Embeddedness of the Solution

of the

Plateau Problem

by

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ABSTRACT

In this thesis we will study minimal surfaces and the Plateau problem. We will first give the proof of Douglas solution to Plateau problem. Then we will study the paper of Meeks and Yau in which they prove the embeddedness of the solution to the Plateau problem under some conditions. We will give the proofs of most of the theorems in detail.

Minimal surfaces received their name according to the property to minimize area for prescribed boundary values. Minimal surface theory is a branch of differential geometry which studies problems related to minimal surfaces. The basic problem that leads to minimal surface theory is the Plateau problem. The Plateau problem asks the existence of an area minimizing disk for a given simple closed curve in a manifold M. The existence was proven in 1930, by Douglas.

After the existence is proven, the regularity was also shown. In the following years, the question of embeddedness of the solution has been studied. It is not necessarily true that for any Jordan curve, any area minimizing surface is embedded. So under what conditions the solution to Plateau problem is embedded was an interesting question. In their paper Meeks and Yau proved that if the Jordan curve is on the boundary of a convex manifold and is contractible, then the solution is embedded. They used topological techniques to solve the problem.

ÖZET

Bu tezde minimal yüzeyleri ve Plateau problemini çalışacağız. İlk olarak Douglas'ın Plateau problemine çözümünü vereceğiz. Daha sonra Meeks ve Yau'nun bazı şartlar altında Plateau probleminin çözümünün gömülü olduğunu ispatladıkları makaleyi çalışacağız. Teoremlerin bir çoğunun ispatını ayrıntılı bir şekilde vereceğiz.

Minimal yüzeyler, adını verilen sınır değerine sahip alanı minimize etme özelliğinden alır. Minimal yüzeyler teorisi, diferansiyel geometrinin minimal yüzeyler ile ilgili problemleri çalışan bir koludur. Minimal yüzeyler teorisine yol açan ana problem Plateau problemidir. Plateau problemi verilen sınır değerleri için en küçük alanlı diskin var olup olmadığını sorar. Bu diskin varlığı 1930 yılında Douglas tarafından ispatlandı.

Bu diskin varlığı ispatlandıktan sonra düzgünlüğü, yani çatallı olmadığı gösterildi. Takip eden yıllarda, bu çözümün gömülü olup olmadığı çalışıldı . Herhangi bir Jordan eğrisi için en küçük alanlı herhangi bir disk gömülü olmak zorunda değildir. Öyleyse hangi şartlar altında gömülü olduğu ilginç bir soruydu. Meeks ve Yau makalelerinde eğer dışbükey bir manifoldun kenarındaki Jordan eğrisi büzülebilirse, çözümün gömülü olduğunu kanıtladılar. Problemi çözmek için topolojik teknikler kullandılar.

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Contents

NOMENCLATURE

D°	interior of D
Δf	Laplacian of a function f
∇f	gradient of a function f
∂	boundary
C_c	real valued functions of compact support
$C^k(U,V)$	k-many differentiable functions with domain U and range V
C^∞	real valued infinitely differentiable functions
f_*.	the map induced by f takes k-forms to k-forms f

1. INTRODUCTION

Minimal surface theory is a branch of differential geometry. The main problem of minimal surface theory is the Plateau problem.

The Classical Plateau problem: Given a Jordan curve γ in a three dimensional space does there exist a minimal disk $f : D \to \mathbb{R}^3$ of least area with $f(\partial D) = \gamma$?

The existence of a minimal surface with a given boundary, was first asked by Lagrange in 1760. It was called as Plateau problem after Plateau had considered the problem for soap films. There are other versions of the Plateau problem. Considering a surface or an orientable surface which has minimal area among all surfaces with a given boundary are other types of the Plateau problem. The existence and regularity issues for these types of Plateau problem are parts of geometric measure theory. For the details see [7]. We will consider only the Plateau problem for disks in this thesis.

In 1930, Douglas and Rado independently solved the Plateau problem for disks. Douglas used new techniques for the solution of the problem. He was awarded the Fields medal in 1936.

In minimal surface theory the terms minimal surface and area minimizing surface are frequently used. A surface is minimal if and only if it has mean curvature zero everywhere. A surface is area minimizing in a class of surfaces if it has least area among all maps in that class. An area minimizing surface is minimal but the converse is not true. Minimal surfaces are critical points for the area functional. While area minimizing surfaces are the global minima for the area functional. Given a Jordan curve γ in a three dimensional space, the question of whether there is a function from the disk into that space with boundary γ that realizes the infimum of such functions is, as mentioned above, called the Plateau problem. After Douglas [6] solved the problem in the affirmative way in 1930, interior and boundary regularity, and also embeddedness of the Douglas solution deserved special interest. In 1948, Morrey [17] solved the Plateau problem for the homogeneously regular Riemannian manifolds. When it comes to regularity (nonexistence of branch points) questions, Morrey also showed that when the manifold is real analytic then the map is also real analytic, and when the manifold is regular then the map has interior regularity. In 1970, Osserman [21] proved that it has no interior true branch points. If a surface has no interior true branch point then it must be an immersed surface in the interior. Afterwards, in 1973, Gulliver [9] proved that it has in fact no interior false branch points. This means that any parametrization of the disk has no interior branch points. Moreover, Osserman and Gulliver showed that the Morrey solution is an immersion in the interior for the three dimensional manifold.

For the boundary regularity of the Douglas and Morrey solution there are some results. In 1951, Lewy [13] proved that when the manifold M and the Jordan curve γ is real analytic then any minimal surface with boundary γ is real analytic up to the boundary. Moroever, it follows from the 1970 work of Hildebrandt and Heinz [12] that for smooth and regular Jordan curve γ in a general Riemannian manifold, any minimal surface with boundary γ is smooth and regular up to the boundary. For minimal surface with smooth boundary in $\mathbb{R}^3$, Nitsche [20] in 1969, showed that there are finitely many boundary branch points.

Another interesting question is the embeddedness of the solution of the Plateau problem. Almgren and Thurston [1] constructed an unknotted Jordan curve which does not bound any embedded minimal disk. So it was seen that when the curve is embedded it does not necessarily follow that it bounds an embedded area minimizing disk. Some advances has been made related to the question of for which curves are the area minimizing disks embedded. Rado [22] proved that if the Jordan curve has a one to one and convex projection onto the plane, then the Douglas solution is unique, embedded minimal disk which is a graph over the plane. An extremal Jordan curve means that it lies on the boundary of its convex hull. Gulliver and Spruck [10] showed that when the Jordan curve is extremal and has total curvature less than or equal to 4π, Douglas solution is again embedded. Dehn [5] has a lemma which states that in a three dimensional manifold, a homotopically trivial Jordan curve on the boundary of the manifold bounds an embedded disk. Whitehead and Shapiro [23] gave a proof of Dehn's Lemma in which they use partial covering space arguments. Meeks and Yau [16] proved the generalized Dehn's lemma in which they use arguments similar to the arguments of Whitehead and Shapiro [23] in determining the singularities of the Douglas-Morrey solution of the Plateau problem. They prove the following theorem:

Theorem 1.0.1. *If a Jordan curve γ on the boundary of a three dimensional convex manifold is homotopically trivial, then every Morrey solution to the Plateau problem for γ is embedded.*

Dehn's lemma says that when the Jordan curve is homotopically trivial, then it bounds an embedded disk. But it does not guarantee that this disk is area minimizing. But the above theorem says that any area minimizing disk is embedded. So it says more.

In this thesis, we will basically focus on the proof of the above theorem. We will follow Meeks and Yau [16].

1.1. Synopsis.

Section 2: In this section a brief introduction and some basic concepts related to the minimal surface theory will be given. The main references for this section are the book by Colding Minicozzi [4], the notes by Grosse- Brauckmann [8] and the paper by Meeks and Perez [15].

Section 3: This section gives Douglas solution to the Plateau problem. In this part, the book of Colding and Minicozzi [4] and the notes of Grosse-Brauckmann [8] are followed. It is convenient to consider an area minimizing sequence of maps with boundary γ, which is a Jordan curve, and take the infimum of the minimizing sequence. It will be observed that it is logical to minimize energy rather than to minimize area and they give the same value when the map is almost conformal. However the convergence of the energy minimizing sequence does not follow directly, because the space of conformal maps on disk is a non-compact space. When we compose some parametrization of a fixed disk with these conformal maps we will get another parametrization of that disk. So any fixed disk has too many parametrizations. Using three-point condition we will get a smaller space which is compact and contains unique parametrization for every disk. This will guarantee the convergence of the energy minimizing sequence and the limit will be the solution of the Plateau problem.

Section 4: The main reference for this section is the paper of Meeks and Yau [16]. The main result in this paper is that, if a Jordan curve on the boundary of a three dimensional convex manifold is null-homotopic then every solution to Plateau problem is embedded.

Section 4.1: This section gives the proof of the theorem, which states that given a Jordan curve on the boundary of a convex manifold, either the Plateau solution for this curve stays completely on the boundary of the manifold or the interior of the solution stays in the interior of M. The main reference for this section is Section 1 of the paper of Meeks and Yau [16].

Section 4.2: This section gives the necessary theorems and lemmas that will be useful in the remaining part of the thesis. The main reference for this section is Section 2 of the paper of Meeks and Yau [16].

Section 4.3: In this section the embeddedness result is assumed to be true for the real analytic case and the smooth case is proven depending on that hypothesis. This will be done by reducing the smooth case to the real analytic case by approximating a smooth metric by real analytic metrics. The main reference for this section is the paper of Meeks and Yau [16] Section 3.

Section 4.4: The hypothesis assumed in section 4.3 is proven. That is, the embeddedness result is shown for the real analytic case. Considering a real analytic Jordan curve and a real analytic manifold is more advantageous because the map from the disk D into real analytic manifold M is real anaytic by Lewy and it follows that it is simplicial with respect to some triangulations of D and M. A simplicial regular neighborhood of the image is considered. Then covering space techniques are used to reduce the problem to the case where there are no triple self-intersection points. This is done by constructing a tower of two-sheeted covering spaces of the simplicial neighborhoods. This process continues up to a covering space such that when the original map is lifted to that space, we get that the lifted Jordan curve stays at the boundary of that space, and the boundary is a disjoint union of spheres. The lifted Jordan curve separate the boundary component into two disks. A retraction from the space which is at the top of the tower to the lifted disk is defined. Then, it is observed that the areas of the retraction restricted to these two disks equal to the area of the lifted disk. Unless this lifted disk is an embedding, this retraction restricted to one of the disks has a folding curve. Since the lifted disk is the Douglas solution for the Jordan curve, it must be embedded. By composing this map with the projection map, we get the map sending the unit disk to the previous level of tower. Then one can get a surface with double self-intersections. These self-intersections consist of a pair of identified simple closed curves. By exchanging their areas we get another map of same area with the original map. But this map also has a folding curve. This is a contradiction which shows that this tower does not exist. Hence the original map is an embedding. The main reference for this section is the paper of Meeks and Yau [16] Section 4.

Section 4.5: The techniques used in section 4.4 are generalized to planar domains and a similar theorem is given for planar domains when the three dimensional manifold is orientable. The main reference for this section is the paper of Meeks and Yau [16] Section 5.

Section 4.: In this section, it is proven that in a convex three dimensional manifold any two Morrey solutions either represents the same disk or intersect only at the boundary. This can be done by using cut and paste arguments. The main reference for this section is the paper of Meeks and Yau [16] Section 6.

2. PRELIMINARIES

In this section we we will give information related to the minimal surfaces. This section is based on the notes of Grosse-Brauckmann [8] and the paper [15] of Meeks and Perez.

Definition 2.0.1. *If $f : U \to \mathbb{R}^m$ is a parametrized surface, and $p \in U$,* ***the first fundamental form*** *is the bilinear form on T_pU*

$$g_p(X, Y) = \langle df_p(X), df_p(Y)\rangle.$$

With respect to the standard basis $e_1, .., e_n$, the matrix notation for the first fundamental form is given by

$$g_{ij} = g_p(e_i, e_j) = \langle f_{x_i}, f_{x_j}\rangle.$$

Definition 2.0.2. *Let $f : U \to \mathbb{R}^{n+1}$ be a parametrized hypersurface. A continuous and differentable map $N : U \to \mathbb{S}^n$ with $\langle N(p), df_p(X)\rangle = 0$ for all $X \in \mathbb{R}^n$ is called* ***the Gauss map*** *or* ***normal mapping****.*

For $n = 2$, the normal map is given by

$$N = \frac{f_x \times f_y}{|f_x \times f_y|}.$$

Definition 2.0.3. *The bilinear form on $\mathbb{R}^n$*

$$b(X, Y) = \langle N, d^2f(X, Y)\rangle$$

is called ***the second fundamental form****.*

With respect to the standard basis $e_1, .., e_n$, the matrix notation for the second fundamental form is given by

$$b_{ij} = b(e_i, e_j) = -\langle N_{x_i}, f_{x_j}\rangle = \langle N, f_{x_i x_j}\rangle.$$

Definition 2.0.4. *The mean curvature of a surface is given by $H = \frac{1}{n}trace(g^{-1}b)$ where g is the matrix of the first fundamental form and b is the matrix of the second fundamental form of that surface.*

We will give equivalent definitions of minimal surfaces.

Theorem 2.0.5. *The following definitions of the minimal surfaces are equivalent.*

Definition 2.0.6. *A surface is minimal if and only if its mean curvature vanishes identically.*

Definition 2.0.7. *A surface is minimal if and only if it can be expressed locally as the graph of the solution of the quasilinear, second order, elliptic partial differential equation*

$$(1+u_x^2)u_{yy} - 2u_xu_yu_{xy} + (1+u_y^2)u_{xx} = 0 \tag{1}$$

Definition 2.0.8. *Let $X = (x_1, x_2, x_3) : M \to \mathbb{R}^3$ be an isometric immersion of a Riemannian surface M into $\mathbb{R}^3$. X is said to be minimal if x_i is a harmonic function on M for each i.*

Definition 2.0.9. *A surface is minimal if and only if every point has a neighborhood with least area relative to its boundary.*

We will give the proof of some of the equivalences.

(2.0.6) $\Leftrightarrow$ (2.0.7)
After rotation, any regular surface $M \subset \mathbb{R}^3$ can be expressed locally as a graph of a function $u = u(x, y)$. So it is enough to show that any minimal graph satisfies equation (1). Let $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ be a C^2 function. Then the upward pointing unit normal is

$$N = \frac{(1, 0, u_x) \times (0, 1, u_y)}{|(1, 0, u_x) \times (0, 1, u_y)|} = \frac{(-u_x, -u_y, 1)}{\sqrt{1 + |\nabla u|^2}} \tag{2}$$

and the mean curvature is

$$H = \frac{1}{n} trace(g^{-1}b) = \sum_{ij} \frac{1}{n} g^{ij} b_{ij} = \frac{1}{2detg}(g_{22}b_{11} - 2g_{12}b_{12} + g_{11}b_{22}) \tag{3}$$

where

$$g = \begin{pmatrix} 1 + u_x^2 & u_x u_y \\ u_y u_x & 1 + u_y^2 \end{pmatrix}$$

.

Then

$$g^{-1} = \frac{1}{1 + u_x^2 + u_y^2} \begin{pmatrix} 1 + u_y^2 & -u_x u_y \\ -u_y u_x & 1 + u_x^2 \end{pmatrix}$$

and b is given by

$$b = \frac{1}{1 + |\nabla u|^2} \begin{pmatrix} u_{xx} & u_{xy} \\ u_{yx} & u_{yy} \end{pmatrix}$$

so we have

$$H = \frac{1}{2(1 + |\nabla u|^2)} \frac{(1 + u_y^2)u_{xx} - 2u_x u_y u_{xy} + (1 + u_x^2)u_{yy}}{\sqrt{1 + |\nabla u|^2}}$$

if and only if

$$2H(1 + |\nabla u|^2)^{3/2} = (1 + u_y^2)u_{xx} - 2u_x u_y u_{xy} + (1 + u_x^2)u_{yy}$$

So the result follows.

(2.0.6) $\Leftrightarrow$ (2.0.8)
Let H be the mean curvature function of X and let $N : M \to S^2$ be the normal (Gaussian) map. Then we have the vector-valued formula $\Delta X = 2HN$. Clearly the converse is also true. So the result follows. (This result will be proven explicitly in the following theorem.)

Definition 2.0.10. *A parametrization $f : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ of a surface is conformal if*

$$\langle f_x, f_y \rangle = 0 \quad \text{and} \quad |f_x| = |f_y| = \lambda.$$

In $\mathbb{R}^3$, we know the existence of isothermal coordinates so any surface can be parametrized conformally. We will see that a conformally parametrized surface is minimal iff $\Delta f = 0$. This means that, each component function is harmonic. This follows from the following theorem.

Theorem 2.0.11. *Suppose $f : U \to \mathbb{R}^3$ is a two dimensional space. If $det(f_x, f_y, N) > 0$ and if f is conformal, it satisfies the following equation*

$$\Delta f = 2H f_x \times f_y \tag{4}$$

for all $p \in U$.

$$(Here \quad \Delta f = f_{xx} + f_{yy}).$$

Proof. From definitions (2.0.1), (2.0.3) and equation (3) we have

$$H = \frac{|f_y|^2\langle f_{xx}, N\rangle - 2\langle f_x, f_y\rangle\langle f_{xy}, N\rangle + |f_x|^2\langle f_{yy}, N\rangle}{2(|f_x|^2|f_y|^2 - \langle f_x, f_y\rangle^2)}.$$

Since f is conformal by replacing the values in the definition (2.0.10), we get

$$H = \frac{1}{2\lambda^2}\langle f_{xx} + f_{yy}, N\rangle = \frac{1}{2\lambda^2}\langle \Delta f, N\rangle.$$

Differentiating the conformality conditions we get

$$\frac{\partial}{\partial x}|f_x|^2 = \frac{\partial}{\partial x}|f_y|^2 \quad \Rightarrow \quad \langle f_{xx}, f_x\rangle = \langle f_{xy}, f_y\rangle$$

$$\frac{\partial}{\partial y}\langle f_x, f_y\rangle = 0 \quad \Rightarrow \quad \langle f_{yy}, f_x\rangle = -\langle f_{xy}, f_y\rangle$$

.

Adding them we get

$$\langle f_{xx} + f_{yy}, f_x\rangle = 0$$

and similarly

$$\langle f_{xx} + f_{yy}, f_y\rangle = 0.$$

f_x, f_y and N are orthogonal, so we have $\Delta f \parallel N$. So it follows that

$$\Delta f = \langle \Delta f, N\rangle N = 2H\lambda^2 N$$

Also

$$\lambda^2 N = |f_x||f_y|N = f_x \times f_y.$$

□

The first fundamental form gives the area of a surface $f : U \to \mathbb{R}^3$. The formula is

$$A_U(f) = \int_U \sqrt{\det g}\, dx. \tag{5}$$

So we have

$$A_U(f) = \int_U \sqrt{g_{11}g_{22} - g_{12}^2}\, dxdy = \int_U \sqrt{|f_x|^2|f_y|^2 - \langle f_x, f_y\rangle^2}\, dxdy. \tag{6}$$

Definition 2.0.12. *Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be a surface with Gauss map N, and $u \in C_0^1(U, \mathbb{R})$ be differentiable with compact support V. Then the normal variation determined by u is given by*

$$f^t : f + tuN.$$

For small $|t|$, f^t is a regular parametrized surface with

$$f_x^t = f_x + tuN_x + tu_x N$$

and

$$f_y^t = f_y + tuN_y + tu_y N.$$

Theorem 2.0.13. *Let $f : U \to \mathbb{R}^3$ be a regular parametrized surface. Then f is a minimal surface iff*

$$\frac{d}{dt} A_V(f + tuN)|_{t=0} = 0$$

for all normal variations $u \in C_0^1(U, \mathbb{R})$ with $V = supp \quad u$.

Proof. By equation (6)

$$A_V(f + tuN) = \int_V \sqrt{g_{11}^t g_{22}^t - {g_{12}^t}^2} dxdy = \int_V \sqrt{|f_x^t|^2 |f_y^t|^2 - \langle f_x^t, f_y^t \rangle^2} dxdy$$

By calculating we get

$$\det g^t = \det g[1 - 4tuH + t^2 u^2 (4H^2 + 2K) + t^2 \|\nabla u\|^2] + O(t^3).$$

So we have

$$\frac{d}{dt} A_V(f + tuN)|_{t=0} = -2 \int_V uH ds = 0$$

for all $u \in C_0^1(U, \mathbb{R})$ iff $H = 0$ by the fundamental lemma of calculus of variations.

□

So we have another equivalent definition for minimal surfaces.

Definition 2.0.14. *A surface is minimal iff it is a critical point of the area functional for all compactly supported variations. (ie, for all u which is zero on the boundary of the domain of the definition.)*

The second variation is computed considering the second derivative of $Area(f + tuN)$. When it is positive then the minimal surface is a local minimum. That is in a neighborhood of it, it has least area among all surfaces which has the same boundary. A global minimum of the area functional gives the area minimizing surface having fixed boundary values. So we have in general, an area minimizing surface and a minimal surface are not same. To show the difference between them is not straightforward.

We will give statements of some theorems. For the proof see [8].

Theorem 2.0.15. *(Weak Maximum Principle) Let U be a bounded domain and $u \in C^2(U, \mathbb{R}) \cap C^0(\overline{U}, \mathbb{R})$ be harmonic. Then*

$$sup_{\partial U} u = sup_U u$$

and

$$inf_{\partial U} u = inf_U u.$$

Theorem 2.0.16. *(Maximum Principle for Minimal Surfaces) Let U be bounded and $u, v \in C^2(U, \mathbb{R}) \cap C^0(\overline{U}, \mathbb{R})$ describe two graphs of mean curvature $H \in C^0(U, \mathbb{R})$ with respect to the upper normal. Then we have:*
(i) Interior maximum principle: If $u \leq v$ and $u(p) = v(p)$ at some interior point $p \in U$, then $u \equiv v$.
(ii) Boundary maximum principle: Let ∂U be smooth and $u, v \in C^2(U) \cap C^0(\overline{U})$. Suppose that u and v have a normal derivative at $p \in \partial U$. Moreover, let $u \leq v$ for all interior points $x \in U$ and $u(p) = v(p)$ for some $p \in \partial U$. Suppose that $\frac{\partial u}{\partial N}(p) = \frac{\partial v}{\partial N}(p)$. Then $u \equiv v$.

Theorem 2.0.17. *(Convex Hull Property)*
(i) A bounded minimal hypersurface M is contained in the convex hull of its boundary values, $M \subset conv(\partial M)$.
(ii) If M touches the boundary $\partial conv(\partial M)$ of its conver hull at a point interior to M, then M is contained in a plane.

Some Examples of Minimal Surfaces:

(1) A plane which is given by $z = 0$.
(2) The helicoid, which is given by the equation $z = \tan^{-1}(\frac{y}{x})$ and in parametric form $(x, y, z) = (t\cos s, t\sin s, s)$ where $s, t \in \mathbb{R}$.
(3) The catenoid, which is given by the equation $z = \cosh^{-1}\sqrt{x^2 + y^2}$. That is the surface obtained by rotating the curve $x = \cosh z$ around the z axis.
(4) Scherk's surface, which is the union of the closure of the surfaces

$$\Sigma_{k,l} = \{(x, y, z) : |x - k| < 1, |y - l| < 1, \quad \text{and} \quad z = \log \frac{\cos \frac{\pi}{2}(y - l)}{\cos \frac{\pi}{2}(x - k)}\},$$

where k, l are even and $k + l \equiv 0 \pmod 4$.

3. SOLUTION OF THE PLATEAU PROBLEM

In this section we will give the Douglas solution to the Plateau problem.

Plateau Problem: Given a closed curve Γ, find an area minimizing surface with boundary Γ.

We shall consider the solution of the Plateau Problem for parametrized disks.

Theorem 3.0.18. *Let $\Gamma \subset \mathbb{R}^3$ be a piecewise C^1, closed Jordan curve. Then there exists a piecewise C^1 map $u : D \subseteq \mathbb{R}^2 \to \mathbb{R}^3$ with $u(\partial D) \subset \Gamma$ such that the image has the minimum area among all disks with boundary Γ.*

It seems reasonable to consider a sequence of maps whose areas are going to infimum and take a convergent subsequence. But there are some problems with this idea. First problem is related to noncompactness of the diffeomorphism group of disk. Since the diffeomorphism group of the disk is not compact we can not guarantee that the area minimizing sequence is convergent. We can take a sequence Φ_k in the diffeomorphism group of a disk which is not convergent. For some fixed map $u : D \to \mathbb{R}^3$, $u(\Phi_k)$ has the same image but it is not convergent. The second problem is that bounding the area will not guarantee that the sequence of surfaces will converge to a surface. For instance, consider the area of a sequence of surfaces which are disks with thin tubes such that the tubes get thinner but longer as index of the sequence increases. The area of this sequence of surfaces converges to the area of the disk but the surfaces do not converge to a surface. To eliminate the last problem we will consider minimizing energy instead of minimizing area. Then we will show that minimizing energy also minimizes the area.

Let (x, y) be coordinates on $\mathbb{R}^2$ and suppose that $u = (u^1, u^2, u^3)$ is a map from $\mathbb{R}^2$ to $\mathbb{R}^3$.

The energy is defined by

$$E(u) = \int_D |\nabla u|^2 dxdy = \int_D (|u_x|^2 + |u_y|^2) dxdy.$$

and the area is given by

$$Area(u) = \int_D \sqrt{detg_{ij}} dx$$

where g is the Riemannian metric defined on the surface:

$$g = \begin{pmatrix} \langle u_x, u_x \rangle & \langle u_x, u_y \rangle \\ \langle u_y, u_x \rangle & \langle u_y, u_y \rangle \end{pmatrix}$$

Then

$$Area(u) = \int_D (|u_x|^2 |u_y|^2 - \langle u_x, u_y \rangle^2)^{1/2} dxdy.$$

Arithmetic and geometric mean gives

$$\frac{a+b}{2} \geq \sqrt{ab} \geq \sqrt{ab-c}$$

for all $a, b \geq 0$ and $0 \leq c \leq ab$ with the equality precisely when $a = b$ and $c = 0$. So

$$(7)\quad \frac{1}{2}E(u) = \frac{1}{2}\int_D (|u_x|^2 + |u_y|^2)dxdy \geq \int_D (|u_x|^2|u_y|^2 - \langle u_x, u_y\rangle^2)^{1/2}dxdy = Area(u)$$

with equality precisely when $|u_x| = |u_y|$ and $\langle u_x, u_y\rangle = 0$ that is, when u is conformal. By a theorem in differential geometry any surface can be parametrized conformally because of the existence of the isothermal coordinates. So there exists a diffeomorphism $\phi : D \to D$ such that $u(\phi) : D \to \mathbb{R}^2$ is almost conformal and $Area(u(\phi)) = Area(u)$(They have the same image).

Let u be an energy minimizing map and v be an area minimizing map. Then v can be considered as a conformal map by the above statement. Then $Area(u) \geq Area(v) = E(v)/2 \geq E(u)/2 \geq Area(u)$ Thus the two variational problems of minimizing energy and minimizing area have the same solutions. Namely, we have the lemma following the Definition 3.0.20.

Definition 3.0.19. *A map $f : \partial D \to \Gamma$ is monotone if the inverse image of the every connected set is connected.*

Define the set
χ_Γ=$\{\Psi : D \to \mathbb{R}^3 : \Psi$ is piecewise C^1 and $\Psi|_{\partial D}$ is a monotone map onto $\Gamma\}$

Definition 3.0.20.

$$A_\Gamma = inf_{\Psi \in \chi_\Gamma} Area(\Psi)$$

and

$$E_\Gamma = inf_{\Psi \in \chi_\Gamma} E(\Psi)$$

Lemma 3.0.21. $A_\Gamma = 1/2E_\Gamma$ *and* $E(\psi) = E_\Gamma$ *imply that* $A(\psi) = A_\Gamma$

We will solve the Plateau problem in 3 steps.

1) We will show that for each parametrization of the boundary there is an energy minimizing map from the disk with these boundary values. Each such map is harmonic. This is the standard Dirichlet problem.

2) We will consider all possible parametrizations of the boundary. When we compose a map on the disk with a conformal diffeomorphism of the disk we will get a map with the same energy. We know that the conformal diffeomorphism group of the disk is not compact. We will consider three point condition to get a compact set.

3) We will take an energy minimizing sequence of harmonic maps over all possible parametrizations of the boundary. Then we will reparametrize them to satisfy the three-point condition which will guarantee convergence on the boundary.

Now we will start with the first step.

Dirichlet Problem: Find a solution $u \in C^2(D, \mathbb{R}) \cap C^0(\overline{D}, \mathbb{R})$ and $\triangle u = 0$ in U with $u|_{\partial D} = \varphi$.

Solution:

Proposition 3.0.22. *If a map $f \in C^2(U,V)$ is conformal then it satisfies Cauchy-Riemann Equations or*

$$\partial_1 f_1 = -\partial_2 f_2 \quad \textit{and} \quad \partial_1 f_2 = \partial_2 f_1.$$

That is, we have the following equation:

$$(\partial_1 f_1)^2 = (\partial_2 f_2)^2 \quad \textit{and} \quad (\partial_1 f_2)^2 = (\partial_2 f_1)^2 .$$

Proof. If f is conformal then

$$\mid \partial_1 f \mid = \mid \partial_2 f \mid \quad \text{and} \quad \langle \partial_1 f, \partial_2 f \rangle = 0.$$

then

$$(\partial_1 f_1)^2 + (\partial_1 f_2)^2 = (\partial_2 f_1)^2 + (\partial_2 f_2)^2$$

and

$$(\partial_1 f_1)(\partial_2 f_1) + (\partial_1 f_2)(\partial_2 f_2) = 0 \quad \Rightarrow \quad \partial_2 f_1 = -\frac{(\partial_1 f_2)(\partial_2 f_2)}{\partial_1 f_1}.$$

which means

$$(\partial_1 f_1)^2 + (\partial_1 f_2)^2 = \left(\frac{(\partial_1 f_2)(\partial_2 f_2)}{\partial_1 f_1}\right)^2 + (\partial_2 f_2)^2$$

implies

$$(\partial_1 f_1)^2 + (\partial_1 f_2)^2 = (\partial_2 f_2)^2 \, \frac{(\partial_1 f_2)^2 + (\partial_1 f_1)^{2}{}^2}{\partial_1 f_1}$$

then

$$(\partial_1 f_1)^2 = (\partial_2 f_2)^2 .$$

So

$$\partial_1 f_1 = \partial_2 f_2 \quad \Rightarrow \quad \partial_1 f_2 = -\partial_2 f_1 \qquad \text{C-R Equations}$$

or

$$\partial_1 f_1 = -\partial_2 f_2 \quad \Rightarrow \quad \partial_1 f_2 = \partial_2 f_1$$

□

Lemma 3.0.23. *Let U and $V \subset \mathbb{R}$. Suppose $h \in C^2(U,\mathbb{R})$ is harmonic and $f \in C^2(U,V)$ is conformal. Then $h \circ f$ is harmonic.*

Proof. f is conformal then

$$\partial_1 f_1 = \partial_2 f_2 \qquad \text{and} \qquad \partial_2 f_1 = -\partial_1 f_2.$$

Differentiate with respect to first and second component,

$$\partial_{11} f_1 = \partial_{12} f_2$$
$$\partial_{22} f_1 = -\partial_{21} f_2$$

$\Longrightarrow \partial_{11} f_1 + \partial_{22} f_1 = 0$

Therefore f_1 is harmonic, and similarly f_2 is harmonic.

$$\partial_i (h \circ f) = (\partial_1 h \circ f)\, \partial_i f_1 + (\partial_2 h \circ f)\, \partial_i f_2$$

$$\partial_{ii}(h\circ f) = \partial_{11}h(\partial_i f_1)^2 + 2\partial_{12}h\partial_i f_1\partial_i f_2$$
$$+\ \partial_1 h\partial_{ii}f_1 + \partial_{22}h(\partial_i f_2)^2 + \partial_2 h\partial_{ii}f_2$$

$$\begin{aligned}
\triangle(h\circ f) &= \partial_{11}(h\circ f) + \partial_{22}(h\circ f)\\
&= \partial_{11}(h\circ f)(\partial_1 f_1)^2 + \partial_{22}(h\circ f)(\partial_2 f_2)^2\\
&+ \partial_{11}(h\circ f)(\partial_2 f_1)^2 + \partial_{22}(h\circ f)(\partial_1 f_2)^2\\
&+ (2\partial_{12}(h\circ f))\partial_1 f_1\partial_1 f_2 + (2\partial_{12}(h\circ f))\partial_2 f_1\partial_2 f_2\\
&+ \partial_1(h\circ f)\partial_{11}f_1 + \partial_1(h\circ f)\partial_{22}f_1\\
&+ \partial_2(h\circ f)\partial_{11}f_2 + \partial_2(h\circ f)\partial_{22}f_2\\
&= (\partial_{11}h + \partial_{22}h)\circ f(\partial_1 f_1)^2 + (\partial_{11}h + \partial_{22}h)\circ f(\partial_2 f_1)^2\\
&+ 2(\partial_{12}h\circ f)\partial_1 f_1\partial_1 f_2 - (2\partial_{12}h\circ f)\partial_1 f_2\partial_1 f_1\\
&+ (\partial_1 h\circ f)(\partial_{11}f_1 + \partial_{22}f_1) + (\partial_2 h\circ f)(\partial_{11}f_2 + \partial_{22}f_2)\\
&= 0.
\end{aligned}$$

(since h is harmonic first terms are zero and also f_1 and f_2 are harmonic).

□

Let us denote the volume of the n-dimensional unit ball by $V(B^n) = \omega_n$. Then by the divergence theorem, the bounding unit sphere has volume $V(S^{n-1}) = n\omega_n$. The following theorem gives a formula for harmonic functions.

Theorem 3.0.24. *(Mean Value Formula) Let $u \in C^2(U,\mathbb{R})$ be harmonic. Then for any ball $B_\rho(y)$ for $\rho \le R$ and $B_R(y) \subset U$*

$$u(y) = \frac{1}{nw_{nR^{n-1}}}\int_{\partial B_\rho} u(x)dS$$

and

$$u(y) = \frac{1}{w_{nR^n}}\int_{B_\rho} u(x)dx.$$

Proof. We know from Divergence Theorem that

$$\int_U divudx = \int_{\partial U} u.\nu dS \quad \Rightarrow \quad \int_U \Delta udx = \int_{\partial U}\nabla u.\nu dS = \int_{\partial U}\frac{\partial u}{\partial \nu}dS$$

where ν is the normal vector to ∂U.

Let $|x-y| = \rho$ and $\frac{x-y}{\rho} = w$. Since u is harmonic

$$0 = \int_{B_\rho}\Delta udx = \int_{\partial B_\rho}\frac{\partial u(x)}{\partial \nu}dS = \rho^{n-1}\int_{\partial B_\rho}\frac{\partial u(\rho w + y)}{\partial \rho} = \rho^{n-1}\frac{\partial}{\partial \rho}\int_{|w=1|} u(\rho w + y)dw$$

Then

$$\int_{|w=1|} u(\rho w + y)dw \quad \text{is independent of} \quad \rho.$$

implies

$$\int_{|w=1|} u(\rho w + y)dw = \int_{|w=1|} u(y)dw = nw_n u(y)$$

So

$$u(y) = \frac{1}{nw_n\rho^{n-1}} \int_{\partial B_\rho} u(\rho w + y)dS = \frac{1}{nw_n\rho^{n-1}} \int_{\partial B_\rho} u(x)dS$$

Now

$$\int_{B_\rho} u(x)dx = \int_0^R (\int_{\partial B_\rho} u(x)dS)d\rho = \int_0^R nw_n\rho^{n-1}u(y)d\rho = w_nR^nu(y)$$

Result follows. □

Theorem 3.0.25. *(Poisson Formula) Let φ be a continuous function on ∂D. Then*

(8) $$u(x) := \begin{cases} \frac{1-|x|^2}{2\pi} \int_{\partial D} \frac{\varphi(y)}{|x-y|^2} dS_y & \text{for} \quad x \in D. \\ \varphi(x) & \text{for} \quad x \in \partial D. \end{cases}$$

belongs to $C^2(D, \mathbb{R}) \cap C^0(\overline{D}, \mathbb{R})$ and is harmonic in D.
Define the Poisson kernel

(9) $$\mathbb{K}(x, y) := \frac{1- \mid x \mid^2}{2\pi \mid x - y \mid^2}, \qquad x \in D, \qquad y \in \partial D$$

the Poisson integral then becomes

(10) $$u(x) = \int_{\partial D} \mathbb{K}(x, y)\varphi(y)dS_y.$$

Proof. φ is independent of x and $\mathbb{K}$ is differentiable in x. So u is differentiable which means $u \in C^2(D, R)$.

Claim 1: If $u \in C^0(\overline{D}, R)$ and $u|_{\partial D} = \varphi(x)$ and $\Delta u = 0$ on D then

$$u(x) = \int_{\partial D} \mathbb{K}(x, y)\varphi(y)dS_y$$

for all $x \in D$.

Proof of Claim 1: Consider for $|x| \leq 1$ an automorphism w_x of D which is defined as $w_x(z) = \frac{x-z}{1-\overline{x}z}$. From Lemma 3.0.23 $u \circ w_x$ is harmonic. So by Thm 3.0.24 (Mean Value Theorem) for all $0 < r < 1$ we have

$$u(x) = u(w_x(0)) = \frac{1}{2\pi} \int_{\partial B_r} u \circ w_x(z)dS_z$$

But u is continuous on D by assumption. So as $r \to 1$

$$u(x) = u(w_x(0)) = \frac{1}{2\pi} \int_{\partial D} \varphi \circ w_x(z)dS_z$$

$$w_x(w_x(z)) = \frac{x - \frac{x-z}{1-\overline{x}z}}{1 - \overline{x}\frac{x-z}{1-\overline{x}z}} = \frac{x - x\overline{x}z - x + z}{1 - \overline{x}z - \overline{x}x + \overline{x}z} = \frac{(1 - |x|^2)z}{1 - |x|^2} = z$$

So w_x is its own inverse. Let $w_x(z) = y \Rightarrow w_x(y) = z$ Then

(11)
$$u(x) = \frac{1}{2\pi}\int_{\partial D} \varphi \circ w_x(z) dS_z = \frac{1}{2\pi}\int_{\partial D} \varphi \circ w_x(w_x(y)) dS_z = \frac{1}{2\pi}\int_{\partial D} \varphi(y)|w_x^{'}(y)| dS_y$$

But
$$w_x^{'}(y) = \frac{-(1-\overline{x}y)+(x-y)\overline{x}}{(1-\overline{x}y)^2} = \frac{|x|^2-1}{(1-\overline{x}y)^2}$$

We need to find $|w_x^{'}(y)|$. For $y \in S^1$ we have
$$|1-\overline{x}y|^2 = (1-\overline{x}y)(1-x\overline{y}) = 1-\overline{x}y - x\overline{y} + |x|^2 = (y-x)(\overline{y}-\overline{x}) = |y-x|^2$$

so $|w_x^{'}(y)| = \frac{1-|x|^2}{|y-x|^2}$. Plug this in (8) claim is proven.
i.e.
$$u(x) = \frac{1}{2\pi}\int_{\partial D} \varphi(y)\frac{1-|x|^2}{|y-x|^2} dS_y.$$

Claim 2: u is harmonic.

Proof of Claim 2:
$$\Delta u(x) = \int_{\partial D} (\Delta K(x,y))\varphi(y) dS_y.$$

Apply the Claim 1 for $u = 1$. Then
$$1 = \frac{1}{2\pi}\int_{\partial D} K(x,y) dS_y \Rightarrow 2\pi K(x,y) = \frac{1-|x|^2}{|y-x|^2} = \frac{y\overline{y} - y\overline{x} + y\overline{x} - x\overline{x}}{(y-x)(\overline{y}-\overline{x})} = \frac{y}{y-x} + \frac{\overline{x}}{\overline{y}-\overline{x}}$$

which is the sum of the holomorphic function $\frac{y}{y-x}$ and the antiholomorphic function $\frac{\overline{x}}{\overline{y}-\overline{x}}$. So this summation is harmonic. Then clearly u is also harmonic. Now we will show that u is continuous at ∂D.

Let $x_0 \in \partial D$ and $\epsilon > 0$ be given. We know that φ is continuous. So there is $\delta > 0$ such that $|x - x_0| < \delta$ implies that $|\varphi(x) - \varphi(x_0)| < \epsilon$. Also φ is continuous on the compact set ∂D there is $M > 0$ such that $|\varphi(x)| \leq M$.

Now for $|x - x_0| < \delta/2$ we have
$$|u(x) - u(x_0)| = \left| \int_{\partial D} \mathbb{K}(x,y)(\varphi(y) - \varphi(x_0)) dS_y \right|$$

$$\leq \left| \int_{y \in \partial D | y - x_0| \leq \delta} \mathbb{K}(x,y)(\varphi(y) - \varphi(x_0)) dS_y \right| + \left| \int_{y \in \partial D | y - x_0| > \delta} \mathbb{K}(x,y)(\varphi(y) - \varphi(x_0)) dS_y \right|$$

We have
$$\left| \int_{y \in \partial D | y - x_0| \leq \delta} \mathbb{K}(x,y)(\varphi(y) - \varphi(x_0)) dS_y \right| < \epsilon$$

since
$$\int_{\partial D} \mathbb{K}(x,y) dS_y = 1$$

(Apply equation (9) to the harmonic function $u = 1$.)

$$\left| \int_{y\in\partial D|y-x_0|>\delta} \mathbb{K}(x,y)(\varphi(y)-\varphi(x_0))dS_y \right| < 2M \left| \int_{\partial D|y-x_0|>\delta} \mathbb{K}(x,y)dS_y \right|$$

since $|x-x_0| < \delta/2$ and $|y-x_0| > \delta$ it must be true that $|x-y| > \delta/2$ and

$$\mathbb{K}(x,y) = \frac{1-|x|^2}{2\pi|x-y|^2} < \frac{1-|x|^2}{2\pi(\delta/2)^2}$$

so

(12)

$$\left| \int_{y\in\partial D|y-x_0|>\delta} \mathbb{K}(x,y)(\varphi(y)-\varphi(x_0))dS_y \right| < 2M\frac{1-|x|^2}{2\pi(\delta/2)^2}\int_{y\in\partial D} dS_y < 2M\frac{1-|x|^2}{(\delta/2)^2}$$

when $|x-x_0|$ is sufficiently small $|x|$ will be close to 1 so the numerator will be sufficiently small to make the left hand side of the equation (12) smaller than ϵ. Thus u is continuous at x_0. □

Result: So for any parametrization of the boundary Γ, Poisson Formula gives us a harmonic function which has the same boundary values as the parametrization. We will now show that a harmonic function has the minimum energy among all functions which has the same boundary values.

Proposition 3.0.26. *Suppose $h \in C^2(D,\mathbb{R}^n) \cap C^0(\overline{D},\mathbb{R}^n)$ is harmonic in D. If f is piecewise $C^1(\overline{D},\mathbb{R}^n)$ and has the same boundary values $f|_{\partial D} = h|_{\partial D}$ and satisfies $E(f) < \infty$ then $E(f) \geq E(h)$.*

To prove this proposition we need some theorems and lemmas.

Theorem 3.0.27. *(Arzela-Ascoli) Let $K \subset \mathbb{R}^n$ be a compact set and $u_k \in C^0(K)$ be a sequence which is uniformly bounded and equicontinuous. Then u_k has a uniformly convergent subsequence on K.*

Proof. Let $\epsilon > 0$ be given. u_k is equicontinuous so there exists $\delta > 0$ such that $|x-y| < \delta$ implies that $|u(x)-u(y)| < \epsilon/3$ for all k. Select $j \in \mathbb{N}$ such that $1/j < \delta$. Also $K \subset \bigcup_{z\in K} B_{1/j}(z)$. Since K is compact $K \subset B_{1/j}(z_1) \cup ... \cup B_{1/j}(z_n)$ for some $n \in N$. u_k is uniformly bounded $\Rightarrow$ $u_k(z_i)$ is bounded for all $i \in \{1,..,n\}$. Consider $u_{k_n}(z_2)$ which is also bounded so has a convergent subsequence say $u_{k_{n_m}}$. Then consider $u_{k_{n_m}}(z_3)$. It is also bounded so has a convergent subsequence. Continue this way. After finitely many steps we can find a subsequence of u_k which converges pointwise at all points z_i for all $i \in \{1,..,n\}$ call this subsequence as u_k. $u_k(z_i)$ is Cauchy for all $i \in \{1,..,n\}$. So there is N_i such that $|u_k(z_i)-u_l(z_i)| < \epsilon/3$ for all $k,l > N_i$. Choose $N = max\{N_1,...,N_n\}$

Let $x \in K$. Then $x \in B_{1/j}(z_i)$ for some i. Which means $|x-z_i| < 1/j < \delta$. By equicontinuity it follows that $|u_k(x)-u_k(z_i)| < \epsilon/3$ for all $k \in \mathbb{N}$. For $k,l \geq N$

$$|u_k(x)-u_l(x)| \leq |u_k(x)-u_k(z_i)|+|u_k(z_i)-u_l(z_i)|+|u_l(z_i)-u_l(x)| < \epsilon/3+\epsilon/3+\epsilon/3 = \epsilon$$

Thus u_k is uniformly convergent. □

Theorem 3.0.28. *Let $u \in C^{\infty}(U, \mathbb{R})$ be harmonic, $K \subset U$ be compact and α be a multi-index. Then there exists a constant C such that*

$$sup_U |\partial^{\alpha} u| \leq C sup_U |u|$$

Proof: If u is harmonic then $\Delta \nabla u = \nabla \Delta u = 0$ so ∇u is also harmonic. For $y \in K$ and

$$R = 1/2 dist(K, \partial U)$$

and $B = B_R(y)$. Applying Thm 3.0.24 (Mean Value Formula) we get

$$\nabla u(y) = \frac{1}{w_n R^n} \int_B \nabla u(x) dx = \frac{1}{w_n R^n} \int_{\partial B} \nabla u \nu dS_x$$

(second equality is from divergence theorem in the above equation) $\Rightarrow$

$$\begin{aligned} |\nabla u(y)| \leq \frac{1}{w_n R^n} \int_{\partial B} |u| dS_x &\leq \frac{1}{w_n R^n} sup_{\partial B} |u| \int_{\partial B} dS_x \\ &= \frac{1}{w_n R^n} sup_{\partial B} |u| \frac{n w_n R^{n-1}}{w_n R^n} = \frac{n}{R} sup_{\partial B} |u| \leq \frac{n}{R} sup_U u \end{aligned}$$

For higher derivatives the above reasoning is iterated but it is important to choose radii to satisfy $R \to 0$ as $\alpha \to \infty$.

Theorem 3.0.29. *Any bounded sequence of harmonic functions $u_k \in C^{\infty}(U, R)$, $k \in \mathbb{N}$, contains a subsequence that converges uniformly on each compact set $K \subset U$ to a harmonic function, all derivatives converge as well.*

Proof. By above theorem(3.0.28) the first derivatives are uniformly bounded over K.

$$\begin{aligned} \frac{u_k(x) - u_k(y)}{x - y} &= \partial_i u_k(c) \quad \text{for some} \quad c \in [x, y] \qquad \textit{by Mean-Value Theorem} \\ &\leq C sup u_k \quad C \quad \text{is independent of} \quad k \qquad \textit{by Theorem 3.0.28} \end{aligned}$$

Since u_k is a bounded sequence there exists M such that $sup u_k \leq M$. Choose $| x - y | < \frac{\epsilon}{CM}$ for given ϵ.
Then

$$| u_k(x) - u_k(y) | = C sup_k u_k \leq | x - y | MC < \epsilon$$

so u_k is equicontinuous. By Arzela-Ascoli Theorem (*by Theorem 3.0.27*), we can obtain a convergent subsequence $u_k \to u$. Now consider ∇u_k for u_k convergent subsequence since by the same theorem third derivatives also bounded similarly. ∇u_k also converges so $\nabla u_k \to \nabla u$ then u is harmonic. We can iterate the argument for higher derivatives. ☐

Lemma 3.0.30. *Let $(h_n)_{n \in \mathbb{N}}$ be a sequence of functions harmonic in $\mathbb{D}$ which converges to a harmonic function h uniformly on all compact subsets $K \subset D$. Then $E(h) = \lim_{n \to \infty} E(h_n)$ on D.*

Proof. By theorem (3.0.21) on compact subsets $K \subset D$ all derivatives of u converge uniformly so $\nabla h_n \to \nabla h$. So $\lim_{n \to \infty} E_k(h_n) = E_k(h)$. Take the limit of compact subsets $K_n \subset D$ so $\lim_{n \to \infty} E(h_n) = E(h)$. ☐

Now we will define some sequence of functions which we will use in the proof of the Proposition 3.0.26.

Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous, 2π-periodic function. The Fourier Series of f is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n cos(nx) + b_n sin(nx) \qquad \text{where}$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx, \qquad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)cos(nx)dx, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)sin(nx)dx$$

Now define the sequence of functions s_n which are extensions of f to D, $\forall n$.

$$s_n : \overline{D} \to \mathbb{R} \qquad s_n(r\exp(i\theta)) = \sum_{k=1}^{n} r^k(a_k cos(k\theta) + b_k sin(k\theta))$$

Each s_n is smooth on $\overline{D}$ also they are harmonic. To see this

$$\begin{aligned}
\triangle s_n(r\exp(i\theta)) &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}s_n)\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}s_n \\
&= \frac{1}{r}\frac{\partial}{\partial r}\left(\sum_{k=1}^{n} kr^k(a_k cos(k\theta) + b_k sin(k\theta)\right) \\
&- \frac{k^2}{r^2}\sum_{k=1}^{n} r^k(a_k cos(k\theta) + b_k sin(k\theta)) \\
&= \frac{1}{r}k^2\sum_{k=1}^{n} r^{k-1}(a_k cos(k\theta) + b_k sin(k\theta)) \\
&- \frac{k^2}{r^2}\sum_{k=1}^{n} r^k(a_k cos(k\theta) + b_k sin(k\theta)) \\
&= 0
\end{aligned}$$

$\mid s_n \mid \leq f$ on $\overline{D}$ since f is convergent, s also converges uniformly on all compact subsets of D. $s_n \to s$ where

$$s = \frac{1}{2}a_0 + \sum_{k=1}^{\infty} r^k(a_k cos(k\theta) + b_k sin(k\theta))$$

which is clearly continuous $s : \overline{D} \to \mathbb{R}$. By theorem (3.0.29) s is also harmonic.

Proof. (Proof of the proposition 3.0.26) $s_n \in C^2\left(\overline{D}, R^n\right)$ where s_n are as defined above.

$$\begin{aligned}
E(f) &= E(s_n + (f - s_n)) = \frac{1}{2}\int_D |\nabla s_n|^2 + 2\langle\nabla s_n, \nabla(f - s_n)\rangle + |\nabla(f - s_n)|^2 dx \\
&= E(s_n) + E(f - s_n) + \int_D \langle\nabla s_n, \nabla(f - s_n)\rangle dS.
\end{aligned}$$

Consider the divergence theorem

$$\int_U divX dx = \int_{\partial U}\langle X, \nu\rangle dS$$

if $X = \nu\nabla u$ then

$$\int_{\partial U} div\,(\nu\nabla u)\,dS = \int_U \langle\nabla\nu,\nabla u\rangle\,dx + \int_U \nu\triangle u dx$$
$$= \int_{\partial U}\langle\nu\nabla u,\nu\rangle\,dS$$

so

$$\int_{\partial D}\langle(f-s_n)\nabla s_n,\nu\rangle\,dS = \int_U\langle\nabla s_n,\nabla(f-s_n)\rangle\,dx + \int_D (f-s_n)\triangle s_n dx$$

s_n is harmonic, so

$$\int_U\langle\nabla s_n,\nabla(f-s_n)\rangle\,dx = \int_{\partial D}\langle(f-s_n)\nabla s_n,\nu\rangle\,dS$$

and this is equal to

$$\int_{\partial D}(f-s_n)\langle\nabla s_n,\nu\rangle\,dS = \int_{\partial D}\left\langle(f-s_n),\frac{\partial s_n}{\partial r}\right\rangle dS$$

We have

$$f(\exp i\theta) - s_n(\exp i\theta) = \sum_{k=n+1}^{\infty}(a_k\cos k\theta + b_k\sin k\theta)$$

and

$$\frac{\partial s_n}{\partial r}(\exp i\theta) = \sum_{k=1}^{\infty} k\,(a_k\cos k\theta + b_k\sin k\theta)$$

Then

$$\int_0^{2\pi}\left\langle(f-s_n),\frac{\partial s_n}{\partial r}\right\rangle d\theta$$

$$= \int_0^{2\pi}\left\langle\sum_{k=n+1}^{\infty}(a_k\cos k\theta + b_k\sin k\theta),\sum_{k=1}^{n} k\,(a_k\cos k\theta + b_k\sin k\theta)\right\rangle$$

$= 0$ (by usual computations)

so $E(f)\geqslant E(s_n)$ for all $n\in\mathbb{N}$ implying that $E(f)\geqslant\liminf E(s_n)$. Taking limit we get $E(f)\geqslant E(s)$. But s should be equal to h since $s-h$ equals to non-zero on the boundary so it should be identically zero by theroem (2.0.15) weak maximum principle.

Hence $E(f)\geqslant E(h)$. □

We have found for each parametrization of the boundary a unique energy minimizing harmonic map. Consider all possible parametrizations of the boundary.

Now we will take a minimizing sequence over all possible parametrizations of the boundary say $f_k\in C^1$. We need to find a limit f with $E(f)=\inf\{E(f_k)\}$.
First, we need to show that the sequence f_n is convergent.
Second, we need to show that f actually converges to a surface with boundary Γ.

Definition 3.0.31. *Let $\mathbb{D} \subset \mathbb{R}^2$ be the open disk. We call*

$$Aut(\mathbb{D}) = \left\{ w(z) \mid w(z) = w_{a,\varphi}(z) = \exp i\varphi \frac{a-z}{1-\overline{a}z}, \quad \varphi \in \mathbb{R} \quad a \in \mathbb{D} \right\}$$

the conformal automorphism group of the unit disk or the group of Möbius transformations.

Since the automorphism group of the disk is not compact we will consider the three-point condition: Given three distinct points q_l in Γ and three distinct points p_l in $\partial\mathbb{D}$ we define the class of mappings

$$\chi_{\Gamma'} = \{\psi \in \chi_\Gamma \mid \psi(p_l) = q_l\}.$$

We will prove that any element of χ_Γ is conformally equivalent to an element of $\chi_{\Gamma'}$.

Lemma 3.0.32. *Any $w \in Aut(\mathbb{D})$ maps $\mathbb{D}$ to $\mathbb{D}$ conformally and $\partial\mathbb{D}$ to $\partial\mathbb{D}$.*

Proof. First $\mid \overline{a}z \mid \leq \mid a \mid\mid z \mid < 1$ implies that the denominator of w does not vanish on $\mathbb{D}$.
Next

$$\mid w(z) \mid^2 = w(z)\overline{w(z)} = \frac{(a-z)(\overline{a}-\overline{z})}{(1-\overline{a}z)(1-a\overline{z})} = \frac{\mid a \mid^2 + \mid z \mid^2 - a\overline{z} - \overline{a}z}{1 + \mid a \mid^2 \mid z \mid^2 - a\overline{z} - \overline{a}z}$$

if $\mid z \mid = 1$ then $\mid w(z) \mid = 1$
if $z \in \mathbb{D}$

$$0 < (1 - \mid a \mid^2)(1 - \mid z \mid^2) \quad \Rightarrow \quad \mid a \mid^2 + \mid z \mid^2 < 1 + \mid a \mid^2 \mid z \mid^2 \quad \text{so} \quad \mid w(z) \mid < 1$$

since $a \in \mathbb{D}$, $w(z)$ is complex differentiable on $\overline{\mathbb{D}}$ so it is holomorphic so conformal. ☐

Proposition 3.0.33. *For any two sets of triples $0 \leq v_1, v_2, v_3 < 2\pi$ and $0 \leq \alpha_1, \alpha_2, \alpha_3 < 2\pi$, there exists a unique Möbius transformation $w \in Aut(\mathbb{D})$ of the unit disk such that $w(\exp iv_j) = \exp i\alpha_j$ for $j = 1, 2, 3$.*

To prove this proposition we will use the upper half plane $\mathbb{H} = \{(x, y) \in \mathbf{C}, y > 0\}$ instead of $\mathbb{D}$ which is conformally equivalent to $\mathbb{D}$.

Lemma 3.0.34. *The map $\eta : \mathbf{C} \setminus \{i\} \to \mathbf{C} \setminus \{1\}$ $\quad \eta(z) = \frac{z-i}{z+i}$ is a conformal diffeomorphism. It maps bijectively $\mathbb{H}$ to $\mathbb{D}$, $\overline{\mathbb{H}}$ to $\overline{\mathbb{D}}$ and the real axis to the unit circle without the point 1.*

$$\begin{aligned} w = \eta(z) = \frac{z-i}{z+i} \quad &\Rightarrow \quad zw + iw = z - i \\ &\Rightarrow \quad z(1-w) = iw + i \\ &\Rightarrow \quad z = \frac{iw+i}{1-w} = \zeta(w) \qquad \text{the inverse of } \eta \end{aligned}$$

w and η are differentiable $\Rightarrow \eta$ is a diffeomorphism.
For $z \in \mathbb{H}$, $\mid z-i \mid < \mid z+i \mid$ so $w(z) \in \mathbb{D}$ and for $w \in \mathbb{D}$

$$2Im\zeta(w) = \frac{1}{i}(\zeta(w) - \overline{\zeta(w)}) = \frac{1+w}{1-w} + \frac{1+\overline{w}}{1-\overline{w}} = \frac{(1+w)(1-\overline{w}) + (1+\overline{w})(1-w)}{(1+w)(1-\overline{w}}$$

$$= \frac{2 - 2 \mid w \mid^2}{\mid 1-w \mid^2} > 0$$

Others are similar.
Also $\eta : \overline{\mathbb{H}} \cup \{\infty\} \to \overline{D}$ is a diffeomorphism.

Lemma 3.0.35. *For every triple of real numbers $a < b < c$ there is a conformal diffeomorphism $f : \overline{\mathbb{H}} \cup \{\infty\} \to \overline{\mathbb{H}} \cup \{\infty\}$ such that $f(a) = 0, \quad f(b) = 1, \quad f(c) = \infty$*

Proof. Let $f : \mathbb{H} \Rightarrow \mathbb{H}$, $f(z) = \frac{z-a}{z-c}\frac{b-c}{b-a}$.
f is complex differentiable so holomorphic so conformal and f maps $\mathbb{H}$ to $\mathbb{H}$. Also f maps extended real line to itself. □

Proof. (Proof of the proposition 3.0.24) Set $a = \eta^{-1}(\exp(iv_1))$, $b = \eta^{-1}(\exp(iv_2))$, $c = \eta^{-1}(\exp(iv_3))$. Then $f \circ \eta^{-1}$ maps $\exp(iv_1)$, $\exp(iv_2)$, $\exp(iv_3)$ to $0, 1, \infty$ in $\partial\mathbb{H} \cup \{\infty\}$. Since mappings are diffeomorphisms any triple $\exp(iv_1)$, $\exp(iv_2)$, $\exp(iv_3)$ can be mapped conformally to any other triple. □

Lemma 3.0.36. *Let $f \in C^1(V, U)$ be conformal and $h \in C^1(U, \mathbb{R}^n)$. Then $E_V(h \circ f) = E_U(h)$*

Proof. We have

$$\partial_i(h \circ f) = (\partial_1 h \circ f)\partial_i f_1 + (\partial_2 h \circ f)\partial_i f_2.$$

Taking the square of this expression, we get

$$|\partial_i(h \circ f)|^2 = |\partial_1 h|^2(\partial_i f_1)^2 + |\partial_2 h|^2(\partial_i f_2)^2 + 2\langle\partial_1 h, \partial_2 h\rangle\partial_i f_1 \partial_i f_2.$$

where $i = 1, 2$ and on the right side $\partial_i h$ is always evaluated at f. This gives

$$|\partial_1(h \circ f)|^2 + |\partial_2(h \circ f)|^2 = |\partial_1 h|^2(\partial_1 f_1^2 + \partial_2 f_1^2) + |\partial_2 h|^2(\partial_1 f_2)^2 + \partial_2 f_2)^2)$$

$$+2\langle\partial_1 h, \partial_2 h\rangle(\partial_1 f_1 \partial_1 f_2 + \partial_2 f_1 \partial_2 f_2)$$

$$= |\partial_1 h|^2 |\det df| + |\partial_2 h|^2 |\det df|.$$

The last step follows from the conformality of f and Proposition (3.0.22).
By making a change of variables in the integration we get:

$$E_V(h \circ f) = \frac{1}{2}\int_V |\partial_1(h \circ f)|^2 + |\partial_2(h \circ f)|^2 dx_V = \frac{1}{2}\int_V (|\partial_1 h \circ f|^2 + |\partial_2 h \circ f|^2)|\det df| dx_V$$

$$= \frac{1}{2}\int_U (|\partial_1 h|^2 + |\partial_2 h|^2) dx_U = E_U(h)$$

□

Lemma 3.0.37. *Any element of* χ_Γ *is conformally equivalent to an element of* $\chi_{\Gamma'}$. *It follows that for any element in* χ_Γ *there is an element in* $\chi_{\Gamma'}$, *which is of same image and same energy.*

Proof. Let $f \in \chi_\Gamma$. Then $f : \mathbb{D} \Rightarrow \mathbb{R}^3$ is piecewise $\mathbb{C}^1$ and $f\mid_{\partial\mathbb{D}}$ is a monotone map onto Γ. $f : \partial\mathbb{D} \to \Gamma$, there exist $\alpha_1, \alpha_2, \alpha_3$ distinct such that $f(\exp(i\alpha_j)) = q_j \quad \text{for} \quad j = 1, 2, 3$. Let $\beta_1, \beta_2, \beta_3$ so that $f(\exp(i\beta_j)) = p_j$. By proposition(3.0.33), there is a Möbius transformation w_k with $w_k(\exp i\beta_j) = \exp(i\alpha_j)$.
Now consider $f \circ w_k = \tilde{f} \in \chi_{\Gamma'}$. w_k is conformal, $E(f \circ w_k) = E(f)$ by lemma 3.0.36. □

Now we will prove *Courant-Lebesgue Lemma* which we will use in the proof of compactness of $\chi_{\Gamma'}$.

Given any point $\rho \in \mathbb{D}$ for each $\rho > 0$, we define $\mathbb{C}_\rho = \{q \in \mathbb{D} \mid \mid p - q \mid = \rho\}$, $d(\mathbb{C}_\rho)$ to be the diameter of the image of the curve $\mathbb{C}_\rho$ and $\mathbb{L}(\mathbb{C}_\rho)$ to be the length of the image of the curve $\mathbb{C}_\rho$.

Lemma 3.0.38. *(Courant-Lebesgue Lemma) Let* $u \in \mathbb{C}^1(\mathbb{D}, \mathbb{R}^3) \cap \mathbb{C}^0(\overline{\mathbb{D}}, \mathbb{R}^3)$ *with* $E(u) \leq \mathbb{K}$. *For each positive* $\delta < 1$ *there exists some* $\rho \in \left[\delta, \sqrt{\delta}\right]$ *such that* $(d(\mathbb{C}_\rho))^2 \leq 2\pi\epsilon_\delta$ *where* $\epsilon_\delta = \frac{4\pi\mathbb{K}}{-\log\delta}$ *when* $\delta \to 0$.

We will bound $\mathbb{L}(\mathbb{C}_\rho)$ and this implies a bound on $d(\mathbb{C}_\rho)$. Define $p(r) = r\int_{\mathbb{C}_r} \mid \nabla u \mid^2 ds$. We have

$$\int_\delta^{\sqrt{\delta}} p(r)d(\log r) = \int_\delta^{\sqrt{\delta}} p(r)\frac{dr}{r} = \int_\delta^{\sqrt{\delta}} \int_{\mathbb{C}_r} \mid \nabla u \mid^2 dsdr \;\leq\; \int_0^1 \int_{\mathbb{C}_r} \mid \nabla u \mid^2 dsdr$$
$$= \int_{\mathbb{D}} \mid \nabla u \mid^2 dxdy \;\leq\; \mathbb{K}(13)$$

by Mean-Value Theorem for integrals for some $\rho \in \left[\delta, \sqrt{\delta}\right]$

$$p(\rho)\int_\delta^{\sqrt{\delta}} d(\log r) = \int_\delta^{\sqrt{\delta}} p(r)d(\log r)$$

$$\Rightarrow \quad p(\rho) = \int_\delta^{\sqrt{\delta}} p(r)d(\log r)\frac{2}{-\log\delta} \leq \frac{2\mathbb{K}}{-\log\delta} \tag{14}$$

Consider in polar coordinates. Then

$$\mathbb{L}(C_r) = \int_{C_r} \sqrt{u_1'(\theta)^2 + u_2'(\theta)^2 u_3'(\theta)^2}d\theta \leq \int_{C_r} \mid \nabla u \mid ds \quad \text{by C-S Inequality}$$

$$\leq \left(\int_{C_r} 1^2 ds\right)^{\frac{1}{2}} \left(\int_{C_r} \mid \nabla u \mid^{\frac{1}{2}} ds\right) \leq (2\pi_r)^{\frac{1}{2}} \left(\frac{p(r)}{r}\right)^{\frac{1}{2}} = (2\pi p(r))^{\frac{1}{2}} \tag{15}$$

Combining (14) and (15) we get that

$$(\mathbb{L}(C_\rho))^2 \leq \frac{4\pi\mathbb{K}}{-\log\delta}$$

then

$$(d(C_\rho))^2 \leq \frac{4\pi\mathbb{K}}{-\log\delta}$$

follows.

Lemma 3.0.39. *For any constant $\mathbb{K}$, the family of functions*

$$\mathbb{F} = \{\psi \mid_{\partial\mathbb{D}} \mid \psi \in \chi_{\Gamma'} \quad and \quad E(\psi) \leq \mathbb{K}\}$$

is equicontinuous on $\partial\mathbb{D}$. Hence by the Arzela - Ascoli Theorem, $\mathbb{F}$ is compact in the topology of uniform convergence.

Proof. Let $\psi \in \chi_{\Gamma'}$ with $E(\psi) \leq \mathbb{K}$ and $\epsilon > 0$ where $\epsilon < \min \mid q_i - q_j \mid$ be given.
For any $p \in \mathbb{D}$ we need to find some $\rho > 0$ such that the neighborhood of p which has diameter less than ρ is mapped to a neighborhood of $\psi(p)$ which has diameter less than ϵ.

Claim: Let Γ be a closed Jordan curve. Then given $\epsilon > 0$, $\exists d > 0$ such that for any points p, q which has distance in $\mathbb{R}^3$ less than d separates Γ such that at least one component has length in Γ less than ϵ.

Proof of the Claim: Assume for a contradiction the claim is not true. Given ϵ, as $d \to 0$ two points with distance d in $\mathbb{R}^3$ get close to each other and the intrinsic distance gets close to the extrinsic distance. So for d small enough their intrinsic distance will be small than ϵ. Contradiction.

As $\log\frac{1}{\delta} \to \infty$ for $\delta \to 0$ we can choose $\delta = \delta(\epsilon) \in (0,1)$ small enough to ensure $\sqrt{2\pi\epsilon_\delta} < d$ where $\epsilon_\delta = \frac{4\pi\mathbb{K}}{\log\frac{1}{\delta}}$ and such that given any $p \in \partial\mathbb{D}$ at least two of the p_i are not in the ball of radius $\sqrt{\delta}$ about p.
Now given any $p \in \partial\mathbb{D}$, by *Courant - Lebesgue Lemma(Lemma 3.0.38)* there exists $\rho \in \left[\delta, \sqrt{\delta}\right]$ such that $d(\mathbb{C}_\rho) \leq \sqrt{2\pi\epsilon_\delta} < d$. Let m_1 and m_2 be the intersection points of the $\mathbb{C}_\rho$ and $\mathbb{D}$. $\mathbb{A}_1$ be the component of $\mathbb{D}$ separated by m_1 and m_2 which contains p and $\mathbb{A}_2$ be the remaining component. Also let $\mathbb{B}_1 = \psi(\mathbb{A}_1)$ and $\mathbb{B}_2 = \psi(\mathbb{A}_2)$. $\psi(m_1)$ and $\psi(m_2)$ has extrinsic distance less than α so one of the components $\mathbb{B}_1$ and $\mathbb{B}_2$ which are separated by $\psi(m_1)$ and $\psi(m_2)$ must have length less than ϵ. $\mathbb{B}_1$ should have length less than ϵ because $\mathbb{A}_1$ contains at most one of the points p_i and $\mathbb{B}_1$ contains at most one of the points q_i and ψ satisfies the 3-point condition.

Hence by the Arzela - Ascoli Theorem (theorem 3.0.27), F is compact in the topology of uniform convergence. Result follows. □

Proof. (Proof of theorem 3.0.18:) For any parametrization of Γ, we can find a harmonic map (by proposition 3.0.26 and the result following poisson formula) which is energy minimizing. Considering all parametrizations of Γ we can find an energy minimizing sequence $\{\tilde{u_j}\}$ of harmonic maps in χ_Γ so that $E(u_j) \to E_\Gamma$. But by the three-point condition, we can change our sequence $\{\tilde{u_j}\}$ with $\{u_j\}$ such that all functions $\{u_j\}$ satisfy the three-point condition, they are harmonic and for all j, u_j is a conformal reparametrization of $\tilde{u_j}$ (from lemma 3.0.37), so they have the same energy (lemma 3.0.36) so $\{u_j\} \subseteq \chi_{\Gamma'}$.
The bounded sequence u_j of harmonic functions contains a subsequence u_n that converges uniformly on each component subset all derivatives converge as well to say u which is harmonic(by theorem 3.0.29). Then by Lemma 3.0.30. $E(u) = \lim E(u_n)$ on

$\mathbb{D}$. Now consider $u\mid_{\partial\mathbb{D}}=\varphi_n$. φ_n is compact by Lemma 3.0.39, so φ_n has a convergent subsequence $\varphi_n \to \varphi$ so u_n converges on the boundary to a monotone function. Hence we have $E(u) = E_\Gamma$. By lemma 3.0.21, we have $\frac{1}{2}E_\Gamma = A_\Gamma$. So the area minimizing map is a conformal and harmonic map.

$\square$

4. MEEKS-YAU EMBEDDEDNESS THEOREM

In this section we will follow Meeks-Yau [16].

4.1. THE EXISTENCE OF MINIMAL SURFACES IN A CONVEX RIEMANNIAN MANIFOLD.

Definition 4.1.1. *A smooth manifold M is said to be strictly convex if the second fundamental form of its boundary is positive definite.*

Definition 4.1.2. *A smooth manifold M is convex if it is a subset of a strictly convex manifold N with the following properties.*

1. *There is a convex function g which is 0 on ∂M and nonpositive on M.*
2. *There is a bi-Lipschitz homeomorphism φ from $\partial M \times [-1,1]$ to a neighborhood of ∂M such that for $x \in \partial M$ and $1 \geq t_2 \geq t_1 \geq -1$ we have $\varphi(x,0) = x$ and $-c(t_2 - t_1) \geq g[\varphi(x,t_2)] - g[\varphi(x,t_1)]$ for some constant c.*
3. *There is a smooth function g defined on N which is strictly convex in a neighborhood of the closure of N/M.*

Consider a compact convex set M in Euclidean space with the usual definition. For the above definition if we consider N as a large ball containing M which is strictly convex. For g consider the distance to the boundary for outside M and negative of the distance to the boundary for the points in the closure of M. Consider origin as in M then take φ as radial deformation and g as the square of the distance from the origin.So this definition is same as the standard convex set definition in the Euclidean space.

Definition 4.1.3. *Let $\Gamma = \{\Gamma_1, ...\Gamma_k\}$ be a collection of Jordan curves in a convex manifold M. Let Γ^i be a subcollection of curves selected from $\{\Gamma_1, ...\Gamma_k\}$ such that $\Gamma^1 \cup ... \cup \Gamma^p = \{\Gamma_1, ...\Gamma_k\}$ where $p > 1$ and $\Gamma^i \cap \Gamma^j = \emptyset$.Then*

1. *$d_M(\Gamma)$: The infimum of areas of all possible maps from a plane domain bounded by k disjoint circles so that the restriction of the maps to the circle i gives a parametrization of the Jordan curve Γ_i.*
2. *$d^*_M(\Gamma)$: For $k > 1$ define as min $\sum_{i=1}^{p} d_M(\Gamma^i)$ where letting Γ^i vary to get the minimum. For $k = 1$ define $d^*_M(\Gamma) = \infty$.*

Theorem 4.1.4. *Suppose that M is a convex manifold as defined above and $\Gamma = \{\Gamma_1, ...\Gamma_k\}$ be a collection of Jordan curves in M. If $d_M(\Gamma) < d^*_M(\Gamma)$ then there exists a connected plane domain B which is bounded by k circles and there exists a map f which is conformal and harmonic on B° with area $d_M(\Gamma)$ that maps each boundary circle C_i of B to Γ_i monotonically. Moreover either f maps B° to M° or f maps B to ∂M.*

(When $d_M(\Gamma) < d^*_M(\Gamma)$ B should be connected since all unconnected possibilities are considered when calculating $d^*_M(\Gamma)$. From a connected plane domain to a homogeneously regular manifold Morrey solution exists [17]. So the important part of this theorem is that either this solution stays completely in ∂M or interior of the solution is mapped into the interior of M.)

Proof. We know that M is a subset of some strictly convex manifold N. We will first prove the theorem for a strictly convex manifold N. Consider N as a subset of some smooth Riemannian manifold $\tilde{N}$ There is a strictly convex function h which is negative on the interior of N, positive on $\tilde{N}\backslash N$ and zero on the boundary of N. We will define a new homogeneously regular metric by multiplying the original metric with some function. Homogeneously regular means that curvature and injectivity radius is bounded. An equivalent definion from [11] is :

Definition 4.1.5. *A Riemannian manifold is homogeneously regular if there exist positive constants k and K such that every point of the manifold lies in the image of a chart φ with domain the unit ball B(0,1) in R^3 such that*

$$k\|v\|^2 \leq g_{ij}(\varphi(x))v_i v_j \leq K\|v\|^2 \quad \textit{for all} \quad x \quad inB(0,1),$$

where v is any tangent vector to x, g is the metric on M and g_{ij} its components.

Homogeneous regularity is important because Morrey solution exists in $\tilde{N}$ when the metric is homogeneously regular. Define a smooth function $\tilde{h}$ on $\tilde{N}$ so that $\tilde{h} = 1$ on N and $1 + \exp(-1/h)$ on $N\backslash\tilde{N}$. We can consider as $\tilde{h}|_{\partial\tilde{N}} = a > 1$. Now define a new metric by multiplying the original metric by the function $(a-1)^2/(a-\tilde{h})^2$. Since $a - \tilde{h} \to 0$ as $x \to \partial\tilde{N}$, so we have $\partial\tilde{N}$ pushed to infinity. On N metric stays the same.

Claim 1: This new metric is homogeneously regular.

Proof of Claim 1: Consider $x \in \tilde{N}$ with distance $\geq \epsilon$ from the boundary and the geodesic ball with center x and radius $\epsilon/2$. Since $\tilde{N}$ is a manifold, this ball is diffeomorphic to the Euclidean ball of radius $\epsilon/2$, so that under this diffeomorphism the metric has the form $\sum_{i,j} g_{ij}dx^i \otimes dx^j$. Since it is homogeneously regular g_{ij} has eigenvalues bounded from above and below by positive constants independent of x. Using radial deformation, map the unit ball onto the $\epsilon/2$ ball. So the pulled back metric on the unit ball becomes $(\epsilon^2/4)\sum_{i,j} g_{ij}dx^i \otimes dx^j$. Now we will multiply this metric by $(a-1)^2/(a-\tilde{h})^2$. h is strictly convex so

$$|\nabla\tilde{h}| = |< exp(-1/h)h^{-2}h_x, exp(-1/h)h^{-2}h_y > | = (exp(-1/h)^2h^{-4}(h_x^2 + h_y^2))^{1/2} > 0$$

so $\frac{a-\tilde{h}}{d(x,\partial N)}$ is bounded below and above by a positive constant in a neighborhood of $\partial\tilde{N}$. Therefore the new metric on the unit ball is obtained by multiplying $(\epsilon^2/4)\sum_{i,j} g_{ij}dx^i \otimes dx^j$ with a function bounded from above and below, which means that it is equivalent to the Euclidean metric. So $\tilde{N}$ is homogeneously regular.

We know Morrey solution exists in $\tilde{N}$. We will show that it is in fact stays in N by considering the function $h \circ f$. We have $\Delta(h \circ f) \geq -b|\nabla(h \circ f)|^2$ on neighborhood of $\tilde{N}\backslash N$ where b is a positive constant. [[16]Theorem 1]

Lemma 4.1.6. *Let h be continuous on a bounded open set B such that Dirichlet integral of h is finite and for some constant b we have $\Delta h \geq -b|\nabla h|^2$. Then $sup_B h \leq sup_{\partial B} h$.*

Proof. Assume on the contrary that $sup_B h < sup_{\partial B} h$. Let c be a number such that

$$sup_B h < c < sup_{\partial B} h \quad \text{and} \quad b(sup_B h - c) < 1$$

Define the set $B_c = \{x|h(x) \geq c\}$. Then

$$\partial B_c = \{x|h(x) = c\}.$$

We have

$$\Delta h \geq -b|\nabla h|^2$$

by the hypothesis. Now multiply both sides with $h-c$. Then

$$\Delta h(h-c) \geq -b(h-c)|\nabla h|^2.$$

Take the integral over the set B$_c$.

$$\int_{B_c} \Delta h(h-c) \geq \int_{B_c} -b(h-c)|\nabla h|^2.$$

By integration by parts we know,

$$\int_D u\Delta u = \int_{\partial D} u.\overrightarrow{n}\, d\sigma - \int_D |\nabla u|^2 = -\int_D |\nabla u|^2$$

the second equality holds if u is zero on ∂U.
So we have

$$\int_{B_c} (h-c)\Delta h = \int_{\partial B_c} (h-c).\overrightarrow{n} - \int |B_c|\nabla h|^2 = -\int |B_c|\nabla h|^2.$$

since $h-c=0$ on ∂B_c. Then we have

$$-\int |B_c|\nabla h|^2 \geq \int_{B_c} -b(h-c)|\nabla h|^2 \Rightarrow -\int |B_c(1-b(h-c))|\nabla h|^2 \leq 0$$

so $|\nabla h|^2 = 0$. But h is strictly convex. Contradiction. □

To see that f maps B into N define the set

$$\Omega_\epsilon = \{x \in \overline{B} | h \circ f(x) > \epsilon\}$$

if $\epsilon > 0$ then for $x \in \overline{\Omega_\epsilon}$, $h \circ f(x) \geq \epsilon$; which means $x \notin \partial B$. So $\overline{\Omega_\epsilon} \cap \partial B = \emptyset$.
Now for some ϵ_1, $\Delta(h \circ f) \geq -b|\nabla(h \circ f)|^2$ on Ω_{ϵ_1}. So by lemma (4.1.6)

$$sup_{\partial\Omega_{\epsilon_1}} h \circ f \geq sup_{\Omega_{\epsilon_1}} h \circ f.$$

For $\epsilon_2 < \epsilon_1$,

$$\Omega_{\epsilon_1} \subseteq \Omega_{\epsilon_2} \quad \text{and} \quad sup_{\partial\Omega_{\epsilon_2}} h \circ f \geq sup_{\Omega_{\epsilon_2}} h \circ f \geq sup_{\partial\Omega_{\epsilon_1}} h \circ f \geq sup_{\Omega_{\epsilon_1}} h \circ f \geq \epsilon_1$$

Let $\epsilon = \frac{\epsilon_1 - \epsilon_2}{2} > 0$. So $\sup_{\partial\Omega_{\epsilon_2}} h \circ f - \epsilon = sup_{\partial\Omega_{\epsilon_2}} h \circ f - \frac{\epsilon_1-\epsilon_2}{2} \leq h \circ f(x_\epsilon)$ for some $x_\epsilon \in \partial\Omega_{\epsilon_2}$.

$\epsilon_1 < \epsilon \leq h \circ f(x_\epsilon)$ so $x_\epsilon \in \Omega_{\epsilon_1}$. Continue similarly we get a sequence ϵ_n when $\epsilon_n \to 0$ $\exists x_n \in \partial\Omega_{\epsilon_n}$ at the same time $x_n \in \Omega_{\epsilon_1}$. But as $\epsilon_n \to 0$ we have $\partial\Omega_{\epsilon_n} \to \partial B$ so $h \circ f(x_\epsilon) \to 0$ which is a contradiction. Therefore Ω_ϵ should be empty for all $\epsilon > 0$ which means $h \circ f \leq 0$ on B and f maps B into N.

Now we will show that f maps B° to N° or ∂N. h is strictly convex in a neighborhood of ∂N and $h \circ f$ is subharmonic in a neighborhood of $\{x | h \circ f(x) = 0\}$. We know $h \circ f \leq 0$ on B. By the strong maximum principle $h \circ f$ takes its maximum on the boundary, if it takes its maximum on the interior then it must be constant. So $0 \geq \sup_{\partial B}(h \circ f) > \sup_{B^\circ}(h \circ f)$. If it takes sup in the interior then it must be constant which means that $h \circ f = 0$ so f maps B to ∂N. But this is not possible since $h \circ f$ constant means f constant since h is strictly convex. This proves theorem 4.1.4 for strictly convex manifolds in case $k = 1$.

For $k > 1$ we have two possible cases.
If $d_{\tilde{N}}(\Gamma) < d^*_{\tilde{N}}(\Gamma)$ then B should be connected, since when computing $d^*_{\tilde{N}}(\Gamma)$, we take the minimum over all possible unconnected partitions of Γ. If B is connected then the argument for $k = 1$ also applies here. We similarly consider the function $h \circ f$ and prove that f maps B° to N°.

If $d_{\tilde{N}}(\Gamma) \geq d^*_{\tilde{N}}(\Gamma)$ then we can find $p > 1$ such that $\Gamma^1 \cup ... \cup \Gamma^p = \{\Gamma_1, ...\Gamma_k\}$ and $\sum_{i=1}^p d_{\tilde{N}}(\Gamma^i) \leq d_{\tilde{N}}(\Gamma)$. Let p be the largest integer that can be chosen in such a way.

Claim 2: $d_{\tilde{N}}(\Gamma^i) < d^*_{\tilde{N}}(\Gamma^i)$

Proof of Claim 2: If $d_{\tilde{N}}(\Gamma^i) \leq d^*_{\tilde{N}}(\Gamma^i)$ for some i, then there is some partition of Γ^i which has lesser area. So we can split Γ^i and get a number q greater than p satisfying $\sum_{i=1}^q d_{\tilde{N}}(\Gamma^i) \leq d_{\tilde{N}}(\Gamma)$ and get a contradiction.

So we have $d_{\tilde{N}}(\Gamma^i) < d^*_{\tilde{N}}(\Gamma^i)$ which means the domain for Γ^i connected for all i. So we can solve Plateau problem for each Γ^i. According to previous arguments the solution must stay in N which means $d_{\tilde{N}}(\Gamma^i) = \mathrm{d}_N(\Gamma^i)$. So $\sum_{i=1}^p d_{\tilde{N}}(\Gamma^i) = \sum_{i=1}^p d_N(\Gamma^i) \leq d_{\tilde{N}}(\Gamma) \leq d_N(\Gamma)$. So we have $d^*_N(\Gamma) < d_N(\Gamma)$ which contradicts with the hypothesis of the theorem. So we have proved Theroem 4.1.4 for strictly convex manifolds.

Now we will consider the theorem for general convex manifold M. By the definition of M we know that M stays in N. By the above arguments we also know that the solution stays in N. To show that the solution also stays in M we will use similar arguments. The problematic part is to show that $g \circ f$ is continuous and subharmonic. This is proven by approximating g by smooth functions. For the details see [[16] Theorem 1]. □

4.2. LOCAL PROPERTIES OF MINIMAL SURFACES.

Lemma 4.2.1. *Let B be an open plane domain and $f : B \to M$ be a minimal immersion where M is a three dimensional manifold. If for some $p \neq q$ $f(p) = f(q)$ then there exists neighborhoods U and V of p and q such that either $f(U) = f(V)$ or $f(U)$ and $f(V)$ intersects along finite number of curves and the intersection is transversal other than the points of $f(p)$.*

Proof. If the images of some small neighborhoods of these point intersects then nothing to prove. If for any neighborhoods the images the intersection of the images does not contain an open set, and the intersection is transversal at $f(p)$ then $f(U) \cap f(V)$ must be clearly the intersection of finite curves. The nontrivial case is when the intersection is not transversal at $f(p)$. In this case we consider the common tangent plane at p and q. We consider local coordinates (x_1, x_2, x_3) such that $f(p)$ represent the point $x_1 = 0, x_2 = 0, x_3 = 0$ and the common tangent plane represents the plane $(x_1, x_2, 0)$. Then we can consider small neighborhoods U and V of p and q respectively such that, locally $f(U)$ and $f(V)$ as graphs of functions φ_1 and φ_2 over U and V. Since φ_1 and φ_2 satisfies minimal surface equation, their difference $\varphi_1 - \varphi_2$ should satisfy a linear homogeneous second order elliptic equation.

By [[4]Lemma 4.31 and 4.32] we can make a C^1 change of the coordinates x^1 and x^2such that $\varphi_1 - \varphi_2$ is given by $p_N(x)$, where $p_N(x)$ is a polynomial of degree $N \geq 2$ satisfying a second order linear homogeneous elliptic equation. So we can take a neghborhood small enough, so that zero is the only critical point in that neighborhood. $p_N(x)$ takes the value zero on the union of finite curves so the intersection of the graphs over the U is finite number of smooth curves intersecting at $f(p)$.

□

Lemma 4.2.2. *If $f : D \to M$ is a conformal harmonic immersion from the disk into a three dimensional manifold M such that f is continuous on ∂D, for $x \in D^\circ$ $f(x) \notin \partial D$ and $f|_{\partial D}$ is one to one. Then the image of any two disjoint open sets is not equal.*

Proof. Assume for a contradiction that for some open sets U_1 and U_2 we have $f(U_1) = f(U_2)$ and also assume wlog that f is an embedding on U_2. For $x \in U_1$ define the conformal map $h : U_1 \to U_2$ such that $h(x) = f^{-1}(f(x))$. Also define for $\epsilon > 0$,$D_\epsilon = \{x : |x| < 1 - \epsilon\}$ such that $U_1 \subset D_\epsilon$. Let $U \subset D_\epsilon$ be the largest open disk such that there exits a locally one to one and conformal map $k : U \to D$ satisfying $k|_{U_1} = h$.

Claim : $U = D$

Proof of Claim: If $U \neq D_\epsilon$ then $\exists x \in \partial U \cup D_\epsilon$. So $f(x) \notin f(\partial D)$ and f is an immersion on D implies that f is a diffeomorphism in a neighborhood of x. Consider $f^{-1}(f(x)) \subset \overline{D}$ which is compact being a closed subset of a compact set. But it is a discrete set because f is a diffeomorphism around each point in it. So $f^{-1}(f(x))$ must be a finite set, say $\{x_1, ...x_n\}$ and each x_i has a neighborhood N_i such that f is an embedding on it. Consider a disk around $x; = x_1$ and call it $D_x \subset N_i$. Then $k(D_x \cap U) \subset N_i$ for some i.Since $f(x) = f(k(x))$ by lemma 3.2.1 we have $f(D_x \cap U)$ and $f(N_i)$ intersects along finite number of curves or $f(D_x) \subset f(N_i)$ when we shrink D_x if necessary. The intersection contains $f(k(D_x \cap U))$ which is an open set since k is one to one conformal map. So we must have $f(D_x) \subset f(N_i)$. Then we can extend k to D_x in a continuous manner so that $f(x) = f(k(x))$. We can consider the above procedure for each $x \in \partial U \cap D_\epsilon$, so we can extend k on each boundary point U. Since U is the largest open set in D_ϵ that we can extend k, we can extend k to D. Since $\epsilon > 0$ is arbitrarily small, by unique continuation of conformal maps we can extend k to D. k is continous on the boundary so it must satisfy $k(x) = x$ because f is one to one on the boundary. Hence k is identity map so $U_1 \subset U_2$ which is a contradiction. □

Corollary 4.2.3. *If $f : D \to M$ is a conformal harmonic immersion from the disk into the three dimensional Riemannian manifold such that f is continuous on D, $f|_{\partial D}$ for $x \in D^\circ$, then the self-intersection set of f can not be a point, a curve with an end point in the interior of D or a set with nonempty interior.*

Proof. By lemma 4.2.2 the self intersection set can not be a set with nonempty interior. By lemma 4.2.1 the self intersection set can not be a point, a curve with an end point in the interior of D. □

Lemma 4.2.4. *Let $f : D \to M$ and $\overline{f} : D \to M$ be two conformal harmonic immersions from the disk into three dimensional manifold M satisfying $f(\partial D) = \overline{f}(\partial D)$, f and $\overline{f}$ continuous on D and one to one on the boundary. Suppose also that for $x \in D^\circ$ $f(x) \notin f(\partial D)$ and there exists two open sets U and V such that $f(U) = \overline{f}(V)$. Then $f(D) \subset \overline{f}(D)$ and there exists a continuous one to one conformal map satisfying $f(x) = \overline{f}(k(x))$ for $x \in D$. When f is a solution to Plateau problem or for $x \in D^\circ$ we have $\overline{f}(x) \notin \overline{f}(\partial D)$, the map k is surjective.*

Proof. **Claim 1:** $f(D) \subset \overline{f}(D)$

Proof of Claim 1: Let $\mathcal{O}$ be the interior of $f^{-1}(\overline{f}(D))$ in D°. We will show that $\mathcal{O} = D^\circ$. Let $x \in \partial\mathcal{O} - \partial D$ and $\overline{x}$ be a point such that $\overline{f}(\overline{x}) = f(x)$. Then by inverse function theorem we can find two disks D_x and $D_{\overline{x}}$ around x and $\overline{x}$ such that $f|_{D_x}$ and $\overline{f}|_{D_{\overline{x}}}$ are embeddings. By lemma 4.2.2 either $f(D_x) \cap \overline{f}(D_{\overline{x}})$ is a union of finite curves or $f(D_x) \subset \overline{f}(D_{\overline{x}})$. But $x \in \partial\mathcal{O}$ so $\mathcal{O} \cap D_x$ contains an open set around x. Similarly $\mathcal{O} \cap D_{\overline{x}}$ contains an open set around $\overline{x}$. So $f(D_x) \cap \overline{f}(D_{\overline{x}})$ contains an open set around $f(x) = \overline{f}(\overline{x})$. So we have $f(D_x) \subset \overline{f}(D_{\overline{x}})$ which means $D_x \subset f^{-1}\overline{f}(D_{\overline{x}})$.This means that we can enlarge $\mathcal{O}$ unless $\mathcal{O} = D^\circ$. Result follows.

Claim 2: There is a function k as stated by theorem.

Proof of Claim 2: Let $k : U \to V$ be the conformal map defined by $f(x) = \overline{f}(k(x))$. Define $D_\epsilon = x||x| < 1 - \epsilon$ so that $U \subset D_\epsilon$. Let W be the maximal open disk in D_ϵ so that $U \subset W$ and we can extend the locally one to one conformal map k to W. Then $k : W \to D$ is a map such that $f(x) = \overline{f}(k(x))$. For $x \in \partial W$ $f(x) \notin \overline{f}(\partial D)$ so $f(x) \in \overline{f}(D^\circ)$ which means $\overline{f}^{-1}(f(x)) \in D^\circ$ so $k(x) \in D^\circ \Rightarrow$ we have $f(x) = \overline{f}(k(x))$ for $k(x) \in D^\circ$ then by lemma 4.2.2 there exists neighborhoods D_x and $D_{k(x)}$ of x and $k(x)$ such that $f(D_x) \cap \overline{f}(D_{k(x)})$ is a finite union of curves or $f(D_x) \subset \overline{f}(D_{k(x)})$. Since $x \in \partial W$, $D_x \cap W$ contains an open set O such that $O \subset D_x \cap W$ and $k(O) \subset D_{k(x)} \cap k(W)$ which implies $f(O) = \overline{f}(k(O)) \subset \overline{f}(D_x)$ so $f(D_x) \cap \overline{D_{k(x)}}$ contains an open set which excludes the option that $f(D_x) \cap \overline{f}(D_{k(x)})$ is a finite union of curves. So we are left with $f(D_x) \subset \overline{f}(D_{k(x)})$. This means that we can extend k D_x. We can consider same way for all $x \in \partial W$ so W maximal makes $W = D_\epsilon$. Letting $\epsilon \to 0$ we obtain a locally one to one conformal map k from D° to D° so that $f(x) = \overline{f}(k(x))$ for $x \in D$.

Claim 3: k is continuous on ∂D.

Proof of Claim 3: Let $x \in \partial D$. Then $f(x) \in f(\partial D) = \overline{f}(\partial D)$. $\overline{f}$ is one to one on ∂D so $\overline{f}^{-1}(f(x))$ is either a finite set or a sequence of points converging to a point on the boundary. Take a neighborhood N of $f(x)$. Then take sufficiently small meigborhoods $U_1, .., U_n$of these points such that these are mapped to N under $\overline{f}$. Then k maps some neighborhood of x into one of these sets U_i. Which means that k is continuous.

Claim 4:k is one to one on D.

Proof of Claim 4: f is one to one and continuous on ∂D then k must be also one to one on ∂D. k is a conformal map so by argument principle k must be one to one on D.

Claim 5: If $\overline{f}(x) \notin \overline{f}(\partial D)$ for $x \in D^\circ$or $\overline{f}$ is a solution to Plateu problem then k is surjective.

Proof of Claim 5: If $\overline{f}(x) \notin \overline{f}(\partial D)$ for $x \in D^\circ$ if $k(D) \neq D$ then we have for $x \in \partial D$ $k(x) \in D^\circ$ which leads clearly a contradiction. If $\overline{f}$ is a solution to Plateau problem for $\overline{f}(\partial D)$ and $k(D) \neq D$ the restriction of $\overline{f}$ to $k(D)$ gives us smaller area with the same boundary values. Bt Jordan curve theorem we must have $k(D^\circ) = D^\circ$ and $k(\partial D) = \partial D$. Result follows. □

Lemma 4.2.5. *Let Ω and $\Omega^{'}$ be two planar domains bounded by circles $\gamma_1, .., \gamma_m$ and $\gamma_1^{'}, .., \gamma_m^{'}$ respectively. Suppose also that f and $\overline{f}$ be two conformal harmonic immersions from Ω and $\Omega^{'}$ to a three dimensional Riemannian manifold M such that $f|\bigcup_{i=1}^m \gamma_i$ and $\overline{f}|\bigcup_{i=1}^m \gamma_i^{'}$ are one to one maps and $f(\Omega^\circ) \cap f(\bigcup_{i=1}^m \gamma_i) = \emptyset$ and $\overline{f}$ is a solution to Plateau's problem for $\bigcup_{i=1}^m f(\gamma_i)$. If there are nonempty open sets U and V so that $f(U) = f(V)$, then there exists a one to one conformal map $k : \Omega \to \Omega^{'}$ which is continous and satisfies $f(x) = \overline{f}(k(x))$.*

Lemma 4.2.6. *Let Σ_1 and Σ_2 be two minimal surfaces in a three dimensional manifold M so that $\partial\Sigma_1 = \partial\Sigma_2$ and some part of the boundary is a smooth curve σ. Suppose that at each point of σ where Σ_1 and Σ_2 are immersed the tangent planes coincide and the inward normals agree along σ. Then some nonempty open subsets of these sets must agree.*

Proof. Choose local coordinates (x_1, x_2, x_3) around a point p of σ such that p represents the point 0 and σ represents the line x_2. (that is $x_1 = 0$, $x_3 = 0$). Let φ_1 and φ_2 be functions such that Σ_1 and Σ_2 represent their graph in a small neighborhood of σ. The tangent planes of the sets are equal along σ means that $\partial\varphi_1/\partial x_2 = \partial\varphi_2/\partial x_2$ along σ. φ_1 and φ_2 satisfies minimal surface equations so $\varphi_1 - \varphi_2$ also satisfies a linear homogeneous elliptic equation. We know $\varphi_1 - \varphi_2 = 0$ along σ. Also by differentiating we see that all derivatives are zero along σ. By the unique continuation property [2]$\varphi_1 - \varphi_2$ is zero in a nighborhood where the functions are defined. So some open sets of Σ_1 and Σ_2 must be equal. □

Definition 4.2.7. *Let f be A Lipschiz map from the disk into a three dimensional manifold M such that the resriction of f to either right or left disk is C^1 up to the boundary and is an immersion. Then f has a **folding curve** along the image of the y axis if one of the following holds.*

(1) *If for each point $(0, y)$, the plane spanned by*

$$f_*(\partial/\partial y)|_{(0,y)}$$

and

$$\lim_{x>0, x\to 0} f_*(\partial/\partial x)|_{(x,y)}$$

is transversal to the plane spanned by

$$f_*(\partial/\partial y)|_{(0,y)}$$

and

$$\lim_{x<0, x\to 0} f_*(-(\partial/\partial x))|_{(x,y)}$$

(there is some angle between the image of the right hand disk and the image of the left hand disk.)

(2) *If for each point $(0, y)$,*

$$\lim_{x<0, x\to 0} f_*((\partial/\partial x))|_{(x,y)}$$

is a positive multiple of

$$\lim_{x>0, x\to 0} f_*(-(\partial/\partial x))|_{(x,y)}$$

.

(the image folded such that the image of the right hand disk and the image of the left hand disk coincide.)

We will give another description of a situation where the folding curve arises. Let f be a Lipschitz map form the unit disk into a three dimenasional manifold M such that the restriction of f to right hand disk and left hand disk is a C^1 up to the boundary and is an immersion. Choose a local coordinate (x^1, x^2, x^3)around the image such that the image of the y axis represents the x^3 and $f(0)$ is the point where $x^1 = 0, x^2 = 0, x^3 = 0$. Consider the planes $P_1, ..., P_n$ which passes through the x^3 axis and a small ball B around the origin. Suppose we have the sequence of Lipschitz maps $f_j : D \to M$ satisfying:

(1) Each f_j is C^1 on both right and left closed half disks.
(2) $f_j(D) \cap \bigcup_{i=1}^n P_i = \emptyset$.
(3) $\{f_j\}$ converges in C^1 both in right and left closed half disks to f.

Then we have the following lemma.

Lemma 4.2.8. *If $f(D) \cap B \subset \bigcup_{i=1}^n P_i$. Then we have one of the following:*

(1)

$$\lim_{x<0, x\to 0} f_*((\partial/\partial x))|_{(x,y)}$$

is a positive multiple of

$$\lim_{x>0, x\to 0} f_*(-(\partial/\partial x))|_{(x,y)}$$

(2) *there are distinct planes P_i and P_j so that for y small* $(0, y)$,

$$\lim_{x<0, x\to 0} f_*((\partial/\partial x))|_{(x,y)} \in P_i$$

and

$$\lim_{x>0, x\to 0} f_*(-(\partial/\partial x))|_{(x,y)} \in P_j$$

.

Particularly when we restrict f to a small disk around the origin, it has a folding curve along the x_3 axis.

Proof. Assume for a contradiction that $\lim_{x<0, x\to 0} f_*((\partial/\partial x))|_{(x,y)}$ is not a positive multiple of $\lim_{x>0, x\to 0} f_*(-(\partial/\partial x))|_{(x,y)}$ and they are elements of the same plane P_i. Then the image of the x axis is a nontrivial curve in P_i and its projection into the (x^1, x^2) plane is a line segment containing the origin. For n big enough consider f_n and the projection of its image of the x axis onto the (x^1, x^2) plane. Then this will same with the projection of f except for a set of very small neighborhood of the origin. So this line segment must intersect one of the planes different from P_i which is a contradiction with the definition of f_n. ☐

Lemma 4.2.9. *Let $f : D \to M$ be a Lipschiz map which has a folding curve defined above. Then f can not have minimal area among all Lipschiz C^1 maps which have the same boundary values as f.*

Proof. Assume for a contradiction that f has minimal area among all maps that have the same boundary values. Since f is an immersion both on the right disk and left disk the mean curvature is zero on that parts. Since f has a folding curve along y axis we can find a vector field such that

$$\langle E, f_*(\frac{\partial}{\partial y})\rangle|_{(0,y)} = 0$$

,

$$\lim_{x>0, x\to 0} \langle E, f_*(\frac{\partial}{\partial x})\rangle|_{(x,y)} > 0$$

and

$$\lim_{x<0, x\to 0} \langle E, f_*(-\frac{\partial}{\partial x})\rangle|_{(x,y)} < 0$$

By the first variation formula the first variation on the right hand disk is given by the integral of

$$-\langle E, f_*(\frac{\partial}{\partial x})\rangle$$

along the folding curve, since at other points we have mean curvature zero. Similarly the first variation on the left hand disk is given by the integral of

$$-\langle E, f_*(-\frac{\partial}{\partial x})\rangle$$

along the folding curve. When f is deformed by E then area is decreasing which is a contradiction. □

Remark 4.2.10. We can say that the self-intersection set of a minimal immersed disk can not be nontrivial under some conditions.

Proof. Assume for a contradiction that it is nontrivial. Let p be a point of self intersection of a minimal immersed disk. Then we know that $f^{-1}(p) = \{p_1, .., p_k\}$ such that each point has neighborhoods $U_1, .., U_k$ such that the restriction of f each one of them is an embedding. By Corollary 4.2.3 we know that they can not intersect on an open set so by lemma 4.2.1 they must be mutually transversal to each other. If we know that f is real analytic then f is simplicial with respect to some triangulations of D and M. We can triangulate U_i's such that all $f(U_i)$'s pass through a real analytic curve containing p. If the self intersection set is a compact subset of D then the inverse image of every point must be finite since f is an immersion. So $f(D)$ seems in a small neighborhood of $f(p)$ as a union of $f(U_i)$'s. So this small neighborhood is similar to the intersection of the planes P_i's. Locally consider the boundary of some small neighborhood is in $f(D)$ then by Lemma 4.2.9 this part of the surface can not have minimal area locally which mean globally it is not a minimal disk. □

In section 4.4 we will use smooth metrics, when passing from the real analytic metric to smooth metric we will use the following lemma.

Lemma 4.2.11. *Let $f : D \to M$ be a Douglas- Morrey solution to Plateau problem for the disjoint union of Jordan curves $\{f(\gamma_i)\}$ where D is a planar domain bounded by disjoint union of Jordan curves $\{\gamma_i\}$ and M is a three dimensional manifold. Suppose also that D' is a proper subset of D which is diffeomorphic to D and f restricted to the each boundary component of D' is one to one. Assume also that there is a smooth curve σ which is on the boundary of D' and on the interior of D. If g is any Douglas-Morrey solution to the Plateau problem for $f(D')$ and $f(\partial D') \cap f(D'^{\circ}) = \emptyset$ then g is equal to f up to a conformal reparametriazation of D'.*

Proof. By nonconformal parametrization we may assume that $f(x) = g(x)$ for all $x \in \partial D'$. By [12], f and g are smooth on σ . Also the theorems of Nitche and Heinz-Hildebrant show that there are only finite number of branch points of f or g on σ we may assume that f and g are immersions in a neighborhood of σ. We will define a new map $\overline{f} : D \to M$ such that $\overline{f}(x) = f(x)$ for $x \in D \backslash D'$ and $\overline{f}(x) = g(x)$ for $x \in D'$.

Claim: $\overline{f}$ does not have a folding curve along σ.

Proof of Claim: Assume for a contradiction that $\overline{f}$ has a folding curve along σ. Then by lemma 4.2.9 we have another map from D into M with smaller area and same boundary values. The area of g is equal to the area of $f|_{D'}$. Because otherwise one will be smaller and this will contradict their being Plateau solution to the boundary of the domains. So the area of $\overline{f}$ is equal to the area of f. If there is a map with smaller area than $\overline{f}$ and have the same boundary values then this will contradict f being a Plateau solution for the boundary of D.

$\overline{f}$ does not have a folding curve along σ implies that the tangent planes of $f(D')$ and $g(D')$ agree along σ then by lemma 4.2.6 some nonempty set of $f(D')$ must be equal to some nonemty set of $g(D')$. Then by lemma 4.2.5 f is equal to g up to a conformal reparametrization of D'. □

Lemma 4.2.12. *Let $f : D \to M$ be from a bounded plane domain D with smooth boundary into a three dimensional smooth manifold M. Then we have f is one to one on ∂D if it satisfies the following conditions:*

(1) $f \in C(D) \cap C^2(D^\circ)$
(2) *f restricted to each properly oriented boundary component of D describes a monotonic representation of an oriented Jordan curve.*
(3) *f is harmonic*
(4) *f is conformal.*

For the proof see [16]

4.3. APPROXIMATING A SMOOTH METRIC BY REAL ANALYTIC METRICS.

We will prove there is an embedded solution to the Plateau problem for a Jordan curve in a smooth convex manifold. In this section we will show that we can prove this, if we know the embeddedness of any solution to Plateau problem for a real analytic curve in a convex real analytic manifold. We will use the following hypothesis in this section and prove it in the next sections.

Hypothesis H: Suppose that M is a three dimensional real analytic manifold with real analytic convex boundary. Also assume that γ is a real analytic Jordan curve in ∂M. Then any solution to Plateau problem for γ is embedded.

Theorem 4.3.1. *Let M be a three dimensional smooth convex manifold, γ be a Jordan curve in ∂M. Then σ bounds an embedded solution to Plateau problem.*

Proof. Let $f : D \to M$ be a solution to Plateau problem for γ. Then by Theorem 4.1.4 either f maps D° to M° or f maps D to ∂M. If the image of the interior of D is completely contained in ∂M then γ is null homotopic on ∂M. By the Jordan curve theorem γ must separate ∂M into two components Ω_1 and Ω_2, where Ω_1 is a disk. f is Plateau solution so is conformal then by open mapping theorem f must be an open map. By the continuity and and being an open map $f(D)$ must be equal to Ω_1 or Ω_2 or $\Omega_1 \cup \Omega_2$. If $f(D)$ is equal to $\Omega_1$or Ω_2 then by open mapping theorem $f(D^\circ) \cap \gamma = \emptyset$. So $f|D^\circ$ is covering projection. The area of the disk is finite so the number of covering sheets is finite. When we consider deck transformations we see that a finite group must have a free action on disk but this is not possible by Brouwer fixed point theorem. This gives us that f is a homeomorphism. The case $\Omega_1 \cup \Omega_2$ is not possible, since f is a solution to Plateau problem, the fact that Ω_1 is a disk with less area gives a contradiction. So when $f(D)$ is a subset of ∂M we have embedding.

Assume that $f(D)$ is a not a subset of ∂M. By definition of a convex manifold there is a strictly convex manifold N, M is a subset of which. By [18], [19] we know that N admits a real analytic structure and real analytic metric. We can approximate the original smooth metric, say $\sum_{i,j} g_{ij}(x)dx^i \otimes dx^j$ by real analytic metric sequence $\sum_{i,j} g^n_{ij}(x)dx^i \otimes dx^j$ such that $g^n_{ij} \to g_{ij}$ in smooth norm when $n \to \infty$. We will consider real analytic manifolds M_n which have real analytic boundary and they are very close to M for n large. By definition of a convex manifold we have some function g which is convex in a neighborhood of $N \setminus M$ such that $M = g^{-1}((-\infty, 0])$ and also some function $\overline{g}$ which is strictly convex in a neighborhood of $N \setminus M$. Take a sequence of real numbers $\epsilon_n \to 0$ such that the functions $g + \epsilon_n \overline{g}$ are strictly convex with respect to the metric $\sum_{i,j} g^n_{ij}(x)dx^i \otimes dx^j$ in a fixed neighborhood of $N \setminus M$. According to [16] by using the Heat kernel of the metric $\sum_{i,j} g^n_{ij}(x)dx^i \otimes dx^j$, we can approximate $g + \epsilon_n \overline{g}$ by a real analytic function g_n which is strictly convex with respect to the metric $\sum_{i,j} g^n_{ij}(x)dx^i \otimes dx^j$ in a fixed neighborhood of ∂M. Since $\epsilon_n \to 0$ we have $g + \epsilon_n \overline{g} \to g$. But for $x \in \partial M$ $g(x) = 0$ so we may assume that $\sup_{x \in \partial M} |g_n(x)| \to 0$ as $n \to \infty$.

By Sard's theorem the points δ making the set $g_n^{-1}(-\infty, \delta)$ real analytic is dense so we may choose a sequence $\delta_n \to 0$ such that $\Sigma_n = \{x \in M : g_n(x) = -sup_{x \in \partial M}|g_n(x)| - \delta_n\}$ is real analytic. Then $M_n = \{x \in M : g_n(x) \leq -sup_{x \in \partial M}|g_n(x)| - \delta_n\}$ is an analytic convex manifold with analytic boundary $\sum_n$.

Let φ be the function as given in definition 3.1.2. φ sends a the neighborhood $\partial M \times (-1, 1)$ of ∂M some neigborhood of ∂M. Since g_n is a smooth approximation of $g + \epsilon_n \overline{g}$ we may assume that for a smaller neigborhood of ∂M the same inequalities hold when we use g_n instead of g. i.e for all $x \in \partial M$ and for all n and for $1/2 \geq t_2 \geq t_1 \geq -1/2$ we have the following:

Recall equation (1): $1 \geq t_2 \geq t_1 \geq -1$ we have $\varphi(x, 0) = x$ and

$$-c(t_2 - t_1) \geq g[\varphi(x, t_2)] - g[\varphi(x, t_1)]$$

for some constant c

$$\frac{-c(t_2 - t_1)}{2} \geq g_n[\varphi(x, t_2)] - g_n[\varphi(x, t_1)] \tag{16}$$

then from equation (1) it follows that for $t_2 = 1/2$ and $t_1 = 0$ we have

$$\frac{-c}{4} + \sup_{x \in \partial M} |g_n(x)| \geq g_n[\varphi(x, 1/2)] \tag{17}$$

$$\frac{c}{4} - \sup_{x \in \partial M} |g_n(x)| \leq g_n[\varphi(x, -1/2)] \tag{18}$$

When n large enough $\sup_{x \in \partial M} |g_n(x)|$ and δ_n is small eough, so we have

$$\frac{-c}{4} + \sup_{x \in \partial M} |g_n(x)| \leq - \sup_{x \in \partial M} |g_n(x)| - \delta_n \leq \frac{c}{4} - \sup_{x \in \partial M} |g_n(x)|$$

so $g_n[\varphi(x,t)] = -\sup_{x \in \partial M} |g_n(x)| - \delta_n$ for some $t \in [-1/2, 1/2]$. which means the line segment $\{\varphi(x,t)| -1/2 \leq t \leq 1/2\}$ intersects Σ_n at least one point. But by inequality (16) the intersection is a single point. This gives us a one to one correspondence between ∂M and Σ_n.

Claim 1: The map $\Psi : \partial M \to \Sigma_n$ defined by $\Psi(x) = \varphi(x,t)$, where $\varphi(x,t)$ is the intersection point in Σ_n is a continuous.

Proof of Claim 1: Consider the map Ψ^{-1}. It is given by the projection of $\varphi^{-1}(\Sigma_n) \subset \partial M \times (-1/2, 1/2)$ onto ∂M. Inverse map is homeomorphism so Ψ is continuous.

Under Ψ the Jordan curve σ on ∂M is mapped onto a Jordan curve $\tilde{\sigma_n}$ on Σ_n.

By [16] being a Jordan curve on a Riemannian surface $\tilde{\sigma_n}$ can be approximated by real analytic Jordan curve σ_n on Σ_n such that σ_n is uniformly close to $\tilde{\sigma_n}$ and they bound an annulus of arbitrary small area with respect to the induced metrics $\sum_{i,j} g^n_{ij} dx^i \otimes dx^j$ on Σ_n.

We will assume that σ is Lipschitz. We can define the annulus between Σ_n and σ as $\{\varphi(x,t) : 0 < t \leq t_x\}$ where t_x is the time that annulus intersects Σ_n. This set is clearly subset of $\{\varphi(x,t) : -1/2 \leq t \leq 1/2\}$ which has finite area with respect to the induced metric. So when Σ_n is very close to σ annulus has arbitrary small area with respect to all metricss that we are considering.

Define $\chi = \{\psi : D \to \mathbb{R}^3 : \psi|_{\partial D} = \sigma\}$ and $\chi_n = \{\psi : D \to \mathbb{R}^3 : \psi|_{\partial D} = \sigma_n\}$. Let $A_n = inf_{\psi \in \chi_n} Area(\psi)$ with respect to the metric $\sum_{i,j} g^n_{ij} dx^i \otimes dx^j$ and let $A = inf_{\psi \in \chi} Area(\psi)$ with respect to the metric $\sum_{i,j} g_{ij} dx^i \otimes dx^j$. Then $\lim_{n \to \infty} supA_n \leq A$. Since otherwise as $n \to \infty$ the annulus has arbitrary small area would give us a contradiction.

Let f_n be a solution to the Plateau problem for Σ_n in M_n. Since they are real analytic by hypotesis H we know that f_n is an embedding.

Claim 2: f_n has a subsequence which converges to a solution to Plateau problem for σ.

Proof of Claim 2: Let p^n_1, p^n_2 and p^n_3 in M_n be a sequence of points such that $p^n_1 \to p_1$, $p^n_2 \to p_2$ and $p^n_3 \to p_3$ where $p_i \in \sigma$ and $p^n_i \in \sigma_n$ for all n and for all i. We have a conformal map on disk sending 0 to $f_n^{-1}(p^n_1)$, $\sqrt[2]{-1}$ to $f_n^{-1}(p^n_2)$ and -1 to $f_n^{-1}(p^n_3)$. So by composing with this map we may assume that $f_n(0) = (p^n_1)$, $f_n(\sqrt[2]{-1}) = (p^n_2)$ and $f_n(-1) = (p^n_3)$ for all n. Since we have $\lim_{n \to \infty} supA_n \leq A$ there is a uniform upper bound for the areas of f_n. By equation (7) in section 3, when f_n is conformal energy and area are equal. So we have also upper bound for the energies of f_n. Since the metrics $\sum_{ij} g^n_{ij} dx^i \otimes dx^j$ are uniformly equivalent to the metric $\sum_{ij} g_{ij} dx^i \otimes dx^j$the energy of f_n will have a uniform upper bound when considered in the metric $\sum_{ij} g_{ij} dx^i \otimes dx^j$. Also we know f_n satisfies three point condition. It can be shown that $f_n|_{\partial D}$ is equicontinuous, so by Arzela-Ascoli Thorem it has a convergent subsequence which converges uniformly on ∂D.

Now we will show that it converges in the interior of D. Let $x \in D^\circ$ be a point such that $B(x,r) \subset D$. Then we know that by the proof of the Courant-Lebesgue Lemma (Lemma 3.0.38) that there is some $r^2 < r_n < r$ such that

$$L(f_n(\partial B(x,r_n))) \leq K(\log 1/r)^{-1} \tag{19}$$

where $L(f_n(\partial B(x,r_n)))$ is the length of the curve $B(x,r_n)$ and K is a constant independent of n.

Let ρ be a number such that any geodesic ball of the metric $\sum_{ij} g_{ij} dx^i \otimes dx^j$ with center in M and radius ρ is is smooth and strictly convex with respect to all metrics $\sum_{ij} g_{ij} n dx^i \otimes dx^j$. We can cover M by finitely many such balls $B_1, ..., B_l$. Choose some $\epsilon > 0$ such that any geodesic ball with respect to the metric $\sum_{ij} g^n_{ij} dx^i \otimes dx^j$ with radius less than ϵ is a proper subset of some B_i and has distance to ∂B_i greater than ϵ.

Now we want $f_n(\partial B(x,r_n))$ to be a proper subset of some B_i and distance of $f_n(\partial B(x,r_n))$ to ∂B_i is greater than ϵ. To obtain this using equation (19), pick some r satisfying

$$K(\log 1/r)^{-1} < \epsilon$$

Claim 3: The energy of f_n over $B(x,r_n)$ is less than $aL(f_n(\partial B(x,r_n)))$ where a is a constant depending on M.

Proof of Claim 3: We may assume that all metrics are uniformly equivalent, so we may assume that B_i is diffeomorphic to Euclidean unit ball (All metrics are uniformly equivalent to the Euclidean metric.) In this unit ball with respect to the euclidean metric, for each Jordan curve $f_n(\partial B(x,r_n))$ we can find a Plateau solution h_n. By [3] it is well known that $Area(h_n) \leq (1/4)(L(f_n(\partial B(x,r_n))))^2$. Since f_n is a solution to Plateau problem for another metric and metrics are uniformly equivalent there is some constant a such that the energy of f_n over $B(x,r_n)$ is bounded above by $aL(f_n(\partial B(x,r_n)))$. Similarly the area is also bounded.

Claim 4: For r small eough we have $f_n(B(x,r_n)) \subset B_i$.

Proof of Claim 4: Assume for a contradiction that $f_n(B(x,r_n))$ is not a subset of B_i. Then there is some point $y \in f_n(B(x,r_n)) \cap \partial B_i$. But we know that the distance of $f_n(\partial B(x,r_n))$ from ∂B is greater than ϵ. So the distance of y to $f_n[\partial B(x,r_n)]$ is greater than ϵ. Then by [[16] Appendix] the area of $f_n(B(x,r_n))$ is greater than some constant depending on ϵ and M. But we can make $K(\log 1/r)^{-1}$ very small such that the area is less than the above constant by choosing r sufficiently small. Contradiction.

Now we have $f_n(B(x,r_n)) \subset B_i$. By Theorem (2.0.17) we have the convex hull property. So $f_n[B(x,r_n)] \subset conv(f_n(\partial B(x,r_n)))$. But when r is small the convex hull of $f_n[\partial B(x,r_n)]$ is arbitrarily small, so the diameter of $f_n(B(x,r_n))$ is uniformly small which gives us the equicontinuity of $\{f_n\}$ for all compact subsets of D. We know that f_n converges uniformly on ∂D. So it converges on D to a continuous map $f : D \to M$ such that $f|_{\partial D} = \sigma$.

Claim 5: f_n converges in smooth norm on compact subsets of D and the limit is smooth in the interior of D, which is the Plateau solution.

Proof of Claim 5: Using L_p estimate and Sobolev inequality the proof is given in [16] page 424–425

Claim 6: f is embedding.

Proof of Claim 6: Assume for a contradiction that there are two distinct points $x, y \in D$ such that $f(x) = f(y)$. We know that f is a homeomorphism on ∂D and f maps D° to M°, so we must have $x, y \in D^\circ$. By lemma 4.2.2 we have convex neighborhoods U and V of x and y, either $f(U)$ and $f(V)$ intersects in an open set or they intersects transversally and the resriction of f to either U or V is an embedding. First one is not possible by lemma 4.2.3 so we have transversal intersection. Now choose o local coordiante system (x_1, x_2, x_3) such that $f(U)$ is on the (x_1, x_2) plane which contains the unit disk on that plane, and $f(V)$ is on the (x_2, x_3) plane which contains the unit disk on that plane. Let z_1 and z_2 be two points on V such that $f(z_1)$ is on the positive x_3 axis and $f(z_2)$ is on the negative x_3 axis. We also have $\overline{f_n(z_1, z_2)} < 1$. For n large consider $f_n(U)$ over the disk with center 0 and radius 1/2. This is very close to f so we can consider it as a graph over the disk with radius 1/2. We may assume that $f_n(z_1)$ is always above these graphs and $f_n(z_2)$ is always below these graphs. Consider $\overline{f_n(z_1, z_2)}$. Since it has Euclidean length less than 1 it must intersect $f_n(U)$,because the curves which do not intersect $f_n(U)$ and connecting $f_n(z_1)$ and $f_n(z_2)$ has length greater than 1. This contradicts with f_n's being embedding. So when we assume that the curve σ is Lipschitz the theorem is proven.

We are left with to prove the theorem for a general σ. We have some map from Σ_n to M which is Lipschitz homeomorphism. Consider the inverse image of σ on $\Sigma_n$under this map. Then approximate it with a smooth Jordan curve so that they bound an annulus with small area. Since the homeomorphism is Lipschitz this curve is mapped to a Lipschitz curve on M and they bound an annulus of small area with σ. So the result follows. □

We will give another hypothesis which we will prove in the next section.

Hypothesis K: Let σ be a real analytic Jordan curve in a three dimensional compact real analytic manifold with real analytic metric and real analytic convex boundary. Let f be a solution for the Plateau problem for σ in M.If f is embedding in a neighborhood of ∂D and for $x \in D^\circ$, $f(x) \notin f(\partial D)$ then f is an embedding from D into M.

Theorem 4.3.2. *Let σ be a C^2 regular Jordan curve in a three dimensional compact manifold M with convex boundary. Let $f : D \to M$ be a Douglas-Morrey solution for σ. If f has no boundary branch point and if $f(x) \notin f(\partial D)$ for $x \in D^\circ$ then f is an embedding from D to M. Also if $\sigma \subset R^3$ then it is enough to assume that σ is C^1 regular.*

Proof. By [12] f is C^1 in a neighborhood of ∂D. f has no branch points so f is an immersion in a neighboRhood of the boundary of D. Since f is a Plateau solution f is conformal and harmonic, also it satisfies other hypothesis of lemma 4.2.12, so $f|\partial D$ is one to one. Hence f is embedding in a neigborhood of ∂D. Define $N_\epsilon = \{x \in D : 1 - \epsilon \leq |x| \leq 1\}$. Then f is an embedding on N_ϵ for small ϵ.

Claim 1: $f(N_\epsilon) \cap f(x||x| < 1-\epsilon) = \emptyset$ for small ϵ.

Proof of Claim 1: Assume for a contradiction that there is an element in $f(N_\epsilon) \cap f(x||x| < 1-\epsilon\}$ for each $\epsilon > 0$. So we can find sequences $\epsilon_i \to 0$ and x_i, y_i in D such that $|x_i| < 1-\epsilon_i$, $|y_i| \geq 1-\epsilon_i$ and $f(x_i) = f(y_i)$. We may assume that x_i converges to a point $x_0 \in \partial D$. Assume that $y_i \to y_0$. If $y_0 \in \partial D$, then $f(x_0) = f(y_0)$ implies that $x_0 = y_0$ because f is an embedding in a neighborhood of ∂D. $y_0 \in D\circ$ is not possible, because $f(x_0) = f(y_0) \in f(\partial D)$ contradicts with the assumption of the theorem. x_i and y_i converges to the same point in the boundary means that eventually both x_i and y_i belongs to the neighborhood of ∂D where f is an embedding. But $f(x_i) = f(y_i)$ gives us a contradiction so result follows.

Now define $S = \{x : |x| = 1-\epsilon/2\}$ and $D_{\epsilon/2} = \{x : |x| < \epsilon/2\}$. Then $f(S)$ is a smooth regular Jordan curve.Also we have $S \subset N_\epsilon$ so f is an embedding in a neighborhood of S from this it follows that also for $x \in D_{\epsilon/2}$, $f(x) \notin f(S)$, because by claim for $|x| < 1-\epsilon$ it is satisfied, for $1-\epsilon \geq |x| \leq 1-\epsilon/2$ we know that f is an embedding on N_ϵ. Then by lemma 4.2.11 any solution for the Plateau problem of $f(S)$ is equal to f up to a conformal parametrization. Now we will show that f is an embedding. In the definition of the convex manifold N, we have a strictly convex manifold N which contains M as a subdomain. By shrinking N a little, we consider N has a real analytic structure and real analytic boundary. We will approximate the smooth metric by real analytic metrics ds_n^2 in smooth norm and approximate the smooth curve $f(S)$ by real analytic curves C_n in smooth norm. Now we can find Plateau solutions f_n for the curves C_n with the metrics ds_n^2 such that $f_n : D_{\epsilon/2} \to M$ and f_n restricted to S parametrizes C_n.

Similar to theorem 4.3.1 we can show that f_n converges uniformly on $D_{\epsilon/2}$. We may assume that f_n has a subsequence converging smoothly on $\overline{D_{\epsilon/2}}$ to a function $\tilde{f}$ which is a solution to Plateau problem for $f(S)$.

f_n converges smoothly on $\overline{D_{\epsilon/2}}$, which implies that $\tilde{f}$ is smooth. Also $\tilde{f}$ is a Plateau solution so by lemma 4.2.12 we have $\tilde{f}$ is a homeomorphism on S. By the three-point condition we may assume that $\tilde{f} = f$ at three distinct points on S. But as stated above by lemma 4.2.11, $\tilde{f}$ is equal to f up to a conformal parametrization. So there is conformal automorphism k on $\overline{D_{\epsilon/2}}$ satisfying $f(x) = \tilde{f}(k(x))$. By the three-point condition $k(x) = x$. for all $x \in D_{\epsilon/2}$. So $f = \tilde{f}$ which means f_n converges to f smoothly on $\overline{D_{\epsilon/2}}$.

Claim 2: For n large enough f_n has no branch points on S.

Proof of Claim 2: f has no branch points on $S = \partial D_{\epsilon/2}$, since it is an embedding in a neighborhood of it.f_n converges to f smoothly on $\overline{D_{\epsilon/2}}$. So f_n has no branch point on S for large n.

Claim 3: For n large enough $f_n(D_{\epsilon/2}) \cap f_n(S) = \emptyset$.

Proof of Claim 3: Assume for a contradiction that this is not true. So for all n there exists $x_n \in D_{\epsilon/2}$ and $y_n \in S$ satisfying $f_n(x_n) = f_n(y_n)$. Assume that $x_n \to x_0$ Since f_n converges uniformly to f we have $\lim_{n\to\infty} f_n(x_n) = f_n(y_n)$ which implies $f(x) = f(y)$ for x and y are the limits of the sequences x_n and y_n so we must have $x = y \in S$ since $\{y_n\} \subset S$ and $f(D_{\epsilon/2}) \cap f(S) = \emptyset$. Take a fixed coordinate neighborhood of $f(x)$ such that for all n, f_n maps a fixed neighborhood of x into that coordinate neighborhood. Define the constant vector field sequence $X_n = \frac{y_n - x_n}{|y_n - x_n|}$ defined on D. By passing to a subsequence converging to some unit vector field X we can consider X_n as a convergent vector field. With respect to the coordinate chart, assume f_n^1 is the first component of f_n. By mean value theorem we have $0 = \frac{f_n^1(x_n) - f_n^1(y_n)}{x_n - y_n} = (f_n^1)'(\overline{x_n})$ for some $\overline{x_n}$ on the line segment joining x_n and y_n. Then we have $(df_n^1)_{\overline{x_n}}(X_n) = X_n(f_n^1)(\overline{x_n}) = 0$. The smooth convergence of f_n to f on $\overline{D_{\epsilon/2}}$ implies that $X(f^1)(x) = 0$. Similarly we have $X(f^2)(x) = 0$ and $X(f^3)(x) = 0$. So $X(f) = df(X)$ is equal to the zero matrix. So the differential of f is not full rank at x. But f is conformal which implies that df is identically zero at x. This means f has a brach point at $x \in S$ which is a contradiction.

We know that f_n has no branch point on S , $f_n(D_{\epsilon/2}) \cap f_n(S) = \emptyset$ and also f_n is an embedding in a neighborhood of S, so by hypothesis K we know that f_n is an embedding on $\overline{D_{\epsilon/2}}$. Arguments similar to Thm 4.3.1 can be used to show that f is also an embedding on $\overline{D_{\epsilon/2}}$. But we already know that f is embedding on remaining parts and the intersection is empty. So result follows.

4.4. DEHN'S LEMMA FOR ANALYTIC MANIFOLDS.

In section 4.3 we proved embeddedness problem when we know embeddedness in real analytic case. Now we will prove for real analytic case.

Topological analysis in real analytic case is simpler, because

1) An analytic solution $f : D \to M$ is simplicial with respect to some triangulations of D and M. This follows from [14]

2) The image of D under f is embedded near the boundary of M. This follows from boundary regularity theorem.

Let $f : M^2 \to M^3$ be a mapping from a surface into a three dimensional manifold.

Definition 4.4.1. *The self intersection set of f is defined by*

$$S(f) = \{x \in M^2 : \exists y \neq x \in M^2 \quad with \quad f(x) = f(y)\}$$

.

Theorem 4.4.2. *Assume that $f : D \to M$ is a solution to Plateau problem from the disk into a three dimensional manifold M satisfying the following properties:*

(1) *$S(f)$ is disjoint from ∂D.*
(2) *f is simplicial with respect to some triangulations of D and M.*
(3) *The image of the interior of f is disjoint from the boundary of M.*

Then f is an embedding.

(When we prove this theorem then we also prove that the following hypothesis H, which was given in section 4.3.

Hypothesis H: Suppose that M is a three dimensional real analytic manifold with real analytic convex boundary. Also assume that γ is a real analytic Jordan curve in ∂M. Then any solution to Plateau problem for γ is embedded. The proof is given in the last part of this section, which is given as analytic version of Dehn's lemma.)

Proof. If necessary by restricting the range space M we can assume that the Jordan curve $f(D) = \gamma$ lies on the boundary of M. We will now construct the following tower of covering spaces. Let N_1 be a regular neighborhood of $f(D)$. If $H_1(N_1, Z_2)$ is nonzero, then by sending nonzero elements to 1 then one can write a surjective homomorphism

$\overline{\rho} : H_1(N_1, Z_2) \to Z_2$, and this induces a homomorphism $\rho : \pi_1(N_1) \to Z_2$. By main isomorphism theorem the kernel of this map has index 2, so $\pi_1(N_1)$ has a subgroup of index 2, which gives us a 2-sheeted covering space $P_1 : \tilde{N}_1 \to N_1$. When we restrict the range of $f(D)$ to the regular neighborhood N_1 we get a new map $f_1 : D \to N_1$. Since D is a simply connected space, we can lift the map f_1 to the space $\tilde{N}_1$ and get a map $\tilde{f}_1 : D \to \tilde{N}_1$. Then restricting the range space $\tilde{f}_1(D)$ to a regular neighorhood N_2, we get another map $f_2 : D \to N_2$. If $H_1(N_2, Z_2)$ is nonzero, then we can repeat the same procedure and get a 2-sheeted cover $P_2 : \tilde{N}_2 \to N_2$ and a lift $\tilde{f}_2 : D \to \tilde{N}_2$ of f_2. After restricting the lift $\tilde{f}_2$ to a regular neighborhood N_3 of $\tilde{f}_2(D)$ we get $f_3 : D \to N_3$. Repeating the same procedure n times we will get a tower of covering spaces such that $P_i : N_{i+1} \to N_i$.

Claim 1: Each of the lifts $f_i : D \to N_i$ is a solution to Plateau's problem for the Jordan curve $f_i(\partial D)$.

Proof of Claim 1: Assume on the contrary it is not a solution for the Plateau problem. Then there is a map $g : D \to N_i$ which is a solution to Plateau's problem for $f_i(\partial D)$ and with respect to the pulled back metric to D we have $Area(g) < Area(f_i)$. When we compose all covering maps and consider in the first space then we have $Area(P_1 \circ ... \circ P_{i-1} \circ g) \leq Area(g) < Area(f_i) \leq Area(f)$. This gives us a contradiction. The result follows.

Claim 2: The maps $f_1, f_2, ..., f_n$ can be made simultaneously simplicial with respect to a fixed triangulation of D that includes $S(f)$ as its 1-complex.

Proof of Claim 2: Let T and K be triangulations of D and M respectively such that for which f is simplicial. Then $f(D) = |L|$ for some subcomplex $L \subset K$. By adding the barycenters of the simplices of $K - L$ we will get a subcomplex K' of K. L is a subcomplex of K'. So similarly add the barycenters of the simplices of $K' - L$ and get a subcomplex K'' of K. Define $N(K, K'') = \{\overline{\sigma} \in K'' : \overline{\sigma} \cap L \neq 0$ where $\overline{\sigma}$ is a closed simplex of K'' $\}$. This is a regular neighborhood of $f_1(D)$ which is a subset of M since we consider the simplices which has a common point with the image. Making two iterations guarantees that this neighborhood does not intersect itself. We can consider this regular neighborhood as N_1. When we restrict the triangulation K'' to N_1 then we will get a triangulation of N_1 and lifting it with the covering map we will get a triangulation K_2 of $\tilde{N}_1$. Then $\tilde{f}_1 : D \to \tilde{N}_1$ is a simplicial map which sends the triangulation T of D to the triangulation K_2 of $\tilde{N}_1$. We may iterate the same procedure to get triangulations for $\tilde{N}_i$ and we have $f_1, f_2, ..., f_n$ are simultaneously simplicial with respect to the triangulation T of D.

Claim 3: For some k we will have N_k with $H_1(N_k, Z_2) = 0$.

Proof of Claim 3: Define $X(f_j) = \{(\sigma, \tau) \in T \times T : \sigma, \tau$ are open simplices, $\sigma \neq \tau$ and $f_j(\sigma) \cap f_j(\tau) \neq \emptyset\}$. Then, clearly $X(f_j)$ are finite sets and we have $X(f_{j+1}) \subset X(f_j)$. We will prove that $X(f_{j+1}) \neq X(f_j)$. Pick a base point p for N_j in $f_j(D) \subset N_j$. Also pick another base point $\tilde{p}$ for $\tilde{N}_j$ which is in the $\tilde{f}(D) \cap P_j^{-1}(p)$. We have $f_j(D) \subset N_j$ so the inclusion map induces the isomorphism $i_* : \pi_1(f_j(D), p) \to \pi_1(N_j, p)$. For any $[\alpha] \in \pi_1(N_j, p)$ is represented by a loop $\alpha : [0, 1] \to f_j(D)$ with

$\alpha(0) = \alpha(1) = p$. Assume for a contradiction that $X(f_{j+1}) = X(f_j)$. Then the restriction of the covering map P_j to the image of $\tilde{f}_j$ must be one to one, because otherwise the number of intersecting simplices under f_j will be greater than the number of intersecting simplices under f_{j+1} which will make $X(f_j)$ greater than $X(f_{j+1})$. The restriction of the covering map P_j to the image of $\tilde{f}_j$ is one to one implies that the loop α will lift to a loop $\tilde{\alpha} : [0,1] \to \tilde{N}_j$. This means that the map $P_{j^*} : \pi_1(\tilde{N}_j, \tilde{p}) \to \pi_1(N_j, p)$ induced by the covering map P_j is onto. This is a contradiction because, $P_{j^*}(\pi_1(\tilde{N}_j, \tilde{p})) \subset \pi_1(N_j, p)$ has index 2. So we have $X(f_{j+1}) \neq X(f_j)$. After k many steps where k is the number of elements in $X(f_1)$, we have $X(f_k) = 0$ so we can not go further to the tower construction. We must have $H_1(N_k, Z_2) = 0$, otherwise we can construct another 2-sheeted covering space.

Let $f_k : D \to N_k$ be the lift of f to the space which is on top of the tower. The pairing between homology and cohomology groups is nondegenerate and $H_1(N_k, Z_2) = 0$ implies that $H^1(N_k, Z_2) = 0$. $H_2(N_k, \partial N_k, Z_2) = 0$ by the Poincare duality for manifolds with boundary. The long exact sequence for the pair $(N_k, \partial N_k)$ is

$$\to^{j_*} H_2(N_k, \partial N_k, Z_2) \to^{\partial} H_1(\partial N_k, Z_2) \to^{i_*} H_1(N_k, Z_2) \to ..$$

It follows that $H_1(\partial N_k, Z_2) = 0$. For each boundary component of N_k the first homology group with Z_2 coefficients is zero. By the classification theorem for compact surfaces it follows that each component of the boundary is a sphere.

Claim 4: $f_k : D \to N_k$ is an embedding.

Proof of Claim 4:

Claim 4.1: Since N_k is a simplicial regular neigborhood after a possible subdivision there is a simplicial retraction $S : N_k \to f_k(D)$.

Proof of Claim 4.1: By the definition of regular neighborhood, $f_k(D)$ is obtained from N_k by sequentially collapsing the three simplices of N_k which has free face or faces that is not contained in $f_k(D)$. For the collapsing process there are three cases to be considered.

Case 1: Suppose σ is a three simplex $[ABCD]$ with vertices A, B, C, D and exactly one free face $[ABC]$ that is not contained in $f_k(D)$. Then let v be the barycenter of the free face $[ABC]$ and L is the straight line joining v to the vertex D. By projecting the face $[ABC]$ linearly along L, onto the other faces, the simplex can be collapsed.

Case 2: Now assume that the simplex has two free faces $[ABC]$ and $[ACD]$ that is not contained in $f_k(D)$. Let L be the line passing through the barycenter of $[AC]$ and barycenter of $[BD]$. By projecting the free faces along L linearly the simplex can be collapsed.

Case 3: Assume that the simplex has three free faces $[ABC]$, $[ACD]$ and $[ABD]$ that is not contained in $f_k(D)$. Let L be the line passing through the barycenter of $[BCD]$ and the vertex A. The simplex can be collapsed by linearly projecting the free faces along L onto $[BCD]$.

Clearly these maps are simplicial when we add the barcenters of the faces. If there are n such simplices after n sequential collapsing we get a piecewise linear map $S|_{\partial N_k} = R : \partial N_k \to f_k(D)$.

Claim 4.2: The simplicial retraction defined in Claim 4.1 has a restriction $R = S|_{\partial N_k} : \partial N_k \to f_k(D)$ such that R covers each open 2-simplex of $f_k(D)$ exactly two times and $R|_{(\partial N_k \setminus f_k(\partial D))}$ is locally one to one.

Proof of the claim 4.2: In the above collapsing process we project the free face that is not contained in $f_k(D)$ onto the other faces in a one to one manner so $R|_{(\partial N_k \setminus f_k(\partial D))}$ is locally one to one. Each open two simplex of $f_k(D)$ is a face of exactly two three simplexes and $R|_{(\partial N_k \setminus f_k(\partial D))}$ is locally one to one, so two simplex is collapsed on it. The result follows.

$\gamma = f_k(D)$ is a Jordan curve lying in ∂N_k. So it is on one of the boundary spheres of N_k. By the Jordan curve theorem it separates the sphere into two connected disks D_1 and D_2. By claim 4.2 we know that the retraction on the boundary N_k covers the two simplices on $f_k(D)$ at most two times. So we have $Area(R|_{D_1}) + Area(R|_{D_2}) \leq 2Area(f_k)$. If one 2-simplex on $f_k(D)$ is not covered by one of the maps $(R|_{D_1})$ or $(R|_{D_2})$ then the inequality is strict. But then one of the maps $(R|_{D_1})$ or $(R|_{D_2})$ has area less than $Area(f_k)$. We know that they bound the same curve γ and f_k is a solution to Plateau problem, which gives us a contradiction. So we have every two simplex on $f_k(D)$ is covered by one of the maps $(R|_{D_1})$ or $(R|_{D_2})$ and we have the equality $Area(R|_{D_1}) = Area(R|_{D_2}) = Area(f_k)$.

By lemma 4.2.1 and 4.2.2 the self intersection set can not be a point and an open set but it can be finite curves, so there is one simplex in $f_k(S(f_k))$. Take a 2-simplex σ having an edge E which is one of the one simplexes in $f_k(S(f_k))$. Then by the last paragraph σ is covered by one of the maps $(R|_{D_1})$ or $(R|_{D_2})$, say $(R|_{D_1})$. f_k is real analytic then by lemma 4.2.1 we may assume that have after subdivision f_k is transverse to itself at points other than the vertices of the triangulation of $f_k(D)$. Because we can make the triangulation such that the vertices is satisfies that property.

Since R is the restriction of a retraction to ∂N_k there are maps $R_i : \partial N_k \to (N_k - f_k(D))$ such that each R_i is an embedding and R_i converges smoothly on each closed simplex to R. (We can consider maps that send the part ∂N_k that stays in above $f_k(D)$ to surface that is stays above it and the part below to a surface below it such that as $i \to \infty$ these surfaces get closer to $f_k(D)$) . Let $R^{-1}(\sigma) = \sigma_1 \subset D_1$ be a two simplex. Let σ_2 be another two simplex which has $R^{-1}(E) \cap \sigma_1$ as an edge. Then $\sigma_1 \cup \sigma_2$ forms a disk and we have the situation described before lemma 4.2.8.(We have maps converging to R that does not intersect the image of R. Also other hypothesis of the lemma satisfied.) So by lemma 4.2.8 R has a folding curve along E. By lemma 4.2.9 we can decrease the area of $R|_{D_1}$ with the same boundary. But $R|_{D_1}$ has the same area with f_k, and f_k is a solution to Plateau's problem, which is a contradiction. So we must have f_k is an embedding. If we show that $k = 1$ the theorem will be proven.

Claim 5: $k = 1$.

Lemma 4.4.3. *Suppose that $f : D \to M$ is a minimal immersion from the disk into a three dimensional manifold, simplicial with respect to some triangulations of M and D with $S(f) \neq \emptyset$ and $S(f) \cap \partial D = \emptyset$. Then there exists a Jordan curve γ_1 on D which bounds a subdisk D_1 with $\partial D_1 = D_1 \cap S(f)$.*

Proof. $S(f)$ is one complex with every vertex in it is joined by at least two edges in it by corollary 4.2.3

By induction it can be shown that a finite one-dimensional complex with these properties have a simple closed curve in each path component.

Hence the collection C of all Jordan curves in $S(f)$ is nonemtpty. Define the Jordan curve $\gamma \in C$ such that γ is the boundary of a subdisk D_1 of D such that $(int(D_1))\cap S(f)$ contains the smallest number of open one-simplexes of $S(f)$. There are two cases.

i) Every closed one-simplex of $S(f)$ that is contained in $D_1 \cap S(f)$ and which intersects γ is a subset of γ.

ii) There is a one simplex in $D_1 \cap S(f)$ which intersects γ at a point p and which is not contained in γ.

In case i) we must have $int(D_1 \cap S(f) = \emptyset$. Assume on the contrary. Then any closed simplex of $S(f)$ must stay on $int(D_1)$. So there is a path component of $S(f)$ staying in $intD_1$. Any path component contains a Jordan curve, so there is a Jordan curve α in D_1 different than γ. α bounds some disk D_2 such that $D_2 \subset D_1$ and $(int(D_2)) \cap S(f)$ has fewer open simplices than $(int(D_1)) \cap S(f)$. So there is no Jordan curve different from γ and contained in $D_1 \cap S(f)$. Contradiction. For this case lemma is proven.

We will see that case ii) is not possible. In the last paragraph we have seen that there is no Jordan curve different from γ and contained in $(D_1 \cap S(f)$. So there is a longest Jordan arc $\tau : [0, 1] \to D_1 \cap S(f)$ with $\tau(0) = p$ and $\tau((0, 1))$ is contained in $int(D_1)$. Because $\tau((0, 1))$ is the longest arc $\tau(1)$ must be a vertex of $S(f)$. We have every vertex in $int(D_1 \cap S(f)$ is joined to an even number of edges of $S(f)$. If $\tau(1) \notin \gamma$, then there is another edge in $(D_1 \cap S(f)$ to which $\tau(1)$ is joined. But τ is the longest arc this is not possible, so we must have $\tau(1) \in \gamma$. The Jordan arc together with one of the arcs on γ joining $\tau(0)$ and $\tau(1)$ gives us a Jordan curve in $D_1 \cap S(f)$ different than γ. We know that this is not possible. So lemma is proven. □

Proof of Claim 5: Assume for a contradiction that $k > 1$ and $S(f_{k-1})$ is nonempty. We know that $f_k : D \to N_k$ is an embedding by the above claim. So it is one to one. Consider the order two deck transfromation $\sigma : \tilde{N_{k-1}} \to \tilde{N_{k-1}}$. Then any point and its image is sent to same point under the covering map P_{k-1}. If a point p in $f_{k-1}(D)$ is an intersection point then there are two points in $\tilde{N_{k-1}}$ sent to that point. That is $P_{k-1}(q) = P_{k-1}(\sigma(q)) = p$. f_k is one to one so there are $q_1, q_2 \in D$ such that $f_k(q_1) = q$ and $f_k(q_2) = \sigma(q)$. So q_1 and q_2 are sent to p under the map f_{k-1}. So $S(f_{k-1})$ consists of entirely double points. By lemma 4.4.3 there is a Jordan curve $\gamma_1 : S1 \to D$ bounding a subdisk D_1 with $D_1 \cap S(f_{k-1}) = \partial D_1$. Since $S(f_{k-1})$ consists of entirely double points there is another curve $\gamma_2 : S1 \to D$. Then we have $f_k(\gamma_2) = \sigma(f_k(\gamma_1))$. As f_k is an embedding and $\gamma_2 = f_k^{-1} \circ \sigma \circ f_k(\gamma_1)$, γ_2 is a continuous Jordan curve.

Then γ_2 bounds a subdisk D_2 of D. Suppose that $Area(D_1) \leq Area(D_2)$. Choose a diffeomorphism $h : D_2 \to D_1$ with $h(\alpha_2(t)) = \alpha_1(t)$. We have $h(\gamma_2(t)) = \gamma_1(t)$. Now define a map $g : D \to N_{k-1}$ by

$$g(x) = \begin{cases} f_{k-1}(x) & if \quad x \in (D - D_2) \\ f_{k-1} \circ h(x) & if \quad x \in D_2 \end{cases}$$

which places the image of the disk D_1 instead of the image of the disk D_2. g is a continuous piecewise smooth map with $Area(g) \leq Area(f_{k-1})$. We will prove that g has a folding curve, then by lemma 4.2.9 $Area(g)$ can be decreased and this will contradict the least area property of f_{k-1}. $S(f_{k-1})$ is compact, so by lemma 4.2.2 and 4.2.3, f_{k-1} crosses itself transversely except at finitely many points which are vertices in the triangulation of D. Pick a point $p \in \gamma_1(S^1)$ and $q = \sigma(p) \in \gamma_2(S^1)$ which corresponds to a point of transverse self-intersection. Since f_{k-1} is an immersion we have disjoint neighborhoods U_1 and U_2 of p and q respectively such that f_{k-1} is an embedding on U_1 and U_2 and $f_{k-1}(U_1)$ and $f_{k-1}(U_2)$ intersect transversely along an arc $\alpha : [0,1] \to N_k$.

For any $x \in \partial D_1$ or ∂D_2 define t_x as the tangent vector of the oriented curves ∂D_1 or ∂D_2 at x and also define n_x as the outer normal vector of the oriented curves ∂D_1 or ∂D_2 at x. At the point $f_{k-1}(p) = f_{k-1}(q)$ the two planes are transversal so the plane spanned by $(f_{k-1})_*(t_p)$ and $(f_{k-1})_*(n_p)$ intersects transversally the plane spanned by $(f_{k-1})_*(t_q)$ and $(f_{k-1})_*(-n_q)$. We know $(f_{k-1})_*(t_p) = g_*(t_p)$, $(f_{k-1})_*(n_p) = g_*(n_p)$, $(f_{k-1})_*(t_q) = g_*(t_p)$ and $(f_{k-1})_*(-n_q) = g_*(-n_p)$. When we replace this in the last sentence we will get the plane spanned by $g_*(t_p)$ and $g_*(n_p)$ intersects transversally the plane spanned by $g_*(t_p)$ and $g_*(-n_p)$. So by the definiton of the folding curve g has a folding curve along $g(\partial D_1)$. This is impossible. So result follows. □

Theorem 4.4.4. *(Analytic Version of Dehn's Lemma:) Suppose M is a three dimensional convex analytic manifold and γ is an analytic Jordan curve in ∂D which is nullhomotopic in M. Then γ bounds an embedded solution to Plateau's problem and each such solution is embedded.*

Proof. As in the proof of Theorem 4.3.1 we may assume that a solution f maps an the interior of D to the interior of M. The case where the solution is mapped onto the boundary can be solved similar method in Theorem 4.3.1. f has no boundary branch point by [13], $f|_{\partial D}$ is one to one, f is an immersion at the boundary and there is a neighborhood N of ∂D where f is an embedding. Since $f(D - N)$ is compact and and disjoint from the boundary of N, $f(D - N)$ stays a positive distance from the boundary. So f is embedded near the boundary of M.

By a theorem of Morrey $f : D \to M$ is an analytic mapping. By the triangulability theorem [14] f is simplicial with respect to some triangulations of D and M. So all conditions of theorem 4.4.2 is satisfied. f is an embedding.

□

4.5. THE EMBEDDING THEOREM FOR PLANAR DOMAINS.

We will say that a continous map g which maps a compact smooth surface Ω into a three dimensional manifold bounds a collection of disjoint Jordan curves $\{\gamma_1, ..., \gamma_n\}$ if $g|_{\partial\Omega}$ is a homeomorphism onto $\bigcup_{i=1}^{n} \gamma_i$.

Theorem 4.5.1. *Let $\Gamma = \{\gamma_1, ..., \gamma_n\}$ be a collection of disjoint unoriented Jordan curves on the boundary of a three dimensional orientable convex manifold M. Assume that these Jordan curves bound a continous mapping g from a compact plane domain Ω. Then there exists a branched minimal immersion $f : \Omega \to M$ which bounds Γ and has least area among such maps. Also all such maps must be embeddings.*

(sketch of proof) First for the anlaytic case is proven. This is done by similar method to section 4.4. The tower construction is done. However this is more complicated since there are more curves. Then it is similarly found that any path component of $\partial N_k \setminus \cup_{i=1}^{n} \gamma_i$ is either a sphere or a planar surface bounded by the curves. Then it is shown that f_k, which is the map on the top of the tower, is embedding. Then by using cut and paste arguments, it is proven that in fact $k = 1$.

For the smooth case the approximation procedure in the proof of 4.3.1 is used. Then for any given Plateau solution, embeddedness is proven by using the given lemmas and theorems.

4.6. THE GENERAL CASE OF DEHN'S LEMMA.

Theorem 4.6.1. *Suppose M is a three dimensional convex manifold. If γ is a Jordan curve on the boundary which is contractible in M, then*

(1) *There exists a solution with finite area to Plateau problem for γ.*
(2) *Any solution to Plateau problem for γ is embedded.*
(3) *Any two solutions to the Plateau problem either represents the same disk or the images intersect only at the boundary.*

Proof. (1) follows from theorem 4.1.4. In section 4.4 we see that the hypotesis is proven. For the smooth case it is proven by approximating the curve with real analytic curves and the metric with real analytic metrics similar to the technique used in the proof of theorem 4.3.1. Alternatively the method given in the last part(before corollary) of the section 5 of [16] may be used. So (2) also follows.

Now we will prove (3). Let f_1 and f_2 be two solutions to Plateau problem for the curve γ which does not represent the same disk.

Claim 1: $f_1(D) \cap f_2(D) = \gamma$

Proof of Claim 1: First we will give the proof for analytic Jordan curve γ and analytic metric on M. Define the sets $S(f_1, f_2) = \{x \in D | \exists y \in D \text{ with } f_1(x) = f_2(y)\}$ and $\{S(f_1, f_2) = x \in D | \exists y \in D \text{ with } f_2(x) = f_1(y)\}$. These are analytic subsets of D and $f_1(D)$ and $f_2(D)$ are embedded in M.

Assume for a contradiction that $f_1(D) \cap f_2(D) \neq \gamma$. By the same technique used in the proof of Lemma 4.4.3 we can see that there is a Jordan curve $\gamma_1 \subset S(f_1, f_2)$. Let γ_2 be the corresponding curve in $S(f_2, f_1)$. Let D_1 and D_2 be the subdisks bounded by γ_1 and γ_2 respectively. $f_1(\gamma_1) = f_2(\gamma_2)$ and f_1 and f_2 are solutions to Plateau problem for the curve $f_1(\gamma_1)$. By lemma 4.2.11 any solution to the Plateau problem for the curve $f_1(\gamma_1)$ must be a conformal reparametrization of the other solution. So we must have $f_1(D_1) = f_2(D_2)$. But this contradicts with lemma 4.2.2. So we must have $f_1(D) \cap f_2(D) = \gamma$.

Let γ be an arbitrary Jordan curve on ∂M. Assume again that $f_1(D) \cap f_2(D) \neq \gamma$ and f_1 is not the conformal reparametrization of f_2. Since by theorem 4.1.4 we have either the interior of D is mapped onto the interior of M, or the disk is mapped completely onto ∂M. If one of $f_1(D)$ or $f_2(D)$ is mapped onto the boundary then clearly we have $f_1(D) \cap f_2(D) = \gamma$. Then assume that neither of the two is not contained in ∂M. By lemma 4.2.1 $f_1(D)$ and $f_2(D)$ intersects transversely. Let $k : [0, 1] \to M^\circ$ be the arc of transverse intersection. Consider a small ball B centered at $k(1/2)$ such that $k(0)$ and $k(1)$ are on ∂B. Let the intersection of B with $f_1(D)$ and $f_2(D)$ be F and E respectively. So E and F also intersects transversely along k. Let the intersection of E with ∂B be α and the intersection of F with ∂B be β. One point intersections of α and β separates α into two subarcs say α_1 and α_2 and β into two subarcs say β_1 and β_2. Let E_1 and E_2 be two subdisks of E separated by k and with boundary α_1 and α_2 respectively. Similarly let F_1 and F_2 be two subdisks of F separated by k and with boundary β_1 and β_2 respectively.

Define $A_{ij} = Area(E_i \cup F_j)$ and B_{ij} is the solution of the Plateau problem for the Jordan curve $\alpha_i \beta_j^{-1}$. Also define $\epsilon = inf\{(A_{ij} - B_{ij}) | 1 \leq i, j \leq 2\}$. Since the $E_i \cup F_j$ has a folding curve along the arc k. So by lemma 4.2.9 we have $\epsilon > 0$.

Now construct a sequence of analytic Jordan curves $\gamma_i : S^1 \to \partial M$. which converge uniformly to $\gamma : S^1 \to \partial M$ and that are disjoint from γ. Assume also that for all i

the area of the annulus bounded by γ_i and γ is less than $min(\epsilon/5, AreaE)$. For the annular area bounded by γ and α $f_2|_{D-f_2^{-1}(E^\circ)}$ say F, is the unique solution to Plateau's problem. Similar to proof of theorem 4.3.1 when we take the maps $F_i : \Omega \to M$ which are annular solutions to Plateau problem for the annular area Ω which is the area bounded by γ_i and α, they converges uniformly to F. We have F is transverse to $f_1(D)$ near the boundary of B. Since the convergence is uniform in C^∞ norm we may assume that F_i is embedded near α and it is transverse to $f_1(D)$ near the boundary of B and $F_i(\Omega^\circ)$ is different from B. By [[16] remark(given in the proof of theorem 5)] it follows that F_i is an embedding for large i.

Now consider large i satisfying the above properties. Glue the embedded annulus $F_i(\Omega)$ along the boundary α to the embedded disk E and let the map represents this area is f_3 which is a piecewise differentiable map. When the metric is real analytic on M, then the sets $S(f_1, f_3) = \{x \in D | \exists y \in D$ with $f_1(x) = f_3(y)\}$ and $\{S(f_3, f_1) = x \in D | \exists y \in D$ with $f_3(x) = f_1(y)\}$ are finite 1-complexes which have even number of edges since the intersection is two to one. The same applies by lemma 4.2.2 when the metric is not real analytic.

Claim 2: There is a Jordan curve $\delta_1 \subset S(f_1, f_3)$ such that $f_1^{-1} \subset \delta_1$.

Proof of Claim 2: Consider the arc $\kappa = f_1^{-1}(k) \subset S(f_1, f_3)$ with end points p_1 and p_2. If we show that there is a path joining p_1 to p_2 in the set $X = (S(f_1, f_3) - \kappa) \cup \{p_1, p_2\}$ then we are done. Assume for a contradiction that p_1 and p_2 lie in different path components P_1 and P_2 respectively. By induction argument on the number of edges in a finite one-complex it can be seen that a finite one-complex can not have odd number of vertices where odd number of edges meet. So this is true in $(S(f_1, f_3)$. ie in $(S(f_1, f_3)$ there is an even number of vertices where odd number of edges meet. When we delete κ, p_1 is left with an odd number of vertices. So P_1 has odd number of edges where odd number of edges meet. We have a contradiction. So we have an arc joining p_1 to p_2. Let σ be the shortest of these arcs and define $\delta_1 = \sigma\kappa$. Result follows.

By the above claim we have a Jordan curve $\delta_1 \subset S(f_1, f_3)$ such that $f_1^{-1} \subset \delta_1$. Let $\delta_2 \subset S(f_3, f_1)$ be the corresponding Jordan curve. So we have $f_1(\delta_1) = f_2(\delta_2)$. Let the disk bounded by δ_1 and δ_2 be D_1 and D_2 respectively. f_1 is a solution to Plateau problem, so by lemma 4.2.11 any Plateau solution to the curve $f_1(\delta_1) = f_2(\delta_2)$ is a conformal reparametrization of the restriction of f_1 to D_1. ie represent the same disk. So we have $Area(f_1|_{D_1}) \leq Area(f_3|_{D_2})$.

We have chosen i large enough so that the area of the annulus between the curves γ and γ_i is less than $\epsilon/5$. So we have $Area(f_3) \leq Area(f_1) + \epsilon/5$. By interchanging the disk $f_3(D_2)$ with the disk $f_1(D_1)$ we get a new map $f_4 : D \to M$ such that $Area(f_4) \leq Area(f_3) \leq Area(f_1) + \epsilon/5$.

We have chosen as the infimum of difference between the Plateau solution to the curve $\alpha_i\beta_j^{-1}$ so we may decrease the area of f_4 at least ϵ. So there is a solution f_5 to Plateau problem for γ_i, so we have $Area(f_5) \leq Area(f_1) - 4\epsilon/5$. We have the area of γ_i and γ is less than $\epsilon/5$. So there is another map $f_6 : D \to M$ with boundary γ such that $Area(f_6) \leq Area(f_1) - 3\epsilon/5$. This contradicts with the fact that f_1 is area minimizing. So the theorem is proven. □

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