The Residue Theorem and an Explicit Duality for Smooth Projective Curves

by

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This is to certify that I have examined this copy of a master's thesis by

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and have found that it is complete and satisfactory in all respects,
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ABSTRACT

The aim of this thesis is to present an explicit duality for smooth projective curves. More standard Serre duality achieves the final results exhibited in the thesis with far greater generality, using powerful homological machinery. The shortcoming of such methods, however, is that they do not yield an explicit dualizing sheaf but only show its existence. The general form of the treatment of the subject we follow was given by Serre in 1959 ([Ser]). A cornerstone of this treatment is the residue theorem, which essentially states that the sum of the residues for a given differential at all points of a regular projective curve is 0. Tate improved upon Serre's treatement (in [Tat]), giving an elegant, characteristic independent proof of the residue theorem by defining residues in a novel way. Hence, our account generally follows Tate, drawing parallels to Serre where appropriate (and occasionally to Chevalley's 1951 work [Che] which contains some seeds of the ideas involved, albeit from a purely algebraic viewpoint). Having developed duality, to demonstrate its power, we conclude the thesis with the proof of a form of the Riemann-Roch Theorem.

ÖZ

Bu tez çalışmasının amacı düz projektif eğriler için açık bir ikilemin sunumunu yapmaktır. Daha standart Serre ikilemi, kuvvetli homolojik yöntemler kullanarak bu tezde gösterilen sonuçların çoğuna -üstelik çok daha genel formda- ulaşabilir. Bu yaklaşımın eksik kaldığı taraf, ortaya çıkan ikilemin açık bir şekilde ifade edilememesidir. Bu tezdeki yaklaşım, en ana hatlarıyla Serre tarafından 1959 anlatılmıştır ([Ser]). Bu yaklaşımın yapıtaşlarından birisi residue teoremidir. Serre'den sonra, Tate residue teoreminin daha doğal, karakteristiğe bağlı olmayan güzel bir kanıtını vermiştir ([Tat]). Bu sebeple, bu tezde daha çok Tate'in yaklaşımı takip edilecektir. Uygun olan yerlerde Serre'in yöntemiyle (ve daha az sıklıkla, buradaki bazı fikirlerin tohumlarını taşıyan Chevalley'in 1951 çalışmasıyla) bağlantılara işaret edilecektir. Projektif düz eğriler için Serre ikilemini kanıtladıktan sonra, bu sonucun gücünü göstermek için tezin sonunda Riemann-Roch teoreminin bir formu kanıtlanacaktır.

Contents

Introduction

The existence of a dualizing sheaf for projective varieties of all dimensions is a standard result of Serre duality (that can for instance be found in [Har], Chpt. III, Sect. 7). The value of the dualizing sheaf is that, given a coherent sheaf on a projective nonsingular variety, it lets one find natural functorial isomorphisms between certain Ext groups and homological groups of various dimensions that are similar in form to the Poincaré duality. If in addition the variety is smooth, the dualizing sheaf is also known to be isomorphic to the canonical sheaf, which is an exterior product of the sheaf of differentials on our variety.

While the existence of this isomorphism can indeed be proved using cohomological methods, it is more difficult to exhibit an explicit isomorphism in the general case. The main purpose of this thesis is to give an account of how an explicit dualizing sheaf can be given in the case of smooth projective curves, in an as self-contained manner as possible. While the result was before presented by Serre (in [Ser]), and in a less recognizable form by Chevalley (in [Che]), we mainly follow the account of Tate (in [Tat]), and occasionally mention the works of Serre and Chevalley to draw parallels.

The unique feature of working with nonsingular projective curves over a field k, is that this category is equivalent to the category of function fields of dimension 1 over k. As such, most of the results presented in the thesis could be dealt with purely in the realm of commutative algebra (which is what Chevalley does).

However, it is much more expedient to use scheme theory as needed. The preliminaries section of the thesis is aimed at stating, and developing where appropriate, results that will let us skip back and forth between commutative rings and schemes.

In the following three sections, we present the theory of residues on curves, and finally prove the residue theorem. The residue theorem is the most important step in our goal of constructing an explicit isomorphism between a dualizing sheaf and the canonical sheaf (which, in dimension 1, is just the sheaf of differentials). The proof of the residue theorem we give is Tate's novelty, the major focus of this thesis, and the main reason we follow Tate's account as opposed to Serre's.

Finally, in section 5, we use the residue theorem in completing our goal of viewing the sheaf of differentials as the dualizing sheaf via an explicit isomorphism.

At this point, we have developed all of the machinery needed to prove a form of the Riemann-Roch theorem for curves. So, while the explicit isomorphism we have constructed is not really needed for it, we conclude the thesis with a proof of this result to demonstrate the power of duality.

1. Preliminaries

In this section, we prove some standard theorems to be used later on. One can defer reading these results until needed later, especially the subsection on commutative algebra.

An important thread running through the first three subsections is how closely our schemes (which are of dimension 1 and "nice") turn out to be related to their underlying rings and fields. Indeed, it is possible (but cumbersome) to give a treatment of the residue theorem using only commutative algebra as Chevalley does in [Che].

Subsection-1.4 introduces divisors on "nice" schemes (that are not necessarily of dimension 1), and concludes by explicating the natural isomorphism between the invertible sheaves on a scheme and a certain quotient group of the divisors on the scheme. Besides being interesting in its own right, this result will let us easily step back and forth between the land of schemes and the land of commutative algebra when working on duality.

Finally, in Subsection-1.5 we prove a simple version of the Riemann-Roch Theorem. This serves a two-fold purpose. First, we use it in the section on duality to prove that the dualizing sheaf we define is isomorphic to the sheaf of differentials on a smooth curve. In turn, we use this isomorphism together with the simple version of the Riemann-Roch Theorem to prove a more general version of the Riemann-Roch Theorem in the final section of the paper.

1.1. **Commutative algebra.**

1.1.1. *Valuation rings.* Here, we give a basic account of valuation rings. In the subsections 1.2 and 1.3, we will explore the ties between the points of a scheme and the valuation rings of its function field. For a proper scheme over a field, we will show that there is a one-to-one correspondence between them. In the case of curves, we will also show that the image of a point under a surjective morphism is determined entirely by the corresponding inclusion of function fields.

Let R be an integral domain with field of fractions K. We say that R is a *valuation ring* if for all $x \in K$, $x \notin R \Rightarrow x^{-1} \in R$. A simple example is $\mathbb{Z}_{(p)}$ for any prime number p.

Theorem 1. *Let K be a field, A a subring of K, and p a prime ideal of A. Then there exists a valuation ring R of K such that $A \subseteq R$, and $m_R \cap A = p$ (where m_R is the maximal ideal of R).*

Proof. [Mat1], Thm. 10.2, p.72. □

The integral closure of A in K is the set of elements of K that are roots of polynomials with leading coefficient 1 and all coefficients in A. It turns out that the integral closure of a valuation ring is just itself.

Theorem 2. *A valuation ring is integrally closed.*

Proof. Let A be a valuation ring of K. Suppose $x \in K$ is integral over A, satisfying $x^n + a_{n-1}x^{n-1} + \ldots + a_0 = 0$, and $x \notin A$. Then $x^{-1} \in m_A$. Multiplying the equality by $x^{-(n-1)}$,

$$x = -a_{n-1} - a_{n-2}x^{-1} - \ldots - a_0 x^{n-1} \in A,$$

which is a contradiction. □

If R and S are local rings with $S \subseteq R$ and $m_R \cap S = m_S$, we say that R *dominates* S. In Subsection-1.3, we will show that the image of a point under a surjective morphism of curves is determined by such a domination relation in a certain way.

Theorem 3. *Let K be a field, and A a subring of K. Then the integral closure B of A in K is the intersection of all the valuation rings of K containing A. If in addition A is a local ring, then B equals the intersection of all valuation rings of K dominating A.*

Proof. By Thm.-2,

$$B \subseteq \bigcap R \quad \text{(taken over valuation rings R containing } A\text{)}$$

Conversely, suppose $x \in K$ is not integral over A. $1 \notin x^{-1}A[x^{-1}]$ (if $1 = a_1x^{-1} + \ldots + a_nx^{-n}$, then $x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_n = 0$, making x integral over A), so $x^{-1}A[x^{-1}] \subseteq m$ for some maximal ideal m of $A[x^{-1}]$. By Thm.-1, there is a valuation ring R of K such that $A \subseteq R$ and $m_R \cap A = m$. But then $x^{-1} \in m_R$, $x \notin R$, and so

$$x \notin \bigcap_{A \subseteq R} R$$

If A is a local ring, then $1 \notin (x^{-1}A[x^{-1}], m_A)$ (if $1 = r + a_1x^{-1} + \ldots + a_nx^{-n}$ for some $r \in m_A$, then $1 - r$ is a unit of A and $1 = (a_1x^{-1} + \ldots + a_nx^{-n})(1-r)^{-1}$). As before $(x^{-1}A[x^{-1}], m_A)$ is contained in some maximal ideal m of $x^{-1}A[x^{-1}]$, so that $m_R \bigcap A[x^{-1}] = m$. Hence $m_R \bigcap A = m \bigcap A = m_A$, i.e. R dominates A. □

1.1.2. *Discrete valuation rings.* Discrete valuation rings (DVRs) are an easier to study subset of all valuation rings. Happily, since we are working in dimension 1 and with "nice" schemes, the valuation rings we encounter will be mostly DVRs.

Definition 1. Let K be a field. A *discrete valuation* on K is a group homomorphism ν of the multiplicative group of K onto the additive group $\mathbb{Z}$ such that $\nu(x+y) \geq \min\{\nu(x), \nu(y)\}$. A subring R of K is a *discrete valuation ring* (DVR) if it equals $\{x|\nu(x) \geq 0\}$ for some discrete valuation ν on K.

We state first the characterization of DVRs among valuation rings, and then their characterization among all rings.

Theorem 4. *Let R be a valuation ring. Then the following conditions are equivalent:*

- *R is a DVR*
- *R is a principal ideal domain*
- *R is noetherian*

Proof. [Mat1], Thm. 11.1, p. 78. □

Theorem 5. *Let R be a ring. Then the following conditions are equivalent:*

- *R is a DVR.*
- *R is a local principal ideal domain, and not a field.*
- *R is a noetherian local ring, $dim R > 0$, and the maximal ideal m_R is principal.*
- *R is a normal noetherian local ring of Krull dimension 1.*

Proof. [Mat1], Thm. 11.2, p. 79. □

For example, it follows immediately from the theorem that $\mathbb{Z}_{(p)}$ is a DVR for any prime p. We could also show this directly by defining an explicit discrete valuation on $\mathbb{Z}_{(p)}$. Given $a \in \mathbb{Z}$, let $\nu(a)$ be the highest integer c such that p^c divides a, and extend this to the whole of $\mathbb{Q}$ in the obvious way. Then the set of elements of $\mathbb{Q}$ with non-negative valuation is precisely $\mathbb{Z}_{(p)}$.

Claim 1. *Let A be a complete discrete valuation ring that is a k-algebra with residue field also k, and let K be the field of fractions of A. Then $A \cong k[[x]]$ and $K \cong k((x))$.*

Proof. Let π be a uniformizer of A. Since $k \hookrightarrow A$, we readily have representatives for the cosets A/m. Take any $a \in A$. Since $A/m = k$, there is an $s_0 \in k$ such that $a - s_0 \in m$, i.e. we can write $a = s_0 + a_1\pi$ for some $a_1 \in A$. Similarly, we can write $a_1 = s_1 + a_2\pi$, so that $a = s_0 + s_1\pi + a_2\pi^2$. Thus, the series $\sum_{n=0}^{\infty} s_n\pi^n$ converges to a. This series representation is clearly unique, since $\sum_{n=0}^{\infty} s_n\pi^n = 0$ only if all of the s_n are 0. This shows that the k-algebra homomorphism $k[[x]] \to A$, determined by the choice of the uniformizer for A, is an isomorphism. $k((x)) \cong K$ follows immediately. □

Lemma 1. *If R is a discrete valuation ring and K its field of fractions, then there do not exist any proper intermediate rings between R and K.*

Proof. Suppose $R \subsetneq S \subset K$. The inverse of some $u\pi^n$ must be in S, where u is a unit in R, and π is a uniformizer of R. Then $\pi^{-1} = \pi^{-n}\pi^{n-1} \in S$, so $S = K$. □

1.1.3. *Regular local rings.* See the beggining of the next subsection for a summary of the results from this section that we will use often. Essentially, the "niceness" of our schemes correspond to the fact that their local rings are regular.

Definition 2. Let A be a noetherian local ring with maximal ideal m and residue field k. We say A is *regular* if $\dim_k(m/m^2) = \dim A$ (where the first $\dim_k$ is dimension as a vector space and the second dim is the Krull dimension).

For example, any discrete valuation ring is a regular local ring. Let A be a DVR, and fix a uniformizer t. We have $m = (t)$, $m^2 = (t^2)$. Denote $A/(t)$ by k. Then multiplication by t is a k-vector space isomorphism between $A/(t)$ and $(t)/(t^2)$. Hence $\dim_k(m/m^2) = 1$, which is the Krull dimension of A.

Lemma 2 (Krull's principal ideal theorem)**.** *Let A be a noetherian ring and let x be an element of A which is neither a zero-divisor nor a unit. Then every minimal prime ideal p containing x has height 1.*

Proof. [Ati], Corollary-11.18, p.122. □

Theorem 6. *A regular local ring is a UFD. In particular, a regular local ring is an integrally closed integral domain.*

Proof. [Mat2], Thm. 48, p.142. For a simpler proof of regular rings being integral domains: [Ati], Lemma 11.23, p. 123. □

Corollary 1. *A regular local ring of Krull dimension 1 is a discrete valuation ring.*

Proof. This follows immediately from the characterization of DVRs among rings (Thm.-5). □

1.1.4. *Integral closure.* Regarding integral closures, we will only need the following lemma, which will be useful in the context of morphisms of curves.

Lemma 3. *Let A be a subring of a field K, S a multiplicative subset of A, and B the integral closure of A in K. Then the integral closure of $S^{-1}A$ is $S^{-1}B$.*

Proof. Suppose x is integral over $S^{-1}A$, satisfying $x^n + a_{n-1}/s_{n-1}x^{n-1} + \ldots + a_0/s_0 = 0$. Let $s = s_1 \cdots s_n$. Multiplying by s^n, we have $(sx)^n + a'_{n-1}(sx)^{n-1} + \ldots + a'_1(sx) + a'_0 = 0$ with the $a'_i \in A$. Hence sx is integral over $S^{-1}A$, so that $sx \in B$ and $x \in S^{-1}B$.

Conversely, take some $b/s \in S^{-1}B$ with b satisfying $b^n + \ldots + a_0 = 0$. Multiplying this integral dependence relation by s^{-n} we have $(b/s)^n + (a_{n-1}/s)(b/s)^{n-1} + \ldots + (a_0/s^n) = 0$, so that b/s is integral over $S^{-1}A$. □

1.1.5. *Kähler differentials.* Let K be a k-algebra. Here, we merely note that $\Omega^1_{K/k}$, the *module of differential forms of K over k*, can be constructed by taking the free K-module generated by $\{df : f \in K\}$, and dividing it by the submodule generated by all elements of the form $d(f + f') - df - df'$ and $d(ff') - fdf' - f'df$ with $f, f' \in K$, and dr with $r \in k$. Then, the map $K \otimes_k K \to \Omega_{K/k},\ f \otimes g \mapsto fdg$ is a surjective K-module homomorphism, with kernel generated by all elements of the form $1 \otimes ff' - f \otimes f' - f' \otimes f$ with $f, f' \in K$ (elements of the form $1 \otimes r$ with $r \in k$ are included in this).

1.1.6. *Trace map on a finite seperable field extension.*

Lemma 4. *Let K be a finite seperable field extension of k. Then there exists some $w \in K$ such that $Tr_{K/k}(w) \neq 0$.*

Proof. By the well-known primitive element theorem, $K = k(\alpha) = k[x]/(f(x))$ for some $\alpha \in K$ and some irreducible, seperable polynomial $f(x) \in k[x]$. Let $\bar{k}$ be the algebraic closure of k (containing K). Then,

$$\begin{aligned} K \otimes_k \bar{k} \cong k[x]/(f(x)) \otimes_k \bar{k} &= \bar{k}[x]/(f(x)) \\ &\cong \bar{k}[x]/(x-\alpha_1)\cdots(x-\alpha_n) \\ &\cong \prod \bar{k} \quad \text{by the Chinese remainder theorem} \end{aligned}$$

Let $e_1, \ldots, e_n$ be the obvious $\bar{k}$-basis for $\prod \bar{k}$. The discriminant $det(\{Tr(e_i e_j)\})$ of $K \otimes_k \bar{k} \cong \prod \bar{k}$ over $\bar{k}$ is $1 \neq 0$, so the discriminant of $K \otimes_k \bar{k}$ will be nonzero independent of the choice of basis.

Given a basis $w_1, \ldots, w_n$ of K over k, $w_1 \otimes 1, \ldots, w_n \otimes 1$ is a basis for $K \otimes_k \bar{k}$ over $\bar{k}$. Hence,

$$det(\{Tr(w_i w_j)\}) = det(\{Tr(w_i \otimes 1 \cdot e_j \otimes 1)\})$$

is nonzero, so that we must have $\mathrm{Tr}_{K/k}(w_i w_j) \neq 0$ for some $w_i w_j \in K$.

□

1.1.7. *Completion.* Completion is a useful tool for localizing a scheme "just enough". The following lemma will be needed in section-5, when considering how residues behave with respect to surjective morphisms of curves.

Lemma 5. *Let A be a local ring, A_p its completion with respect to its maximal ideal p, K its field of fractions and K_p the completion of K with respect to the valuation whose valuation ring is A. Then K_p equals the field of fractions of A_p.*

Proof. Note that K_p is the completion of K with respect to the system of neighborhoods $pA \supseteq p^2A \supseteq \ldots$ of 0, and A_p is the completion of A with respect to the subset topology of A. So, $0 \to A \to K \to K/A \to 0$ induces $0 \to A_p \to K_p \to K/A \to 0$ (where the completion of K/A is again K/A, since the quotient topology on it is the discrete topology). Since K is a flat A-algebra, we get

$$0 \to A_p \otimes K \to K_p \otimes K = K_p \to K/A \otimes K = 0.$$

□

1.2. **Schemes.** The eventual object of interest in this paper is a "regular curve proper over a field k". In this subsection, we explore the implications of properness in a slightly more general setting. The most important result here is lemma-6, which gives a relation between the points of a scheme X and the field extension $K(X)/k$.

We start with two basic properties of schemes. A scheme X is *reduced* if its local rings have no nilpotent elements, and it is *integral* if $\mathcal{O}_X(U)$ is an integral domain for every open subset U of X.

Proposition 1. *A scheme is integral if and only if it is reduced and irreducible.*

Proof. Since all local rings of an integral scheme are integral domains (but the converse does not hold), clearly an integral scheme is reduced. If an integral scheme X is not irreducible, we can pick disjoint open sets U and V contained in X. Then $\mathcal{O}_X(U \cup V) = \mathcal{O}_X(U) \times \mathcal{O}_X(V)$ is not an integral domain. Conversely, suppose there exist nonzero elements $f, g \in \mathcal{O}_X(U)$ such that $fg = 0$. Let $Y_f = \{x \in U | f_x \in m_x\}$, the complement of $D(f) \cap U$ in U, and $Y_g = \{x \in U | g_x \in m_x\}$, the complement of $D(g) \cap U$ in U. We have $Y_f \cup Y_g = U$. But since X is irreducible, U is irreducible. So $Y_f = U$ or $Y_g = U$, say $Y_f = U$. Then, given any open affine subset $W = \mathrm{SpecR}$ of U, $f|_W$ is in the nilradical of R. Since X is reduced, covering U by open affine subsets, we have $f = 0$. □

Definition 3. A morphism $f : X \to Y$ is of *finite type* if for every open affine subset $U = SpecR$ of Y, $f^{-1}(U)$ can be covered by finitely many open affine subsets $\{U_i\} = \mathrm{Spec}R_i$ with each R_i a finitely generated R-algebra.

Definition 4. A scheme X is noetherian if it can be covered by finitely many open affine subsets $\{U_i\} = \mathrm{Spec}R_i$ with each R_i a noetherian ring.

In particular, if X is of finite type over $Y = \mathrm{Spec}k$, then X can be covered by finitely many open affine subsets $\{U_i\} = \mathrm{Spec}R_i$ where each R_i is a finitely generated k-algebra, i.e. isomorphic to some $k[x_1, \ldots, x_n]/I$. Hence X is noetherian in this case.

Definition 5. X is *proper* over k if the morphism $X \to \mathrm{Spec}k$ is separated, of finite type, and universally closed.

We will mainly use two consequences of properness: The following lemma, and the fact that a proper curve is projective (next section).

Theorem 7. *Let $f : X \to Y$ be a morphism of finite type, with X noetherian. Then f is proper if and only if for every valuation ring R with field of fractions K and for every morphism of $U = SpecK$ to X and $T = SpecR$ to Y forming a commutative diagram*

$$\begin{tikzcd} U \arrow[r] \arrow[d, "i"'] & X \arrow[d, "f"] \\ T \arrow[r] & Y \end{tikzcd}$$

there exists a unique morphism $T \to X$ making the following diagram commute

$$\begin{tikzcd} U \arrow[r] \arrow[d] & X \arrow[d] \\ T \arrow[r] \arrow[ur] & Y \end{tikzcd}$$

Proof. [Har], Chpt. 2, Thm. 4.7. □

This characterization of properness is much more handy than the raw definition. It gives us our first important bridge between schemes and their underlying rings, in the form of the following lemma.

Lemma 6. *Let X be an integral noetherian scheme over a field k, and let K be its function field. We say that a valuation ring R of K/k has* center *x on X if R dominates $\mathcal{O}_{X,x}$. If X is proper over k, then every valuation ring of K/k has a unique center on X.*

Proof of Lemma. Given a valuation ring R of K/k, we have the morphisms

- $T = SpecR \to Speck$, induced by $k \hookrightarrow R$, and
- $U = SpecK \to X$, taking the only point of $SpecK$ to the generic point of X,

adding up to the commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d, "i"'] & X \arrow[d] \\ T \arrow[r] & Speck \end{tikzcd}$$

which means there is a unique morphism $f : SpecR \to X$ by the valuative criterion of properness. Let p denote the only closed point of T. If $f(p) = x$, we have

$$\begin{tikzcd} \mathcal{O}_{X,x} \arrow[rr, "\varphi"] \arrow[dr] & & T_p = R \arrow[dl] \\ & K & \end{tikzcd}$$

where φ is an injective local homomorphism, so that R dominates $\mathcal{O}_{X,x}$. On the other hand, suppose R had another center x' on X. Then, we have the inclusion maps of rings $\mathcal{O}_{X,x'} \hookrightarrow R \hookrightarrow K$, inducing the morphisms $U \to SpecR \to Spec\mathcal{O}_{X,x'} \to X$. This gives another morphism $f' : T \to X$ commuting with the diagram. R dominates $\mathcal{O}_{X,x'}$, so we know f' takes p to x'. Hence f' is a different morphism than f, contradicting the valuative criterion of properness.

□

Leaving commutative algebra aside for a bit, we state a basic but very useful lemma about the cohomology of schemes. While simple, it provides a large chunk of the benefit of our working with schemes.

Lemma 7. *Let X be any scheme. A flasque sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules has zero cohomology in dimension 1. In particular, if X is irreducible, then a constant sheaf of $\mathcal{O}_X$-modules has zero cohomology in dimension 1.*

Proof. Embed $\mathcal{F}$ in an injective $\mathcal{O}_X$-module $\mathcal{I}$. We have

$$0 \to \mathcal{F} \to \mathcal{I} \to \mathcal{F}/\mathcal{I} \to 0,$$

and the induced long exact sequence

$$0 \to \Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{I}) \to \Gamma(X, \mathcal{I}/\mathcal{F}) \to H^1(X, \mathcal{F}) \to H^1(X, \mathcal{I}) = 0$$

But since $\mathcal{F}$ is flasque, we also have

$$0 \to \Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{I}) \to \Gamma(X, \mathcal{I}/\mathcal{F}) \to 0$$

□

Remark 1. Indeed, a flasque sheaf has zero cohomology in all positive dimensions, as follows easily from the above by induction, after noting that quotients of flasque sheaves and injective $\mathcal{O}_X$-modules are flasque.

1.3. **Curves.** Now, we narrow our focus to the case of curves. A *curve* is a scheme of dimension 1, of finite type over some field k. A curve is then Noetherian, because any scheme of finite type over a field is Noetherian as noted in the previous section.

The first important result of this Subsection is Lemma-8, which gives a relation between a surjective morphism of schemes and the corresponding inclusion of function fields. Often one wants to compute the residue in an easy case, and then pass to another curve using this lemma. For example, Serre proves the residue theorem by direct calculation in the easy case of the projective line, and then generalizes the result using surjective morphisms of curves.

In this paper, we are interested in connected regular curves proper over some field k. The other important result in the Subsection, Theorem-8, is a standard result that basically means our curves are projective as well. This will sanction us to use some standard cohomological results formulated for projective schemes, and say things about the k-dimensions of the global sections and the first cohomological groups of our schemes. The fact that these dimensions are finite will make the formulation of the rudimentary Riemann-Roch theorem in the final subsection make sense, as well as help directly in proving the residue theorem later on.

From here on, unless noted otherwise, X denotes a curve as described in the previous paragraph.

Definition 6. A scheme X is *regular* if all of its local rings are regular.

Note that being regular is an absolute notion, i.e. independent of the base field k. Summing up the results in the previous sections about regular rings of Krull dimension 1, we note that the local rings of X at its closed points are integral domains; integrally closed; discrete valuation rings; unique factorization domains.

Proposition 2. *A regular connected noetherian scheme X of dimension 1 is irreducible.*

Proof. Suppose X is not irreducible and let $C_1, \dots, C_s$ be the irreducible components of X. Since X is connected there are distinct irreducible components of X whose intersection contains a closed point, say $x \in C_1 \cap C_2$. Let U = SpecR be an affine neighborhood of x. Then $C_1 \cap U$ and $C_2 \cap U$ are irreducible subsets of U, say $C_1 \cap U = V(\mathfrak{p}_1)$ and $C_2 \cap U = V(\mathfrak{p}_2)$ for prime ideals $\mathfrak{p}_1, \mathfrak{p}_2$ of R. We have $\mathfrak{p}_1, \mathfrak{p}_2 \subseteq \mathfrak{m}_x$. But X is regular, so $\mathcal{O}_{U,x}$ is a DVR and has only m_x and 0 as prime ideals. Hence, without loss of generality, we have $\mathfrak{p}_1 \subseteq \mathfrak{p}_2$. Since $\{x\} \subseteq V(\mathfrak{p}_2) \subseteq V(\mathfrak{p}_1)$ and U is of dimension 1, we either have $C_2 \cap U = \{x\}$ or $C_2 \cap U = C_1 \cap U$. The former contradicts the irreducibility of C_2, and the latter means $C_2 = (C_2 \cap U)^- = (C_1 \cap U)^- = C_1$. We conclude that X is irreducible. □

Since X is regular, it is reduced, and hence an integral scheme by Proposition-1.

Thus, X consists of a generic point and some closed points. We denote the local ring at the generic point, or the *function field* of X, by K(X) or just K. Since X is integral, we can think of all local rings $\mathcal{O}_{X,p}$ of X as subrings of $K(X) = \mathcal{O}_{X,\eta}$, each local ring having K(X) as its field of fractions. The morphism $X \to Speck$ induces $k \hookrightarrow \mathcal{O}_{X,p} \hookrightarrow K(X)$, giving a discrete valuation ring of K/k. In light of Lemma-6, we conclude that the closed points of X correspond bijectively to the set of valuation rings of K/k.

Next, we make some comments about a surjective morphism of two curves of the type we are interested in

$$\begin{tikzcd} X' \arrow[rr] \arrow[dr] & & X \arrow[dl] \\ & k & \end{tikzcd}$$

inducing a commutative diagram of function fields:

$$\begin{tikzcd} K(X) \arrow[rr] \arrow[dr] & & K(X') \arrow[dl] \\ & k & \end{tikzcd}$$

Since $K(X)$ and $K(X')$ are both function fields of dimension 1 over k, $K(X')$ is a finite extension field of $K(X)$. Suppose $p' \mapsto p$, where p, p' are closed points. Then, we have the commutative diagram (with all maps injective, and where $\mathcal{O}_{p'}$, $\mathcal{O}_p$ are discrete valuation rings)

$$\begin{tikzcd} \mathcal{O}_{p'} \arrow[d] & \mathcal{O}_p \arrow[l] \arrow[d] \\ K' & K \arrow[l] \end{tikzcd}$$

Claim 2. *$\mathcal{O}_{p'}$ dominates $\mathcal{O}_p$.*

Proof. Note that $\mathcal{O}_{p'} \cap K = \mathcal{O}_p$ by (Lemma-1). $r \in m_p$ implies $r \in m_{p'} \cap \mathcal{O}_p$, since otherwise $r^{-1} \in \mathcal{O}_{p'} \cap K$, but $r^{-1} \notin \mathcal{O}_p$. On the other hand, $r \in m_{p'} \cap \mathcal{O}_p$ implies $r \in m_p$, since otherwise $r^{-1} \in m_p \subseteq \mathcal{O}_{p'}$. □

Conversely, suppose R is a valuation ring of $K(X')/k$ that dominates $\mathcal{O}_p$ considered as a subring of $K(X')$. By (Lemma-6), R has a unique center p' on X. Then, $p' \mapsto p$: Suppose $p' \mapsto x$. Then R dominates both $\mathcal{O}_p$ and $\mathcal{O}_x$ by the claim, so that

$\mathcal{O}_p = R \cap K = \mathcal{O}_x$ by (Lemma-1). Thus $p = x$ (since $\mathcal{O}_p = \mathcal{O}_x$ has a unique center on X). This concludes the proof of the following lemma:

Lemma 8. *Let $f : X' \to X$ be a morphism of two connected regular schemes of dimension 1 over a field k. Then, $f(p') = p$ if and only if $\mathcal{O}_{p'}$ dominates $\mathcal{O}_p$.*

The rest of this subsection, except for one short result at the very end, is devoted to proving that a regular proper curve over a field is projective. The result holds, but is somewhat more complicated to prove, without the assumption of regularity. Since our curves are regular anyway, we will keep regularity as a condition.

Once we have constructed projective morphisms for small pieces, the following two lemmas will help us combine them.

Lemma 9. *Let X be a reduced scheme over S, Y a separated scheme over S. If f and g are two S-morphisms that agree on an open dense subset of X, then f=g.*

Proof. [Liu], Prop. 3.11, p. 102. ☐

Lemma 10. *Let Y be a proper scheme over a field k, X a normal proper curve over k, and U an open subset of X. Then any morphism $f : U \to Y$ extends to a morphism $X \to Y$.*

Proof. The generic point η of X is contained in U, so f induces a morphism f_η : $\mathrm{Spec}K(X) \to Y$ (given by $\mathcal{O}_Y(U) \to \mathcal{O}_{Y,f(\eta)} \to k(f(\eta)) \to K(X)$). For any closed point $x \in X$ (not necessarily contained in U), $\mathcal{O}_{X,x}$ is a discrete valuation ring with field of fractions $K(X)$, so we have

$$\begin{tikzcd} \mathrm{Spec}K(X) \arrow[r] \arrow[d] & Y \arrow[d] \\ \mathrm{Spec}\mathcal{O}_{X,x} \arrow[r] \arrow[ur, "f_x"'] & k \end{tikzcd}$$

where f_x is the unique map of the valuative criterion of properness. Next, we show that f_x extends to a morphism from an open neighborhood of x.

Let $V = \mathrm{Spec}R$ be an open affine neighborhood of $f_x(x)$ for a finitely generated k-algebra R, say $R = k[\alpha_1, \ldots, \alpha_n]$. Then the whole image of f_x is contained in V, and f_x factors into $\mathrm{Spec}\mathcal{O}_{X,x} \to V \to Y$, with the first morphism corresponding to some ring homomorphism $\varphi : R \to \mathcal{O}_{X,x}$. For each i, pick some

$s_i \in \mathcal{O}_X(V_i)$ having the germ $\varphi(\alpha_i) \in \mathcal{O}_{X,x}$. We thus have a commutative diagram

$$\begin{tikzcd} k[\alpha_1, \ldots, \alpha_n] \arrow[r] \arrow[dr, "\varphi"] & \mathcal{O}_X(U_x) \arrow[d] \\ & \mathrm{Spec}\mathcal{O}_{X,x} \end{tikzcd}$$

(where U_x is an open affine neighborhood of x contained in $\mathcal{O}_X(\bigcap_i V_i)$), corresponding to an extension g_x of f_x to U_x.

Now, let W be an open affine neighborhood of $g_x(x)$, and consider

$$U' = f^{-1}(W) \bigcap {g_x}^{-1}(W) \subseteq U \bigcap U_x$$

U', as it contains η, is nonempty. The diagram

$$\begin{tikzcd} \mathcal{O}_Y(W) \arrow[r, "f|_{U'}"] \arrow[d, "g_x|_{U'}"] & \mathcal{O}_X(U') \arrow[d] \\ \mathcal{O}_X(U') \arrow[r] & K(X) \end{tikzcd}$$

is commutative, with $\mathcal{O}_X(U') \to K(X)$ injective, so $f|_{U'}$ and $g_x|_{U'}$ induce the same homomorphism $\mathcal{O}_Y(W) \to \mathcal{O}_X(U')$. Since morphisms from a scheme to an affine scheme correspond bijectively to ring homomorphisms of global sections in the reverse direction, we conclude $f|_{U'} = g_x|_{U'}$. Then, by Lemma-9, f and g_x agree on $U \bigcap U_x$. Likewise, g_x and $g_{x'}$ agree on $U_x \bigcap U_{x'}$ for any other closed point x'. Gluing all these g_x, we get the extension of f to X. $\square$

Proposition 3. *A normal proper curve X over a field k is projective.*

Proof. Let $\{U_i\}_i$ be a finite open cover of X by affine open sets $U_i = SpecR_i$ with each R_i a finitely generated k-algebra. Then for each i there is an open immersion $\varphi : U_i \to Y_i$ for some projective variety Y_i over k, and so a morphism $\bigcap_i U_i \to Y$ where Y is the fiber product of all the Y_i (which is projective). By Lemma-10, this morphism extends uniquely to $f : X \to Y$. Since X and Y are proper over k, f is surjective onto the scheme-theoretic image Z of X and the induced maps $Z_{f(x)} \to \mathcal{O}_{X,x}$ are surjective for each $x \in X$. We now have the following diagram which we will subsequently show to be commutative:

$$\begin{tikzcd} U_i \arrow[r] \arrow[drr, dashed, "\varphi_i"] & C \arrow[r, "f"] & Y \arrow[d] \\ & Z & Y_i \end{tikzcd}$$

Let $g : U_i \to Y_i$ be the composition of the three maps in the diagram. We do know that *this* diagram is commutative

$$\begin{tikzcd} \bigcap_i U_i \arrow[r] \arrow[drrr, "g"'] & U_i \arrow[r] & X \arrow[r] & Y \arrow[d] \\ & & & Y_i \end{tikzcd}$$

because f is just the extension of $\bigcap_i U_i \to Y$ to all of X. So φ_i and g are two morphisms that agree on the dense open subset $\bigcap_i U_i$ of U_i, and hence they are equal on the whole of U_i by Lemma-9. But φ is an isomorphism, so the induced maps $Z_{f(x)} \to \mathcal{O}_{X,x}$ are isomorphisms.

Now, Z is projective and hence proper over k. The only point of Z whose local ring is a field is $f(\eta)$, so it is irreducible. As it is also reduced, it is an integral scheme. All of its local rings are discrete valuation rings, so it is a regular curve. Suppose $f(x) = z = f(y)$ for $x, y \in X$. Then $\mathcal{O}_{X,x}$ and $\mathcal{O}_{X,y}$ both dominate the valuation ring $\mathcal{O}_{Z,z}$ (all considered as subrings of K(X)). Hence $\mathcal{O}_{X,x} = \mathcal{O}_{Z,z} = \mathcal{O}_{X,y}$, and so x = y (because each valuation ring of K/k has a unique center on X). This shows that f is a bijection of the underlying sets of X and Z, taking the generic point of X to the generic point of Z. Since X and Z both have the cofinite topology, f is a homeomorphism. Thus X is isomorphic to the projective scheme Z.

□

Now that we know our curves to be projective, we will make extensive use of the following standard theorem.

Theorem 8. *Let X be a projective scheme over a noetherian ring A. Then for any coherent sheaf $\mathcal{F}$ on X, $H^i(X, \mathcal{F})$ is a finitely generated A-module. In particular, if X is a projective curve over some field k, then $H^0(X, \mathcal{O}_X)$ and $H^1(X, \mathcal{O}_X)$ are finite dimensional vector spaces over k.*

Proof. ([Har], III.5, Thm. 5.2) □

1.4. **Divisors.** We define (Weil) *divisors* on a noetherian integral seperated regular scheme X over some field k (that is not necessarily of dimension 1) and mention their connection to the group PicX of isomorphism classes of invertible sheaves on X. The material here is a recap of ([Har], II.6) simplified to the types of schemes we are interested in.

The first few results are a general introduction to divisors, and will be useful for proving the residue theorem. Lemma-13 is a basic result that we will use often towards the end of the paper, when dealing with duality, after proving the residue theorem.

The final result in the subsection, which explicates an isomorphism between the group of invertible sheaves on a scheme X and a certain quotient group of the divisors on X, will be useful when proving the Riemann-Roch theorem, in addition to being interesting in its own right.

Definition 7. A *prime divisor* of X is a closed irreducible subset of codimension 1. DivX is the free abelian group generated by the prime divisors. A *divisor* of X is an element of DivX.

In our case with dimX equal to 1, the prime divisors are just the closed points of X.

Each prime divisor has a corresponding generic point η. Since the codimension of the closure of η is 1, the Krull dimension of $\mathcal{O}_\eta$ is 1. Moreover, since X is integral, the field of fractions of $\mathcal{O}_\eta$ is K = K(X). Since X is a scheme over Speck, k is contained in $\mathcal{O}_\eta$. Finally, X is regular, so $\mathcal{O}_\eta$ is a discrete valuation ring of K/k. Now, for any $f \in K(X)$ and any η corresponding to a prime divisor Y, the valuation $v_Y(f)$ of f with respect to $\mathcal{O}_\eta$ is defined.

Definition 8. For any $f \in K^*$ we define the divisor (f) of f by $(f) = \sum v_Y(f) \cdot Y \in$ DivX (with the sum taken over all prime divisors Y). An element of DivX that can be expressed as (f) for some $f \in K$ is called a *principal divisor*. The sum $\sum v_Y(f) \cdot Y$ is indeed finite (see lemma below).

The principal divisors form an additive subgroup of DivX: For any $f, g \in K$, we have $(f) - (g) = \sum (v_Y(f) - v_Y(g)) \cdot Y = \sum v_Y(f/g) \cdot Y$, so $(f) - (g)$ is a principal divisor; $(1) = 0$ is the identity element.

Lemma 11. *For any $f \in K^*$, $v_Y(f)$ is nonzero for at most finitely many prime divisors Y of X.*

Proof. Let $U = \text{SpecA}$ be an open affine subset of X satisfying $f \in \mathcal{O}_X(U)$. The generic point η of X is contained in U since X is integral. Then each prime divisor Y of X contained in $Z = X - U$ is an irreducible component of Z (since if there were some closed irreducible subset W of Z containing Y, we would have $Y \subsetneqq W \subsetneqq X$ which is impossible as the codimension of Y in X is one). On the other hand, if Y is not contained in Z, then the generic point of Y must be in U. Since f is a regular function on U, $v_Y(f) \geqslant 0$. If $v_Y(f) > 0$, we have $f \in \eta A_\eta$, and so $f \in \eta$ (as $f \in A$ also). The converse is also true, so $v_Y(f) > 0$ if and only if Y is contained in $V(Af)$. Now $V(Af)$ is a proper closed subset of U since $f \neq 0$, so each prime divisor of U contained in $V(Af)$ is also an irreducible component of $V(Af)$. But $V(Af)$ is noetherian and can only have finitely many irreducible components. □

Definition 9. The quotient of DivX by the subgroup of principal divisors is called the *divisor class group* of X, and is denoted by ClX.

Definition 10. A scheme is called *normal* if all its local rings are integrally closed.

Lemma 12. *Let A be a noetherian integral domain. Then A is a UFD if and only if every prime ideal of A of height 1 is principal.*

Proof. [Mat2], Thm. 47, p. 141. □

Proposition 4. *Let A be a noetherian integral domain. Then A is a UFD if and only if $X = SpecA$ is normal and $ClX = 0$.*

Proof. If A is a UFD, then it is integrally closed, so all of its localizations are integrally closed, and Spec$A$ is a normal scheme. This and the previous lemma reduce the problem to: In an integrally closed domain A, every prime ideal of height 1 is principal if and only if Cl(SpecA)=0.

Suppose every prime ideal of height 1 is principal. Each prime divisor Y corresponds to a prime ideal p_Y of height 1, say generated by f_Y. Then $v_Y(f_Y) = 1$. If we had $f_Y \in p_Z$ for a different prime divisor Z, we would have $p_Y \subsetneqq p_Z$, and so $Z \subsetneqq Y$, contradicting codim(Z,X)=1. For any prime divisor $Z \neq Y$, since $f_Y \notin p_Z$, f_Y is a unit in A_{p_Z}, and so $v_Z(f_Y) = 0$. Hence $(f_Y) = 1 \cdot Y$. By multiplying such f_Y or their inverses we see that any divisor can be written in the form (f) for some $f \in K$.

Conversely, suppose ClX=0. Let p be a prime ideal of height 1. Then $1 \cdot Y = 1 \cdot V(p)$ is a divisor of X, so that we can write $Y = (f)$ for some $f \in K$. This means $v_Y(f) = 1$ (so that $f \in pA_p$) and $v_{Y'}(f) = 0$ (so that $f \in A_{p'}$) for $Y' \neq Y$. Hence,

$f \in A$. Then $f \in A \cap pA_p = p$. It only remains to show that an element f of p with $v_Y(f) = 1$ and $v_{Y'}(f) = 0$ for $Y' \neq Y$ in fact generates p.

Take any $g \in p$, say with $v_Y(g) = n$. Then $v_Y(g/f^n) = v_Y(g) - nv_Y(f) = 0$ $(g/f^n \in K)$ and $v_{Y'}(g/f^n) = v_{Y'}(g) \geq 0$, so that $g/f^n \in A_p$ for all prime ideals p of A of height one. Hence $g/f^n \in A$, and so $g \in (f)$ in A.

□

Note that ClX = 0 means every divisor on X can be expressed as a principal divisor. In the rest of this section, we look at the relation between invertible sheaves and divisors.

Proposition 5. *The isomorphism classes of invertible sheaves on a ringed space X form a group under the operation* $\otimes$*.*

Proof. Given any $x \in$ X, $\mathcal{L}|_{U_x} \cong \mathcal{O}_X|_{U_x}$ and $\mathcal{G}|_{U_x} \cong \mathcal{O}_X|_{U_x}$ for a sufficiently small neighorbhood U_x of x. Then $\mathcal{L} \otimes \mathcal{G}|_{U_x} \cong \mathcal{O}_X \otimes \mathcal{O}_X|_{U_x} \cong \mathcal{O}_X|_{U_x}$, which shows $\mathcal{L} \otimes \mathcal{G}$ is invertible on X. The inverse of an invertible sheaf $\mathcal{L}$ is $\mathcal{L}^\vee = \mathcal{HOM}(\mathcal{L}, \mathcal{O}_X)$:

$$\mathcal{HOM}_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{L} \cong \mathcal{HOM}_{\mathcal{O}_X}(\mathcal{L}, \mathcal{L}) \cong \mathcal{O}_X$$

by the following lemma. □

Lemma 13. *For a locally free sheaf* $\mathcal{L}$ *and any* $\mathcal{O}_X$*-module* $\mathcal{F}$*,* $\mathcal{HOM}_{\mathcal{O}_X}(\mathcal{L}, \mathcal{F}) \cong \mathcal{L}^\vee \otimes_{\mathcal{O}_X} \mathcal{F}$*.*

Proof. Let R be a ring, M a finitely generated free R-module, and N some R-module. The map

$$\psi : HOM_R(M, R) \otimes_R N \to HOM_R(M, N)$$

defined by

$$[\psi(\lambda \otimes n)](m) = \lambda(m) \cdot n$$

is an isomorphism (and independent of the basis chosen for M).
Now, let U be an open set such that $\mathcal{O}_X|_U \cong \mathcal{L}|_U$. For any open affine set $SpecR = V \subseteq U$, noting that $\mathcal{HOM}_{\mathcal{O}_X|_V}(\mathcal{L}, \mathcal{F}) = HOM_R(\mathcal{L}(V), \mathcal{F}(V))$, and using the isomorphism ψ, we have

$$HOM_{\mathcal{O}_X(V)}(\mathcal{L}(V), \mathcal{O}_X(V)) \otimes_{\mathcal{O}_X(V)} \mathcal{F}(V) \cong HOM_{\mathcal{O}_X(V)}(\mathcal{L}(V), \mathcal{F}(V)).$$

Moreover, this isomorphism is independent of the basis chosen for $\mathcal{L}(V)$ (i.e. independent of the isomorphism $\mathcal{O}_X|_V \cong \mathcal{L}|_V$), so they can be glued together to give the canonical isomorphism

$$\mathcal{HOM}_{\mathcal{O}_X}(\mathcal{L}, \mathcal{F})|_U \cong (\mathcal{L}^\vee \otimes_{\mathcal{O}_X} \mathcal{F})|_U.$$

Covering X by such open sets U and glueing these isomorphisms, we get the isomorphism of the lemma. □

Definition 11. The group of Proposition-5 is called the *Picard group* of X, and denoted by PicX.

Theorem 9. *On a noetherian integral separated regular scheme X, the groups ClX and PicX are isomorphic in a natural way.*

Proof. See Chpt.2, Corollary 6.16 [Har] for a proof.
Here, we briefly describe the isomorphism, given by $D \mapsto \mathcal{O}_X(D)$.
For any $x \in X$, D induces a divisor D_x on Spec$\mathcal{O}_{X,x}$ (keeping the coefficients of prime divisors passing through x and making all others 0). Since X is regular, Spec$\mathcal{O}_{X,x}$ is a UFD, and so D_x is a principal divisor, say $D_x = (f_x)$ on Spec$\mathcal{O}_{X,x}$ (where $f_x \in K(X)$). Throwing out the prime divisors that don't pass through x, we get an open set U_x of X on which D_x and (f_x) agree. Then, using the collection $\{(U_x, f_x)\}$ thus obtained (with the $\{U_x\}$ covering X), let $\mathcal{O}_X(D)$ be the subsheaf of $\underline{K(X)}$ defined by

$$\mathcal{O}_X(D)|_{U_x} = {f_x}^{-1}\mathcal{O}_X|_{U_x}.$$

□

Remark 2. In our case with X a curve, given

$$D = \sum_{\text{p closed}} c_p \cdot p,$$

$\mathcal{O}_X(D)$ is the subsheaf of the constant sheaf $\underline{K(X)}$ defined by

$$f \in [\mathcal{O}_X(D)](U) \Leftrightarrow v_p(f) \geq ord_p(-D) \text{ for all closed points } p \in U.$$

1.5. **Rudimentary Riemann-Roch.** We prove an easy version of the Riemann-Roch theorem, which we will need later first to determine the dimension of the *dualizing sheaf*, and then to prove a more powerful version of the Riemann-Roch theorem in the last section of the paper.

Let X be a regular, integral scheme of dimension 1 proper over a field k. For any $\mathcal{O}_X$-module $\mathcal{F}$, denote $dim_k(H^i(X,\mathcal{F}))$ by $h^i(\mathcal{F})$ and $h^0(\mathcal{F}) - h^1(\mathcal{F})$ by $\chi(\mathcal{F})$. These numbers are all finite by Theorem-8.

Definition 12. For a divisor D of X, let $degD = \sum_p dim_k(k(p)) \cdot v_p(D)$.

$dim_k(k(p))$ is indeed finite for each closed point p: $k(p) = \mathcal{O}_p/m_p = R/\mathfrak{p}$, where R is a finitely generated k-algebra such that $U = SpecR$ is an open affine neighborhood of p and $\mathfrak{p}$ is the maximal ideal of R corresponding to p. Then $k(p)$ is a field that is finitely generated as a k-algebra, whence it is a finite extension of k.

Proposition 6. *For any divisor D on X,*

$$\chi(\mathcal{O}_X(D)) = degD + \chi(\mathcal{O}_X)$$

Proof. We use induction on D. For $D = 0$ the equality is obviously true. Suppose it holds for some D. For any closed point p,

$$deg(D+p) + \chi(\mathcal{O}_X) = degD + dim_k(k(p)) + \chi(\mathcal{O}_X),$$

so we need to show the left hand side increases by $dim_k(k(p))$ as well. Note that this suffices, as it also shows that if the equality holds for some D, then it holds for $D - p$ as well. Because $\mathcal{O}_X(D)$ is a subsheaf of $\mathcal{O}_X(D+p)$, we have the exact sequence

$$0 \to \mathcal{O}_X(D) \to \mathcal{O}_X(D+p) \to \mathcal{Q} \to 0,$$

which gives rise to the long exact sequence

$$0 \to H^0(X,\mathcal{O}_X(D)) \to H^0(X,\mathcal{O}_X(D+p)) \to H^0(X,\mathcal{Q})$$

$$\to H^1(X,\mathcal{O}_X(D)) \to H^1(X,\mathcal{O}_X(D+p)) \to H^1(X,\mathcal{Q}) \to 0$$

But $\mathcal{Q}_q = 0$ for $q \neq p$, so $H^1(X, \mathcal{Q}) = 0$, and

$$\begin{aligned} H^0(X, \mathcal{Q}) = \mathcal{O}_X(D+p)_p / \mathcal{O}_X(D)_p &\cong t^{v_p(D+p)} \mathcal{O}_p / t^{v_p(D)} \mathcal{O}_p \\ &= t^{v_p(D)+1} \mathcal{O}_p / t^{v_p(D)} \mathcal{O}_p \\ &\cong k(p). \end{aligned}$$

Applying the general property of vector spaces to the exact sequence above, we get

$$h^0(\mathcal{O}_X(D)) - h^0(\mathcal{O}_X(D+p)) + dim_k(k(p)) - h^1(\mathcal{O}_X(D)) + h^1(\mathcal{O}_X(D+p)) - 0 = 0,$$

and rewriting it

$$\chi(\mathcal{O}_X(D+p)) = \chi(\mathcal{O}_X(D)) + dim_k(k(p))$$

□

Note that, as $ClX \cong PicX$, we could have shown $\chi(\mathcal{L}) = deg\mathcal{L} + \chi(\mathcal{O}_X)$ for *any* invertible $\mathcal{O}_X$-module $\mathcal{L}$, defining $deg\mathcal{L}$ to be the the degree of the divisor corresponding to $\mathcal{L}$. We now know that this is indeed a legitimite definition, since it is an immediate corollary of the proposition that principal divisors have degree 0 (because $\mathcal{O}_X((f)) \cong \mathcal{O}_X$ for any principal divisor (f)).

We easily get the following corollary, which says that the global sections of certain invertible sheaves are trivial.

Corollary 2. *If $degD < 0$, then $H^0(X, \mathcal{O}_X(D)) = 0$*

Proof. Suppose $f \in K(X)$ is a nonzero global section of $\mathcal{O}_X(D)$. Then $deg(f) > 0$, which is a contradiction since $deg(f) = 0$ by the remark above. □

Remark 3. The fact that X is projective was essential to our proof, since otherwise $h^i(X, \mathcal{O}_X)$ may not be finite dimensional over k. Indeed, for instance Spec$k[x]$ is a regular, separated scheme of dimension 1 over k, but it will have many principal divisors that are not of degree 0.

2. Traces

In this section we extend the usual definition of the trace of an endomorphism of a finite dimensional vector space to finite-potent endomorphisms of infinite dimensional vector spaces. This new kind of trace is the main tool for defining the residue in a different way, allowing for Tate's elegant proof of the residue theorem.

Definition 13. Let k be a field and V a vector space over k. Then $\theta \in \mathrm{HOM}_k(V, V)$ is *finite-potent* if $\theta^n(V)$ is finite-dimensional for some $n \in \mathbb{N}$. We then define $Tr_V(\theta)$ to be $Tr_{\theta^n V}(\theta)$.

To give a more concrete feeling to this definition, consider the filtration $\theta^n(V) \subsetneq \theta^{n-1}(V) \subsetneq \cdots \subsetneq V$. Then we can construct a basis $\mathcal{B}$ for V by starting with a basis for $\theta^n(V)$ and extending to a basis for $\theta^{n-1}(V)$, and so on. Attempting to calculate Tr_V with respect to this basis in the usual way, we naturally arrive at the above definition.

When V is finite dimensional, the same basis construction shows $\mathrm{Tr}_V(\theta) = \mathrm{Tr}_{\theta^n V}(\theta)$. Also, since $\theta^{i+1}V = \theta^i(\theta V) \subseteq \theta^i V$, we have $\theta^n(V) = \theta^{n+1}(V)$ for n large enough. Thus, in the definition, we can as well take $Tr_V(\theta)$ to be $Tr_{\theta^n V}(\theta)$ for n large enough.

- **(T1):** If V is finite dimensional, $Tr_V(\theta)$ is the usual trace.
- **(T2):** For a subspace W of V satisfying $\theta(V) \subseteq W$, we have $Tr_V(\theta) = Tr_W(\theta) + Tr_{V/W}(\theta)$.
- **(T3):** If θ is nilpotent, then $Tr_V(\theta) = 0$.

(T3) follows immediately from the definition of Tr_V. (T1) follows immediately from writing out $Tr_V(\theta)$ using the base $\mathcal{B}$. To show (T2), note that $Tr_{V/W}(\theta) + Tr_W(\theta) = Tr_{(\theta^n V + W)/W}(\theta) + \mathrm{Tr}_{\theta^n W}(\theta) = \mathrm{Tr}_{\theta^n V/(\theta^n V \cap W)}(\theta) + \mathrm{Tr}_{\theta^n V \cap W}(\theta)$ for n large enough (since $\theta^n(\theta^n V \cap W) = \theta^n W$ for n large enough that $\theta^n V = \theta^{n+1} V$ and $\theta^n W = \theta^{n+1} W$). This last quantity equals $\mathrm{Tr}_{\theta^n V}(\theta)$, as $(T2)$ holds in the finite case.

Remark 4. The properties (T1), (T2), (T3) give an equivalent definition of Tr_V:

$$
\begin{aligned}
Tr_V(\theta) &= Tr_{\theta^n} V(\theta) + Tr_{V/\theta^n V}(\theta) && \text{(by } T_2\text{)} \\
&= Tr_{\theta^n V}(\theta) && \text{(by } T_1 \text{ and } T_3\text{)}
\end{aligned}
$$

In extending the definition of the trace we have sacrificed its linearity, as the sum of two finite potent maps need not even be finite potent. Even if it is, the trace of the sum might not be the sum of the traces. There is a counterexample where the trace of the sum of *three* finite potent maps is not the sum of their traces, even though the sum of the three of them is finite potent (but the sum of any two of them is not finite potent) ([T1]). However the case of two maps appears to be still an open question for which a similar counterexample will not work ([T2]). Tate himself notes that he doubts linearity for two maps holds in general ([Tat]).

Luckily, we can recover linearity for certain subspaces of $End(V)$. We say F is a *finite potent subspace* of End_V if there is an $n \in \mathbb{N}$ such that $\theta_1 \cdots \theta_n(V)$ is finite dimensional for any $\theta_1, \ldots, \theta_n \in F$.

(T4): For a finite potent k-subspace F of $End(V)$, $Tr_V : F \to k$ is linear.

It is immediate from the definition that any element of F is itself finite potent, so Tr_V is defined on all of F. To show $Tr_V(\theta_1 + \theta_2) = Tr_V(\theta_1) + Tr_V(\theta_2)$, we may assume F is spanned by θ_1 and θ_2. Then $F^n(V)$ is finite dimensional, being generated by the union of at most 2^n finite dimensional vector spaces. For any $\theta \in F$, $Tr_V(\theta) = Tr_{F^nV} + Tr_{V/F^nV}(\theta) = Tr_{F^nV}(\theta)$ since θ is nilpotent on V/F^nV. Now $Tr_V(\theta_1 + \theta_2) = Tr_{F^nV}(\theta_1 + \theta_2) = Tr_{F^nV}(\theta_1) + Tr_{F^nV}(\theta_2) = Tr_V(\theta_1) + Tr_V(\theta_2)$.

(T5): Suppose $\varphi : V' \to V$ and $\psi : V \to V'$ are vector-space homomorphisms and $\varphi\psi : V \to V$ is finite potent. Then $\psi\varphi$ is finite potent and $Tr_V(\varphi\psi) = Tr_{V'}(\psi\varphi)$

Proof. Let n be large enough that $(\varphi\psi)^n V$ is finite dimensional. Then $\psi\varphi$ is finite potent since $\varphi(\psi\varphi)^n V' = (\varphi\psi)^n \varphi V'$ is finite dimensional, and so $\psi(\varphi\psi)^n \varphi V = (\psi\varphi)^{n+1} V$ is also finite dimensional. Now, let n be large enough that $W = (\varphi\psi)^n V = (\varphi\psi)^{n+1} V$ and $W' = (\psi\varphi)^n V' = (\psi\varphi)^{n+1} V'$. Then $\psi W = \psi(\varphi\psi)^n V = (\psi\varphi)^n \psi V \subseteq W'$, and similarly $\varphi W' \subseteq W$. We have $W' = \psi\varphi W' \subseteq \psi W \subseteq W'$, and so $\psi W = W'$; similarly, $\varphi W' = W$. Hence ϕ and ψ induce isomorphisms of W and W'. In particular, fixing a base $\{e_1, \ldots, e_r\}$ for W, we have a base $\{\psi(e_1), \ldots, \psi(e_r)\}$ for W'. Let $< v, e_i >$ denote the i^{th} coordinate of v in the basis $e_1, \ldots, e_n$. Then, $Tr_V(\varphi\psi) = Tr_W(\varphi\psi) = \sum_i < \varphi\psi(e_i), e_i >$. Also, $Tr_{V'}(\psi\varphi) = Tr_{W'}(\psi\varphi) = \sum_i < \psi\varphi \circ \psi(e_i), \psi(e_i) >$. Write $\varphi\psi(e_i) = c_{1,i}e_1 + \ldots + c_{r,i}e_r$, so that $< \varphi\psi(e_i), e_i >= c_{i,i}$ and $< \psi\varphi \circ \psi(e_i), \psi(e_i) >= c_{i,i}$. Hence, $Tr_V(\varphi\psi) = \sum_i c_{i,i} = Tr_{V'}(\psi\varphi)$. $\square$

In the remainder of this section, we fix a k-vector space V, and investigate how we can use Tr_V on certain subspaces of End(V).

We say a k-subspace A of V is *not much bigger than* a k-subspace B, and write $A < B$ if $(A + B)/B$ is finite dimensional.

Claim 3. *This definition is equivalent to requiring $A \subseteq B + W$ for some finite dimensional subspace W of V.*

Proof. If $A \subseteq B + W$, $(A + B)/B \subseteq (B + W)/B$ is clearly finite dimensional. Conversely, if $(A + B)/B$ has a finite basis $\bar{v_1}, \ldots, \bar{v_s}$, A is contained in $B + Span_k\{v_1, \ldots, v_s\}$. □

We say A is *about the same size as* B, and write $A \sim B$, if $A < B$ and $B < A$. Here are three basic properties of these relations:

- $A < B$ and $B < C \Rightarrow A < C$.

  Proof. $A \subseteq B + W_1$ and $B \subseteq C + W_2 \Rightarrow A \subseteq B + (W_1 + W_2)$ □

- For any k-linear map φ, $A < B \Rightarrow \varphi(A) < \varphi(B)$.

  Proof. $A \subseteq B + W \Rightarrow \varphi(A) \subseteq \varphi(B + W) = \varphi(B) + \varphi(W)$, where $\varphi(W)$ is of course finite dimensional. □

- $\sum_{i=1}^{m} A_i < \cap_{j=1}^{n} B_j$ if and only if $A_i < B_j$ for all i,j

  Proof. The obvious map $A_i + \cap B_j \to \bigoplus_{j=1}^{n} (A_i + B_j)/B_j$ has kernel $\cap B_j$, and so we have $(A_i + \cap B_j)/\cap B_j \hookrightarrow \bigoplus_j (A_i + B_j)/B_j$. The latter is finite dimensional by the assumption $A_i < B_j$. This shows $A_i < \cap B_j$ for all i, or equivalently that $A_i \subseteq \cap B_j + W_i$ for each i, for some finite dimensional subspace W_i. Then $\sum A_i \subseteq \cap B_j + (W_1 + \ldots + W_m)$ as wanted. The other direction is immediate. □

Now we fix a k-subspace A of V and define some subspaces of End(V) with respect to this A:

$$\begin{aligned}
\text{E} &= \{\theta \in \text{End(V)} | \theta A < A\} \\
E_1 &= \{\theta \in \text{End}(V) | \theta V < A\} \\
E_2 &= \{\theta \in \text{End}(V) | \theta A < (0)\} \\
E_0 &= \{\theta \in \text{End}(V) | \theta V < A \text{ and } \theta A < (0)\}
\end{aligned}$$

Proposition 7.
(1) *E is a k-subalgebra (with identity) of End(V)*
(2) *E_0, E_1, E_2 are two-sided ideals of E*
(3) *The E's depend only on the $\sim$-class of A*
(4) *$E_1 \cap E_2 = E_0$ and $E_1 + E_2 = E$*
(5) *E_0 is finite potent.*

Proof. (1) For any $\theta_1, \theta_2 \in E$, $\theta_1\theta_2 A \subseteq \theta_1(A+W) = \theta_1(A) + \theta_1(W)$, so $(\theta_1\theta_2)A < A$ For any $\lambda \in k$, $(\lambda\theta)A = \lambda(\theta A) = \theta A < A$ $(\theta_1+\theta_2)A \subseteq \theta_1 A + \theta_2 A < A$ by N_3

(2) let $\varphi \in E$, $\theta, \theta_1, \theta_2, \in E_i$. $E_1 : (\theta_1+\theta_2)V \subseteq \theta_1(V) + \theta_2(V) < A.\theta\varphi(V) \subseteq \theta(V) < A$. $(\varphi\theta)V \subseteq \varphi(A+W) = \varphi(A) + \varphi(W) < \varphi(A) < A$. E_2 : $(\theta_1+\theta_2)A \subseteq \theta_1(A)+\theta_2(A) < (0)$. $(\theta\varphi)A \subseteq \theta(A+W) = \theta(A)+\theta(W) < (0)$. $(\varphi\theta)A \subseteq \varphi(W) < (0)$. E_0 : This is just the intersection of the two ideals E_1 and E_2.

(3) Suppose $A \sim B$. Let $E_\cdot^B$ be the same $E_\cdot$'s defined for B in place of A. Let $\theta \in E_\cdot^A = E_\cdot$. $E : \theta B < \theta A < A < B$ since $B < A$ and $A < B$, so $\theta \in E^B$. By symmetry, $E^A = E^B$. $E_1 : \theta V < A < B$, so $\theta \in E_1^B$. E_2 : $\theta B < \theta A < (0)$, so $\theta \in E_2^B$. $E_0 : E_0 = E_1 \cap E_2 = E_1^B \cap E_2^B = E_0^B$.

(4) $E_1 \cap E_2 = E_0$ is immediate from the definitions. To prove $E = E_1 + E_2$, let π be a linear projection of V onto A (i.e. a linear map $V \to A$ whose restriction to A is the identity). $\pi(V) = A < A$, so $\pi \in E_1$. Also, $(1-\pi)(a) = 0$ for all $a \in A$, so $(1-\pi)(A) = 0$ and $(1-\pi) \in E_2$. Now for any $\theta \in E$, we have $\theta = 1 \cdot \theta = ((1-\pi)+\pi)\theta = (1-\pi)\theta + \pi\theta$, where $\pi\theta \in E_1$ and $(1-\pi)\theta \in E_2$ since E_1 and E_2 are two sided ideals of E.

(5) For any $\theta_1, \theta_2 \in E_0$, we have $\theta_1\theta_2(V) < \theta_1 A < (0)$.

□

Importantly, (5) means our map Tr_V restricted to E_0 is linear (by T_4).

Proposition 8. *Suppose that either $\varphi \in E_0$ and $\psi \in E$, or $\varphi \in E_1$ and $\psi \in E_2$. Then, $[\varphi, \psi] = \varphi\psi - \psi\varphi \in E_0$ and $Tr_V([\varphi, \psi]) = 0$.*

Proof. In either case, $\varphi\psi, \psi\varphi$, and $[\varphi, \psi]$ are elements of the finite potent subspace E_0, and so we have

$$Tr_V(\varphi\psi - \psi\varphi) = Tr_V(\varphi\psi) - Tr_V(\psi\varphi) = 0 \qquad \text{(by } T_4 \text{ and } T_5).$$

□

3. Abstract residues

Using these basic properties of infinite traces, we now define a residue map in a certain algebraic setting, to be used later for the curves we are interested in. Throughout this section, K denotes a commutative k-algebra with identity, V denotes a K-module (which is also a k-vector space in the obvious way), and A denotes a k-subspace of V satisfying $fA < A$ for all $f \in K$. For any element $f \in K$, multiplication by f induces a k-endomorphism of V. Moreover, $fA < A$, so the k-endomorphism induced by f is in fact in E. This gives a (possibly non-injective) homomorphism of k-algebras $K \to E$. In the rest of this thesis we don't distinguish between $f \in K$ and the induced endomorphism $f \in E$.

Let c denote the K-linear map

$$K \otimes_k K \to \Omega^1_{K/k}, \quad f \otimes g \mapsto fdg,$$

which is surjective with kernel generated as a K-module by $\{f \otimes gh - fg \otimes h - fh \otimes g | f, g, h \in K\}$.

3.1. **Definition of residue.** :

The following lemma gives the definition of the abstract residue, and shows that it is well-defined.

Lemma 14. *With K, k, V, A as above, there is a unique k-linear map*

$$res^V_A : \Omega^1_{K/k} \to k,$$

such that for any $f, g \in K$

$$res^V_A(fdg) = Tr_V([f_1, g_1])$$

for any $f_1, g_1 \in E$ satisfying

i: $f \equiv f_1(\text{mod} E_2)$ and $g \equiv g_1(\text{mod} E_2)$

ii: $f_1 \in E_1$ or $g_1 \in E_1$

Proof. Given any $f, g \in K$ we can write $f = f_1+f_2$, $g = g_1+g_2$ since $E = E_1+E_2$, so $res^V_A(fdg)$ is meaningful for all $f, g \in K$. Moreover, we can always pick both f_1 and g_1 from E_1. Suppose f_1', g_1' is any other pair satisfying (i) and (ii), say with $f_1' \in E_1$. Then $f_1 - f_1' \in E_0$, so $Tr_V([f_1 - f_1', g_1']) = 0$. Since $[f_1', g_1'], [f_1 - f_1', g_1'] \in E_0$, we have $Tr_V([f_1', g_1']) = Tr_V([f_1, g_1'])$ by T_4. Likewise, $[f_1, g_1 - g_1'] \in E_0$ and its trace is 0, so $Tr_V([f_1, g_1']) = Tr_V([f_1, g_1])$. Thus $Tr_V([f_1, g_1])$ does not depend on the choice of f_1 and g_1.

Given f_1, g_1 satisfying (i) and (ii), we have $[f_1, g_1] \in E_1$ and

$$f_1g_1 - g_1f_1 \equiv [f, g] = 0(\text{mod} E_2)$$

since K is commutative. Hence $[f_1, g_1] \in E_0$. So, by T_4,

$$Tr_V([f_1, (g+h)_1]) = Tr_V([f_1, g_1 + h_1]) = Tr_V([f_1, g_1]) + Tr_V([f_1, h_1]).$$

Thus the map $K \times K \to k$ defined by $(f, g) \mapsto Tr_V([f_1, g_1])$ is k-bilinear, and so induces the k-linear map $r : K \otimes_k K \to k$, $r(f \otimes g) = Tr_V([f_1, g_1])$.

Now, all that remains is to show that r factors through the map $c : K \otimes_k K \to \Omega^1_{K/k}$. Then res^V_A will be the factoring map $\Omega^1_{K/k} \to k$. The uniqueness of res^V_A is immediate, since c is surjective and $\mathrm{res}^V_A \circ c = r$.

Indeed, r vanishes on the kernel of c:

$r(f \otimes gh - fg \otimes h - fh \otimes g) = r(f \otimes gh) - r(fg \otimes h) - r(fh \otimes g) = \mathrm{Tr}_V([f_1, (gh)_1]) - \mathrm{Tr}_V([(fg)_1, h_1]) - \mathrm{Tr}_V([(fh)_1, g_1]) = \mathrm{Tr}_V([f_1, g_1h_1]) - \mathrm{Tr}_V([f_1g_1, h_1]) - \mathrm{Tr}_V([f_1h_1, g_1])$ (where the pair $((fg)_1 = f_1g_1, h_1)$ satisfies (i) and (ii) once we choose the appropriate $f_1, g_1, h_1 \in E_1$). By T_4, this last quantity equals $Tr_V([f_1, g_1h_1] - [f_1g_1, h_1] - [f_1h_1, g_1]) = Tr_V(0) = 0$. □

3.2. **A lemma for calculating residues.** Of course, the definition of the previous subsection is difficult to use in practice, as it involves traces on infinite dimensional vector spaces. The following lemma expresses the residue of some fixed element as a finite trace (on a vector space that is determined by the fixed element we want to evaluate the residue at).

Let f, g be fixed elements of K.

Let

$$B = gA + A \ (< A)$$

$$C = B \cap f^{-1}(A) \cap (fg)^{-1}(A) = \{v \in B | fv \in A \text{ and } (fg)v \in A\},$$

and suppose $\pi : A + fA + fgA \to A$ is a k-linear projection onto A.

Then,

i: dim(B/C) is finite

ii: $res^V_A(fdg) = Tr_{B/C}([\pi f, g])$

Proof. Extend π to a projection of all of V onto A (eg. pick a basis $\mathfrak{B}$ for $A + fA + fgA$, extend it to a basis for V; then send every base element not in $\mathfrak{B}$ to 0 and every $e \in \mathfrak{B}$ to $\pi(e)$). Then $\pi f \in E_1$, and $\pi f \equiv f (\mathrm{mod} E_2)$ since $(\pi - 1) \in E_2$ and E_2 is an ideal of E. Hence $\mathrm{res}^V_A(fdg) = \mathrm{Tr}_V([\pi f, g])$ by definition.

For any $c \in C$, $f(c), fg(c) \in A$, so

$$[\pi f, g]c = \pi fg(c) - g\pi f(c) = fg(c) - gf(c) = 0 \in C$$

So, by T_2,

$$\mathrm{Tr}_V([\pi f, g]) = \mathrm{Tr}_{V/C}([\pi f, g]) + \mathrm{Tr}_C([\pi f, g]) = \mathrm{Tr}_{V/C}([\pi f, g]).$$

Also, $[\pi f, g](V) \subseteq B$. Applying T_2 once more,

$$\mathrm{Tr}_{V/C}([\pi f, g]) = \mathrm{Tr}_{B/C}([\pi f, g]) + \mathrm{Tr}_{(V/C)/(B/C)}([\pi f, g]) = \mathrm{Tr}_{B/C}([\pi f, g]).$$

To prove dim(B/C) is finite it suffices to show $\dim(A/A \cap f^{-1}A \cap (fg)^{-1}A)$ is finite, since $B = A + gA \subseteq A + W$ for some finite dimensional subspace W of V. As $A \cap f^{-1}A \cap (gf)^{-1}A$ is a k-subspace of A, we can write $A = A' \bigoplus (A \cap f^{-1}A \cap (gf)^{-1}A)$. For each $0 \neq a' \in A'$, we either have $fa' \notin A$ or $gfa' \notin A$.

Now, suppose we had an infinite k-basis $e_1, \ldots$ for A'. Then, we either have $fe_i \notin A$ or $gfe_i \notin A$ (or both) for infinitely many i. Assume without loss of generality that $fe_i \notin A$ for infinitely many i. We can once again decompose A as $A'' \bigoplus (A \bigcap f^{-1}A)$ with A'' having an infinite basis $e_1, \ldots$. As $fa'' \in A$ (with $a'' \in A''$) if and only if $a'' = 0$, we see that f is injective on A'', the vectors $fe_1, \ldots$ are linearly independent, and that the k-span of $fe_1, \ldots$ intersects A at 0 only. This contradicts $fA < A$, so A' must be finite dimensional. □

3.3. **Some properties of $\mathbf{res}_A^V$.** Having established a way to calculate the abstract residue, we now prove some basic properties. All of these will come in handy in the next section, when we work with residues on curves.

(R1). **i:** *If $A \subseteq V' \subseteq V$ and $KV' = V'$ then $res_A^V = res_A^{V'}$.*
ii: *If $A \sim A'$, then $res_A^V = res_{A'}^V$.*

Proof. **i:** For $f, g \in K$, $\mathrm{res}_A^V(fdg) = \mathrm{Tr}_{B/C}([\pi f, g]) = \mathrm{res}_A^{V'}(fdg)$ by applying the lemma above twice.

ii: E, E_0, E_1, E_2 depend only on the equivalance class of A, and by definition res_A^V depends only on V and these subspaces of End(V).

□

Thanks to (R1), from now on we will usually omit the V on res_A^V.

(R2). *If $fA + fgA + fg^2A \subseteq A$, then $res_A(fdg) = 0$*

Proof. Let $b = a + a' \in B$. Then $fb = fa + fga' \in A$ and $fgb = fga + fg^2a' \in A$, so $b \in B \cap f^{-1}(A) \cap (fg)^{-1}(A) = C$. Hence $B = C$ and $\mathrm{res}_A(fdg) = \mathrm{Tr}_{B/C}([\pi f, g]) = 0$. $\square$

In particular, $\mathrm{res}_A(fdg) = 0$ if $fA \subseteq A$ and $gA \subseteq A$. If A is a K-submodule of V, then res_A^V is identically 0.

The following two properties hint at an alternate definition of the residue, to be seen in the next section. Let R be a discrete valuation ring that is also complete, K its field of fractions, and g a fixed uniformizer of R. Then, given some $f \in K$, we can write it in the form of a Laurent series in the variable g. Loosely speaking, in the next section we will apply the two following properties in a situation like this and see that the residue of f equals the coefficient of the -1^{th} term (much like in complex analysis).

(R3). *Let $g \in K$. Then $res_A(g^n dg) = 0$ for all $n \geq 0$; and for all $n \leq -2$ in case g is invertible in K. In particular, $res_A(dg) = 0$ for all $g \in K$.*

Proof. Choose $g_1 \in E_1$ satisfying $g_1 \equiv g (\mathrm{mod}\ E_2)$ (eg. let $g_1 = \pi g = (\pi - 1)g + g$). Note that $g_1^n \equiv g^n (\mathrm{mod}\ E_0)$ for $n \geq 0$. Then $\mathrm{res}_A(g^n dg) = \mathrm{Tr}_V([g_1^n, g_1]) = 0$.
If g is invertible we have $0 = d1 = d(gg^{-1}) = gdg^{-1} + g^{-1}dg$, i.e. $-gdg^{-1} = g^{-1}dg$. Then,

$$g^{-n}g^{-2}dg = g^{-n-1}(g^{-1}dg) = g^{-n-1}(-g)dg^{-1} = -(g^{-1})^n dg^{-1},$$

whose residue is 0 for $n \geq 0$ by the first part of the statement. $\square$

(R4). *If g is invertible in K, and $h \in K$ is such that $hA \subseteq A$, then $res_A(hg^{-1}dg) = Tr_{A/(A\cap gA)}(h) - Tr_{gA/(A\cap gA)}(h)$. In particular, if g is invertible and $gA \subseteq A$, then $res_A(g^{-1}dg) = dim_k(A/gA)$.*

Proof. Let $f = hg^{-1} (= g^{-1}h)$ and let B, C be as in the lemma above. For a projection π of V onto A, we have

$$\begin{aligned}
\mathrm{res}_A^V(fdg) = \mathrm{Tr}_{B/C}([\pi f, g]) &= \mathrm{Tr}_{B/C}(\pi f g - g\pi f) \\
&= \mathrm{Tr}_{B/C}(\pi h - g\pi g^{-1}h) \\
&= \mathrm{Tr}_{B/C}(\pi h - \pi_1 h) \ (\pi_1 \text{ some projection of V onto gA}) \\
&= \mathrm{Tr}_{B/C}(\pi h) - \mathrm{Tr}_{B/C}(\pi_1 h)
\end{aligned}$$

Note that πh and $\pi_1 h$ are both endomorphism of B/C, so that the last line of the equality follows from linearity of the trace on the finite dimensional vector space B/C.

Since $\pi h(C) \subseteq A \cap gA \subseteq C$, we have

$$\mathrm{Tr}_{B/A\cap gA}(\pi h) = \mathrm{Tr}_{B/C}(\pi h) + \mathrm{Tr}_{C/A\cap gA}(\pi h) = \mathrm{Tr}_{B/C}(\pi h) \text{ by } T_2.$$

In the exact same way,

$$\mathrm{Tr}_{B/C}(\pi_1 h) = \mathrm{Tr}_{A+gA/A\cap gA}(\pi_1 h).$$

Hence,

$$\begin{aligned}\mathrm{Tr}_{B/C}(\pi h) - \mathrm{Tr}_{B/C}(\pi_1 h) &= \mathrm{Tr}_{A+gA/A\cap gA}(\pi h) - \mathrm{Tr}_{A+gA/A\cap gA}(\pi_1 h)\\ &= \mathrm{Tr}_{A/A\cap gA}(\pi h) - \mathrm{Tr}_{gA/A\cap gA}(\pi_1 h) \quad \text{by } T_2\\ &= \mathrm{Tr}_{A/A\cap gA}(h) - \mathrm{Tr}_{gA/A\cap gA}(h)\end{aligned}$$

Now suppose $gA \subseteq A$ and let $h = 1$ in the above. Then,

$$\mathrm{res}_A^V(f dg) = \mathrm{Tr}_{A/gA}(1) - \mathrm{Tr}_{gA/gA}(1) = \mathrm{Tr}_{A/gA}(1) = \dim_k(A/gA).$$

$\square$

The following property -with an easy but arduous proof- is the key tool for evaluating an important residue in the next section.

(R5). *Suppose B is another k-subspace of V satisfying $fB < B$ for all $f \in K$. Then,*

$$f(A+B) < A+B, f(A\cap B) < A\cap B \quad \textit{for all } f \in K,$$

and we have

$$res_A + res_B = res_{A+B} + res_{A\cap B}$$

Proof. The first two statements follow immediately from the properties of $<$.

To prove the last statement, we first construct projection maps of V onto A, B, A+B and $A \cap B$ satisfying $\pi_A + \pi_B = \pi_{A+B} + \pi_{A\cap B}$.

We have $A \cap B \subseteq A \subseteq A+B \subseteq V$, which we will use to construct an appropriate basis for V. Let $\mathfrak{B}$ be a basis for $A\cap B$. Let $\mathfrak{B}_A$ and $\mathfrak{B}_B$ be extensions of $\mathfrak{B}$ to A and B respectively. Then $\mathfrak{B}_A \cup \mathfrak{B}_B$ is a basis for A+B. Finally, extend this to a basis $\mathfrak{B}_V$ of V. Now let $\pi_A : V \to A$ be the projection map that takes every basis element in

$\mathfrak{B}_A$ to itself and takes $\mathfrak{B}_V - \mathfrak{B}_A$ to 0, and so on. Clearly, $\pi_A + \pi_B = \pi_{A+B} + \pi_{A\cap B}$. We have

$$\mathrm{res}_A(fdg) - \mathrm{res}_{A+B}(fdg) = \mathrm{Tr}_V([\pi_A f, g]) - \mathrm{Tr}_V([\pi_{A+B} f, g]).$$

Claim. *$\phi_1 = [\pi_A f, g]$ and $\phi_2 = [\pi_{A+B} f, g]$ generate a finite potent subspace of End(V):*

Proof. **i:** $\phi_1(V) = (\pi_A g f - g\pi_A f)(V) \subseteq (\pi_A g - g\pi_A)(V) \subseteq A + gA < A < A + B$

Similarly, $\phi_2(V) < A + B$.

ii: $\phi_1(A+B) < \phi_1(V) < A$

$\phi_2(A+B) = (\pi_{A+B} g - g\pi_{A+B}) f(A+B) < (\pi_{A+B} g - g\pi_{A+B})(A+B) = (\pi_{A+B} g - g)(A+B) = (\pi_{A+B} - 1)g(A+B) < (\pi_{A+B} - 1)(A+B) = (0) < A$

iii: $\phi_1(A) < (\pi_A g - g\pi_A)A = (\pi_A g - g)A = (\pi_A - 1)gA < (\pi_A - 1)A = (0)$

$\phi_2(A) < \phi_2(A+B) < (0)$

□

Substituting $\pi_{A\cap B}$ and π_B for π_A and π_{A+B} respectively in the proof above, we see that $[\pi_B f, g]$ and $[\pi_{A\cap B} f, g]$ also generate a finite potent subspace of End(V). By the linearity of Tr_V on finite potent subspaces, we have for any $f, g \in K$,

$$\begin{aligned}
\mathrm{res}_A(fdg) - \mathrm{res}_{A+B}(fdg) &= \mathrm{Tr}_V([\pi_A f, g]) - \mathrm{Tr}_V([\pi_{A+B} f, g]) \\
&= \mathrm{Tr}_V([(\pi_A - \pi_{A+B}) f, g]) \\
&= \mathrm{Tr}_V([(\pi_{A\cap B} - \pi_B) f, g]) \\
&= \mathrm{Tr}_V([\pi_{A\cap B} f, g]) - \mathrm{Tr}_V([\pi_B f, g]) \\
&= \mathrm{res}_{A\cap B}(fdg) - \mathrm{res}_B(fdg)
\end{aligned}$$

□

The final property we will prove is useful for calculating the residue in a simple case, and then passing to a more general curve by a base extension. A corollary of this result is how Serre generalizes the residue theorem, after manually calculating the residue in the simple case of $\mathbb{P}^1_k$.

(R6). *Let K' be a commutative K-algebra which is a free K-module of finite rank. Let $V' = K' \otimes_K V$ and $A' = \sum_i x_i \otimes A = \sum_i^n \{x_i \otimes a_i | a_i \in A\} \subseteq V'$ for a K-basis $\{x_1, \ldots, x_n\}$ of K'. Then,*

- **i:** *$f'A' < A'$ for all $f' \in K'$.*
- **ii:** *The $\sim$-equivalance class of A' depends only on that of A, not on the choice of the basis $\{x_i\}$, and*
- **iii:** *$res_{A'}(f'dg) = res_A((Tr_{K'/K}f')dg)$ for any $f' \in K$, $g \in K$.*

Proof. We can write $f' = h_1x_1 + \ldots + h_nx_n$ where $h_i \in K$ for each i. Also, for each j, we have

$$f'x_j = r_{1j}x_1 + \ldots + r_{nj}x_n \text{ with } r_{ij} \in K.$$

Then,

$$\begin{aligned} f'(\sum_i x_i \otimes a_i) = \sum_i f'x_i \otimes a_i &= \sum_i((\sum_j r_{ij}x_j) \otimes a_i) \\ &= \sum_i \sum_j (x_j \otimes r_{ij}a_i) \\ &= \sum_j x_j \otimes (\sum_i r_{ij}a_i) \end{aligned}$$

Since $r_{ij}A < A$ for each r_{ij}, there is a finite dimensional subspace W of V satisfying $r_{ij}A \subseteq A + W$ for each r_{ij}. Hence, $\sum_j x_j \otimes (\sum_i r_{ij}a_i) \in \sum_j x_j \otimes (A + W) = \sum_j x_j \otimes A + \sum_j x_j \otimes W < \sum_j x_j \otimes A = A'$ (note that $\sum_j x_j \otimes W$ is a finite dimensional k-vector space with basis $\{x_j \otimes e\}$ as $j = 1, \ldots, n$ and e runs through the elements of a basis for W). This concludes the proof of (i).

(ii): Let $\{z_j\}$ be another basis for K' over K. For each j, write $z_j = r_{1j}x_1 + \ldots + r_{nj}x_n$. Then we have $A'' = \sum_i z_i \otimes A = \sum_j x_j \otimes (\sum_i r_{ij}A) < A'$ as in the proof of *(i)*. By symmetry, we also have $A' < A''$, showing that the $\sim$-equivalance class of A' does not depend on the choice of basis. On the other hand, suppose B is another subspace of V satisfying $A \sim B$. Then $B' = \sum_i x_i \otimes B \subseteq \sum_i x_i \otimes (A + W) = \sum_i x_i \otimes A + \sum_i x_i \otimes W < A'$. Again by symmetry $A' \sim B'$, showing that A' depends only on the equivalance class of A and not A itself.

(iii)

Note that $K' \otimes V \cong K^n \otimes V = V^n$ as K-modules, where the first isomorphism is determined by the choice of basis $\{x_i\}$. Thus, let us denote elements of V' by n-tuples of elements of V, with $\begin{pmatrix} v_1 \\ \dots \\ \dots \\ v_n \end{pmatrix}$ corresponding to $\sum_i x_i \otimes v_i$. Now, given a k-endomorphism φ of V', we can decompose it as $\varphi = \varphi_1 + \ldots + \varphi_n$, where φ_j equals φ on $x_j \otimes V$ and 0 on $x_i \otimes V$ for $i \neq j$. Then, let $\varphi_{ij} = \pi_i \circ \varphi_j$. Thus,

$$\varphi \begin{pmatrix} v_1 \\ \dots \\ \dots \\ v_n \end{pmatrix} = \varphi(\sum_j x_j \otimes v_j) = \sum_j (\varphi_j(x_j \otimes v_j)) = \sum_j (\sum_i \varphi_{ij}(x_j \otimes v_j)).$$

By abuse of notation, let φ_{ij} also denote the k-endomorphism of V taking v to v' if $\varphi_{ij}(x_j \otimes v) = x_i \otimes v'$. Thus, we can write

$$\varphi \begin{pmatrix} v_1 \\ \dots \\ \dots \\ v_n \end{pmatrix} = \begin{pmatrix} \varphi_{11} & \dots & \dots & \varphi_{1n} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \varphi_{n1} & \dots & \dots & \varphi_{nn} \end{pmatrix} \begin{pmatrix} v_1 \\ \dots \\ \dots \\ v_n \end{pmatrix}$$

On the other hand, any such $n \times n$ matrix of k-endomorphisms of V defines a k-endomorphism of V':

$$\begin{pmatrix} \varphi_{11} & \dots & \dots & \varphi_{1n} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \varphi_{n1} & \dots & \dots & \varphi_{nn} \end{pmatrix} \begin{pmatrix} v_1 \\ \dots \\ \dots \\ v_n \end{pmatrix} = \sum_i (\sum_j x_i \otimes \varphi_{ij}(v_j)) = \sum_{i,j} x_i \otimes \varphi_{ij}(v_j)$$

Suppose F is a finite potent subspace of End(V). Then, $n \times n$-matrices with entries from F form a finite potent subspace F' of End(V'). Given $M = \varphi \in F'$, write $M = M' + M_1 + M_2$ with M' diagonal, and M_1 lower triangular, M_2 upper triangular with 0's on the diagonal. Since F' is finite potent, we have (even though the notation looks like the trace on the matrices, we are still in the infinite dimensional setting)

$$\begin{aligned}
\mathrm{Tr}_{V'}(M) &= \mathrm{Tr}_{V'}(M') + \mathrm{Tr}_{V'}(M_1) + \mathrm{Tr}_{V'}(M_2) \\
&= \mathrm{Tr}_{V'}(M') \text{ (since } M_1 \text{ and } M_2 \text{ are nilpotent)} \\
&= \mathrm{Tr}_{V'} \begin{pmatrix} \varphi_{11} & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 \end{pmatrix} + \ldots + \mathrm{Tr}_{V'} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & \varphi_{nn} \end{pmatrix}
\end{aligned}$$

Next, let us relate $\mathrm{Tr}_{V'} \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & \varphi_{ii} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{pmatrix}$ with $\mathrm{Tr}_V(\varphi_{ii})$.

Let $\varphi'_{ii} : V' \to V'$ denote the map corresponding to $\mathrm{Tr}_{V'} \begin{pmatrix} \varphi_{11} & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{pmatrix}$.

$$\begin{aligned}
\mathrm{Tr}_{V'}(\varphi'_{ii}) &= \mathrm{Tr}_{V'/x_i \otimes \varphi^r_{ii} V}(\varphi'_{ii}) + \mathrm{Tr}_{x_i \otimes \varphi^r_{ii} V}(\varphi'_{ii}) \\
&= \mathrm{Tr}_{x_i \otimes \varphi^r_{ii} V}(\varphi'_{ii}) \\
&= \mathrm{Tr}_{\varphi^r_{ii} V}(\varphi_{ii})
\end{aligned}$$

Hence, $\mathrm{Tr}_{V'}(M') = \sum_{i=1}^n \mathrm{Tr}_{\varphi^r_{ii} V}(\varphi_{ii}) = \sum_{i=1}^n \mathrm{Tr}_V(\varphi_{ii})$.

Write $f' x_j = \sum_{i=1}^n f_{ij} x_i$ with $f_{ij} \in K$. Let $\pi : V \to V$ be a k-linear projection, and let $\pi' : V' \to A'$ be the k-linear projection defined by $\pi(\sum x_i \otimes v_i) = \sum x_i \otimes \pi v_i$.

Before finally proving our result, we consider what the ij^{th} entry of the matrix $\langle [f'\pi', g] \rangle$ representing $[f'\pi', g]$ is.

$$\begin{aligned}
\langle f'\pi' \rangle (v_1, \ldots, v_n) = \langle f' \rangle (\pi v_1, \ldots, \pi v_n) &= \langle \{f_{ij}\} \rangle (\pi v_1, \ldots, \pi v_n) \\
&= \langle \{f_{ij} \pi\} \rangle (v_1, \ldots, v_n)
\end{aligned}$$

Also, $\langle g\rangle$ is just the diagonal matrix with all its nonzero entries equal to g, since $g \in K$. Hence,

$$\langle [f'\pi', g]\rangle = \langle \{f_{ij}\pi g\}\rangle - \langle \{g f_{ij}\pi\}\rangle = \langle \{[f_{ij}\pi, g]\}\rangle.$$

Thus,

$$\begin{aligned}
\mathrm{res}_{A'}^{V'}(f'dg) &= \mathrm{Tr}_{V'}([f'\pi', g]) \\
&= \sum_{i=1}^{n} \mathrm{Tr}_V([f'\pi', g]_{ii}) \\
&= \sum_{i=1}^{n} \mathrm{Tr}_V([f_{ii}\pi, g]) \\
&= \mathrm{Tr}_V([\sum_{i=1}^{n} f_{ii}\pi, g]) \quad \text{(the } [f_{ii}\pi, g] \text{ form a finite potent subspace)} \\
&= \mathrm{Tr}_V([(\mathrm{Tr}_{K'/K} f')\pi, g]) \\
&= \mathrm{res}_A^V((\mathrm{Tr}_{K'/K} f')dg)
\end{aligned}$$

□

4. Algebraic curves

Finally, we apply our results about abstract residues to curves. This section is the gist of the thesis, bringing to conclusion Tate's elegant proof of the residue theorem. The actual use of the residue theorem for duality, which is similar in the accounts of Tate and Serre, is left to the next section.

Throughout this section, X denotes a connected regular scheme of dimension 1, proper over a ground field k and K = K(X) denotes its function field (see Section-1.3 for some standard implications of these hypotheses).

4.1. **Residues on the curve X.** For each closed point p of X, let A_p be the completion of $\mathcal{O}_p$ and let K_p be the field of fractions of A_p. Then K_p is equal to the completion of K with respect to the valuation defined by $\mathcal{O}_p$ (Lemma-5).

Now, for any closed point p of X, we define

$$\mathrm{res}_p : \Omega_{K_p/k} \to k, \mathrm{res}_p(fdg) = \mathrm{res}_{A_p}^{K_p}(fdg).$$

This definition makes sense: For any generator t_p of the maximal ideal of A_p, $A_p/t_pA_p \cong k(p)$ and so is finite dimensional as a k-vector space. This can be seen by taking an affine neighorbood $U \cong \mathrm{Spec}R$ of p, with R a finitely generated k-algebra. Then $k(p) = R/m_p$ is a field that is finitely generated as a k-algebra, so that it is algebraic over k. Being finitely generated and algebraic over k, k(p) is finite dimensional over k.

Let $\varphi : t_p^i A_p/t_p^{i+1}A_p \to t_p^{i+1}A_p/t_p^{i+2}A_p$ be the k vector space homomorphism induced by multiplication by t_p (in K_p). Indeed, φ is an isomorphism because if $t_p^{i+1}a \in t_p^{i+2}A_p$, then $t_p^{i+1}(a - t_p a') = 0$ in A_p for some $a' \in A_p$, so that $a = a' t_p$ and $t_p^i a \in t_p^{i+1}A_p$. Then, for any $n \in \mathbb{Z}, {t_p}^n A_p \sim {t_p}^{n+1}A_p$ since $({t_p}^n A_p + {t_p}^{n+1}A_p)/{t_p}^n A_p$ is 0 and $({t_p}^n A_p + {t_p}^{n+1}A_p)/{t_p}^{n+1}A_p = {t_p}^n/{t_p}^{n+1}A_p$ is finite dimensional by induction. Thus, $A_p \sim t_p^n A_p$ for all $n \in \mathbb{Z}$.

Now, given $f \in K_p$, we can write $f = t_p^n u$ for some unit u of A_p. Hence, for $f \in K_p$, $fA_p = {t_p}^n A_p$ for some n, and so $fA_p < A_p$. Thus $\mathrm{res}_{A_p}^{K_p}$ is defined.

The following theorem shows that, at a k-rational point, the residue can be expressed very neatly. It provides the bridge between the residue definitions of Serre and Tate (see also the remark following the theorem).

Theorem 10. *Let p be a* k-rational point *of X (i.e. let p be such that $\mathcal{O}_p/m_p \cong k$). Then, $A_p \cong k[[t]]$ and $K_p \cong k((t))$. If $f = \sum_{\nu \ggg -\infty} a_\nu t^\nu$ and $g = \sum_{\mu \ggg -\infty} b_\mu t^\mu$ are elements of K (or K_p), then $res_p(fdg)$ equals the coefficient of t^{-1} in $f(t)g'(t)$ ($= \sum_{\mu+\nu=0} \mu a_\nu b_\mu$).*

Proof. The first statement is just Claim-1.

Let ν', μ' be the smallest values that ν and μ take in the above sums. Write

$$f = f_1 + f_2 = \sum_{\nu \ggg -\infty}^{-2\mu'} (a_\nu t^\nu) + \sum_{\nu=-2\mu'+1}^{\infty} (a_\nu t^\nu).$$

Then, $f_2 A_p + f_2 g A_p + f_2 g^2 A_p \subseteq A_p$, so by (R_2), $\mathrm{res}_{A_p}^{K_p}(f_2 dg) = 0$. Hence,

$$\begin{aligned}\mathrm{res}_p(fdg) = \mathrm{res}_p(f_1 dg) &= -\mathrm{res}_p(g df_1) \quad \text{(by the definition of abstract residues)}\\ &= -\mathrm{res}_p(g_1 df_1)\\ &= \mathrm{res}_p(f_1 dg_1),\end{aligned}$$

where f_1 and g_1 have only finitely many non-zero coefficients.

So we can assume that f and g have only finitely many non-zero terms, and then

$$fdg = f(t)g'(t)dt \in \Omega_{K_p/k}.$$

By the k-linearity of $\mathrm{res}_{A_p}^{K_p}$, and (R_3), we have

$$\begin{aligned}\mathrm{res}_{A_p}^{K_p}(f(t)g'(t)dt) = \mathrm{res}_{A_p}^{K_p}(c_{-1}t^{-1})dt &= c_{-1}\mathrm{res}_{A_p}^{K_p}(t^{-1}dt) \\ &= c_{-1}dim_k(A_p/tA_p) \quad \text{by } R_4 \\ &= c_{-1} \quad \text{(since p is a k-rational point)}\end{aligned}$$

□

Remark 5. In (II.7, [Ser]), Serre works over an algebraically closed ground field k, in which case all closed points are k-rational, and *defines* $\mathrm{res}_p(fdg)$ to be the coefficient of t^{-1}, which he later has to prove to be independent of the choice of the uniformizing parameter t of A_p. The fact that the residue does not depend on the choice of the parameter t is built into our definition.

Next, we give a convenient formula for summing up residues at various closed points of X, and finally the residue theorem follows as a simple corollary.

Theorem 11. *Let S be a subset of the closed points of X. Put*

$$O(S) = \bigcap_{p\in S} \mathcal{O}_p \subset K$$

Then for $\omega \in \Omega_{K/k}$*,*

$$\sum_{p\in S} res_p(\omega) = res_{O(S)}^K(\omega).$$

Corollary 3. *(Residue theorem)*

$$\sum_{p \text{ closed in } X} res_p(\omega) = 0$$

Proof of Corollary. Since X is projective over k, $\Gamma(X, \mathcal{O}_X)$ is finite dimensional as a k-vector space. Since X is an integral scheme, the set of elements of K(X) that are defined globally is just the set of elements that are in every local ring of X, i.e. O(X). Hence $O(X) \sim (0)$, and we have $\mathrm{res}_{O(X)}^K = 0$ by (R_1). □

Proof of Theorem. Define

$$A_S = \prod_{p\in S} A_p$$

$$V_S = \{(f_p)_{p\in S} | f_p \in A_p \text{ for almost all p}\} \subseteq \prod_{p\in S} K_p$$

The case $S = \emptyset$ is trivially true: $\sum_{p\in S} res_p(\omega) = 0 = \mathrm{res}_K^K = \mathrm{res}_{O(S)}^K$.

We have $K \hookrightarrow V_S$ defined by $f \mapsto (f)$. This is well defined, as $f \in A_p$ for all but finitely many $p \in S$: $K \cap A_p = \mathcal{O}_p$, with the intersection taken in K_p, so it suffices to show $f \in \mathcal{O}_p$ for all but finitely many $p \in S$. Let U be an open subset of X on which f is defined. Then $f \notin \mathcal{O}_p \Rightarrow p \in X - U$, but $X - U$ is finite.

Noting $A_S \subseteq V_S$ and $K \cap A_S = \{f \in K | f \in A_p \text{ for all } p \in S\} = O(S)$, we have the following picture:

$$\begin{tikzcd}[column sep=small]
 & V_S \arrow[d, dash] & \\
 & K + A_S \arrow[dl, dash] \arrow[dr, dash] & \\
K \arrow[dr, dash] & & A_S \arrow[dl, dash] \\
 & K \cap A_S \arrow[d, dash] & \\
 & (0) &
\end{tikzcd}$$

This translates into our notation for abstract residues with:

$V_S \sim V, A \sim K, B \sim A_S$

We let V_S have the obvious K-module structure. For $f \in K$, $fK \subseteq K$ and $fA_S = \prod_{p\in S} fA_p \subseteq \prod_{f\in\mathcal{O}_p} A_p \times \prod_{f\notin\mathcal{O}_p} fA_p$. Since there are only finitely many $p \in S$ for which $f \notin \mathcal{O}_p$, and $fA_p < A_p$, we have $fA_S < A_S$.

By (R_5), we have:

$$\mathrm{res}_{A_S}^{V_S} + \mathrm{res}_K^{V_S} = \mathrm{res}_{O(S)}^{V_S} + \mathrm{res}_{K+A_S}^{V_S}.$$

But $\mathrm{res}_K^{V_S} = 0$ by (R_2), and $\mathrm{res}_{K+A_S}^{V_S} = 0$ by (R_1), since $V_S \sim (K + A_S)$ ($V_S/(K + A_S)$ is finite dimensional by Corollary-4 below). Thus,

$$\begin{aligned}
\mathrm{res}_{A_S}^{V_S} &= \mathrm{res}_{O(S)}^{V_S} \\
&= \mathrm{res}_{O(S)}^{K} \quad \text{by } (R_1)
\end{aligned}$$

It only remains to show $\mathrm{res}_{A_S}^{V_S}(\omega) = \sum_{p\in S} \mathrm{res}_p(\omega)$. Fix $\omega = fdg \in \Omega_{K/k}$; let $S' = \{p \in S | f \notin \mathcal{O}_p \text{ or } g \notin \mathcal{O}_p\}$ be the set of poles of f or g, and let $T = S - S'$.

Since S' is finite, we have

$$V_S = V_T \times \prod p \in S' K_p,$$
$$\text{and } A_S = A_T \times \prod p \in S' A_p$$

For any $f \in K$, $f \cdot A_T \times \{0\} < A_T \times \{0\}$ and $f \cdot A_{S'} \times \{0\} < A_{S'} \times \{0\}$, so we can apply (R_5) to get

$$\text{res}^{V_S}_{A_S}(fdg) + \text{res}^{V_S}_{A_S}(fdg) = \text{res}^{V_S}_{A_T + A_{S'}}(fdg) + \text{res}^{V_S}_{A_T \cap A_{S'}}(fdg).$$

Obviously $A_T + A_{S'} = A_S$, $A_T \cap A_{S'} = 0$, and $\text{res}^{V_S}_{A_T} = 0$ (since $fA_T \subseteq A_T$ and $gA_T \subseteq A_T$). Hence,

$$\begin{aligned}
\text{res}^{V_S}_{A_S}(fdg) &= \text{res}^{V_S}_{A_{S'}}(fdg) = \text{res}^{V_{S'}}_{A_{S'}}(fdg) \\
&= \text{res}^{V_{S'}}_{A_q}(fdg) + \text{res}^{V_{S'}}_{A_{S'-\{q\}}}(fdg) - \text{res}^{V_{S'}}_{0}(fdg) \quad \text{by } (R_5) \\
&= \text{res}^{K_q}_{A_q}(fdg) + \text{res}^{V_{S'-\{q\}}}_{A_{S'-\{q\}}}(fdg) \\
&= \text{res}_q(fdg) + \text{res}^{V_{S'-\{q\}}}_{A_{S'-\{q\}}}(fdg) \\
&= \dots \\
&= \sum_{q \in S'} \text{res}_q(fdg) \\
&= \sum_{p \in S} \text{res}_p(fdg) \quad (\text{for } p \notin S',\ \text{res}_p(fdg) = 0 \text{ since } fA_p, gA_p \subseteq A_p)
\end{aligned}$$

□

To show that $V_S/(K + A_S)$ is indeed finite dimensional, we prove a slightly more general lemma that we will need later. Let $V = V_X$ (where X stands for all *closed* points of X). Then, define

$$V(D) = \{(f_p) \in V | \text{ord}_p f_p \geq -\text{ord}_p D \text{ for all } p \in X\},$$

and the invertible $\mathcal{O}_X$-module $\mathcal{O}_X(D)$ by

$$[\mathcal{O}_X(D)](U) = \{f \in K | \nu_q(f) \geq -\nu_q(D) \text{ for all } q \in U\}$$

for any open subset U of X, as before (see Thm.-9).

Lemma 15. $H^1(X, \mathcal{O}_X(D)) \cong V/(K + V(D))$

Proof. Let $\mathcal{G}$ denote the cokernel of $\mathcal{O}_X(D) \hookrightarrow \underline{K(X)}$. Consider the exact sequence

$$0 \to \mathcal{O}_X(D) \to \underline{K(X)} \to \mathcal{G} \to 0,$$

which induces the long exact sequence

$$0 \to H^0(X, \mathcal{O}_X(D)) \to H^0(X, \underline{K(X)}) \to H^0(X, \mathcal{G})$$
$$\to H^1(X, \mathcal{O}_X(D)) \to H^1(X, \underline{K(X)}) \to \dots$$

Since X is irreducible, $H^0(X, \underline{K(X)}) = K(X)$. On an irreducible scheme any constant sheaf is flasque, and hence has zero homology groups in dimensions greater than 0, so $H^1(X, \underline{K(X)}) = 0$ (Lemma-7). Hence

$$H^1(X, \mathcal{O}_X(D)) \cong H^0(X, \mathcal{G})/K(X).$$

$\mathcal{O}_X(D)_p$ is just the sub-module of K of elements with valuation greater than or equal to $-ord_p(D)$. So,

$$\bigoplus_p \mathcal{G}_p = \bigoplus_p K(X)/\mathcal{O}_X(D)_p = V/V(D),$$

and all that remains to do is to show that the global sections of $\mathcal{G}$ is a direct sum of its stalks at its closed points. Let $\mathcal{G}'_p$ be the *skyscraper sheaf* defined by $\mathcal{G}'_p(U) = 0$ if $p \notin U$ and $\mathcal{G}'_p = \mathcal{G}_p$ if $p \in U$. We first define a map $\psi_p : \mathcal{G}'_p \to \mathcal{G}$:

Given $g_p \in (\mathcal{G}'_p)_p = \mathcal{G}_p$, g_p comes from some section $g' \in \mathcal{G}(U)$, whose restriction to some smaller neighorhood V' of p must be an element of $K(X)/[\mathcal{O}_X(D)](V')$. Let $g \in K(X)$ be a representative for this element. Let V be the subset of V' obtained by throwing away all points where D is nonzero and all poles of g, and adding back p if necessary (both sets that are thrown away are finite, so V is open). Now, for $g_p \in \mathcal{G}'_p(W)$, $\psi_p(g_p)$ is the element of $\mathcal{G}(W)$ whose restriction to $V \bigcap W$ is g, and restriction to $W - \{p\}$ is 0.

Next, taking the direct sum of the ψ_p we have a map

$$\bigoplus_p \mathcal{G}'_p \to \mathcal{G},$$

that is obviously an isomorphism at each stalk. Hence $\Gamma(X, \mathcal{G}) = \bigoplus_p \mathcal{G}_p$ as wanted. □

Corollary 4. *For any subset S of the closed points of X, $V_S/(K + A_S)$ is finite dimensional.*

Proof. The projection $V_X/(K+A_X) \to V_S/(K+A_S)$ is surjective, so it suffices to show this for S = X. But $V_X/(K+A_X) \cong H^1(X, \mathcal{O}_X)$ by taking D=0 in the lemma. Since X is projective over k, $H^1(X, \mathcal{O}_X)$ is finite dimensional. □

The final theorem of this section is the same result Serre utilizes to generalize (by means of base extensions) his manual calculation of residues on the projective line ([Ser], Chpt. 2, Sect. 3).

Theorem 12. *Let $X' \to X$ be a surjective morphism inducing the inclusion of function fields $K(X) = K \hookrightarrow K' = K(X')$.*

i: *For $f' \in K', g \in K$, and $p \in X$,*

$$\sum_{p' \mapsto p} res_{p'}(f'dg) = res_p((Tr_{K'/K} f)dg).$$

ii: *For $p' \in X'$ mapping to $p \in X$, $f' \in K'_{p'}$, and $g \in K_p$,*

$$res_{p'}(f'dg) = res_p((Tr_{K'_p/K_p} f')dg).$$

Proof. **i:**

$$\begin{aligned}
\sum_{p' \mapsto p} \mathrm{res}_{p'}(f'dg) &= \mathrm{res}^{K'}_{\bigcap_{p' \mapsto p} \mathcal{O}_{p'}}(f'dg) \\
&= \mathrm{res}^{K'}_{\bigcap_{\{\mathcal{O}_{p'} \text{ dominates } \mathcal{O}_p\}} \mathcal{O}_{p'}}(f'dg) \quad \text{by Lemma-8} \\
&= \mathrm{res}^{K'}_{\bigcap_{\{R \text{ val. ring. dominating } \mathcal{O}_p\}} R}(f'dg) \\
&= \mathrm{res}^{K'}_B(f'dg) \quad (\text{with B the integral closure of } \mathcal{O}_p \text{ in } K' \text{ by Thm.-3}) \\
&= \mathrm{res}^{K}_{\mathcal{O}_p}((\mathrm{Tr}_{K'/K} f')dg) \quad \text{by (R6)}
\end{aligned}$$

Here is how we were able to apply (R6) in the last line: B is a finitely generated free $\mathcal{O}_p-$module (finitely generated since it is the integral closure of the finitely generated k-algebra $\mathcal{O}_p$, and free since it is torsion free over the PID $\mathcal{O}_p$), say $x_1, \ldots, x_r$ is an $\mathcal{O}_p-$basis for B. Since localization commutes with taking the integral closure (Lemma-3), $K \cdot B = K'$, and $x_1, \ldots, x_r$ is also a K-basis for K' . Thus in the notation of (R_6), $A' = \sum_i x_i \otimes \mathcal{O}_p = B$ and $V' = K' \otimes_K K = K'$.

ii: This follows exactly as in (i), since the integral closure of A_p in $K'_{p'}$ is just $A'_{p'}$.

□

This concludes our treatment of residues. The next section is devoted to the application of these results to exhibit an explicit *dualizing sheaf* in the case of smooth curves.

5. Duality

In this section, we first construct a natural "dualizing sheaf" $J_{X/k}$ for a regular curve proper over an arbitrary field k. We show that it is an invertible sheaf, and has the desired properties. However, the dualizing sheaf is somewhat contrived, is not a familiar object at all. So, we employ the residue theorem to show that the dualizing sheaf is actually the same as $\Omega_{X/k}$ in the case of smooth curves.

To begin with, let $J(D) = \{\lambda \in HOM_k(V,k) | \lambda(K + V(D)) = HOM_k(V/(K + V(D)), k)$ $(\cong H^1(X, \mathcal{O}_X(D))^\vee$ by Lemma-15).

We have $D \leq D' \Rightarrow V(D) \subseteq V(D') \Rightarrow J(D') \subseteq J(D)$. Given any two divisors, there is another divisor that is smaller than both of them, so $\bigcup_D (J(D))$ as D runs through all divisors D of $(X, \mathcal{O}_X)$ is a k-module. Let $J_{K/k} = \bigcup_D (J(D))$, and $J_p = \{\lambda \in J_{K/k} | \lambda(A_p) = 0\}$. Finally, for each open $U \subseteq X$, define $J_{X/k}(U) = \bigcap_{p \in U} J_p \subseteq J_{K/k}$. The stalk of $J_{X/k}$ at each closed point p is J_p, and its generic stalk is $J_{K/k}$.

Now, $J_{X/k}$ has a natural $\mathcal{O}_X -$ module structure given by $f \cdot \lambda(v) = \lambda(f \cdot v)$, where $v \in V$, $f \in \mathcal{O}_X(U)$, and $\lambda \in J_{X/k}(U)$. Since $f \in \mathcal{O}_X(U)$, $f \cdot A_p \subseteq A_p$ and so $f \cdot \lambda(A_p) = 0$ for all $p \in U$. Also, $f(K + V(D)) = K + V(D - (f))$, so $f \cdot \lambda \in J(D - (f))$. Hence, $f \cdot \lambda \in J_{X/k}(U)$.

That $J_{X/k}$ has the dualizing property is a simple consequence of Lemma-15:

$$H^0(X, J_{X/k}(-D)) = H^0(X, J_{X/k} \otimes \mathcal{O}_X(-D)) = HOM_k(V/(K + V(D), k))$$
$$\cong H^1(X, \mathcal{O}_X(D))^\vee$$

The following lemma will help prove that $J_{X/k}$ is an invertible sheaf.

Lemma 16. *$J_{X/k}$ is coherent.*

Proof. Let $U = \mathrm{Spec}A$ be some open affine subset of X, and $M = J_{X/k}(U)$. We show that for any distinguished open subset $D(f)$ of $\mathrm{Spec}A$ $J_{X/k}(D(f)) = M_f = A_f M$ (where the last term is an A_f submodule of $J_{K/k}$).

If $\lambda \in J_{X/k}(D(f))$, then $\lambda(A_p) = 0$ for all $p \in U$ satisfying $f \notin m_p\mathcal{O}_p$. There are finitely many points in $U - D(f)$, and $v_q(f) \gneq 0$ for each $q \in U - D(f)$. Now, since $\lambda(V(D)) = 0$ for some divisor D, we have $\lambda({t_q}^{n_q} A_q) = 0$ for some integer n_q for each q. Let n be the maximum of these n_q. Then $f^n A_q \subseteq {t_q}^n A_q \subseteq {t_q}^{n_q} A_q$ for each q, so that $f^n \cdot \lambda(A_q) = \lambda(f^n A_q) = 0$. Hence $J_{X/k}(D(f)) \subseteq A_f M$. The converse follows immediately, since $M \subseteq J_{X/k}(D(f))$ and $J_{X/k}$ is an $\mathcal{O}_X$-module. □

Now that we know it is coherent, showing $J_{X/k}$ to be invertible is essentially a matter of considering dimensions. This is where the rudimentary Riemann-Roch theorem (Prop.-6) plays its part.

Proposition 9. *$J_{X/k}$ is an invertible $\mathcal{O}_X$-module.*

Proof. It suffices to show that J_p is a free $\mathcal{O}_p$-module of rank 1 for each point p. We start with the generic stalk showing $dim_K(J_{K/k}) = 1$, and the rest will follow easily.

Step 1. $dim_K(J_{K/k}) \leq 1$.

We suppose α, α' are two linearly independent elements of $J_{K/k}$, and reach a contradiction. First, we must have $\alpha, \alpha' \in J(D)$ for some D, since given any two divisors D_1, D_2 we can always find some divisor D smaller than both of them, so that $J(D_1) \subseteq J(D)$ and $J(D_2) \subseteq J(D)$. Let Δ_n be a divisor of degree n.

Given $f \in H^0(X, \mathcal{O}_X(\Delta_n))$, consider $f \cdot \alpha \in J_{K/k}$. If $x \in V(D+(f))$(or equivalently $fx \in V(D)$), we have $0 = \alpha(fx) = f \cdot \alpha(x)$, so $f \cdot \alpha \in J(D + (f))$. But $-\Delta_n \leq (f)$, so $f \cdot \alpha \in J(D - \Delta_n)$. Now, we have a mapping

$$H^0(X, \mathcal{O}_X(\Delta_n)) \times H^0(X, \mathcal{O}_X(\Delta_n)) \to J(D - \Delta_n), (f, g) \mapsto f\alpha + g\alpha'$$

This map is injective by the assumption that α, α' are linearly independent over K, and we must have

$$dim_k J(D - \Delta_n) \geq 2 \cdot h^0(X, \mathcal{O}_X(\Delta_n)) \quad (*)$$

By Lemma-15, $dim_k J(D - \Delta_n) = h^1(\mathcal{O}_X(D - \Delta_n))$. By Prop.-6, this last quantity equals $h^0(\mathcal{O}_X(D - \Delta_n)) + h^1(\mathcal{O}_X) - deg(D - \Delta_n) - 1$. If $deg(D - \Delta_n)$ is less than 0, then $H^0(X, \mathcal{O}_X(D - \Delta_n)) = 0$ (Corollary to Prop.-6), so for n large enough the left hand side of (*) equals $deg(\Delta_n) + c_0 = n + c_0$ (where $c_0 = h^1(\mathcal{O}_X) - deg(D) - 1$ does not depend on n).

On the other hand $h^0(\mathcal{O}_X(\Delta_n)) = h^1(\mathcal{O}_X(\Delta_n)) + deg(\Delta_n) + c_1$, where $c_1 = 1 - h^1(\mathcal{O}_X)$ does not depend on n. But then the right hand side of (*) is at least $2n + 2c_1$, so that by (*) $n + c_0 \geq 2n + 2c_1$, which cannot be true for very large n.

Step 2. $dim_K J_{K/k}$ is exactly 1.

It suffices to exhibit a nonzero element of $V/(V(D) + K)$ for some divisor D, or equivalently that $V(D) + K \neq V$. Let $D = -2 \cdot p$ for some closed point p and take $(f_q) \in V$ with $f_q = 1$ for $q = p$ and $f_q = 0$ for $q \neq p$. Suppose we can write $(f_q) = (g_q) + h$ with $(g_q) \in V(D)$ and $h \in K$. $v_p(g_p + h) = v_p(f_p) = 1$ and $v_p(g_p) \geq 2$, so we must have $v_p(h) = 1$. But as we noted before the degree of the principal divisor defined by h must be 0, so we must have $v_q(h) < 0$ for some q. Then $v_q(g_q + h) < 0 \leq v_q(f_q)$, contradicting $(f_q) = (g_q) + h$.

Step 3. For each closed point p, J_p is a free $\mathcal{O}_{X,p}$ module of rank 1.

It will suffice to show J_p is generated by a single element over $\mathcal{O}_{X,p}$ (because $\mathcal{O}_{X,p} \subseteq K$, $J_p \subseteq J_{K/k}$, and $J_{K/k}$ is a K-vector space). Let $\lambda \in J_p - t_p J_p$. For any $\varphi \in J_p$, we have $f \cdot \lambda = \varphi$ for some $f \in K$, since $J_{K/k}$ is a 1-dimensional K-vector space. Suppose $f \notin \mathcal{O}_p$. Then ${t_p}^{-1} A_p \subseteq f A_p$ and $0 = \varphi(A_p) = f \cdot \lambda(A_p) = \lambda(f A_p)$, so that $\lambda({t_p}^{-1} A_p) = 0$. But then $t_p \cdot \lambda \in J_p$, which is a contradiction.

□

Remark 6. Step-2 of the proof of the proposition is in line with Chevalley's approach in ([Che], Chpt.2). The question he investigates is: given $x_p \in K$ and integers n_p for each valuation ring R_p of K/k, when can we find $x \in K$ satisfying $v_p(x - x_p) \geq n_p$ for each p. Then he restricts to the case where $v_p x_p \geq 0$ for all but finitely many p and $n_p = 0$ for all but finitely many p, so that the question becomes about $V/(V(D) + K)$. Then using purely algebraic arguements he investigates the dimension of $V/(V(D)+K)$ over k and thereby proves a form of the Riemann-Roch theorem.

The fact that X is proper is essential to Step-2. If we had $X = \mathrm{Spec} A$ for a Dedekind domain A for instance, the approximation lemma assures that given finitely many $x_p \in A_p$ and integers n_p, one can find $x \in K$ such that $v_p(x - x_p) \geq n_p$ for each p and $v_q(x) \geq 0$ for all other prime ideals of A ([Ser2], p.12), which shows that $V/(K + V(D)) = 0$, making $J_{K/k}$ trivial. In the proper case a similar result holds, but without the guarantee $v_q(x) \geq 0$ ([Che], Chpt.1, Thm.3), which would be impossible because of the fact that $deg x = 0$ (see Remark-3).

Finally, we can relate $J_{X/k}$ with $\Omega^1_{X/k}$. Consider the canonical map

$$c : \Omega^1_{K/k} \to J_{K/k}$$

taking $w \in \Omega^1_{K/k}$ to the linear map (cw) defined by

$$(cw)(f) =< f, w >= \sum_{p \in X} \mathrm{res}_p(f_p w) \quad \text{for all } f \in V.$$

The image of c is indeed contained in $J_{K/k}$: By Corollary-3, $(cw)(K) = 0$. Write $w = hdg$ with $h, g \in K$. Define a divisor D by:

$$ord_p D = max\{v_p(h), v_p(h) + v_p(g), v_p(h) + 2v_p(g)\} \text{ for each p}$$

Then $v_p(f_p h), v_p(f_p hg), v_p(f_p hg^2)$ are all nonnegative, so $(f_p h + f_p hg + f_p hg^2)A_p \subseteq A_p$ and by (R_2) $\mathrm{res}_{A_p}(f_p hdg) = 0$ for all p. Hence $(cw)(K + V(D)) = 0$, and $(cw) \in J_{K/k}$.

Note that we can easily glue morphisms that extend this homomorphism to whole open sets of X: If φ_1 and φ_2 are homomorphisms on U_1 and U_2 that agree at the generic stalk, then $\varphi_1 - \varphi_2|_{U_1 \bigcap U_2}$ induces the 0 map on the generic stalk, so that $\varphi_1|_{U_1 \bigcap U_2} - \varphi_2|_{U_1 \bigcap U_2}$ is itself the 0 map.

Now, we show that c can be extended to a morphism on any open affine subset, say $U = \mathrm{Spec} R$. $\Omega^1_{X/k}|_U \cong (\Omega^1_{R/k})^{\sim}$, so we only need to give an R-module homomorphism $\Omega^1_{R/k} \to J_{X/k}(U) = \{\lambda \in J_{K/k} | \lambda(A_p) = 0 \text{ for all } p \in U\}$ that agrees with c. The restriction of c to $\Omega^1_{R/k}$ works: For $w = hdg \in \Omega^1_{R/k}$ and $(f) \in A_p$ (i.e. $f_q = 0$ for $q \neq p$ and $f_p \in A_p$), we have $(cw)(f) = \mathrm{res}_p(f_p w) = 0$ ($f_p hA_p, f_p gA_p \subseteq A_p$, as h and g are both regular at p).

In the case of smooth curves the canonical map c is actually an isomorphism, as the next theorem asserts.

Theorem 13. *Suppose X is smooth over k (the residue fields k(p) at the closed points are separable over k and K(X) is separably generated over k). Then the homomorphism $c : \Omega^1_{X/k} \to J_{X/k}$ is an isomorphism.*

Proof. At any closed point p, J_p and Ω^1_p are free $\mathcal{O}_p$-modules of rank 1, with Ω^1_p generated by dt_p for some uniformizer t_p of $\mathcal{O}_p$ and J_p generated by any element not in $t_p J_p$. For $\lambda \in J_p$, $\lambda \notin t_p J_p$ if and only if ${t_p}^{-1}\lambda \notin J_p$, if and only if $\lambda({t_p}^{-1} A_p) \neq 0$.

We show that $c(dt_p)$ has this property:

$\lambda = c(dt_p)$ is defined by

$$\lambda((f_q)) = \sum_{q \in X} \mathrm{res}_q(f_q dt_q).$$

Let $(f_q) = u{t_p}^{-1}dt_p$ when $q = p$, and $(f_q) = 0$ when $q \neq p$, where u is some unit of A_p to be determined. Then,

$$\begin{aligned}[c(dt_p)]((f_q)) = \mathrm{res}_p(u{t_p}^{-1}dt_p) &= \mathrm{res}_{A_p}(u{t_p}^{-1}dt_p) \\ &= \mathrm{Tr}_{A_p/(A_p \bigcap t_pA_p)}(u) - \mathrm{Tr}_{t_pA_p/(A_p \bigcap t_pA_p)}(u) \quad ((R4)) \\ &= \mathrm{Tr}_{A_p/(t_pA_p)}(u) = \mathrm{Tr}_{k(p)/k}(u)\end{aligned}$$

Now, we can find $u \in k(p)$ such that $\mathrm{Tr}_{k(p)/k}(u) \neq 0$ by Lemma-4 (strictly speaking, the lemma tells us we can find such $u' \in k(p) = A_p/t_pA_p$, and then we can pick any u whose image is u').

Since $\Omega_{K/k}$ and $J_{K/k}$ are 1-dimensional vector spaces and c is nonzero, it is an isomorphism at the generic point as well. □

Corollary 5. *If X/k is smooth (eg. if k is perfect), then for any invertible sheaf $\mathcal{L}$ on X, we have*

$$H^1(X, \mathcal{L}) \cong H^0(X, \Omega^1_{X/k} \otimes \mathcal{L}^\vee)^\vee.$$

Proof. $\mathcal{L} \cong \mathcal{O}_X(D)$ for some divisor D, and so

$$\begin{aligned}H^0(X, \Omega^1_{X/k} \otimes \mathcal{L}^\vee) &= H^0(X, J_{X/k} \otimes \mathcal{O}_X(-D)) \\ &= Hom_k(V/(K + V(D)), k) \cong H^1(X, \mathcal{O}_X(D))^\vee = H^1(X, \mathcal{L})^\vee\end{aligned}$$

□

6. Riemann-Roch

At this point, we have concluded our main mission. We have proved the residue theorem using Tate's elegant method, and then used it to show that the dualizing sheaf $J_{X/k}$ is the same thing as the sheaf of differentials for smooth curves.

In finishing, we prove a stronger form of the Riemann-Roch theorem to demonstrate the power of duality. For some related applications, see ([Har], IV.1).

As before, let X be a regular proper curve over some field k. In addition, let X be smooth over k. Since $\Omega_{X/k}$ is an invertible $\mathcal{O}_X$-module, $\Omega_{X/k} \cong \mathcal{O}_X(D)$ for some divisor D. It is not necessarily unique, but we call any divisor in its equivalance class (in the group $DivX$ divided by the principal divisors of X) a *canonical divisor*, and denote it by K. Note that $\mathcal{O}_X(K)$ and $degK$ are well-defined since $\mathcal{O}_X(K+(f)) \cong \mathcal{O}_X(K) \otimes_{\mathcal{O}_X} \mathcal{O}_X((f)) \cong \mathcal{O}_X(K) \otimes_{\mathcal{O}_X} \mathcal{O}_X \cong \mathcal{O}_X(K)$ and $deg((f)) = 0$ for any principal divisor (f).

Definition 14. The *genus* of the curve X is $g = h^0(\Omega_{X/k})$.

Theorem 14. *Let D be a divisor on the curve X. Then,*

$$h^0(\mathcal{O}_X(D)) - h^0(\mathcal{O}_X(K-D)) = deg(D) + h^0(\mathcal{O}_X) - g$$

Proof. $\mathcal{O}_X(K-D) \cong \mathcal{O}_X(K) \otimes \mathcal{O}_X(-D) \cong \Omega_{X/k} \otimes \mathcal{O}_X(-D)$, and by Corollary-5, $h^0(\Omega_{X/k} \otimes \mathcal{O}_X(-D)) = h^1(\mathcal{O}_X(D))$. Now, the equation we wanted to prove is equivalent to

$$h^0(\mathcal{O}_X(D)) - h^1(\mathcal{O}_X(D)) = deg(D) + h^0(\mathcal{O}_X) - g$$

But $g = h^0(\Omega_{X/k}) = h^0(J_{X/k}) = dim_k(Hom_k(V/(K + V(0)))) = h^1(\mathcal{O}_X)$ by Lemma-15, so we are reduced to proving

$$\chi(\mathcal{O}_X(D)) = deg(D) + \chi(\mathcal{O}_X),$$

which is just Proposition-6. $\square$

Remark 7. In case k is algebraically closed, $h^0(\mathcal{O}_X) = 1$, so the equation simplifies to $h^0(\mathcal{O}_X(D)) - h^0(\mathcal{O}_X(K-D)) = deg(D) + 1 - g$.

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