2+1 DIMENSIONAL GRAVITY THEORIES

by

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This is to certify that I have examined this copy of a master's thesis by

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and that any and all revisions required by the final

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To my friends

ABSTRACT

(2+1) dimensional Topologically Massive Gravity (TMG) and New Massive Gravity (NMG) theories are studied. These theories have higher order derivatives of metric tensor in their field equations which make them both renormalizable and unitary. Field equations of TMG are derived using variational methods. A null frame is introduced in general and TMG field equations are written in the null frame. pp-wave solutions of TMG are given. NMG field equations in the null frame are also derived. The pp-wave and recently found bending AdS wave solutions of NMG are given.

ÖZETÇE

(2+1) boyutta topolojik kütleli gravitasyon (TMG) ve yeni kütleli gravitasyon (NMG) teorileri incelenmiştir. Bu teorilerin alan denklemleri metrik tensörü bileşenlerinin yüksek mertebeden türevlerini içermekte olup bu nedenle üniterlikleri gösterilebilmekte ve renormalize edilebilmektedirler. TMG alan denklemleri varyasyon yöntemleriyle çıkarılmıştır. Genel bir ışıksal referans çerçevesi verilmiş ve TMG alan denklemleri ışıksal çerçevede yazılmıştır. TMG teorisinin pp-dalga çözümleri verilmiştir. NMG alan denklemleri de ışıksal çerçevede çıkarılmıştır. pp-dalga çözümleri ve yakınlarda bulunmuş bükülmüş AdS çözümleri verilmiştir.

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Chapter 1

INTRODUCTION

Einstein's theory of General Relativity (GR) defines the gravitational behavior of matter as a geometric property of the space-time through Einstein's field equations.

$$\mathcal{R}_{ab} - \frac{1}{2} g_{ab} \mathcal{R} = \frac{8\pi G}{c^4} T_{ab} \tag{1.1}$$

In this equation, $\mathcal{R}_{ab}$ is the Ricci tensor and $\mathcal{R}$ is the curvature of the space-time manifold, g is the metric tensor and T_{ab} is the energy-momentum tensor. I will provide more information about these tensors in the following section. In particular, as Einstein's description of Mach's principle, matter tells space how to curve and space tells matter how to move. Here "matter" is understood more generally through T_{ab} as a combination of matter, energy, and pressure densities. When there is no matter present, i.e. $T_{ab} = 0$, (1.1) reduces to *vacuum field equations* with zero curvature constant, $\mathcal{R} = 0$.

Later, Einstein introduced a cosmological constant Λ to find a static universe solution.

$$\mathcal{R}_{ab} - \frac{1}{2} g_{ab} \mathcal{R} + \Lambda g_{ab} = \frac{8\pi G}{c^4} T_{ab} \tag{1.2}$$

Besides giving a static, but unstable, universe, Λ also defines a vacuum pressure. However, when Hubble found out our universe is not static, but expanding, Einstein abandoned the cosmological constant and called it " the biggest blunder of his life". Though, cosmological constant was reconsidered again when it is realized that the universe is not only expanding,but also this expansion is accelerating. It is worth to mention the *Lambda-CDM Model*, regarded as the standard model for big-bang cosmology, is led by adding the cosmological constant to FLRW (Friedman-Lemaître-Robertson-Walker) metric describing a simply-connected, homogeneous, isotropic, and expanding universe. With cosmological constant, (1.2) for vacuum gives

$$\mathcal{R} = \frac{2D}{D-2}$$

in D-dimensions

Einstein's field equations (1.1) are nonlinear differential equations unlike many famous equations in physics, like Maxwell's equations or Schrödinger equation. It is very hard to find exact solutions to nonlinear equations like (1.1). Usually linearization methods are used to find solutions to (1.1). At this point, I should say that 'finding a solution to (1.1)' means finding the metric g that defines the space-time manifold for a given matter distribution T_{ab}. Also note that in the *weak-field approximation* (small curvature, slow motion limit), Einstein's field equations (1.1) give the Newtonian limit as required.

GR has passed many tests since Einstein heuristically proposed it. For example, it calculates the perihelion precession of Mercury perfectly. Furthermore, GR showed that calculations based on Newton's laws gives only the half of the value for the deflection of light by sun, which was proved in 1919 by Eddington. Another prediction of GR was the gravitational redshift, which was precisely tested and accounted for by Pound-Rebka experiment conducted in 1959.

Problems of GR

Although GR passed these tests, it carries its own limitations. In 1916, Schwarzschild found an exact solution to (1.1) with a spatial singularity, namely a black hole, called the *Schwarzschild metric*. GR allowed space-time singularities, producing black holes, big-bang, or big-crunch. These singularities may damage the causal structure of space-time, and hence the basic principles of GR. In 1969, Roger Penrose conjectured the *cosmic censorship hypothesis*, which states that any singularity, except the big-bang, is hidden away from the rest of the universe by an event horizon. This secures the causal structure, but GR still doesn't work near the singularity. Also this conjecture is not fully developed, having counterexamples.

Another problem with GR arose after the formulation of quantum theory in 1920s by Heisenberg, Schrödinger, Dirac, and others. Quantum field theory (QFT) aims to quantize the fundamental interactions (electromagnetic, weak, strong, gravitational) in order to explain how they behave at very small scales (Planck scale). QFT was very successful with three fundamental interactions rather than gravitation. We are still lack a well-defined theory of *quantum gravity*. The obscurity in defining quantum gravity is due to the fact that gravitation is nonrenormalizable. Compared to other quantum theories, one predicts a mediator particle to mediate the gravitational force, like photons mediate EM, gluons mediate strong interactions, and W and Z bosons mediate weak interaction. Elementary particle that mediates gravitation is predicted to be a massless, spin-2 particle called *graviton*. QFT collapses if one tries to calculate Feynman diagrams for two or more gravitons; ultraviolet divergences appear and they cannot be removed by renormalization since GR is not renormalizable.

Higher Order Modifications to GR

Long before the nonrenormalizability of GR was showed (by 't Hooft, and Veltman in 1975), scientists tried to modify Einstein's theory. Hermann Weyl, in 1919, in order to unify gravitation and electromagnetism, introduced a Lagrangian which is a polynomial function of the curvature and conformally invariant. Though the theory was not successful, Weyl introduced the important concepts of gauge and *gauge theory*. As it turned out 4D GR is not gauge-invariant, rendering conformal invariance unattainable for Weyl [2].

Later, the curvature^2 (i.e. fourth order in derivatives of the metric tensor) theories drove attraction. In 1947, Gregory showed that a purely quadratic Lagrangian of the form $L = a\mathcal{R}^2 + b\mathcal{R}_{ab}\mathcal{R}^{ab}$ is scale invariant, but doesn't give the Newtonian limit. [17] This problem was solved in the same paper by adding the usual Einstein-Hilbert term $\kappa\mathcal{R}$ to the quadratic Lagrangian breaking the scale invariance. The desired Newtonian limit was recovered up to an exponentially small term [17].

Several decades later, Utiyama and DeWitt showed that higher-order curvature terms (i.e. higher order derivatives of metric in the field equations) in the Lagrangian are necessary in order the theory to be one-loop renormalizable [14]. In his 1977 paper, Stelle showed that 4D fourth-order gravity is renormalizable [15]. However, the theory has massive gravitons with negative kinetic energy (ghosts). If one redefines the theory such that massive gravitons would have positive-definite energies, then their norm becomes negative. This negative norm cannot be eliminated unless the unitarity of the S-matrix is destroyed [15]. Thus 4D fourth order gravity is renormalizable but not unitary. This is not good enough for the quantum theory of gravity.

GR In (2+1) Dimensions

Problems of reconciling unitarity and renormalizability in gravity theories seems to be solved in lower dimensions. It is obvious that calculations for GR will be easier in 3D. Pure GR theory in 3D has trivial solutions and no propagating degrees of freedom. So it seems we have nothing to gain. However, in 1982, Deser, Jackiw, and Templeton showed that when we add a Chern-Simons term to usual Einstein-Hilbert term (with a wrong sign) in the Lagrangian, the theory has massive spin-2 particles with helicities ± 2 [7]. This theory is called *Topologically Massive Gravity* (TMG), and its Lagrangian is

$$L = \frac{1}{2\kappa^2}\mathcal{R} * 1 + \frac{1}{4\mu}\left(\omega^a{}_b \wedge d\omega^b{}_a + \frac{2}{3}\omega^a{}_c \wedge \omega^c{}_b \wedge \omega^b{}_a\right) \tag{1.3}$$

Here μ is the mass parameter. Vacuum field equations (will be derived in the next chapter) are

$$\frac{1}{\kappa^2}G_a + \frac{1}{\mu}C_a = 0 \tag{1.4}$$

where

$$C_a = D\left(Ric_a - \frac{1}{4}\mathcal{R}e_a\right) \tag{1.5}$$

is the Cotton tensor. Field equations are third order in derivatives of the metric, hence more manageable compared to fourth order 4D gravity. There are no acausalities and ghosts in this theory, so that it is unitary and also superrenormalizable. Topological Chern-Simons term is gauge invariant, but parity-violating. TMG was also shown to have black hole solutions (known as BTZ black holes) [11] which is very important to show that the theory is non-trivial. With these properties, TMG seemed to be a good tool to understand quantum gravity in 4D.

Recently, a new theory of massive gravity (NMG) in 3D is proposed by Bergshoeff, Hohm and Townsend [3]. They add a particular fourth order to (wrong-sign) Einstein-Hilbert term in the gravitational Lagrangian.

$$L = \frac{1}{2\kappa^2}\mathcal{R} * 1 + \frac{1}{2m^2}\left(Ric_a \wedge *Ric^a - \frac{3}{8}\mathcal{R}^2 * 1\right) \tag{1.6}$$

where m is a mass parameter analogous to μ in TMG. Field equations (which will be derived in the 3rd chapter) are fourth order in the derivatives of the metric. The theory is shown to be unitary and renormalizable. NMG addition to EH Lagrangian is gauge invariant, and also parity-preserving. Gravitons in the theory are massive and have two polarization states both with helicities ± 2.

With these two massive gravity theories at hand, 3D gravity is a useful tool to investigate further into quantum gravity. In order to do that, we should first understand what other properties these theories has. Though in 3D, there are very few exact solutions to TMG, and NMG. In this thesis, I will explain some of these exact solutions of TMG and NMG.

1.1 Mathematical Preliminaries

In General Relativity, space-time is defined to be a differentiable manifold equipped with a (pseudo)-Riemannian metric and connections found uniquely from Cartan structure equations, shown as $(\mathcal{M}, g, \nabla)$. It is essential to learn the differential structure of space-time, and the algebra defined on the space-time manifold, *exterior algebra.* In this section I will give a brief introduction to the topological structure of the manifolds, and exterior algebra. For further reading, one can consult to [1].

Definition 1 For a topological space $\mathcal{M}$, a *chart* is a homeomorphism $\varphi : U \to \mathbb{R}^n$ where $U \subseteq \mathcal{M}$ is an open subset.The pair (U, φ) is called a chart.

For two charts (U_i, φ_i) and (U_j, φ_j), the function $\varphi_j \circ \varphi_i^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ is called the *transition function* and it defines a coordinate transformation.

The collection of charts $\bigcup_i (U_i, \varphi_i)$, where $\mathcal{M} = \bigcup_i U_i$, such that the transition functions $\varphi_j \circ \varphi_i^{-1}$ are continuous is called an *atlas.*

Definition 2 An *n-dimensional manifold* $\mathcal{M}$ is a topological space equipped with an atlas. So that locally $\mathcal{M}$ resembles $\mathbb{R}^n$.

If the first derivatives of the transition functions exist, $\mathcal{M}$ is called a C^1-manifold or a differentiable manifold. If all derivatives exist, $\mathcal{M}$ is called a C^∞-manifold or a smooth manifold. From here on, $\mathcal{M}$ will always represent an n- dimensional differentiable manifold unless indicated otherwise.

Boundary of $\mathcal{M}$, if it exists, is an (n-1) dimensional manifold and is shown by $\partial\mathcal{M}$. Note that boundary of a boundary is always empty, i.e. it doesn't exist.

Definition 3 Space of all tangent vectors at a point $p \in \mathcal{M}$ is called the *tangent space* and shown as $T_p\mathcal{M}$. It is an n-dimensional vector space. The collection of tangent spaces at each point $p \in \mathcal{M}$ is called the *tangent bundle* $T\mathcal{M} = \bigcup_{p\in\mathcal{M}} T_p\mathcal{M}$. Tangent bundle is a 2n-dimensioanl vector space. An element $X \in T\mathcal{M}$ is called a *vector field*. Vector fields are contravariant quantities.

Definition 4 $T_p^*\mathcal{M}$ is called the *cotangent space* at point $p \in \mathcal{M}$. It is the dual vector space to $T_p\mathcal{M}$. Cotangent space is also an n-dimensional vector space. *Cotangent bundle* is defined similarly as $T^*\mathcal{M} = \bigcup_{p\in\mathcal{M}} T_p^*\mathcal{M}$. $T^*\mathcal{M}$ is a 2n-dimensional vector space. An element $\omega \in T^*\mathcal{M}$ is called a *covector field* or a *1-form*. It is a covariant quantity. By definition $\omega(X) \in \mathbb{R}$.

Suppose $\{x^k\}, k = 1, 2, ..., n$, is a local coordinate chart near a point p. Then $\{\partial_k\} = \{\frac{\partial}{\partial x^k}\}$ is a natural basis for $T\mathcal{M}$. This means that $\forall X \in T\mathcal{M}$ we can write the expansion

$$X = \xi^k(x)\partial_k.$$

Here Einstein summation convention is understood for repeated covariant and contravariant indices. ξ^k's are called the *coordinate components* of a vector field.

The dual basis $\{dx^l\}$ is a natural basis for $T^*\mathcal{M}$, i.e.

$$dx^l(p)\partial_k(p) = \delta^l{}_k$$

where $\delta^i{}_j$ is defined as

$$\delta^i{}_j = \begin{cases} 1, & \text{if } i \neq j \\ 0, & \text{if } i = j \ . \end{cases}$$

An expansion of a covector would be

$$\omega = \eta_l(x)dx^l$$

The choice of a basis for a vector field is not unique. We can take any set of n linearly independent vector fields $\{X_a\}, a = 1, ..., n$. This is is called a *frame*. These frame vectors can be expanded in natural basis as $X_a = E^k_a\partial_k$. $\{E^k_a\}$ is called the *inverse n-bein*. Dual basis of $T^*\mathcal{M}$ induced by $\{X_a\}$ is $\{e^a\}$ such that $e^aX_b = \delta^a{}_b$. $\{e^a\}$ is called the *coframe*. Similarly they can be expanded as $e^a = e^a_k dx^k$. $\{e^a_k\}$ is called the *n-bein*.

Definition 5 A *tensor of type (p,q)* is a multilinear map

$$R : \underbrace{T^*\mathcal{M} \otimes \cdots \otimes T^*\mathcal{M}}_{p\ times} \otimes \underbrace{T\mathcal{M} \otimes \cdots \otimes T\mathcal{M}}_{q\ times} \to \mathbb{R}$$

p is the contravariant degree and q is the covariant degree of the tensor R. In particular, (1,0) tensors are vectors, (0,1) tensors are covectors, and, by definition, (0,0) tensors are functions. Let us define the Levi-Civita tensor for we will use it very much:

$$\epsilon_{i_1 \dots i_n} = \begin{cases} +1, & \text{if } i_1 \dots i_n \text{ is an even permutation} \\ -1, & \text{if } i_1 \dots i_n \text{ is an odd permutation .} \\ 0 & \text{if there is repeated indices} \end{cases}$$

Tensors are indexed according to their contravariant (upper) and covariant (lower) degree.

$$\begin{aligned} R : (dx^{i_1}, \dots, dx^{i_p}, \partial_{j_1}, \dots, \partial_{j_q}) \rightarrow & R(dx^{i_1}, \dots, dx^{i_p}, \partial_{j_1}, \dots, \partial_{j_q}) \in \mathbb{R} \\ & = R^{i_1 \dots i_p}{}_{j_1 \dots j_q} \in \mathbb{R} \end{aligned}$$

$R^{i_1 \dots i_p}{}_{j_1 \dots j_q}$ is coordinate components of the tensor. We can raise or lower indices using a metric tensor.

Definition 6 A *metric* g is a (0,2) type tensor with the following properties:

1. It is non-degenerate, i.e. for

$$g(X_a, X_b) = g_{ab},$$

   which is an nxn matrix, $Det(g_{ab}) \neq 0$

2. It is symmetric, i.e. $\forall X, Y \in T\mathcal{M}$, $g(X,Y) = g(Y,X)$.

A *Riemannian manifold* is a smooth manifold equipped with a positive-definite metric tensor. For a *pseudo-Riemannian manifold*, the metric need not to be positive-definite. Lorentzian manifolds are an example of a pseudo-Riemannian manifold with the metric signature $(n-1,1)$.

Exterior Product

Let $\{e^i\}, i = 1, \dots, n$, be a basis for $T^*\mathcal{M}$. We know that e^i's are (0,1) type tensors called 1-forms. The *wedge (exterior) product* is the completely antisymmetric tensor product such that

$$e^i \wedge e^j = -e^j \wedge e^i := e^i \otimes e^j - e^j \otimes e^i \tag{1.7}$$

A *p-form* $\omega \in \Omega^p(\mathcal{M})$ on $\mathcal{M}$ is the wedge product of p 1-forms, It is a (0,p) type tensor field. $\Omega^p(\mathcal{M})$ is the space of all p-forms.

Properties of wedge product

For $\omega \in \Omega^p(\mathcal{M}), \mu \in \Omega^q(\mathcal{M}), \nu \in \Omega^s(\mathcal{M})$, wedge product is

(i) associative,

$$\omega \wedge (\mu \wedge \nu) = (\omega \wedge \mu) \wedge \nu$$

(ii) bilinear,

$$(a\omega + b\mu) \wedge \nu = a\omega \wedge \nu + b\mu \wedge \nu \quad \text{for} \quad a, b \in \mathbb{R}$$

(iii) graded commutative,

$$\omega \wedge \mu = (-1)^{pq} \mu \wedge \omega$$

By definition, $\Omega^0(\mathcal{M}) = C^\infty(\mathcal{M})$ and $\Omega^p(\mathcal{M}) = \emptyset$ for $p > n$. Any p-form, $0 \leq p \leq n$, can be written as

$$\omega = \frac{1}{p!} \omega_{i_1 \ldots i_p} e^{i_1} \wedge e^{i_2} \wedge \cdots \wedge e^{i_p} \tag{1.8}$$

Exterior Derivative

Exterior derivative of a p-form on $\mathcal{M}$ is a linear map $d : \Omega^p(\mathcal{M}) \to \Omega^{p+1}(\mathcal{M})$ satisfying the following properties

(i) $d(\omega + \mu) = d\omega + d\mu$ for $\omega, \mu \in \Omega^p(\mathcal{M})$

$d(c\omega) = c\ d\omega$ for $c \in \mathbb{R}$

(ii) Graded Leibniz rule:

$$d(\omega \wedge \mu) = d\omega \wedge \mu + (-1)^p \omega \wedge d\mu$$

for $\omega \in \Omega^p(\mathcal{M})$

(iii) Poincaré lemma: $d(d\omega) = d^2\omega = 0$ for any p-form.

For a p-form given in (1.8), its exterior derivative is

$$d\omega = \frac{1}{p!} \frac{\partial}{\partial x^j} \omega_{i_1 \ldots i_p} dx^j \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p} \tag{1.9}$$

in natural basis $\{dx^i\}$.

A p-form ω is called *closed* if $d\omega = 0$. It is called *exact* if there exists a (p-1)-form α such that $\omega = d\alpha$. It is clear that every exact form is closed, but converse is not always true.

Interior Product (Contraction)

Let $X \in T\mathcal{M}$ be a vector field on $\mathcal{M}$. The interior product with respect to X is the linear map $\imath_X : \Omega^p(\mathcal{M}) \to \Omega^{p-1}(\mathcal{M})$ such that for a given p-form ω on $\mathcal{M}$, the contraction of ω with X is

$$\imath_X \omega := \frac{1}{p!} \sum_{k=1}^{p} X^{i_k} \omega_{i_1 \ldots i_k \ldots i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_{k-1}} \wedge dx^{i_{k+1}} \wedge \cdots \wedge dx^{i_p} \tag{1.10}$$

Our notation will be such that, if X_a is a frame vector, we write $\imath_a$ instead of $\imath_{X_a}$.

The interior product satisfies

(i) $\imath_X(\omega \wedge \mu) = \imath_X \omega \wedge \mu + (-1)^p \omega \wedge \imath_X \mu$ for a p-form ω and a q-form μ.

(ii) $\imath_X(\imath_X \omega) = (\imath_X)^2 \omega = 0$

These rules make the interior product, like the exterior product, a graded derivation, namely an antiderivation.

Lie Derivative

Lie derivative of a p-form $\omega \in \Omega^p(\mathcal{M})$ with respect to a vector field $X \in T\mathcal{M}$ is a type-preserving linear map $\mathcal{L}_X : \Omega^p(\mathcal{M}) \to \Omega^p(\mathcal{M})$ given by the Cartan's formula:

$$\mathcal{L}_X \omega = d\imath_X \omega + \imath_X d\omega \tag{1.11}$$

It can be interpreted as the rate of change of ω along the flow generated by the vector field X. It has the following properties:

(i) $\mathcal{L}_X(\omega + \mu) = \mathcal{L}_X \omega + \mathcal{L}_X \mu$

(ii) $\mathcal{L}_X(\omega \wedge \mu) = (\mathcal{L}_X \omega) \wedge \mu + \omega \wedge (\mathcal{L}_X \mu)$

Hodge Star Map

The Hodge star map is an isomorphism $* : \Omega^p(\mathcal{M}) \to \Omega^{n-p}(\mathcal{M})$. For a given p-form ω, its Hodge dual $*\omega$ is defined as

$$*\omega := \frac{1}{p!(n-p)!} \omega_{i_1 \ldots i_p} \epsilon^{i_1 \ldots i_p}{}_{j_1 \ldots j_{n-p}} e^{j_1} \wedge \cdots \wedge e^{j_{n-p}} \tag{1.12}$$

It has the following properties

(i) $** \, \omega = (-1)^{p(n-p)} s \, \omega$ where s is the sign of the determinant of the metric.

(ii) $*1 = \frac{1}{n!}\epsilon_{i_1 \dots i_n} e^{i_1} \wedge \cdots \wedge e^{i_n}$. This is called the *volume element* on $\mathcal{M}$. The orientation on $\mathcal{M}$ is fixed by this volume form.

(iii) $\omega \wedge *\mu = \mu \wedge *\omega$ for two p-forms ω and μ.

The *codifferential* is a linear map $\delta \equiv *d* : \Omega^p(\mathcal{M}) \to \Omega^{p-1}(\mathcal{M})$. Through this we can define the degree-preserving, second order, linear differential operator

$$\triangle := d\delta + \delta d \tag{1.13}$$

called the *Laplace-Beltrami operator*.

Integration of p-forms

Let $\mathcal{M}$ be an n-dimensional manifold with boundary, and ω be an n-form on $\mathcal{M}$. Then we have a chart $\varphi_i : U_i \to \mathbb{R}^n$. Note that, we can always find a partition of unity $\{f_i\} \in C^\infty(\mathcal{M})$ subordinate to U_i on $\mathcal{M}$, such that we can write

$$\omega = \sum_i f_i \omega$$

where $f_i \omega$ vanishes outside of U_i. So we have

$$f_i \omega = h_i(x^1, \dots, x^n) dx^1 \wedge \cdots \wedge dx^n$$

where $\{x^i\}$ are local coordinates given by the chart φ_i, and the function h_i is zero outside of U_i. Now we can write

$$\int_{\mathcal{M}} \omega = \sum_i \int_{U_i} h_i(x^1, \dots, x^n) dx^1 \wedge \cdots \wedge dx^n \tag{1.14}$$

Stoke's Theorem

Let ω be an (n-1)-form on $\mathcal{M}$ with a compact support. Then

$$\int_{\mathcal{M}} d\omega = \int_{\partial \mathcal{M}} \omega \tag{1.15}$$

Connection ∇

A *connection* on a differentiable manifold $\mathcal{M}$ is a rule for parallel transport of tensor fields. There are equivalent ways of defining a connection, like a covariant derivative operator, or through Christoffel symbols in the case of a Levi-Civita connection, which will be defined in the following section. Christoffel symbols, which are affine connection coefficients, $\{\Gamma^a_{bc}\}$ are defined as

$$\nabla_{X_a} X_b = \Gamma^c_{ab} X_c \tag{1.16}$$

A metric compatible connection is defined as

$$\nabla_X g(Y,Z) = X(g(Y,Z)) + g(\nabla_X Y, Z) + g(Y, \nabla_X Z) = 0 \tag{1.17}$$

where $X, Y, Z \in T\mathcal{M}$.

I will use *connection 1-forms* $\{\omega^a{}_b\}$ to define a connection, actually a Levi-Civita connection, such that

$$\omega^a{}_b(X_c) = \Gamma^a_{bc} \tag{1.18}$$

We can write *covariant derivative* D in terms of connection 1-forms as follows

$$De^a = de^a + \omega^a{}_b \wedge e^b = T^a \tag{1.19}$$

where T^a are called *torsion 2-forms*.

Levi-Civita Connection

A torsion free, metric compatible connection ∇ is uniquely determined by g, from the 1^{st} Cartan structure equation

$$de^a + \omega^a{}_b \wedge e^b = 0 \tag{1.20}$$

and called the *Levi-Civita connection*. A smooth manifold $\mathcal{M}$, with a Riemannian metric g, and a Levi-Civita connection ∇, $(\mathcal{M}, g, \nabla)$, is called a *Riemannian manifold*. If the metric g is Lorentzian, it is called a *pseudo-Riemannian manifold*.

Curvature

Curvature 2-forms, $R^a{}_b$, can be computed from 2^{nd} Cartan structure equation

$$D\omega^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b = R^a{}_b \tag{1.21}$$

A space-time is called *flat* if $R^a{}_b = 0$. Minkowski space-time is flat.

We can also write

$$R^a{}_b = \frac{1}{2} R^a{}_{b,cd} e^c \wedge e^d \tag{1.22}$$

where $R^a{}_{b,cd}$ is the *Riemann tensor* of type (1,3). *Ricci tensor* $\mathcal{R}_{ab}$ is the contraction of the Riemann tensor

$$\mathcal{R}_{bc} = R^a{}_{b,ac} \tag{1.23}$$

Further contraction of Ricci tensor gives the *curvature scalar* $\mathcal{R}$:

$$\mathcal{R} = \mathcal{R}^a{}_a \tag{1.24}$$

Einstein Field Equations

Einstein's field equations, formulated by Albert Einstein in 1916, relates the curvature of the space-time and the matter distribution of this space-time. The equation is

$$G_{ab} = -\frac{8\pi G}{c^4} T_{ab} \tag{1.25}$$

where G_{ab} is the Einstein tensor and T_{ab} is the energy-momentum tensor. Einstein tensor is given in terms of Ricci tensor and curvature scalar as follows:

$$G_{ab} = \mathcal{R}_{ab} - \frac{1}{2} \eta_{ab} \mathcal{R} \tag{1.26}$$

So that $DG_{ab} = 0$ identically, which gives $DT_{ab} = 0$. This is the energy-momentum conservation.

Chapter 2

TOPOLOGICALLY MASSIVE GRAVITY

As discussed before, it is reasonable to think that studying 3D gravity will give important insights about 4D quantum gravity. 3D gravity has some significant properties, such as pure GR in 3D has no propagating degrees of freedom and no Newtonian limit. Riemann tensor is proportional to Ricci tensor in 3D, and hence pure Einstein theory admits only flat space as a solution. However, if a gauge-invariant, parity-violating topological Chern-Simons term is added to the EH term in the action, (1.3), we obtain a dynamical theory of massive gravitons with single polarization having helicities ± 2 [7]. The gravitons have causal propagation, and for a special value of μ, there are no ghosts. Hence the theory is unitary as well as renormalizable.

Pure GR in 3D has no black hole solutions. However, in 1992, Banados, Teitelboim, and Zanelli showed that in the presence of a negative cosmological constant, (2+1) dimensional GR has black holes [11]. These black holes, named BTZ black holes, have inner and outer horizons, and no hair. Black hole solutions are very important in the discussion of quantum gravity, since defining the entropy of a black hole is a crucial problem in quantum gravity. TMG has also AdS and black hole solutions [19].

Thus, TMG suggests a safe and solid ground to study quantum gravity. However, there are very few known exact solutions of TMG. Chow, Pope, and Sezgin showed in [19] that, almost all known solutions of TMG belong to one of the following metrics:

(i) Timelike-squashed AdS_3

(ii) Spacelike-squashed AdS_3

(iii) AdS pp-waves

They also found Kundt solutions of TMG in [10].

2.1 TMG Field Equations

Let us remember TMG Lagrangian without cosmological constant. We include also Lagrange multiplier one-forms λ_a in order to variate the Lagrangian with the torsion-free constraint [6].

$$L = \frac{1}{2\kappa^2} R_{ab} \wedge *(e^a \wedge e^b) + \frac{1}{4\mu}\left(\omega^a{}_b \wedge d\omega^b{}_a + \frac{2}{3}\omega^a{}_c \wedge \omega^c{}_b \wedge \omega^b{}_a\right) + (de^a + \omega^a{}_b \wedge e^b) \wedge \lambda_a \tag{2.1}$$

$$L = L_{EH} + L_{CS} + (\text{Lagrange multiplier}) \tag{2.2}$$

The last term in this Lagrangian is torsion wedged with Lagrange multiplier one-forms λ_a. From now on I will use the notation $e^{ab} = e^a \wedge e^b$, etc.

Einstein field equations can be derived from a minimum action principle from the action

$$S[e^a] = \int_{\mathcal{M}} L[e^a, \omega^a{}_b] \tag{2.3}$$

where L is the Lagrangian density 3-form, and $\mathcal{M}$ is a manifold. Action principle says that $\delta S = 0$. If we take $\mathcal{M}$ with no boundary, we get

$$\delta S = \int_{\mathcal{M}} \delta L + \int_{\partial\mathcal{M}} L = \int_{\mathcal{M}} \delta L \tag{2.4}$$

This shows that minimizing the action with δS=0 is equivalent to $\delta L = 0$.Thus we should find the coframe variations of our Lagrangian.

$$\begin{aligned}\delta L = &\frac{1}{2\kappa^2}(\delta R_{ab} \wedge *e^{ab} + R_{ab} \wedge \delta * e^{ab}) \\ &+ \frac{1}{4\mu}(\delta\omega^a{}_b \wedge d\omega^b{}_a + \omega^a{}_b \wedge d\delta\omega^b{}_a + 2\delta\omega^a{}_c \wedge \omega^c{}_b \wedge \omega^b{}_a) \\ &+ (d\delta e^a + \delta\omega^a{}_b \wedge e^b + \omega^a{}_b \wedge \delta e^b) \wedge \lambda_a\end{aligned} \tag{2.5}$$

Now I will look into each term separately in this variation.

$$\delta R_{ab} = D\delta\omega_{ab} \tag{2.6}$$

$$\delta R_{ab} \wedge *e^{ab} = D\delta\omega_{ab} \wedge *e^{ab} \tag{2.7}$$

$$= \underbrace{d(\delta\omega_{ab} \wedge *e^{ab})}_{\text{integrates to 0}} + \delta\omega_{ab} \wedge D * e^{ab} \tag{2.8}$$

$$= \delta\omega_{ab} \wedge D * e^{ab} \tag{2.9}$$

$$= 0 \tag{2.10}$$

Last step follows from the fact that in 3D, $*e^{ab}$ is a one-form and we have torsion free connection. On the other hand,

$$\delta * e^{ab} = \delta(\epsilon^{ab}{}_c e^c) = \epsilon^{ab}{}_c \delta e^c \tag{2.11}$$

So,

$$R_{ab} \wedge \delta * e^{ab} = \epsilon^{ab}{}_c R_{ab} \wedge \delta e^c \tag{2.12}$$

$$= \delta e^c \wedge \epsilon_{abc} R^{ab} \tag{2.13}$$

The next term

$$\omega^a{}_b \wedge d\delta\omega^b{}_a = -\underbrace{d(\omega^a{}_b \wedge \delta\omega^b{}_a)}_{\text{integrates to 0}} + d\omega^a{}_b \wedge \delta\omega^b{}_a \tag{2.14}$$

$$= \delta\omega^b{}_a \wedge d\omega^a{}_b \tag{2.15}$$

And also

$$\delta\omega^a{}_b \wedge d\omega^b{}_a + \delta\omega^b{}_a \wedge d\omega^a{}_b = 2\delta\omega^a{}_b \wedge d\omega^b{}_a \tag{2.16}$$

Using

$$d\delta e^a \wedge \lambda_a = d(\delta e^a \wedge \lambda_a) + \delta e^a \wedge d\lambda_a = \delta e^a \wedge d\lambda_a \tag{2.17}$$

we get

$$d\delta e^a \wedge \lambda_a + \delta\omega^a{}_b \wedge e^b \wedge \lambda_a + \omega^a{}_b \wedge \delta e^b \wedge \lambda_a \tag{2.18}$$

$$= \delta e^a \wedge d\lambda_a - \delta e^a \wedge \omega^b{}_a \wedge \lambda_b + \delta\omega^a{}_b \wedge e^b \wedge \lambda_a \tag{2.19}$$

$$= \delta e^a \wedge \underbrace{(d\lambda_a - \omega^b{}_a \wedge \lambda_b)}_{D\lambda_a} + \delta\omega^a{}_b \wedge \left(\frac{1}{2}(e^b \wedge \lambda_a - e^a \wedge \lambda_b)\right) \tag{2.20}$$

$$= \delta e^a \wedge D\lambda_a + \delta\omega^a{}_b \wedge \left(\frac{1}{2}(e^b \wedge \lambda_a - e^a \wedge \lambda_b)\right) \tag{2.21}$$

Now we have all we need to write the coframe variation of L:

$$\delta L = \frac{1}{2\kappa^2}\delta e^a \wedge \epsilon_{abc} R^{bc} + \frac{1}{2\mu}\delta\omega^a{}_b \wedge \underbrace{(d\omega^b{}_a + \omega^b{}_c \wedge \omega^c{}_a)}_{R^b{}_a}$$

$$+ \delta e^a \wedge D\lambda_a + \delta\omega^a{}_b \wedge \frac{1}{2}(e^b \wedge \lambda_a - e^a \wedge \lambda_b) \tag{2.22}$$

$$\delta L = \delta e^a \wedge \left(\frac{1}{2\kappa^2}\epsilon_{abc}R^{bc} + D\lambda_a\right) + \delta\omega^a{}_b \wedge \left(\frac{1}{2\mu}R^b{}_a + \frac{1}{2}(e^b \wedge \lambda_a - e^a \wedge \lambda_b)\right) \quad (2.23)$$

In order to minimize the action, we should have $\delta L = 0$. This gives two equations:

$$\frac{1}{2\mu}R^b{}_a + \frac{1}{2}(e^b \wedge \lambda_a - e^a \wedge \lambda_b) = 0 \quad (2.24)$$

$$\frac{1}{2\kappa^2}\epsilon_{abc}R^{bc} + D\lambda_a = 0 \quad (2.25)$$

We should first solve (2.24) for λ_a, and then insert them into (2.25). Thus, solve (2.24):

$$i_a/ \quad e^a \wedge \lambda^b - e^b \wedge \lambda^a = \frac{1}{\mu}R^{ba} \quad (2.26)$$

$$3\lambda^b - \underbrace{e^a \wedge (i_a\lambda^b)}_{\lambda^b} - \lambda^b + e^b \wedge (i_a\lambda^a) = \frac{1}{\mu}i_aR^{ba} \quad (2.27)$$

$$i_b/ \quad \lambda^b + e^b(i_a\lambda^a) = -\frac{1}{\mu}Ric^a \quad (2.28)$$

$$i_b\lambda^b + 3i_a\lambda^a = -\frac{1}{\mu}\mathcal{R} \quad (2.29)$$

$$i_a\lambda^a = -\frac{1}{4\mu}\mathcal{R} \quad (2.30)$$

$$\Rightarrow \lambda^a = -\frac{1}{\mu}\left(Ric^a - \frac{\mathcal{R}}{4}e^a\right) \quad (2.31)$$

In the field equations (2.25), we have covariant derivative of λ_a, which is

$$D\lambda^a = -\frac{1}{\mu}D\left(Ric^a - \frac{\mathcal{R}}{4}e^a\right) \quad (2.32)$$

$$D\lambda^a = -\frac{1}{\mu}DY^a = -\frac{1}{\mu}C^a \quad (2.33)$$

Y^a is called the Schouten tensor and C^a is the Cotton tensor. In (2+1)-dimensions Weyl tensor vanishes identically. Cotton tensor gives the conformal properties in (2+1)-dimensions. If it vanishes, the metric is conformally flat. Cotton tensor is (i) symmetric, (ii) traceless, (iii) covariantly constant.

The field equations (**??**) can be written in terms of the Cotton tensor as:

$$\frac{1}{2\kappa^2}\epsilon_{abc}R^{bc} - \frac{1}{\mu}C_a = 0 \quad (2.34)$$

$$\frac{1}{\kappa^2}G_a + \frac{1}{\mu}C_a = 0 \quad (2.35)$$

where

$$G_a = -\frac{1}{2}\epsilon_{abc}R^{bc} \tag{2.36}$$

is the Einstein tensor in (2+1)-dimensions.

2.2 Null Frame in (2+1)-dimensions

A Lorentzian metric is written as

$$g = \eta_{\alpha\beta}e^{\alpha} \otimes e^{\beta} \quad \alpha, \beta = 0, 1, 2 \tag{2.37}$$

where $\eta_{\alpha\beta} = diag(-1, 1, 1)$ is the diagonal metric tensor. Here we are using the mostly plus signature, (-++). Then metric is

$$g = -e^0 \otimes e^0 + e^1 \otimes e^1 + e^2 \otimes e^2 \tag{2.38}$$

We now change to null coordinates defined by

$$\left\{\ell = \frac{e^1 + e^0}{\sqrt{2}}, \quad n = \frac{e^1 - e^0}{\sqrt{2}}, \quad e^2\right\} \tag{2.39}$$

We use a null basis, because we will study wave solutions of TMG an NMG, and these metrics admit null Killing vector fields. Thus, in the null basis calculations become easier.

In the null basis, the metric in (2.37) becomes

$$g = \ell \otimes n + n \otimes \ell + e^2 \otimes e^2 \tag{2.40}$$

Interior products become

$$i_\ell = \frac{i_1 + i_0}{\sqrt{2}}, \quad i_n = \frac{i_1 - i_0}{\sqrt{2}}, \quad i_2 \tag{2.41}$$

satisfying

$$i_\ell \ell = i_n n = 0, \ i_2 e^2 = 1$$

and

$$i_\ell n = i_n \ell = 0$$

Hodge duals of these null coordinates are

$$* e^2 = \epsilon^2{}_{01} e^{01} = \epsilon_{201} e^0 \wedge e^1 = \frac{1}{2}(\ell - n) \wedge (\ell + n) = \ell \wedge n \tag{2.42a}$$

$$* \ell = \frac{*e^1 + *e^0}{\sqrt{2}} = \frac{1}{\sqrt{2}}(\epsilon^1{}_{20} e^{20} + \epsilon^0{}_{12} e^{12}) = \frac{1}{\sqrt{2}}(e^2 \wedge e^0 - e^1 \wedge e^2) = -\ell \wedge e^2 \tag{2.42b}$$

$$* n = \frac{*e^1 - *e^0}{\sqrt{2}} = \frac{1}{\sqrt{2}}(\epsilon^1{}_{20} e^{20} - \epsilon^0{}_{12} e^{12}) = \frac{1}{\sqrt{2}}(e^2 \wedge e^0 + e^1 \wedge e^2) = n \wedge e^2 \tag{2.42c}$$

$$* 1 = e^0 \wedge e^1 \wedge e^2 = \frac{\ell - n}{\sqrt{2}} \wedge \frac{\ell + n}{\sqrt{2}} \wedge e^2 = \ell \wedge n \wedge e^2 \tag{2.42d}$$

In order to write structure equations in a compact form, we also redefine connection one-forms and curvature 2-forms in the following way

$$\omega_0 = \omega^0{}_1, \quad \omega_+ = \frac{\omega^0{}_2 + \omega^1{}_2}{\sqrt{2}}, \quad \omega_- = \frac{\omega^0{}_2 - \omega^1{}_2}{\sqrt{2}} \tag{2.43a}$$

$$R_0 = R^0{}_1, \quad R_+ = \frac{R^0{}_2 + R^1{}_2}{\sqrt{2}}, \quad R_- = \frac{R^0{}_2 - R^1{}_2}{\sqrt{2}} \tag{2.43b}$$

We can also write the curvature scalar in terms of null coordinates and R_0, R_+, R_-:

$$\begin{aligned}
\mathcal{R} * 1 &= R_{ab} \wedge *e^{ab} = 2(R_{01} \wedge *e^{01} + R_{02} \wedge *e^{02} + R_{12} \wedge *e^{12}) \\
&= 2(R^0{}_1 \wedge e^2 - R^0{}_2 \wedge e^1 + R^1{}_2 \wedge e^0) \\
&= 2\left(R_0 \wedge e^2 - \frac{1}{2}(R_+ + R_-) \wedge (\ell + n) + \frac{1}{2}(R_+ - R_-) \wedge (\ell - n)\right)
\end{aligned}$$

$$\mathcal{R} * 1 = 2\left(R_0 \wedge e^2 - R_+ \wedge n - R_- \wedge \ell\right) \tag{2.44}$$

Without torsion, 1st Cartan structure equations in (2+1)-dimensions are as follows:

$$de^0 + \omega^0{}_1 \wedge e^1 + \omega^0{}_2 \wedge e^2 = 0 \tag{2.45a}$$

$$de^1 + \omega^1{}_0 \wedge e^0 + \omega^1{}_2 \wedge e^2 = 0 \tag{2.45b}$$

$$de^2 + \omega^2{}_0 \wedge e^0 + \omega^2{}_1 \wedge e^1 = 0 \tag{2.45c}$$

Let us first add up (2.45b) and (2.45a)

$$d(e^1 + e^0) + \omega^0{}_1 \wedge (e^0 + e^1) + (\omega^1{}_2 + \omega^0{}_2) \wedge e^2 = 0$$

$$d\left(\frac{e^1 + e^0}{\sqrt{2}}\right) + \omega^0{}_1 \wedge \left(\frac{e^0 + e^1}{\sqrt{2}}\right) + \left(\frac{\omega^1{}_2 + \omega^0{}_2}{\sqrt{2}}\right) \wedge e^2 = 0$$

$$d\ell + \omega_0 \wedge \ell + \omega_+ \wedge e^2 = 0 \tag{2.46}$$

Now subtract (2.45a) from (2.45b)

$$
\begin{aligned}
& d(e^1 - e^0) + \omega^0{}_1 \wedge (e^0 - e^1) + (\omega^1{}_2 - \omega^0{}_2) \wedge e^2 = 0 \\
& d\left(\frac{e^1 - e^0}{\sqrt{2}}\right) + \omega^0{}_1 \wedge \left(\frac{e^0 - e^1}{\sqrt{2}}\right) + \left(\frac{\omega^1{}_2 - \omega^0{}_2}{\sqrt{2}}\right) \wedge e^2 = 0 \\
& dn - \omega_0 \wedge n - \omega_- \wedge e^2 = 0 \qquad (2.47)
\end{aligned}
$$

We can write the last equation (2.45c) as follows

$$
\begin{aligned}
& de^2 + \omega^0{}_2 \wedge \left(\frac{\ell - n}{\sqrt{2}}\right) - \omega^1{}_2 \wedge \left(\frac{\ell + n}{\sqrt{2}}\right) = 0 \\
& de^2 + \left(\frac{\omega^0{}_2 - \omega^1{}_2}{\sqrt{2}}\right) \wedge \ell - \left(\frac{\omega^0{}_2 + \omega^1{}_2}{\sqrt{2}}\right) \wedge n = 0 \\
& de^2 + \omega_- \wedge \ell - \omega_+ \wedge n = 0 \qquad (2.48)
\end{aligned}
$$

Thus the following are the first group of Cartan structure equations (2.45) in null basis:

$$d\ell + \omega_0 \wedge \ell + \omega_+ \wedge e^2 = 0 \qquad (2.49a)$$

$$dn - \omega_0 \wedge n - \omega_- \wedge e^2 = 0 \qquad (2.49b)$$

$$de^2 + \omega_- \wedge \ell - \omega_+ \wedge n = 0 \qquad (2.49c)$$

Given a metric, these equations are solved for connection one-forms. After we find the connection one-forms, we can use 2$^{\text{nd}}$ Cartan structure equations to find curvature two-forms. These equations in (2+1)-dimensions are

$$R^0{}_1 = d\omega^0{}_1 + \omega^0{}_2 \wedge \omega^2{}_1 \qquad (2.50a)$$

$$R^0{}_2 = d\omega^0{}_2 + \omega^0{}_1 \wedge \omega^1{}_2 \qquad (2.50b)$$

$$R^1{}_2 = d\omega^1{}_2 + \omega^1{}_0 \wedge \omega^0{}_2 \qquad (2.50c)$$

Using the definitions in (2.43), we can write

$$R_0 = d\omega_0 - \frac{\omega_+ + \omega_-}{\sqrt{2}} \wedge \frac{\omega_+ - \omega_-}{\sqrt{2}} = d\omega_0 - \omega_- \wedge \omega_+ \qquad (2.51)$$

$$R_+ = d\left(\frac{\omega^0{}_2 + \omega^1{}_2}{\sqrt{2}}\right) + \omega_0 \wedge \left(\frac{\omega^0{}_2 + \omega^1{}_2}{\sqrt{2}}\right) = d\omega_+ + \omega_0 \wedge \omega_+ \qquad (2.52)$$

$$R_- = d\left(\frac{\omega^0{}_2 - \omega^1{}_2}{\sqrt{2}}\right) - \omega_0 \wedge \left(\frac{\omega^0{}_2 - \omega^1{}_2}{\sqrt{2}}\right) = d\omega_- - \omega_0 \wedge \omega_- \qquad (2.53)$$

Thus equations (2.50) become

$$R_0 = d\omega_0 + \omega_+ \wedge \omega_- \tag{2.54a}$$

$$R_+ = d\omega_+ + \omega_0 \wedge \omega_+ \tag{2.54b}$$

$$R_- = d\omega_- - \omega_0 \wedge \omega_- \tag{2.54c}$$

Finally we can rewrite the Einstein tensor. Using the fact that in 3D, we have

$$G_a = -\frac{1}{2}\epsilon_{abc}R^{bc}$$

we get

$$G_0 = -R^{12}, \quad G_1 = R^{02}, \quad G_2 = -R^{01} \tag{2.55}$$

Using these we define the following

$$G_- = \frac{G_1 + G_0}{\sqrt{2}} = R_-, \quad G_+ = \frac{G_1 - G_0}{\sqrt{2}} = R_+, \quad G_2 = -R_0 \tag{2.56}$$

2.3 TMG Field Equations In Null Frame

Now we will write the TMG field equations in (2.35) in the null frame. We do this because field equations are simpler to deal with in this form.

We should write the Cotton tensor in terms of the null frame tensors. If we consider (3.6), we have

$$*G_a = -Ric_a + \frac{1}{2}\mathcal{R}e_a \tag{2.57}$$

And also using (2.33), Cotton tensor can be written as

$$C_a = D\left(Ric_a - \frac{\mathcal{R}}{4}e_a\right) = D\left(- * G_a + \frac{\mathcal{R}}{4}e_a\right) = -D * G_a + \frac{1}{4}d\mathcal{R}\wedge e_a \tag{2.58}$$

Explicitly writing we have

$$C_0 = -D * G_0 + \frac{1}{4}d\mathcal{R}\wedge e_0 \tag{2.59a}$$

$$C_1 = -D * G_1 + \frac{1}{4}d\mathcal{R}\wedge e_1 \tag{2.59b}$$

$$C_2 = -D * G_2 + \frac{1}{4}d\mathcal{R}\wedge e_2 \tag{2.59c}$$

Let us define the following

$$C_+ := \frac{C_1 - C_0}{\sqrt{2}} = -D * R_+ + \frac{1}{4} d\mathcal{R} \wedge \ell \tag{2.60a}$$

$$C_- := \frac{C_1 + C_0}{\sqrt{2}} = -D * R_- + \frac{1}{4} d\mathcal{R} \wedge n \tag{2.60b}$$

$$C_2 = D * R_0 + \frac{1}{4} d\mathcal{R} \wedge e^2 \tag{2.60c}$$

TMG field equations (2.35) with a negative cosmological constant are

$$\frac{1}{\kappa^2} G_a + \frac{1}{\mu} C_a + \Lambda * e_a = 0 \tag{2.61}$$

$$\Longrightarrow \quad \frac{1}{\kappa^2} G_0 + \frac{1}{\mu} C_0 + \Lambda * e_0 = 0 \tag{2.62a}$$

$$\frac{1}{\kappa^2} G_1 + \frac{1}{\mu} C_1 + \Lambda * e_1 = 0 \tag{2.62b}$$

$$\frac{1}{\kappa^2} G_2 + \frac{1}{\mu} C_2 + \Lambda * e_2 = 0 \tag{2.62c}$$

If we follow the procedures as before we find that these equations can be written as

$$\frac{1}{\kappa^2} R_+ - \frac{1}{\mu} D * R_+ + \frac{1}{4\mu} d\mathcal{R} \wedge \ell - \Lambda(\ell \wedge e^2) = 0 \tag{2.63a}$$

$$\frac{1}{\kappa^2} R_- - \frac{1}{\mu} D * R_- + \frac{1}{4\mu} d\mathcal{R} \wedge n + \Lambda(n \wedge e^2) = 0 \tag{2.63b}$$

$$-\frac{1}{\kappa^2} R_0 + \frac{1}{\mu} D * R_0 + \frac{1}{4\mu} d\mathcal{R} \wedge e^2 + \Lambda(\ell \wedge n) = 0 \tag{2.63c}$$

2.4 pp-wave Solutions of TMG

The term *pp-wave* stands for plane-fronted waves with parallel propagation. pp-waves are exact solutions of GR, modeling radiation (EM, gravitation, etc) moving with the speed of light. They were first introduced by Brinkman in 1925 [16]. Actually, they are a special case of a more general class of spacetimes called *Kundt spacetimes*. Kundt spacetimes are defined as spacetimes admitting a null geodesic vector field which is expansion free, shear free and twist free. In 3D, a null geodesic vector field is already shear free, and twist free [10]. Here it is worth to note also Robinson-Trautman spacetimes which admits a null geodesic vector field that is expanding, shear-free and twist-free.

Kundt metric in 3D is

$$g = 2H(u, v, \rho) du^2 + 2W(u, v, \rho) du d\rho + 2du dv + d\rho^2 \tag{2.64}$$

pp-waves are the special case of this metric, such that null Killing vector is covariantly constant. pp-wave metric is

$$g = 2H(u,\rho)du^2 + 2dudv + d\rho^2 \tag{2.65}$$

Changing to null coordinates we have

$$\ell = du, \quad n = dv + Hdu, \quad e^2 = d\rho$$

So the first Cartan structure equations in (2.49) becomes

$$\omega_0 \wedge \ell + \omega_+ \wedge e^2 = 0 \tag{2.66a}$$

$$H'e^2 \wedge \ell - \omega_0 \wedge n - \omega_- \wedge e^2 = 0 \tag{2.66b}$$

$$\omega_- \wedge \ell - \omega_+ \wedge n = 0 \tag{2.66c}$$

Here $H' = \frac{\partial H}{\partial \rho}$. Solving these equations we find the connection 1-forms as

$$\bullet\ \omega_0 = 0 \qquad \bullet\ \omega_+ = 0 \qquad \bullet\ \omega_- = -H'\ell \tag{2.67}$$

Here we can directly see why we changed to null coordinates. Calculations are much easier in this way.

We find the curvature 2-forms from (2.54) as follows

$$\bullet\ R_0 = 0 \qquad \bullet\ R_+ = 0 \qquad \bullet\ R_- = H''\ell \wedge e^2 \tag{2.68}$$

The curvature constant $\mathcal{R}$ is zero.

Components of the Cotton tensor can be found from (2.60)

$$\bullet\ C_2 = 0 \qquad \bullet\ C_+ = 0 \qquad \bullet\ C_- = H'''\ell \wedge e^2 \tag{2.69}$$

Field equations from (2.63) gives the following

$$\begin{aligned} \frac{1}{\kappa^2}R_- - \frac{1}{\mu}D * R_- &= 0 \\ \frac{1}{\kappa^2}H''\ell \wedge e^2 + \frac{1}{\mu}H'''\ell \wedge e^2 &= 0 \\ \implies \frac{1}{\mu}H''' + \frac{1}{\kappa^2}H'' &= 0 \end{aligned} \tag{2.70}$$

The solution to this equation is

$$H(u,\rho) = \frac{\kappa^4}{\mu^2} \mathrm{e}^{-\frac{\mu}{\kappa^2}\rho} f(u) \tag{2.71}$$

where $f(u)$ is an arbitrary function of u. pp-waves in 3D progress in time with $f(u)$, but at any given time, their wavefronts decay exponentially.

Chapter 3

NEW MASSIVE GRAVITY

Recently, a new theory of massive gravity in 3D is introduced by Bergshoeff, Hohm, and Townsend [3]. They add a particular, parity preserving quadratic term to EH-term in the Lagrangian:

$$L = \frac{1}{2\kappa^2}\mathcal{R} * 1 + \frac{1}{2m^2}\left(Ric_a \wedge *Ric^a - \frac{3}{8}\mathcal{R}^2 * 1\right) \tag{3.1}$$

where m is a mass parameter. The theory is renormalizable, and unitary for massive spin-2 particles with two polarization states of helicities ± 2.

NMG caught immediate attention since it is the only ghost-free, super-renormalizable theory. Exact solutions were worked out[4], besides BTZ black holes [4] and Kundt solutions [9]. Investigation of AdS/CFT correspondence is also in progress [20].

It is worth to note that, very recently Gullu, Sisman, and Tekin showed that NMG action can be extracted from a Born-Infeld action [8]

$$S_{BI} = -\frac{1}{2\kappa^2}\int d^3x\left(\sqrt{-det\left(g - \frac{1}{m^2}G\right)} - \sqrt{-detg}\right) \tag{3.2}$$

with the small curvature expansion. (G is the matrix form the Einstein tensor). This expansion readily extends to higher order massive theories.

One can combine Chern-Simons term in TMG and NMG Lagrangian as

$$L = \frac{1}{2\kappa^2}\mathcal{R} * 1 + \frac{1}{4\mu}\left(\omega^a{}_b \wedge d\omega^b{}_a + \frac{2}{3}\omega^a{}_c \wedge \omega^c{}_b \wedge \omega^b{}_a\right) + \frac{1}{2m^2}\left(Ric_a \wedge *Ric^a - \frac{3}{8}R^2 * 1\right) \tag{3.3}$$

Following Bergshoeff, *et al.*, we call this theory *general massive gravity* (GMG). It has two massive ($m_\pm$) spin-2 modes with helicities ± 2. The two masses are defined as

$$m^2 = m_+ m_- \qquad\qquad \mu = \frac{m_+ m_-}{m_- - m_+} \tag{3.4}$$

The limit $m_- \to \infty$ for fixed m_+ gives the usual TMG theory. Extension of TMG and NMG with a cosmological constant Λ is straightforward. We will use the cosmological GMG action instead of NMG action, since it is the most general case, and one can always recover NMG limit for $\mu \to \infty$ and $\Lambda = 0$. [3].

3.1 NMG Field Equations

Consider the NMG Lagrangian with CS-term and a negative cosmological constant Λ:

$$
\begin{aligned}
L = \frac{1}{2\kappa^2} R_{ab} \wedge *(e^a \wedge e^b) + \frac{1}{4\mu}\left(\omega^a{}_b \wedge d\omega^b{}_a + \frac{2}{3}\omega^a{}_c \wedge \omega^c{}_b \wedge \omega^b{}_a\right) - \Lambda * 1 + (de^a + \omega^a{}_b \wedge e^b) \wedge \lambda_a \\
+ \frac{1}{2m^2}\left(Ric_a \wedge *Ric^a - \frac{3}{8} R^2 * 1\right)
\end{aligned}
\tag{3.5}
$$

Again we will use the Lagrange multiplier one-forms for torsion-free variations. This Lagrangian will give fourth order differential equations for metric tensor. We will find the field equations from a variational principle.

Let us first write $Ric_a \wedge *Ric^a$ term in terms of R^{ab} and $\mathcal{R}$, which we know how to vary. Einstein tensor is

$$
G_a = *\left(Ric_a - \frac{1}{2}\mathcal{R} e_a\right) \tag{3.6}
$$

Also in (2+1) dimensions we have

$$
G_a = -\frac{1}{2}\epsilon_{abc} R^{bc}
$$

So

$$
\begin{aligned}
-\frac{1}{2}\epsilon_{abc} R^{bc} &= *\left(Ric_a - \frac{1}{2}\mathcal{R} e_a\right) \\
\frac{1}{2}\epsilon_{abc} R^{bc} &= Ric_a - \frac{1}{2}\mathcal{R} e_a \\
\Rightarrow Ric_a &= \frac{1}{2}\mathcal{R} e_a + \frac{1}{2}\epsilon_{abc} * R^{bc}
\end{aligned}
\tag{3.7}
$$

$$
\begin{aligned}
Ric_a \wedge *Ric^a &= \left(\frac{1}{2}\mathcal{R}e_a + \frac{1}{2}\epsilon_{abc} * R^{bc}\right) \wedge \left(\frac{1}{2}\mathcal{R} * e^a - \frac{1}{2}\epsilon^{adf} * R_{df}\right) \\
&= -\frac{1}{4}\epsilon_{abc}\epsilon^{adf} * R^{bc} \wedge R_{df} + \frac{1}{4}\mathcal{R}^2 \underbrace{e_a \wedge *e^a}_{*1} + \frac{1}{4}\mathcal{R}\epsilon_{abc} * R^{bc} \wedge *e^a - \frac{1}{4}\mathcal{R}\epsilon^{adf} e_a \wedge R_{df} \\
&= -\frac{1}{4}(\eta^d{}_b\eta^f{}_c - \eta^d{}_c\eta^f{}_b) * R^{bc} \wedge R_{df} + \frac{\mathcal{R}^2}{4} * 1 + \frac{\mathcal{R}}{4} * R^{bc} \wedge * \underbrace{(\epsilon_{abc}e^a)}_{*e_{bc}} - \frac{\mathcal{R}}{4} R_{df} \wedge \underbrace{(\epsilon^{adf} e_a)}_{*e^{df}} \\
&= -\frac{1}{2} * R^{bc} \wedge R_{bc} + \frac{\mathcal{R}^2}{4} * 1 - \frac{\mathcal{R}}{4} \underbrace{*R^{bc} \wedge e_{bc}}_{R^{bc}\wedge *e_{bc} = \mathcal{R}*1} - \frac{\mathcal{R}^2}{4} * 1
\end{aligned}
$$

$$
Ric_a \wedge *Ric^a = -\frac{1}{2}R_{bc} \wedge *R^{bc} - \frac{1}{4}\mathcal{R}^2 * 1 \tag{3.8}
$$

NMG term in the Lagrangian becomes

$$
L_{NMG} = \frac{1}{2m^2}\left(-\frac{1}{2}R_{bc} \wedge *R^{bc} - \frac{5}{8}\mathcal{R}^2 * 1\right) = \frac{1}{2m^2}L' \tag{3.9}
$$

Variations of this Lagrangian are

$$
\delta L' = \delta\left(-\frac{1}{2}R_{bc} \wedge *R^{bc}\right) - \frac{5}{8}\mathcal{R}^2\delta * 1 - \frac{5}{4}\mathcal{R}\delta\mathcal{R} * 1
$$

First term in this variation is like $F \wedge *F$ term in case of electromagnetic field, so is the variation:

$$
\begin{aligned}
&\delta\left(-\frac{1}{2}R_{bc} \wedge *R^{bc}\right) = -\delta R_{ab} \wedge *R^{ab} + \frac{1}{2}\delta e^a \wedge \left(\imath_a R_{bc} \wedge *R^{bc} - R_{bc} \wedge \imath_a * R^{bc}\right) \\
&\delta R_{ab} \wedge *R^{ab} = D\delta\omega_{ab} \wedge *R^{ab} = \delta\omega_{ab} \wedge D * R^{ab}
\end{aligned}
$$

$$
\delta\left(-\frac{1}{2}R_{bc} \wedge *R^{bc}\right) = -\delta\omega_{ab} \wedge D * R^{ab} + \frac{1}{2}\delta e^a \wedge \left(\imath_a R_{bc} \wedge *R^{bc} - R_{bc} \wedge \imath_a * R^{bc}\right) \tag{3.10}
$$

Variation of *1, $\delta * 1$ is

$$
\delta * 1 = \delta e^a \wedge *e_a \tag{3.11}
$$

Last term needs more attention:

$$
\begin{aligned}
&\mathcal{R} * 1 = R_{ab} \wedge *e^{ab} \\
&\delta(\mathcal{R} * 1) = \delta(R_{ab} \wedge *e^{ab}) \\
\mathcal{R}/ \;\; &\delta\mathcal{R} * 1 + \mathcal{R}\delta * 1 = \delta R_{ab} \wedge *e^{ab} + R_{ab} \wedge \delta * e^{ab} \\
&\mathcal{R}\delta\mathcal{R} * 1 + \mathcal{R}^2\delta e^a \wedge *e_a = \underbrace{D\delta\omega_{ab} \wedge \mathcal{R} * e^{ab}}_{\delta\omega_{ab}\wedge D\mathcal{R}*e^{ab} = \delta\omega_{ab}\wedge d\mathcal{R}\wedge *e^{ab}} + \mathcal{R}R_{ab} \wedge \delta e^c \wedge *e^{ab}{}_c
\end{aligned}
$$

$$
\mathcal{R}\delta\mathcal{R} * 1 = \delta\omega_{ab} \wedge d\mathcal{R} \wedge *e^{ab} + \delta e^a \wedge (\mathcal{R}R^{bc} \wedge *e_{abc} - \mathcal{R}^2 * e_a) \tag{3.12}
$$

Combining these results we have the desired variation for L'

$$\delta L' = -\delta\omega_{ab} \wedge D * R^{ab} + \frac{1}{2}\delta e^a \wedge \left(\imath_a R_{bc} \wedge *R^{bc} - R_{bc} \wedge \imath_a * R^{bc}\right) - \frac{5}{8}\mathcal{R}^2 \delta e^a \wedge *e_a - \frac{5}{4}\delta\omega_{ab} \wedge d\mathcal{R} \wedge *e^{ab}$$
$$-\frac{5}{4}\delta e^a \wedge (\mathcal{R}\epsilon_{abc} R^{bc} - \mathcal{R}^2 * e_a)$$

$$\begin{aligned}\delta L' &= \delta\omega_{ab} \wedge \left(-D * R^{ab} - \frac{5}{4} d\mathcal{R} \wedge *e^{ab}\right) \\ &+ \delta e^a \wedge \left(\frac{1}{2}\left(\imath_a R_{bc} \wedge *R^{bc} - R_{bc} \wedge \imath_a * R^{bc}\right) - \frac{5}{8}\mathcal{R}^2 * e_a - \frac{5}{4}\mathcal{R}\epsilon_{abc} R^{bc}\right)\end{aligned} \tag{3.13}$$

Now we can fully variate the Lagrangian in (3.5)

$$\begin{aligned}\delta L =& \delta\omega_{ab} \wedge \left\{\frac{2}{\mu} R^{ba} + \frac{1}{2}(e^b \wedge \lambda^a - e^a \wedge \lambda^b) - \frac{1}{2m^2} D * R^{ab} - \frac{5}{8m^2} d\mathcal{R} \wedge *e^{ab}\right\} + \\ & \delta e^a \wedge \left\{\frac{1}{2\kappa^2}\epsilon_{abc} R^{bc} + D\lambda_a - \Lambda * e_a - \frac{5\mathcal{R}}{8m^2}\epsilon_{abc} R^{bc} + \frac{5\mathcal{R}^2}{16m^2} * e_a + \frac{1}{2m^2}\tau_a\right\}\end{aligned} \tag{3.14}$$

where we will call

$$\tau_a = \frac{1}{2}\left(\imath_a R_{bc} \wedge *R^{bc} - R_{bc} \wedge \imath_a * R^{bc}\right) \tag{3.15}$$

the stress tensor.

Minimum action principle says we have $\delta L = 0$, so we have two equations

$$e^a \wedge \lambda^b - e^b \wedge \lambda^a = \frac{1}{\mu} R^{ba} - \frac{1}{m^2} D * R^{ab} - \frac{5}{4m^2} d\mathcal{R} \wedge *e^{ab} \tag{3.16}$$

$$\left(\frac{1}{2\kappa^2} - \frac{5\mathcal{R}}{8m^2}\right)\epsilon_{abc} R^{bc} + \left(-\Lambda + \frac{5\mathcal{R}^2}{16m^2}\right) * e_a + \frac{1}{2m^2}\tau_a + D\lambda_a = 0 \tag{3.17}$$

We should first solve (3.16) for λ_a, then insert it into the field equation (3.17).

$$\begin{aligned}\imath_a/ \quad & e^a \wedge \lambda^b - e^b \wedge \lambda^a = \Sigma^{ab} = \frac{1}{\mu} R^{ba} - \frac{1}{m^2} D * R^{ab} - \frac{5}{4m^2} d\mathcal{R} \wedge *e^{ab} \\ & 3\lambda^b - \lambda^b - \lambda^b + e^b \imath_a \lambda^a = \imath_a \Sigma^{ab} \\ \imath_b/ \quad & \lambda^b + e^b i_a \lambda^a = \imath_a \Sigma^{ab} \\ & 4\imath_a \lambda^a = \imath_b \imath_a \Sigma^{ab}\end{aligned}$$

$$\Rightarrow \lambda^a = -\imath_b \Sigma^{ab} - \frac{1}{4} e^a \imath_b \imath_c \Sigma^{cb} \tag{3.18}$$

$$\lambda^a = -\frac{1}{\mu} Ric^a + \frac{1}{m^2}\imath_b D * R^{ab} - \frac{5}{4m^2}\imath_b d\mathcal{R} \wedge *e^{ab} + e^a \left(\frac{\mathcal{R}}{4\mu} + \frac{1}{4m^2}\imath_b \imath_c D * R^{cb}\right)$$

$$\lambda^a = -\frac{1}{\mu}\left(Ric^a - \frac{1}{4}\mathcal{R} e^a\right) + \frac{1}{m^2}\left(\imath_b D * R^{ab} + \frac{5}{4}\imath_b d\mathcal{R} \wedge *e^{ab} + \frac{e^a}{4}\imath_b \imath_c D * R^{cb}\right) \tag{3.19}$$

Covariant derivative of λ^a is

$$D\lambda^a = -\frac{1}{\mu}C^a + \frac{1}{m^2}\left(D\imath_b D * R^{ab} + \frac{5}{4}\underbrace{D\imath_b d\mathcal{R} \wedge *e^{ab}}_{-D\imath^a * d\mathcal{R}} + \frac{e^a}{4} d\imath_b \imath_c D * R^{cb}\right)$$

$$D\lambda^a = -\frac{1}{\mu}C^a + \frac{1}{m^2}\left(D\imath_b D * R^{ab} - \frac{5}{4}D\imath^a * d\mathcal{R} + \frac{e^a}{4} d\imath_b \imath_c D * R^{cb}\right) \tag{3.20}$$

where C^a is the Cotton tensor defined in (2.33).

Now we can write the field equations in (3.17) as follows

$$\frac{1}{\kappa^2}G_a + \frac{1}{\mu}C_a + \Lambda * e_a + \frac{1}{m^2}K_a = 0 \tag{3.21}$$

where

$$K_a = \frac{5}{8}\epsilon_{abc}\mathcal{R}R^{bc} - \frac{5}{16}\mathcal{R}^2 * e_a - \frac{\tau_a}{2} - D\imath^b D * R_{ab} + \frac{5}{4}D\imath_a * d\mathcal{R} - \frac{e_a}{4} d\imath_b \imath_c D * R^{cb} \tag{3.22}$$

3.2 NMG Field Equations in Null Coordinates

In this part we will write (3.21) in terms of null coordinates and null frame tensors. First let us look at the stress tensor

$$\tau_a = \frac{1}{2}\left(\imath_a R_{bc} \wedge *R^{bc} - R_{bc} \wedge \imath_a * R^{bc}\right) \tag{3.23}$$

Explicitly we have

$$\tau_0 = i_0 R_{01} \wedge *R^{01} - R_{01} \wedge i_0 * R^{01} + i_0 R_{02} \wedge *R^{02} - R_{02} \wedge i_0 * R^{02} + i_0 R_{12} \wedge *R^{12} - R_{12} \wedge i_0 * R^{12} \tag{3.24a}$$

$$\tau_1 = i_1 R_{01} \wedge *R^{01} - R_{01} \wedge i_1 * R^{01} + i_1 R_{02} \wedge *R^{02} - R_{02} \wedge i_1 * R^{02} + i_1 R_{12} \wedge *R^{12} - R_{12} \wedge i_1 * R^{12} \tag{3.24b}$$

$$\tau_2 = i_2 R_{01} \wedge *R^{01} - R_{01} \wedge i_2 * R^{01} + i_2 R_{02} \wedge *R^{02} - R_{02} \wedge i_2 * R^{02} + i_0 R_{12} \wedge *R^{12} - R_{12} \wedge i_2 * R^{12} \tag{3.24c}$$

Following same procedures as in Chapter 2, we have the following

$$\tau_+ = -i_n R_0 \wedge *R_0 + R_0 \wedge i_n * R_0 - i_n R_+ \wedge *R_- + R_+ \wedge i_n * R_- - i_n R_- \wedge *R_+ + R_- \wedge i_n * R_+ \tag{3.25a}$$

$$\tau_- = -i_\ell R_0 \wedge *R_0 + R_0 \wedge i_\ell * R_0 - i_\ell R_+ \wedge *R_- + R_+ \wedge i_\ell * R_- - i_\ell R_- \wedge *R_+ + R_- \wedge i_\ell * R_+ \tag{3.25b}$$

$$\tau_2 = -i_2 R_0 \wedge *R_0 + R_0 \wedge i_2 * R_0 - i_2 R_+ \wedge *R_- + R_+ \wedge i_2 * R_- - i_2 R_- \wedge *R_+ + R_- \wedge i_2 * R_+ \tag{3.25c}$$

where

$$\tau_+ = \frac{\tau_1 - \tau_0}{\sqrt{2}}, \quad \tau_- = \frac{\tau_1 + \tau_0}{\sqrt{2}} \tag{3.26}$$

Now let us look (3.22). First, define

$$\beta = \frac{1}{2}\imath_b \imath_c D * R^{cb} \tag{3.27}$$

$$\begin{aligned}
&= \imath_1\imath_0 D * R^{01} + \imath_2\imath_0 D * R^{02} + \imath_2\imath_1 D * R^{12} \\
&= \left(\frac{\imath_\ell + \imath_n}{\sqrt{2}}\right)\left(\frac{\imath_\ell - \imath_n}{\sqrt{2}}\right) D * R_0 + \imath_2 \left(\frac{\imath_\ell - \imath_n}{\sqrt{2}}\right) D * \left(\frac{R_+ + R_-}{\sqrt{2}}\right) + i_2 \left(\frac{i_\ell + i_n}{\sqrt{2}}\right) D * \left(\frac{R_+ - R_-}{\sqrt{2}}\right) \\
&= \imath_n \imath_l D * R_0 + \frac{1}{2}\imath_2 \left(\imath_\ell D * R_+ + \imath_\ell D * R_- - \imath_n D * R_+ - \imath_n D * R_-\right) \\
&\qquad + \frac{1}{2}\imath_2 \left(\imath_\ell D * R_+ - \imath_\ell D * R_- + \imath_n D * R_+ - \imath_n D * R_-\right)
\end{aligned}$$

$$\beta = \imath_n \imath_\ell D * R_0 + \imath_2 \imath_\ell D * R_+ - \imath_2 \imath_n D * R_- \tag{3.28}$$

So we can write the components of this K_a tensor:

$$K_0 = \frac{5\mathcal{R}}{4} R^1{}_2 - \frac{5\mathcal{R}^2}{16} * e_0 - \frac{\tau_0}{2} + D\imath_1 D * R^0{}_1 + D\imath_2 D * R^0{}_2 + \frac{5}{4} D\imath_0 * d\mathcal{R} - \frac{1}{2} e_0 \wedge d\beta \tag{3.29a}$$

$$K_1 = -\frac{5\mathcal{R}}{4} R^0{}_2 - \frac{5\mathcal{R}^2}{16} * e_1 - \frac{\tau_1}{2} + D\imath_0 D * R^0{}_1 - D\imath_2 D * R^1{}_2 + \frac{5}{4} D\imath_1 * d\mathcal{R} - \frac{1}{2} e_1 \wedge d\beta \tag{3.29b}$$

$$K_2 = \frac{5\mathcal{R}}{4} R^0{}_1 - \frac{5\mathcal{R}^2}{16} * e_2 - \frac{\tau_2}{2} + D\imath_0 D * R^0{}_2 + D\imath_1 D * R^1{}_2 + \frac{5}{4} D\imath_2 * d\mathcal{R} - \frac{1}{2} e_2 \wedge d\beta \tag{3.29c}$$

Let us apply the same tricks: first subtract (3.29a) from (3.29b):

$$\begin{aligned}
\frac{K_1 - K_0}{\sqrt{2}} &= -\frac{5\mathcal{R}}{4}\left(\frac{R^0{}_2 + R^1{}_2}{\sqrt{2}}\right) - \frac{5\mathcal{R}^2}{16} * \left(\frac{e_1 - e_0}{\sqrt{2}}\right) - \frac{1}{2}\left(\frac{\tau_1 - \tau_0}{\sqrt{2}}\right) + D\left(\frac{\imath_0 - \imath_1}{\sqrt{2}}\right) D * R^0{}_1 \\
&\quad - D\imath_2 D * \left(\frac{R^1{}_2 + R^0{}_2}{\sqrt{2}}\right) + \frac{5}{4} D \left(\frac{\imath_1 - \imath_0}{\sqrt{2}}\right) * d\mathcal{R} - \frac{1}{2}\left(\frac{e_1 - e_0}{\sqrt{2}}\right) \wedge d\beta
\end{aligned}$$

$$K_+ = -\frac{5\mathcal{R}}{4} R_+ - \frac{5\mathcal{R}^2}{16} * \ell - \frac{\tau_+}{2} - D\imath_n D * R_0 - D\imath_2 D * R_+ + \frac{5}{4} D\imath_n * d\mathcal{R} - \frac{1}{2}\ell \wedge d\beta \tag{3.30}$$

Then add (3.29a) and (3.29b):

$$\begin{aligned}
\frac{K_1 + K_0}{\sqrt{2}} &= -\frac{5\mathcal{R}}{4}\left(\frac{R^0{}_2 - R^1{}_2}{\sqrt{2}}\right) - \frac{5\mathcal{R}^2}{16} * \left(\frac{e_1 + e_0}{\sqrt{2}}\right) - \frac{1}{2}\left(\frac{\tau_1 + \tau_0}{\sqrt{2}}\right) + D\left(\frac{\imath_0 + \imath_1}{\sqrt{2}}\right) D * R^0{}_1 \\
&\quad + D\imath_2 D * \left(\frac{R^0{}_2 - R^1{}_2}{\sqrt{2}}\right) + \frac{5}{4} D \left(\frac{\imath_1 + \imath_0}{\sqrt{2}}\right) * d\mathcal{R} - \frac{1}{2}\left(\frac{e_1 + e_0}{\sqrt{2}}\right) \wedge d\beta
\end{aligned}$$

$$K_- = -\frac{5\mathcal{R}}{4}R_- - \frac{5\mathcal{R}^2}{16} * n - \frac{\tau_-}{2} + D\imath_\ell D * R_0 + D\imath_2 D * R_- + \frac{5}{4}D\imath_\ell * d\mathcal{R} - \frac{1}{2}n \wedge d\beta \tag{3.31}$$

And (3.29c) gives

$$K_2 = \frac{5\mathcal{R}}{4}R^0{}_1 - \frac{5\mathcal{R}^2}{16} * e_2 - \frac{\tau_2}{2} + D\left(\frac{\imath_\ell - \imath_n}{\sqrt{2}}\right) D * \left(\frac{R_+ + R_-}{\sqrt{2}}\right) + D\left(\frac{\imath_\ell + \imath_n}{\sqrt{2}}\right) D * \left(\frac{R_+ - R_-}{\sqrt{2}}\right)$$
$$+ \frac{5}{4}D\imath_2 * d\mathcal{R} - \frac{1}{2}e_2 \wedge d\beta$$

$$K_2 = \frac{5\mathcal{R}}{4}R_0 - \frac{5\mathcal{R}^2}{16} * e_2 - \frac{\tau_2}{2} + D\imath_\ell D * R_+ - D\imath_n D * R_- + \frac{5}{4}D\imath_2 * d\mathcal{R} - \frac{1}{2}e_2 \wedge d\beta \tag{3.32}$$

After these calculations, field equations in (3.21) simply turns into the following forms

$$\frac{1}{\kappa^2}G_+ + \frac{1}{\mu}C_+ + \frac{1}{m^2}K_+ + \Lambda * \ell = 0 \tag{3.33a}$$
$$\frac{1}{\kappa^2}G_- + \frac{1}{\mu}C_- + \frac{1}{m^2}K_- + \Lambda * n = 0 \tag{3.33b}$$
$$\frac{1}{\kappa^2}G_2 + \frac{1}{\mu}C_2 + \frac{1}{m^2}K_2 + \Lambda * e_2 = 0 \tag{3.33c}$$

3.3 pp-wave Solutions of NMG

We already calculated the curvatures and the Cotton tensor for a pp-wave metric

$$g = 2H(u,\rho)du^2 + 2dudv + d\rho^2$$

Let us remember the results:

- $R_0 = 0$ • $R_+ = 0$ • $R_- = H''\ell \wedge e^2$
- $C_2 = 0$ • $C_+ = 0$ • $C_- = H'''\ell \wedge e^2$

The stress tensor τ, curvature constant $\mathcal{R}$, and the cosmological constant Λ are zero for this metric. After an easy calculation, we find the K-tensor as

- $K_2 = 0$ • $K_+ = 0$ • $K_- = -H^{(4)}\ell \wedge e^2$

So that (3.33b) gives the following fourth order differential equation for $H(u,\rho)$

$$\frac{H^{(4)}}{m^2} - \frac{H'''}{\mu} - \frac{H''}{\kappa^2} = 0 \tag{3.34}$$

Solution of this equation is

$$H(u,\rho) = \left\{ C_1 e^{\frac{m\rho}{2\mu\kappa}\left(m\kappa - \sqrt{m^2\kappa^2+4\mu^2}\right)} + C_2 e^{\frac{m\rho}{2\mu\kappa}\left(m\kappa + \sqrt{m^2\kappa^2+4\mu^2}\right)} \right\} f(u) \tag{3.35}$$

where $f(u)$ is an arbitrary function, and C_1, C_2 are constants depending on initial conditions.

If we take $\mu \to \infty$ limit, i.e. without the Chern-Simons term, the solution becomes

$$H(u,\rho) = \cosh\left(\frac{m}{\kappa}\rho\right) f(u) \tag{3.36}$$

again, $f(u)$ being an arbitrary function. Cosmological constant Λ is zero for pp-wave solutions of NMG, like TMG.

3.4 Bending AdS Wave Solutions of NMG

AdS waves are gravitational waves in the presence of a negative cosmological constant. They are generalizations of gravitational wave solutions, i.e. Kundt or Robinson-Trautman solutions, for a cosmological constant. Siklos in 1985, defined the metric in D-dimensions

$$g = \frac{\alpha^2}{y^2}(F(u,y,x^i)du^2 - 2dudv + dy^2 + dx_i dx^i) \tag{3.37}$$

This metric is a conformal transformation of the pp-wave metric. Kundt and Robinson-Trautman spacetimes has a multiple principle null direction corresponding to their Weyl tensor. If, for a negative cosmological constant, this null direction is also a Killing vector, then they are diffeomorphic to Siklos spacetimes [21]. If $F = 0$, (3.37) reduces to AdS metric, where in 4D $\mathcal{R} = 6\Lambda = -6/\alpha^2$. So that α is the radius of AdS spacetime.

Now let us consider the AdS wave metric in 3D as in [5]

$$g = \frac{\alpha^2}{y^2}(2F(u,y)du^2 + 2dudv + dy^2) \tag{3.38}$$

Null coordinates are

$$\ell = \frac{\alpha}{y}du \qquad n = \frac{\alpha}{y}(dv + Fdu) \qquad e^2 = \frac{\alpha}{y}dy$$

Now let us calculate exact derivatives of coordinate 1-forms in order to use them in structure equations:

$$d\ell = -\frac{\alpha}{y^2} dy \wedge du = -\frac{\alpha}{y^2}\left(\frac{y}{\alpha}e^2\right) \wedge \left(\frac{y}{\alpha}\ell\right) = \frac{1}{\alpha}\ell \wedge e^2 \tag{3.39a}$$

$$\begin{aligned} dn &= d\left(\frac{\alpha}{y}dv\right) + d\left(\frac{\alpha}{y}F du\right) = -\frac{\alpha}{y^2}dy \wedge dv + \left(-\frac{\alpha}{y^2}F + -\frac{\alpha}{y}F'\right) dy \wedge du \\ &= -\frac{\alpha}{y^2}\left(\frac{y}{\alpha}e^2\right) \wedge \left(\frac{y}{\alpha}n - F\frac{y}{\alpha}\ell\right) - \frac{\alpha}{y^2}(F - yF')\left(\frac{y}{\alpha}e^2\right) \wedge \left(\frac{y}{\alpha}\ell\right) \\ &= -\frac{1}{\alpha}e^2 \wedge n + \frac{F}{\alpha}e^2 \wedge \ell + \frac{1}{\alpha}(yF' - F)e^2 \wedge \ell \\ &= \frac{1}{\alpha}n \wedge e^2 - \frac{y}{\alpha}F'\ell \wedge e^2 \end{aligned} \tag{3.39b}$$

$$de^2 = 0 \tag{3.39c}$$

Here $'$ means derivations with respect to y.

With these, structure equations in (2.49) can be written as follows

$$-\frac{1}{\alpha}e^2 \wedge \ell + \omega_0 \wedge \ell + \omega_+ \wedge e^2 = 0 \tag{3.40a}$$

$$\frac{1}{\alpha}e^2 \wedge n + \frac{y}{\alpha}F'e^2 \wedge \ell - \omega_0 \wedge n - \omega_- \wedge e^2 = 0 \tag{3.40b}$$

$$\omega_- \wedge \ell - \omega_+ \wedge n = 0 \tag{3.40c}$$

Solving these equations we find Levi-Civita connection 1-forms as

$$\bullet\ \omega_0 = 0 \qquad \bullet\ \omega_+ = -\frac{1}{\alpha}\ell \qquad \bullet\ \omega_- = -\frac{y}{\alpha}F'\ell + \frac{1}{\alpha}n \tag{3.41}$$

Now let us calculate derivatives of ω's since we will use them to find curvature 2-forms.

$$d\omega_0 = 0 \tag{3.42a}$$

$$d\omega_+ = -\frac{1}{\alpha}d\ell = -\frac{1}{\alpha^2}\ell \wedge e^2 \tag{3.42b}$$

$$\begin{aligned} d\omega_- &= d\left(-\frac{y}{\alpha}F'\frac{\alpha}{y}du\right) + \frac{1}{\alpha}dn = -F''dy \wedge du + \frac{1}{\alpha}\left(-\frac{1}{\alpha}e^2 \wedge n + \frac{y}{\alpha}F'e^2 \wedge \ell\right) \\ &= -\frac{y^2}{\alpha^2}F''e^2 \wedge \ell - \frac{1}{\alpha^2}e^2 \wedge n + \frac{y}{\alpha^2}F'e^2 \wedge \ell \\ d\omega_- &= \frac{1}{\alpha^2}n \wedge e^2 + \frac{1}{\alpha^2}(y^2F'' - yF')\ell \wedge e^2 \end{aligned} \tag{3.42c}$$

Thus with the Cartan equations in (2.54), we have the curvature 2-forms:

$$R_0 = -\frac{1}{\alpha^2}\ell \wedge n \tag{3.43a}$$

$$R_+ = -\frac{1}{\alpha^2}\ell \wedge e^2 \tag{3.43b}$$

$$R_- = \frac{1}{\alpha^2}n \wedge e^2 + \frac{1}{\alpha^2}(y^2F'' - yF')\ell \wedge e^2 \tag{3.43c}$$

Using (2.44) we can find the curvature constant:

$$\mathcal{R} = -\frac{6}{\alpha^2} \tag{3.44}$$

so that $d\mathcal{R} = 0$. And from (3.25) we can find the stress tensor as follows:

$$\tau_2 = \frac{1}{\alpha^4}\ell \wedge n \tag{3.45a}$$

$$\tau_+ = -\frac{1}{\alpha^4}\ell \wedge e^2 \tag{3.45b}$$

$$\tau_- = \frac{1}{\alpha^4}n \wedge e^2 - \frac{4}{\alpha^4}(y^2F'' - yF')\ell \wedge e^2 \tag{3.45c}$$

Cotton tensor follows from (2.60)

$$\begin{aligned}
C_2 &= D * R_0 = D\left(\frac{1}{\alpha^2}e^2\right) = 0\\
C_+ &= -D * R_+ = D\left(\frac{1}{\alpha^2}\ell\right) = 0\\
C_- &= -D * R_- = D\left(\frac{1}{\alpha^2}n\right) - D\left(\frac{1}{\alpha^2}(y^2F'' - yF')\ell\right) = -\frac{1}{\alpha^2}d(y^2F'' - yF') \wedge \ell\\
&= -\frac{1}{\alpha^2}(2yF'' + y^2F''' - F' - yF'')dy \wedge \ell\\
&= \frac{1}{\alpha^3}(y^3F''' + y^2F'' - yF')\ell \wedge e^2
\end{aligned}$$

Now calculate the K-tensor from (3.22):

$$\begin{aligned}
K_2 &= -\frac{15}{2\alpha^2}\left(-\frac{1}{\alpha^2}\ell \wedge n\right) - \frac{45}{4\alpha^2}\ell \wedge n - \frac{1}{\alpha^2}\ell \wedge n\\
&= -\frac{17}{4\alpha^4}\ell \wedge n && (3.46)\\
K_+ &= \frac{15}{2\alpha^2}\left(-\frac{1}{\alpha^2}\ell \wedge n\right) + \frac{45}{4\alpha^2}\ell \wedge n + \frac{1}{\alpha^2}\ell \wedge n\\
&= \frac{17}{4\alpha^4}\ell \wedge n && (3.47)\\
K_- &= \frac{15}{2\alpha^2}\left(\frac{1}{\alpha^2}n \wedge e^2 + \frac{1}{\alpha^2}(y^2F'' - yF')\ell \wedge e^2\right) - \frac{45}{4\alpha^2}n \wedge e^2 - \frac{1}{2\alpha^2}n \wedge e^2 + \frac{2}{\alpha^4}(y^2F'' - yF')\ell \wedge e^2\\
&\qquad + D\imath_2 D * R_-\\
&= -\frac{17}{4\alpha^4}n \wedge e^2 + \frac{19}{2\alpha^4}(y^2F'' - yF')\ell \wedge e^2 + \frac{1}{\alpha^3}d(y^3F''' + y^2F'' - yF') \wedge \ell\\
&= -\frac{17}{4\alpha^4}n \wedge e^2 - \frac{1}{\alpha^4}y^4F^{(4)}\ell \wedge e^2 - \frac{4}{\alpha^4}y^3F'''\ell \wedge e^2 + \frac{17}{2\alpha^4}(y^2F'' - yF')\ell \wedge e^2 && (3.48)
\end{aligned}$$

With these, we calculated everything we need to write the field equations. Following the convention of [5], we take $m^2 \to -m^2$ in the NMG Lagrangian, and concequencely in the field equations.The equations (3.33a) and (3.33c) reduce to the same equation, and give the fixed value of cosmological constant:

$$\Lambda = -\frac{1}{\alpha^2}\left(\frac{1}{\kappa^2} + \frac{17}{4m^2\alpha^2}\right) \tag{3.49}$$

The limit $m^2 \to \pm\infty$ gives the usual GR AdS vacuum case $\Lambda = -\frac{1}{\alpha^2\kappa^2}$.

Equation (3.33b) results a single fourth order equation for $F(u, y)$ to solve:

$$y^4 F^{(4)} + \left(4 + \frac{m^2\alpha}{\mu}\right) y^3 F''' + \left(\frac{m^2\alpha^2}{\kappa^2} - \frac{17}{2} + \frac{m^2\alpha}{\mu}\right)(y^2 F'' - yF') = 0 \tag{3.50}$$

We can see that u dependence of F is arbitrary. Solutions for this equation are of type $F \propto y^r$. So we get the following equation

$$r(r-2)\left(r^2 + \frac{m^2\alpha}{\mu} r - \frac{19}{2} + \frac{m^2\alpha^2}{\kappa^2}\right) = 0 \tag{3.51}$$

Hence the wave function is

$$F(u, y) = F_+(u) y^{-\frac{m^2\alpha}{2\mu} + \sqrt{\frac{m^4\alpha^2}{4\mu^2} + \frac{19}{2} - \frac{m^2\alpha^2}{\kappa^2}}} + F_-(u) y^{-\frac{m^2\alpha}{2\mu} - \sqrt{\frac{m^4\alpha^2}{4\mu^2} + \frac{19}{2} - \frac{m^2\alpha^2}{\kappa^2}}} \tag{3.52}$$

Here $F_+(u)$ and $F_-(u)$ are arbitrary functions of u [5].

If we eliminate the topological Chern-Simons term taking $\mu \to \infty$, we get

$$F(u, y) = F_+(u) y^{\sqrt{\frac{19}{2} - \frac{m^2\alpha^2}{\kappa^2}}} + F_-(u) y^{-\sqrt{\frac{19}{2} - \frac{m^2\alpha^2}{\kappa^2}}} \tag{3.53}$$

which is NMG solution.

Chapter 4

CONCLUSION

In this thesis, I investigated (2+1) dimensional massive gravity theories, TMG and NMG. Higher order theories are important because pure 4D gravity is not renormalizable, and pure 3D gravity is not dynamic. Higher order 4D gravity is renormalizable, but not unitary. 3D gravity gained interest when it was shown that TMG, which is third order, is renormalizable and unitary. However it is parity-violating. Recently suggested NMG, which is fourth order, is renormalizable, unitary, as well as parity preserving. Both theories have massive spin-2 gravitons, unlike massless gravitons of pure 4D. 3D higher order theories are good toy models for studying quantum gravity. Thus, finding exact solutions of TMG and NMG are very important. Both theories have BTZ black holes. Kundt solutions of TMG and NMG were shown in [10] and [9], respectively. In this thesis, I have studied pp-wave solutions, which are a special case of Kundt solutions, of TMG and NMG. I also studied bending AdS-wave solutions of NMG. AdS solutions are important for the study of AdS/CFT correspondence on the AdS boundary.

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