Exchange Stability: An Analysis of the Roommate Problem in the Presence of Room Scarcity

by

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A Thesis Submitted to the

Graduate School of Social Sciences & Humanities

in Partial Fulfillment of the Requirements for

the Degree of

Master of Arts

in

Economics

Koç University

August, 2009

Koç University

Graduate School of Social Sciences and Humanities (GSSS)

This is to certify that I have examined this copy of a master's thesis by

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To my family

ABSTRACT

This paper analyzes exchange stability for the roommate problem under the restriction that the rooms are in a scarce supply. While the central axioms in matching problems are Gale-Shapley stability (Gale and Shapley, 1962) and Pareto optimality, recently a new property, exchange stability is proposed. We demonstrate that the set of exchange stable solutions, which can be empty, is a subset of Pareto optimal solutions. We also prove an impossibility result that there is no well-defined solution which coincides with the set of exchange stable matchings whenever it is non-empty and satisfies consistency. Moreover, we show that Gale-Shapley stability and exchange stability are independent concepts, thus, algorithms to find Gale-Shapley stable solutions cannot be used for identifying exchange stable solutions. In addition, a necessary condition for a matching to satisfy both Gale-Shapley stability and exchange stability is found for the marriage problem, which is a special case of the roommate problem.

Keywords: Matching Problems, Roommate Problem, Marriage Problem, Stability, Exchange Stability, Consistency.

ÖZET

Bu makalede takas istikrarı, oda kısıtı bulunan oda arkadaşı problemleri için incelenmektedir. Takas istikrarı, Gale ve Shapley (1962) tarafından tanımlanan istikrar ve Morrill (2007) tarafindan önerilen Pareto optimum ile kıyaslandığında problemin çözümü için daha uygun bir özellik olduğu görülmektedir. Bu makalede takas istikrarını sağlayan eşleşmelerin kümesinin boş küme olabileceği gösterilmiştir. Ayrıca bu kümenin Pareto optimum eşleşmelerinin kümesinin alt kümesi olduğu kanıtlanmıştır. Kanıtlanan önemli sonuçlardan biri ise takas istikrarı ve tutarlılık hakkında bir imkansızlık sonucudur. Buna göre takas istikrarını sağlayan eşleşmelerin kümesinin boş küme olmadığı durumlarda bu küme ile örtüşen ve aynı zamanda tutarlı bir çözüm yoktur. Bunun yanısıra Gale-Shapley istikrarı ve takas istikrarının bağımsız olduğu kanıtlanmıştır. Bu durum Gale-Shapley istikrarını sağlayan eşleşmelerin bulunmasi için kullanılan algoritmaların takas istikrarını sağlayan çözümleri bulmada kullanılamayacağını vurgular. Ayrıca, oda arkadaşı probleminin özel bir durumu olan evlilik problemi için bir eşleşmenin hem Gale-Shapley istikrarını ve takas istikrarını sağlaması için gerekli koşullar belirtilmiştir.

Anahtar Kelimeler: Eşleşme Problemleri, Oda Arkadaşı Problemi, Evlilik Problemi, İstikrar, Takas İstikrarı, Tutarlılık.

ACKNOWLEDGEMENTS

I am greatly indebted to my supervisor, Özgür Yılmaz, whose expertise, understanding and patience contributed considerably to my graduate experience. I would like to express my gratitude to Levent Koçkesen and Seda Ertaç for their invaluable advices and being role models during my graduate studies with their understanding, kindness and patience. I am grateful to Yalçın Akçay for spending his valuable time to listen my thesis defense and be one of my committee members. I am also indebted to Erol Hasan Çakmak and Halil Baha Karabudak from Middle East Technical University for providing me direction and support and for being invaluable mentors during my academic studies. I owe a lot to my family for their love, encouragement and support they provided me through my entire life. I owe very special thanks to Ramazan Bora for his support and encouragement, without him I would not have finished this thesis. I must acknowledge Özlem Tonguç for being a precious friend and making valuable comments about the thesis. I would also thank to all of my friends at Department of Economics of Koç University for unforgettable two years. I am also grateful to TUBITAK and Vehbi Koç Foundation for financial assistance.

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1 Introduction

Lionel Robbins (1945) defines economics as "the science which studies human behavior as a relationship between ends and scarce means which have alternative uses". In this regard, matching theory, which deals with the problems of allocation, matching and exchange of resources or objects, is one of the central fields of study in economics.

There are various lines of interest in matching theory, few of which we would like to illustrate. One of the main problems of matching theory is the allocation or exchange of indivisible objects or resources among a group of people. Assigning houses or rooms to residents, jobs to workers, or transplant organs to patients can be enumerated as examples of this class of problems. There are also two-sided matching problems. These problems involve markets with two sides such as matching men with women, workers with firms, students with schools (Sönmez and Ünver, 2008). Another matching problem and which this paper primarily deals with is related to matchings in one-sided markets. Assigning a set of agents within themselves in pairs, triples, or in groups pertains to one-sided matching. For example, a well known one-sided matching problem is the roommate problem, in which n agents are assigned into rooms in pairs.

The matching literature on one-sided and two-sided matching is pioneered by the research of Gale and Shapley (1962). They introduced both two-sided problems, i.e., the marriage and the college admission problems and one-sided problem, i.e., the *roommate problem*[1]. In their work, they proposed *stability*[2] as the central axiom for both of these problems. They also proved that there is always at least one GS-stable matching for the two sided matching problem through a simple iterative algorithm known as *the deferred*

[1]The marriage problem and two-sided matching, and the roommate problem and one-sided matching are terms used interchangeably throughout the research.

[2]Henceforth stability will be referred to as GS-stability.

acceptance algorithm.[3] On the other hand, for the one-sided matching problem, they provided an example where a GS-stable matching does not exist.

Let's first consider the one-sided matching problem. In the roommate problem, a *matching* is the assignment of a group of agents into the rooms in pairs. According to Gale and Shapley's definition, a matching is *GS-stable* if there is no *blocking pair*, a pair of agents who prefer to be matched together to their current matches. What is remarkable about this definition is that if two agents want to be matched together, then they can become roommates. If there exists a blocking pair in a prescribed assignment, they can leave their existing roommates and become roommates as moving into a new room together, so the pair violates stability.

In order to evaluate whether a given roommate problem has a GS-stable solution, Irving (1985) proposed an algorithm that would find a stable matching if there is one or would report that none exists. Later, Tan (1991) specified the necessary and sufficient conditions for the existence of a stable matching for the roommate problem, namely, the non-existence of any odd party. [4] Therefore, we can identify whether a given roommate problem has a GS-stable solution. If there exists a GS-stable matching, then, according to the studies of Roth and Vande Vate (1990), Chung (2000) and Diamantoudi et al (2004), starting with an unstable matching, there is a random path to stability, namely, allowing randomly chosen blocking pairs to match converges to a GS-stable matching with probability one. This result is of importance because even if there is no centralized decision making system, the roommate market will eventually reach a GS-stable solution.

Another line of research in one-sided matching theory is the characterization of GS-stable matchings. One of the key properties in the characterization of the solutions, espe-

[3] Detailed explanation of the algorithm will be given in the third section.

[4] For a roommate problem with strict preferences, odd party is defined as a subset of agents $(a_1, ..., a_K)$ such that $K \geq 3$ is odd and $a_{k+1} \succ_{a_k} a_{k-1} \succ_{a_k} a_k$ for all $k \in \{1, ..., K\}$.

cially for the problems with dynamic populations, is *consistency*. To understand consistency, suppose that for a given roommate problem a solution is applied and a GS-stable matching is prescribed. After an assignment is made for given a roommate problem, some agents, which can be in pairs or singles, leave the market with their roommates and the remaining agents decide to apply the same solution to the restricted problem. The solution is said to be *inconsistent* if it prescribes a matching that is not GS-stable. Ozkal-Sanver (2008), and Klaus and Nichifor (2009) analyze consistency for the roommate problem in order to characterize GS-stable solutions. Yet, they both establish an impossibility result stating that there is no solution that satisfies consistency and coincides with the set of GS-stable solutions whenever there exists a GS-stable matching.

This research mainly focuses on a special situation where another source of scarcity is imposed into the roommate problem. Suppose that there are n agents and $n/2$ rooms so that the rooms are in scarce supply. For this case, the stability concept defined above does not apply. To understand this, consider that there exists a blocking pair for a matching, namely, there are two agents in two different rooms who prefer each other to their current roommates. They want to be matched with each other but they cannot move into a new room together since all rooms are occupied. Therefore, the existence of a blocking pair may not violate stability if there exists room scarcity.

On the other hand, think of a case in which there are two agents in two different rooms such that each of them prefers other's roommate to his current match for a given roommate assignment with room scarcity. So, they can be better off by exchanging their rooms and roommates. This situation motivates the introduction of a new stability notion when there is scarcity of rooms.

This modified problem was studied by Morrill (2007) too. Morrill also realized that Gale and Shapley's stability concept ignores the physical constraint and therefore, he claimed that this stability concept is too restrictive. Another axiom is *Pareto Optimality* as a proper solution concept for the modified roommate problem. A matching is defined to be Pareto optimal if any change to make any agent better off would make at least one agent worse off. However, this research shows that Pareto Optimality is not a proper solution concept for the modified problem.

Another property was introduced by Alcalde (1995). Alcalde stated that each allocation in one-sided matching markets gives property rights to the agents. Therefore, given an initial allocation, third parties may be affected by the actions of any pair of agents. Since the classical notion of stability introduced by Gale and Shapley (1962) does not take into consideration of such cases, Alcalde suggested exchange stability as a complement stability concept to the problem.

Alcalde explained exchange stability as follows. Given a roommate problem, suppose each matching, once achieved, endows agents with property rights. Therefore, after an initial allocation is made, roommates become “endowments” of each other. So, considering the roommate problem as a class of exchange markets, agents can be better off by exchanging their endowments. A matching is *exchange stable* if there exists no such exchanges made by any group of agents.

Cechlárová (2002) investigated the complexity of the problem of deciding whether an exchange stable matching exists for a given preference profile. She proved that the problem is nondeterministic polynomial complete (NP-complete).[5] In addition, Cechlárová and Manlove (2005) proved that the problem of deciding whether an exchange stable matching exists for a given marriage or roommate problem is NP-complete. They also studied the

[5]NP-completeness is a complexity class of decision problems for which answers can be checked for correctness by an algorithm in polynomial time. Solving time of a NP-complete problem increases exponentially with an increase in the problem size. A supposed solution to a NP-complete problem is easy to verify for correctness, but it is unknown if there is a significantly faster way to solve the problem than to try every single possible subset.

computational complexity of the problem of deciding whether there exists a stable matching that is also exchange stable and found that the problem is NP-complete for a roommate market. Moreover, Irving (2008) studied stable matching problems with exchange restrictions. His main contribution is that the problem of deciding whether there exists a stable matching with an additional property that no two men would prefer to exchange partners for a marriage market is NP-complete.

In this work we analyze exchange stability for the modified roommate problem and compare GS-stability and exchange stability in various aspects. Our main results show that the set of exchange stable matchings may be empty and this set, if it is non-empty, is a subset of the set of Pareto Optimal matchings. Moreover, we prove an impossibility result that there exists no solution which corresponds to the set of exchange stable matchings whenever it is non-empty and satisfies consistency. We also adduce that GS-stability and exchange stability are independent concepts. In addition, unlike GS-stable solutions, the set of agents may not be same for each of the exchange stable matching and there may be no random paths to exchange stability. Moreover, we introduce a necessary condition for achieving both the classical stability concept and exchange stability.

The remainder of this paper is organized as follows. Section 2 is devoted to the roommate problem. It begins with an introduction of the problem and discusses the stability concepts mentioned above. The main results are presented in this section. Section 3 is about marriage problem. After the introduction of the marriage problem, the necessary conditions for obtaining a solution that satisfies both GS-stability and exchange stability will be specified. Section 4 concludes the paper and discusses the further possible research.

2 The Roommate Problem Revisited

This section analyzes exchange stability for the roommate problem with scarcity of rooms. We begin with a formal definition of both the classical roommate problem and the problem with room scarcity. Then we discuss GS-stability, Pareto optimality and exchange stability. Later, the results about exchange stability are presented.

The roommate problem is about matching a set of agents among themselves in pairs or singletons so that the pairs share a room. Each agent has a preference ordering over all of the other agents and the prospect of remaining single. Formally, the *roommate problem* (Gale and Shapley, 1962) is a pair (A, P) where A is a finite set of agents and P is the preference profile of the agents in A. For each agent $a \in A$, preferences over all other agents, $P(a)$, are strict, complete and transitive. A solution to a roommate problem is called a matching, a partition of A in pairs and singletons.

Definition 1. Let (A, P) be a roommate problem. A **matching** μ is a function from the set of agents onto itself of order two, i.e., $\mu : A \to A$ such that for all $a \in A$, $\mu(\mu(a)) = a$.

Gale and Shapley (1962) proposed stability as the solution concept for the problem. The formal definition is as follows.

Definition 2. Agents a_1 and a_2 form a **blocking pair** to a matching μ if $\mu(a_1) \neq a_2$ and $a_1 \succ_{a_2} \mu(a_2)$ and $a_2 \succ_{a_1} \mu(a_1)$. A matching is **GS-stable** if there exists no blocking pair.

A roommate problem is *solvable* if the set of GS-stable matchings is non-empty. Gale and Shapley (1962) showed that a GS-stable matching may not exist for a roommate problem. They presented a roommate problem which is not solvable. The following example is similar to their counter example. Consider a roommate problem with three agents: a_1, a_2, a_3 with $P(a_1) = a_2, a_3; P(a_2) = a_3, a_1; P(a_3) = a_1, a_2$. Each agent prefers being matched to remaining single. Therefore, if all of the agents remain single then any two of the agents form a blocking pair. Thus, at least one pair should be formed. However, if that is the case, then the remaining agent is the most preferable roommate for one of the other two agents, so, they form a blocking pair.

Regarding the same example, now let's turn our attention to a case in which the rooms are in scarce supply. Suppose, there is only one room to allocate. When the room is allocated to a pair, say a_1 and a_2, it is known that the remaining single agent, a_3, and one of the matched agents, for this case a_2, will form a blocking pair. However, since the other agent, a_1, occupies the room, a_2 and a_3 cannot be matched. Therefore, even though the blocking pair violates GS-stability, this violation is vacuous when the rooms are scarce. Yet, the problem and the stability notion are different when there exists scarcity of rooms or, as Alcalde (1995) stated, in case of property rights.

The roommate problem with room scarcity, *the modified roommate problem*, was introduced by Morrill. It was an extension of Gale and Shapley's version. The problem involves assigning n agents into m rooms in pairs according to agents' strict, complete and transitive preferences. Thus, we could define the modified roommate problem with a triple (A, P, m), where A is the set of agents, P is the set of preference lists, and m is the number of rooms. We know that the traditional solution concept is GS-stability; however, as it is shown above, GS-stability might be too strong in the case of scarcity of rooms. Morrill suggested *Pareto optimality* as the central axiom for the modified roommate problem.

Definition 3. A matching μ is **inefficient** if there exists a different matching μ' such that

$$\mu'(a) \succeq_a \mu(a) \text{ for all } a \in A \quad \text{and}$$
$$\mu'(a) \succ_a \mu(a) \text{ for some } a \in A.$$

A matching is **Pareto optimal** if it is not inefficient.

A matching is Pareto optimal if any attempt to make any agent to better off would make someone worse off. The previous example with three agents may serve as a guide to understand why Morrill suggested Pareto optimality instead of GS-stability for the modified roommate problem. For each of the matching with one pair and one singleton, there is always one blocking pair. However, the blocking pair cannot be matched without having the will of the other agent. The agent would not vacate the room voluntarily because leaving the room and being single makes him worse off. Therefore, the blocking pair cannot be matched without making the other agent worse off. The following example motivates the introduction of a new stability notion where there is scarcity of rooms.

Example 1. *Let (A, P, m) be a modified roommate problem with four agents and two rooms and the preferences of the agents are as follows:*

$P(1) = 2, 3, 4$

$P(2) = 3, 4, 1$

$P(3) = 4, 1, 2$

$P(4) = 1, 2, 3$

There are three different candidate matchings:

$$\mu = \begin{matrix} 1-2 \\ \\ 3-4 \end{matrix}, \qquad \mu' = \begin{matrix} 1-4 \\ \\ 2-3 \end{matrix}, \qquad \mu'' = \begin{matrix} 1-3 \\ \\ 2-4 \end{matrix}.$$

The matching μ is a Pareto optimal Agent2 prefers 3 to 1, and agent 4 prefers 1 to 3, i.e. $3 \succ_2 1 = \mu(2)$ and $1 \succ_4 3 = \mu(4)$. On the other hand, μ matches 1 with 2 and 3 with 4. So, 2 and 4 can be better off by changing their rooms. If that is the case, μ' is reached. μ' is also Pareto optimal. Note that agent 1 prefers 2 to 4, and agent 3 prefers 4 to 2, i.e. $2 \succ_1 4 = \mu(1)$ and $4 \succ_3 2 = \mu(3)$. As μ' matches 1 with 4 and 2 with 3, 1 and 3 can be better off by exchanging their rooms. After this exchange, μ is reached again.

The matching μ'' is also Pareto optimal. It matches all agents with the second most preferred mates. There is no blocking pair, so μ'' is GS-stable matching. Also, there exists no group of agents who wants to exchange their rooms (or mates). For example, agent 1 prefers 2 to his mate in μ'', 3. However, agent 2's mate in μ, 4, prefers to be matched with 1 instead of his mate, 2. Therefore, agent 1 cannot change his room to be better off. The situation is similar for all agents.

The idea of blocking of a matching via exchanging mates is introduced by Alcalde (1995).

Definition 4. $R = (a_1, a_2, ..., a_r)$ is an **exchange-blocking ring** with length r to a matching μ if $\mu(a_2) \succ_{a_1} \mu(a_1)$, $\mu(a_3) \succ_{a_2} \mu(a_2)$, ..., $\mu(a_r) \succ_{a_{r-1}} \mu(a_{r-1})$, $\mu(a_1) \succ_{a_r} \mu(a_r)$, and $\mu(a_n) \neq a_m$ for all $n, m \in \{1, 2, ..., r\}$ where $r \geq 2$. A matching μ is **exchange stable** if there is no exchange-blocking ring.

It is known that there are problems with a solution, for example, there is an exchange stable matching for the modified roommate problem given in Example 1. However, an exchange stable matching may not exist for a modified roommate problem.

Proposition 2.1. *An exchange stable matching may not exist for a modified roommate problem,* (A, P, m).

Proof. Let the preferences of the agents of a modified roommate problem with four agents and two rooms be given by:

$P(1) = 2, 3, 4$

$P(2) = 3, 4, 1$

$P(3) = 4, 1, 2$

$P(4) = 1, 3, 2$

There are three different matchings:

$$\mu = \begin{array}{c} 1-2 \\ \\ 3-4 \end{array}, \qquad \mu' = \begin{array}{c} 1-4 \\ \\ 2-3 \end{array}, \qquad \mu'' = \begin{array}{c} 1-3 \\ \\ 2-4 \end{array}.$$

The matching μ matches 1 with 2 and 3 with 4. But, agent 2 prefers 3 to his mate in μ, 1, and agent 4 prefers 1 to his mate, 3, i.e. $3 \succ_2 1 = \mu(2)$ and $1 \succ_4 3 = \mu(4)$. So, agent 2 and 4 can be better off by exchanging their rooms and roommates as a result of which μ' is reached. On the other hand, agent 1 and 3 are worse off due to this exchange. They were matched with the most preferred mates in μ. Thus, they form a coalition to exchange their rooms, as a result the matching μ is achieved. So, neither μ nor μ' is exchange stable. The matching μ'' matches 1 with 3 and 2 with 4. But, agent 1 prefers 2 to his mate in μ'', 3, and agent 4 prefers 3 to his mate, 2, i.e. $2 \succ_1 3 = \mu(1)$ and $3 \succ_4 2 = \mu(4)$. Therefore, 1 and 4 form an exchange-blocking ring and so μ'' is not exchange stable. Hence, there is no exchange stable matching for this problem. $\square$

The arguments so far imply that GS-stability and exchange stability are actually different notions. In fact, they are independent concepts. This claim is proved in the following proposition.

Proposition 2.2. *The set of matchings that satisfy GS-stability and the set of matchings that satisfy exchange stability for a given roommate problem are independent.*

Proof. Consider a problem with four agents and two rooms, and let the preferences of the agents be given by:

$P(1) = 4, 2, 3$

$P(2) = 1, 3, 4$

$P(3) = 4, 1, 2$

$P(4) = 1, 3, 2$

The matching $\mu = \begin{array}{c} 1-2 \\ 3-4 \end{array}$ is exchange stable, since there is no exchange-blocking ring. However, since $4 \succ_1 \mu(1)$ and $1 \succ_4 \mu(4)$, agent 1 and 4 forms a blocking pair for GS-stability.

On the other hand, consider a problem with six agents and three rooms, and let the preferences of agents over other agents be given by:

$P(1) = 4, 2, 5, 3, 6$

$P(2) = 1, 5, 3, 4, 6$

$P(3) = 6, 4, 2, 1, 5$

$P(4) = 3, 1, 6, 2, 5$

$P(5) = 2, 6, 1, 3, 4$

$P(6) = 5, 3, 4, 2, 1$

The matching $\mu = \begin{array}{c} 1-2 \\ 3-4 \\ 5-6 \end{array}$ satisfies GS-stability. Nevertheless, agents 1, 3 and 5 form an exchange-blocking ring, since $4 \succ_1 2 = \mu(1)$, $6 \succ_3 4 = \mu(3)$ and $2 \succ_5 6 = \mu(5)$. Therefore, the matching μ is GS-stable but not exchange stable. Therefore, GS-stability and exchange stability are independent. □

The solutions to the classical roommate problem, the GS-stable matchings, also satisfy Pareto optimality. A simple proof can be found in Morrill's (2007) paper. As, we have shown in Example 1, Pareto optimality does not imply exchange stability. However, as we prove next, exchange stability implies Pareto optimality.

Proposition 2.3. *For a given roommate problem, if a matching is exchange stable, then it is Pareto optimal.*

Proof. The contrapositive of the statement will be proven, i.e. if μ is not Pareto efficient, then μ is not exchange stable.

Suppose μ is not Pareto efficient. Then, this means that there exists a matching μ' such that

$$\mu'(a) \succeq_a \mu(a) \text{ for all } a \in A \text{ and}$$
$$\mu'(a) \succ_a \mu(a) \text{ for some } a \in A.$$

Now, let a_1 be such that $\mu'(a_1) \neq \mu(a_1)$. We know that there exists such a_1 since $\mu'(a) \succ_a \mu(a)$ for some $a \in A$. Therefore, $\mu'(a_1) \succ_{a_1} \mu(a_1)$. Let $\mu'(a_1) = b_1$ and $\mu(b_1) = a_2$. Since $\mu'(a_1) = b_1$, $\mu'(a_2) \neq \mu(a_2) = b_1$. μ' is Pareto improvement, hence $\mu'(a_2) \succ_{a_2} \mu(a_2)$. Let $\mu'(a_2) = b_2$ and $\mu(b_2) = a_3$. Since $\mu'(a_2) = b_2$, $\mu'(a_3) \neq \mu(a_3) = b_2$. If $a_3 = a_1$, then an exchange-blocking ring with $r = 2$ is found. If $a_3 \neq a_1$, since μ is Pareto improvement, $\mu'(a_3) \succ_{a_3} \mu(a_3)$. Let $\mu'(a_3) = b_3$ and $\mu(b_3) = a_4$. If $a_4 = a_2$ or $a_4 = a_1$, the process identifies an exchange blocking ring with $r = 2$ or $r = 3$ respectively. Otherwise, continue the process. If $a_i = a_k$ where $k < i - 1$, an exchange-blocking ring with $r = i - k + 1$ and $R = (a_k, a_k + 1, ..., a_i)$. If $a_i \neq a_k$, then continue the process with a_{i+1} and b_i. Since A is finite, the process will eventually terminate and identify an exchange-blocking ring. $\square$

We have established that exchange stable matchings are also Pareto optimal. Similarly, our next result is an attempt to characterize the set of exchange stable solutions and tries to answer the following question: “Let (A, P) be a roommate problem with an exchange stable solution μ. Suppose a group of agents leaves with their matched roommates. Would the restriction of μ to the remaining agents be exchange stable?” The answer is “not necessarily” since there is no solution which coincides with the set of exchange stable matchings whenever it is non-empty and satisfies consistency.

A similar impossibility result about the set of GS-stable matchings and consistency is proved by Ozkal-Sanver (2008), and by Klaus and Nichifor (2009) independently. Ozkal-Sanver (2008) tries to characterize the set of GS-stable solutions for the class of one-sided matching problems regardless of the problem is solvable or not. Klaus and Nichifor (2009), on the other hand, analyze various desirable properties of solutions of the class of solvable roommate problems. Yet they share an impossibility result on the class of all one-sided assignment problems, there exists no solution satisfies consistency and coincides with the set of GS-stable matchings whenever the set is non-empty.

Let $(\mathfrak{A}, \mathfrak{P})$ denote the set of all roommate problems and $\mathcal{M}(A)$ denote the set of all matchings for the set of agents, A. A ***solution*** φ is a correspondence that associates each problem (A, P) with a non-empty subset of $\mathcal{M}(A)$. Given a roommate problem (A, P) and a subset of agents $A' \subset A$, a new problem on A' with preferences restricted to agents in A' is defined as a ***reduced problem*** of (A, P) with respect to A'. Formally; the reduced problem of (A, P) with respect to A' is $(A', P|_{A'}) \in (\mathfrak{A}, \mathfrak{P})$ for all $(A, P) \in (\mathfrak{A}, \mathfrak{P})$ and all $A' \subset A$.

Definition 5. A solution φ satisfies ***consistency*** if for all $(A, P) \in (\mathfrak{A}, \mathfrak{P})$ and all $\mu \in \varphi(A, P)$, $\mu|_{A'} \in \varphi(A' \cup \mu(A'), P|_{A' \cup \mu(A')})$ for all proper subsets $A' \subset A$.

Theorem 2.1. *For the roommate problem, there is no well-defined solution φ that coincides with the set of exchange stable solutions whenever it is non-empty and satisfies consistency.*

Proof. Let $\xi(A, P)$ define the set of exchange stable matchings. Let φ be a solution for any roommate problem $(A, P) \in (\mathfrak{A}, \mathfrak{P})$ such that if $\xi(A, P) \neq \emptyset$ then $\varphi(A, P) = \xi(A, P)$ and φ satisfies consistency.

The following example can be found in Ozkal-Sanver (2008). Let (A, P) be a roommate problem with five agents and the preference lists of the agents be as follows.

$P(1) = 3, 4, 5, 2$

$P(2) = 1, 5, 3, 4$

$P(3) = 4, 2, 1, 5$

$P(4) = 5, 3, 2, 1$

$P(5) = 2, 1, 4, 3$

Define the set of single agents for a matching μ as $I_\mu = \{a_i \in A \quad | \quad \mu(a_i) = a_i\}$. Since $|A| = 5$, for any matching $\mu \in \varphi(A, P)$, $1 \leq |I_\mu| \leq 5$. First, consider any $\mu \in \varphi(A, P)$ such that $1 < |I_\mu| \leq 5$. There exists some $A' = \{a_i, a_j\} \subseteq I_\mu$. Let $(A', P|_{A'})$ be the reduced problem. Since $\varphi(A', P|_{A'}) = \mu' = \xi(A', P|_{A'})$ such that $\mu'(a_i) = a_j$, $\mu|_{A'} \notin \varphi(A', P|_{A'})$. This contradicts with the consistency of φ.

Now, consider any $\mu \in \varphi(A, P)$ such that $|I_\mu| = 1$. The following table shows 15 different matchings with $|I_\mu| = 1$. μ, A', $\mu|_{A'}$ and $\varphi(A' \cup \mu(A'), P|_{A' \cup \mu(A')})$ are shown in the columns respectively. It can be observed from the table, for each $\mu \in \varphi(A, P)$ with $|I_\mu| = 1$, there exists a subset A' such that $\mu|_{A'} \notin \varphi(A' \cup \mu(A'), P|_{A' \cup \mu(A')})$. This contradicts with the consistency of φ.

μ	A'	$\mu\|_{A'}$	$\varphi(A' \cup \mu(A'), P\|_{A' \cup \mu(A')})$
$\{(1, 2), (3, 4), (5)\}$	$\{3, 4, 5\}$	$\{(3, 4), (5)\}$	$\{(4, 5), (3)\}$
$\{(1, 3), (2, 4), (5)\}$	$\{1, 3, 5\}$	$\{(1, 3), (5)\}$	$\{(1, 5), (3)\}$
$\{(1, 4), (2, 3), (5)\}$	$\{1, 4, 5\}$	$\{(1, 4), (5)\}$	$\{(1, 5), (4)\}$
$\{(1, 2), (3, 5), (4)\}$	$\{3, 4, 5\}$	$\{(3, 5), (4)\}$	$\{(4, 5), (3)\}$
$\{(1, 3), (2, 5), (4)\}$	$\{2, 4, 5\}$	$\{(2, 5), (4)\}$	$\{(4, 5), (2)\}$
$\{(1, 5), (2, 3), (4)\}$	$\{2, 3, 4\}$	$\{(2, 3), (4)\}$	$\{(3, 4), (2)\}$
$\{(1, 2), (4, 5), (3)\}$	$\{1, 2, 3\}$	$\{(1, 2), (3)\}$	$\{(2, 3), (1)\}$
$\{(1, 4), (2, 5), (3)\}$	$\{1, 3, 4\}$	$\{(1, 4), (3)\}$	$\{(3, 4), (1)\}$
$\{(1, 5), (2, 4), (3)\}$	$\{2, 3, 4\}$	$\{(2, 4), (3)\}$	$\{(3, 4), (2)\}$
$\{(1, 3), (4, 5), (2)\}$	$\{1, 2, 3\}$	$\{(1, 3), (2)\}$	$\{(2, 3), (1)\}$
$\{(1, 4), (3, 5), (2)\}$	$\{2, 3, 5\}$	$\{(3, 5), (2)\}$	$\{(2, 3), (5)\}$
$\{(1, 5), (3, 4), (2)\}$	$\{1, 2, 5\}$	$\{(1, 5), (2)\}$	$\{(1, 2), (5)\}$
$\{(2, 3), (4, 5), (1)\}$	$\{1, 4, 5\}$	$\{(4, 5), (1)\}$	$\{(1, 5), (4)\}$
$\{(2, 4), (3, 5), (1)\}$	$\{1, 2, 4\}$	$\{(2, 4), (1)\}$	$\{(1, 2), (4)\}$
$\{(2, 5), (3, 4), (1)\}$	$\{1, 2, 5\}$	$\{(2, 5), (1)\}$	$\{(1, 2), (5)\}$

$\square$

Before continuing to the next result, we need to underscore the following issue for the variable population problems: A roommate problem with at least one exchange stable matchings can be extended to a new problem with no exchange stable matching by an introduction of new agents to the original problem. This can be seen from the previous example given in the proof of the impossibility result. There are exchange stable assignments for the three agent problem, yet there is no exchange stable matchings for the roommate problem with five agents. Thus, we can conclude that a roommate problem with no exchange stable assignment may have reduced problems with an exchange stable matching.

The roommate problem is about matching agents in pairs or singletons. If the set of GS-stable matchings is non-empty, then the singletons are the same for all GS-stable matchings. This result is presented in Proposition 2.4. The single agents are the ones who want to be alone or the ones that nobody wants to be matched with. However, circumstances are different in the case of scarcity of rooms and property rights. An agent may be compelled to be matched by his less preferred agent if all rooms are assigned and there is no other room available. Therefore, the singles may be different for each of the exchange stable matchings. This claim is given in Proposition 2.5. The following lemma is used for Proposition 2.4.

Lemma 2.1. ***Decomposition Lemma for the Roommate Problem.*** *Let μ and μ' be GS-stable matchings in a roommate problem (A, P), and all preferences are strict. Let $A(\mu)$ be the set of agents who prefer μ to μ', and let $A(\mu')$ be the set of agents who prefer μ' to μ. Then μ and μ' map $A(\mu)$ onto $A(\mu')$.*

Proof. Suppose $a \in A(\mu')$. Then $\mu'(a) \succ_a \mu(a) \succeq_a a$, so $\mu'(a) \neq a$. Now, let $b = \mu'(a)$. Then we cannot have $\mu'(b) \succ_b \mu(b)$ because if that is the case, then (a, b) would be a blocking pair for μ. Therefore, since preferences are strict, $b \in A(\mu)$. Thus, $\mu'(A(\mu')) \subseteq A(\mu)$.

Similarly, if $b \in A(\mu)$, then $\mu(b) \succ_b \mu'(b) \succeq_b b$. This implies that $\mu(b) \neq b$. Now let $c = \mu(b)$. Then we cannot have $\mu(c) \succ_c \mu'(c)$ because it that is the case, then (b, c) would be a blocking pair for μ'. Since preferences are strict, $c \in A(\mu')$. Therefore $\mu(A(\mu)) \subseteq A(\mu')$.

Since μ and μ' are one-to-one and $A(\mu)$ and $A(\mu')$ are finite, μ and μ' map $A(\mu)$ onto $A(\mu')$. □

Proposition 2.4. *For a roommate problem with (A, P), the set of agents who are single is the same for all GS-stable matchings if the problem has at least one solution.*

Proof. If there is only one solution to the problem, then the conclusion follows trivially.

If there are multiple solutions, then let μ and μ' be different stable matchings. Suppose a is matched under μ' but not under μ, i.e. $\mu(a) = a$. Since all preferences are strict, $a \in A(\mu')$. By Lemma 2.1, μ maps $A(\mu)$ onto $A(\mu')$. This means a is also matched under μ which contradicts with a being single in μ. □

Remark. A marriage problem is a special case of a roommate problem (A, P) where the set of agents A is the union of two disjoint sets, women (W) and men (M), and each agent in W prefers being single to being matched with any agent in W and, similarly, any agent in M prefers remaining single to being matched with any agent in M. Since the roommate problem is a generalization of the marriage problem, Lemma 2.1 and Proposition 2.4 also hold for the marriage problem. These results will be used in the Marriage Problem section.

Proposition 2.5. *For a roommate problem, (A, P, m), the agents that remain as singles may not be same for all exchange stable matchings.*

Proof. Let (A, P, m) a roommate problem with five agents and three rooms. Let the preferences of the agents be given by:

$P(1) = 2, 3, 4, 5$

$P(2) = 1, 5, 4, 3$

$P(3) = 1, 4, 5, 2$

$P(4) = 1, 2, 3, 5$

$P(5) = 2, 3, 1, 4$

Now, consider the matchings

$$\mu = \begin{array}{l} 1-2 \\ 3-4 \\ 5 \end{array} \text{ and } \mu' = \begin{array}{l} 1-3 \\ 2-5 \\ 4 \end{array}.$$

The rooms are allocated to two pairs and a single agent. There is no exchange-blocking coalition for both of the matchings. Therefore, each of them is exchange stable; however, the single agents in each matching are different.

□

Another distinction between GS-stability and exchange stability is about random paths to stability. Diamantoudi et al. (2004) generalized the results about random paths to stability of Roth and Vande Vate (1990) and Chung (2000) under strict preferences. They proved that for all roommate problems with strict preferences, if a problem is solvable, i.e. a GS-stable matching exists, then there exists a random path to GS-stability for any unstable matching. This means that, for any unstable matching, there is a finite sequence of consecutive blocking pairs leading to a GS-stable matching if at least one GS-stable matching exists. Therefore, starting from any unstable matching, the process of allowing a randomly chosen blocking pair to match converges to a GS-stable matching with probability one. This result is crucial because it implies that the decentralized decision making process always leads to a GS-stable matching if the problem has a GS-stable solution. However, unfortunately, this property does not hold for exchange stability. Even if it is known that there exists an exchange stable matching, there is no guarantee that the process of allowing a randomly chosen exchange-blocking ring to exchange their room leads to an exchange stable solution for any exchange-unstable matching. It should be emphasized that if an exchange-blocking coalition is formed and all coalition members' roommates are worse off due to this exchange, then the roommates always have a chance to reverse this situation by forming a coalition to exchange their rooms. Formally, let $R = (a_1, a_2, ..., a_r)$ be an exchange-blocking ring for a matching μ and $\mu(a_i) = b_i$ for all $i \in \{1, 2, ..., r\}$. So, $\mu(a_{i+1}) = b_{i+1} \succ_{a_i} \mu(a_i) = b_i$ for all $i \in \{1, 2, ..., r\}$. Let μ' be the matching achieved after allowing the exchange-blocking ring to exchange their rooms. Thus, $\mu'(a_i) = b_{i+1}$. Assume, for all $i \in \{1, 2, ..., r\}$, b_i's are worse off due to this exchange, i.e., $\mu(b_i) \succ_{b_i} \mu'(b_i)$. Since $\mu(b_i) = a_i = \mu'(b_{i+1})$ for all $i \in \{1, 2, ..., r\}$, $\mu'(b_{i+1}) \succ_{b_i} \mu'(b_i)$ holds for all $i \in \{1, 2, ..., r\}$. Therefore, $R' = (b_1, b_2, ..., b_r)$ is an exchange-blocking ring for μ' and if exchange is allowed for R', then μ is reached. Since there may be no random path to exchange stability even if there exists an exchange stable matching, the initial matching becomes very important to reach an equilibrium for the modified roommate problem and the decentralized decision making process can not be utilized if the aim is to reach an exchange stable matching. Depending on the initial matching, a roommate problem can be stuck in a disequilibrium situation although there exists an exchange stable solution. The following proposition summarizes the above observations.

Proposition 2.6. *For any roommate problem, there may be no random paths to exchange stability even if there exists an exchange stable matching.*

Proof. We will provide a counterexample. Recall Example 1. This is a four agents and two rooms problem with the following preference lists of the agents:

$P(1) = 2, 3, 4$

$P(2) = 3, 4, 1$

$P(3) = 4, 1, 2$

$P(4) = 1, 2, 3$

We know that there exists an exchange stable matching $\mu'' = \begin{matrix} 1-3 \\ 2-4 \end{matrix}$. If μ'' is the initial matching, then equilibrium is achieved automatically.

Now, consider the matching $\mu = \begin{matrix} 1-2 \\ 3-4 \end{matrix}$. It can be observed that the agents 2 and 4 form an exchange-blocking pair, since $\mu(4) = 3 \succ_2 \mu(2) = 1$ and $\mu(2) = 1 \succ_4 \mu(4) = 3$. If 2 and 4 exchange their rooms, $\mu' = \begin{matrix} 1-4 \\ 2-3 \end{matrix}$. is reached. Yet, as we stated earlier, 1 and

3 form an exchange-blocking pair for μ' since they are both worse off due to the previous exchange. If 1 and 3 exchange their room, μ is reached again. However, we know that μ is not exchange stable and this process will go on indefinitely. $\square$

3 The Marriage Problem

This section specifies the necessary condition for achieving both GS-stability and exchange stability for a given marriage problem.

A marriage market involves two distinct sets, men and women. Each agent has a preference ordering on the members of the opposite sex. More formally, we can define the *marriage problem* as a triple (M, W, P) where M is the set of men, W is the set of women and P is the set of preference lists of the agents. For each agent, preferences over the agents on the other side of the market and the prospect of remaining single, are strict, complete and transitive. A matching, which is a set of man-woman pairs and the remaining singletons, is a solution to the marriage problem. The formal definition is as follows.

Definition 6. Let (M, W, P) be a marriage market. A **matching** μ is one-to-one correspondence from the set of agents M $\cup$ W onto itself of order two such that if $\mu(m) \neq m$ then $\mu(m) \in W$ and if $\mu(w) \neq w$ then $\mu(w) \in M$.

As it is stated earlier, Gale and Shapley (1962) proposed GS-stability as the solution concept for the problem. They also proved that every marriage problem has a GS-stable solution through a simple iterative algorithm known as the *deferred acceptance algorithm*. Next, we define GS-stability for the marriage problem, then discuss the deferred acceptance algorithm.

Definition 7. Let (M, W, P) be a marriage market. For a given matching μ, a man m and a woman w forms a **blocking pair** if $\mu(m) \neq w$, and $w \succ_m \mu(m)$ and $m \succ_w \mu(w)$. A matching μ is **GS-stable** if there is no blocking pairs.

The deferred acceptance algorithm proposed by Gale and Shapley (1962) assures the existence of a GS-stable matching for the marriage problem. There are men and women proposing versions of the algorithm. The following procedure describes the men proposing version. The women proposing version can be obtained by interchanging the roles of men and women in the algorithm. The algorithm involves a sequence of proposals from men to women. Each man proposes to his most preferred woman, namely, first woman in his preference list. Each woman rejects the proposal if the man is *unacceptable*, i.e., she prefers to be single rather being matched to this man, or if she receives more than one proposals, she rejects all but her most preferred one among these men. All men with accepted proposals are kept engaged. Any man who was rejected at the previous step proposes to the next woman in his preference list. This procedure continues as long as there is an acceptable woman to whom he has not proposed yet. Each woman rejects proposals from any unacceptable men. If she is engaged and received one or more new proposals, then she accepts her most preferred man among these. If no man is rejected in any step of the procedure, then the algorithm stops. After the algorithm terminates, every man either is engaged to a woman or rejected by all women in his preference list. The engaged pairs is married, and women who do not get any acceptable proposals and men whose proposals are all rejected remain single. At the end of the deferred acceptance algorithm with men proposing, the matching μ_M is obtained.

Gale and Shapley (1962) proved that μ_M always exists and is GS-stable. In addition, they proved that μ_M is M-optimal GS-stable matching, i.e., every man likes at least as any other GS-stable matching. Roth and Sotomayor (1990) defined a man and a woman to be *achievable* to each other for a given marriage market (M, W, P) if they are matched at some GS-stable matching. In the light of this definition, they concluded that μ_M matches each man with his most preferred achievable woman. Similarly, μ_W is produced by the deferred acceptance algorithm with women proposing. μ_W is W-optimal matching, that is, every woman likes it as least as any other GS-stable matching. In addition, μ_W matches each woman with her most preferred achievable man.

The definition of exchange stability for the marriage problem is given next.

Definition 8. Let (M, W, P) be a marriage market. For a given matching μ, $R = (m_1, m_2, ..., m_r)$ is an **exchange-blocking ring** formed with men with length r if $\mu(m_2) \succ_{m_1} \mu(m_1)$, $\mu(m_3) \succ_{m_2} \mu(m_2)$, ..., $\mu(m_r) \succ_{m_{r-1}} \mu(m_{r-1})$, $\mu(m_1) \succ_{m_r} \mu(m_r)$, and $m_k \neq m_n$ for all $k, n \in \{1, 2, ..., r\}$ where $r \geq 2$. Exchange blocking ring formed by women is defined similarly. A matching μ is **exchange stable** if there is no exchange-blocking ring formed by men or women.

Now, we are ready to give a necessary condition for a matching to satisfy both GS-stability and exchange stability.

Proposition 3.1. *For any marriage market with more than one GS-stable solution, there is no GS-stable matching that is also exchange stable. If there exists only one GS-stable solution for the marriage market, then it is the only candidate for satisfying both GS-stability and exchange stability.*

Proof. The aim of the proof is to show that there exists at least one exchange blocking coalition for each of the GS-stable matching for marriage problems with more than one solution.

Since more than one GS-stable matchings exists, we know that W-optimal and M-optimal GS-stable matchings exist and are different from each other.

Consider any GS-stable matching, μ, that is different than W-optimal GS-stable matching, including M-optimal GS-stable matching. Since, μ_W is W-optimal matching, every woman likes it at least as well as any other matching. Thus, for each woman w, either $\mu_W(w) = \mu(w)$ or $\mu_W(w) \succ_w \mu(w)$ holds. If the former holds, the mate of the woman, w, is same for both of the GS-stable matchings or she can be single for all GS-stable matchings, i.e. her mate is herself for all GS-stable matchings. If the latter holds, w is matched with a less preferred match under μ. Let A be the set of women for whom the latter inequality hold. Given Proposition 2.4, the set of people who is single is the same for all GS-stable matchings. Therefore, the set of agents can be decomposed into three groups: the singles, the people matched with the same mates in both of the matchings, and the remaining agents, which consists of the set A and their mates. Thus, $\mu_W(A) = \mu(A)$. Now, consider any woman, say w_1 from A. She is matched with her most achievable man in W-optimal matching. However, since it is known that her mate in μ_W is not single in μ as a consequence of Proposition 2.4, then he is matched another woman, say w_2, in μ. That is, $\mu(w_2)$ $= \mu_W(w_1) \succ_{w_1} \mu(w_1)$. Thus, we can conclude that w_2 is also an element of A, since she is not matched with the same mate in μ_W. Therefore, there exists w_3 such that $\mu(w_3) = \mu_W(w_2) \succ_{w_2} \mu(w_2)$ holds. If $w_3 = w_1$, w_1 and w_2 forms an exchange blocking coalition. If $w_3 \neq w_1$, then $w_3 \in A$ and there exists $w_4 \in A$ such that $\mu(w_4) = \mu_W(w_3) \succ_{w_3} \mu(w_3)$ holds. If $w_4 = w_1$, then w_1, w_2, w_3 forms an exchange blocking coalition. If $w_4 \neq w_1$, we repeat the process. Since W is finite, A is also finite. Eventually, the process will identify an exchange blocking coalition, formed by the women.

A similar argument holds for W-optimal GS-stable matching. However, in this case, men form an exchange blocking coalition. Define the set, B, of the men who strictly prefers M-optimal matching to W-optimal matching, μ_M to μ_W. Thus, $\mu_M(m) \succ_m \mu_W(m)$ for all $m \in B$. Also, similar to the argument above, $\mu_M(B) = \mu_W(B)$. Let $m_1 \in B$. Then, there exists m_2 such that $\mu_W(m_2) = \mu_M(m_1) \succ_{m_1} \mu_W(m_1)$. Therefore, $m_2 \in B$, because he is matched with different mates in μ_M and μ_W. So, there exits m_3 such that $\mu_W(m_3) = \mu_M(m_2) \succ_{m_2} \mu_W(m_2)$. Thus, m_3 is also an element of B. If $m_3 = m_1$, then m_1 and m_2 forms an exchange blocking coalition. If $m_3 \neq m_1$, then there exists m_4 such that $\mu_W(m_4) = \mu_M(m_3) \succ_{m_3} \mu_W(m_3)$. Thus, $m_4 \in B$. If $m_4 = m_1$, then m_1, m_2 and

m_3 forms an exchange blocking coalition. If $m_4 \neq m_1$, then the process is repeated. Since M is finite, B is also finite. Eventually, the process will identify an exchange blocking coalition, formed by the men. $\square$

The theorem above assures that there is no exchange stable solution that belongs to the set of stable matchings if there are more than one GS-stable matching. Therefore, the only candidate for an exchange stable solution among the GS-stable matchings is for the problems with only one stable solution. For these markets, deferred acceptance algorithm with men proposing and women proposing produce the same solution. Therefore, M-optimal and W-optimal matchings are the same, i.e., $\mu_M = \mu_W$. This means that all men and all women are matched with their most preferred achievable mates. However, this does not ensure that the only GS-stable matching is also exchange stable. In the following example, there is only one GS-stable matching which is not exchange stable.

Example 2. *(Roth and Sotomayor, 1990) Let $M = \{m_1, m_2, m_3\}$ and $W = \{w_1, w_2, w_3\}$ with preferences over the acceptable people given by:*

$$\begin{array}{ll} P(m_1) = w_2, w_1, w_3 & P(w_1) = m_1, m_2, m_3 \\ P(m_2) = w_1, w_2, w_3 & P(w_2) = m_3, m_1, m_2 \\ P(m_3) = w_1, w_2, w_3 & P(w_3) = m_1, m_2, m_3. \end{array}$$

Then,

$$\mu_M = \mu_W = \begin{array}{ccc} w_1 & w_2 & w_3 \\ m_1 & m_3 & m_2 \end{array}.$$

However,

$$\mu_1 = \begin{array}{ccc} w_1 & w_2 & w_3 \\ m_3 & m_1 & m_2 \end{array}.$$

can be achieved by exchanging m_1 and m_3. m_1 and m_3 are better off by exchanging their mates; whereas, w_1 and w_2 are worse off under μ_1 compared to the μ_M and μ_W.

Therefore, for this marriage problem, there is no GS-stable matching that is also exchange stable. Nonetheless, there is an exchange stable matching for this problem, i.e.,

$$\mu_2 = \begin{array}{ccc} w_1 & w_2 & w_3 \\ & & \\ m_2 & m_3 & m_1 \end{array} .$$

is exchange stable.

Proposition 3.1 and Example 2 adduce the incapability of the deferred acceptance algorithm in identifying the exchange stable solutions even if there is only one GS-stable solution for the problem.

4 Conclusion and Further Discussion

The primary concern of this paper is the roommate problem under the restriction that the rooms are in a scarce supply. Our findings suggest that the set of exchange stable solutions may be empty for a given roommate problem with room scarcity. Moreover, the set of exchange stable matchings is a subset of the set of Pareto optimal matchings.

Moreover, we establish an impossibility result that there is no solution which coincides with the set of exchange stable matchings whenever there exists at least one exchange stable matching and satisfies consistency. Moreover, we demonstrate with an example that a roommate problem with an exchange stable solution can be extended to a roommate problem with no exchange stable solution. This result is crucial for roommate problems with variable population.

We also show that the concepts of GS-stability and exchange stability are independent for a roommate problem. Therefore, disregarding scarce room supply and using Irving (1985) and Tan's (1991) algorithms for finding a solution for the roommate problem may lead to a matching that is not exchange stable. Therefore, one of the possible lines of further research can be on an algorithm that identifies an exchange stable solution, and on the characterization of the necessary and sufficient conditions to have an exchange stable solution for a given roommate problem.

In addition, for a given marriage problem, if the set of GS-stable matchings is a singleton, this matching is the only candidate for a matching that satisfies both GS-stability and exchange stability. For marriage problems with more than one GS-stable solution, it is impossible to obtain a solution that satisfy both GS-stability and exchange stability. Therefore, it is important to decide on which stability notion is more appropriate for the market.

One important distinction between GS-stability and exchange stability is the need for centralized decision making. Previous research shows that given a GS-unstable initial matching, there is always a decentralized decision making process that lead to a GS-stable solution. However, our research shows that this process may not be applicable for exchange stability. Even if there is an exchange stable solution, a decentralized decision making process may not result in achieving one.

On the other hand, if there is more than one exchange stable solutions, the social planner faces the problem of deciding among different solutions to implement. In GS-stable matchings, the single agents are the ones who wants to be alone or who nobody wants to be matched with. However, this result does not extend to exchange stable matchings, since we prove that the single agents may be different in each of the exchange stable solutions.

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