Risk Management in the Newsvendor Model with Random Supply using Financial Instruments

by

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A Thesis Submitted to the
Graduate School of Engineering
in Partial Fulfillment of the Requirements for
the Degree of

Master of Science

in

Industrial Engineering

Koç University

July, 2010

Koç University

Graduate School of Sciences and Engineering

This is to certify that I have examined this copy of a master's thesis by

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ABSTRACT

In this thesis we study a single-period, single-item inventory (newsvendor) problem. We analyze the opportunities of financial hedging to mitigate inventory risks when the demand and/or supply processes are correlated with the price of a financial asset. The risk or uncertainty in a classical newsvendor model is often generated by random demand. This randomness forces decision makers to determine their managing policies while facing risk. If the demand exceeds or falls short of expectations, the decision maker will face a shortage or a loss. However, apart from the uncertainty of demand we also incorporate supply uncertainty as a source of randomness. Supply uncertainty implies that the quantity received is not equal to the quantity ordered due to problems encountered during production or transportation. The combined randomness of demand and supply enhances the level of uncertainty, thus leading to an increased risk for the manager. Apart from the uncertainty levels, the majority of the literature on common inventory models are based upon two important assumptions. Primarily, a risk-neutral setting for the decision maker. Secondarily, independence of the demand and supply from any kind of financial instrument. So, the decision problem is often formulated as the minimization or maximization of the expected cost or profit. Then, the optimal inventory management policy is determined by solving the resulting optimization problem. Besides supply and demand there are other forms of risk as well such as interest rate, currency risk, catastrophe, etc.. Hence, we provide a general framework of decision making in a risky environment by categorizing our model under three different approaches. In the first one, we analyze the conventional newsvendor model with shortage cost. This model is extended by adding different types of supply uncertainty, while the assumption of independence between demand and market still holds. In the second one, we use financial instruments like options, bonds, futures, etc. to hedge the risks associated

with the revenue or the cash flow by assuming perfect correlation between demand/supply and the market. The manager or the decision maker now has to determine the optimal portfolio of these hedging instruments as well as the optimal ordering quantity. For the last approach, we characterize a setting for hedging the risk when there is partial correlation between demand/supply and the market. In such a scenario, forming a replicating portfolio will not be possible since there is no perfect correlation. So instead, a minimum variance type approach is used.

ÖZETÇE

Envanter, tedarik zincirinin her halkasında sıklıkla kullanılmaktadır. Dolayısıyla endüstri mühendisliği ve işletme yönetiminde en çok ele alınan konulardan biridir. Envanter modellerinde talep ve arz dengesi çok önemlidir. Ancak bu faktörlerin rassal olması sebebiyle karar yönetimi güçlükler içermektedir. Rassallık beraberinde risk getirir ve doğal olarak karar veren kişiler, kararlarını bu riskli ortam içerisinde vermek durumundadırlar. Talebin beklenenden az ya da çok olması ürünün elde kalmasına, yahut arzın karşılanamamasına yol açacaktır. Sonuç olarakda nakit akışı ile maliyet ve kararlar etkilenecektir. Talep ve arzdaki bu rassallık karar veren kişi riske karşı duyarsız (risk-neutral) olmadığı sürece göz ardı edilemez. Dolayısıyla envanter problemi sadece maliyet veya karların beklenen değerinin enazlanması veya ençoklanması olmaktan çıkar. Bu gerçek finansal modellerde iyice ortaya çıkmaktadır. Böyle modellerde riski kontrol altında tutabilmek için çeşitli yöntemlerin ve finansal ürünlerin kullanıldığı bilinmektedir.

Bu tezde tek dönem ve tek ürün içeren envanter (newsvendor) modelleri incelenmektedir. Amaç bu tip modellere finansal perspektiften bakarak arz ve talepten doğan rassallığı risk yönetim politikaları oluşturarak, kontrol altında tutmaktır. Araştırmada üç yaklaşım kullanılmıştır. Birincisinde karar verenlerin riske karşı duyarsız oldukları düşünülüp arzın rassal olduğu durumlar incelenecektir. İkincisinde karar verenin riske karşı duyarlı olduğunu kabul edip elde edilecek gelirin veya nakit akışının taşıdığı riski azaltmak amacı ile vadeli işlemler ve türev ürünler kullanılacaktır. Rassallığı meydana getiren talep veya arz gibi değerlerin finansal marketteki çeşitli ürünlerle korelasyonu olduğu bilinmektedir. Bu korelasyon kullanılarak oluşturulacak olan portföyde alınan pozisyon, dönem sonu nakit akışının riskini azaltabilir. Bu yaklaşımda karar veren kişi sipariş miktarlarının yanında portföy pozisyonunu da dikkate alacaktır. Üçüncü olarakda talep ve arz gibi rassalların finansal ürünlerle

olan korelasyonunun mükemmel olmadığı durumlar incelenecektir. Ancak, korelasyonun mükemmel olmaması nakit akışının finansal ürünler kullanılarak portföye çevrilmesini engellemektedir. Dolayısla ikinci bölümde kullandığımız metodlar geçersiz kalmaktadır. Bu sebepten dolayı sapmayı enazlama olarak adlandırılan başka bir yöntem kullanılacaktır. Bu yöntem iki aşamadan oluşmaktadır. Öncelikle, karar veren sabit bir sipariş miktarı için en uygun portföyü bulacaktır. Daha sonra, bu portföy kullanılarak beklenen kar ençoklanacaktır.

ACKNOWLEDGMENTS

First of all, I would like to express my gratitude to my supervisors, Prof. Süleyman Özekici and Prof. Fikri Karaesmen. Their expertise, understanding and patience contributed considerably to my thesis and graduate experience. I consider myself very lucky to have adbvisors such as them and I am also very proud of being one of their students. I am also very grateful to Asst. Prof Sibel Salman, Asst. Prof. Yalçın Akçay and Assoc. Prof. Metin Türkay for taking part in my thesis committee and for their valuable suggestions and comments. I would like to thank to my office mates Fatma, Filiz, Bahriye, Selin, Ezgi, Gökçen and Yeliz for their support and kindness. I would like to thank Gözde Gencer especially for being more than a sister to me. And I would like to thank my parents for believing in me, guiding me and supporting me my whole life.

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NOMENCLATURE

K	:	Random capacity
U	:	Random yield
D	:	Random demand
y	:	Order quantity
$Q(y)$	:	Amount of inventory recieved from ordering y units of inventory
c	:	Purchasing cost of one unit of inventory
h	:	Shortage cost of one unit of inventory
s	:	Salvage cost of one unit of inventory
p	:	Selling price of one unit of inventory
$\hat{p}$	:	Critical ratio or the probability of satisfying the demand on time
S_0	:	Price of a tradable asset in the market at time 0
S	:	Price of a tradable asset in the market at time T
F_X	:	Cumulative distribution function of X
f_X	:	Probability density function corresponding to F_X
r_X	:	Failure rate of X
Q	:	Risk-neutral probability measure
P_Q	:	Probability under risk neutral probability measure Q

Chapter 1

INTRODUCTION

Every business, whether it is a retailer, manufacturer or a simple office, has some form of inventory that someone has to keep track of. There are always items and products that must be ordered periodically for the continuation of the business. These items need to be managed efficiently or else, the business will lose money. To avoid such undesirable situations companies pay a lot of attention to inventory and its management. In a nut shell, inventory management can be referred to as the planned course of action against random consumption of the items, products, goods, etc. The scope of inventory management stretches from physical holding, lead times, holding costs, replenishments, defective goods to pricing, quality control and inventory visibility. Hence, inventory models can be regarded as one of the most widely studied topics in industrial engineering and operations management. Due to the uncertain nature of the environment, these models are known to have a complex structure.

Risk exposure refers to the undesirable outcome of a random prospect. In conventional newsvendor models, the random prospect is typically called demand and is modeled by an exogenous probability distribution. Randomness forces decision makers to determine their managing policies while facing a risk. If the demand exceeds or falls short of expectations, the decision makers will face shortage or loss. Moreover, the uncertainty of demand is not necessarily the only source of randomness. In fact in recent years, many studies emphasized models with supply uncertainty as well. Supply uncertainty implies that the quantity received is not equal to the quantity ordered due to problems encountered during production or transportation. The combined randomness of demand and supply enhances the level of uncertainty, thus leading to an increased risk for the manager.

Apart from the uncertainty levels, the majority of the literature on common inventory models are based upon two important assumptions. Primarily, a risk-neutral setting for the decision maker. This implies that the decision maker is completely indifferent to the risk involved in an investment and is only concerned about expected return. The second assumption, a more indirect but equally important one, is the independence of the demand and supply from any kind of financial instrument. So, the decision problem is often formulated as the minimization or maximization of the cost or profit based on the decision maker's experience of the past and expectations of the future. Then, the optimal inventory management policy is determined by solving the resulting optimization problem. Evidently, not all inventory managers are risk-neutral and this fact takes an important role in describing the risk attitude of investors in financial models. In fact, many planners are willing to trade lower expected profit for downside protection against possible losses. In other words they tend to be risk averse. Furthermore, in most business practices it is possible to find a correlation between the random parameters and the financial market. Disregarding such correlations could lead to arbitrage opportunities for careful eyes.

Besides supply and demand there are other risks as well such as interest rate, currency risk, etc.. According to a research conducted by International Swaps and Derivatives Association in August 2003, world's top 500 companies hedge most of their interest rate and currency risk, but only little of their commodity and equity risk. That's why there were more than \$221.3 bn losses incurred from 1984 to 2004 due to unexpected catastrophic events. Recalling the terrible disaster Katrina, one of the five deadliest hurricanes in United States in 2005, almost everybody lost a lot; however, some traders who bet on oil prices, exchange rates or SP500 made considerable profits. This extreme example illustrates the correlation between the financial market and a stochastic event. In our context, we consider stochastic events related to demand and supply uncertainty. Hence, by exploiting the correlation between demand/supply and the financial market one can hedge some portion of the risks associated with the random event. Nevertheless, conventional inventory models aren't able to provide methods to use such correlation and thus, meet the needs of risk-averse planners. Therefore, in this thesis, we study the opportunities of financial hedging to mitigate inventory risks when the demand and/or supply processes are correlated with the price of a financial asset. We start by providing important and related works that have been done until today in Chapter 2. We group them under four headings; satisficing probability maximization, utility functions, value at risk and random supply. In this thesis, we propose an effective framework of financial hedging that allows a manager to exploit various financial securities for mitigation of inventory carrying risks. Then, we determine optimal inventory and risk management policies by focusing on single-period newsvendor model. In order to provide a general setting we categorize our newsvendor model under three different approaches. In the first one, Chapter 3, we analyze the conventional newsvendor model with shortage cost. The model is extended by adding different models of supply uncertainty, but the assumption of independence between demand and market still holds. However, it is known that the demand or supply for the product is often correlated with some financial assets or economic indices. Therefore, one can manage the risks involved in an inventory model by taking positions in the futures or derivatives markets for such instruments. In the second one, Chapter 4, we use financial instruments like options, bonds, futures, etc. to hedge the risks associated with the revenue or the cash flow. At this point we continue our analysis by making the assumption of perfect correlation between demand/supply and the market. The manager or the decision maker now has to determine the optimal portfolio of these hedging instruments as well as the optimal ordering quantity. The process of finding an optimal portfolio entails the calculation of replicating portfolio. We utilize representations of difference of convex (DC) functions to generate the replicating portfolio. For the last approach, Chapter 5, we generate a framework of decision making for hedging the risk when there is partial correlation between demand/supply and the market. In such a scenario, forming a replicating portfolio will not be possible since there is no perfect correlation. So instead, a minimum variance type approach is introduced where we first determine the best possible hedged cash flow in terms of variance and then maximize the expected hedged cash flow by choosing the order quantity. Chapter 6 illustrates the results of our study and concludes the paper with directions to future research.

Our approach which combines inventory management and financial hedging is not commonly used in industrial engineering and operations management. Hence, we believe that our research in inventory management using risk management tools will constitute a novel and interesting project that also has practical implications. But one should note that, hedging obviously requires the presence of uncertainty and its standard objective is to reduce risk, not to make money.

Chapter 2

LITERATURE REVIEW

In the literature there are a lot of studies related to inventory management problems, specifically newsvendor models due to the fact that such models are the basic building blocks of many multi-period dynamic inventory, capacity-planning, and contract design problems (see the excellent textbook by Zipkin [47] for details). However, the majority of these studies are focused on risk-neutral decision makers who are concerned about expected profit or cost criteria. Although using expected values is very helpful in situations involving different uncertainties, it models decision makers to behave risk-neutrally, which in reality, is not true. That's why models with risk-neutrality assumptions have limited viability in practice. In recent years, the risk-averse behavior of the decision maker is addressed implicitly through other criteria such as satisficing probability maximization, utility functions, Value-at-Risk (VaR) (A technique which uses the statistical analysis of historical market trends and volatilities to estimate the likelihood that a given portfolio's losses will exceed a certain amount) and other risk measures.

Apart from the risk attitude of the decision maker, another concern for the newsvendor problem is the supplier's replenishment power. Generally the source of randomness in a newsvendor model is demand, yet in real life there is also supply uncertainty as well. We now discuss some of the literature in these areas.

2.1 Satisficing Probability Maximization

Satisficing probability maximization refers to probability of achieving a certain level of profit. These types of models are best suited for practices in which the dis-utility resulting from not achieving a certain level of profit is much larger than the rewards of over-achieving it. Lau [24] examines this issue by solving the satisficing probability maximization problem for a single-product model with shortage and salvage costs. Sankasubramanian and Kumaraswamy [39] also address the same problem for single-period inventory models. Then, Lau and Lau [23] use the some methodology for two-product newsvendor models, which is extended by Li et al. [26] and [27] for two-product newsvendor models for uniformly and exponentially distributed demands. Additionally single-period models are analyzed as multiple criteria maximization by Parlar and Weng [36] in order to find tradeoff with satisficing probability maximization. Their work also concludes that the optimal order quantity of satisficing probability maximization is less than the optimal order quantity for expected profit maximization in the conventional newsvendor problem.

2.2 Utility Functions

Starting with Lau [24], the utility function approach is commonly used for modelling the risk in inventory management. The conventional objective function is maximizing the expected utility of the decision maker. The structure of the utility function may vary, but usually exponential, quadratic and power functions can be seen in the literature. Lau [24], analyzes this issue for a single-period model where the utility function of the decision maker is quadratic, in other words, the mean-variance approach. Bouakiz and Sobel [5] use exponential functions for multi-period models and they conclude that a base-stock policy is optimal. Eeckhoudt et al. [12], examine the effects of risk aversion in single period problem by analyzing the changes in various price and cost parameters using piecewise-linear, kinked payoff function and exponential utility function. By comparing and contrasting the optimal ordering quantity for conventional newsvendor problem with the utility maximization problem and with different risk attitudes, Schweitzer and Cachon [40] investigate the decision bias when the demand is deterministic. The paper concludes that optimal ordering quantities are smaller than ordering quantities maximizing the expected profit for high-profit products and the opposite for low-profit products (a product is a high-profit product if the optimal shortage probability is below 0.5).

Among all the literature on inventory models, few worked on controlling the inventory risk through borrowing and trading in financial markets. Anvari [4], is one of the early papers written on this subject. According to the paper, the capital asset pricing model is used to solve the single-period newsvendor model with no setup costs by investing some portion of capital in inventory and other on financial assets. The resulting optimal policy is characterized and compared with the classical expected utility maximization structure. Gaur and Seshadri [16], another pioneer paper on this subject, use SP500 index to construct static hedging strategies using both mean-variance and utility-maximization frameworks. They establish a financial hedging approach when facing stochastic demand. They also put forward that SP500 index has high correlation with the demand process as long as the products have discretionary demand. For a single-period problem, in which there is a linear dependence between demand uncertainty with the market, they derive the hedged-cash flow for a perfectly-correlated arbitrage-free complete market; showing that it is possible to make riskless profits from non-financial operations of a firm by using financial instruments. But since perfect correlation is not completely realistic in practice, they extend their framework to fit partially-correlated markets using expected utility-maximization. An important aspect they pointed out is, the risk of inventory carrying can be replicated as a financial portfolio by using simple instruments like bonds, futures and options. According to their research, a risk-averse decision maker orders more inventory when hedging is applied. Caldantey and Hough [6] extend the hedging methods for continuous-time models. Their paper views the non-financial operations and facilities of a corporation, as assets in the corporation's portfolio; thus, turning the problem into a financial hedging problem in incomplete markets. By dynamically hedging the profits of a corporation, when these profits are correlated with returns in the financial markets, they propose a framework for modelling the operations of a non-financial corporation that also trades in the financial markets. A solution for the more general problem of simultaneously optimizing over both the operating and hedging policies of the corporation is provided. A more recent paper, Chu et al. [9] examines a continuously reviewed inventory model with uncertain demand by developing a continuously reviewed inventory model and a mean-variance criterion. Then, a financial hedging approach is established for hedging the inventory carrying risks. To our knowledge, these are the only papers that research hedging the risk through establishing a financial portfolio.

There are also other papers examining the control of risk with various other approaches. For example, Agrawal and Seshadri [1] consider a single-period inventory model in which a risk-averse retailer faces uncertain customer demand and makes a purchasing-order-quantity and a selling-price decision with the objective of maximizing expected utility facing uncertain customer demand. This problem is in many ways similar to the classic newsvendor problem, except the distribution of demand being a function of the selling price, which is determined by the retailer and the objective being utility maximization. Two separate approaches are considered in which price affects the distribution of demand. In the first model, the change in price affects the scale of the distribution and in the second model, the change in price only affects the location of the distribution. A methodology is presented by reducing two decision variables problem to a single variable problem. The results are then compared with a risk-neutral setting. According to the results, the first model will charge a higher price and order less; whereas, in the second model a risk-averse retailer will charge a lower price. Ding et al. [10], on the other hand, study the integrated operational and financial hedging of currency exchange rate risk using a mean-variance utility function to model the firm's risk aversion in decision making when there are multiple products and suppliers. They purpose a model within a two-stage decision framework; the first stage being the financial hedging for the exchange rate risk and the second stage being the production allocation for operational hedging. They show that the firm's financial hedging strategy is closely related to the firm's operational strategy. According to their paper the use, or lack of use, of financial hedges can alter global supply chain's structural choices, such as the desired location and number of production facilities to be employed to meet global demand. An extension of risk-averse, singe-item, multi-period inventory model is analyzed, with objective function being a coherent risk measure, by Ahmed et al. [2]. Their model also extends to infinite-horizon including fixed ordering costs. A recent paper, Wang et al. [45], also analyze the relationship between a risk-averse newsvendor's optimal order quantity and selling price. They conclude that for bounded decreasing absolute risk aversion utility functions, a risk-averse newsvendor tends to order less as selling price gets larger than a threshold value. A different approach, loss-aversion, is used to characterize the outcomes of the bias in decision making process by Wang and Webster [44]. They use a kinked piecewise-linear loss-aversion utility function to study the single-period newsvendor model. They find that a loss-averse newsvendor will order less than the risk-neutral newsvendor when facing low shortage cost and vice versa. The contributions of this paper to literature are the quantification of the newsvendor bias with loss-averse structure, and the influence of loss-aversion to supply chain inefficiency. Following this work, Wu et al. [46] also study the risk-averse newsvendor model under mean-variance objective function with stockout cost. But instead of the loss-averse model, they use the mean-variance trade-off and analyzed the effects of stockout cost. An explicit form of the variance of the profit function is derived and shown that the variance of the profit function is no longer a monotone increasing function. Furthermore, under the assumption that the demand function follows the power distribution, the set of optimal ordering quantities are found. Based on the results, they show that with stockout the risk-averse newsboy doesn't order less than the risk-neutral as long as a power distribution is used for the demand. Chen et al. [8] propose a framework for incorporating risk aversion in multi-period inventory models as well as multi-period models that coordinate inventory and pricing strategies. In each case, a characterization of the optimal policy for various measures of risk is provided. In particular, they show that the structure of the optimal policy for a decision maker with exponential utility functions is almost identical to the structure of the optimal risk-neutral inventory (and pricing) policies. These structural results are extended to models in which the decision maker has access to a complete or partially complete financial market.

2.3 Value-at-Risk

Other approaches for controlling the risk involves VaR and conditional VaR (CVaR) analyses. VaR is widely used in financial mathematics and financial risk management as a measure of the risk of loss on a specific portfolio of financial assets. For a given portfolio, probability and time horizon, VaR is defined as a threshold value such that the probability that the mark-to-market loss on the portfolio over the given time horizon exceeds this value (assuming normal markets and no trading in the portfolio) is the given probability level. Furthermore VaR can also be regarded as the maximum potential change in value of a portfolio of financial instruments over a defined horizon. CVaR on the other hand, defines the conditional expected loss exceeding VaR. In other words, CVaR accounts for the risk beyond the VaR value. For detailed reviews about the VaR models see Simons [42], Jorion [21], and Jorion and Dowd [11].

Tapiero [43] analyzes an asymmetric valuation between ex-ante expected cost above an appropriate target cost and the expected costs below that same target level. An explanation for the VaR criterion is provided when it is used as a tool for VaR efficiency design. This approach is used in a single-period stochastic inventory problem. Some of their ideas are also extended to multi-period problems as well. By using VaR criteria, Gan et al. [15], examine the inventory coordination problem between retailer and supplier. They incorporate VaR concept to newsvendor model with a downside risk constraint. Ozler et al. [31] study single-period problem with downside risk constaints while utilizing VaR for a multi-product newsvendor model. They derive the exact distribution function for the two-product newsvendor problem and develop an approximation method for the profit distribution of the multi-product case. Additionally, a mathematical programming approach is implemented to determine the solution of the newsvendor problem with a VaR constraint.

Rockafellar and Uryasev [38] derive fundamental properties of CVaR, as a measure of risk with significant advantages over VaR for loss distributions in finance that can involve discreetness. Gotoh and Takano [18] examine a single-period newsvendor model under CVaR criteria maximization problem. However, none of these works considered reducing risk via financial instruments.

2.4 Random Supply

Demand in inventory models is not the only the source of uncertainty. The process of determining an optimal ordering policy and actual replenishment from the suppliers involves lots of uncertainties as well. During suppliers production or the procurement phase, planned or unplanned events (such as maintenance, machine failures, problems, shortage of input materials, reprocessing, economical changes, trend shifts, disasters, managerial changes etc.) may occur. Due to these unforeseen events, from the retailer's point of view, the amount received could be different from the amount ordered. Furthermore, problems still continue to occur even after the production is finished due to transportation issues, accidents, scrapped goods, depreciation, etc.. Thus, in reality all of these random events contribute to a general supply uncertainty. And this randomness constitutes a different type of risk for the decision makers. In history there are a lot of tragic examples concerning losses made due to this supply risk. For example, according to Norrman and Jansson [30], Ericsson lost $400 m because of a fire occurred in one of its suppliers developing radio-frequency chips in 2001. The abundance of such examples encouraged the development of sparse literature on random supply models. The earliest model of a random supply in inventory model with random demand was developed by Karlin [22]. This is followed by Shih [41], Noori and Keller [29] and Lee and Yano [25], among many others. Karlin [22] assumes that the only decision available is whether to order, and that if an order is placed, a random quantity is delivered. It is also shown that if the inventory holding and shortage cost functions are convex increasing, then there is a single critical initial on-hand inventory below which an order should be placed, otherwise it is optimal not to order. Shih [41] assumes that inventory holding and shortage costs are linear and that the distribution of the fraction defective is invariant with the production level. The optimal production/order quantity can be found using a variant of the newsvendor model. Noori and Keller [29] extend Shih's study by providing closed form solutions for the optimal order quantity for uniform and exponential demand distributions and for various distributions of the quantity received. Gerchak et al. [17] obtain the same result for the profit maximization objective by assuming continuous demand and yield, and they consider a model with initial stock. According to their work, there is a critical level of initial stock above which no order will be placed, and this level is the same as the certain yield case. They also show that when initial stock is below that critical level, the expected yield corresponding to the amount ordered will in general not be simply equal to the difference. Henig and Gerchak [20] discuss single and multi-period models with more general assumptions about the random replenishment distribution and the cost structure. They prove that for a single-period model there exists an optimal order point that is independent of replenishment randomness. A substantial amount of research effort (for example, Parlar and Berkin [33], Parlar and Wang [35], Anupindi and Akella [3], Gupta [19], Parlar and Perry [34], Parlar [32], etc.) has been taken to model the supplier uncertainty phenomenon. Based on all the literature, it is possible to enumerate representations of supply randomness in three groups: Random capacity, random yield and binomial models. For the sake of better understanding, let y be the amount ordered and $Q(y)$ be the amount received from ordering y units.

- Random Capacity: In random capacity models, the supplier has a replenishment power which is a random variable, represented by K. In other words, the supplier has a random upper bound of units that are available to ship. This random capacity model can be represented as

  $$Q(y) = \min\{K, y\}. \tag{2.1}$$

  When an order is placed for y units, the suppliers will ship y if the total amount of on hand inventory they poses, K, is greater than y. Or else, they will send all the inventory they poses, which is K. Furthermore, this random capacity K may have some degree of correlation with demand. For example, Erdem and Özekici [13] consider a periodically reviewed single-item inventory model in a random environment where the yield is random due to the random capacity. By analyzing this problem in single, multiple and infinite horizons they show that a base-stock policy is optimal.

- Random Yield: In random yield models, it is assumed that the amount ordered could be different from the amount received so that only a fraction enters the stockpile. The randomness in this case is represented by a random variable U. The random yield model can be written as

$$Q(y) = yU. \tag{2.2}$$

When an order is placed for y units, the amount received will be Uy. For example Henig and Gerchak [20] considers a random availability model and show that a non-order-up-to policy is optimal.

- Binomial: In reality some of the goods ordered may be damaged, scrapped or lost. Binomial models are used to represent these kinds of uncertainties. Let p be the probability of successfully delivering a single order and $q = 1 - p$ be the probability of not being able to deliver a single order successfully. This randomness can be modeled so that the random supply has the binomial distribution

$$P\{Q(y) = x\} = \binom{y}{x} p^x q^{y-x}$$

for $x = 0, 1, 2, \cdots, y$. We will not analyze binomial models.

In this thesis, we generate a framework of decision making in a risk sensitive environment by examining a single-period newsvendor problem. Our work is closely related to Gaur and Seshadri [16] in the sense that we also examine static-hedging decisions to a single-period newsvendor, with risk-averse decision makers facing uncertainty. And we also analyze the use of financial securities to manage the inventory carrying risk. However, we extend their work by using non-linear functions to represent the dependence between the demand and the market. Furthermore we also incorporate different models of random supply models into the hedging framework. As in Gaur and Seshadri [16], in order to characterize hedging decision we analyze cases with different levels of correlation between supply/demand and the market; no correlation, perfect correlation and partial correlation. However, another major difference of our thesis is in the partially correlation case. Instead of using a utility maximization approach, we utilize the minimium-variance approach. This is a two step approach which starts by finding an optimal portfolio for reducing the risk and then maximizing the profit by choosing the order quantity using the optimal portfolio.

Chapter 3

NEWSVENDOR PROBLEM WITH RANDOM SUPPLY

In this chapter we first summarize some of the existing results of newsvendor problem in Section (3.1). Then, through Sections (3.2)-(3.4), we present new contributions, by extending this problem, by adding random supply models where demand and supply has some joint probability distribution.

3.1 Standard Newsvendor Problem

The newsvendor problem is a well-known single-item, single-period inventory problem in which the decision maker (or newsboy) has to decide on how much to order. The replenishment decision is critical because if he orders too many, purchase cost will be unnecessarily high; on the contrary, there will be a missed opportunity for additional profit if he orders too few. In daily life, it is very common to encounter examples of newsvendor models, that's the foremost reason why these models are studied extensively. In standard models there is a continuos non-negative stochastic demand D with a known distribution function $F_D(x) = P\{D \leq x\}$ that has a density $f_D(x)$. From this point on, we assume that all marginal and joint distributions have continuous probability distribution functions. Moreover we suppose that there is a fixed sales price p, a fixed purchase cost c, a fixed shortage penalty h and a fixed salvage value s which satisfies $p > c > s > 0$ and $h > c$. The aim of the newsboy is maximizing the expected cash flow by choosing an ordering quantity y, or

$$\max_y E[CF(D, y)] \tag{3.1}$$

where $CF(D,y)$ is the random cash flow and it can be written as

$$\begin{aligned} CF(D,y) &= -cy + p\min\{D,y\} + s\max\{y-D,0\} - h\max\{D-y,0\} \\ &= -cy + sy + p + \min\{D,y\} - s\min\{D,y\} + h\min\{y,D\} - hD \\ &= (s-c)\,y + (p+h-s)\min\{D,y\} - hD \end{aligned} \tag{3.2}$$

so that

$$E[CF(D,y)] = (s-c)\,y + (p+h-s)\,E[\min\{D,y\}] - hE[D]\,.$$

Note that for any random variable X with a continuously differentiable probability density function f, we can write

$$E[\min\{X,y\}] = \int_0^y xf(x)dx + y\int_y^\infty f(x)dx$$

and one can easily show that

$$\frac{dE[\min\{X,y\}]}{dy} = P\{X > y\}\,. \tag{3.3}$$

In our analysis, we will use (3.3) repeatedly. Particularly, in order to solve (3.1), we take the derivative with respect to y and set it equal to 0 where X is D. Hence, we obtain

$$\frac{d}{dy}E[CF(D,y)] = (s-c) + (p+h-s)\,(1 - P\{D > y\}) = 0.$$

The second order condition is trivially satisfied since

$$\frac{d^2E[CF(D,y)]}{dy^2} = -(p+h-s)\,f_D(dy) \leq 0$$

and the objective function is concave. Recall that the randomness in (3.2) is generated by D only and the newsboy makes replenishment decisions based on his expectations of $CF(D,y)$. Then, the optimal order quantity y^* satisfies

$$P\{D \leq y^*\} = \frac{p+h-c}{p+h-s} = \hat{p}. \tag{3.4}$$

Note that (3.4) is the optimality condition and $\hat{p}$ denotes the critical ratio which clearly satisfies $0 \leq \hat{p} \leq 1$. The solution is unique if $P\{D \leq y\}$ is strictly increasing in y.

This characterizes the optimal order quantity decision when the source of uncertainty is only the demand. However in reality, there are many different forms of randomness, each contributing to a collective uncertainty. One of these is the randomness is in the supply. Due to the economical changes, trends, shifts, disasters, governmental changes, etc. the supplier's ability to supply could change as well as the customers demand. So, one must also consider these changes in supplier's behavior. In Section (2.4), we summarized supply uncertainty in three categories. In the following sections, starting with random capacity models, we'll analyze the newsvendor problem when supply is also a part of the collective uncertainty

3.2 Random Capacity

This section deals with the effects of the supply uncertainty when it is caused by random capacity (see Section (2.4) for more details). In order to model the variability in the capacity, we define a random variable K as the available capacity to ship. This amount K determines the maximum number of units that the supplier can ship. It can be bigger or smaller then the replenishment order y. Thus we model the random supply representing the relation between supplier and the order as $Q(y) = \min\{K, y\}$. In short, when we order y, we will receive $Q(y)$. Let $P\{K \leq z\} = F_K(z) > 0$ for all z, denote the distribution function of K. Also suppose that D and K have some joint distribution function $P\{K \leq z, D \leq x\} = F_{KD}(z, x)$. Moreover, let $F_{K|x}(z) = P\{K \leq z \mid D = x\}$ represent the conditional distribution function of K given $D = x$.

Note that $CF(D, y)$ in (3.1) becomes the new cash flow $CF(D, K, y)$ since there is a new random component in the function. Therefore, (3.2) should be modified to include the random capacity condition. The payoff function or the cash flow can now be represented as,

$$CF(D, K, y) = (s - c)\min\{K, y\} + (p + h - s)\min\{D, K, y\} - hD. \tag{3.5}$$

Then the objective function becomes

$$E[CF(D, K, y)] = (s - c)E[\min\{K, y\}] + (p + h - s)E[\min\{D, K, y\}] - hE[D].$$

Note that for any random variable X and Z with continuously differentiable probability density functions f_X and f_Z, we can write

$$\begin{aligned} E\left[\min\left\{X, Z, y\right\}\right] &= \int_0^\infty f_Z\left(dz\right)\left(\int_0^{\min\{z,y\}} x f_{X|z}\left(x\right)dx + \min\left\{z, y\right\}\int_{\min\{z,y\}}^\infty f_{X|z}\left(x\right)dx\right) \\ &= \left\{\begin{array}{ll} \int_0^z x f_{X|z}\left(x\right)dx + z\int_z^\infty f_{X|z}\left(x\right)dx & z \leq y \\ \int_0^y x f_{X|z}\left(x\right)dx + y\int_y^\infty f_{X|z}\left(x\right)dx & z > y \end{array}\right\} \end{aligned}$$

where $f_{X|z}\left(x\right)$ is the conditional density of X give $\{Z = z\}$. One can also show that

$$\begin{aligned} \frac{dE\left[\min\left\{X, Z, y\right\}\right]}{dy} &= \left\{\begin{array}{ll} 0 & z \leq y \\ \int_y^\infty F_{X|z}\left(x\right)dx & z > y \end{array}\right\} = \int_y^\infty f_Z\left(dz\right)\int_y^\infty f_{X|z}\left(x\right)dx \\ &= P\left\{X > y, Z > y\right\}. \end{aligned} \tag{3.6}$$

We will also use (3.6) repeatedly in our analysis. Particularly, in order to obtain first order optimality condition we need to take the derivative with respect to y and set it equal to 0. Using (3.3) and (3.6) where X represents D and Z denotes K we obtain the optimality condition as

$$\frac{dE\left[CF\left(D, K, y\right)\right]}{dy} = \left(s - c\right)P\left\{K > y\right\} + \left(p + h - s\right)P\left\{D > y, K > y\right\} = 0 \tag{3.7}$$

which can also be written written as

$$g\left(y\right) = P\left\{K > y\right\}\left(\left(s - c\right) + \left(p + h - s\right)P\left\{D > y \mid K > y\right\}\right) = 0. \tag{3.8}$$

Since by our assumption $P\left\{K > y\right\} > 0$ for all y, (3.8) can be written as

$$\left(s - c\right) + \left(p + h - s\right)P\left\{D > y \mid K > y\right\} = 0.$$

Rearranging the terms, we obtain the optimality condition

$$1 - P\left\{D > y \mid K > y\right\} = 1 - \frac{c - s}{p + h - s}$$

which finally yields

$$P\left\{D \leq y^* \mid K > y^*\right\} = \frac{p + h - c}{p + h - s} = \hat{p}. \tag{3.9}$$

Note that we obtained the same critical ratio on the right-hand side of (3.4). However we have a different probability on the left-hand side of (3.9). For further analysis, let

$h(y) = P\{D \leq y \mid K > y\}$ denote this probability. The existence and uniqueness of a solution of (3.9) depends on the structure of $h(y)$. If $h(0) \leq \hat{p} \leq h(\infty)$ and $h(y)$ is strictly increasing in y, then there is a unique optimal solution y^* that satisfies (3.9). In case $h(0) \geq \hat{p}$ we have $y^* = 0$, and if $h(\infty) \leq \hat{p}$ we have $y^* = \infty$. From this point on we assume that $h(0) \leq \hat{p} \leq h(\infty)$ is always true without loss of generality. Moreover, the objective function is not necessarily concave, so we need to check the second order condition for optimality. Assuming $h(y)$ is an increasing function in y, the objective function becomes a pseudo concave function. In (3.8), $s - c < 0$ and $p + h - s > 0$; hence, since $P\{D > y \mid K > y\} = 1 - h(y)$ is decreasing in y, $g(y)$ must be concave increasing on $[0, y^*]$ where $g(y^*) = 0$ or $h(y^*) = \hat{p}$. On (y^*, ∞), $g(y)$ must be negative and the objective function is decreasing. Therefore, (3.9) is a necessary and sufficient condition of optimality.

Next, we analyze some special cases of random capacity models.

3.2.1 Infinite capacity

Intuitively when the supplier's capacity is infinite or always sufficiently large the model should revert back to the standard model (3.4). In other words, $K = \infty$ and $K > y$ is a always true for all $y \geq 0$. Hence, the optimality condition becomes

$$P\{D \leq y^* \mid K > y^*\} = P\{D \leq y^*\} = \frac{p + h - c}{p + h - s} = \hat{p}.$$

As our intuition suggested the capacity randomness became redundant and the model reverted back to (3.4).

3.2.2 No dependence between demand and capacity

If there is no dependence between K and D,

$$P\{D \leq y \mid K > y\} = P\{D \leq y\}.$$

Then the optimality condition (3.9) becomes

$$P\{D \leq y^*\} = \frac{p + h - c}{p + h - s} = \hat{p}.$$

Again the model reverts back to (3.4).

3.2.3 Perfect dependence between demand and capacity

When it is assumed that the dependence between random variables are perfect, then for some deterministic function A, $K = A(D)$ and $P\{A(D) \leq z\} = F_K(z)$. Assuming A is an increasing, differentiable function that has an inverse, (3.9) becomes

$$\begin{aligned} P\{D \leq y \mid K > y\} &= P\{D \leq y \mid A(D) > y\} \\ &= P\left\{D \leq y \mid D > A^{-1}(y)\right\} \\ &= \frac{P\left\{D \leq y, D > A^{-1}(y)\right\}}{P\{D > A^{-1}(y)\}}. \end{aligned}$$

Moreover, (3.7) becomes

$$\begin{aligned} \frac{dE[CF(D,K,y)]}{dy} &= (s-c)(1-F_K(y)) \\ &\quad + (p+h-s)\int_y^\infty F_D(dx)\, P\{A(D) > y \mid D = x\} \\ &= (s-c)(1-F_K(y)) + (p+h-s)\int_y^\infty F_D(dx)\, 1_{\{x > A^{-1}(y)\}} \\ &= (s-c)\, P\{A(D) > y\} + (p+h-s)\int_{\min\{y, A^{-1}(y)\}}^\infty F_D(dx) \\ &= (s-c)\, P\left\{D > A^{-1}(y)\right\} + (p+h-s)\, P\left\{D > \min\left\{y, A^{-1}(y)\right\}\right\}. \end{aligned}$$

From here, it is possible to analyze the following two cases.

- Case 1: $A(y) \leq y \quad$ for all y. This implies that $\min\left\{y, A^{-1}(y)\right\} = y$ and

$$\begin{aligned} \frac{dE[CF(D,K,y)]}{dy} &= ((s-c) + (p+h-s))\, P\left\{D > A^{-1}(y)\right\} \\ &= (p+h-c)\, P\left\{D > A^{-1}(y)\right\} > 0. \end{aligned}$$

  Since the objective function has positive slope, $y^* = \infty$.

- Case 2: $A(y) > y \quad$ for all y. This implies that $\min\left\{y, A^{-1}(y)\right\} = A^{-1}(y)$ and

$$\begin{aligned}\frac{dE\left[CF\left(D,K,y\right)\right]}{dy} &= (s-c)\,P\left\{D > A^{-1}(y)\right\} + (p+h-s)\,P\left\{D > y\right\} \\ &= (s-c)\,P\left\{K > y\right\} + (p+h-s)\,P\left\{K > A(y)\right\} \\ &= (s-c)\left(1 - F_K(y)\right) + (p+h-s)\left(1 - F_K\left(A(y)\right)\right)\end{aligned}$$

and the optimality condition is obtained by setting this equal to 0, so that

$$(s-c) + (p+h-s)\,\frac{1 - F_K\left(A(y)\right)}{1 - F_K(y)} = 0 \tag{3.10}$$

since $1 - F_K\left(A(y)\right) > 0$, we can now write

$$1 - \frac{1 - F_K\left(A(y)\right)}{1 - F_K(y)} = 1 - \frac{c-s}{p+h-s}$$

Thus, we finally get the optimally condition as

$$\frac{F_K\left(A(y^*)\right) - F_K(y^*)}{1 - F_K(y^*)} = \frac{p+h-c}{p+h-s} = \hat{p}. \tag{3.11}$$

In order to comment on the existence and uniqueness of a solution, we analyze

$$g(y) = \frac{1 - F_K\left(A(y)\right)}{1 - F_K(y)}$$

from (3.10) and conclude that there is indeed a unique optimal solution under the assumption that $A'(y) \geq 1$. Then, to check the second order condition we take the derivative of g with respect to y, and obtain

$$\begin{aligned}\frac{dE\left[g(y)\right]}{dy} &= \frac{-A'(y)\,f_K\left(A(y)\right)\left(1 - F_K(y)\right) + \left[1 - F_K\left(A(y)\right)\right] f_K(y)}{\left[1 - F_K(y)\right]^2} \\ &= \frac{1}{1 - F_K(y)}\left(r_K\left(A(y)\right)\left(1 - F_K\left(A(y)\right)\right) - A'(y)\,f_K\left(A(y)\right)\right)\end{aligned} \tag{3.12}$$

where f_K is the probability density function corresponding to F_K and r_K is the corresponding failure rate function defined as

$$r_K(y) = \frac{f_K(y)}{1 - F_K(y)}.$$

From here, we show that

$$r_K\left(A\left(y\right)\right)\left(1-F_K\left(A\left(y\right)\right)\right)-A'\left(y\right)f_K\left(A\left(y\right)\right)\leq 0$$

or

$$r_K\left(A\left(y\right)\right)-A'\left(y\right)r_K\left(A\left(y\right)\right)\leq 0$$

since $1\geq 1-F_K\left(x\right)\geq 0$ is true for all x and $r_K\left(A\left(y\right)\right)\leq A'\left(y\right)r_K\left(A\left(y\right)\right)$ is true for all y. This shows that $g'\left(y\right)\leq 0$ implying that $g\left(y\right)$ is decreasing. Therefore, we can conclude that there exist an optimal y^* satisfying (3.10) which can be characterized as (3.11). Moreover, the structure of y^* depends on the constant term $\hat{p}$. If $g\left(0\right)\geq\hat{p}$, then $y^*=0$ and $y^*=\infty$ if $g\left(\infty\right)<\hat{p}$.

3.2.4 Perfect linear dependence

In this case, $K=a+bD$ and

$$F_K\left(z\right)=P\left\{a+bD\leq z\right\}=P\left\{D\leq\left(z-a\right)/b\right\}=F_D\left(\left(z-a\right)/b\right).$$

Moreover,

$$F_{D|z}\left(y\right)=P\left\{D\leq y\mid K=z\right\}=P\left\{\left(z-a\right)/b\leq y\right\}=1_{\{z\leq a+by\}}.$$

Using the results in previous Section (3.1.3), we define two different scenarios:

- Case 1: $A\left(y\right)=a+by\leq y$ for all y $\left(a\leq 0,\ b\leq 1\right)$

  The results of previous Section (3.1.3) apply and $y^*=\infty$

- Case 2: $A\left(y\right)=a+by\geq y$ for all y $\left(a\geq 0,\ b\geq 1\right)$

  The results of previous Section (3.1.3) again apply, using $\left(1-F_K\left(y\right)\right)\geq\left(1-F_K\left(a+by\right)\right)$ and $p+h-c<p+h-s$. The optimality condition can be found as

$$\left(s-c\right)+\left(p+h-s\right)\frac{1-F_K\left(a+by\right)}{1-F_K\left(y\right)}=0$$

which leads to the characterization

$$\frac{F_K(a+by^*)-F_K(y^*)}{1-F_K(y^*)}=\frac{p+h-c}{p+h-s}=\hat{p}. \tag{3.13}$$

For the existence and uniqueness of the solution we analyze the structure of

$$g(y)=\frac{1-F_K(a+by)}{1-F_K(y)}.$$

The derivative of $g(y)$ with respect to y is

$$\frac{dg(y)}{dy}=\frac{1}{1-F_K(y)}\left(r_K(a+by)(1-F_K(a+by))-bf_K(a+by)\right).$$

As shown in the previous section

$$r_K(a+by)(1-F_K(a+by))-bf_K(a+by)\leq 0$$

and $g(y)$ is decreasing. So, there exists an optimal y^* satisfying (3.13). If $g(0)\geq\hat{p}$, then $y^*=0$ and $y^*=\infty$ if $g(\infty)<\hat{p}$.

For the random capacity case we enumerated explicit solutions for general and all the special cases. Based on these results, we conclude that it is possible to find a characterization for y, equals a constant "critical ratio" $\hat{p}$. Some of these characterizations requires certain assumptions on the structure of the optimality conditions. In the next section we continue to examine random supply models, but with different source of randomness.

3.3 Random Yield

Random capacity is not the only possible source of randomness in the context of supply uncertainty. Another widely used setting is random yield. In these models it is assumed that due to various factors such as defects, errors, transportation problems, etc., the amount ordered will be different than the amount received. Let $U\geq 0$ be a random variable representing the proportion of the ordered quantity that will be received in good condition. Then, in random yield models we have $Q(y)=Uy$ where $P\{U\leq v\}=F_U(v)$. For generality, let us assume that U and D are not necessarily independent and have a joint distribution $F_{DU}(x,v) = P\{D \leq x, U \leq v\}$, and with a conditional distribution function $F_{D|v}(x) = P\{D \leq x \mid U = v\}$. If we rewrite our payoff function in (3.2) with these parameters, we get

$$CF(D,U,y) = (s-c)\,Uy + (p+h-s)\min\{D,Uy\} - hD. \tag{3.14}$$

Hence the objective function becomes

$$E[CF(D,U,y)] = (s-c)\,yE[U] + (p+h-s)\,E[\min\{D,Uy\}] - hE[D]. \tag{3.15}$$

Note that for any random variable X and Z with continuously differentiable probability density functions f_X and f_Z, we can write

$$E[\min\{X,Zy\}] = \int_0^\infty f_Z(z)\,dz \left(\int_0^{zy} x f_{X|z}(x)\,dx + zy\int_{zy}^\infty f_{X|z}(x)\,dx\right)$$

where $f_{X|z}(x)$ is the conditional density. One can show that

$$\begin{aligned}\frac{dE[\min\{X,Zy\}]}{dy} &= \int_0^\infty z f_Z(z)\,dz \int_{zy}^\infty f_{X|z}(x)\,dx \\ &= E\left[Z1_{\{X>Zy\}}\right]. \end{aligned}\tag{3.16}$$

By using (3.16), we take the derivative of (3.15) and set it equal to zero. Hence, the optimality condition is

$$(s-c)\,E[U] + (p+h-s)\,E\left[U1_{\{D>Uy\}}\right] = 0$$

which can also be expressed as

$$(p+h-c)\,E[U] + (p+h-s)\,E\left[U1_{\{D\leq Uy\}}\right] = 0$$

by using the fact that $1_{\{D>Uy\}} = 1 - 1_{\{D\leq Uy\}}$. Finally, the optimality condition becomes

$$\frac{E\left[U1_{\{D\leq Uy^*\}}\right]}{E[U]} = \frac{p+h-c}{p+h-s} = \hat{p}. \tag{3.17}$$

Note that the objective function $E[CF(D,U,y)]$ is concave since $E\left[U1_{\{D\leq Uy^*\}}\right]$ is trivially increasing in y and $U \geq 0$. Therefore, there is an optimal y^* which satisfies (3.17) for some $0 \leq \hat{p} \leq 1$, provided that the solution is not at the boundary. If this function is strictly increasing, then this solution is unique.

3.3.1 No dependence between demand and yield

If there is no dependence, than the optimality condition can be written as

$$\frac{E\left[UP\left\{D \leq Uy^* \mid U\right\}\right]}{E\left[U\right]} = \frac{E\left[UF_D\left(Uy^*\right)\right]}{E\left[U\right]} = \frac{p+h-c}{p+h-s} = \hat{p}.$$

The function on the left-hand side is increasing in terms of y. Therefore, there exists an optimal solution y^* which satisfies the optimality condition.

3.3.2 Perfect dependence between demand and yield

Suppose that there is perfect dependence so that $U = W\left(D\right)$ for some non-negative function W. We can then write the optimality condition (3.17) as

$$\frac{E\left[W\left(D\right)1_{\{D \leq W(D)y^*\}}\right]}{E\left[W\left(D\right)\right]} = \frac{p+h-c}{p+h-s} = \hat{p}.$$

There exists at least one optimal solution since the function on the left-hand side is increasing in y.

3.3.3 Perfect linear dependence

Now further suppose that $U = W\left(D\right) = a + bD$ where $a \geq 0$ and $b \geq 0$. Then, (3.17) becomes

$$\frac{E\left[\left(a+bD\right)1_{\left\{D \leq \frac{ay^*}{1-by^*}\right\}}\right]}{a+bE\left[D\right]} = \frac{p+h-c}{p+h-s} = \hat{p}.$$

For the random yield case, due to the fact that the objective function (3.15) is concave, we find simple and explicit characterizations for optimal ordering quantities for general and special cases. Our results again suggest that, a characterization for y with a constant critical ratio $\hat{p}$ is easily obtained without extra assumptions. In the next chapter, we further examine random supply models by combining capacity and yield uncertainties.

3.4 Random Yield and Capacity

We've analyzed both yield and capacity models separately; however, in reality these two uncertainties may coexist. Now suppose the supply uncertainty has two elements: capacity and yield. Again, let U be a random variable representing the uncertainty in the yield and K be the supplier's random replenishment capacity. For generality lets again assume that U, K and D have a joint distribution function $F_{DKU}(x, z, v) = P\{D \le x, K \le z, U \le v\}$ and conditional distribution functions

$$F_{K|v}(z) = P\{K \le z \mid U = v\}$$

and

$$F_{D|zv}(x) = P\{D \le x \mid K = z, U = v\}.$$

The supply uncertainty is modeled as $Q(y) = U \min\{K, y\}$. Then, our cash flow becomes

$$CF(D, K, U, y) = (s - c) U \min\{y, K\} + (p + h - s) \min\{D, U \min\{K, y\}\} - hD. \quad (3.18)$$

Using the above cash flow, the objective function can be written as

$$E[CF(D, K, U, y)] = (s - c) E[U \min\{y, K\}] + (p + h - s) E[\min\{D, UK, Uy\}] - hE[D]. \quad (3.19)$$

Note that for any random variable X, Z and V with continuously differentiable probability density functions f_X, f_Z and f_V

$$\begin{aligned} E[\min\{X, ZV, Vy\}] &= \int_0^\infty f_V(v)\,dv \int_0^\infty f_{Z|v}(z)\,dz \int_0^{v\min\{y,z\}} x f_{X|vz}(x)\,dx \\ &\quad + \int_0^\infty f_V(v)\,dv \int_0^\infty f_{Z|v}(z)\,dz \min\{y, z\} \int_{v\min\{y,z\}}^\infty f_{X|vz}(x)\,dx \end{aligned}$$

where $f_{Z|v}(z)$ is the conditional density of Z given $\{V = v\}$, and $f_{X|vz}(x)$ is the conditional density of X given $\{V = v, Z = z\}$. One can show that

$$\begin{aligned} \frac{dE[\min\{X, ZV, Vy\}]}{dy} &= \int_0^\infty v f_V(v)\,dv \int_y^\infty f_{Z|v}(z)\,dz \int_{vy}^\infty f_{X|vz}(x)\,dx \\ &= E\left[V 1_{\{Z>y, X>Vy\}}\right]. \end{aligned} \quad (3.20)$$

By using (3.16) and (3.20) we take the derivative of (3.19) and set it equal to zero. Hence, the optimality condition is

$$\frac{dE\left[CF\left(D,K,U,y\right)\right]}{dy} = \left(p+h-s\right)E\left[U1_{\{K>y,D>Uy\}}\right] + \left(s-c\right)E\left[U1_{\{K>y\}}\right] = 0$$

which can be written as

$$\begin{aligned}\frac{dE\left[CF\left(D,K,U,y\right)\right]}{dy} &= \left(s-c\right)E\left[U1_{\{K>y\}}\right] + \left(p+h-s\right)E\left[U1_{\{K>y\}} - U1_{\{K>y,D\leq Uy\}}\right] \\ &= \left(p+h-c\right)E\left[U1_{\{K>y\}}\right] + \left(p+h-s\right)E\left[U1_{\{K>y,D\leq Uy\}}\right] \\ &= 0.\end{aligned}$$

Finally, the optimal order quanitity satisfies

$$\frac{E\left[U1_{\{D\leq Uy^*,K>y^*\}}\right]}{E\left[U1_{\{K>y^*\}}\right]} = \frac{p+h-c}{p+h-s} = \hat{p}. \tag{3.21}$$

The cash flow function (3.18) is concave for $UK > D$ and non-concave for $UK < D$. Consequently, in order to define a characterization for the optimal y we need to make some assumptions. Assuming

$$g\left(y\right) = \frac{E\left[U1_{\{D\leq Uy,K>y\}}\right]}{E\left[U1_{\{K>y\}}\right]}$$

is increasing, there exist at least one optimal y^* for which (3.21) is satisfied provided that the optimal solution is not at the boundaries.

3.4.1 No dependence between demand, supply and capacity

When the random variables involved D, K and U are independent, (3.21) becomes

$$\frac{E\left[UP\left\{D\leq Uy^* \mid U\right\}\right]}{E\left[U\right]} = \frac{E\left[UF_D\left(Uy^*\right)\right]}{E\left[U\right]} = \frac{p+h-c}{p+h-s} = \hat{p}.$$

Interestingly, the model then reverts back to the random yield case.

3.4.2 Perfect dependence between demand, supply and capacity

In the case of perfect dependence, any knowledge about any of the three random variables can be applied directly to the others. In other words, we can write all the random variables in terms of demand so that $K = T(D)$, and $U = J(D)$. By rewriting these functions with respect to D, we get

$$\frac{E\left[U1_{\{T(D)>y^*, D\leq J(D)y^*\}}\right]}{E\left[U1_{\{T(D)>y^*\}}\right]} = \frac{p+h-c}{p+h-s} = \hat{p}.$$

3.4.3 Perfect linear dependence

When the variables are linearly dependent, using the structure in Section (3.4.2), we rewrite random variables in terms of linear functions as $K = a_1 + b_1 D$, $U = a_2 + b_2 D$. Assuming $a_1 \geq 0$, $b_1 \geq 0$, $a_2 \geq 0$ and $b_2 \geq 0$, the optimality condition is

$$\frac{a_2 P\left\{\frac{y^*-a_1}{b_1} < D \leq \frac{y^* a_2}{1-b_2 y^*}\right\} + b_2 E\left[D1_{\left\{\frac{y^*-a_1}{b_1} < D \leq \frac{y^* a_2}{1-b_2 y^*}\right\}}\right]}{a_2 P\left\{D > \frac{y^*-a_1}{b_1}\right\} + b_2 E\left[D1_{\left\{D > \frac{y^*-a_1}{b_1}\right\}}\right]} = \frac{p+h-c}{p+h-s} = \hat{p}.$$

Models with random yield and capacity have non-concave objective functions. Therefore, obtaining an explicit solution for the general problem is hard, if not impossible. Nevertheless, we derive a complex characterization for the general problem and show the structure of the optimal solution under certain assumptions. Moreover, we further analyzed some special cases of random yield and capacity models to better understand the behavior of the optimal solution.

We examined the behavior of the optimal policy for various types of supply uncertainty. Up to this point our decision maker has been assumed to be risk-neutral; however, from this point on we'll examine the implications of having a risk-sensitive attitude and analyze the framework of decision making with financial hedging opportunities.

Chapter 4

NEWSVENDOR PROBLEM WITH PERFECT HEDGING

The profit (cash flow) in a newsvendor problem is random due to the stochastic nature of demand and supply. In Chapter 3, in order to deal with this stochasticity, we implicitly made two important assumptions: (1) there is a risk neutral decision maker, and demand and/or supply are not correlated with the financial market. The expected profit function was calculated based on newsboy's experience, by disregarding the negative or positive deviations. Therefore, the only aim was maximizing the expected profit by choosing the ordering quantity, without caring about the variance. Moreover, since there is no market correlation, there can neither be a replicating portfolio nor a hedging opportunity. However in real life there is a market, which makes the assumption of risk-neutrality unrealistic. Evidence suggests that decision makers tend to care a lot about risk; especially the downside potential (unexpected loss). That's why corporations are forced to systematically manage their risk.

In literature there are different methods of hedging which can be categorized into two groups: Operational and financial. In operational hedging, as Mieghem [28] puts forward, decisions makers try mitigating risk by counterbalancing actions in a processing network that do not involve financial instruments. Utilizing financial instruments, in particular futures, derivatives and options, for risk management is called financial hedging. Note that, in this paper we are not concerned with operational hedging; thus, all the terms, ideas and methods used are referred to as financial hedging from here on.

In this chapter, we define a framework of decision making in a risk-sensitive environment where the decision maker is risk-averse and there is a complete arbitrage free market along with risk neutral probability measures. We characterize explicit solutions for two different environments, where there is perfect correlation and where there is partial correlation between the market and our random variables. Before continuing with the hedging strategies and discuss its effects on the order quantity and profit, we first take a look at a general framework of decision making using instruments in hedging.

4.1 General Framework

Due to the stochastic nature of the environment decision makers face risky situations in their process of decision making. These risks are caused by various reasons, such as weather, economy, government, legislations and many more. Some of these variables can be predicted and estimated; however, some of them can't. In this chapter we present a general framework for decision making in a risk sensitive environment. In real life decision makers decide on multiple things correlated with each other. But for now lets assume that we have a single decision variable y. Also, let the randomness in the environment we live be caused by only one source of randomness X. Now suppose that they together form a random payoff

$$Z_T = g\left(X, y\right)$$

which will be received at time T. Thus, as a decision maker our job becomes optimizing the payoff function. But since the payoff function is random, we optimize its expected value. Then, the optimization problem for the risk-neutral decision maker becomes

$$\max_y E\left[g\left(X, y\right)\right].$$

Let S be a financial variable that denotes the price of a tradable asset at time T. Suppose further that there is a perfect deterministic relationship between random variable and the financial variable S, so that $X = G\left(S\right)$ for some function G. Let $H_i\left(S, y\right)$ denote the payoff of derivative i on financial security S and suppose that there are n such derivatives in the market. Then, there will be a perfect hedge provided that we can write

$$Z_T = g\left(X, y\right) = g\left(G\left(S\right), y\right) = \sum_{i=1}^{n} C_i H_i\left(S, y\right) \tag{4.1}$$

where $\{C_1, C_2, \cdots, C_n\}$ denotes the unit of the derivative $\{H_1, H_2, \cdots, H_n\}$ used in the replicating portfolio. Therefore the time 0 value of the payoff is

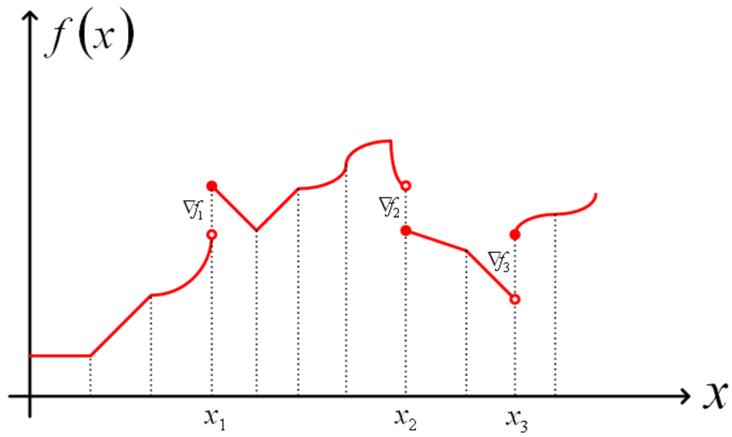

Figure 4.1: Typical Payoff Function

$$Z_0 = \sum_{i=1}^{n} C_i p_i (S_0, y)$$

where $p_i (S_0, y)$ denotes the current market price of H_i when the current price of asset is S_0. It is possible find a replicating portfolio for complicated payoff functions.

For fixed y, let $f(x) = g(G(x, y))$. More generally, the function f can be represented as in Figure 4.1. We suppose that the function f has m jumps at $\{x_1, x_2, \cdots, x_m\}$ with magnitudes $\{\Delta f_1, \Delta f_2, \cdots, \Delta f_m\}$ respectively. Moreover, f is twice differentiable over each interval (x_i, x_{i+1}) where the derivative $f'(x)$ is a function of bounded variation.

As long as $f : R_+ \to R_+$ is as described in Figure 4.1, the cash flow can be replicated by taking positions in futures, digital claims, European calls and cash bonds. This follows by noting that we can write

$$f(x) = f(0) + \sum_{k=1}^{m} \nabla f_k 1_{\{x \le x_k\}} + f'_+(0)\, x + \int_0^\infty (x - z)^+ f'(dz)\,. \tag{4.2}$$

The function (4.1) and (4.2) are identical. In fact, from a financial perspective, (4.2) is a portfolio consisting of various financial instruments that replicates (4.1). The first term represents the amount of bonds, the second term is for digital claims, the third term denotes the number of futures and the last term depicts the position of European call options the portfolio contains (for more details, see Protter [37]).

4.2 Demand Perfectly Correlated with the Price of a Market Security

Each decision maker faces a certain amount of risk by investing in inventory. Risk can basically be defined as the mismatch of demand and supply. Thus, decision makers are forced to manage the risk in order to prevent unexpected loses. Up until now we only focused on the correlation between demand and supply, completely neglecting the effects of the financial market. And since there was no market, there would be no replicating portfolio as well. However if there is a market, disregarding it will lead to arbitrage opportunities. In reality, it is possible to form portfolios that have correlation with demand and supply. Via the help of such portfolios it will be possible for the decision makers to hedge the risks of carrying inventory and shortage. In this chapter we assume risk-neutral probability measures exists and contingent claims can be priced with these measures. Additionally, for this chapter we suppose that demand and supply are perfectly correlated with the market.

4.2.1 Demand Uncertainty

In traditional newsvendor models the only uncertainty is generated by demand. In this chapter we are going to assume this uncertainty can be perfectly replicated in the market, because there is some deterministic function $\mathcal{D}$ such that $D = \mathcal{D}(S)$ where $\mathcal{D}$ is a twice differentiable increasing function with an inverse. In Chapter 3, we did not use discounting or compounding while calculating the cash flow since the rates are unclear and based on decision maker preferences. However, in this chapter we will use the risk-free interest rate r. Let S_0 be the current price of the financial asset and $S = S_T$ be the price at time T. In this setup, $S_0 = e^{-rT} E_Q[S]$ where E_Q is the expectation under risk-neutral probability measure (RNPM). Furthermore, recall that p is the selling price of each inventory, s is the salvage value, ce^{rT} is the compounded cost of purchasing one unit of inventory and h is the shortage penalty which satisfies $p > ce^{rT} > s > 0$ and $h > ce^{rT}$.

At time 0, the firm invests money on inventory and pays ordering cost $ce^{rT}y$ and its time T value is

$$CF(y) = -ce^{rT}y.$$

At time T, as a result of the investment, the cash flow becomes

$$CF(S,y) = \left(s - ce^{rT}\right)y + (p+h-s)\min\{\mathcal{D}(S), y\} - h\mathcal{D}(S). \tag{4.3}$$

We can now apply (4.2) for finding a replicating portfolio and use it to hedge the risks associated with the inventory model. It follows from (4.3) that for fixed y

$$\begin{aligned}
f(0) &= \left(s - ce^{rT}\right)y + (p+h-s)\min\{\mathcal{D}(0), y\} - h\mathcal{D}(0) \\
f'(0) &= (p+h-s)\mathcal{D}'(0)\,1_{\{y>\mathcal{D}(0)\}} - h\mathcal{D}'(0) \\
f'(dz) &= (p+h-s)\mathcal{D}''(z)\,1_{\{\mathcal{D}(z)<y\}}dz - (p+h-s)\mathcal{D}'(z)\,1_{\{z=\mathcal{D}^{-1}(y)\}} - h\mathcal{D}''(z)\,dz.
\end{aligned}$$

Then the replicating portfolio can therefore be characterized as

$$\begin{aligned}
f(S) &= \left[\left(s - ce^{rT}\right)y + (p+h-s)\min\{\mathcal{D}(0), y\} - h\mathcal{D}(0)\right] \\
&\quad + \left[(p+h-s)\mathcal{D}'(0)\,1_{\{y>\mathcal{D}(0)\}} - h\mathcal{D}'(0)\right]S \\
&\quad -h\int_0^\infty \max\{S-z,0\}\,\mathcal{D}''(z)\,dz \\
&\quad + (p+h-s)\int_0^{\mathcal{D}^{-1}(y)} \max\{S-z,0\}\,\mathcal{D}''(z)\,dz \\
&\quad - (p+h-s)\mathcal{D}'\left(\mathcal{D}^{-1}(y)\right)\max\left\{S-\mathcal{D}^{-1}(y),0\right\}.
\end{aligned} \tag{4.4}$$

Since $f(S) = CF(S,y)$ for any y, we can replicate the cash flow $CF(S,y)$ by a portfolio consisting of

$$\left(s - ce^{rT}\right)y + (p+h-s)\min\{\mathcal{D}(0), y\} - h\mathcal{D}(0)$$

cash bonds, $(p+h-s)\mathcal{D}'(0)\,1_{\{y>\mathcal{D}(0)\}} - h\mathcal{D}'(0)$ in futures and

$$\begin{aligned}
&(p+h-s)\left[\int_0^{\mathcal{D}^{-1}(y)} \max\{S-z,0\}\,\mathcal{D}''(z)\,dz - \mathcal{D}'\left(\mathcal{D}^{-1}(y)\right)\max\{S-z,0\}\right] \\
&\quad -h\int_0^\infty \max\{S-z,0\}\,\mathcal{D}''(z)\,dz
\end{aligned}$$

European call options. Now that we found the replicating portfolio we can price it using risk-neutral probability measures to avoid arbitrage opportunities. We suppose that the risk-neutral or arbitrage probability measure is Q and $P_Q\{S \le x\} = F_S(x)$. The expected hedged cash flow at time T is

$$\begin{aligned}
E_Q[f(S)] &= \left[\left(s - ce^{rT}\right)y + (p+h-s)\min\{\mathcal{D}(0), y\} - h\mathcal{D}(0)\right] \\
&\quad + \left[(p+h-s)\mathcal{D}'(0)1_{\{y>\mathcal{D}(0)\}} - h\mathcal{D}'(0)\right]S_0 e^{rT} \\
&\quad - (p+h-s)\mathcal{D}'\left(\mathcal{D}^{-1}(y)\right)\int_{\mathcal{D}^{-1}(y)}^{\infty} F_S(dx)\left(x - \mathcal{D}^{-1}(y)\right) \\
&\quad + (p+h-s)\left[\int_0^{\mathcal{D}^{-1}(y)} dz \int_z^{\infty} F_S(dx)(x-z)\mathcal{D}''(z)\right] \\
&\quad - h\int_0^{\infty} dz \int_z^{\infty} F_S(dx)(x-z)\mathcal{D}''(z)
\end{aligned}$$

Similarly the cash flow at time 0 can be found as

$$\begin{aligned}
E_Q[Z_0] &= e^{-rT}E_Q[f(S)] = \left[\left(se^{-rT} - c\right)y + e^{-rT}\left((p+h-s)\min\{\mathcal{D}(0), y\} - h\mathcal{D}(0)\right)\right] \\
&\quad + \left[(p+h-s)\mathcal{D}'(0)1_{\{y>\mathcal{D}(0)\}} - h\mathcal{D}'(0)\right]S_0 \\
&\quad - e^{-rT}(p+h-s)\mathcal{D}'\left(\mathcal{D}^{-1}(y)\right)\int_{\mathcal{D}^{-1}(y)}^{\infty}\left(x - \mathcal{D}^{-1}(y)\right)F_S(dx) \\
&\quad + e^{-rT}(p+h-s)\left[\int_0^{\mathcal{D}^{-1}(y)} dz \int_z^{\infty}(x-z)F_S(dx)\mathcal{D}''(z)\right] \\
&\quad - e^{-rT}h\int_0^{\infty}\int_z^{\infty}(x-z)\mathcal{D}''(z)\,dz
\end{aligned}$$

From here on we will characterize our pricing solutions using the expected hedged cash flow at time T. To simplify the analysis we will assume without the loss of generality that $y \ge \mathcal{D}(0)$. Then,

$$\begin{aligned}
\frac{dE_Q[f(S)]}{dy} &= \left(s - ce^{rT}\right) - (p+h-s)\frac{\mathcal{D}''\left(\mathcal{D}^{-1}(y)\right)}{\mathcal{D}'\left(\mathcal{D}^{-1}(y)\right)}\int_{\mathcal{D}^{-1}(y)}^{\infty} F_S(dx)\left(x - \mathcal{D}^{-1}(y)\right) \\
&\quad + (p-s+h)\frac{\mathcal{D}''\left(\mathcal{D}^{-1}(y)\right)}{\mathcal{D}'\left(\mathcal{D}^{-1}(y)\right)}\int_{\mathcal{D}^{-1}(y)}^{\infty} F_S(dx)\left(x - \mathcal{D}^{-1}(y)\right) \\
&\quad - (p+h-s)\int_{\mathcal{D}^{-1}(y)}^{\infty} F_S(dx),
\end{aligned}$$

and the optimality condition becomes

$$\left(s - ce^{rT}\right) + (p + h - s) \int_{\mathcal{D}^{-1}(y)}^{\infty} F_S\left(dx\right) = 0$$

which yields

$$P_Q\left\{S \leq \mathcal{D}^{-1}\left(y^*\right)\right\} = \frac{p + h - ce^{rT}}{p + h - s} = \hat{p}. \tag{4.5}$$

Note that the optimality condition is same as (3.4), the only difference is the risk-neutral probability measure Q and the fact that the cost c is compounded to time T. Hence, as long as there is perfect correlation between market and demand, instead of pricing the replicating portfolio we can utilize the corresponding characterizations in Chapter 3. In this case, (3.4) is used to obtain y^* with hedging by substituting $\mathcal{D}\left(S\right)$ with D we get

$$P_Q\left\{\mathcal{D}\left(S\right) \leq y^*\right\} = \frac{p + h - ce^{rT}}{p + h - s} = \hat{p}.$$

Linear dependence between demand and market

If there is linear dependence we can replace demand with $\mathcal{D}\left(S\right) = a + bS$, and (4.5) becomes

$$P_Q\left\{S \leq \frac{y^* - a}{b}\right\} = \frac{p + h - ce^{rT}}{p + h - s} = \hat{p}.$$

In random demand models we used the perfect correlation between the demand and the market to replicate the random cash flow. Then, we priced this cash flow according to RNPM and show that the structure of the optimal solution is the same as the optimal order quantity without hedging (see Chapter 3).

4.2.2 Supply Uncertainty

We characterized optimal ordering policies when the only uncertainty was the random demand using (4.2). Now we will investigate the decision making process when supply is also random by analyzing random capacity and yield models, as we did in Chapter 3.

Random Capacity

We now model the supply uncertainty as we did in Section (3.2). Recall that $D = \mathcal{D}(S)$, similarly let K be perfectly correlated with the same financial instrument S, so that $K = \mathcal{K}(S)$. Since both $\mathcal{D}(S)$ and $\mathcal{K}(S)$ have perfect dependence with the financial instrument S, they also have perfect dependence among themselves. Suppose $\mathcal{D}(S)$ and $\mathcal{K}(S)$ are increasing, twice differentiable functions that have an inverse, then our payoff function can be represented as

$$\begin{aligned} CF(S,y) &= \left(s - ce^{rT}\right) \min\{\mathcal{K}(S), y\} - h\mathcal{D}(S) \\ &\quad + (p+h-s)\min\{\mathcal{D}(S), \mathcal{K}(S), y\}. \end{aligned} \tag{4.6}$$

The calculation of the replicating portfolio for the cash flow above is quite involved due to the fact that the objective function is non-concave. Thus, instead of pricing the replicating portfolio we will use the solution in (3.9) to find the characterization of optimal ordering quantity. Since the correlation between the financial asset, demand and supply is perfect, we conclude that the optimal solution becomes

$$P_Q\left\{S \le \mathcal{D}^{-1}(y^*) \mid S > \mathcal{K}^{-1}(y^*)\right\} = \frac{p+h-ce^{rT}}{p+h-s}. \tag{4.7}$$

Note that by modifying (3.9), we implicitly made the assumption that demand and capacity are perfectly correlated as well; however, in (3.9) this is not the case. Moreover this transformation also requires the change of probability measure. In (4.7), risk-free measure Q is utilized.

As stated before, finding a general setting for the price of the replicating portfolio explicitly is quite involved. That's why we examine this issue by considering three special cases.

- Case 1: $\mathcal{K}(x) > \mathcal{D}(x)$ for all $x \geq 0$. One can show that

$$\begin{aligned}
f(0) &= \left(s - ce^{rT}\right) \min\{\mathcal{K}(0), y\} + (p+h-s)\min\{\mathcal{D}(0), y\} - h\mathcal{D}(0) \\
f'(0) &= \left(s - ce^{rT}\right) \mathcal{K}'(0)\, 1_{\{\mathcal{K}(0)<y\}} + (p+h-s)\,\mathcal{D}'(0)\, 1_{\{\mathcal{D}(0)<y\}} - h\mathcal{D}'(0) \\
f'(dz) &= \left(s - ce^{rT}\right) \mathcal{K}''(z)\, 1_{\{\mathcal{K}(z)<y\}} dz - \left(s - ce^{rT}\right) \mathcal{K}'(z)\, 1_{\{z=\mathcal{K}^{-1}(y)\}} \\
&\quad + (p+h-s)\,\mathcal{D}''(z)\, 1_{\{\mathcal{D}(z)<y\}} dz - (p+h-s)\,\mathcal{D}'(z)\, 1_{\{z=\mathcal{D}^{-1}(y)\}} \\
&\quad - h\mathcal{D}''(z)\, dz.
\end{aligned}$$

Then, the cash flow becomes

$$\begin{aligned}
f(S) &= \left(s - ce^{rT}\right) \min\{\mathcal{K}(0), y\} + (p+h-s)\min\{\mathcal{D}(0), y\} - h\mathcal{D}(0) \\
&\quad + \left(\left(s - ce^{rT}\right) \mathcal{K}'(0)\, 1_{\{\mathcal{K}(0)<y\}} + (p+h-s)\,\mathcal{D}'(0)\, 1_{\{\mathcal{D}(0)<y\}} - h\mathcal{D}'(0)\right) S \\
&\quad + \left(s - ce^{rT}\right) \int_0^{\mathcal{K}^{-1}(y)} \max\{S-z, 0\}\, \mathcal{K}''(z)\, dz \\
&\quad - \left(s - ce^{rT}\right) \max\left\{S - \mathcal{K}^{-1}(y), 0\right\} \mathcal{K}'\left(\mathcal{K}^{-1}(y)\right) \\
&\quad + (p+h-s) \int_0^{\mathcal{D}^{-1}(y)} \max\{S-z, 0\}\, \mathcal{D}''(z)\, dz \\
&\quad - (p+h-s) \max\left\{S - \mathcal{D}^{-1}(y), 0\right\} \mathcal{D}'\left(\mathcal{D}^{-1}(y)\right) \\
&\quad - h \int_0^{\infty} \max\{S-z, 0\}\, \mathcal{D}''(z)\, dz.
\end{aligned}$$

By taking the expectation under risk-neutral probability measure Q, we get

$$\begin{aligned}
E_Q[f(S)] &= \left(s - ce^{rT}\right) \min\{\mathcal{K}(0), y\} + (p+h-s)\min\{\mathcal{D}(0), y\} - h\mathcal{D}(0) \\
&\quad + \left(\left(s - ce^{rT}\right) \mathcal{K}'(0)\, 1_{\{\mathcal{K}(0)<y\}} + (p+h-s)\,\mathcal{D}'(0)\, 1_{\{\mathcal{D}(0)<y\}}\right) S_0 e^{rT} \\
&\quad - h\mathcal{D}'(0)\, S_0 e^{rT} \\
&\quad + \left(s - ce^{rT}\right) \int_0^{\mathcal{K}^{-1}(y)} dz \int_z^{\infty} F_S(dx)\,(x-z)\,\mathcal{K}''(z) \\
&\quad - \left(s - ce^{rT}\right) \mathcal{K}'\left(\mathcal{K}^{-1}(y)\right) \int_{\mathcal{K}^{-1}(y)}^{\infty} F_S(dx)\left(x - \mathcal{K}^{-1}(y)\right) \\
&\quad - (p+h-s)\,\mathcal{D}'\left(\mathcal{D}^{-1}(y)\right) \int_{\mathcal{D}^{-1}(y)}^{\infty} F_S(dx)\left(x - \mathcal{D}^{-1}(y)\right) \\
&\quad + (p+h-s) \int_0^{\mathcal{D}^{-1}(y)} dz \int_z^{\infty} F_S(dx)\,(x-z)\,\mathcal{D}''(z) \\
&\quad - h \int_0^{\infty} dz \int_z^{\infty} F_S(dx)\,(x-z)\,\mathcal{D}''(z).
\end{aligned}$$

Suppose that $y > \max\{K(0), D(0)\}$, then by taking the derivative of the expected cash flow we get

$$\begin{aligned}
\frac{dE_Q[f(S)]}{dy} &= \left(s - ce^{rT}\right)\frac{\mathcal{K}''\left(\mathcal{K}^{-1}(y)\right)}{\mathcal{K}'\left(\mathcal{K}^{-1}(y)\right)}\int_{\mathcal{K}^{-1}(y)}^{\infty}\left(x - \mathcal{K}^{-1}(y)\right)F_S(dx) \\
&\quad - \left(s - ce^{rT}\right)\frac{\mathcal{K}''\left(\mathcal{K}^{-1}(y)\right)}{\mathcal{K}'\left(\mathcal{K}^{-1}(y)\right)}\int_{\mathcal{K}^{-1}(y)}^{\infty}\left(x - \mathcal{K}^{-1}(y)\right)F_S(dx) \\
&\quad + \left(s - ce^{rT}\right)\int_{\mathcal{K}^{-1}(y)}^{\infty}F_S(dx) + (p+h-s)\int_{\mathcal{D}^{-1}(y)}^{\infty}F_S(dx) \\
&\quad + (p+h-s)\frac{\mathcal{D}''\left(\mathcal{D}^{-1}(y)\right)}{\mathcal{D}'\left(\mathcal{D}^{-1}(y)\right)}\int_{\mathcal{D}^{-1}(y)}^{\infty}\left(x - \mathcal{D}^{-1}(y)\right)F_S(dx) \\
&\quad - (p+h-s)\frac{\mathcal{D}''\left(\mathcal{D}^{-1}(y)\right)}{\mathcal{D}'\left(\mathcal{D}^{-1}(y)\right)}\int_{\mathcal{D}^{-1}(y)}^{\infty}\left(x - \mathcal{D}^{-1}(y)\right)F_S(dx)
\end{aligned}$$

so that the optimality condition becomes

$$\left(s - ce^{rT}\right)\int_{\mathcal{K}^{-1}(y)}^{\infty}F_S(dx) + (p+h-s)\int_{\mathcal{D}^{-1}(y)}^{\infty}F_S(dx) = 0$$

which can be written as

$$\left(s - ce^{rT}\right)P_Q\{\mathcal{K}(S) > y\} + (p+h-s)P_Q\{\mathcal{D}(S) > y\} = 0$$

Rewriting this equation yields the following optimality condition

$$\frac{P_Q\{\mathcal{K}(S) > y\} - P_Q\{\mathcal{D}(S) > y\}}{P_Q\{\mathcal{K}(S) > y\}} = \frac{p+h-ce^{rT}}{p+h-s} = \hat{p}. \tag{4.8}$$

Note that when $\mathcal{K}(x) > \mathcal{D}(x)$ for all x,

$$\begin{aligned}
P_Q\{\mathcal{D}(S) \le y\} &= P_Q\{\mathcal{D}(S) \le y \mid \mathcal{K}(S) > y\}P_Q\{\mathcal{K}(S) > y\} \\
&\quad + P_Q\{\mathcal{D}(S) \le y \mid \mathcal{K}(S) \le y\}P_Q\{\mathcal{K}(S) \le y\} \\
&= P_Q\{\mathcal{D}(S) \le y \mid \mathcal{K}(S) > y\}P_Q\{\mathcal{K}(S) > y\} \\
&\quad + 1 - P_Q\{\mathcal{K}(S) > y\}
\end{aligned}$$

since $P_Q\{\mathcal{D}(S) \le y \mid \mathcal{K}(S) \le y\} = 1$ and $P_Q\{\mathcal{K}(S) \le y\} = 1 - P_Q\{\mathcal{K}(S) > y\}$. Therefore,

$$P_Q\{\mathcal{D}(S) \le y\} = P_Q\{\mathcal{K}(S) > y\}\left(P_Q\{\mathcal{D}(S) \le y \mid \mathcal{K}(S) > y\} - 1\right) + 1. \tag{4.9}$$

Using the fact that

$$P_Q\left\{S \leq \mathcal{D}^{-1}(y) \mid S > \mathcal{K}^{-1}(y)\right\} = \frac{P_Q\{\mathcal{D}(S) \leq y, \mathcal{K}(S) > y\}}{P_Q\{\mathcal{K}(S) > y\}}$$

(4.9) becomes

$$P_Q\{\mathcal{D}(S) \leq y\} = 1 + P_Q\{\mathcal{D}(S) \leq y, \mathcal{K}(S) > y\} - P_Q\{\mathcal{K}(S) > y\}.$$

Then, (4.8) becomes

$$\frac{P_Q\{\mathcal{K}(S) > y\} - 1 + P_Q\{\mathcal{D}(S) \leq y\}}{P_Q\{\mathcal{K}(S) > y\}} = \frac{P_Q\{\mathcal{D}(S) \leq y, \mathcal{K}(S) > y\}}{P_Q\{\mathcal{K}(S) > y\}}$$

which can be written as

$$P_Q\{\mathcal{D}(S) \leq y \mid \mathcal{K}(S) > y\} = \frac{p + h - ce^{rT}}{p + h - s} = \hat{p}. \tag{4.10}$$

Recall that, (4.10) is equal to (4.7) since

$$P_Q\left\{S \leq \mathcal{D}^{-1}(y) \mid S > \mathcal{K}^{-1}(y)\right\} = \frac{P_Q\{\mathcal{D}(S) \leq y, \mathcal{K}(S) > y\}}{P\{\mathcal{K}(S) > y\}}.$$

This shows that the transformation we utilized above holds when $\mathcal{K}(x) > \mathcal{D}(x)$ for all x.

- Case 2: $\mathcal{D}(x) > \mathcal{K}(x)$ for all $x \geq 0$. In this case,

$$\begin{aligned}
f(0) &= \left(s - ce^{rT}\right)\min\{\mathcal{K}(0), y\} + (p + h - s)\min\{\mathcal{K}(0), y\} - h\mathcal{D}(0) \\
f'(0) &= \left(s - ce^{rT}\right)\mathcal{K}'(0)\,1_{\{\mathcal{K}(0)<y\}} + (p + h - s)\,\mathcal{K}'(0)\,1_{\{\mathcal{K}(0)<y\}} - h\mathcal{D}'(0) \\
f'(dz) &= \left(s - ce^{rT}\right)\mathcal{K}''(z)\,1_{\{\mathcal{K}(z)<y\}}dz - \left(s - ce^{rT}\right)\mathcal{K}'(z)\,1_{\{z=\mathcal{K}^{-1}(y)\}} \\
&\quad + (p + h - s)\,\mathcal{K}''(z)\,1_{\{\mathcal{K}(z)<y\}}dz - (p + h - s)\,\mathcal{K}'(z)\,1_{\{z=\mathcal{K}^{-1}(y)\}} \\
&\quad - h\mathcal{D}''(z)\,dz.
\end{aligned}$$

The replicating cash flow is written as

$$\begin{aligned}
f(S) &= \left(p+h-ce^{rT}\right)\min\{\mathcal{K}(0),y\} - h\mathcal{D}(0) \\
&\quad + \left(\left(p+h-ce^{rT}\right)\mathcal{K}'(0)\,1_{\{\mathcal{K}(0)<y\}} - h\mathcal{D}'(0)\right)S \\
&\quad + \left(p+h-ce^{rT}\right)\int_0^{\mathcal{K}^{-1}(y)} \max\{S-z,0\}\,\mathcal{K}''(z)\,dz \\
&\quad - \left(p+h-ce^{rT}\right)\max\left\{S-\mathcal{K}^{-1}(y),0\right\}\mathcal{K}'\left(\mathcal{K}^{-1}(y)\right) \\
&\quad - h\int_0^{\infty}\max\{S-z,0\}\,\mathcal{D}''(z)\,dz.
\end{aligned}$$

The expected cash flow is

$$\begin{aligned}
E_Q[f(S)] &= \left(p+h-ce^{rT}\right)\min\{\mathcal{K}(0),y\} - h\mathcal{D}(0) \\
&\quad + \left(\left(p+h-ce^{rT}\right)\mathcal{K}'(0)\,1_{\{\mathcal{K}(0)<y\}} - h\mathcal{D}'(0)\right)S_0e^{rT} \\
&\quad - \left(p+h-ce^{rT}\right)\mathcal{K}'\left(\mathcal{K}^{-1}(y)\right)\int_{\mathcal{K}^{-1}(y)}^{\infty}\left(x-\mathcal{K}^{-1}(y)\right)F_S(dx) \\
&\quad + \left(p+h-ce^{rT}\right)\int_0^{\mathcal{K}^{-1}(y)}\int_z^{\infty}(x-z)\,\mathcal{K}''(z)\,F_S(dx)\,dz \\
&\quad - h\int_0^{\infty}\max\{S-z,0\}\,\mathcal{D}''(z)\,dz
\end{aligned}$$

Supposing $\mathcal{K}(0) \le y$, the optimality condition can be found by taking the derivative of the expected cash flow with respect to y, so that

$$\begin{aligned}
\frac{dE_Q[f(S)]}{dy} &= \left(p+h-ce^{rT}\right)\frac{\mathcal{K}''\mathcal{K}^{-1}(y)}{\mathcal{K}'\left(\mathcal{K}^{-1}(y)\right)}\int_{\mathcal{K}^{-1}(y)}^{\infty}\left(x-\mathcal{K}^{-1}(y)\right)F_S(dx) \\
&\quad - \left(p+h-ce^{rT}\right)\frac{\mathcal{K}''\mathcal{K}^{-1}(y)}{\mathcal{K}'\left(\mathcal{K}^{-1}(y)\right)}\int_{\mathcal{K}^{-1}(y)}^{\infty}\left(x-\mathcal{K}^{-1}(y)\right)F_S(dx) \\
&\quad + \left(p+h-ce^{rT}\right)\int_{\mathcal{K}^{-1}(y)}^{\infty}F_S(dx) \\
&= \left(p+h-ce^{rT}\right)\int_{\mathcal{K}^{-1}(y)}^{\infty}F_S(dx) \\
&= \left(p+h-ce^{rT}\right)P_Q\{\mathcal{K}(S)>y\}. \qquad (4.11)
\end{aligned}$$

Since the derivative is always greater than 0, the optimal solution is $y^* = \infty$. Also, note that when $\mathcal{K}(x) < \mathcal{D}(x)$ for all x, the optimal solution is $y^* = \infty$ since (4.7) gives

$$P_Q\{\mathcal{D}(S) \le y \mid \mathcal{K}(S) > y\} = 0$$

for all y. Since characterizations in (4.7) and (4.11) leads to some conclusion, our transformation again holds. Intuitively if the capacity is always less than the demand, the logical action would be ordering very large quantities; because, it is a fact that all the inventory on hand will be sold.

- Case 3: $\mathcal{K}(x) > \mathcal{D}(x)$ for all $\bar{y} > x \geq 0$ and $\mathcal{K}(x) < \mathcal{D}(x)$ for all $\bar{y} < x$ when there is a unique $\bar{y}$ for which $\mathcal{K}(\bar{y}) = \mathcal{D}(\bar{y})$

  We've analyzed the case if functions $\mathcal{K}$ and $\mathcal{D}$ do not intercept. Now we will analyze the case when these function have only one cross-section point $\bar{y}$. Recall that the profit function is

$$CF(S, y) = \left(s - ce^{rT}\right) \min\{\mathcal{K}(S), y\} + (p + h - s) \min\{\mathcal{K}(S), \mathcal{D}(S), y\} - h\mathcal{D}(S).$$

  Following the same steps as in previous examples, we obtain

$$\begin{aligned}
f(0) &= \left(s - ce^{rT}\right) \min\{\mathcal{K}(0), y\} + (p + h - s) \min\{\mathcal{D}(0), y\} - h\mathcal{D}(0) \\
f'(0) &= \left(s - ce^{rT}\right) \mathcal{K}'(0) 1_{\{\mathcal{K}(0) < y\}} + (p + h - s) \mathcal{D}'(0) 1_{\{\mathcal{D}(0) < y\}} - h\mathcal{D}'(0) \\
f'(dz) &= \left(s - ce^{rT}\right) \mathcal{K}''(z) 1_{\{\mathcal{K}(z) < y\}} dz - \left(s - ce^{rT}\right) \mathcal{K}'(z) 1_{\{\mathcal{K}(z) = y\}} \\
&\quad + (p + h - s) \mathcal{D}''(z) 1_{\{z \leq \bar{y}, \mathcal{D}(z) \leq y\}} dz + (p + h - s) \mathcal{K}''(z) 1_{\{z > \bar{y}, \mathcal{K}(z) \leq y\}} dz \\
&\quad - (p + h - s) \mathcal{D}'(z) 1_{\{z = \bar{y}\}} + (p + h - s) \mathcal{K}'(z) 1_{\{z = \bar{y}\}} \\
&\quad - (p + h - s) \mathcal{D}'(z) 1_{\{\mathcal{D}(z) = y\}} dz - (p + h - s) \mathcal{K}'(z) 1_{\{\mathcal{K}(z) = y\}} dz \\
&\quad - h\mathcal{D}''(z) dz
\end{aligned}$$

Using (4.2), we characterize the replicating portfolio as

$$\begin{aligned} f(S) \;=\;& \left(s - ce^{rT}\right) \min\left\{\mathcal{K}(0), y\right\} + (p+h-s)\min\left\{\mathcal{D}(0), y\right\} - h\mathcal{D}(0) \\ &+ \left[\left(s - ce^{rT}\right)\mathcal{K}'(0)\, 1_{\{\mathcal{K}(0)<y\}} + (p+h-s)\,\mathcal{D}'(0)\, 1_{\{\mathcal{D}(0)<y\}} - h\mathcal{D}'(0)\right] S \\ &+ (p+h-s)\max\left\{S - \mathcal{K}^{-1}(y), 0\right\}\mathcal{K}'\left(\mathcal{K}^{-1}(y)\right) 1_{\{\mathcal{K}(S)\le y, S>\bar{y}\}} \\ &- \left(s - ce^{rT}\right)\max\left\{S - \mathcal{K}^{-1}(y), 0\right\}\mathcal{K}'\left(\mathcal{K}^{-1}(y)\right) \\ &+ (p+h-s)\int_0^{\min\left\{\mathcal{D}^{-1}(y), \bar{y}\right\}} \max\left\{S - z, 0\right\}\mathcal{D}''(z)\, dz \\ &- (p+h-s)\max\left\{S - \mathcal{D}^{-1}(y), 0\right\}\mathcal{D}'\left(\mathcal{D}^{-1}(y)\right) 1_{\{\mathcal{D}^{-1}(y)\le y\}} \\ &+ (p+h-s)\int_{\bar{y}}^{\mathcal{K}^{-1}(y)} \max\left\{S - z, 0\right\}\mathcal{K}''(z)\, dz 1_{\{\mathcal{K}(S)\le y, S>\bar{y}\}} \\ &+ (p+h-s)\max\left\{S - \bar{y}, 0\right\}\mathcal{K}'(\bar{y})\, 1_{\{\mathcal{K}(S)\le y, S>\bar{y}\}} \\ &- (p+h-s)\max\left\{S - \bar{y}, 0\right\}\mathcal{D}'(\bar{y})\, 1_{\{\mathcal{D}^{-1}(y)>y\}} \\ &+ \left(s - ce^{rT}\right)\int_0^{\mathcal{K}^{-1}(y)} \max\left\{S - z, 0\right\}\mathcal{K}''(z)\, dz \\ &- h\int_0^{\infty} \max\left\{S - z, 0\right\}\mathcal{D}''(z)\, dz. \end{aligned}$$

Then the objective function becomes

$$
\begin{aligned}
E_Q\left[f\left(S\right)\right] = {} & \left(s-ce^{rT}\right)\min\left\{\mathcal{K}\left(0\right),y\right\}+\left(p+h-s\right)\min\left\{\mathcal{D}\left(0\right),\mathcal{K}\left(0\right),y\right\}-h\mathcal{D}\left(0\right) \\
& +\left[\left(s-ce^{rT}\right)\mathcal{K}'\left(0\right)_{\{\mathcal{K}(0)<y\}}+\left(p+h-s\right)\mathcal{D}'\left(0\right)_{\{\mathcal{K}(0)<y\}}-h\mathcal{D}'\left(0\right)\right]S_0e^{rT} \\
& -\left(p+h-s\right)\int_{\mathcal{D}^{-1}(y)}^{\infty}F_S\left(dx\right)\left(x-\mathcal{D}^{-1}\left(y\right)\right)\mathcal{D}'\left(\mathcal{D}^{-1}\left(y\right)\right)1_{\{\mathcal{D}^{-1}(y)\leq y\}} \\
& +\left(p+h-s\right)\int_{\mathcal{K}^{-1}(y)}^{\infty}F_S\left(dx\right)\left(x-\mathcal{K}^{-1}\left(y\right)\right)\mathcal{K}'\left(\mathcal{K}^{-1}\left(y\right)\right)1_{\{\mathcal{K}(S)\leq y,S>\bar{y}\}} \\
& +\left(s-ce^{rT}\right)\int_{0}^{\mathcal{K}^{-1}(y)}dz\int_{z}^{\infty}F_S\left(dx\right)\left(x-z\right)\mathcal{K}''\left(z\right) \\
& -\left(s-ce^{rT}\right)\int_{\mathcal{K}^{-1}(y)}^{\infty}F_S\left(dx\right)\left(x-\mathcal{K}^{-1}\left(y\right)\right)\mathcal{K}'\left(\mathcal{K}^{-1}\left(y\right)\right) \\
& +\left(p+h-s\right)\int_{0}^{\min\left\{\mathcal{D}^{-1}(y)\bar{y}\right\}}dz\int_{z}^{\infty}F_S\left(dx\right)\left(x-z\right)\mathcal{D}''\left(z\right) \\
& +\left(p+h-s\right)\int_{\bar{y}}^{\mathcal{K}^{-1}(y)}dz\int_{z}^{\infty}F_S\left(dx\right)\left(x-z\right)\mathcal{K}''\left(z\right)1_{\{\mathcal{K}(S)\leq y,S>\bar{y}\}} \\
& +\left(p+h-s\right)\int_{\bar{y}}^{\infty}F_S\left(dx\right)\left(x-\bar{y}\right)\mathcal{K}'\left(\bar{y}\right)1_{\{\mathcal{K}(S)\leq y,S>\bar{y}\}} \\
& -\left(p+h-s\right)\int_{\bar{y}}^{\infty}F_S\left(dx\right)\left(x-\bar{y}\right)\mathcal{D}'\left(\bar{y}\right)1_{\{\mathcal{D}^{-1}(y)>y\}} \\
& -h\int_{0}^{\infty}\int_{z}^{\infty}F_S\left(dx\right)\left(x-z\right)\mathcal{D}''\left(z\right)dz.
\end{aligned}
$$

Supposing that $y\geq\max\left\{\mathcal{D}\left(0\right),\mathcal{K}\left(0\right)\right\}$, we can observe two different scenarios: (1) $\min\left\{\mathcal{D}^{-1}\left(y\right),\bar{y}\right\}=\mathcal{D}^{-1}\left(y\right)$ and (2) $\min\left\{\mathcal{D}^{-1}\left(y\right),\bar{y}\right\}=\bar{y}$. If the first scenario is true, in order to find the optimality condition we take the derivative with respect to y, which yields

$$
\begin{aligned}
\frac{dE_Q\left[f\left(S\right)\right]}{dy} &= \left(s-ce^{rT}\right)\int_{\mathcal{K}^{-1}(y)}^{\infty}F_S\left(dx\right)+\left(p+h-s\right)\int_{\mathcal{D}^{-1}(y)}^{\infty}F_S\left(dx\right) \\
&= \left(s-ce^{rT}\right)P_Q\left\{\mathcal{K}\left(S\right)>y\right\}+\left(p+h-s\right)P_Q\left\{\mathcal{D}\left(S\right)>y\right\}=0.
\end{aligned}
$$

The optimal solution can be characterized as

$$
\frac{P_Q\left\{\mathcal{D}\left(S\right)\leq y^*\right\}}{P_Q\left\{\mathcal{K}\left(S\right)>y^*\right\}}=\frac{p+h-ce^{rT}}{p+h-s}. \tag{4.12}
$$

Note that $\min\left\{\mathcal{D}^{-1}\left(y\right),\bar{y}\right\}=\mathcal{D}^{-1}\left(y\right)$ implies that $\mathcal{D}^{-1}\left(y\right)>\mathcal{K}^{-1}\left(y\right)$ which further implies that $\mathcal{K}(y) > \mathcal{D}(y)$. Under these conditions

$$\begin{aligned} P_Q\{\mathcal{D}(S) \leq y\} &= P_Q\{\mathcal{D}(S) \leq y \mid \mathcal{K}(S) \leq y\} P_Q\{\mathcal{K}(S) \leq y\} \\ &\quad + P_Q\{\mathcal{D}(S) \leq y \mid \mathcal{K}(S) > y\} P_Q\{\mathcal{K}(S) > y\} \\ &= P_Q\{\mathcal{D}(S) \leq y \mid \mathcal{K}(S) > y\} P_Q\{\mathcal{K}(S) > y\} \\ &= P_Q\{\mathcal{D}(S) \leq y, \mathcal{K}(S) > y\}. \end{aligned}$$

Hence optimality condition in (4.12) can be rewritten as

$$\begin{aligned} \frac{P_Q\{\mathcal{D}(S) \leq y^*\}}{P_Q\{\mathcal{K}(S) > y^*\}} &= \frac{P_Q\{\mathcal{D}(S) \leq y^*, \mathcal{K}(S) > y^*\}}{P_Q\{\mathcal{K}(S) > y^*\}} \\ &= P_Q\{\mathcal{D}(S) \leq y^* \mid \mathcal{K}(S) > y^*\} = \hat{p}. \end{aligned}$$

This solution shows that our transformation holds for this sub-case.

However, if the second scenario is true, meaning $\min\left\{\mathcal{D}^{-1}(y), \bar{y}\right\} = \bar{y}$, then the optimal condition becomes

$$\begin{aligned} \frac{dE_Q[f(S)]}{dy} &= \left(s - ce^{rT}\right) \int_{\mathcal{K}^{-1}(y)}^{\infty} F_S(dx) + (p + h - s) \int_{\mathcal{K}^{-1}(y)}^{\infty} F_S(dx) \\ &= \left(p + h - ce^{rT}\right) P_Q\{\mathcal{K}(S) > y\} > 0 \end{aligned}$$

so that the optimal solution is $y^* = \infty$. Note that the characterization reverts back to Case 2, meaning our transformation holds again.

Based on all these 3 cases that we considered our transformation checks out. It is also possible to show the same results for more complex cases including functions with multiple cross-section points, but we are not going to analyze them any further.

Random Yield

Apart from capacity model there are also other methods of modeling supply uncertainty; one of which is the random yield model. In these types of models supply uncertainty is generated by using a random variable U which represents the variability of the shipped order quantity. U is assumed to have a correlation with financial stocks. In this chapter, this correlation is perfect; thus, $D = \mathcal{D}(S)$ and $U = \mathcal{U}(S)$.

Due to the uncertainty in supply when we order y units we would get $Q(y) = Uy$. Furthermore, we suppose that $\mathcal{D}(S)$ is increasing and $\mathcal{U}(S)$ is decreasing and both are twice differentiable functions. The cash flow becomes

$$CF(S,y) = \left(s - ce^{rT}\right)\mathcal{U}(S)\,y + (p - s + h)\min\{\mathcal{U}(S)\,y, \mathcal{D}(S)\}. \tag{4.13}$$

The cash flow is a concave function in terms of y for all S. The expectation with respect to S doesn't change the concavity, therefore the derivative of the cash flow is a decreasing function. Moreover, let $\bar{y}(y)$ be the point where $\mathcal{U}(S)\,y$ and $\mathcal{D}(S)$ intersects each other so that $\mathcal{U}(\bar{y}(y)) = \mathcal{D}(\bar{y}(y))$. Then,

$$\begin{aligned}
f(0) &= \left(s - ce^{rT}\right)\mathcal{U}(0)\,y + (p - s + h)\min\{\mathcal{U}(0)\,y, \mathcal{D}(0)\} - h\mathcal{D}(0) \\
f'(0) &= \left(s - ce^{rT}\right) y\mathcal{U}'(0) + (p - s + h)\, y\mathcal{U}'(0)\, 1_{\{\mathcal{U}(0)y<\mathcal{D}(0)\}} \\
&\quad + (p - s + h)\,\mathcal{D}'(0)\, 1_{\{\mathcal{U}(0)y>\mathcal{D}(0)\}} - h\mathcal{D}'(0) \\
f'(dz) &= \left(s - ce^{rT}\right) y\mathcal{U}''(z)\,dz + (p - s + h)\, y\mathcal{U}''(z)\, 1_{\{\mathcal{U}(z)y<\mathcal{D}(z)\}} dz \\
&\quad + (p - s + h)\,\mathcal{D}''(z)\, 1_{\{\mathcal{U}(z)y>\mathcal{D}(z)\}} dz \\
&\qquad + (p - s + h)\left(\mathcal{D}'(\bar{y}(y)) - y\mathcal{U}'(\bar{y}(y))\right) - h\mathcal{D}''(z)\,dz.
\end{aligned}$$

The replicating portfolio becomes

$$\begin{aligned}
f(S) &= \left(s - ce^{rT}\right)\mathcal{U}(0)\,y + (p + h - s)\min\{\mathcal{U}(0)\,y, \mathcal{D}(0)\} - h\mathcal{D}(0) \\
&\quad + \left(s - ce^{rT}\right) y\mathcal{U}'(0)\,S - h\mathcal{D}'(0)\,S \\
&\quad + (p + h - s)\left(y\mathcal{U}'(0)\, 1_{\left\{\mathcal{U}(0)<\frac{\mathcal{D}(0)}{y}\right\}} + \mathcal{D}'(0)\, 1_{\left\{\mathcal{U}(0)>\frac{\mathcal{D}(0)}{y}\right\}}\right) S \\
&\quad + \left(s - ce^{rT}\right)\int_0^\infty \max\{S - z, 0\}\, y\mathcal{U}''(z)\,dz \\
&\quad + (p + h - s)\max\{S - \bar{y}(y), 0\}\left(\mathcal{D}'(\bar{y}(y)) - y\mathcal{U}'(\bar{y}(y))\right) \\
&\quad + (p + h - s)\int_0^\infty \max\{S - z, 0\}\, y\mathcal{U}''(z)\, 1_{\left\{y<\frac{\mathcal{D}(z)}{\mathcal{U}(z)}\right\}} dz \\
&\quad + (p + h - s)\int_0^\infty \max\{S - z, 0\}\, y\mathcal{D}''(z)\, 1_{\left\{y>\frac{\mathcal{D}(z)}{\mathcal{U}(z)}\right\}} dz.
\end{aligned} \tag{4.14}$$

Pricing (4.14) is hard, due to the complex structure of the replicating cash flow. Instead we utilize the perfect correlation to derive the pricing formulation using (3.17). Recall that the optimality condition of (3.17) is

$$\frac{E\left[U1_{\{D\leq Uy^*\}}\right]}{E\left[U\right]}=\frac{p+h-c}{p+h-s}.$$

Substituting $\mathcal{D}(S)$ and $\mathcal{U}(S)$ into the equation we obtain

$$\frac{E_Q\left[\mathcal{U}(S)\,1_{\{\mathcal{D}(S)\leq\mathcal{U}(S)y^*\}}\right]}{E_Q\left[\mathcal{U}(S)\right]}=\frac{p+h-ce^{rT}}{p+h-s} \tag{4.15}$$

by noting that time is an issue in financial models we replace c with ce^{rT}.

As we depicted in Section (3.4), this function is an increasing function in terms of y. Thus there exists an optimal solution so that the optimality condition holds. And since the objective function is concave, the second order condition also holds. Again we manage to obtain the same optimal y^* characterization as in Section (3.4) but with zero risk.

Random Yield and Capacity

Random yield and capacity models often coexist, thus to be more realistic we are going to combine the both models. Meaning, the supply uncertainty now has two elements, capacity and variability. We are going to represent this new uncertainty as follows, $Q(y)=\mathcal{U}(S)\min\{y,\mathcal{K}(S)\}$, assuming $P\{\mathcal{D}(S)<0\}=0$. Then, our cash flow becomes

$$\begin{aligned}CF(S,y) &= \left(s-ce^{rT}\right)\mathcal{U}(S)\min\{y,\mathcal{K}(S)\}+(p+h-s)\min\{\mathcal{U}(S)\min\{y,\mathcal{K}(S)\},\mathcal{D}(S)\}\\ &\quad -h\mathcal{D}(S).\end{aligned}$$

Finding the replicating portfolio and its time zero value is involved. So, we modify the optimal y characterization in (3.21) so that

$$\frac{E_Q\left[\mathcal{U}(S)\,1_{\{\mathcal{K}(S)>y^*,\mathcal{D}(S)\leq\mathcal{U}(S)y^*\}}\right]}{E_Q\left[\mathcal{U}(S)\,1_{\{\mathcal{K}(S)>y^*\}}\right]}=\frac{p+h-ce^{rT}}{p+h-s} \tag{4.16}$$

Reaching useful results is not very easy since the optimality contion (4.16) is complex. The complexity depends on the relation between demand, capacity and yield and the structure of

$$g(y)=\frac{E_Q\left[\mathcal{U}(S)\,1_{\{\mathcal{K}(S)>y,\mathcal{D}(S)\leq\mathcal{U}(S)y\}}\right]}{E_Q\left[\mathcal{U}(S)\,1\right]}.$$

As y increases $P_Q\left\{\mathcal{K}\left(S\right)>y\right\}$ will decrease, however, $P_Q\left\{\mathcal{K}\left(S\right)>y,\mathcal{D}\left(S\right)\leq\mathcal{U}\left(S\right)y\right\}$ may decrease or increase. If it increases, then, $g\left(y\right)$ is an increasing function in terms of y and there exists an optimal solution. If $P_Q\left\{\mathcal{K}\left(S\right)>y,\mathcal{D}\left(S\right)\leq\mathcal{U}\left(S\right)y\right\}$ decreases; as y increases, then the important issue becomes which one decreases faster, $P_Q\left\{\mathcal{K}\left(S\right)>y\right\}$ or $P_Q\left\{\mathcal{K}\left(S\right)>y,\mathcal{D}\left(S\right)\leq\mathcal{U}\left(S\right)y\right\}$. If $P_Q\left\{\mathcal{K}\left(S\right)>y,\mathcal{D}\left(S\right)\leq\mathcal{U}\left(S\right)y\right\}$ decreases faster, then again, $g\left(y\right)$ is an increasing function in terms of y and there exists an optimal solution.

Chapter 5

NEWSVENDOR PROBLEM WITH IMPERFECT HEDGING

In the previous chapter, we covered a decision making framework in an environment with a complete arbitrage-free market where there is a perfect dependence between random demand, supply and a financial product. This enabled us to obtain useful insights with respect to the decision-making process but it is clear that the perfect dependence assumption is not realistic. In reality, we would exploit an imperfect dependence structure between the supply and demand processes and the financial market. This would prevent us from obtaining a replicating portfolio, thus limiting our ability to reduce the variability of the expected profit. The inexistence of such a portfolio implies that the realized profit will be random, making the analysis more challenging.

Since a replication approach cannot be employed, other methods could be applied. A general approach would be to maximize the expected utility of the portfolio as in Özekici and Çanakoglu [7]. However, the general expected utility maximization problem is challenging and falls outside the scope of this thesis. Instead, as done frequently in the literature, we investigate trade-offs between the expected value and variance criteria. These methods strive to achieve a trade-off between the expected profit and the variance of the profit with the underlying assumption that decision-makers prefer lower variance for the same level of expected returns. Below we describe two alternative approaches to address the trade-off:

- Mean-Variance Hedging: The aim here is to choose a portfolio of financial securities to maximize a weighted sum of the of expected cash flow with the variance of the cash flow for a given order quantity y. Let $\mathbf{X}$ denote the vector of random variables corresponding to demand and supply uncertainties, S be the price of a primary asset in the market, $f_i(S)$ be the payoff of the ith derivative security, α_i denote the amount from this security i, and $CF(\mathbf{X}, y)$ denote the unhedged cash flow. Although we let S denote the price of a single asset, our analysis actually holds as well when S is indeed a vector representing the price of a number of primary assets in the market. The total cash flow is given by

$$CF_\alpha(\mathbf{X}, S, y) = CF(\mathbf{X}, y) + \sum_{i=1} \alpha_i f_i(S_i).$$

To express the objective function, let θ denote the relative weight of the variance criterion. The mean-variance objective function can be written as

$$\max_\alpha E\left[CF_\alpha(\mathbf{X}, S, y)\right] - \theta Var\left(CF_\alpha(\mathbf{X}, S, y)\right).$$

- Minimum Variance Hedging: The aim here is to minimize the variance of the hedged cash flow for different return levels by holding a portfolio of financial securities. This could be viewed as a special case of the mean-variance criterion above as θ becomes large. The goal now is to find the optimal α_i amounts to minimize the variance of the total cash flow for a given order quantity y. The optimization problem can be rewritten as

$$\min_\alpha Var\left(CF(\mathbf{X}, y) + \sum_{i=1}^{n} \alpha_i f_i(S)\right). \tag{5.1}$$

Once the optimal solution $\alpha^*(y)$ is determined for any order quantity y, the decision maker chooses the optimal order quantity by solving

$$\max_y E\left[CF(\mathbf{X}, y) + \sum_{i=1}^{n} \alpha_i^*(y) f_i(S)\right].$$

The minimum variance (min-var) criterion appears to be more analytically tractable than the mean-variance criterion. Therefore, in the rest of the thesis, we focus on min-var hedging models.

5.1 *Minimum Variance Hedging*

In our context, decision makers select the order quantity y and the financial portfolio simultaneously. This makes the minimum variance (min-var) approach a two-step characterization method. First, the decision maker attempts to reduce the volatility of the returns by investing in inventory and a portfolio consisting of different securities. After the optimal amount of each security is calculated for a given ordering quantity, decision maker may aim to either maximize the expected profit or to further minimize the variance by choosing the ordering quantity. In the remainder of this chapter, we will concentrate on maximizing the expected profit for the second step.

As before, we start the investigation with the case of random demand and no supply uncertainty. Further, we first investigate the special case of a portfolio consisting of a single security and later generalize it to the case of multiple securities. This analysis is later carried out in the cases with supply uncertainty due to both random capacity and uncertain yield.

5.1.1 Random Demand

Let us suppose that the supply is certain and the randomness of the profit comes only from demand. Recall that the hedged cash flow in this situation is given by

$$CF_{\alpha}\left(D,S,y\right)=\left(s-ce^{rT}\right)y+\left(p+h-s\right)\min\left\{D,y\right\}-hD+\sum_{i=1}^{n}\alpha_{i}f_{i}\left(S\right).$$

The objective is to solve the optimal portfolio determination problem given in (5.1). Next, we explore the solution of this problem for a single security first and for multiple securities later.

Hedging with only one financial security

There are many financial products available in the market and the decision maker may keep multiple securities in his/her portfolio. Intuitively as the number of instrument types invested in increases, the reduction in the variability will also increase. But for now let us assume that the decision maker prefers to hedge the risks associated with inventory operations, carrying only one type of instrument. Thus, the optimization problem becomes

$$\min_{\alpha}Var\left(CF\left(D,y\right)+\alpha f\left(S\right)\right).$$

In the first step, we will use the objective function to obtain the optimal α^* value for given y. Then we will plug in the α^* to find the optimal y^*. The objective function can be written as follows

$$\begin{aligned} Var\left(CF\left(D,y\right)+\alpha f\left(S\right)\right) &= \alpha^2 Var\left(f\left(S\right)\right)+2\alpha\left(p+h-s\right)Cov\left(f\left(S\right),\min\left\{D,y\right\}\right) \\ &\quad -2\alpha h Cov\left(f\left(S\right),D\right)+Var\left(CF\left(D,y\right)\right). \qquad (5.2) \end{aligned}$$

Differentiating with respect to α, we obtain

$$\begin{aligned} \frac{\partial Var\left(CF\left(D,y\right)+\alpha f\left(S\right)\right)}{\partial\alpha} &= 2\alpha Var\left(f\left(S\right)\right)+2\left(p+h-s\right)Cov\left(f\left(S\right),\min\left\{D,y\right\}\right) \\ &\quad -2hCov\left(S,D\right). \end{aligned}$$

Differentiating (5.2) for the second time in terms of α again

$$\frac{\partial^2 Var\left(CF\left(D,y\right)+\alpha f\left(S\right)\right)}{\partial\alpha^2}=2Var\left(f\left(S\right)\right)\geq 0$$

which shows that the variance of the total cash flow is convex in α. Hence, to find the optimal value of α, we use the first order condition,

$$\alpha Var\left(f\left(S\right)\right)+\left(p+h-s\right)Cov\left(f\left(S\right),\min\left\{D,y\right\}\right)-hCov\left(f\left(S\right),D\right)=0.$$

This yields the following for the optimal hedging position α for a given y value

$$\alpha^*\left(y\right)=\frac{-\left(p+h-s\right)Cov\left(f\left(S\right),\min\left\{D,y\right\}\right)+hCov\left(f\left(S\right),D\right)}{Var\left(S\right)}.$$

Moreover, by letting

$$\beta_D\left(y\right)=\frac{Cov\left(f\left(S\right),\min\left\{D,y\right\}\right)}{Var\left(f\left(S\right)\right)}$$

one can also show that the derivative of β with respect to y is

$$\beta'_D\left(y\right)=\frac{Cov\left(f\left(S\right),1_{\{D>y\}}\right)}{Var\left(f\left(S\right)\right)}.$$

We can explicitly write the optimal hedging position as

$$\alpha^*\left(y\right)=-\left(p+h-s\right)\beta_D\left(y\right)+h\beta_D\left(\infty\right). \qquad (5.3)$$

To make further progress, we need assumptions about the relationship between D and S. To this end, let us employ the definition of "positive association" between two random variables. According to Esary et al. [14] two random variables, such as D and S, are positively associated if $Cov\left(g\left(D\right), h\left(S\right)\right) \geq 0$ is true for all pairs of non-decreasing functions g and h. Moreover, they point out that this association becomes the strongest, when $Cov\left(g\left(D, S\right), h\left(D, S\right)\right) \geq 0$ is true for all pairs of non-decreasing functions g and h. Variables having such a property are called positively associated (PA). Additionally, Esary et al. [14] claims that PA also implies $P\left\{D > d \mid S = s\right\}$ is a non-decreasing function of d for fixed s. Therefore, assuming D and S are PA variables implies that $Cov\left(f\left(S\right), 1_{\{D>y\}}\right) > 0$, as long as both $f\left(S\right)$ and $1_{\{D>y\}}$ are non-decreasing functions of S and D. Hence, $\beta_D\left(y\right)$ becomes an increasing function of y. Based on this we observe that $\alpha^*\left(0\right) = h\beta_D\left(\infty\right) > 0$, $\alpha^*\left(\infty\right) = -\left(p+h-s\right)\beta_D\left(\infty\right) < 0$ and as y increases $\alpha^*\left(y\right)$ decreases. This means that a lower amount of investment in the security is needed when the order quantity is higher.

Utilizing (5.3) we can start the second step. Depending on the manager's decision, the optimal hedging position $\alpha^*\left(y\right)$ can be used to maximize profits or to minimize variance. Suppose the goal is to maximize the expected total profit, then our objective function becomes

$$\begin{aligned} E\left[CF\left(D, y\right) + \alpha^*\left(y\right) f\left(S\right)\right] &= \left(s - ce^{rT}\right) y + \left(p+h-s\right) E\left[\min\left\{D, y\right\}\right] \\ &\quad -hE\left[D\right] + \alpha^*\left(y\right) E\left[f\left(S\right)\right]. \end{aligned}$$

By substituting optimal $\alpha^*\left(y\right)$ into objective function we get

$$\begin{aligned} E\left[CF\left(D, y\right) + \alpha^*\left(y\right) f\left(S\right)\right] &= \left(s - ce^{rT}\right) y + \left(p+h-s\right) E\left[\min\left\{D, y\right\}\right] - hE\left[D\right] \\ &\quad -\left(\left(p+h-s\right)\beta_D\left(y\right) - h\beta_D\left(\infty\right)\right) E\left[f\left(S\right)\right]. \end{aligned} \tag{5.4}$$

Recall that in (3.3) we pointed out that

$$\frac{dE\left[\min\left\{D, y\right\}\right]}{dy} = P\left\{D > y\right\}.$$

Hence, by differentiating (5.4 with respect to y we obtain,

$$\begin{aligned} \frac{dE\left[CF\left(D, y\right) + \alpha^*\left(y\right) f\left(S\right)\right]}{dy} &= \left(s - ce^{rT}\right) + \left(p+h-s\right) P\left\{D > y\right\} \\ &\quad -\left(p+h-s\right)\beta_D'\left(y\right) E\left[f\left(S\right)\right]. \end{aligned}$$

And the optimality condition becomes

$$\left(s - ce^{rT}\right) + (p + h - s)\left(1 - P\left\{D \leq y\right\}\right) - \beta_D'(y)\, E\left[f(S)\right] = 0.$$

By rewriting the optimality condition can be found as

$$P\left\{D \leq y^*\right\} + \beta_D'(y^*)\, E\left[f(S)\right] = \frac{p + h - ce^{rT}}{p + h - s} = \hat{p}. \tag{5.5}$$

Recall that the right-hand side of (5.5), $\hat{p}$, is the critical ratio. The left-hand side is a probability plus a covariance term multiplied by a positive constant. Hence, the optimal solution depends on the covariance term's sign and shape, which makes the structure of $Cov\left(f(S), 1_{\{D>y\}}\right)$ very important. First of all, as long as the covariance term is strictly increasing there exists a unique optimal y^*. Moreover, assuming $f(S)$ is non-negative and increasing,

- If $Cov\left(f(S), 1_{\{D>y\}}\right) > 0$, the optimal order quantity with hedging will be smaller than the optimal order quantity without hedging.

- If $Cov\left(f(S), 1_{\{D>y\}}\right) < 0$, the optimal order quantity with hedging will be larger than the optimal order quantity without hedging.

- If $Cov\left(f(S), 1_{\{D>y\}}\right) = 0$, they will be equal.

Also note that the assumption of non-decreasing covariance in terms of y could be relaxed. What we need is a non-decreasing left-hand side in terms of y in (5.5). Furthermore, if there is no correlation between S and D, $\beta_D(y)$ will be zero; thus, the optimality condition in (5.5) reverts back to classical newsvendor model (3.4).

Hedging with multiple financial securities

In general hedging is achieved by using multiple financial securities. Intuitively, as the number of securities increases, the variance of the expected return will decrease. The variance of the cash flow can be written as,

$$\begin{aligned}
Var\left(CF_{\alpha}\left(D,S,y\right)\right) &= Var\left(CF\left(D,y\right)+\sum_{i=1}^{n}\alpha_i f_i\left(S\right)\right)\\
&= \sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_i\alpha_j Cov\left(f_i\left(S\right),f_j\left(S\right)\right)\\
&\quad +2\sum_{i=1}^{n}\alpha_i Cov\left(f_i\left(S\right),CF\left(D,y\right)\right)\\
&\quad +Var\left(CF\left(D,y\right)\right).
\end{aligned}$$

We can rewrite this equation in matrix notation as

$$Var\left(CF_{\alpha}\left(D,S,y\right)\right)=\boldsymbol{\alpha}^{\mathbf{T}}\mathbf{C}\boldsymbol{\alpha}+2\boldsymbol{\alpha}^{\mathbf{T}}\boldsymbol{\mu}\left(y\right)+Var\left(CF\left(D,y\right)\right) \tag{5.6}$$

where $\boldsymbol{\alpha}=\{\alpha_1,\alpha_2,\cdots,\alpha_n\}$ is the column vector, $\boldsymbol{\alpha}^T$ is the transpose of $\boldsymbol{\alpha}$, $\mathbf{C}$ is the covariance matrix with entries

$$C_{ij}=Cov\left(f_i\left(S\right),f_j\left(S\right)\right)$$

and

$$\mu_i\left(y\right)=Cov\left(f_i\left(S_i\right),CF\left(D,y\right)\right)$$

denotes the covariance between the financial securities and the cash flow. By taking the gradient with respect to $\boldsymbol{\alpha}$ and setting it equal to 0, the optimal order quantities can be characterized as

$$\boldsymbol{\alpha}^*\left(y\right)=-\mathbf{C}^{-1}\boldsymbol{\mu}\left(y\right). \tag{5.7}$$

The second order condition also checks out since the Hessian matrix of (5.6) is the covariance matrix $\mathbf{C}$ which is positive semi-definite. Hence, the optimal hedging quantity can be used to maximize the expected profit, the objective function is

$$\begin{aligned}
E\left[CF\left(D,y\right)+\boldsymbol{\alpha}^*\left(y\right)^{\mathbf{T}}\mathbf{f}\left(S\right)\right] &= \left(s-ce^{rT}\right)y+\left(p+h-s\right)E\left[\min\{D,y\}\right]\\
&\quad -hE\left[D\right]-\boldsymbol{\mu}\left(y\right)^{\mathbf{T}}\mathbf{C}^{-1}E\left[\mathbf{f}\left(S\right)\right]
\end{aligned}$$

where $\mathbf{f}\left(S\right)=\{f_1\left(S\right),f_2\left(S\right),\cdots,f_n\left(S\right)\}$ denotes the column vector of derivative securities. By taking the derivative with respect to y, we obtain the optimality condition

$$\left(s-ce^{rT}\right)+\left(p+h-s\right)P\{D>y\}-\boldsymbol{\mu}'\left(y\right)^{\mathbf{T}}\mathbf{C}^{-1}E\left[\mathbf{f}\left(S\right)\right]=0$$

which can be simplified to

$$P\left\{D \leq y\right\} + \frac{\boldsymbol{\mu}'\left(y\right)^{\mathbf{T}} \mathbf{C}^{-1} E\left[\mathbf{f}\left(S\right)\right]}{\left(p+h-s\right)} = \frac{p+h-ce^{rT}}{p+h-s} = \hat{p}. \tag{5.8}$$

Here $\boldsymbol{\mu}'\left(y\right)$ is the gradient vector obtained by setting

$$\mu_i'\left(y\right) = \frac{d\mu_i\left(y\right)}{dy}.$$

Recall that, as (3.3) pointed out,

$$\frac{dE\left[\min\left\{D, y\right\}\right]}{dy} = P\left\{D > y\right\}.$$

In (5.8), $\mu_i\left(y\right) = \left(p+h-s\right) Cov\left(f_i\left(S\right), \min\left\{D, y\right\}\right) - hCov\left(f_i\left(S\right), D\right)$, thus

$$\mu_i'\left(y\right) = \left(p+h-s\right) \frac{\partial Cov\left(f_i\left(S\right), \min\left\{y, D\right\}\right)}{\partial y} = \left(p+h-s\right) Cov\left(f_i\left(S\right), 1_{\{D>y\}}\right).$$

Consequently,

$$\boldsymbol{\mu}'\left(y\right) = \hat{\boldsymbol{\mu}}\left(y\right)\left(p+h-s\right)$$

where $\hat{\mu}_i\left(y\right) = Cov\left(f_i\left(S\right), 1_{\{D>y\}}\right)$. Hence, the optimality condition can be rewritten as

$$P\left\{D \leq y^*\right\} + \hat{\boldsymbol{\mu}}\left(y^*\right)^{\mathbf{T}} \mathbf{C}^{-1} E\left[\mathbf{f}\left(S\right)\right] = \frac{p+h-ce^{rT}}{p+h-s} = \hat{p}.$$

The right-hand side of the optimality condition is the critical ratio. The left-hand side consists up of a probability plus an addition term, $\hat{\boldsymbol{\mu}}\left(y\right)^{\mathbf{T}} \mathbf{C}^{-1} E\left[\mathbf{f}\left(S\right)\right]$; note that this term's sign and behavior is crucial. If it is non-decreasing and positive, then the optimal ordering quantity will be smaller compared to unhedged optimal ordering quantity. If it is non-decreasing and negative, then the optimal ordering quantity will be larger compared to the unhedged optimal ordering quantity. Furthermore, if there is no correlation between S and D, $\boldsymbol{\mu}\left(y\right)$ will be zero, thus, the optimality condition in (5.8) reverts back to classical newsvendor model (3.4).

5.1.2 Random Capacity

By adding supply uncertainty into our model we further increase the variability of the profit function. As usual, we define K to be the capacity of the supplier. Thus, when we order

y we will get $Q(y) = \min\{y, K\}$. Moreover this random variable K is assumed to have a positive correlation with S. So, the objective function becomes

$$\begin{aligned} CF_\alpha(D,K,S,y) &= CF(D,K,y) + \boldsymbol{\alpha}^{\mathbf{T}}\mathbf{f}(S) = \left(s - ce^{rT}\right)\min\{K,y\} \\ &\quad + (p+h-s)\min\{D,K,y\} \\ &\quad -hD + \sum_{i=1}^{n}\alpha_i f_i(S). \end{aligned} \tag{5.9}$$

Hedging with only one security

We assume that the decision maker is allowed to use only one financial instrument in the market. Then, the objective function is

$$\begin{aligned} Var(CF_\alpha(D,K,S,y)) &= \alpha^2 Var(f(S)) + 2\alpha Cov(f(S), CF(D,K,y)) \\ &\quad + Var(CF(D,K,y)). \end{aligned} \tag{5.10}$$

We find the optimal α by taking the derivative and setting it equal to zero so that

$$\frac{\partial Var(CF_\alpha(D,K,S,y))}{\partial\alpha} = 2\alpha Var(f(S)) + 2Cov(f(S), CF(D,K,y)) = 0 \tag{5.11}$$

and the optimal solution is

$$\begin{aligned} \alpha^*(y) &= -\frac{Cov(f(S), CF(D,K,y))}{Var(f(S))} \\ &= \frac{hCov(f(S),D) - \left(s - ce^{rT}\right)Cov(f(S), \min\{K,y\})}{Var(f(S))} \\ &\quad -\frac{(p+h-s)Cov(f(S), \min\{D,K,y\})}{Var(f(S))}. \end{aligned} \tag{5.12}$$

Since

$$\frac{\partial^2 Var(_\alpha(D,S,K,y))}{\partial\alpha^2} = 2Var(f(S)) \geq 0$$

the first order condition (5.11) is sufficient for optimality. Note that (5.12) can be simplified as

$$\alpha^*(y) = h\beta_D(\infty) - \left(s - ce^{rT}\right)\beta_K(y) - (p+h-s)\beta_{D,K}(y)$$

where

$$\begin{aligned}\beta_D(y) &= Cov\left(f(S), \min\{D, y\}\right)\\ \beta_K(y) &= Cov\left(f(S), \min\{K, y\}\right)\\ \beta_{D,K}(y) &= Cov\left(f(S), \min\{D, K, y\}\right).\end{aligned}$$

Thus, given the optimal hedging quantity, we can maximize the expected total profit by plugging in α from (5.12) and obtain the objective function

$$\begin{aligned}E\left[CF_{\alpha^*}(D,S,K,y)\right] &= \left(s - ce^{rT}\right)\left(E\left[\min\{K,y\}\right] - \beta_K(y)\,E\left[f(S)\right]\right)\\ &\quad + (p+h-s)\left(E\left[\min\{D,K,y\}\right] - \beta_{D,K}(y)\,E\left[f(S)\right]\right)\\ &\quad - h\left(E[D] - \beta_D(\infty)\,E\left[f(S)\right]\right)\end{aligned} \tag{5.13}$$

Recall that, in (3.3) and (3.6) we shown that

$$\frac{dE\left[\min\{K,y\}\right]}{dy} = P\{K > y\}$$

and

$$\frac{dE\left[\min\{D,K,y\}\right]}{dy} = P\{D > y, K > y\}.$$

Hence, the derivative of (5.13) with respect to y yields

$$\begin{aligned}\frac{dE\left[CF_{\alpha^*}(D,S,K,y)\right]}{dy} &= \left(s - ce^{rT}\right)\left(P\{K>y\} - \beta'_K(y)\,E\left[f(S)\right]\right)\\ &\quad + (p+h-s)\,P\{D>y, K>y\}\\ &\quad - (p+h-s)\,\beta'_{D,K}(y)\,E\left[f(S)\right].\end{aligned} \tag{5.14}$$

By rewriting (5.14), the optimality condition becomes

$$P\{D \le y^* \mid K > y^*\} + \frac{\left(\left(s - ce^{rT}\right)\beta'_K(y^*) + (p+h-s)\,\beta'_{D,K}(y^*)\right)E\left[f(S)\right]}{(p+h-s)\,P\{K > y^*\}} = \frac{p+h-ce^{rT}}{p+h-s}.$$

In this case the optimal y characterization shows resemblance to (3.9), or

$$P\{D \le y \mid K > y\} = \hat{p}$$

in the sense that we now have

$$P\{D \le y^* \mid K > y^*\} + A(y^*) = \hat{p}$$

where

$$A(y) = \frac{\left(\left(s - ce^{rT}\right)\beta'_K(y) + (p + h - s)\beta'_K(y)\right)E[f(S)]}{(p + h - s)P\{K > y\}}$$

If $A(y)$ is positive increasing, then the optimal solution satisfying this equation will be smaller than the optimal order quantity without hedging. On the other hand, if it is negative increasing, then the optimal solution satisfying this equation will be larger than the optimal order quantity quantity without hedging. When the correlations between random variables D, K and S is zero, $\beta_K(y)$ and $\beta_K(y)$ becomes zero, hence, the model reverts back to random capacity model (3.9).

Hedging with multiple securities

If the decision maker wants to use multiple instruments for hedging, the objective function becomes

$$Var\left(CF_\alpha(D,K,S,y)\right) = \boldsymbol{\alpha}^{\mathbf{T}}\mathbf{C}\boldsymbol{\alpha} + 2\boldsymbol{\alpha}^{\mathbf{T}}\boldsymbol{\mu}(y) + Var\left(CF(D,K,y)\right)$$

where

$$\boldsymbol{\mu}_i(y) = Cov\left(f_i(S), CF(D,K,y)\right)$$

now denotes the covariance between the financial securities and the new cash flow. By taking the gradient with respect to $\boldsymbol{\alpha}$ vector, the optimal hedging order quantities can be characterized as follows

$$\boldsymbol{\alpha}^*(y) = -\mathbf{C}^{-1}\boldsymbol{\mu}(y). \tag{5.15}$$

Adding multiple instruments does not effect the second order condition since the covariance matrix is always positive semi-definite. Therefore, the optimal hedging quantity can be used to maximize expected cash flow, or the objective function is

$$\begin{aligned} E\left[CF_{\alpha^*}(D,K,S,y)\right] &= \left(s - ce^{rT}\right)E\left[\min\{K,y\}\right] \\ &\quad + (p+h-s)E\left[\min\{D,\min\{K,y\}\}\right] \\ &\quad - hE[D] - \boldsymbol{\mu}(y)^{\mathbf{T}}\mathbf{C}^{-1}E\left[\mathbf{f}(S)\right]. \end{aligned}$$

By taking the derivative with respect to y we obtain the optimality condition

$$\left(s - ce^{rT}\right) P\left\{K > y\right\} + \left(p + h - s\right) P\left\{D > y, K > y\right\} - \boldsymbol{\mu}'\left(y\right)^{\mathbf{T}} \mathbf{C}^{-1} E\left[\mathbf{f}\left(S\right)\right] = 0$$

or

$$\left(s - ce^{rT}\right) + \left(p + h - s\right) P\left\{D > y \mid K > y\right\} - \frac{\boldsymbol{\mu}'\left(y\right)^{\mathbf{T}} \mathbf{C}^{-1} E\left[\mathbf{f}\left(S\right)\right]}{P\left\{K > y\right\}} = 0$$

where we used the function that

$$\begin{aligned} \frac{dE\left[\min\left\{K, y\right\}\right]}{dy} &= P\left\{K > y\right\} \\ \frac{dE\left[\min\left\{D, K, y\right\}\right]}{dy} &= P\left\{D > y, K > y\right\}. \end{aligned}$$

The optimality condition which can be rewritten as

$$P\left\{D \le y^* \mid K > y^*\right\} + \frac{\boldsymbol{\mu}'\left(y^*\right)^{\mathbf{T}} \mathbf{C}^{-1} E\left[\mathbf{f}\left(S\right)\right]}{\left(p + h - s\right) P\left\{K > y^*\right\}} = \frac{p + h - ce^{rT}}{p + h - s} = \hat{p}. \tag{5.16}$$

As in the previous case, this characterization shows resemblance to the random capacity model. We have a probability plus an addition term

$$A\left(y\right) = \frac{\boldsymbol{\mu}'\left(y\right)^{\mathbf{T}} \mathbf{C}^{-1} E\left[\mathbf{f}\left(S\right)\right]}{\left(p + h - s\right) P\left\{K > y\right\}}$$

on the left-hand side and critical ration on the right-hand side. If $A\left(y\right)$ is a positive increasing function, then the optimal ordering quantity satisfying (5.16) will be smaller than the optimal order quantity without hedging and vice versa. Moreover, if the correlation between random variables and the market is zero, the model reverts back to random capacity model in Chapter 3.

5.1.3 Random Yield

In this chapter, we assume that the randomness of the supply comes from the yield. Let U be a random variable which represents the variability in the supply so that $Q\left(y\right) = Uy$. Moreover, U is assumed to have some degree of correlation with financial stocks. Then, the cash flow becomes

$$\begin{aligned} CF_\alpha(D,U,S,y) &= CF(D,U,y) + \boldsymbol{\alpha}^{\mathbf{T}}\mathbf{f}(S) = \left(s - ce^{rT}\right)Uy \\ &\quad + (p+h-s)\min\{D,Uy\} \\ &\quad -hD + \sum_{i=1}^{n}\alpha_i f_i(S) \end{aligned} \tag{5.17}$$

Hedging with only one security

When the use of only one instrument is permitted, the objective function can be written as

$$Var\left(f_\alpha(D,U,S,y)\right) = Var\left(CF(D,U,y) + \alpha^* f(S)\right)$$

We find the optimal α^* by taking the derivative with respect to α and setting it equal to zero, so that

$$\frac{\partial Var\left(CF_\alpha(D,U,S,y)\right)}{\partial\alpha} = 2\alpha Var\left(f(S)\right) + 2Cov\left(f(S), CF(D,K,y)\right) = 0$$

and the optimal solution is

$$\alpha(y^*) = -\frac{Cov\left(f(S), CF(D,U,y)\right)}{Var\left(f(S)\right)}$$

or

$$\alpha^* = h\beta_D(\infty) - \left(s - ce^{rT}\right)y\beta_U - (p+h-s)\beta_{D,U}(y)$$

where

$$\begin{aligned} \beta_{D,U}(y) &= \frac{Cov\left(f(S), \min\{D,Uy\}\right)}{Var\left(f(S)\right)} \\ \beta_D(y) &= \frac{Cov\left(f(S), \min\{D,y\}\right)}{Var\left(f(S)\right)} \\ \beta_U &= \frac{Cov\left(f(S), U\right)}{Var\left(f(S)\right)}. \end{aligned}$$

The first order optimality condition is sufficient since the second order condition

$$\frac{\partial^2 Var\left(CF_\alpha(D,U,S,y)\right)}{\partial\alpha^2} = 2Var\left(f(S)\right) \geq 0$$

is satisfied. Given the optimal hedging quantity, we can now maximize the expected total profit over y, or

$$\begin{aligned} E\left[CF_{\alpha^*}(D,U,S,y)\right] &= \left(s-ce^{rT}\right)yE\left[U\right]+(p+h-s)E\left[\min\left\{D,Uy\right\}\right] \\ &\quad -hE\left[D\right]+h\beta_D(\infty)E\left[f(S)\right]-\left(s-ce^{rT}\right)y\beta_U E\left[f(S)\right] \\ &\quad -(p+h-s)\beta_{D,U}(y)E\left[f(S)\right] \\ &= \left(s-ce^{rT}\right)y\left(E\left[U\right]-\beta_U E\left[f(S)\right]\right)-h\left(E\left[D\right]-\beta_D(\infty)E\left[f(S)\right]\right) \\ &\quad +(p+h-s)\left(E\left[\min\left\{D,Uy\right\}\right]-\beta_{D,U}(y)E\left[f(S)\right]\right). \end{aligned}$$

The derivative with respect to y yields

$$\begin{aligned} \frac{dE\left[CF_{\alpha^*}(D,U,S,y)\right]}{dy} &= \left(s-ce^{rT}\right)\left(E\left[U\right]-\beta_U E\left[f(S)\right]\right) \\ &\quad +(p+h-s)\left(E\left[U1_{\{D>Uy\}}\right]-\beta'_{D,U}(y)E\left[f(S)\right]\right) \\ &= \left(s-ce^{rT}\right)\left(E\left[U\right]-\beta_U E\left[f(S)\right]\right) \\ &\quad +(p+h-s)\left(E\left[U\right]-E\left[U1_{\{D\le Uy\}}\right]-\beta'_{D,U}(y)E\left[f(S)\right]\right) \\ &= \left(p+h-ce^{rT}\right)E\left[U\right]-\left(s-ce^{rT}\right)\beta_U E\left[f(S)\right] \\ &\quad -(p+h-s)E\left[U1_{\{D\le Uy\}}\right]-\beta'_{D,U}(y)E\left[f(S)\right]. \end{aligned}$$

By setting it to zero the optimality condition can be written as

$$\frac{E\left[U1_{\{D\le Uy^*\}}\right]}{E\left[U\right]}+\frac{\beta'_{D,U}(y^*)E\left[f(S)\right]+\left(s-ce^{rT}\right)\beta_U E\left[f(S)\right]}{(p+h-s)E\left[U\right]}=\frac{p+h-ce^{rT}}{p+h-s}=\hat{p}. \quad (5.18)$$

Recall that the optimal solution resembles (3.17), or

$$\frac{E\left[U1_{\{D\le Uy^*\}}\right]}{E\left[U\right]}=\hat{p}.$$

In (5.18) there is an extra term

$$A(y)=\frac{\beta'_{D,U}(y)E\left[f(S)\right]+\left(s-ce^{rT}\right)\beta_U E\left[f(S)\right]}{(p+h-s)E\left[U\right]},$$

whose structure depends $\beta'_{D,U}(y)$ and β_U. If $A(y)$ is a positive increasing function, then the optimal order quantity will be smaller than the optimal order quantity without hedging. On the other hand if it is negative increasing then the optimal order quantity will be larger than the optimal order quantity without hedging. Lastly if there is no correlation at all, meaning $A(y)=0$, the model reverts back to random yield case in Chapter 3.

Hedging with multiple securities

When the decision maker is allowed to invest in many different securities, the objective function becomes,

$$Var\left(CF_{\alpha}\left(D,U,S,y\right)\right)=\boldsymbol{\alpha}^{\mathbf{T}}\mathbf{C}\boldsymbol{\alpha}+2\boldsymbol{\alpha}^{\mathbf{T}}\boldsymbol{\mu}\left(y\right)+Var\left(CF\left(D,U,y\right)\right). \tag{5.19}$$

where

$$\mu_i\left(y\right)=Cov\left(f_i\left(S\right),CF\left(D,U,y\right)\right).$$

By taking the gradient with respect to $\boldsymbol{\alpha}$, the optimal hedging order quantities can again be characterized as

$$\boldsymbol{\alpha}^*\left(y\right)=-\mathbf{C}^{-1}\boldsymbol{\mu}\left(y\right) \tag{5.20}$$

and the second order condition is also satisfied. The optimal hedging quantity can be used to maximize the expected cash flow, thus, the objective function is

$$\begin{aligned} E\left[CF_{\alpha^*}\left(D,U,S,y\right)\right] \quad &= \quad \left(s-ce^{rT}\right)E\left[U\right]y+\left(p+h-s\right)E\left[\min\left\{D,Uy\right\}\right] \\ &\quad -hE\left[D\right]-\boldsymbol{\mu}\left(y\right)^{\mathbf{T}}\mathbf{C}^{-1}E\left[\mathbf{f}\left(S\right)\right]. \end{aligned}$$

By taking the derivative with respect to y and setting it to zero, we obtain,

$$\left(s-ce^{rT}\right)E\left[U\right]+\left(p+h-s\right)E\left[U1_{\{D>Uy\}}\right]-\boldsymbol{\mu}'\left(y\right)^{\mathbf{T}}\mathbf{C}^{-1}E\left[\mathbf{f}\left(S\right)\right]=0$$

or

$$\left(p+h-ce^{rT}\right)E\left[U\right]-\left(p+h-s\right)E\left[U1_{\{D\leq Uy\}}\right]-\boldsymbol{\mu}'\left(y\right)^{\mathbf{T}}\mathbf{C}^{-1}E\left[\mathbf{f}\left(S\right)\right]=0.$$

Then, the optimality condition becomes

$$\frac{E\left[U1_{\{D\leq Uy^*\}}\right]}{E\left[U\right]}+\frac{\boldsymbol{\mu}'\left(y^*\right)^{\mathbf{T}}\mathbf{C}^{-1}E\left[\mathbf{f}\left(S\right)\right]}{\left(p+h-s\right)E\left[U\right]}=\frac{p+h-ce^{rT}}{p+h-s}=\hat{p}. \tag{5.21}$$

The optimality condition above, provides a characterization for y^* which is again similar to random yield model in previous chapter. However, this characterization has an addition term on the left-hand side, which is

$$A\left(y\right)=\frac{\boldsymbol{\mu}'\left(y\right)^{\mathbf{T}}\mathbf{C}^{-1}E\left[\mathbf{f}\left(S\right)\right]}{\left(p+h-s\right)E\left[U\right]}.$$

If $A(y)$ is positive and increasing, then the optimal order quantity will be smaller than the optimal order quantity without hedging and vice versa. Moreover, if there is no correlation between demand, supply and the market, $A(y)$ becomes zero making the model equal to random yield case in Chapter 3.

5.1.4 Random Yield and Random Capacity

In this chapter we are going to work on models with both random yield and random capacity. Thus, when we order y we will get $Q(y) = U \min\{y, K\}$. Moreover, random variables K and U assumed to have some degree of correlation with S. Then, the cash flow becomes

$$\begin{aligned} CF_\alpha(D,K,U,S,y) &= CF(D,K,U,y) + \boldsymbol{\alpha}^{\mathbf{T}}\mathbf{f}(S) = \left(s - ce^{rT}\right) U \min\{K,y\} \\ &\quad + (p+h-s)\min\{Uy, UK, D\} \\ &\quad - hD + \sum_{i=1}^{n} \alpha_i f_i(S) \end{aligned} \tag{5.22}$$

Hedging with only one security

The objective function can be written as

$$\begin{aligned} Var(CF_\alpha(D,K,U,S,y)) &= \alpha^2 Var(f(S)) + 2\alpha Cov(f(S), CF(D,K,U,y)) \\ &\quad + Var(CF(D,K,U,y)) \end{aligned}$$

We find the optimal α by taking the derivative with respect to α and setting it to 0, so that the objective function is

$$2\alpha Var(f(S)) + 2Cov(f(S), CF(D,K,U,y)) = 0$$

and we obtain

$$\begin{aligned} \alpha^*(y) &= -\frac{Cov(f(S), CF(D,K,U,y))}{Var(f(S))} \\ &= h\beta_D(\infty) - \left(s - ce^{rT}\right)\beta_{K,U}(y) - (p+h-s)\beta_{D,K,U}(y) \end{aligned} \tag{5.23}$$

where

$$\begin{aligned}
\beta_D(y) &= \frac{Cov\left(f(S), \min\{D, y\}\right)}{Var\left(f(S)\right)} \\
\beta_{K,U}(y) &= \frac{Cov\left(f(S), U\min\{K, y\}\right)}{Var\left(f(S)\right)} \\
\beta_{D,K,U}(y) &= \frac{Cov\left(f(S), \min\{Uy, UK, D\}\right)}{Var\left(f(S)\right)}.
\end{aligned}$$

The first order condition is sufficient since the second order condition is satisfied. Given the optimal hedging quantity, we can maximize the expected total profit by plugging in $\alpha^*(y)$, so that the objective function is

$$\begin{aligned}
E\left[CF_{\alpha^*}(D, K, U, S, y)\right] = {} & \left(s - ce^{rT}\right) E\left[U\min\{K, y\}\right] \\
& + (p + h - s)\, E\left[\min\{Uy, UK, D\}\right] \\
& - \left(s - ce^{rT}\right) \beta_{K,U}(y)\, E\left[f(S)\right] \\
& - (p + h - s)\, \beta_{D,K,U}(y)\, E\left[f(S)\right] \\
& - h\left(E[D] - \beta_D(\infty)\, E\left[f(S)\right]\right)
\end{aligned} \tag{5.24}$$

Recall that, in (3.20) we show that

$$\frac{dE\left[\min\{Uy, UK, D\}\right]}{dy} = E\left[U1_{\{D > Uy, K > y\}}\right]$$

and the optimality condition is obtained by taking the derivative of (5.24) and setting this equal to zero so that

$$\begin{aligned}
& (p + h - s)\, E\left[U1_{\{D > Uy, K > y\}}\right] - (p + h - s)\, \beta'_{D,K,U}(y)\, E\left[f(S)\right] \\
& + \left(s - ce^{rT}\right) E\left[U1_{\{K > y\}}\right] - \left(s - ce^{rT}\right) \beta'_{K,U}(y)\, E\left[f(S)\right] = 0
\end{aligned}$$

which yields

$$\frac{E\left[U1_{\{D > Uy^*, K > y^*\}}\right]}{E\left[U1_{\{K > y^*\}}\right]} - \frac{\beta'_{D,K,U}(y^*)\, E\left[f(S)\right]}{E\left[U1_{\{K > y^*\}}\right]} - \frac{\left(s - ce^{rT}\right)\beta'_{K,U}(y^*)\, E\left[f(S)\right]}{(p + h - s)\, E\left[U1_{\{K > y^*\}}\right]} = 1 - \hat{p}.$$

or

$$\frac{E\left[U1_{\{D \le Uy^*, K > y^*\}}\right]}{E\left[U1_{\{K > y^*\}}\right]} + \frac{\beta'_{D,K,U}(y^*)\, E\left[f(S)\right]}{E\left[U1_{\{K > y^*\}}\right]} + \frac{\left(s - ce^{rT}\right)\beta'_{K,U}(y^*)\, E\left[f(S)\right]}{(p + h - s)\, E\left[U1_{\{K > y^*\}}\right]} = \hat{p}. \tag{5.25}$$

As in previous chapter, the optimal y characterization highly resembles models without hedging. In (3.21), the optimality condition is

$$\frac{E\left[U1_{\{D\leq Uy^*,K>y^*\}}\right]}{E\left[U1_{\{K>y^*\}}\right]}=\hat{p},$$

In (5.25) we have an additional term

$$A(y)=\frac{\beta'_{D,K,U}(y)\,E[f(S)]}{E\left[U1_{\{K>y\}}\right]}+\frac{\left(s-ce^{rT}\right)\beta'_{K,U}(y)\,E[f(S)]}{(p+h-s)\,E\left[U1_{\{K>y\}}\right]}$$

If $A(y)$ is positive and increasing, then the optimal order quantity will be smaller than the optimal order quantity without hedging and vice versa. Moreover, if there is no correlation between demand, supply and the market, $A(y)$ will be equal to zero, once again, the model reverts to random yield case in Chapter 3.

Hedging with multiple securities

The objective function can be written as

$$Var\left(CF_\alpha(D,K,U,S,y)\right)=\boldsymbol{\alpha}^{\mathbf{T}}\mathbf{C}\boldsymbol{\alpha}+2\boldsymbol{\alpha}^{\mathbf{T}}\boldsymbol{\mu}(y)+Var\left(CF(D,K,U,y)\right)$$

where we now have

$$\mu_i(y)=Cov\left(f_i(S),CF(D,K,U,y)\right)$$

denotes the covariance between the financial securities and the cash flow. By taking the gradient with respect to $\boldsymbol{\alpha}$, the optimal hedging order quantities can be characterized as

$$\boldsymbol{\alpha}^*(y)=-\mathbf{C}^{-1}\boldsymbol{\mu}(y)\,. \tag{5.26}$$

The optimal hedging quantity can be used to maximize expected cash flows. Then, the objective function is

$$\begin{aligned}E\left[CF(D,K,U,y)+\boldsymbol{\alpha}^*(y)^{\mathbf{T}}\mathbf{f}(S)\right] &= \left(s-ce^{rT}\right)E\left[U\min\{K,y\}\right]\\ &\quad+(p+h-s)\,E\left[\min\{D,UK,Uy\}\right]\\ &\quad-hE[D]-\boldsymbol{\mu}(y)^{\mathbf{T}}\mathbf{C}^{-1}E[\mathbf{f}(S)]\end{aligned}$$

By taking the derivative with respect to y, we obtain the optimality condition as

$$\left(s - ce^{rT}\right) E\left[U1_{\{K>y\}}\right] + (p+h-s) E\left[U1_{\{D>Uy, K>y\}}\right] - \boldsymbol{\mu}'(y)^{\mathbf{T}} \mathbf{C}^{-1} E\left[\mathbf{f}(S)\right] = 0$$

The optimality condition can be written as

$$\begin{aligned} &\left(s - ce^{rT}\right) E\left[U1_{\{K>y\}}\right] + (p+h-s)\left(E\left[U1_{\{K>y\}}\right] - E\left[U1_{\{D\leq Uy, K>y\}}\right]\right) \\ &\quad -\boldsymbol{\mu}'(y)^{\mathbf{T}} \mathbf{C}^{-1} E\left[\mathbf{f}(S)\right] = 0 \end{aligned}$$

or

$$\left(p + h - ce^{rT}\right) E\left[U1_{\{K>y\}}\right] - (p+h-s) E\left[U1_{\{D\leq Uy, K>y\}}\right] - \boldsymbol{\mu}'(y)^{\mathbf{T}} \mathbf{C}^{-1} E\left[\mathbf{f}(S)\right] = 0.$$

Hence,

$$\frac{E\left[U1_{\{D\leq Uy^*, K>y^*\}}\right]}{E\left[U1_{\{K>y^*\}}\right]} + \frac{\boldsymbol{\mu}'(y^*)^{\mathbf{T}} \mathbf{C}^{-1} E\left[\mathbf{f}(S)\right]}{(p+h-s) E\left[U1_{\{K>y^*\}}\right]} = \frac{p+h-ce^{rT}}{p+h-s} = \hat{p}. \tag{5.27}$$

This optimal y characterization is similar to random yield and capacity model in previous chapter. The only difference is

$$A(y) = \frac{\boldsymbol{\mu}'(y)^{\mathbf{T}} \mathbf{C}^{-1} E\left[\mathbf{f}(S)\right]}{(p+h-s) E\left[U1_{\{K>y\}}\right]}.$$

Again, if this term is positive and increasing, then the optimal order quantity will be smaller than the optimal order quantity without hedging and vice versa. Moreover, if there is no correlation between demand, supply and the market, $A(y)$ will be equal to zero and the model reverts to random yield case in Chapter 3.

Chapter 6

CONCLUSIONS

In this thesis, we study a single-period, single-item inventory (newsvendor) problem. We analyze the opportunities of financial hedging to mitigate inventory risks when the demand and/or supply processes are correlated with the price of a financial asset. The risk or uncertainties in our model is generated by random demand. Apart from the uncertainty of demand, we also incorporate supply uncertainty as a source of randomness. The combined randomness of demand and supply enhances the level of uncertainty, thus leading to an increased risk for the manager. Hence, we provide a general framework of decision making in a risky environment by categorizing our model under three different approaches. In the first one, the conventional newsvendor model with shortage cost is analyzed. This model is extended by adding different designs of supply uncertainty, while the assumption of independence between demand and market still holds. In the second one financial instruments like options, bonds, futures, etc. is used to hedge the risks associated with the revenue or the cash flow by assuming perfect correlation between demand/supply and the market. The manager or the decision maker now has to determine the optimal portfolio of these hedging instruments as well as the optimal ordering quantity. For the last approach, a setting for hedging the risk is characterized when there is partial correlation between demand/supply and the market. In such a scenario, forming a replicating portfolio will not be possible since there is no perfect correlation. So instead, a minimum-variance type approach is introduced.

In the first part of the thesis the focus is on the classic newsvendor model. We analyze three different types of supply uncertainty, random capacity, random yield and both. By incorporating shortage costs we conclude that for random capacity case it is possible to find a characterization for the optimal order quantity. The characterization is such that the optimal probability of satisfying the demand is equal to a critical ratio. However, these characterizations require certain assumptions on the structure of the optimality conditions which we discussed explicitly in Chapter 3. For random yield case we find simple and explicit characterizations for the optimal ordering quantities, due to the fact that, the objective function is concave. Our results again suggest that the characterization for the optimal order quantity is similar. Models with random yield and capacity have non-concave objective functions. Therefore, obtaining an explicit solution for the general problem is very hard, if impossible. Nevertheless, we derive a complex characterization for the general problem and show the structure of the optimal solution under certain assumptions. Moreover, we further analyzed some special cases of random yield and capacity models to better understand the behavior of the optimal solution.

For the second part, Chapter 4, we examine the implications of having a risk-sensitive attitude and analyze the framework of decision making with hedging opportunities. Each decision maker faces a certain amount of risk by investing in inventory. Thus, decision makers are forced to manage the risk in order to prevent unexpected loses. An efficient way of controlling the risk is using market correlations. In reality, it is possible to form portfolios that have correlation with demand and supply. Via the help of such portfolios it is possible for the decision makers to hedge the risks of carrying inventory and shortage. Moreover, in this part of the thesis it is assumed that risk-neutral probability measures exists, contingent claims can be priced with these measures and demand/supply are perfectly correlated with the market. The same line of reasoning is followed as in the first part. For random demand models a replicating portfolio of random cash flow is generated and priced using RNPM. The structure of the optimal solution is same as the optimal order quantity without hedging. For the random capacity, the calculation of the replicating portfolio of the cash flow above is quite involved since the objective function is non-concave. Thus, instead of pricing the replicating portfolio, the previous solutions in the first part is modified by exploiting the perfect correlation. Furthermore, we prove this modification is indeed correct for various cases of capacity models. In random yield, the optimal solution for order quantity is obtained explicitly. Reaching useful results is quite hard in combined yield and capacity models, but under certain assumptions we provide some insights about the optimal characterization of order quantity. Bottom line in the second part of the thesis, a hedging framework is shown to reduce the variance of the expected profit when it is possible to invest in a portfolio perfectly correlated with random factors.

In the last part, the decision making framework in a risky environment is covered with a complete arbitrage-free market where there is a partial correlation between random demand, supply and a financial product. The inexistence of perfect correlation prevents us from constructing a replicating portfolio, thus, limiting our ability to reduce the variability of the profit. Moreover, the inexistence of such a portfolio implies that the realized profit will remain random, making the analysis more challenging. Since a replication approach cannot be employed, we utilize another method: Minimum-variance hedging. The aim here is to minimize the variance of the hedged cash flow for different return levels by holding a portfolio of financial securities. Specifically, in our context, decision makers select the order quantity and the financial portfolio simultaneously. This makes the minimum variance (min-var) approach a two-step characterization method. First, the decision maker attempts to reduce the volatility of the returns by investing in inventory and a portfolio consisting of different securities. After the optimal amount of each asset is calculated for a given ordering quantity, decision maker may aim to either maximize the expected profit or to further minimize the variance by choosing the ordering quantity. The same line of reasoning is followed as in parts 1 and 2. We derive the optimal ordering policy for random demand case for a portfolio consisting of a single security and later generalize it to the case of multiple securities. This analysis is later carried out in the cases with supply uncertainty due to both random capacity and uncertain yield. Based on our results, as in previous parts, the optimal ordering policies are always equal to the same critical ratio. However, in all of the cases we face a complex covariance term whose structure depends on the relation between financial security and random variables. The sign and shape of this covariance directly effects the behavior of the optimal solution.

Financial hedging is a vast concept, in our analysis we manage to cover some portion of it. There are many suitable areas for extentions, such as, using utility functions for financial hedging, further extending covariance analysis for partial correlation, making same analysis for multi-product, multi-period newsvendor with many suppliers, etc..

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