle, riskli hisse senetleri içeren bir portföy servet düzeyine dayanmaz. Fakat, bu tezde risk toleransı rassal olduğu zaman bunun doğru olmadığı gösterilmiştir. Ayrıca en iyi politika hakkında bir kaç ilginç karakterizasyon elde edilmiştir.

Tezin ilk kısmında riskli varlığın dağılımı rasgele kabul edilmiş ve en iyi politikanın özelikleri incelenmiştir. Tek varlık durumu değerlendirilmiş ve riskli varlığın satın alınma veya açığa satılması kararının riskli varlığın "fazla ortalama getirisine" bağlı olduğu bulunmuştur. Ayrıca en iyi politikanın sınırlı, ve fazla ortalama getirisi pozitif olduğunda varlık seviyesine bağlı olarak arttığı, negatif ise azaldığı ispat edilmiştir.

Tezin ikinci kısmında daha özel bir durumu, riskli varlığın getirisinin normal dağılım olduğu durumu incelenmiştir. Normal ve çok değişkenli normal dağılım olduğu durumlar ayrı analiz edilmiştir. Bu düzenlemede çok değişkenli normal dağılım durumunun tek değişkenli normal dağılıma indirgenebildiği ispatlanmıştır. Dahası, riskli varlığın satın alınması veya açığa satılması kararının "ayarlanmış fazla ortalama getirisine" bağlı olduğu gösterilmiştir.

Son kısımda ise, araştırmamız sırasında edindiğimiz diğer sonuçları incelenmiştir. Üstel dağılım durumunda genel dağılımdaki sonuçların aynısını elde etmek için doğrudan bir yaklaşım izlenmiştir. Son olarak da üstel fayda fonksiyonları rasgele içbükey fayda fonksiyonlarına genelleştirilmiş ve en iyi politikanın bazı karakteristik özellikleri bulunmuştur.

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NOMENCLATURE

$U(i,x)$	: Utility function given wealth is x and the risk tolerance is i
m	: Number of utilities
n	: Number of risky assets
β_i	: Risk aversion parameter for utility i
r_f:	: Return of the riskless asset
$\mathbb{R}$	: The set of all real numbers
W	: The wealth level after one period
R	: The return vector of the risky assets
R^e	: The excess return vector
μ	: The mean excess return vector
u	: The investment policy vector
u^*	: The optimal investment policy vector
$g(x,u)$	: Expected utility using policy u with the available money for investment x
σ_{ij}	: Covariance matrix for multivariate normal distribution
M_{ij}	: (i,j) minor of covariance matrix σ_{ij}
C_{ij}	: (i,j) cofactor matrix of σ_{ij}
P_j	: Probability that the utility of the investor is j
$E[.]$	: Expectation
σ^2	: Variance of the risky asset
$\bar{\mu}$	: Adjusted mean excess return vector
λ	: Rate parameter of the exponential distribution

Chapter 1

INTRODUCTION

Decision making under uncertainty is an important problem in a variety of applications. Modeling the behaviors of agents in an uncertain setting is difficult, so there are many approaches to analyze this issue. Expected Utility Theory (EUT) is the standard choice for explaining such decision making problems. But, EUT is subject to criticism as well. Allais [22] claims that von Neumann-Morgenstern axiomatization of expected utility is flawed by showing a counterexample to the independence axiom. Starmer [25] reviews non-expected utility theories and explains that violation of the independence axiom empirically is not the only reason for their development. To mention a few, Segal [24], Wakker [30] and Mohammed Abdellaoui [1] propose axiomatizations of rank-dependent expected utility; Kahneman and Tversky [15] uses prospect theory. Blavatskyy [4] suggests the concept of stochastic expected utility introducing a stochastic component as a part of the decision making process.

Moreover, the idea that the agent behaves exactly the same when he is facing the same decision problem under uncertainty had also been subject to criticism as well. Camerer [6] describes an experiment where 31.6% of subjects reversed their preferences in a test of the reliability of subjects' responses by seeing how often they expressed the same preference for the same gamble. Starmer and Sugden [26] show that individual's preferences reverse on the second repetition of the decision problem 26.5% of the time. Hey and Orme [14] find that even when the subjects are allowed to declare indifference, 25% prefer the other choice. Wu [31] reports that in the case of repeated decision problems, 5% to 45% of the subjects reverse their choices.

Blavatskyy [5] briefly reviews decision theories under risk and he outlines three different approaches to the problem. The first one is called "the tremble model" introduced by Harless and Camerer [12]. In their setting, the individuals choose an alternative choice according to a random tremble. They suppose that individuals have a unique preference on the set of risky lotteries, but a "tremble" occurs and they switch preferences with some probability p. With probability $1-p$, the individuals act according to their first preference. The second model is called "Fechner model" of random errors proposed by Fechner [10]. It suggests that individuals have unique preferences like in the tremble model, but there is a random error attached to each preference having zero mean and constant deviation. In mathematical terms, the utility U is defined on the set of risky lotteries, and a relative advantage function d is used. This relative advantage $d = U(L_1) - U(L_2)$ stands for the advantage of lottery L_1 over another lottery L_2. If there are no random errors, the individual will choose L_1 over L_2 whenever $d > 0$. But since there is a random error involved, the individual will choose L_1 over L_2 only if $d + \varepsilon > 0$ where ε is assumed to have a normal distribution with zero mean and constant deviation in the classic Fechner model. Finally, in the third model called "random preference model" proposed by Loomes and Sugden [17], the authors analyze the role of randomness in decision theory. In their paper, they propose a new axiomatic setting different from EUT and show this model's consistency. In another paper by Loomes et al. [16], they refer to the random preference model and apply it to a stochastic setting. They claim that when facing a decision problem, the individuals go through three different stages. In the first stage called "preference selection stage", the individuals are assumed to be uncertain about their preference (while they are certain about some general principles required by EUT as core theory). Hence, they model this setting probabilistically using random preferences in such a way that the utility function is stochastic. As a matter of fact, it is this approach that we will follow to model stochastic choice of investors.

One of the application fields of decision theory under uncertainty is in the world of finance. In today's financial world, there are various methods to model the financial markets. There are numerous approaches that try to describe this uncertainty; ranging from utility based approach finding its roots in the book by Von Neumann and Morgenstern [29] to portfolio optimization using dynamic programming approach as in Çanakoğlu and Özekici [8]. Among all, stochastic modeling is now a well-accepted approach within the research community where the aim is to find an optimal portfolio among risky and risk-free assets. In a survey done by Steinbach [27], there are 208 papers analyzing the mean-variance models in financial portfolio analysis. On the other hand, previous research by Mossin [20], Merton [19] and Hakansson [11] concentrates on using expected utility maximization. Moussin [20] examined utility functions leading to myopic policies, and Merton [19] considered special utility functions with logarithmic and power structures. Hakansson [11] analyzes portfolio optimization using logarithmic and power utility in discrete time and random market setting. More recently, Dokuchaev [9] considers an optimization problem in which the expected utility of the terminal wealth is maximized in a discrete-time market with serial correlations.

In this thesis, motivated by the idea presented by Loomes and Sugden [17], we aim to apply the idea of random preference decision making in the portfolio selection problem. We will use exponential utility functions which are widely used both in practice and theory. In our setting, we assume that at the beginning of a period, the investor needs to choose among a risk-free and risky assets.

It can be empirically observed that the states of the market varies throughout time. Following the market trends and identifying such trends accurately can be considered to be of the investor's best interest. It is intuitively true that the returns of the assets in the market vary significantly depending on the current state of the market. Starting with Pye [23], there has been a growing interest in modeling a stochastic financial market by a Markov chain. In the continuous-time setting, Norberg [21] proposes a Markov process interest model with applications to insurance. On the other hand, there is growing interest in the literature to use a stochastic market process in order to modulate various parameters of the financial model to make it more realistic. Hernández-Hernández and Marcus [13], Bielecki et al. [2], Bielecki and Pliska [3], Di Massi and Stettner [18], and Stettner [28] provide examples on risk-sensitive portfolio optimization with observed, unobserved and partially observed states in Markovian markets. Hence, it can be deduced that the state of the market is uncertain for the next period. Out of this uncertainty an important question rises: Is it logical to assume that the investor's risk preference is the same for all states of the market? In this thesis, we will assume that the risk preferences of the investor is not constant, but it is random like the state of the market. To clarify this by an example, if the market is in a "bearish" trend, the risk preference of the investor will be different than the one in a "bullish" market trend.

The organization of this thesis be as follows: In Chapter 2, we introduce the main problem and describe our random risk preference model. We also give some general results. In Chapter 3, we state our results on the structure of the optimal portfolio when the returns of the risky asset are assumed to have an arbitrary distribution. In Chapter 4, we consider a specific case of Chapter 3 when the returns of the risky assets have the normal distribution. In Chapter 5, we complete our analysis by considering some further extensions. We first suppose that the returns of the risky assets are exponentially distributed, and then consider a general model with an arbitrary concave utility function. Finally, Chapter 6 includes our conclusions and suggestions for future research.

Chapter 2

PORTFOLIO SELECTION WITH EXPONENTIAL UTILITY FUNCTIONS

We consider an investor with an initial wealth x who chooses a portfolio among n risky and one risk-free asset. Moreover, there are m random utilities or preferences and the investor's preference is the ith one with some probability $P_i > 0$ where $\sum_{i=1}^{m} P_i = 1$. We assume that the investor's utility is described by the exponential utility functions

$$U(i, x) = K_i - C_i \exp(-x/\beta_i) \tag{2.1}$$

when the risk preference of the investor is i. Here, K_i and $C_i > 0$ denotes the parameters of the exponential utility function for risk preference i and x represents the wealth level. Also, note that Pratt-Arrow's measure of absolute risk aversion is $-U''(i, x)/U'(i, x) = 1/\beta_i$ or β_i is the risk tolerance. The previous discussion on random risk preference gets in the picture here. The preference dependent measure of absolute risk aversion indicates that the investor has a random risk preference. In fact, the risk tolerance is β_i with probability P_i. We suppose without loss of generality that the risk tolerances are ordered so that $0 \leq \beta_1 \leq \beta_2 \leq \cdots \leq \beta_m$. Our aim is to determine the optimal portfolio of risky and risk-free assets where the investor has a random risk preference given by exponential utility function (2.1) with probability P_i.

In this thesis, unless stated otherwise, a vector z is a column vector so that its transpose, denoted by z', is always a row vector. We let $\mathbb{R} = (-\infty, +\infty)$ denote the set of all real numbers. Moreover, because of the involvement of different β values for different states, the value function can not be evaluated analytically while considering the next period. Hence, only a single period will be analyzed. Let $u = [u_1, u_2, \cdots, u_n]$ denote the amount of wealth that is invested in n risky assets. Since the returns of the risky assets are random let $R = [R_1, R_2, \cdots, R_n]$ denote the return vector of the risky assets. Also, let r_f denote the return of risk-free asset. Therefore, after one period, the wealth is

$$\begin{aligned} W &= r_f\left(x - 1'u\right) + R'u \\ &= r_f x + (R^e)'\, u \end{aligned} \tag{2.2}$$

where $1 = (1, \cdots, 1)$ and $R^e = R - r_f$ is the excess return. Throughout this thesis, we let $\bar{r} = E[R]$ denote the mean return vector and $\mu = E[R - r_f] = \bar{r} - r_f$ denote the mean excess return vector. We suppose that $R^e \neq 0$ trivially since this case implies that $W = r_f x$ independent of u and any policy is optimal. In case $n = 1$, there is only a single risky asset, and both $\bar{r}$ and μ are scalars.

Let $g(x, u)$ denote the expected utility using the investment policy u when the amount of money available for investment is x. Hence,

$$\begin{aligned} g(x,u) &= \sum_{j=1}^{m} P_j E\left[U\left(j, r_f x + (R^e)'\, u\right)\right] \\ &= \sum_{j=1}^{m} P_j E\left[K_j - C_j \exp(-\left(r_f x + (R^e)'\, u\right)/\beta_j)\right] \\ &= \sum_{j=1}^{m} P_j K_j - \sum_{j=1}^{m} P_j C_j \exp(-\frac{r_f x}{\beta_j}) E\left[\exp\left(-\frac{(R^e)'\, u}{\beta_j}\right)\right]. \end{aligned} \tag{2.3}$$

To find the optimum portfolio of risky assets and risk free asset, note that gradient of $g(x, u)$ is

$$\nabla_k g(x,u) = \frac{\partial g(x,u)}{\partial u_k} = \sum_{j=1}^{m} P_j C_j \exp(-\frac{r_f x}{\beta_j}) E\left[\frac{R_k^e}{\beta_j} \exp\left(-\frac{(R^e)'\, u}{\beta_j}\right)\right]$$

and the Hessian is

$$H_{k,l} = \frac{\partial^2 g(x,u)}{\partial u_k \partial u_l} = -\sum_{j=1}^{m} P_j C_j \exp(-\frac{r_f x}{\beta_j}) E\left[\frac{R_k^e R_l^e}{\beta_j^2} \exp\left(-\frac{(R^e)'\, u}{\beta_j}\right)\right].$$

For any vector $z = [z_1, z_2, \cdots, z_n]$, note that

$$z'Hz = -\sum_{j=1}^{m} P_j C_j \exp(-\frac{r_f x}{\beta_j}) E\left[\frac{1}{\beta_j^2}\left(\sum_{k=1}^{n} z_k R_k^e\right)^2 \exp\left(-\frac{(R^e)'\, u}{\beta_j}\right)\right] \leq 0.$$

This implies that the Hessian is negative semi-definite and $g(x, u)$ is concave in u. Hence, setting $\nabla g(x, u) = [0, \cdots, 0]$ would give the optimal policy.

As a summary, the optimal policy can be found by solving the first order condition

$$\nabla_k g(x,u) = \sum_{j=1}^{m} \frac{P_j C_j}{\beta_j} \exp(-\frac{r_f x}{\beta_j}) E\left[R_k^e \exp\left(-\frac{(R^e)'\, u}{\beta_j}\right)\right] = 0 \tag{2.4}$$

for all $k = 1, 2, \cdots, n$. Before conducting further analysis, define

$$A_j^k(x,u) = \frac{P_j C_j}{\beta_j} \exp(-\frac{r_f x}{\beta_j}) E\left[R_k^e \exp\left(-\frac{(R^e)' u}{\beta_j}\right)\right]$$

so that the optimality condition (2.4) can be rewritten as

$$A^k(x,u) = \sum_{j=1}^{m} A_j^k(x,u) = 0.$$

Note that $A_j^k(x,u)$ and, therefore, $A^k(x,u)$ are strictly decreasing in u since

$$\frac{\partial A^k(x,u)}{\partial u_k} = -\frac{P_j C_j}{\beta_j^2} \exp(-\frac{r_f x}{\beta_j}) E\left[(R_k^e)^2 \exp\left(-\frac{(R^e)' u}{\beta_j}\right)\right] < 0$$

for all k.

Now, for a fixed point $(x^1, u^1) \in \mathbb{R}^{1+n}$ with $A(x^1, u^1) = 0$ consider the gradient of A^k at (x^1, u^1) so that

$$\frac{\partial A^k}{\partial u_k}(x^1, u^1) = -\sum_{j=1}^{m} \frac{P_j C_j}{\beta_j^2} \exp(-\frac{r_f x}{\beta_j}) E\left[(R_k^e)^2 \exp\left(-\frac{(R^e)' u^1}{\beta_j}\right)\right] < 0$$

for all k. Then, by the implicit function theorem, there exists an open set $U \subset \mathbb{R}$ containing x^1, an open set $V \subset \mathbb{R}^n$ containing u^1, and a unique continuously differentiable function $g : U \to V$ such that

$$\{(x, g(x))\} = \left\{(x, u_1, u_2, \cdots, u_n) : A^k(x, u_1, u_2, \cdots, u_n) = 0\right\} \cap U \times V.$$

The above result is true for all points satisfying $A^k(x,u) = 0$, and we also know that the optimal decision $u(x)$ satisfying (2.4) is unique when it exists by the concavity analysis of g, we conclude that the optimal policy $u^*(x) = [u_1^*(x), u_2^*(x), \cdots, u_n^*(x)]$ is a unique continuously differentiable function in x whenever it exists.

Chapter 3

GENERAL DISTRIBUTION MODEL

In this setting, the distributions of the risky asset returns are assumed to be arbitrary. Our analysis in this chapter is presented in two sections. In the first section, we assume that the risk tolerance is constant and there are n risky assets. In the second section, the risk tolerance is random, hence there are m utilities for the investor, but there is only $n = 1$ risky asset. The second section is further divided into two parts. In the first part of the single asset model, the expected excess return $E[R^e] = \mu$ will be greater than zero; and in the second part, expected excess return will be less than zero.

3.1 Constant risk tolerance with n risky assets and m utilities

If we were to assume that the risk preference of the investor is constant rather than random so that $\beta_j = \beta$ for all j, the expected utility becomes

$$\begin{aligned} g(x,u) &= \sum_{j=1}^{m} P_j K_j - \sum_{j=1}^{m} P_j C_j \exp(-\frac{r_f x}{\beta}) E\left[\exp\left(-\frac{(R^e)' u}{\beta}\right)\right] \\ &= \sum_{j=1}^{m} P_j K_j - \exp(-\frac{r_f x}{\beta}) E\left[\exp\left(-\frac{(R^e)' u}{\beta}\right)\right] \sum_{j=1}^{m} P_j C_j. \end{aligned}$$

A similar argument as in Chapter 2 implies that $g(x,u)$ is still concave and equating the gradient of $g(x,u)$ to the zero vector will give the optimal decision. Therefore,

$$\nabla_k g(x,u) = \frac{\partial g(x,u)}{\partial u_k} = \exp(-\frac{r_f x}{\beta}) E\left[\frac{R_k^e}{\beta} \exp\left(-\frac{(R^e)' u}{\beta}\right)\right] \sum_{j=1}^{m} P_j C_j = 0.$$

But since all the terms in the summation are positive, $\nabla_k g(x,u)$ is equal to zero if and only if $E\left[R_k^e \exp\left(-(R^e)' u/\beta\right)\right]$ is zero. Therefore, in this setting the optimal decision u^* is the vector that satisfies

$$E\left[R_k^e \exp\left(-(R^e)' u/\beta\right)\right] = 0 \tag{3.1}$$

for all k. It can be noticed from this that u^* is independent of the wealth level of the investor as noted by Çanakoğlu and Özekici [7] for exponential utility function in a generalized multiperiod model in a stochastic market.

3.2 Random risk tolerance with $n = 1$ risky asset and m utilities

For this case, since the distribution of the risky asset is general, we need to consider the generalized optimality condition (2.4). We know that $A(x, u)$ is decreasing in u from Chapter 2. Note that

$$A(x,0) = \sum_{j=1}^{m} A_j(x,0) = \sum_{j=1}^{m} \frac{P_j C_j}{\beta_j} \exp(\frac{-r_f x}{\beta_j}) E\left[R^e\right] = \left(\sum_{j=1}^{m} \frac{P_j C_j}{\beta_j} \exp(\frac{-r_f x}{\beta_j}) \right) \mu \quad (3.2)$$

where μ is the mean of the excess return. Therefore, the sign of $A(x, 0)$ depends on the sign of μ. If $\mu > 0$, then $A(x, 0) > 0$ and since $A(x, u)$ is strictly decreasing, the optimal decision $u^*(x)$ making $A(x, u) = 0$ is greater than zero for all x. Moreover, when $\mu < 0$, the optimal decision $u^*(x)$ is less than zero for all x by a similar argument. Finally, it follows from (3.2) that $u^*(x) = 0$ for all x when $\mu = 0$.

3.2.1 Mean excess return greater than zero ($\mu > 0$)

The assumption $\mu > 0$ implies that $\bar{r} > r_f$ which indicates that the expected return of the risky asset is greater than the return of the risk-free asset. We now show that in this setting the optimal decision is increasing with respect to x.

Theorem 1 *The optimal decision $u^*(x)$ is bounded and positively increasing in x if $\mu > 0$.*

Proof. *If $\mu > 0$, it is clear that $u^*(x) > 0$ since $A(x, u)$ is decreasing in u with $A(x, 0) > 0$. For any j, let $\hat{u}_j > 0$ be the unique solution of $A_j(x, u) = 0$ for all x. Observe that $A_j(x, \hat{u}_j) = 0$ if and only if $E\left[R^e \exp\left(-R^e \hat{u}_j/\beta_j\right)\right] = 0$ and it is independent of x. Then, note that*

$$E\left[R^e \exp\left(-\frac{R^e \hat{u}_j}{\beta_{j+1}}\right)\right] = E\left[R^e \exp\left(-\frac{R^e \hat{u}_j}{\beta_j}\right)^{\frac{\beta_j}{\beta_{j+1}}}\right]$$

for any $j = 1, 2, \cdots m-1$. This can be rewritten as

$$f(\alpha) = E\left[R^e \exp\left(-\frac{R^e \hat{u}_j}{\beta_j}\right)^{\alpha}\right]$$

where $\alpha = \beta_j/\beta_{j+1} \leq 1$. Observe that $f(1) = 0$ by our assumption. Now consider the derivative

$$\frac{df(\alpha)}{d\alpha} = -E\left[\frac{(R^e)^2 \hat{u}_j}{\beta_j} \exp\left(-\frac{R^e \hat{u}_j}{\beta_j}\right)^{\alpha}\right] \leq 0$$

since β_j *is always positive and* $\hat{u}_j$ *is positive when* $\mu > 0$*. Therefore, we can conclude that* $f(\alpha) \geq f(1) = 0$ *for all* $0 \leq \alpha \leq 1$*. This further implies that* $E\left[R^e \exp\left(-R^{e\prime}\hat{u}_j/\beta_{j+1}\right)\right] \geq 0$ *by taking* $\alpha = \beta_j/\beta_{j+1} \leq 1$*. Since* $A_{j+1}(x,u)$ *is decreasing in* u *one can conclude that* $\hat{u}_j \leq \hat{u}_{j+1}$ *because* $A_{j+1}(x,\hat{u}_{j+1}) = 0$*.*

Next fix x_1 *and let* u_1^* *be the optimum decision making* $A(x_1,u_1^*) = 0$*. Because* $A(x_1,u)$ *is strictly decreasing in* u *and the feasible region is always positive, there exist* $1 \leq k \leq m-1$ *such that*

$$\hat{u}_k \leq u_1^* < \hat{u}_{k+1}.$$

Moreover, $A_j(x_1,u_1^*) \leq 0$ *for all* $j \leq k$ *since* $u_1^* \geq \hat{u}_j$ *for all* $j \leq k$ *and* $A_j(x,u)$ *is strictly decreasing in* u*. Similarly,* $A_j(x_1,u_1^*) > 0$ *for all* $j > k$*. Therefore,* $A\left(x_1,u^*\right) = 0$ *implies that*

$$\sum_{j=1}^{k} A_j(x_1,u_1^*) = -\sum_{j=k+1}^{m} A_j(x_1,u_1^*) \leq 0. \tag{3.3}$$

Consider the point $x_1 + \Delta x > x_1$ *for some* $\Delta x > 0$*. We now have*

$$\begin{aligned}
\sum_{j=1}^{m} A_j(x_1+\Delta x,u_1^*) &= \sum_{j=1}^{k} A_j(x_1,u_1^*)\exp(-r_f\frac{\Delta x}{\beta_j}) + \sum_{j=k+1}^{m} A_j(x_1,u_1^*)\exp(-r_f\frac{\Delta x}{\beta_j}) \\
&\geq \exp(-r_f\frac{\Delta x}{\beta_k})\sum_{j=1}^{k} A_j(x_1,u_1^*) + \exp(-r_f\frac{\Delta x}{\beta_{k+1}})\sum_{j=k+1}^{m} A_j(x_1,u_1^*) \\
&= -\exp(-r_f\frac{\Delta x}{\beta_k})\sum_{j=k+1}^{m} A_j(x_1,u_1^*) + \exp(-r_f\frac{\Delta x}{\beta_{k+1}})\sum_{j=k+1}^{m} A_j(x_1,u_1^*) \\
&= \left[\exp(-r_f\frac{\Delta x}{\beta_{k+1}}) - \exp(-r_f\frac{\Delta x}{\beta_k})\right]\sum_{j=k+1}^{m} A_j(x_1,u_1^*) \geq 0
\end{aligned}$$

The first inequality follows from the fact that $\exp(-x)$ *is a nonnegative decreasing function. The second equality follows from (3.3) and the final term is positive since* $\exp(-x)$ *is a decreasing function and* $\sum_{j=k+1}^{m} A_j(x_1,u^*)$ *is positive by (3.3).*

Since $\sum_{j=1}^{m} A_j(x_1+\Delta x,u_1^*) \geq 0$*, and* $\sum_{j=1}^{m} A_j(x_1+\Delta x,u_1)$ *decreases in* u*, it can be concluded that the optimal decision* $u_1^*(x_1+\Delta x)$ *is greater than* $u_1^*(x)$ *indicating that* $u^*(x)$ *is increasing in* x*.*

Finally to show that $u^*(x)$ *is bounded, consider* $\hat{u}_j > 0$*. Since* $\hat{u}_j's$ *are ordered, we know that* $\hat{u}_1 < \hat{u}_j < \hat{u}_m$ *for all* j*. Now, if* $u^*(x) < \hat{u}_1$ *then, since* $A_j(x,u)$ *is strictly decreasing in* u*,* $A_j(x,u^*) > A_j(x,\hat{u}_1) = 0$ *for all* j*. But this is a contradiction since* $\sum_{j=1}^{m} A_j(x,u^*) = 0$*.*

Similarly, if $\hat{u}_m < u^*(x)$ *then, by the same reason* $A_j(x, u^*) < A_j(x, \hat{u}_m) = 0$. *This also leads to a contradiction. Therefore we conclude that* $0 < \hat{u}_1 < u^*(x) < \hat{u}_m$ *for all* x. ■

3.2.2 Mean excess return less than zero ($\mu < 0$)

The assumption $\mu < 0$ implies that $\bar{r} < r_f$ which indicates that the expected return of the risky asset is less than the return of the risk-free asset. Next, we show that in this setting the optimal decision is decreasing with respect to x.

Theorem 2 *The optimal decision* $u^*(x)$ *is bounded and negatively decreasing in* x *if* $\mu < 0$.

Proof. *If* $\mu < 0$, *it is clear that* $u^*(x) > 0$ *since* $A(x, u)$ *is decreasing in* u *with* $A(x, 0) < 0$. *For any* j, *let* $\hat{u}_j < 0$ *be the unique solution of* $A_j(x, u) = 0$ *for all* x. *Observe that* $A_j(x, \hat{u}_j) = 0$ *if and only if* $E\left[R^e \exp\left(-R^e \hat{u}_j/\beta_j\right)\right] = 0$ *and it is independent of* x. *Using a proof similar to that of Theorem 1, note that* $f(1) = 0$ *but* $df(\alpha)/d\alpha \geq 0$ *since* β_j *is always positive and* $\hat{u}_j$ *is negative when* $\mu < 0$. *Therefore, we can conclude that* $f(\alpha) \leq f(1) = 0$ *for all* $0 \leq \alpha \leq 1$. *This further implies that* $E\left[R^e \exp\left(-R^e \hat{u}_j/\beta_{j+1}\right)\right] \leq 0$. *Since* $A_{j+1}(x, u)$ *is decreasing in* u *one can conclude that* $\hat{u}_j \geq \hat{u}_{j+1}$ *because* $A_{j+1}(x, \hat{u}_{j+1}) = 0$.

Fix the initial wealth x_2 *and let* u_2^* *be the optimal decision for* x_2. *By a similar argument from the positive mean excess return case, we can conclude that there exists* k *such that*

$$\hat{u}_{k+1} \leq u_2^* < \hat{u}_k.$$

Next, $A_j(x_2, u_2^*) \geq 0$ *for all* $j \leq k$ *since* $u^* \geq \hat{u}_j$ *for all* $j \leq k$ *and* $A_j(x, u)$ *is strictly decreasing in* u. *Similarly* $A_j(x_1, u^*) \leq 0$ *for all* $j > k$. *Therefore,* $A(x_2, u_2^*) = 0$ *implies that*

$$\sum_{j=1}^{k} A_j(x_1, u^*) = -\sum_{j=k+1}^{m} A_j(x_1, u^*) \geq 0. \tag{3.4}$$

Consider the point $x_2 + \Delta x > x_2$ for some $\Delta x > 0$. We now have

$$\begin{aligned}
\sum_{j=1}^{m} A_j(x_2 + \Delta x, u_2^*) &= \sum_{j=1}^{k} A_j(x_2, u_2^*) \exp(-r_f \frac{\Delta x}{\beta_j}) + \sum_{j=k+1}^{m} A_j(x_2, u_2^*) \exp(-r_f \frac{\Delta x}{\beta_j}) \\
&\leq \exp(-r_f \frac{\Delta x}{\beta_k}) \sum_{j=1}^{k} A_j(x_2, u_2^*) + \exp(-r_f \frac{\Delta x}{\beta_{k+1}}) \sum_{j=k+1}^{m} A_j(x_2, u_2^*) \\
&= -\exp(-r_f \frac{\Delta x}{\beta_k}) \sum_{j=k+1}^{m} A_j(x_2, u_2^*) + \exp(-r_f \frac{\Delta x}{\beta_{k+1}}) \sum_{j=k+1}^{m} A_j(x_2, u_2^*) \\
&= \left[\exp(-r_f \frac{\Delta x}{\beta_{k+1}}) - \exp(-r_f \frac{\Delta x}{\beta_k})\right] \sum_{j=k+1}^{m} A_j(x_2, u_2^*) \leq 0
\end{aligned}$$

The first inequality follows from the fact that $\exp(-x)$ is a nonnegative decreasing function. The second equality follows from (3.4) and similarly the final term is negative since $\exp(-x)$ is a decreasing function and $\sum_{j=k+1}^{m} A_j(x_2, u_2^*)$ is negative by (3.4).

Since $\sum_{j=1}^{m} A_j(x_2 + \Delta x, u_2^*) \leq 0$, and $\sum_{j=1}^{m} A_j(x_1 + \Delta x, u)$ decreases in u, it can be concluded that the optimal decision $u_2^*(x_2 + \Delta x)$ is less than $u^*(x_2)$ indicating that $u^*(x)$ is decreasing in x.

Finally to show that $u^*(x)$ is bounded, consider $\hat{u}_j < 0$. Since $\hat{u}_j'$s are ordered, we know that $\hat{u}_m \leq \hat{u}_j \leq \hat{u}_1$ for all j. Now, if $u^*(x) > \hat{u}_1$ then, since $A_j(x, u)$ is strictly decreasing in u, $A_j(x, u^*) \leq A_j(x, \hat{u}_1) = 0$ for all j. But this is a contradiction since $\sum_{j=1}^{m} A_j(x, u^*) = 0$. Similarly, if $\hat{u}_m > u^*(x)$ then, by the same reason $A_j(x, u^*) > A_j(x, \hat{u}_m) = 0$. This also leads to a contradiction. Therefore we conclude that $\hat{u}_m \leq u^*(x) \leq \hat{u}_1 < 0$ for all x. ■

The characteristics obtained in this section on the structure of the optimal policy are quite intuitive. If $\mu > 0$ and the mean return of the risky asset exceeds that of the risk-free asset, then $u^*(x) > 0$ indicating that some positive amount of current wealth is invested in the risky asset. It should be noted that it is possible that $u^*(x) > x$ which implies that the extra amount $u^*(x) - x$ is obtained by short selling the risk-free asset. A similar argument can be made for the case when $\mu < 0$. Now, since the return of the risk-free asset exceeds that of the risky asset $u^*(x) < 0$ and the risky asset is sold short to invest $x - u^*(x) > x$ in the riskless asset.

Chapter 4

NORMAL DISTRIBUTION MODEL

This chapter is primarily divided into three sections. In the first section, the investor's risk preference is assumed to be constant indicating that $\beta_j = \beta$ for all j; and in the second section, the investor's risk preference is random. The third section is dedicated to illustrations of the results found in the first two sections.

In the normal distribution model, if there is only one risky asset ($n = 1$) to be considered, then the excess return of the risky asset is assumed to be normal and denoted by $R^e = (R - r_f) \sim$ Norm (μ, σ^2) where $\mu = \bar{r} - r_f$. Here, σ^2 denotes the variance of the risky asset. If there are n risky assets to be considered, then the excess return of the risky assets are assumed to be multivariate normal and denoted by $R^e = (R - r_f) \sim$ Multi-Norm (μ, σ) where $\mu = \bar{r} - r_f = [\mu_1, \mu_2, \cdots, \mu_n]$ is the excess return vector of the risky assets and σ is the covariance matrix. We also let $\|\sigma\| = \det(\sigma)$ which is positive since a covariance matrix is always positive definite and nonsingular.

We start by observing following identities which will be useful throughout this section. If there is only one risky asset having a normal distribution, then

$$E\left[e^{-\alpha R^e}\right] = \exp(-\mu\alpha + \frac{\sigma^2\alpha^2}{2})$$

so that

$$\begin{aligned} E\left[R^e e^{-\alpha R^e}\right] &= -\frac{\partial\left(\exp(-\mu\alpha + \frac{\sigma^2\alpha^2}{2})\right)}{\partial\alpha} \\ &= \exp(\frac{\alpha^2\sigma^2}{2} - \alpha\mu)\left(\mu - \alpha\sigma^2\right) \end{aligned} \tag{4.1}$$

and if there are n risky assets having multivariate normal distribution, then

$$E\left[e^{-\alpha' R^e}\right] = \exp(-\mu'\alpha + \frac{1}{2}\alpha'\sigma\alpha)$$

so that

$$\begin{aligned} E\left[R_k^e e^{-\alpha' R^e}\right] &= \frac{\partial\left(E\left[e^{-\alpha' R^e}\right]\right)}{\partial \alpha_k} \\ &= \exp(-\mu'\alpha + \frac{1}{2}\alpha'\sigma\alpha)\left(\mu_k - \sum_{i=1}^{n}\sigma_{ki}\alpha_i\right). \end{aligned} \tag{4.2}$$

4.1 Constant risk tolerance

4.1.1 Constant risk tolerance with $n = 1$ risky asset and m utilities

Suppose that the investor's risk preference is constant so that $\beta_j = \beta$ for all j. Then, (2.4) becomes

$$\sum_{j=1}^{m} \frac{P_j C_j}{\beta} \exp\left(-\frac{r_f x}{\beta}\right) E\left[R^e \exp\left(-\frac{R^e u}{\beta}\right)\right] = 0$$

or

$$\exp\left(-\frac{r_f x}{\beta}\right) E\left[R^e \exp\left(-\frac{R^e u}{\beta}\right)\right] \sum_{j=1}^{m} \frac{P_j C_j}{\beta} = 0. \tag{4.3}$$

Since all the terms in (4.3) are positive; by using (4.1) one would have

$$E\left[R^e \exp\left(-\frac{R^e u}{\beta}\right)\right] = 0$$

or

$$\exp\left(\frac{u^2\sigma^2 - 2u\mu\beta}{2\beta^2}\right)\left(\mu - \frac{u}{\beta}\sigma^2\right) = 0$$

and this implies that the optimal amount of money invested in the risky asset is

$$u^* = \left(\frac{\mu}{\sigma^2}\right)\beta. \tag{4.4}$$

It can be observed from (4.4) that the optimal amount of money invested in the risky asset depends only on the mean and variance of the excess return of the risky asset and the risk tolerance of the investor. The decision on buying or short selling of the asset relies on the sign of the excess return of the asset. If the mean excess return of the asset is positive, then the investor is advised to buy the asset and short sell it otherwise. Similarly, if the variance of the excess return of the risky asset increases, indicating that the risk level of the risky asset increases, then the absolute value of the optimal decision decreases. This also indicates that if the variance of the risky asset increases, then the investor is advised to take less risk by buying or short selling less of the risky asset. Moreover, it can be inferred from (4.4) that the amount of money invested in the risky asset is independent of wealth indicating that the investor is memoryless in wealth. Note that the result is consistent with the result found in Çanakoğlu and Özekici [7].

4.1.2 *Constant risk tolerance with n risky assets and m utilities*

In this setting, we assume that investor's risk preference is still constant so that $\beta_j = \beta$ for all j. Moreover, there are n different risky assets. The excess returns of the risky assets are assumed to be driven by a multivariate normal distribution with a return vector $\mu' = [\mu_1, \mu_2, \cdots, \mu_n]$ and a covariance matrix σ.

Then, (2.4) is

$$\sum_{j=1}^{m} \frac{P_j C_j}{\beta} \exp\left(-\frac{r_f x}{\beta}\right) E\left[R_k^e \exp\left(-\frac{(R^e)' u}{\beta}\right)\right] = 0 \tag{4.5}$$

or

$$\exp\left(-\frac{r_f x}{\beta}\right) E\left[R_k^e \exp\left(-\frac{(R^e)' u}{\beta}\right)\right] \sum_{j=1}^{m} \frac{P_j C_j}{\beta} = 0 \tag{4.6}$$

for all k. Similarly, since all the terms in (4.6) are positive; by using (4.2) one would have

$$E\left[R^e \exp\left(-\frac{(R^e)' u}{\beta}\right)\right] = 0$$

or

$$\exp(-\frac{1}{\beta}\mu' u + \frac{1}{2\beta^2} u' \sigma u) \left(\mu_k - \frac{1}{\beta} \sum_{i=1}^{n} \sigma_{ki} u_i\right) = 0$$

for all k. This implies that the optimal amount of money invested in the risky assets is given by the solution of the system of linear equation

$$\sigma u = \beta \mu. \tag{4.7}$$

Since σ is a positive definite matrix, the system of linear equations in (4.7) has a unique solution explicitly given by

$$u^* = \beta \sigma^{-1} \mu.$$

Moreover, again by (4.7), it can be concluded that the optimal decision is independent of the wealth level of the investor. This result is harmonious with results found in Çanakoğlu and Özekici [7].

4.2 Random risk tolerance

4.2.1 Random risk tolerance with $n = 1$ risky asset and $m = 2$ utilities

In this section, suppose that the risk preference is random with two risk tolerances β_1 and β_2. The investor needs to form a portfolio using one risk-free and one risky asset. This section will be divided into two subsections with the mean excess return of the risky asset being less or greater than zero.

As a side remark, first consider the case when mean excess return $\mu = 0$. Then, (2.4) using (4.1) is

$$\sum_{j=1}^{2} \frac{P_j C_j}{\beta_j} \exp(-\frac{r_f x}{\beta_j}) \left[\exp(\frac{u^2\sigma^2}{2\beta_j^2}) \left(\frac{u}{\beta_j}\sigma^2 \right) \right] = 0$$

and the equality holds only if $u^* = 0$. This indicates that when there is only one risky asset having a mean excess return equal to zero, the investor buys only the risk-free asset which is quite reasonable.

Mean excess return greater than zero ($\mu > 0$).

Assume $\mu > 0$ so that $\bar{r} > r_f$ which indicates that the expected return of the risky asset is greater than the return of the risk-free asset. In this situation, it is reasonable to expect that the optimal amount of money invested in the risky asset would be greater than zero. To show this, further assume without loss of generality that $\beta_1 < \beta_2$, then the optimality condition in (2.4) becomes

$$\frac{P_1 C_1}{\beta_1} \exp\left(\frac{u^2\sigma^2 - u\mu\beta_1}{2\beta_1^2} - \frac{r_f x}{\beta_1} \right) \left(\mu - \frac{u}{\beta_1}\sigma^2 \right) = -\frac{P_2 C_2}{\beta_2} \exp\left(\frac{u^2\sigma^2 - u\mu\beta_2}{2\beta_2^2} - \frac{r_f x}{\beta_2} \right) \cdot \left(\mu - \frac{u}{\beta_2}\sigma^2 \right)$$

so that

$$\frac{P_1 C_1 \beta_2}{P_2 C_2 \beta_1} \exp\left(r_f x \frac{\beta_1 - \beta_2}{\beta_1 \beta_2} \right) = -\exp\left(u^2\sigma^2 \frac{\beta_1^2 - \beta_2^2}{2\beta_1^2\beta_2^2} + u\mu \frac{\beta_2 - \beta_1}{\beta_1\beta_2} \right) \frac{\left(\mu - \frac{u}{\beta_2}\sigma^2 \right)}{\left(\mu - \frac{u}{\beta_1}\sigma^2 \right)}. \quad (4.8)$$

Then, this equation reduces to

$$g(x) = h(u) \quad (4.9)$$

where

$$g(x) = \frac{P_1 C_1}{P_2 C_2} \frac{\beta_2}{\beta_1} \exp\left(r_f x \frac{\beta_1 - \beta_2}{\beta_1 \beta_2} \right) \quad (4.10)$$

and

$$h(u) = -\exp\left(u^2\sigma^2(\frac{1}{2\beta_2^2} - \frac{1}{2\beta_1^2}) + \left(\frac{1}{\beta_1} - \frac{1}{\beta_2}\right)\mu u\right)\frac{\left(\mu - \frac{u}{\beta_2}\sigma^2\right)}{\left(\mu - \frac{u}{\beta_1}\sigma^2\right)}.$$

$$g(0) = \frac{P_1C_1\beta_2}{P_2C_2\beta_1} \geq 0$$

and

$$\lim_{x\to\infty} g(x) = 0.$$

For $h(u)$, divide it into two parts so that

$$h(u) = h_1(u)h_2(u)$$

where

$$h_1(u) = -\exp\left(u\left(\beta_1 - \beta_2\right)\frac{u\sigma^2\left(\beta_1 + \beta_2\right) - 2\mu\beta_1\beta_2}{2\beta_1^2\beta_2^2}\right)$$

and

$$h_2(u) = \frac{\left(\mu - \frac{u}{\beta_2}\sigma^2\right)}{\left(\mu - \frac{u}{\beta_1}\sigma^2\right)}.$$

Considering $h_1(u)$, it can be observed that $h_1(u) < 0$ for all u. So, for (4.9) to hold, one would need $h_2(u) < 0$ or

$$\frac{\left(\mu - \frac{u}{\beta_2}\sigma^2\right)}{\left(\mu - \frac{u}{\beta_1}\sigma^2\right)} < 0.$$

In order to have this inequality to hold, there are two possible cases:

$$\begin{aligned} 1.\ \ \mu - \frac{u}{\beta_2}\sigma^2 &< 0 \text{ and } \mu - \frac{u}{\beta_1}\sigma^2 > 0. \\ 2.\ \ \mu - \frac{u}{\beta_2}\sigma^2 &> 0 \text{ and } \mu - \frac{u}{\beta_1}\sigma^2 < 0. \end{aligned} \tag{4.11}$$

The first case implies $\mu < \left(\sigma^2/\beta_2\right)u$ and $\mu > \left(\sigma^2/\beta_1\right)u$, or $\left(\mu/\sigma^2\right)\beta_2 < u < \left(\mu/\sigma^2\right)\beta_1$. The assumption $\beta_1 < \beta_2$ therefore implies that the first case is not possible. For the second case, $\mu > \left(\sigma^2/\beta_2\right)u$ and $\mu < \left(\sigma^2/\beta_1\right)u$, or $\left(\mu/\sigma^2\right)\beta_1 < u < \left(\mu/\sigma^2\right)\beta_2$. Denote this region by $I = \left(\mu/\sigma^2\right)[\beta_1, \beta_2] \subset [0, +\infty)$. Note that $h(\mu\beta_1/\sigma^2) = +\infty$ and $h(\mu\beta_2/\sigma^2) = 0$. Then, it can can be observed that $g(x) = h(u)$ has a solution $u \in I$ for any x.

For the analysis of $g(x)$ and $h(u)$, $g(x) = h(u)$ implies that the optimal decision is $u^*(x) = h^{-1}(g(x))$. For that, first consider

$$\begin{aligned} h'(u) &= \frac{d\left(-\exp\left(u^2\sigma^2(\frac{1}{2\beta_2^2}-\frac{1}{2\beta_1^2})+\left(\frac{1}{\beta_1}-\frac{1}{\beta_2}\right)\mu u\right)\frac{\left(\mu-\frac{u}{\beta_2}\sigma^2\right)}{\left(\mu-\frac{u}{\beta_1}\sigma^2\right)}\right)}{du} \\ &= \frac{1}{\beta_1\beta_2^3}\exp\left(\frac{u^2\sigma^2}{2}(\frac{1}{\beta_2^2}-\frac{1}{\beta_1^2})+u\mu(\frac{1}{\beta_1}+\frac{1}{\beta_2})\right)\frac{\beta_2-\beta_1}{(\mu\beta_1-u\sigma^2)^2}k(u) \end{aligned} \tag{4.12}$$

where

$$k(u) = (\beta_1+\beta_2)\,\sigma^6u^3 - \mu\sigma^4(\beta_1^2+3\beta_1\beta_2+\beta_2^2)u^2 + 2\mu^2\sigma^2\beta_1\beta_2(\beta_1+\beta_2)u - \beta_1^2\beta_2^2\mu(\mu^2+\sigma^2) \tag{4.13}$$

so that

$$k'(u) = 3\,(\beta_1+\beta_2)\,\sigma^6u^2 - 2\mu\sigma^4(\beta_1^2+3\beta_1\beta_2+\beta_2^2)u + 2\mu^2\sigma^2\beta_1\beta_2(\beta_1+\beta_2).$$

Theorem 3 *The optimal decision $u^*(x) = h^{-1}(g(x))$ is bounded and positively increasing in x if $\mu > 0$.*

Proof. *First note that $u^*(x) \in \left(\mu/\sigma^2\right)[\beta_1, \beta_2] \subset [0, +\infty)$ implies that u^* is bounded and positive. Since $g(x)$ is convex decreasing, it suffices to show that $h(u)$ decreases in I because if $h(u)$ decreases and $g(x)$ decreases then $h^{-1}(g(x))$ increases. For that, note that*

$$\frac{1}{\beta_1\beta_2^3}\exp\left(\frac{u^2\sigma^2}{2}(\frac{1}{\beta_2^2}-\frac{1}{\beta_1^2})+u\mu(\frac{1}{\beta_1}+\frac{1}{\beta_2})\right)\frac{\beta_2-\beta_1}{(\mu\beta_1-u\sigma^2)^2} > 0$$

for all u in I. Then, to show that $h(u)$ decreases in I, it suffices to prove that $k(u)$ is negative in I. It is clear that $k(u)$ is a polynomial of third degree. With basic calculus, it can be observed that a third degree polynomial can have at most two turning points. Here, turning points indicate that the polynomial $k(u)$ changes from increasing to decreasing or decreasing to increasing. This follows from the fact that $k'(u)$ is a polynomial of degree two and $k'(u)$ can have at most two roots. Observe the following facts:

1. $k(\left(\mu/\sigma^2\right)\beta_1) = k(\left(\mu/\sigma^2\right)\beta_2) = -\beta_1^2\beta_2^2\mu\sigma^2 < 0.$

2. $k'(\left(\mu/\sigma^2\right)\beta_1) = \beta_1^2\mu^2\sigma^2(\beta_1-\beta_2) < 0$, $k'(\left(\mu/\sigma^2\right)\beta_2) = \beta_2^2\mu^2\sigma^2(\beta_2-\beta_1) > 0$ *and* $k'(0) = 2\mu^2\sigma^2\beta_1\beta_2(\beta_1+\beta_2) > 0.$

The intermediate value theorem indicates that if a function f is continuous on a interval $I = [a, b]$ and $f(a) < 0$ and $f(b) > 0$, then there exits c in I such that $f(c) = 0$. So, considering this theorem and observation two, it can be concluded that there is a turning point in $[0, (\mu/\sigma^2)\,\beta_1]$ and another one in I. Since $k(u)$ is a third degree polynomial these are the only possible turning points.

Finally the first observation shows that the value of k at the end points are negative. Also, from the above argument, there is only one turning point $\bar{u}$ in I at which $k(\bar{u}) < k((\mu/\sigma^2)\beta_1)$ since $k'((\mu/\sigma^2)\beta_1) < 0$. This indicates that for every u in I, $k(u)$ can not be greater than $k((\mu/\sigma^2)\,\beta_1) = k((\mu/\sigma^2)\,\beta_2) < 0$, hence $k(u)$ is negative for all u in R. ■

By Theorem 3, $k(u)$ is negative implies that $h(u)$ is decreasing in the feasible region I. Therefore, one can conclude that as x increases, $u(x)$ must increase as well. This fact together with the assumption $\mu = \bar{r} - r_f > 0$ basically shows that if the investor has stochastic exponential utility then the money invested in the risky asset, which has a better return than the risk-free asset, has a direct relationship with the wealth level of the investor. First, $u > 0$ for all x implies that there is always money invested in the risky asset when the risky asset's return is better than the risk-free one. Second, Theorem 1 implies that as the wealth level of the investor increases the amount of money that is invested in risky asset increases.

This result is important because in previous work by Çanakoğlu and Özekici [7], it is known that when the investor has exponential utility, the optimal decision is independent of the wealth level of the investor. But, if the investor has stochastic exponential utility, then the money invested on the risky asset depends on the wealth level of the investor.

Another important thing to note is $u(x)$ depends on the risk tolerances $\{\beta_1, \beta_2\}$ of the investor. Since the feasible region I depends on $\{\beta_1, \beta_2\}$, the investor having a greater risk tolerance would invest more money in the risky asset.

Mean excess return less than zero ($\mu < 0$).

Next assume $\mu < 0$ so that $\bar{r} < r_f$. Considering the optimality condition again one needs (4.9) to hold. By a similar argument there are two possibilities like in (4.11).

Note that now only the first region is possible since $\mu < 0$. Hence, the feasible region is $I = (\mu/\sigma^2)\,[\beta_2, \beta_1] \subset (-\infty, 0]$. Observe that $u \in I$ is always negative indicating that when the mean of the asset is less than zero, the investor short sells the risky asset and invests in the risk-free asset. According to intuition, one can conjecture that as x increases u must decrease meaning that the investor would invest more money in the risk-free asset and short sells more of the risky asset when the investors wealth level increases by a similar argument above. Therefore, this time we will show that $h(u)$ is increasing on I. Similarly, considering (4.12), in order to show that $h(u)$ is increasing on I, it suffices to show that $k(u)$ in (4.13) is positive on I.

Theorem 4 *The optimal decision $u^*(x) = h^{-1}(g(x))$ is bounded and negatively decreasing in x if $\mu < 0$.*

Proof. *The theorem will be proven using a similar argument. The negativeness and boundedness of u^* follows trivially since $u^*(x) \in I = \left(\mu/\sigma^2\right)[\beta_2, \beta_1] \subset (-\infty, 0]$. Observe now that*

1. $k(\left(\mu/\sigma^2\right)\beta_2) = k(\left(\mu/\sigma^2\right)\beta_1) = -\beta_1^2\beta_2^2\mu\sigma^2 > 0.$

2. $k'(\left(\mu/\sigma^2\right)\beta_2) = \beta_2^2\mu^2\sigma^2(\beta_2 - \beta_1) > 0$, $k'(\left(\mu/\sigma^2\right)\beta_1) = \beta_1^2\mu^2\sigma^2(\beta_1 - \beta_2) < 0$ *and* $k'(0) = 2\mu^2\sigma^2\beta_1\beta_2(\beta_1 + \beta_2) > 0.$

Again by intermediate value theorem and observation two it can be concluded that there are two turning points; one in I and the other in $[\left(\mu/\sigma^2\right)\beta_1, 0]$. By the first observation, $k(u)$ is positive at the end points of I; hence, $k(u)$ is greater than $k(\left(\mu/\sigma^2\right)\beta_2) = k(\left(\mu/\sigma^2\right)\beta_1) > 0$ for all u in I. Therefore, $k(u)$ is positive in I. ■

By Theorem 2, $k(u)$ is positive implies that $h(u)$ is increasing on the feasible region I. Therefore one can conclude that as x increases, u must decrease. Similarly this fact together with the assumption $\mu = \bar{r} - r_f < 0$ shows that when the risky asset's expected return is lower than the risk-free asset, the investor should short sell the risky asset and invest more in the risk-free asset. Also, the fact that as x increases $u(x)$ must decrease shows that as the wealth level of the investor increases, he short sells more of the risky asset and invest more in the risk-free asset.

Moreover, note also that like in the case of $\mu > 0$, the amount of money invested in risky asset depends on the risk tolerance of investor. This indicates that an investor who is more tolerant to risk would short sell more of the risky asset and invest on the risk-free asset.

4.2.2 Random risk tolerance with n risky assets and $m = 2$ utilities

In this subsection, there are n different risky assets with a multivariate normal distribution. The optimality condition for a market consisting of two different states is

$$\frac{\partial g(x,u)}{\partial u_k} = \sum_{j=1}^{2} \frac{P_j C_j}{\beta_j} \exp\left(-\frac{r_f x}{\beta_j}\right) E\left[R_k^e \exp\left(-\frac{(R^e)' u}{\beta_j}\right)\right] = 0 \tag{4.14}$$

for all k.

Using (4.2), the expectation term in (4.14) is

$$E\left[R_k^e \exp\left(-\frac{(R^e)' u}{\beta_j}\right)\right] = \exp(-\frac{1}{\beta_j}\mu' u + \frac{1}{2\beta_j^2} u'\sigma u) \left(\mu_k - \frac{1}{\beta_j} (\sigma u)_k\right). \tag{4.15}$$

Therefore, (4.14) becomes

$$\sum_{j=1}^{2} \frac{P_j C_j}{\beta_j} \exp\left(-\frac{r_f x}{\beta_j}\right) \left[\exp(-\frac{1}{\beta_j}\mu' u + \frac{1}{2\beta_j^2} u'\sigma u) \left(\mu_k - \frac{1}{\beta_j} (\sigma u)_k\right)\right] = 0. \tag{4.16}$$

First consider the case when $\mu = [0, 0, \cdots, 0]$. Then, (4.16) becomes

$$(\sigma u)_k \sum_{j=1}^{2} \frac{P_j C_j}{\beta_j^2} \exp\left(-\frac{r_f x}{\beta_j}\right) \exp(\frac{1}{2\beta_j^2} u'\sigma u) = 0 \tag{4.17}$$

for all k. Note that (4.17) is equal to zero only when $(\sigma u)_k = 0$. Going one step further if $\mu = [0, 0, \cdots, 0]$ then $(\sigma u)_k = 0$ for all k. Therefore, since σ is a positive definite matrix the only solution of $\sigma u = 0$ is $u = 0$. Summarizing, if $\mu = [0, 0, \cdots, 0]$ then $u = 0$. Hence, suppose that $\mu \neq [0, 0, \cdots, 0]$ in the remainder of the section. This further implies that $u^* \neq 0$ because $\mu = 0$ whenever $u = 0$ in (4.16).

Now, (4.14) implies

$$\begin{aligned} &\frac{P_1 C_1}{\beta_1} \left[\exp(-\frac{r_f x}{\beta_1} - \frac{1}{\beta_1}\mu' u + \frac{1}{2\beta_1^2} u'\sigma u) \left(\mu_k - \frac{(\sigma u)_k}{\beta_1}\right)\right] \\ &\quad + \frac{P_2 C_2}{\beta_2} \left[\exp(-\frac{r_f x}{\beta_2} - \frac{1}{\beta_2}\mu' u + \frac{1}{2\beta_2^2} u'\sigma u) \left(\mu_k - \frac{(\sigma u)_k}{\beta_2}\right)\right] = 0. \end{aligned}$$

By separating two equations, one would get

$$\frac{P_1 C_1}{P_2 C_2} \frac{\beta_2}{\beta_1} \exp\left(r_f x \frac{\beta_1 - \beta_2}{\beta_1 \beta_2}\right) = -\exp\left(\left(\frac{1}{\beta_1} - \frac{1}{\beta_2}\right) \mu' u + \frac{\beta_1^2 - \beta_2^2}{2\beta_2^2 \beta_1^2} u'\sigma u\right) \frac{\left(\mu_k - \frac{1}{\beta_2} (\sigma u)_k\right)}{\left(\mu_k - \frac{1}{\beta_1} (\sigma u)_k\right)}. \tag{4.18}$$

Note that (4.18) can be rewritten as

$$g(x) = h_1(u) h_2^k(u) \tag{4.19}$$

where $g(x)$ is given by (4.10) and

$$\begin{aligned} h_1(u) &= -\exp\left(\left(\frac{1}{\beta_1}-\frac{1}{\beta_2}\right)\mu' u + \left(\frac{1}{2\beta_2^2}-\frac{1}{2\beta_1^2}\right)u'\sigma u\right) \\ h_2^k(u) &= \frac{\left(\mu_k - \frac{1}{\beta_2}(\sigma u)_k\right)}{\left(\mu_k - \frac{1}{\beta_1}(\sigma u)_k\right)} \end{aligned}$$

for $k = 1, 2, \cdots, n$.

Notice that in (4.18), left hand side of the equation does not depend on k. Hence, it can be concluded that

$$\frac{\left(\mu_j - \frac{1}{\beta_2}(\sigma u)_j\right)}{\left(\mu_j - \frac{1}{\beta_1}(\sigma u)_j\right)} = \frac{\left(\mu_k - \frac{1}{\beta_2}(\sigma u)_k\right)}{\left(\mu_k - \frac{1}{\beta_1}(\sigma u)_k\right)} \tag{4.20}$$

for all j, k and $\mu \neq [0, 0, \cdots, 0]$.

Moreover, (4.20) implies that

$$\mu_k (\sigma u)_j = \mu_j (\sigma u)_k \tag{4.21}$$

for all j, k. Now, define

$$\bar{\mu}_j = \left(\sum_{t=1}^{n} \mu_t (-1)^{j+t} M_{jt}\right) \tag{4.22}$$

where M_{ij} denotes the (i, j) minor of the covariance matrix σ for all i, j. For technical reasons, we assume that $\bar{\mu}_j u_j \neq 0$ for all j. We will show that this assumption is reasonable by validating it for $n = 2$. The difficulty comes from the fact that it is hard to find a relationship between $\bar{\mu}_j$ and u_j. For the case when $n = 2$, assume that $\bar{\mu}_j u_j = 0$ for $j = 1, 2$. Then, for $j = 1$,

$$\begin{aligned} \bar{\mu}_1 u_1 &= \sum_{t=1}^{2} \mu_t (-1)^{1+t} M_{1t} u_1 \\ &= (\mu_1 \sigma_{22} - \mu_2 \sigma_{12})\, u_1 = 0 \end{aligned} \tag{4.23}$$

and similarly

$$\bar{\mu}_2 u_2 = (\mu_1 \sigma_{12} - \mu_2 \sigma_{11})\, u_2 = 0 \tag{4.24}$$

for $j = 2$.

There are three possible cases to consider: either $(u_1 = 0,\, u_2 \neq 0)$, or $(u_1 \neq 0,\, u_2 = 0)$, or $(u_1,\, u_2 \neq 0)$. If $u_1 = 0$, then $u_2 \neq 0$ since the optimal decision is different than the zero vector. This further implies that

$$\mu_1 \sigma_{12} = \mu_2 \sigma_{11} \tag{4.25}$$

from (4.24). Now, taking $k = 1$ and $j = 2$, (4.21) reduces to

$$\mu_1(\sigma_{21}u_1 + \sigma_{22}u_2) = \mu_2(\sigma_{11}u_1 + \sigma_{12}u_2)$$

or

$$\mu_1\sigma_{22}u_2 = \mu_2\sigma_{12}u_2$$

since $\mu_1\sigma_{12} = \mu_1\sigma_{21} = \mu_2\sigma_{11}$. This finally leads to

$$\mu_1\sigma_{22} = \mu_2\sigma_{12}. \tag{4.26}$$

Note that (4.25) and (4.26) together imply that $\mu_1,\ \mu_2 \neq 0$ since $\mu_1 = 0$ implies $\mu_2 = 0$ from (4.25), and $\mu_2 = 0$ implies $\mu_1 = 0$ from (4.26) since $\sigma_{11},\ \sigma_{22} > 0$. Now, (4.25) and (4.26) lead to $\sigma_{12}^2 = \sigma_{11}\sigma_{22}$ which contradicts the fact that σ is positive definite. A similar argument leads to the same conclusion when $u_2 = 0$. The final case with $u_1,\ u_2 \neq 0$ leads to (4.25) and (4.26) from (4.23) and (4.24) trivially and we reach the same contradiction.

Theorem 5 *The relation between the optimal investments in any asset j and k is*

$$\bar{\mu}_k u_j = \bar{\mu}_j u_k \tag{4.27}$$

for all j and k.

Proof. *We suppose that $\bar{\mu}_j u_j \neq 0$ for all j. Consider the following identity which will be useful throughout the proof:*

$$\sum_{t=1}^{n} \mu_t \sum_{j=1}^{n} \sigma_{kj}(-1)^{j+t} M_{jt} = \mu_k \left\|\sigma\right\| \tag{4.28}$$

for any k. In order to prove (4.28), first let $\bar{\sigma}(l,m)$ denote the matrix formed by first removing lth column and then replacing it with mth column of the matrix σ. Also, it is known from basic linear algebra that $\|\sigma\| = \sum_{j=1}^{n}\sigma_{jk}(-1)^{j+k}M_{jk}$ for any k.

Consider any asset k for the which $\bar{\mu}_k \neq 0$. It clearly exists by our assumption. Rewriting the left-hand side of (4.28) we get

$$\begin{aligned}\sum_{t=1}^{n} \mu_t \sum_{j=1}^{n} \sigma_{kj}(-1)^{j+t} M_{jt} &= \sum_{\substack{t=1\\t\neq k}}^{n} \mu_t \sum_{j=1}^{n} \sigma_{kj}(-1)^{j+t} M_{jt} + \mu_k \sum_{j=1}^{n} \sigma_{kj}(-1)^{j+k} M_{jk} \\ &= \sum_{\substack{t=1\\t\neq k}}^{n} \mu_t \sum_{j=1}^{n} \sigma_{jk}(-1)^{j+t} M_{jt} + \mu_k \sum_{j=1}^{n} \sigma_{jk}(-1)^{j+k} M_{jk} \end{aligned} \tag{4.29}$$

by the symmetry of σ. *Next, consider* $\sum_{j=1}^{n} \sigma_{jk}(-1)^{j+t} M_{jt}$ *in (4.29) when* $t \neq k$. *Note also that*

$$\det \bar{\sigma}(t,k) = \sum_{j=1}^{n} \bar{\sigma}_{jl}(t,k)(-1)^{j+l} \bar{M}_{jl}(t,k) \tag{4.30}$$

for any l *where* $\bar{M}_{jl}(t,k)$ *is the* (j,l) *minor of* $\bar{\sigma}(t,k)$. *For* $l = t \neq k$, *note that* $\bar{M}_{jl}(t,k) = M_{jt}$ *and* $\bar{\sigma}_{jl}(t,k) = \sigma_{jk}$. *Therefore, (4.30) is equal to* $\sum_{j=1}^{n} \sigma_{jk}(-1)^{j+t} M_{jt}$ *in (4.29). Moreover,* $\det \bar{\sigma}(t,k) = 0$ *since the* k*th column and* t*th column of* $\bar{\sigma}(t,k)$ *are the same.*

Therefore the first sum on the right hand side of (4.29) vanishes and we obtain

$$\begin{aligned} \sum_{t=1}^{n} \mu_t \sum_{j=1}^{n} \sigma_{kj}(-1)^{j+t} M_{jt} &= \mu_k \sum_{j=1}^{n} \sigma_{jk}(-1)^{j+k} M_{jk} \\ &= \mu_k \left\| \sigma \right\|. \end{aligned} \tag{4.31}$$

Now, (4.21) can be written as

$$\sum_{t=1}^{n} \left(\mu_k \sigma_{jt} - \mu_j \sigma_{kt} \right) u_t = 0 \tag{4.32}$$

for all j. *We can rewrite (4.32) as the system of linear equations* $C(k)u = 0$, *where we define the matrix* $C(k)$ *for a fixed* k *as*

$$C_{jt}(k) = \mu_k \sigma_{jt} - \mu_j \sigma_{kt}.$$

To complete the proof, we need to show that (4.27) solves the system of linear equations $C(k)u = 0$. *Since* $\bar{\mu}_k \neq 0$ *and* $u_k \neq 0$, *plugging (4.27) into* $C(k)u$, *together with (4.31), gives the* j*th entry*

$$\begin{aligned} (C(k)u)_j &= \sum_{i=1}^{n} \left(\mu_k \sigma_{ji} - \mu_j \sigma_{ki} \right) \frac{\bar{\mu}_i}{\bar{\mu}_k} u_k \\ &= \frac{u_k}{\bar{\mu}_k} \sum_{i=1}^{n} \left(\mu_k \sigma_{ji} - \mu_j \sigma_{ki} \right) \bar{\mu}_i \\ &= \frac{u_k}{\bar{\mu}_k} \sum_{t=1}^{n} \mu_t \sum_{i=1}^{n} M_{it}(-1)^{i+t} \left(\mu_k \sigma_{ji} - \mu_j \sigma_{ki} \right) \\ &= \frac{u_k}{\bar{\mu}_k} \left(\mu_k \sum_{t=1}^{n} \mu_t \sum_{i=1}^{n} \sigma_{ji}(-1)^{i+t} M_{it} - \mu_j \sum_{t=1}^{n} \mu_t \sum_{i=1}^{n} \sigma_{ki}(-1)^{i+t} M_{it}) \right) \\ &= \frac{u_k}{\bar{\mu}_k} \left(\mu_k \mu_j \left\| \sigma \right\| - \mu_j \mu_k \left\| \sigma \right\| \right) \\ &= 0 \end{aligned}$$

for all j. *Hence, (4.27) is a solution to the system of linear equations* $C(k)u = 0$.

It was shown in Chapter 3 that the objective function $g(x,u)$ is strictly concave in u for any distribution. It is also known that for strictly concave function the maximizer is unique if it exists. Since the solution obtained from (4.27) satisfies the first order condition (2.4), then it can be concluded that it is the unique solution to the problem because of the uniqueness of the maximizer. ■

On the other hand, left-hand side of (4.18) is always positive. Therefore, in order to have the equality in (4.19)

$$\frac{\left(\mu_k - \frac{1}{\beta_2}(\sigma u)_k\right)}{\left(\mu_k - \frac{1}{\beta_1}(\sigma u)_k\right)} < 0$$

must hold. To obtain this, either

$$(\mu_k - (1/\beta_2)(\sigma u)_k) < 0 \text{ and } (\mu_k - (1/\beta_1)(\sigma u)_k) > 0 \tag{4.33}$$

or

$$(\mu_k - (1/\beta_2)(\sigma u)_k) > 0 \text{ and } (\mu_k - (1/\beta_1)(\sigma u)_k) < 0. \tag{4.34}$$

Considering (4.33), one would get $(\sigma u)_k > \beta_2\mu_k$ and $(\sigma u)_k < \beta_1\mu_k$ implying

$$\beta_2\mu_k < (\sigma u)_k < \beta_1\mu_k. \tag{4.35}$$

Similarly, (4.34) implies

$$\beta_1\mu_k < (\sigma u)_k < \beta_2\mu_k. \tag{4.36}$$

Next, consider (4.27) to determine the feasible region for u_k. Replacing u_j with $\bar{\mu}_j u_k/\bar{\mu}_k$ in $(\sigma u)_k$, we obtain

$$\begin{aligned}
(\sigma u)_k &= \sum_{j=1}^{n} \sigma_{kj} u_j \\
&= \sum_{j=1}^{n} \sigma_{kj} \frac{\bar{\mu}_j}{\bar{\mu}_k} u_k \\
&= \frac{u_k}{\bar{\mu}_k} \sum_{j=1}^{n} \sigma_{kj} \bar{\mu}_j \\
&= \frac{u_k}{\bar{\mu}_k} \sum_{t=1}^{n} \mu_t \sum_{j=1}^{n} \sigma_{kj} (-1)^{j+t} M_{jt} \\
&= \frac{\mu_k \|\sigma\|}{\bar{\mu}_k} u_k
\end{aligned} \tag{4.37}$$

by (4.28). Substituting $(\sigma u)_k$ given by (4.37) in (4.35) and (4.36), we obtain the feasible region

$$\beta_1 < \frac{\|\sigma\|}{\bar{\mu}_k} u_k < \beta_2$$

which implies

$$l_1 = \frac{\bar{\mu}_k}{\|\sigma\|}\beta_1 < u_k < \frac{\bar{\mu}_k}{\|\sigma\|}\beta_2 = l_2 \tag{4.38}$$

when $\bar{\mu}_k > 0$; or

$$\beta_2 < \frac{\|\sigma\|}{\bar{\mu}_k} u_k < \beta_1$$

which now implies

$$l_2 = \frac{\bar{\mu}_k}{\|\sigma\|}\beta_2 < u_k < \frac{\bar{\mu}_k}{\|\sigma\|}\beta_1 = l_1 \tag{4.39}$$

when $\bar{\mu}_k < 0$.

It can be inferred from (4.38) and (4.39) that since $\|\sigma\|$ is always positive, the feasible region is determined by the value of $\bar{\mu}_k = \sum_{t=1}^{n} \mu_t (-1)^{k+t} M_{kt}$. If $\bar{\mu}_k$ is positive, then the feasible region for the kth asset is given by (4.38); otherwise, it is given by (4.39). Denote these regions as $I_1 = (l_1, l_2) \subset (0, +\infty)$ and $I_2 = (l_2, l_1) \subset (-\infty, 0)$ respectively.

Moreover, $(\sigma u)_k$ can be replaced with $(\mu_k \|\sigma\| / \bar{\mu}_k)\, u_k$ in (4.18) to get

$$\frac{P_1 C_1}{P_2 C_2}\frac{\beta_2}{\beta_1}\exp\left(r_f x \frac{\beta_1 - \beta_2}{\beta_1 \beta_2}\right) = -h_k(u_k)$$

where

$$h_k(u_k) = \exp\left(\frac{1}{\bar{\mu}_k}\left(\frac{1}{\beta_1} - \frac{1}{\beta_2}\right)\mu' \bar{\mu} u_k + \frac{1}{\bar{\mu}_k^2}\left(\frac{1}{2\beta_2^2} - \frac{1}{2\beta_1^2}\right)\bar{\mu}'\sigma\bar{\mu} u_k^2\right)\frac{\left(\mu_k - \frac{1}{\beta_2}\left(\mu_k \|\sigma\| / \bar{\mu}_k\right) u_k\right)}{\left(\mu_k - \frac{1}{\beta_1}\left(\mu_k \|\sigma\| / \bar{\mu}_k\right) u_k\right)}$$

by using (4.27) and (4.37). Like in the case of one risky asset, consider the derivative

$$h_k'(u_k) = \frac{dh_k(u_k)}{du_k} = \frac{\left(\frac{1}{\beta_1} - \frac{1}{\beta_2}\right)\exp\left(\frac{u_k}{\bar{\mu}_k}\left(\frac{1}{\beta_1} - \frac{1}{\beta_2}\right)\mu'\bar{\mu} + \frac{u_k^2}{\bar{\mu}_k^2}\left(\frac{1}{2\beta_2^2} - \frac{1}{2\beta_1^2}\right)\bar{\mu}'\sigma\bar{\mu}\right)}{\left(\mu_k - \frac{1}{\beta_1}\left(\mu_k \|\sigma\| / \bar{\mu}_k\right) u_k\right)^2} k(u_k)$$

where

$$k(u_k) = \left(\frac{\mu_k \|\sigma\|}{\bar{\mu}_k \beta_2} u_k - \mu_k\right)\left(\frac{u_k}{\bar{\mu}_k^2}\left(\frac{1}{\beta_2} + \frac{1}{\beta_1}\right)\bar{\mu}'\sigma\bar{\mu} + \frac{1}{\bar{\mu}_k}\mu'\bar{\mu}\right)\left(\frac{\mu_k \|\sigma\|}{\bar{\mu}_k \beta_1} u_k - \mu_k\right) - \left(\mu_k^2 \|\sigma\| / \bar{\mu}_k\right).$$

Note that the sign of $h'_k(u_k)$ depends on the sign of $k(u_k)$. If $k(u_k)$ is positive, then $h'_k(u_k)$ is negative and it is positive otherwise. Moreover, note also that

$$\begin{aligned} k(l_1) &= k(l_2) = -\frac{\mu_k^2 \|\sigma\|}{\bar{\mu}_k} \\ k(0) &= \frac{\mu_k^2}{\bar{\mu}_k}(\mu'\bar{\mu} - \|\sigma\|) = \frac{\mu_k^2}{\bar{\mu}_k}\mu'\bar{\mu} + k(l_1). \end{aligned}$$

The derivative of k is

$$\begin{aligned} k'(u_k) &= \frac{\mu_k \|\sigma\|}{\bar{\mu}_k \beta_2}\left(\frac{1}{\bar{\mu}_k^2}\left(\frac{1}{\beta_2} + \frac{1}{\beta_1}\right)\bar{\mu}'\sigma\bar{\mu}u_k + \frac{1}{\bar{\mu}_k}\mu'\bar{\mu}\right)\left(\frac{\mu_k \|\sigma\|}{\bar{\mu}_k \beta_1}u_k - \mu_k\right) \\ &\quad + \frac{1}{\bar{\mu}_k^2}\left(\frac{1}{\beta_2} + \frac{1}{\beta_1}\right)\bar{\mu}'\sigma\bar{\mu}\left(\frac{\mu_k \|\sigma\|}{\bar{\mu}_k \beta_2}u_k - \mu_k\right)\left(\frac{\mu_k \|\sigma\|}{\bar{\mu}_k \beta_1}u_k - \mu_k\right) \\ &\quad + \frac{\mu_k \|\sigma\|}{\bar{\mu}_k \beta_1}\left(\frac{\mu_k \|\sigma\|}{\bar{\mu}_k \beta_2}u_k - \mu_k\right)\left(\frac{1}{\bar{\mu}_k^2}\left(\frac{1}{\beta_2} + \frac{1}{\beta_1}\right)\bar{\mu}'\sigma\bar{\mu}u_k + \frac{1}{\bar{\mu}_k}\mu'\bar{\mu}\right) \end{aligned}$$

and

$$\begin{aligned} k'(l_1) &= -\frac{\mu_k^2 \|\sigma\|}{\bar{\mu}_k^2 \beta_1}\left(1 - \frac{\beta_1}{\beta_2}\right)\left(\left(\frac{1}{\beta_2} + \frac{1}{\beta_1}\right)\bar{\mu}'\sigma\bar{\mu}\frac{\beta_1}{\|\sigma\|} + (\mu'\bar{\mu})\right) \\ k'(l_2) &= \frac{\mu_k^2 \|\sigma\|}{\bar{\mu}_k^2 \beta_2}\left(\frac{\beta_2}{\beta_1} - 1\right)\left(\left(\frac{1}{\beta_2} + \frac{1}{\beta_1}\right)\bar{\mu}'\sigma\bar{\mu}\frac{\beta_2}{\|\sigma\|} + (\mu'\bar{\mu})\right). \end{aligned}$$

Note that $\mu'\bar{\mu}$ can be rewritten as

$$\begin{aligned} \mu'\bar{\mu} &= \sum_{j=1}^{n} \mu_j \bar{\mu}_j \\ &= \sum_{j=1}^{n}\sum_{t=1}^{n} \mu_j \mu_t C_{jt} \end{aligned}$$

where $C_{jt} = (-1)^{j+t}M_{jt}$ is the cofactor of the matrix σ. For any positive definite matrix σ, it is known that $\sigma C^T = \|\sigma\| I$ where I is the identity matrix and C^T is the transpose of the cofactor matrix of σ. It is also known that if σ is positive definite, then σ^{-1} is also positive definite. Therefore, it can be concluded $C = \|\sigma\| \sigma^{-1}$ is also positive definite concluding that $\mu'\bar{\mu}$ is always positive. Moreover, since σ is positive definite, $\bar{\mu}'\sigma\bar{\mu}$ is also always positive. Therefore, it can be concluded that $k'(l_1) < 0$ and $k'(l_2) > 0$. Finally, since $\mu'\bar{\mu} > 0$, $k(0) > k(l_1) = k(l_2)$ when $\bar{\mu}_k > 0$, and $k(0) < k(l_1) = k(l_2)$ when $\bar{\mu}_k < 0$.

Adjusted mean excess return is greater than zero ($\bar{\mu}_k > 0$).

In this case assume that adjusted mean excess return $\bar{\mu}_k$ is greater than zero. It should be noted that when $\bar{\mu}_k$ is greater than zero, μ_k is not necessarily greater than zero. From

Section 4.2, we know that if $\bar{\mu}_k$ is greater than zero, then the feasible region for optimal decision for kth asset is positive. So the decision of whether buying or short selling the kth asset relies on the sign of $\bar{\mu}_k$ rather than the sign of $\mu_{k.}$ In the illustrations, we will provide examples of the cases where the mean excess return of the kth asset is positive, but in the optimal decision, kth asset is advised to short sell because of the sign of $\bar{\mu}_k$.

Theorem 6 *The optimal decision $u_k^*(x) = h_k^{-1}(g(x))$ is bounded and positively increasing in x if $\bar{\mu}_k > 0$.*

Proof. *It is already shown by (4.38) that when $\bar{\mu}_k > 0$, the feasible region is $I_1 = [l_1, l_2] = [(\bar{\mu}_k / \|\sigma\|)\,\beta_1, (\bar{\mu}_k / \|\sigma\|)\,\beta_2] \subset (0, +\infty)$, therefore $u_k^*(x)$ is bounded and positive. As in the case of single asset with $m = 2$ utilities, it suffices to show that $h_k(u_k)$ is decreasing in u_k. First observe that the sign of the derivative of $h_k(u_k)$ depends on the sign of $k(u_k)$. Therefore, we need to show that $k(u_k)$ is negative in the feasible region.*

Since $k(u_k)$ is a third degree polynomial in u_k, it can have at most two turning points. Moreover, note that $k(l_1) = k(l_2) < 0$. As in the case of single asset, $k'(l_1) < 0$, $k'(l_2) > 0$ indicates that there is a turning point in I_1. Moreover, $k'(l_1) < 0$ indicates that there is another turning point in $(-\infty, l_1]$ since $\lim_{u_k \to -\infty} k(u_k) = -\infty$, $k(u_k)$ is continuous for all u_k and $k(0)$ is a finite number. Since these are the only two possibilities for turning points, $k(u_k)$ is less than $k(l_1) = k(l_2) < 0$ for every u_k in I_1 concluding that $k(u_k)$ is less than zero in I_1. Therefore, $h'_k(u_k)$ is negative on I_1, and $u_k^*(x)$ is increasing in x. ■

Adjusted mean excess return is less than zero ($\bar{\mu}_k < 0$).

In this setting, since $\bar{\mu}_k$ is less than zero, we know that the optimal decision for the kth asset is negative. Like in the previous case, even if the the mean excess return μ_k is positive, there might be cases where the optimal decision for the kth asset is negative because of the adjusted mean excess return $\bar{\mu}_k$ being less than zero. The examples of such cases will be illustrated in the illustration section.

Theorem 7 *The optimal decision $u_k^*(x) = h_k^{-1}(g(x))$ is bounded and negatively decreasing in x if $\bar{\mu}_k < 0$.*

Proof. *It is already shown by (4.39) that when $\bar{\mu}_k < 0$, the feasible region is $I_2 = [l_2, l_1] = [(\bar{\mu}_k / \|\sigma\|)\,\beta_2, (\bar{\mu}_k / \|\sigma\|)\,\beta_1] \subset (-\infty, 0)$, therefore $u_k^*(x)$ is bounded and negative.*

As in the case of single asset with $m = 2$ utilities, it suffices to show that $h(u_k)$ is increasing in u_k. First observe that the sign of the derivative of $h_k(u_k)$ depends on the sign of $k(u_k)$. Therefore, we need to show that $k(u_k)$ is positive in the feasible region.

Since $k(u_k)$ is a third degree polynomial in u_k, it can have at most two turning points. Moreover, note that $k(l_1) = k(l_2) > 0$. As in the case of single asset, $k'(l_2) > 0$, $k'(l_1) < 0$ indicates that there is a turning point in I_2. Also $k'(l_1) < 0$ and $\lim_{u_k \to +\infty} k(u_k) = +\infty$ together implies that there must be another turning point in $[l_1, +\infty)$. Since these are the only two possibilities for turning points, $k(u_k)$ is greater then $k(l_1) = k(l_2) > 0$ for every u_k in I_2, concluding that $k(u_k)$ is greater than zero in I_2. Therefore $h'_k(u_k)$ is positive on I_2, and $u_k^*(x)$ is decreasing in x. ■

4.2.3 Random risk tolerance with $n = 1$ risky asset and m utilities

In this setting assume that the number of the market states are $m > 2$ instead of two. Consider the optimality condition

$$\sum_{j=1}^{m} \frac{P_j C_j}{\beta_j} \exp\left(-\frac{r_f x}{\beta_j}\right) E\left[R^e \exp\left(-\frac{R^e u}{\beta_j}\right)\right] = 0$$

or

$$\sum_{j=1}^{m} \frac{P_j C_j}{\beta_j} \exp\left(-\frac{r_f x}{\beta_j}\right) \exp\left(\frac{u^2\sigma^2 - 2u\mu\beta_j}{2\beta_j^2}\right)\left(\mu - \frac{u}{\beta_j}\sigma^2\right) = 0 \tag{4.40}$$

where R^e has normal distribution with mean μ and variance σ^2. First, suppose $\mu = 0$, then (4.40) becomes

$$\sum_{j=1}^{m} \frac{P_j C_j}{\beta_j} \exp\left(-\frac{r_f x}{\beta_j}\right) \exp\left(\frac{u^2\sigma^2}{2\beta_j^2}\right)\left(-\frac{u}{\beta_j}\sigma^2\right) = 0$$

or

$$u \sum_{j=1}^{m} \frac{P_j C_j}{\beta_j} \frac{\sigma^2}{\beta_j} \exp\left(-\frac{r_f x}{\beta_j}\right) \exp\left(\frac{u^2\sigma^2}{2\beta_j^2}\right) = 0.$$

Note that since all the terms in the summation are positive, the equality holds only if $u^* = 0$.

Next, consider

$$A_j(x, u) = \frac{P_j C_j}{\beta_j} \exp\left(-\frac{r_f x}{\beta_j}\right) \exp\left(\frac{u^2\sigma^2 - 2u\mu\beta_j}{2\beta_j^2}\right)\left(\mu - \frac{u}{\beta_j}\sigma^2\right).$$

The partial derivative of A_j with respect to u is

$$\frac{\partial A_j(x,u)}{\partial u} = -\frac{P_j C_j}{\beta_j^2} \exp\left(-\frac{r_f x}{\beta_j}\right) \exp\left(\frac{u^2\sigma^2 - 2u\mu\beta_j}{2\beta_j^2}\right)\left(\left(\mu - \frac{\sigma^2}{\beta_j}u\right)^2 + \sigma^2\right) < 0.$$

Therefore, $A_j(x,u)$ is strictly decreasing in u for all j and $A_j(x,u) = 0$ when $u = \left(\mu/\sigma^2\right)\beta_j$. Next, without loss of generality, assume that $\left\{\beta_j\right\}$ is ordered as $0 < \beta_1 \le \beta_2 \le \cdots \le \beta_m$. Since $A_j(x,u)$ is strictly decreasing in u for all j, $\sum_{j=1}^m A_j(x,u)$ also strictly decreases in u. Denoting

$$A(x,u) = \sum_{j=1}^{m} A_j(x,u)$$

the optimality condition can be written as $A(x,u) = 0$. Moreover, there is a unique $u^*(x)$ that satisfies $A(x,u^*(x)) = 0$ for any x since $A(x,u)$ strictly decreases in u and $A(x,+\infty) = +\infty$ and $A(x,-\infty) = -\infty$.

Also note that for any initial wealth x, the optimal decision $u^*(x)$ is always in the region $\left(\mu/\sigma^2\right)(\beta_1,\beta_m)$ when $\mu > 0$ and in $\left(\mu/\sigma^2\right)(\beta_m,\beta_1)$ when $\mu < 0$. To show this, assume that $\left(\mu/\sigma^2\right)\beta_1 > u^*(x)$ or $\left(\mu/\sigma^2\right)\beta_m < u^*(x)$ when $\mu > 0$ for some x. But, under these assumptions, (4.40) will not hold due to the fact that $\left(\mu - \left(\sigma^2/\beta_j\right)u^*(x)\right) < 0$ or $\left(\mu - \left(\sigma^2/\beta_j\right)u^*(x)\right) > 0$ for all j. The case when $\mu < 0$ can be shown similarly. Therefore, the feasible region is

$$0 < \left(\frac{\mu}{\sigma^2}\right)\beta_1 < u < \left(\frac{\mu}{\sigma^2}\right)\beta_m \tag{4.41}$$

if $\mu > 0$, and

$$\left(\frac{\mu}{\sigma^2}\right)\beta_m < u < \left(\frac{\mu}{\sigma^2}\right)\beta_1 < 0 \tag{4.42}$$

if $\mu < 0$. To simplify the notation, we define $\hat{u}_i = (\mu/\sigma^2)\beta_i$.

Mean excess return greater than zero ($\mu > 0$).

Assume $\mu > 0$ so that $\bar{r} > r_f$ which indicates that the expected return of the risky asset is greater than the return of the risk-free asset. We have already shown that the optimal amount of money invested in the risky asset is greater than zero.

Theorem 8 *The optimal decision $u^*(x)$ is bounded and positively increasing in x if $\mu > 0$.*

Proof. *In this setting the feasible region is given by (4.41) and this clearly indicates that $u^*(x) > 0$ is bounded. Fix the initial wealth level x_1. Let u_1^* be the optimal decision for x_1 so that $A(x_1,u_1^*) = 0$. Since the feasible region is given by (4.41), one can deduce that there exist $1 \le k \le m-1$ such that*

$$\hat{u}_k \le u_1^* < \hat{u}_{k+1}.$$

Moreover, $A_j(x_1, u_1^*) \leq 0$ *for all* $j \leq k$ *since* $u_1^* \geq \hat{u}_j$ *for all* $j \leq k$. *Similarly,* $A_j(x_1, u_1^*) > 0$ *for all* $j > k$. *Therefore,* $A(x_1, u_1^*) = 0$ *implies that*

$$\sum_{j=1}^{k} A_j(x_1, u^*) = -\sum_{j=k+1}^{m} A_j(x_1, u^*) \leq 0. \tag{4.43}$$

Consider the point $x_1 + \Delta x > x_1$ *for some* $\Delta x > 0$. *We now have*

$$\begin{aligned}
\sum_{j=1}^{m} A_j(x_1 + \Delta x, u_1^*) &= \sum_{j=1}^{k} A_j(x_1, u_1^*) \exp(-r_f \frac{\Delta x}{\beta_j}) + \sum_{j=k+1}^{m} A_j(x_1, u_1^*) \exp(-r_f \frac{\Delta x}{\beta_j}) \\
&\geq \exp(-r_f \frac{\Delta x}{\beta_k}) \sum_{j=1}^{k} A_j(x_1, u_1^*) + \exp(-r_f \frac{\Delta x}{\beta_{k+1}}) \sum_{j=k+1}^{m} A_j(x_1, u_1^*) \\
&= -\exp(-r_f \frac{\Delta x}{\beta_k}) \sum_{j=k+1}^{m} A_j(x_1, u_1^*) + \exp(-r_f \frac{\Delta x}{\beta_{k+1}}) \sum_{j=k+1}^{m} A_j(x_1, u_1^*) \\
&= \left[\exp(-r_f \frac{\Delta x}{\beta_{k+1}}) - \exp(-r_f \frac{\Delta x}{\beta_k}) \right] \sum_{j=k+1}^{m} A_j(x_1, u_1^*) \geq 0.
\end{aligned}$$

The first inequality follows from the fact that $\exp(-x)$ *is a nonnegative decreasing function. The second equality follows from (4.43). Similarly, the final term is positive since* $\exp(-x)$ *is a decreasing function and* $\sum_{j=k+1}^{m} A_j(x_1, u_1^*)$ *is positive by (4.43).*

Since $\sum_{j=1}^{m} A_j(x_1 + \Delta x, u_1^*) \geq 0$, *and* $\sum_{j=1}^{m} A_j(x_1 + \Delta x, u)$ *decreases in* u, *it can be concluded that the optimal decision for* $x_1 + \Delta x$ *is greater than* u_1^* *indicating that* $u^*(x)$ *is increasing in* x. ■

Mean excess return less than zero ($\mu < 0$*).*

Assume $\mu < 0$ so that $\bar{r} < r_f$ which indicates that the expected return of the risky asset is less than the return of the risk-free asset. In this situation, we have already shown that the optimal amount of money invested in the risky asset would be less than zero.

Theorem 9 *The optimal decision* $u^*(x)$ *is bounded and positively decreasing in* x *if* $\mu < 0$.

Proof. *In this setting the feasible region is given by (4.42) and this clearly indicates that* $u^*(x) < 0$ *is bounded. Fix the initial wealth level* x_1. *Let* u_1^* *be the optimal decision for* x_1 *so that* $A(x_1, u_1^*) = 0$. *Since the feasible region is given by (4.42), one can deduce that there exist* $1 \leq k \leq m - 1$ *such that*

$$\hat{u}_{k+1} < u_1^* \leq \hat{u}_k.$$

Moreover, $A_j(x_1, u_1^*) \geq 0$ *for all* $j \leq k$ *since* $u_1^* \geq \hat{u}_j$ *for all* $j \leq k$*. Similarly* $A_j(x_1, u_1^*) < 0$ *for all* $j > k$*. Therefore,* $A(x_1, u_1^*) = 0$ *implies that*

$$\sum_{j=1}^{k} A_j(x_1, u_1^*) = -\sum_{j=k+1}^{m} A_j(x_1, u_1^*) \geq 0. \tag{4.44}$$

Consider the point $x_1 + \Delta x > x_1$ *for some* $\Delta x > 0$*. We now have*

$$\begin{aligned}
\sum_{j=1}^{m} A_j(x_1 + \Delta x, u_1^*) &= \sum_{j=1}^{k} A_j(x_1, u_1^*) \exp(-r_f \frac{\Delta x}{\beta_j}) + \sum_{j=k+1}^{m} A_j(x_1, u_1^*) \exp(-r_f \frac{\Delta x}{\beta_j}) \\
&\leq \exp(-r_f \frac{\Delta x}{\beta_k}) \sum_{j=1}^{k} A_j(x_1, u_1^*) + \exp(-r_f \frac{\Delta x}{\beta_{k+1}}) \sum_{j=k+1}^{m} A_j(x_1, u_1^*) \\
&= -\exp(-r_f \frac{\Delta x}{\beta_k}) \sum_{j=k+1}^{m} A_j(x_1, u_1^*) + \exp(-r_f \frac{\Delta x}{\beta_{k+1}}) \sum_{j=k+1}^{m} A_j(x_1, u_1^*) \\
&= \left[\exp(-r_f \frac{\Delta x}{\beta_{k+1}}) - \exp(-r_f \frac{\Delta x}{\beta_k})\right] \sum_{j=k+1}^{m} A_j(x_1, u_1^*) \leq 0.
\end{aligned}$$

The first inequality follows from the fact that $\exp(-x)$ *is a nonnegative decreasing function. The second equality follows from (4.44). Similarly, the final term is negative since* $\exp(-x)$ *is a decreasing function and* $\sum_{j=k+1}^{m} A_j(x_1, u_1^*)$ *is negative by (4.44).*

Since $\sum_{j=1}^{m} A_j(x_1 + \Delta x, u_1^*) \leq 0$*, and* $\sum_{j=1}^{m} A_j(x_1 + \Delta x, u)$ *decreases in* u*, it can be concluded that the optimal decision for* $x_1 + \Delta x$ *is less than* u_1^* *indicating that* $u^*(x)$ *is decreasing in* x*.* ■

4.2.4 Random risk tolerance with n risky assets and m different utilities

Suppose that there are m utilities and there are n risky asset. In this setting the optimality condition (2.4) turns out to be

$$\sum_{j=1}^{m} \frac{P_j C_j}{\beta_j} \exp\left(-\frac{r_f x}{\beta_j}\right) \left[\exp(-\frac{1}{\beta_j}\mu' u + \frac{1}{2\beta_j^2} u' \sigma u) \left(\mu_k - \frac{1}{\beta_j} (\sigma u)_k\right)\right] = 0 \tag{4.45}$$

for all $k = 1, 2, \cdots, n$. Next denote

$$\bar{A}_j^k(x, u) = \frac{P_j C_j}{\beta_j} \exp\left(-\frac{r_f x}{\beta_j}\right) \left[\exp(-\frac{1}{\beta_j}\mu' u + \frac{1}{2\beta_j^2} u' \sigma u) \left(\mu_k - \frac{1}{\beta_j} (\sigma u)_k\right)\right].$$

Therefore, the optimality condition is $\sum_{j=1}^{m} \bar{A}_j^k(x, u) = 0$. Taking derivative with respect to u_k we get

$$\frac{\partial \bar{A}_j^k(x, u)}{\partial u_k} = -\frac{P_j C_j}{\beta_j^2} \exp\left(-\frac{r_f x}{\beta_j}\right) \exp(-\frac{1}{\beta_j}\mu' u + \frac{1}{2\beta_j^2} u' \sigma u) \left[\left(\mu_k - \frac{1}{\beta_j} (\sigma u)_k\right)^2 + \sigma_{kk}\right] < 0$$

Hence, $\bar{A}_j^k(x,u)$ is strictly decreasing in u_k for all j. Moreover, sum of strictly decreasing functions is strictly decreasing, therefore $\sum_{j=1}^m \bar{A}_j^k(x,u)$ is also a strictly decreasing function in $u_{k.}$ Rewrite (4.45) to get

$$\begin{aligned}\mu_k \sum_{j=1}^m \frac{P_jC_j}{\beta_j} \exp\left(-\frac{r_f x}{\beta_j}\right) \exp(-\frac{\mu' u}{\beta_j} + \frac{u'\sigma u}{2\beta_j^2}) &= (\sigma u)_k \sum_{j=1}^m \frac{P_jC_j}{\beta_j^2} \exp\left(-\frac{r_f x}{\beta_j}\right) \\ &\quad \cdot \exp(-\frac{\mu' u}{\beta_j} + \frac{u'\sigma u}{2\beta_j^2})\end{aligned} \tag{4.46}$$

or

$$\frac{\mu_k}{(\sigma u)_k} = \frac{\sum_{j=1}^m \frac{P_jC_j}{\beta_j^2} \exp\left(-\frac{r_f x}{\beta_j}\right) \exp(-\frac{1}{\beta_j}\mu' u + \frac{1}{2\beta_j^2} u'\sigma u)}{\sum_{j=1}^m \frac{P_jC_j}{\beta_j} \exp\left(-\frac{r_f x}{\beta_j}\right) \exp(-\frac{1}{\beta_j}\mu' u + \frac{1}{2\beta_j^2} u'\sigma u)}. \tag{4.47}$$

When $\mu_k \neq 0$, $(\sigma u)_k$ cannot be equal to zero; otherwise, (4.45) will not hold. If $\mu_k = 0$, then $(\sigma u)_k = 0$ from (4.45). If $\mu_k = 0$ for all k, then (4.45) holds only if $(\sigma u)_k = 0$ for all k. This would induce to the system of linear equations $\sigma u = 0$. Since σ is positive definite, the only solution is $u^*(x) = [0, 0, \cdots, 0]$. Therefore, without loss of generality, assume that $\mu_k \neq 0$ for some k making $(\sigma u)_k \neq 0$. Moreover, right-hand side of (4.47) is independent of k; hence, it can be stated that (4.21) holds for this setting too. Therefore, there is a linear relationship between risky assets given by equation (4.27). Therefore, in this setting, the optimal decision satisfies

$$\begin{aligned} u^*(x) &= [u_1^*(x), u_2^*(x), \cdots, u_n^*(x)] \\ &= \frac{u_k(x)}{\bar{\mu}_k}[\bar{\mu}_1, \bar{\mu}_2, \cdots, \bar{\mu}_n] \end{aligned}$$

for any k with $\bar{\mu}_k \neq 0$.

Moreover, in Section 4.2.2 it was shown that monotonicity of u_k^* is also determined by the sign of $\bar{\mu}_k$. It was also further shown that $(\sigma u)_k$ can be written in terms of u_k. It is useful to do the same substitution here in this setting because it would reduce the n risky asset setting to a single risky asset setting. Therefore, substituting $\mu_k \left\|\sigma\right\| / \bar{\mu}_k u_k$ with $(\sigma u)_k$, and also using (4.27) in (4.45), we get

$$\begin{aligned} \sum_{j=1}^m \frac{P_jC_j}{\beta_j} \exp\left(-\frac{r_f x}{\beta_j}\right) \left[\exp(-\frac{(\mu'\bar{\mu})\, u_k}{\beta_j \bar{\mu}_k} + \frac{(\bar{\mu}'\sigma\bar{\mu})\, u_k^2}{2\beta_j^2 \bar{\mu}_k^2}) \left(\mu_k - \frac{\mu_k \left\|\sigma\right\|}{\bar{\mu}_k} u_k\right)\right] &= 0 \\ \mu_k \sum_{j=1}^m \frac{P_jC_j}{\beta_j} \exp\left(-\frac{r_f x}{\beta_j}\right) \left[\exp(-\frac{(\mu'\bar{\mu})\, u_k}{\beta_j \bar{\mu}_k} + \frac{(\bar{\mu}'\sigma\bar{\mu})\, u_k^2}{2\beta_j^2 \bar{\mu}_k^2}) \left(1 - \frac{\left\|\sigma\right\|}{\beta_j \bar{\mu}_k} u_k\right)\right] &= 0 \\ \sum_{j=1}^m \frac{P_jC_j}{\beta_j} \exp\left(-\frac{r_f x}{\beta_j}\right) \left[\exp(-\frac{(\mu'\bar{\mu})\, u_k}{\beta_j \bar{\mu}_k} + \frac{(\bar{\mu}'\sigma\bar{\mu})\, u_k^2}{2\beta_j^2 \bar{\mu}_k^2}) \left(1 - \frac{\left\|\sigma\right\|}{\beta_j \bar{\mu}_k} u_k\right)\right] &= 0 \end{aligned}$$

since $\mu_k \neq 0$. By using the same argument in Section 4.2.3, one can conclude that

$$0 \leq \frac{\bar{\mu}_k}{\|\sigma\|}\beta_1 < u_k < \frac{\bar{\mu}_k}{\|\sigma\|}\beta_m \tag{4.48}$$

if $\bar{\mu}_k > 0$ and

$$\frac{\bar{\mu}_k}{\|\sigma\|}\beta_m < u_k < \frac{\bar{\mu}_k}{\|\sigma\|}\beta_1 \leq 0 \tag{4.49}$$

if $\bar{\mu}_k < 0$. Note that (4.48) and (4.49) implies that the optimal decision u_k is bounded. Here, note that $\sum_{j=1}^{m} \bar{A}_j^k(x,u)$ decreasing in u_k is essential for (4.48) and (4.49) to hold. Again, for simplification let

$$\hat{u}_j = \bar{\mu}_k / \|\sigma\| \, \beta_j$$

for a fixed k and

$$\bar{A}^k(x,u) = \sum_{j=1}^{m} \bar{A}_j^k(x,u).$$

Adjusted mean excess return is greater than zero ($\bar{\mu}_k > 0$).

In this setting since we have reduced the n risky asset case to a single asset case, we will perform an analysis similar to the previous ones. In the previous case, the monotonicity of the optimal decision for kth asset ($u_k^*(x)$) depended on the sign of mean excess return whereas in this case the monotonicity of the optimal decision for kth asset depends on the sign of the adjusted mean excess return $\bar{\mu}_k$.

Theorem 10 *The optimal decision $u_k^*(x)$ is bounded and positively increasing in x if $\bar{\mu}_k > 0$.*

In this setting the feasible region is given by (4.48) and this clearly indicates that $u_k^*(x) > 0$ is bounded. The proof will be similar to the single asset case. Fix the initial wealth level x_1. Let u_k^* be the optimal decision for x_1 so that $\bar{A}^k(x_1, u_k^*) = 0$. Since the feasible region is given by (4.48), one can deduce that there exist a $1 \leq l \leq m-1$ such that

$$\hat{\beta}_l \leq u_k^* < \hat{\beta}_{l+1}.$$

Moreover, $\bar{A}_j(x_1, u_k^*) \leq 0$ for all $j \leq l$ since $u_k^* \geq \hat{u}_j$ for all $j \leq l$. Similarly $\bar{A}_j^k(x_1, u_k^*) > 0$ for all $j > l$. Therefore, $A\left(x_1, u_1^*\right) = 0$ implies that

$$\sum_{j=1}^{l} \bar{A}_j^k(x_1, u_k^*) = -\sum_{j=l+1}^{m} \bar{A}_j^k(x_1, u_k^*) < 0. \tag{4.50}$$

Consider the point $x_1 + \Delta x > x_1$ for some $\Delta x > 0$. We now have

$$\begin{aligned}
\sum_{j=1}^{m} \bar{A}_j^k(x_1 + \Delta x, u_k^*) &= \sum_{j=1}^{l} \bar{A}_j^k(x_1, u_k^*) \exp(-r_f \frac{\Delta x}{\beta_j}) + \sum_{j=l+1}^{m} \bar{A}_j^k(x_1, u_k^*) \exp(-r_f \frac{\Delta x}{\beta_j}) \\
&\geq \exp(-r_f \frac{\Delta x}{\beta_l}) \sum_{j=1}^{l} \bar{A}_j^k(x_1, u_k^*) + \exp(-r_f \frac{\Delta x}{\beta_{l+1}}) \sum_{j=l+1}^{m} \bar{A}_j^k(x_1, u_k^*) \\
&= -\exp(-r_f \frac{\Delta x}{\beta_l}) \sum_{j=l+1}^{m} \bar{A}_j^k(x_1, u_k^*) + \exp(-r_f \frac{\Delta x}{\beta_{l+1}}) \sum_{j=l+1}^{m} \bar{A}_j^k(x_1, u_k^*) \\
&= \left[\exp(-r_f \frac{\Delta x}{\beta_{l+1}}) - \exp(-r_f \frac{\Delta x}{\beta_l})\right] \sum_{j=l+1}^{m} \bar{A}_j^k(x_1, u_k^*) \geq 0.
\end{aligned}$$

The first inequality follows from the fact that $\exp(-x)$ is a nonnegative decreasing function. The second equality follows from (4.50). Similarly, the final term is positive since $\exp(-x)$ is a decreasing function and $\sum_{j=k+1}^{m} A_j(x_1, u_1^*)$ is positive by (4.50).

Since $\sum_{j=1}^{m} \bar{A}_j^k(x_1 + \Delta x, u_1^*) \geq 0$, and $\sum_{j=1}^{m} \bar{A}_j^k(x_1 + \Delta x, u_k)$ decreases in u_k, it can be concluded that the optimal decision for $x_1 + \Delta x$ is greater than u_k^* indicating that $u_k^*(x)$ is increasing in x.

Adjusted mean excess return is less than zero ($\bar{\mu}_k < 0$).

In this setting assume that the adjusted mean excess return $\bar{\mu}_k$ is less than zero. In the previous section we showed that under this assumption the feasible region is negative therefore the investor is advised to short sell the kth asset.

Theorem 11 *The optimal decision $u_k^*(x)$ is bounded and negatively decreasing in x if $\bar{\mu}_k < 0$.*

In this setting the feasible region is given by (4.49) and this clearly indicates that $u_k^*(x) < 0$ is bounded. The proof will be similar to the single asset case. Fix the initial wealth level x_1. Let u_k^* be the optimal decision for x_1 so that $\bar{A}(x_1, u_k^*) = 0$. Since the feasible region is given by (4.49), one can deduce that there exist a $1 \leq l \leq m-1$ such that

$$\hat{\beta}_{l+1} < u_k^* \leq \hat{\beta}_l.$$

Moreover, $\bar{A}_j^k(x_1, u_k^*) > 0$ for all $j \leq l$ since $u_k^* \geq \hat{u}_j$ for all $j \leq l$. Similarly $\bar{A}_j^k(x_1, u_k^*) \leq 0$ for all $j > l$. Therefore, $\bar{A}\,(x_1, u_1^*) = 0$ implies that

$$\sum_{j=1}^{l} \bar{A}_j^k(x_1, u_k^*) = -\sum_{j=l+1}^{m} \bar{A}_j^k(x_1, u_k^*) > 0. \tag{4.51}$$

Consider the point $x_1 + \Delta x > x_1$ for some $\Delta x > 0$. We now have

$$\begin{aligned}
\sum_{j=1}^{m} \bar{A}_j^k(x_1 + \Delta x, u_k^*) &= \sum_{j=1}^{l} \bar{A}_j^k(x_1, u_k^*) \exp(-r_f \frac{\Delta x}{\beta_j}) + \sum_{j=l+1}^{m} \bar{A}_j^k(x_1, u_k^*) \exp(-r_f \frac{\Delta x}{\beta_j}) \\
&\leq \exp(-r_f \frac{\Delta x}{\beta_l}) \sum_{j=1}^{l} \bar{A}_j^k(x_1, u_k^*) + \exp(-r_f \frac{\Delta x}{\beta_{l+1}}) \sum_{j=l+1}^{m} \bar{A}_j^k(x_1, u_k^*) \\
&= -\exp(-r_f \frac{\Delta x}{\beta_l}) \sum_{j=l+1}^{m} \bar{A}_j^k(x_1, u_k^*) + \exp(-r_f \frac{\Delta x}{\beta_{l+1}}) \sum_{j=l+1}^{m} \bar{A}_j^k(x_1, u_k^*) \\
&= \left[\exp(-r_f \frac{\Delta x}{\beta_{l+1}}) - \exp(-r_f \frac{\Delta x}{\beta_l})\right] \sum_{j=l+1}^{m} \bar{A}_j^k(x_1, u_k^*) < 0.
\end{aligned}$$

The first inequality follows from the fact that $\exp(-x)$ is a nonnegative decreasing function. The second equality follows from (4.51) and similarly the final term is negative since $\exp(-x)$ is a decreasing function and $\sum_{j=k_1+1}^{m} \bar{A}_j^k(x_1, u_k^*)$ is negative by (4.51).

Since $\sum_{j=1}^{m} \bar{A}_j^k(x_1 + \Delta x, u_k^*) < 0$, and $\sum_{j=1}^{m} \bar{A}_j^k(x_1 + \Delta x, u_k)$ decreases in u_k, it can be concluded that the optimal decision for $x_1 + \Delta x$ is less than u_k^* indicating that $u_k^*(x)$ is decreasing in x.

So far we have shown that for the fixed kth asset, the optimal decision $u_k^*(x)$ is nonnegatively increasing when adjusted mean excess return $\bar{\mu}_k > 0$ and it is nonpositively decreasing otherwise. For any other asset $j \neq k$, we have shown that there is a linear relationship between any two assets by (4.27). Therefore, the optimal decision for asset $j \neq k$ can be obtained from (4.27). For example, assume that $\bar{\mu}_k$ is positive for an asset k. Then, for any asset $j \neq k$, the optimal amount to be invested in jth asset is equal to $\bar{\mu}_j u_k / \bar{\mu}_k$. If $\bar{\mu}_j$ is also positive, then u_j is also increasing with respect to x and it is decreasing otherwise. To summarize,

1. If $\bar{\mu}_k > 0$ so that $u_k^*(x) > 0$ and it is increasing in x, then

   (a) $\bar{\mu}_j > 0$ implies $u_j^*(x) = \bar{\mu}_j u_k^*(x)/\bar{\mu}_k > 0$ and it is increasing in x,

   (b) $\bar{\mu}_j < 0$ implies $u_j^*(x) = \bar{\mu}_j u_k^*(x)/\bar{\mu}_k < 0$ and it is decreasing in x.

j	P_j	K_j	C_j	$\boldsymbol{\beta}_j$
1	0.25	3	2	3
2	0.30	2	1	8
3	0.35	4	2	9
4	0.10	4	2	17

Table 4.1: Parameters when $m = 4$

j	P_j	K_j	C_j	$\boldsymbol{\beta}_j$
1	0.60	3	2	3
2	0.40	2	1	8

Table 4.2: Parameters when $m = 2$

2. If $\bar{\mu}_k < 0$ so that $u_k^*(x) < 0$ and it is decreasing in x, then

   (a) $\bar{\mu}_j > 0$ implies $u_j^*(x) = \bar{\mu}_j u_k^*(x)/\bar{\mu}_k < 0$ and it is decreasing in x,

   (b) $\bar{\mu}_j < 0$ implies $u_j^*(x) = \bar{\mu}_j u_k^*(x)/\bar{\mu}_k > 0$ and it is increasing in x.

The consistency of the results can be observed from (4.27) and (4.47).

4.3 Illustrations

In this section, numerical illustrations of the previous results will be presented. We will first start with the illustrations on the case with constant risk tolerance discussed in Section 3.1. We will discuss both single and multiple risky assets cases. Then, the next subsection will be divided into two main parts. The first part will be on one risky asset model and the other one will be on multiple risky assets model.

The numerical examples in this section uses generated data. Throughout the section, suppose that $r_f = 1.10$ and the initial wealth level x of the investor varies between -100 and 100 with five increments. Moreover, if there are more than two utilities, then we arbitrarily take $m = 4$. Assume also that utilities have the following probabilities and parameters: If the number of the utilities is two, then we use the following parameters:

Moreover, if there is only one risky asset to be considered, then it has a normal distribution with variance $\sigma^2 = 0.004$. If the mean excess return of the asset is positive, then we take $\mu = 0.08$; otherwise, it is $\mu = -0.04$. Finally, if there are n risky assets to be considered, then we consider $n = 3$ and asset returns have a multivariate normal distribution with

$$\mu = \left[\begin{array}{ccc} 0.08 & -0.04 & 0.01 \end{array}\right], \quad \sigma = \left[\begin{array}{ccc} 4.000 & 0.664 & 3.492 \\ 0.664 & 4.000 & 1.258 \\ 3.492 & 1.258 & 25.136 \end{array}\right] \tag{4.52}$$

so that the correlation matrix becomes

$$\rho = \left[\begin{array}{ccc} 1.000 & 0.166 & 0.348 \\ 0.166 & 1.000 & 0.126 \\ 0.348 & 0.126 & 1.000 \end{array}\right].$$

Note that in the covariance matrix the values are obtained by multiplying the actual values by 1,000 to simplify the values.

4.3.1 Constant risk tolerance

Throughout this section $\beta_j = \beta = 0.03$ for all j.

Example 12 ***One risky asset and four utilities*** ($n = 1$, $m = 4$). *Using the results from Section 3.1, the optimal solution is*

$$u^*(x) = \left(\frac{\mu}{\sigma^2}\right)\beta = \left(\frac{0.08}{0.004}\right)0.03 = 0.6$$

when $\mu = 0.08 > 0$, *and*

$$u^*(x) = \left(\frac{\mu}{\sigma^2}\right)\beta = \left(\frac{-0.04}{0.004}\right)0.03 = -0.3$$

when $\mu = -0.04 < 0$.

When $\mu = 0.08$, $u^*(x) = 0.6$ *implies that the investor is advised to invest* 0.60 *in the risky asset and his remaining wealth* $x - 0.40$ *in the risk-free asset. Note that in case if* $x < 0.6$, *then the investor borrows (or short sells the risk-free asset)* $0.60 - x$ *at the risk-free rate. Similarly, when* $\mu = -0.04$, *the investor is advised to short sell the risky asset by an amount of* 0.30, *and invest the proceeds as well as the initial wealth in the risk-free asset. Moreover, observe that* u^* *is independent of the values of* x.

Example 13 ***Three risky assets and four utilities*** $(n = 3,\ m = 4)$. *Using parameters from Table* 1 *and (4.52), the optimal amount of money invested in the three assets are found by solving the system of linear equations in (4.7). Hence, the optimal decision is*

$$u^*(x) = \beta\sigma^{-1}\mu = \begin{bmatrix} 0.73 & -0.40 & -0.07 \end{bmatrix}$$

for all x. *In this scenario, the investor is advised to short sell the second and third risky assets by amounts of* 0.40 *and* 0.07 *respectively. On the other hand, the investor is advised to buy the first asset for an amount of* 0.73. *The remaining sum* $x + 0.40 + 0.07 - 0.73 = x - 0.26$ *is compensated by borrowing money at the risk-free rate. To summarize, in this setting the investor is advised to*

Buy 1^{st} *asset*	0.73
Shortsell 2^{nd} *asset*	0.40
Shortsell 3^{rd} *asset*	0.07
Borrow/Lend money	$x - 0.26$

Moreover, observe similarly that u^* *is independent of the values of* x. *It can also be noted that although the mean excess return of the third asset is positive* (0.01), *the investor is advised to short sell it. This is due to the reason that the variance of the third asset* $\left(25.136 \times 10^{-3}\right)$ *is high and the third asset is positively correlated with the other two assets. Therefore, instead of taking risk with the third asset, which also has low mean excess return, the investor is advised to invest his money in the first asset. The second asset is also sold short because of high negative mean excess return and positive correlation with the other assets.*

4.3.2 Random risk tolerance

In this section, we will use the parameters from Table 2. Similarly, this section will be divided into parts depending on the number of assets to be considered. We used MATLAB to find the optimal policy u^* for the single asset cases and we used GAMS for multiple asset cases.

Example 14 ***One risky asset with positive mean excess return and two utilities*** $(n = 1,\ m = 2,\ \mu > 0)$. *If* $\mu = 0.08 > 0$, *then the optimal policy is found by using (4.8) and it is given in Figure* 4.1.

Example 15 ***One risky asset with negative mean excess return and two utilities*** *($n = 1$, $m = 2$, $\mu < 0$). If $\mu = -0.04 < 0$, then the optimal policy is found by using (4.8) and it is given in Figure* 4.2.

Comparing Examples 14 and 15, it can be deduced that the optimal decisions are both bounded from above and below. Especially a closer look shows that in both cases the optimal decision is bounded below by $(\mu/\sigma^2)\beta_1 = 60$ and above by $(\mu/\sigma^2)\beta_2 = 160$ when $\mu = 0.08 > 0$, and similarly bounded below by $(\mu/\sigma^2)\beta_2 = -80$ and above by $(\mu/\sigma^2)\beta_1 = -30$ when $\mu = -0.04 < 0$.

We can interpret the optimal policies as follows. For $\mu > 0$, if the initial wealth of the investor is say $x = 20$, then the investor should invest $u^*(x) = 125.05$ in the risky asset. The remaining $125.05 - 20 = 105.05$ is acquired by borrowing money at the risk-free rate. For $\mu < 0$, if the investor has an initial wealth of $x = 20$, then the investor is advised to short sell the risky asset by an amount of $|u^*(x)| = 68.42$ and invest the money acquired together with the initial amount in the risk-free asset. The total money invested in the risk-free asset will be $68.42 + 20 = 88.42$.

Example 16 ***One risky asset with positive mean excess return and four utilities*** *($n = 1$, $m = 4$, $\mu > 0$). In this example we want to extend the results that we found in Example 14. The optimal policy is found by using (4.40) and it is given in Figure* 4.3.

Example 17 ***One risky asset with negative mean excess return and four utilities*** *($n = 1$, $m = 2$, $\mu < 0$). If $\mu = -0.04 < 0$, then the optimal policy is again found by using (4.40) and it is given in Figure* 4.4.

Comparing Examples 16 and 17, one can observe that the optimal policy is similar to the previous examples. They are both bounded below and above by $(\mu/\sigma^2)\beta_1 = 60$ and $(\mu/\sigma^2)\beta_2 = 160$ respectively, but as we change the number of utilities the range of the optimal decision varies significantly. Contrary to two the utilities case, the region -100 to 100 for the wealth level x is not sufficient to show proper boundaries of the optimal policy in both Examples 16 and 17. To generalize these results we foresee that as the number of different utilities increases, the region for which the optimal decision vary will extend as well. For the analysis of the investor's policy, the same line of reasoning can used as in Examples 14 and 15.

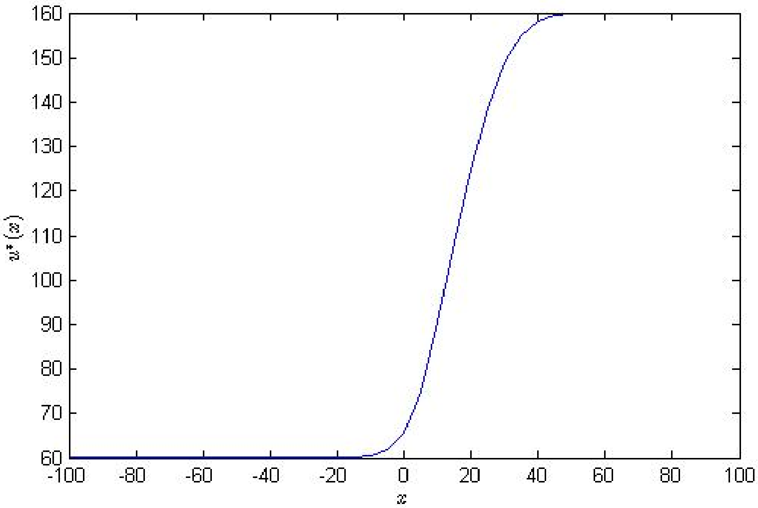

Figure 4.1: Optimal Policy for Example 14

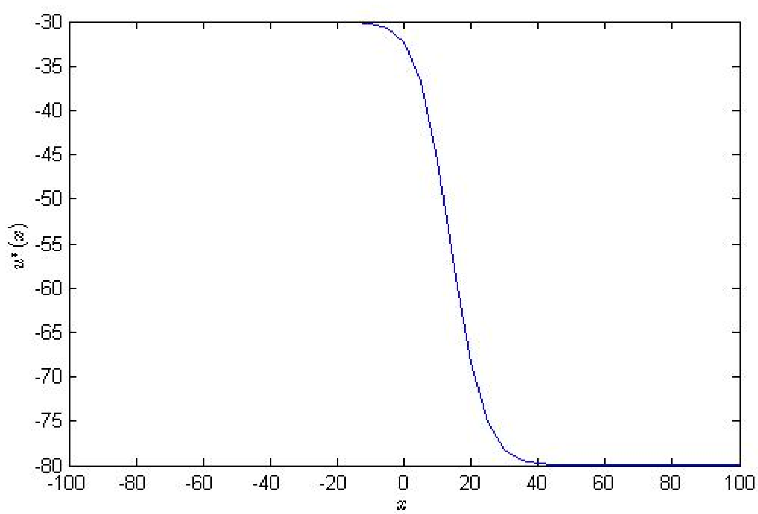

Figure 4.2: Optimal Policy for Example 15

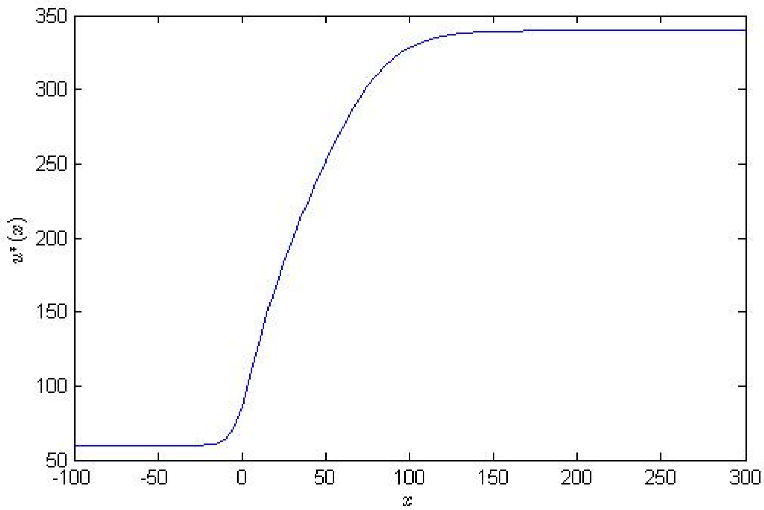

Figure 4.3: Optimal Policy for Example 16

Example 18 ***Three risky assets and two utilities*** $(n = 3,\ m = 2)$. *In this example, we will use the parameters from (4.52) together with Table* 2. *The optimal policy is found by using (4.16) and given in Figures* 4.5, 4.6 *and* 4.7 *for assets 1,2 and 3 respectively.*

Example 19 ***Three risky assets and four utilities*** $(n = 3,\ m = 4)$. *We will use the parameters from (4.52) together with Table* 2. *This time, the optimal policy can be found by using (4.45) and it is given in Figure* 4.8, 4.9, *and* 4.10 *for assets 1,2 and 3 respectively.*

For both Examples 18 and 19, we can calculate the adjusted mean excess returns as $\bar{\mu} = [\ 0.8277 \quad -0.4545 \quad -0.0786\ ] \times 10^{-5}$ *using (4.22). This indicates that, as expected, the first asset is advised to buy while the other two are advised to short sell irrespective of the wealth level.*

Comparing Examples 18 and 19, it can be observed that same kind of characteristics occur. In the single asset case of Examples 16 and 17, we observed from Figures 4.3 *and* 4.4 *that as the number of preferences increase, the region* $[-100, 100]$ *for the initial wealth* x *is not sufficient to show the proper boundaries of the optimal decision for the asset. The same observation is true here.*

To illustrate how this policy is advised, consider Figues 4.5, 4.6 and 4.7. Suppose that the initial wealth of the investor is $x = 20$. Then, the investor is advised to buy the first

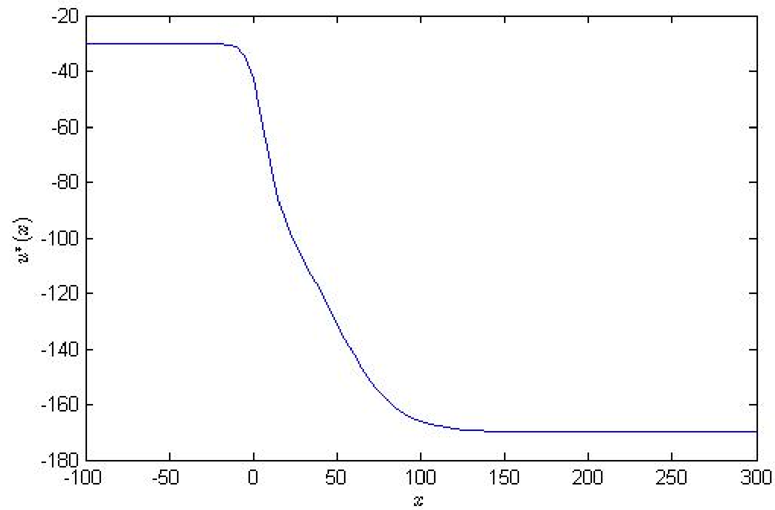

Figure 4.4: Optimal Policy for Example 17

asset for an amount of 144.88, and short sell the second and the third assets for an amount of 79.55 and 13.76 respectively. The remaining sum $144.88 - (79.55 + 13.76) - 20 = 31.57$ is compensated by borrowing money at the risk-free rate.

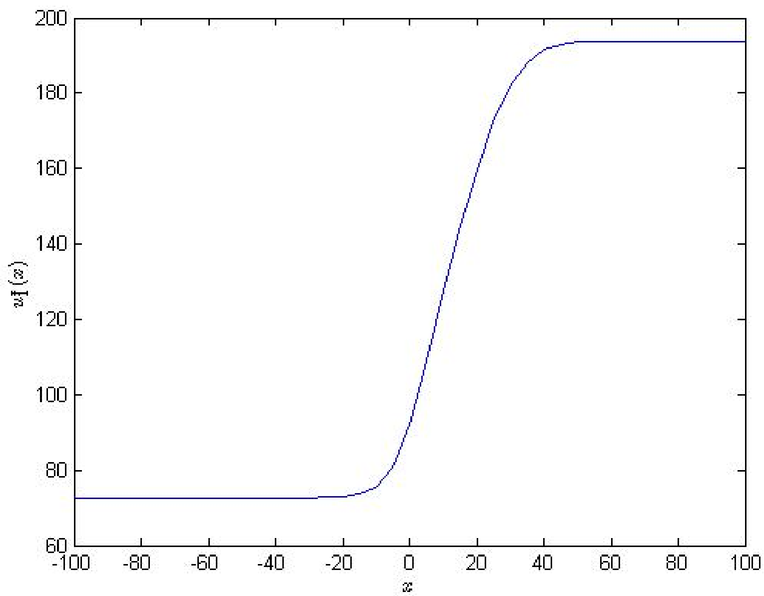

Figure 4.5: Optimal Policy for the First Asset in Example 18

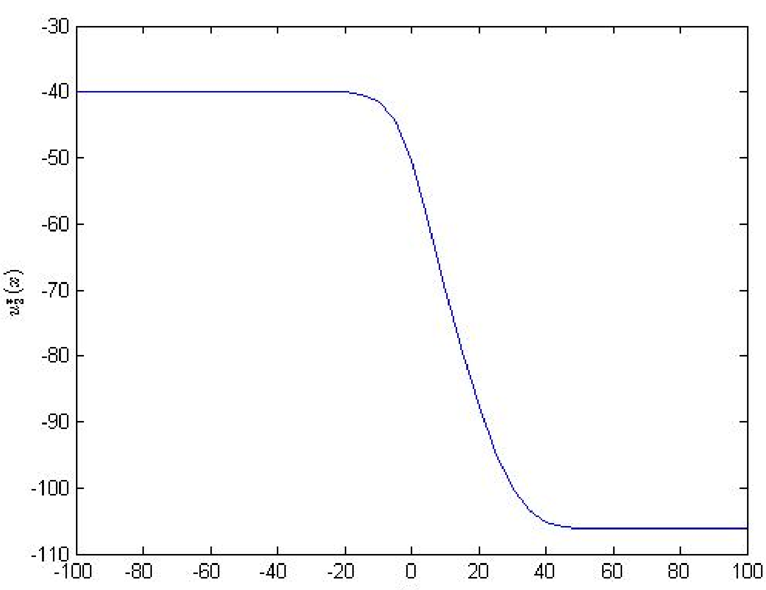

Figure 4.6: Optimal Policy for the Second Asset in Example 18

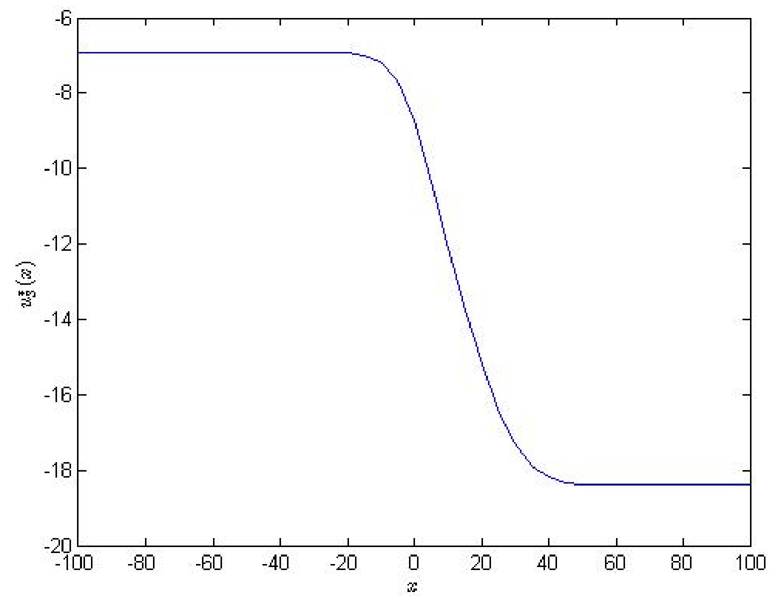

Figure 4.7: Optimal Policy for the Third Asset in Example 18

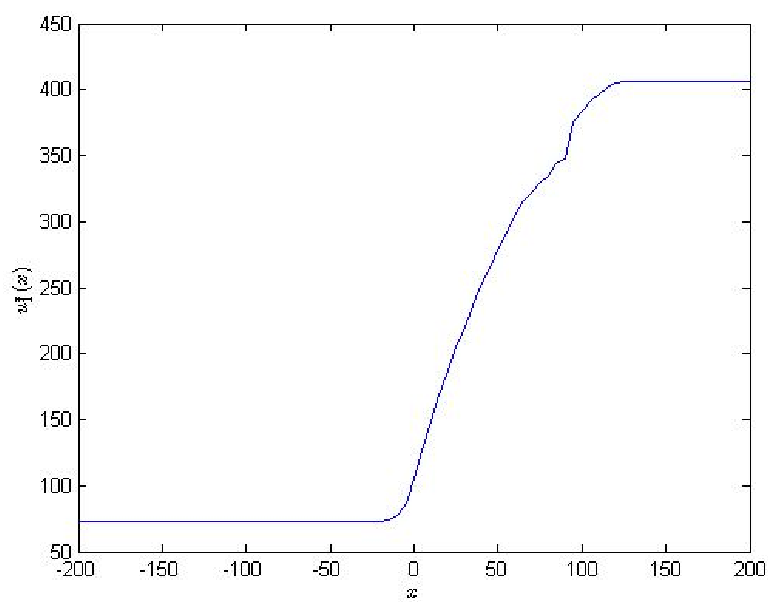

Figure 4.8: Optimal Policy for the First Asset in Example 19

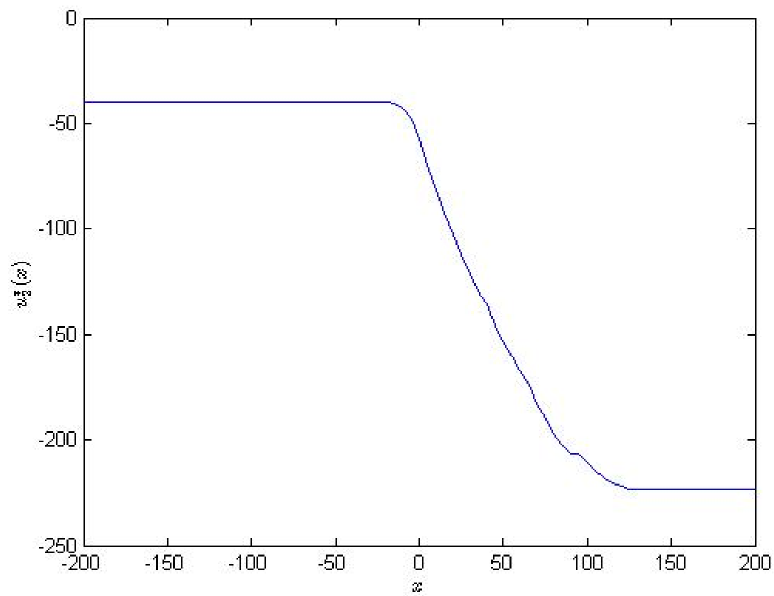

Figure 4.9: Optimal Policy for the Second Asset in Example 19

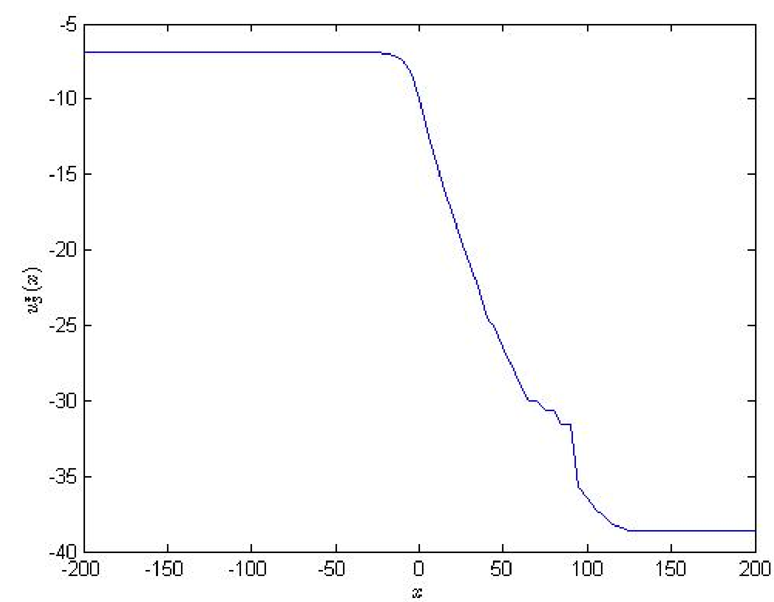

Figure 4.10: Optimal Policy for the Third Asset in Example 19

Chapter 5

EXTENSIONS

In this chapter, we will present some of the other results that we have found during this research. The exponential distribution model is a special case of general distribution model where there is only a single asset with exponentially distributed return. Although all of the results found in Chapter 3 are valid, here we will use a different, more direct approach. With this approach we will be able to identify the feasible region of the optimal policy explicitly. In the second section, we will generalize the utility function. Instead of using exponential utility functions, we will consider any concave utility functions and obtain some preliminary results.

5.1 Exponential Distribution Model

In this setting, all the assumptions in the normal model at Chapter 4 are the same except for the distribution of the risky asset. It is now exponential instead of normal so that $R^e = R - r_f$ where $R \sim \text{Exp}(\lambda)$ with $\lambda > 0$. Similarly, this model will also have two subsections. In the first subsection, the risk preference of the investor is constant. In the second subsection, the investor has $m = 2$ risk preferences. This subsection is further divided into two parts. In the first part, the expected excess return $E[R^e] = \mu = (1/\lambda) - r_f > 0$ indicating that $\lambda r_f < 1$, and, in the second part, expected excess return will be less than zero so that $\lambda r_f > 1$. Throughout this section, we denote

$$\begin{aligned} E\left[e^{-\alpha R}\right] &= \frac{\lambda}{\alpha + \lambda} \\ E\left[R e^{-\alpha R}\right] &= \frac{\lambda}{(\alpha + \lambda)^2} \end{aligned} \tag{5.1}$$

for $\alpha \geq 0$.

5.1.1 Constant risk tolerance

The optimality condition is

$$\sum_{j\in E}\frac{P_jC_j}{\beta}\exp\left(-\frac{r_fx}{\beta}\right)E\left[R^e\exp\left(-\frac{R^eu}{\beta}\right)\right]=0.$$

If $\beta_j=\beta$ for all j, then it is sufficient and necessary that

$$E\left[(R-r_f)\exp\left(-\frac{(R-r_f)u}{\beta}\right)\right]=0$$

since all the coefficients are positive. This can be rewritten as

$$\exp\left(\frac{r_fu}{\beta}\right)E\left[(R-r_f)\exp\left(-\left(\frac{u}{\beta}\right)R\right)\right]=0$$

so that

$$E\left[(R-r_f)\exp\left(-\left(\frac{u}{\beta}\right)R\right)\right]=0$$

which implies

$$E\left[R\exp\left(-\left(\frac{u}{\beta}\right)R\right)\right]=r_fE\left[\exp\left(-\left(\frac{u}{\beta}\right)R\right)\right]$$

or

$$\int_0^{+\infty}\lambda x\exp(-x\left(\lambda+\frac{u}{\beta}\right))dx=r_f\int_0^{+\infty}\lambda\exp(-x\left(\lambda+\frac{u}{\beta}\right))dx. \tag{5.2}$$

The integrals in (5.2) exists only if $(\lambda+(u/\beta))>0$. Hence, for $u>-\lambda\beta$, it can be concluded that the above integral would reduce to

$$\frac{\lambda}{\left(\left(\frac{u}{\beta}\right)+\lambda\right)^2}=\frac{\lambda r_f}{\left(\frac{u}{\beta}\right)+\lambda}$$

or

$$\frac{1}{\left(\frac{u}{\beta}\right)+\lambda}=r_f$$

which is the same as

$$\left(\frac{u}{\beta}\right)+\lambda=\frac{1}{r_f}$$

and the optimal solution is

$$u^*=\left(\frac{1}{r_f}-\lambda\right)\beta=(1-\lambda r_f)\left(\frac{\beta}{r_f}\right).$$

We can also observe that

$$u^* = -\lambda\beta + \frac{\beta}{r_f} > -\lambda\beta$$

and the integrals in (5.2) exist. Here note that if $\lambda r_f \leq 1$ or $\mu \geq 0$ then u^* is positive. For $\lambda r_f > 1$, since β / r_f is always greater than zero, optimal decision exists and it is negative.

5.1.2 Random risk tolerance with $n = 1$ risky asset and $m = 2$ utilities

Next, assume that the risk preference is not constant and the investor has two different risk preferences. Hence, the new optimality condition is

$$\sum_{j=1}^{2} \frac{P_j C_j}{\beta_j} \exp\left(-\frac{r_f x}{\beta_j}\right) E\left[R^e \exp\left(-\frac{R^e u}{\beta_j}\right)\right] = 0 \tag{5.3}$$

which equals

$$\sum_{j=1}^{2} \frac{P_j C_j}{\beta_j} \exp\left(\frac{r_f}{\beta_j}(u-x)\right) \left[E\left[R \exp(-\frac{u}{\beta_j}R)\right] - r_f E\left[\exp(-\frac{u}{\beta_j}R)\right]\right] = 0 \tag{5.4}$$

or

$$\sum_{j=1}^{2} \frac{P_j C_j}{\beta_j} \exp\left(\frac{(u-x) r_f}{\beta_j}\right) \left[\int_0^{+\infty} \lambda x \exp\left(-x \frac{\lambda\beta_j + u}{\beta_j}\right) dx \right.$$
$$\left. - r_f \int_0^{+\infty} \lambda \exp\left(-x \frac{\lambda\beta_j + u}{\beta_j}\right) dx\right] = 0.$$

Here note that both of the integrals exists only if $\left(\lambda + u/\beta_j\right) > 0$ for $j = 1, 2$. Therefore, $u > -\lambda\beta_j$ for $j = 1$ and 2. By the assumption $\beta_1 < \beta_2$, it can be concluded that the integrals exist for

$$u > -\lambda\beta_1. \tag{5.5}$$

For any $u > -\lambda\beta_1$, (5.4) together with (5.1) would reduce to

$$\sum_{j=1}^{2} \frac{P_j C_j}{\beta_j} \exp\left(\frac{r_f}{\beta_j}(u-x)\right) \left[\left(\frac{\lambda}{\left(\left(\frac{u}{\beta_j}\right) + \lambda\right)^2}\right) - \left(\frac{\lambda r_f}{\left(\frac{u}{\beta_j}\right) + \lambda}\right)\right]$$

which equals

$$\frac{P_1 C_1}{P_2 C_2} \frac{\beta_2}{\beta_1} \exp\left(-r_f x \frac{\beta_2 - \beta_1}{\beta_1 \beta_2}\right) = -\exp\left(-r_f u \frac{\beta_2 - \beta_1}{\beta_1 \beta_2}\right) \frac{\left[\left(\frac{\lambda}{\left(\left(\frac{u}{\beta_2}\right)+\lambda\right)^2}\right) - \left(\frac{\lambda r_f}{\left(\frac{u}{\beta_2}\right)+\lambda}\right)\right]}{\left[\left(\frac{\lambda}{\left(\left(\frac{u}{\beta_1}\right)+\lambda\right)^2}\right) - \left(\frac{\lambda r_f}{\left(\frac{u}{\beta_1}\right)+\lambda}\right)\right]}$$

so that

$$\frac{P_1C_1}{P_2C_2}\frac{\beta_2}{\beta_1}\exp\left(-r_f x\frac{\beta_2-\beta_1}{\beta_1\beta_2}\right) = -\exp\left(-r_f u\frac{\beta_2-\beta_1}{\beta_1\beta_2}\right)\frac{\left(\frac{\beta_2}{u+\lambda\beta_2}\right)^2-\left(\frac{r_f\beta_2}{u+\lambda\beta_2}\right)}{\left(\frac{\beta_1}{u+\lambda\beta_1}\right)^2-\left(\frac{r_f\beta_1}{u+\lambda\beta_1}\right)}$$

and the optimality condition is

$$\frac{P_1C_1}{P_2C_2}\frac{\beta_2}{\beta_1}\exp\left(-r_f x\frac{\beta_2-\beta_1}{\beta_1\beta_2}\right) = -\exp\left(-r_f u\frac{\beta_2-\beta_1}{\beta_1\beta_2}\right)\left(\frac{u+\lambda\beta_1}{u+\lambda\beta_2}\right)^2\left(\frac{(1-\lambda r_f)\beta_2-r_f u}{(1-\lambda r_f)\beta_1-r_f u}\right).$$

Like in the normal model, divide the above equation into two parts so that

$$g(x) = h(u) \tag{5.6}$$

where

$$\begin{aligned} g(x) &= \frac{P_1C_1}{P_2C_2}\frac{\beta_2}{\beta_1}\exp\left(-r_f x\frac{\beta_2-\beta_1}{\beta_1\beta_2}\right)\\ h(u) &= -\exp\left(-r_f u\frac{\beta_2-\beta_1}{\beta_1\beta_2}\right)\left(\frac{u+\lambda\beta_1}{u+\lambda\beta_2}\right)^2\left(\frac{(1-\lambda r_f)\beta_2-r_f u}{(1-\lambda r_f)\beta_1-r_f u}\right) \end{aligned}$$

First note that $g(x)$ is always positive and, in order for (5.6) to hold,

$$\left(\frac{(1-\lambda r_f)\beta_2-r_f u}{(1-\lambda r_f)\beta_1-r_f u}\right) < 0$$

and for that to happen, there are two cases:

$$\begin{aligned} &1.\ (1-\lambda r_f)\beta_2 - r_f u > 0 \text{ and } (1-\lambda r_f)\beta_1 - r_f u < 0, \\ &2.\ (1-\lambda r_f)\beta_2 - r_f u < 0 \text{ and } (1-\lambda r_f)\beta_1 - r_f u > 0. \end{aligned} \tag{5.7}$$

Each case has different outcomes depending on $(1-\lambda r_f)$ being negative or positive. Also, note that $(1-\lambda r_f)$ determines whether the expected excess return is greater than or less than zero. Both cases will be considered separately.

Excess return greater than zero ($\lambda r_f < 1$).

Returning to (5.6), in order for this equality to hold, (5.7) must be negative and this can be achieved only if $(1-\lambda r_f)\beta_2 - r_f u > 0$ and $(1-\lambda r_f)\beta_1 - r_f u < 0$ or $(1-\lambda r_f)\beta_1/r_f < u < (1-\lambda r_f)\beta_2/r_f$. Denote this region by $I = ((1-\lambda r_f)/r_f)\,[\beta_1, \beta_2]$. Therefore, (5.6) holds only on I. Note also that u is always positive hence (5.5) is also satisfied.

For the analysis of $g(x)$ and $h(u)$, $g(x) = h(u)$ implies that the optimal decision is $u^*(x) = h^{-1}(g(x))$. For that, first consider

$$\frac{\partial h(u)}{\partial u} = \exp(r_f u(\frac{1}{\beta_2} - \frac{1}{\beta_1}))(u + \lambda\beta_1)(\beta_2 - \beta_1)\frac{Au^4 + Bu^3 - Cu^2 + Du - E}{\beta_1\beta_2(u + \lambda\beta_2)^3(r_f u - (1 - \lambda r_f)\beta_1)^2} \quad (5.8)$$

where

$$\begin{aligned} A &= r_f^3 \\ B &= r_f^2(\beta_2 + \beta_1)(2r_f\lambda - 1) \\ C &= r_f^2\lambda((1 - \lambda r_f)(\beta_1^2 + 4\beta_1\beta_2 + \beta_2^2) + \beta_1\beta_2) \\ D &= 2\lambda r_f\beta_1\beta_2(\beta_1 + \beta_2)(1 - \lambda r_f)^2 \\ E &= \beta_1^2\beta_2^2\lambda(1 - \lambda r_f)((1 - \lambda r_f)^2 + 1). \end{aligned}$$

Theorem 20 *The optimal decision $u^*(x) = h^{-1}(g(x))$ is bounded and positively increasing in x.*

Proof. *Since the feasible region is $I = ((1 - \lambda r_f)/r_f)[\beta_1, \beta_2] \subset [0, +\infty)$, $u^*(x) > 0$ is bounded. For proving $u^*(x)$ is increasing in x, it is sufficient to show that (5.8) is less than zero. First observe that every term in (5.8) is always positive except $P(u) = Au^4 + Bu^3 - Cu^2 + Du - E$. So, in order to show that $h(u)$ is decreasing on I, one needs to show that $P(u)$ is negative on I. Denote the end points of I as $I_1 = ((1 - \lambda r_f)/r_f)\beta_1$ and $I_2 = ((1 - \lambda r_f)/r_f)\beta_2$. Then*

$$\begin{aligned} P(I_1) &= -\frac{\beta_1^2\beta_2(1 - \lambda r_f)((1 - \lambda r_f)\beta_1 + \lambda r_f\beta_2)}{r_f} < 0 \\ P(I_2) &= -\frac{\beta_1\beta_2^2(1 - \lambda r_f)((1 - \lambda r_f)\beta_2 + \lambda r_f\beta_1)}{r_f} < 0 \end{aligned}$$

In this proof a method similar to the one in the normal case will be used. A polynomial of fourth degree can have at most three turning points since

$$P'(u) = 4Au^3 + 3Bu^2 - 2Cu + D$$

is a polynomial of order three.

Next, observe the following:

1. *$\lim_{u \to -\infty} P'(u) = -\infty$ and $P'(0) = D > 0$ implies that there exists at least one turning point in $(-\infty, 0]$.*

2. *$P'(I_1) = -\beta_1^2(1-\lambda r_f)((1-\lambda r_f)(\beta_2-\beta_1)+2\beta_2) < 0$ and $P'(0) = D > 0$ implies that there exists at least one turning point in $[0, I_1]$.*

3. *Observe that $P'(I_2) = \beta_2^2(1-\lambda r_f)\left((1-\lambda r_f)(\beta_2-\beta_1)-2\beta_1\right)$. Moreover, $P'(I_2) < 0$ when $(1-\lambda r_f)(\beta_2-\beta_1)-2\beta_1 < 0$ and $P'(I_2) \geq 0$ otherwise.*

   - *If $\left((1-\lambda r_f)(\beta_2-\beta_1)-2\beta_1\right) < 0$ then, $P'(u) < 0$ and $\lim_{u \to +\infty} P'(u) = +\infty$ implies that there exists at least one turning point in $[I_2, +\infty)$*
   - *If $\left((1-\lambda r_f)(\beta_2-\beta_1)-2\beta_1\right) \geq 0$ then, $P'(u) \geq 0$. This fact together with $P'(I_1) < 0$ implies that there exists at least one turning point in $[I_1, I_2]$.*

Since $P'(u)$ is a third degree polynomial by the above observations, it can be concluded that $P(u)$ has three turning points either in $(-\infty, 0]$, $[0, I_1]$ and $[I_2, +\infty)$ or $(-\infty, 0]$, $[0, I_1]$ and $[I_1, I_2]$. In the first case since $P(I_1) < 0$ and there is no turning point in I, it can be concluded that for every u in I, $P(u)$ is negative. In the second case, there is a turning point in I but $P'(u)$ is decreasing at I_1 and it starts increasing at some point in I but it increases up to $P(I_2)$ which is negative. This means for all u in I, $P(u)$ is negative also in this case. Hence, in any case, it is shown that $P(u)$ is negative. Therefore the optimal decision $u^*(x) = h^{-1}(g(x))$ is increasing in x. ■

As a result it can be concluded that $h(u)$ is decreasing on the feasible region. This result also indicates that as x increases, $u(x)$ also increases. This follows from the fact that $x = g^{-1}(h(u))$. Notice that this result is exactly the same as the one obtained in the normal model. Therefore, all of the results in the normal distribution model of Chapter 4 are also true for the exponential distribution as well.

Excess return less than zero ($\lambda r_f > 1$).

Motivated by the normal model, we tried to prove that the optimal decision $u^*(x)$ is decreasing when $\lambda r_f > 1$. It can be easily shown that $u^*(x)$ is negative when $\lambda r_f > 1$ but when we tried to prove that it is decreasing, we were supposed to show that $P(u)$

$= Au^4 + Bu^3 + Cu^2 + Du + E$ with the same parameters from the previous section, is positive on the feasible region. Since $P(u)$ is a polynomial of fourth degree we tried to find its roots and tried to order them. We used various methods from abstract algebra, but we couldn't prove the desired result using this line of analysis. However, since the optimal decision is negatively decreasing when the mean excess return is negative for any distribution, the following theorem follows trivially from Theorem 3.2.2.

Theorem 21 *The optimal decision $u^*(x) = h^{-1}(g(x))$ is bounded and negatively decreasing in x.*

5.2 General Distribution with Concave Utility Functions

In this section we want to generalize the concepts we introduced so far. Instead of using exponential utility functions, we will generalize the model to any concave utility function. Concave utility functions are widely used in the literature and they are mostly used to explain the behaviors of risk averse investors. We assume that the investor will have m different utilities each of which will stand for a different risk preference. In mathematical terms, the jth utility of the investor with the given wealth level x will be represented as $U(j, x)$. The expected utility for m different risk preferences is

$$g(x, u) = \sum_{j=1}^{m} P_j E\left[U\left(j, r_f x + (R^e)' u\right)\right].$$

But, $U(j, x)$ is now any concave function, twice differentiable in x for utility j. For technical reasons, assume that $U(j, x)$ is positive for all j. If the chosen class of utility functions are exponential utility functions, then

$$U(j, x) = K_j - C_j \exp(-x/\beta_j)$$

where $-U''(j, x)/U'(j, x) = 1/\beta_j$ denotes the risk preference for the market state j.

For concavity analysis, consider the gradient

$$\nabla_k g(x, u) = \frac{\partial g(x, u)}{\partial u_k} = \sum_{j=1}^{m} P_j E\left[R^e_k U'(j, r_f x + (R^e)' u)\right]$$

so that the Hessian is

$$H_{k,l} = \frac{\partial^2 g(x, u)}{\partial u_k \partial u_l} = \sum_{j=1}^{m} P_j E\left[R^e_k R^e_l U''(j, r_f x + (R^e)' u)\right].$$

For any vector $z = [z_1, z_2, \cdots, z_n]$, note that

$$z'Hz = \sum_{j=1}^{m} P_j E\left[\left(\sum_{k=1}^{n} z_k R_k^e\right)^2 U''(j, r_f x + (R^e)' u)\right] \leq 0$$

since we assumed that $U(j, x)$ is a concave function in x for all j. Therefore, the optimality condition is given by

$$\frac{\partial g(x,u)}{\partial u_k} = \sum_{j=1}^{m} A_j^k(x,u) = A^k(x,u) = 0$$

for all k where

$$A_j^k(x,u) = P_j E\left[R_k^e U'(j, r_f x + (R^e)' u)\right].$$

By a similar argument in Chapter 3, one can show that he optimal decision $u^*(x) = [u_1^*(x), u_2^*(x), \cdots, u_n^*(x)]$ is a unique continuously differentiable function in x whenever it exists.

For the single asset case ($n = 1$), observe that

$$\begin{aligned} A(x,0) &= \sum_{j=1}^{m} A_j(x,0) = \sum_{j=1}^{m} P_j E\left[R^e U'(j, r_f x)\right] \\ &= E\left[R^e\right] \sum_{j=1}^{m} U'(j, r_f x) P_j \end{aligned}$$

since the utility is deterministic as a function of x. The sign of $A(x,0)$ depends on the sign of $E\left[R^e\right] = \mu$ due to the fact that $U(j,x)$ is decreasing in x. Consider the derivative of $A(x,u)$ with respect to u. It satisfies

$$\frac{\partial A(x,u)}{\partial u} = \sum_{j=1}^{m} P_j E\left[(R^e)^2 U''(j, r_f x + R^e u)\right] \leq 0$$

since $U(j,x)$ is concave. Therefore, $A(x,u)$ is decreasing in u. Hence, if $\mu > 0$, then the optimal decision $u^*(x)$ is positive for all x since $A(x,u)$ is decreasing and $A(x,0)$ is positive. With a similar reasoning we can deduce that if $\mu < 0$, then the optimal decision $u^*(x)$ is negative for all x since $A(x,u)$ is decreasing and $A(x,0)$ is negative.

For the monotonicity structure of the optimal decision $u^*(x)$, we tried to obtain similar results with the exponential utility functions case. But since we lacked the characteristic attributes of exponential utility functions, we could not obtain a similar characterization.

Chapter 6

CONCLUSIONS

Decision making under uncertainty has always been important line of research. Although Expected Utility Theory (EUT) is widely used in the research community, there is growing interest in non-Expected Utility Theories due to shortcomings of certain assumptions of EUT. The idea that the investors behave exactly the same under uncertainty was criticized with empirical evidence. Therefore, for modeling the behaviors of the investors in such setting, it is reasonable to add a random component to the structure of the model.

There have been a number of suggestions to model stochastic choice and in this thesis we use the so called "random preference approach". Loomes and Sugden [17] claim that when facing a decision problem, individuals are often uncertain about their preference. It basically means that the decision makers do not have a unique preference for their utilities, but there is a randomness involved about their preferences. Motivated with this idea, we used the random preference approach in portfolio optimization. Utility based portfolio selection is a popular approach used in the research community and we followed this line of research in this thesis. We basically combined the random preference approach with expected utility maximization. We mainly used exponential utility functions to represent the behavior of investors facing investment decisions. It is widely used both in theory and practice. Although the utility of the investor is always exponential, the risk preference parameter or the risk tolerance is random. If this parameter is not random, Çanakoğlu and Özekici [7] showed that the optimal decision of the investor is independent of the initial wealth. But, we proved that this is no longer true if the risk preference of the investor is random.

We obtained cthe haracterization of the optimal policy and the expected utility. We showed that the expected utility function is a concave function of the decision variables and, therefore, the optimal decision is acquired by setting the gradient equal to zero. We also showed that the optimal policy is unique and continuous. In Chapter 3, we assumed that the return of the risky assets have an arbitrary distribution, and we showed that the optimal decision is positively increasing in the wealth level if the mean excess return is positive, and it is negatively decreasing otherwise. In Chapter 4, we analyzed the case where the returns of the risky assets are described by a multivariate normal distribution. The results we found in Chapter 3 are still valid and the structure of the optimal policy depends on the "adjusted mean excess return". We showed that if the adjusted mean excess return is positive, then the optimal decision is positively increasing in the wealth level, and it is negatively decreasing otherwise. We also illustrated our results by some examples.

In Chapter 5, we analyzed some extensions including a risky asset with exponential return distribution and showed its consistency with the results found in Chapter 3. We also analyzed a generalization where the utility of the investor is described by any concave utility function rather than the exponential utility function. For the single asset setting, we showed that the optimal decision is positive if the mean excess return is positive, and it is negative otherwise.

Finally, for future research, one can extend the idea we presented in this thesis in several directions. Here, we assumed that the behavior of the investor is random but the returns of the risky assets are independent of states of the market. This idea can be extended by assuming that the returns of the risky assets are dependent on the market state as well as the risk preferences. Another idea might be extending this problem to the multiperiod setting. Although we already showed that the value function can not be evaluated analytically while considering the next period, a numerical analysis of multiperiod case can be performed. Most of our research concentrated on the normal distribution model, considering cases with other important distributions may also end up with useful characterizations on the structure of the optimal policy.

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VITA

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