Geometric Measure Theory

And

The Double Bubble Conjecture

by

Metin Alper Gür

A Thesis Submitted to the

Graduate School of Sciences and Engineering

in Partial Fulfillment of the Requirements for

the Degree of

Master of Science

in

Mathematics

Koç University

September 2009

Koç University

Graduate School of Sciences and Engineering

This is to certify that I have examined this copy of a master's thesis by

Metin Alper Gür

and have found that it is complete and satisfactory in all respects,

and that any and all revisions required by the final

examining committee have been made.

Committee Members:

Barış Coşkunüzer, Ph. D. (Advisor)

A. Muhammed Uludağ, Ph. D.

Sinan Ünver, Ph. D.

Date: 10 September, 2009

ABSTRACT

In this thesis we will study the geometric measure theory and the double bubble conjecture. Our treatment of the geometric measure theory will be introductory and yet the theorems and proofs will be studied thoroughly. We will then, as an application of the geometric measure theory, present the double bubble conjecture in $\mathbb{R}^2$ and the double bubble conjecture in $\mathbb{R}^3$.

The geometric measure theory is a branch of differential geometry which deals with maps and surfaces that are not necessarily smooth. It extends the notions of differential geometry with the use of measure theory. One of the most important problems of the geometric measure theory, that has served as the starting point of this field, is to find a spanning set of a given boundary in the Euclidean space with the least area and to decide whether this set has any geometric significance and whether it is unique.

The double bubble conjecture in $\mathbb{R}^2$ asserts that the standard double bubble in $\mathbb{R}^2$ is the unique perimeter minimizing enclosure of two given quantities of area in $\mathbb{R}^2$. The double bubble conjecture in $\mathbb{R}^2$ has been proven jointly by J. Foisy, M. Alfaro, J. Brock, N. Hodges, and J. Zimba. We will give a complete proof of this conjecture and before giving a sketch of the proof of the double bubble conjecture in $\mathbb{R}^3$ we will study the structure theorem which has been proven by M. Hutchings. The structure theorem asserts that except for the standard double bubble there is only one hypersurface that may be a candidate for an area minimizing set enclosing two quantities of volume in $\mathbb{R}^3$. M. Hutchings, F. Morgan, M. Ritoré, and A. Ros, by using an original stability argument, have ruled out the possibility of any minimizer other than the standard double bubble and hence they have showed that the standard double is the unique area minimizing double bubble enclosing and separating two given quantities of volume in $\mathbb{R}^3$.

ÖZET

Bu tezde geometrik ölçü teorisini ve çift kabarcık zanını çalışacağız. Geometrik ölçü teorisini işleyişimiz başlangıç düzeyinde olacak ancak teoriler ve kanıtları detaylı bir şekilde çalışılacaktır. Daha sonra, geometrik ölçü teorisinin bir uygulaması olarak, iki ve üç boyutlu Öklid uzaylarında ki çift kabarcık zanlarını tanıtacağız.

Geometrik ölçü teorisi diferansiyel geometrinin, düzgün olmayabilen fonksiyon ve yüzeyler ile uğraşan, bir koludur. Bu alan diferansiyel geometrinin fikirlerini, ölçü teorisini kullanarak, genişletmektedir. Geometrik ölçü teorisinin başlangıç noktasını oluşturan en önemli sorularından biri, Öklid uzayında verilen bir sınıra sahip en düşük alanlı kümeyi bulmak, bu kümenin herhangi bir geometrik özelliği olup olmadığına ve tek olup olmadığına karar vermektir.

İki boyutlu Öklid uzayında ki çift kabarcık zanı standart çift kabarcığın verilen iki alanı kaplayan ve ayıran en düşük çeper uzunluğuna sahip tek küme olduğunu öne sürmektedir. İki boyutlu Öklid uzayında ki çift kabarcık zanı J. Foisy, M. Alfaro, J. Brock, N. Hodges ve J. Zimba tarafından beraberce gösterilmiştir. Bu zanın tam kanıtını vereceğiz ve üç boyutlu Öklid uzayında ki çift kabarcık zanının kanıtının taslağını vermeden önce M. Hutchings tarafından kanıtlanan yapı teorisini çalışacağız. Yapı teorisi, standart çift kabarcık haricinde üç boyutlu Öklid uzayında iki hacmi kaplayp, sınır yüzeyinin alanı en küçük olabilecek tek bir alternatifin olduğunu öne sürmektedir. M. Hutchings, F. Morgan, M. Ritoré ve A. Ros, orjinal bir stabilite savı kullanarak, standart çift kabarcık haricinde ki olasılığı yok saymışlar ve böylece standart çift kabarcığın üç boyutlu Öklid uzayında verilen iki hacmi kaplayan ve ayıran minimum yüzey alanlı tek küme olduğunu göstermişlerdir.

ACKNOWLEDGEMENTS

I would like to express my gratitude to my advisor Asst. Prof. Coşkunüzer for helping me find my true calling in mathematics. Many capable minds have perished and many others will, in the vast and unforgiving maze of mathematics because of misguidance.

I would like to thank Assoc. Prof. Uludağ and Asst. Prof. Ünver for their participation in my thesis committee.

Last but not least I would like to thank my family for their limitless support throughout my education.

Contents

List of Tables

List of Figures

NOMENCLATURE

$\mathcal{B}_{\mathbb{R}^n}$	Borel σ-algebra on $\mathbb{R}^n$
$\mathcal{L}^n$	$n-$ dimensional Lebesque measure in $\mathbb{R}^n$
$\mathcal{H}^s$	$s-$ dimensional Hausdorff measure
$\alpha(n)$	$\mathcal{L}^n$-measure of the unit ball in $\mathbb{R}^n$
Γ	Gamma function
$\mathcal{H}_{\dim}$	Hausdorff dimension
P_a	plane through the origin perpendicular to a
S_a	Steiner symmetrization of A with respect to the plane P_a
Θ^m	m-dimensional density
$\mathcal{H}^n \llcorner A$	restriction of measure to a set $A \subseteq \mathbb{R}^n$
χ	characteristic function
ap lim	approximate limit
ap D	approximate differential
$\mathrm{OP}(\mathbb{R}^n)$	collection of open sets in $\mathbb{R}^n$
$\mathbb{H}^k$	upper half space in $\mathbb{R}^k$
$\mathbb{H}^k_+$	open upper half space in $\mathbb{R}^k$
∂	boundary
spt f	support of a function f
Lip f	Lipschitz constant of a function f
C_c	real valued functions of compact support
C_c^∞	real valued infinitely differentiable functions of compact support
$[\![L]\!]$	jacobian of a linear transformation L
$Jf(x)$	jacobian of a differentiable function f at x
$\mathrm{Tan}(E,a)$	tangent cone of E at a
$\mathrm{Tan}^m(E,a)$	approximate tangent cone of E at a
$\mathcal{T}_x(\mathbb{R}^n)$	tangent space to $\mathbb{R}^n$ at x
$\mathrm{L}^k(V)$	vector space of k-linear maps on vector space V
$\mathrm{A}^k(V)$	vector space of k-linear alternating maps on vector space V
$\mathcal{D}^m$	C^∞ differential m-forms with compact support
$\mathcal{D}_m$	dual space of $\mathcal{D}^m$, m-dimensional currents
$\mathcal{E}_m$	m-dimensional currents of compact support
$\mathcal{R}_m$	rectifiable currents
$\mathcal{P}_m$	integral polyhedral chains
$\mathbf{I}_m$	integral currents
$\mathcal{F}_m$	integral flat chains
$\mathbf{M}$	mass norm

$\mathbf{F}$	flat norm
F	generalized flat norm
$\mathbf{N}_m$	normal currents
$\mathbf{F}_m$	real flat chains
$\mathbf{R}_m$	real flat chains with finite mass
$\mathbf{P}_m$	real polyhedral chains
$\langle T, u, r+\rangle$	slice of a current T by a Lipschitz function u
$\mathcal{F}_m^{\mathrm{loc}}$	locally integral flat chains
$\mathcal{R}_m^{\mathrm{loc}}$	locally rectifiable currents
$\mathbf{I}_m^{\mathrm{loc}}$	locally integral currents
δ	first variation
A_n	least area function
P	least perimeter function
A_n^0	surface area of the standard double bubble in $\mathbb{R}^n$ $(n \geq 3)$
P_0	perimeter of the standard double bubble in $\mathbb{R}^2$

1. INTRODUCTION

Geometric measure theory is a branch of differential geometry which deals with maps and surfaces, that are not necessarily smooth, by using the techniques of measure theory. The most notable mathematicians who have worked on the subject include Herbert Federer, Wendell Fleming, Fred Almgren, and Ennio De Giorgi.

The problem that serves as an archetype for problems in geometric measure theory is to find the surface of least area which spans a given boundary in $\mathbb{R}^n$. The greatest challenge to surmount towards solving the problem is to find a workable space of surfaces.

Towards solving the problem, originally, one considered the two dimensional surfaces, defined as mappings of the disk. In 1930's J.Douglas in [8] and T.Rado in [24] showed that every smooth Jordan curve bounds a disk of least area and then Brian White showed in [28] that in higher dimensional surfaces the geometric measure theory solution also solves the mapping problem. Even though considering surfaces as mappings had substantial success, one of its greatest drawbacks was that the natural topology lacks compactness properties.

The brute force method for finding a surface of least area with a given boundary has the following steps:

(1) take a sequence of surfaces with areas decreasing to the infimum,
(2) choose a convergent subsequence,
(3) show that the limit surface is the surface of least area.

Since the space of surfaces realized as mappings of the disk is not compact, by sending out thin tentacles toward every point of rational coordinates, the sequence could include all of $\mathbb{R}^n$ in its closure.

A rectifiable current is an m-dimensional oriented surface of the geometric measure theory. The applicable functions $f\colon \mathbf{R}^m \to \mathbf{R}^n$ need not be smooth, but merely Lipschitz,

$$|f(x) - f(y)| \le C|x - y| \text{ for all } x, y \in \mathbb{R}^m,$$

for some constant $C > 0$.

There is an m-dimensional measure on $\mathbb{R}^n$, called Hausdorff measure, $\mathcal{H}^m$ which agrees with the classical notion of a surface area of an embedded manifold, but it is defined on all subsets of $\mathbb{R}^n$.

A Borel subset B of $\mathbb{R}^n$ is called $(\mathcal{H}^m, m)$-rectifiable if B is a countable union of Lipschitz images of bounded subsets of $\mathbb{R}^m$ with finite Hausdorff measure. A rectifiable set is a good generalization of a surface of differential geometry because it has a tangent plane at almost every point.

A rectifiable current is an oriented rectifiable set with integer multiplicites, finite area, and compact support. We can integrate a smooth differential form φ over an oriented rectifiable set S, and hence view S as a current : a linear functional on differential forms,

$$\varphi \mapsto \int_S \varphi.$$

This approach induces a topology on the space of surfaces that is dual to a topology on differential forms. The compactness theorem shows that this new topology has useful compactness properties.

There is also a boundary operator ∂ from m-dimensional rectifiable currents to $(m-1)$-dimensional rectifiable currents given by

$$(\partial S)(\varphi) = S(d\varphi)$$

where $d\varphi$ is the differential of φ. By Stoke's theorem, this definition of boundary, ∂S, coincides with the usual notion of boundary for smooth, compact manifold with boundary.

The compactness theorem asserts that the set of all m-dimensional rectifiable currents T in a closed ball in $\mathbb{R}^n$, such that the boundary ∂T is also rectifiable and such that the area of both T and ∂T are bounded by a positive constant c, is compact in a weak topology.

The regularity theorem states that the area minimizing rectifiable currents are not arbitrary objects and have geometric significance. Namely, a two dimensional area minimizing rectifiable current in $\mathbb{R}^3$ is a smooth embedded manifold and for $m \leq 6$, an m-dimensional area minimizing rectifiable current in $\mathbb{R}^{m+1}$ is a smooth embedded manifold.

The Double Bubble Conjecture in $\mathbb{R}^2$ [14] states that the set having the least perimeter such that it encloses and separates two given quantities of area is the standard double bubble. The Double Bubble Conjecture in $\mathbb{R}^2$ was jointly proved in 1990 by J. Foisy, M. Alfaro, J. Brock, N. Hodges, and J. Zimba. In 1997 Michael Hutchings in his paper [17] showed that minimal double bubbles in $\mathbb{R}^n$ have no empty chambers and the enclosed regions are connected in certain cases. He concluded his paper with the proof of the structure theorem which asserts that any minimal double bubble in $\mathbb{R}^n$ is either the standard double bubble or a surface of revolution about some line consisting of a topological sphere with a single tree of annular bands attached. In the year of 2000 Michael Hutchings, Frank Morgan, Manuel Ritoré, and Antonio Ros have jointly proven in [18] that the standard double bubble in $\mathbb{R}^3$ is the unique area minimizing double bubble enclosing two volumes in $\mathbb{R}^3$.

1.1. Synopsis.

Section 2: We will define the s-dimensional Hausdorff measure in $\mathbb{R}^n$. We will introduce the s-dimensional Hausdorff dimension of a set in $\mathbb{R}^n$ and point out in section 4, after some preliminary work, that the Hausdorff dimension generalizes the dimension of an embedded manifold. Hausdorff measure in $\mathbb{R}^n$ is a generalization of the Lebesque measure $\mathcal{L}^n$ in $\mathbb{R}^n$. In particular, $\mathcal{H}^n$-measure of a set in $\mathbb{R}^n$ coincides with its $\mathcal{L}^n$-measure. Hausdorff measure also agrees with the classical mapping volume of an embedded manifold. Namely, $\mathcal{H}^s$-measure of an s-dimensional embedded manifold in $\mathbb{R}^n$ equals to its volume. The main references for this section are the books by Evans & Gariepy [9], Krantz & Parks [19], and Morgan [22].

Section 3: This section gives a quick overview of manifolds in the Euclidean space and the integral of a scalar function over a manifold. First we will present the k-dimensional volume of a parallelepiped in $\mathbb{R}^n$ and move on to describe the k-dimensional manifold in $\mathbb{R}^n$ and its volume. Many of the definitions and results obtained in this section will be used throughout this thesis. The main reference for this section is the book by Munkres [23].

Section 4: In this section we will study the Lipschitz functions. Lipschitz functions in geometric measure theory serve a purpose similar to smooth functions in manifold theory. In particular, by Rademacher's theorem (4.2.1) Lipschitz function is differentiable a.e. and hence it can be used as a coordinate map of an embedded (Lipschitz) manifold. We will then study the area formula which connects the manifold theory with the geometric measure theory. In particular, by using the area formula we can show that the Hausdorff measure of an embedded manifold is equal to its volume and that its Hausdorff dimension coincides with its classical dimension. We will introduce the coarea formula, which is a penultimate ("curvilinear") generalization of the Fubini's theorem. The coarea formula implies that the $\mathcal{L}^n$-measure of a measurable set $A \subseteq \mathbb{R}^n$ is equal to the integral of the Hausdorff measure of the restriction of A to various level sets. Finally we will define the rectifiable sets, which are the generalized surfaces of the geometric measure theory and present some of their properties. The main references for this section are the books by Evans & Gariepy [9] and Federer [10].

Section 5: This section gives a quick overview of differential forms and the integral of forms over manifolds. The culmination of our effort in the section is the generalized Stoke's theorem, which will be further generalized in section 6. The generalized Stoke's theorem relates the integral of a differential form over the boundary of the manifold with the integral of its differential over the manifold itself. The main reference for this section is the book by Munkres, [23].

Section 6: In this section we will study the currents. The space of currents is the dual of space of differential forms with compact support. A current can be seen as a generalized surface since every oriented submanifold of $\mathbb{R}^n$ of compact support defines by integration a linear functional on forms. Being a generalized surface, a current admits a new definition of boundary and by Stoke's theorem this new definition coincides with the classical notion of the boundary of a manifold. We will define the slice of a current and present some theorems that give an ultimate generalization of the coarea formula and hence of the Fubini's theorem. The slice of a current, when it is a submanifold of the Euclidean space, is the restriction of the manifold to the image set of a Lipschitz map. We will introduce the deformation theorem which implies, when restricted to C^1 manifolds with compact support, that these manifolds can be approximated by simplices. We will prove the closure theorem which gives useful connections between different types of currents and the compactness theorem which asserts that a certain collection of "nice" currents is compact in the weak topology. Then using the compactness theorem we will prove one of the most important theorems in geometric measure theory asserting that an area minimizing "nice" current with a given boundary exists. We will give a quick overview of Calculus of variations, just enough for our present purpose, and derive the minimal surface equation. We will give survey of results asserting, in certain cases, that the solution of the area minimization is a smooth embedded manifold, except perhaps for a set of small Hausdorff dimension. We will generalize our definition of currents to admit surfaces that are not compact; these currents locally coincide with the previously defined currents with compact support. In the last subsection we will prove the most famous result in the calculus of variations:

that a straight line is the shortest distance between two points. The main references for this section are the books by Federer [10], Gelfand & Fomin [15], and Morgan [22].

Section 7: In this section we will present the least perimeter function and the proof of the double bubble conjecture in $\mathbb{R}^2$. One of the consequences of F.J. Almgren's work [2] is the existence of a set in $\mathbb{R}^n$ enclosing and separating a given m quantities of volume in $\mathbb{R}^n$ with the minimal surface area. Thus we can define the least perimeter function of two variables giving the value of the least perimeter of an enclosure of two quantities of area in $\mathbb{R}^2$. We will then present our proof of a known fact that this function is continuous. J. Taylor in [27] has showed that an area minimizing bubble cluster B in $\mathbb{R}^3$ consists of real analytic constant mean curvature surfaces meeting smoothly in threes at $120°$ angles along smooth curves. Then F. Morgan has showed in [20] that a perimeter minimizing bubble cluster in $\mathbb{R}^2$ consists of arcs of circles (or line segments) meeting in threes at angles of $120°$. Using the existence of the least perimeter function and the result of Morgan, J. Foisy, M. Alfaro, J. Brock, N. Hodges, and J. Zimba have jointly showed in [14] that the standard double bubble is the unique perimeter minimizing enclosure of two areas in $\mathbb{R}^2$.

Section 8: In this section we will study the structure of area minimizing double bubbles in the Euclidean space. We will present a rough sketch of the proof of the double bubble conjecture in $\mathbb{R}^3$, and give a survey of double bubble problems in general ambient space. Almgren has showed in [2] that given m volumes in $\mathbb{R}^n$, one can find an enclosure of least surface area. M. Hutchings has proved in [17] that for the case $m = 2$ the enclosure is either the standard double bubble or a surface of revolution consisting of a topological sphere with a single tree of annular bands attached. Hutchings has also presented several properties of the least area function and introduced their implications, the most important of which is that in a least area enclosure the exterior must be connected. We will give a rough sketch of the proof of the double bubble conjecture in $\mathbb{R}^3$ which has been shown jointly by M. Hutchings, F. Morgan, M. Ritoré, and A. Ros in [18]. In the proof of the double bubble conjecture in $\mathbb{R}^3$ the major difficulty is ruling out the surface of revolution consisting of a topological sphere with a single tree of annular bands attached, which is the only alternative of the standard double bubble according to [17]. This has been accomplished by an original stability argument (8.4.1) and consequently the standard double bubble in $\mathbb{R}^3$ is the unique area minimizing set enclosing and separating two volumes in $\mathbb{R}^3$. Finally we will give a survey of double bubble problems in general ambient space.

2. HAUSDORFF MEASURE, ISODIAMETRIC INEQUALITY, DENSITY

In this section we will introduce the s-dimensional Hausdorff measure of objects in $\mathbb{R}^n$. The Hausdorff measure was invented by Felix Hausdorff out of necessity for measuring lower dimensional objects in the Euclidean space. We will define the Hausdorff dimension, but postpone until the fourth section the observation of how it generalizes our usual notion of dimension of a set in the Euclidean space. We will introduce the Steiner symmetrization which is a procedure of taking a symmetrization of an object with respect to a hyperplane in the Euclidean space. And as a result of our study of Steiner symmetrization we will be able to prove the Isodiametric inequality which states that among all objects of fixed diameter the sphere has the largest volume. The ultimate result of this section is the proof that n-dimensional Hausdorff measure in $\mathbb{R}^n$ coincides with the n-dimensional Lebesque measure in $\mathbb{R}^n$. In the last part of this section we will generalize the notion of continuity and differentiability of functions with the use of density of sets and show that every Lebesque measurable function is almost continuous at almost every point in its domain. The main references for this section are the books by Evans & Gariepy [9], Krantz & Parks [19], and Morgan [22].

2.1. **Measures And Measurable Functions.**

Definition 2.1.1. *Let X denote a set, and 2^X the collection of all subsets of X. A mapping $\mu\colon 2^X \to [0,\infty]$ is called a measure on X if*

(1) $\mu(\emptyset) = 0$, *and*

(2) $\mu(A) \leq \sum_{k=1}^{\infty} \mu(A_k)$ *whenever* $A \subset \bigcup_{k=1}^{\infty} A_k$.

A real-valued set function satisfying the property (2) *is called countably subadditive.*

Definition 2.1.2. *A set $A \subset X$ is called μ-measurable if for each set $B \subset X$,*

$$\mu(B) = \mu(B \cap A) + \mu(B \cap A^c).$$

Definition 2.1.3. *A collection $\mathcal{A} \subset 2^X$ is called a σ-algebra if*

(1) $\emptyset$, $X \in \mathcal{A}$;

(2) $A \in \mathcal{A}$ *implies* $X - A \in \mathcal{A}$;

(3) $A_k \in \mathcal{A}(k = 1, 2, \ldots)$ *implies* $\bigcup_{k=1}^{\infty} A_k \in \mathcal{A}$.

Definition 2.1.4. *The Borel σ-algebra of $\mathbb{R}^n$ is the smallest σ-algebra of $\mathbb{R}^n$ containing the open subsets of $\mathbb{R}^n$. The elements of the Borel σ-algebra are called Borel sets.*

Definition 2.1.5.

(1) *A measure μ on X is regular if for each set $A \subseteq X$ there exists a μ-measurable set B such that $A \subseteq B$ and $\mu(A) = \mu(B)$.*

(2) *A measure μ on $\mathbb{R}^n$ is called Borel if every Borel set is μ-measurable.*

(3) *A measure μ on $\mathbb{R}^n$ is Borel regular if μ is Borel and for each $A \subseteq \mathbb{R}^n$ there exists a Borel set B such that $A \subseteq B$ and $\mu(A) = \mu(B)$.*

(4) *A measure μ on $\mathbb{R}^n$ is a Radon measure if μ is Borel regular and $\mu(K) < \infty$ for each compact set $K \subset \mathbb{R}^n$.*

Definition 2.1.6. *Let X be a set and Y a topological space. Assume μ is a measure on X. A function $f\colon X \to Y$ is called μ-measurable if for each open subset $U \subset Y$, $f^{-1}(U)$ is μ-measurable.*

Theorem 2.1.7. *Let $f\colon \mathbb{R}^n \to [0,\infty]$ be $\mathcal{L}^n$ (Lebesque)-measurable, then*

$$A = \{(x,y)\colon x \in \mathbb{R}^n,\ 0 \le y \le f(x)\}$$

is $\mathcal{L}^{n+1}$-measurable.

2.2. **Hausdorff Measure.**

Definition 2.2.1.

(1) *Let $A \subseteq \mathbb{R}^n$, $0 \le s < \infty$, $0 < \delta \le \infty$. Define*

$$\mathcal{H}^s_\delta(A) = \inf\left\{\sum_{j=1}^{\infty} \alpha(s)\left(\frac{diam\ C_j}{2}\right)^s : A \subseteq \bigcup_j C_j,\ diam\ C_j \le \delta\right\}$$

where

$$\alpha(s) = \frac{\pi^{s/2}}{\Gamma(s/2+1)}, \quad \Gamma(s) = \int_0^\infty e^{-x}x^{s-1}\,dx.$$

Since any subset $A \subseteq \mathbb{R}^n$ can be covered by countably many subsets of $\mathbb{R}^n$ with diameter $\le \delta$, $\mathcal{H}^s_\delta(A)$ exists for each $\delta > 0$.

(2) *$\mathcal{H}^s(A) = \lim_{\delta\to 0} \mathcal{H}^s_\delta(A) = \sup_{\delta>0} \mathcal{H}^s_\delta(A)$ exists and it is called the s-dimensional Hausdorff measure on $\mathbb{R}^n$.*

Remark 2.2.2.

- $\delta \to 0$ is required so that the coverings approximate the local geometry of the set A.
- Note that
$$\alpha(n) = \frac{\pi^{n/2}}{\Gamma(n/2+1)} = \mathcal{L}^n(B(x,1))$$
is the Lebesque measure of the unit ball in $\mathbb{R}^n$.

Theorem 2.2.3. *$\mathcal{H}^s$ is a Borel regular measure* $(0 \le s \le \infty)$.

Proof.

(1) Claim #1: $\mathcal{H}^s_\delta$ is a measure.

Proof of Claim # 1. Choose $\{A_k\}_{k\in\mathbb{N}} \subseteq \mathbb{R}^n$, suppose that $A_k \subseteq \bigcup_j C_j^k$, diam $C_j^k \leq \delta$ and

$$\mathcal{H}^s_\delta(A_k) + \frac{\epsilon}{2^k} > \sum_{j=1}^{\infty} \alpha(s) \left(\frac{\text{diam } C_j^k}{2}\right)^s$$

hold for eack $k \in \mathbb{N}$. WLOG we may assume that for each $k \in \mathbb{N}$, $\mathcal{H}^s_\delta(A_k) < \infty$. Then

$$\bigcup_k A_k \subseteq \bigcup_{j,k} C_j^k, \quad \text{diam } C_j^k \leq \delta$$

$$\begin{aligned} \mathcal{H}^s_\delta\left(\bigcup_k A_k\right) &\leq \sum_{j,k} \alpha(s) \left(\frac{\text{diam } C_j^k}{2}\right)^s \\ &\leq \sum_k \mathcal{H}^s_\delta(A_k) + \frac{\epsilon}{2^k} \leq \sum_k \mathcal{H}^s_\delta(A_k) + \epsilon. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary $\mathcal{H}^s_\delta$ is countably subadditive.

(2) Claim #2: $\mathcal{H}^s$ is a measure.

Proof of Claim # 2. We need to show that $\mathcal{H}^s$ is countably subadditive. Let $\{A_k\}_{k\in\mathbb{N}} \subseteq \mathbb{R}^n$, then for every $\delta > 0$

$$\mathcal{H}^s_\delta\left(\bigcup_{k=1}^{\infty} A_k\right) \leq \sum_k \mathcal{H}^s_\delta(A_k) \leq \sum_k \mathcal{H}^s(A_k).$$

$\Rightarrow \mathcal{H}^s\left(\bigcup_k A_k\right) \leq \sum_k \mathcal{H}^s(A_k)$.

(3) Claim # 3: $\mathcal{H}^s$ is a Borel measure.

Proof of Claim # 3. We want to show that every Borel set is measurable and in order to accomplish this we need to show that $\mathcal{H}^s$ satisfies the Carathéodory's criterion. In particular, we need to show that given two sets $A, B \subseteq \mathbb{R}^n$ such that dist$\,(A, B) > 0$, $\mathcal{H}^s$ satisfies

$$\mathcal{H}^s(A \cup B) = \mathcal{H}^s(A) + \mathcal{H}^s(B).$$

Choose A, $B \subseteq \mathbb{R}^n$, dist$(A, B) > 0$ and let $0 < \delta < d/4$ where $d = \text{dist}\,(A, B)$. Let $\{C_k\}_{k\in\mathbb{N}}$ be a sequence of subsets of $\mathbb{R}^n$ such that $A \cup B \subseteq \bigcup_k C_k$ with diam $C_k \leq \delta$ for each $k \in \mathbb{N}$. Define

$$\mathcal{A} = \{C_j \mid A \cap C_j \neq \emptyset\} \quad \mathcal{B} = \{C_j \mid B \cap C_j \neq \emptyset\}.$$

Then

$$A \subseteq \bigcup_{C_j \in \mathcal{A}} C_j, \quad B \subseteq \bigcup_{C_j \in \mathcal{B}}, \quad \mathcal{A} \cap \mathcal{B} = \emptyset.$$

$$\begin{aligned} &\sum_{j=1}^{\infty} \alpha(s) \left(\frac{\text{diam } C_j}{2}\right)^s \\ &\geq \sum_{C_j \in \mathcal{A}} \alpha(s) \left(\frac{\text{diam } C_j}{2}\right)^s + \sum_{C_j \in \mathcal{B}} \alpha(s) \left(\frac{\text{diam } C_j}{2}\right)^s \\ &\geq \mathcal{H}^s_\delta(A) + \mathcal{H}^s_\delta(B). \end{aligned}$$

$$\Rightarrow \mathcal{H}^s_\delta(A \cup B) \geq \mathcal{H}^s_\delta(A) + \mathcal{H}^s_\delta(B)$$
$$\Rightarrow \mathcal{H}^s(A \cup B) \geq \mathcal{H}^s_\delta(A) + \mathcal{H}^s_\delta(B) \text{ where } \delta > 0 \text{ is arbitrary}$$
$$\Rightarrow \mathcal{H}^s(A \cup B) \geq \mathcal{H}^s(A) + \mathcal{H}^s(B).$$

So $\mathcal{H}^s$ is a metric outer measure and by the Caratéodory's criterion every Borel set is measurable.

(4) Claim # 4: $\mathcal{H}^s$ is a Borel regular measure.

Proof of Claim # 4. Since diam $C =$ diam $\overline{C}$ for every $C \subseteq \mathbb{R}^n$ we can define $\mathcal{H}^s_\delta$ as follows:

$$\mathcal{H}^s_\delta(A) = \inf \left\{ \sum_j \alpha(s) \left(\frac{\operatorname{diam} C_j}{2} \right)^s : A \subseteq \bigcup_j C_j,\ \operatorname{diam} C_j \leq \delta,\ C_j \text{ closed} \right\}.$$

Let $A \subseteq \mathbb{R}^n$, $\mathcal{H}^s(A) < \infty$ then $\mathcal{H}^s_\delta(A) < \infty$ for every $\delta > 0$. For each $k \in \mathbb{N}$ choose closed sets $\{C^k_j\}_{j \in \mathbb{N}}$ so that diam $C_j \leq 1/k$ for every $j \in \mathbb{N}$,

$$A \subseteq \bigcup_j C^k_j \text{ and } \sum_j \alpha(s) \left(\frac{\operatorname{diam} C^k_j}{2} \right)^s < \mathcal{H}^s_{1/k}(A) + 1/k.$$

Define $A_k = \bigcup_j C^k_j$, $B = \bigcap_k A_k$ then $A \subseteq B$ is a Borel set containing A. Now we need to show that their Hausdorff measure coincide.

$$\mathcal{H}^s_{1/k}(B) \leq \sum_j \alpha(s) \left(\frac{\operatorname{diam} C^k_j}{2} \right)^s \leq \mathcal{H}^s_{1/k}(A) + 1/k$$

holds for every $k \in \mathbb{N}$,

$$\mathcal{H}^s(B) = \lim_{k \to \infty} \mathcal{H}^s_{1/k}(B) \leq \lim_{k \to \infty} \mathcal{H}^s_{1/k}(A) + 1/k = \mathcal{H}^s(A).$$ □

Theorem 2.2.4 (Elementary Properties Of Hausdorff Measure)**.**

(1) $\mathcal{H}^0$ *is a counting measure.*

(2) $\mathcal{H}^1 = \mathcal{L}^1$ *on* $\mathbb{R}$.

(3) $\mathcal{H}^s = 0$ *on* $\mathbb{R}^n$ *when* $s > n$.

(4) $\mathcal{H}^s(\lambda A) = \lambda^s \mathcal{H}^s(A) \quad \lambda > 0,\ A \subseteq \mathbb{R}^n$.

(5) $\mathcal{H}^s(T(A)) = \mathcal{H}^s(A) \quad T\colon \mathbb{R}^n \to \mathbb{R}^n$ *isometry,* $A \subseteq \mathbb{R}^n$.

Proof.

(1) We need to show that $\mathcal{H}^0$ assigns 1 to each singleton.

$$\mathcal{H}^0(\{a\}) = \lim_{\delta \to 0} \inf \left\{ \sum_j \alpha(0) \left(\frac{\operatorname{diam} C_j}{2} \right)^0 : \{a\} \subseteq \bigcup_j C_j,\ \operatorname{diam} C_j \leq \delta \right\}$$
$$= 1.$$

Then $\mathcal{H}^0(A) = \sum_{x \in A} 1$, $A \subseteq \mathbb{R}^n$ is the counting measure.

(2) We need to show that 1-dimensional Hausdorff measure and 1-dimensional Lebesque measure coincide.
Let $A \subseteq \mathbb{R}$, $\delta > 0$.

$$\begin{aligned}\mathcal{L}^1(A) &= \inf\left\{\sum_j \text{diam } C_j \mid A \subseteq \bigcup_j C_j\right\}\\ &\leq \inf\left\{\sum_j \text{diam } C_j \mid A \subseteq \bigcup C_j, \text{ diam } C_j \leq \delta\right\}\end{aligned}$$

since $\alpha(1) = 2$

$$= \mathcal{H}^1_\delta(A) \leq \mathcal{H}^1(A).$$

Define $I_k = [k\delta, (k+1)\delta]\ k \in \mathbb{Z}$, then diam $(C_j \cap I_k) \leq \delta$ and

$$\sum_{k=-\infty}^{\infty} \text{diam } (C_j \cap I_k) \leq \text{diam } C_j$$

$$\begin{aligned}\mathcal{L}^1(A) &= \inf\left\{\sum_j \text{diam } C_j \mid A \subseteq \bigcup_j C_j\right\}\\ &\geq \inf\left\{\sum_j \sum_{k=-\infty}^{\infty} \text{diam } (C_j \cap I_k) \mid A \subseteq \bigcup_j C_j\right\}\\ &\geq \mathcal{H}^1_\delta(A).\end{aligned}$$

Since $\delta > 0$ is arbitrary $\mathcal{L}^1(A) \geq \mathcal{H}^1(A)$ and hence $\mathcal{H}^1 = \mathcal{L}^1$.

(3) We want to show that the s-dimensional Hausdorff measure on $\mathbb{R}^n$ is the zero function whenever $s > n$. Let $m \geq 1$, the unit cube Q in $\mathbb{R}^n$ can be decomposed into m^n cubes of side length $1/m$ and diameter $\sqrt{n}/m$.

$$\mathcal{H}^s_{\sqrt{n}/m}(Q) \leq \sum_{i=1}^{m^n} \alpha(s)\left(\frac{\sqrt{n}}{m}\right)^s = \alpha(s) m^{n-s} n^{s/2}$$

as $m \to \infty$ $\mathcal{H}^s_{\sqrt{n}/m}(Q) \to 0$, then $\mathcal{H}^s(Q)$ and hence $\mathcal{H}^s = 0$.

(4) We want to show that given $\lambda > 0$, $\mathcal{H}^s(\lambda A) = \lambda^s \mathcal{H}^s(A)$ for every $A \subseteq \mathbb{R}^n$. Let $A \subseteq \mathbb{R}^n$, $\delta > 0$ and choose $\{C_j\}_{j\in\mathbb{N}} \subseteq \mathbb{R}^n$ such that $A \subseteq \bigcup_j C_j$, diam $C_j \leq \delta$ for each $j \in \mathbb{N}$. Then

$$\lambda A \subseteq \bigcup_j \lambda C_j \quad \text{diam } \lambda C_j \leq \lambda\delta, \text{ and}$$

$$\begin{aligned}\lambda^s \sum_j \alpha(s)\left(\frac{\text{diam } C_j}{2}\right)^s &= \sum_j \alpha(s)\left(\frac{\text{diam } \lambda C_j}{2}\right)^s\\ &\geq \mathcal{H}^s_{\lambda\delta}(\lambda A)\end{aligned}$$

$$\Rightarrow \lambda^s \mathcal{H}^s_\delta(A) \geq \mathcal{H}^s_{\lambda\delta}(\lambda A) \text{ and } \lambda^s \mathcal{H}^s(A) \geq \mathcal{H}^s_{\lambda\delta}(\lambda A).$$

Since $\delta > 0$ is arbitrary $\lambda^s \mathcal{H}^s(A) \geq \mathcal{H}^s(\lambda A)$, and similarly we can show that $1/\lambda^s \mathcal{H}^s(\lambda A) \geq \mathcal{H}^s(A)$.

(5) Since the Hausdorff measure is not affected by affine tranformations the last property is trivial.

□

Lemma 2.2.5. *Suppose $A \subseteq \mathbb{R}^n$ and $\mathcal{H}^s_\delta(A) = 0$ for some $0 < \delta \leq \infty$. Then $\mathcal{H}^s(A) = 0$.*

Proof. The conclusion holds for $s = 0$. Assume that $s > 0$ and fix $\epsilon > 0$, then there exists a sequence of sets $\{C_j\}_{j\in\mathbb{N}}$ in $\mathbb{R}^n$, $A \subseteq \bigcup_j C_j$ such that

$$\sum_j \alpha(s) \left(\frac{\text{diam } C_j}{2}\right)^s \leq \epsilon, \quad \text{diam } C_j \leq \delta \text{ for each } j \in \mathbb{N}.$$

Then for each j,

$$\text{diam } C_j \leq 2\left(\frac{\epsilon}{\alpha(s)}\right)^{1/s} \equiv \delta(\epsilon)$$

hence

$$\mathcal{H}^s_{\delta(\epsilon)}(A) \leq \epsilon.$$

As $\epsilon \to 0$ $\delta(\epsilon) \to 0$ so $\mathcal{H}^s(A) = 0$. □

Lemma 2.2.6. *Let $A \subseteq \mathbb{R}^n$, $0 \leq s < t < \infty$.*

(1) *If $\mathcal{H}^s(A) < \infty$ then $\mathcal{H}^t(A) = 0$.*

(2) *If $\mathcal{H}^t(A) > 0$ then $\mathcal{H}^s(A) = \infty$.*

Proof.

(1) Let $\mathcal{H}^s(A) < \infty$ and $\delta > 0$, then there exists a sequence of sets $\{C_j\}_{j\in\mathbb{N}}$ in $\mathbb{R}^n$ such that diam $C_j \leq \delta$, $A \subseteq \bigcup_j C_j$ and

$$\sum_j \alpha(s) \left(\frac{\text{diam } C_j}{2}\right)^s \leq \mathcal{H}^s_\delta(A) + 1 \leq \mathcal{H}^s(A) + 1.$$

On the other hand,

$$\begin{aligned} \mathcal{H}^t_\delta(A) &\leq \sum_j \alpha(t) \left(\frac{\text{diam } C_j}{2}\right)^t \\ &= \frac{\alpha(t)}{\alpha(s)} 2^{s-t} \sum_j \alpha(s) \left(\frac{\text{diam } C_j}{2}\right)^s (\text{diam} C_j)^{t-s} \\ &\leq \frac{\alpha(t)}{\alpha(s)} 2^{s-t} \delta^{t-s} (\mathcal{H}^s(A) + 1). \end{aligned}$$

As $\delta \to 0$, $\mathcal{H}^t(A) = 0$.

(2) This assertion follows from (1).

□

Definition 2.2.7. *The Hausdorff dimension of a set $A \subseteq \mathbb{R}^n$ is defined to be*

$$\begin{aligned}\mathcal{H}_{\text{dim}}(A) &= \inf\{0 \le s < \infty \mid \mathcal{H}^s(A) = 0\}\\ &= \inf\{0 \le s < \infty \mid \mathcal{H}^s(A) < \infty\}\\ &= \sup\{0 \le s < \infty \mid \mathcal{H}^s(A) > 0\}\\ &= \sup\{0 \le s < \infty \mid \mathcal{H}^s(A) = \infty\},\end{aligned}$$

where the equivalent definitions follow from (2.2.6).

Remark 2.2.8. If $A \subseteq \mathbb{R}^n$ then $\mathcal{H}^s(A) = 0$ for every $s > n$, which follows from (2.2.4), and hence $\mathcal{H}_{\text{dim}}(A) \le n$. If $s = \mathcal{H}_{\text{dim}}(A)$, then for every $t > s$ $\mathcal{H}^t(A) = 0$ and for every $t < s$ $\mathcal{H}^t(A) = \infty$. If there exists $m \ge 0$ for which $0 < \mathcal{H}^m(A) < \infty$, then $m = \mathcal{H}_{\text{dim}}(A)$. By using the last observation we will see that the Hausdorff dimension generalizes our usual notion of dimension.

Example 2.2.9 ([19])**.** In this example we will calculate the Hausdorff dimension of a $C(\lambda)$-set. Fix $0 < \lambda < 1/2$. Set $I_0 = [0,1]$ and let $I_{1,1} = [0,\lambda]$ and $I_{1,2} = [1-\lambda, 1]$. Now from the remaining intervals remove two intervals of length $(1-2\lambda)\lambda = \lambda - 2\lambda^2$ obtaining the four intervals

$$\begin{aligned} I_{2,1} &= [0,\lambda^2] & I_{2,2} &= [\lambda-\lambda^2,\lambda]\\ I_{2,3} &= [1-\lambda, 1-\lambda+\lambda^2] & I_{2,4} &= [1-\lambda^2, 1].\end{aligned}$$

Inductively, if the 2^{k-1} intervals $I_{k-1,1},\dots,I_{k-1,2^{k-1}}$, each having length λ^{k-1}, have been constructed, then define $I_{k,1},\dots,I_{k,2^k}$ by deleting an interval of length $(1-2\lambda)\cdot$ diam $I_{k-1,j} = (1-2\lambda)\lambda^{k-1}$ from the middle of each $I_{k-1,j}$. All of the 2^k intervals obtained in the kth stage have length λ^k, and hence

$$\mathcal{H}^1\left(\bigcup_{j=1}^{2^k} I_{k,j}\right) = \mathcal{L}^1\left(\bigcup_{j=1}^{2^k} I_{k,j}\right) = (2\lambda)^k.$$

Define the set $C(\lambda)$ to be

$$C(\lambda) = \bigcap_{k=0}^{\infty}\bigcup_{j=1}^{2^k} I_{k,j}.$$

So for $\lambda = 1/3$ the $C(\lambda)$-set is the usual Cantor middle-thirds set. $C(\lambda) \subseteq \bigcup_{j=1}^{2^k} I_{k,j}$ for each $k \ge 0$, then

$$\mathcal{H}^m_{\lambda^k}(C(\lambda)) \le \sum_{j=1}^{2^k} \text{diam }(I_{k,j})^m = 2^k\lambda^{km}$$

If we choose $m \in \mathbb{R}$ so that $2\lambda^m = 1$ and hence $m = \frac{\log 2}{\log(1/\lambda)}$, then

$$\mathcal{H}^m(C(\lambda)) = \lim_{k\to\infty} \mathcal{H}^m_{\lambda^k}(C(\lambda)) \le 1.$$

Hence $\mathcal{H}_{\text{dim}}(C(\lambda)) \le m$. Now if we can show that $\mathcal{H}^m(C(\lambda)) > 0$ then we can conclude, by the help of (2.2.8), that $\mathcal{H}_{\text{dim}}(C(\lambda)) = \frac{\log 2}{log(1/\lambda)}$. So our aim is to find a lower bound $k > 0$ such that $\mathcal{H}^m(C(\lambda)) \ge k > 0$. Let $\{C_j\}_{j\in\mathbb{N}}$ be a covering of $C(\lambda)$ such that diam $C_j \le \delta$ for each j. Choose, for every $j \in \mathbb{N}$, x_j, $y_j \in \mathbb{R}$ with diam $C_j = y_j - x_j$ and

$[x_j, y_j] \supseteq C_j$. By slightly enlarging each interval $[x_j, y_j]$ we can choose an open interval $I_j \supseteq C_j$ such that $C(\lambda) \subseteq \bigcup_j C_j \subseteq \bigcup_j I_j$ and

$$\begin{aligned}
\sum_j \alpha(m)\left(\frac{\text{diam } C_j}{2}\right)^m &\le \sum_j \alpha(m)\left(\frac{\text{diam } I_j}{2}\right)^m \\
&< \sum_j \alpha(m)\left(\frac{\text{diam } C_j}{2}\right)^m + \epsilon \\
\Rightarrow \frac{\alpha(m)}{2^m}\sum_j (\text{diam } I_j)^m &\le \sum_j \alpha(m)\left(\frac{\text{diam } C_j}{2}\right)^m + \epsilon\frac{\alpha(m)}{2^m}.
\end{aligned}$$

If we can show that $1/16 \le \sum_j (\text{diam } I_j)^m$ whenever the I_j's are open intervals covering $C(\lambda)$, then

$$\frac{\alpha(m)}{2^m \cdot 16} \le \mathcal{H}^m_\delta(C(\lambda)) + \epsilon\frac{\alpha(m)}{2^m} \le \mathcal{H}^m(C(\lambda)) + \epsilon\frac{\alpha(m)}{2^m}$$

and since $\epsilon > 0$ is arbitrary, $k \equiv \frac{\alpha(m)}{2^m \cdot 16}$, $k \le \mathcal{H}^m(C(\lambda)) \le 1$. Thus $\mathcal{H}_{\text{dim}}(C(\lambda)) = m$. Now we need to show that $1/16 \le \sum_j (\text{diam } I_j)^m$ whenever the I_j's are open intervals covering $C(\lambda)$. Since $C(\lambda)$ is compact we may assume that $I_1, \ldots, I_n$ cover $C(\lambda)$. Since $C(\lambda)$ has no interior, by slightly enlarging each I_j, we can assume that the endpoints of each I_j lie outside $C(\lambda)$. Then we may select $\delta > 0$ such that the Euclidean distance from the set of all endpoints of the I_j's to $C(\lambda)$ is at least δ. Choose $k \in \mathbb{N}$ large enough so that $\delta > \lambda^k = \text{diam } I_{k,i}$, then each interval $I_{k,i}$ is contained in some I_j. Let's assume that for any open interval I and any fixed index $l \in \mathbb{N}$, we have the equality

$$\sum_{I_{l,i} \subseteq I} (\text{diam } I_{l,i})^m \le 16(\text{diam } I)^m \tag{2.2.9.1}$$

then

$$16\sum_{j=1}^n (\text{diam } I_j)^m \ge \sum_{j=1}^n \sum_{I_{k,i} \subseteq I_j} (\text{diam } I_{k,i})^m \ge \sum_{i=1}^{2^k} (\text{diam } I_{k,i})^m = 1.$$

Now it remains to show (2.2.9.1). Suppose that there are some intervals $I_{l,i}$ that lie inside I and let n be the smallest integer for which I contains $I_{n,i}$. Let $\{I_{n,j_1}, \cdots, I_{n,j_p}\}$ be the complete list of all nth generation intervals contained in I, then $p \le 4$ and

$$\begin{aligned}
16(\text{diam } I)^m &\ge \sum_{s=1}^p 4(\text{diam } I_{n,j_s})^m \ge \sum_{s=1}^p \sum_{I_{l,i} \subseteq I_{n,j_s}} (\text{diam } I_{l,i})^m \\
&\ge \sum_{I_{l,i} \subseteq I} (\text{diam } I_{l,i})^m \\
&\Rightarrow \mathcal{H}_{\text{dim}}(C(\lambda)) = \frac{\log 2}{\log(1/\lambda)}.
\end{aligned}$$

2.3. **Isodiametric Inequality, $\mathcal{H}^n = \mathcal{L}^n$.**

Definition 2.3.1 (Steiner Symmetrization)**.** *Fix* $a,\ b \in \mathbb{R}^n,\ |a| = 1$. *Define*

$$L^a_b = \{b + ta \mid t \in \mathbb{R}\} \text{ the line through } b \text{ in the direction } a.$$

$$P_a = \{x \in \mathbb{R}^n \mid x \cdot a = 0\} \text{ the plane through the origin perpendicular to } a.$$

Let $A \subseteq \mathbb{R}^n$. The Steiner symmetrization of A with respect to the plane P_a is the set

$$S_a(A) = \bigcup_{\substack{b \in P_a \\ A \cap L_b^a \neq \emptyset}} \left\{ b + ta \colon |t| \leq \frac{\mathcal{H}^1(A \cap L_b^a)}{2} \right\}.$$

Remark 2.3.2. If $A \subseteq \mathbb{R}^2$ and $a \in \mathbb{R}^2$ then the Steiner symmetrization of A with respect to the line through the origin and perpendicular to a is the symmetric image of A with respect to this line.

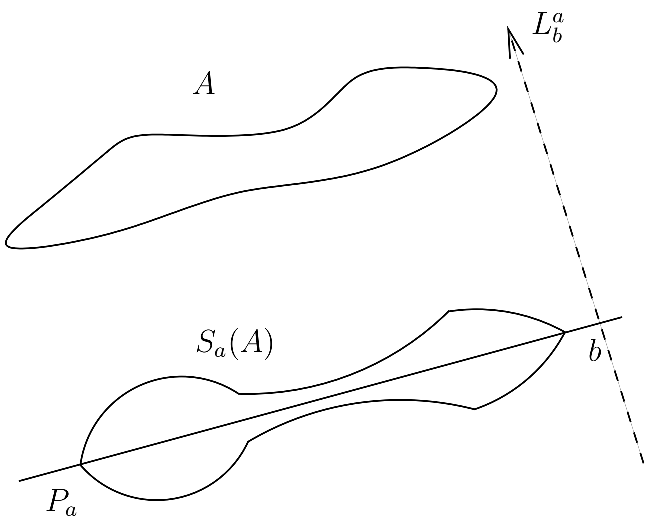


FIGURE 1. Steiner Symmetrization

Lemma 2.3.3 (Properties Of Steiner Symmetrization)**.**

(1) diam $S_a(A) \leq$ diam A.

(2) *If A is $\mathcal{L}^n$-measurable, then so is $S_a(A)$, and $\mathcal{L}^n(S_a(A)) = \mathcal{L}^n(A)$.*

Proof.

(1) WLOG we may assume that diam $A < \infty$. By considering $\overline{A}$ we may also assume that A is closed. Fix $\epsilon > 0$ and choose x, $y \in S_a(A)$ such that diam $S_a(A) < |x - y| + \epsilon$. Define $b = x - (x \cdot a)a$ and $c = y - (y \cdot a)a$, then b, $c \in P_a$. Set

$$r = \inf\{t \mid b + ta \in A\},$$
$$u = \inf\{t \mid c + ta \in A\},$$
$$s = \sup\{t \mid b + ta \in A\},$$
$$v = \sup\{t \mid c + ta \in A\}.$$

WLOG we may assume that $v - r \geq s - u$. Then

$$v - r \geq \frac{v - r}{2} + \frac{s - u}{2}$$

$$= \frac{s-r}{2} + \frac{v-u}{2}$$
$$\geq \frac{\mathcal{H}^1(A \cap L_b^a)}{2} + \frac{\mathcal{H}^1(A \cap L_c^a)}{2}.$$

Since $|x \cdot a| \leq 1/2\, \mathcal{H}^1(A \cap L_b^a)$, $|y \cdot a| \leq 1/2\, \mathcal{H}^1(A \cap L_c^a)$, and consequently,

$$v - r \geq |x \cdot a| + |y \cdot a| \geq |x \cdot a - y \cdot a|.$$

Therefore,

$$\begin{aligned}(\text{diam } S_a(A) - \epsilon)^2 &\leq |x-y|^2 \\ &= |b-c|^2 + |x \cdot a - y \cdot a|^2 \\ &\leq |b-c|^2 + (v-r)^2 \\ &= |(b+ra) - (c+va)|^2 \\ &\leq (\text{diam } A)^2 \text{ since } A \text{ is closed.}\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, diam $S_a(A) \leq$ diam A.

(2) Let $A \subseteq \mathbb{R}^n$ and $a \in \mathbb{R}^n$. We want to show that $S_a(A)$ is $\mathcal{L}^n$-measurable and that n-dimensional Lebesque measure is invariant under Steiner symmetrization. Since $\mathcal{L}^n$ is rotation invariant we can assume that $a = e_n = (0, \ldots, 1)$. Then $P_a = P_{e_n} = \mathbb{R}^{n-1}$. By Fubini's theorem the map $f\colon \mathbb{R}^{n-1} \to \mathbb{R}$ defined by $f(b) = \mathcal{H}^1(A \cap L_b^a)$ is $\mathcal{L}^{n-1}$-measurable and $\mathcal{L}^n(A) = \int_{\mathbb{R}^{n-1}} f(b)\, d\mathcal{L}^{n-1}b$. Hence

$$S_a(A) = \left\{(b,y)\colon\ -\frac{f(b)}{2} \leq y \leq \frac{f(b)}{2}\right\} - \{(b,0)\colon L_b^a \cap A = \emptyset\}$$

is $\mathcal{L}^n$-measurable by theorem (2.1.7).
And $\mathcal{L}^n(S_a(A)) = \int_{\mathbb{R}^{n-1}} f(b)\, d\mathcal{L}^{n-1}b = \mathcal{L}^n(A)$.

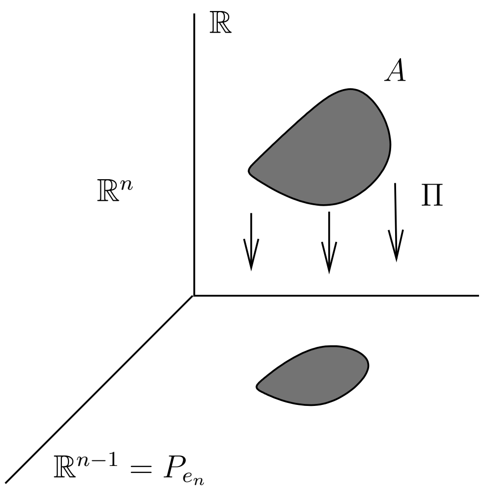


FIGURE 2. Steiner Symmetrization Preserves The Lebesque Measure

□

Theorem 2.3.4 (Isodiametric Inequality)**.** *Let $A \subseteq \mathbb{R}^n$. Then*

$$\mathcal{L}^n(A) \leq \alpha(n) \left(\frac{diam\ A}{2}\right)^n.$$

Remark 2.3.5. The isodiametric inequality states that among all sets of the same diameter the sphere has the largest the volume.

Proof. WLOG we may assume that diam $A < \infty$. Let $\{e_1, \ldots, e_n\}$ be the standard basis for $\mathbb{R}^n$, and $A_1 = S_{e_1}(A),\ A_2 = S_{e_2}(A_1), \ldots, A_n = S_{e_n}(A_{n-1})$ and let $A^* = A_n$.

(1) Claim # 1: A^* is symmetric with respect to the origin.

Proof Of Claim # 1. A_1 is symmetric with respect to P_{e_1}. Let $1 \leq k < n$ and suppose A_k is symmetric with respect to $P_{e_1}, \ldots, P_{e_k}$. We want to show that A_{k+1} is symmetric with respect to $P_{e_1}, \ldots, P_{e_{k+1}}$. By definition $A_{k+1} = S_{e_{k+1}}(A_k)$ is symmetric with respect to $P_{e_{k+1}}$. For each $1 \leq j \leq k$ let $S_j \colon \mathbb{R}^n \to \mathbb{R}^n$ be the reflection through P_{e_j}. So we need to show that for each $1 \leq j \leq k$ $S_j(A_{k+1}) = A_{k+1}$. Let $b \in P_{e_{k+1}}$. Since $S_j(A_k) = A_k$,

$$\mathcal{H}^1(A_k \cap L_b^{e_{k+1}}) = \mathcal{H}^1(A_k \cap L_{S_j b}^{e_{k+1}});$$

consequently

$$\{t \mid b + te_{k+1} \in A_{k+1}\} = \{t \mid S_j b + te_{k+1} \in A_{k+1}\}. \tag{2.3.4.1}$$

Since

$$S_j(A_{k+1}) = \bigcup_{\substack{b \in P_{k+1} \\ L^b_{e_{k+1}} \cap A_k \neq \emptyset}} \left\{ S_j b + te_{k+1} \colon |t| \leq \frac{\mathcal{H}^1\left(A_k \cap L_b^{e_{k+1}}\right)}{2} \right\}$$

then

$$A_{k+1} \subseteq S_j(A_{k+1}).$$

And (2.3.4.1) implies $A_{k+1} \supseteq S_j(A_{k+1})$. Thus A_{k+1} is symmetric with respect to P_{e_j} and by induction $A_n = A^*$ is symmetric with respect to each P_{e_i} $1 \leq i \leq n$. So A^* is symmetric with respect to the origin.

(2) Claim # 2: $\mathcal{L}^n(A^*) \leq \alpha(n)(1/2 \text{ diam } A^*)^n$.

Proof Of Claim # 2. Since A^* is symmetric with respect to the origin $A^* \subseteq B(0, \text{diam } A^*/2)$ and hence

$$\mathcal{L}^n(A^*) \leq \mathcal{L}^n\left(B\left(0, \frac{\text{diam } A^*}{2}\right)\right) = \alpha(n)\left(\frac{\text{diam } A^*}{2}\right)^n.$$

(3) Claim # 3: $\mathcal{L}^n(A) \leq \alpha(n)((\text{ diam } A)/2)^n$.

Proof Of Claim # 3. Since $\overline{A}$ is measurable, (2.3.1) implies that $\mathcal{L}^n((\overline{A})^*) = \mathcal{L}^n(\overline{A})$ and diam $(\overline{A})^* \leq$ diam $\overline{A}$. Thus

$$\begin{aligned} \mathcal{L}^n(A) &\leq \mathcal{L}^n(\overline{A}) = \mathcal{L}^n((\overline{A})^*) \\ &\leq \alpha(n)\left(\frac{\text{diam } (\overline{A})^*}{2}\right)^n \quad \text{by Claim \# 2} \\ &\leq \alpha(n)\left(\frac{\text{diam } \overline{A}}{2}\right)^n \end{aligned}$$

$$= \alpha(n)\left(\frac{\text{diam } A}{2}\right).$$

□

Theorem 2.3.6. *$\mathcal{H}^n = \mathcal{L}^n$ on $\mathbb{R}^n$.*

Proof. Fix $\delta > 0$. Choose sets $\{C_j\}_{j\in N}$ such that $A \subseteq \bigcup_j C_j$ diam $C_j \le \delta$ for every $j \in \mathbb{N}$. Then by the isodiametric inequality

$$\mathcal{L}^n(A) \le \sum_j \mathcal{L}^n(C_j) \le \sum_j \alpha(n)\left(\frac{\text{diam } C_j}{2}\right)^n,$$

and

$$\mathcal{L}^n(A) \le \mathcal{H}^n_\delta(A) \le \mathcal{H}^n(A) \text{ since the covering is arbitrary.}$$

(1) Claim # 1: $\mathcal{H}^n \ll \mathcal{L}^n$.

Proof Of Claim # 1. Set $C_n = \alpha(n)(\sqrt{n}/2)^n$. Then for each cube $Q \subseteq \mathbb{R}^n$

$$\alpha(n)\left(\frac{\text{diam } Q}{2}\right)^n = C_n \mathcal{L}^n(Q).$$

Thus

$$\mathcal{H}^n_\delta(A) \le \inf\left\{\sum_i \alpha(n)\left(\frac{\text{diam } Q_i}{2}\right)^n : Q_i \text{ cubes, } A \subseteq \bigcup_i Q_i\right\}$$

where each cube Q_i has diameter $\le \delta$

$$= C_n \mathcal{L}^n(A) \text{ by definition of } \mathcal{L}^n.$$

So if $\mathcal{L}^n(A) = 0$ then by letting $\delta \to 0$ we see that $\mathcal{H}^n(A) = 0$.

(2) Claim # 2: $\mathcal{H}^n(A) \le \mathcal{L}^n(A)$ for every $A \subseteq \mathbb{R}^n$.

Proof. Fix $\delta > 0$, $\epsilon > 0$. We can choose a sequence of cubes $\{Q_i\}_{i\in\mathbb{N}}$ such that $A \subseteq \bigcup_i Q_i$, diam $Q_i < \delta$, and

$$\sum_{i=1}^{\infty} \mathcal{L}^n(Q_i) \le \mathcal{L}^n(A) + \epsilon.$$

By the Vitali Covering Theorem for each i there exists a disjoint sequence of closed ball $\{B^i_k\}_{k\in\mathbb{N}}$ contained in Q_i° such that

$$\text{diam } B^i_k \le \delta,\ \mathcal{L}^n\left(Q_i - \bigcup_k B^i_k\right) = \mathcal{L}^n\left(Q_i^\circ - \bigcup_k B^i_k\right) = 0.$$

By Claim # 2, $\mathcal{H}^n(Q_i - \bigcup_k B^i_k) = 0$ for each $i \in \mathbb{N}$. Thus

$$\mathcal{H}^n_\delta(A) \le \sum_i \mathcal{H}^n_\delta(Q_i) = \sum_i \mathcal{H}^n_\delta\left(\bigcup_k B^i_k\right)$$

$$\le \sum_i \sum_k \mathcal{H}^n_\delta(B^i_k) \le \sum_i \sum_k \alpha(n)\left(\frac{\text{diam } B^i_k}{2}\right)^n$$

$$= \sum_i \sum_k \mathcal{L}^n(B_k^i) = \sum_i \mathcal{L}^n(Q_i) \le \mathcal{L}^n(A) + \epsilon$$

Since $\delta > 0$ and $\epsilon > 0$ are arbitrary $\mathcal{H}^n(A) \le \mathcal{L}^n(A)$.

□

2.4. **Density Of Sets And Measures.**

Definition 2.4.1 (Densities)**.**

(1) *Let $A \subseteq \mathbb{R}^n$. For $1 \le m \le n$, $a \in \mathbb{R}^n$ the m-dimensional density $\Theta^m(A,a)$ of A at a is defined by*

$$\Theta^m(A,a) = \lim_{r\to 0} \frac{\mathcal{H}^m(A \cap B^n(a,r))}{\alpha_m r^m} \quad \textit{when the limit exists}$$

where $B^n(a,r)$ is the closed ball in $\mathbb{R}^n$ with center a and radius $r > 0$.

(2) *If μ is a measure on $\mathbb{R}^n$, $1 \le m \le n$, $a \in \mathbb{R}^n$ then m-dimensional density $\Theta^m(\mu,a)$ of μ at a is defined by*

$$\Theta^m(\mu,a) = \lim_{r\to 0} \frac{\mu(B^n(a,r))}{\alpha_m r^m} \quad \textit{when the limit exists.}$$

Remark 2.4.2. Note that for any $A \subseteq \mathbb{R}^n$, $\Theta^m(A,a) = \Theta^m(\mathcal{H}^m \llcorner A, a)$ where $\mathcal{H}^m \llcorner A$ is the measure defined by

$$(\mathcal{H}^m \llcorner A)(E) = \mathcal{H}^m(A \cap E) \quad E \subseteq \mathbb{R}^n.$$

Hence the density of measures generalizes the notion of density of sets.

Theorem 2.4.3. *If $A \subseteq \mathbb{R}^n$ is $\mathcal{L}^n$-measurable, then $\Theta^n(A,x) = \chi_A(x)$ $\mathcal{L}^n$-a.e $x \in \mathbb{R}^n$.*

Proof. Since for every $A \subseteq \mathbb{R}^n$ which is $\mathcal{L}^n$-measurable and $x \in \mathbb{R}^n$, $\Theta^n(A^c,x) = 1$ implies $\Theta^n(A,x) = 0$, it suffices to show that when $A \subseteq \mathbb{R}^n$ is $\mathcal{L}^n$-measurable $\Theta^n(A,x) = \chi_A(x)$ $\mathcal{L}^n$-a.e. $x \in A$. Assume not. By taking the restriction of $\mathcal{L}^n$ we can also assume that $0 < \mathcal{L}^n(A) < \infty$. We may further assume that for some $0 < \delta < 1$

$$(\dagger) \qquad \Theta^n_*(A,a) = \liminf_{r\to 0} \frac{\mathcal{L}^n(A \cap B^n(a,r))}{\alpha_n r^n} < \delta \quad \text{for every } a \in A$$

by first choosing $0 < \delta < 1$ such that

$$\mathcal{L}^n(\{a \in A \colon \Theta_*(A,a) < \delta\}) > 0.$$

(otherwise $\Theta^n_*(A,a) = 1$ $\mathcal{L}^n$-a.e. $a \in A$ implies $\Theta^n(A,a) = 1$ $\mathcal{L}^n$-a.e. $a \in A$, which contradicts the assumption) and then replacing A by $\{x \in A \colon \Theta_*(A,a) < \delta\}$. We can choose an open set $U \supseteq A$ such that

$$\mathcal{L}^n(A) > \delta \mathcal{L}^n(U).$$

Let $\mathcal{F}$ be the collection of closed balls $B \equiv B^n(x,r)$ defined by

$$\mathcal{F} = \{B(x,r) \colon x \in A,\ r > 0,\ B \subseteq U, \text{ and } \mathcal{L}^n(A \cap B) \le \delta \mathcal{L}^n(B)\}.$$

Then by (†) we can say that $\mathcal{F}$ is a fine covering of A. By the Besicovitch's Covering Theorem there exists a countable disjoint collection $\mathcal{G} \subseteq \mathcal{F}$ covering almost all of A and hence

$$\mathcal{L}^n(A) = \sum_{B \in \mathcal{G}} \mathcal{L}^n(A \cap B) \leq \delta \sum_{B \in \mathcal{G}} \mathcal{L}^n(B) \leq \delta \mathcal{L}^n(U),$$

which is a contradiction. □

Definition 2.4.4. *$A \subseteq \mathbb{R}^m$, a function $f\colon A \to \mathbb{R}^n$ has approximate limit $y \in \mathbb{R}^n$ at $a \in \mathbb{R}^m$ if for every $\epsilon > 0$, $\mathbb{R}^m - \{x \in A\colon |f(x) - y| < \epsilon\}$ has m-dimensional density 0 at a, which is denoted by*

$$y = \operatorname{ap} \lim_{x \to a} f(x).$$

Remark 2.4.5.

- Note that if $A \subseteq \mathbb{R}^m$ and $f\colon A \to \mathbb{R}^n$ such that f has an approximate limit $y \in \mathbb{R}^n$ at $a \in \mathbb{R}^m$ then A must have m-dimensional density 1 at a. In particular,
$$\frac{\mathcal{H}^m(\mathbb{R}^m \cap B^m(a,r))}{\alpha_m r^m} \leq \frac{\mathcal{H}^m(A \cap B^m(a,r))}{\alpha_m r^m} + \frac{\mathcal{H}^m(A^c \cap B^m(a,r))}{\alpha_m r^m}$$
$$\leq \frac{\mathcal{H}^m(A \cap B^m(a,r))}{\alpha_m r^m} + \frac{\mathcal{H}^m(\mathbb{R}^m - \{x \in A\colon |f(x) - y| < \epsilon\} \cap B^m(a,r))}{\alpha_m r^m}$$
then
$$1 = \lim_{r \to 0} \frac{\mathcal{H}^m(\mathbb{R}^m \cap B^m(a,r))}{\alpha_m r^m} \leq \lim_{r \to 0} \frac{\mathcal{H}^m(A \cap B^m(a,r))}{\alpha_m r^m} \leq 1.$$
- If a function $f\colon A \to \mathbb{R}^n$ has the limit $y \in \mathbb{R}^n$ at $a \in \mathbb{R}^m$ then it has the approximate limit $y \in \mathbb{R}^n$ at $a \in \mathbb{R}^m$.

Theorem 2.4.6. *A function $f\colon A \subseteq \mathbb{R}^m \to \mathbb{R}^n$ has an approximate limit $y \in \mathbb{R}^n$ at $a \in \mathbb{R}^m$ iff there is set $B \subseteq A$ such that B^c has m-dimensional density* 0 *at a and $f|_B$ has the limit y at a.*

Proof. (⇒) Assume that $y = \operatorname{ap} \lim_{x \to a} f(x) = 0$, then

$$\lim_{r \to 0} \frac{\mathcal{H}^m(\mathbb{R}^m - \{x \in A\colon |f(x)| < 1/i\} \cap B^m(a,r))}{\alpha_m r^m} = 0$$

holds for each $i \geq 1$. Let $A_i = \mathbb{R}^m - \{x \in A\colon |f(x)| < 1/i\}$, hence each A_i has density 0 at a. For $i = 1$ choose $r_1 > 0$ such that for every $0 < r \leq r_1$

$$\frac{\mathcal{H}^m(A_1 \cap B^m(a,r))}{\alpha_m r^m} \leq 2^{-1},$$

for $i = 2$ choose $0 < r_2 < r_1$ such that for every $0 < r \leq r_2$

$$\frac{\mathcal{H}^m(A_2 \cap B^m(a,r))}{\alpha_m r^m} \leq 2^{-2}$$

Now assuming that $r_1 > \cdots > r_{k-1}$ have been chosen, we can choose $0 < r_k < r_{k-1}$ such that for every $0 < r \leq r_k$

$$(*) \qquad \frac{\mathcal{H}^m(A_{k+1} \cap B^m(a,r))}{\alpha_m r^m} \leq 2^{-k}$$

and

$$A_k = \mathbb{R}^m - \{x \in A \colon |f(x)| < 1/k\} \supseteq A_{k-1}.$$

Thus by induction we can find a strictly decreasing sequence $\{r_k\}_{k\in\mathbb{N}}$ and an increasing sequence $\{A_k\}_{k\in\mathbb{N}}$ satisfying $(*)$ for each $k \in \mathbb{N}$. Let $B^c = \bigcup_i (A_i \cap B^m(a, r_i))$. We need to show that $f|_B$ has the limit y at a. By construction $x \in B$ iff for each $i \in \mathbb{N}$, $x \in A_i^c$ or $x \in (B^m(a,r))^c$. Given $\epsilon > 0$ we can choose $j \in \mathbb{N}$ and $\delta > 0$ such that $1/j < \epsilon$ and $0 < \delta < r_j$. If $x \in B$ and $|x - a| < \delta < r_j$ then $x \in B^m(a, r_j)$ and $x \in A_j^c$. Thus $f|_B(x) \to y$ as $x \to a,\ x \in B$.
Now we need to show that B^c has density 0 at a. Choose $s \in \mathbb{R}$ such that $r_{i+1} < s < r_i$, then

$$\begin{aligned}\mathcal{H}^m(B^c \cap B^m(A, s)) &\leq \mathcal{H}^m(A_i \cap B^m(a,s)) + \mathcal{H}^m(A_{i+1} \cap B^m(a, r_{i+1})) \\ &\quad + \mathcal{H}^m(A_{i+2} \cap B^m(a, r_{i+2})) + \cdots \\ &\leq \alpha_m s^m \sum_{k=i}^{\infty} \frac{1}{2^k} = \alpha_m s^m 2^{-(i-1)} \\ \frac{\mathcal{H}^m(B^c \cap B^m(a,s))}{\alpha_m s^m} &\leq \frac{1}{2^{i-1}}.\end{aligned}$$

So B^c has density 0 at a.
$(\Leftarrow)$ Assume that there exists a set $B \subseteq A$ such that $\Theta^m(B^c, a) = 0$ and $\lim_{x\to a} f|_B(x) = y$. Given $\epsilon > 0$ there exists $\delta > 0$ such that for every $0 < r < \delta$

$$(*) \qquad \{x \in B \colon |f(x) - y| \geq \epsilon\} \cap B^m(a, r) = \emptyset.$$

$B \subseteq A$ implies that $\Theta^m(A^c, a) \leq \Theta^m(B^c, a) = 0$ and

$$\mathbb{R}^m - \{x \in A \colon |f(x) - y| < \epsilon\} \subseteq B^c \cup \{x \in A - B \colon |f(x) - y| \geq \epsilon\}$$

implies that

$$\begin{aligned}(**) \quad & \mathcal{H}^m(\mathbb{R}^m - \{x \in A \colon |f(x) - y| < \epsilon\} \cap B^m(a, r)) \\ & \leq \mathcal{H}^m(B^c \cap B^m(a,r)) + \mathcal{H}^m(\{x \in A - B \colon |f(x) - y| \geq \epsilon\} \cap B^m(A, r)) \\ & \leq 2\mathcal{H}^m(B^c \cap B^m(a, r)).\end{aligned}$$

If $0 < r < \delta$ then by using $(*)$ and $(**)$ we can conclude that

$$\frac{\mathcal{H}^m(\mathbb{R}^m - \{x \in A \colon |f(x) - y| < \epsilon\} \cap B^m(a, r))}{\alpha_m r^m} \leq 2\frac{\mathcal{H}^m(B^c \cap B^m(a,r))}{\alpha_m r^m} \to 0$$

as $\delta \to 0$. □

Definition 2.4.7.

(1) *The function $f \colon A \subseteq \mathbb{R}^m \to \mathbb{R}^n$ is approximately continuous at $a \in A$ if for every $\epsilon > 0$*

$$\lim_{r\to 0} \frac{\mathcal{H}^m(\mathbb{R}^m - \{x \in A \colon |f(x) - f(a)| < \epsilon\} \cap B^m(a, r))}{\alpha_m r^m} = 0$$

(2) *The function* $f\colon A \subseteq \mathbb{R}^m \to \mathbb{R}^n$ *is approximately differentiable at* a *if there is a linear transformation* $L\colon \mathbb{R}^m \to \mathbb{R}^n$ *such that*

$$\operatorname{ap}\lim_{x\to a} \frac{|f(x)-f(a)-L(x-a)|}{|x-a|} = 0.$$

The approximate derivative of f *at* a *is denoted by* $L \equiv \operatorname{ap} Df(a)$.

Remark 2.4.8.

- If a function $f\colon A \to \mathbb{R}^n$ is continuous at $a \in A$ then it is approximately continuous $a \in A$.
- Note that the theorem (2.4.6) has an obvious extension to the approximately continuous functions.
- If a function $f\colon A \to \mathbb{R}^n$ is differentiable at $a \in A$ then it is approximately differentiable at $a \in A$.
- When there is no confusion about the ambient space we will use $B(x,r)$ instead of $B^n(x,r)$ to denote a closed ball in $\mathbb{R}^n$ of radius $r > 0$ and center at $x \in \mathbb{R}^n$.

Theorem 2.4.9. *Let* $f\colon \mathbb{R}^n \to \mathbb{R}^m$ *be* $\mathcal{L}^n$*-measurable. Then* f *is approximately continuous* $\mathcal{L}^n$*-a.e.*

Proof.

(1) Claim # 1: There exists a sequence of pairwise disjoint compact sets $\{K_i\}_{i\in\mathbb{N}} \subseteq \mathbb{R}^n$ such that

$$\mathcal{L}^n\left(\mathbb{R}^n - \bigcup_i K_i\right) = 0 \text{ and } f|_{K_i} \text{ is continuous for each } i \in \mathbb{N}.$$

Proof Of Claim # 1. For each $m \in \mathbb{N}$, set $B_m = B(0,m)$. By Lusin's Theorem, there exists a compact set $K_1 \subseteq B_1$ such that $\mathcal{L}^n(B_1 - K_1) \leq 1$ and $f|_{K_1}$ is continuous. Now applying the Lusin's Theorem to $B_2 - K_1$ we can choose a compact set $K_2 \subseteq B_2 - K_1$ such that $\mathcal{L}^n(B_2 - \bigcup_{i=1}^2 K_i) \leq 1/2$ and $f|_{K_2}$ is continuous. Now assuming that $K_1, \ldots K_{m-1}$ have been chosen accordingly, we can choose a compact set

$$K_m \subseteq B_m - \bigcup_{i=1}^{m-1} K_i \text{ such that } \mathcal{L}^n\left(B_m - \bigcup_{i=1}^{m} K_i\right) \leq \frac{1}{m+1} \tag{$*$}$$
$$\text{and } f|_{K_m} \text{ is continuous.}$$

Thus by induction we can choose a sequence of compact sets $\{K_m\}_{m\in\mathbb{N}}$ such that K_m satisfies $(*)$ for each $m \in \mathbb{N}$ and

$$\begin{aligned}\mathcal{L}^n\left(\mathbb{R}^n - \bigcup_{i=1}^{\infty} K_i\right) &= \mathcal{L}^n\left(\bigcup_{i=1}^{\infty} B_i - \bigcup_{i=1}^{\infty} K_i\right)\\ &= \lim_{i\to\infty} \mathcal{L}^n\left(B_i - \bigcup_{i=1}^{\infty} K_j\right)\end{aligned}$$

$$\leq \lim_{i\to\infty} \frac{1}{i+1} = 0.$$

(2) Claim # 2: f is approximately continuous $\mathcal{L}^n$-a.e.

Proof Of Claim # 2. For each $i \in \mathbb{N}$ and $\mathcal{L}^n$-a.e. $x \in K_i$

$$\lim_{r\to 0} \frac{\mathcal{L}^n(B(x,r) - K_i)}{\mathcal{L}^n(B(x,r))} = 0. \tag{$**$}$$

Define $A = \{x \in \mathbb{R}^n \mid \text{for some } i \in \mathbb{N},\ x \in K_i, \text{and } (**) \text{ holds}\}$, then $\mathcal{L}^n(\mathbb{R}^n - A) = 0$. Now we want to show that f is approximately continuous at each $x \in A$. Fix $x \in A$. Then $x \in K_i$ for some $i \in \mathbb{N}$ and given $\epsilon > 0$ there exists $s > 0$ such that if $y \in K_i$ and $|x - y| < s$ then $|f(x) - f(y)| < \epsilon$. If $0 < r < s$ then $B(x,r) \cap \{y\colon |f(x) - f(y)| \geq \epsilon\} \subseteq B(x,r) - K_i$ so ap $\lim_{y\to x} f(y) = f(x)$.

□

3. MANIFOLDS IN $\mathbb{R}^n$ AND GENERALIZATION OF CONCEPTS IN CALCULUS

In this section we will first introduce the k-dimensional parallelepiped in $\mathbb{R}^n$ and its k-dimensional volume. We will state the generalized Phytagorean Theorem and postpone its proof until the next section where the proof becomes easier in a more general setting. We will introduce k-dimensional analogues of curves and surfaces; they are called k-manifolds in $\mathbb{R}^n$. We will define the k-dimensional volume of such objects and introduce the integral of a scalar function over k-manifold with respect to k-volume, generalizing concepts defined in Calculus for curves and surfaces. The main reference for this section is the book by Munkres [23].

3.1. **The Volume Of A Parallelepiped.**

Definition 3.1.1. *Let* $\{a_1, \dots, a_k\}$*be a linearly independent collection of vectors in* $\mathbb{R}^n$, $1 \le k \le n$. *The* k*-dimensional parallelepiped in* $\mathbb{R}^n$ *is defined as*

$$\mathcal{P}(a_1, \dots, a_k) = \{x \in \mathbb{R}^n \colon x = x_1 a_1 + \cdots + x_k a_k,\ 0 \le x_i \le 1,\ 1 \le i \le k\}.$$

Remark 3.1.2. We want to define the k-dimensional volume of $\mathcal{P}(a_1, \dots, a_k)$ in $\mathbb{R}^n$. When $1 \le k < n$, the n-dimensional volume ($\mathcal{L}^n$-measure) is zero because the parallelepiped is contained in a k-dimensional subspace of $\mathbb{R}^n$ which has measure zero in $\mathbb{R}^n$.

We will state two lemmas from linear algebra, that will be used in the proof of the existence and the uniqueness of the volume function. The proofs of these lemmas can be found in Munkres [23].

Lemma 3.1.3. *Let* W *be a linear subspace of* $\mathbb{R}^n$ *of dimension* k. *Then there is an orthonormal basis for* $\mathbb{R}^n$ *whose first* k *elements form a basis for* W.

Lemma 3.1.4. *Let* W *be a* k*-dimensional linear subspace of* $\mathbb{R}^n$. *There is an orthogonal transformation* $h \colon \mathbb{R}^n \to \mathbb{R}^n$ *that carries* W *onto the subspace* $\mathbb{R}^k \times 0$ *of* $\mathbb{R}^n$.

Theorem 3.1.5 (The Volume Function)**.** *There is a unique function* V *that assigns, to each* k*-tuple* $(x_1, \dots, x_k)$ *of elements of* $\mathbb{R}^n$, *a non-negative number such that*

(1) *If* $h \colon \mathbb{R}^n \to \mathbb{R}^n$ *is an orthogonal transformation, then*

$$V(h(x_1), \dots, h(x_k)) = V(x_1, \dots, x_k).$$

(2) *If* $y_1, \dots, y_k$ *belong to the subspace* $\mathbb{R}^k \times 0$ *of* $\mathbb{R}^n$ *so that*

$$y_i = \begin{bmatrix} z_i \\ 0 \end{bmatrix}$$

for $z_i \in \mathbb{R}^k$, *then*

$$V(y_1, \dots, y_k) = |\det[z_1, \dots, z_k]|.$$

The function V vanishes iff the vectors $x_1, \dots, x_k$ are dependent. It satisfies the equation

$$V(x_1, \dots, x_k) = [\det(X^t X)]^{1/2},$$

where X is the $n \times k$ matrix with columns $x_1, \dots, x_k$.

Proof. Let $X = [x_1, \dots, x_k]$ be an $n \times k$ matrix and define $F(X) = \det(X^t X)$.

(1) Step # 1: Let $h\colon \mathbb{R}^n \to \mathbb{R}^n$ be an orthogonal transformation defined by $h(x) = Ax$, where A is an orthonormal matrix. Then

$$(*) \qquad F(AX) = \det((AX)^t (AX)) = \det(X^t X) = F(X)$$

and hence the function F is not affected by orthogonal transformations, $F(h(x_1), \dots, h(x_k)) = F(x_1, \dots, x_k)$.
If Z and Y are respectively $k \times k$ and $n \times k$ matrices defined by

$$Y = \begin{bmatrix} Z \\ 0 \end{bmatrix}$$

then

$$(**) \qquad \begin{aligned} F(Y) &= \det\left(\begin{bmatrix} Z^t & 0 \end{bmatrix} \cdot \begin{bmatrix} Z \\ 0 \end{bmatrix}\right) \\ &= \det(Z^t Z) = (\det Z)^2. \end{aligned}$$

So $F(y_1, \dots, y_k) = (\det[z_1, \dots, z_k])^2$.

(2) Step # 2: We can see that $(*)$ and $(**)$, and lemma (3.1.4) together imply that F is a nonnegative function. Thus $\sqrt{F}$ makes sense. In particular let $x_1, \dots, x_k \in \mathbb{R}^n$ be vectors in $\mathbb{R}^n$ and W a k-dimensional subspace of $\mathbb{R}^n$ containing them. By using lemma (3.1.4) let $h\colon \mathbb{R}^n \to \mathbb{R}^n$ be an orthogonal transformation carrying W onto $\mathbb{R}^k \times 0$ defined by $h(x) = Ax$. Then

$$h(X) = h([x_1 \cdots x_k]) = A[x_1 \cdots x_k] = [Ax_1 \cdots Ax_k] = \begin{bmatrix} Z \\ 0 \end{bmatrix}$$

where Z is a $k \times k$ matrix. Then $F(X) = F(AX) = (\det Z)^2 \geq 0$ and $F(X) = 0$ iff the columns of Z are linearly dependent iff the columns $x_1, \dots, x_k$ of X are linearly dependent.

(3) Step # 3: Let $V(X) = \sqrt{F(X)}$. The function satisfies the conditions of the theorem and Step # 2 shows that there is a unique function satisfying these properties.

□

Definition 3.1.6.

(1) *If $x_1, \dots, x_k$ are linearly independent vectors in $\mathbb{R}^n$, the k-dimensional volume of $\mathcal{P}(x_1, \dots, x_k)$ is defined by*

$$V(\mathcal{P}(x_1, \dots, x_k)) = \sqrt{\det(X^t X)} \text{ where } X = [x_1 \cdots x_k].$$

(2) *If $x_1, \dots, x_n$ are linearly independent vectors in $\mathbb{R}^n$, the volume of the parallelepiped spanned by these vectors equals to*

$$V(\mathcal{P}(x_1, \dots, x_n)) = |\det X| \text{ where } X = [x_1 \cdots x_n].$$

Definition 3.1.7. *Let $x_1, \dots, x_k$ be vectors in $\mathbb{R}^n$ and X the matrix whose columns are these vectors. If $I = (i_1, \dots, i_k) \in \{1, \dots, n\}^k$ and $1 \leq i_1 < \cdots < i_k \leq n$, then define X_I as the $k \times k$ submatrix of X consisting of rows $i_1, \dots, i_k$ of X.*

Theorem 3.1.8. *Let $1 \leq k \leq n$. Let X be the $n \times k$ matrix, then*

$$V(X) = \left\{ \sum_{(I)} (\det X_I)^2 \right\}^{1/2}$$

where the summation extends over all ascending k-tuples from $\{1, \dots, n\}$.

Remark 3.1.9.

- Theorem (3.1.8) states that the square of the k-dimensional volume of a k-parallelepiped in $\mathbb{R}^n$ is equal to the sum of the squares of the volumes of the k-parallelepipeds obtained by projecting it onto the various k-planes of $\mathbb{R}^n$. This is the generalized Phytagorean Theorem.

- For a proof see Munkres [23]. Binet-Cauchy formula (4.3.8), whose proof is simpler, is a generalization of (3.1.8).

3.2. **The Volume Of A Parameterized Manifold.**

Definition 3.2.1. *Let $1 \leq k \leq n$. Let $A \in \mathrm{OP}(\mathbb{R}^k)$ (A is open in $\mathbb{R}^k$), and let $\alpha \colon A \to \mathbb{R}^n$ be of class C^r. The set $Y = \alpha(A)$ together with the map α is called a parameterized manifold of dimension k in $\mathbb{R}^n$. We denote this parameterized manifold by Y_α and define the k-dimensional volume by*

$$V(Y_\alpha) = \int_A V([D\alpha(x)]),$$

whenever the integral exists.

Remark 3.2.2. Note that if $Y = \mathcal{P}(x_1, \dots, x_k)$ is a k-parallelepiped in $\mathbb{R}^n$ then Y is a parameterized k-manifold and

$$V(\mathcal{P}(x_1, \dots, x_k)) = \int_Q V([D\alpha(x)]),$$

where Q is the unit cube in $\mathbb{R}^k$ and $\alpha \colon \mathbb{R}^k \to \mathbb{R}^n$ is the linear map taking Q onto $\mathcal{P}(x_1, \dots, x_k)$ and $[D\alpha(x)] = [\alpha]$ for each $x \in Q$.

Definition 3.2.3. *Let $A \in \mathrm{OP}(\mathbb{R}^k)$, let $\alpha \colon A \to \mathbb{R}^n$ be of class C^r, and let $Y = \alpha(A)$. If $f \colon Y \to \mathbb{R}$ is continuous, then we define the integral of f over Y_α by*

$$\int_{Y_\alpha} f \, dV = \int_A (f \circ \alpha) \, V(D\alpha).$$

Theorem 3.2.4. *Let $g\colon A \to B$ be a diffeomorphism of open sets in $\mathbb{R}^k$. Let $\beta\colon B \to \mathbb{R}^n$ be of class C^r, let $Y = \beta(B)$ and $\alpha = \beta \circ g$. If $f\colon Y \to \mathbb{R}$ is continuous, then f is integrable over Y_β iff it is integrable over Y_α, and in this case*

$$\int_{Y_\alpha} f\,dV = \int_{Y_\beta} f\,dV.$$

In particular, $V(Y_\alpha) = V(Y_\beta)$.

Remark 3.2.5. Note that theorem (3.2.4) states that the integral of a continuous function over a parameterized manifold and the volume of a parameterized manifold are independent of the parametrization

Proof. We need to show that

$$\int_A (f\circ\alpha)\,V(D\alpha) = \int_B (f\circ\beta)\,V(D\beta)$$

where one integral exists iff the other exists. The change of variables theorem states that $(f\circ\beta)\,V(D\beta)$ is integrable over B iff $[(f\circ\beta)\circ g]\,V(D\beta\circ g)\,|\det Dg|$ is integrable over A and in this case

$$\int_B (f\circ\beta)\,V(D\beta) = \int_A [(f\circ\beta)\circ g]\,V(D\beta\circ g)\,|\det Dg|.$$

So it suffices to show that $V(D\beta\circ g)\,|\det Dg| = V(D\alpha)$. Let $x\in A,\ y = g(x)$, then

$$\begin{aligned}
D\alpha(x) &= D(\beta\circ g)(x) = D\beta(g(x))\circ Dg(x)\\
[V(D\alpha(x))]^2 &= \det\Big([Dg(x)]^t[D\beta(y)]^t[D\beta(y)][Dg(x)]\Big)\\
&= \Big(\det[Dg(x)]\Big)^2\Big(V(D\beta(y))\Big)^2\\
&\Rightarrow V(D\alpha) = V(D\beta\circ g)|\det Dg|.
\end{aligned}$$

Then

$$V(Y_\alpha) = \int_A 1\,V(D\alpha) = \int_B 1\circ\beta\,V(D\beta) = V(Y_\beta).$$

□

In the next two examples we will show how the volume of a parameterized k-manifold generalizes the length of a curve and the area of a surface.

Example 3.2.6. Let A be an open interval in $\mathbb{R}$, let $\alpha\colon A\to\mathbb{R}^n$ be a map of class C^r, $\alpha(t) = (\alpha_1(t),\ldots,\alpha_n(t)),\ Y = \alpha(A)$. Then

$$\begin{aligned}
[D\alpha(t)] &= [\alpha_1'(t)\cdots\alpha_n'(t)]\\
V(D\alpha(t)) &= \{(\alpha_1'(t))^2+\cdots+(\alpha_n'(t))^2\}^{1/2}
\end{aligned}$$

and hence

$$V(Y_\alpha) = \int_A V(D\alpha) = \int_A \{(\alpha_1'(t))^2+\cdots(\alpha_n'(t))^2\}^{1/2}\,dt$$

is the 1-dimensional volume (length) of Y_α.

Example 3.2.7. Let $A \in \mathrm{OP}(\mathbb{R}^2)$ and $\alpha\colon A \to \mathbb{R}^3$ be of class C^r, $Y = \alpha(A)$. Let $\alpha(x,y) = (\alpha_1(x,y), \alpha_2(x,y), \alpha_3(x,y))$. Then

$$V(D\alpha) = \left|\frac{\partial \alpha}{\partial x} \times \frac{\partial \alpha}{\partial y}\right|$$

and the two-dimensional volume of Y_α is given by

$$V(Y_\alpha) = \int_A \left|\frac{\partial \alpha}{\partial x} \times \frac{\partial \alpha}{\partial y}\right|.$$

In particular if α has the form $\alpha(x,y) = (x, y, f(x,y))$ where $f\colon A \to \mathbb{R}$ is a C^r map, we get the familiar equality

$$V(Y_\alpha) = \int_A \sqrt{1 + (\partial f/\partial x)^2 + (\partial f/\partial y)^2}.$$

3.3. **Manifolds In $\mathbb{R}^n$.**

Definition 3.3.1.

(1) *Let $\mathbb{H}^k$ denote the upper half space in $\mathbb{R}^k$ defined by*

$$\mathbb{H}^k = \{(x_1, \ldots, x_k) \in \mathbb{R}^k \colon x_k \geq 0\}.$$

(2) *Let $\mathbb{H}^k_+$ denote the open upper half space in $\mathbb{R}^k$ defined by*

$$\mathbb{H}^k_+ = \{(x_1, \ldots, x_k) \in \mathbb{R}^k \colon x_k > 0\}.$$

Definition 3.3.2. *Let $k > 0$. A k-manifold in $\mathbb{R}^n$ of class C^r is a topological subspace M of $\mathbb{R}^n$ having the property that for every $p \in M$, there is an open set $V \subseteq M$ containing p and an open set U in $\mathbb{R}^k$ or $\mathbb{H}^k$ such that there exists a map $\alpha\colon U \to V$*

(1) *α is bijective and of class C^r,*

(2) *$\alpha^{-1}\colon V \to U$ is continuous,*

(3) *$D\alpha(x)$ has rank k for every $x \in U$.*

The map α is called a coordinate map.
If $k = 0$, any discrete collection of points in $\mathbb{R}^n$ is defined as a 0-manifold in $\mathbb{R}^n$.

The next theorem, which we will state without proof, has many consequences one of which is its use in the precise definition of the boundary of a k-manifold in $\mathbb{R}^n$. For a proof see Munkres [23].

Theorem 3.3.3. *Let M be a k-manifold in $\mathbb{R}^n$, of class C^r. Let $\alpha_0\colon U_0 \to V_0$ and $\alpha_1\colon U_1 \to V_1$ be coordinate maps on M, with $W = V_0 \cap V_1 \neq \emptyset$. Let $W_i = \alpha_i^{-1}(W)$ $i = 0, 1$. Then the map*

$$\alpha_1^{-1} \circ \alpha_0\colon W_0 \to W_1$$

is of class C^r, and its derivative is nonsingular.

Remark 3.3.4. Note that the change of coordinates map defined in the theorem (3.3.3) is actually a diffeomorphism, which can be shown with the help of the Inverse Function Theorem.

Definition 3.3.5. *Let M be a k-manifold in $\mathbb{R}^n$, let $p \in M$. If there is a coordinate map $\alpha\colon U \to V$ on M about p such that $U \in \mathrm{OP}(\mathbb{R}^k)$ then p is called an interior point of M. Otherwise, p is called a boundary point of M. The set of all boundary points of M is called the boundary of M and is denoted by ∂M. The interior of M is denoted by $M - \partial M$.*

The next lemma, which can be shown by the use of theorem (3.3.3), gives a complete classification of interior and boundary points of a k-manifold M in $\mathbb{R}^n$. For a proof see Munkres [23].

Lemma 3.3.6. *Let M be a k-manifold in $\mathbb{R}^n$, let $\alpha\colon U \to V$ be a coordinate map about $p \in M$.*

(1) *If $U \in \mathrm{OP}(\mathbb{R}^k)$, then p is an interior point of M.*

(2) *If $U \in \mathrm{OP}(\mathbb{H}^k)$, and $\alpha(x_0) = p$, $x_0 \in \mathbb{H}^k_+$, then p is an interior points of M.*

(3) *If $U \in \mathrm{OP}(\mathbb{H}^k)$, and $\alpha(x_0) = p$, $x_0 \in \mathbb{R}^{k-1} \times \{0\}$, then p is a boundary point of M.*

Theorem 3.3.7. *Let M be a k-manifold in $\mathbb{R}^n$, of class C^r. If $\partial M \neq \emptyset$, then ∂M is a $(k-1)$-manifold without boundary in $\mathbb{R}^n$ of class C^r.*

Proof. Let $p \in \partial M$. Let $\alpha\colon U \to V$ be a coordinate map about p. Then $U \in \mathrm{OP}(\mathbb{H}^k)$, $\alpha(x_0) = p$ and $x_0 \in \partial\mathbb{H}^k$. By the lemma (3.3.6) each point of $U \cap \mathbb{H}^k_+$ is mapped by α to an interior point of M, and each point $U \cap \partial\mathbb{H}^k$ is mapped to a point of ∂M. Thus

$$\alpha|_{U\cap\partial\mathbb{H}^k} : U \cap \partial\mathbb{H}^k \to V \cap \partial M$$

is bijective and $V_0 = V \cap \partial M$ is open in ∂M. Let $\Pi\colon \mathbb{R}^k \to \mathbb{R}^{k-1}$ be the projection onto the first $(k-1)$ coordinates and let $U_0 = \Pi(U)$ be open in $\mathbb{R}^{k-1}$, then $U \cap \partial\mathbb{H}^k = U \cap (\mathbb{R}^{k-1} \times \{0\}) = U_0 \times \{0\}$. If $x \in U_0$, then let $\alpha_0\colon U_0 \in \mathrm{OP}(\mathbb{R}^{k-1}) \to V_0$ be a map defined by $\alpha_0(x) = \alpha(x, 0)$. So α_0 is of class C^r map because α is, $D\alpha_0(x)$ has rank $(k-1)$, and $\alpha_0^{-1} = \Pi \circ \alpha^{-1}\colon V_0 \to U_0$ is continuous. Consequently, α_0 is a coordinate map and since $p \in \partial M$ is arbitrary ∂M is a $(k-1)$-manifold without boundary in $\mathbb{R}^n$. □

3.4. **Integrating A Scalar Function Over A Manifold.**

Definition 3.4.1. *Let M be a compact k-manifold in $\mathbb{R}^n$, of class C^r, let $f\colon M \to \mathbb{R}$ be a continuous function, and $C = \operatorname{spt} f$. Assume that there exists a coordinate map $\alpha\colon U \to V$ such that $C \subseteq V$, then the integral of f over M is defined by*

$$\int_M f\,dV = \int_{U^\circ} (f \circ \alpha) V(D\alpha)$$

where U° is the interior of U in $\mathbb{R}^k$.

Remark 3.4.2.

- If M is both a parameterized manifold and a manifold in $\mathbb{R}^n$ then in both cases the definitions of the integral of a continuous real-valued function f over M coincide.

- Considering M to be a compact subset is not a restriction because the general case follows from this case by using the techniques of integration.

Lemma 3.4.3. *If the support of f can be covered by a single coordinate map, then $\int_M f\,dV$ is independent of the choice of coordinate map.*

Proof. We need to show that the definition of the integral of a scalar funtion is independent of the chosen coordinate map when the support of the function can be covered by a single coordinate map. Let $\alpha\colon U \to V$ be a coordinate map containing $\operatorname{spt} f$. Then we can choose $W \subseteq U$ open and $\alpha(W) \supseteq \operatorname{spt} f$, and

$$(*) \qquad \int_{W^\circ} (f \circ \alpha)\, V(D\alpha) = \int_{U^\circ} (f \circ \alpha)\, V(D\alpha).$$

Now assume that $\alpha_0\colon U_0 \to V_0$ and $\alpha_1\colon U_1 \to V_1$ are coordinate maps on M and $V_0 \cap V_1 \supseteq \operatorname{spt} f$. We want to show that

$$\int_{U_0^\circ} (f \circ \alpha_0)\, V(D\alpha_0) = \int_{U_1^\circ} (f \circ \alpha_1)\, V(D\alpha_1).$$

Let $W_0 = \alpha_0^{-1}(W),\ W_1 = \alpha_1^{-1}(W)$, where $W = V_0 \cap V_1$. By $(*)$ it suffices to show that

$$\int_{W_0} (f \circ \alpha_0)\, V(D\alpha_0) = \int_{W_1} (f \circ \alpha_1)\, V(D\alpha_1)$$

and this equality follows by (3.2.4) since $\alpha_1^{-1} \circ \alpha_0\colon W_0 \to W_1$ is a diffeomorphism. $\square$

The next lemma, whose proof can be found in Munkres [23], shows the existence of a partition of unity on a k-manifold M in $\mathbb{R}^n$. A partition of unity on a manifold is used to define the integral of a scalar function when the support of the function can not be covered by a single coordinate map.

Lemma 3.4.4. *Let M be a compact k-manifold in $\mathbb{R}^n$, of class C^r. Given a covering of M by coordinate maps, there exists a finite collection of C^∞ functions $\phi_1, \ldots, \phi_l$ mapping $\mathbb{R}^n$ into $\mathbb{R}$ such that*

(1) *$\phi_i(x) \geq 0$ for every $x \in M$,*
(2) *given $i \in \{1, \ldots, l\}$ $\operatorname{spt} \phi_i$ is compact, and there is a coordinate map $\alpha_i\colon U_i \to V_i$ belonging to the given covering such that $\operatorname{spt} \phi_i \cap M \subseteq V_i$,*
(3) *$\sum_i \phi_i(x) = 1$ for every $x \in M$.*

Definition 3.4.5. *Let M be a compact k-manifold in $\mathbb{R}^n$, of class C^r. Fix a covering $\alpha_j\colon U_j \to V_j,\ 1 \leq j \leq t$ of M by coordinate maps. Let $f\colon M \to \mathbb{R}$ be a continuous function. Choose a partition of unity $\phi_1, \ldots, \phi_l$ on M. The integral of f over M is defined by*

$$\int_M f\,dV = \sum_{i=1}^{l} \int_M \phi_i f\,dV,$$

where

$$\int_M \phi_i f\,dV = \int_{U_j^\circ} (\phi_i f) \circ \alpha_j\, V(D\alpha_j),$$

where $\operatorname{spt} \phi_i \subseteq V_j$ for some $1 \leq j \leq t$

Remark 3.4.6. A procedure similar to the one used in the proof of lemma (3.4.3) can be used to show that the definition of the integral does not depend on the chosen partition of unity.

4. RADEMACHER'S THEOREM, AREA-COAREA FORMULAS, RECTIFIABLE SETS

In this section we will study the Lipschitz functions. In geometric measure theory the Lipschitz functions serve the same purpose as the smooth functions in manifold theory. We will show that Lipschitz functions are differentiable almost everywhere and hence their Jacobians exist almost everywhere. Jacobians are the corrective factors that relate the volume of the domain and the volume of the range of differentiable functions. We will introduce the area formula which shows how the notions of the manifold theory are generalized to geometric measure theory by the techniques of the measure theory. We will show that the Hausdorff measure of a k-manifold is its volume and that its Hausdorff dimension is k. We will introduce the coarea formula, which is a generalization of the Fubini's theorem. The coarea formula asserts, roughly, that the measure of a Lebesque measurable subset A of $\mathbb{R}^n$ is the integral of the Hausdorff measure of the restriction of A to various level sets. In the last part we will define the rectifiable sets, which are the generalized surfaces of the geometric measure theory, and present some of their basic properties. The main references for this section are the books by Evans & Gariepy [9] and Federer [10].

4.1. **Lipschitz Functions.**

Definition 4.1.1.

(1) *Let $A \subseteq \mathbb{R}^n$. A function $f\colon A \to \mathbb{R}^m$ is called Lipschitz if*

$$|f(x) - f(y)| \leq C|x - y|$$

for some constant C and all x, $y \in A$. The smallest constant C such that the above inequality holds for all x and y is denoted

$$\begin{aligned} \mathrm{Lip}\,(f) &= \inf\{C \geq 0\colon |f(x) - f(y)| \leq C|x - y|\ x,\, y \in A\} \\ &= \sup\left\{\frac{|f(x) - f(y)|}{|x - y|}\colon x,\, y \in A,\, x \neq y\right\}. \end{aligned}$$

(2) *A function $f\colon A \to \mathbb{R}^m$ is called locally Lipschitz if for each compact $K \subseteq A$, there exists a constant C_K such that*

$$|f(x) - f(y)| \leq C_K|x - y|$$

for all x, $y \in K$.

Theorem 4.1.2. *Let $f\colon \mathbb{R}^n \to \mathbb{R}^m$ be Lipschitz, $A \subset \mathbb{R}^n$, $0 \leq s < \infty$. Then*

$$\mathcal{H}^s(f(A)) \leq (\mathrm{Lip}\, f)^s \mathcal{H}^s(A).$$

Proof. Fix $\delta > 0$ and choose sets $\{C_i\}_{i\in\mathbb{N}} \subseteq \mathbb{R}^n$ such that diam $C_i \leq \delta$ for each $i \in \mathbb{N}$ and $A \subseteq \bigcup_i C_i$, then diam $f(C_i) \leq (\mathrm{Lip}\, f)$diam $C_i \leq (\mathrm{Lip}\, f)\delta$ and $f(A) \subseteq \bigcup_i f(C_i)$. Then

$$\begin{aligned} \mathcal{H}^s_{(\mathrm{Lip}\, f)\delta}(f(A)) &\leq \sum_{i=1}^{\infty} \alpha(s) \left(\frac{\mathrm{diam}\ f(C_i)}{2}\right)^s \\ &\leq (\mathrm{Lip}\, f)^s \sum_{i=1}^{\infty} \left(\frac{\mathrm{diam}\ C_i}{2}\right)^s. \end{aligned}$$

Since the sets $\{C_i\}_{i\in\mathbb{N}}$ are arbitrary, $\mathcal{H}^s_{(\mathrm{Lip}\, f)\delta}(f(A)) \leq (\mathrm{Lip}\, f)^s \mathcal{H}^s_\delta(A) \leq (\mathrm{Lip}\, f)^s \mathcal{H}^s(A)$. Now by letting $\delta \to 0$, we get $\mathcal{H}^s(f(A)) \leq (\mathrm{Lip}\, f)^s \mathcal{H}^s(A)$. □

Corollary 4.1.3. *Suppose $n > k$. Let $P \colon \mathbb{R}^n \to \mathbb{R}^k$ be the projection map, $A \subseteq \mathbb{R}^n$, $0 \leq s < \infty$. Then*

$$\mathcal{H}^s(P(A)) \leq \mathcal{H}^s(A).$$

Theorem 4.1.4 (Extension Of Lipschitz Functions)**.** *Assume $A \subseteq \mathbb{R}^n$, and let $f \colon A \to \mathbb{R}^m$ be Lipschitz. Then there exists a Lipschitz function $\overline{f} \colon \mathbb{R}^n \to \mathbb{R}^m$ such that*

(1) $\overline{f} = f$ *on* A.

(2) $\mathrm{Lip}\,(\overline{f}) \leq \sqrt{m}\,\mathrm{Lip}\,(f)$.

Proof. Assume that $f \colon A \to \mathbb{R}$. Define

$$\overline{f}(x) = \inf_{a\in A}\{f(a) + \mathrm{Lip}\,(f)|x-a|\}, \ x \in \mathbb{R}^n.$$

If $b \in A$ then $\overline{f}(b) = f(b)$. If x, $y \in \mathbb{R}^n$, then

$$\begin{aligned}\overline{f}(x) &\leq \inf_{a\in A}\{f(a) + \mathrm{Lip}\,(f)(|y-a| + |x-y|)\}\\ &= \overline{f}(y) + \mathrm{Lip}\,(f)|x-y|\end{aligned}$$

and similarly

$$\overline{f}(y) \leq \overline{f}(x) + \mathrm{Lip}\,(f)|x-y|.$$

$$|\overline{f}(x) - \overline{f}(y)| \leq \mathrm{Lip}\,(f)|x-y| \ \Rightarrow \ \mathrm{Lip}\,\overline{f} \leq \mathrm{Lip}\, f.$$

Now for the general case let $f \colon A \to \mathbb{R}^m$, $f = (f_1, \ldots f_m)$ and define $\overline{f} = (\overline{f_1}, \ldots, \overline{f_m})$. Then

$$|\overline{f}(x) - \overline{f}(y)|^2 = \sum_{i=1}^m |\overline{f_i}(x) - \overline{f_i}(y)|^2 \leq \sum_{i=1}^m (\mathrm{Lip}\, f_i)^2|x-y|^2 \leq m(\mathrm{Lip}\, f)^2|x-y|^2.$$

□

Definition 4.1.5. *The function $f \colon \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $x \in \mathbb{R}^n$ if there exists a linear map $L \colon \mathbb{R}^n \to \mathbb{R}^m$ such that*

$$\lim_{y\to x} \frac{|f(y) - f(x) - L(y-x)|}{|y-x|} = 0.$$

The linear map $L \equiv Df(x)$ is called the derivative of f at x.

Theorem 4.1.6. *If a locally Lipschitz function $f \colon \mathbb{R}^n \to \mathbb{R}^m$ is approximately differentiable at $a \in \mathbb{R}^n$ then it is differentiable at $a \in \mathbb{R}^n$.*

Proof. Suppose that the function $f \colon \mathbb{R}^n \to \mathbb{R}^m$ is approximately differentiable at $a \in \mathbb{R}^n$ but not differentiable at a. For notational convenience we may assume that $a = 0$, $f(a) = 0$, $\mathrm{ap}\, Df(a) = 0$. Since f is not differentiable at $a = 0$, for some $0 < \epsilon < 1$ we can choose a sequence of point $\{a_i\}_{i\in\mathbb{N}} \subseteq \mathbb{R}^n$ such that $a_i \to 0$ and $|f(a_i)| \geq \epsilon|a_i|$ for each $i \in \mathbb{N}$. Let $C = \max\{\operatorname{Lip} f, 1\}$. If $x \in B(a_i, \epsilon|a_i|/3C)$, then

$$|f(x)| \geq |f(a_i)| - |f(a_i) - f(x)| \geq \epsilon|a_i| - \frac{\epsilon|a_i|}{3} \geq \frac{\epsilon|x|}{2}.$$

And hence

$$x \in E = \bigcup_{i=1}^{\infty} B\left(a_i, \frac{\epsilon|a_i|}{3C}\right) \text{ implies } |f(x)| \geq \frac{\epsilon|x|}{2}$$

But E does not have density 0 at $a = 0$ because there exists a sequence $\{|a_i| + (\epsilon|a_i|)/3C\}_{i\in\mathbb{N}}$ converging to 0 and

$$\begin{aligned}\frac{\mathcal{L}^n(E \cap B(0, |a_i| + (\epsilon|a_i|)/3C))}{\alpha_n(|a_i| + (\epsilon|a_i|)/3C)^n} &\geq \frac{\mathcal{L}^n(B(a_i, \epsilon|a_i|/3C))}{\alpha_n((4|a_i|)/3)^n} \\ &\geq \frac{(\epsilon|a_i|/3C)^n}{(4|a_i|/3)^n} = \underbrace{\frac{\epsilon^n}{4^nC^n}}_{\text{independent of } r} > 0.\end{aligned}$$

$$\Theta^{n*}(E, 0) = \limsup_{r\to 0} \frac{\mathcal{L}^n(E \cap B(0, r))}{\alpha_n r^n} \geq \frac{\epsilon^n}{4^nC^n},$$

so E does not have density 0 at $a = 0$ and hence

$$\operatorname{ap}\lim_{x\to 0} \frac{|f(x)|}{|x|} \neq 0$$

because

$$\limsup_{r\to 0} \frac{\mathcal{L}^n(\{x\colon |f(x)| > \epsilon|x|/2\} \cap B(0, r))}{\alpha_n r^n} \geq \frac{\epsilon^n}{4^nC^n} > 0,$$

which is a contradiction. □

4.2. **Rademacher's Theorem.**

Theorem 4.2.1 (Rademacher's Theorem)**.** *Let $f\colon \mathbb{R}^n \to \mathbb{R}^m$ be a locally Lipschitz function. Then f is differentiable $\mathcal{L}^n$-a.e.*

Proof. We may assume that $m = 1$ because f is differentiable iff each of its coordinate functions is differentiable. Since differentiability is a local property we can assume (by considering $\tilde{f}_k(x) = f\chi_{Q_k}(x)$ where Q_k is an open cube centered at the origin with side length $2k$, $k \in \mathbb{N}$) that f is Lipschitz.
Fix $v \in \mathbb{R}^n$ with $|v| = 1$, and define

$$D_v f(x) = \lim_{t\to 0} \frac{f(x + tv) - f(x)}{t} \quad x \in \mathbb{R}^n,$$

whenever this limit exists.

(1) Claim # 1: $D_v f(x)$ exists for $\mathcal{L}^n$-a.e. $x \in \mathbb{R}^n$.

Proof Of Claim # 1. Since f is continuous,

$$\overline{D}_v f(x) = \limsup_{t\to 0} \frac{f(x + tv) - f(x)}{t}$$

$$= \inf_{\epsilon > 0} \sup_{0<|t|<\epsilon} \frac{f(x+tv)-f(x)}{t} = \inf_{k \geq 1} \sup_{\substack{0<|t|<1/k \\ t\in\mathbb{Q}}} \frac{f(x+tv)-f(x)}{t}.$$

Hence $\overline{D}_v f(x)$ is Borel measurable, and similarly

$$\underline{D}_v f(x) = \liminf_{t\to 0} \frac{f(x+tv)-f(x)}{t}$$

is measurable. Thus

$$\begin{aligned} A_v &= \{x \in \mathbb{R}^n \mid D_v f(x) \text{ does not exist}\} \\ &= \{x \in \mathbb{R}^n \mid \underline{D}_v f(x) < \overline{D}_v f(x)\} \end{aligned}$$

is Borel measurable. Our aim is to show that $\mathcal{L}^n(A_v) = 0$. Given $x \in \mathbb{R}^n$, define $\varphi\colon \mathbb{R} \to \mathbb{R}$ by

$$\varphi(t) = f(x+tv).$$

Then φ is Lipschitz, and hence differentiable $\mathcal{L}^1$-a.e. Since

$$\begin{aligned} D_v f(x) &= \lim_{t\to 0} \frac{f(x+tv)-f(x)}{t} \\ &= \lim_{t\to 0} \frac{\varphi(t)-\varphi(0)}{t}, \end{aligned}$$

$D_v f(x)$ exists a.e on $L_x = \{x+tv\colon t \in \mathbb{R}\}$ and consequently $\mathcal{H}^1(A_v \cap L_x) = \mathcal{L}^1(A_v \cap L_x) = 0$ for every $x \in \mathbb{R}^n$, namely f has a directional derivative at almost all points on L_x in the direction of v.

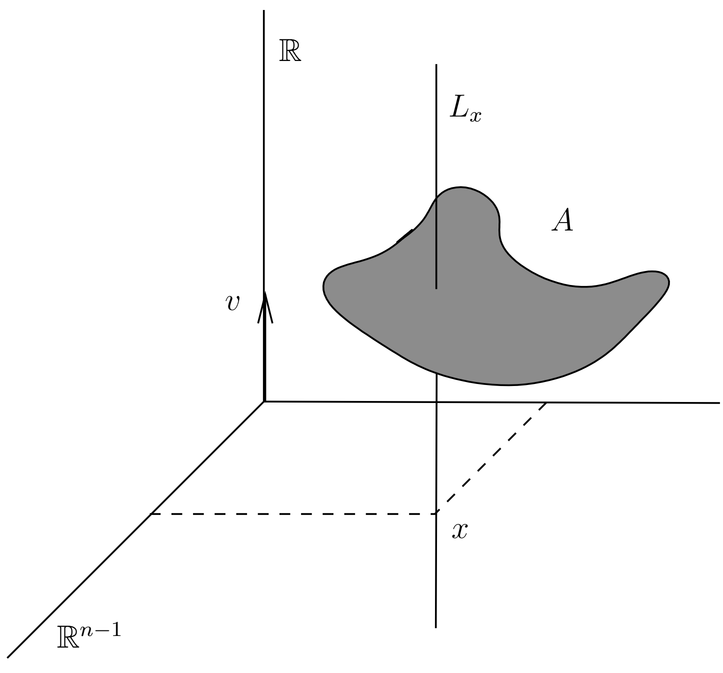


FIGURE 3. $D_v f(x)$ exists $\mathcal{L}^1$-a.e.

By Fubini's theorem

$$\mathcal{L}^n(A_v) = (\mathcal{L}^{n-1} \times \mathcal{L}^1)(A_v) = \int_{\mathbb{R}^{n-1}} \mathcal{L}^1((A_v)_x)\, d\mathcal{L}^{n-1}x$$

$$= \int_{\mathbb{R}^{n-1}} \mathcal{H}^1((A_v)_x)\, d\mathcal{L}^{n-1}x$$
$$= \int_{\mathbb{R}^{n-1}} \mathcal{H}^1(A_v \cap L_x)\, d\mathcal{L}^{n-1}x = 0.$$

Now as a consequence of Claim # 1 the gradient function

$$\operatorname{grad} f(x) = \left[\frac{\partial f}{\partial x_1}(x), \dots, \frac{\partial f}{\partial x_n}(x)\right]$$

exists $\mathcal{L}^n$-a.e. $x \in \mathbb{R}^n$.

(2) Claim # 2: $D_v f(x) = v \cdot \operatorname{grad} f(x)$ for $\mathcal{L}^n$-a.e. $x \in \mathbb{R}^n$.

Proof Of Claim # 2. Let $\zeta \in C_c^\infty(\mathbb{R}^n)$. Then

$$\int_{\mathbb{R}^n} \left[\frac{f(x+tv) - f(x)}{t}\right] \zeta(x)\, d\mathcal{L}^n x = -\int_{\mathbb{R}^n} f(x) \left[\frac{\zeta(x) - \zeta(x-tv)}{t}\right] d\mathcal{L}^n x,$$

by the change of variables. Let $t = 1/k$ for $k \in \mathbb{N}$, then

$$\left|\frac{f(x+v/k) - f(x)}{1/k}\right| \le (\operatorname{Lip} f)|v| = \operatorname{Lip} f.$$

If we let $g_k(x) = \frac{f(x+v/k)-f(x)}{1/k}\zeta(x)$, then $g_k(x) \to D_v f(x)\zeta(x)$ $\mathcal{L}^n$-a.e. Hence the Dominated Convergence Theorem (DCT) implies that

$$\int_{\mathbb{R}^n} D_v f(x)\zeta(x)\, d\mathcal{L}^n x = \lim_k \int_{\mathbb{R}^n} \frac{f(x+v/k) - f(x)}{1/k}\zeta(x) d\mathcal{L}^n x$$
$$= -\lim_k \int_{\mathbb{R}^n} \frac{\zeta(x) - \zeta(x - v/k)}{1/k} f(x) d\mathcal{L}^n x$$
$$= -\int_{\mathbb{R}^n} f(x) D_v \zeta(x)\, d\mathcal{L}^n x$$
$$= -\sum_{i=1}^n v_i \int_{\mathbb{R}^n} f(x) \frac{\partial \zeta}{\partial x_i}(x)\, d\mathcal{L}^n x.$$

The absolute continuity of f on almost every line parallel to coordinate axis implies that

$$= \sum_{i=1}^n v_i \int_{\mathbb{R}^n} \frac{\partial f}{\partial x_i}(x)\zeta(x)\, d\mathcal{L}^n x$$
$$= \int_{\mathbb{R}^n} (v \cdot \operatorname{grad} f(x))\zeta(x)\, d\mathcal{L}^n x.$$

Since the equality holds for each $\zeta \in C_c(\mathbb{R}^n)$, $D_v f(x) = v \cdot \operatorname{grad} f(x)$ $\mathcal{L}^n$-a.e. $x \in \mathbb{R}^n$.

Let $\mathcal{D} = \{v_k\}_{k \in \mathbb{N}}$ be a countable dense subset of $\partial B(0,1)$. Define for each $k \in \mathbb{N}$ the set A_k as

$$A_k = \{x \in \mathbb{R}^n \colon D_{v_k} f(x) = v_k \cdot \operatorname{grad} f(x)\}$$

and let

$$A = \bigcap_{k=1}^{\infty} A_k.$$

By claim # 1 and claim # 2 we have $\mathcal{L}^n(\mathbb{R}^n - A) = 0$.

(3) Claim # 3 f is differentiable at each point $x \in A$.

Proof Of Claim # 3. Fix $x \in A$. Choose $v \in \partial B(0,1)$, $t \in \mathbb{R} - \{0\}$ and define

$$Q(x,v,t) = \frac{f(x+tv) - f(x)}{t} - v \cdot \text{grad } f(x).$$

If $v' \in \partial B(0,1)$, then

$$\begin{aligned} |Q(x,v,t) - Q(x,v',t)| &\leq \left| \frac{f(x+tv) - f(x+tv')}{t} \right| + |(v-v') \cdot \text{grad } f(x)| \\ &\leq (\text{Lip } f)|v - v'| + |\text{grad } f(x)||v - v'| \\ &\leq (\sqrt{n}+1)(\text{Lip } f)|v - v'|. \end{aligned}$$

Fix $\epsilon > 0$, since $\partial B(0,1)$ is compact we can choose $N \in \mathbb{N}$ large enough so that if $v \in \partial B(0,1)$, then

$$|v - v_k| \leq \frac{\epsilon}{2(\sqrt{n}+1)(\text{Lip } f)}$$

$v_k \in \mathcal{D}$ for some $k \in \{1, \ldots N\}$. Since

$$\lim_{t \to 0} Q(x, v_k, t) = 0 \quad (k = 1, \ldots, N)$$

there exists $\delta > 0$ with

$$|Q(x, v_k, t)| < \epsilon/2 \quad \text{for } 0 < |t| < \delta \text{ and } k = 1, \ldots, N.$$

Consequently, for each $v \in \partial B(0,1)$, there exists $k \in \{1, \ldots, N\}$ such that

$$|Q(x,v,t)| \leq |Q(x,v_k,t)| + |Q(x,v,t) - Q(x,v_k,t)| < \epsilon$$

whenever $0 < |t| < \delta$. Thus we can conclude that

$$\lim_{t \to 0} Q(x,v,t) = \lim_{t \to 0} \frac{f(x+tv) - f(x)}{t} - v \cdot \text{grad } f(x)$$

holds for every $v \in \partial B(0,1)$ and that the partial derivative of f at $x \in A$ in each direction exists. Choose $y \in \mathbb{R}^n$ with $y \neq x$ and let $v = \frac{y-x}{|y-x|}$, then $y = x + tv$ where $t = |x - y|$. Then

$$f(y) - f(x) - \text{grad } f(x) \cdot (y - x) = f(x+tv) - f(x) - tv \cdot \text{grad } f(x)$$

$$\begin{aligned} \frac{|f(y) - f(x) - \text{grad } f(x) \cdot (y-x)|}{|y-x|} &= \left| \frac{f(x+tv) - f(x)}{t} - v \cdot \text{grad } f(x) \right| \\ &\to 0 \end{aligned}$$

as $y \to x$ and $t \to 0$. Thus f is differentiable at $x \in A$, $Df(x)(y-x) = \text{grad } f(x) \cdot (y-x)$ and $[Df(x)] = \text{grad } f(x)$.

$\square$

Now we will prove a corollary of Rademacher's Theorem which shows the similarities in behavior between Lipschitz functions and differentiable functions.

Corollary 4.2.2.

(1) *Let* $f\colon \mathbb{R}^n \to \mathbb{R}^m$ *be locally Lipschitz,and*
$$Z = \{x \in \mathbb{R}^n \mid f(x) = 0\}.$$
Then $Df(x) = 0$ *for* $\mathcal{L}^n$*-a.e.* $x \in Z$*.*

(2) *Let* $f, g\colon \mathbb{R}^n \to \mathbb{R}^n$ *be locally Lipschitz, and*
$$Y = \{x \in \mathbb{R}^n \mid g(f((x)) = x\}.$$
Then $Dg(f((x))Df(x) = I$ *for* $\mathcal{L}^n$*-a.e.* $x \in Y$*.*

Proof.

(1) We may assume that $m = 1$. WLOG we may assume that $\mathcal{L}^n(Z) > 0$ and that we can choose $x \in Z$ such that $Df(x)$ exists and
$$\lim_{r\to 0} \frac{\mathcal{L}^n(Z \cap B(x,r))}{\mathcal{L}^n(B(x,r))} = 1$$
as a consequence of Lebesque-Besicovitch Differentiation Theorem. Then
$$f(y) = Df(x)(y - x) + \circ(|y - x|) \text{ as } y \to x.$$
Assume that $[Df(x)] = a \neq 0$, and set
$$S = \{v \in \partial B(0,1) \mid a \cdot v \geq |a|/2\}.$$

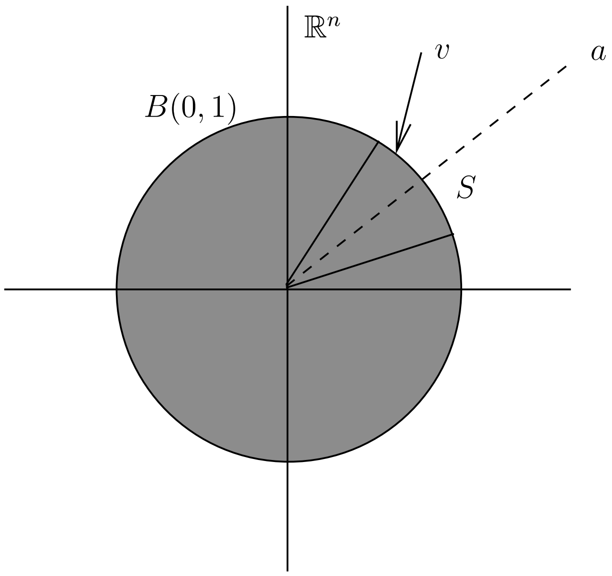


FIGURE 4. A constant Lipschitz function has the zero differential almost everywhere.

For each $v \in S$ and $t > 0$, set $y = x + tv$, then
$$\begin{aligned} f(x+tv) &= a \cdot tv + \circ(|tv|) \\ &\geq \frac{t|a|}{2} + \circ(t) \quad \text{as } t \to 0. \end{aligned}$$
Dividing by t we obtain
$$\frac{f(x+tv)}{t} \geq \frac{|a|}{2} + \epsilon_t, \text{ where } \epsilon_t \to 0 \text{ as } t \to 0.$$

So there exists $t_0 > 0$ such that $f(x+tv) > 0$ for $0 < t < t_0$ and for all $v \in S$. Thus $\mathcal{L}^n(Z \cap B(x,r)) \leq \alpha \mathcal{L}^n(B(x,r))$ for all $0 < r < t_0$ and for some $\alpha < 1$. But this implies

$$\lim_{r\to 0} \frac{\mathcal{L}^n(Z \cap B(x,r))}{\mathcal{L}^n(B(x,r))} < 1,$$

which is a contradiction.

(2) Define

$$\begin{aligned} \text{dom } Df &= \{x \mid Df(x) \text{ exists}\}, \\ \text{dom } Df &= \{x \mid Dg(x) \text{ exists}\}. \end{aligned}$$

Let

$$X = Y \cap \text{dom } Df \cap f^{-1}(\text{dom } Dg)$$

then

$$Y - X \subseteq (\mathbb{R}^n - \text{dom } Df) \cup g(\mathbb{R}^n - \text{dom } Dg).$$

By Rademacher's Theorem $\mathcal{L}^n(\mathbb{R}^n - \text{dom}Df) = 0$, $\mathcal{L}^n(\mathbb{R}^n - \text{dom}Dg) = 0$ and

$$\begin{aligned} \mathcal{L}^n(g(\mathbb{R}^n - \text{dom } Dg)) &= \mathcal{H}^n(g(\mathbb{R}^n - \text{dom } Dg)) \\ &\leq (\operatorname{Lip} g)^n \mathcal{H}^n(\mathbb{R}^n - \text{dom } Dg) \\ &= (\operatorname{Lip} g)^n \mathcal{L}^n(\mathbb{R}^n - \text{dom } Dg) \\ &= 0, \end{aligned}$$

so $\mathcal{L}^n(Y - X) = 0$. If $x \in X$ then $Df(x)$, $Dg(f(x))$, and $D(g\circ f)(x)$ exist. Since

$$(g \circ f)(x) - x = 0 \quad \text{for every } x \in X \subseteq Y$$

then

$$D(g \circ f)(x) - DI(x) = 0 \quad \text{for every } x \in X \subseteq Y$$

and hence

$$D(g \circ f)(x) = I \quad \mathcal{L}^n\text{-a.e. } x \in Y$$

□

4.3. **Linear Maps And Jacobians.**

Definition 4.3.1.

(1) *A linear map $O\colon \mathbb{R}^n \to \mathbb{R}^m$ is called orthogonal if $(Ox)\cdot(Oy) = x \cdot y$ for all x, $y \in \mathbb{R}^n$, that is a linear map is orthogonal if it preserves the dot product.*

(2) *A linear map $S\colon \mathbb{R}^n \to \mathbb{R}^n$ is called symmetric if $x \cdot (Sy) = (Sx)\cdot y$ for all x, $y \in \mathbb{R}^n$, that is a linear map is called symmetric if its matrix representation with respect to standard basis is a symmetric matrix.*

(3) *Let $A\colon \mathbb{R}^n \to \mathbb{R}^m$ be linear. The adjoint of A is the linear map $A^*\colon \mathbb{R}^m \to \mathbb{R}^n$ defined by $x \cdot (A^*y) = (Ax)\cdot y$ for all $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$.*

Now we will recall some useful properties of the linear maps.

Theorem 4.3.2. *Let A, B be linear maps.*

(1) $A^{**} = A$.

(2) $(A \circ B)^* = B^* \circ A^*$.

(3) *If $O\colon \mathbb{R}^n \to \mathbb{R}^n$ is orthogonal then $O^* = O^{-1}$.*

(4) *If $S\colon \mathbb{R}^n \to \mathbb{R}^n$ is symmetric then $S^* = S$.*

(5) *If $O\colon \mathbb{R}^n \to \mathbb{R}^m$, then $n \le m$ and*

$$\begin{aligned} O^* \circ O &= I \quad \text{on } \mathbb{R}^n, \\ O \circ O^* &= I \quad \text{on } O(\mathbb{R}^n). \end{aligned}$$

Theorem 4.3.3 (Polar Decomposition)**.** *Let $L\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear map.*

(1) *If $n \le m$, there exist a symmetric map $S\colon \mathbb{R}^n \to \mathbb{R}^n$ and an orthogonal map $O\colon \mathbb{R}^n \to \mathbb{R}^m$ such that*
$$L = O \circ S.$$

(2) *If $n \ge m$, there exist a symmetric map $S\colon \mathbb{R}^m \to \mathbb{R}^m$ and an orthogonal map $O\colon \mathbb{R}^m \to \mathbb{R}^n$ such that*
$$L = S \circ O^*.$$

Definition 4.3.4. *Let $L\colon \mathbb{R}^n \to \mathbb{R}^m$ be linear.*

(1) *If $n \le m$, let $L = O \circ S$ be a polar decomposition of L, then the Jacobian of L is defined to be*
$$[\![L]\!] = |\det S|.$$

(2) *If $n \ge m$ then $L = S \circ O^*$ be a polar decomposition of L, then the Jacobian of L is defined to be*
$$[\![L]\!] = |\det S|.$$

Remark 4.3.5.

- It seems that the definition of the jacobian is dependent on the polar decomposition of the linear map, but the next theorem will show that the jacobian is actually well-defined.

- It is clear from the definition that
$$[\![L]\!] = [\![L^*]\!].$$

Theorem 4.3.6.

(1) *If $n \le m$,*
$$[\![L]\!]^2 = \det(L^* \circ L).$$

(2) *If $n \ge m$,*
$$[\![L]\!]^2 = \det(L \circ L^*).$$

Proof.

(1) Let $L = O \circ S$ be a polar decomposition of L, then $L^* = S \circ O^*$, and $L^* \circ L = (S \circ O^*) \circ (O \circ S) = S^2$. Thus $\det(L^* \circ L) = (\det S)^2 = [\![L]\!]^2$.

(2) The proof of (2) is similar.

□

Definition 4.3.7.

(1) *If $n \leq m$, then define*

$$\Lambda(m,n) = \{\lambda\colon \{1,\dots,n\} \to \{1,\dots,m\} \mid \lambda \textit{ is increasing}\}.$$

(2) *For each $\lambda \in \Lambda(m,n)$, define a linear map $P_\lambda\colon \mathbb{R}^m \to \mathbb{R}^n$ by*

$$P_\lambda(x_1,\dots,x_m) = (x_{\lambda(1)},\dots,x_{\lambda(n)}).$$

There exists an n-dimensional subspace

$$S_\lambda = \langle\{e_{\lambda(1)},\dots,e_{\lambda(n)}\}\rangle \subset \mathbb{R}^m$$

such that P_λ is the projection of $\mathbb{R}^m$ onto S_λ.

Theorem 4.3.8 (Binet-Cauchy Formula)**.** *Assume $n \leq m$ and $L\colon \mathbb{R}^n \to \mathbb{R}^m$ is linear. Then*

$$[\![L]\!]^2 = \sum_{\lambda \in \Lambda(m,n)} (\det[P_\lambda \circ L])^2.$$

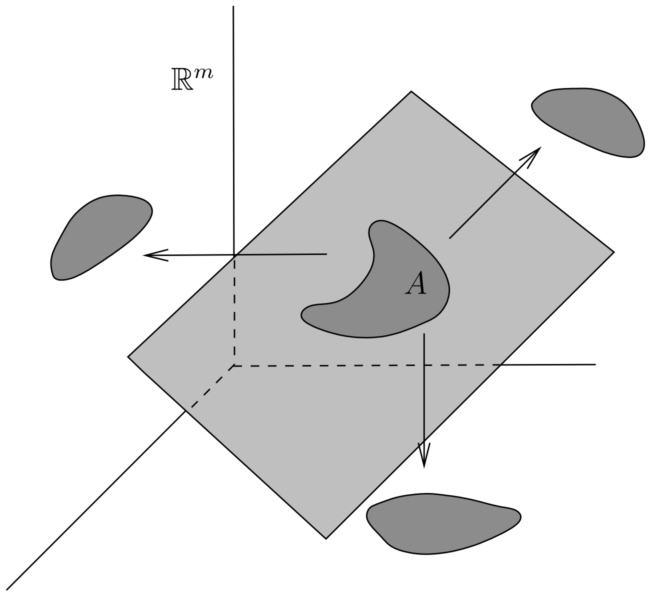


FIGURE 5. The square of the $\mathcal{H}^n$-measure of A equals the sum of the squares of the $\mathcal{H}^n$ measure of the projections of A onto the coordinate planes.

Remark 4.3.9.

- $[\![L]\!]^2$ is equal to the sum of the squares of the determinants of each $(n \times n)$-submatrix of the $(m \times n)$-matrix representing L with respect to the standard bases of $\mathbb{R}^n$ and $\mathbb{R}^m$.

- If L is an injective linear transformation then the columns of the matrix representation $[L]$ with respect to the standard bases span the n-dimensional parallelepiped in $\mathbb{R}^m$ whose n-dimensional volume is defined to be $[\![L]\!]$. Thus the Binet-Cauchy formula is a generalization of the higher dimensional version of the Pythagorean theorem (3.1.8).

Proof. After we identify the linear maps with their matrix representation with respect to the standard bases of $\mathbb{R}^n$ and $\mathbb{R}^m$ we can write

$$L_{m\times n} = (l_{ij})_{i,j}, \quad A_{n\times n}= L^* \circ L = (a_{ij})_{i,j}$$

and

$$a_{ij} = \sum_{k=1}^{m} l^*_{ik}\, l_{kj} = \sum_{k=1}^{m} l_{ki}\, l_{kj} \quad i, j \in (1, \ldots n).$$

Then

$$[\![L]\!]^2 = \det A = \sum_{\sigma\in\Sigma} \text{sgn } (\sigma) \prod_{i=1}^{n} a_{i\sigma(i)},$$

where Σ is the set of all permutations of $\{1, \ldots , n\}$.

$$[\![L]\!]^2 = \sum_{\sigma\in\Sigma} \text{sgn } (\sigma) \prod_{i=1}^{n} \sum_{k=1}^{m} l_{ki}\, l_{k\sigma(i)},$$

after taking into account the cancellations due to sgn (σ) we obtain

$$= \sum_{\sigma\in\Sigma} \text{sgn } (\sigma) \sum_{\varphi\in\Phi} \prod_{i=1}^{n} l_{\varphi(i)i}\, l_{\varphi(i)\sigma(i)},$$

where Φ is the set of all injective mappings of $\{1, \ldots , n\}$ into $\{1, \ldots , m\}$.

Claim # 1: For each $\varphi \in \Phi$ there exist unique $\theta \in \Sigma$ and $\lambda \in \Lambda(m, n)$ such that $\varphi = \lambda \circ \theta$

Proof Of Claim # 1.

(1) **Uniqueness:** Let $\varphi = \lambda_1 \circ \theta_1 = \lambda_2 \circ \theta_2$ and assume that $\lambda_1 \neq \lambda_2$ and that k_0 is the smallest number in $\{1, \ldots , n\}$ with $\lambda_1(k_0) < \lambda_2(k_0)$. Then

$$(*) \qquad \lambda_1(k_0) \notin \{\lambda_2(k_0), \ldots , \lambda_2(n)\}$$

and

$$\lambda_1(1) = \lambda_2(1), \ldots , \lambda_1(k_0 - 1) = \lambda_2(k_0 - 1).$$

$\lambda_1(k_0) = \lambda_1(\theta_1(\tilde{k}_0)) = \varphi(\tilde{k}_0) = \lambda_2(\theta_2(\tilde{k}_0)) = \lambda_1(\theta_2(\tilde{k}_0))$, where $\tilde{k}_0 \in \{1, \ldots, n\}$, then $(*)$ implies that $\theta_2(\tilde{k}_0) < k_0$, which is a contradiction because λ_1 is increasing. So $\lambda_1 = \lambda_2$ and $\theta_1 = \theta_2$.

(2) **Existence:** Let $\varphi\colon \{1, \ldots, n\} \to \{1, \ldots, m\}$ be injective. Let $i_1, \ldots, i_n$ be an ordering of $1, \ldots, n$ such that $\varphi(i_1) < \cdots < \varphi(i_n)$ and $\{i_1, \ldots, i_n\} = \{1, \ldots, n\}$. Define maps

$$\lambda\colon \{1, \ldots, n\} \to \{1, \ldots, m\} \quad \lambda(k) = \varphi(i_k)$$

and

$$\theta\colon \{1, \ldots, n\} \to \{1, \ldots, n\} \quad \theta(i_j) = j,$$

then

$$\lambda \in \Lambda(m, n),\ \theta \in \Sigma \text{ and } \varphi = \lambda \circ \theta.$$

Consequently,

$$\begin{aligned}
[\![L]\!]^2 &= \sum_{\sigma\in\Sigma} \operatorname{sgn}(\sigma) \sum_{\lambda\in\Lambda(m,n)} \sum_{\theta\in\Sigma} \prod_{i=1}^{n} l_{\lambda\circ\theta(i),i}\, l_{\lambda\circ\theta(i),\sigma(i)} \\
&= \sum_{\sigma\in\Sigma} \operatorname{sgn}(\sigma) \sum_{\lambda\in\Lambda(m,n)} \sum_{\theta\in\Sigma} \prod_{i=1}^{n} l_{\lambda(i),\theta^{-1}(i)}\, l_{\lambda(i),\sigma\circ\theta^{-1}(i)} \\
&= \sum_{\lambda\in\Lambda(m,n)} \sum_{\theta\in\Sigma} \sum_{\sigma\in\Sigma} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} l_{\lambda(i),\theta(i)}\, l_{\lambda(i),\sigma\circ\theta(i)} \\
&= \sum_{\lambda\in\Lambda(m,n)} \sum_{\rho\in\Sigma} \sum_{\sigma\in\Sigma} \operatorname{sgn}(\theta)\operatorname{sgn}(\rho) \prod_{i=1}^{n} l_{\lambda(i),\theta(i)}\, l_{\lambda(i),\rho(i)} \\
&= \sum_{\lambda\in\Lambda(m,n)} \left(\sum_{\theta\in\Sigma} \operatorname{sgn}(\theta) \prod_{i=1}^{n} l_{\lambda(i),\theta(i)} \right)^2 \\
&= \sum_{\lambda\in\Lambda(m,n)} (\det[P_\lambda \circ L])^2. \qquad \square
\end{aligned}$$

Definition 4.3.10. *Let $f\colon \mathbb{R}^n \to \mathbb{R}^m$ be Lipschitz. By Rademacher's theorem* (4.2.1)*, f is differentiable $\mathcal{L}^n$-a.e. and hence $Df(x)$ is an $\mathcal{L}^n$-a.e. defined linear map of $\mathbb{R}^n$ into $\mathbb{R}^m$. The matrix representation of $Df(x)$ is*

$$[Df(x)] = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x) & \cdots & \frac{\partial f_1}{\partial x_n}(x) \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1}(x) & \cdots & \frac{\partial f_m}{\partial x_n}(x) \end{bmatrix}$$

The Jacobian of f is defined to be

$$Jf(x) = [\![Df(x)]\!] \quad \mathcal{L}^n\text{-a.e.}$$

4.4. **The Area Formula And Its Applications.**

Lemma 4.4.1. *Suppose* $L\colon \mathbb{R}^n \to \mathbb{R}^m$ *is linear,* $n \leq m$*, then* $\mathcal{H}^n(L(A)) = [\![L]\!]\mathcal{L}^n(A)$ *for every* $A \subseteq \mathbb{R}^n$*.*

Remark 4.4.2. This property of the Hausdorff measure is the generalization of a property of the Lebesque measure, namely for every $A \subseteq \mathbb{R}^n$ which is Lebesque measurable and for every $L\colon \mathbb{R}^n \to \mathbb{R}^n$ linear, $\mathcal{L}^n(L(A)) = (\det L)\mathcal{L}^n(A)$.

Proof. Let $L = O \circ S$ be a polar decomposisiton of L, where $S\colon \mathbb{R}^n \to \mathbb{R}^n$ is symmetric and $O\colon \mathbb{R}^n \to \mathbb{R}^m$ is orthogonal, and $[\![L]\!] = |\det S|$.

(1) Assume that $[\![L]\!] = |\det S| = 0$.
Since S is singular $\dim S(\mathbb{R}^n) \leq n-1$ and $\dim L(\mathbb{R}^n) \leq n-1$, thus we can think of $L(\mathbb{R}^n)$ as a subset of $\mathbb{R}^{n-1}$. Consequently, $\mathcal{H}^n(L(\mathbb{R}^n)) = 0$ and we have proved the lemma for this case.

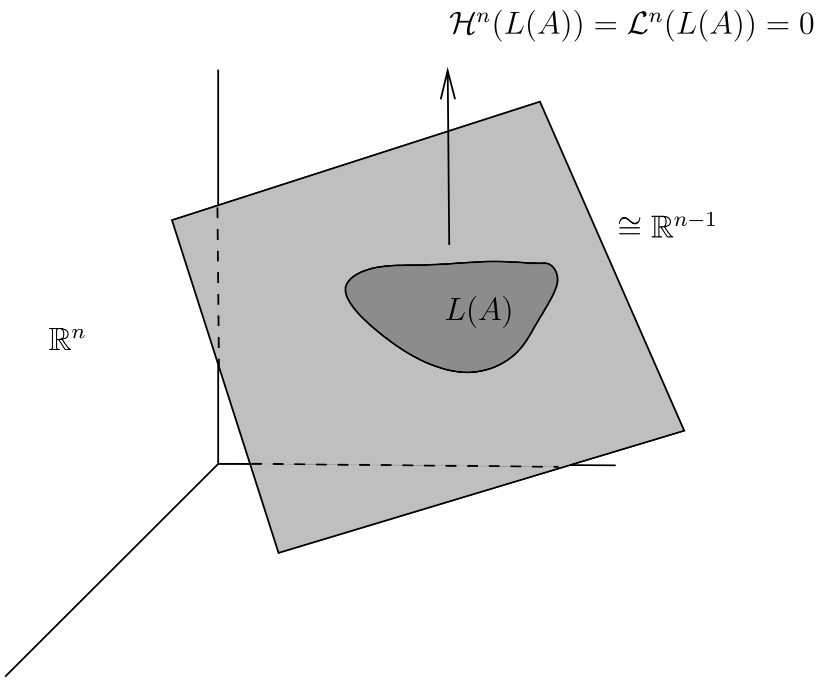


FIGURE 6. Lower dimensional set in $\mathbb{R}^n$ have $\mathcal{L}^n$-measure zero and hence its $\mathcal{H}^n$-measure is also zero.

(2) Now assume that $[\![L]\!] > 0$.

$$\frac{\mathcal{H}^n(L(B(x,r)))}{\mathcal{L}^n(B(x,r))} = \frac{\mathcal{H}^n(S(B(x,r)))}{\mathcal{L}^n(B(x,r))} \quad \text{since O is an isometry}$$
$$= \frac{\mathcal{H}^n(O^* \circ L(B(x,r)))}{\mathcal{L}^n(B(x,r))}$$

since $O^* \circ L$ is a linear operator on $\mathbb{R}^n$

$$= \frac{\mathcal{L}^n(O^* \circ L(B(x,r)))}{\mathcal{L}^n(B(x,r))}$$

$$= |\det S| \frac{\mathcal{L}^n(B(0,1))}{\alpha(n)} = [\![L]\!].$$

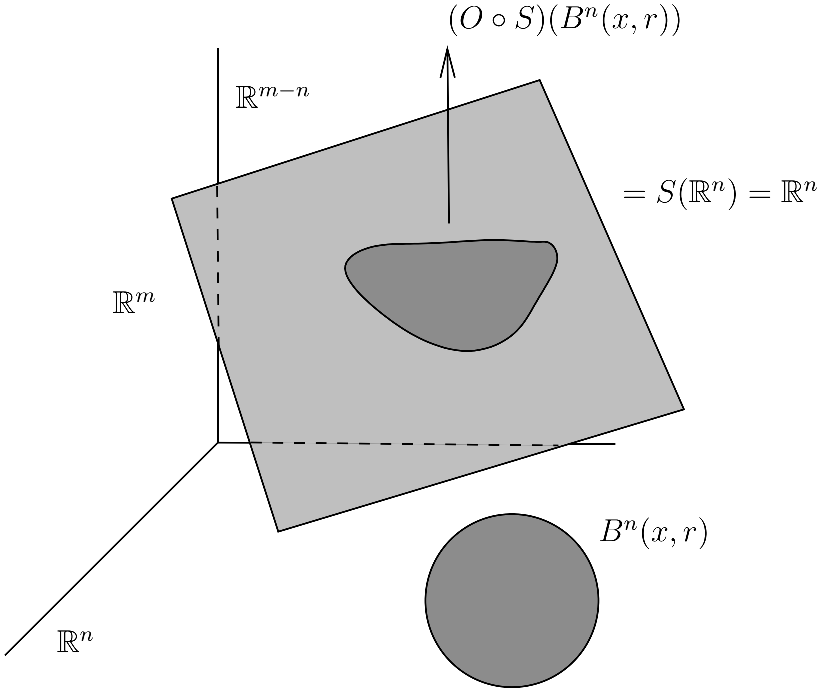


FIGURE 7. $\mathcal{H}^n$-measure is invariant under orthogonal transformations

Define $\nu(A) = \mathcal{H}^n(L(A))$ for every $A \subseteq \mathbb{R}^n$. Then ν is a Radon measure that is absolutely continuous with respect to the $\mathcal{L}^n$ measure. Then

$$D_{\mathcal{L}^n}\nu(x) = \lim_{r\to 0} \frac{\nu(B(x,r))}{\mathcal{L}^n(B(x,r))} = [\![L]\!]$$

$\mathcal{L}^n$-a.e. $x \in \mathbb{R}^n$. A generalization of the fundamental theorem of Calculus to the Radon measures implies that for every $B \in \mathcal{B}_{\mathbb{R}^n}$ (Borel σ-algebra on $\mathbb{R}^n$)

$$\nu(B) = \mathcal{H}^n(L(B)) = \int_B D_{\mathcal{L}^n}\nu(x)\, d\mathcal{L}^n x = [\![L]\!]\mathcal{L}^n(B).$$

Since ν and $\mathcal{L}^n$ are Radon measures $\nu(A) = \mathcal{H}^n(L(A)) = [\![L]\!]\mathcal{L}^n(A)$ holds for every $A \subseteq \mathbb{R}^n$.

□

Lemma 4.4.3. *Assume $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is Lipschitz, $n \le m$, $A \subseteq \mathbb{R}^n$ is $\mathcal{L}^n$-measurable. Then*

(1) *$f(A)$ is $\mathcal{H}^n$-measurable.*

(2) *The mapping $y \to \mathcal{H}^0(A \cap f^{-1}(\{y\}))$ is $\mathcal{H}^n$-measurable on $\mathbb{R}^n$. This mapping is called the multiplicity function.*

(3) $\int_{\mathbb{R}^m} \mathcal{H}^0(A \cap f^{-1}(\{y\}))\, d\mathcal{H}^n y \le (\operatorname{Lip} f)^n \mathcal{L}^n(A)$.

Proof.

(1) Proof of # 1: WLOG we may assume that A is bounded. Since $\mathcal{L}^n$ is a Radon measure there exists a sequence of compact subsets $\{K_i\}_{i\in\mathbb{N}}$ of A such that $\mathcal{L}^n(A - \bigcup_i K_i) = 0$. Since $f(\bigcup_i K_i)$ is $\mathcal{H}^n$-measurable and

$$\mathcal{H}^n\left(f(A) - f\left(\bigcup_i K_i\right)\right) \leq (\operatorname{Lip} f)^n \mathcal{L}^n(A - \bigcup_i K_i) = 0,$$

then $f(A)$ is $\mathcal{H}^n$-measurable.

(2) Proof of # 2: For each $k \in \mathbb{N}$ let

$$\mathcal{B}_k = \left\{Q = (a_1, b_1] \times \cdots \times (a_n, b_n] \;\middle|\; a_i = \tfrac{c_i}{k},\ b_i = \tfrac{c_i+1}{k},\ c_i \in \mathbb{Z}\right\}$$

then $\mathbb{R}^n = \bigcup_{Q\in\mathcal{B}_k} Q$ is a pairwise disjoint union. For each $k \in \mathbb{N}$ define a function

$$g_k(y) = \sum_{Q\in\mathcal{B}_k} \chi_{f(A\cap Q)}(y)$$

which counts the number of cubes $Q \in \mathcal{B}_k$ such that $f^{-1}(\{y\}) \cap (A \cap Q) \neq \emptyset$. For each $k \in \mathbb{N}$, g_k is $\mathcal{H}^n$-measurable and since $g_k(y) \nearrow \mathcal{H}^0(A \cap f^{-1}(\{y\}))$ as $k \to \infty$

$$\lim_{k\to\infty} g_k(y) = \mathcal{H}^0(A \cap f^{-1}(\{y\})) \text{ is } \mathcal{H}^n\text{-measurable.}$$

(3) Proof of # 3: By the Monotone Convergence Theorem (MCT)

$$\begin{aligned}
\int_{\mathbb{R}^m} \mathcal{H}^0(A \cap f^{-1}(\{y\})\, d\mathcal{H}^n y &= \lim_{k\to\infty} \int_{\mathbb{R}^m} g_k(y)\, d\mathcal{H}^n y \\
&= \lim_{k\to\infty} \sum_{Q\in\mathcal{B}_k} \mathcal{H}^n(f(A\cap Q)) \\
&\leq \limsup_{k\to\infty} \sum_{Q\in\mathcal{B}_k} (\operatorname{Lip} f)^n \mathcal{H}^n(A\cap Q) \\
&= (\operatorname{Lip} f)^n \limsup_{k\to\infty} \sum_{Q\in\mathcal{B}_k} \mathcal{L}^n(A\cap Q) \\
&= (\operatorname{Lip} f)^n \mathcal{L}^n(A).
\end{aligned}$$

□

The next lemma is used in the proof of the Area Formula (4.4.5). See Evans & Gariepy [9] for a proof.

Lemma 4.4.4. *Let $t > 1$ and $B = \{x \mid Df(x)$ exists, $Jf(x) > 0\}$, $f\colon \mathbb{R}^n \to \mathbb{R}^m$ Lipschitz, $n \leq m$. Then there exists a countable collection $\{E_k\}_{k\in\mathbb{N}}$ of Borel subsets of $\mathbb{R}^n$ such that*

(1) $B = \bigcup_k E_k$

(2) *$f|_{E_k}$ is injective for each $k \in \mathbb{N}$*

(3) *For each $k \in \mathbb{N}$ there exists a symmetric automorphism $T_k\colon \mathbb{R}^n \to \mathbb{R}^n$ such that*

$$\operatorname{Lip}(f|_{E_k} \circ T_k^{-1}) \leq t \qquad \operatorname{Lip}(T_k \circ (f|_{E_k})^{-1}) \leq t$$
$$t^{-n}|\det T_k| \leq Jf|_{E_k} \leq t^n |\det T_k|$$

Theorem 4.4.5 (The Area Formula)**.** *Let $f\colon \mathbb{R}^n \to \mathbb{R}^m$ be Lipschitz, $n \le m$. Then for each $\mathcal{L}^n$-measurable subset $A \subseteq \mathbb{R}^n$*

$$\int_A Jf(x)\, d\mathcal{L}^n x = \int_{\mathbb{R}^m} \mathcal{H}^0(A \cap f^{-1}(\{y\}))\, d\mathcal{H}^n y.$$

Proof.

- By Rademacher's Theorem (4.2.1) and lemma (4.4.3) we may assume that $Df(x)$, $Jf(x)$ exist for every $x \in A$ and we may also assume that $\mathcal{L}^n(A) < \infty$.

- Case # 1: $A \subseteq \{Jf > 0\}$.
  Fix $t > 1$. We can choose a sequence of pairwise disjoint Borel sets $\{E_j\}_{j\in\mathbb{N}}$ as in lemma (4.4.4). For each $k \in \mathbb{N}$ let

  $$\mathcal{B}_k = \left\{ Q = (a_1, b_1] \times \cdots \times (a_n, b_n] \;\middle|\; a_i = \tfrac{c_i}{k},\ b_i = \tfrac{c_i+1}{k},\ c_i \in \mathbb{Z} \right\}$$

  and

  $$F_j^i = E_j \cap Q_i \cap A \quad Q_i \in \mathcal{B}_k\ i, j \in \mathbb{N}.$$

  Thus

  $$\text{the sets } F_j^i \text{ are pairwise disjoint and } A = \bigcup_{i,j} F_j^i$$

  Claim # 1:

  $$\lim_{k\to\infty} \sum_{i,j} \mathcal{H}^n f(F_j^i)) = \int_{\mathbb{R}^m} \mathcal{H}^0(A \cap f^{-1}(\{y\}))\, d\mathcal{H}^n y$$

  Proof Of Claim # 1. For each $k \in \mathbb{N}$ define a function g_k by

  $$g_k = \sum_{i,j} \chi_{f(F_j^i)}.$$

  Then each g_k if $\mathcal{L}^n$-measurable by lemma (4.4.3). $g_k(y)$ is the number of sets F_j^i such that $F_j^i \cap f^{-1}(\{y\}) \neq \emptyset$, then $g_k(y) \nearrow \mathcal{H}^0(A \cap f^{-1}(\{y\}))$ as $k \to \infty$. By the MCT we obtain

  $$\begin{aligned}\int_{\mathbb{R}^m} \mathcal{H}^0(A \cap f^{-1}(\{y\}))\, d\mathcal{H}^n y &= \lim_{k\to\infty} \int_{\mathbb{R}^m} g_k(y)\, d\mathcal{H}^n y \\ &= \lim_{k\to\infty} \sum_{i,j} \mathcal{H}^n(f(F_j^i))\end{aligned}$$

  By lemma (4.4.4)

  $$\begin{aligned}\mathcal{H}^n(f(F_j^i)) &= \mathcal{H}^n[(f|_{E_j} \circ T_j^{-1} \circ T_j)(F_j^i)] \\ &\le (\operatorname{Lip}(f|_{E_j} \circ T_j^{-1}))^n \mathcal{H}^n(T_j(F_j^i)) \le t^n \mathcal{L}^n(T_j(F_j^i))\end{aligned}$$

  and

  $$\begin{aligned}\mathcal{L}^n(T_j(F_j^i)) &= \mathcal{L}^n(T_j \circ (f|_{E_j})^{-1} \circ f(F_j^i)) = \mathcal{H}^n(T_j \circ (f|_{E_j})^{-1} \circ f(F_j^i)) \\ &\le (\operatorname{Lip}(T_j \circ (f|_{E_j})^{-1}))^n \mathcal{H}^n(f(F_j^i)) \\ &= t^n \mathcal{H}^n(f(F_j^i)).\end{aligned}$$

Hence again by (4.4.4)

$$\begin{aligned} t^{-2n}\mathcal{H}^n(f(F_j^i)) &\le t^{-n}\mathcal{L}^n(T_j(F_j^i)) = t^{-n}|\det T_j|\mathcal{L}^n(F_j^i) \\ &\le \int_{F_j^i} Jf|_{E_j}(x)\, d\mathcal{L}^n x = \int_{F_j^i} Jf(x)\, d\mathcal{L}^n x \\ &\le t^n|\det T_j|\mathcal{L}^n(F_j^i) = t^n\mathcal{L}^n(T_j(F_j^i)) \\ &\le t^{2n}\mathcal{H}^n(f(F_j^i)). \end{aligned}$$

If we sum on $i, j \in \mathbb{N}$, then

$$t^{-2n}\sum_{i,j}\mathcal{H}^n(f(F_j^i)) \le \int_A Jf(x)\, d\mathcal{H}^n x \le t^{2n}\sum_{i,j}\mathcal{H}^n(f(F_j^i)).$$

If we let $k \to \infty$ then by Claim # 1 we obtain

$$\begin{aligned} t^{-2n}\int_{\mathbb{R}^m}\mathcal{H}^0(A\cap f^{-1}(\{y\}))\, d\mathcal{H}^n y &\le \int_A Jf(x)\, d\mathcal{L}^n x \\ &\le t^{2n}\int_{\mathbb{R}^m}\mathcal{H}^0(A\cap f^{-1}(\{y\}))\, d\mathcal{H}^n y, \end{aligned}$$

and then send $t \to 1^+$

$$\int_A Jf(x)\, d\mathcal{L}^n x = \int_{\mathbb{R}^m}\mathcal{H}^0(A\cap f^{-1}(\{y\}))\, d\mathcal{H}^n y.$$

- Case # 2: $A \subseteq \{Jf = 0\}$.
  Fix $0 < \epsilon \le 1$ and factor $f = p \circ g$ where

$$\begin{aligned} &g\colon \mathbb{R}^n \to \mathbb{R}^m \times \mathbb{R}^n \quad g(x) = (f(x), \epsilon x) \quad x \in \mathbb{R}^n \\ &p\colon \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m \quad g(y,z) = y \quad y \in \mathbb{R}^m, z \in \mathbb{R}^n \end{aligned}$$

Claim # 2: There exists a constant C such that $0 < Jg(x) \le C\epsilon^2$ for all $x \in A$.

Proof Of Claim # 2. Let $g = (f_1, \ldots, f_m, \epsilon x_1, \ldots, \epsilon x_n)$, then

$$[Dg(x)] = \begin{bmatrix} [Df(x)] \\ [\epsilon I] \end{bmatrix}_{(m+n)\times n} \quad \text{where } [Df(x)]_{m\times n} \text{ and } [\epsilon I]_{n\times n}$$

By (4.3.8) $[Jg(x)]^2 = [\![Dg(x)]\!]^2$ equals the sum of the squares of the $(n \times n)$-subdeterminants of $Dg(x)$, hence $[Jg(x)]^2 \ge \epsilon^{2n} > 0$ holds for every $x \in A$. The differential of f is the map $Df\colon \mathbb{R}^n \to L(\mathbb{R}^n, \mathbb{R}^m)$ defined by

$$Df\colon a \mapsto Df(a) \text{ for every } a \in \mathbb{R}^n.$$

$$\begin{aligned} |Df(a)(h)| &\le |f(a+h) - f(a) - Df(a)(h)| + |f(a+h) - f(a)| \\ &\le |f(a+h) - f(a) - Df(a)(h)| + \operatorname{Lip} f|h| \end{aligned}$$

$$\left|Df(a)\left(\frac{h}{|h|}\right)\right| \le \underbrace{\frac{|f(a+h) - f(a) - Df(a)(h)|}{|h|}}_{\to 0 \text{ as } h\to 0} + \operatorname{Lip} f$$

since h is arbitrary

$$(*) \qquad \|Df(a)\| \le \operatorname{Lip} f \text{ holds for all } a \in \mathbb{R}^n.$$

We can employ theorem (4.3.8) and $(*)$ to compute

$$
\begin{aligned}
[Jg(x)]^2 &= [Jf(x)]^2 + [\text{sum of squares of terms with at least one } \epsilon] \\
&\leq C\epsilon^2 \text{ for some constant } C > 0.
\end{aligned}
$$

Since $p\colon \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$ is a projection then by theorem (4.1.2)

$$\mathcal{H}^n(f(A)) = \mathcal{H}^n(p(g(A))) \leq (\operatorname{Lip} p)^n \mathcal{H}^n(g(A)) = \mathcal{H}^n(g(A)).$$

Since the map g is injective $\mathcal{H}^0(A \cap g^{-1}(\{y,z\})) = 1$ on $g(A)$,

$$\mathcal{H}^n(g(A)) = \int_{g(A)} \mathcal{H}^0(A \cap g^{-1}(\{y,z\}))\, d\mathcal{H}^n(y,z)$$

and hence

$$\mathcal{H}^n(g(A)) \leq \int_{\mathbb{R}^{n+m}} \mathcal{H}^0(A \cap g^{-1}(\{y,z\}))\, d\mathcal{H}^n(y,z). \tag{$**$}$$

Since $A \subseteq \{Jg > 0\}$ then Claim # 1 implies that

$$\int_{\mathbb{R}^{n+m}} \mathcal{H}^0(A \cap g^{-1}(\{y,z\})) = \int_A Jg(x)\, d\mathcal{L}^n x \leq C\epsilon^2 \mathcal{L}^n(A)$$

so using $(**)$ we obtain

$$\mathcal{H}^n(f(A)) \leq C\epsilon^2 \mathcal{L}^n(A).$$

Let $\epsilon \to 0$ so $\mathcal{H}^n(f(A)) = 0$. Since $\operatorname{spt}[\mathcal{H}^0(A \cap f^{-1}(\{y\}))] \subseteq f(A)$,

$$\int_{\mathbb{R}^m} \mathcal{H}^0(A \cap f^{-1}(\{y\}))\, d\mathcal{H}^n y = \int_{f(A)} \mathcal{H}^0(A \cap f^{-1}(\{y\}))\, d\mathcal{H}^n y = 0.$$

And theorem follows for this case because

$$\int_A Jf(x)\, d\mathcal{L}^n x = 0 = \int_{\mathbb{R}^m} \mathcal{H}^0(A \cap f^{-1}(\{y\}))\, d\mathcal{H}^n y.$$

- General Case: Let $A = (A \cap \{Jf = 0\}) \cup (A \cap \{Jf > 0\})$. Then using Claim # 1 and Claim #2 we get

$$
\begin{aligned}
\int_A Jf(x)\, d\mathcal{L}^n x &= \int_{A \cap \{Jf=0\}} Jf(x)\, d\mathcal{L}^n x + \int_{A \cap \{Jf>0\}} Jf(x)\, d\mathcal{L}^n x \\
&= \int_{\mathbb{R}^m} \mathcal{H}^0(A \cap f^{-1}(\{y\}))\, d\mathcal{H}^n y.
\end{aligned}
$$

$\square$

Theorem 4.4.6 (Change Of Variables Formula). *Let $f\colon \mathbb{R}^n \to \mathbb{R}^m$ be Lipschitz, $n \leq m$. Then for each integrable function $g\colon \mathbb{R}^n \to \mathbb{R}$,*

$$\int_{\mathbb{R}^n} g(x) Jf(x)\, d\mathcal{L}^n x = \int_{\mathbb{R}^m} \sum_{x \in f^{-1}(\{y\})} g(x)\, d\mathcal{H}^n y.$$

Remark 4.4.7.

- $f^{-1}(\{y\})$ is at most countable for $\mathcal{H}^n$-a.e. $y \in \mathbb{R}^m$. In particular, let A_M be a cube in $\mathbb{R}^n$ centered at the origin with side length $2M$. Since
$$|f^{-1}(\{y\})| = \lim_{M\to\infty} \mathcal{H}^0(A_M \cap f^{-1}(\{y\}))$$
it suffices to show that for each $M \in \mathbb{N}$, $\mathcal{H}^0(A_M \cap f^{-1}(\{y\}))$ is at most countable $\mathcal{H}^n$-a.e. $y \in \mathbb{R}^m$. If $A_M \subseteq \{Jf > 0\}$, then using the sequence of functions $\{g_k\}_{k\in\mathbb{N}}$ defined in the proof of the area formula (4.4.5) we get $\mathcal{H}^0(A_M \cap f^{-1}(\{y\})) = \lim_k g_k(y)$ where each $g_k(y)$ is countable for every $y \in \mathbb{R}^m$. If $A_M \subseteq \{Jf = 0\}$ then by again using the area formula we get $\mathcal{H}^0(A_M \cap f^{-1}(\{y\})) = 0$ $\mathcal{H}^n$-a.e. $y \in \mathbb{R}^m$.
- If $A \subseteq \mathbb{R}^n$ and $\mathcal{L}^n(A) < \infty$ let $g(x) = \chi_A(x)$, then the change of variables theorem reduces to the area formula.

Proof.

- Case # 1: $g \geq 0$. We can write $g = \sum_i \frac{1}{i}\chi_{A_i}$ where each A_i is $\mathcal{L}^n$-measurable. By MCT
$$\begin{aligned}
\int_{\mathbb{R}^n} g(x) Jf(x)\, d\mathcal{L}^n x &= \int_{\mathbb{R}^n} \sum_i \frac{1}{i}\chi_{A_i}(x) Jf(x)\, d\mathcal{L}^n x \\
&= \sum_i \frac{1}{i} \int_{\mathbb{R}^n} \chi_{A_i}(x) Jf(x)\, d\mathcal{L}^n x \\
&= \sum_i \frac{1}{i} \int_{\mathbb{R}^m} \mathcal{H}^0(A_i \cap f^{-1}(\{y\}))\, d\mathcal{H}^n y \\
&= \int_{\mathbb{R}^m} \sum_i \frac{1}{i} \sum_{x\in f^{-1}(\{y\})} \chi_{A_i}(x)\, d\mathcal{H}^n y \\
&= \int_{\mathbb{R}^m} \sum_{x\in f^{-1}(\{y\})} g(x)\, d\mathcal{H}^n y.
\end{aligned}$$
- Case # 2: If g is an arbitrary integrable function then apply Case # 1 to the positive and negative parts of g.

□

Now we will give several examples showing the various applications of the area formula. The area formula serves as a bridge between measure theory and manifold theory. It is used to show that the Hausdorff measure gives the expected value for "nice" sets, namely for k-manifolds in the Euclidean space. In the examples below we will see that the Hausdorff measure of a curve is its length and the Hausdorff measure of a surface is its area. And in general the Hausdorff measure of a k-manifold is its volume. These examples will also show, by using the observation in (2.2.8), that the Hausdorff dimension of a curve is one or more generally the Hausdorff dimension of a k-manifold is k, and hence it generalizes our notion of dimension coming from the manifold theory.

Applications:

Example 4.4.8 (Length Of A Curve)**.** Let $f\colon \mathbb{R} \to \mathbb{R}^m$ be Lipschitz and injective. Write

$$f(x) = (f_1(x), \dots, f_m(x)) \quad [Df(x)] = [f_1'(x) \cdots f_m'(x)]$$

Then using the Binet-Cauchy formula

$$Jf(x) = \left\{ \sum_i (f_i'(x))^2 \right\}^{1/2} = |f'(x)|$$

Let $C = f([a,b]) \subseteq \mathbb{R}^m$, then by using the area formula

$$\begin{aligned} \int_a^b Jf(x)\,dx = \int_a^b |f'(x)|\,dx &= \int_C \overbrace{\mathcal{H}^0([a,b] \cap f^{-1}(\{y\}))}^{1}\,d\mathcal{H}^1 y \\ &= \mathcal{H}^1(C) \text{ is the length of the curve.} \end{aligned}$$

When $0 < \mathcal{H}^1(C) < \infty$, $\mathcal{H}_{\dim}(C) = 1$.

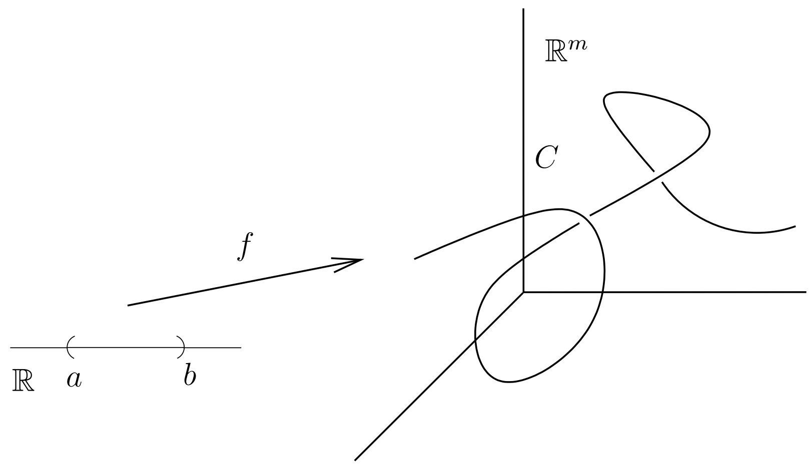


FIGURE 8. Length of a curve is equal to its $\mathcal{H}^1$-measure.

Example 4.4.9 (Surface Area Of A Graph)**.** Let $g\colon \mathbb{R}^n \to \mathbb{R}$ be Lipschitz and define $f\colon \mathbb{R}^n \to \mathbb{R}^{n+1}$ by

$$f(x) = (x, g(x)).$$

Then

$$[Df(x)] = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & 1 \\ \frac{\partial g}{\partial x_1}(x) & \cdots & \cdots & \frac{\partial g}{\partial x_n}(x) \end{bmatrix}$$

and by using the Binet-Cauchy formula we obtain $[Jf(x)]^2 = 1 + |Dg(x)|^2$. Let $U \in \mathrm{OP}(\mathbb{R}^n)$ and $G = \{(x, g(x)) \colon x \in U\} \subseteq \mathbb{R}^{n+1}$. Then by the area formula

$$\begin{aligned}\int_U (1 + |Dg(x)|)^{1/2}\, d\mathcal{L}^n x &= \int_G \mathcal{H}^0(U \cap f^{-1}(\{x, y\}))\, d\mathcal{H}^n((x, y))\\ &= \mathcal{H}^n(G) \text{ is the area of the surface.}\end{aligned}$$

When $0 < \mathcal{H}^n(G) < \infty$, $\mathcal{H}_{\dim}(G) = n$.

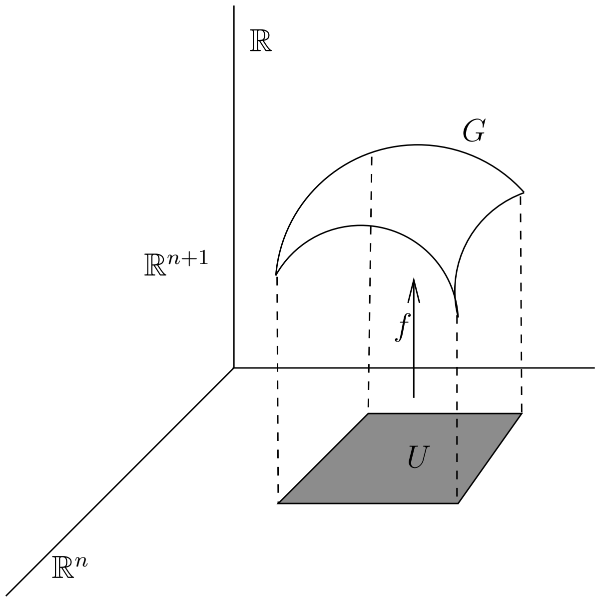


Figure 9. Area of a surface is equal to its $\mathcal{H}^2$-measure.

Example 4.4.10 (Volume Of A k-Manifold)**.** Let $M \subseteq \mathbb{R}^n$ be a Lipschitz, k-manifold in $\mathbb{R}^n$. Suppose that $U \in \mathrm{OP}(\mathbb{R}^k)$ and $f \colon U \to M$ is a (Lipschitz) coordinate map for M. Let $f(U) \supseteq A$ be Borel and $B = f^{-1}(A)$. Define

$$g = (g_{ij})_{i,j} \quad g_{ij} = \frac{\partial f}{\partial x_i} \cdot \frac{\partial f}{\partial x_j}\ 1 \le i, j \le k$$

then

$$[Df(x)] \cdot [Df(x)]^t = g \text{ and } \det\Big([Df(x)]^t \cdot [Df(x)]\Big) = \det g.$$

So the Jacobian of f is $Jf(x) = \sqrt{\det g}$ and applying the area formula we obtain

$$\begin{aligned}\int_B \sqrt{\det g(x)}\, d\mathcal{L}^k(x) &= \int_A \mathcal{H}^0(B \cap f^{-1}(\{y\}))\, d\mathcal{H}^n y\\ &= \mathcal{H}^k(A) \text{ volume of } A.\end{aligned}$$

When $0 < \mathcal{H}^k(A) < \infty$, $\mathcal{H}_{\dim}(A) = k$.

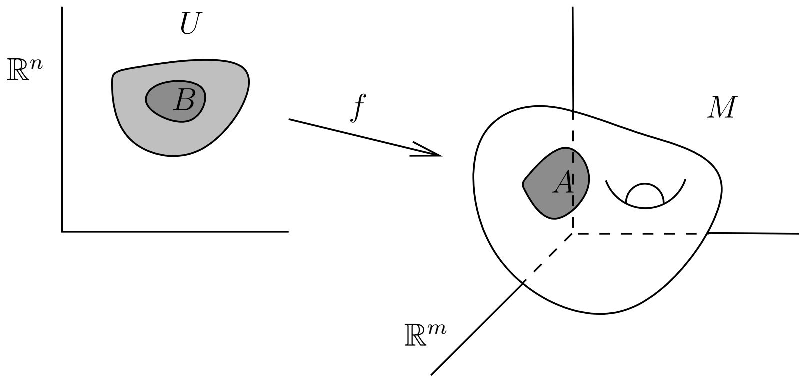


FIGURE 10. Volume of an n-manifold in $\mathbb{R}^m$ is equal to its $\mathcal{H}^n$-measure.

4.5. **The Coarea Formula And Its Applications.**

Lemma 4.5.1. *Suppose $L\colon \mathbb{R}^n \to \mathbb{R}^m$ is linear, $n \geq m$ and $A \subseteq \mathbb{R}^n$ is $\mathcal{L}^n$-measurable, then*

(1) *The mapping $y \mapsto \mathcal{H}^{n-m}(A \cap L^{-1}(\{y\}))$ is $\mathcal{L}^m$-measurable.*

(2) $\int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap L^{-1}(\{y\}))\, d\mathcal{L}^m y = [\![L]\!]\mathcal{L}^n(A)$.

Proof.

- Case # 1: $\dim L(\mathbb{R}^n) < m$.
  $A \cap L^{-1}(\{y\}) = \emptyset$ $\mathcal{L}^m$-a.e. $y \in \mathbb{R}^m$ because
  $$\{y \in \mathbb{R}^m \colon A \cap L^{-1}(\{y\}) \neq \emptyset\} \subseteq L(A) \text{ and } \mathcal{L}^m(L(A)) = 0.$$
  Consequently, $\mathcal{H}^{n-m}(A \cap L^{-1}(\{y\})) = 0$ $\mathcal{L}^m$-a.e. $y \in \mathbb{R}^m$, and
  $$y \mapsto \mathcal{H}^{n-m}(A \cap L^{-1}(\{y\})) \text{ is } \mathcal{L}^m\text{-measurable.}$$
  The linear map $L\colon \mathbb{R}^n \to \mathbb{R}^m$ has a polar decomposition $L = S \circ O^*$, where $O^*\colon \mathbb{R}^n \to \mathbb{R}^m$, O is orthogonal and $S\colon \mathbb{R}^m \to \mathbb{R}^m$, S is symmetric. Since O^* is surjective $L(\mathbb{R}^n) = S(\mathbb{R}^m)$, then $\dim S(\mathbb{R}^m) < m$, S is singular, and $|\det S| = [\![L]\!] = 0$. Consequently,
  $$[\![L]\!]\mathcal{L}^n(A) = 0 = \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap L^{-1}(\{y\}))\, d\mathcal{L}^m y$$
  so the lemma is proved for the case $\dim L(\mathbb{R}^n) < m$.

- Case # 2: $L = P =$ orthogonal projection of $\mathbb{R}^n$ onto $\mathbb{R}^m$.
  For each $y \in \mathbb{R}^m$, $P^{-1}(\{y\})$ is an (n-m)-dimensional affine subspace of $\mathbb{R}^n$ and a translate of $P^{-1}(\{0\})$, in particular $P^{-1}(\{y\}) = \tilde{y} + P^{-1}(\{0\})$, where $\tilde{y} \in \mathbb{R}^n$, $P(\tilde{y}) = y$ and it has all other entries zero. Since A is $\mathcal{L}^n$-measurable $f(x,y) = \chi_A(x,y)$ is $\mathcal{L}^n$-measurable where $x \in X = \mathbb{R}^{n-m}$, $y \in Y = \mathbb{R}^m$ and
  $$y \mapsto \int_{\mathbb{R}^{n-m}} \chi_{A^y}(x)\, d\mathcal{L}^{n-m}x \text{ is } \mathcal{L}^m\text{-measurable.}$$
  Since $A^y = \{x \in \mathbb{R}^{n-m} \colon (x,y) \in A\}$ and $\mathcal{L}^{n-m}$ is translation invariant
  $$\begin{aligned}\int_{\mathbb{R}^{n-m}} \chi_{A^y}(x)\, d\mathcal{L}^{n-m}x &= \int_{\mathbb{R}^{n-m}} \chi_{A \cap P^{-1}(\{y\})}(x)\, d\mathcal{L}^{n-m}x \\ &= \mathcal{L}^{n-m}(A \cap P^{-1}(\{y\}))\end{aligned}$$

$$= \mathcal{H}^{n-m}(A \cap P^{-1}(\{y\})).$$

So $y \mapsto \mathcal{H}^{n-m}(A \cap P^{-1}(\{y\}))$ is $\mathcal{L}^m$-measurable and

$$\begin{aligned}\mathcal{L}^n(A) &= (\mathcal{L}^m \times \mathcal{L}^{n-m})(A)\\ &= \int_{\mathbb{R}^m} \mathcal{L}^{n-m}(A \cap P^{-1}(\{y\}))\, d\mathcal{L}^m y\\ &= \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap P^{-1}(\{y\}))\, d\mathcal{L}^m y.\end{aligned}$$

Since $L = P =$ is an orthogonal projection $[\![L]\!] = 1$ and hence

$$[\![L]\!]\mathcal{L}^n(A) = \mathcal{L}^n(A) = \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap P^{-1}(\{y\}))\, d\mathcal{L}^m y.$$

So the lemma holds in the case where L is an orthogonal projection.

- Case # 3:$L\colon \mathbb{R}^n \to \mathbb{R}^m$, $\dim L(\mathbb{R}^n) = m$.
  L has a polar decomposition $L = S \circ O^*$, where

$$\begin{aligned}S&\colon \mathbb{R}^m \to \mathbb{R}^m \text{ is a symmetric automorphism,}\\ O&\colon \mathbb{R}^m \to \mathbb{R}^n \text{ orthogonal,}\\ [\![L]\!] &= |\det S| > 0.\end{aligned}$$

Claim: $O^* = P \circ Q$ where P is the orthogonal projection of $\mathbb{R}^n$ onto $\mathbb{R}^m$, and $Q\colon \mathbb{R}^n \to \mathbb{R}^n$ is orthogonal.

Proof Of Claim. Let $O\colon \mathbb{R}^m \to \mathbb{R}^n$ be an orthogonal map and $\{e_1, \dots, e_n\}$ be the standard basis of $\mathbb{R}^n$. For $1 \le i \le m$ let $\tilde{e}_i$ be the standard basis element of $\mathbb{R}^m$, and $e_i = \tilde{e}_i \times 0_m$. Then

$$\langle O\tilde{e}_i, O\tilde{e}_j\rangle = \langle \tilde{e}_i, \tilde{e}_j\rangle = \langle e_i, e_j\rangle = \delta_{ij} \quad 1 \le i, j \le m.$$

Then $\{O\tilde{e}_1, \dots, O\tilde{e}_m\}$ is an orthonormal set of vectors in $\mathbb{R}^n$. Let $\{\alpha_{m+1}, \dots, \alpha_n\}$ be an orthonormal set of vectors in $\mathbb{R}^n$ such that $\{O\tilde{e}_1, \dots, O\tilde{e}_m, \alpha_{m+1}, \dots, \alpha_n\}$ is orthonormal. Let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ be the linear map defined by

$$\begin{aligned}Te_i &= O\tilde{e}_i \quad 1 \le i \le m\\ Te_j &= \alpha_j \quad m+1 \le j \le n,\end{aligned}$$

then T is an orthogonal map such that $T(x_1, \dots, x_m, 0, \dots, 0) = O(x_1, \dots, x_m)$. Let $T^{-1} = T^* \equiv Q\colon \mathbb{R}^n \to \mathbb{R}^n$ be the orthogonal map, and let $P\colon \mathbb{R}^n \to \mathbb{R}^m$ be the orthogonal projection onto the first m coordinates. $P^*\colon \mathbb{R}^m \to \mathbb{R}^n$ is the linear map defined by $\langle y, Px\rangle = \langle P^*y, x\rangle$ for every $x \in \mathbb{R}^n,\ y \in \mathbb{R}^m$. Then

$$\begin{aligned}\langle e_i, P^*x\rangle &= \langle Pe_i, x\rangle = x_i \quad 1 \le i \le m\\ \langle e_i, P^*x\rangle &= 0 \qquad\qquad\qquad i > m.\end{aligned}$$

So $P^*(x_1, \dots, x_m) = (x_1, \dots, x_m, 0, \dots, 0) \in \mathbb{R}^n$.
If $x = (x_1, \dots, x_m) \in \mathbb{R}^m$ then

$$\begin{aligned}&(Q^* \circ P^*)(x) = Q^*(x_1, \dots, x_m, 0, \dots, 0) = O(x_1, \dots, x_m)\\ &\Rightarrow O^* = P \circ Q, \text{ and hence we proved the claim.}\end{aligned}$$

$L^{-1}(\{0\}) = \ker L$ is an $(n-m)$-dimensional subspace of $\mathbb{R}^n$. Let $X = \ker L \cong \mathbb{R}^{n-m}$, $Y = \mathbb{R}^m$, then $L^{-1}(\{y\}) = \ker L + x_0$ where $Lx_0 = y$, $x_0 \in \mathbb{R}^n$ and $L^{-1}(\{y\})$ is an $(n-m)$-dimensional affine subspace of $\mathbb{R}^n$ which is a translate of $\ker L$. Let $P\colon \mathbb{R}^n \to \ker L$ be an orthogonal projection onto the kernel of L defined by $P((x,y)) = x \in \ker L$.

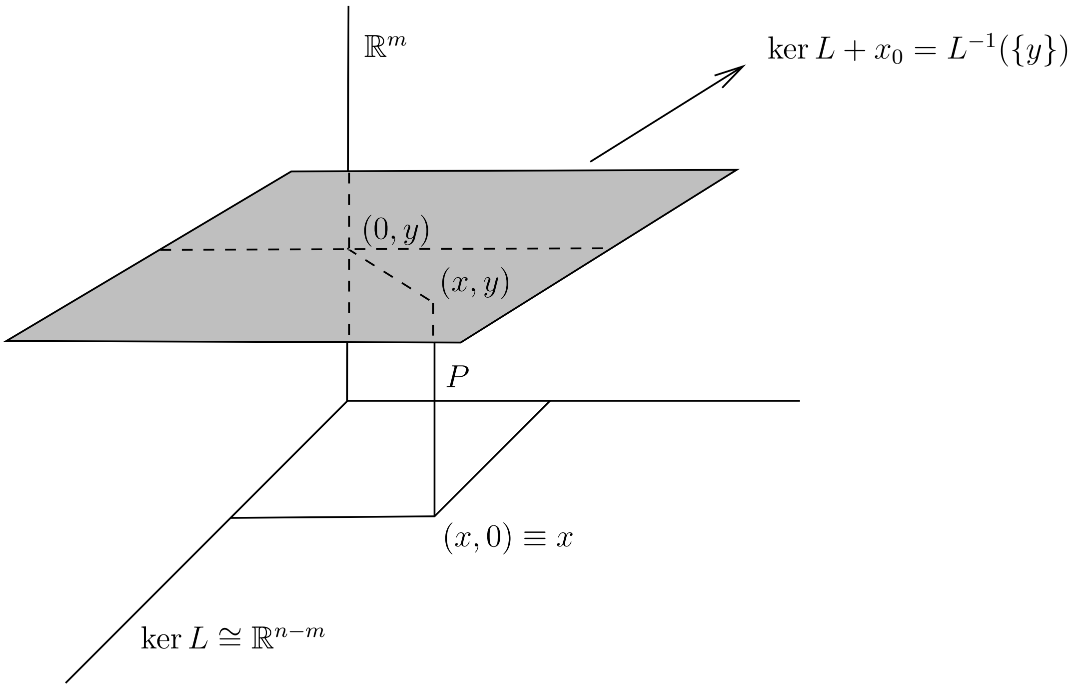


FIGURE 11. $\ker L$ is an $(n-m)$-dimensional subspace of $\mathbb{R}^n$ and P is a projection onto $\ker L$.

Since $A \subseteq \mathbb{R}^n$ is $\mathcal{L}^n$-measurable

$$
\begin{aligned}
y \mapsto \int_{\mathbb{R}^{n-m}} & \chi_{A^y}(x)\, d\mathcal{L}^{n-m}x \text{ is } \mathcal{L}^m\text{-measurable.} \\
&= \int_{\mathbb{R}^{n-m}} \chi_{\{x\in X\colon (x,y)\in A\}}(x)\, d\mathcal{L}^{n-m}x \\
&= \mathcal{L}^{n-m}(\{x \in X\colon (x,y) \in A\}) \\
&= \mathcal{H}^{n-m}(\{x \in X\colon (x,y) \in A\})
\end{aligned}
$$

since $\mathcal{H}^{n-m}$ is not affected by affine transformations

$$= \mathcal{H}^{n-m}(A \cap L^{-1}(\{y\})).$$

So $y \mapsto \mathcal{H}^{n-m}(A \cap L^{-1}(\{y\}))$ is $\mathcal{L}^m$-measurable. Since $Q\colon \mathbb{R}^n \to \mathbb{R}^n$ is an isometry

$$
\begin{aligned}
\mathcal{L}^n(A) &= \mathcal{L}^n(Q(A)) \\
&= \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(Q(A) \cap P^{-1}(\{y\}))\, d\mathcal{L}^m y \\
&= \int_{\mathbb{R}^m} \mathcal{H}^{n-m}[A \cap (Q^{-1} \circ P^{-1})(\{y\})]\, d\mathcal{L}^m y
\end{aligned}
$$

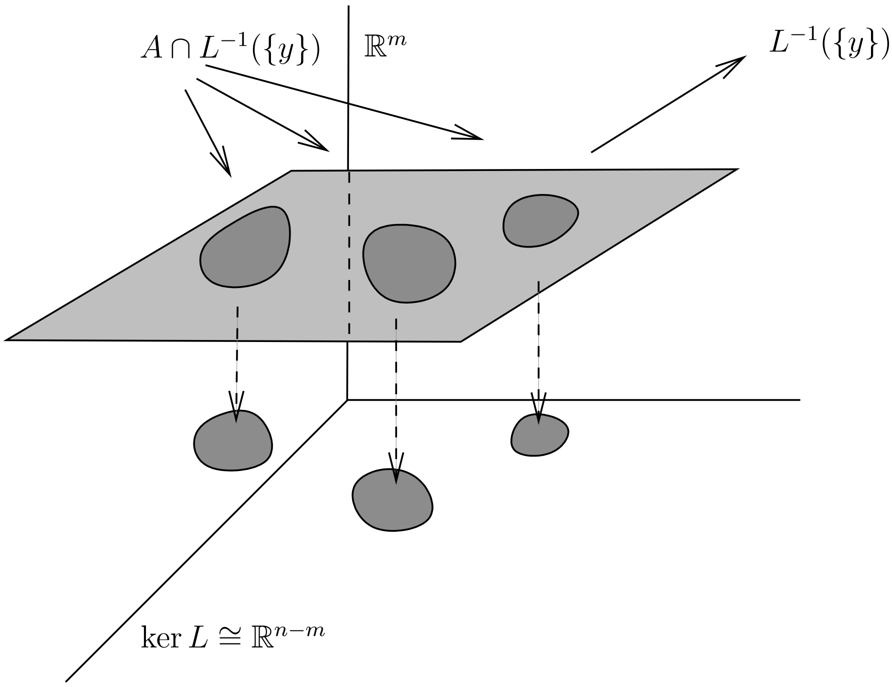


FIGURE 12. Since Hausdorff measure is not affected by affine transformations, $\mathcal{H}^{n-m}(A \cap L^{-1}(\{y\})) = \mathcal{H}^{n-m}(\{x \in \ker L \colon (x, y) \in A\})$

Since $S\colon \mathbb{R}^m \to \mathbb{R}^m$ is a symmetric automorphism it is a Lipschitz map defined by

$$S\colon y \mapsto S(y) = z.$$

WLOG we may assume that $\mathcal{L}^n(A) < \infty$, and then $y \mapsto \mathcal{H}^{n-m}(A \cap Q^{-1} \circ P^{-1}(\{y\}))$ is $\mathcal{L}^m$-integrable. Since $JS(y) = [\![DS(y)]\!] = [\![S]\!] = |\det S|$ we can apply the change variables formula (4.4.6) to compute

$$\begin{aligned} |\det S|\mathcal{L}^n(A) &= \int_{\mathbb{R}^m} |\det S|\mathcal{H}^{n-m}(A \cap (Q^{-1} \circ P^{-1})(\{y\}))\, d\mathcal{L}^m y \\ &= \int_{\mathbb{R}^m} \sum_{y \in S^{-1}(\{z\})} \mathcal{H}^{n-m}(A \cap (Q^{-1} \circ P^{-1})(\{y\}))\, d\mathcal{H}^m z \\ &= \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap (Q^{-1} \circ P^{-1} \circ S^{-1})(\{z\}))\, d\mathcal{L}^m z. \end{aligned}$$

L has a polar decomposition $L = S \circ O^* = S \circ P \circ Q$, which follows from the Claim, and then

$$[\![L]\!]\mathcal{L}^n(A) = \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap L^{-1}(\{y\}))\, d\mathcal{L}^m y.$$

□

The next lemma, which we will state without proof, establishes the necessary prerequisites for the proof of the Coarea Formula. For a proof see Evans & Gariepy [9].

Lemma 4.5.2. *Let $A \subseteq \mathbb{R}^n$ be $\mathcal{L}^n$-measurable, and $f\colon \mathbb{R}^n \to \mathbb{R}^m$ be a Lipschitz map $n \geq m$. Then*

(1) *$A \cap f^{-1}(\{y\})$ is $\mathcal{H}^{n-m}$-measurable $\mathcal{L}^m$-a.e. $y \in \mathbb{R}^m$.*

(2) *$y \mapsto \mathcal{H}^{n-m}(A \cap f^{-1}(\{y\}))$ is $\mathcal{L}^m$-measurable.*

(3) *$\int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap f^{-1}(\{y\}))\, d\mathcal{L}^m y \leq \frac{\alpha(n-m)\alpha(m)}{\alpha(n)}(\operatorname{Lip} f)^m \mathcal{L}^n(A)$.*

The next lemma is used in the proof of the Coarea Formula just as the lemma (4.4.4) is used in the proof of the Area Formula. But this time the sequence of Borel sets cover $B = \{x \mid Dh(x) \text{ exists },\ Jh(x) > 0\}$ $\mathcal{L}^n$-a.e. For a proof see Evans & Gariepy [9].

Lemma 4.5.3. *Let $t > 1$, assume $h\colon \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz, and let*

$$B = \{x \mid Dh(x) \text{ exists, } Jh(x) > 0\}.$$

Then there exists a countable collection of Borel subsets $\{D_k\}_{k\in\mathbb{N}}$ of $\mathbb{R}^n$ such that

(1) *$\mathcal{L}^n(B - \bigcup_k D_k) = 0$*

(2) *$h|_{D_k}$ is injective for each $k \in \mathbb{N}$*

(3) *For every $k \in \mathbb{N}$ there exists a symmetric automorphism $S_k\colon \mathbb{R}^n \to \mathbb{R}^n$ such that*

$$\operatorname{Lip}(S_k^{-1} \circ (h|_{D_k})) \leq t \quad \operatorname{Lip}((h|_{D_k})^{-1} \circ S_k) \leq t$$
$$t^{-n}|\det S_k| \leq Jh|_{D_k} \leq t^n |\det S_k|$$

Theorem 4.5.4 (The Coarea Formula)**.** *Let $f\colon \mathbb{R}^n \to \mathbb{R}^m$ be Lipschitz, $n \geq m$, then for each $\mathcal{L}^n$-measurable subset $A \subseteq \mathbb{R}^n$,*

$$\int_A Jf(x)\, d\mathcal{L}^n x = \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap f^{-1}(\{y\}))\, d\mathcal{L}^m y.$$

Remark 4.5.5. The coarea formula with the change of variables formula, coming next, together generalize the Fubini's theorem. In particular, let $P\colon \mathbb{R}^n \to \mathbb{R}^m$ be the projection onto the first m coordinates, then P is a Lipschitz map with $\operatorname{Lip} P = 1$, and $JP(z) = [\![DP(z)]\!] = [\![P]\!] = 1$. Assume that $\mathcal{L}^n(A) < \infty$, then

$$\begin{aligned}
\int_A JP(z)\, d\mathcal{L}^n(z) = \mathcal{L}^n(A) &= \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap P^{-1}(\{y\}))\, d\mathcal{L}^m y \\
&= \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A^y)\, d\mathcal{L}^m y \\
&= \int_{\mathbb{R}^m} \mathcal{L}^{n-m}(A^y)\, d\mathcal{L}^m y \\
&= \int_{\mathbb{R}^m} \int_{\mathbb{R}^{n-m}} \chi_{A^y}(x,y)\, d\mathcal{L}^{n-m} x\, d\mathcal{L}^m y,
\end{aligned}$$

which is exactly the assertion of the Fubini's theorem when the integrable function is the characteristic function $f(x,y) = \chi_A(x,y)$ of A, and for the most general integrable function we need to use the change of variables formula.

Proof. From lemma (4.5.2) we know that $A \cap f^{-1}(\{y\})$ is $\mathcal{H}^{n-m}$ measurable $\mathcal{L}^m$-a.e. $y \in \mathbb{R}^m$ and $y \mapsto \mathcal{H}^{n-m}(A \cap f^{-1}(\{y\}))$ is $\mathcal{L}^m$-measurable. Since f is Lipschitz $Df(x)$ and $Jf(x)$ exist $\mathcal{L}^n$-a.e. $x \in A$ and hence there exists $A \supseteq B$ $\mathcal{L}^n$-measurable, $\mathcal{L}^n(B) = 0$ such that f is not differentiable in B. From lemma (4.5.2) $\int_{\mathbb{R}^m} \mathcal{H}^{n-m}(B \cap f^{-1}(\{y\}))\, d\mathcal{L}^m y = 0$ and hence

$$\begin{aligned}\int_A Jf(x)\, d\mathcal{L}^n x &= \int_{A-B} Jf(x)\, d\mathcal{L}^n x \\ &= \int_{\mathbb{R}^m} \mathcal{H}^{n-m}((A-B) \cap f^{-1}(\{y\}))\, d\mathcal{L}^m y \\ &= \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap f^{-1}(\{y\}))\, d\mathcal{L}^m y.\end{aligned}$$

So WLOG we may assume that $Df(x)$ and $Jf(x)$ exist for every $x \in A$, and $\mathcal{L}^n(A) < \infty$.

- Case # 1: $A \subseteq \{Jf > 0\}$.
  For each $\lambda \in \Lambda(n, n-m)$, $\lambda\colon \{1, \dots, n-m\} \to \{1, \dots n\}$ increasing, write

  $$\begin{aligned} &h_\lambda\colon \mathbb{R}^n \to \mathbb{R}^m \times \mathbb{R}^{n-m},\ h_\lambda(x) = (f(x), P_\lambda(x)) \quad (x \in \mathbb{R}^n) \\ &q\colon \mathbb{R}^m \times \mathbb{R}^{n-m} \to \mathbb{R}^m,\ \ q(y,z) = y \qquad\qquad (y \in \mathbb{R}^m, z \in \mathbb{R}^{n-m}),\end{aligned}$$

  and $P_\lambda(x) = P_\lambda(x_1, \dots, x_n) = (x_{\lambda(1)}, \dots, x_{\lambda(n-m)})$. Set $A_\lambda = \{x \in A\colon \det Dh_\lambda \neq 0\}$, then

  $$\begin{aligned}\det[Dh_\lambda(x)] \neq 0 &\Leftrightarrow Dh_\lambda(x)\colon \mathbb{R}^n \to \mathbb{R}^n \text{ is invertible} \\ &\Leftrightarrow (Dh_\lambda(x))^{-1}(\{0\}) = \{0\} \\ &\Leftrightarrow \{y \in \mathbb{R}^n\colon Dh_\lambda(x)(y) = 0\} = \{0\} \\ &\Leftrightarrow \{y \in \mathbb{R}^n\colon P_\lambda(y) = 0\} \cap \{y \in \mathbb{R}^n\colon Df(x)(y) = 0\} = \{0\}\end{aligned}$$

  $$\begin{aligned} &P_\lambda|_{(Df(x))^{-1}(\{0\})} \text{ is injective} \\ &\Leftrightarrow \{y \in \mathbb{R}^n\colon P_\lambda|_{\{z \in \mathbb{R}^n\colon Df(x)(z)=0\}}(y) = 0\} = \{0\} \\ &\Leftrightarrow \{y \in \mathbb{R}^n\colon P_\lambda(y) = 0\} \cap \{y \in \mathbb{R}^n\colon Df(x)(y) = 0\} = \{0\}.\end{aligned}$$

  Thus

  $$A_\lambda = \{x \in A\colon P_\lambda|_{(Df(x))^{-1}(\{0\})} \text{ is injective}\}.$$

  By definition it is clear that $\cup_{\lambda \in \Lambda(n,n-m)} A_\lambda \subseteq A$, now we want to show that $A \subseteq \cup_{\lambda \in \Lambda(n,n-m)} A_\lambda$. Let $x \in A$, so $Jf(x) > 0$ and then

  $$\begin{aligned} &0 < [Jf(x)]^2 = [\![(Df(x))^*]\!]^2 = \sum_{\tilde{\lambda} \in \Lambda(n,m)} (\det[\tilde{P}_{\tilde{\lambda}} \circ (Df(x))^*])^2 \\ &\Rightarrow 0 < \det[\tilde{P}_{\tilde{\lambda}}] \cdot [Df(x)]^t \text{ for some } \tilde{\lambda} \in \Lambda(n,m) \\ &\Rightarrow 0 < \det \left[\frac{\partial f}{\partial x_{\tilde{\lambda}(1)}} \cdots \frac{\partial f}{\partial x_{\tilde{\lambda}(m)}} \right] \\ &\Rightarrow \left\{ \frac{\partial f}{\partial x_{\tilde{\lambda}(1)}}, \cdots, \frac{\partial f}{\partial x_{\tilde{\lambda}(m)}} \right\} \text{ is linearly independent.}\end{aligned}$$

  $\tilde{\lambda}\colon \{1, \dots, m\} \to \{1, \dots, n\}$ increasing, $\tilde{\lambda}$ induces a unique

$\lambda\colon \{1,\ldots,n-m\} \to \{1,\ldots,n\}$ increasing and

$$\{1,\ldots n\} = \tilde{\lambda}(\{1,\ldots,m\}) \cup \lambda(\{1,\ldots,n-m\}).$$

Then

$$\begin{aligned} P_\lambda|_{(Df(x))^{-1}(\{0\})} \text{ is injective} &\Leftrightarrow P_\lambda|_{(Df(x))^{-1}(\{0\})}(y) = 0 \Rightarrow y = 0 \\ &\Leftrightarrow \text{for every } y \in \mathbb{R}^n \text{ with } Df(x)(y) = 0 \\ &\quad\ y_{\lambda(1)} = \cdots = y_{\lambda(n-m)} = 0 \Rightarrow y = 0. \end{aligned}$$

Since the set $\left\{\frac{\partial f}{\partial x_{\tilde{\lambda}(1)}}, \cdots, \frac{\partial f}{\partial x_{\tilde{\lambda}(m)}}\right\}$ is linearly independent then $P_\lambda|_{(Df(x))^{-1}(\{0\})}$ is injective, and $x \in A_\lambda$. Since each A_λ is measurable we may assume for simplicity that $A = A_\lambda$ for some $\lambda \in \Lambda(n, n-m)$. Fix $t > 1$, $A = A_\lambda = \{\det Dh_\lambda \neq 0\}$ where $\lambda \in \Lambda$ is fixed and $h = h_\lambda\colon \mathbb{R}^n \to \mathbb{R}^n$ is a Lipschitz map. Then by using lemma (4.5.3) we may choose a pairwise disjoint collection of Borel sets $\{D_k\}_{k\in\mathbb{N}}$ such that

(1) $\mathcal{L}^n(A - \bigcup_k D_k) = 0$

(2) $h_\lambda|_{D_k}$ is injective for each $k \geq 1$

(3) For each $k \geq 1$ there exists a symmetric automorphism $S_k\colon \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\mathrm{Lip}\,(S_k^{-1} \circ (h_\lambda|_{D_k})) \leq t \quad \mathrm{Lip}\,((h|_{D_k})^{-1} \circ S_k) \leq t$$
$$t^{-n}|\det S_k| \leq Jh_\lambda|_{D_k} \leq t^n|\det S_k|$$

Set $G_k = A \cap D_k$.
Claim # 1: $t^{-n}[\![q \circ S_k]\!] \leq Jf|_{G_k} \leq t^n[\![q \circ S_k]\!]$.

Proof Of Claim # 1. Since $f = q \circ h$ we have $\mathcal{L}^n$-a.e. equalities

$$\begin{aligned} Df &= D(q \circ h) \\ &= q \circ Dh \\ &= q \circ S_k \circ S_k^{-1} \circ Dh \\ &= q \circ S_k \circ D(S_k^{-1} \circ h) = q \circ S_k \circ C, \end{aligned}$$

where $C = D(S_k^{-1} \circ h)$. From lemma (4.5.3) we obtain,

$$t^{-1} \leq \mathrm{Lip}\,(S_k^{-1} \circ h) = \mathrm{Lip}\,(D(S_k^{-1} \circ h)) \leq t.$$

The linear maps $Df\colon \mathbb{R}^n \to \mathbb{R}^m$ and $q \circ S_k\colon \mathbb{R}^n \to \mathbb{R}^m$ have polar decompositions

$$\begin{aligned} Df &= S \circ O^* \\ q \circ S_k &= T \circ P^*, \end{aligned}$$

where S, $T\colon \mathbb{R}^m \to \mathbb{R}^m$ are symmetric automorphisms, and O, $P\colon \mathbb{R}^m \to \mathbb{R}^n$ are orthogonal. Then,

$$(*) \qquad \begin{aligned} &S \circ O^* = T \circ P^* \circ C \\ &\Rightarrow S \circ O^* \circ O = T \circ P^* \circ C \circ O \\ &\Rightarrow S = T \circ P^* \circ C \circ O \end{aligned}$$

Since $G_k \subseteq A \subseteq \{Jf > 0\}$ and $Jf(x) = |\det S|$ then $\det S \neq 0$ and hence $\det T \neq 0$. If $v \in \mathbb{R}^m$,

$$\begin{aligned}|(T^{-1} \circ S)(v)| &= |(P^* \circ C \circ O)(v)| \\ &\leq |(C \circ O)(v)| \\ &\leq t|Ov| = t|v|.\end{aligned}$$

Therefore $(T^{-1} \circ S)(B(0,1)) \subseteq B(0,t)$. Since $T^{-1} \circ S$ is a linear operator on $\mathbb{R}^m$, then

$$\begin{aligned}V[(T^{-1} \circ S)B(0,1)] &= |\det(T^{-1} \circ S)| V(B(0,1)), \text{ and} \\ (T^{-1} \circ S)B(0,1) &\subseteq tB(0,1) \text{ imply that} \\ \alpha(m)|\det(T^{-1} \circ S)| &= \mathcal{L}^m((T^{-1} \circ S)B(0,1)) \leq \mathcal{L}^m(tB(0,1)) \\ &= t^m \mathcal{L}^m(B(0,1)) = \alpha(m)t^m.\end{aligned}$$

And hence

$$\begin{aligned}|\det T|^{-1}|\det S| &\leq t^m \leq t^n, \\ Jf = |\det S| &\leq t^n |\det T| = t^n [\![q \circ S_k]\!], \text{ so} \\ Jf|_{G_k} &\leq t^n [\![q \circ S_k]\!].\end{aligned}$$

Since $C = S_k^{-1} \circ Dh$ and Dh is invertible then by the inverse function theorem $C^{-1} = D(h^{-1} \circ S_k)$, and lemma (4.5.3) implies that $\operatorname{Lip} C = \operatorname{Lip}(D(h^{-1} \circ S_k)) = \operatorname{Lip}(h^{-1} \circ S_k) \leq t$. From $(*)$ we obtain $S^{-1} \circ T = O^* \circ C^{-1} \circ P$. If $v \in \mathbb{R}^m$, then

$$\begin{aligned}|(S^{-1} \circ T)(v)| &= |(O^* \circ C^{-1} \circ P)(v)| \\ &\leq |(C^{-1} \circ P)(v)| \\ &\leq t|Pv| = t|v|.\end{aligned}$$

Therefore $(S^{-1} \circ T)(B(0,1)) \subseteq B(0,t)$ and similar calculations yield

$$\begin{aligned}|\det S|^{-1}|\det T| &\leq t^n \\ [\![q \circ S_k]\!] = |\det T| &\leq t^n |\det S| = t^n Jf.\end{aligned}$$

Consequently, $t^{-n}[\![q \circ S_k]\!] \leq Jf|_{G_k} \leq t^n [\![q \circ S_k]\!]$ holds for each $k \in \mathbb{N}$, and hence we proved the claim.

Now using all the information we have gathered so far we can calculate

$$\begin{aligned}& t^{-3n+m} \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(G_k \cap f^{-1}(\{y\}) \, d\mathcal{L}^m y \\ (**) \qquad & = t^{-3n+m} \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(h^{-1}(h(G_k) \cap q^{-1}(\{y\}))) \, d\mathcal{L}^m y.\end{aligned}$$

Since $\underbrace{(h^{-1} \circ S_k)}_{\operatorname{Lip}(h^{-1} \circ S_k) \leq t} \circ S_k^{-1}(h(G_k) \cap q^{-1}(\{y\}))$, then

$$\begin{aligned}&\mathcal{H}^{n-m}(h^{-1}(h(G_k) \cap q^{-1}(\{y\}))) \\ &\leq (\operatorname{Lip}(h^{-1} \circ S_k))^{n-m} \mathcal{H}^{n-m}(S_k^{-1}(h(G_k)) \cap q^{-1}(\{y\})) \\ &\leq t^{n-m} \mathcal{H}^{n-m}(S_k^{-1}(h(G_k)) \cap q^{-1}(\{y\})).\end{aligned}$$

Now if we continue the calculation from $(**)$ and using the above observation

$$\leq t^{-2n}\int_{\mathbb{R}^m}\mathcal{H}^{n-m}(S_k^{-1}(h(G_k)\cap q^{-1}(\{y\})))\,d\mathcal{L}^m y$$
$$= t^{-2n}\int_{\mathbb{R}^m}\mathcal{H}^{n-m}((S_k^{-1}\circ h)(G_k)\cap(q\circ S_k)^{-1}(\{y\})))\,d\mathcal{L}^m y$$

from lemma (4.5.1)

$$= t^{-2n}[\![q\circ S_k]\!]\mathcal{L}^n((S_k^{-1}\circ h)(G_k))$$
$$\leq t^{-2n}[\![q\circ S_k]\!](\mathrm{Lip}\,(S_k^{-1}\circ h))^n\mathcal{L}^n(G_k)$$
$$\leq t^{-n}[\![q\circ S_k]\!]\mathcal{L}^n(G_k)$$
$$\leq \int_{G_k} Jf(x)\,d\mathcal{L}^n x$$
$$\leq t^{n}[\![q\circ S_k]\!]\mathcal{L}^n(G_k)$$

since $\mathcal{L}^n(G_k)\leq t^n\mathcal{L}^n((S_k^{-1}\circ h)G_k)$

$$\leq t^{2n}[\![q\circ S_k]\!]\mathcal{L}^n((S_k^{-1}\circ h)S_k)$$

again by lemma (4.5.1)

$$= t^{2n}\int_{\mathbb{R}^m}\mathcal{H}^{n-m}((S_k^{-1}\circ h)(G_k)\cap(q\circ S_k)^{-1}(\{y\}))\,d\mathcal{L}^m y$$

since $S_k^{-1}(h(G_k)\cap q^{-1}(\{y\}))=(S_k^{-1}\circ h)h^{-1}(h(G_k)\cap q^{-1}(\{y\}))$

and $\mathrm{Lip}\,(S_k^{-1}\circ h)\leq t$, then

$$\leq t^{3n-m}\int_{\mathbb{R}^m}\mathcal{H}^{n-m}(h^{-1}(h(G_k)\cap q^{-1}(\{y\})))\,d\mathcal{L}^m y$$
$$= t^{3n-m}\int_{\mathbb{R}^m}\mathcal{H}^{n-m}(G_k\cap f^{-1}(\{y\}))\,d\mathcal{L}^m y.$$

If we sum on $k\in\mathbb{N}$ we obtain

$$t^{-3n+m}\sum_{k=1}^{\infty}\int_{\mathbb{R}^m}\mathcal{H}^{n-m}(G_k\cap f^{-1}(\{y\}))\,d\mathcal{L}^m y$$
$$\leq \sum_{k=1}^{\infty}\int_{G_k} Jf(x)\,d\mathcal{L}^n x$$
$$\leq t^{3n-m}\sum_{k=1}^{\infty}\int_{\mathbb{R}^m}\mathcal{H}^{n-m}(G_k\cap f^{-1}(\{y\}))\,d\mathcal{L}^m y.$$

By the MCT

$$t^{3n-m}\int_{\mathbb{R}^m}\mathcal{H}^{n-m}\Big(\bigcup_k G_k\cap f^{-1}(\{y\})\Big)\,d\mathcal{L}^m y$$
$$\leq \int_{\bigcup_k G_k} Jf(x)\,d\mathcal{L}^n x$$

$$\leq t^{3n-m} \int_{\mathbb{R}^m} \mathcal{H}^{n-m}\Bigg(\bigcup_k G_k \cap f^{-1}(\{y\})\Bigg).$$

From lemma (4.5.3) we have $\mathcal{L}^n(A - \bigcup_k G_k) = 0$, and then

$$\int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap f^{-1}(\{y\}))\, d\mathcal{L}^m y$$
$$= \int_{\mathbb{R}^m} \mathcal{H}^{n-m}\Bigg(\bigcup_k G_k \cap f^{-1}(\{y\})\Bigg)\, d\mathcal{L}^m y.$$

As $t \to 1^+$, we get

$$\int_A Jf(x)\, d\mathcal{L}^n x = \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap f^{-1}(\{y\}))\, d\mathcal{L}^m y$$

and hence we proved the Coarea Formula for the case $A \subseteq \{Jf > 0\}$.

- Case # 2: $A \subseteq \{Jf = 0\}$.
Fix $0 < \epsilon \leq 1$, and define

$$g\colon \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m,\ g(x,y) = f(x) + \epsilon y$$
$$p\colon \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m,\ p(x,y) = y \qquad (x \in \mathbb{R}^n,\ y \in \mathbb{R}^m).$$

Then

$$[Dg(x,y)] = \begin{bmatrix} Dg(x,y)(e_1) & \cdots & Dg(x,y)(e_{n+m}) \end{bmatrix}$$
$$= \begin{bmatrix} Df(x)(\tilde{e}_1) & \cdots & Df(x)(\tilde{e}_n) & \epsilon I_{m\times m} \end{bmatrix}$$

where $\{e_1, \dots, e_{n+m}\}$ is the standard basis of $\mathbb{R}^{n+m}$ and $\{\tilde{e}_1, \dots, \tilde{e}_n\}$ is the standard basis of $\mathbb{R}^n$, and as in the proof of the area formula

$$\epsilon^m \leq Jg \leq [\![Dg]\!] \leq [\![(Dg)^*]\!] \leq C\epsilon$$

for some constant $C > 0$.
Define a function $k(y) = \mathcal{H}^{n-m}(A \cap f^{-1}(\{y\}))$ which is nonnegative and $\mathcal{L}^m$-measurable a.e., then

$$\int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap f^{-1}(\{y\}))\, d\mathcal{L}^m y = \int_{\mathbb{R}^m} k(y)\, d\mathcal{L}^m y$$

since Lebesque integral is invariant under translation

$$= \int_{\mathbb{R}^m} k(y - \epsilon w)\, d\mathcal{L}^m y$$
$$= \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap f^{-1}(\{y - \epsilon w\}))\, d\mathcal{L}^m y.$$

Thus

$$= \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap f^{-1}(\{y\}))\, d\mathcal{L}^m y.$$

$$(***) \qquad \frac{1}{\alpha(m)} \int_{B(0,1)} \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap f^{-1}(\{y - \epsilon w\}))\, d\mathcal{L}^m y\, d\mathcal{L}^m w.$$

Claim # 2: Fix $y \in \mathbb{R}^m$, $w \in \mathbb{R}^m$. and set $B = A \times B(0,1) \subseteq \mathbb{R}^{n+m}$. Then

$$B \cap g^{-1}(\{y\}) \cap p^{-1}(\{w\}) = \begin{cases} \emptyset & \text{if } w \notin B(0,1) \\ (A \cap f^{-1}(\{y - \epsilon w\})) \times \{w\} & \text{if } w \in B(0,1). \end{cases}$$

Proof Of Claim # 2.

$$\begin{aligned}
&(x,z) \in B \cap g^{-1}(\{y\}) \cap p^{-1}(\{w\}) \\
&\Leftrightarrow x \in A,\ z \in B(0,1),\ f(x) + \epsilon z = y,\ w = z \\
&\Leftrightarrow x \in A,\ z = w \in B(0,1),\ f(x) = y - \epsilon w \\
&\Leftrightarrow w \in B(0,1) \text{ and } (x,z) \in A \cap f^{-1}(\{y - \epsilon w\}) \times \{w\}
\end{aligned}$$

Now using Claim # 2 continue the calculation from $(***)$

$$\frac{1}{\alpha(m)} \int_{B(0,1)} \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap f^{-1}(\{y - \epsilon w\}))\, d\mathcal{L}^m y\, d\mathcal{L}^m w$$

since $\mathcal{H}^{n-m}$ is invariant under affine transformations

$$\begin{aligned}
&= \frac{1}{\alpha(m)} \int_{B(0,1)} \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap f^{-1}(\{y - \epsilon w\}) \times \{w\})\, d\mathcal{L}^m y\, d\mathcal{L}^m w \\
&= \frac{1}{\alpha(m)} \int_{\mathbb{R}^m} \int_{B(0,1)} \mathcal{H}^{n-m}(A \cap f^{-1}(\{y - \epsilon w\}) \times \{w\})\, d\mathcal{L}^m w\, d\mathcal{L}^m y \\
&= \frac{1}{\alpha(m)} \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(B \cap g^{-1}(\{y\}) \cap p^{-1}(\{w\}))\, d\mathcal{L}^m w\, d\mathcal{L}^m y \\
&\leq \frac{\alpha(n-m)}{\alpha(n)} \int_{\mathbb{R}^m} \mathcal{H}^{n}(B \cap g^{-1}(\{y\}))\, d\mathcal{L}^m y
\end{aligned}$$

since $Jg > 0$ on $\mathbb{R}^{n+m}$, then by Case # 1

$$\begin{aligned}
&= \frac{\alpha(n-m)}{\alpha(n)} \int_{B} Jg(x,z)\, d\mathcal{L}^n x\, d\mathcal{L}^m z \\
&\leq \frac{\alpha(n-m)}{\alpha(n)} \mathcal{L}^{n+m}(A \times B(0,1)) \sup_B Jg(x,z) \\
&\leq \frac{\alpha(n-m)\alpha(m)}{\alpha(n)} \mathcal{L}^{n}(A) \sup_B Jg(x,z) \\
&\leq C' \mathcal{L}^n(A)\epsilon, \text{ where } C' > 0 \text{ is a constant.}
\end{aligned}$$

As $\epsilon \to 0$, we obtain

$$\int_A Jf(x)\, d\mathcal{L}^n x = 0 = \int_{\mathbb{R}^m} \mathcal{H}^{n-m}(A \cap f^{-1}(\{y\})))\, d\mathcal{L}^m y$$

and hence we proved the Coarea Formula for the case $A \subseteq \{Jf = 0\}$.

- General Case: Let $A = A_1 \cap A_2$ where $A_1 = A \cap \{Jf > 0\}$ and $A_2 = A \cap \{Jf = 0\}$, then A_1 and A_2 are pairwise disjoint measurable sets. Now apply Case # 1 and Case # 2.

□

Theorem 4.5.6. *Let $f\colon \mathbb{R}^n \to \mathbb{R}^m$ be Lipschitz, $n \geq m$. Then for each integrable function $g\colon \mathbb{R}^n \to \mathbb{R}$,*

$$g|_{f^{-1}(\{y\})} \text{ is } \mathcal{H}^{n-m}\text{-integrable } \mathcal{L}^m\text{-a.e.} y \in \mathbb{R}^m$$

and

$$\int_{\mathbb{R}^n} g(x,y)Jf(x,y)\,d\mathcal{L}^n(x,y) = \int_{\mathbb{R}^m}\int_{f^{-1}(\{y\})} g(x,y)\,d\mathcal{H}^{n-m}x\,d\mathcal{L}^m y$$

Remark 4.5.7.

- The change of variables formula is a generalization of the coarea formula. In particular if $g(x,y)=\chi_A(x,y)$, $A\subseteq\mathbb{R}^n$ is $\mathcal{L}^n$-measurable and $\mathcal{L}^n(A)<\infty$ then the change of variables formula implies that $\chi_A|_{f^{-1}(\{y\})}$ is $\mathcal{H}^{n-m}$ integrable a.e. and
$$\begin{aligned}\int_A Jf(x,y)\,d\mathcal{L}^n(x,y) &= \int_{\mathbb{R}^n}\chi_A(x,y)Jf(x,y)\,d\mathcal{L}^n(x,y)\\ &= \int_{\mathbb{R}^m}\int_{f^{-1}(\{y\})}\chi_A(x,y)\,d\mathcal{H}^{n-m}x\,d\mathcal{L}^m y\\ &= \int_{\mathbb{R}^m}\mathcal{H}^{n-m}(A\cap f^{-1}(\{y\}))\,d\mathcal{L}^m y.\end{aligned}$$
- The coarea formula together with the change of variables formula generalize the Fubini's theorem. In particular if
$P\colon\mathbb{R}^n\to\mathbb{R}^m$ is a projection map onto the last m coordinates and $g\colon\mathbb{R}^n\to\mathbb{R}$ is an integrable function, then the map $y\mapsto g(x,y)$ is $\mathcal{H}^{n-m}$ integrable $\mathcal{L}^m$-a.e. $y\in\mathbb{R}^m$ and
$$(*)\qquad\begin{aligned}\int_{\mathbb{R}^n} g(x,y)\,\mathcal{L}^n(x,y) &= \int_{\mathbb{R}^m}\int_{P^{-1}(\{y\})} g(x,y)\,d\mathcal{H}^{n-m}x\,d\mathcal{L}^m y\\ &= \int_{\mathbb{R}^m}\int_{P^{-1}(\{y\})} g(x,y)\,d\mathcal{L}^{n-m}x\,d\mathcal{L}^m y\\ &= \int_{\mathbb{R}^m}\int_{\mathbb{R}^{n-m}} g(x,y)\,d\mathcal{L}^{n-m}x\,d\mathcal{L}^m y\end{aligned}$$
where the second equality holds for every measurable characteristic function and hence for every integrable function. From $(*)$ we see that the map $y\mapsto \int_{\mathbb{R}^{n-m}} g(x,y)\,d\mathcal{L}^{n-m}x$ is $\mathcal{L}^{n-m}$ integrable a.e. and the assertion of the Fubini's theorem holds.

Proof.

(1) Case # 1: $g\geq 0$.
We can write $g=\sum_i \frac{1}{i}\chi_{A_i}$ where each A_i is $\mathcal{L}^n$-measurable. Then by the MCT
$$\begin{aligned}\int_{\mathbb{R}^n} g(x,y)Jf(x,y)\,\mathcal{L}^n(x,y) &= \sum_i\frac{1}{i}\int_{A_i} Jf(x,y)\,d\mathcal{L}^n(x,y)\\ &= \sum_i\frac{1}{i}\int_{\mathbb{R}^m}\mathcal{H}^{n-m}(A_i\cap f^{-1}(\{y\}))\,d\mathcal{L}^m y\\ &= \int_{\mathbb{R}^m}\sum_i\frac{1}{i}\mathcal{H}^{n-m}(A_i\cap f^{-1}(\{y\}))\,d\mathcal{L}^m y\\ &= \int_{\mathbb{R}^m}\int_{f^{-1}(\{y\})} g(x,y)\,d\mathcal{H}^{n-m}x\,d\mathcal{L}^m y\end{aligned}$$

(2) Case # 2: If g is an arbitrary integrable function then apply Case # 1 to the positive and negative parts of g.

$\square$

The coarea formula can be used to show that the integral of a real-valued integrable function is equal to the integral with respect to the spherical coordinates. The coarea formula also implies that the integral of the jacobian of a real-valued Lipschitz function f is equal to the integral of the Hausdorff measure of each level set $\{f = t\}$.

Applications:

Proposition 4.5.8 (Polar Coordinates)**.** *Let $g\colon \mathbb{R}^n \to \mathbb{R}$ be $\mathcal{L}^n$-integrable. Then*

$$\int_{\mathbb{R}^n} g(x,r)\, d\mathcal{L}^n(x,r) = \int_0^\infty \int_{\partial B(0,r)} g(x,r)\, d\mathcal{H}^{n-1}x\, dr.$$

Proof. Let x and r denote the general point of $\mathbb{R}^{n-1}$ and $\mathbb{R}$, respectively, and $f\colon \mathbb{R}^n \to \mathbb{R}$ be defined by $f(x,r) = |(x,r)|$, then $Df(x,r)\colon \mathbb{R}^n \to \mathbb{R}$ exists for every $(x,r) \in \mathbb{R}^n - \{0\}$ and

$$Df(x,r)(h) = \frac{(x,r)}{|(x,r)|} \cdot h, \text{ then } Jf(x,r) = 1, \text{ for every } (x,r) \in \mathbb{R}^n - \{0\}.$$

Then by using the change of variables formula we obtain

$$\begin{aligned}\int_{\mathbb{R}^n} g(x,r)\, d\mathcal{L}^n(x,r) &= \int_{\mathbb{R}^{\geq 0}} \int_{f^{-1}(\{r\})} g(x,r)\, d\mathcal{H}^{n-m}x\, dr \\ &= \int_0^\infty \int_{\partial B(0,r)} g(x,r)\, d\mathcal{H}^{n-1}x\, dr.\end{aligned}$$

$\square$

Theorem 4.5.9. *Assume $f\colon \mathbb{R}^n \to \mathbb{R}$ is Lipschitz, then*

$$\int_{\mathbb{R}^n} |Df(x,t)|\, d\mathcal{L}^n(x,t) = \int_{-\infty}^\infty \mathcal{H}^{n-1}(\{f = t\})\, dt.$$

Proof. Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be Lipschitz and Q_k be the cube in $\mathbb{R}^n$ centered at the origin with side length $2k \in \mathbb{N}$, and let x and t denote the general point in $\mathbb{R}^{n-1}$ and $\mathbb{R}$, respectively, then $\chi_{Q_k}\colon \mathbb{R}^n \to \mathbb{R}$ is integrable and the change of variables formula implies that

$$\begin{aligned}\int_{\mathbb{R}^n} \chi_{Q_k}(x,t) Jf(x,t)\, d\mathcal{L}^n(x,t) &= \int_{\mathbb{R}} \int_{f^{-1}(\{t\})} \chi_{Q_k}(x,t)\, d\mathcal{H}^{n-1}x\, dt \\ &= \int_{-\infty}^\infty \mathcal{H}^{n-1}(Q_k \cap \{f = t\})\, dt\end{aligned}$$

And the MCT gives

$$\int_{\mathbb{R}^n} |Df(x,t)|\, d\mathcal{L}^n(x,t) = \int_{-\infty}^\infty \mathcal{H}^{n-1}(\{f = t\})\, dt.$$

$\square$

4.6. **Rectifiable Sets.**

Definition 4.6.1.

(1) *The Tangent Cone of $E \subseteq \mathbb{R}^n$ at $a \in \mathbb{R}^n$ consists of the tangent vectors of E at a*

$$\operatorname{Tan}(E,a) = \{r \in \mathbb{R} \colon r \geq 0\} \left[\bigcap_{\epsilon>0} \operatorname{Clos}\left\{ \frac{x-a}{|x-a|} : x \in E, 0 < |x-a| < \epsilon \right\} \right]$$

(2) *The cone of approximate tangent vectors of E at a is defined to be*

$$\operatorname{Tan}^m(E,a) = \bigcap \{\operatorname{Tan}(S,a) \colon \Theta^m(E-S,a) = 0\}.$$

Remark 4.6.2. We can think of the approximate tangent cone as the subset of the tangent cone but without the lower dimensional pieces of the tangent cone.

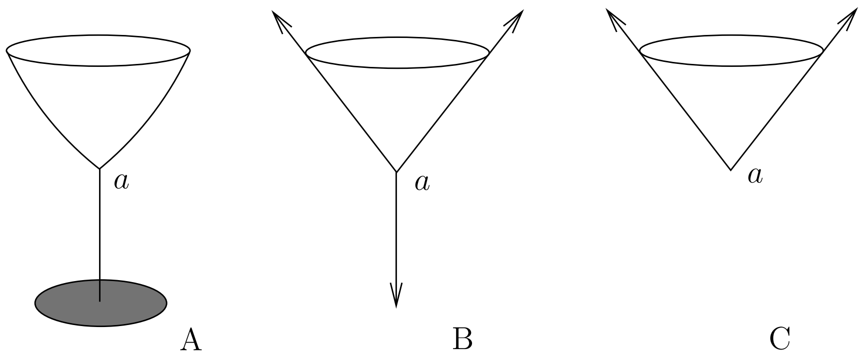


FIGURE 13. (A) is the set, (B) it its tangent cone at a, and (C) is its approximate tangent cone at a.

Definition 4.6.3. *A set $E \subseteq \mathbb{R}^n$ is called $(\mathcal{H}^m, m)$ rectifiable if $\mathcal{H}^m(E) < \infty$ and $\mathcal{H}^m$-a.e. $x \in E$ is contained in the union of the images of countably many Lipschitz functions from $\mathbb{R}^m$ to $\mathbb{R}^n$.*

These sets are the generalized surfaces of geometric measure theory. $E \subseteq \mathbb{R}^n$ is called an m-dimensional rectifiable set if it is $(\mathcal{H}^m, m)$-rectifiable and $\mathcal{H}^m$-measurable.

The next proposition, whose proof can be found in Federer [10], shows that if an $\mathcal{H}^m$-measurable set $E \subseteq \mathbb{R}^n$ is m-rectifiable then $\mathcal{H}^m$-a.e. $x \in E$ is contained in a countable union of C^1 embedded manifolds.

Proposition 4.6.4. *If a set $E \subseteq \mathbb{R}^n$ is m-rectifiable, then there exist a sequence $\{K_i\}_{i\in\mathbb{N}}$ of compact subsets of $\mathbb{R}^m$ and a sequence $\{f_i\}_{i\in\mathbb{N}}$ of C^1 maps such that each f_i has domain K_i and $\{f_i(K_i)\}_{i\in\mathbb{N}}$ is a collection pairwise disjoint subsets of E, and*

$$\mathcal{H}^m\left(E - \bigcup_i f_i(K_i)\right) = 0.$$

The following proposition, whose proof can be found in Federer [10], shows that a rectifiable set has a tangent plane $\mathcal{H}^m$-a.e.

Proposition 4.6.5. *If $E \subseteq \mathbb{R}^n$ is an m-dimensional rectifiable subset, then for $\mathcal{H}^m$-a.e. $x \in E$, $\Theta^m(E, x) = 1$ and $\operatorname{Tan}^m(E, x)$ is an m-dimensional plane.*

Definition 4.6.6. *An orientation of an m-dimensional rectifiable set $E \subseteq \mathbb{R}^n$ is a choice of orientation for each $\operatorname{Tan}^m(E, x)$.*

5. DIFFERENTIAL FORMS, INTEGRATING FORMS OVER MANIFOLDS, GENERALIZED STOKE'S THEOREM

In this section we will introduce the differential forms and list some of their properties. We will give the definition of the integral of a differential form over a parameterized manifold in $\mathbb{R}^n$. The definition of the integral of a differential form over a manifold is a generalization of the integral over a parameterized manifold. We can think of a manifold as formed by patches of parameterized manifolds and hence the integral of a differential form over a manifold is the sum of integrals over various parameterized manifolds covering our initial manifold. The culmination of our labor in this section is the Generalized Stokes' Theorem which relates the integral of a differential form over the boundary of the manifold with the integral of its differential over the manifold. The main reference for this section is the book by Munkres [23].

5.1. **Tangent Vectors And Vector Fields.**

Definition 5.1.1. *Given $x \in \mathbb{R}^n$, a tangent vector to $\mathbb{R}^n$ at x is a pair $(x; v)$ where $v \in \mathbb{R}^n$. The set of all tangent vectors to $\mathbb{R}^n$ at x forms a vector space under the operations*

$$(x; v) + (x; w) = (x; v + w)$$
$$c(x; v) = (x; cv)$$

It is called the tangent space to $\mathbb{R}^n$ at x and is denoted $\mathcal{T}_x(\mathbb{R}^n)$. The tangent space can be thought as the set of all arrows emanating from x.

Definition 5.1.2. *Let $\alpha\colon (a, b) \to \mathbb{R}^n$ be a map of class C^r, the velocity vector of α at t is defined to be $(\alpha(t); \alpha'(t))$.*

Definition 5.1.3. *Let A be open in $\mathbb{R}^k$ or $\mathbb{H}^k$, let $\alpha\colon A \to \mathbb{R}^n$ be of class C^r. Let $x \in A$, and let $p = \alpha(x)$, and define a linear map $\alpha_*\colon \mathcal{T}_x(\mathbb{R}^k) \to \mathcal{T}_p(\mathbb{R}^n)$ by $\alpha_*(x; v) = (\alpha(x); D\alpha(x)(v))$. It is called the transformation induced by α. Using the chain rule it is easy to see that $\alpha_*(x, v)$ is the velocity vector of $\beta(t) = \alpha(x + tv)$ at $t = 0$.*

The next lemma, whose proof can be found in Munkres [23], states that the star operation distributes over composition.

Lemma 5.1.4. *Let A be open in $\mathbb{R}^k$ or $\mathbb{H}^k$ and let $\alpha\colon A \to \mathbb{R}^m$ be of class C^r. Let B be an open set in $\mathbb{R}^m$ or $\mathbb{H}^m$ containing $\alpha(A)$, let $\beta\colon B \to \mathbb{R}^n$ be of class C^r, then*

$$(\beta \circ \alpha)_* = \beta_* \circ \alpha_*.$$

Definition 5.1.5. *Let $A \in \mathrm{OP}(\mathbb{R}^n)$, a tangent vector field in A is a continuous function $F\colon A \to \mathbb{R}^n \times \mathbb{R}^n$ such that $F(x) \in \mathcal{T}_x(\mathbb{R}^n)$ for every $x \in A$. Thus F has the form $F(x) = (x; f(x))$ where $f\colon A \to \mathbb{R}^n$. If F is of class C^r then we call it a tangent vector field of class C^r.*

Definition 5.1.6.

(1) *Let M be a k-manifold of class C^r in $\mathbb{R}^n$. If $p \in M$ choose a coordinate map $\alpha\colon U \to V$ about p, where U is open in $\mathbb{R}^k$ or $\mathbb{H}^k$. Let $x \in U$ be such that $\alpha(x) = p$. The set of all vectors of the form $\alpha_*(x; v)$, $v \in \mathbb{R}^k$, is called the tangent space to M at p and is denoted $\mathcal{T}_p(M)$.*

$$\mathcal{T}_p(M) = \alpha_*(\mathcal{T}_x(\mathbb{R}^k))$$

is a subspace of $\mathcal{T}_p(\mathbb{R}^n)$ that does not depend on the choice of the coordinate map.

$$\mathcal{T}_p(M) = \{(p; D\alpha(x)(v))\colon v \in \mathbb{R}^k\}$$

is spanned by

$$\left\{\left(p; \frac{\partial\alpha}{\partial x_1}(x)\right), \ldots, \left(p; \frac{\partial\alpha}{\partial x_k}(x)\right)\right\}$$

where $\{e_1 \ldots, e_k\}$ is the standard basis for $\mathbb{R}^k$. Since $D\alpha(x)$ has rank k, these vectors are linearly independent.

(2) *The union of the tangent spaces $\mathcal{T}_p(M)$ for $p \in M$ is called the tangent bundle of M.*

(3) *A tangent vector field to M is a continuous function $F\colon M \to \mathcal{T}_p(M)$ defined by $p \mapsto F(p)$.*

5.2. **Tensor Fields And Differential Forms.**

Definition 5.2.1.

(1) *Let $A \in \mathrm{OP}(\mathbb{R}^n)$, a k-tensor field in A is a function ω assigning, to each $x \in A$, a k-tensor defined on the vector space $\mathcal{T}_x(\mathbb{R}^n)$.*

$$\begin{aligned} \omega\colon A \to& \mathrm{L}^k(\mathcal{T}.(\mathbb{R}^n)) \\ x \mapsto& \omega(x) \in \mathrm{L}^k(\mathcal{T}_x(\mathbb{R}^n)) \end{aligned}$$

(2) *The value of $\omega(x)$ at $((x; v_1), \ldots, (x; v_k))$ is denoted by $\omega(x)((x; v_1), \ldots, (x; v_k))$ and we require ω to be a continuous function with respect $(x, v_1, \ldots, v_k)$. If it is of class C^r then we call it a tensor field of class C^r.*

(3) *ω is called differential k-form on A if for every $x \in A$, $\omega(x) \in \mathrm{A}^k(\mathcal{T}_x(\mathbb{R}^n))$, alternating k-tensor on A.*

(4) *If M is an m-manifold in $\mathbb{R}^n$, then a k-tensor field on M is a function $\omega\colon p \in M \mapsto \omega(p) \in \mathrm{L}^k(\alpha_*(\mathcal{T}_x(\mathbb{R}^m)))$, where α is a coordinate map about $p = \alpha(x)$. If $\omega(p) \in \mathrm{A}^k(\mathcal{T}_p(M))$ for every $p \in M$ then ω is called a differential k-form on M.*

Definition 5.2.2.

(1) *Let* $\{e_1, \ldots, e_n\}$ *be the usual basis for* $\mathbb{R}^n$. *Then* $\{(x; e_1), \ldots (x; e_n)\}$ *is a basis for* $\mathcal{T}_x(\mathbb{R}^n)$. *For each* $1 \leq i \leq n$ *define the elementary* 1*-form* $\tilde{\phi}_i$ *on* $\mathbb{R}^n$ *by*

$$\begin{aligned} \tilde{\phi}_i \colon \mathbb{R}^n &\to \mathrm{A}^1(\mathcal{T}.(\mathbb{R}^n)) \\ x \quad &\mapsto \tilde{\phi}_i(x)(x; e_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \end{aligned}$$

(2) *Let* $I = (i_1, \ldots, i_k) \in \{1, \ldots, n\}^k$ *be ascending, then define the elementary* k*-form* $\tilde{\psi}_I$ *on* $\mathbb{R}^n$ *by*

$$\begin{aligned} \tilde{\psi}_I \colon \mathbb{R}^n &\to \mathrm{A}^k(\mathcal{T}.(\mathbb{R}^n)) \\ x \quad &\mapsto \tilde{\psi}_I(x) = \tilde{\phi}_{i_1}(x) \wedge \cdots \wedge \tilde{\phi}_{i_k}(x) \end{aligned}$$

where

$$\begin{aligned} \tilde{\psi}_I(x)((x; v_1), \ldots, (x, v_k)) &= \sum_{\sigma \in S_k} (\operatorname{sgn}(\sigma)) \tilde{\phi}_{i_1}(x; v_{\sigma(1)}) \cdots \tilde{\phi}_{i_k}(x)(x; v_{\sigma(k)}) \\ &= \sum_{\sigma \in S_k} (\operatorname{sgn}(\sigma)) v_{i_1, \sigma(1)} \cdots v_{i_k, \sigma(k)} = \det X_I \end{aligned}$$

and $X = [v_1 \cdots v_k]$.

(3) *If* ω *is a* k*-form defined on an open set* A *of* $\mathbb{R}^n$, *then the alternating* k*-tensor* $\omega(x)$ *can be written uniquely in the form*

$$\omega(x) = \sum_{(I)} b_I(x) \tilde{\psi}_I(x)$$

where the summation extends over all $I \in \{1, \ldots, n\}^k$ *ascending.*

Remark 5.2.3. From now on we will require that a k-form on $A \in \mathrm{OP}(\mathbb{R}^n)$ is an alternating k-tensor field of class C^∞, that is the component functions b_I are all of class C^∞. And by convention we will define a 0-form on A as a real-valued C^∞ function on A.

5.3. **Differential Operator.**

Definition 5.3.1. *Let* $A \in \mathrm{OP}(\mathbb{R}^n)$, *let* $f \colon A \to \mathbb{R}$ *be a function of class* C^∞. *The* 1*-form* df *on* A *is defined by*

$$\begin{aligned} df \colon A \to & \mathrm{A}^1(\mathcal{T}.(\mathbb{R}^n)) \\ x \mapsto & df(x) \in \mathrm{A}^1(\mathcal{T}_x(\mathbb{R}^n)) \end{aligned}$$

where $df(x)(x; v) = Df(x)(v)$.

Lemma 5.3.2. *Let* $\tilde{\phi}_1, \ldots, \tilde{\phi}_n$ *be the elementary* 1*-forms in* $\mathbb{R}^n$. *Let* $\pi_i \colon \mathbb{R}^n \to \mathbb{R}$ *be the projection onto the* i*th coordinate, then* $d\pi_i = \tilde{\phi}_i$ *for each* $1 \leq i \leq n$.

Proof. Since π_i is linear, $d\pi_i$ is a 1-form on $\mathbb{R}^n$ and $d\pi_i(x)(x;v) = D\pi_i(x)(v) = v_i$. Then

$$\begin{aligned}\tilde{\phi}_i(x)(x;v) &= \tilde{\phi}_i(x)\left(x; \sum_j v_j e_j\right)\\ &= \sum_j v_j \tilde{\phi}_i(x)(x;e_j) = v_i = d\pi_i(x)(x;v).\end{aligned}$$

□

Remark 5.3.3. If x is the general point of $\mathbb{R}^n$, we denote the ith projection mapping of $\mathbb{R}^n$ onto $\mathbb{R}$ by x_i. Then dx_i equals the elementary 1-form $\tilde{\phi}_i$ in $\mathbb{R}^n$. If $I = (i_1, \dots, i_k) \in \{1, \dots, n\}^k$ is ascending, then $dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_k}$, which is the elementary k-form $\tilde{\psi}_I$ in $\mathbb{R}^n$. And the general k-form will be written as

$$w = \sum_{(I)} b_I dx_I.$$

Then the 1-form df where f is of class C^∞ can be written as $df = (D_1 f)dx_1 + \cdots + (D_n f)dx_n$.

Definition 5.3.4. *Let $A \in \mathrm{OP}(\mathbb{R}^n)$, and $\Omega^k(A)$ be the set of all k-forms on A of class C^∞, then $\Omega^k(A)$ is a vector space.*

The next theorem, whose proof can be found in Munkres [23], shows that there exists a unique linear transformation with certain properties such that it sends each k-form into a $(k+1)$-form.

Theorem 5.3.5. *Let $A \in \mathrm{OP}(\mathbb{R}^n)$, then there exists a unique linear transformation*

$$d\colon \Omega^k(A) \to \Omega^{k+1}(A)$$

defined for $k \geq 0$, such that:

(1) *If f is a 0-form, then df is the 1-form*

$$df(x)(x;v) = Df(x)(v).$$

(2) *If ω and η are forms of order k and l, respectively, then*

$$d(\omega \wedge \eta) = (d\omega) \wedge \eta + (-1)^k \omega \wedge d\eta.$$

(3) *For every form ω*

$$dd\omega = 0.$$

We call d the differential operator and $d\omega$ is called the differential of ω, and from the above properties the differential of $\omega = \sum_{(I)} b_I dx_I$ equals $d\omega = \sum_{(I)} db_I \wedge dx_I$.

5.4. **The Action Of A Differentiable Map.**

Definition 5.4.1. *Let $A \in \mathrm{OP}(\mathbb{R}^k)$, $B \in \mathrm{OP}(\mathbb{R}^n)$ and let $\alpha\colon A \to \mathbb{R}^n$ be a map of class C^∞ such that $\alpha(A) \subseteq B$. The transformation $\alpha^*\colon \Omega^l(B) \to \Omega^l(A)$ is defined by*

$$(\alpha^* f)(x) = f(\alpha(x))$$

whenever f is a 0-form on B,

$$(\alpha^*\omega)(x)((x;v_1),\ldots,(x;v_l)) = \omega(\alpha(x))(\alpha_*(x;v_1),\ldots,\alpha_*(x;v_l))$$

whenever ω is an $l > 0$ form on B. Hence $\alpha^* f$ is a 0-form on A and $\alpha^*\omega$ is an l-form on A.

Remark 5.4.2. There is a relationship between α^* and α_*, namely assume that $\alpha\colon A \in \mathrm{OP}(\mathbb{R}^k) \to \mathbb{R}^n$ is a map of class C^∞ and $\alpha(x) = y$. Then α induces a linear transformation

$$T = \alpha_*\colon \mathcal{T}_x(\mathbb{R}^k) \to \mathcal{T}_y(\mathbb{R}^n);$$

this transformation induces a dual transformation of alternating tensors $T^*\colon \mathrm{A}^l(\mathcal{T}_y(\mathbb{R}^n)) \to \mathrm{A}^l(\mathcal{T}_x(\mathbb{R}^k))$ defined by

$$(T^*\beta)((x;v_1),\ldots,(x;v_l)) = \beta(T[(x;v_1),\ldots,(x;v_l)]),$$

where $\beta \in \mathrm{A}^l(\mathcal{T}_y(\mathbb{R}^n))$ and $v_1,\ldots,v_l \in \mathbb{R}^k$. Now if ω is an l-form on B, $\omega(y) \in \mathrm{A}^l(\mathcal{T}_y(\mathbb{R}^n))$, then $T^*(\omega(y)) \in \mathrm{A}^l(\mathcal{T}_x(\mathbb{R}^k))$, and

$$\begin{aligned}
&T^*(\omega(y))((x;v_1),\ldots,(x;v_l)) = \omega(y)(T(x;v_1),\ldots,T(x;v_l))\\
&= \omega(y)(\alpha_*(x;v_1),\ldots,(x;v_l))\\
&= \omega(y)((\alpha(x);D\alpha(x)(v_1)),\ldots,(\alpha(x);D\alpha(x)(v_l)))\\
&= (\alpha^*\omega)(x)((x;v_1),\ldots,(x;v_l)).
\end{aligned}$$

To compute the integral of a differential form over a manifold we will need a formula for $\alpha^*\omega$. The next theorem gives the formula when ω is an elementary 1-form or an elementary k-form. This is all we need because the transfomation α^* is linear, preserves wedge products, and $\alpha^* f$ equals $f \circ \alpha$ when f is a 0-form.

Theorem 5.4.3. *Let $A \in \mathrm{OP}(\mathbb{R}^k)$, let $\alpha\colon A \to \mathbb{R}^n$ be a map of class C^∞. Let x denote the general point of $\mathbb{R}^k$ and let y denote the general point of $\mathbb{R}^n$. Then*

(1) $\alpha^*(dy_i) = d\alpha_i$, $1 \le i \le n$

(2) $I = (i_1,\ldots,i_k) \in \{1,\ldots,n\}^k$ *is ascending*

$$\begin{aligned}
\alpha^*(dy_I) &= \alpha^*(dy_{i_1} \wedge \cdots \wedge dy_{i_k})\\
&= \left(\det \frac{\partial \alpha_I}{\partial x}\right) dx_1 \wedge \cdots \wedge dx_k
\end{aligned}$$

where

$$\frac{\partial \alpha_I}{\partial x} = \frac{\partial(\alpha_{i_1},\ldots,\alpha_{i_k})}{\partial(x_1,\ldots,x_k)}.$$

Proof.

(1) Let $\alpha\colon A \in \mathrm{OP}(\mathbb{R}^k) \to \mathbb{R}^n$ be a map of class C^∞, $\alpha(A) \subseteq B \in \mathrm{OP}(\mathbb{R}^n)$

$$\alpha^*\colon \Omega^1(B) \to \Omega^1(A).$$

We want to show that $\alpha^*(dy_i) = d\alpha_i$. Let $y = \alpha(x)$,

$$\begin{aligned}\alpha^*(dy_i)(x)(x;v) &= dy_i(\alpha(x))(\alpha(x); D\alpha(x)(v))\\ &= Dy_i(y)(D\alpha(x)(v))\\ &= D\alpha_i(x)(v)\\ &= \sum_{j=1}^{k} D_j\alpha_i(x)v_j = \sum_{j=1}^{k} \frac{\partial \alpha_i}{\partial x_j}(x)dx_j(x)(x;v)\end{aligned}$$

(2) Since $dy_I = dy_{i_1}\wedge\cdots\wedge dy_{i_k}$ is a k-form on B then $\alpha^*(dy_I)$ is a k-form on $A \subseteq \mathbb{R}^k$ and hence it has the form

$$\alpha^*(dy_I) = h\, dx_1 \wedge \cdots \wedge dx_k$$

for some scalar function h.

$$\begin{aligned}&\alpha^*(dy_I)(x)((x;v_1),\ldots,(x;v_k))\\ &= h(x)\, dx_1(x) \wedge \cdots \wedge dx_k(x)((x;v_1),\ldots,(x;v_k)).\end{aligned}$$

Since $h(x) = h(x)\, dx_1(x) \wedge \cdots dx_k(x)((x;e_1),\ldots,(x;e_k))$, then

$$\begin{aligned}h(x) &= \alpha^*(dy_I)(x)((x;e_1),\ldots,(x;e_k))\\ &= dy_I(y)(y)\left(\left(y;\frac{\partial\alpha}{\partial x_1}(x)\right),\ldots,\left(y;\frac{\alpha}{\partial x_k}(x)\right)\right)\\ &= \det[D\alpha(x)]_I = \det\frac{\partial\alpha_I}{\partial x}\\ &\Rightarrow \alpha^*(dy_I) = \left(\det\frac{\partial\alpha_I}{\partial x}\right) dx_1\wedge\cdots\wedge dx_k.\end{aligned}$$

□

Remark 5.4.4. If $\omega = \sum_{(I)} b_I dy_I$ is a k-form defined in an open subset of $\mathbb{R}^n$ containing $\alpha(A)$ where α is a map of class C^∞ then theorem (5.4.3) gives $\alpha^*(\omega)$ as

$$\begin{aligned}\alpha^*(\omega) &= \alpha^*\left(\sum_{(I)} b_I dy_I\right) = \sum_{(I)} \alpha^*(b_I \wedge dy_I)\\ &= \sum_{(I)} \alpha^* b_I \wedge \alpha^* dy_I\\ &= \sum_{(I)} (b_I\circ\alpha)\det\frac{\partial\alpha_I}{\partial x} dx_1\wedge\cdots\wedge dx_k\end{aligned}$$

Another property of the transformation α^* that we will use frequently is commutativity with the differential operator. The next theorem, whose proof can be found in Munkres [23], shows that the α^* transformation commutes with d.

Theorem 5.4.5. *Let $A \in \mathrm{OP}(\mathbb{R}^k)$ and $\alpha\colon A \to \mathbb{R}^n$ be of class C^∞. If ω is an l-form defined in an open set of $\mathbb{R}^n$ containing $\alpha(A)$, then*

$$\alpha^*(d\omega) = d(\alpha^*\omega).$$

5.5. **Integrating Forms Over Parameterized Manifolds.**

Definition 5.5.1. *Let $A \in \mathrm{OP}(\mathbb{R}^k)$, let η be a k-form defined in A, then η can be written uniquely in the form*

$$\eta = f\, dx_1 \wedge \cdots \wedge dx_k,$$

and the integral of η over A is defined by

$$\int_A \eta = \int_A f,$$

whenever the latter integral exists.

Remark 5.5.2. The definition of the integral of a k-form over an open set $A \subseteq \mathbb{R}^k$ does not depend on the choice of basis as long as it has the same orientation.

Definition 5.5.3. *Let $A \in \mathrm{OP}(\mathbb{R}^k)$ and $\alpha\colon A \to \mathbb{R}^n$ be of class C^∞, then $Y_\alpha = (Y, \alpha)$ is a parameterized manifold. If ω is a k-form defined in an open set of $\mathbb{R}^n$ containing Y, then the integral of ω over Y_α is defined by*

$$\int_{Y_\alpha} \omega = \int_A \alpha^*\omega$$

where $\alpha^*\omega$ is a k-form defined on A.

The next theorem, which can be shown using the change Of variables theorem of the manifold theory, asserts that the integral is invariant under reparametrization, up to sign.

Theorem 5.5.4. *Let $g\colon A \to B$ be a diffeomorphism of open sets in $\mathbb{R}^k$. Assume that* $\det Dg$ *does not change sign on A. Let $\beta\colon B \to \mathbb{R}^n$ be a map of class C^∞, $Y = \beta(B)$, and let $\alpha = \beta \circ g$. If ω is a k-form defined in an open set of $\mathbb{R}^n$ containing Y, then ω is integrable over Y_β iff it is integrable over Y_α and*

$$\int_{Y_\alpha} \omega = \pm \int_{Y_\beta} \omega$$

where the sign agrees with the sign on $\det Dg$.

Theorem 5.5.5. *Let $A \in \mathrm{OP}(\mathbb{R}^k)$, $\alpha\colon A \to \mathbb{R}^n$ be a map of class C^∞ and $Y = \alpha(A)$. Let x denote the general point of $\mathbb{R}^k$ and z denote the general point of $\mathbb{R}^n$. If $\omega = f\, dz_I$ is a k-form defined in an open set of $\mathbb{R}^n$ containing Y, then*

$$\int_{Y_\alpha} \omega = \int_A (f \circ \alpha) \det \frac{\partial \alpha_I}{\partial x}.$$

Proof. Applying theorem (5.4.3)

$$\alpha^*\omega = \alpha^*(f\, dz_I) = (f\circ\alpha)\left[\det\frac{\partial\alpha_I}{\partial x}\right] dx_1\wedge\cdots dx_k$$

implies that

$$\int_{Y_\alpha}\omega = \int_A \alpha^*\omega = \int_A (f\circ\alpha)\det\frac{\partial\alpha_I}{\partial x}$$

□

Remark 5.5.6. For a general k-form ω defined on an open subset of $\mathbb{R}^n$ containing Y the integral of ω over a parameterized manifold Y_α can be computed as

$$\begin{aligned}\int_{Y_\alpha}\omega = \int_A \alpha^*\omega &= \int_A \sum_{(I)}\alpha^*(f_I dz_I)\\ &= \sum_{(I)}\int_A (f_I\circ\alpha)\det\frac{\partial\alpha_I}{\partial x}.\end{aligned}$$

5.6. **Orientable Manifolds.**

Definition 5.6.1.

(1) *Let $g\colon A\to B$ be a diffeomorphism of open sets in $\mathbb{R}^k$, then g is said to be orientation preserving if* $\det Dg > 0$ *on A or orientation reversing if* $\det Dg < 0$ *on A.*

(2) *Let M be a k-manifold in $\mathbb{R}^n$. Two coordinate maps $\alpha_i\colon U_i\to V_i,\ i=0,1$ are said to overlap positively if $V_0\cap V_1\neq\emptyset$ and $\alpha_1^{-1}\circ\alpha_0$ is orientation preserving. If M can be covered by such a collection of coordinate maps, then M is called orientable, and a maximal collection of such coordinate maps is called an orientation of M.*

The next theorem, whose proof can be found in Munkres [23], and the definition following it will be used in the proof of the Generalized Stoke's Theorem.

Theorem 5.6.2. *Let $k>1$. If M is an orientable k-manifold with boundary, then ∂M is an orientable $(k-1)$-manifold.*

Definition 5.6.3. *Let M be an orientable k-manifold with boundary, then the induced orientation on ∂M is defined by:*

(1) *If k is even the orientation is obtained by restricting the coordinate maps belonging to the orientation of M.*

(2) *If k is odd it is the opposite of the orientation defined in* (1).

5.7. **Integrating Forms Over Orientable Manifolds.**

Definition 5.7.1. *Let M be a compact k-manifold in $\mathbb{R}^n$ and let ω be a k-form defined in an open set containing M. Let $C = M \cap \operatorname{spt} \omega$. If there exists a coordinate map $\alpha\colon U \to V$ with $C \subseteq V$, then the integral of ω over M is defined to be*

$$\int_M \omega = \int_{U^\circ} \alpha^* \omega$$

where U° denotes the interior of U in $\mathbb{R}^k$.

Remark 5.7.2.

- The integral defined in (5.7.1) always exists whenever the conditions are satisfied. The definition of the integral does not depend on the choice of the coordinate map.
- If $-M$ denotes the k-manifold with the opposite orientation, then
$$\int_{-M} \omega = -\int_M \omega.$$
In fact, if $\alpha\colon U \to \alpha(U)$ belongs to the given orientation and $\beta\colon V \to \beta(V)$ belongs to the opposite orientation such that $M \cap \operatorname{spt} \omega \subseteq \alpha(U) \cap \beta(V) = W$ and $W_0 = \alpha^{-1}(W)$, $W_1 = \beta^{-1}(W)$, then
$$\int_M \omega = \int_{W_0} \alpha^* \omega = -\int_{W_1} \beta^* \omega = -\int_{-M} \omega,$$
where the second equality follows from theorem (5.5.4).

Definition 5.7.3. *Let M be a compact oriented k-manifold in $\mathbb{R}^n$. Let ω be a k-form defined in an open set containing M. Fix an orientation for M. Choose a partition of unity $\{\phi_i\}_{i \in \mathbb{N}}$ on this orientation for which all but finitely many $\{\phi_1, \ldots, \phi_l\}$ of C^∞ functions do not vanish on M. Then the integral of ω over M is defined by*

$$\int_M \omega = \sum_{i=1}^{l} \int_M \phi_i \omega.$$

Remark 5.7.4.

- The definition of the integral does not depend on the choice of partition of unity.
- The integral changes sign when the orientation is reversed.

5.8. **Generalized Stokes' Theorem.**

Lemma 5.8.1. *Let $k > 1$. Let η be a $(k-1)$-form defined in an open set U of $\mathbb{R}^k$ containing $I^k = [0,1]^k$. Assume that η vanishes at all points of ∂I^k except possibly at points of $(I_{k-1})^\circ \times \{0\}$. Then*

$$\int_{(I^k)^\circ} d\eta = (-1)^k \int_{(I^{k-1})^\circ} b^* \eta$$

where $b\colon I^{k-1} \to I^k$ is defined by $(u_1, \ldots, u_{k-1}) \mapsto (u_1, \ldots, u_{k-1}, 0)$.

Proof. Let $x \in \mathbb{R}^k$ be the general point of $\mathbb{R}^k$ and $u \in \mathbb{R}^{k-1}$ be the general point of $\mathbb{R}^{k-1}$. For each $1 \leq j \leq k$ define $I_j = (1, \dots, \hat{j}, \dots, k)$, then a typical elementary $(k-1)$-form in $\mathbb{R}^k$ has the form $dx_{I_j} = dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_k$. Since d, b^*, and the integral are linear we can assume that $\eta = f\, dx_{I_j}$.

(1) Calculate $\int\limits_{(I^k)^\circ} d\eta$.

$$d\eta = d(f \wedge dx_{I_j}) = df \wedge dx_{I_j} = \left(\sum_{i=1}^{k} D_i f dx_i\right) \wedge dx_{I_j}$$
$$= (-1)^{j-1} D_j f\, dx_1 \wedge \cdots \wedge dx_k$$

then

$$\int_{(I^k)^\circ} d\eta = (-1)^{j-1} \int_{(I^k)^\circ} D_j f$$

since $D_j f$ is continuous, bounded and vanishes a.e on ∂I^k

$$= (-1)^{j-1} \int_{I^k} D_j f$$

for every $v \in (x_1, \dots, \widehat{x_j}, \dots, x_k)$ the Fubini's theorem implies that

$$= (-1)^{j-1} \int_{v \in I^{k-1}} \int_{x_j \in I} D_j f(x_1, \dots, x_k).$$

Using the FTC we can calculate the inner integral as

$$\int_{x_j \in I} D_j f(x_1, \dots, x_k) = f(x_1, \dots, 1, \dots, x_k) - f(x_1, \dots, 0, \dots, x_k)$$

where 1 and 0 appear in the jth place.
If $j < k$, then

$$f(x_1, \dots, 0, \dots, x_k) = 0 = f(x_1, \dots, 1, \dots, x_k).$$

If $j = k$, then

$$f(x_1, \dots, x_{k-1}, 1) - f(x_1, \dots, x_{k-1}, 0) = -f(x_1, \dots, x_{k-1}, 0).$$

Consequently,

$$\int_{(I^k)^\circ} d\eta = \begin{cases} 0 & \text{if } j < k \\ (-1)^k \int_{I^{k-1}} (f \circ b) & \text{if } j = k. \end{cases}$$

(2) Calculate $\int\limits_{(I^{k-1})^\circ} b^*\eta$.

Since $b\colon \mathbb{R}^{k-1} \to \mathbb{R}^k$ is defined by $(u_1, \dots, u_{k-1}) \mapsto (u_1, \dots, u_{k_1}, 0)$

$$[Db(u)] = \begin{bmatrix} I_{k-1} \\ 0 \end{bmatrix}$$

By theorem (5.4.3) we can compute

$$b^*(dx_{I_j}) = b^*(dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_k)$$
$$= \det \frac{\partial b_{I_j}}{\partial x}\, du_1 \wedge \cdots \wedge du_{k-1}$$

$$= \begin{cases} 0 & \text{if } j < k \\ du_1 \wedge \cdots \wedge du_{k-1} & \text{if j = k} \end{cases}$$

and

$$\int_{(I^{k-1})^\circ} b^*\eta = \begin{cases} 0 & \text{if } j < k \\ \int_{(I^{k-1})^\circ} (f \circ b) & \text{if } j = k \end{cases}$$

(1) and (2) imply that

$$\int_{(I^k)^\circ} d\eta = (-1)^k \int_{(I^{k-1})^\circ} b^*\eta.$$

□

Theorem 5.8.2 (Stokes' Theorem). *Let $k > 1$. Let M be a compact oriented k-manifold in $\mathbb{R}^n$, give ∂M the induced orientation if $\partial M \neq \emptyset$. Let ω be a $(k-1)$-form defined in an open set containing M, then*

$$\int_M d\omega = \int_{\partial M} \omega \textit{ if } \partial M \neq \emptyset$$
$$\int_M d\omega = 0 \qquad \textit{otherwise.}$$

Proof. We will cover M by special coordinate maps so that we can apply the lemma (5.8.1). Let $p \in M - \partial M$. Choose a coordinate map $\alpha \colon U \to V$ belonging to the orientation of M such that $U \in \mathrm{OP}(\mathbb{R}^k)$, $I^k \subseteq U$, and α carries a point of $(I^k)^\circ$ to the point p. Let $W = (I^k)^\circ$, $Y = \alpha(W)$, then $\alpha \colon W \to Y$ is a coordinate map about p belonging to the orientation of M.
Let $p \in \partial M$. Choose a coordinate map $\alpha \colon U \to V$ belonging to the orientation of M such that $U \in \mathrm{OP}(\mathbb{H}^k)$, $I^k \subseteq U$, and α carries a point of $(I^{k-1})^\circ \times \{0\}$ to the point p. Let $W = (I^k)^\circ \cup ((I^{k-1})^\circ \times \{0\})$, $Y = \alpha(W)$, then $\alpha \colon W \to Y$ is a coordinate map about p belonging to the orientation of M.
Since the operator d and the integrals involved are linear and from the definition (5.7.3), it suffices to prove the theorem for the special case where ω is a $(k-1)$-form such that $C = M \cap \operatorname{spt} \omega$ can be covered by a single one of the coordinate maps. Since the support of $d\omega$ is contained in the support of ω, the set $M \cap \operatorname{spt} d\omega$ is contained in C, so it is covered by a single coordinate map.
The form $\eta = \alpha^*\omega$ can be extended, if necessary, to a $(k-1)$-form on an open subset of $\mathbb{R}^k$ containing I^k. Then η vanishes at all points of ∂I^k, except possibly at points of $(I^{k-1})^\circ \times \{0\}$, and the hypotheses of the lemma (5.8.1) are satisfied.
Assume that $p \in M - \partial M$. Then there exists a coordinate map $\alpha \colon W \to Y$ where $W = (I^k)^\circ$ and $\alpha(W) = Y$. Since $\alpha^* d\omega = d\alpha^*\omega = d\eta$,

$$\int_M d\omega = \int_{(I^k)^\circ} \alpha^* d\omega = \int_{(I^k)^\circ} d\eta = (-1)^k \int_{(I^{k-1})^\circ} b^*\eta = 0,$$

since η vanishes outside $(I^k)^\circ$. And since the support of ω is disjoint from ∂M,

$$\int_{\partial M} \omega = 0.$$

Assume that $p \in \partial M$. Then there exists a coordinate map $\alpha\colon W \to Y$, where $W = (I^k)^\circ \cup ((I^{k-1})^\circ \times \{0\})$, $Y \cap \partial M \neq \emptyset$, and $W^\circ = (I^k)^\circ$. Thus we have

$$\int_M d\omega = \int_{(I^k)^\circ} \alpha^* d\omega = (-1)^k \int_{(I^{k-1})^\circ} b^*\eta.$$

Since $M \cap \operatorname{spt}\omega$ is covered by $\alpha\colon W \to Y$, $\partial M \cap \operatorname{spt}\omega$ is covered by the coordinate map $\beta = \alpha \circ b\colon (I^{k-1})^\circ \to Y \cap \partial M$ obtained by restricting α. β belongs to the induced orientation of ∂M if k is even, or it belongs to the opposite orientation if k is odd. So we must multiply the integral of ω over ∂M by $(-1)^k$ when we use the coordinate map β. Thus we have

$$\int_{\partial M} \omega = (-1)^k \int_{(I^{k-1})^\circ} \beta^*\omega.$$

Since $\beta^*\omega = b^*(\alpha^*\omega) = b^*\eta$. $\square$

6. CURRENTS, CLOSURE AND COMPACTNESS THEOREMS, AREA MINIMIZING SURFACES, REGULARITY

In this section we will introduce the currents which are the duals of smooth m-forms with compact support. Since every oriented k-manifold in $\mathbb{R}^n$ defines by integration a linear functional on forms, currents can be seen as generalized surfaces. Being a generalized surface, a current admits a definition of boundary and by Stoke's theorem the boundary of a current coincides with the boundary of a manifold. We will define the slice of a current which generalizes the level sets and present some theorems from the slicing theory. We will then move to the deformation theorem, which asserts, when restricted to C^1 manifolds with compact support, that these manifolds can be approximated by simplices. The closure and compactness theorems will be introduced next and will be used to show that a "nice" boundary B in $\mathbb{R}^n$ admits an area minimizing surface with boundary B. We will list some theorems asserting, in certain cases, that the solution of area minimization is a smooth embedded manifold except perhaps, for a set of small Hausdorff dimension. We will generalize our definition of currents to admit surfaces that are not compact; these currents locally ressemble the previously defined currents with compact support. In the last subsection we will prove, as a subcase of a regularity result of a hypersurface in $\mathbb{R}^2$, that the line segment is the shortest path connecting two points in the plane. The main references for this section are the books by Federer [10], Gelfand & Fomin [15], and Morgan [22].

6.1. **Currents.**

Definition 6.1.1.

(1) *Let the ambient space be $\mathbb{R}^n$ and define*
$$\mathcal{D}^m = \{C^\infty \text{ differential } m\text{-forms with compact support}\}.$$

(2) *The dual space $(\mathcal{D}^m)^*$ of $\mathcal{D}^m$ is denoted $\mathcal{D}_m$ and it is called the space of m-dimensional currents.*

(3) *The weak topology on $\mathcal{D}_m$ is generated by the basis elements of the form*
$$N(T_0, B, \epsilon) = \{T \in \mathcal{D}_m \colon |T(\varphi) - T_0(\varphi)| < \epsilon, \text{ for all } \varphi \in B\}$$
where $B \subseteq \mathcal{D}_m$ is finite and $\epsilon > 0$.

(4) *A sequence $\{T_j\}_{j\in\mathbb{N}}$ in $\mathcal{D}_m$ is said to converge to T under the weak topology if for every $\varphi \in \mathcal{D}^m$, $T_j(\varphi) \to T(\varphi)$.*

Remark 6.1.2. Any oriented m-dimensional rectifiable set S may be viewed as a current. Namely, let $\vec{S}(x)$ denote the unit m-vector associated with the tangent plane to S at x. For any differential m-form in $\mathcal{D}^m$, define
$$S(\varphi) = \int_S \langle \vec{S}(x), \varphi(x)\rangle \, d\mathcal{H}^m x.$$
If a Lipschitz map $\alpha\colon A \subseteq \mathbb{R}^m \to S$ covers S, $\mathcal{H}^m$-a.e. then
$$= \int_A (\alpha^*\varphi)(x)(\vec{S}(x))\, d\mathcal{L}^m x.$$

We will also require S to carry a positive integer multiplicity function $\mu(x)$, with $\int_S \mu(x)\, d\mathcal{H}^m x < \infty$, and define

$$S(\varphi) = \int_S \langle \vec{S}(x), \varphi(x) \rangle \mu(x)\, d\mathcal{H}^m x.$$

Finally, we require that S have compact support. Such currents are called rectifiable currents.

Definition 6.1.3.

(1) *The support of a current S is the smallest closed set C such that for each $\varphi \in \mathcal{D}^m$*

$$(\operatorname{spt} \varphi) \cap C = \emptyset \quad \Rightarrow \quad S(\varphi) = 0.$$

(2) *The boundary of an m-dimensional current $T \in \mathcal{D}_m$ is the $(m-1)$-dimensional current $\partial T \in \mathcal{D}_{m-1}$ defined by*

$$\partial T(\varphi) = T(d\varphi)$$

where $\varphi \in \mathcal{D}^{m-1}$.

Remark 6.1.4.

- By Stokes' theorem the definition of the boundary of a current generalizes the notion of boundary in the manifold theory. Namely, let T be a smooth compact oriented Lipschitz manifold with boundary, then T and ∂T are rectifiable currents and from remark (6.1.2)

$$\partial T(\varphi) = \int_{\partial T} \varphi = \int_T d\varphi = T(d\varphi),$$

  and hence the boundary of the current T is the boundary of the manifold T.

- Generally the boundary of a rectifiable current need not be a rectifiable current. If it happens to be, then the original current is called an integral current.

Definition 6.1.5. *Now we give the list of important spaces of currents used in geometric measure theory.*

$$\begin{aligned}
\mathcal{D}_m &= \{m\text{-}\textit{dimensional currents in } \mathbb{R}^n\},\\
\mathcal{E}_m &= \{T \in \mathcal{D}_m \colon \operatorname{spt} T \textit{ is compact}\},\\
\mathcal{R}_m &= \{\textit{rectifiable currents}\}\\
&= \{T \in \mathcal{E}_m \colon T \textit{ is associated with oriented rectifiable sets,}\\
&\qquad \textit{with integer multiplicities and with finite total measure}\},\\
\mathcal{P}_m &= \{\textit{integral polyhedral chains}\}\\
&= \textit{additive subgroup of } \mathcal{E}_m \textit{ generated by classically oriented simplices,}\\
\mathbf{I}_m &= \{\textit{integral currents}\}\\
&= \{T \in \mathcal{R}_m \colon \partial T \in \mathcal{R}_{m-1}\},\\
\mathcal{F}_m &= \{\textit{integral flat chains}\}\\
&= \{T + \partial S \colon T \in \mathcal{R}_m,\ S \in \mathcal{R}_{m+1}\}.
\end{aligned}$$

Definition 6.1.6. *The two important seminorms on the space of currents $\mathcal{D}_m$ are the mass norm* $\mathbf{M}$ *and the flat norm* $\mathbf{F}$.

$$\mathbf{M}(T) = \sup\left\{T(\varphi)\colon \sup_x \|\varphi(x)\|^* \leq 1\right\},$$

where $\|\varphi(x)\|^* = \sup\{\,|\varphi(x)(\xi)|\colon \xi$ *is a unit* m*-vector*$\}$

$$\mathbf{F}(T) = \inf\{\mathbf{M}(A) + \mathbf{M}(B)\colon T = A + \partial B,\ A \in \mathcal{R}_m,\ B \in \mathcal{R}_{m+1}\}.$$

Theorem 6.1.7. *Let* $S \in \mathcal{R}_m$*, then* $\mathbf{M}(S) = \int_S \mu(x)\, d\mathcal{H}^m x$.

Remark 6.1.8. If the multiplicty function equals 1 $\mathcal{H}^m$-a.e. then the mass of S is equal to the Hausdorff measure of the associated rectifiable set S.

Proof. S is an m-dimensional rectifiable current with an associated rectifiable set S such that for every $\varphi \in \mathcal{D}^m$

$$S(\varphi) = \int_S \varphi(x)(\vec{S}(x))\mu(x)\, d\mathcal{H}^m x,$$

where $\vec{S}(x)$ denotes the unit m-vector associated with the tangent plane to S at x, and $\mu(x) \in \mathbb{N}$, $\int_S \mu(x)\, d\mathcal{H}^m x < \infty$.

$$\mathbf{M}(T) \leq \int_S \mu(x)\, d\mathcal{H}^m x$$

because

$$\begin{aligned} S(\varphi) &\leq \int_S |\varphi(x)(\vec{S}(x))|\mu(x)\, d\mathcal{H}^m x \\ &\leq \int_S \mu(x)\, d\mathcal{H}^m x \text{ for every } \varphi \in \mathcal{D}^m. \end{aligned}$$

On the other hand, if for every x in its domain $\varphi(x)(\xi) = 1$ for every unit m-vector, then $\|\varphi(x)\|^* = 1$ and hence

$$\int_S \mu(x)\, d\mathcal{H}^m x \leq \mathbf{M}(T).$$

□

Remark 6.1.9.

- The flat norm gives a good indication of when two surfaces are geometrically close together in the ambient space. In particular, let $\mathbb{R}^3$ be the ambient space and let D_1 and D_2 be two unit disks such that D_2 lies on the xy-plane with center at the origin and D_1 is at arbitrarily small distance form D_2 in the positive z-direction. Then $T = D_1 - D_2$ is a two-manifold in $\mathbb{R}^3$ with mass

$$\mathbf{M}(T) = \int_T d\mathcal{H}^2 = \mathcal{H}^2(T) = 2\pi.$$

  On the other hand $T = \partial S$ where S is the right cylinder whose base is D_2 and whose apex is D_1 and hence $\mathbf{F}(T) \leq \mathbf{M}(S) = V(S)$.

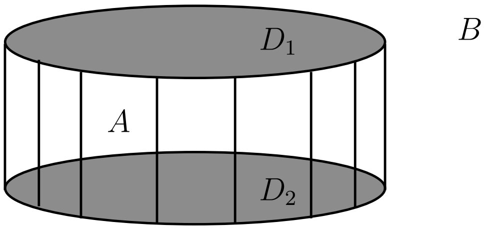


FIGURE 14. The unit discs D_1, D_2 are close together in the flat norm $\mathbf{F}$ because $T = D_1 - D_2$, together with a thin band A, is the boundary of a cylinder B of small mass (volume).

- As a result of closure theorem (6.4.3) we will see that

$$\mathbf{I}_m = \{T \in \mathcal{R}_m \colon \mathbf{M}(\partial\partial T) < \infty\}$$
$$\mathcal{R}_m = \{T \in \mathcal{F}_m \colon \mathbf{M}(\partial T) < \infty\}.$$

Thus only by having infinite boundary mass can a rectifiable current fail to be an integral current.

- To obtain a rectifiable current which is not an integral current, choose an underlying rectifiable set with infinite boundary. For example,

$$E = \bigcup_{k=1}^{\infty} \{(x, y, z) \colon x^2 + y^2 \le k^{-2},\ z = k^{-1}\}$$

has finite total area but infinite boundary length. In this example E is rectifiable, ∂E is not rectifiable but it is an integral flat chain.

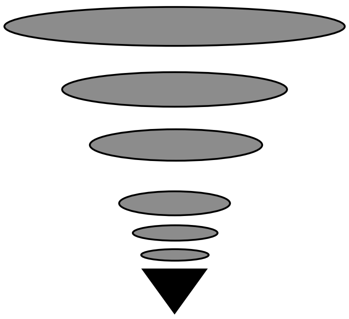


FIGURE 15. This infinite collection of discs is a rectifiable current which is not an integral current. The total area is finite, but the total perimeter is infinite

Theorem 6.1.10. *The flat norm topology on $\mathcal{F}_m(\mathbb{R}^n)$ is stronger than the weak topology on $\mathcal{F}_m(\mathbb{R}^n)$.*

Proof. Note that the seminorms $\mathbf{M}$ and $\mathbf{F}$, when restricted $\mathcal{F}_m(\mathbb{R}^n)$, become norms. Our objective is to show that the weak topology on $\mathcal{F}_m$ is a subset of the flat norm topology on $\mathcal{F}_m$.

Let $T_0 \in \mathcal{F}_m$ and let $N(T_0, B, \epsilon)$ be an arbitrary basis element of the weak topology, where $B = \{\varphi_1, \ldots, \varphi_l\} \subseteq \mathcal{D}^m$ and $\epsilon > 0$. We want to show that $N(T_0, B, \epsilon)$ is open in the flat norm topology. Let $T \in N(T_0, B, \epsilon)$, we need to show that there exists $\delta > 0$ such that $B_{\mathbf{F}}(T, \delta) \subseteq N(T_0, B, \epsilon)$. Define K and ξ respectively as

$$K = \max_{1 \le i \le l} \{\|\varphi_i\|^*, \|d\varphi\|^*\} \text{ and}$$
$$\xi = \max_{1 \le i \le l} |(T - T_0)(\varphi_i)|.$$

If we let $\delta = (\epsilon - \xi)/K$, then $\tilde{T} \in B_{\mathbf{F}}(T, \delta)$ implies that there exist $A \in \mathcal{R}_m$ and $B \in \mathcal{R}_{m+1}$ such that

$$T - \tilde{T} = A + \partial B \text{ and}$$
$$\mathbf{M}(A) + \mathbf{M}(B) < (\epsilon - \xi)/K.$$

Now for each $1 \le i \le l$

$$\begin{aligned} |(\tilde{T} - T_0)(\varphi_i)| &\le |(T - \tilde{T})(\varphi_i)| + |(T - T_0)(\varphi_i)| \\ &\le |A(\varphi_i)| + |\partial B(\varphi_i)| + \xi \\ &\le K[\mathbf{M}(A) + \mathbf{M}(B)] + \xi \\ &< \epsilon. \end{aligned}$$

Thus $\tilde{T} \in N(T_0, B, \epsilon)$ and $B_{\mathbf{F}}(T, \delta) \subseteq N(T_0, B, \epsilon)$. □

Definition 6.1.11. *We want to define the image of a compactly supported current under a C^∞ map. Let $\alpha \colon \mathbb{R}^n \to \mathbb{R}^v$ be a C^∞ map. Let T be a compactly supported current in $\mathcal{D}_m(\mathbb{R}^n)$, then we define $\alpha^* T \in \mathcal{D}_m(\mathbb{R}^v)$ by*

$$(\alpha^* T)(\varphi) = T(\alpha^* \varphi) \quad \varphi \in \mathcal{D}^m(\mathbb{R}^v),$$

where for each $x \in \mathbb{R}^n$ such that $\alpha(x)$ is in the domain of φ

$$\begin{aligned} &(\alpha^*\varphi)(x)((x; u_1), \ldots, (x; u_m)) \\ &= \varphi(\alpha(x))((\alpha(x); D\alpha(x)(u_1)), \ldots, (\alpha(x); D\alpha(x)(u_m))). \end{aligned}$$

Definition 6.1.12. *A current $T \in \mathcal{D}_m$ is said to be representable by integration if there are a Borel regular measure $\|T\|$ on $\mathbb{R}^n$, finite on all compact subsets, and a function $\vec{T} \colon \mathbb{R}^n \to (\mathcal{T}(\mathbb{R}^n))^m$ with $\|\vec{T}(x)\| = 1$ for $\|T\|$-a.e x such that*

$$T(\varphi) = \int_{\mathbb{R}^n} \langle \vec{T}(x), \varphi(x) \rangle \, d\|T\|x.$$

The mass $\mathbf{M}(T)$ is equal to $\|T\|(\mathbb{R}^n)$. We denote such current as $T = \|T\| \wedge \vec{T}$.

Remark 6.1.13. Every rectifiable current is representable by integration. If E is the associated set with multiplicity function l, then $d\|S\| = l\, d(\mathcal{H}^m \llcorner E)$ is a Radon measure and $\vec{S}$ is the unit m-vectorfield orienting E.

$$S = l(\mathcal{H}^m \llcorner E) \wedge \vec{S} = \mathcal{H}^m \llcorner E \wedge \eta$$

where $\eta = l\vec{S}$. The mass is equal to

$$\mathbf{M}(S) = \|S\|(\mathbb{R}^n) = \int_{\mathbb{R}^n} l\, d(\mathcal{H}^m \llcorner E) = \int_E l\, d\mathcal{H}^m,$$

which coincides with our previous observation in theorem (6.1.7).

The following theorem, whose proof can be found in Federer [10], gives two equivalent definitions of a rectifiable current.

Theorem 6.1.14. *The following are equivalent definitions for $T \in \mathcal{D}_m$ to be a rectifiable current.*

(1) *Given $\epsilon > 0$, there are an integral polyhedral chain $P \in \mathcal{P}_m(\mathbb{R}^v)$ and a Lipschitz function $f\colon \mathbb{R}^v \to \mathbb{R}^n$ such that $\mathbf{M}(T - f^*P) < \epsilon$.*

(2) *There are a rectifiable set B and an $\mathcal{H}^m \llcorner B$-integrable m-vectorfield η such that, $|\eta(x)|$ is an integer, $\operatorname{Tan}^m(B, x)$ is associated with $\eta(x)$, and*

$$T(\varphi) = \int_B \langle \eta(x), \varphi(x) \rangle \, d\mathcal{H}^m x.$$

Remark 6.1.15. The one of the equivalent definitions (6.1.14.1) of a rectifiable current asserts that $\mathcal{P}_m$ is $\mathbf{M}$-dense in $\mathcal{R}_m$.

Definition 6.1.16.

(1) *Now we will present a more general flat norm that is defined for all currents $T \in \mathcal{D}_m$.*

$$\begin{aligned}\mathsf{F}(T) &= \sup\{T(\varphi)\colon \varphi \in \mathcal{D}^m,\ \|\varphi(x)\|^* \leq 1,\ \textit{and } \|d\varphi(x)\|^* \leq 1 \textit{ for all } x\} \\ &= \min\{\mathbf{M}(A) + \mathbf{M}(B)\colon T = A + \partial B,\ A \in \mathcal{E}_m,\ \mathcal{E}_{m+1}\}\end{aligned}$$

(2) *Now continuing the definitions of the spaces of currents from (6.1.5), let*

$$\begin{aligned}\mathbf{N}_m &= \{\textit{normal currents}\} \\ &= \{T \in \mathcal{E}_m\colon \mathbf{M}(T) + \mathbf{M}(\partial T) < \infty\} \\ \mathbf{F}_m &= (\overline{\mathbf{N}}_m)^{\mathsf{F}}\ \mathsf{F}\textit{-closure of } \mathbf{N}_m \textit{ in } \mathcal{E}_m, \\ \mathbf{R}_m &= \{T \in \mathbf{F}_m\colon \mathbf{M}(T) < \infty\}, \\ \mathbf{P}_m &= \{\textit{real linear combination of elements of } \mathcal{P}_m\}.\end{aligned}$$

Example 6.1.17. Let $A = \{(x, y)\colon 0 \leq x \leq 1,\ 0 \leq y \leq 1\}$ be the unit square in the plane, then $S_1 = \sqrt{2}(\mathcal{H}^2 \llcorner A) \wedge x_{12}$, where $x_{12} = x_1 \wedge x_2$, is a two dimensional normal current which is not an integral current.

(1) S_1 is not an integral current because the multiplicity function is not integer-valued.

(2) In order to show that S_1 is a normal current we need to show that $\mathbf{M}(S_1) + \mathbf{M}(\partial S_1) < \infty$. By definition S_1 is representable by integration, namely

$$\begin{gathered} S_1(\varphi) = \int_{\mathbb{R}^2} \sqrt{2} \langle x_{12}, \varphi(x) \rangle \, d(\mathcal{H}^2 \llcorner A)x \\ \langle x_{12}, \varphi(x) \rangle = \varphi(x)(x_1 \wedge x_2) = b_{12}(x), \end{gathered}$$

where $b_{12}(x)$ is the component function in the representation of $\varphi(x)$. Then $S_1(\varphi) = \sqrt{2} \int_A b_{12}(x) \, d\mathcal{H}^2 x$ and hence $\mathbf{M}(S_1) = \sqrt{2}$.

∂S_1 is a 1-dimensional current in $\mathbb{R}^2$ with compact support. Let φ be a 1-form in $\mathbb{R}^2$ which can be written as $\varphi = b_1 dx_1 + b_2 dx_2$.

$$\partial S_1(\varphi) = S_1(d\varphi) = S_1\left(\left[\frac{\partial b_2}{\partial x_1} - \frac{\partial b_1}{\partial x_2}\right] dx_1 \wedge dx_2\right)$$

$$
\begin{aligned}
&= \sqrt{2}\int_A \frac{\partial b_2}{\partial x_1}(x,y) - \frac{\partial b_1}{\partial x_2}(x,y)\, d\mathcal{H}^2(x,y) \\
&= \sqrt{2}\int_0^1\int_0^1 \frac{\partial b_2}{\partial x_1}(x,y)\, dxdy - \sqrt{2}\int_0^1\int_0^1 \frac{\partial b_1}{\partial x_2}(x,y) dydx \\
&= \sqrt{2}\left\{\int_0^1 b_1(x,0)\, dx + \int_0^1 b_2(1,y)\, dy\right\} - \sqrt{2}\int_0^1 b_1(x,1)\, dx \\
&\qquad - \sqrt{2}\int_0^1 b_2(0,y)\, dy,
\end{aligned}
$$

then $\mathbf{M}(\partial S_1) = 4\sqrt{2}$, and hence S_1 is a normal current.

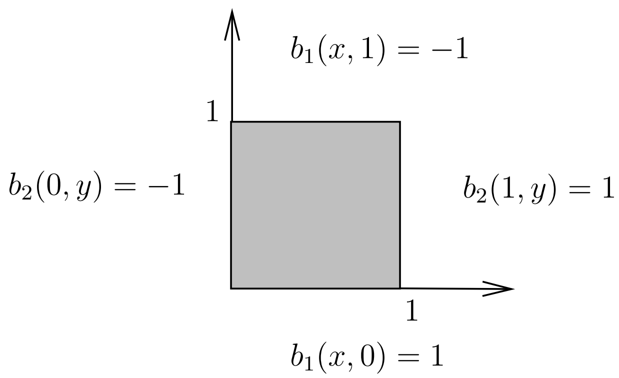


FIGURE 16. We choose the functions b_1 and b_2 accordingly to show that the mass of ∂S_1 is finite.

Example 6.1.18. $S_2 = (\mathcal{H}^2 \llcorner A) \wedge x_1$ is a 1-dimensional current in $\mathbb{R}^2$ with compact support $A = [0,1]^2$, then S_2 is not an integral current because the vectorfield is 1-dimensional but it is a normal current. The mass of S_2 equals

$$
\mathbf{M}(S_2) = \int_{\mathbb{R}^2} d(\mathcal{H}^2 \llcorner A)x = \mathcal{H}^2(A) = 1.
$$

∂S_2 is a 0-dimensional current in $\mathbb{R}^2$,

$$
\begin{aligned}
\partial S_2(f(x_1,x_2)) &= S_2(df(x_1,x_2)) = S_2\left(\frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2\right) \\
&= \int_A \left\langle x_1, \frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2\right\rangle\, d\mathcal{H}^2(x_1,x_2) \\
&= \int_A \frac{\partial f}{\partial x_1}(x_1,x_2)\, d\mathcal{H}^2(x_1,x_2) = \int_0^1\int_0^1 \frac{\partial f}{\partial x_1}(x_1,x_2)dx_1dx_2 \\
&= \int_0^1 f(1,x_2)\, dx_2 - \int_0^1 f(0,x_2)\, dx_2,
\end{aligned}
$$

then $\mathbf{M}(\partial S_2) = 2$, and hence S_2 is a normal current.

Remark 6.1.19. Assume that S_1 and S_2, A are defined as in (6.1.17) and (6.1.18), respectively.

- If $B = \{(x,0)\colon 0 \le x \le 1\}$, $T = \mathcal{H}^1 \llcorner B \wedge x_1$ and let $\tau_{(x,y)}$ denote translation by (x,y), then

$$\begin{aligned} S_2(\varphi) = S_2(b_1 dx_1 + b_2 dx_2) &= \int_A b_1(x_1,x_2)\, d\mathcal{H}^2(x_1,x_2) \\ &= \int_0^1 \int_0^1 b_1(x_1,x_2)\, dx_1 dx_2. \end{aligned}$$

The inner integral equals

$$\int_0^1 b_1(x_1,x_2)\, dx_1 = (\tau^*_{(0,y)}T)(\varphi).$$

In particular,

$$\begin{aligned} (\tau^*_{(0,y)}T)(\varphi) = T(\tau^*_{(0,y)}\varphi) &= (\mathcal{H}^1 \llcorner B) \wedge x_1(\tau^*_{(0,y)}\varphi) \\ &= \int_B \langle x_1, \tau^*_{(0,y)}\varphi\rangle\, d\mathcal{H}^1 x \end{aligned}$$

since $(\tau^*_{(0,y)}\varphi)(x,0) = b_1(x_1,x_2)$

$$\Rightarrow \int_B \langle x_1, \tau^*_{(0,y)}\varphi\rangle\, d\mathcal{H}^1 x = \int_0^1 b_1(x_1,x_2)\, dx_1.$$

So S_2 is an integral of integral currents $\tau^*_{(0,y)}T$.

- More generally, if T is an m-dimensional integral current in $\mathbb{R}^n$ and if f is a function of compact support with $\int |f|\, d\mathcal{L}^n < \infty$, then the weighted smoothing of T

$$S = \int_{\mathbb{R}^n} f(x)(\tau_x^* T)\, d\mathcal{L}^n x$$

is a normal current, and

$$\partial S = \int_{\mathbb{R}^n} f(x)(\tau_x^* \partial T)\, d\mathcal{L}^n x.$$

Proposition 6.1.20. *The space $\mathbf{R}_m$ is the $\mathbf{M}$-closure of $\mathbf{N}_m$ in $\mathcal{E}_m$.*

Proof. From the definitions of the spaces of currents we can see that $\mathbf{N}_m \subseteq \mathbf{F}_m$, $\mathbf{N}_m \subseteq \mathbf{R}_m$ and $\mathbf{R}_m = \overline{\mathbf{R}}_m^{\mathbf{M}}$. Let $T \in \mathbf{R}_m$ and $\epsilon > 0$. Choose $S \in \mathbf{N}_m$ such that $F(T-S) < \epsilon$. So there are currents $A \in \mathcal{E}_m$ and $B \in \mathcal{E}_{m+1}$ such that $T-S = A+\partial B$ and $\mathbf{M}(A)+\mathbf{M}(B) < \epsilon$. Since $\partial B = T-S-A$ and $\mathbf{M}(\partial B) = \mathbf{M}(T-S-A) < \infty$, $\partial B \in \mathbf{N}_m$. Hence $S+\partial B \in \mathbf{N}_m$, and $\mathbf{M}(T-(S+\partial B)) = \mathbf{M}(A) < \epsilon$, and since $\epsilon > 0$ is arbitrary, T is in the $\mathbf{M}$-closure of $\mathbf{N}_m$. ∎

Theorem 6.1.21. *If $T \in \mathbf{F}_m(\mathbb{R}^n)$ and $\mathcal{I}^m(\operatorname{spt} T) = 0$, then $T = 0$.*

Remark 6.1.22. $\mathcal{I}^m(\operatorname{spt} T) = 0$ means that the m-dimensional Hausdorff measure of the projection of $\operatorname{spt} T$ onto m-dimensional planes is zero for almost every m-plane. Then theorem (6.1.21) asserts that if a current $T \in \mathbf{F}_m(\mathbb{R}^n)$, when projected onto m-planes, has m-dimensional Hausdorff measure zero for almost every projection, then this current must be the zero current.

Outline Of Proof.

(1) A smooth normal current in $\mathbb{R}^n$ is one of the form $\mathcal{L}^n \wedge \xi$, where ξ is a smooth m-vectorfield of compact support. Any normal current T can be approximated in the flat norm by a smooth normal current $T_\epsilon = \mathcal{L}^n \wedge \xi$, by letting f to be a smooth approximation to the delta function at 0 and $T_\epsilon = \int_{\mathbb{R}^n} f(x)(\tau_x^* T)\, d\mathcal{L}^n x$.

(2) If $T \in \mathbf{F}_n(\mathbb{R}^n)$, then T can be written of the form $\mathcal{L}^n \wedge \xi$ for some n-vectorfield ξ.
Since codimension is zero, the norms $\mathbf{M}$ and F coincide. Therefore, T can be $\mathbf{M}$-approximated by a normal current, by definition, and hence by smoothing by $\mathcal{L}^n \wedge \xi_1$, where ξ_1 is a smooth n-vectorfield of compact support, $\mathbf{M}(T - \mathcal{L}^n \wedge \xi_1) < 2^{-1}$. Given a differential form $\varphi = b\, dx_1 \wedge \cdots \wedge dx_n \in \mathcal{D}^n$, $\|\varphi\|^* \leq 1$ and by using the definitions (3.1.6) and (5.2.2) we can calculate

$$\begin{aligned}
(\mathcal{L}^n \wedge \xi_1)(\varphi) &= \int_{\mathbb{R}^n} \langle \xi_1(x), \varphi(x) \rangle \, d\mathcal{L}^n x \\
&\leq \int_{\mathbb{R}^n} |b(x)|\, |\det[\xi_1(x)]|\, d\mathcal{L}^n x \\
&\leq \int_{\mathbb{R}^n} |\xi_1(x)|\, d\mathcal{L}^n x
\end{aligned}$$

and if we let $b = 1$ be constant, then

$$\int_{\mathbb{R}^n} |\xi_1(x)|\, d\mathcal{L}^n x \leq \mathbf{M}(\mathcal{L}^n \wedge \xi_1).$$

Consequently,

$$\begin{aligned}
\mathbf{M}(\mathcal{L}^n \wedge \xi_1) &= \int_{\mathbb{R}^n} |\xi_1(x)|\, d\mathcal{L}^n x \\
&< \mathbf{M}(T) + 2^{-1}.
\end{aligned}$$

Since $T - (\mathcal{L}^n \wedge \xi_1) \in \mathbf{F}_n(\mathbb{R}^n)$, it can be $\mathbf{M}$-approximated by $\mathcal{L}^n \wedge \xi_2$, with $\mathbf{M}(T - (\mathcal{L}^n \wedge \xi_1 + \mathcal{L}^n \wedge \xi_2)) < 2^{-2}$, and

$$\mathbf{M}(\mathcal{L}^n \wedge \xi_2) = \int_{\mathbb{R}^n} |\xi_2|\, d\mathcal{L}^n < 2^{-1} + 2^{-2}.$$

So by induction, for every $m \geq 2$ we can choose a smooth normal current $\mathcal{L}^n \wedge \xi_m$ with ξ_m being a smooth n-vectorfield with compact support such that

$$\mathbf{M}(\mathcal{L}^n \wedge \xi_m) = \int_{\mathbb{R}^n} |\xi_m|\, d\mathcal{L}^n < \sum_{k=1}^{m} \left(\frac{1}{2}\right)^k$$

and hence

$$\int_{\mathbb{R}^n} \sum_j |\xi_j|\, d\mathcal{L}^n \leq \mathbf{M}(T) + \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k = \mathbf{M}(T) + 2 < \infty.$$

By DCT, $\sum_j \xi_j$ converges under the L^1-metric, we can let $\xi = \sum_j \xi_j$, $\mathcal{L}^n \wedge \xi = \sum_j \mathcal{L}^n \wedge \xi_j$ and hence $T = \mathcal{L}^n \wedge \xi$.

(3) If $m = n$ then $T = \mathcal{L}^n \wedge \xi$ and $\mathcal{L}^n(\operatorname{spt} T) = 0$ implies that $T = 0$. Let $m < n$. Since $\mathcal{I}^m(\operatorname{spt} T) = 0$, $\operatorname{spt} T$ projects to sets of measure 0 in the m-dimensional coordinate axis planes. For notational convenience let's assume that $m = 1$, then $T \in \mathbf{F}_1(\mathbb{R}^n)$ and let φ be a 1-form defined by

$$\varphi = f_1 dx_1 + \cdots f_n dx_n.$$

We want to show that $T(\varphi) = \sum_{j=1}^n T(f_j dx_j) = 0$, so it suffices to show that $T(f_j dx_j) = 0$ for each $j \in \{1, \ldots, n\}$. Let x_j denote the projection onto the jth coordinate axis, and let $T \llcorner f$ denote the current defined by $(T \llcorner f)(\varphi) = T(f\varphi)$. Then

$$T(f_j dx_j) = (T \llcorner f_j)(dx_j) = (T \llcorner f_j)(x_j^* I) = (x_j^*(T \llcorner f_j))(I),$$

where I is the identity one-form on $\mathbb{R}$ defined by

$$\begin{aligned} I \colon \mathbb{R} \to & \mathrm{A}^1(\mathcal{T}_\cdot(\mathbb{R})) = (\mathcal{T}_\cdot(\mathbb{R}))^* \\ x \mapsto & I(x) \in (\mathcal{T}_x(\mathbb{R}))^* \\ I(x) \colon (x; v) \mapsto & v. \end{aligned}$$

Since $x_j^*(T \llcorner f_j) \in \mathbf{F}_1(\mathbb{R}^1)$ is of the form $\mathcal{L}^1 \wedge \xi$ by part (2), and its support has measure zero, then it must be the zero current, and hence $T(f_j dx_j) = 0$.

□

The Constancy Theorem, whose proof can be found in Federer [10], will be used to show that the straight line is the shortest path connecting two points (6.8.1).

Theorem 6.1.23 (Constancy Theorem)**.** *Suppose B is an m-dimensional connected, C^1 manifold with boundary in $\mathbb{R}^n$, oriented by ξ. If a real flat chain $T \in \mathbf{F}_m$ is supported in B and its boundary is supported in the boundary of B, then, for some $r \in \mathbb{R}$, $T = r(\mathcal{H}^m \llcorner B) \wedge \xi$.*

6.2. **Slicing Theory.** The coarea formula relates the Lebesque measure of a measurable subset $A \subseteq \mathbb{R}^n$ to lower dimensional Haudorff measure of its slices, namely its projections onto the various level sets,

$$\mathcal{L}^n(A) = \int_{\mathbb{R}} \mathcal{H}^{n-1}(A \cap P^{-1}(\{y\}))\, dy,$$

where $P \colon \mathbb{R}^n \to \mathbb{R}$ is a projection map. Now we will define $(m-1)$-dimensional slices of m-dimensional normal currents by hyperplanes or by hypersurfaces $\{u(x) = r\}$, where u is a Lipschitz function. Throughout this subsection we will assume that T is a normal current and $\|T\|$ is a Radon measure.

Definition 6.2.1.

(1) *If $T \in \mathcal{D}_m$ and η is a k-form , a current $T \llcorner \eta \in \mathcal{D}_{m-k}$ is defined by*

$$(T \llcorner \eta)(\varphi) = T(\eta \wedge \varphi).$$

(2) *If f is a 0-form and T is representable by integration, then $T \llcorner f$ is also representable by integration whenever $\int_{\mathbb{R}^n} |f|\, d\|T\| < \infty$, and $T \llcorner f = f\|T\| \wedge \vec{T}$.*

(3) *If $A \subseteq \mathbb{R}^n$, then we define the current $T \llcorner A$ as $T \llcorner A = T \llcorner \chi_A$.*

Definition 6.2.2.

(1) *Let $T \in \mathbf{N}_m(\mathbb{R}^n)$, $u\colon \mathbb{R}^n \to \mathbb{R}$ Lipschitz, $r \in \mathbb{R}$, we define the slice*
$$\langle T, u, r+\rangle = (\partial T)\llcorner\{x\colon u(x) > r\} - \partial(T\llcorner\{x\colon u(x) > r\}).$$

(2) *An equivalent definition of $\langle T, u, r+\rangle$ is given by*
$$\begin{aligned}&(\partial T)\llcorner\{u > r\}(\varphi) - \partial(T\llcorner\{u > r\})(\varphi) = \partial T(\chi_{\{u>r\}}\varphi) - T(\chi_{\{u>r\}}d\varphi)\\ &= \partial T([1-\chi_{\{u\le r\}}]\varphi) - T([1-\chi_{\{u\le r\}}]d\varphi)\\ &= \partial T(\varphi) - \partial T(\chi_{\{u\le r\}}\varphi) - T(d\varphi) + T(\chi_{\{u\le r\}}d\varphi)\\ &= (T\llcorner\chi_{\{u\le r\}})(d\varphi) - (\partial T)\llcorner\chi_{\{u\le r\}}(\varphi)\\ &= \partial(T\llcorner\{u \le r\})(\varphi) - (\partial T)\llcorner\{u\le r\}(\varphi),\end{aligned}$$
then
$$\langle T, u, r+\rangle = \partial(T\llcorner\{u \le r\}) - (\partial T)\llcorner\{u \le r\},$$
and it follows that $\partial\langle T, u, r+\rangle = -\langle \partial T, u, r+\rangle$.

Proposition 6.2.3.
$$\mathbf{M}\langle T, u, r+\rangle \le (\operatorname{Lip} u)\liminf_{h\to 0^+}\frac{\|T\|\{r < u < r+h\}}{h}$$
If $f(r) = \|T\|(B(0,r))$ then for almost all r,
$$\mathbf{M}\langle T, u, r+\rangle \le (\operatorname{Lip} u)f'(r).$$

Proof. Let χ be the characteristic function of $\{x\colon u(x) > r\}$, then $\langle T, u, r+\rangle = (\partial T)\llcorner\chi - \partial(T\llcorner\chi)$. For small, positive h, approximate χ by a C^∞ satisfying
$$f(x) = \begin{cases}0, & \text{if } u(x) \le r\\ 1, & \text{if } u(x) \ge r+h\end{cases}$$
and $\operatorname{Lip} f \lesssim \operatorname{Lip} u/h$. Then
$$\begin{aligned}\mathbf{M}\langle T, u, r+\rangle &\approx \mathbf{M}((\partial T\llcorner f) - \partial(T\llcorner f))\\ &= \mathbf{M}(T\llcorner df)\\ (*)\qquad &\le (\operatorname{Lip} f)\|T\|\{x\colon r < u(x) < r+h\}.\end{aligned}$$
The last inequality follows because T is representable by integration. In particular whenever $\varphi \in \mathcal{D}^{m-1}$ and $\|\varphi\|^* \le 1$, then
$$\begin{aligned}|(T\llcorner df)(\varphi)| &\le \int_{\mathbb{R}^n} |(df\wedge\varphi)(x)(\vec{T}(x))|\, d\|T\|x\\ &\le \int_{\mathbb{R}^n} |\bigtriangledown f(x)|\, d\|T\|x\\ &\le (\operatorname{Lip} f)\int_{\{r<u(x)<r+h\}} d\|T\|x\\ &= (\operatorname{Lip} f)\|T\|\{x\colon r < u(x) < r+h\},\end{aligned}$$
and the inequality follows because φ is arbitrary.
Continuing from $(*)$ we have
$$(\operatorname{Lip} f)\|T\|\{x\colon r < u(x) < r+h\} \lesssim (\operatorname{Lip} u)\frac{\|T\|\{x\colon r < u(x) < r+h\}}{h}.$$

Consequently,

$$\mathbf{M}\langle T, u, r+\rangle \le (\operatorname{Lip} u)\frac{\|T\|\{x\colon r < u(x) < r+h\}}{h},$$

since $h > 0$ is arbitrary

$$\mathbf{M}\langle T, u, r+\rangle \le (\operatorname{Lip} u)\liminf_{h\to 0^+}\frac{\|T\|\{x\colon r < u(x) < r+h\}}{h}.$$

If $f(r) = \|T\|B(0,1)$ then f is monotonically increasing and its derivative exists a.e. $r > 0$. If we let $u(x) = |x|$ we get $\|T\|\{x\colon r < |x| < r+h\} = f(r+h) - f(r)$ and

$$f'(r) = \liminf_{h\to 0^+}\frac{\|T\|\{r < u(x) < r+h\}}{h}$$

implies that $\mathbf{M}\langle T, u, r+\rangle \le (\operatorname{Lip} u)f'(r)$ a.e. $r > 0$. □

Proposition 6.2.4.

$$\int_a^b \mathbf{M}\langle T, u, r+\rangle\, dr \le (\operatorname{Lip} u)\|T\|\{x\colon a \le u(x) \le b\}.$$

Proof. Consider the function $f(r) = \|T\|\{u < r\}$. Since f is monotonically increasing its derivative exists a.e.

$$\begin{aligned}(\operatorname{Lip} u)\|T\|\{a \le u(x) < b\} &= (\operatorname{Lip} u)\Big(f(b) - \lim_{x\to a} f(x)\Big)\\ &= (\operatorname{Lip} u)\int_a^b f'(r)\, dr\\ &\ge \int_a^b \mathbf{M}\langle T, u, r+\rangle\, dr,\end{aligned}$$

where the last inequality follows from (6.2.3). □

Corollary 6.2.5. *If $T \in \mathbf{N}_m(\mathbb{R}^n)$ then $\langle T, u, r+\rangle \in \mathbf{N}_{m-1}(\mathbf{R}^n)$ for almost all r.*

Proof. Need to show that $\mathbf{M}\langle T, u, r+\rangle + \mathbf{M}\partial\langle T, u, r+\rangle < \infty$. $\mathbf{M}\langle T, u, r+\rangle < \infty$ for almost all r, otherwise we can choose $a < b$ such that

$$\int_a^b \mathbf{M}\langle T, u, r+\rangle\, dr = \infty,$$

contradicting the proposition (6.2.4). Since $\partial\langle T, u, r+\rangle = -\langle \partial T, u, r+\rangle$ and $\partial T \in \mathbf{N}_{m-1}(\mathbf{R}^n)$, then $\mathbf{M}\partial\langle T, u, r+\rangle = \mathbf{M}\langle \partial T, u, r+\rangle$ and hence $\mathbf{M}\langle \partial T, u, r+\rangle < \infty$. □

Proposition 6.2.6. *If T is a normal current, then*

$$\int_a^b \mathsf{F}(T \llcorner \{u(x) \le r\})\, dr \le (b - a + \operatorname{Lip} u)\mathsf{F}(T),$$

where $u\colon \mathbb{R}^n \to \mathbb{R}$ is a Lipschitz function.

Proof. Since T is a normal current, we can find currents $A \in \mathcal{E}_m$ and $B \in \mathcal{E}_{m+1}$ such that $\infty > \mathbf{M}(T) \geq \mathsf{F}(T) = \mathbf{M}(A) + \mathbf{M}(B)$, and $T = A + \partial B$. $\partial T = \partial(A + \partial B) = \partial A$, $\mathbf{M}(\partial A) = \mathbf{M}(\partial T)$ so A is a normal current, and $\partial B = T - A$ implies that B is also a normal current. Let $u\colon \mathbb{R}^n \to \mathbb{R}$ be the given Lipschitz map, then

$$\begin{aligned}
T \llcorner \{u(x) \leq r\} &= (A + \partial B) \llcorner \{u(x) \leq r\} \\
&= A \llcorner \{u(x) \leq r\} + (\partial B) \llcorner \{u(x) \leq r\} \\
&= A \llcorner \{u(x) \leq r\} + \partial(B \llcorner \{u(x) \leq r\}) - \langle B, u, r+\rangle \\
\Rightarrow \mathsf{F}(T \llcorner \{u(x) \leq r\}) &\leq \mathsf{F}(A \llcorner \{u(x) \leq r\}) + \mathsf{F}(\partial(B \llcorner \{u(x) \leq r\})) \\
&\quad + \mathsf{F}\langle B, u, r+\rangle \\
&\leq \mathbf{M}(A) + \mathbf{M}(B) + \mathbf{M}\langle B, u, r+\rangle.
\end{aligned}$$

Then,

$$\begin{aligned}
\int_a^b \mathsf{F}(T \llcorner \{u(x) \leq r\})\, dr &\leq (b - a)(\mathbf{M}(A) + \mathbf{M}(B)) + (\operatorname{Lip} u)\mathsf{F}(T) \\
&= (b - a + \operatorname{Lip} u)\mathsf{F}(T).
\end{aligned}$$

□

The following lemma, whose proof can be found in Federer [10], considers slices of T by the function $u(x) = |x - a|$. If T has no boundary, then

$$\langle T, u, r+\rangle = \partial(T \llcorner \{u(x) \leq r\}) = \partial(T \llcorner B(a, r)).$$

The lemma asserts that if almost all such slices by spheres are rectifiable, then T is also rectifiable.

Lemma 6.2.7. *It T is a normal current without boundary and if for each $a \in \mathbb{R}^n$, $\partial(T \llcorner B(a, r))$ is rectifiable for almost all $r \in \mathbb{R}$, then T is rectifiable.*

The next lemma, whose proof can be found in Federer [10], implies that the measure theoretic slice of a current corresponds to our usual geometric notion of a slice of an object in the Euclidean space.

Proposition 6.2.8. *Let W be an m-dimensional rectifiable set in $\mathbb{R}^n$ and $u\colon W \to \mathbb{R}^k$ be a Lipschitz map. Then for almost all $z \in \mathbb{R}^k$, $W \cap u^{-1}\{z\}$ is rectifiable and the associated current is the slice $\langle T, u, r+\rangle$ of the current T associated with W.*

6.3. **The Deformation Theorem.**

Theorem 6.3.1 (Deformation Theorem)**.** *Given $T \in \mathbf{I}_m(\mathbb{R}^n)$ and $\epsilon > 0$, there exist $P \in \mathcal{P}_m(\mathbb{R}^n)$, $Q \in \mathbf{I}_m(\mathbb{R}^n)$, and $S \in \mathbf{I}_{m+1}(\mathbb{R}^n)$ such that the following conditions hold for $\gamma = 2n^{2m+2}$:*

(1) $$T = P + Q + \partial S.$$

(2) $$\begin{aligned}
\mathbf{M}(P) &\leq \gamma[\mathbf{M}(T) + \epsilon\mathbf{M}(\partial T)], \\
\mathbf{M}(\partial P) &\leq \gamma[\mathbf{M}(\partial T)], \\
\mathbf{M}(Q) &\leq \epsilon\gamma\mathbf{M}(\partial T), \\
\mathbf{M}(S) &\leq \epsilon\gamma\mathbf{M}(T).
\end{aligned}$$

Consequently, $\mathbf{F}(T-P) \leq \mathbf{M}(Q)+\mathbf{M}(S) \leq \epsilon\gamma[\mathbf{M}(T)+\mathbf{M}(\partial T)]$.

(3) $\operatorname{spt} P \subseteq m-$ *dimensional* 2ϵ *grid.*

(4) $\operatorname{spt} \partial P \subseteq (m-1)-$ *dimensional* 2ϵ *grid.*

(5) $\operatorname{spt} P \cup \operatorname{spt} Q \cup \operatorname{spt} S \subseteq \{x\colon d(x, \operatorname{spt} T) \leq 2n\epsilon\}$.

Remark 6.3.2. According to Deformation Theorem an m-dimensional integral current in $\mathbb{R}^n$ can be approximated by an integral polyhedral chain with an error term $Q+\partial S$ where Q and S have arbitrarily small mass. The resulting polyhedral chain has support which is arbitrarily close to the support of T and it is contained in the m-dimensional 2ϵ grid, that is given $x \in \operatorname{spt} P$ at least $n-m$ of the coordinates of x are even multiples of ϵ.

Sketch Of Proof. We will give a sketch of the proof for the case $m=1$, $n=3$. Let W_k denote the k-dimensional ϵ grid, $k=0,1,2$

$$W_k = \{(x_1,x_2,x_3)\colon \text{at least } 3-k \text{ of the } x_j \text{ are even multiples of } \epsilon\}.$$

We can think of the support of T as a compact 1-manifold of $\mathbb{R}^3$. In order to gain some geometric insight we can further assume that the support of T is a C^1-curve with finite length. Hence our aim is to approximate this curve by simplices in $\mathbb{R}^3$.
Project the curve T radially outward from the center of the cubes onto W_2. Let S_1 be the surface swept out by T during this projection, let Q_1 be the curve swept out by ∂T, and let T_1 be the image of T in W_2. Then T can be decomposed as $T=T_1+Q_1+\partial S_1$, and $\mathbf{M}(T_1)\approx\mathbf{M}(T)$, $\mathbf{M}(\partial T_1)\approx\mathbf{M}(\partial T)$, $\mathbf{M}(Q_1)\approx\epsilon\mathbf{M}(\partial T)$, $\mathbf{M}(S_1)\approx\epsilon\mathbf{M}(T)$.
Now project the curve T_1 radially outward from the centers of the squares onto W_1. Let S_2 be the surface swept out by T_1, Q_2 be the curve swept out by ∂T_1, and let T_2 be the image of T_1 in W_2. Then T_1 can be decomposed as $T_1=T_2+Q_2+\partial S_2$, and $\mathbf{M}(T_2)\approx\mathbf{M}(T_1)\approx\mathbf{M}(T)$, $\mathbf{M}(\partial T_2)\approx\mathbf{M}(\partial T)$, $\mathbf{M}(Q_2)\approx\epsilon\mathbf{M}(\partial T)$, and $\mathbf{M}(S_2)\approx\epsilon\mathbf{M}(T)$.
Let Q_3 consists of line segments in W_1 from each point of ∂T_2 to the nearest point of W_0, and let $P=T_2-Q_3$, then P is an integral polyhedral chain supported in W_1, and ∂P lies in W_0. The masses satisfy

$$\begin{aligned}\mathbf{M}(Q_3) &\approx \epsilon\mathbf{M}(\partial T)\\ \mathbf{M}(P) &= \mathbf{M}(T_2)+\mathbf{M}(Q_3)\approx\mathbf{M}(T)+\epsilon\mathbf{M}(\partial T)\\ \mathbf{M}(\partial T) &\approx \mathbf{M}(\partial T_2)\approx\mathbf{M}(\partial T).\end{aligned}$$

Let $Q=Q_1+Q_2+Q_3$, $S=S_1+S_2$, then $T=P+Q+\partial S$ and the masses satisfy $\mathbf{M}(Q)\leq\mathbf{M}(Q_1)+\mathbf{M}(Q_2)+\mathbf{M}(Q_3)\approx\epsilon\mathbf{M}(\partial T)$, and $\mathbf{M}(S)\approx\epsilon\mathbf{M}(T)$. □

Corollary 6.3.3. *The set*

$$\mathcal{A}=\{T\in\mathbf{I}_m(\mathbb{R}^n)\colon \operatorname{spt} T\subseteq B(0,c_1),\ \mathbf{M}(T)\leq c_2,\ \mathbf{M}(\partial T)\leq c_3\}$$

is totally bounded under $\mathbf{F}$.

Proof. Given $\epsilon>0$, we need to show that $\mathcal{A}$ can be covered by finitely many ϵ-balls. Given $T\in\mathcal{A}$, there exists a polyhedral chain in the $\tilde\epsilon$-grid with $\mathbf{M}(P)\leq\gamma[\mathbf{M}(T)+\tilde\epsilon\mathbf{M}(\partial T)]$ where

$$0<\tilde\epsilon<\frac{\epsilon}{\gamma[c_2+c_3]} \text{ and } \operatorname{spt} T\subseteq B(0,c_1+2n\epsilon).$$

Consequently, $\mathbf{F}(T-P)<\epsilon$. Since there are only finitely many polyhedral chains P with uniformly bounded mass and supported in the $2\tilde{\epsilon}$-grid, $\mathcal{A}$ is totally bounded. ☐

Theorem 6.3.4 (Isoperimetric Inequality)**.** *If $T\in\mathbf{I}_m(\mathbb{R}^n)$ with $\partial T=0$, then there exists $S\in\mathbf{I}_{m+1}(\mathbb{R}^n)$ with $\partial S=T$ and*

$$(\mathbf{M}(S))^{m/(m+1)}\le\gamma\mathbf{M}(T),\ \ \textit{where } \gamma=2n^{2m+2}.$$

Remark 6.3.5.

- The Isoperimetric inequality relates the k-volume of the compact oriented k-manifold in $\mathbb{R}^n$ with the $(k-1)$-volume of its boundary.
- A rectifiable current bounded by T can be found by taking the cone over T.

Sketch Of Proof. Since $T\in\mathbf{I}_m(\mathbb{R}^n)$, then $\mathbf{M}(T)<\infty$ and hence we can choose $\epsilon>0$ such that $\gamma\mathbf{M}(T)=\epsilon^m$. Now we can apply the Deformation Theorem to T to obtain $T=P+Q+\partial S$ where $P\in\mathcal{P}_m(\mathbb{R}^n)$, $Q\in\mathbf{I}_m(\mathbb{R}^n)$, and $S\in\mathbf{I}_{m+1}(\mathbb{R}^n)$ so that the masses satisfy

$$\mathbf{M}(Q)\le\epsilon\gamma\mathbf{M}(\partial T)=0$$

so $Q=0$ and

$$\mathbf{M}(P)\le\gamma\mathbf{M}(T)=\epsilon^m.$$

Since P lies in the m-dimensional 2ϵ-grid, the mass $\mathbf{M}(P)$ must be an integral multiple of $(2\epsilon)^m$ which exceeds $\gamma\mathbf{M}(T)$ by the choice of ϵ. Then $P=0$, $T=\partial S$, and $\mathbf{M}(S)\le\epsilon\gamma\mathbf{M}(T)=\epsilon^{m+1}=[\gamma\mathbf{M}(T)]^{(m+1)/m}$. ☐

6.4. Closure And Compactness Theorems.

Lemma 6.4.1. *$\mathbf{I}_m$ is $\mathbf{M}$-dense in $\mathcal{R}_m$ and $\mathbf{F}$-dense in $\mathcal{F}_m$.*

Proof. Since $(\overline{\mathcal{P}}_m)^{\mathbf{M}}=\mathcal{R}_m$ and $\mathcal{P}_m\subseteq\mathbf{I}_m\subseteq\mathcal{R}_m$, then $(\bar{\mathbf{I}}_m)^{\mathbf{M}}=\mathcal{R}_m$, and hence $\mathbf{I}_m$ is $\mathbf{M}$-dense in $\mathcal{R}_m$. Now we want to show that $(\bar{\mathbf{I}}_m)^{\mathbf{F}}=\mathcal{F}_m$. Since $\mathbf{I}_m\subseteq\mathcal{F}_m$ and $\mathcal{F}_m$ is $\mathbf{F}$-closed, it is enough to show that $\mathcal{F}_m\subseteq(\bar{\mathbf{I}}_m)^{\mathbf{F}}$. Let $T\in\mathcal{F}_m$, then there exist $A\in\mathcal{R}_m$ and $B\in\mathcal{R}_{m+1}$ such that $T=A+\partial B$. Choose $T_1\in\mathbf{I}_m$, $T_2\in\mathbf{I}_{m+1}$ such that $\mathbf{M}(A-T_1)<\epsilon/2$, and $\mathbf{M}(B-T_2)<\epsilon/2$, then $T_1+\partial T_2\in\mathbf{I}_m$ and $T-(T_1+\partial T_2)=(A-T_1)+\partial(B-T_2)$.

$$\begin{aligned}\mathbf{F}(T-(T_1+\partial T_2))&=\mathbf{F}((A-T_1)+\partial(B-T_2))\\&\le\mathbf{M}(A-T_1)+\mathbf{M}(B-T_2)\\&<\epsilon,\end{aligned}$$

which implies that $T\in(\bar{\mathbf{I}}_m)^{\mathbf{F}}$. ☐

Lemma 6.4.2. *$\mathcal{A}=\{T\in\mathcal{F}_m\colon \operatorname{spt}T\subseteq B(0,r)\}$ is $\mathbf{F}$-complete.*

Proof. Let $\{R_j\}_{j\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{A}$. By taking a subsequence, if necessary, we can assume that $\mathbf{F}(R_{j+1}-R_j) < 2^{-j}$ for each $j \geq 1$. Then by definition $R_{j+1}-R_j = T_j + \partial S_j$ where $T_j \in \mathcal{R}_m$ and $S_j \in \mathcal{R}_{m+1}$ and hence $\mathbf{M}(T_j) + \mathbf{M}(S_j) < 2^{-j}$. Since $\mathcal{A}$ is $\mathbf{M}$-complete $\sum_j T_j$ converges to a rectifiable current T and $\sum_j S_j$ converges to a rectifiable current S under $\mathbf{M}$. Then for each $n \in \mathbb{N}$

$$R_{n+1} - R_1 = \sum_{j=1}^{n}(R_{j+1} - R_j) = \sum_{j=1}^{n} T_j + \partial \sum_{j=1}^{n} S_j,$$

taking the limit as $n \to \infty$

$$\lim_{n\to\infty}(R_{n+1} - R_1) = \lim_{n\to\infty}\sum_{j=1}^{n} T_j + \partial \lim_{n\to\infty}\sum_{j=1}^{n} S_j,$$

where the last equality holds because $\mathbf{M}$-convergence implies $\mathbf{F}$-convergence and ∂ is $\mathbf{F}$-continuous. Thus

$$\lim_n R_n = R_1 + (T + \partial S) \in \mathcal{A}, \text{ where } R_n \xrightarrow{\mathbf{F}} R_1 + (T + \partial S)$$

□

Theorem 6.4.3 (Closure Theorem)**.**

(1) $\mathbf{I}_m$ *is* F*-closed in* $\mathbf{N}_m$*.*

(2) $\mathbf{I}_{m+1} = \{T \in \mathcal{R}_{m+1} \colon \mathbf{M}(\partial T) < \infty\}$*.*

(3) $\mathcal{R}_m = \{T \in \mathcal{F}_m \colon \mathbf{M}(T) < \infty\}$*.*

(4) $\mathcal{A} = \{T \in \mathbf{I}_m \colon \operatorname{spt} T \subseteq B(0,R),\ \mathbf{M}(T) \leq c,\ \mathbf{M}(\partial T) \leq c\}$ *is* $\mathbf{F}$*-complete.*

Proof. Step # 1: Show that for every $m \in \mathbb{N}$, (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4).
Let $m \in \mathbb{N}$, assuming that $\mathbf{I}_m$ is F-closed in $\mathbf{N}_m$ we want to show that $\mathbf{I}_{m+1} = \{T \in \mathcal{R}_{m+1} \colon \mathbf{M}(\partial T) < \infty\}$. It is clear from the definition of an integral current that $\mathbf{I}_{m+1} \subseteq \{T \in \mathcal{R}_{m+1} \colon \mathbf{M}(\partial T) < \infty\}$. From lemma (6.4.1) $\mathcal{R}_{m+1} = (\bar{\mathbf{I}}_{m+1})^{\mathbf{M}}$ and then $T \in \mathcal{R}_{m+1}$ implies that there exists a sequence $\{T_j\}_{j\in\mathbb{N}} \subseteq \mathbf{I}_{m+1}$ such that $T_j \xrightarrow{\mathbf{M}} T$. So we want to show that if $T \in \mathcal{R}_{m+1}$ and $\mathbf{M}(\partial T) < \infty$ then $T \in \mathbf{I}_{m+1}$. $T_j \xrightarrow{\mathbf{M}} T$ implies that $\mathbf{M}(T_j - T) \to 0$ as $j \to \infty$. The definition of F implies that $\mathsf{F}(\partial T_j - \partial T) = \mathsf{F}(\partial(T_j - T)) \leq \mathbf{M}(T_j - T) \to 0$ as $j \to \infty$. Then $\partial T_j \xrightarrow{\mathsf{F}} \partial T$, and $\partial T \in (\bar{\mathbf{I}}_m)^{\mathsf{F}} = \mathbf{I}_m \subseteq \mathbf{N}_m$ holds by hypothesis. So $\mathbf{M}(\partial T) < \infty$, $\partial T \in \mathbf{I}_m$ and $T \in \mathcal{R}_{m+1}$ imply that $T \in \mathbf{I}_{m+1}$ so we have shown that (1) implies (2).
Now we want to show that $\mathcal{R}_m = \{T \in \mathcal{F}_m \colon \mathbf{M}(T) < \infty\}$ assuming that $\mathbf{I}_{m+1} = \{T \in \mathcal{R}_{m+1} \colon \mathbf{M}(\partial T) < \infty\}$. It is clear from the definition of a rectifiable current and an integral flat chain that $\mathcal{R}_m \subseteq \{T \in \mathcal{F}_m \colon \mathbf{M}(T) < \infty\}$. If $U \in \mathcal{F}_m$ with $\mathbf{M}(U) < \infty$, then $\mathbf{M}(U) = \mathbf{M}(T + \partial S)$ where $T \in \mathcal{R}_m$ and $S \in \mathcal{R}_{m+1}$. $\mathbf{M}(\partial S) = \mathbf{M}(U - T) \leq \mathbf{M}(U) + \mathbf{M}(T) < \infty$, then $S \in \mathbf{I}_{m+1}$ by hypothesis and $U = T + \partial S \in \mathcal{R}_m$, so we have shown that (2) implies (3).
We want to prove that $\mathcal{A} = \{T \in \mathbf{I}_m \colon \operatorname{spt} T \subseteq B(0,R),\ \mathbf{M}(T) \leq c,\ \mathbf{M}(\partial T) \leq c\}$ is $\mathbf{F}$-complete assuming that (3) holds. Let $\{T_j\}_{j\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{A}$, then by lemma (6.4.2) $T_j \xrightarrow{\mathbf{F}} T$ and $\partial T_j \xrightarrow{\mathbf{F}} \partial T$, where $T \in \mathcal{F}_m$ and $\operatorname{spt} T \subseteq B(0,R)$. Since $\mathbf{M}(T) \leq \liminf_{j\to\infty} \mathbf{M}(T_j)$ and $\mathbf{M}(\partial T) \leq \liminf_{j\to\infty} \mathbf{M}(\partial T_j)$, then $\mathbf{M}(T) \leq c$ and

$\mathbf{M}(\partial T) \leq c$. And by hypothesis $T \in \mathcal{R}_m$ and $\partial T \in \mathcal{R}_{m-1}$, so $T \in \mathbf{I}_m$, thus we have shown that (3) implies (4).
Step # 2: By induction on $m \in \mathbf{N}$ we want to show that (1) holds for every $m \in \mathbb{N}$. Since the assertion trivially holds for $m = 0$, let's assume that (1) holds for $m-1$ and show that it also holds for m. Suppose that a sequence $\{Q_i\}_{i\in\mathbb{N}}$ of integral currents converges in the F-norm to a normal current $T \in \mathbf{N}_m$. We want to show that $T \in \mathbf{I}_m$. Since $Q_i \xrightarrow{\mathsf{F}} T$ implies $\partial Q_i \xrightarrow{\mathsf{F}} \partial T$ where $\{\partial Q_i\}_{i\in\mathbb{N}} \subseteq \mathbf{I}_{m-1}$ and $\partial T \in \mathbf{N}_{m-1}$, then by the induction assumption we can assume that $\partial T \in \mathbf{I}_{m-1}$. Since $\partial T \in \mathbf{I}_{m-1}$ and $\partial(\partial T) = 0$, by the Isoperimetric Inequality (6.3.4) there exists $T_1 \in \mathbf{I}_m$ with $\partial T = \partial T_1$, then $Q_i - T_1 \xrightarrow{\mathsf{F}} T - T_1 \in \mathbf{N}_m$. In order to show that T is an integral current it is enough to show that $T - T_1$ is an integral current and hence by replacing T with $T - T_1$ we can assume that $\partial T = 0$. So given a sequence $\{Q_i\}_{i\in\mathbb{N}}$ of integral currents converging in F-norm to a normal current $T \in \mathbf{N}_m$ with $\partial T = 0$ we want to show that $T \in \mathbf{I}_m$. Since we have assumed that $\partial T = 0$, by lemma (6.2.7) it is enough to show that for every $p \in \mathbb{R}^n$, $\partial(T \llcorner B(p,r))$ is rectifiable a.e. $r > 0$. Now by taking a subsequence, if necessary, we can assume that

$$\sum_{i=1}^{\infty} \mathsf{F}(Q_i - T) < \infty.$$

Let $0 < a < b$ and $u(x) = |x - p|$, since $Q_i - T \in \mathbf{N}_m$ for each $i \in \mathbb{N}$

$$\int_a^b \mathsf{F}[(Q_i - T)\llcorner\{u(x) \leq r\}]\, dr \leq [(b-a) + \operatorname{Lip} u]\mathsf{F}(Q_i - T)$$

follows from proposition (6.2.6) and summing on $i \in \mathbb{N}$ we obtain

$$\sum_{i=1}^{\infty} \int_a^b \mathsf{F}[(Q_i - T)\llcorner\{u(x) \leq r\}]\, dr \leq [(b-a) + \operatorname{Lip} u] \sum_{i=1}^{\infty} \mathsf{F}(Q_i - T) < \infty.$$

The Monotone Convergence Theorem implies that

$$\int_a^b \sum_{i=1}^{\infty} \mathsf{F}[(Q_i - T)\llcorner B(p,r)]\, dr < \infty.$$

Since the integrand is nonnegative

$$\sum_{i=1}^{\infty} \mathsf{F}[(Q_i - T)\llcorner B(p,r)] < \infty \quad \text{a.e. } r > 0$$

and

$$Q_i \llcorner B(p,r) \xrightarrow{\mathsf{F}} T\llcorner B(p,r) \quad \text{a.e. } r > 0$$

and from the definition of F-convergence

$$\partial(Q_i \llcorner B(p,r)) \xrightarrow{\mathsf{F}} \partial(T\llcorner B(p,r)) \quad \text{a.e. } r > 0.$$

Since $Q_i \llcorner B(p,r)$ and $T\llcorner B(p,r)$ are normal currents $\partial(Q_i \llcorner B(p,r))$ and $\partial(T\llcorner B(p,r))$ are also normal currents. By induction assumption for $m-1$ applied to (2) $Q_i \llcorner B(p,r)$ and $\partial(Q_i \llcorner B(p,r))$ are integral currents. By induction assumption for $m-1$ applied to (1) we know that $\mathbf{I}_{m-1}$ if F-closed in $\mathbf{N}_{m-1}$ and hence $\partial(T\llcorner B(p,r))$ is an integral current a.e. $r > 0$. Since T is a normal current without boundary and $\partial(T \llcorner B(p,r)) \in \mathcal{R}_{m-1}$ for every $p \in \mathbb{R}^n$ and a.e. $r > 0$, T is a rectifiable current as follows from (6.2.7). Since $\partial T = 0$, then T is an integral current and hence we have proved (1). $\square$

Theorem 6.4.4 (Compactness Theorem)**.** *Let K be a closed ball in $\mathbb{R}^n$, $0 \leq c < \infty$, then*

$$\mathcal{A} = \{T \in \mathbf{I}_m(\mathbb{R}^n) \colon \operatorname{spt} T \subseteq K,\ \mathbf{M}(T) \leq c \text{ and } \mathbf{M}(\partial T) \leq c\}$$

is $\mathbf{F}$-compact.

Proof. The set $\mathcal{A}$ is totally bounded under the $\mathbf{F}$-norm by corollary (6.3.3) and it is complete under the $\mathbf{F}$-norm by the closure theorem (6.4.3). Thus it is $\mathbf{F}$-compact. $\square$

Remark 6.4.5.

- As a consequence of theorem (6.1.10) the flat norm topology on $\mathcal{F}_m \supseteq \mathbf{I}_m$ is stronger than the weak topology and hence the $\mathbf{F}$-compactness of $\mathcal{A}$ implies the weak compactness. Thus under a suitable topology we attain compactness.

- We will use this compactness property in theorem (6.4.6) to show that given a "nice" boundary there exists an area minimizing surface spanning this boundary.

Theorem 6.4.6 (Existence Of Area Minimizing Surfaces)**.** *Let B be an $(m-1)$-dimensional rectifiable current in $\mathbb{R}^n$ with $\partial B = 0$. Then there is an m-dimensional area minimizing rectifiable current S with $\partial S = B$.*

Remark 6.4.7. A rectifiable current S is called area minimizing if for every rectifiable current T with $\partial T = \partial S$, $\mathbf{M}(S) \leq \mathbf{M}(T)$.

Proof. By taking the cone over B we can show that B bounds some rectifiable current. Let $B(0,r)$ be a large ball containing $\operatorname{spt} B$. Let $\{S_j\}_{j\in\mathbb{N}}$ be a sequence of rectifiable currents such that $\partial S_j = B$ for each $j \in \mathbb{N}$ and masses decreasing to $\inf\{\mathbf{M}(S) \colon \partial S = B\}$. Even though each $\operatorname{spt} S_j$ is bounded, the set $\{\operatorname{spt} S_j \colon j \in \mathbb{N}\}$ may not be bounded. Let Π be the projection map which keeps $B(0,r)$ fixed and projects each point outside of $B(0,r)$ onto $\partial B(0,r)$. Since Π is distance non-increasing it is mass non-increasing and hence by replacing each S_j with $\Pi^* S_j$, we may assume that for each $j \in \mathbb{N}$, $\operatorname{spt} S_j \subseteq B(0,r)$. Note that each S_j is an integral current, so by using the compactness theorem (6.4.4) we can extract a subsequence $\{S_{j_k}\}_{k\in\mathbb{N}}$ such that $S_{j_k} \xrightarrow{\mathbf{F}} S$, where S is a rectifiable current. Since ∂ is $\mathbf{F}$-continuous $\partial S_{j_k} \xrightarrow{\mathbf{F}} \partial S$, so $\partial S = B$ and by the lower semicontinuity of the mass $\mathbf{M}(S) \leq \liminf_{k\to\infty} \mathbf{M}(S_k) = \inf\{\mathbf{M}(T) \colon \partial T = B\}$. Therefore S is the area minimizing surface. $\square$

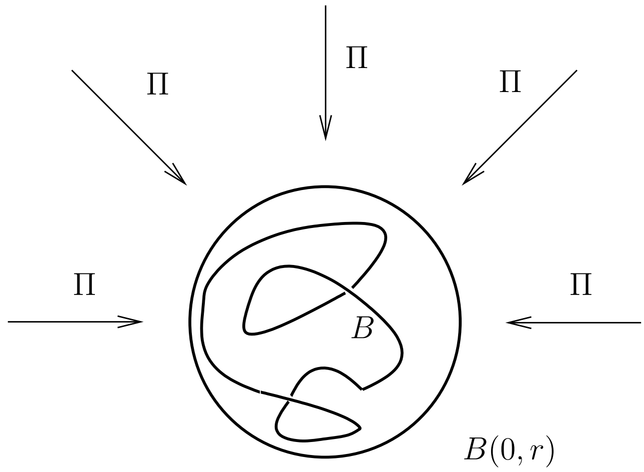


FIGURE 17. The map Π keeps the closed ball $B(0, r)$ intact and project everything else onto its surface in a manner such that it does not increase mass.

6.5. **Calculus Of Variations And The Minimal Surface Equation.** The next lemma, whose proof can be found in Gelfand & Fomin [15], will be used in the derivation of the Euler's Equation.

Lemma 6.5.1. *If $\alpha(x, y)$ is a fixed real-valued function which is continuous in a closed region R and if the integral*

$$\iint_R \alpha(x, y)h(x, y)\, dxdy = 0$$

for every real-valued function h of class C^2 in R and vanishes on $\Gamma = \partial R$, then $\alpha = 0$ on R.

Remark 6.5.2.

- Let J be a functional defined on the space of all C^2 class maps on R defined by
$$J[z] = \iint_R F(x, y, z, z_x, z_y)\, dxdy$$
where $z = z(x, y)$.

- We want to find a function z such that
  (1) z is of class C^2 in R,

  (2) z takes the given values on the boundary,

  (3) The functional has an extremum for z.

- The necessary condition for J to have an extremum for z is that its variation vanishes. Let h be a map of class C^2 in R which vanishes on Γ, then
$$\begin{aligned} \triangle J[z] &= J[z + h] - J[z] \\ &= \iint_R F(x, y, z + h, z_x + h_x, z_y + h_y) - F(x, y, z, z_x, z_y)\, dxdy \end{aligned}$$

$$= \iint_R (F_z h + F_{z_x} h_x + F_{z_y} h_y)\, dxdy + R_h,$$

where R_h goes to zero as $\|h\| = \sum_{i=0}^{2} \sup_{x\in R} |h^{(i)}(x)|$ goes to zero.
The variation of $J[z]$ is equal to

$$\delta J[h] = \iint_R (F_z h + F_{z_x} h_x + F_{z_y} h_y)\, dxdy.$$

$$\begin{aligned}
&\iint_R (F_{z_x} h_x + F_{z_y} h_y) \\
&= \iint_R \frac{\partial}{\partial x}(F_{z_x} h) + \frac{\partial}{\partial y}(F_{z_y} h)\, dxdy - \iint_R \left(\frac{\partial}{\partial x} F_{z_x} + \frac{\partial}{\partial y} F_{z_y} \right) h\, dxdy \\
&= \int_\Gamma F_{z_x} h\, dy - F_{z_y} h\, dx - \iint_R \left(\frac{\partial}{\partial x} F_{z_x} + \frac{\partial}{\partial y} F_{z_y} \right) h\, dxdy,
\end{aligned}$$

where the last equality follow from the Green's Theorem. Since h vanishes on Γ,

$$\delta J[h] = \iint_R \left(F_z - \frac{\partial}{\partial x} F_{z_x} - \frac{\partial}{\partial y} F_{z_y} \right) h\, dxdy.$$

So the condition $\delta J = 0$ implies that $\delta J[h] = 0$ for all admissible $h = h(x, y)$, and hence from the lemma (6.5.1), the Euler's Equation is of the form

$$(*) \qquad F_z - \frac{\partial}{\partial x} F_{z_x} - \frac{\partial}{\partial y} F_{z_y} = 0.$$

Thus the necessary condition for the functional J to have an extremum is that $(*)$ holds.

Now using the observation in (6.5.2) we will classify a certain type of functions whose graph is area minimizing.

Theorem 6.5.3 (Minimal Surface Equation)**.** *Let $f\colon A \in \mathrm{OP}(\mathbb{R}^2) \to \mathbb{R}$, be a C^2 class map, such that the graph of f, $G(f) = \{(x, f(x))\colon x \in A\}$ is area minimizing. Then f satisfies the minimal surface equation:*

$$(1 + f_x^2) f_{xx} - 2 f_x f_y f_{xy} + (1 + f_x^2) f_{yy} = 0.$$

Conversely, if f satisfies the minimal surface equation and A is convex, then its graph is area minimizing.

Proof. Let $f\colon A \in \mathrm{OP}(\mathbb{R}^2) \to \mathbb{R}$ be of class C^2, then $G_\alpha = (G(f), \alpha)$ is a parameterized 2-manifold in $\mathbb{R}^3$, where $\alpha(x, y) = (x, y, f(x, y))$. Then the area of G_α is given by

$$\int_{G_\alpha} dV = \int_A V(D\alpha),$$

and then

$$[D\alpha(x, y)] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix}$$

implies that

$$V(D\alpha) = \left\{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2\right\}^{1/2}.$$

Let $J[f]$ be the area functional on the space of all C^2 class maps defined on A and given by

$$J[f] = \iint_A \sqrt{1 + f_x^2 + f_y^2}\, dxdy.$$

If f is area minimizing then f is an extremum for J and hence it is necessary that the Euler's Equation must satisfy

$$\begin{aligned} 0 &= F_f - \frac{\partial}{\partial x} F_{f_x} - \frac{\partial}{\partial y} F_{f_y} \\ &= f_{xx}(1 + f_x^2) - 2 f_x f_y f_{xy} - f_{yy}(1 + f_x^2). \end{aligned}$$

To show the converse, assume that A is convex and f satisfies the minimal surface equation. Given $f\colon A \to \mathbb{R}$, define a 2-form φ on $A \times \mathbb{R}$ by

$$\begin{aligned} \varphi\colon A \times \mathbb{R} \to& \mathrm{A}^2(\mathcal{T}_{\cdot}(\mathbb{R}^3)) \\ p \mapsto& \varphi(p) \in \mathrm{A}^2(\mathcal{T}_p(\mathbb{R}^3)) \end{aligned}$$

$$\varphi(p) = \frac{-f_x(p) dy(p) \wedge dz(p) - f_y(p) dz(p) \wedge dx(p) + dx(p) \wedge dy(p)}{(f_x^2(p) + f_y^2(p) + 1)^{1/2}}.$$

By Cauchy-Schwartz inequality we can conclude that $|\varphi(p)(\xi(p))| \leq 1$ for every unit 2-vectorfield ξ and if $\xi(p)$ is tangent to the graph of f at p, then $|\varphi(p)(\xi(p))| = 1$. Since f satisfies the minimal surface equation it can be checked that $d\varphi = 0$. Let G_f denote the graph of f and let T be any other rectifiable current with the same boundary. Since A is convex, we may assume $\operatorname{spt} T \subseteq A \times \mathbb{R} = \operatorname{dom} \varphi$, by projecting T into $A \times \mathbb{R}$ if necessary without increasing the area T and hence we may assume that $\operatorname{spt} T \subseteq \operatorname{dom} \varphi$. From the definition of the volume of a parameterized manifold we have

$$\operatorname{area} G_f = \int_{G_f} \varphi$$

because $|\varphi(p)(\xi(p))| = 1$ whenever $\xi(p)$ is tangent to G_f at p. Since $G_f - T$ bounds and φ is closed, then the generalized Stokes' theorem implies that

$$\int_{G_f} \varphi = \int_T \varphi.$$

Since $\varphi(p)(\xi(p)) \leq 1$ for all 2-planes $\xi(p)$, then

$$\int_T \varphi \leq \operatorname{area} T$$

and hence

$$\operatorname{area} G_f = \int_{G_f} \varphi = \int_T \varphi \leq \operatorname{area} T.$$

Therefore G_f is area minimizing. $\square$

Definition 6.5.4. *A calibration is a differential form φ which is closed and $\sup_x \varphi(x)(\xi(x)) = 1$ for all unit k-planes $\xi(x)$. A surface is said to be calibrated by φ if each oriented tangent plane ξ satisfies $\varphi(\xi) = 1$.*

Corollary 6.5.5. *A calibrated surface is area minimizing.*

Proof. The 2-form defined in the second part of the theorem (6.5.3) is a calibration and the surface G_f is calibrated by φ. The second part also shows that a calibrated surface is area minimizing among all rectifiable surfaces with the same boundary. □

6.6. **Survey Of Regularity Results.** In this subsection we will list several results concerning the regularity of the area minimizing rectifiable currents. In subsection (6.4) we showed the existence of area minimizing rectifiable currents. We will now present several important theorems asserting that under certain circumstances the area minimizing surface has geometric significance.

W. Fleming proved in [12] that a two dimensional, area minimizing rectifiable current is a smooth manifold in $\mathbb{R}^3$.

Theorem 6.6.1 (Fleming, [12])**.** *A two dimensional area minimizing rectifiable current T in $\mathbb{R}^3$ is a smooth, embedded manifold on the interior.*

The regularity theorem (6.6.1) was generalized to three dimensional surfaces in $\mathbb{R}^4$ by F.J. Almgren in [1] and up through six dimensional surfaces in $\mathbb{R}^7$ by J. Simons in [26].

H. Federer proved in [11] that an area minimizing hypersurface is a smooth manifold in $\mathbb{R}^n$ on the interior except for a set of Hausdorff dimension at most $n - 8$.

Theorem 6.6.2 (Federer, [11])**.** *An $(n-1)$-dimensional area minimizing rectifiable current T in $\mathbb{R}^n$ is a smooth, embedded manifold on the interior except for a singular set of Hausdorff dimension at most $n - 8$.*

Regularity in higher codimension, for an m-dimensional area minimizing rectifiable current T in $\mathbb{R}^n$, with $m < n - 1$ is proved by Fred Almgren in [3].

Theorem 6.6.3 (Almgren, [3])**.** *An m-dimensional area minimizing rectifiable current in $\mathbb{R}^n$ is a smooth embedded manifold on the interior except for a singular set of Hausdorff dimension at most $m - 2$.*

Boundary Regularity : In 1979 Hardt and Simon proved the boundary regularity theorem for area minimizing hypersurfaces.

Theorem 6.6.4 (Hardt, Simon, [16])**.** *Let T be an $(n-1)$-dimensional area minimizing rectifiable current in $\mathbb{R}^n$, bounded by a C^2, oriented submanifold with multiplicity one. Then at every point,* $\operatorname{spt} T$ *is a C^1, embedded manifold with boundary.*

6.7. **Monotonicity And Oriented Tangent Cones.**

Definition 6.7.1. *We will generalize our definitions to include noncompact surfaces such as oriented planes.*

(1) *The space $\mathcal{F}_m^{\text{loc}}$ is the space of locally integral flat chains that do not necessarily have compact support and locally coincide with integral flat chains.*

$$\mathcal{F}_m^{\text{loc}} = \{T \in \mathcal{D}_m \colon \textit{for all } x \in \mathbb{R}^n \textit{ there exists } S \in \mathcal{F}_m,\ x \notin \operatorname{spt}(T - S)\}.$$

(2) *The space $\mathcal{R}_m^{\text{loc}}$ is the space of locally rectifiable currents that do not necessarily have compat support and locally coincide with rectifiable currents.*

$$\mathcal{R}_m^{\text{loc}} = \{T \in \mathcal{D}_m \colon \textit{for all } x \in \mathbb{R}^n \textit{ there exists } S \in \mathcal{R}_m,\ x \notin \operatorname{spt}(T - S)\}.$$

(3) *Alternative definitions of $\mathcal{R}_m^{\text{loc}}$ are given by*

$$\begin{aligned}\mathcal{R}_m^{\text{loc}} &= \{T \in \mathcal{D}_m \colon T \llcorner B(0,r) \in \mathcal{R}_m,\ \textit{for all } r > 0\}\\ &= \{T \in \mathcal{D}_m \colon T \llcorner B(a,r) \in \mathcal{R}_m \textit{ for each } a \in \mathbb{R}^n,\ \textit{for all } r > 0\}.\end{aligned}$$

(4) *The space $\mathbf{I}_m^{\text{loc}}$ is the space of locally integral currents that do not necessarily have compact support and locally coincide with integral currents.*

$$\mathbf{I}_m^{\text{loc}} = \{T \in \mathcal{D}_m \colon \textit{for all } x \in \mathbb{R}^n \textit{ there exists } S \in \mathbf{I}_m,\ x \notin \operatorname{spt}(T - S)\}.$$

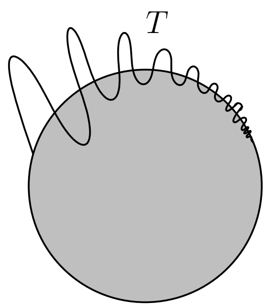


FIGURE 18. T is an integral current, but its restriction to the inside of the circle has infinite mass and hence it is not an integral current.

These spaces satisfy $\mathbf{I}_m^{\text{loc}} \subseteq \mathcal{R}_m^{\text{loc}} \subseteq \mathcal{F}_m^{\text{loc}}$.

Definition 6.7.2.

(1) *For the local flat topology, a typical neighborhood U_δ at $0 \in \mathbb{R}^n$ has the form*

$$\begin{gathered}U_\delta = \{T \in \mathcal{F}_m^{\text{loc}} \colon \operatorname{spt}(T - (A + \partial B)) \cap U(0,r) = \emptyset \textit{ for some}\\ r > 0,\ A \in \mathcal{R}_m,\ B \in \mathcal{R}_{m+1},\ \mathbf{M}(A) + \mathbf{M}(B) < \delta\}\end{gathered}$$

where $U(0,r)$ is the open ball in $\mathbb{R}^n$ with radius r and center at the origin.

(2) *Similar definition can be made for the local topology of $\mathcal{R}_m^{\text{loc}}$ and $\mathbf{I}_m^{\text{loc}}$.*

Remark 6.7.3. The collection $\{U_\delta\}_{\delta>0}$ of neighborhoods together with their translates

$$\begin{gathered}U_\delta^B = \{T \in \mathcal{F}_m^{\text{loc}} \colon \text{for each } x \in B,\ \operatorname{spt}(T - (A + \partial B)) \cap U(x,r) = \emptyset\\ \text{for some } r > 0,\ A \in \mathcal{R}_m,\ B \in \mathcal{R}_{m+1},\ \mathbf{M}(A) + \mathbf{M}(B) < \delta\},\end{gathered}$$

where $B \subseteq \mathbb{R}^n$ is a finite set, form a basis for the local flat topology. In particular we need to check the two axioms for basis:

(1) Given $B_1,\ B_2$ finite subsets of $\mathbb{R}^n$ and $\delta_1,\ \delta_2 > 0$ let $B = B_1 \cup B_2$ and $0 < \delta \leq \min\{\delta_1, \delta_2\}$, then $U_\delta^B \subseteq U_{\delta_1}^{B_1} \cap U_{\delta_2}^{B_2}$.

(2) Given $T_0 \in \mathcal{F}_m^{\text{loc}}$ and $x_0 \in \mathbb{R}^n$ we need to show that there exists U_δ^B such that $T_0 \in U_\delta^B$. Since $T_0 \in \mathcal{F}_m^{\text{loc}}$, then for $x_0 \in \mathbb{R}^n$ there exist $A_0 \in \mathcal{R}_m$ and $B_0 \in \mathcal{R}_{m+1}$ such that $S_0 = A_0 + \partial B_0 \in \mathcal{F}_m$ and $x_0 \notin \operatorname{spt}(T_0 - S_0)$. Let $\delta > \mathbf{M}(A_0) + \mathbf{M}(B_0)$ and $B = \{x_0\}$, then $T_0 \in U_\delta^B$.

Definition 6.7.4. *A locally rectifiable current T is area minimizing if for all $a \in \mathbb{R}^n$, $r > 0$, $T \llcorner B(a,r)$ is area minimizing.*

Definition 6.7.5. *Let $T \in \mathcal{R}_m^{\text{loc}}$ and $a \in \mathbb{R}^n$, the mass ratio is defined by*

$$\Theta^m(T,a,r) = \frac{\mathbf{M}(T \llcorner B(a,r))}{\alpha_m r^m}.$$

The density of T at a is defined by

$$\Theta^m(T,a) = \lim_{r \to \infty} \Theta^m(T,a,r),$$

whenever the limit exists.

Theorem 6.7.6. *Let T be an area minimizing locally rectifiable current in $\mathcal{R}_m^{\text{loc}}$. Let $a \in \operatorname{spt} T$. Then for $0 < r < \operatorname{dist}(a, \operatorname{spt} \partial T)$, the mass ratio is a monotonically increasing function of r.*

Proof. For $0 < r < \operatorname{dist}(a, \operatorname{spt} \partial T)$ let $f(r) = \mathbf{M}(T \llcorner B(a,r))$. Then f is monotonically increasing and hence it is differentiable a.e. in its domain. Let $u(x) = |x - a|$, then

$$\begin{aligned}
\langle T, u, r+\rangle &= \partial(T \llcorner \{u(x) \le r\}) - (\partial T) \llcorner \{u(x) \le r\} \\
&= \partial(T \llcorner B(a,r)) - (\partial T) \llcorner B(a,r) \\
&= \partial(T \llcorner B(a,r)). \\
\Rightarrow \mathbf{M}\langle T, u, r+\rangle &= \mathbf{M}(\partial(T \llcorner B(a,r))) \le f'(r),
\end{aligned}$$

where the last inequality follows from proposition (6.2.3) and the observation $\mathbf{M}(T \llcorner B(a,r)) = \|T\|(B(a,r))$. Since T is area minimizing $M(T \llcorner B(a,r))$ is less than or equal to the area of the cone C over $\partial(T \llcorner B(a,r))$ to a. Then

$$f(r) \le \mathbf{M}(C) = \frac{r}{m} \mathbf{M}(\partial(T \llcorner B(a,r))) \le \frac{r}{m} f'(r)$$

and

$$\begin{aligned}
\frac{d}{dr} \alpha_m \Theta^m(T,a,r) &= \frac{d}{dr}(r^{-m} f(r)) = r^{-m} f'(r) - m r^{-m-1} f(r) \\
&= \frac{m}{r^{m+1}} \left[\frac{r}{m} f'(r) - f(r)\right] \ge 0.
\end{aligned}$$

Thus $\Theta^m(T,a,r)$ is an increasing function of r in its domain. □

Corollary 6.7.7. *Suppose $T \in \mathcal{R}_m^{\text{loc}}$ is area minimizing. Then $\Theta^m(T,a)$ exists for every $a \in \operatorname{spt} T - \operatorname{spt} \partial T$.*

Proof. From theorem (6.7.6) the mass ratio $\Theta^m(T,a,r)$ is monotonically increasing in its domain $0 < r < \operatorname{dist}(a, \operatorname{spt} \partial T)$ whenever $a \in \operatorname{spt} T - \operatorname{spt} \partial T$ and hence

$$\Theta^m(T,a) = \lim_{r \to 0} \Theta^m(T,a,r) = \inf_{r > 0} \Theta^m(T,a,r)$$

exists. □

Corollary 6.7.8. *Suppose $T \in \mathcal{R}_m^{\mathrm{loc}}$ is area minimizing and $a \in \operatorname{spt} T - \operatorname{spt} \partial T$. Then for $0 < r < \operatorname{dist}(a, \operatorname{spt} \partial T)$,*

$$\mathbf{M}(T \llcorner B(a,r)) \geq \Theta^m(T,a)\alpha_m r^m.$$

Corollary 6.7.9. *Let T be an area minimizing rectifiable current in $\mathcal{R}_m(\mathbb{R}^n)$. Then for all $a \in \operatorname{spt} T - \operatorname{spt} \partial T$, $\Theta^m(T,a) \geq 1$.*

Proof. Since T is a rectifiable current there exists an associated rectifiable set $E \supseteq \operatorname{spt} T$ such that the m-dimensional density of E at $\mathcal{H}^m$-a.e. $a \in E$

$$\Theta^m(E,a) = \lim_{r \to 0} \frac{\mathcal{H}^m(E \cap B(a,r))}{\alpha_m r^m} = 1.$$

Since T is area minimizing the multiplicity function equals 1 $\mathcal{H}^m$-a.e. and hence $\mathbf{M}(T \llcorner B(a,r)) = \mathcal{H}^m(E \cap B(a,r))$ which implies that

$$\Theta^m(E,a) = \lim_{r \to 0} \frac{\mathbf{M}(T \llcorner B(a,r))}{\alpha_m r^m} = \Theta^m(T,a) = 1$$

$\mathcal{H}^m$-a.e. $a \in E$. Let $a \in \operatorname{spt} T - \operatorname{spt} \partial T$, then we can choose a sequence $\{a_j\}_{j \in \mathbb{N}} \subseteq E$ such that $a_j \to a$ and $\Theta^m(T,a_j) = \Theta^m(E,a_j) \geq 1$ for each $j \in \mathbb{N}$. Let $0 < r < \operatorname{dist}(a, \operatorname{spt} \partial T)$, $r_j = \operatorname{dist}(a, a_j)$, we can assume that $r_j < r$ for each $j \in \mathbb{N}$, then

$$\mathbf{M}(T \llcorner B(a,r)) \geq \mathbf{M}(T \llcorner B(a_j, r - r_j)).$$

By monotonicity of the mass ratio we have

$$\begin{aligned} \mathbf{M}(T \llcorner B(a_j, r - r_j)) &\geq \Theta^m(T,a_j)\alpha_m(r - r_j)^m \\ &\geq \alpha_m(r - r_j)^m, \end{aligned}$$

hence

$$\begin{aligned} &\mathbf{M}(T \llcorner B(a,r)) \geq \alpha_m(r - r_j)^m \text{ holds for each } j \in \mathbb{N} \text{ and} \\ &\mathbf{M}(T \llcorner B(a,r)) \geq \alpha_m r^m. \end{aligned}$$

Since $0 < r < \operatorname{dist}(a, \operatorname{spt} \partial T)$ is arbitrary

$$\Theta^m(T,a) \geq 1.$$

□

Definition 6.7.10. *A locally integral flat chain C is called a cone if for every $r > 0$, $\mu_r^* C = C$, where μ_r is defined by*

$$\begin{aligned} \mu_r \colon \mathbb{R}^n &\to \mathbb{R}^n \\ x &\mapsto rx. \end{aligned}$$

If $T \in \mathcal{F}_m^{\mathrm{loc}}$, such a cone C is called an oriented tangent cone to T at 0 if there is a decreasing sequence $\{r_j\}_{j \in \mathbb{N}}$ tending to zero such that $\mu_{r_j^{-1}}^* T$ converges to C in the local flat topology.

Lemma 6.7.11. *If U is a rectifiable current in $\mathcal{R}_m(\mathbb{R}^n)$, then $\mathbf{M}(\mu_{r^{-1}}^* U) = r^{-m}\mathbf{M}(U)$.*

Proof. Let U be a rectifiable current with the associated rectifiable set E. By definition $\mathbf{M}(\mu^*_{r^{-1}}U) = \sup\{(\mu^*_{r^{-1}}U)(\varphi) \colon \sup_x \|\varphi(x)\|^* \leq 1\}$ and

$$(\mu^*_{r^{-1}}U)(\varphi) = U(\mu^*_{r^{-1}}\varphi) = \int_E \langle \xi(x), (\mu^*_{r^{-1}}\varphi)(x)\rangle l(x)\, d\mathcal{H}^m x.$$

If we expand the integrand we obtain

$$\langle \xi(x), (\mu^*_{r^{-1}}\varphi)(x)\rangle = (\mu^*_{r^{-1}}\varphi)(x)((x;a_1(x)), \dots (x;a_m(x))),$$

where $\xi \colon x \mapsto ((x;a_1(x)), \dots (x;a_m(x)))$ is a unit m-plane orienting E

$$\begin{aligned}
&= \varphi(r^{-1}x)((r^{-1}x; r^{-1}a_1), \dots, (r^{-1}x; r^{-1}a_m)) \\
&= \sum_{(I)} b_I(r^{-1}x) dx_I(r^{-1}x)((r^{-1}x; r^{-1}a_1), \dots, (r^{-1}x; r^{-1}a_m)) \\
&= r^{-m}\varphi(r^{-1}x)(\tilde{\xi}(x)),
\end{aligned}$$

where $\tilde{\xi}$ is a unit m-plane. Then

$$\begin{aligned}
(\mu^*_{r^{-1}}U)(\varphi) &= r^{-m}\int_E \langle \tilde{\xi}(x), \varphi(r^{-1}x)\rangle l(x)\, d\mathcal{H}^m x \\
&\leq r^{-m}\mathbf{M}(U).
\end{aligned}$$

Since φ is arbitrary

$$\mathbf{M}(\mu^*_{r^{-1}}U) \leq r^{-m}\mathbf{M}(U).$$

And by symmetry we have

$$\mathbf{M}(U) \leq r^m \mathbf{M}(\mu^*_{r^{-1}}U).$$

□

Theorem 6.7.12. *Let T be an area minimizing rectifiable current in $\mathcal{R}_m(\mathbb{R}^n)$. Suppose $0 \in \operatorname{spt} T - \operatorname{spt} \partial T$. Then T has an oriented tangent cone C at $0 \in \mathbb{R}^n$.*

Partial Proof. We will show that there exist a rectifiable current C supported in $B(0,1)$ and a decreasing sequence $\{r_j\}_{j\in\mathbb{N}}$ tending to zero such that the sequence $\{\mu^*_{r_j^{-1}}(T \llcorner B(0,r_j))\}_{j\in\mathbb{N}}$ converges to C in the local flat topology. First note that $\partial \circ \mu^*_r = \mu^*_r \circ \partial$. In particular , if $T \in \mathcal{D}_m$, $\varphi \in \mathcal{D}^{m-1}$ and $\mu_r \colon x \mapsto rx$, then

$$\begin{aligned}
[(\partial \circ \mu^*_r)T](\varphi) &= (\mu^*_r T)(d\varphi) = T(\mu^*_r d\varphi) \\
&= T((\mu^*_r \circ d)\varphi) = T((d \circ \mu^*_r)\varphi) \\
&= (\partial T)(\mu^*_r \varphi) = [(\mu^*_r \circ \partial)T](\varphi).
\end{aligned}$$

Fix $0 < r_0 < \operatorname{dist}(0, \operatorname{spt}\partial T)$, then for $0 < r \leq r_0$ monotonicity of the mass ratio and lemma (6.7.11) imply

$$\begin{aligned}
\mathbf{M}(\mu^*_{r^{-1}}(T \llcorner B(0,r))) &= \mathbf{M}(T \llcorner B(0,r)) r^{-m} \\
&\leq \mathbf{M}(T \llcorner B(0,r_0)) r_0^{-m} \equiv c.
\end{aligned}$$

Let $u(x) = |x|$, then

$$\langle T \llcorner B(0,r), u, s+\rangle = \partial(T \llcorner [B(0,r) \cap \{u(x) \leq s\}]) - \partial(T \llcorner B(0,r)) \llcorner \{u(x) \leq s\}.$$

If $0 < s < r_0$ then $s < \operatorname{dist}(0, \operatorname{spt} \partial T)$ and hence

$$\langle T \llcorner B(0,r), u, s+\rangle = \partial(T \llcorner [B(0,r) \cap \{|x| \le s\}]).$$

If we apply the proposition (6.2.4) for $0 < s < r_0$, then

$$\int_{s/2}^{s} \mathbf{M}(\partial(T \llcorner B(0,r)))\, dr \le \mathbf{M}(T \llcorner B(0,s)) \le cs^m.$$

Consequently, for some $s/2 < r < s$,

$$\mathbf{M}(\partial(T \llcorner B(0,r))) \le \frac{cs^m}{s/2} \le 2^m c r^{m-1}$$

and

$$\mathbf{M}(\mu^*_{r^{-1}} \partial(T \llcorner B(0,r))) \le 2^m c,$$

where the last inequality follows from the closure theorem (6.4.3) and lemma (6.7.11). So whenever $0 < r \le r_0$, $\tilde{c} \ge \max\{c, 2^m c\}$

$$(*) \qquad \begin{aligned} \mathbf{M}(\mu^*_{r^{-1}}(T \llcorner B(0,r))) &\le c \le \tilde{c} \\ \mathbf{M}(\mu^*_{r^{-1}}(\partial(T \llcorner B(0,r)))) &\le 2^m c \le \tilde{c} \end{aligned}$$

and hence we can choose a sequence $\{r_j\}_{j\in\mathbb{N}}$ converging to zero such that $(*)$ holds for each $j \in \mathbb{N}$. In order to use the Compactness Theorem (6.4.4) we need to show that each $T_k = \mu^*_{r_k^{-1}}(T \llcorner B(0,r_k))$ is an integral current. First of all each T_k is a rectifiable current whose support is in some closed ball K because each $T \llcorner B(0,r_k)$ is a rectifiable current whose support is a subset of $B(0,r_k)$ with $r_k \to 0$. For each $k \in \mathbb{N}$, ∂T_k is a rectifiable current because

$$\begin{aligned} \partial T_k = \partial(\mu^*_{r_k^{-1}}(T \llcorner B(0,r_k))) &= \mu^*_{r_k^{-1}} \partial(T \llcorner B(0,r_k)) \\ &= \mu_{r_k^{-1}} \langle T \llcorner B(0,r_k), u, r_k+\rangle \end{aligned}$$

is rectifiable and hence each T_k is an integral current. Now by using the Compactness Theorem (6.4.4) we can extract a subsequence $\{s_j\}_{j\in\mathbb{N}}$ of $\{r_j\}_{j\in\mathbb{N}}$ such that

$$\mu^*_{s_j^{-1}}(T \llcorner B(0,s_j)) \xrightarrow{\mathbf{F}} C$$

where C is a rectifiable current supported in K. Since for each $k \in \mathbb{N}$, $\mu^*_{s_j^{-1}}(T \llcorner B(0,s_j))$ and C are rectifiable $\mathbf{F}$-convergence implies convergence in the local flat topology. $\square$

Theorem 6.7.13. *Let T be an area minimizing rectifiable current in $\mathcal{R}_m(\mathbb{R}^n)$ and $0 \in \operatorname{spt} T - \operatorname{spt} \partial T$. Let C be an oriented tangent cone to T at 0, then $\Theta^m(C,0) = \Theta^m(T,0)$.*

Proof. After replacing the sequence $\{r_j\}_{j\in\mathbb{N}}$ such that $\mu^*_{r_j^{-1}}(T - C) \to 0$ with a subsequence, if necessary, for each $j \in \mathbb{N}$ we can choose currents $A_j \in \mathcal{R}_m(\mathbb{R}^n)$ and $B_j \in \mathcal{R}_{m+1}(\mathbb{R}^n)$ such that

$$\begin{aligned} \operatorname{spt}(\mu^*_{r_j^{-1}}(T - C) - (A_j + \partial B_j)) \cap U(0,2) &= \emptyset \\ \mathbf{M}(A_j) + \mathbf{M}(B_j) &\le 1/j^2 \end{aligned}$$

Let $u(x) = |x|$, then proposition (6.2.4) implies

$$\int_1^{1+1/j} \mathbf{M}\langle B_j, u, r+\rangle\, dr \le \mathbf{M}(B_j).$$

For each $j \in \mathbb{N}$ we can choose $1 < s_j < 1 + 1/j \leq 2$ with $\mathbf{M}\langle B_j, u, s_j+\rangle \leq j\mathbf{M}(B_j) \leq 1/j$.

$$\begin{aligned}(\mu^*_{r_j^{-1}}(T - C))\llcorner B(0, s_j) &= (A_j + \partial B_j)\llcorner B(0, s_j)\\ (\mu^*_{r_j^{-1}}T)\llcorner B(0, s_j) &= C\llcorner B(0, s_j) + A_j\llcorner B(0, s_j) + (\partial B_j)\llcorner B(0, s_j)\\ &= C\llcorner B(0, s_j) + A_j\llcorner B(0, s_j) + \partial(B_j\llcorner B(0, s_j))\\ &\qquad - \langle B_j, u, s_j+\rangle\end{aligned}$$

Since

$$\begin{aligned}&\mathbf{F}((\mu^*_{r_j^{-1}}T)\llcorner B(0, s_j) - C\llcorner B(0, 1))\\ &\leq \mathbf{F}(C\llcorner[B(0, s_j) - B(0, 1)]) + \mathbf{F}(A_j\llcorner B(0, s_j)) + \mathbf{F}(\partial(B_j\llcorner B(0, s_j)))\\ &\qquad + \mathbf{F}\langle B_j, u, s_j+\rangle,\end{aligned}$$

where the RHS goes to 0 as $j \to \infty$, then $(\mu^*_{r_j^{-1}}T)\llcorner B(0, s_j) \to C\llcorner B(0, 1)$ in the flat norm.
By the lower semicontinuity of $\mathbf{M}$, $\mathbf{M}(C\llcorner B(0, 1)) \leq \liminf_{j\to\infty} \mathbf{M}(\mu^*_{r_j^{-1}}(T\llcorner B(0, s_j)))$ and $\Theta^m(C, 0) \leq \Theta^m(T, 0)$.
Since $(\mu^*_{r_j^{-1}}T)\llcorner B(0, s_j)$ is area minimizing and it has the same boundary as

$$C\llcorner B(0, s_j) + A_j\llcorner B(0, s_j) - \langle B_j, u, s_j+\rangle,$$

then $\mathbf{M}((\mu^*_{r_j^{-1}}T)\llcorner B(0, s_j)) \leq \mathbf{M}(C\llcorner B(0, s_j)) + 2/j$ and hence $\Theta^m(T, 0) \leq \Theta^m(C, 0)$. □

6.8. **The Regularity Of Area Minimizing Hypersurfaces.**

Theorem 6.8.1. *Let T be an area minimizing rectifiable current in $\mathcal{R}_1(\mathbb{R}^2)$. Then* $\operatorname{spt} T - \operatorname{spt} \partial T$ *consists of disjoint line segments.*

Partial Proof. In each case we will show that each $a \in \operatorname{spt} T - \operatorname{spt} \partial T$ has a neighborhood $U(a, r)$ such that $\operatorname{spt} T \cap U(a, r)$ is a straight line segment.

(1) Case # 1: If ∂T consists of two oppositely oriented points, then T is the oriented line segment between them.

Proof Of Case # 1. So case #1 implies that the shortest path between two points is the line segment. Let's assume that $\partial T = \delta_{(1,0)} - \delta_{(0,0)}$. Let T_0 be the oriented line segment between $(0, 0)$ and $(1, 0)$, then T_0 is represented by

$$T_0 = [(0, 0), (1, 0)] = \mathcal{H}^1\llcorner\{0 \leq x \leq 1, y = 0\} \wedge \vec{i}.$$

Let $\varphi \in \mathcal{D}^1$ be a 1-form in $\mathbb{R}^2$ of compact support given by $\varphi(p) = b_1(p)dx_1(p) + b_2(p)dx_2(p)$, then

$$\begin{aligned}T_0(\varphi) &= \int_{\mathbb{R}^2} \langle \vec{i}(p), \varphi(p)\rangle \, d\mathcal{H}^1\llcorner\{(x, 0)\colon 0 \leq x \leq 1\}p\\ &= \int_{\{(x,0)\colon 0\leq x\leq 1\}} \varphi(p)(\vec{i}(p)) \, d\mathcal{H}^1 p\\ &= \int_{\{(x,0)\colon 0\leq x\leq 1\}} b_1(p) \, d\mathcal{H}^1 p.\end{aligned}$$

In order to show that the line segment T_0 is the shortest path connecting $(0,0)$ and $(1,0)$ it is enough to show that T_0 uniquely minimizes mass (length) among all normal currents $N \in \mathbf{N}_1(\mathbb{R}^2)$ with the same boundary as T_0.
Let $N \in \mathbf{N}_1(\mathbb{R}^2)$, then N can be represented by integration as

$$N(\varphi) = \int_{\mathbb{R}^2} \langle \vec{N}(p), \varphi(p) \rangle \, d\|N\|p$$

where $\vec{N} \colon \mathbb{R}^2 \to \mathcal{T}(\mathbb{R}^2)$ is a 1-vectorfield, $|\vec{N}(p)| = 1$ $\|N\|$-a.e. and $\|N\|$ is Radon measure on $\mathbb{R}^2$.

$$\begin{aligned}\mathbf{M}(N) = \|N\|(\mathbb{R}^2) &= \int_{\mathbb{R}^2} d\|N\|p \geq \int_{\mathbb{R}^2} N_1(p) \, d\|N\|p \\ &= N(dx_1) = \partial N(x_1) = 1 = \mathbf{M}(T_0),\end{aligned}$$

then T_0 minimizes length.
If $\mathbf{M}(N) = 1$, then

$$\begin{aligned}&\int_{\mathbb{R}^2} \langle \vec{N}(p), dx_1(p) \rangle \, d\|N\|p = \int_{\mathbb{R}^2} \langle \vec{i}(p), dx_1(p) \rangle \, d\|N\|p = 1 \\ \Rightarrow &\int_{\mathbb{R}^2} \langle \vec{N}(p) - \vec{i}(p), dx_1(p) \rangle \, d\|N\|p = \int_{\mathbb{R}^2} (N_1(p) - 1) \, d\|N\|p = 0,\end{aligned}$$

and hence $N_1(p) = 1$, $\vec{N}(p) = \vec{i}(p)$ $\|N\|$-a.e. Now we will show that if $\mathbf{M}(N) = 1$, then $\operatorname{spt} N \subseteq \{y = 0\}$. If not, for some $\epsilon > 0$ there is a C^∞ map $0 \leq f(p) \leq 1$ such that $f(p) = 1$ for $|p| \leq \epsilon$ and $\mathbf{M}(N \llcorner f) < 1$.

$$\begin{aligned}\partial(N \llcorner f)(\varphi) = (N \llcorner f)(d\varphi) &= N(f d\varphi) \\ &= N(d(f\varphi)) - N(\varphi df) \\ &= (\partial N) \llcorner f(\varphi) - (N \llcorner df)(\varphi) = \partial N(\varphi),\end{aligned}$$

where the last equality follows because $\vec{N} = \vec{i}$ $\|N\|$-a.e. and $df(p)(\vec{i}(p)) = 0$. Since $\partial(N \llcorner f) = \partial N = \partial T_0$ and T_0 is mass minimizing, then $\mathbf{M}(N \llcorner f) \geq 1$. But this contradicts the assumption and hence $\operatorname{spt} N \subseteq \{y = 0\}$. Since $\partial(N - T_0) = 0$, by Constancy Theorem (6.1.23) $N - T_0$ is a constant multiple of $\mathcal{H}^1 \wedge \vec{i}$. Since $N - T_0$ has a compact support this constant must be zero and $N = T_0$.

(2) Case # 2: If the density $\Theta^1(T, a)$ equals 1, then $\operatorname{spt} T$ is a straight line segment in some neighborhood $U(a, r)$ of a.

Proof Of Case # 2. By lemma (6.2.5) for almost all s, $0 < s < \operatorname{dist}(0, \operatorname{spt} \partial T)$, the slice $\partial(T \llcorner B(a, s)) = \langle T, u, s+ \rangle$ is a 0-dimensional rectifiable current and a boundary containing an even number of points. $\mathbf{M}(\partial(T \llcorner B(a, s))) \geq 2$, otherwise $\tilde{T} = 0$ would have the same boundary as $T \llcorner B(a, s)$ and less mass because

$$\Theta^1(T, a) = \lim_{r \to 0} \frac{\mathbf{M}(T \llcorner B(a, s))}{2r} = 1$$

implies $\mathbf{M}(T \llcorner B(a, s)) > 0$ for all sufficiently small $s > 0$. If we let E be the associated rectifiable set, then by using proposition (6.2.4) we get

$$\begin{aligned}\int_0^s \mathbf{M}(\partial(T \llcorner B(a, r))) \, dr &\leq \|T\|\{u \leq s\} \leq (\mathcal{H}^1 \llcorner E)\{u \leq s\} \\ &= \mathcal{H}^1(E \cap B(a, s)) = \mathbf{M}(T \llcorner B(a, s)).\end{aligned}$$

Then by dividing each side by s we get

$$s^{-1}\int_0^s \mathbf{M}(\partial(T\llcorner B(a,r)))\,dr \le s^{-1}\mathbf{M}(T\llcorner B(a,s))$$

and

$$\frac{\mathbf{M}(T\llcorner B(a,s))}{s} \searrow \alpha_1\Theta^1(T,a) = 2.$$

We already know that $\mathbf{M}(\partial(T\llcorner B(a,r))) \ge 2$ for $0 < r < \operatorname{dist}(a, \operatorname{spt}\partial T)$. Therefore for some small $r > 0$ $\mathbf{M}(\partial(T\llcorner B(a,r)))$ $= 2$ and hence $\partial(T\llcorner B(a,r))$ consists of two points. From case # 1, $\operatorname{spt} T\llcorner B(a,r)$ is a straight line segment between the boundary points and hence T is a line segment in a neighborhood $U(a,r)$ of a.

(3) For the general case see Federer [10].

$\square$

The next theorem, whose proof can be found in Federer [10], proves the regularity of the area minimizing rectifiable currents in $\mathcal{R}_{n-1}(\mathbb{R}^n)$ for $2 \le n \le 7$.

Theorem 6.8.2 (Regularity For Area Minimizing Hypersurfaces)**.** *Let T be an area minimizing rectifiable current in $\mathcal{R}_{n-1}(\mathbb{R}^n)$ for $2 \le n \le 7$. Then $\operatorname{spt} T - \operatorname{spt}\partial T$ is a smooth, embedded manifold.*

Remark 6.8.3. E. Bombieri, E. De Giorgi, and E. Giusti, in [5], gave an example of a seven dimensional, area minimizing rectifiable current T in $\mathbb{R}^8$ with an isolated singularity at $0 \in \mathbb{R}^8$. This current T is the oriented truncated cone over $B = \mathbb{S}^3(0, 1/\sqrt{2}) \times \mathbb{S}^3(0, 1/\sqrt{2}) \subset \mathbb{S}^7(0,1) \subset \mathbb{R}^8$ and $\partial T = B$.

7. THE LEAST PERIMETER FUNCTION AND THE DOUBLE BUBBLE CONJECTURE IN $\mathbb{R}^2$

In this section we will present the least perimeter function and the proof of the double bubble conjecture in $\mathbb{R}^2$. The double bubble conjecture, which is proved jointly by J. Foisy, M. Alfaro, J. Brock, N. Hodges, J. Zimba in [14], states that the standard double bubble uniquely minimizes perimeter of an enclosure of any two given areas in $\mathbb{R}^2$. The double bubble conjecture has its roots from the works of F.J. Almgren [2], J. Taylor [27], and F. Morgan [20] who established the existence and the structure of an area minimizing bubble cluster in general dimensions. Their results admit the possibility of disconnected bubbles and exterior. One of the consequences of Almgren's paper [2] is the existence of a set S in $\mathbb{R}^n$ with the least surface area and separating the given m volumes in $\mathbb{R}^n$. Taylor's article [27] gives a rigorous proof of one of the observations made by physicist Plateau more than 100 years ago. Namely, an area minimizing soap bubble cluster B in $\mathbb{R}^3$ consists of real analytic constant mean curvature surfaces meeting smoothly in threes at 120° angles along smooth curves. Later Morgan showed in [20] that a perimeter minimizing bubble cluster in $\mathbb{R}^2$ consists of arcs of circles (or line segments) meeting in threes at angles of 120°. As a consequence of the works by Almgren, Taylor, and Morgan we can define the least perimeter function, namely a real-valued non-negative function giving the value of the least perimeter of an enclosure of two given quantities of area. We will present our proof of a known fact that the least perimeter function is continuous.

7.1. **The Least Perimeter Function.**

Definition 7.1.1.

(1) *A cluster of bubbles (or bubble cluster) in $\mathbb{R}^2$ is a collection of finitely many pairwise disjoint open sets $B_1, \dots, B_k$.*
(2) *A cluster of exactly two bubbles is called a double bubble. A double bubble B will be denoted as $B = (B_1, B_2)$, where B_1 and B_2 are the two bubbles.*
(3) *In $\mathbb{R}^2$, a standard double bubble is a double bubble consisting of three circular arcs (a line segment is understood to be a circular arc) meeting at two vertices at angles of 120°.*
(4) *The perimeter of a cluster is given by the one-dimensional Hausdorff measure of the topological boundaries of the bubble:*

$$\mathcal{H}^1\left(\bigcup_i \partial B_i\right)$$

(5) *A cluster is said to be perimeter minimizing if no other cluster enclosing the same area has less perimeter.*

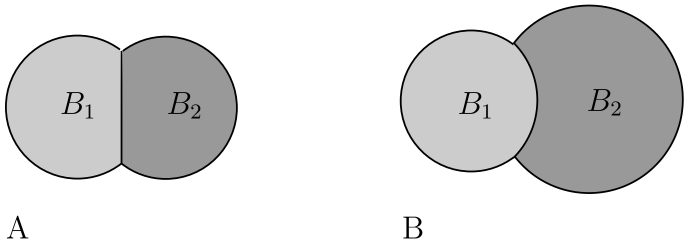


FIGURE 19. The standard double bubbles. (A) Bubbles have the same area. (B) Bubbles have different areas.

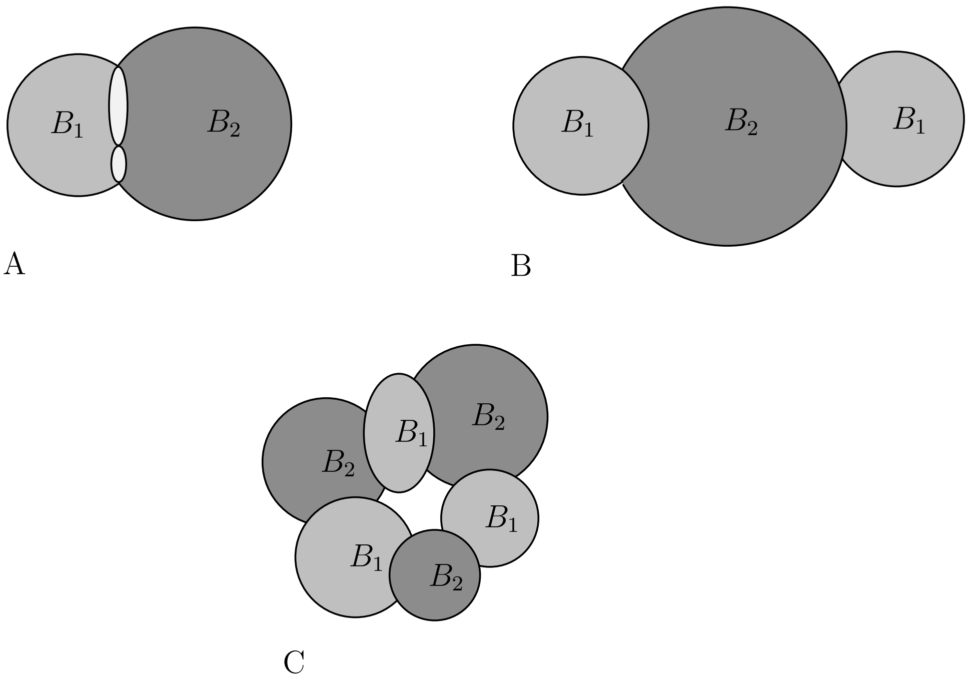


FIGURE 20. The non standard double bubbles. (A) has connected bubbles but the exterior is disconnected. (B) has a connected exterior, but the bubble B_1 is disconnected. (C) has both disconnected bubbles and a disconnected exterior.

Definition 7.1.2 (The Least Perimeter Function)**.** *For two prescribed quantities of area, A_1 and A_2, let $P(A_1, A_2)$ be the perimeter of the least perimeter double bubble enclosing the areas A_1 and A_2. Let $P(A_1, 0) = P(A_1)$ be the perimeter of the least perimeter enclosure of a region of area A_1, so $P(A_1)$ is the perimeter of the disk of area A_1. Let $P_0(A_1, A_2)$ be the perimeter of the standard double bubble enclosing areas of size A_1 and A_2.*

Remark 7.1.3.

- We will show in (7.2.3) that there exists a unique standard double bubble enclosing any two given areas in $\mathbb{R}^2$.
- Let $\lambda \geq 0$, then for any A_1, $A_2 \geq 0$, $P(\lambda A_1, \lambda A_2) = \sqrt{\lambda} P(A_1, A_2)$, where $\lambda \geq 0$ is the scaling factor.

In order to show that the least perimeter function is continuous as a function of two variables we will first show that it is continuous with respect to each variable separately when the other variable is held fixed. Then we will use the conclusions obtained in this preliminary stage to prove the most general case of continuity.

Lemma 7.1.4. *For any fixed A_1, $A_2 \geq 0$, the function $f\colon [A_1, \infty) \to \mathbb{R}$ defined by $f(A) = P(A, A_2)$ is continuous.*

Proof. Let $\tilde{A}_1 > A_1$, we want to show that f is continuous at $\tilde{A}_1$, or equivalently we need to show that for a given $\epsilon > 0$ there exists $0 < \delta_0 < \tilde{A}_1 - A_1$ such that

$$\begin{cases} -\epsilon & \leq f(\tilde{A}_1) - f(\tilde{A}_1 - \delta) \leq \epsilon \\ -\epsilon & \leq f(\tilde{A}_1) - f(\tilde{A}_1 + \delta) \leq \epsilon, \text{ for every } 0 \leq \delta \leq \delta_0. \end{cases}$$

From the definitions we get the inequality $f(\tilde{A}_1) = f((\tilde{A}_1 - \delta) + \delta) \leq f(\tilde{A}_1 - \delta) + P(\delta)$, and $f(\tilde{A}_1) - f(\tilde{A}_1 - \delta) \leq P(\delta)$. Using the scaling factor $\lambda = \frac{\tilde{A}_1 - \delta}{\tilde{A}_1}$ we obtain

$$\begin{aligned} P(\tilde{A}_1 - \delta, A_2) &= P\left(\left(\frac{\tilde{A}_1 - \delta}{\tilde{A}_1}\right)\tilde{A}_1, \left(\frac{\tilde{A}_1 - \delta}{\tilde{A}_1}\right)A_2\left(\frac{\tilde{A}_1}{\tilde{A}_1 - \delta}\right)\right) \\ &= \sqrt{\frac{\tilde{A}_1 - \delta}{\tilde{A}_1}}P\left(\tilde{A}_1, \left(\frac{\tilde{A}_1}{\tilde{A}_1 - \delta}\right)A_2\right) \\ &\leq P\left(\tilde{A}_1, \left(\frac{\tilde{A}_1}{\tilde{A}_1 - \delta}\right)A_2\right) \\ &\leq P(\tilde{A}_1, A_2) + P\left(\left(\frac{\delta}{\tilde{A}_1 - \delta}\right)A_2\right). \end{aligned}$$

Hence

$$\begin{aligned} P(\tilde{A}_1 - \delta, A_2) - P(\tilde{A}_1, A_2) &\leq P\left(\frac{\delta}{\tilde{A}_1 - \delta}A_2\right) \\ \Rightarrow f(\tilde{A}_1) - f(\tilde{A}_1 - \delta) &\geq -P\left(\frac{\delta}{\tilde{A}_1 - \delta}A_2\right) \\ \Rightarrow -P\left(\frac{\delta}{\tilde{A}_1 - \delta}A_2\right) &\leq f(\tilde{A}_1) - f(\tilde{A}_1 - \delta) \leq P(\delta). \end{aligned}$$

Since $P(A, 0) = P(A) \searrow 0$ as $A \searrow 0$ we can find $0 < \delta_1 < \tilde{A}_1 - A_1$ satisfying the condition $-\epsilon \leq f(\tilde{A}_1) - f(\tilde{A}_1 - \delta) \leq \epsilon$ for every $0 \leq \delta \leq \delta_1$.
$f(\tilde{A}_1 + \delta) \leq f(\tilde{A}_1) + P(\delta) \Rightarrow -P(\delta) \leq f(\tilde{A}_1) - f(\tilde{A}_1 + \delta)$ holds for every $\delta \geq 0$. If we replace $\tilde{A}_1$ with $(\tilde{A}_1 + \delta) - \delta$ then, from the calculations above, we get

$$\begin{aligned} f(\tilde{A}_1) = f((\tilde{A}_1 + \delta) - \delta) &\leq P(\tilde{A}_1 + \delta, A_2) + P\left(\frac{\delta}{A_1}A_2\right) \\ \Rightarrow f(\tilde{A}_1) - f(\tilde{A}_1 + \delta) &\leq P\left(\frac{\delta}{A_1}A_2\right) \\ \Rightarrow -P(\delta) \leq f(\tilde{A}_1) - f(\tilde{A}_1 + \delta) &\leq P\left(\frac{\delta}{A_1}A_2\right). \end{aligned}$$

Similarly as above we can find a $0 < \delta_2 < \tilde{A}_1 - A_1$ satisfying the condition $-\epsilon \leq f(\tilde{A}_1) - f(\tilde{A}_1 + \delta) \leq \epsilon$ for every $0 \leq \delta \leq \delta_2$. Hence for $0 < \delta_0 \leq \min\{\delta_1, \delta_2\}$ the continuity condition will be satisfied. The the continuity at $\tilde{A}_1 = A_1$ can be shown similarly. $\square$

Corollary 7.1.5. *The function* $h\colon [0,\infty)\times[0,\infty)\to[0,\infty)$ *defined by*

$$h(A_1, A_2) = P(A_1, A_2)$$

is continuous.

Proof. Let $(A_1, A_2) \in (0,\infty)^2$. We want to show that given $\epsilon > 0$ there exists $\delta > 0$ such that for every $(B_1, B_2) \in [0,\infty)^2$, $|B_1 - A_1| < \delta$ and $|B_2 - A_2| < \delta$ imply $|h(A_1, A_2) - h(B_1, B_2)| < \epsilon$. Let $\delta_1, \delta_2 > 0$.

$$\begin{aligned}
P(A_1-\delta_1, A_2-\delta_2) &= P\left(\frac{A_1-\delta_1}{A_1}A_1, \frac{A_2-\delta_2}{A_2}A_2\right)\\
&= \sqrt{\frac{(A_1-\delta_1)(A_2-\delta_2)}{A_1A_2}}\,P\left(\frac{A_2}{A_2-\delta_2}A_1, \frac{A_1}{A_1-\delta_1}A_2\right)\\
&\le P\left(\left[\frac{A_2}{A_2-\delta_2}\right]A_1, A_2\right) + P\left(\frac{\delta_1}{A_1-\delta_1}A_2\right)\\
&\le P(A_1, A_2) + P\left(\frac{\delta_2}{A_2-\delta_2}A_1\right) + P\left(\frac{\delta_1}{A_1-\delta_1}A_2\right).\\
&\Rightarrow P(A_1,A_2) - P(A_1-\delta_1, A_2-\delta_2) \ge\, -P\left(\frac{\delta_1}{A_1-\delta_1}A_2\right) - P\left(\frac{\delta_2}{A_2-\delta_2}A_1\right)
\end{aligned}$$

and

$$P(A_1,A_2) - P(A_1-\delta_1, A_2-\delta_2) \le P(\delta_1) + P(\delta_2)$$

imply that

$$-P\left(\frac{\delta_1}{A_1-\delta_1}A_2\right) - P\left(\frac{\delta_2}{A_2-\delta_2}A_1\right) \le P(A_1,A_2) - P(A_1-\delta_1, A_2-\delta_2) \le P(\delta_1)+P(\delta_2)$$

Then we can find $\delta^{(1)} > 0$ such that $P(\delta_1) + P(\delta_2) < \epsilon$ and $P\left(\frac{\delta_1}{A_1-\delta_1}A_2\right) + P\left(\frac{\delta_2}{A_2-\delta_2}A_1\right) < \epsilon$ whenever $0 < \delta_1,\ \delta_2 < \delta^{(1)}$. Similarly for each sector (in $\mathbb{R}^2$ there are four of them) we can find $\delta^{(i)} > 0$ such that the absolute value of the difference will be less than ϵ. Now let $0 < \delta \le \min_{1\le i\le 4}\delta^{(i)}$ then for every $\delta_1,\ \delta_2 \in [0,\delta)$

$$-\epsilon < P(A_1, A_2) - P(A_1 \pm \delta_1, A_2 \pm \delta_2) < \epsilon,$$

and hence $-\epsilon < h(B_1,B_2) - h(A_1,A_2) < \epsilon$ for every $(B_1,B_2) \in [0,\infty)^2$ with $|B_1 - A_1| < \delta$ and $|B_2 - A_2| < \delta$. The continuity in the case $A_1 = 0$ or $A_2 = 0$ can be shown similarly. $\square$

7.2. **The Double Bubble Conjecture In $\mathbb{R}^2$.**

Proposition 7.2.1. [14, Proposition 2.1.] *Let S be an edge of a perimeter minimizing bubble cluster in $\mathbb{R}^2$. Define C to be the distance between the endpoints of S, with θ the angle between S and the line segment connecting its endpoints. Then the radius of* *curvature R of S, the area A of the region between S and the line segment connecting its endpoints, and the length L of S are given by*

$$R(\theta, C) = \frac{C}{2\sin\theta}, \qquad A(\theta, C) = \frac{C^2(\theta - \sin(\theta)\cos(\theta))}{4\sin^2(\theta)}$$
$$L(\theta, C) = \frac{C\theta}{\sin\theta}.$$

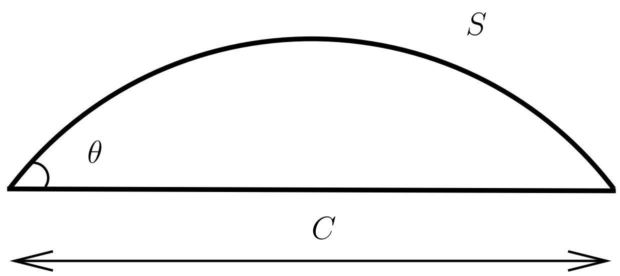


FIGURE 21. The circular arc S has radius of curvature R, area A and length L.

Proposition 7.2.2. [14, Proposition 2.2.] *If a perimeter minimizing double bubble in $\mathbb{R}^2$ has connected bubbles and a connected exterior, then it is a standard double bubble.*

Proof. A perimeter minimizing double bubble in $\mathbb{R}^2$ corresponds to a planar graph. The Euler's formula concerning the vertices, edges, and faces of a planar graph gives $V - E + F = 1$. From the article of Morgan [20] we can conclude that each vertex has three edges connecting to it, then $2E = 3V$, and $F = 2$ comes from the assumption of connected bubbles. Solving these equations yields $V = 2$ and $E = 3$. The cluster consists of three circular arcs all meeting in two points at angles of 120°, and hence it is a standard double bubble. □

Theorem 7.2.3. [14, Theorem 2.3.] *For any two prescribed quantities of area, there exists a unique standard double bubble.*

Proof. Our aim is to show for every $0 < \lambda \leq 1$ there exists a unique standard double bubble $B = (B_1, B_2)$, up to an affine transformation in $\mathbb{R}^2$, such that the ratio $\frac{\text{area}(B_1)}{\text{area}(B_2)}$ of the two areas it encloses is λ.

Consider a standard double bubble with the distance between its two vertices fixed to be 1. Since the edges are circular arcs that meet at these two points, the area underneath an arc of our double bubble is given by

$$A(\theta) = A(\theta, 1) = \frac{\theta - \sin(\theta)\cos(\theta)}{4\sin^2(\theta)}.$$

A simple calculation yields that

$$A'(\theta) = \frac{\sin\theta - \theta\cos\theta}{2\sin^3(\theta)}$$
$$A''(\theta) = \frac{2\theta(2 + \cos(2\theta)) - 3(\sin(2\theta))}{4\sin^4\theta}.$$

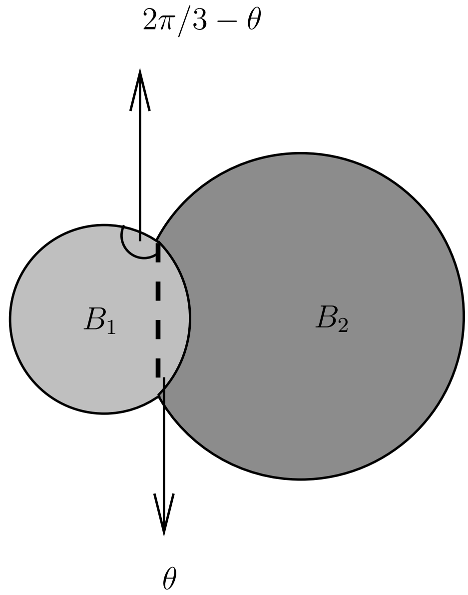


FIGURE 22. For any θ, one can construct a standard double bubble.

It can be shown that $A'(\theta) > 0$ and $A''(\theta) > 0$ on $(0, \pi)$, so $A(\theta)$ and $A'(\theta)$ are strictly increasing on $(0, \pi)$.

For any $\theta \in [0, \frac{\pi}{3})$, an angle formed by a circular arc and the line segment of length 1 that joins its endpoint, one can construct a standard double bubble. The enclosed areas must satisfy

$$\text{area } B_1(\theta) = A((\tfrac{2\pi}{3} - \theta)) + A(\theta)$$

$$\text{area } B_2(\theta) = A((\tfrac{2\pi}{3} + \theta)) - A(\theta).$$

Claim: For $\theta \in [0, \frac{\pi}{3})$, any ratio of A_1 to A_2, where A_1 and A_2 are two quantities of area, will be uniquely represented.

Let $F(\theta) = \frac{(\text{area } B_1(\theta))}{(\text{area } B_2(\theta))}$. Since $A''(\theta) > 0$ for $\theta \in (0, \pi)$, in $(0, \frac{\pi}{3})$, area $B_1(\theta) = A(\frac{2\pi}{3} - \theta) + A(\theta)$ is strictly decreasing (the magnitude of decrease of the LHS is strictly greater than the magnitude of increase of the RHS), and area $B_2(\theta) = A(\frac{2\pi}{3} + \theta) - A(\theta)$ is strictly increasing (the magnitude of increase of the LHS is stricly greater than the magnitude of decrease of the RHS). So increasing the angle θ will decrease the area enclosed by the smaller region and increase the area enclosed by the larger region. Thus $F(\theta)$ is stricly decreasing on the interval $[0, \frac{\pi}{3})$. In addition, $F(0) = 1$ and $F(\theta) \to 0$ as $\theta \to \frac{\pi}{3}$. So $F\colon [0, \frac{\pi}{3}) \to (0, 1]$ is bijective.

Since F is bijective, for any ratio $\lambda \in (0, 1]$ of areas enclosed by the two regions there exists a unique $\theta \in [0, \frac{\pi}{3})$, an angle between the line segment joining the vertices and the middle circular arc, and a unique pair of circular arcs which all together constitute the unique standard double bubble. The actual combination of quantities of area (A_1, A_2) can be obtained by scaling the line segment joining the two vertices. $\square$

Lemma 7.2.4. [14, Lemma 2.4.] *A perimeter minimizing double bubble whose exterior is connected must be standard.*

Proof. Let U be a perimeter minimizing double bubble with connected exterior. If U is not standard, by Proposition (7.2.2), U has a disconnected bubble. Our aim is to show that U can not be perimeter minimizing.

Consider a graph formed by placing a vertex inside each bubble component of U, with an edge between vertices of adjacent components. For any U with connected exterior, the corresponding graph has no cycles. If not, then there is a loop in the graph with a vertex v_0 being its basepoint. This loop is an n-gon and homeomorphic to a circle and hence divides the plane into two disconnected regions, by the Jordan-Brouwer Separation Theorem. Since the bubble cluster is a double bubble there is a point in the inner region of the loop which belongs to the exterior of the cluster, hence the exterior is disconnected, but this contradicts the hypothesis. Thus there will be a component of U that lies at an endpoint of the corresponding graph. It must have exactly two edges and two vertices.

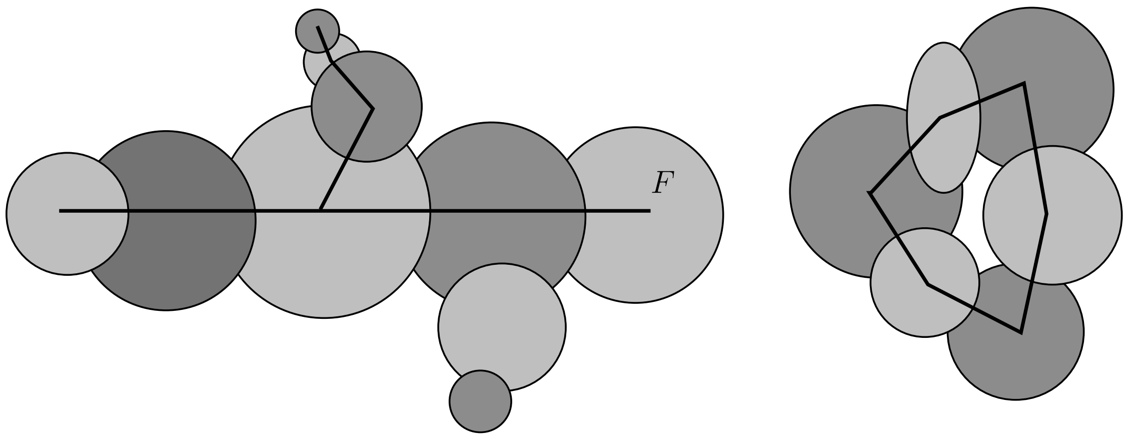


FIGURE 23. Since the exterior is assumed to be connected as in (A), the associated graph has an endpoint in a component with the edges and two vertices. If the exterior were disconnected, then a cycle as in (B) could result.

Let F be a component of U that has exactly two edges and exactly two vertices, r and q. Let t be a vertex of U that is adjacent to r but is not a vertex of F. Let S be the edge connecting r and t. Let p be a point on the edge S and define a new bubble cluster U_p by replacing the component F by its reflection across the perpendicular bisector of the line segment qp. As p moves continuously along S towards the vertex t the bubble cluster U_p with an extreme component F_p preserves the initial perimeter and the area.

As p varies from r to t, two things could happen: either there will be a point p_0 for which the reflection will result in the touching of another bubble component and the creation of a new vertex with four edges leading to it, or p will eventually coincide with t, and four edges meet at a vertex. Both of these cases will lead to a contradiction to the regularity theorem [20] since reflection preserves the perimeter minimizing bubble cluster.

□

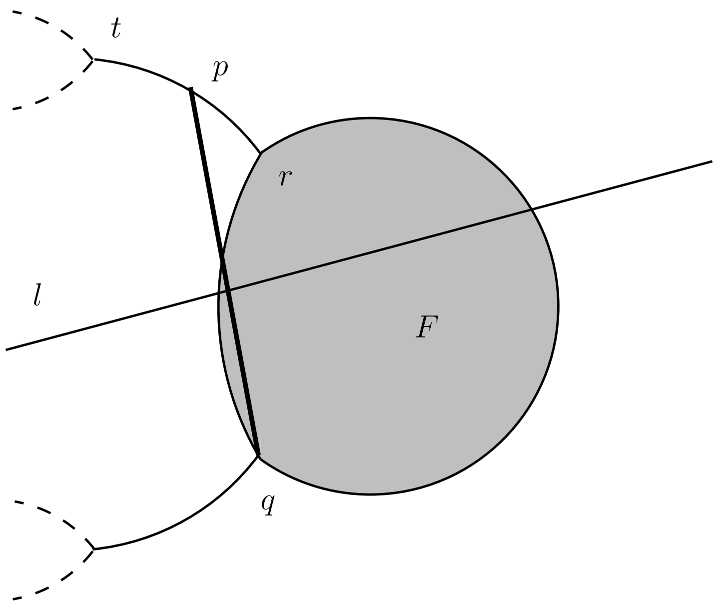


FIGURE 24. The reflection of F with respect to l may touch another component, contradicting regularity.

Lemma 7.2.5. [14, Lemma 2.5.] *Increasing the larger of the two prescribed areas enclosed by a standard double bubble will increase the total perimeter.*

Proof. From Proposition (7.2.1), the length function for a circular arc with endpoints distance 1 apart and with θ, the angle between the segment connecting the endpoints and the arc, is

$$L(\theta) = \frac{\theta}{\sin(\theta)}.$$

The perimeter of the standard double bubble with angle θ between the line segment connecting its vertices (distance 1 apart) is given by

$$\text{perim }(\theta) = L(\theta) + L\big(\tfrac{2\pi}{3} + \theta\big) + L\big(\tfrac{2\pi}{3} - \theta\big).$$

It can be shown that

$$L'(\theta) = \frac{\sin\theta - \theta\cos\theta}{\sin^2\theta} \text{ and}$$
$$L''(\theta) = \frac{\theta\sin^2\theta + 2\theta\cos^2\theta - \sin 2\theta}{\sin^3\theta}.$$

A simple calculation shows that $L'(\theta) > 0$ on $(0, \pi)$, and thus $L(\theta)$ is increasing on $\big[0, \frac{\pi}{3}\big)$. In addition, $L''(\theta) > 0$ on $(0, \pi)$; thus on $\big[0, \frac{\pi}{3}\big)$, $L\big(\frac{2\pi}{3} + \theta\big) + L\big(\frac{2\pi}{3} - \theta\big)$ is increasing (the magnitude of increase of the LHS is strictly greater than the magnitude of decrease of the RHS).

If the area enclosed by the larger bubble, B_2, increases then $\frac{\text{area } B_1}{\text{area } B_2}$ decreases with an accompanying increase in θ, where $\theta \in \big[0, \frac{\pi}{3}\big)$. Since $L(\theta)$ increases as $\theta \in \big[0, \frac{\pi}{3}\big)$ increases we can conclude that increasing the area enclosed by the larger region will increase the perimeter of the standard double bubble. $\square$

Lemma 7.2.6. [14, Lemma 2.7.] *For any fixed A_1, $A_2 \geq 0$, the function $P(A, A_2)$ has a minimum for $A \in [A_1, \infty)$.*

Proof. Given A_1, $A_2 \geq 0$ let the function $f(A)\colon [A_1, \infty) \to \mathbb{R}$ be defined by $f(A) = P(A, A_2)$. By the isoperimetric inequality, $f(A) \geq$ perim (D_A), where D_A is a disk of area A. Hence $f(A) \to \infty$ as $A \to \infty$. There exists $\tilde{A}_1 \geq A_1$ such that for every $A \geq \tilde{A}_1$, $\text{perim}(D_{A_1}) + \text{perim}(D_{A_2}) \leq f(A)$. Since f is continuous, f has a minimum for $A \in [A_1, \tilde{A}_1]$. Let $f(A_0)$ be the minimum value of f in $[A_1, \tilde{A}_1]$, and since $f(A_0) \leq f(A_1) \leq \text{perim}(D_{A_1}) + \text{perim}(D_{A_2})$, $f(A_0)$ is the absolute minimum value of f in $[A_1, \infty)$. □

As lemma (7.1.4) has an extension (7.1.5) so does lemma (7.2.6) has an extension (7.2.7) whose proof will become trivial once we show that the least perimeter function is an increasing function of both variables.

Corollary 7.2.7. *For any fixed A_1, $A_2 \geq 0$, the function $h\colon [A_1, \infty) \times [A_2, \infty) \to [0, \infty)$ defined by $h(A, B) = P(A, B)$ has a minimum for $(\tilde{A}_1, \tilde{A}_2) \in [A_1, \infty) \times [A_2, \infty)$.*

Proposition 7.2.8. [14, Proposition 2.8.] *The exterior of a perimeter minimizing double bubble must be connected.*

Proof. Given two quantities of area, A_1 and A_2, without loss of generality we may assume that $A_1 \geq A_2$. Suppose that the exterior of a perimeter minimizing double bubble $B = (B_1, B_2)$ enclosing A_1 and A_2 is disconnected. By Lemma (7.2.6), we can choose some $A_1' \in [A_1, \infty)$ that minimizes $P(A, A_2)$, $A \in [A_1, \infty)$. In particular, $P(A_1', A_2) \leq P(A_1, A_2)$.

We want to show that if $B' = (B_1', B_2')$ is a perimeter minimizing enclosure of the quantities of area A_1' and A_2, respectively, then the exterior must be connected. Suppose, to obtain a contradiction, that the exterior is disconnected. One of the components of the bubble B_1' must share a boundary with a bounded component of the exterior. Remove this boundary and incorporate the exterior into the bubble B_1'. Thus a new bubble is formed which has a strictly less perimeter than B_1' and encloses a region of area $A_1' + \epsilon$ where $\epsilon > 0$. A perimeter minimizing double bubble enclosing regions of area $A_1' + \epsilon > A_1' \geq A_1$ and A_2 will have a total perimeter ($= f(A_1' + \epsilon)$) strictly less $P(A_1', A_2) = f(A_1')$, which is a contradiction.

Thus $B_1 \neq B_1'$, otherwise $A_1 = A_1'$ implies that the initial double bubble (B_1, B_2) has connected exterior. Thus $A_1 < A_1'$. By lemma (7.2.5), $P_0(A_1, A_2) < P_0(A_1', A_2)$. By lemma (7.2.4) a perimeter minimizing double bubble with connected exterior is standard, $P_0(A_1', A_2) = P(A_1', A_2)$ and by definition, $P(A_1, A_2) \leq P_0(A_1, A_2)$. In summary:

$$P(A_1, A_2) \leq P_0(A_1, A_2) < P_0(A_1', A_2) = P(A_1', A_2) \leq P(A_1, A_2).$$

This is a contradiction. Therefore, we conclude that the exterior must be connected. □

Main Theorem. [14, Main Theorem 2.9.] *For any two prescribed quantities of area, the standard double bubble is the unique perimeter minimizing enclosure of the prescribed quantities of area.*

Proof. F.J. Almgren and F. Morgan have shown that a perimeter minimizing double bubble exists. By Proposition (7.2.8), the exterior of this double bubble must be connected. Then, by Lemma (7.2.4), the bubble cluster must be standard. Therefore, only a standard double bubble is perimeter minimizing. Given any two quantities of area A_1 and A_2 with $A_1 \leq A_2$ there exists a unique $\theta \in \left[0, \frac{\pi}{3}\right)$, angle between the line segment, of length 1, connecting the two vertices of the standard double bubble and the middle circular arc, leading to a unique standard double bubble such that the ratio of the areas enclosed equals the ratio of A_1 to A_2. This bubble will have two components with area A_1' and A_2' such that $\frac{A_1}{A_2} = \frac{A_1'}{A_2'}$ and the actual combination (A_1, A_2) can be obtained by scaling the line segment connecting the two vertices. Therefore, there exists a unique standard double bubble enclosing the two prescribed quantities of area. $\square$

Corollary 7.2.9. [14, Corollary 2.10.] *Increasing either given area A_1, A_2 increases the perimeter of the perimeter minimizing double bubble.*

Proof. If the conclusion is false, then for some perimeter minimizing standard double bubble (B_1, B_2) enclosing regions with respective quantities of area A_1 and A_2, an increase in, say $A_1 \leq A_2$, decreases the least perimeter of the minimizing double bubble. By decreasing the angle $\theta \in \left[\frac{\pi}{3}, \frac{2\pi}{3}\right)$, between the circular arc of B_1' and the line segment connecting the two vertices we can decrease the area continuously back to its original value A_1. The decrease in θ also decreases the total perimeter of the double bubble and contradicts the minimizing property of the original standard double bubble.

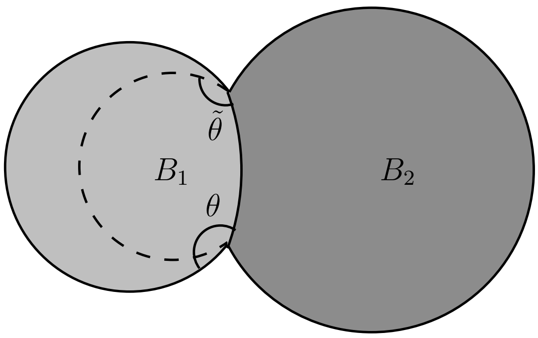


FIGURE 25. After we decrease the angle θ between the bubble B_1 and the separating arc we get a double bubble with less perimeter and B_1 has less area.

$\square$

8. THE STRUCTURE OF AREA MINIMIZING DOUBLE BUBBLES AND THE DOUBLE BUBBLE CONJECTURE IN $\mathbb{R}^3$

In this section we will study the structure of area minimizing double bubbles in the Euclidean Space and the double bubble conjecture in $\mathbb{R}^3$. Fred Almgren has shown in [2] that given any m volumes in $\mathbb{R}^n$ one can find an enclosure of least surface area. M. Hutchings has proved in [17] that for the case $m = 2$ the enclosure is either the standard double bubble or a surface or revolution consisting of a topological sphere with a single tree of annular bands attached. Hutchings has also presented the properties of the least area function and introduced their implications, the most important of which is that in a least area enclosure the exterior must be connected. He has shown the Decomposition Theorem which implies that a least area enclosure of two volumes must separate the space into finitely many components and in the case $n = 3$, a least area enclosure of two volumes must be connected.

M. Hutchings, F. Morgan, M. Ritoré, and A. Ros have jointly proved in [18] that the set in $\mathbb{R}^3$ with the least surface area enclosing the two given volumes in $\mathbb{R}^3$ is the standard double bubble. We will give a rough sketch of this proof and conclude the section with a survey of articles related to the double bubble problem in general dimension. For the sketch of the proof of the double bubble conjecture in $\mathbb{R}^3$ the main reference is the Morgan's book [22].

8.1. **The Least Area Function.**

Remark 8.1.1. Given prescribed volumes $v_1, \ldots, v_m$, Fred Almgren has shown that there exists a set $B \subseteq \mathbb{R}^n$ of smallest $\mathcal{H}^{n-1}$ measure (area) such that B encloses and separates the m disjoint sets $R_1, \ldots, R_m$ with $\mathcal{H}^n(R_i) = \mathrm{vol}\,(R_i) = v_i$.

Definition 8.1.2.

(1) *The least area function is a map*

$$A_n \colon \mathbb{R}^{\geq 0} \times \cdots \times \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$$

giving the least area required to enclose and separate m objects in $\mathbb{R}^n$ with prescribed volumes. The least area function is denoted by $A_n(v_1, \ldots, v_m)$.

(2) *The exterior region R_0 of the cluster consisting of a separating surface B and m pairwise disjoint sets R_i is defined as*

$$R_0 = \mathbb{R}^n - \left(B \cup \bigcup_{i=1}^{m} R_i \right).$$

A cluster is said to contain an empty chamber if the exterior region is disconnected. An empty chamber can be thought as an enclosure containing a vacuum and contributing nothing to the total amount of volume enclosed.

(3) *A standard double bubble in $\mathbb{R}^n$, $n \geq 3$, consists of two exterior spherical pieces and a separating surface (which is either spherical or planar) meeting at $120°$. An argument similar to the one in theorem (7.2.3) can be used to show that there exists a unique standard double bubble in $\mathbb{R}^n$ ($n \geq 3$) enclosing any two prescribed volumes.*

The next lemma is a trivial extension of the corollary (7.1.5) to m-volumes. For completeness we will give an outline of its proof.

Lemma 8.1.3. [17, Lemma 3.1] *For any m, $n \in \mathbb{N}$, the least area function $A_n(v_1, \ldots, v_m)$ is continuous.*

Partial Proof. We will show the continuity along a line with $v_2, \ldots, v_m$ constant; the general proof is similar to the proof of corollary (7.1.5). Let $v_1 > 0$, given an enclosure of volumes $v_1, \ldots, v_m$ we can increase v_1 by δ with a controlled increase in area by creating a sphere disjoint from the cluster with volume δ and incorporating this volume into R_1. The controlled increase means that the area fluctuation does not exceed a predetermined $\epsilon > 0$, and hence we have proved the continuity when the volume is increased. Now we need to show that the area change can be controlled when we decrease the volume of R_1. To decrease the volume of R_1 we can scale the entire cluster by $\lambda = (v_1 - \delta)/v_1$ so that R_1 has volume $v_1 - \delta$ and then we add spheres to other bubbles to restore their volumes. The resulting area increase will be controlled uniformly on some interval containing v_1, for small enough $\delta > 0$. Thus continuity for the case $v_1 > 0$ is proved. Let $v_1 = 0$, then

$$A_n(0, v_2, \ldots, v_m) \leq A_n(\delta, v_2, \ldots, v_m) \leq A_n(0, v_2, \ldots, v_m) + A_n(\delta)$$

holds for all $\delta > 0$. Since $A_n(\delta) \searrow 0$ as $\delta \searrow 0$, the continuity in the case $v_1 = 0$ is proved. $\square$

The next three lemmas, which will be stated without proof, will be used in the proof of the strict concavity of the least area function. The proof of these lemmas can be found in Hutchings [17].

Lemma 8.1.4. [17, Lemma 2.5.] *Let B be an area minimizing enclosure of m volumes in $\mathbb{R}^n$. Let $H \subseteq \mathbb{R}^n$ be a hyperplane and let B_1, B_2 be two symmetrizations of B about H. Suppose B_1 and B_2 minimize area for the volume they enclose. Let $A_1, A_2 \subseteq H$ be nonempty affine subspaces of dimension at most $n-2$. Suppose B_i is symmetric about A_i $i = 1, 2$. Then $A_1 \cap A_2 \neq \emptyset$ and B is symmetric about $A_1 \cap A_2$.*

Lemma 8.1.5. [17, Lemma 2.9.] *If $n \geq 3$, any minimal double bubble in $\mathbb{R}^n$ is symmetric about some line.*

Lemma 8.1.6. [17, Lemma 2.10] *Let B be a minimal double bubble in $\mathbb{R}^n$, $n \geq 3$, and let $H \subseteq \mathbb{R}^n$ be a hyperplane. Suppose each symmetrization of B across H is area minimizing for the volume it encloses, then B is symmetric about a line in H.*

Theorem 8.1.7 (Strict Concavity)**.** [17, Theorem 3.2.] *If $n \geq 3$, $v, w \in [0, \infty)^2$ are two pairs of nonnegative volumes, and if $0 < t < 1$, then*

$$A_n(tv + (1-t)w) > tA_n(v) + (1-t)A_n(w).$$

Proof. Suppose not, then there exists $0 < t < 1$ such that

$$A_n(tv + (1-t)w) - tA_n(v) - (1-t)A_n(w) \leq 0.$$

Let the function $f\colon [0,1] \to \mathbb{R}$ be defined by

$$f(t) = A_n(tv + (1-t)w) - tA_n(v) - (1-t)A_n(w).$$

From lemma (8.1.3) we can conclude that the function f is continuous and hence takes its minimum on $[0,1]$ at some $t_0 \in (0,1)$. Let $B = (R_1, R_2)$ be a minimal cluster enclosing volumes $t_0 v + (1-t_0)w = (t_0 v_1 + (1-t_0)w_1, t_0 v_2 + (1-t_0)w_2)$, then by lemma (8.1.5) B is symmetric about a line L. By discarding unnecessary points we may assume that B is compact.

We can parameterize the set of oriented hyperplanes in $\mathbb{R}^n$ by $\mathbb{S}^{n-1} \times \mathbb{R}$. In particular, each oriented hyperplane is determined by a normal direction and an oriented distance from the origin in that direction. Let the volume map $g\colon \mathbb{S}^{n-1} \times \mathbb{R} \to \mathbb{R}^2$ be defined by

$$g\colon P \mapsto (\mathrm{vol}\,(R_1 \cap U), \mathrm{vol}\,(R_2 \cap U)),$$

where U is the upper half space determined by P. Since B is compact, g is continuous because small variation in the hyperplane (small changes in the parameterization) leads to small variations in $\mathrm{vol}\,(R_1 \cap U)$ and $\mathrm{vol}\,(R_2 \cap U)$.

We can assume that the origin lies on L. Choose $x \in \mathcal{S}^{n-1}$ orthogonal to L, so that the hyperplane described by (x,r) contains L if $r=0$ and is parallel to L otherwise. Since B is symmetric with respect to L we have

$$g(x,r) + g(x,-r) = t_0 v + (1-t_0)w \text{ for every } r \in \mathbb{R}.$$

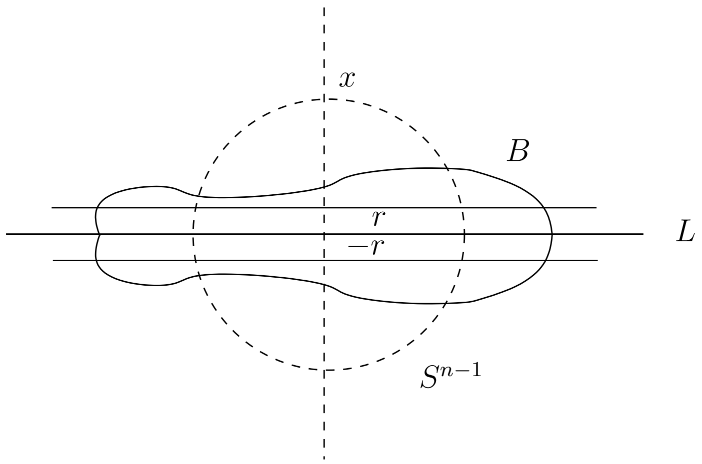


FIGURE 26. Bubble cluster B is symmetric about a line L passing through the origin

Consider the line segment in the volume plane

$$K = \left\{ \frac{tv + (1-t)w}{2} \,\middle|\, t,\ 2t_0 - t \in (0,1) \right\} \subseteq \mathbb{R}^2,$$

then $g(x,0) \in K$ and K contains a line segment in the volume plane passing through $w/2$ in the direction of $(v-w)/2$. Either $g(x,r) \in K$ for some $r \neq 0$, or else by continuity of g and the symmetry observation

$$g(x,0) = t_0 \left(\frac{v-w}{2} \right) + \frac{w}{2} \in K,$$

$g(y \times \mathbb{R})$ must contain an element of K for every $y \in \mathbb{S}^{n-1}$ close to x. In both cases we can find a hyperplane H, not containing L, with

$$g(H) = \frac{tv + (1-t)w}{2} \in K.$$

Let U and V be, respectively, the upper and lower half-spaces determined by H; let $a_0 = \text{area}\,(B \cap H)$, $a_1 = \text{area}\,(B \cap U)$, $a_2 = \text{area}\,(B \cap V)$. If we replace $B \cap V$ with the reflection of $B \cap U$ across H we obtain

$$\begin{aligned} a_0 + 2a_1 &\geq A_n(tv + (1-t)w) \\ &= f(t) + tA_n(v) + (1-t)A_n(w) \\ &\geq f(t_0) + tA_n(v) + (1-t)A_n(w) \\ &= A_n(t_0 v + (1-t_0)w) + (t - t_0)A_n(v) + (t_0 - t)A_n(w). \end{aligned}$$

If we symmetrize in the other direction, namely if we replace $B \cap U$ with the reflection of $B \cap V$ across H we obtain

$$\begin{aligned} &a_0 + 2a_2 \geq A_n((2t_0 - t)v + (1 - (2t_0 - t))w) \\ &= f(2t_0 - t) + (2t_0 - t)A_n(v) + (1 - 2t_0 + t)A_n(w) \\ &\geq f(t_0) + (2t_0 - t)A_n(v) + (1 - 2t_0 + t)A_n(w) \\ &\geq A_n(t_0 v + (1-t_0)w) + ((2t_0 - t) - t_0)A_n(v) + (t_0 - (2t_0 - t))A_n(w). \end{aligned}$$

Adding the two inequalities obtained by each symmetrization, we get

$$2(a_0 + a_1 + a_2) \geq 2A_n(t_0 v + (1-t_0)w) \Rightarrow (a_0 + a_1 + a_2) \geq A_n(t_0 v + (1-t_0)w),$$

but we know that this inequality is an equality, hence

$$a_0 + 2a_1 = A_n(tv + (1-t)w)$$

and

$$a_0 + 2a_2 = A_n((2t_0 - t)v + (1 - (2t_0 - t))w).$$

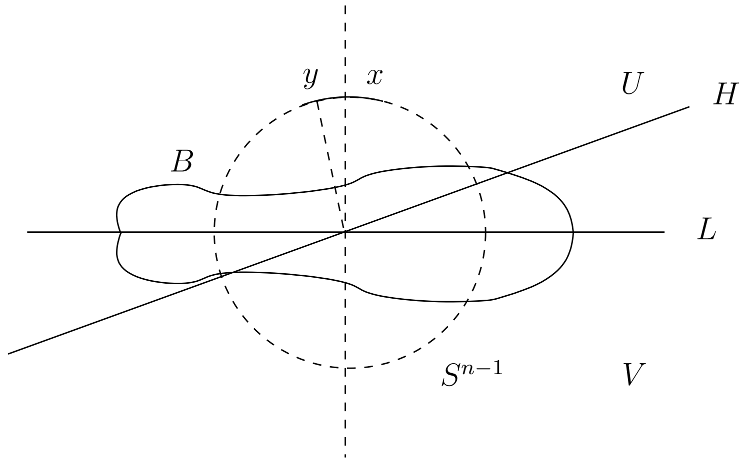


FIGURE 27. There exist a hyperplane H not containing L with $g(H) \in K$.

Thus each symmetrization of B across H is area minimizing for the volume it encloses. By lemma (8.1.6), B is symmetric about a line $\tilde{L} \subseteq H$. Since B is compact and hence bounded, L and $\tilde{L}$ must intersect. But $L \neq \tilde{L}$, so by applying lemma (8.1.4) to a hyperplane containing L and $\tilde{L}$ we can conclude that B must be symmetric about $L \cap \tilde{L} = \{p\}$, and hence it must be a union of concentric spheres. Since the number of spheres is finite, we can move one of the spheres without affecting the minimization property and violate the symmetry. So B must contain only one sphere and hence one volume, meaning that $t_0 v + (1-t_0)w$ must lie on one of the coordinate axes. Then v and w must both lie on the same coordinate axis. WLOG let's assume that $v = (v_1, 0),\ w = (w_1, 0)$ and $0 < v_1 < w_1$, then for every $t \in [0,1]$

$$\begin{aligned} A_n(tv + (1-t)w) &= A_n(\lambda_t(v_1, 0)) \\ &= \sqrt[(n-1)/n]{\lambda_t} A_n(v_1, 0), \end{aligned}$$

where $\lambda(t)$ equals

$$\lambda(t) = t\left(\frac{w_1}{v_1} - 1\right) + 1.$$

Let $K = A_n(v_1, 0) = A_n(v_1)$ be the surface area of the sphere with volume v_1, then $A_n(tv + (1-t)w) = K \sqrt[(n-1)/n]{\lambda_t}$ for each $t \in [0,1]$ is a strictly concave function, thus we get a contradiction. $\square$

Theorem 8.1.8 (Balancing)**.** [17, Theorem 3.5.] *If $v_1 > 2v_2$, then in any least area enclosure of volumes v_1 and v_2 in $\mathbb{R}^n$, R_1 is connected.*

Proof. If $n \geq 3$, then A_n is strictly concave along the line $x + y = c$ where $v_1 + v_2 = c$ and $A_n(v_1, v_2) = A_n(v_2, v_1)$. Since for every $0 \leq a, b$ and $a + b = c$, $A_n(a, b) = A_n(b, a)$ then $A_n(c/2, c/2) > A_n(a, b)$. We want to show that A_n increases strictly as (v_1, v_2) gets closer to $(c/2, c/2)$ along the line $x + y = c$. Fix $0 \leq a, b$ and $a + b = c$ and let $a' + b' = c,\ (a', b')$ be closer to $(c/2, c/2)$, then by using the strict concavity of the least are function (8.1.7) we get

$$\begin{aligned} A_n(a', b') &> tA_n(a, b) + (1-t)A_n(c/2, c/2) \\ &> tA_n(a, b) + (1-t)A_n(a, b) \\ \Rightarrow A_n(a', b') &> A_n(a, b). \end{aligned}$$

Now suppose that R_1 is disconnected in a minimal double bubble enclosing volumes v_1 and v_2. We can find a nonempty union Q of connected components of R_1 whose volume is at most $v_1/2 < v_1 - v_2$. If we declare Q to be a part of R_2, we obtain a cluster with the same area whose volumes (more balanced) are closer to $(c/2, c/2)$ along the line $x + y = c$, a contradiction to the above observation.

When $n = 2$, Frank Morgan has shown in [20] that a minimal cluster is a union of finitely many arcs of circles and line segments meeting at 120° angles. If Q does not have an edge in common with R_2 then it has no vertices and it is floating in R_0. We can then move Q without affecting the minimization property until it first touches the cluster, creating an illegal singularity which contradicts Morgan's theorem. So Q must have an edge in common with R_2. If we remove this edge and declare Q to be a part of R_2, the length decreases while the enclosed area remains the same, contradicting the assumption that the initial bubble cluster is length minimizing. $\square$

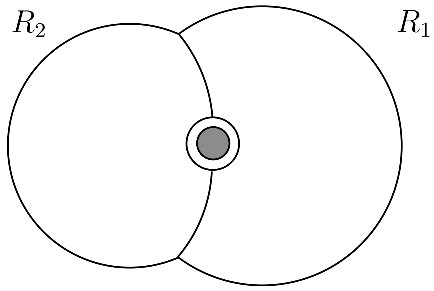


FIGURE 28. Moving the innermost circle will result in an illegal singularity.

Corollary 8.1.9 (Strictly Increasing)**.** [17, Corollary 3.3.] *For a fixed $n \in \mathbb{N}$, the function $A_n(v_1, v_2)$ is strictly increasing in each v_i, $i = 1, 2$.*

Proof. Suppose that $\tilde{v}_2 < v_2$ and $A_n(v_1, v_2) \le A_n(v_1, \tilde{v}_2)$, then by concavity $A_n(v_1, v_2) \ge A_n(v_1, w)$ for every $w > v_2$. In particular, if we let $u = (v_1, \tilde{v}_2)$, $v = (v_1, w)$, and $tu + (1-t)v = (v_1, v_2)$ for some $t \in (0, 1)$, then by the strict concavity of the least area function (8.1.7) we get

$$(*) \qquad \begin{aligned} A_n(tu + (1-t)v) = A_n(v_1, v_2) &> tA_n(v_1, \tilde{v}_2) + (1-t)A_n(v_1, w) \\ &\ge tA_n(v_1, v_2) + (1-t)A_n(v_1, w). \end{aligned}$$

Consequently, $A_n(v_1, v_2) > A_n(v_1, w) \ge A_n(w)$ holds for every $w > v_2$ and $A_n(w) \to \infty$ as $w \to \infty$ which implies that $A_n(v_1, v_2) = \infty$, a contradiction. □

Theorem 8.1.10 (No Empty Chamber)**.** [17, Theorem 3.4.] *Minimal double bubbles in $\mathbb{R}^n$ do not have empty chambers.*

Proof. Assume that a minimal double bubble contains an empty chamber. If we "fill up" the empty chamber with volume and declare it to be a part of R_1, then the volume v_1 is increased but the total area is the same, contradicting the corollary (8.1.9). □

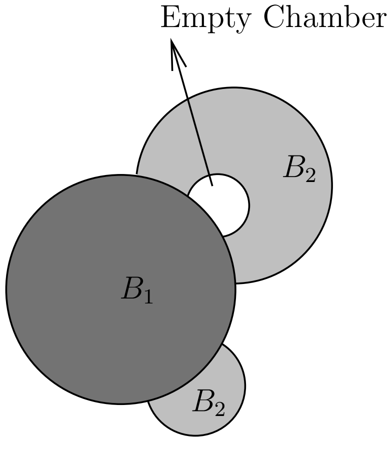


FIGURE 29. A minimal double bubble in $\mathbb{R}^n$ does not have an empty chamber.

8.2. **The Decomposition Lemma.**

Lemma 8.2.1 (Decomposition)**.** [17, Lemma 4.1.] *Suppose that in a minimal enclosure of volumes v_1, v_2 in $\mathbb{R}^n$, R_2 has a connected component with volume x. Then,*

$$2A_n(v_1, v_2) \geq A_n(x) + A_n(v_1, v_2 - x) + A_n(v_1 + x, v_2 - x).$$

Proof. We can think of this enclosure of regions R_1, R_2, and R_3 with volumes v_1, x, and $v_2 - x$, respectively. Let $S_{ij} = \partial R_i \cap \partial R_j$, and $a_{ij} = \text{area}\,(S_{ij})$, then we can decompose the enclosing surface and add up the surface area of parts in this decomposition to obtain

$$\begin{aligned}
&2A_n(v_1, v_2)\\
&= (a_{02} + a_{12}) + (a_{01} + a_{12} + a_{13} + a_{03}) + (a_{01} + a_{02} + a_{03} + a_{13})\\
&= \text{area}\,(\partial R_2) + \text{area}\,(\partial R_1 \cup \partial R_2) + \text{area}\,(\partial(\overline{R_1} \cup \overline{R_2}) \cup \partial R_3)\\
&\geq A_n(x) + A_n(v_1, v_2 - x) + A_n(v_1 + x, v_2 - x).
\end{aligned}$$

□

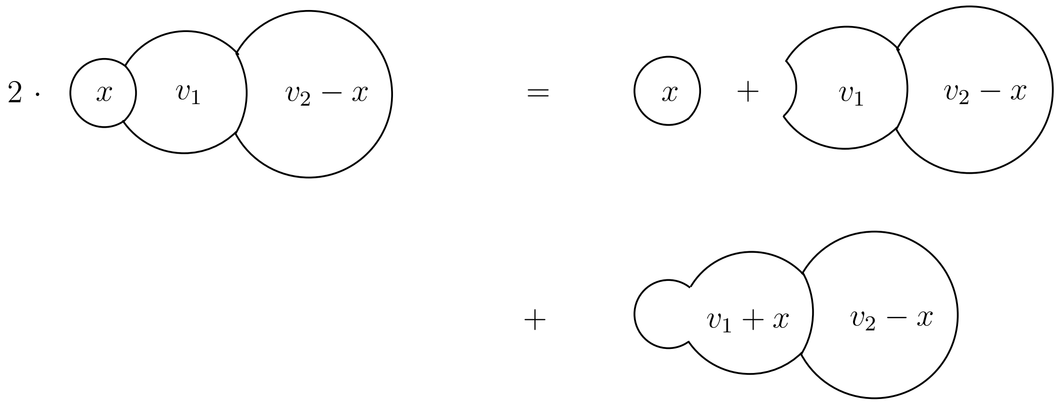


FIGURE 30. Schematic for the proof of (8.2.1).

Theorem 8.2.2 (Basic Estimate)**.** [17, Theorem 4.2.] *Suppose that in a minimal enclosure of volumes v_1 and v_2 in $\mathbb{R}^n$, R_2 has a connected component with volume $x > 0$. Then*

$$\frac{2A_n(v_1, v_2)}{A_n(1)} \geq v_2 x^{-1/n} + v_1^{(n-1)/n} + (v_1 + v_2)^{(n-1)/n}.$$

Proof. By decomposition lemma (8.2.1)

$$(*) \qquad 2A_n(v_1, v_2) \geq A_n(x) + A_n(v_1, v_2 - x) + A_n(v_1 + x, v_2 - x)$$

$$A_n(v_1, v_2 - x) = A_n\left(\frac{v_2 - x}{v_2}(v_1, v_2) + \frac{x}{v_2}(v_1, 0)\right)$$

by concavity it follows

$$\geq \frac{v_2 - x}{v_2} A_n(v_1, v_2) + \frac{x}{v_2} A_n(v_1).$$

$$A_n(v_1 + x, v_2 - x) = A_n\left(\frac{v_2 - x}{v_2}(v_1, v_2) + \frac{x}{v_2}(v_1 + v_2, 0)\right)$$

by concavity it follows

$$\geq \frac{v_2 - x}{v_2} A_n(v_1, v_2) + \frac{x}{v_2} A_n(v_1 + v_2).$$

Going back to $(*)$ and using the two inequalities obtained above, we get

$$2A_n(v_1, v_2) \geq 2\left(1 - \frac{x}{v_2}\right) A_n(v_1, v_2) + \frac{x}{v_2} A_n(v_1) + \frac{x}{v_2} A_n(v_1 + v_2) + A_n(x)$$

$$A_n(x) = A_n\left(v_2 \frac{x}{v_2}\right) = \left(\frac{x}{v_2}\right)^{1-1/n} A_n(v_2) = \frac{x}{v_2}\left(A_n(v_2)\left(\frac{v_2}{x}\right)^{1/n}\right).$$

Then using the new equality for $A_n(x)$, we obtain

$$\begin{aligned} 2A_n(v_1, v_2) \geq 2A_n(v_1, v_2) - \frac{2x}{v_2} A_n(v_1, v_2) + \frac{x}{v_2} A_n(v_1) + \frac{x}{v_2} A_n(v_1 + v_2) \\ + \frac{x}{v_2}\left(A_n(v_2)\left(\frac{v_2}{x}\right)^{1/n}\right). \end{aligned}$$

Equivalently we get

$$\begin{aligned} &[2A_n(v_1, v_2)]\frac{x}{v_2} \geq \frac{x}{v_2}\left(A_n(v_1) + A_n(v_1 + v_2) + A_n(v_2)\left(\frac{v_2}{x}\right)^{1/n}\right) \\ &\Leftrightarrow 2A_n(v_1, v_2) \geq A_n(v_1) + A_n(v_1 + v_2) + A_n(v_2)\left(\frac{v_2}{x}\right)^{1/n} \\ &\Leftrightarrow \frac{2A_n(v_1, v_2)}{A_n(1)} \geq v_2 x^{-1/n} + v_1^{(n-1)/n} + (v_1 + v_2)^{(n-1)/n}. \end{aligned}$$

□

Corollary 8.2.3. [17, Corollary 4.3.] *A minimal enclosure of two volumes in $\mathbb{R}^n$ separates $\mathbb{R}^n$ into finitely many components.*

Proof. Let $B = (R_1, R_2)$ be a minimal enclosure of volumes $v_1 = \mathrm{vol}\,(R_1)$ and $v_2 = \mathrm{vol}\,(R_2)$ in $\mathbb{R}^n$. If $x > 0$ is the volume of the connected component of R_2, then by the Basic Estimate (8.2.2)

$$\frac{v_2}{x} \leq \left(\frac{2A_n(v_1, v_2) - A_n(v_1) - A_n(v_1 + v_2)}{A_n(1)}\right)^n.$$

Hence there is an upper bound on the number of components of R_2 and similarly there is an upper bound on the number of components of R_1. Thus the cluster separates $\mathbb{R}^n$ into finitely many components. □

Corollary 8.2.4. *Let $A_n^0(v_1, v_2)$ be the area of the standard double bubble in $\mathbb{R}^n$ enclosing volumes v_1 and v_2. Consider a minimizing double bubble of volumes v_2 and $1 - v_2$, $0 < v_2 < 1$. Then the second region has at most k components where*

$$A_n(v_2)k^{1/n} = 2A_n^0(v_2, 1 - v_2) - A_n(1) - A_n(1 - v).$$

Proof. Let $x > 0$ be the volume of the smallest component of the second region and Let $K(v_2)$ be defined by

$$K(v_2) = \left(\frac{2A_n^0(v_2, 1 - v_2) - A_n(1) - A_n(1 - v_2)}{A_n(v_2)}\right)^n,$$

then $A_n(v_2)K(v_2)^{1/n} = 2A_n^0(v_2, 1-v_2) - A_n(1) - A_n(1-v)$ and the Basic Estimate (8.2.2) implies that

$$K(v_2) \geq \left(\frac{2A_n(v_2, 1-v_2) - A_n(1) - A_n(1-v_2)}{A_n(v_2)}\right)^n \geq \frac{v_2}{x}$$

is an upper bound on the number of the components of R_2. □

The table showing the bounds on the number of components in a minimizing double bubble in various dimensions will be used in the sketch of the proof of the double bubble conjecture in $\mathbb{R}^3$.

	$\mathbb{R}^2$	$\mathbb{R}^3$	$\mathbb{R}^4$	$\mathbb{R}^5$	$\mathbb{R}^n$
Bounds on the number of components in larger or equal region	1	1	1	2	3
Bound on the number of components in smaller region	1	2	4	6	2^n

TABLE 1. [22, Bounds On The Number Of Components]

Corollary 8.2.5. [17, Corollary 4.4.] *In any least area enclosure of two equal or almost equal volumes in $\mathbb{R}^3$, each R_i, $i = 0, 1, 2$ is connected.*

Proof. Let $B = (R_1, R_2)$ be a least area enclosure of two unit volumes in $\mathbb{R}^3$. By theorem (8.1.10) we know that the exterior, R_0, of B is connected. Assume that R_2 is not connected, then we can find a connected component of R_2 with volume $0 < x \leq 1/2$. The Basic Estimate (8.2.2) then gives

$$\frac{2A_3(1,1)}{A_3(1)} \geq x^{-1/3} + 1 + \sqrt[3]{4}.$$

By calculating the surface area of the standard double bubble enclosing two unit volumes in $\mathbb{R}^3$ we can conclude that

$$\frac{2A_3(1,1)}{A_3(1)} \leq 3\sqrt[3]{2},$$

and then

$$x^{-1/3} \leq 3\sqrt[3]{2} - 1 - \sqrt[3]{4},$$

which is false for $x = 1/2$ and for any x smaller that $1/2$. So in the case of equal volumes each bubble must be connected.
Since the least area function and all the calculations performed are continuous we can claim that each bubble is connected in the case when the volumes are very close to each other. □

Now we we give two examples of a possible area minimizing double bubble in $\mathbb{R}^3$ implied by the Structure theorem. Figure (31) is a surface of revolution consisting of three components and figure (32) is a surface of revolution consisting of five components. The trunk of the tree is a topological sphere which is attached to annular bands.

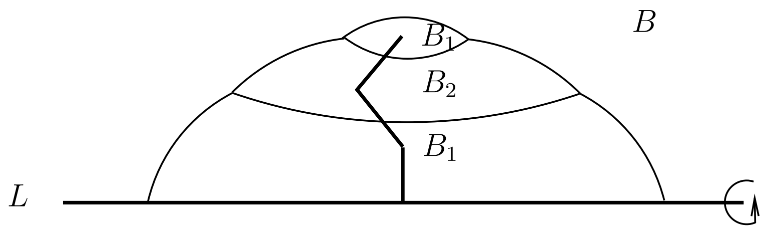


FIGURE 31. B is a surface of revolution about line L consisting of a topological sphere with a single tree of annular bands attached.

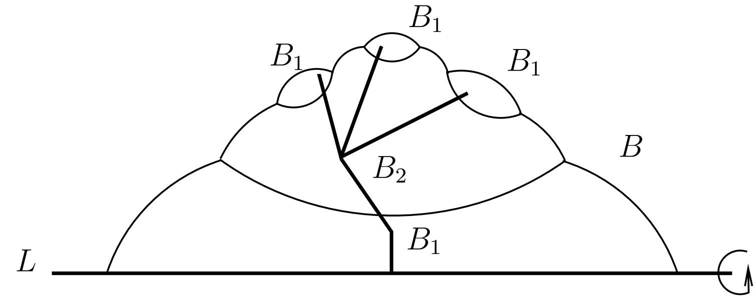


FIGURE 32. B is a surface of revolution about line L.

8.3. **The Structure Theorem.**

Theorem 8.3.1 (Structure Theorem)**.** [17, Theorem 5.1.] *Any minimal double bubble in $\mathbb{R}^n$ that is not the standard double bubble is a surface of revolution about some line, and consists of a topological sphere with a tree of annular bands attached. The two caps are pieces of spheres, and the root of the tree has just one branch. The surfaces are all constant mean curvature surfaces of revolution, meeting in threes at* 120°.

Sketch Of Proof. Any minimal double bubble in $\mathbb{R}^n$, $n \geq 2$ is a hypersurface of revolution about some line. The case $n = 2$ follows from [14], where the unique minimizer is the standard double bubble.

One can adapt the planar regularity theory of Frank Morgan in [20] to show that the generating curves must meet in threes at 120° and they must intersect the symmetry line perpendicularly. The bubble must be connected, otherwise moving components could create an asymmetric minimizer. By comparison with spheres centered on the axis and vertical hyperplanes, pieces of surface meeting the axis must be such spheres or hyperplanes so that minimization property is satisfied.

The number of surfaces intersecting the axis must be either two or three. If it were zero then an argument given by Foisy in [13] shows that the surface area of the bubble can be decreased in a volume preserving manner. If it were one, then the cluster can not separate any region. If the number is three then the bubble must be standard. Now assume that B is not standard and intersects the axis more than twice. Since B is connected and has no empty chamber, some surface S_0 of B must meet the axis with R_1 and R_2 on either side. The surfaces S_1 and S_2 must be spherical because of the minimization property of the sphere. Since B is nonstandard we may assume that S_1 meets some hypersurface other than S_0, S_1, or S_2. We can then roll $B - S_1$, as in figure (33) around the sphere containing S_1 until they first touch in fours. Since the rolling process does not affect the minimization property we get an illegal singularity contradicting [27]. Therefore, every nonstandard minimal double bubble must meet the axis exactly twice and the standard double bubble is the only minimal bubble cluster meeting the axis three times. The same rolling argument can be applied to show that the root of the tree has one branch. □

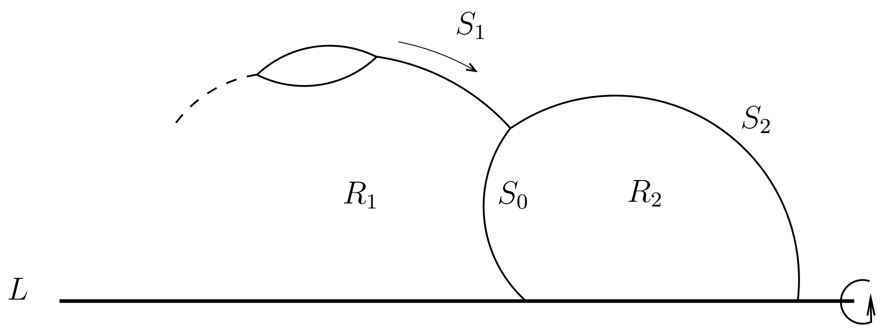
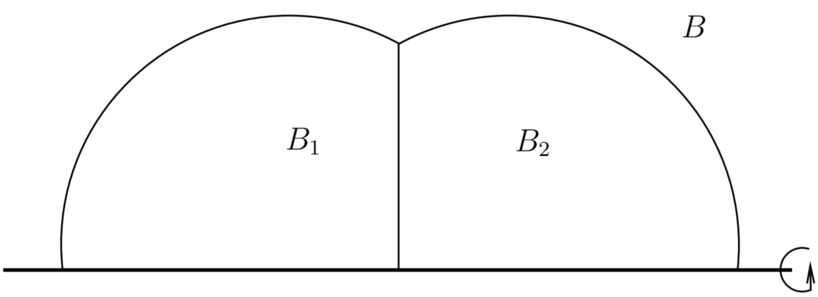


FIGURE 33. Since B is nonstandard S_1 or S_2 must meet some hypersuface other than S_0, S_1, and S_2. We can then roll $B - S_1$ around the sphere containing it until they first touch, creating an illegal singularity.

8.4. **The Double Bubble Conjecture In $\mathbb{R}^3$.** Hutchings, Morgan, Ritoré, and Ros have jointly showed in [18] that the standard double bubble is the unique double bubble enclosing and separating two volumes in $\mathbb{R}^3$ with the least surface area. Before the proof of the conjecture, Hutchings has shown in [17] that any minimal double bubble in $\mathbb{R}^3$ is either a standard double bubble or a surface of revolution about some line consisting of a topological sphere with a single tree of annular bands attached.

Our objective is to show that a non standard double bubble in $\mathbb{R}^3$ can not be area minimizing and hence by the structure theorem the standard double bubble is the unique minimizing double bubble enclosing and separating two quantities of volume. In order to accomplish our objective we will list several theorems appearing in [18] and show how these theorems are used to prove the conjecture.

Now we give the two forms of the standard double bubble in $\mathbb{R}^3$. Both bubbles are a hypersurface of revolution about some line as shown in [17]. Figure (34) is a standard double bubble such that both bubbles have the same volume and figure (35) is a standard double bubble such that the bubbles have different volumes.



L

FIGURE 34. B is a surface or revolution about L, both bubbles have the same volume and the separating surface is planar.

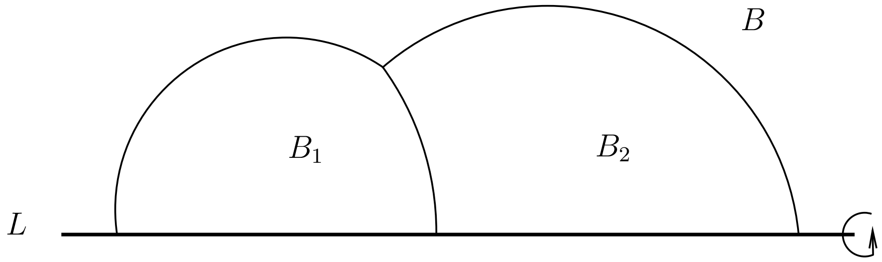


FIGURE 35. B is a surface of revolution about L, bubbles have different volumes and the mean curvature of the separating surface is the difference of the mean curvatures of outer caps.

Consider a minimizing double bubble of revolution about the x-axis L in $\mathbb{R}^n$ $(n \geq 3)$, with cross-section Γ consisting of circular arcs $\overline{\Gamma}_0$ meeting the axis, and other arcs $\overline{\Gamma}_i$ meeting in threes, wit interiors Γ_i. Define a map

$$\begin{aligned} f\colon \Gamma - L \to & L \cup \{\infty\} \\ p \quad \mapsto & L(p) \cap L \text{ if } L(p) \text{ meets } L \\ & \infty \text{ otherwise.} \end{aligned}$$

Proposition 8.4.1. [18, Proposition 5.2.] *Consider a minimizing double bubble of revolution about the x-axis L in $\mathbb{R}^n$ $(n \geq 3)$. Suppose that there is a minimal set of points $\{p_1, \ldots, p_k\}$ in $\bigcup \Gamma_i$ with $x = f(p_1) = \cdots = f(p_k)$ which separates Γ. Then every component of the regular set which contains some p_i, $i \in \{1, \ldots, k\}$ is part of a sphere centered at x (if $x \in L$ or part of a hyperplane othogonal to L (in the case $x = \infty$.*

Corollary 8.4.2. [18, Corollary 5.3.] *There is no nonstandard minimizing double bubble in $\mathbb{R}^n$ in which both regions and the exterior are connected.*

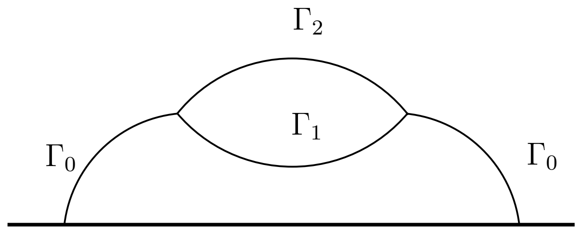


FIGURE 36. As corollary (8.4.2) shows, there is no nonstandard minimizing double bubble of revolution with connected bubbles and connected exterior.

Proposition 8.4.3. [18, Proposition 5.8.] *There is no minimizing double bubble in $\mathbb{R}^n$ in which the region of smaller or equal pressure is connected, the other region has two components, and the exterior is connected.*

Lemma 8.4.4. [18, Lemma 6.4.] *In a minimizing double bubble in $\mathbb{R}^n$ enclosing two unequal volumes, the smaller region has larger pressure.*

Theorem 8.4.5. [18, Theorem 7.1.] *The standard double bubble in* $\mathbb{R}^3$ *is the unique area minimizing double bubble for prescribed volumes.*

Proof. Consider a minimizing double bubble of volumes v and $1-v$, then the first region has at most k components where the value of k is given by (8.2.4)

$$k = \left\{ \frac{2A_3^0(v, 1-v) - A_3(1) - A_3(v)}{A_3(v)} \right\}^3 .$$

For $n = 3$ the table (1) gives the bounds as

- bound on the number of components in larger or equal region **1**

- bound on the number of components in smaller region **2**.

So either both regions are connected or one of larger volume and smaller pressure is connected and the other of smaller volume and larger pressure has two components.

If both regions are connected then by (8.4.2) the area minimizing double bubble should be standard. On the other hand, using the lemma (8.4.4) and the table (1) we can conclude that the larger region with smaller pressure is connected and the smaller region with larger pressure has two components, but in this case the proposition (8.4.3) asserts that the double bubble can not be a minimizing double bubble. Thus any minimizing double bubble should be standard. □

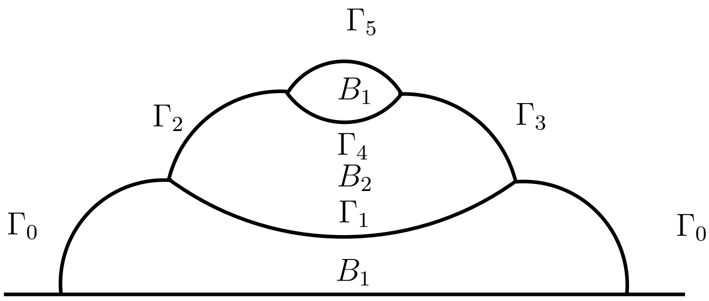


FIGURE 37. As a result of theorem (8.4.5) we can conclude that a double bubble with three components can not be minimizing.

8.5. Survey Of Double Bubble Problems.

- F. Morgan and W. Wichiramala have shown jointly in [21] that the standard double bubble is the unique double bubble satisfying the critical and stable points of the perimeter functional.

- Ben W. Reichardt has proven in [25] the conclusive double bubble conjecture which asserts that the standard double bubble is the unique area minimizing double bubble in $\mathbb{R}^n$ enclosing and separating any two volumes in $\mathbb{R}^n$.

- J. Cornelli, N. Hoffman, P. Holt, G. Lee, N. Leger, S. Moseley, and E. Schoenfeld have proven jointly in [6] the double bubble conjecture in $\mathbb{S}^3$ and $\mathbb{H}^3$ in the following cases:
  (1) in $\mathbb{S}^3$, when each enclosed volume and the complement occupy at least 10% of the volume of $\mathbb{S}^3$;
  (2) in $\mathbb{H}^3$, when the smaller volume is at least 85% that of the larger.

- J. Cornelli, P. Holt, G. Lee, N. Leger, E. Schoenfeld, and B. Steinhurst have shown in [7] that the standard double bubble is the unique perimeter minimizing double bubble on the two-torus.

- M.C. Álvarez, J. Cornelli, G. Walsh, and S. Beheshti have shown jointly in [4] that the only area minimizing double bubble enclosing two small volumes in 3- or 4-manifold is the standard double bubble.

Bibliography

[1] F.J. Almgren, *Some Interior Regularity Theorems For Minimal Surfaces And An Extension Of Bernstein's Theorem*, Ann. of Math. **84** (1966), 277 – 292. 6.6

[2] F.J. Almgren, *Existence And Regularity Almost Everywhere Of Solutions To Elliptic Variational Problems With Constraints*, Mem. AMS **4** (1976), no. 165. 1.1, 7, 8

[3] F.J. Almgren, *Q-valued Functions Minimizing Dirichlet's Integral And The Regularity Of Area Minimizing Rectifiable Currents Up To Codimension Two*, Bull. AMS **8** (1983), no. 2, 327 – 328. 6.6, 6.6.3

[4] M.C. Álvarez, J. Cornelli, G. Walsh, S. Beheshti, *Double Bubbles In The Three-Torus*, Experiment. Math. **12** (2003), no. 1, 79 – 89. 8.5

[5] E. Bombieri, E. De Giorgi, E. Giusti, *Minimal Cones And The Bernstein Problem*, Invent. Math. **7** (1969), 243 – 268. 6.8.3

[6] J. Cornelli, N. Hoffman, P. Holt, G. Lee, N. Leger, S. Moseley, E. Schoenfeld, *Double Bubbles In $\mathbb{S}^3$ and $\mathbb{H}^3$*, J. Geom. Anal. **17** (2007), no. 2, 189 – 212. 8.5

[7] J. Cornelli, P. Holt, G. Lee, N. Leger, E. Schoenfeld, B. Steinhurst, *The Double Bubble Problem On The Flat Two-Torus*, Trans. AMS **356** (2004), no. 9, 3769 – 3820 (electronic). 8.5

[8] J. Douglas, *Solution Of The Problem Of Plateau*, Trans. AMS **33** (1931), no. 1, 263 – 321. 1

[9] L.C. Evans and R.F. Gariepy, *Measure Theory And Fine Properties of Functions*, CRC Press, Boca Raton, FL, USA, 1992. 1.1, 2, 4, 4.4, 4.5, 4.5

[10] H. Federer, *Geometric Measure Theory*, Reprint of the 1969 ed, Springer-Verlag, Berlin, GERMANY, 1996. 1.1, 4, 4.6, 4.6, 6, 6.1, 6.1, 6.2, 6.2, 3, 6.8

[11] H. Federer, *The Singular Sets Of Area Minimizing Rectifiable Currents With Codimension One And Of Area Minimizing Flat Chains Modulo Two With Arbitrary Codimension*, Bull. AMS **76** (1970), 767 – 771. 6.6, 6.6.2

[12] W. Fleming, *On The Oriented Plateau Problem*, Rend. Circ. Mat. Palermo (2) **11**, (1962), 1 – 22. 6.6, 6.6.1

[13] J. Foisy, *Soap Bubble Clusters in $\mathbb{R}^2$ and $\mathbb{R}^3$*, undergraduate thesis, Williams College, 1991. 8.3

[14] J. Foisy, M. Alfaro, J. Brock, N. Hodges, and J. Zimba, *The Standard Double Soap Bubble In $\mathbb{R}^2$ Uniquely Minimizes Perimeter*, Pacific J. Math. **159** (1993), no. 1, 47–59. 1, 1.1, 7, 7.2.1, 7.2.2, 7.2.3, 7.2.4, 7.2.5, 7.2.6, 7.2.8, 7.2, 7.2.9, 8.3

[15] I.M. Gelfand and S.V. Fomin, *Calculus Of Variations*, Prentice-Hall, Englewood Cliffs, NJ, USA, 1963. 1.1, 6, 6.5

[16] R. Hardt, L. Simon, *Boundary Regularity And Embedded Solutions For The Oriented Plateau Problem*, Ann. of Math. **110** (1979), no. 3, 439 – 486. 6.6.4

[17] M. Hutchings, *The Structure Of Area Minimizing Double Bubbles*, J. Geom. Anal. **7** (1997), no. 2, 285–304. 1, 1.1, 8, 8.1.3, 8.1, 8.1.4, 8.1.5, 8.1.6, 8.1.7, 8.1.8, 8.1.9, 8.1.10, 8.2.1, 8.2.2, 8.2.3, 8.2.5, 8.3.1, 8.4

[18] M. Hutchings, F. Morgan, M. Ritoré, A. Ros, *Proof Of The Double Bubble Conjecture*, Ann. of Math. **155** (2002), no. 2, 459 – 489. 1, 1.1, 8, 8.4, 8.4.1, 8.4.2, 8.4.3, 8.4.4, 8.4.5

[19] S.G. Krantz and H.R. Parks, *Geometric Integration Theory*, Birkhäuser, Boston, MA, USA, 2008. 1.1, 2, 2.2.9

[20] F. Morgan, *Soap Bubbles In $\mathbb{R}^2$ And In Surfaces*, Pac. J. Math. **165** (1994), no. 2, 347 – 361. 1.1, 7, 7.2, 7.2, 8.1, 8.3

[21] F. Morgan, W. Wichiramala, *The Standard Double Bubble Is The Unique Stable Double Bubble In $\mathbb{R}^2$*, Proc. AMS **130** (2002), no. 9, 2725 – 2751 (electronic). 8.5

[22] F. Morgan, *Geometric Measure Theory. A Beginner's Guide*, fourth edition, Academic Press, San Diego, CA, USA, 2009. (document), 1.1, 2, 6, 8, 1

[23] J.R. Munkres, *Analysis On Manifolds*, Westview Press, USA, 1991. 1.1, 3, 3.1, 3.1.9, 3.3, 3.3, 3.4, 5, 5.1, 5.3, 5.4, 5.6

[24] T. Rado, *On The Problem Of Plateau*, Springer-Verlag, New York, NY, USA, 1933. 1

[25] Ben W. Reichardt, *Proof Of The Double Bubble Conjecture In $\mathbb{R}^n$*, J. Geom. Anal. **18** (2008), no. 1, 172 – 192. 8.5

[26] J. Simons *Minimal Varieties In Riemannian Manifolds*, Ann. of Math. **88** (1968), 62 – 105. 6.6

[27] Jean E. Taylor, *The Structure Of Singularities In Soap-Bubble-Like And Soap-Film-Like Minimal Surfaces*, Ann. of Math. **103** (1976), no. 3, 489 – 539. 1.1, 7, 8.3

[28] B. White, *Existence of least area mappings of N-dimensional domains.*, Ann. of Math. **118** (1983), no. 1, 179 – 185. 1